Centre Vortices Underpin Dynamical Chiral Symmetry Breaking in SU(3) Gauge Theory

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To Phiala
Abstract

The dynamical breaking of chiral symmetry is one of the key phenomena in QCD. It plays a vital role in explaining the QCD spectrum. The dynamical mass generation associated with the breaking of chiral symmetry is responsible for almost all of the mass of ground-state hadrons. The fundamental objects responsible for dynamical chiral symmetry breaking, however, remain elusive.

In this thesis, we investigate the role of a class of topological objects, centre vortices, in dynamical chiral symmetry breaking in SU(3) gauge theory using lattice gauge theory. We describe in detail an algorithm for identifying centre vortices on the lattice and isolating their effects. This allows us to define ensembles with centre vortices removed and ensembles consisting solely of centre vortices. We study the effects of smoothing algorithms on centre vortices, and find a strong connection to instantons, topological objects known to be important in dynamical chiral symmetry breaking. Then, we use the quark propagator to probe dynamical chiral symmetry breaking. We show a loss of dynamical chiral symmetry breaking after vortex removal, and correspondingly an ability to recreate it using the centre-vortex information alone. We then study the ground-state hadron spectrum, and show that the removal of centre vortices results in a spectrum consistent with restored chiral symmetry. Moreover, we show how chiral symmetry remains broken on the vortex-only ensembles.

Having examined multiple measures of dynamical chiral symmetry breaking and found them to be in agreement, we conclude that centre vortices are the fundamental objects underlying dynamical chiral symmetry breaking in SU(3) gauge theory.
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Daniel Trewartha
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Chapter 1

Introduction

Quantum chromodynamics (QCD) is now widely accepted as the theory of the strong interaction. It has met with remarkable success in describing nature. Despite this, fundamental aspects of it remain poorly understood.

At the Lagrangian level, QCD is a theory of the behaviour of quarks and gluons. The QCD spectrum, however, at face value bears little resemblance to a theory of quarks; free quarks are never seen, only a diverse spectrum of hadrons: bound states of multiple quarks with masses far in excess of the quarks composing them. Two peculiar properties of QCD are responsible: confinement, which forbids isolated quarks at low energies, and dynamical chiral symmetry breaking, which generates almost all of the mass of hadrons. It is easy to show that these properties exist in QCD, but a deep understanding of their origin is still lacking.

While the origin of confinement and dynamical chiral symmetry breaking is not yet fully understood, much progress has been made. It is now widely believed that the topological structure of the QCD vacuum is key. In this thesis, we focus on one variety of these topological objects, centre vortices and their role in dynamical chiral symmetry breaking. Centre vortices are related to the fundamental centre degree of freedom of QCD and so are an appealing candidate: they cannot be understood in terms of any more basic aspect of QCD. Great success has been achieved in understanding the simpler SU(2)-colour model in terms of centre vortices; it is known that centre vortices are responsible for both confinement and dynamical chiral symmetry
CHAPTER 1. INTRODUCTION

breaking there. However in the gauge group of QCD, SU(3), studies of centre vortices have been more ambiguous, with no clear link to either confinement or dynamical chiral symmetry breaking.

In this thesis, we aim to resolve this ambiguity. Are centre vortices the fundamental objects underlying SU(3) dynamical chiral symmetry breaking, as they are in SU(2)? Can we explain the ambiguity seen in previous studies?

Here, we use the tools of lattice QCD. Lattice QCD discretises space-time, and addresses the QCD equations directly on the resulting grid using supercomputers. This allows a first principles study of QCD at low energies impervious to any other first principles technique. The use of algorithms which isolate topological objects allows us to isolate their effects and study them on the lattice.

We begin in Chapter 2 with a brief summary of the lattice techniques used in this thesis. Then, in Chapter 3 we give an introduction to topology in QCD, as well as to confinement and dynamical chiral symmetry breaking.

In Chapter 4, we summarise the study of centre vortices on the lattice, and describe in detail the algorithm used in this thesis to isolate them. This allows us to study the effects of smoothing on centre vortices in Chapter 5, where we find a deep connection between the centre vortices and instantons, a topological object known to be important to dynamical chiral symmetry breaking.

Then in Chapter 6, we use the quark propagator as a probe of dynamical chiral symmetry breaking, and are able to show a clear link between centre vortices and dynamical chiral symmetry breaking. Here we are also able to explain the inability of previous studies in SU(3) to detect this link.

In Chapter 7, we study dynamical chiral symmetry breaking using the low-lying hadron spectrum. Again, we are able to show a clear link between centre vortices and dynamical chiral symmetry breaking.

In the final chapter, we present the conclusion: centre vortices underpin SU(3) dynamical chiral symmetry breaking and outline avenues for further investigation.
Chapter 2

Lattice Gauge Field Theories

Lattice QCD has met with remarkable success, and has developed into a diverse field of study. Here we focus mainly on defining the numerical techniques used in this thesis, and refer to standard texts for an introductory approach [1–3].

The lattice discretises spacetime onto a $N_s^3 \times N_t$ hypercube, with $N_s$ sites in the spatial dimensions and $N_t$ in the temporal, of volume $V$ and isotropic lattice spacing $a$. Fermion fields are typically discretised onto lattice sites, and gauge fields onto parallel transport operators between sites. Here we use an ensemble of lattice dimensions $20^3 \times 40$, with $a = 0.125$ fm.

2.1 Gauge Action

Gauge fields on the lattice are stored as link variables, with $U_\mu(x)$ denoting the link at lattice site $x$ in the $\hat{\mu}$ direction. These are interpreted as parallel transport operators between lattice sites, i.e.

\[ U_\mu(x) = \exp(iaA_\mu(x)). \]  

(2.1)

A key object used to define lattice actions is the ordered product of links forming a closed path. The simplest of these is the $1 \times 1$ plaquette, $P_{\mu\nu}(x)$,

\[ P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U_{-\mu}(x + \hat{\mu} + \hat{\nu})U_{-\nu}(x + \hat{\nu}). \]  

(2.2)
Plaquettes are desirable for their gauge transformation properties. Under a local gauge transformation $\Omega(x)$, $P_{\mu \nu}(x)$ transforms as

$$P_{\mu \nu}(x) \rightarrow P'_{\mu \nu}(x) = \Omega(x) P_{\mu \nu}(x) \Omega^\dagger(x).$$

(2.3)

This means that gauge invariant quantities can be defined in terms of plaquettes in relatively simple ways: either with fermion fields, $\bar{\psi}(x) P_{\mu \nu}(x) \psi(x)$;

(2.4)

or by using the cyclical properties of the trace,

$$\text{Tr} [P_{\mu \nu}(x)].$$

(2.5)

These properties hold true for all closed loops, allowing the freedom to use larger loops in the definition of lattice actions.

The simplest discretised gauge action is defined in terms of $1 \times 1$ plaquettes, $P_{\mu \nu}(x)$ as [4]

$$S^G(x) = \frac{6}{g^2} \sum_{\mu < \nu} \frac{1}{3} \text{ReTr} [1 - P_{\mu \nu}(x)].$$

(2.6)

Higher order error terms can be reduced by choosing appropriate combinations of larger loops. The ensemble used in this thesis consists of pure gauge field configurations, using an $O(a^2)$ mean-field improved action [5]. This is given by

$$S^G(x) = \frac{6}{g^2} \sum_{\mu < \nu} \frac{1}{3} \text{ReTr} \left[ \frac{5}{3} (1 - P_{\mu \nu}(x)) - \frac{1}{12} (1 - R_{\mu \nu}(x)) \right],$$

(2.7)

where $R_{\mu \nu}(x)$ is the $2 \times 1$ plaquette. We have used mean-field improved lattice links [6], given by

$$\frac{1}{u_0} U_{\mu}(x),$$

(2.8)

where the mean-field improvement factor, $u_0$, is given by

$$u_0 = \left( \frac{1}{3} \text{Tr} P_{\mu \nu}(x) \right)^{\frac{1}{4}}$$

(2.9)
2.2 Fermion Action

While the ensemble used in this thesis consists of pure gauge configurations, which we discuss in Chapter 4, we briefly introduce fermion actions on the lattice, as they are needed for the calculation of propagators. The simplest fermion action which solves the fermion doubling problem, the Wilson action, is given for $N_f$ flavours by \[ S_F(x) = \sum_{i=1}^{N_f} \left[ \bar{\psi}(x) \left( \nabla + \frac{a}{2} \sum_{\mu} \Delta_{\mu} + m \right) \psi(x) \right], \] (2.10)

where we have defined the lattice central difference operator $\nabla$ by

\[
\nabla_{\mu} \psi(x) = \frac{1}{2a} \left( U_{\mu}(x) \psi(x + \hat{\mu}) - U_{\mu}^{\dagger}(x - \hat{\mu}) \psi(x - \hat{\mu}) \right),
\] (2.11)

$m$ as the quark mass matrix, and the lattice Laplacian operator $\Delta$ by

\[
\Delta_{\mu} \psi(x) = \frac{1}{a^2} \left[ 2 - U_{\mu}(x) \psi(x + \hat{\mu}) - U_{\mu}^{\dagger}(x - \hat{\mu}) \psi(x - \hat{\mu}) \right].
\] (2.12)

While the introduction of the $\Delta$ term is necessary to avoid fermion doublers, it explicitly breaks chiral symmetry. As discussed in Section 3.3, anticommutation of the Dirac matrix with $\gamma_5$ is required in order to respect chiral symmetry, and the addition of a term proportional to $\mathbf{I}$ breaks this relation.

As for the gluon action, careful choices of combinations of higher order terms can reduce errors. Since higher order terms are physically irrelevant, we can also perform smearing (discussed in detail in chapter 5) on the gauge field before calculating their contribution. This leads to the class of actions known as fat link actions. For the purposes of this thesis, we use the fat link irrelevant clover (FLIC) action, given by \[ S^F(x) = \sum_{i=1}^{N_f} \left[ \bar{\psi}(x) \left( \nabla + \frac{a}{2} \sum_{\mu} \Delta_{\mu}^{fl} - \frac{1}{4} \sigma_{\mu\nu} F_{\mu\nu}^{fl} + m \right) \psi(x) \right], \] (2.13)

where the superscript 'fl' indicates the term is calculated on a smeared gauge field, and $\sigma_{\mu\nu}$ are defined as

\[
\sigma_{\mu\nu} = \frac{[\gamma_{\mu}, \gamma_{\nu}]}{2i}.
\] (2.14)

In this thesis, when the FLIC action is calculated four sweeps of stout link smearing (defined in chapter 5) are performed.
2.3 Overlap Action

As a consequence of the infamous Nielsen-Ninomiya no-go theorem [8], the fermion actions in the previous section all explicitly break chiral symmetry. As we are primarily interested in chiral effects, this is an unacceptable drawback. This can be resolved by the use of a fermion action which minimally breaks chiral symmetry, encapsulated in the Ginsparg-Wilson relation [9],

$$\gamma_5 D + D \gamma_5 = 2 a D \gamma_5 D.$$  \hspace{1cm} (2.15)

This is equivalent to an exact symmetry of the form [10]

$$\delta \psi(x) = \gamma_5 (1 - \frac{1}{2} a D) \psi(x) \text{ , } \delta \bar{\psi}(x) = \bar{\psi}(x) (1 - \frac{1}{2} a D) \gamma_5.$$  \hspace{1cm} (2.16)

Since this deviates from chiral symmetry in the continuum only by $\mathcal{O}(a)$ factors, we expect it to have the correct continuum limit. We note that this symmetry correctly reproduces the axial anomaly [11]; the fermion determinant does not transform trivially under this transformation, just as for the continuum chiral symmetry. For Wilson-like actions, the axial anomaly arises differently, from the explicit chiral symmetry breaking terms [12]. Chiral symmetry is discussed in more detail in section 3.3.

Here we use the overlap operator, given in the massless case by [13–16]

$$D_o = \frac{1}{2} (1 + \gamma_5 \epsilon(\gamma_5 D)),$$  \hspace{1cm} (2.17)

where $\epsilon$ is the matrix sign function, $\epsilon(A) = A (A^2)^{-\frac{1}{2}}$, and $D$ is any Hermitian Dirac operator, referred to as the overlap kernel. While the choice of overlap kernel does not affect results, it has significant effects on computational intensity. We use the FLIC action as the overlap kernel, as studies have shown it to have superior spectral properties, accelerating calculation of the overlap operator, and reduced lattice discretisation errors [17,18]. We project low modes of the kernel and calculate their contribution to the matrix sign function explicitly, greatly reducing the condition number of the matrix square root.

The massive overlap operator is then given by

$$D_o(\mu) = (1 - \mu)D_o(0) + \mu,$$  \hspace{1cm} (2.18)
where the overlap mass parameter $\mu$ is defined to represent a bare quark mass

$$m_0 = 2m_w\mu,$$

(2.19)

where $m_w$ is the negative mass parameter of $D$, a regulator parameter in the overlap operator.

### 2.4 Field Strength Tensor

For the purposes of smoothing (discussed in chapter 5) it is necessary to use a highly improved definition of the field strength tensor. Define the $n \times m$ clover improved field strength tensor by

$$F_{\mu
\nu}^{n \times m}(x) = \frac{1}{2i g a^2} \left[ C_{\mu\nu}^{n \times m}(x) - C_{\mu\nu}^{n \times m}(x)^\dagger \right],$$

(2.20)

where the clover term, $C_{\mu\nu}^{n \times m}(x)$, is the sum of the four $n \times m$ loops centered around $x$ in the $\tilde{\mu} - \tilde{\nu}$ plane, appropriately normalised. i.e.;

$$C^{1 \times 1}_{\mu \nu}(x) = P_{\mu \nu}(x) + P_{\nu \mu}(x) + P_{-\mu -\nu}(x) + P_{-\nu \mu}(x),$$

(2.21)

and similarly for larger loops.

Then we can choose linear combinations of these such that $O(a^2)$ and $O(a^4)$ errors vanish, giving [19,20]

$$F^{\text{imp.}}_{\mu\nu}(x) = \frac{3}{2u_0^2} F^{1 \times 1}_{\mu\nu}(x) - \frac{3}{20u_0^2} F^{2 \times 2}_{\mu\nu}(x) + \frac{1}{90u_0^2} F^{3 \times 3}_{\mu\nu}(x).$$

(2.22)

### 2.5 Landau Gauge Fixing

For the study of the quark propagator (presented in chapter 6) it is necessary to gauge fix. We choose Landau gauge, defined in the continuum by

$$\partial_\mu A_\mu(x) = 0.$$  

(2.23)

On the lattice, this is achieved by maximising the functional

$$\mathcal{F}(G) = a^2 \sum_x \sum_\mu \text{Tr} \left( U_\mu(x) + U^\dagger_\mu(x) \right).$$

(2.24)

We use the Fourier-accelerated $O(a^2)$ improved algorithm derived in Ref. [21].
Chapter 3

Topology in QCD

At low energies, QCD is dominated by two main features; confinement of quarks in hadrons, and dynamical chiral symmetry breaking, with the associated dynamical generation of mass. Although these can be easily shown to exist, there exists no first principles derivation of these. The question of the mechanism behind these two effects, as well as whether the two mechanisms share a common origin, is an open one.

The general view is that these two phenomena are caused by some topological object in the QCD vacuum which dominates at large distance scales. A wide variety of candidates have been considered, including instantons [22–30], Abelian monopoles [22,25,31], merons [32,33], calorons [34–36], and centre vortices [37–58].

Here we will discuss two of these objects relevant to this work; centre vortices and instantons. We begin with discussions of confinement, in Section 3.1, motivating the introduction of centre vortices, in Section 3.2. We then discuss dynamical chiral symmetry breaking, in Section 3.3, followed by a discussion of instantons, in Section 3.4.

In this thesis our focus will be on the role of SU(3) centre vortices. To what extent can the key features of dynamical chiral symmetry breaking and confinement be explained using the centre vortex degrees of freedom? And what connection do they have to instantons?
CHAPTER 3. TOPOLOGY IN QCD

3.1 Confinement

The fundamental fermionic degrees of freedom in QCD are the quark fields. Despite this, free quarks have never been experimentally observed, only being seen only bound into colour singlet hadrons (for a review, see Ref. [59]). Simply stated, this absence of free quarks at low energies is the phenomenon of confinement. Above a critical temperature of $T_C \approx 160 \text{ MeV}$ in full QCD, confinement breaks down, resulting in the formation of a quark-gluon plasma [60–62].

The most common approach to the study of confinement is through the static quark potential, the strong force potential between a heavy quark/anti-quark pair. If the potential is an increasing function of $R$, at some separation the potential will exceed $2m_q$, and so it will be energetically favourable to create a new quark/anti-quark pair. This gives a maximum `separation` of quarks.

Consider the vacuum expectation value of creating a quark/anti-quark pair at separation $R$ using an operator $Q$, and annihilating after time $T$. This can be expressed as

$$\langle Q^\dagger(T)Q(0) \rangle = \sum_k \langle 0|Q^\dagger|k\rangle \langle k|Q|0\rangle \exp(-TE_k). \tag{3.1}$$

The lowest energy state corresponds to the quark/anti-quark pair remaining at constant separation and propagating purely in the temporal direction. The energy of this state is thus identified with the value of the static quark potential at distance $R$. As the quark masses tend to infinity, this contribution dominates, and is given by the product of links around the $R \times T$ rectangle: the Wilson loop $W(R, T)$. Thus, we can write

$$V(R) = -\lim_{T \to \infty} \langle \log \left[ \frac{W(R, T + 1)}{W(R, T)} \right] \rangle. \tag{3.2}$$

On the lattice, it can be shown that the force between a static quark/anti-quark pair is everywhere attractive, but does not increase with distance [63], i.e.

$$\frac{dV}{dR} > 0, \quad \frac{d^2V}{dR^2} \leq 0. \tag{3.3}$$
Figure 3.1: Lattice data for the static quark potential. This data is described in detail in Sec. 5.2.1
Lattice evidence suggests that at long ranges, the static quark potential saturates these inequalities: it increases linearly. This behaviour is illustrated in Fig. 3.1. In fact, the confinement problem is often stated as showing that at long ranges the static quark potential is linear, i.e.

\[ V(R) \approx \sigma R. \]  

(3.4)

Note that from Eq. 3.2 it is sufficient to show that Wilson loops have an area-law falloff,

\[ W(C) \approx \exp[-\sigma A], \]

(3.5)

where \(A\) is the minimal area enclosed by the Wilson loop over curve \(C\), and \(\sigma\) a constant.

