Maximizing Excess Return Per Unit Variance:

A Novel Investment Management Objective[1]

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Abstract

I propose a novel investment objective for portfolios fully invested in risky assets only. The new objective is based on achieving the highest possible excess return per unit of variance. The optimal portfolio is a linear combination of the tangent portfolio and the minimum variance portfolio where the weights are inversely proportional to the standard deviation of the return of each portfolio. Using a standard factor model of securities' returns, I provide an empirical application of the optimal portfolio and show that it performs quite well out-of-sample relative to the maximum Sharpe ratio portfolio as well as the minimum variance portfolio.

Key Words: Risk premia, tracking error, active return, tangent portfolio weights, minimum variance portfolio weights, factor models of expected returns.

JEL Classification: G11, G12.

1 Introduction

The ground-breaking work of Markowitz (1952) and Sharpe (1964) has been instrumental in establishing the first two moments of risky securities' returns as well as the concept of systematic risk as the crucible of both applied and theoretical finance.[1] The importance of the trade-off between return and systematic risk with the associated risk-return trade-off in terms of the Sharpe ratio has come to dominate investment management performance in both academia as well as practice. The market model of Sharpe (1964),[2] in particular, has proved very useful in terms of implementing the tangent portfolio by serendipitously shrinking the estimates of securities' first and second moments of returns using historical data. More recently, practitioners have noted the excellent out-of-sample performance of the minimum variance portfolio (see Clark et al (2013), among many others). From the perspective of the mean-*variance* trade-off, in the sense of excess return per unit variance, achieved by both the tangent and the minimum variance portfolio, it is well-known that both portfolios offer the same excess return per unit variance. This is true despite the obvious fact that *ex ante* the minimum variance portfolio necessarily has a lower Sharpe ratio than the tangent portfolio. Nevertheless, the fact that both portfolios offer the same excess return per unit variance implies that there is a portfolio located on the mean-variance frontier somewhere between the minimum variance and the tangent portfolio offering a greater excess return for the same amount of risk. This observation is the starting point of the paper. Next, I present a simple motivating example, followed by some analytical results and an empirical implementation of the optimal portfolio. Finally, the last section of the articles offers some concluding thoughts as well as directions for future research.

2 A Motivating Example

Consider a set of two risky assets offering excess returns of $\mu_1 = 0.05$ and $\mu_2 = 0.10$. The standard deviations of the returns of the two assets are given by $\sigma_1 = 0.05$ and $\sigma_2 = 0.10$,

[1]Fully 60 years after Markowitz (1952), the optimality of mean and variance in the framework of portfolio choice has been strongly reaffirmed even if securities returns are moderately non-Gaussian (Markowitz (2012)).

[2]Numerous studies have raised concerns about the validity of the market model most recently in Chen (2015).

respectively. The correlation between the excess returns of the two assets if $\rho_{12} = 0.5$. Note that the risk premium offered by asset one is equal to $0.05/(0.05^2) = 20$ while the risk premium offered by asset two is given by $0.1/(0.1^2) = 10$. The global minimum variance portfolio consists of 100% invested in the first asset and 0% invested in the second asset. The tangent portfolio is comprised of 66.67% invested in the first asset and 33.33% invested in the second asset. Both the minimum variance and the tangent portfolio have an excess return per unit variance ratio, or a risk premium, of 20. A portfolio that is designed to achieve the highest possible ratio of excess return per unit variance contains 84.53% invested in the first asset and 15.47% invested in the second asset. This maximum risk premium portfolio has an excess return per unit variance of 21.547 which exceeds the reward per unit variance of both the tangent and the minimum variance portfolios.

Figure 1 plots the mean-variance frontier along with the three portfolios just described in mean-standard deviation space. It is clearly obvious that the maximal risk premium portfolio has a Sharpe ratio that is lower than the Sharpe ratio of the tangent portfolio. Similarly, the Sharpe ratio of the minimum variance portfolio is always lower than the Sharpe ratio of the maximal risk premium portfolio.

