Two Price Economic Equilibria and Financial Market Bid/Ask prices*

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Abstract

Demand and supply uncertainty lead to a model of markets that set prices to acceptable risk levels for excess supplies and net revenues. The result is a two price partial equilibrium economy. The equilibrium solutions are applied to two price financial market data to infer demand and supply elasticities and log normal volatilities from market quotes on bid and ask prices. Demand elasticities are observed to be higher than supply elasticities as are the volatilities. Normalizing observed volatilities to the volatility of the daily traded volume a market implied duration of the economic equilibrium is inferred. The median level of duration is around a minute and half with an interquartile range from 37 seconds to three minutes. For larger orders, bid and ask prices may be constructed by calibrating the demand and supply volatilities.

Keywords: Acceptable Risks, Distorted Expectations, Minmaxvar Distortion, Convex Risk Measures.

JEL Classification: G10, D53, D58.

1 Introduction

Bid and ask prices have been attributed in the literature to aberrations preventing the ideal formulation of the law of one price assumed to prevail in any equilibrium. The departures from a fundamental law of one price may be explained in different ways. They could result as a consequence of transactions costs as studied in Constantinedes (1986), Jouini and Kallal (1995), Lo, Mamaysky and Wang (2004). The classical model is free of such costs. Alternatively they could reflect the impact of informed traders on market makers (Copeland and Galai (1983), Easley and O'Hara (1987), Glosten and Milgrom (1985), Back and Baruch (2004)). In the classical model all traders are equally and fully informed. Transactions costs were expanded to include order processing and inventory costs in Demsetz (1968), Stoll (1978), Ho and Stoll (1981, 1983). Mean returns and bid-ask spreads are related via liquidity considerations in Ahimud and Mendelson (1986). Once again the classical equilibrium model is free of these considerations.

Madan (2015) approaches the two prices from a no arbitrage perspective, as an alternative to an equilibrium solution. The two prices arise naturally as nonlinear functions of promised outcomes when the set of zero cost traded claims are not closed under negation. Madan (2020) questions the validity of the classical equilibrium model on recognizing that demands, supplies and excess demands at any price are random variables that cannot be equated to zero to solve for an equilibrium price. The market is then modeled as the single risk absorbing agent ensuring that supplies tend to exceed demands and revenues from demanders tend to exceed the expenses associated with payments to suppliers. Importantly, market participants are then not concerned with risks, as their concerns are fully met and are free of risk considerations. The market modeled as a clearing agent is solely subject to risk and operates with a degree of conservatism. There are then two equations for conservatively positive excess supplies and net revenues. These two equations when solved simultaneously, determine the two prices of a two price equilibrium. These are an upper or ask price at which the market sells to demanders and a lower or bid price at which the market buys from suppliers. There are no transactions or processing costs and no ill informed market participants. Madan (2020) considers both a one good partial equilibrium and a more general equilibrium for an economy with many goods. The objective here is to relate the one good partial equilibrium model to data on financial market bid and ask prices.

The data are obtained from the Wharton Research Data Service (WRDS) and consist of closing bid and ask prices for 874 equity assets, across the sectors of the US economy, for 1257 days from January 5, 2015 to December 31, 2019. The risky one good partial equilibrium model introduced in Madan (2020) develops equations for bid and ask prices related to log normal demand and supply uncertainties in the context of constant elasticity demand and supply functions. In general the demand and supply uncertainties may be correlated but in the initial investigation undertaken here they are assumed uncorrelated. The two prices are then deterministic functions of the levels and elasticities of demand and supply, along with their volatilities. The prices also depend on the degree of market conservatism embedded in the modeling of a risky equilibrium. The conservatism is solely that of the market as it is the only agent facing risk. Participants need not have risk attitudes as they do not face risk. They are made whole by the market. The idea behind the risky equilibrium is to aggregate all risk at the level of the market. Economic agents are as they were in a classical Arrow Debreu risk free equilibrium. As a consequence, there is a sharp dichotomy between economic agents and the market. The former is risk free the latter must cope with and manage all the risk.

The outline of the rest of the paper is as follows. Section 2 summarizes the risky partial equilibrium model of Madan (2020). Section 3 goes further into the equilibrium structure and comments on the relationship between mean returns and spreads from a different perspective than liquidity. Section 4 presents the estimation strategy adopted in the empirical exercise. The data is presented in Section 5. A subsample of results is reported in Section 6 as the estimation is too costly to be conducted on a large scale. The results on the equilibrium duration are presented in Section 7. Section 8 applies the model to generate lower and upper prices for longer durations. Section 9 concludes.

2 Risky Partial Equilibria

Classically a partial equilibrium is formulated as a price, ceteris paribus, at which markets clear or demand equals supply. Madan (2020) introduces a model in which both demand and supply are positive random variables at each price level. As a consequence they may not be equated. There is always a probability that demand may exceed the supply or fall short of it, over the period for which one is attempting to determine the equilibrium price. Let $D(p)$ be the demand curve, typically taken to be inversely related to the price p, while $S(p)$ is a supply curve positively related to the price. When the demands and supplies are viewed as such deterministic functions of the price the equilibrium price may be determined by equating demand to supply. However, suppose now that there are positive random perturbations M_D, M_S of unit expectations such that the actual demands D and supplies S are given by

$$\begin{aligned} D &= D(p)M_D \\ S &= S(p)M_S. \end{aligned}$$

Madan (2020) assumed illustratively that M_D, M_S were correlated log normal variates and this formulation will be considered later. In general suppose that there is a randomness in both entities, demand and supply, given the price level. By construction now, the demand and supply curves represent the expected levels for demand and supply. At every price level there is an exposure to the risk of demand exceeding or falling short of supply. Hence the concept of a risky partial equilibrium.