### 3.1.1 Centre Symmetry

Next we turn to an order parameter of confinement, the Polyakov loop. Polyakov loops are Wilson lines which wind around the lattice in the temporal direction, closed by periodic boundary conditions,

\[ P(x) = \text{Tr} \left[ U_0(x, 1) U_0(x, 2) \ldots U_0(x, n_t) \right]. \]

(3.6)

This can be thought of as the world-line of a massive static quark, propagating in the time direction.

\[ \langle P(x) \rangle = \exp[-E_q n_t], \]

(3.7)

where \(E_q\) is the energy of a free quark. In the confinement phase, free quarks are not found, and so this is infinite, but it is finite in the deconfined phase. There are thus two possibilities,

\[ \langle P(x) \rangle = \begin{cases} 0 & \text{confinement phase,} \\ \text{non-zero} & \text{deconfined phase.} \end{cases} \]

(3.8)

This leads to the introduction of a global symmetry intimately related to confinement, the centre symmetry. The centre of a group is the set of elements that commute with all other elements. In the case of SU(3), this is simply the three elements proportional to the identity, i.e.,

\[ Z = \exp \left[ \frac{2\pi i}{3} m \right] I, \quad m \in \{-1, 0, 1\}. \]

(3.9)
Consider a global centre transformation which multiplies all links in the temporal direction at a fixed time, $t_0$, by a centre element,

$$U_0(x, t_0) \rightarrow z U_0(x, t_0), \forall x. \quad (3.10)$$

Any loops not closed by periodicity in the temporal direction contain links in both the $\hat{t}$ and $-\hat{t}$ direction, and so are unaffected by this transformation. This includes all loops contained in the gauge-field action, and so the action is invariant under this transformation. Polyakov loops, however, transform as

$$P(x) \rightarrow z P(x). \quad (3.11)$$

Again, there are two possibilities,

$$\langle P(x) \rangle = \begin{cases} 0 & \text{unbroken centre symmetry,} \\ \text{non-zero} & \text{broken centre symmetry}. \end{cases} \quad (3.12)$$

Combining this with Eq. 3.8, we see that

$$\text{confinement phase} \leftrightarrow \text{unbroken centre symmetry}$$

$$\text{deconfined phase} \leftrightarrow \text{broken centre symmetry}, \quad (3.13)$$

and that these possibilities can be distinguished by the behaviour of the Polyakov loop. In Fig. 3.2 we have reproduced a plot of the phase of Polyakov loops [64], coloured according to the closest centre element at each point. In the confinement phase, Polyakov loops are evenly distributed between centre phases, giving $\langle P(x) \rangle = 0$. In the deconfined phase, a single centre phase comes to dominate, and so $\langle P(x) \rangle \neq 0$.

### 3.2 Centre Vortices

We have seen a close connection between the centre degree of freedom and confinement. This motivates the study of centre vortices: topological objects associated with centre symmetry.

In Eq. 3.10 we defined a centre transformation on a surface which affected all loops which cross the surface only once; in that case the loops which are
Figure 3.2: Visualisation of the phase of the Polyakov loops in the confined (left) and deconfined (right) phases. Each point is coloured according to the centre element closest to the phase of the Polyakov loop at that point. Figure reproduced from Ref. [64].

closed in the temporal direction. These are the loops which are topologically linked to the surface: they cannot be translated an arbitrary distance away from the surface without crossing it. The transformation has created a topological defect; continuous transformations of the gauge field can change the set of loops affected by the transformation, but they cannot remove its effects entirely.

We can generalise this transformation to one creating a topological defect on an arbitrary surface. Under this transformation, loops \( U(C) \) topologically linked to the surface transform as

\[
U(C) \rightarrow z U(C),
\]

for a centre element \( z \). This has created a surface of centre ‘flux’, an object known as a thin centre vortex. The flux can be smeared out over a finite region, a ‘thick’ centre vortex. Note that this is no longer a symmetry of the action, however for sufficiently large centre vortices, this has a small effect on the action. It was shown by ’t Hooft [37] that the vacuum expectation value of this vortex creation operator serves as an order parameter for confinement.
CHAPTER 3. TOPOLOGY IN QCD

The centre vortex model of confinement [37–45] is that the vacuum can be viewed as centre vortices independently superimposed on a non-confining background. Then we can decompose any loop as the product of a centre vortex part, $Z$, and a part due to the non-confining background, $R$,

$$U(C) = Z(C) R(C).$$

3.2.1 Centre Vortex Confinement in SU(2)

We can see how the centre vortex part gives rise to confinement with a simple argument in the case of SU(2). In SU(2), the centre is simply $\pm I$. Consider a planar loop with minimal area $A$ in a centre vortex background, where the density of intersections between vortices and the plane in which the loop lies is given by $\rho$. Denote the probability that the loop is pierced by $k$ vortices by $P(k)$. Then

$$\langle Z(C) \rangle = 1 P(0) + (-1) P(1) + 1 P(2) + \ldots$$

(3.16)

Note that since the locations of centre vortices are uncorrelated, we can describe $P(k)$ using a Poisson distribution with mean $\rho A$, i.e.:

$$P(k) = \frac{(\rho A)^k}{k!} \exp [-\rho A],$$

(3.17)

and so

$$\langle Z(C) \rangle = \sum_{k=0}^{\infty} \frac{(-\rho A)^k}{k!} \exp [-\rho A] = \exp [-\sigma A],$$

(3.18)

where

$$\sigma = 2 \rho.$$  

(3.19)

While the centre vortex model was motivated primarily by confinement, centre vortices also display a connection to dynamical chiral symmetry breaking. Topological charge density is produced by intersections and writhing points of vortices, and the probability density of zero modes of the Dirac operator peaks at these points [65,66].

Centre vortices have been extensively studied on the lattice, and we return to this in chapter 4.
3.3 Dynamical Chiral Symmetry Breaking

Dynamical chiral symmetry breaking is one of the central properties of QCD; it plays an important role in explaining the QCD spectrum, and the associated dynamical mass generation is responsible for around 99% of the proton’s mass.

We begin with a brief overview of chiral symmetry in the continuum. Consider an SU(3) gauge theory with a single massless fermion flavour. The Lagrangian density of this theory is given by

\[ L(\psi, \overline{\psi}, A) = \overline{\psi} D \psi = \overline{\psi} \gamma_\mu (\partial_\mu + i A_\mu) \psi. \]  

(3.20)

Now consider a chiral transformation, given by

\[ \psi \rightarrow \psi' = \exp(i \alpha \gamma_5) \psi, \]  

(3.21)

where \( \alpha \) is a real constant. The Lagrangian density is invariant under this transformation,

\[ L(\psi', \overline{\psi}, A) = \overline{\psi} \exp(i \alpha \gamma_5) \gamma_\mu (\partial_\mu + i A_\mu) \exp(i \alpha \gamma_5) \psi = \overline{\psi} \exp(i \alpha \gamma_5) \exp(-i \alpha \gamma_5) \gamma_\mu (\partial_\mu + i A_\mu) \psi = L(\psi, \overline{\psi}, A) \]  

(3.22)

where we have used \( \gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu \). We can see that for chiral symmetry it is necessary for the Dirac operator to commute with \( \gamma_5 \);

\[ D \gamma_5 + \gamma_5 D = 0. \]  

(3.23)

However, the introduction of a quark mass term explicitly breaks chiral symmetry;

\[ m \overline{\psi} \psi \rightarrow m \overline{\psi} \exp(2i \alpha \gamma_5) \psi. \]  

(3.24)

Chiral symmetry holds exactly only for massless quarks; hence the limit of quark masses tending to zero is often called the chiral limit. However, the light quark masses are small compared to the QCD scale of 1 GeV, with \( m_u \approx 2.3 \text{ MeV}, m_d \approx 4.8 \text{ MeV} \) in the MS scheme at a renormalization scale of 2 GeV [67], and so explicit chiral symmetry breaking is a relatively small effect.
Chiral symmetry decouples right- and left-handed massless fermions; define the right- and left-handed projectors,
\[ P_R = \frac{I + \gamma_5}{2}, \quad P_L = \frac{I - \gamma_5}{2}, \]
(3.25)
and correspondingly right- and left-handed quark fields,
\[ \psi_R = P_R \psi_R, \quad \psi_L = P_L \psi_L. \]
(3.26)

Then we can write the massless Lagrangian density as a sum of right- and left-handed components,
\[ \mathcal{L}(\psi, \bar{\psi}, A) = \bar{\psi}_R D \psi_R + \bar{\psi}_L D \psi_L. \]
(3.27)

The quark mass term, however, mixes the right- and left-handed quark fields;
\[ m \bar{\psi} \psi = m \left( \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \right). \]
(3.28)

Additional quark flavours allow us to also define flavour-mixing chiral transformations. In this thesis, it is sufficient to consider the case of two quark flavours. In this case, the flavour-mixing chiral transformations are given by,
\[ \psi \rightarrow \psi' = \exp(i \alpha \gamma_5 \sigma_i) \psi, \]
(3.29)
where \( \sigma_i \) are the Pauli matrices. This transformation can be performed independently for right- and left-handed quarks, and so, combined the massless Lagrangian has the symmetry
\[ \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_A. \]
(3.30)

If chiral symmetry was broken only explicitly by the small quark masses, one would expect hadron correlators related by a chiral transformation to have approximately degenerate masses. Among others, this would require the nucleon and its opposite parity partner, the \( N^* \), to have degenerate masses. However, the ground state nucleon has a mass of around 940 MeV, while the lowest lying odd-parity state has a mass of around 1535 MeV. This difference is much larger than can be explained by the small symmetry breaking due to the quark masses, and so chiral symmetry must be dynamically broken in
the vacuum state. The dynamical breaking of chiral symmetry plays a major role in the QCD spectrum, splitting the masses of a wide variety of hadrons which are otherwise chirally degenerate.

We have seen earlier that a quark mass term is not invariant under chiral transformations. This allows us to define an order parameter for dynamical chiral symmetry breaking, using a property with the same chiral transformation as a mass term,

\[ \langle \bar{q}q \rangle, \]  

(3.31)

which is known as the chiral condensate.

By Goldstone’s theorem \[68\], the dynamical breaking of chiral symmetry results in the creation of massless particles. For chiral symmetry, these are the pions, which acquire a small mass due to explicit quark-mass chiral symmetry breaking, of around 140 MeV. The state associated with the U(1)\(_A\) symmetry, the \(\eta\) meson, however, has a much larger mass than the pions, around 548 MeV. The explicit breaking of the axial symmetry is much larger than can be explained by non-zero quark masses alone, a phenomenon known as the axial anomaly.

While the massless Lagrangian density is invariant under U(1)\(_A\) transformations, the integration measure for fermions is not invariant, causing a much larger explicit symmetry breaking than from quark masses alone. This leads to our first hint of a connection between chiral symmetry breaking and topology; under an axial transformation, the fermion determinant transforms as \[1\]

\[ D[\psi, \bar{\psi}] \rightarrow D[\psi', \bar{\psi}'] = D[\psi, \bar{\psi}] \left( 1 - 2i \alpha N_f Q + \mathcal{O}(\alpha^2) \right). \]

(3.32)

\(Q\) is the topological charge, an integer-valued function of the gauge field. The value of the topological charge characterises topologically distinct vacuum states; since it is integer-valued, one cannot continuously transform vacuum states to ones of differing topological charge. Explicitly, the topological charge is given by

\[ Q = \int d^4x \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left[ F^{\mu\nu}(x) F^{\rho\sigma}(x) \right], \]

(3.33)

leading naturally to the definition of the local topological charge density,
q(x),
\[
\frac{1}{32 \pi^2} \epsilon_{\mu \nu \rho \sigma} \text{Tr} [F^{\mu \nu}(x) F^{\rho \sigma}(x)].
\]
(3.34)
The QCD action is symmetric for configurations of positive and negative topological charge, and so
\[
\langle Q \rangle = 0.
\]
(3.35)
The distribution of configurations of a given topological charge can be quantified by the topological susceptibility, \( \chi \),
\[
\chi = \frac{1}{V} \langle Q^2 \rangle.
\]
(3.36)

The topological charge displays a deep connection to chiral symmetry. The Atiyah-Singer index theorem [69] connects the topological charge to zero modes of the massless Dirac operator,
\[
Q = n_- - n_+,
\]
(3.37)
where \( n_- \) and \( n_+ \) respectively denote the left- and right-handed zero modes of the massless Dirac operator. The Banks-Casher relation [70],
\[
\langle \bar{q}q \rangle = -\pi \rho(0), \text{ as } m_q \to 0
\]
(3.38)
provides a connection between the chiral condensate and the density of eigenvalues of the Dirac operator near 0, \( \rho(0) \).

This provides a possible model for dynamical chiral symmetry breaking; a vacuum composed of weakly-interacting topological field configurations. While these structures would create exact zero modes in isolation by the Atiyah-Singer theorem, their interactions shift these away from 0, creating a density of near zero eignemodes, which by the Banks-Casher relation creates a chiral condensate, and thus dynamical chiral symmetry breaking.

3.4 Instantons

Naturally, the picture of the dynamical chiral symmetry breaking as being caused by weakly-interacting topological objects leads to the question of what, exactly, these objects should be. The search for topologically non-trivial classical solutions of the Yang-Mills equations first bore fruit with the
discovery of instantons [22]. Instantons have a topological charge of 1, and anti-instantons have a charge of $-1$.

The classical instanton solution in SU(2) is given explicitly by

$$A^a_{\mu}(x, z) = -\frac{2}{\rho} \frac{\eta_{\mu\nu}(x - z)\nu}{g(x - z)^2 + \rho^2} \quad (3.39)$$

$$A_{\mu}(x, z) = A^a_{\mu}(x, z) \frac{\sigma^a}{2}, \quad (3.40)$$

with $\rho$ the instanton radius, $\sigma$ the Pauli matrices and $\eta_{\mu\nu}$ the self-dual 't Hooft symbols [26,27]. We can write the solution as

$$A_{\mu}(x) = \frac{x^2}{x^2 + \rho^2} \frac{i}{g} \partial_{\mu}(S) S^{-1}, \quad (3.41)$$

with $S$ given by

$$S = \frac{x_4 \pm i \vec{x} \cdot \vec{\sigma}}{\sqrt{x^2}}, \quad (3.42)$$

with +/- for instantons/anti-instantons respectively. These provide the only solutions with topological charge $\pm 1$ [71].

The SU(3) instanton solution is then obtained by embedding this in one of the three SU(2) subgroups of SU(3), then performing an arbitrary colour rotation.

This solution is used for a model of the topological vacuum composed of an ensemble of instantons, known as the instanton gas model [72]. This features an instanton background made by ‘stitching together’ instanton solutions at large distances. The model proved unfeasible due to unphysically large instantons dominating, leading to the instanton liquid model [73–75]. This has much closer spacing, and thus instanton radii are stabilised through interactions. The vacuum is modelled as composed of equal numbers of instantons and anti-instantons, with equal radius.

Objects resembling instantons were found on the lattice as early as 1985 [76] through the use of an UV filter. Extensive work has since been carried out on their role in models of the QCD vacuum, and we refer the reader to refs [28–30] for reviews.
Chapter 4

Centre Vortices on the Lattice

The centre vortex model of the QCD vacuum is capable of explaining confinement, and its relevance to vacuum structure has strong support from lattice results.

A procedure for locating centre vortices on the lattice was first described in Refs. [77, 78]. The central idea is to gauge transform to a gauge where the lattice links are as close as possible to centre elements, then project onto the centre. The excitations in this centre gauge field are thin centre vortices, and their locations correspond to the locations of the thick centre vortices in the original configuration. This procedure forms the basis for the work in this thesis, and is described in detail in Section 4.1.

The study of SU(2) centre vortices in this way has enjoyed great success. The thin centre vortices identified are capable of reproducing the full string tension of the original configuration [77–79], and scale as physical objects [80]. This finding was confirmed using an independent method; identifying loops pierced by centre vortices by taking the sign of Wilson loops [81].

It was later shown that upon removing the identified centre vortices, the string tension was correspondingly removed [51]. In the same paper, it was noted that after removing centre vortices both the chiral condensate and the topological charge vanished; dynamical chiral symmetry breaking, as well as confinement, was spoiled with the removal of centre vortices.

Later studies showed that the centre vortex configurations had the same topological susceptibility as the original configurations [82], and that vortex
removal resulted in the density low-lying modes of the Dirac operator tending to zero \cite{83}.

Further study \cite{48–50, 52–54, 84, 85} confirmed these results; in SU(2), centre vortices are the fundamental object underpinning both confinement and dynamical chiral symmetry breaking.

In SU(3) gauge theory, however, the picture is far less clear. While vortex removal remains sufficient to remove the string tension, the centre vortices alone can only reproduce around 67% of the original value \cite{55}. Recent results \cite{46}, however, have suggested the centre-vortex model of confinement is more consistent with lattice results than other currently available models.

Few lattice studies of the connection between centre vortices and dynamical chiral symmetry breaking in SU(3) exist. It has been shown that topological charge density preferentially lies near centre vortices \cite{56}, thus suggesting a similar role for centre vortices to that observed in SU(2). However, study of dynamical mass generation through the quark propagator \cite{57} appeared to show the persistence of dynamical chiral symmetry breaking after vortex removal. In Ref. \cite{58} the ground-state hadron spectrum was studied using a Wilson fermion action. There, vortex removal resulted in an absence of dynamical chiral symmetry breaking, and degeneracy of the ground state meson and baryon spectra, an apparently contradictory result.

We discuss these studies in detail in Chapters 6 and 7, and present novel results, showing for the first time a link between centre vortices and dynamical chiral symmetry breaking in SU(3) gauge theory analogous to that in SU(2).