Insert Figure 1 about here.

It is hard to picture the risk premium in mean-standard deviation space and, hence, it is not immediately obvious that the reward per unit variance is higher for the proposed portfolio compared to the tangent and the minimum variance portfolios. Figure 2 presents the same frontier and the three portfolios in mean-variance space. The slopes of the lines going through the origin are now risk premia rather than Sharpe ratios. Observing the frontier and the relative positions of the three portfolios in this plot it is clear that the maximal excess return per unit risk portfolio indeed offers a greater excess return for the same amount of variance risk than either the minimum variance or the tangent portfolio.

Insert Figure 2 about here.

Furthermore, there are at least two economic reasons to consider excess return per unit variance as an alternative investment management objective. First, the certainty equivalent return to an investor with a constant absolute risk aversion of Γ is $CER = \mu - \frac{1}{2}\Gamma\sigma^2$. This quantity will be positive whenever the excess return per unit variance,

μ/σ^2, exceeds $\frac{1}{2}\Gamma$. Secondly, in deciding how much to allocate to a portfolio consisting of risky assets, an investor with a constant relative risk aversion of γ will invest a fraction of their total wealth equal to $\frac{\mu}{\gamma\sigma^2}$. Once again, the level of overall exposure to risky assets will be determined by the relative magnitude of the excess return per unit variance and the coefficient of constant relative risk aversion.

3 Model

Consider a set of N risky assets with a vector of excess mean returns of μ and a variance-covariance matrix of V. Standard mean-variance portfolio theory provides the exact portfolio weights of the tangent portfolio, w_{tg}, and the minimum variance portfolio, w_{mv}, as follows:

$$w_{tg} = \frac{V^{-1}\mu}{1_N' V^{-1}\mu}, \tag{1}$$

$$w_{mv} = \frac{V^{-1}1_N}{1_N' V^{-1}1_N}, \tag{2}$$

where 1_N is an $(N \times 1)$ column vector of ones.

The expected excess return of the tangent portfolio, μ_{tg}, and the minimum variance portfolio, μ_{mv}, are, respectively:

$$\mu_{tg} = \frac{\mu' V^{-1}\mu}{1_N' V^{-1}\mu}, \tag{3}$$

$$\mu_{mv} = \frac{\mu' V^{-1}1_N}{1_N' V^{-1}1_N}. \tag{4}$$

Similarly, the variance of the excess return of the tangent portfolio, σ^2_{tg}, and the variance of the excess return of the minimum variance portfolio, σ^2_{mv}, are given by:

$$\sigma^2_{tg} = \frac{\mu' V^{-1}\mu}{(1_N' V^{-1}\mu)^2}, \tag{5}$$

$$\sigma^2_{mv} = \frac{1}{1_N' V^{-1}1_N}. \tag{6}$$

Given (3)-(4) and (5)-(6), it is straightforward to show that the risk premia of the tangent portfolio, π_{tg}, and the risk premium of the minimum variance portfolio, π_{mv}, are equal and given by the following:

$$\pi_{tg} \equiv \left(\frac{\mu_{tg}}{\sigma^2_{tg}}\right) = \left(\mu' V^{-1} 1_N\right), \tag{7}$$

$$\pi_{mv} \equiv \left(\frac{\mu_{mv}}{\sigma^2_{mv}}\right) = \left(\mu' V^{-1} 1_N\right). \tag{8}$$

However, as the motivating example from the previous section demonstrates, there is a portfolio on the mean-variance frontier that achieves a higher risk premium than the risk premium of the tangent and minimum variance portfolios. In order to derive explicitly the optimal portfolio weights of the maximum risk premium portfolio, we need to solve the following problem:

$$\max_{w} \quad \left(\frac{w'\mu}{w'Vw}\right) \qquad \text{s.t.} \quad w' 1_N = 1. \tag{9}$$

The solution to the above problem is given by the following:

$$w^\star = a \times w_{tg} + (1-a) \times w_{mv}, \tag{10}$$

where

$$a = \frac{(\mu' V^{-1} 1_N)}{(\mu' V^{-1} 1_N) + \sqrt{(\mu' V^{-1}\mu)\,(1_N' V^{-1} 1_N)}}, \tag{11}$$

with the details of the derivation provided in the Appendix.