In such a context, Madan (2020) introduces the market as a risk absorbing counterparty that sells to demanders at a price p_U and buys from sellers at a price p_L. The excess supply Z is then the random variable

$$Z = S(p_L)M_S - D(p_U)M_D.$$

In addition to this excess supply the market will experience a random net revenue of

$$R = p_U D(p_U)M_D - p_L S(p_L)M_S.$$

The gaps in demand and supply for the good in question, the market may meet by or take into cost free inventory. The same applies to net revenues that are fluctuations in the market's cash reserves. The market, viewed now as an abstract entity interested in market clearing has an interest in the positivity of Z, R. They are risks to held by the market and hence may be modeled by requiring them to be acceptable using a theory of acceptable risks.

Risk acceptability has been modeled quite generally in Artzner, Delbaen, Eber and Heath (1999). Strict nonnegativity is clearly acceptable but too severe and possibly unattainable in practice. More generally, acceptable risks are random outcomes belonging to a convex cone containing the cone of nonnegative outcomes. Clearly positive scalar multiples of nonnegative outcomes are nonnegative, and hence these outcomes have a conic structure. They are also a convex set of outcomes. The more general acceptable risks are then just a larger convex cone permitting some negative exposures as risks that can be taken. The degree of negative exposures that are acceptable depends on the size of the cone of acceptable risks. Artzner, Delbaen, Eber and Heath (1999) show that any set $\mathcal{A}$ of acceptable risks $X \in \mathcal{A}$ is associated with a convex set $\mathcal{M}$ of probability measures $Q \in \mathcal{M}$ such that

$$X \in \mathcal{A} \Longleftrightarrow E^Q[X] \geq 0, \text{ for all } Q \in \mathcal{M},$$

where $E^Q[X]$ is the expectation of X under the probability Q. The set $\mathcal{M}$ of probabilities associated with $\mathcal{A}$ is referred to as the set of supporting measures for acceptability in $\mathcal{A}$.

Acceptable risks were related to distorted expectations in Kusuoka (2001) under two further assumptions about risk acceptability. The first, termed law invariance, required the acceptability of a risk to be defined purely in terms of the distribution function of the risk. The second requires risk valuations

$$V(X) = \inf_{Q \in \mathcal{M}} E^Q[X]$$

to satisfy comonotone additivity, whereby $V(X+Y) = V(X)+V(Y)$, when X, Y are comonotone. Comonotone risks are risks with no negative comovements. Under these assumptions, there exists a concave distribution function $\Psi(u)$, $0 \leq u \leq 1$ with the property that X with distribution function $F_X(x)$ is acceptable or $X \in \mathcal{A}$ only if

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} x d\Psi(F_X(x))dx \geq 0.$$

The operator $\mathcal{E}(X)$ is an expectation of X under the distorted distribution function $\Psi(F_X(x))$ and is termed a distorted expectation. Let P be the original probability measure under which $F_X(x)$ is defined. The set of supporting measures $\mathcal{M}$ is all measures Q satisfying

$$Q(A) \leq \Psi(P(A)), \text{ for all } A.$$

Assuming the differentiability of Ψ, the distorted expectation is also an expectation under a change of probability $\Psi'(F_X(x))$ as

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} x\Psi'(F_X(x))dF_X(x).$$

whereby losses are reweighted upwards while gains are reweighted down by Ψ'.

Following these ideas, for two concave distortions Ψ_Z and Ψ_R let $\mathcal{E}_Z$, $\mathcal{E}_R$ be the associated distorted expectations. The acceptability of the risks Z, R for the market are then defined by

$$\mathcal{E}_Z(Z) = 0 \quad (1)$$

$$\mathcal{E}_R(R) = 0. \quad (2)$$

A risky two-price partial equilibrium is then a pair of prices (p_U, p_L) satisfying the two equations (1) and (2).

Proposition 1 *The two price equilibrium* (p_U, p_L) *satisfies* $p_L \leq p_U$.