Extensive work has also been carried out on centre vortices at finite temperature, and we refer to Ref. \cite{86} for a review.

### 4.1 Identifying Centre Vortices on the Lattice

The strategy for identifying centre vortices on the lattice relies on a simple idea. We seek to decompose gauge links in the form

\[
U_\mu(x) = Z_\mu(x) R_\mu(x), \tag{4.1}
\]
with $Z_\mu(x)$ a centre element,

$$Z_\mu(x) = \exp \left[ \frac{2\pi i}{3} m_\mu(x) \right] \mathbf{I}, \quad (4.2)$$

in such a way that we have captured all vortex information of the original configuration in $Z_\mu(x)$. $Z_\mu(x)$ is a configuration of thin centre vortices, and $R_\mu(x)$ contains all remaining short-range information, and is also responsible for ‘thickening’ the vortices.

Then we can identify the plaquettes that have been pierced by centre vortices through their centre projection,

$$P_{\mu\nu}(x) = Z_\mu(x) Z_\nu(x + \mu) Z_\mu^\dagger(x + \nu) Z_\nu^\dagger(x), \quad (4.3)$$

where there exists a non-trivial centre flux if $P_{\mu\nu}(x) \neq \mathbf{I}$.

The natural choice [77, 78] to perform this decomposition is to find some gauge transformation $\Omega(x)$ which minimizes the distance between gauge links and centre elements; i.e.,

$$\sum_{x,\mu} ||U_{\mu}^\Omega(x) - Z_\mu(x)|| \rightarrow \min, \quad (4.4)$$

minimizing over both the gauge transformation $\Omega(x)$ and choice of centre element $Z_\mu(x)$. Choices of gauge fixing functional which fulfil this condition are collectively known as maximal centre gauge (MCG). Since all centre elements are mapped to the identity in the adjoint representation, this is equivalent to maximising the trace of the links in the adjoint representation. Many possible choices have been investigated: ‘direct’ maximal centre gauge [77, 78, 87, 88]; ‘indirect’ maximal centre gauge, which requires first fixing to maximal Abelian gauge [89], used primarily to investigate the connection between centre vortices and Abelian monopoles [77, 78]; Laplacian centre gauge [53, 90–92]; and ‘direct’ Laplacian centre gauge [93].

Here we use a variant of direct maximal centre gauge, ‘mesonic’ maximal centre gauge [87, 88]. We discuss its advantages over Laplacian centre gauge variants in sec 4.1.3. The mesonic maximal centre gauge fixing functional is given by

$$R = \frac{1}{VN_{\text{dim}}^3} \sum_{x,\mu} |\text{Tr} U_{\mu}^\Omega(x)|^2 \rightarrow \max, \quad (4.5)$$
where the normalisation factor, \(1/(V N_{\text{dim}} 3^2)\), is introduced to guarantee \(|R| \leq 1\). Note that \(U_\mu^\Omega(x)\) denotes the link \(U_\mu(x)\) after performing the gauge transformation \(\Omega(x)\).

We maximize this quantity using the method of Ref. [88], which is inspired by the Cabibbo-Marinari Okawa method for thermalization [94,95].

### 4.1.1 MCG Fixing Procedure

We now describe in detail the procedure for optimising the maximal centre gauge fixing functional, Eq. 4.5. Set the gauge transformation \(\Omega(x) = I\) everywhere except at a single lattice \(x'\). Then, the functional is a local quantity given by,

\[
R_{\text{local}}(x') = \sum_\mu |\text{Tr} \Omega(x') U_\mu(x')|^2 + \sum_\mu |\text{Tr} U_\mu(x' - \mu) \Omega^\dagger(x')|^2.
\]

We restrict \(\Omega(x')\) to be in one of the SU(2) subgroups of SU(3). We parametrize an SU(2) matrix \([\Omega(x')]_{\text{SU}(2)}\) by

\[
[\Omega(x')]_{\text{SU}(2)} = g_4 I - ig_i \sigma_i,
\]

and embed the resulting SU(2) matrix in one of the 3 SU(2) subgroups of SU(3). Then, \(R_{\text{local}}(x')\) can be re-written as

\[
R_{\text{local}}(x') = \sum_{i,j=1}^4 \frac{1}{2} g_i a_{ij} g_j - \sum_i g_i b_i + c.
\]

Where \(a_{ij}\) are elements of a real, symmetric matrix \(A\), \(b_i\) a real vector and \(c\) a real constant, all dependent only on \(U_\mu(x')\) and \(U_\mu(x' - \mu)\). These coefficients are written explicitly in terms of \(U_\mu(x')\) and \(U_\mu(x' - \mu)\) in Appendix A. We find the global maximum of this quantity using the method of Ref. [96], summarised here.

Define the quantities

\[
A' = -A, \quad b' = \frac{1}{2} b.
\]
Then we can write the problem as maximising
\[ g^T A' g - 2 b^T g, \] (4.10)
subject to the condition
\[ g^T g = 1, \] (4.11)
in order to maintain unitarity.

We solve this using the method of Lagrangian multipliers, reducing the problem to one of finding the minimal value of \( \lambda \) satisfying the equations,
\[ A' = \lambda g + b, \]
\[ g^T g = 1. \] (4.12)

Take the eigenvalue decomposition of \( A' \);
\[ A' = Q D Q^T, \] (4.13)
where \( D \) is a diagonal matrix with elements given by the eigenvalues of \( A' \), and \( Q \) a matrix with columns given by the eigenvectors of \( A' \). The Lagrange equations transform (4.12) into
\[ Q D Q^T g = \lambda g + b, \]
\[ g^T g = 1. \] (4.14)

Then define
\[ u = Q^T g \]
\[ d = Q^T b, \] (4.15)
giving
\[ D u = \lambda u + d, \]
\[ u^T u = 1. \] (4.16)

Since the case that \( \lambda \) is an eigenvalue of \( A' \) will never appear numerically,
\[ u = (D - \lambda I)^{-1} d, \] (4.17)
and so the solution is the minimal value of $\lambda$ satisfying

$$u^T u = d^T (D - \lambda I)^{-2} d = \sum_{i=1}^{4} \left( \frac{d_i}{\lambda_i - \lambda} \right)^2 = 1,$$

where $\lambda_i$ are the eigenvalues of $A'$ in order. Note that the function

$$f(\lambda) = \sum_{i=1}^{4} \left( \frac{d_i}{\lambda_i - \lambda} \right)^2,$$

is strictly increasing for $-\infty < \lambda < \lambda_1$, and that

$$\lim_{\lambda \to -\infty} f(\lambda) = 0, \quad \lim_{\lambda \to \lambda_1} f(\lambda) = \infty.$$

There is thus exactly one solution satisfying $\lambda < \lambda_1$. This condition allows us to rapidly find the minimal solution using standard numerical methods. Then the maximum value of the gauge fixing functional is given by,

$$g = Q (D - \lambda I)^{-1} Q^T b'.$$

4.1.2 Identified Vortices

Once we have fixed to maximal centre gauge, we decompose links into polar form as

$$\text{Tr} U_\mu^\Omega(x) = r_\mu(x) \exp \left[ i\theta_\mu(x) \right],$$

and thus select as centre projection the element $Z_\mu(x) = \exp \left[ \frac{2\pi i}{3} m_\mu(x) \right]$, $m_\mu(x) \in \{-1, 0, 1\}$, such that $\frac{2\pi i}{3} m_\mu(x)$ is closest to $\theta_\mu(x)$. In Fig. 4.1 we

In this work results are calculated on 100 pure gauge-field configurations using the Lüscher-Weisz $O(a^2)$ mean-field improved action [5], with a $20^3 \times 40$ volume at a lattice spacing of 0.125 fm.
show a histogram of the values of $\theta_\mu(x)$ found after fixing to maximal centre gauge, showing three clear peaks, corresponding to the three centre phases.

We thus define the three ensembles which will serve as the basis for this work:

- the original, Monte-Carlo generated, ‘untouched’ (UT) configurations,
  
  \[ U_\mu(x); \]  

- the vortex-only (VO) configurations,
  
  \[ Z_\mu(x); \]  

- and the vortex-removed (VR) configurations,
  
  \[ R_\mu(x) = Z_\mu^\dagger(x) U_\mu^{RL}(x). \]  

We note, however, that while we have globally maximised the quantity $R_{\text{local}}(x')$, we have only found a local maximum of the gauge fixing functional. There are multiple gauge transformations locally maximising the functional, known as Gribov copies. We illustrate this issue in Fig. 4.2; we show the
values of $R$ obtained on the same configurations fixed to maximal centre gauge starting from two distinct random gauges. Although we have reached similar values of $R$ both times, there is no correspondence on a configuration-by-configuration basis; variation is well outside of the stability of the gauge fixing algorithm, indicating we have reached distinct local maxima on each occasion.

Attempts were made in SU(2) to reach higher values of the gauge fixing functional using a variety of techniques [97,98]. However, an anti-correlation was seen between the value of the gauge fixing functional and the ability of the identified vortices to reproduce the physical string tension, putting into doubt the physical meaning of the identified vortices. However, it was shown in Ref. [99] that the maximal centre gauge functional is necessarily low on plaquettes pierced by centre vortices, and so higher values correspond to a failure to identify some vortex matter. This was later confirmed in SU(3) by the use of a preconditioner to maximise the value of the gauge fixing functional [100]; again an anti-correlation between the value reached and the quality of the vortex matter identified was shown.

Laplacian centre gauge avoids the issue of Gribov copies, however the vortices found lack an interpretation as physical objects, a point we will later return to in Subsection 4.1.3.

This issue means we are unable to uniquely identify the vortex matter of a configuration; some of the physical centre vortices embedded in the lattice may be missed by the vortex identification procedure. Herein, we have selected the gauge fixing directly following the generation of the ensemble by the Markov chain. This ensemble is denoted by the square blue points in Fig. 4.2.

4.1.3 Physical Properties of Identified Vortices

Now that we have defined an ensemble of thin centre vortices, the question naturally arises: how can we verify that the thin vortices we have identified correspond to the thick vortices on the Monte Carlo generated configurations?

The reproduction of physical quantities from vortex configurations is in itself insufficient; the SU(2) string tension can be fully reproduced from vortex
Figure 4.2: Values of the maximal centre gauge fixing functional, $R$, after gauge fixing on a configuration-by-configuration basis. Square points in blue provide values from the original ensemble generated in the Markov chain. Circular points in red correspond to values found after performing a random gauge transformation before gauge fixing.
configurations produced without gauge fixing, which clearly do not cor-
respond to physical objects [101].

The use of the vortex removed ensemble serves as a powerful consistency
check in this regard, but more general arguments can be made that the
vortices identified by MCG fixing correspond to physical objects.

Firstly the density of objects found should scale with lattice spacing as
expected for a physical quantity. For MCG vortices, this has been shown
to be true in both SU(2) [78, 80] and SU(3) [55]. The vortices found in
Laplacian centre gauge, however, do not scale appropriately, and so lack an
interpretation as physical objects [55].

Secondly, the locations of the thin centre vortices found should correspond
to the centre of the thick centre vortices on the Monte Carlo generated con-
fugurations. This can be checked with a simple test; let $W_n$ be the set of
Wilson loops on the Monte Carlo generated configurations at the same loca-
tion as the loops identified as being pierced $n$ times by centre vortices on the
vortex only configurations. Then if these are also pierced $n$ times by centre
vortices on the untouched configurations, we should have in SU(2)
\[ W_n/W_0 \rightarrow (-1)^n \] (4.26)
as the minimal area covered by the loop increases, and correspondingly in
SU(3),
\[ W_n/W_0 \rightarrow \exp \left( \frac{2\pi in}{3} \right) \] (4.27)
This has been shown to be the case in both SU(2) [78] and SU(3) [87].

Thirdly, if a classical vortex is inserted onto the lattice ‘by hand’, then it
should be among those located. Again, the SU(3) vortices identified in MCG
pass this test [88].

Combined, these tests offer a strong interpretation of identified vortices
as corresponding to physical objects on the untouched configurations. We
can thus interpret the Gribov copy issue simply as an inability of gauge fixing
to detect all physical vortex matter.
Chapter 5

Smoothing

Now that we have obtained a set of ensembles suited to the study of centre vortex effects, we wish to analyse their gauge field structure. To what extent have we captured the long-range behaviour of the gauge fields with the centre vortex degree of freedom? Which aspects of the gauge field topology can be reproduced from these alone?

We begin by examining the effects of smoothing on the ensembles. This is motivated in two ways. Firstly, the process of vortex identification identifies thin centre vortices on lattice configurations, whereas the vortex matter on Monte Carlo generated configurations consists of thick centre vortices. Smoothing will allow us to create thick centre vortices from the identified thin vortex matter. Secondly, the thin centre vortex configurations consist, by definition, solely of centre elements, and so are rough configurations with a much higher action than those generated by Monte-Carlo methods. The overlap fermion action is only well defined on gauge fields which satisfy a smoothness condition, and so in order to examine the behaviour of quarks on vortex only configurations we must reduce their action. This is additionally motivated by evidence that, in SU(2) gauge theory, vortex-only configurations are too rough to reproduce the low-lying modes of the Dirac operator essential to dynamical chiral symmetry breaking, but are able to do so after smearing [84].

Smoothing algorithms reduce the action on gauge field configurations, bringing them closer to the classical solution. Carefully defined, smoothing
algorithms leave topologically non-trivial objects intact, thus isolating their influence. Smoothing is known to produce a gauge-field background resembling a 'liquid' of (anti-)instanton-like objects \cite{76,102}, which is capable of reproducing dynamical mass generation, and thus dynamical chiral symmetry breaking \cite{103}. In this chapter we will thus focus on the instanton degrees of freedom revealed under smoothing.

We will begin in section 5.1 with a brief introduction to the smoothing algorithms used; cooling \cite{20,104–107} and over-improved stout link smearing \cite{108,109}. Both algorithms are tuned to preserve instanton-like objects on the lattice \cite{104,105,108,109}, and so provide an ideal way to test the nature of the instanton-like objects found on the lattice, as well as their robustness under choice of smoothing algorithm.

Then, in section 5.2 we will apply the smoothing algorithms to the untouched, vortex-only, and vortex-removed ensembles, and analyse the gauge field structure obtained. Under smoothing, we find the vortex-only and untouched ensembles showing similar behaviour, with the action, topological charge, and static quark potential all being essentially equivalent between untouched and vortex-only ensembles after smoothing. We also show that under smoothing the vortex-removed backgrounds become essentially empty, almost devoid of long-range structure.

We then focus on the instanton degrees of freedom. We find that centre vortices display a strong connection to instanton-like objects; with vortex removal otherwise stable instanton-like objects are removed under the smoothing algorithms, leading to a ‘trivial’ gauge field background. Conversely, we show that on vortex-only configurations we are able to recreate a background of instanton-like objects essentially equivalent on the ensemble level to that found in the untouched case.

5.1 Smoothing Algorithms

5.1.1 Cooling

Cooling was the earliest smoothing method applied to the lattice, first applied in the context of the O(3) sigma model \cite{76,102,110}. Cooling is conceptually
simple; we systematically sweep over lattice links $U_\mu(x)$, and replace each with a new ‘cooled’ link, $U'_\mu(x)$, minimising the local gluonic action associated with that link. Each iteration over all lattice links constitutes a single sweep of cooling.

Explicitly, for the $1 \times 1$ Wilson action, Eq. 2.6, the gluonic part is given by

$$S_{\text{Wil}}(x) = \frac{6}{g^2} \sum_{\mu<\nu} \frac{1}{3} \Re \text{Tr} [I - P_{\mu\nu}(x)].$$ \hspace{1cm} (5.1)

It is convenient to define the ‘staple’ $\Xi_{\mu\nu}(x)$ as the three links of the plaquette $P_{\mu\nu}(x)$ other than $U_\mu(x)$. i.e.,

$$\Xi_{\mu\nu}(x) = U_\nu(x + \hat{\mu}) U_\mu^\dagger(x) U_\nu^\dagger(x).$$ \hspace{1cm} (5.2)

It can be seen that minimising the gluonic action locally is equivalent to choosing $U_\mu(x)$ to maximise

$$\sum_{\mu<\nu} \Re \text{Tr} [U_\mu(x) \Xi_{\mu\nu}(x)].$$ \hspace{1cm} (5.3)

The method used to do so is based on the Cabbibo-Marinari [94] algorithm for constructing SU(3) gauge field configurations. Just as in the maximal centre gauge fixing algorithm presented in Section 4.1, we consider a single SU(2) subgroup of SU(3).

Denote the parts of $U_\mu(x)$ and $\Xi_{\mu\nu}(x)$ corresponding to the given SU(2) subgroup by $[U_\mu(x)]_{\text{SU}(2)}$ and $[\Xi_{\mu\nu}(x)]_{\text{SU}(2)}$ respectively. Then we define

$$\bar{U}(x) \equiv \frac{1}{k} \sum_{\mu<\nu} [\Xi_{\mu\nu}(x)]_{\text{SU}(2)},$$ \hspace{1cm} (5.4)

where we have defined

$$k \equiv \det \left( \sum_{\mu<\nu} [\Xi_{\mu\nu}(x)]_{\text{SU}(2)} \right),$$ \hspace{1cm} (5.5)

in order to ensure $\bar{U}(x)$ is an element of SU(2).

Maximising the local gluonic action is then equivalent to maximising

$$\Re \text{Tr} \left([U_\mu(x)]_{\text{SU}(2)} \bar{U}(x)\right).$$ \hspace{1cm} (5.6)
CHAPTER 5. SMOOTHING

Which is achieved by

$$\text{ReTr} \left[ [U_\mu(x)]_{\text{SU}(2)} \bar{U}(x) \right] = \text{ReTr} [I].$$  \hspace{1cm} (5.7)

We can see that the SU(2) part of the cooled link, $$[U_\mu'(x)]_{\text{SU}(2)}$$, should be chosen to be

$$[U_\mu'(x)]_{\text{SU}(2)} = U^{-1}(x) = U^\dagger(x).$$  \hspace{1cm} (5.8)

We embed this in an SU(3) matrix, and then iterate over the three SU(2) subgroups. We loop over all three SU(2) subgroups 8 times per link, creating an SU(3) cooled link which brings the local gluonic action as close as possible to a minimum. Then, we sweep over each link on the lattice.