In order to get further intuition regarding the parameter a we can express it equivalently as follows:[3]

$$a = \frac{\left(\frac{1}{\sigma_{tg}}\right)}{\left(\frac{1}{\sigma_{tg}}\right) + \left(\frac{1}{\sigma_{mv}}\right)} = \left(\frac{\sigma_{mv}}{\sigma_{mv} + \sigma_{tg}}\right). \tag{12}$$

This clearly shows that the weight on the MV and TG portfolios is inversely proportional to the standard deviation of return of the MV and TG portfolios, respectively. The

[3] Recalling that $(\mu' V^{-1} 1_N) = \mu_{tg}/\sigma^2_{tg}$, $(\mu' V^{-1}\mu) = \mu^2_{tg}/\sigma^2_{tg}$ as well as $\sigma^2_{mv} = 1/(1_N' V^{-1} 1_N)$ we can simplify the previous expression as $a = \frac{1}{1+\left(\frac{\sigma_{tg}}{\sigma_{mv}}\right)}$ which simplifies to the stated expression.

more risky the TG portfolio is relative to the MV portfolio, the closer the maximum risk premium portfolio will be to the MV portfolio. And, vice versa, the closer σ_{tg} is to σ_{mv} the closer the optimal portfolio will be to an equal-weighted portfolio of MV and TG (as a will get close to $\frac{1}{2}$).

An alternative expression relating a to the mean excess returns[4] of the tangent and minimum variance portfolios is as follows:

$$a = \frac{1}{1+\sqrt{\frac{\mu_{tg}}{\mu_{mv}}}} = \frac{\sqrt{\mu_{mv}}}{\sqrt{\mu_{mv}}+\sqrt{\mu_{tg}}}. \tag{13}$$

A final verification that we do have the portfolio with the highest possible risk premium involves deriving the risk premium, $\pi^\star$, of the optimal portfolio $w^\star$. It is straightforward to show that:

$$\pi^\star = \pi\left(\frac{\sigma_{tg}+\sigma_{mv}}{2\sigma_{mv}}\right), \tag{14}$$

where $\pi = \pi_{tg} = \pi_{mv}$. It is clear that as long as $\sigma_{tg} > \sigma_{mv}$ then we have $\pi^\star > \pi$. In the degenerate case where the tangent portfolio coincides with the minimum variance portfolio we have $\pi^\star = \pi$.[5]

4 Factor Models of Security Returns

The inferior out-of-sample performance of mean-variance portfolios calculated using historical means and variances is well-known in both the practitioner and the academic literature. One particular avenue of improvement that has proved fruitful in the past involves the use of factor models for the first and second moments of securities' returns (see Glabadanidis (2014)). Consider the following version of the market model:

$$\mu = \beta\mu_m, \tag{15}$$

$$V = \beta\beta'\sigma_m^2 + D, \tag{16}$$

[4] Assuming that $\mu_{mv} > 0$ and $\mu_{tg} > 0$.