Proof. As expectations exceed distorted expectations it follows from equations (1) and (2) that

$$E^Q[S] \geq E^Q[D], \ Q \in \mathcal{M},$$

and that

$$E^Q[p_U D] \geq E^Q\left[p_L S\right],\ Q \in \mathcal{M}.$$

Consequently, given that the prices are deterministic,

$$E^Q\left[S\right] \geq \frac{p_L}{p_U} E^Q\left[S\right],\ Q \in \mathcal{M},$$

or that $p_L \leq p_U$. ■

An important characteristic of the interplay between excess supply, net revenue and the price spread is the covariance between Z, R. In the following, it is assumed that all random variables have finite second moments

Proposition 2 *The covariance between Z, R is given by*

$$Cov\left(Z,R\right) = -\frac{p_U + p_L}{2}\sigma_Z^2 + \frac{p_U - p_L}{2}\left(\sigma_S^2 - \sigma_D^2\right).$$

Proof. The expectation of the product is given by

$$\begin{aligned} E\left[ZR\right] &= E\left[\left(S-D\right)\left(p_U D - p_L S\right)\right] \\ &= (p_U + p_L)E(DS) - p_U E\left(D^2\right) - p_L E(S^2). \end{aligned}$$

The product of expectations is

$$\begin{aligned} E(Z)E(R) &= \left(E(S) - E(D)\right)\left(p_U E(D) - p_L E(S)\right) \\ &= (p_U + p_L)E(D)E(S) - p_U E(D)^2 - p_L E(S)^2 \end{aligned}$$

Hence

$$\begin{aligned} Cov\left(Z,R\right) &= (p_U + p_L)\sigma_{DS} - p_U\sigma_D^2 - p_L\sigma_S^2 \\ &= \overline{p}2\sigma_{DS} - \left(\overline{p} + \frac{s}{2}\right)\sigma_D^2 - \left(\overline{p} - \frac{s}{2}\right)\sigma_S^2 \\ &= -\overline{p}\sigma_Z^2 + \frac{s}{2}\left(\sigma_S^2 - \sigma_D^2\right) \end{aligned}$$

where $\overline{p} = (p_U + p_L)/2$, $s = p_U - p_L$, $\sigma_D^2 = Var(D)$, $\sigma_S^2 = Var(S)$, and $\sigma_{DS} = Cov(D,S)$. ■

Raising supply and its volatility over the demand and its volatility by using a positive spread helps in making the covariance between Z, R less negative.

2.1 A Specific Risky Equilibrium

A specific equilibrium is formulated on specifying the demand and supply curves, the distributions for the random effects M_D, M_S and the distortions Ψ_Z, Ψ_R. For the demand and supply curves we adopt a pair of constant elasticity curves. Hence let

$$\begin{aligned} D(p) &= A_D p^{-\beta} \\ S(p) &= A_S p^{\eta}. \end{aligned}$$

For the distributions of M_D, M_S consider the log normal model studied in Madan (2020). Let (Z_D, Z_S) be a standard bivariate normal pair of random variables with correlation ρ. For volatilities of demand and supply σ_D, σ_S define

$$\begin{aligned} M_D &= \exp\left(\sigma_D Z_D - \frac{\sigma_D^2}{2}\right) \\ M_S &= \exp\left(\sigma_S Z_S - \frac{\sigma_S^2}{2}\right). \end{aligned}$$

With regard to distortions we work with the parameterized family Ψ^γ introduced in Cherny and Madan (2009) termed *minmaxvar*. The *minmaxvar* distortion is

$$\Psi^\gamma(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}.$$

The parameter γ is a measure of the stress in the distortion. The higher the level of γ the more concave the distortion and the larger is the set of measures supporting acceptability. The distortion has the property of reweighting losses upwards towards infinity and gains down towards zero as $\Psi'(u)$ tends to infinity as u tends to zero and it tends to zero as u tends to unity. In what follows, the same distortion parameter is applied for Z and R.

3 Further Equilibrium Details

The point of a two price equilibrium is that all the market can deliver is the two prices and there is not a price to be spoken about. As a consequence there is not a clear concept of return or mean returns either. Researchers subscribed to the law of one price by assumption may take as data for their one price the mid quote and then interpret the gap between the mid quote and the upper and lower prices as due to transactions costs, information effects or other matters. However, the equilibrium has failed to deliver a price.

In somewhat greater detail one may wish to enquire into the source of demand and supply curves for stocks or other financial assets. In doing this let us maintain the classical assumption of homogeneous beliefs. However, these cannot be beliefs about returns as prices are not known. Suppose then, in the context of a static one period model that the stock has a random liquidation cash flow X with a distribution that is common to all market participants. Where do the demand and supply curves for stocks come from? Who buys and sells at what prices. For prices p_U and p_L, ignoring time value considerations, or viewing prices as forward prices set by the market, the buyers take on the cash flows

$$X - p_U,$$

while the sellers hold

$$p_L - X.$$

Financial assets are not being bought or sold for purposes of utility maximization over a single period, the questions are who buys and who sells and how much and why. For a simple model one may turn to more general concepts of risk acceptability being applied by market participants. Index the agents by $v \in V$ and suppose we have a measure κ on the agent space V that describes the distribution of agents in the agent space.

Risk acceptability is more generally defined by convex sets containing the non-negative variables and contained in a convex cone containing the non-negative variables. The convex set thereby excludes any strictly negative elements. Each agent v has such a convex set $\mathcal{K}(v)$ defined by pairs (Q, a_Q) $\in \mathcal{C}(v) \subset \mathcal{M} \times \mathbb{R}$ of measures $Q \in \mathcal{M}$ and scalars $a_Q \in \mathbb{R}$ such that a risk $Y \in \mathcal{K}(v)$ only if

$$E^Q[Y] \geq a_Q \text{ for all } (Q, a_Q) \in \mathcal{C}(v).$$

The coefficients a_Q associated with measures Q are referred to as expectations floors for acceptability by Q.