Although this presentation used a gluon action featuring only $$1 \times 1$$ loops, it can be easily extended to actions featuring larger loops, simply by extending the sum over $$1 \times 1$$ loops in Eq. 5.4 to include loops of all sizes.

Cooling presents a numerical inefficiency when parallelised; we cannot simultaneously update links which are too close on the lattice. While the calculation of a cooled link is ongoing, none of the links which are involved in the local action of that link may be updated. Naively sweeping through the lattice is thus numerically untenable. We use the method of ref. [106], partitioning links into the maximal sets which can be simultaneously updated. This allows efficient parallelisation of cooling.

The choice of gluonic action used in the cooling algorithm bears some thought. We wish instanton-like objects to remain stable under cooling, in order to be able to remove action from the lattice without damaging the underlying topology. Taking the explicit single instanton solution from the continuum, Eq. 3.41, and inserting into the $$1 \times 1$$ Wilson gauge action, Eq. 2.6, we obtain [111]

$$S_{\text{Wil.}}^{\text{inst.}} = \frac{8 \pi^2}{g^2} \left[ 1 - \frac{1}{5} \left( \frac{a}{\rho_{\text{inst}}^5} \right)^2 + \mathcal{O} \left( \frac{a}{\rho_{\text{inst}}}^4 \right) \right],$$ \hspace{1cm} (5.9)

where $$\rho_{\text{inst}}$$ is the instanton radius. Although at leading order this is equivalent to the continuum instanton action, the leading error term is both negative and inversely proportional to the instanton radius. Thus, in reducing the action, the cooling algorithm will shrink instanton-like objects, leading to their eventual destruction.
It is therefore essential that we use highly-improved actions, removing lower-order errors, in order to minimise this effect. We cool using an $\mathcal{O}(a^4)$-three-loop improved action. We note that while the leading-order error terms are still negative, they are now $\mathcal{O}\left(\left(\frac{a}{\rho_{\text{inst}}}\right)^6\right)$, and so for a reasonable number of cooling sweeps have minimal effect on instanton-like objects. It is also of note that while isolated (anti-)instantons are stable under cooling, nearby anti-instanton/instanton pairs can still be drawn together and annihilated by the cooling algorithm.

### 5.1.2 Over-Improved Stout-Link Smearing

We have seen an issue with the stability of instanton-like objects under smoothing, arising due to a negative highest-order error term in the single instanton action. While this issue can be ameliorated by using highly-improved actions, another approach naturally suggests itself; we can define a gluonic action in such a way that the highest-order error terms are positive. This is the approach of over-improved stout-link smearing, an idea first implemented in ref. [111], although here we use the more local formulation implemented in refs. [108,109].

We introduce a new free parameter, $\epsilon$, into the action, in such a way that a value of $\epsilon = 1$ gives the $1 \times 1$ Wilson gauge action, and a value of $\epsilon = 0$ gives the two-loop improved action, Eq. 2.7. Explicitly, the new action is given by

$$S(x, \epsilon) = \frac{6}{g^2} \sum_{\mu<\nu} \frac{1}{3} \text{ReTr} \left[ 5 - 2\epsilon \left(1 - P_{\mu\nu}(x)\right) - \frac{1-\epsilon}{12} \left(1 - R_{\mu\nu}(x)\right) \right].$$  

\[ (5.10) \]

Then, inserting the single instanton solution into this action as before, we obtain

$$S_{\text{inst}}(\epsilon) = \frac{8\pi^2}{g^2} \left[ 1 - \frac{\epsilon}{5} \left(\frac{a}{\rho_{\text{inst}}}\right)^2 + \mathcal{O}\left(\frac{a}{\rho_{\text{inst}}}\right)^4\right].$$  

\[ (5.11) \]

Negative values of $\epsilon$ will thus give a positive leading error term, and enlarge instanton-like objects instead of shrinking them. The small negative value of $\epsilon = -0.25$ has been shown to best preserve instanton-like objects on the lattice.
We minimise this new action through the process of smearing; similarly to cooling, we sweep the lattice, systematically replacing each link with a new, smeared link. However, the process of smearing is different to cooling; instead of updating links with a cooled link minimizing the local action, we update links with a smeared link which is ‘averaged’ with neighbours. This allows us to update all links simultaneously, although it does not remove action from the lattice as efficiently per sweep as cooling.

Explicitly, we replace each link simultaneously with a smeared link, $U'_{\mu}(x)$, given by

$$U'_{\mu}(x) = \exp(iQ_{\mu}(x))U_{\mu}(x), \quad (5.12)$$

noting that the use of the exponential is necessary to ensure that the smeared link is an SU(3) matrix. We have defined

$$Q_{\mu}(x) = \frac{i}{2} (\Omega^\dagger_{\mu}(x) - \Omega_{\mu}(x)) - \frac{i}{2N} \text{Tr}(\Omega^\dagger_{\mu}(x) - \Omega_{\mu}(x)), \quad (5.13)$$

with

$$\Omega_{\mu}(x) = C_{\mu}(x)U^\dagger_{\mu}(x), \quad (5.14)$$

and

$$C_{\mu}(x) = \rho \sum_{\nu} \left[ \frac{5 - 2\varepsilon}{3}(\Xi_{\mu\nu}(x)) - \frac{1 - \varepsilon}{12}(L_{\mu\nu}(x)) \right]. \quad (5.15)$$

We have introduced $L_{\mu\nu}(x)$ as the analogue of $\Xi_{\mu\nu}(x)$ for $2 \times 1$ rectangles; it is defined as the 5 links in $R_{\mu\nu}(x)$ other than $U_{\mu}(x)$. We have also introduced the smearing parameter $\rho$, controlling the level of smearing. A value of $\rho = 0.06$ provides rapid smoothing when using Eq. 5.15.

### 5.2 Results

We begin with a plot of the average action on the three ensembles discussed in subsec. 4.1.2 as a function of smoothing sweeps, in Fig. 5.1. In the untouched case, there is a rapid initial loss of action for both cooling and over-improved stout-link (OISL) smearing. This corresponds to the rapid removal of short-range noise; quantum fluctuations away from the classical minimum of the action. After this rapid loss, taking approximately 15 sweeps in the case of
cooling and 40 in the case of over-improved stout-link smearing, the gauge-field configurations come to closely resemble a classical solution, and so further action loss is slow. At this point, the gauge-field background resembles a ‘liquid’ of (anti-)instantons. After this, action is very slowly removed from the lattice as these can only disappear via pair-annihilation. As expected, the rate at which action is removed from the configurations is much higher for cooling than OISL smearing; approximately 3 times as many sweeps of OISL smearing as cooling are required for the same average action.

While the vortex-removed ensemble starts with a similar average action to the untouched, it behaves markedly differently under smoothing. The rapid loss of action seen in the untouched ensemble continues for many more sweeps, and the average action drops far lower. The removal of centre-vortices has destabilised the otherwise topologically non-trivial long-range structure of the vacuum, leading to its destruction by smoothing algorithms.

The vortex-only ensemble initially has a high action, with a relatively small loss of action under smoothing. However, there is a dramatic drop in the action after around 5 sweeps of cooling or 130 of OISL smearing, resulting in an average action on the vortex-only ensemble very similar to that seen on the untouched after a similar amount of smoothing.
the untouched ensemble, the vortex-only ensemble is very distant from a classical solution; this seems to manifest in a very slow initial decrease in the action, where after each smoothing sweep the vortex-only configurations remain distant from a Monte-Carlo generated configuration, followed by a rapid drop. Notably, after dropping to be similar to the untouched ensemble, the vortex-only ensemble behaves very similarly thereafter, even at extremely high levels of smoothing. This suggests a very similar gauge-field background on the vortex-only and untouched, in both cases ‘fuller’ than the near empty background found in the vortex-removed case.

Also of note is the large amount of OISL smearing required to create an action similar to the untouched ensemble on the vortex-only. Whereas it takes only around 15 sweeps of cooling to reduce the action on the vortex-only configurations to a level similar to the untouched, this requires around 180 sweeps of OISL smearing, much higher than the approximately 3:1 ratio of action loss observed on the untouched ensemble. This can be understood in terms of the two ways the two algorithms reduce the action; while each cooled link is simply the link minimising the local action, the OISL smeared links are an average of the original link with nearby neighbours. Local averaging is an efficient way to remove action for configurations which are already close to a minimum of the action; on the untouched configurations smearing is only required to remove short-range noise, which by its nature is local. However, in the vortex-only case, long-range changes are required. Local averaging is unable to efficiently make long-range changes to the gauge-fields.

Next we examine the topological structure of the ensembles under smoothing; in fig. 5.2 we have plotted the average absolute value of the topological charge under both smoothing algorithms. The topological charge is calculated using the gauge-field definition, summing the local topological charge density. We have used an $O(a^4)$-five-loop improved definition of the field-strength tensor, Eq. 2.22.

The gauge-field definition of topological charge is poorly defined without smoothing; the effects of long-range topological structure are overwhelmed by local noise. This is manifested on the untouched ensemble in a very high initial average topological charge. After a small amount of smoothing, however, this settles on a single value, approximately 6 in both cooling and
Figure 5.2: The average absolute value of the topological charge on untouched (squares), vortex-only (circles) and vortex-removed (crosses) ensembles under cooling (a) and over-improved stout-link smearing (b).

OISL smearing, which remains stable up to high levels of smearing. We note that since we have plotted the ensemble average, this need not be an integer. The stability of topological objects under both smoothing algorithms is clear here; destruction of instantons would be manifested as a reduction in the magnitude of topological charge. Although (anti-)instanton pairs can pair-annihilate under smoothing, this results in no net change to the configuration topological charge, and so cannot be seen on this plot.

The vortex-removed ensemble shows similar behaviour under small levels of smoothing to the untouched; an initially high average topological charge, rapidly reducing. However, the vortex-removed settles on a much smaller value, around 2 for both cooling and OISL smearing. The vortex-removed backgrounds are far 'emptier', with much less long-range structure stable under smoothing. The backgrounds are not completely trivial; some topological objects have survived the vortex-removal process, most likely due to the inability to remove all centre vortices from the lattice. This reinforces the behaviour seen in the average action; the vortex-removed configurations become almost trivial under smoothing, with only a few topological objects remaining.

The vortex-only ensemble, just as in the case of the average action, re-
quires a larger amount of smoothing than the other ensembles to display 'smoothed' behaviour; the average topological charge magnitude does not settle on a stable value until around 15 sweeps of cooling or 180 sweeps of OISL smearing. Notably, this is the same amount of smoothing required to reduce the average action to a level comparable to the untouched. The eventual value of the topological charge reached is comparable to the untouched, but slightly higher; around 7.5 compared to around 6 on the untouched. Again, this is stable under further smoothing. Two measures of gauge-field structure are in agreement; cooling and OISL smearing both produce a gauge-field background similar to that seen in the untouched, behaving similarly under further smoothing.

### 5.2.1 Static Quark Potential

We next examine heavy quark confinement, using the static quark potential. It has long been known [55, 113] that while SU(3) vortex-removal results in the removal of the string tension, and thus confinement, vortex-only ensembles, without smoothing, can reproduce only approximately 66\% of the string tension. We have examined the static quark potential on untouched and vortex-only ensembles after 10, 40, and 80 sweeps of cooling, plotted in fig. 5.3.

After 10 sweeps of cooling, short-range effects have been removed from the lattice, and so at short ranges the static quark potential on the untouched ensemble is flat. The long-range structure, however, remains, as reflected by a linearly rising potential at long distances. The slope of this linear rise gives the string tension. The vortex-only ensemble behaves similarly after 10 sweeps of cooling, although the string tension is lower. Under further cooling the gap between the untouched and vortex-only decreases, and eventually, after 80 sweeps of cooling, the two are virtually identical. Both are, however, substantially reduced compared to the full string tension found in the untouched ensemble without smoothing.

We have fit the static quark potential to the form \( V(R) = \sigma R + c \) at long ranges, with fit parameter \( \sigma \) summarised in table 5.1. After 10 sweeps of cooling, our results are consistent with the well-known unsmoothed case;
Figure 5.3: The static quark potential on untouched (blue) and vortex-only (green) configurations after 10, 40, and 80 sweeps of cooling. Note that the 80 sweeps untouched potential is hidden behind the 80 sweeps vortex-only potential.

Table 5.1: Value of the string tension (lattice units) found on untouched (UT) and vortex-only (VO) ensembles after 10, 40, and 80 sweeps of cooling.

<table>
<thead>
<tr>
<th>Cooling Sweeps</th>
<th>( \sigma_{UT} )</th>
<th>( \sigma_{VO} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.12(4)</td>
<td>0.75(5)</td>
</tr>
<tr>
<td>40</td>
<td>0.64(2)</td>
<td>0.59(2)</td>
</tr>
<tr>
<td>80</td>
<td>0.42(1)</td>
<td>0.40(1)</td>
</tr>
</tbody>
</table>
the vortex-only ensemble reproduces 67% of the string tension. This rises to 93% after 40 sweeps, and 97% after 80 sweeps. On measures of heavy quark confinement, after 80 sweeps of cooling the vortex-only and untouched ensemble differ only within error bars.

5.2.2 Instanton Structure

Having seen that the vortex-only and untouched ensembles become similar under smoothing using measures of gauge-field structure at the configuration level, we will now examine them at the level of individual topological objects. We will directly examine the instanton degrees of freedom on the ensembles, and the degree to which the instanton-like objects seen on the vortex-only and untouched ensembles are similar.

We examine the instanton degrees of freedom using the method of ref. [108], scanning the lattice for local maxima of the action density, taking each as a candidate instanton, and fitting the classical instanton solution around them;

\[
S_0(x) = \frac{6}{\pi^2} \left(\frac{\rho^4}{(x - x_0)^2 + \rho^2}\right) \quad .
\] (5.16)

At each local maximum, we fit 6 parameters to the surrounding $3^4$ hypercube; the instanton radius, $\rho$, the location of the instanton centre, $x_0$, not restricted to lie on a lattice site, and a scale parameter $\xi$. We note that this method requires a relatively smooth gauge-field background to be reliable; on the Monte-Carlo generated configurations before smoothing, most local maxima will correspond to noise, not topological structure. Studies of the instanton structure of the lattice in this way have shown that smoothing is able to produce a background of objects closely resembling classical instantons [103].

The average number of local maxima per configuration has been plotted in Fig. 5.4, as a measure of the number of instanton-like objects. On the untouched ensemble, there is a rapid initial drop, corresponding to the removal of local maxima corresponding to short-range noise. Then, a gauge-field background dominated by instanton-like objects has appeared, which remains stable under further smoothing. There continues to be a slow loss of local maxima, due to the pair-annihilation of instantons.
Figure 5.4: The number of local maxima of the action found on untouched (squares), vortex-only (circles) and vortex-removed (crosses) ensembles after cooling (a) and over-improved stout-link smearing (b).

The vortex-removed ensemble shows similar behaviour to that seen in the integrated action and topological charge; initially indistinguishable from the untouched ensemble, but a dramatic drop in the number of local maxima is seen under smoothing. Under high levels of smoothing, there are as few as 3 local maxima of the action per configuration. Under smoothing, the vortex-removed background is almost completely devoid of structure, behaving as a trivial background.

Just as in the case of the action and topological charge, initially smoothing has little effect on the number of local maxima of the action seen on the vortex-only ensemble. There remains a large number of local maxima under small amounts of smoothing, indicating a gauge-field background that remains noisy. However, just as seen previously, after around 15 sweeps of cooling or 180 of OISL smearing the number of local maxima drops to be very similar to that seen in the untouched. By around 40 sweeps of cooling or 200 of OISL smearing the two ensembles are almost identical. The background of instanton-like objects found on the untouched configurations is capable of being reproduced by the vortex-only configurations.

The average fitted radius of instanton candidates has been plotted in fig. 5.5. The untouched, again, shows a rapid initial drop corresponding to the removal of noise, followed by a small, steady increase as the configuration
Figure 5.5: The average radius ($\rho$) of instanton candidates found on untouched (squares), vortex-only (circles) and vortex-removed (crosses) ensembles after cooling (a) and over-improved stout-link smearing (b).

becomes dominated by instanton-like objects. The vortex-only ensemble initially has a much lower average radius than the untouched. This shows an action density with much larger ‘spikes’; the classical instanton solution is proportional to $1/\rho^4$, and so a smaller average radius corresponds to much larger local maxima of the action. Again, after around 40 sweeps of cooling or 200 of OISL smearing, the vortex-only ensemble comes to closely resemble the untouched. In both the number and size of instanton-like objects, under smoothing the vortex-only ensemble is able to replicate the instanton structure of the untouched.

While vortex-removal destabilizes instanton-like objects, it does not directly affect the topological charge density. There remain regions of uniform topological charge density, corresponding to the positions of the instantons destabilized by vortex-removal. The smoothing algorithms destroy these regions by ‘flattening’ them out, enlarging them and shrinking the local topological charge density until they have been removed. This behaviour can be seen in the average radius of instanton candidates in the vortex-removed case; the average radius dramatically increases as these regions are smoothed out. Under cooling, this process is finished, and the average radius shrinks to a stable value, as there remain some stable instanton-like objects. Under OISL smearing, this shrinking is not seen; it seems 600 sweeps of OISL smearing
are insufficient to see this behaviour.

Having seen the configuration-averaged behaviour of instanton-like objects, we now turn to individual objects for additional insight into this behaviour. In fig. 5.6, we have plotted the radius and topological charge density at centre of individual objects on a representative configuration, for two values of cooling, 10 and 40 sweeps. This can be compared to the classical relationship between instanton radius and topological charge density at the centre, \( q(x_0) \),

\[
q(x_0) = Q \frac{6}{\pi^2 \rho^4},
\]

where \( Q = \mp 1 \) for an (anti-)instanton.