[5] To see how (14) comes about we can evaluate the ratio of excess return to volatility for the optimal portfolio which leads to $\pi^\star = \frac{1}{2}\left(\sqrt{(\mu' V^{-1}\mu)}\sqrt{(1_N' V^{-1} 1_N)} + (1_N' V^{-1}\mu)\right) = \frac{1}{2}\left(\pi + \frac{\mu_{tg}}{\sigma_{tg}\sigma_{mv}}\right) = \frac{\pi}{2}\left(1+\frac{\sigma_{tg}}{\sigma_{mv}}\right)$. The latter simplifies to the stated result.

where

$$D = \begin{pmatrix} \sigma^2_{\epsilon,1} & 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{\epsilon,2} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{\epsilon,N-1} & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{\epsilon,N} \end{pmatrix}. \tag{17}$$

This version of the market model is particularly useful in that it simplifies the tangent portfolio weights considerably.[6] Using the Woodbury (1950) matrix inverse formula,[7] we can conveniently express V^{-1} in terms of D^{-1} as follows:

$$V^{-1} = D^{-1} - D^{-1}\beta \left(\frac{1}{\sigma^2_m} + \left(\beta' D^{-1} \beta \right) \right)^{-1} \beta' D^{-1}. \tag{18}$$

Using the results in Glabadanidis (2014) we know that the tangent portfolio weights are proportional to the ratio of factor loadings to the idiosyncratic variance:

$$w_{tg,i} \; \propto \; \frac{\left(\frac{\beta_i}{\sigma^2_{\epsilon,i}} \right) \left(\frac{\mu_m}{\sigma^2_m} \right)}{\left[\left(\frac{1}{\sigma^2_m} \right) + \sum_{j=1}^{j=N} \left(\frac{\beta^2_j}{\sigma^2_{\epsilon,j}} \right) \right]} \propto \left(\frac{\beta_i}{\sigma^2_{\epsilon,i}} \right), \tag{19}$$

where the last step follows from the fact that the market risk premium, μ_m/σ^2_m, is common for all stocks as is, similarly, the term inside the square brackets in the denominator. Note that this ratio can be interpreted as a signal-to-noise ratio. The numerator contains a component (beta) that is proportional to the future expected return of the risky asset while the denominator contains a component that is related to idiosyncratic or security-specific risk. The lower the last quantity to higher the signal-to-noise ratio is. Similarly, the higher the beta of the risky asset, the greater the signal-to-noise ratio. Nevertheless, it is the ratio of beta to idiosyncratic variance that matters rather then either beta or idiosyncratic variance on their own.

In the same spirit, using the Woodbury (1950) matrix inverse formula leads to a

[6] Note that in general D does not need to be diagonal. Nevertheless, I adopt this modeling feature in order to supply more explicit expressions for the individual portfolio weights.

[7] For a matrix $A = bxx' + C$, where b is a scalar, x is a vector, and C is a square, invertible matrix, we have that $A^{-1} = C^{-1} - C^{-1}x\left(\left(\frac{1}{b}\right) + \left(x'C^{-1}x\right)\right)^{-1} x'C^{-1}$. The stated result obtains by setting $C = D$, $x = \beta$ and $b = \sigma^2_m$.

slightly more involved expression for the minimum variance portfolio weights:

$$w_{mv,i} \propto \left(\frac{1}{\sigma_{\epsilon,i}^2}\right)\left\{1-\frac{\left(\frac{\beta_i^2}{\sigma_{\epsilon,i}^2}\right)}{\left[\left(\frac{1}{\sigma_m^2}\right)+\sum_{j=1}^{j=N}\left(\frac{\beta_j^2}{\sigma_{\epsilon,j}^2}\right)\right]}\right\}. \tag{20}$$

Unfortunately, the above expression for the minimum variance portfolio weights does not simplify as nicely as the one for the tangent portfolio weights. Nevertheless, it is useful to ponder the intuition behind the above result. First, it is clear that securities with lower idiosyncratic variances will have larger weights in the minimum variance portfolios. Secondly, because the factor structure is present in both the first and second moments of securities' returns, we have terms relating to the mean appear in the formula for the minimum variance portfolio weights. In particular, note the appearance of the signal-to-noise ratio which works here in the opposite way that it does for the tangent portfolio. Finally, note that for large portfolios, i.e., as $N \to \infty$, then the minimum variance portfolio weight is approximately proportional to the inverse of the idiosyncratic variance. The intuition behind this is that in the limit the minimum variance portfolio approximates a portfolio that equalizes the idiosyncratic risk contribution of each security. This is reminiscent of the equal-risk contribution portfolio literature in, for example, Lee (2011), Roncalli (2014) and Roncalli and Weisang (2015), among others.