Acceptable risks have personalized values that may be defined in terms of the convex set of risk acceptability. Suppose $Y \in \mathcal{K}(v)$. Then one may ask what is the largest constant amount one may subtract from the outcome Y and maintain acceptability. More exactly the value of Y, $W(Y, v)$ is given by

$$W(Y, v) = \sup \left\{ c | E^Q \left[Y - c \right] \geq a_q, \text{ all } (Q, a_Q) \in \mathcal{C}(v) \right\}.$$

For acceptable outcomes this value is positive by construction. Unacceptable outcomes have a negative value and participants have to be paid to take on such risks. For further details and connections with utility theory see Föllmer and Schied (2002, 2004) and Denuit, Dhaene, Goovaerts, Kaas and Laeven (2006).

Agent v chooses a quantity $\alpha(v)$ to buy such that

$$Y = \alpha(v) \left(X - p_U \right) \in \mathcal{K}(v)$$

and the agent may sell a quantity $\beta(v)$ such that

$$Y' = \beta(v) \left(p_L - X \right) \in \mathcal{K}(v).$$

Proposition 3 *No agent will consider both buying and selling an outcome $X \in \mathcal{K}(v)$.*

Proof. Suppose both positions $\alpha(v)(X - p_U)$ and $\beta(v)(p_L - X)$ were acceptable and elements of $\mathcal{K}(v)$. Then by convexity of $\mathcal{K}(v)$

$$\frac{\beta(v)}{\alpha(v) + \beta(v)} \alpha(v)(X - p_U) + \frac{\alpha(v)}{\alpha(v) + \beta(v)} \beta(v)(p_L - X) = -\frac{\beta(v)\alpha(v)}{\alpha(v) + \beta(v)} (p_U - p_L) \in \mathcal{K}(v),$$

but this is a negative element and cannot be in $\mathcal{K}(v)$. ■

Furthermore, both $\alpha(v)$ and $\beta(v)$ are finite provided $E^Q[X - p_U] < 0$ for some Q with $(Q, a_q) \in \mathcal{C}(v)$. Otherwise one has

$$E^Q[\lambda(X - p_U)] \geq a_q$$

for arbitrary large values of λ and then it follows that

$$E^Q[X - p_U] \geq 0$$

for all Q, which contradicts $E^Q[X - p_U] < 0$ for some Q.

The demand and supply are respectively given by

$$\begin{aligned} D(p_U) &= \int_V \alpha(v)\kappa(dv) \\ S(p_L) &= \int_V \beta(v)\kappa(dv). \end{aligned}$$

We thus obtain random demands and supplies under homogeneous beliefs.

In this context one may evaluate returns at mid quote prices by

$$\mu = \frac{2E[X]}{(p_U + p_L)}.$$

We may also define normalized spreads by

$$s = \frac{p_U - p_L}{p_U + p_L}. \tag{3}$$

One may then observe that

$$\frac{p_U}{p_L} = \frac{1+s}{1-s}.$$

It then follows that

$$\mu = \frac{E[X](1-s)}{p_L},$$

and illiquidity is associated in such an economy with lower expected returns for a given upper return $\mu_U = E[X]/p_L$. This result is consistent with the empirical studies of Ahimud and Mendelson (1986). Of course one may equivalently also write

$$\mu = \frac{E[X]}{p_U}(1+s)$$

with mean returns now positively related to illiquidity for a given lower return $\mu_L = E[X]/p_U$. The question turns on the relative stability of the upper or lower return. From a two price perspective the study of returns is problematic as there are two returns and more exactly the lower return should be measured as

$$R_{L,t} = \frac{p_{L,t}}{p_{U,t-1}} - 1$$

and the upper one as

$$R_{U,t} = \frac{p_{U,t}}{p_{L,t-1}} - 1.$$

However, what is available to market participants is the lower return. The upper one would be earned by the market.

A detailed study of the upper and lower returns is a matter for another research study. Here we just report the results of regressing the lower and upper returns on the contemporaneous spread for 864 stocks over the period of 1256 days described later in Section 5. Table 1 presents the quantiles for the t-statistics on regressing lower and upper returns on the current normalized spread as defined in equation (3). The lower returns are significantly negatively related to spreads in line with Ahimud and Mendelson (1986). The relationship is however positive for the upper return.

TABLE 1
t-statistic percentiles for spread coefficients of lower and upper return regressions

Percentile	Lower Return	Upper Return
1	−33.26	16.36
5	−28.30	17.79
10	−26.35	18.84
25	−23.82	20.93
50	−21.59	23.56
75	−19.50	26.51
90	−17.90	29.89
95	−17.18	31.94
99	−15.70	37.34

4 Risky Equilibrium Estimation Strategy

The parameters to be estimated in a risky equilibrium are those of the demand and supply curves, say, A_D, A_S, β, η. In addition we have the demand and supply volatilities and correlations σ_D, σ_S, ρ and finally the distortion parameter γ. First note that in the absence of uncertainties in demand and supply the one price equilibrium, obtained by equating demand and supply, would be

$$p = \left(\frac{A_D}{A_S} \right)^{\beta+\eta}.$$

The observed bid and ask prices are around the mid quote and this depends on the relative values of A_D and A_S. Given β, η, without loss of generality, set $A_S = 1$ and

$$A_D = p^{\frac{1}{\beta+\eta}}$$

where p is the mid quote of the bid and ask prices. In the initial exercise being reported here we take ρ to be zero. Finally the stress levels γ for Z, R are set exogeneously at 0.25. Daily equity returns have acceptability indices reported in Madan, Schoutens and Wang (2017) well below 0.2. A slightly higher stress level like 0.25 is probably conservative enough. The exercise can be repeated for a variety of stress level settings. It remains to estimate the parameters $\beta, \eta, \sigma_D, \sigma_S$.