After just 10 sweeps of cooling, all instanton candidates found on the untouched configuration closely fit the theoretical relationship. This remains true at 40 sweeps of cooling, although their number has been greatly reduced by pair annihilation. After 10 sweeps of cooling, there is very little correlation between the instanton candidates found on the vortex-only configuration and the theoretical relationship. A large number of small objects have been found, which lie far from the theory line, suggesting they do not correspond to instanton-like objects. There are, however, some objects of medium size, with radii around 4 lattice spacings, which lie close to the theory line. It seems that after 10 sweeps of cooling, instanton-like objects have only just begun to appear on the vortex-only configuration. By 40 sweeps of cooling, the vortex-only configurations closely resemble the untouched, with all objects lying close to the theoretical relationship, suggesting a background of instanton-like objects.

On the vortex-removed configuration, a number of large objects have been found at 10 sweeps of cooling. These are in the process of being destroyed by the cooling algorithm, the results of which can be seen after 40 sweeps of cooling. Almost all objects have been removed, with a few remaining at very large fitted radii and low topological charge density at centre. There remains a very small number of stable objects which closely approximate instantons. Notably, these all have low radii. The vortex removal process inherently is imperfect, due to the inability to find a unique global maximum of the maximal centre gauge fixing condition. For a few instantons of size
Figure 5.6: The values of the instanton radius, $\rho$, against the topological charge at the centre, $q(x_0)$. Results are compared to the theoretical relationship between the instanton radius and topological charge at the centre, Eq. (5.17) (solid lines). Results are shown on a typical configuration for the untouched (top), vortex-only (middle), and vortex-removed (bottom) cases at 10 (left) and 40 (right) sweeps of cooling. Note the larger scale for $\rho$ in the vortex-removed case after 40 sweeps of cooling.
2-3 lattice spacings, the centre vortex matter inside the instanton may thus be intact after vortex removal. For larger instantons, it is extremely unlikely that this will be the case, and so only small instantons remain on the vortex-removed configurations.

The objects seen on the vortex-only configuration have no one-to-one correspondence with those seen on the corresponding untouched configuration. While the centre vortices contain the information necessary to recreate the instanton structure of the untouched on the ensemble level, individual objects are not recreated. We have plotted a level set of the topological charge density after 40 sweeps of cooling in fig. 5.7 to illustrate this. In both the untouched and vortex-only cases, instanton-like objects are visible as regions of uniform topological charge density. The two are qualitatively similar, but there is no direct correspondence between the objects on the untouched and vortex-only configurations.

![Figure 5.7](image)

Figure 5.7: A visualisation of the topological charge density after 40 sweeps of cooling on a representative configuration, in both untouched (a) and vortex-only (b) form. Contours of positive topological charge density are plotted in yellow, and negative in blue, for a level set.

### 5.3 Summary

We have analysed the behaviour of the untouched, vortex-removed, and vortex-only ensembles under two different smoothing algorithms, using a number of measures of gauge-field structure; the action, topological charge, static quark potential, and instanton structure.
On the untouched ensemble, smoothing results in the removal of short-range effects, and the creation of a gauge-field background dominated by objects which closely approximate classical instantons. Isolated instantons are then stable under further smoothing, although (anti-)instanton pairs continue to pair-annihilate.

While the vortex-removed ensemble appears similar to the untouched ensemble without smoothing, the process of smoothing rapidly destroys almost all of the gauge-field structure. The average action and topological charge rapidly decrease, well below those on the untouched ensemble. This was explained by the behaviour of the instanton-like objects; after vortex removal these otherwise topologically non-trivial objects are no longer preserved under smoothing, leading to an almost empty gauge-field background. Some small instantons survive the vortex removal process, associated with the Gribov copy induced imperfections in the vortex identification procedure.

The vortex-only ensemble is distant from the Monte-Carlo generated configurations, and so requires a larger amount of smoothing before short-range effects are removed. Once this has occurred, the vacuum structure of the vortex-only ensemble closely resembles that of the untouched ensemble. While individual objects have not been reproduced, the gauge field structures seen across the two ensembles are essentially equivalent. The centre vortices contain the ‘seeds’ of the instanton background; all of the information necessary to recreate the topological structure. Additionally, after smoothing, the vortex-only ensemble is able to recreate the residual string tension seen on the untouched. Both the instanton structure and residual heavy quark confinement can be replicated on the vortex-only backgrounds after smoothing.

These findings are robust under the smoothing method used, and all measures of gauge-field structure are in agreement as to the amount of smoothing required. This raises the possibility that previous studies of SU(3) centre vortex configurations suffered from a lack of smoothness, and so could not correctly discern the role they play in vacuum structure. The centre vortices identified on the lattice are thin centre vortices, whereas those embedded in Monte-Carlo generated configurations are thick centre vortices. It may be that the gauge field dynamics are captured fully only by thick centre vortices.
Vortex removal is sufficient to destroy thick centre vortices, and so confinement and instanton-like objects are spoiled on the vortex-removed ensemble without smoothing.

Although we have found similar gauge field background structure on the untouched and vortex-only configurations after smoothing, this naturally has removed the short-range information of the gauge fields, as well as some of the structure relevant for confinement. It would be desirable to have a method of smoothing that would allow us to gain smooth centre-vortex configurations without removing relevant short-range or confining structure, allowing direct comparison to unsmeared configurations. It would be interesting to explore an SU(3) extension to the technique proposed for the SU(2) case in Ref. [114].
Chapter 6

Landau-Gauge Quark Propagator

In the previous chapter we saw that after a small amount of smoothing the vortex only and untouched gauge field backgrounds had ensemble-average properties which are essentially identical. We naturally wish to see whether this similarity carries through for further aspects of the fundamental non-perturbative features of QCD; confinement and dynamical chiral symmetry breaking.

The QCD Green’s functions provide an ideal testing ground for non-perturbative physics and have long been an active area of research, both through Dyson-Schwinger equations and on the lattice (for reviews, see e.g. [115–118]). A major appeal of the study of the Green’s functions is the ability to study both confinement and dynamical chiral symmetry breaking in the context of light quarks.

Here we use the Landau-gauge quark propagator as a probe of dynamical chiral symmetry breaking. The infrared behaviour of the Dirac scalar part of the quark propagator, commonly referred to as the mass function, provides a clear signal of the presence or absence of dynamical chiral symmetry breaking. The presence of dynamical mass generation, and thus dynamical chiral symmetry breaking is shown through enhancement of the mass function at low momentum. This behaviour persists even in the chiral limit [118].

In the context of centre vortices, the Landau-gauge AsqTad quark propa-
gator \cite{119,120} has been used to study dynamical chiral symmetry breaking in both SU(2) \cite{48}, and SU(3) \cite{57} gauge field theory. In SU(2), the AsqTad quark propagator showed a role of centre vortices in dynamical chiral symmetry in agreement with other studies. Dynamical mass generation breaking was shown to disappear after removal of centre vortices. Conversely, on the vortex-only background, infrared enhancement of the quark propagator was shown, although a large amount of noise was present on these backgrounds. By contrast, in SU(3), the AsqTad quark propagator was unable to show any systematic effects of centre vortex removal; the changes to the gauge field resulted only in an increase in statistical noise.

The AsqTad fermion action, however, explicitly breaks chiral symmetry. It is perhaps unsurprising that the AsqTad quark propagator is then insensitive to chiral effects; the relationship between centre vortex removal and dynamical chiral symmetry breaking may simply be overwhelmed by lattice artefacts. Here, we will thus use the overlap fermion action, described in Section 2.3. The overlap fermion action’s lattice-deformed version of chiral symmetry provides renowned sensitivity to topological effects \cite{121}, and also prevents additive mass renormalization \cite{122}, greatly simplifying propagator analysis \cite{123–125}.

Using the Landau-gauge overlap quark propagator, we are able to show, for the first time, an intimate relationship between centre vortices and dynamical chiral symmetry breaking in SU(3) similar to that shown in SU(2). Upon removal of centre vortices, we show a loss of dynamical mass generation, and we show an ability to recreate the infrared behaviour of the quark propagator within error bars on vortex-only configurations.

We begin in Section 6.1 with a description of the decomposition of the quark propagator into Dirac scalar and vector parts. Then, in Section 6.2 we provide a description of the numerical methods used. Results are then provided in Section 6.3, and a summary in Section 6.4.
6.1 The Mass and Renormalization functions

We begin our analysis of the overlap quark propagator with the definition of the massless overlap propagator in momentum space as \[ S(p)_{m_q=0} \equiv \frac{1}{2m_w} (D_o^{-1}(0) - 1). \] (6.1)

The overlap operator is introduced and discussed in Section 2.3. This ensures that we have the same chiral behaviour as the continuum in the chiral limit, i.e.,

\[ \{ S(p)_{m_q=0}, \gamma_5 \} = 0. \] (6.2)

Then we arrive at the massive overlap quark propagator, \( S(p) \), by noting that we must satisfy

\[ S^{-1}(p) = S_{m_q=0}^{-1}(p) + m_q. \] (6.3)

The massive overlap quark propagator is thus defined as

\[ S(p) \equiv \frac{1}{2m_w(1 - \mu)} (D_0^{-1}(\mu) - 1). \] (6.4)

We can then define \( B \) and \( C \) by \[ S(p) \equiv -iC \gamma \cdot p + B(p), \] (6.5)

where \( B(p) \) has no explicit colour or spin dependence. Taking the trace in colour space, and using \( \text{Tr} (\gamma_\mu) = 0 \),

\[ \text{Tr} (S(p)) = n_s n_c B(p) \]

\[ B(p) = \frac{1}{n_s n_c} \text{Tr} (S(p)). \] (6.6)

Similarly, we acquire

\[ C_\mu(p) = \frac{i}{n_s n_c} \text{Tr} (\gamma_\mu S(p)). \] (6.7)

Thus

\[ S^{-1}(p) = \frac{i\gamma \cdot p + B(p)}{C^2(p) + B^2(p)}. \] (6.8)
Due to the absence of additive mass renormalization, the overlap quark propagator on the lattice will have as its general form

\[ S(p) = \frac{Z(p)}{i\hat{q} + M(p)}, \]  

with \( q \mu \) the kinematic lattice momentum which, for the overlap, is the only tree-level correction necessary [125]. The Dirac scalar part, \( M(p) \), is referred to as the mass function and the Dirac vector part, \( Z(p) \), the renormalization function. We define the inverse as

\[ S^{-1}(p) = i \sum_{\mu} (C_\mu(p) \gamma_\mu) + B(p). \]  

Comparing to Eq. 6.8,

\[ C_\mu(p) = \frac{C_\mu(p)}{C^2(p) + B^2(p)}, \]

\[ B_\mu(p) = \frac{B_\mu(p)}{C^2(p) + B^2(p)}. \]  

Define \( A(p) \) by

\[ S^{-1}(p) = i\hat{q} A(p) + B(p), \]  

and so

\[ A(p) = \sum_{\mu} \left( \frac{q_\mu C_\mu}{q^2} \right). \]  

Similarly, define

\[ \mathcal{A}(p) = \sum_{\mu} \left( \frac{q_\mu C_\mu}{q^2} \right), \]  

such that the mass and renormalisation functions are,

\[ Z(p) = \frac{1}{A(p)} = \frac{C^2(p) + B^2(p)}{\mathcal{A}(p)}, \]

\[ M(p) = \frac{B(p)}{A(p)} = \frac{B(p)}{\mathcal{A}(p)}. \]  

### 6.2 Numerical Methods

We use the fat-link irrelevant clover (FLIC) fermion operator [7,17,18,128], described in section 2.2, as the overlap kernel \( D(-m_w) \). The regulator parameter, \( m_w \), governs the condition number of the overlap kernel, and thus
the computation time. We take the value $m_w = 1$, as it has been predicted to give maximal displacement of the eigenvalues of the Wilson kernel from 0, both theoretically [129] and in practice [103]. The values of the overlap mass parameter, $\mu$, considered, together with the corresponding physical bare quark masses, are given in table 6.1. Here, the lattice spacing $a = 0.125$ fm has been used.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$m_q$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.004</td>
<td>12</td>
</tr>
<tr>
<td>0.008</td>
<td>25</td>
</tr>
<tr>
<td>0.012</td>
<td>40</td>
</tr>
<tr>
<td>0.016</td>
<td>50</td>
</tr>
<tr>
<td>0.022</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 6.1: Values of the overlap mass parameter, $\mu$, considered, with corresponding bare quark masses. Here, the lattice spacing $a = 0.125$ fm has been used.

Quark propagator analysis requires an explicit choice of gauge, and so we choose Landau gauge for its popularity in the Dyson-Schwinger equation community. We rotate to Landau gauge using a Fourier transform accelerated algorithm [130], fixing to the $O(a^2)$ improved gauge fixing functional [21].

For quark propagator analysis, we first fix to maximal centre gauge, and use the centre vortex identification procedure discussed in Chapter 4 to create the vortex only and vortex removed ensembles from the untouched. The untouched and vortex removed ensembles are then gauge fixed to Landau gauge for the propagator analysis. We have investigated using the Landau gauge transformation from the untouched configurations as a preconditioner for the corresponding vortex removed configurations, and found no benefit. The vortex only configurations first have a random gauge transformation performed upon them, and then are cooled for 10 sweeps, as discussed in Chapter 5. These configurations are then fixed to Landau gauge. The untouched configurations with which we compare the cooled vortex only configurations are also cooled before Landau gauge fixing, but after maximal centre gauge fixing.
We have found that without gauge transformation, the centre vortex configurations represent a pathological starting point for the Landau gauge fixing algorithm. We have therefore performed a random gauge transformation before Landau gauge fixing. We have examined several possibilities other than a random gauge transformation, including inverting the maximal centre gauge transformation, and found them to be equivalent. We have also examined the possibility that maximal centre gauge is a pathological starting point for Landau gauge fixing on the untouched and vortex removed ensembles, and found this to not be the case. The difficulties on fixing to Landau gauge in the vortex only case are therefore attributed to the purely diagonal nature of centre elements, which causes numerical instability in common algorithms.

A cylinder cut [131] is performed on the propagator data, and $Z(p)$ is renormalised to be 1 at the highest momentum considered, $p \approx 5.2$ GeV. The matrix sign function is calculated using the Zolotarev rational polynomial approximation [132]. Cooling is performed as in chapter 5, using an $O(a^4)$-three-loop improved action [20,106].

6.3 Results

6.3.1 Vortex-Removed Quark Propagator

We begin by showing results for the untouched and vortex-removed ensembles, plotted in Figs. 6.1 and 6.2. Beginning with the untouched results for the mass function at the highest quark mass, illustrated in Fig. 6.1(a), we see the well-known features reproduced here. In the infra-red region there is strong enhancement of the mass function; this large effective mass is evidence of the presence of dynamical mass generation, and thus dynamical chiral symmetry breaking. At low momenta, the mass function sits around 450 MeV, well above the input bare quark mass of 12 MeV. As we move to higher momenta, the mass function decreases asymptotically toward the bare quark mass. Here short-range effects produce logarithmic corrections associated with the running quark mass. As we increase the quark mass, Figs. 6.1(c), 6.1(e), 6.2(a), and 6.2(c), we see a similar shape, although there is the expected reduction in statistical errors with larger quark masses.
Figure 6.1: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-removed (red crosses) ensembles, at bare quark masses of 12 MeV ((a) & (b)), 25 MeV ((c) & (d)), and 40 MeV ((e) & (f)).
Figure 6.2: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-removed (red crosses) ensembles, at bare quark masses of 50 MeV ((a) & (b)) and 70 MeV ((c) & (d)).
By contrast, the vortex-removed ensemble shows almost no infra-red enhancement. At low momenta, the mass function remains almost flat, with only around 90 MeV of dynamically generated mass, well below the 400 MeV seen on the untouched configurations. This is in agreement with the results of chapter 5; vortex removal has destabilized the otherwise topologically non-trivial instanton-like objects. This led to their destruction under smoothing, and it seems the overlap operator is able to ‘see’ the subtle damage caused to these objects through vortex removal. Dynamical mass generation, and thus dynamical chiral symmetry breaking, have been spoiled with vortex removal. This contrasts with the behaviour of the AsqTad propagator, in Ref. [57], which showed no change in infra-red behaviour after vortex removal. The enhanced chiral sensitivity of the overlap operator is key to detecting these effects.

There remains some residual enhancement of the vortex-removed mass function in the infra-red; this is again in concurrence with the results of chapter 5. There, we saw that the Gribov-related imperfections inherent in the Maximal Centre Gauge fixing procedure resulted in an inability to remove all vortex matter from our gauge field configurations, and thus there remained some topologically non-trivial objects on vortex-removed configurations.

Notably, at higher quark masses, it is clear that the ultra-violet behaviour of the vortex-removed mass function is identical to that of the untouched. The process of removing centre-vortices has spoiled the long-range structure of the vacuum, but the short-range effects remain intact.

Turning now to the renormalization function, plotted in Figs. 6.1 and 6.2, we see again the untouched ensemble reproducing the expected features. The renormalization function is suppressed in the infra-red, and flattens in the ultra-violet limit. The vortex-removed ensemble, again, is identical within error bars in the ultra-violet regime, but significantly different in the infra-red. There is significantly less infra-red suppression of the renormalization function on the vortex-removed ensemble. This ‘tree-like’ behaviour, where $Z(p) \approx 1$, reinforces that we have spoiled the long-range features of the gauge fields.
6.3.2 Vortex-Only Quark Propagator

We turn now to the quark propagator on the vortex-only ensemble. As previously seen in Chapter 5, the nature of the vortex-only configurations, consisting of only centre elements, creates a high-action rough background. The overlap operator, however, has a smoothness requirement in order to be well defined [16]. Additionally, it was seen in the SU(2) case that the vortex-only configurations are too rough to reproduce the low-lying modes of the Dirac operator which are essential to chiral symmetry breaking, but these low-lying modes could be reproduced after smearing [84].

We calculate the vortex-only quark propagator on smoothed backgrounds. We have seen that cooling requires far fewer sweeps than over-improved stout-link smearing to produce a gauge-field background on the vortex-only ensemble similar to that of the untouched, and thus we will consider only smoothing with cooling.