These four parameters are critical to a two price equilibrium. The elasticities of demand and supply measure the responsiveness of the two sides of the market to the upper and lower prices with significant effects of excess demands and net revenues. The volatilities are critical to whole issue of risk in the risky equilibrium. The bid and ask prices observed in the market on any day are here modeled as a two price equilibrium for demand and supply over some time horizon h with respect to which the bid and ask prices are to be relevant. If the daily volatility of demand is v_D while that of supply is v_S then the estimated volatilities σ_D, σ_S over the horizon h are given by

$$\begin{aligned}\sigma_D &= v_D\sqrt{h}\\ \sigma_S &= v_S\sqrt{h}\end{aligned}$$

where h is a fraction or multiple of a day. If we further suppose that

$$\frac{v_D^2 + v_S^2}{2} = v^2$$

where v is volatility of the logarithm of daily volume then one may estimate the horizon as

$$h = \frac{\sigma_D^2 + \sigma_S^2}{2v^2}. \tag{4}$$

On a daily basis one cannot hope to recover four parameters from observations of two bid and ask prices. We take the view that for ten randomly selected adjacent days all four parameters are constant for these ten days. Parameter estimates are then sought that minimize by least squares the distance between observed market bid and ask prices for these ten days and those implied by the model and the four parameters.

For each of the ten days, given the parameters one has to solve ten optimization problems, one for each day, to determine the model bid and ask prices for the day. These are optimization problems over $(\widehat{p}_{Ui}, \widehat{p}_{Li})$, for $i = 1, 2, \cdots, 10$ that seek to minimize the sum of squared distorted expectations $\mathcal{E}_Z(Z_i)$, $\mathcal{E}_R(R_i)$ for day i with the object of getting them towards zero. The solution of these ten optimization problems are the input into an optimization problem over $(\beta, \eta, \sigma_D, \sigma_S)$ which seeks to minimize for market observed bid and ask prices (b_i, a_i) the quantity

$$\zeta = \sum_{i=1}^{10} (b_i - p_{Li})^2 + (a_i - p_{Ui})^2.$$

A single estimation takes around 15 minutes. We report results for 24 stocks at five randomly selected contiguous ten day intervals. Given that the objective function involves the use of internal optimizations, it is not appropriate to employ a gradient descent optimizer. The optimizer used was a global optimization algorithm called patternsearch in MATLAB.

5 Data for Two Price Equilibrium Analysis

The data for the analysis conducted was obtained from WRDS (Wharton Research Data Service) and contains information on daily close prices, the closing bid and ask prices and the volume traded for the day. The time period covers 1257 days from January 5, 2015 to December 13, 2019. There are 864 stocks in the data set for a total of $1,086,048$ stock days. Table 2 presents a sample of percentiles for the percentage spread of the ask price over the bid price. The spreads range from 1.33 percent to 2.69 percent in the inter-quartile range.

TABLE 2
Sample Percentiles for the percentage spread of the ask over the bid price

Percentile	Percentage Spread
1	0.6320
5	0.8539
10	1.0044
25	1.3290
50	1.8599
75	2.6895
90	3.8863
95	4.9299
99	8.0861

A robust regression of the spread on the logarithm of the mid quote and its square indicates a negative convex relationship between the spread and the price of the asset. The result of this regression is given by equation (5).

$$spr = 4.9731 - 1.2163 * \log(p) + 0.1171 * (\log(p))^2 \tag{5}$$

The associated $t - statistics$ are 391.75, -203.02, and 165.86. The $R - square$ is 4.8%.

By way of an illustration Figures 1 and 2 present graphs over time of the bid and ask prices, relative to the mid quote, smoothed using a kernel estimator, for the tickers $AAPL$ and $INTC$ respectively.

6 Sample of Equilibrium Solutions

For 20 stocks and five randomly selected days for each stock, 100 estimations using data for bid and ask prices for ten consecutive days following the selected day, were conducted for the four parameters $\beta, \eta, \sigma_D, \sigma_S$. Table 3a reports the estimates for the four parameters along with the ticker for the stock and the date for some stock days. More extensive Tables may be found in Elliott, Madan, and Siu (2020).