Motivated by the results in chapter 5, we calculate the quark propagator at three levels of cooling; after 10 sweeps, the minimal amount of cooling required to reduce the action on the vortex-only ensembles to an approximately similar level to that on the untouched; after 40 sweeps, the amount of cooling required to produce a similar background of instanton-like objects on the vortex-only and untouched ensembles; and after a large amount of cooling, 80 sweeps. The ordering of the gauge fixing and smoothing algorithms is described in detail in Section 6.2.

The mass and renormalization functions on the untouched and vortex-only ensembles are plotted in Figs. 6.3 and 6.4. As expected at low momenta, the mass function on the untouched configurations shows only a small reduction in dynamical mass generation, while being qualitatively similar to the uncooled results [103]. This is due to the ability of the cooling algorithm to preserve the instanton-like objects responsible for dynamical mass generation, and thus the low momentum behaviour of the mass function. At high momenta, cooling has removed all short-range effects, and so the mass function becomes trivial. Likewise, the renormalization function behaves similarly to that on the uncooled configurations.

The mass and renormalization functions on the vortex-only ensemble are
Figure 6.3: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-only (green circles) ensembles after 10 sweeps of cooling, at bare quark masses of 12 MeV ((a) & (b)), 25 MeV ((c) & (d)), and 40 MeV ((e) & (f)).
Figure 6.4: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-only (green circles) ensembles after 10 sweeps of cooling, at bare quark masses of 50 MeV ((a) & (b)) and 70 MeV ((c) & (d)).
strikingly similar to those on the untouched ensemble at all quark masses considered. Dynamical mass generation, and thus dynamical chiral symmetry breaking, have been completely recreated from the centre vortex information after smoothing. Notably, this is apparent after just 10 sweeps of cooling, far fewer than required to produce a gauge field background of instanton-like objects. Once again, the overlap operator is reactive to subtle gauge field effects, which otherwise require extensive cooling to be made apparent. This remains true up to the highest quark mass considered, with the vortex-only ensemble able to almost exactly reproduce the behaviour seen on the untouched ensemble. The centre vortex degree of freedom robustly contains the information necessary for dynamical chiral symmetry breaking.

The mass and renormalization functions for untouched and vortex-only ensembles with 40 and 80 sweeps of cooling are plotted in Figs. 6.5 and 6.6, and Figs. 6.7 and 6.8 respectively. We have previously seen that after 40 sweeps of cooling the vortex-only gauge field backgrounds resemble an instanton liquid, identical on the ensemble level to that seen on the untouched backgrounds. This background of instanton-like objects is, again, capable of fully reproducing the features of the quark propagator, including dynamical mass generation. Increasing the level of cooling from 40 to 80 sweeps has resulted in the pair annihilation of some instantons, and so a corresponding small loss of dynamical mass generation, but the qualitative features of the quark propagator remain intact. The increase in cooling has a similar effect on both the vortex-only and untouched ensembles; once the gauge field background on each has come to be dominated by instanton-like objects, it behaves similarly under further smoothing.

6.4 Summary

In this chapter, we have used the Landau-gauge overlap quark propagator as a probe of dynamical mass generation, and thus dynamical chiral symmetry breaking. In contrast to previous studies in SU(3) using the AsqTad fermion action, we have been able to show for the first time centre vortices playing a key role in dynamical mass generation. We have shown that removal of centre vortices from gauge field backgrounds correspondingly spoils dynamical mass
Figure 6.5: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-only (green circles) ensembles after 40 sweeps of cooling, at bare quark masses of 12 MeV ((a) & (b)), 25 MeV ((c) & (d)), and 40 MeV ((e) & (f)).
Figure 6.6: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-only (green circles) ensembles after 40 sweeps of cooling, at bare quark masses of 50 MeV ((a) & (b)) and 70 MeV ((c) & (d)).
Figure 6.7: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-only (green circles) ensembles after 80 sweeps of cooling, at bare quark masses of 12 MeV ((a) & (b)), 25 MeV ((c) & (d)), and 40 MeV ((e) & (f)).
Figure 6.8: The mass (left) and renormalization (right) functions on untouched (blue squares), and vortex-only (green circles) ensembles after 80 sweeps of cooling, at bare quark masses of 50 MeV ((a) & (b)) and 70 MeV ((c) & (d)).
generation, and so dynamical chiral symmetry breaking.

We have also shown that, after a small amount of cooling to ensure the smoothness requirement of the overlap action is met, dynamical mass generation can be fully reproduced on vortex-only backgrounds. This reproduction is robust; the vortex-only quark propagator can fully reproduce the behaviour of the untouched ensemble under changes to both the quark mass and the amount of cooling performed. Once the issue of gauge field roughness is dealt with, the vortex-only backgrounds are capable of fully reproducing dynamical mass generation.

The use of the chirally sensitive overlap operator is vital in discerning the results of the changes to the gauge field that come from centre vortex effects. The overlap operator correctly shows that the instanton structure has been damaged by vortex-removal, and so no longer behaves as a topologically non-trivial background. Likewise, we have previously seen that 40 sweeps of cooling are required for the gauge-field background on vortex-only and untouched ensembles to have a similar instanton-liquid structure, but the overlap is able to show similar chiral symmetry breaking behaviour after just 10 sweeps. This is in agreement with the results of Ref. [121], which showed that the low-lying modes of the overlap correctly predicted the number and location of topologically non-trivial objects revealed under cooling. This may also suggest that thick centre vortices are required to reproduce dynamical chiral symmetry breaking; the thin centre vortices of the vortex-only configurations are of themselves insufficient. The characteristic size of thick vortices is around 1 fm. We have found that 10 sweeps of cooling is sufficient to recreate dynamical chiral symmetry breaking from a centre vortex background. Using a random walk hypothesis, this is equivalent to a cooling radius of 1.2 fm.

The behaviour of the quark propagator as we approach the continuum limit will distinguish whether the issue is purely one of gauge-field roughness, or of the necessity to consider thick centre vortices. If it is sufficient to hold the number of smoothing sweeps fixed as the lattice spacing approaches zero, then the issue is one of roughness. If thick centre vortices are necessary to reproduce dynamical chiral symmetry breaking, then smoothing will have to be carried out to a fixed physical length; the characteristic size of thick centre
vortices.

Combined, we have shown for the first time that centre vortices play a similarly important role in SU(3) dynamical chiral symmetry breaking to that shown in SU(2) gauge field theory. The centre vortex degree of freedom robustly contains the information necessary to reproduce the long-range structure of the SU(3) vacuum.

A natural extension to the work here would be to investigate the role of centre vortices in light-quark confinement. While SU(3) centre vortices cannot fully reproduce the untouched, unsmoothed string tension, even after cooling, this may differ in the case of light-quark confinement. The overlap quark propagator in Coulomb gauge would allow us to test this result in the context of light quarks [133].
Chapter 7

Low-lying Hadron Spectrum

In the previous chapter we have seen that SU(3) centre vortices play a key role in dynamical mass generation, a key signal of dynamical chiral symmetry breaking. We now further investigate their role in dynamical chiral symmetry breaking through another channel, the low-lying hadron spectrum.

The breaking of chiral symmetry is an important feature in determining the low-lying QCD spectrum. Under chiral transformations, a large number of hadron currents become degenerate. The dynamical breaking of chiral symmetry, through Goldstone’s theorem, also results in several low-mass hadrons. These features provide a simple test of the presence of dynamical chiral symmetry breaking.

The ground-state hadron spectrum has been previously analysed in the context of SU(3) centre vortices in Ref. [58]. There it was found that upon removal of centre vortices the pion lost its pseudo-Goldstone nature, gaining a much higher mass. Additionally, the pion and rho mesons became degenerate, as did the nucleon and delta baryons. However the use of a non-chirally symmetric fermion action resulted in additive mass renormalisation of quarks, and the changes to the gauge field dynamics under vortex removal lead to uncertainty as to whether the Gell-Mann-Oakes-Renner relation remained valid. It was thus impossible to unambiguously compare vortex-removed and untouched configurations at the same quark mass.

Here we examine the ground-state spectrum using the overlap action. The lattice-deformed version of chiral symmetry gained by this choice offers
both increased sensitivity to chiral effects and the removal of additive mass renormalisation, enabling the comparison of results at identical quark masses. We find evidence of a restoration of chiral symmetry with vortex removal at low quark masses, as well as a replication of the qualitative features of the ground state spectrum on vortex only configurations. This confirms our earlier finding that centre vortices are the key underlying feature behind dynamical chiral symmetry breaking in SU(3).

We begin with a summary of the numerical methods used in Section 7.1. We then present results on the untouched and vortex-only ensembles in Section 7.2. In Section 7.3, we consider the implications of the removal of dynamical chiral symmetry breaking in the hadron spectrum, which provides a framework for the interpretation of the results on the vortex-removed ensemble, presented in Section 7.4.

7.1 Numerical Methods

We use the FLIC operator [7, 17, 18, 128] as the overlap kernel, with negative Wilson mass $m_w = 1$. We have seen in Chapter 6 that 10 sweeps of cooling is sufficient for the vortex-only quark propagator to behave identically to the untouched ensemble propagator. Thus 10 sweeps of cooling will be considered on the vortex-only ensemble herein.

Gauge-invariant Gaussian smearing [134, 135] is performed at the fermion source and sink. We use the value of $n_s = 100$. We use fixed boundary conditions in the temporal direction, and place our source at $n_t = 10$ relative to the lattice length of 40 where a fixed boundary condition is used. Masses are extracted in the standard way.

We have investigated the use of the variational method [136, 137], and found no significant improvement in our ability to discern the ground state signals salient to our investigation. We thus do not use it here.

The meson interpolating fields we consider are listed in Table 7.1, and the baryon in Table 7.2. Note that we consider only isovector mesons in order to avoid disconnected contributions.

We have considered the hadron spectrum over a large range of bare quark masses, given in table 7.3. For mass fits our uncertainties are obtained via a
Table 7.1: Meson interpolators considered herein.

<table>
<thead>
<tr>
<th>Meson</th>
<th>I, J^PC</th>
<th>Operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>π</td>
<td>1, 0^{+-}</td>
<td>\bar{q} \gamma_5 \frac{e_\mu}{2} q</td>
</tr>
<tr>
<td>ρ</td>
<td>1, 1^{--}</td>
<td>\bar{q} \gamma_i \frac{e_\mu}{2} q</td>
</tr>
<tr>
<td>o_0</td>
<td>1, 0^{++}</td>
<td>\bar{q} \gamma_5 \frac{e_\mu}{2} q</td>
</tr>
<tr>
<td>o_1</td>
<td>1, 1^{++}</td>
<td>\bar{q} \gamma_i \gamma_5 \frac{e_\mu}{2} q</td>
</tr>
</tbody>
</table>

Table 7.2: Baryon interpolators considered herein.

<table>
<thead>
<tr>
<th>Baryon</th>
<th>I, J^P</th>
<th>Operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nucleon</td>
<td>\frac{1}{2}, \frac{1}{2}^{+}</td>
<td>[u^T C \gamma_5 d] u</td>
</tr>
<tr>
<td>Δ</td>
<td>\frac{3}{2}, \frac{3}{2}^{+}</td>
<td>[u^T C \gamma_i u] u</td>
</tr>
</tbody>
</table>

second-order single-elimination jackknife analysis.

Table 7.3: Values of the overlap mass parameter, μ, considered, with corresponding bare quark masses in physical units, using a = 0.125 fm.

<table>
<thead>
<tr>
<th>μ</th>
<th>m_q (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.004</td>
<td>13</td>
</tr>
<tr>
<td>0.008</td>
<td>25</td>
</tr>
<tr>
<td>0.012</td>
<td>40</td>
</tr>
<tr>
<td>0.016</td>
<td>50</td>
</tr>
<tr>
<td>0.032</td>
<td>100</td>
</tr>
<tr>
<td>0.040</td>
<td>126</td>
</tr>
<tr>
<td>0.048</td>
<td>151</td>
</tr>
<tr>
<td>0.056</td>
<td>177</td>
</tr>
</tbody>
</table>

7.2 Vortex-Only Spectrum

Results for the lightest quark mass on the untouched and vortex-only ensembles are presented in Fig. 7.1. Results for the untouched spectrum are
as expected. The pion, rho, nucleon, and $\Delta$ all have clear signals, and sit slightly heavier than their physical values. The vortex-only ensemble is able to reproduce all qualitative features of the spectrum, although masses are slightly lower. This is most likely an artefact of cooling; unlike in the quark-propagator results in chapter 6 we have not performed cooling on the untouched ensemble. Notably, in both cases the pion is much lighter than the rho, a clear signal that it retains its nature as a pseudo Goldstone boson, and thus that dynamical chiral symmetry breaking is present on the vortex only ensemble. A clear separation between the nucleon and $\Delta$ baryons is also maintained. Just as seen in the quark-propagator, after smoothing the vortex-only ensemble is able to reproduce the behaviour of the untouched.

The hadron spectrum on the remaining quark masses is plotted in Figs. 7.2, 7.3, and 7.4. The behaviour seen at the lightest quark mass is replicated here. There are clear signals for all four hadrons in both the untouched and vortex-only cases. The masses seen on the vortex-only ensemble are slightly smaller than those in the untouched ensemble, but the qualitative features are reproduced. The pion is much lighter than the rho, behaving as a pseudo-Goldstone boson. The separation between the nucleon and the $\Delta$ is maintained at all quark masses.
Figure 7.2: The low-lying mesons (left) and baryons (right) considered on the untouched (blue) and vortex-only (green) ensembles, at bare quark masses of 25, 38, and 50 MeV.
Figure 7.3: The low-lying mesons (left) and baryons (right) considered on the untouched (blue) and vortex-only (green) ensembles, at bare quark masses of 101 and 126 MeV.
Figure 7.4: The low-lying mesons (left) and baryons (right) considered on the untouched (blue) and vortex-only (green) ensembles, at bare quark masses of 151 and 177 MeV.
These larger quark masses enable an exploration of the vortex only correlators at very large Euclidean times. An interesting artefact in the vortex only effective mass plots appears in the range $t = 17 - 20$, where the effective mass plots rise temporarily before returning to their previous values as the signal degrades. Understanding the nature of this asymptotic behaviour will require further study. If the effect is robust, then possible underlying candidates for the temporary rise include artefacts associated with the finite range of the smoothing algorithm, or the selection of boundary conditions.

Figure 7.5: The $a_0$ and $a_1$ mesons on the untouched (blue) and vortex-only (green) ensembles, at bare quark masses of 126 (a), 151 (b) and 177 (c) MeV.

At low quark masses, the signal for the $a_0$ and $a_1$ mesons is too poor to
allow study. We have plotted these results exclusively at the three heaviest quark masses, 126, 151, and 177 MeV, in Fig. 7.5. At these masses, the $a_0$ and $a_1$ are approximately degenerate on the untouched ensemble, and this is reproduced on the vortex-only ensemble. Again, while the masses are slightly lower, the qualitative features of the hadron spectrum are reproduced.

In pure gauge theory on the lattice, the $\eta'$ meson cannot gain mass from repeated $q\bar{q}$ annihilation due to the lack of disconnected fermion loops, and so does not have the relatively large mass given to it by the axial anomaly in the continuum. This leads to an $\eta' - \pi$ 'ghost' state in the scalar meson channel, which gives a negative contribution to the effective mass [138]. This effect is most pronounced at low quark masses; at higher quark masses this state is much higher energy than the two quark state, and thus does not appear. This phenomenon is responsible for the difficulties in measuring the $a_0$ and $a_1$ at low quark masses.

### 7.3 Restoration of Chiral Symmetry

Before presenting results for the vortex-removed ensemble, it is worth carefully considering our expectations upon removal of dynamical chiral symmetry breaking. Under the restoration of chiral symmetry, we expect baryon currents related by chiral transformations to become degenerate. Recall from Eq. 3.30 that the massless QCD Lagrangian has a SU(2)$_L \times$ SU(2)$_R \times$ U(1)$_A$ symmetry. The U(1)$_A$ symmetry is, however, explicitly broken by the axial anomaly. We must therefore admit the possibility that the U(1)$_A$ and SU(2)$_L \times$ SU(2)$_R$ symmetries are restored separately. The complete restoration of chiral symmetry would imply the following degeneracies [139],

\[
\begin{align*}
\pi & \xrightarrow{U(1)_A} a_0 \\
\rho & \xrightarrow{SU(2)_L \times SU(2)_R} a_1 \\
\text{Nucleon} & \xrightarrow{SU(2)_L \times SU(2)_R} \Delta.
\end{align*}
\]

(7.1)

However in lattice simulations we use a non-zero bare quark mass. Chiral symmetry is thus explicitly broken even in the absence of dynamical chiral
symmetry breaking. At small bare quark masses, the explicit breaking of chiral symmetry is negligible and so we expect these degeneracies to hold. At larger masses, chiral symmetry no longer holds even approximately. We thus expect to see something approximating a non-interacting constituent quark-like model, where the mass of each state is simply the sum of the dressed quark masses composing it, possibly with some momentum.

This is the result seen in Ref. [58]; degenerate pion and rho meson masses were observed, even though the two are not related by a chiral transformation. The mesons had a mass of approximately 2/3 of the mass of the baryons. We have seen in chapter 6 that some amount of remnant dynamical mass generation is present on the vortex-removed ensemble; we expect this constituent quark mass to therefore be larger than the bare quark mass.

![Figure 7.6: The “hairpin” diagram, showing a π-η or ρ-η intermediate state with the quantum numbers of the a₀ or a₁ mesons.](image)

Because we are considering the pure gauge theory, one must also consider multiparticle states contributing to the a₀ and a₁ correlators. In the pure gauge sector, quark flows such as the “hairpin”, illustrated in Fig. 7.6, can carry the quantum numbers of the a₀ or a₁ through π-η or ρ-η intermediate states respectively. Because the sea quark loops vital to generating the mass of the singlet η meson are absent, \( m_\eta = m_\pi \). The associated mass thresholds of these multiparticle states carrying the quantum numbers of the a₀ and a₁ are thus \( 2m_\pi \) and \( m_\pi + m_\rho \) respectively. We will refer to these multiparticle states as π-η and ρ-η states.