We observe that the demand elasticities are systematically nearly double the supply elasticities. This could be a reflection of the fact that sellers are either

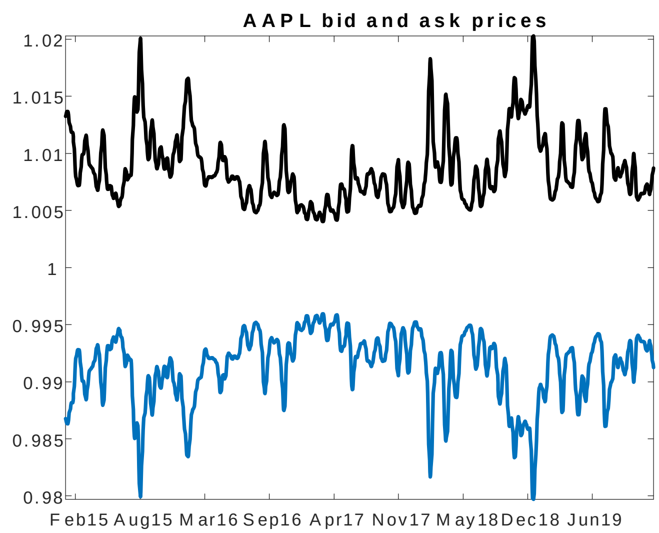

Figure 1: Smoothed bid and ask prices for AAPL relative to the mid quote.

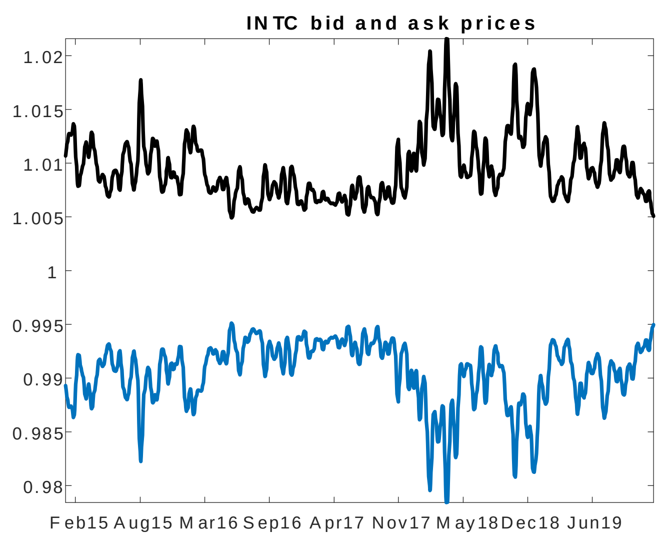

Figure 2: Smoothed bid and ask prices for INTC relative to the mid quote.

		Table 3a			
		Demand	Supply	Demand	Supply
Ticker	Date	Elasticity	Elasticity	Volatility	Volatility
ORCL	20170925	2.2505	1.1289	0.0041	0.0120
ORCL	20160915	2.3320	1.1875	0.0008	0.0178
ORCL	20160926	2.1113	1.0234	0.0085	0.0110
ORCL	20160321	2.6407	1.5313	0.0080	0.0139
ORCL	20180829	2.0859	1.0508	0.0140	0.0075
MSFT	20180928	2.0078	0.9687	0.0264	0.0027
MSFT	20151027	2.1406	1.0313	0.0133	0.0181
MSFT	20151106	2.0938	1.0625	0.0178	0.0088
MSFT	20190422	2.2189	1.1250	0.0022	0.0178
MSFT	20181003	2.1094	1.0625	0.0246	0.0231
KO	20150724	2.0939	1.0000	0.0125	0.0113
KO	20171107	1.9843	1.0000	0.0183	0.0045
KO	20191008	1.9990	1.0000	0.0168	0.0008
KO	20190123	2.0000	1.0078	0.0253	0.0004
KO	20190813	2.0000	0.9766	0.0217	0.0025
XOM	20171019	2.8750	1.7500	0.0041	0.0100
XOM	20180406	2.0000	1.0313	0.0178	0.0012
XOM	20190524	2.1250	1.0000	0.0095	0.0105
XOM	20170912	2.0645	0.9961	0.0071	0.0100
XOM	20160520	2.1870	1.1245	0.0095	0.0080
GE	20170214	2.0313	1.0000	0.0254	0.0029
GE	20161214	2.0614	1.0156	0.0173	0.0105
GE	20160610	2.1094	1.0001	0.0091	0.0120
GE	20160405	2.1396	1.0391	0.0090	0.0120
GE	20160831	2.0610	1.0156	0.0140	0.0078

owners of the stock or those taking short positions and this is a smaller group of individuals compared to the potential buyers. The demand volatilities are also larger than the supply volatilities with the median demand volatility being 1.75% compared to one percent for the supply volatility.

7 Equilibrium Durations

For each of the estimated sets of stock days, we estimated the subsequent ten day volatility of the traded volume. This gave an estimate of v. Equation (4) was then employed to estimate the horizon , which gives an estimate for the equilibrium duration. Allowing for 390 minutes per day the equilibrium duration in minutes was computed for each stock day. Table 5 presents a sample of percentiles for the equilibrium durations or the market implied time period for which the equilibrium is relevant. The median duration is about a minute and a half. The interquartile ranges from 37 seconds to three minutes. It would be interesting to corroborate these durations from other sources, but the magnitudes involved are reasonable if one thinks of the frequency with which prices change.