One might also be concerned about the “double hairpin” graph illustrated in Fig. 7.7 that can provide a negative-metric contribution to the correlators. All of our correlators on the vortex-removed configurations remain positive and therefore there is no evidence that this process contributes significantly.
In summary, if vortex removal does indeed result in the loss of dynamical chiral symmetry breaking, we can make the following predictions:

- In the absence of dynamical chiral symmetry breaking, the pion is no longer a pseudo Goldstone boson, and so there is no a priori reason for it to have a much lower mass than the other mesons.

- In the high quark mass region, the pion should be degenerate with the rho. Likewise, the nucleon and the $\Delta$ should be degenerate. The baryons should have a mass of $\frac{3}{2}$ times that of the mesons.

- At low quark masses, there is no chiral transformation relating the pion and the rho, and so we expect the two to differ in mass.

- The restoration of the $U(1)_A$ symmetry will be shown by the degeneracy of the pion and ground state $a_0$ at low quark masses.

- The restoration of the $SU(2)_L \times SU(2)_R$ symmetry will be shown by the degeneracy of the rho and ground state $a_1$ at low quark masses.

- In the constituent quark regime, there is no way to make the quantum numbers of the $a_0$ with quarks at rest. There are two possibilities in this channel: a $\pi$-$\eta$ state with mass $2m_\pi$, or two dressed quark masses with the lowest non-trivial momentum to provide overlap with an $l = 1$ orbital angular momentum state. The $a_0$ mass should be the lower of these two.

- Similarly, the $a_1$ should be the lower of two possibilities: a $\rho$-$\eta$ state, or two dressed quark masses with the lowest non-trivial momentum.
• The nucleon and $\Delta$ should be degenerate at all quark masses; they are related through the SU$(2)_L \times$ SU$(2)_R$ symmetry at low mass, and are both composed of 3 dressed quarks at high mass.

For the $a_0$ and $a_1$ mesons composed of two dressed quarks at high quark mass at high quark masses, we expect their mass to be given by two dressed quarks with lowest available non-trivial momentum on the lattice. This energy associated with each such quark is given by

$$E_q^2 = \left( m_{q_{\text{cons.}}}^2 + \left( \frac{\pi}{N_S} \right)^2 \right),$$

(7.2)

where $m_{q_{\text{cons.}}}$ is the constituent quark mass. We note that the constituent quark mass may not match with the infrared mass seen in the quark propagator results in chapter 6: the value extracted there is gauge dependent. Instead, we estimate it as half of the fitted ground state rho mass.

### 7.4 Vortex-Removed Hadron Spectrum

![Figure 7.8](image)

(a) m$_{q} = 0.013$ GeV

- Vortex Removed Pion
- Vortex Removed Rho
- Vortex Removed $a_0$
- Vortex Removed $a_1$

(b) m$_{q} = 0.013$ GeV

- Vortex Removed Nucleon
- Vortex Removed Delta

Figure 7.8: The low-lying mesons (left) and baryons (right) considered on the vortex-removed ensemble, at a bare quark mass of 13 MeV.

Results for the lightest quark mass on the vortex removed ensemble are plotted in Fig. 7.8. In the meson spectrum, the pion behaves similarly to the
untouched case: the ground state mass is reduced slightly from around 200 MeV to around 100 MeV. The rho, however, is markedly different, having a mass of around 170 MeV as compared to 1000 MeV in the untouched ensemble. Both the pion and the rho have smaller masses than even their physical values. Just as we have seen in the quark propagator, vortex removal removes almost all dynamical mass generation. The remnant dynamical mass generation present on the vortex-removed ensemble is reflected by both the pion and the rho having masses larger than twice the bare quark mass. We note that while the rho is greatly reduced in mass, it is not degenerate with the pion, indicating that the quark mass is outside of the constituent quark model region.

The $a_0$ behaves dramatically differently to the untouched case. An excited state of higher mass than the rho meson is visible between timeslices 15 and 20, before becoming degenerate with the pion. This degeneracy is clear evidence of the effective restoration of the $U(1)_A$ symmetry. The excited state seen is consistent with a $\pi-\eta$ state, which has the same quantum numbers as the $a_0$.

The $a_1$ behaves similarly to the $a_0$, showing an excited state consistent with a $\rho-\eta$ state, then a ground state consistent with the rho. The $a_1$, however, has much larger error bars and is not shown for $t > 22$.

In the baryon spectrum, we see similar results. The nucleon and the $\Delta$ both have dramatically lower masses than in the untouched cases, both around 220 - 260 MeV. Notably, they are also nearly degenerate, evidence of the restoration of the $SU(2)_L \times SU(2)_R$ symmetry. Again, this mass is larger than three times the bare quark mass.

Results for the intermediate bare quark masses of 25, 38, and 50 MeV are plotted in Fig. 7.9. Similar trends continue for the pion and rho; both have lower masses than in the untouched case, increasing with increasing bare quark mass. The pion continues to be lighter than the rho, although the gap is reduced at higher values of $m_q$. As the bare quark mass is increased, so is the explicit chiral symmetry breaking, and so the results move closer to the predictions for the constituent quark regime. By $m_q = 50$ MeV, the pion and rho are almost degenerate, as expected in the constituent quark regime. We also note that the pion and rho both show a slow approach to the mass
Figure 7.9: The low-lying mesons (left) and baryons (right) considered on the vortex-removed ensemble, at bare quark masses of 25, 38, and 50 MeV. Note the smaller scale on the vertical axis in (a).
plateau, indicating a dense tower of excited states. This echoes results seen in Ref. [58].

The nucleon and $\Delta$ show remarkably similar behaviour, degenerate within error bars at all 3 of the intermediate quark masses. Similar to the pion and the rho, the baryons show a loss of almost all dynamical mass generation, with a much lower plateau reached than in the untouched case. A slow approach to the mass plateau indicates a dense tower of excited states. Unlike in the meson channel, there is no signal of the transition from the chiral regime to the constituent quark regime; the nucleon and $\Delta$ baryons are predicted to be degenerate in both. The non-degeneracy of the pion and rho mesons at these masses, however, shows that we are outside the constituent quark regime still. The degeneracy of the nucleon and $\Delta$ baryons here is consistent with the restoration of chiral symmetry.

Also of note is the stability of the ground state seen in both the mesons and baryons, a reflection of the near-empty gauge field background in the vortex removed case.

At $m_q = 25$ MeV, both the $a_0$ and $a_1$ are too noisy to extract a clean signal. At the other two intermediate quark masses, a similar result is seen to that at the lightest quark mass. There is an excited state with a mass higher than the other two mesons, followed by a ground state plateau similar to the pion and rho. The degeneracy of the $a_0$ with the pion, and the $a_1$ with the rho, is a signal of the restoration of both the $U(1)_A$ and $SU(2)_L \times SU(2)_R$ symmetries. This, combined with the non-degeneracy of the pion and rho mesons, suggests that at $m_q = 50$ MeV, explicit chiral symmetry breaking is still small enough that the predictions of chiral symmetry restoration hold.

Results at the higher quark masses, 101, 126, 151, and 177 MeV, are presented in Figs. 7.10 and 7.11. At these masses, the pion and rho mesons have become degenerate, indicating that explicit chiral symmetry breaking is now large enough that both hadrons behave as though composed of two weakly-interacting constituent quarks. As seen at the lighter masses, the nucleon and $\Delta$ baryons remain degenerate at all masses.

At these higher quark masses, the $a_0$ and $a_1$ no longer reach a plateau degenerate with the other mesons. This also indicates that at these masses chiral symmetry is no longer approximately restored; the lightest state in
Figure 7.10: The low-lying mesons (left) and baryons (right) considered on the vortex-removed ensemble, at bare quark masses of 101 and 126 MeV.
Figure 7.11: The low-lying mesons (left) and baryons (right) considered on the vortex-removed ensemble, at bare quark masses of 151 and 177 MeV.
these channels is now two quarks of lowest non-trivial momentum.

Qualitatively, the results seen suggest agreement with the predictions of chiral symmetry restoration below a bare quark mass of 50 MeV, and above that, agreement with the predictions of a constituent quark-like model. We now turn to quantitative measures of these predictions.

Table 7.4: Fitted masses of the pion, rho, nucleon, and Δ as a function of the bare quark mass, $m_q$.

<table>
<thead>
<tr>
<th>$m_q$ (MeV)</th>
<th>$m_\pi$ (MeV)</th>
<th>$m_\rho$ (MeV)</th>
<th>$m_N$ (MeV)</th>
<th>$m_\Delta$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>85(3)</td>
<td>171(7)</td>
<td>219(6)</td>
<td>260(10)</td>
</tr>
<tr>
<td>25</td>
<td>132(4)</td>
<td>203(5)</td>
<td>272(7)</td>
<td>295(7)</td>
</tr>
<tr>
<td>38</td>
<td>173(4)</td>
<td>228(5)</td>
<td>316(7)</td>
<td>334(6)</td>
</tr>
<tr>
<td>50</td>
<td>213(4)</td>
<td>257(4)</td>
<td>365(5)</td>
<td>378(5)</td>
</tr>
<tr>
<td>100</td>
<td>366(3)</td>
<td>386(3)</td>
<td>572(5)</td>
<td>575(5)</td>
</tr>
<tr>
<td>126</td>
<td>439(3)</td>
<td>453(3)</td>
<td>676(4)</td>
<td>676(4)</td>
</tr>
<tr>
<td>151</td>
<td>510(3)</td>
<td>521(3)</td>
<td>780(4)</td>
<td>779(4)</td>
</tr>
<tr>
<td>177</td>
<td>578(3)</td>
<td>588(3)</td>
<td>881(4)</td>
<td>880(5)</td>
</tr>
</tbody>
</table>

We will first consider the validity of the constituent quark-like model of the hadron spectrum at high bare quark masses. Fits of the ground state masses of the pion, rho, nucleon, and Δ, are summarised in table 7.4. In Fig. 7.12, we have plotted these masses, divided by the number of valence quarks. At masses of $m_q = 101$ MeV and beyond, the constituent quark-like model is highly successful in describing the behaviour of the spectrum, with all hadrons degenerate after division by the number of constituent quarks. At these quark masses, all four hadrons can be accurately modelled as weakly interacting dressed quarks. Below this value, while the rho, nucleon and Δ are still in agreement, the pion is markedly lighter. It is in this region, therefore, that we expect the predictions of the chirally restored theory to be valid.

While the large error bars on the $a_0$ and $a_1$ make fitting a mass value difficult, we can test the predictions for their values in both the chiral and constituent quark regimes using ratios of masses. For the $a_0$, the $U(1)_A$ symmetry predicts degeneracy with the pion in the chiral regime. In the
Figure 7.12: The implied constituent quark mass from each of the hadrons considered as a function of the input bare quark mass.
constituent quark regime, we expect it to be described by two constituent quarks of lowest non-trivial momentum, $2E_q$. At all masses a $\pi$-$\eta$ state has the same quantum numbers as the $a_0$, and thus appears as an excited state, or possibly as the ground state in the constituent quark regime.

We have thus defined the ratio

$$R_1 = \frac{m_{a_0}}{2m_\pi}.$$ \hspace{1cm} (7.3)

In the chiral regime, we expect this to have a value of $1/2$, as $m_{a_0} = m_\pi$. If the $a_0$ is described by a $\pi$-$\eta$ state, it will have a value of 1. At high quark masses, this ratio will be given by $\frac{2E_q}{4m_{q,\text{cons}}}$. For the $a_1$, the SU(2)$_L \times$ SU(2)$_R$ symmetry predicts degeneracy with the rho meson in the chiral regime. In the constituent regime, we again expect it to be described by $2E_q$. The quantum numbers of the $a_1$ can be produced by a $\rho$-$\eta$ state, and so we expect this to appear as an excited state in the chiral regime, and as either an excited state or the ground state in the constituent regime.

We have thus defined the ratio

$$R_2 = \frac{m_{a_1}}{2m_\rho}.$$ \hspace{1cm} (7.4)

Again, in the chiral regime we expect this to have a value of $1/2$, as $m_{a_1} = m_\rho$. In the constituent quark regime, we again expect a value of $\frac{2E_q}{4m_{q,\text{cons}}}$. We note that while a $\rho$-$\eta$ state is required to create the quantum numbers of the $a_1$, in the constituent regime the pion and the rho are degenerate, and so this state will have mass $m_\rho + m_\pi = 2m_\pi$ and once again produce a value of $R_2 = 1$. In the chiral regime, the pion is lighter than the rho, and so $R_2$ for this state will be less than 1, varying from 0.75 at $m_q = 13$ MeV to 0.91 at $m_q = 50$ MeV. This still allows a clean separation from the prediction of restored chiral symmetry, where $R_2 = \frac{1}{2}$.

We have summarised these ratios and their expected values in Table 7.5.

Based on the results in Fig. 7.12, we have defined the constituent quark mass to be half the fitted mass of the rho meson. These values, together with the corresponding energies of constituent quarks with smallest non-trivial momentum, derived from Eq. 7.2, are listed in Table 7.6.
Figure 7.13: The ratio $R_1$ for the $a_0$ meson on the vortex-removed ensemble, at bare quark masses of 13 (a), 25 (b), 38 (c), and 50 MeV (d). Horizontal lines are drawn at $\frac{1}{2}$ ($U(1)_A$ chiral regime), 1 ($\pi$-$\eta$ state), and $\frac{2 E_q}{4 m_q^{cons}}$ (two-quark state) to guide the eye.
Table 7.5: The two mass ratios considered, together with their expected values in the chirally restored and constituent quark regimes.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>Definition</th>
<th>Chiral regime value</th>
<th>Constituent quark regime value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$\frac{m_{a_0}}{2m_\pi}$</td>
<td>$\frac{1}{2}$</td>
<td>Smaller of 1 ($\pi$-$\eta$ state) or $\frac{2E_q}{4m_{q\text{ cons.}}}$ (2 quark state)</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$\frac{m_{a_1}}{2m_\rho}$</td>
<td>$\frac{1}{2}$</td>
<td>Smaller of 1 ($\rho$-$\eta$ state) or $\frac{2E_q}{4m_{q\text{ cons.}}}$ (2 quark state)</td>
</tr>
</tbody>
</table>

Table 7.6: Constituent quark masses ($m_{q\text{ cons.}}$) and energy of a constituent quark with smallest non-trivial momentum ($E_q$) as inferred from the fitted ground-state rho meson masses.

<table>
<thead>
<tr>
<th>$m_q$ (MeV)</th>
<th>$m_{q\text{ cons.}}$ (MeV)</th>
<th>$E_q$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>85</td>
<td>262</td>
</tr>
<tr>
<td>25</td>
<td>101</td>
<td>267</td>
</tr>
<tr>
<td>38</td>
<td>114</td>
<td>273</td>
</tr>
<tr>
<td>50</td>
<td>128</td>
<td>279</td>
</tr>
<tr>
<td>100</td>
<td>193</td>
<td>314</td>
</tr>
<tr>
<td>126</td>
<td>226</td>
<td>336</td>
</tr>
<tr>
<td>151</td>
<td>260</td>
<td>359</td>
</tr>
<tr>
<td>177</td>
<td>294</td>
<td>385</td>
</tr>
</tbody>
</table>

In Fig. 7.13 we have plotted $R_1$ for the $a_0$ meson at the four lightest bare quark masses: 13, 25, 38, and 50 MeV. At the lightest quark mass, $R_1$ touches 1, before dropping down to a stable value at $\frac{1}{2}$. The plateau at $\frac{1}{2}$ shows a restoration of the U(1)$_A$ symmetry; degeneracy of the $a_1$ and pion. There is also evidence of a $\pi$-$\eta$ state in the same channel, reflected by the value of 1 at earlier timeslices.

At $m_q = 38$ and $m_q = 50$ MeV, there remains clear evidence of the multiparticle state; a plateau at 1 is seen at early timeslices. At later timeslices, however, while there is some evidence of the value decreasing, the signal becomes too noisy to see a ground state plateau at $\frac{1}{2}$. It may be that due to the additional explicit symmetry breaking from the axial anomaly, the restoration of the U(1)$_A$ symmetry is particularly sensitive to explicit sym-
metry breaking from the bare quark mass. In this case, the $a_1$ would be in an interim region, where the U(1)$_A$ symmetry is broken, but the hadron spectrum does not yet behave like a constituent quark model. In this case, the lowest-lying state in this channel is a $\pi$-$\eta$ threshold state.

Figure 7.14: The ratio $R_1$ for the $a_0$ meson on the vortex-removed ensemble, at bare quark masses of 100 (a), 126 (b), 151 (c), and 177 MeV (d). Horizontal lines are drawn at $\frac{1}{2}$ (U(1)$_A$ chiral regime), 1 ($\pi$-$\eta$ state), and $\frac{2E_q}{4m_{q\text{ cons.}}}$ (two-quark state) to guide the eye.

We have plotted $R_1$ for the $a_0$ meson at the four highest quark masses in Fig. 7.14. We have seen previously that at these masses the hadrons behave like weakly-interacting constituent quarks, and this is quantified here. $R_1$ lies almost exactly on the line drawn at $\frac{2E_q}{4m_{q\text{ cons.}}}$, indicating the $a_0$ is best described by two constituent quarks with the smallest non-trivial back-to-
back momenta. There is no clear evidence of any other states in this region.

![Graphs showing the ratio R₂ for the a₁ meson at different quark masses.](image)

**Figure 7.15:** The ratio $R_2$ for the $a_1$ meson on the vortex-removed ensemble, at bare quark masses of 13 (a), 25 (b), 38 (c), and 50 MeV (d). Horizontal lines are drawn at $\frac{1}{2}$ (SU(2)$_L$ × SU(2)$_R$ chiral regime), $\frac{m_\pi+m_\rho}{2m_\rho}$, 1, and $\frac{2E_q}{m_{q,cons.}}$ (two-quark state) to guide the eye.

We have plotted the ratio $R_2$ for the $a_1$ meson at the four lowest quark masses in Fig. 7.15. At all four masses, $R_2$ lies on the line $\frac{m_\rho+m_\pi}{2m_\rho}$ initially; this corresponds to the $a_1$ being described by a $\rho$-$\eta$ state. At the lightest two masses, the signal is too poor to provide evidence of any other state.