TABLE 4

Sample Percentiles for the Equilibrium Duration in Minutes

Percentile	Duration in Minutes
1	0.0644
5	0.1631
10	0.2461
25	0.6181
50	1.4831
75	3.0652
90	6.0235
95	10.9349
99	17.6400

Tables 5a presents the duration for a sample of stock days separately. A more extensive set of Tables is presented in Elliott, Madan, and Siu (2020).

With a few exceptions, most of the durations are of the order of a few minutes or less.

8 Duration Specific Bid and Ask Quotations

The model may be applied to generate bid and ask price quotations for larger blocks of trades that require longer time periods for their execution. If the size of order is large and it is anticipated that its execution will take an hour or more, for example, then the median duration of a minute and a half may be inflated by a suitable factor. The volatilities of demand and supply may be enhanced by the square root of this time factor to evaluate the bid and ask

		Table 5a			
		Duration			Duration
Ticker	Date	In Minutes	Ticker	Date	In Minutes
ORCL	20170925	0.1902	IBM	20191002	0.0723
ORCL	20160915	0.2227	IBM	20180208	4.2134
ORCL	20160926	1.9788	IBM	20170407	8.4292
ORCL	20160321	1.0895	IBM	20180518	3.3597
ORCL	20180829	3.1579	IBM	20170512	3.1099
MSFT	20180928	5.7926	AMGN	20150511	2.4018
MSFT	20151027	3.3225	AMGN	20160422	3.5776
MSFT	20151106	1.4943	AMGN	20191118	0.3268
MSFT	20190422	1.4484	AMGN	20181228	1.1032
MSFT	20181003	5.5673	AMGN	20151109	2.9389
KO	20150724	2.6671	GOOG	20190114	13.7509
KO	20171107	0.6598	GOOG	20190617	1.4709
KO	20191008	0.7503	GOOG	20180424	2.3520
KO	20190123	2.5421	GOOG	20170914	0.7463
KO	20190813	1.3183	GOOG	20150626	1.7964
XOM	20171019	0.3079	AAPL	20190822	2.3930
XOM	20180406	1.5212	AAPL	20190712	2.4651
XOM	20190524	0.3323	AAPL	20180227	14.6946
XOM	20170912	0.1416	AAPL	20151221	0.7953
XOM	20160520	0.0949	AAPL	20180125	3.7472
GE	20170214	2.7887	JNJ	20170602	0.7720
GE	20161214	0.1926	JNJ	20180122	2.3969
GE	20160610	0.2210	JNJ	20170113	0.4686
GE	20160405	1.6863	JNJ	20160816	2.4304
GE	20160831	0.6949	JNJ	20171023	0.7199

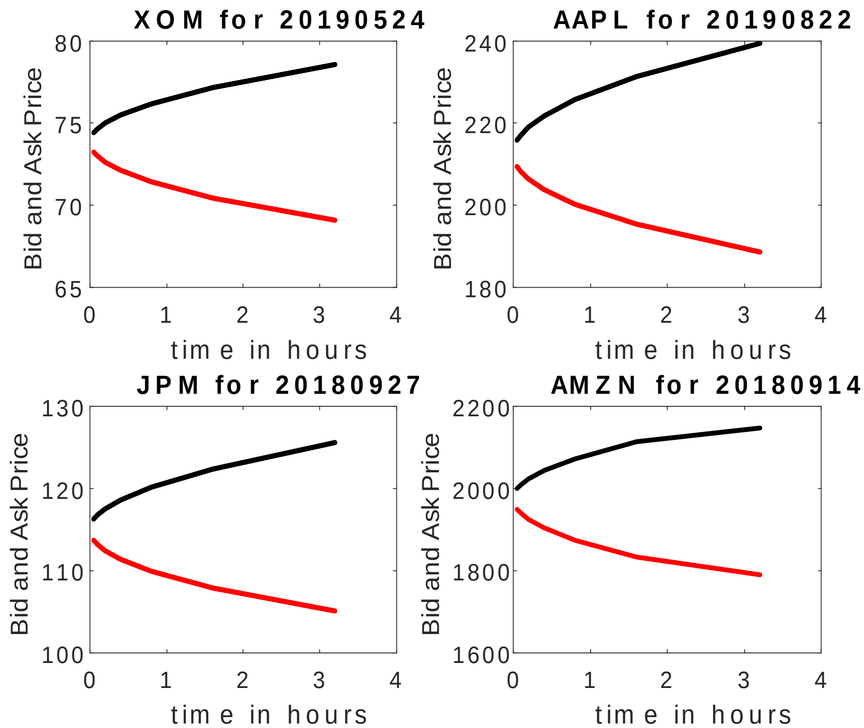

Figure 3: Bid and Ask Prices for a sample of stocks and days as function of the time to execute an order size.

prices appropriate for the longer duration. It is anticipated that the bid prices will fall and the ask prices rise with the duration. The result may be seen as an endogeneous equilibrium based impact factor. There is a considerable literature (for examples see, Bertsimas and Lo (1998)), Almgren and Chriss (2000), Cont and Larrard (2013), Gatheral, Schied and Slynko (2012), Obhizeva and Wang (2013)) modeling price impacts using a variety functional forms. The two price equilibrium model delivers a theoretical solution for such functions. Figure 3 presents the bid and ask prices ask functions of the time to unwind for a sample of stocks and days.

9 Conclusion

A two price equilibrium economy is defined by recognizing uncertainty in demands and supplies and modeling the market as setting prices to acceptable risk levels for excess supplies and net revenues. The equilibrium solutions are applied to two price financial market data to infer demand and supply elasticities and volatilities from a segment of daily bid and ask price quotes. Demand elasticities are observed to be higher than supply elasticities, as are their volatilities for some cases. Normalizing observed volatilities to the volatility of the traded volume the duration of the economic equilibrium is inferred. The median level is around a minute and a half with an interquartile range from 37 seconds to three minutes. Bid and Ask prices for longer durations reflecting the time required to execute larger trades may be constructed by calibrating the demand and supply volatilities.

References

[1] Ahimud, Y. and H. Mendelson (1986), "Asset Pricing and the Bid-Ask Spread," *Journal of Financial Economics*, 17, 223-249.

[2] Almgren, R. and N. Chriss (2000), "Optimal Execution of Portfolio Transactions," *Journal of Risk*, 3, 5-39.

[3] Artzner, P., F. Delbaen, M. Eber, and D. Heath (1999),"Coherent Measures of Risk," *Mathematical Finance* 9, 203-228.

[4] Back, K. and S. Baruch (2004), "Information in Securities Markets: Kyle meets Glosten and Milgrom," *Econometrica*, 72, 433-465.

[5] Bertsimas, D. and A. Lo (1998), "Optimal Control of Execution Costs," *Journal of Financial Markets*, 1, 1-50.

[6] Cherny, A., Madan, D. B. (2009), "New Measures for Performance Evaluation," *Review of Financial Studies* 22, 2571-2606.

[7] Constantinedes, G. M. (1986), "Capital Market Equilibrium with Transactions Costs," *Journal of Political Economy*, 94, 842-862.

[8] Cont, R. and A de Larrard (2013), "Price Dynamics in a Markovian Limit Order Market," *SIAM Journal of Financial Mathematics*, 4, 1-25.

[9] Copeland, T. E., and D. Galai (1983), "Information Effects on the Bid-Ask Spread," *Journal of Finance*, 38, 1457-1469.

[10] Demsetz, H. (1968), "The Costs of Transacting," *Quarterly Journal of Economics,* 82, 33-53.

[11] Denuit, M., J. Dhaene, M. Goovaerts, R. Kaas and R. Laeven (2006), "Risk Measurement with Equivalent Utility Principles," *Statistics and Decisions*, 24, 1-25.

[12] Easley, D., and M. O'Hara (1987), "Price, Trade Size and Information in Securities Markets," *Journal of Financial Economics*, 19, 69-90.

[13] Elliott, R. J., D. B. Madan, and K. Siu (2020), "Two Price Economic Equilibria and Financial Market Bid/Ask Prices," Available at SSRN paper no. 3633351.

[14] Föllmer, H., and A. Schied (2002), Convex measures of Risk and Trading Constraints," *Finance and Stochastics*, 6, 429-447.

[15] Föllmer, H. and A. Schied (2004), *Stochastic Finance*, (2nd Edition), De Gruyter, Berlin.

[16] Gatheral, J., A. Schied and A. Slynko (2012), "Transient Linear Price Impact and Fredholm Integral Equations," *Mathematical Finance* 22, 445-474.

[17] Glosten, L., and P. R. Milgrom (1985), "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14, 71-100.

[18] Ho, T. and H. R. Stoll (1981), "Optimal Dealer Pricing under Transactions Costs and Return Uncertainty," *Journal of Financial Economics*, 9, 47-73.

[19] Ho, T. and H. R. Stoll (1983), The Dynamics of Dealer Markets under Competition," *Journal of Finance*, 38, 1053-1074.

[20] Jouini, E., Kallal, H. (1995), "Martingale and Arbitrage in Securities Markets with Transaction Cost," *Journal of Economic Theory* 66, 178-197.

[21] Kusuoka, S.: On Law Invariant Coherent Risk Measures. Advances in Mathematical Economics 3, 83-95 (2001)

[22] Lo, A. W., H. Mamaysky and J. Wang (2004), "Asset Prices and Trading Volume Under Fixed Transaction Costs," *Journal of Political Economy*, 112, 1054-1090.

[23] Madan, D. B. (2015), "Asset Pricing Theory for Two Price Economies," *Annals of Finance*, 11, 1-35.

[24] Madan, D. B. (2020), "General Financial Economic Equilibria," Available at SSRN paper no. 3570067.

[25] Madan, D. B., W. Schoutens and K. Wang (2017), "Measuring and Monitoring the Efficiency of Markets," *International Journal of Theoretical and Applied Finance*, 20, 8.

[26] Obizhaeva, A. and J. Wang (2013), "Optimal Trading Strategy and Supply/Demand Dynamics," *Journal of Financial Markets*, 16, 1-32.

[27] Stoll, H. R. (1978), "The Pricing of Security Dealer Services: An Empirical Study of NASDAQ Stocks," *Journal of Finance*, 33, 1153-1172.