At $m_q = 38$ MeV and $m_q = 50$ MeV, however, after timeslice 25 another stable plateau is seen at $R_2 = \frac{1}{2}$. This indicates the degeneracy of the $a_1$ with the rho meson, evidence of the restoration of the SU(2)$_L$ × SU(2)$_R$ symmetry. This concurs with the results seen for the onset of degeneracy between the
pion and rho; at $m_q = 50$ MeV, explicit chiral symmetry breaking from the quark mass is still sufficiently small that the symmetry holds.

![Graphs showing the ratio $R_2$ for the $a_1$ meson at different quark masses.](image)

Figure 7.16: The ratio $R_2$ for the $a_1$ meson on the vortex removed ensemble, at bare quark masses of 100 (a), 126 (b), 151 (c), and 177 MeV (d). Horizontal lines are drawn at $\frac{1}{2}$ (SU(2)$_L \times$ SU(2)$_R$ chiral regime), $\frac{m_\pi + m_\rho}{2m_\rho}$, 1, and $\frac{2E_q}{4m_{q,\text{cons}}}$ (two-quark state) to guide the eye.

$R_2$ for the $a_1$ meson is plotted at the four highest quark masses considered in Fig. 7.16. Just as is the case for $R_1$, at high quark masses $R_2$ reaches a clear plateau on the line at $\frac{2E_q}{4m_{q,\text{cons}}}$. There is an onset of degeneracy between the $a_0$ and $a_1$ at high quark masses. While this is a feature seen also in the untouched ensemble, remarkably the masses of both are given within error bars by $2E_q$. At the four highest quark masses, the constituent quark-like model is remarkably successful; all six hadrons considered are in agreement.
with the predictions at all of these masses.

7.5 Summary

We have previously seen evidence through the quark propagator that vortex removal results in the loss of dynamical chiral symmetry breaking, and correspondingly that a vortex-only background is capable of reproducing dynamical chiral symmetry breaking. In this chapter, we have further investigated this through the low-lying hadron spectrum.

It was seen in chapters 5 and 6 that a small amount of cooling is required for a vortex-only background to reproduce the topological features and dynamical mass generation seen on the untouched configurations. We have found here that after 10 sweeps of cooling, the vortex-only backgrounds are capable of recreating the features of the ground-state hadron spectrum. While the ground state masses are slightly lower due to cooling, the qualitative features of the spectrum are intact. In particular, the pion remains much lighter than the other mesons. Its behaviour as a pseudo-Goldstone boson is a clear signal of the presence of dynamical chiral symmetry breaking on the vortex-only ensemble. There is also no sign of chiral symmetry restoration, with a clear separation of the masses of chiral partners maintained. Just as seen in the quark propagator, after only a small amount of smoothing, the overlap operator is able to reproduce dynamical chiral symmetry breaking on the vortex-only ensemble, and the two results are in agreement as to the amount of smoothing required.

In the vortex removed case, we have examined a number of signals of the loss of dynamical chiral symmetry breaking. At low quark masses, there is strong evidence of the restoration of SU(2)$_L \times$ SU(2)$_R$ chiral symmetry. The nucleon and $\Delta$ baryons become degenerate, as do the $a_1$ and rho mesons. The evidence for the restoration of the U(1)$_A$ symmetry at our lowest quark masses is clear; the $a_0$ shows a degeneracy with the pion at the lightest mass considered. However, the signal is less satisfactory at the other light masses. It may be either that this issue would be resolved with more statistics, or that the U(1)$_A$ symmetry is more sensitive to explicit chiral symmetry breaking from the bare quark mass.
At high quark masses, the vortex-removed hadron spectrum behaves as weakly-interacting constituent quarks in a near-trivial background, as seen in previous studies. We have seen from the quark-propagator results that there is some remnant dynamical mass generation on the vortex-removed ensemble, and this is replicated here. The implied constituent quark mass from the pion, rho, nucleon, and ∆ are in agreement, with a value higher than the bare quark mass. Using the constituent quark mass extracted from these, the $a_0$ and $a_1$ mesons can be successfully described as consisting of two constituent quarks with smallest non-trivial back-to-back momentum.

The vortex-removed gauge fields behave as an almost trivial background. Intriguingly, unlike in the quark propagator, previous studies using Wilson-like fermion actions were able to correctly show this at high quark masses, despite explicit chiral symmetry breaking. Here, however, for the first time we have also been able to reproduce symmetries expected in the chirally restored regime. The use of the overlap, which respects chiral symmetry, is vital to revealing this. The vortex removed-ensemble deviates from a trivial background only in that there is a small amount of dynamical mass generation, as evinced by a constituent-quark mass larger than the bare quark mass.

These results have again shown that centre vortices are the key feature behind dynamical chiral symmetry breaking in SU(3) gauge theory; we have successfully both reproduced dynamical chiral symmetry breaking from vortex only backgrounds and removed it with vortex removal.
Chapter 8

Conclusion

8.1 Summary

The goal of this thesis was to use the tools of lattice QCD to determine whether SU(3) centre vortices are the fundamental underlying feature behind dynamical chiral symmetry breaking. After examining multiple signals of dynamical chiral symmetry breaking, we can answer this question in the affirmative.

We have found that with the removal of centre vortices dynamical chiral symmetry breaking is spoiled. At very light quark masses chiral symmetry was manifest in the hadron mass spectrum of the vortex removed ensemble. In addition a background consisting solely of centre vortices can fully reproduce dynamical chiral symmetry breaking in both the quark propagator and the associated hadron mass spectrum.

We began in Chapter 4 by describing in detail the procedure for identifying centre vortices on the lattice. We used this procedure to define ensembles consisting solely of centre vortices, with vortices removed, and the original untouched configurations. We then argued that the centre vortices identified using this procedure correspond to physical objects on the Monte-Carlo generated configurations.

In Chapter 5, we examined the effects of smoothing on the ensembles. It was found that while the vortex-removed ensemble had a similar action and topological charge density to the untouched ensemble, these densities
were rapidly reduced under smoothing. After smoothing, the vortex-removed ensemble resembled a trivial gauge field background, with almost no long-range structure. We also found that after smoothing, the vortex-only ensemble became essentially equivalent to the untouched ensemble, with both the ensemble-averaged action and topological charge being the same.

Here we also touched on the issue of confinement, and found that after smoothing the vortex-only background produced a similar string tension to the untouched after smoothing, although this was reduced from the full string tension in the unsmoothed case. We also successfully replicated the established result that after vortex removal the string tension vanishes.

We then examined the instanton degrees of freedom, and found a strong connection between centre vortices and instanton-like objects. Upon vortex removal, topologically non-trivial instantons are ‘destabilised’, leading to their removal under smoothing. Conversely, despite beginning from a background consisting solely of centre elements, instanton-like objects emerged on the vortex-only ensemble after smoothing. The ensemble average size, topological charge, and density of these objects was identical to those found on the untouched ensemble. This result is robust under choice of smoothing algorithm, thus the centre vortex degree of freedom contains all the information necessary to recreate instanton-like objects.

We then examined dynamical mass generation through the Landau-gauge overlap quark propagator in Chapter 6. Here we found that the overlap fermion action was able to correctly discern that the topological objects on the vortex-removed configurations had been spoiled; after vortex removal very little dynamical mass generation was seen. The quark propagator resembled its tree-level equivalent.

On the vortex-only ensemble, after a small amount of cooling the quark propagator reproduced the infrared behaviour of its untouched equivalent within error bars. Notably, the amount of cooling required was smaller than that needed to produce an instanton background. We found that this behaviour was robust under changes of quark mass and the level of cooling performed. Under all permutations, dynamical mass generation was fully reproduced on the vortex-only ensemble.

Lastly, we examined the low-lying hadron spectrum in Chapter 7. Here
we found agreement with the results obtained in the examination of the quark propagator. The vortex-only ensemble was capable of reproducing dynamical chiral symmetry breaking; chiral partners were not degenerate, and the pion retained its nature as a pseudo-Goldstone boson.

We were also able to show dynamical chiral symmetry breaking spoiled on the vortex-removed background. At low quark masses, the predictions of chiral symmetry restoration were observed, with chiral partners becoming degenerate. We found strong evidence that the chiral SU(2)$_L \times$ SU(2)$_R$ symmetry was restored at low quark masses, and evidence of the restoration of the U(1)$_A$ symmetry. At high quark masses, we successfully described the masses of all hadrons considered using a constituent quark-like model of weakly interacting dressed quarks, again an indication that with vortex removal the gauge field background leaves only a small residual dressing of the bare quark mass.

Combined, all examined measures were in agreement. A centre vortex configuration can fully reproduce SU(3) dynamical chiral symmetry breaking, and the removal of centre vortices spoils it. We can thus conclude that centre vortices are the key fundamental object underpinning SU(3) dynamical chiral symmetry breaking.

8.2 Future Investigations

The results shown herein have offered several lines of investigation for future work. We have found a small amount of smoothing is required on the vortex-only backgrounds in order to reproduce dynamical chiral symmetry breaking, and the measures considered are in agreement as to the extent. We have suggested two possibilities for this result; either this level of smoothing is required in order to create a gauge-field background smooth enough for lattice algorithms to be well defined, or the fundamental object of import is the thick centre vortex, rather than the thin. These two scenarios can be distinguished by the behaviour of these results on the approach to the continuum limit; if it is sufficient to hold the number of smoothing sweeps constant, corresponding to a decreasing physical smoothing radius, the explanation of smoothness is correct. If it is needed to keep the physical smoothing radius constant, then
the explanation of thick centre vortices is correct. It would also be of interest to study this behaviour using a smoothing algorithm explicitly designed for centre vortices, such as an SU(3) extension of the technique used in Ref. [114].

The question of the role of centre vortices in SU(3) confinement also invites further investigation. While we have found that the centre vortex configurations can reproduce the untouched string tension after smoothing, the level of smoothing required is higher than that required to reproduce dynamical chiral symmetry breaking, and the string tension on the untouched configurations is much lower after smoothing. The use of the Coulomb-gauge overlap quark propagator would allow the study of light quark confinement, providing insight into this result.

While the evidence of the restoration of the SU(2)_L × SU(2)_R chiral symmetry in the hadron spectrum after vortex removal was clear, the evidence for the restoration of the U(1)_A symmetry can be strengthened. It may be that the axial anomaly spoils the restoration of this symmetry, even with the removal of dynamical chiral symmetry breaking, or that the restoration of this symmetry is more sensitive to explicit chiral symmetry breaking. Study of chiral partners in the hadron spectrum, that require the calculation of disconnected contributions, would clarify this issue.
Appendix A

Maximal Centre Gauge Fixing Coefficients

For the Maximal Centre Gauge fixing procedure, discussed in section 4.1, we wish to maximize the quantity

$$R_{\text{local}}(x') = \sum_{\mu} |\text{Tr} \Omega(x') U_{\mu}(x')|^2 + \sum_{\mu} |\text{Tr} U_{\mu}(x' - \hat{\mu}) \Omega^\dagger(x')|^2,$$

at a given lattice site $x'$, for a local gauge transformation $\Omega(x')$.

This is achieved by restricting $\Omega(x')$ to be in one of the SU(2) subgroups of SU(3). Then, we parametrize an SU(2) matrix $[\Omega(x')]_{\text{SU}(2)}$ by

$$[\Omega(x')]_{\text{SU}(2)} = g I - ig \sigma_i,$$

and embed the resulting SU(2) matrix in one of the three SU(2) subgroups of SU(3). This allows us to rewrite the gauge fixing condition in terms of the SU(2) parametrization as

$$R_{\text{local}}(x') = \sum_{i,j=1}^{4} \frac{1}{2} g_i a_{ij} g_j - \sum_{i}^{4} g_i b_i + c.$$

Where $a_{ij}$ are elements of a real, symmetric matrix, $b_i$ a real vector and $c$ a real constant, all dependent only on $U_{\mu}(x')$ and $U_{\mu}(x' - \hat{\mu})$ for the four values of $\mu$ as discussed below. Once the coefficients are defined the local quantity $R_{\text{local}}(x')$ can be maximised according to the method of, e.g., Ref. [96].
APPENDIX A. MAXIMAL CENTRE GAUGE FIXING COEFFICIENTS

The gauge fixing coefficients $a_{ij}$, $b_i$, and $c$ are defined in terms of the gauge links $U_\mu(x')$ and $U_\mu(x' - \hat{\mu})$, for any given SU(2) subgroup of SU(3).

It is convenient to define

$$U = U_\mu(x'),$$
$$V = U_\mu(x - \hat{\mu}),$$

where we are suppressing the index $\mu$. Then we define complex $2 \times 2$ matrices, $U$ and $V$, as the components of $U$ and $V$ corresponding to the given SU(2) subgroup, and complex scalars, $u$ and $v$, as the remaining diagonal element. The other elements of $U$ and $V$ do not contribute to the coefficients, and so for convenience are set to 0.

As an example, for the first SU(2) subgroup, we set

$$U = \begin{pmatrix} U & 0 \\ 0 & 0 \\ 0 & 0 & u \end{pmatrix},$$

and

$$V = \begin{pmatrix} V & 0 \\ 0 & 0 \\ 0 & 0 & v \end{pmatrix}.$$
APPENDIX A. MAXIMAL CENTRE GAUGE FIXING COEFFICIENTS

Then we have for the real symmetric matrix $a_{ij},$

\[
\begin{align*}
a_{11} &= |U_{12}|^2 + |U_{21}|^2 + U_{12}U_{21} + U_{21}U_{12} + |V_{12}|^2 + |V_{21}|^2 + V_{12}V_{21} + V_{21}V_{12}, \\
a_{12} &= 2i \left(U_{12}U_{21}^* - U_{21}U_{12}^* + V_{12}V_{21}^* - V_{21}V_{12}^* \right), \\
a_{13} &= +U_{11}U_{12}^* + U_{12}U_{11}^* + U_{11}U_{21}^* + U_{21}U_{11}^* - U_{22}U_{12}^* - U_{12}U_{22}^* - U_{22}U_{12}^* - U_{12}U_{22}^* \\
&\quad + V_{11}V_{12}^* + V_{12}V_{11}^* + V_{11}V_{21}^* + V_{21}V_{11}^* - V_{22}V_{12}^* - V_{12}V_{22}^* - V_{22}V_{12}^* - V_{12}V_{22}^*, \\
a_{14} &= i \left(U_{11}U_{12}^* - U_{12}U_{11}^* + U_{11}U_{21}^* - U_{21}U_{11}^* + U_{22}U_{12}^* - U_{12}U_{22}^* + U_{22}U_{12}^* - U_{12}U_{22}^* \\
&\quad - V_{11}V_{12}^* + V_{12}V_{11}^* - V_{11}V_{21}^* + V_{21}V_{11}^* - V_{22}V_{12}^* + V_{12}V_{22}^* - V_{22}V_{12}^* + V_{12}V_{22}^*, \\
a_{22} &= |U_{12}|^2 + |U_{21}|^2 - U_{12}U_{21}^* - U_{21}U_{12}^* + |V_{12}|^2 + |V_{21}|^2 - V_{12}V_{21}^* - V_{21}V_{12}^*, \\
a_{23} &= i \left(-U_{11}U_{12}^* + U_{12}U_{11}^* + U_{11}U_{21}^* - U_{21}U_{11}^* + U_{22}U_{12}^* - U_{12}U_{22}^* - U_{22}U_{12}^* + U_{22}U_{12}^* \\
&\quad - V_{11}V_{12}^* + V_{12}V_{11}^* + V_{11}V_{21}^* + V_{21}V_{11}^* + V_{22}V_{12}^* + V_{12}V_{22}^* + V_{22}V_{12}^* + V_{12}V_{22}^*, \\
a_{24} &= +U_{11}U_{12}^* + U_{12}U_{11}^* - U_{11}U_{21}^* - U_{21}U_{11}^* + U_{12}U_{12}^* + U_{12}U_{22}^* - U_{22}U_{21}^* - U_{22}U_{21}^* \\
&\quad - V_{11}V_{12}^* - V_{12}V_{11}^* + V_{11}V_{21}^* + V_{21}V_{11}^* - V_{22}V_{12}^* - V_{12}V_{22}^* + V_{22}V_{12}^* + V_{22}V_{12}^*, \\
a_{33} &= |U_{11}|^2 + |U_{22}|^2 - U_{11}U_{22}^* + U_{22}U_{11}^* + |V_{11}|^2 + |V_{22}|^2 - V_{11}V_{22}^* - V_{22}V_{11}^*, \\
a_{34} &= 2i \left(-U_{11}U_{22}^* + U_{22}U_{11}^* + V_{11}V_{22}^* - V_{22}V_{11}^* \right), \\
a_{44} &= |U_{11}|^2 + |U_{22}|^2 + U_{11}U_{22}^* + U_{22}U_{11}^* + |V_{11}|^2 + |V_{22}|^2 + V_{11}V_{22}^* + V_{22}V_{11}^*,
\end{align*}
\]

and for the real vector $b_i,$

\[
\begin{align*}
b_1 &= i \left(-U_{12}u^* + U_{12}u - U_{21}u^* + U_{21}u + V_{12}v^* - V_{12}v + V_{21}v^* - V_{21}v \right), \\
b_2 &= U_{12}u^* + U_{12}u - U_{21}u^* - U_{21}u - V_{12}v^* - V_{12}v + V_{21}v^* + V_{21}v, \\
b_3 &= i \left(-U_{11}u^* + U_{11}u + U_{22}u^* - U_{22}u + V_{11}v^* - V_{11}v - V_{22}v^* + V_{22}v \right), \\
b_4 &= U_{11}u^* + U_{11}u + U_{22}u^* + U_{22}u + V_{11}v^* + V_{11}v + V_{22}v^* + V_{22}v,
\end{align*}
\]

and
\[
\begin{align*}
c &= |u|^2 + |v|^2.
\end{align*}
\]

Recalling the definition of $U$ and $V$ in Eq. A.4, the definitions of $a, b,$ and $c$ above all require a sum over the suppressed $\mu$ index.
Bibliography


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