5 Empirical Investigation

5.1 Industry Portfolios

In the section I present the distribution of out-of-sample performance of the three portfolios discussed previously over a very long time period using thirty industry portfolios. I use daily industry portfolio and market factor returns which are available for download from Ken French's online Data Library.[8] The data covers the period between July 1, 1926 until December 31, 2014. I use the historical daily return data from 1926 to estimate the market model and compute the optimal tangent, minimum variance, and maximal risk premium portfolio weights. I then track the out-of-sample performance of the three

[8] I am grateful to Ken French for making the portfolios and factors historical returns available on his website.

portfolios during 1927. I repeat this exercise one more time using 1927 as the in-sample parameter estimation period and using 1928 as the out-of-sample test period. This exercise is repeated until the last available out-of-sample period in 2014. This leaves us with 88 out of sample periods which cover a very significant range of bullish and bearish stock market periods. I keep track of the top 10%, median 50% and bottom 10% performance indicators for various sizes of the tangent portfolio ranging from one all the way up to 30. Figure 3 plots these three quantiles across different number of securities included in the tangent portfolio regarding the correlation between the portfolio and the market return, the annualized tracking error in per cent, the annualized Sharpe ratio, the realized active return in per cent, the annualized α in per cent, as well as the β of tangent portfolio.

Insert Figure 3 about here.

Several key findings emerge from the plots. First, the more securities we add to the tangent portfolio the higher the ex post correlation between the market return and the tangent portfolio return. Second, the tracking errors declines very quickly as we add more securities to the tangent portfolio and, furthermore, the range of possible outcomes (top 10% to bottom 10%) shrinks in size as well. Third, the median ex post annualized Sharpe ratio is approximately equal to one though the range of possible outcomes tends to be rather large, from -1 all the way up to 2.5. Similarly, the ex post median cumulative active return is positive for most portfolio sizes but hovering close to zero with a wide range of possible outcome. However, the range of possible outcomes does indeed decrease in size as we add more securities to the tangent portfolio. Next, the ex post median annualized α of the tangent portfolio is mostly positive though close to zero with a shrinking range of possible outcomes as the number of securities in the tangent portfolio increases. Finally, the ex post median β of the tangent portfolio does increase in value as we add more securities to the tangent portfolio getting very close to 1 once all 30 industry portfolios are included in the tangent portfolio. The range of possible outcomes for the ex post beta also shrinks as we add more securities to the tangent portfolio. However, the bottom 10% of outcomes do have β values that are quite a bit lower than 1 reaching 0.8 once all 30 industry portfolio have been included in the tangent portfolio.

Insert Figure 4 about here.

Figure 4 plots the three quantiles of the same out-of-sample performance indicators for the minimum variance portfolio constructed using the same 30 industry portfolios and the short-cut formula using the market model from Section 4. The ex post performance of the minimum variance portfolio is quite similar to the performance of the tangent portfolio reported in Figure 3 previously. The only significant difference is the lower ex post β of the portfolio with median values starting around 0.8 when only a few industry portfolios are included in the minimum variance portfolio going all the way up a little over 0.9 when all 30 industries are included. If anything, this is a testament to how well the factor models works in terms of describing securities' returns. Furthermore, from a statistical point of view, the factor model provides a kind of a shrinkage estimator of securities' means, variances and covariances. These shrunk estimates provide a less noisy estimate of both the tangent as well as the minimum variance portfolio weights.

Insert Figure 5 about here.

Finally, Figure 5 presents the out-of-sample performance indicators for the optimal maximum risk premium portfolio proposed in Section 3 with the factor model short-cut from Section 4. Since the maximum risk premium portfolio is a weighted average combination of the tangent portfolio and the minimum variance portfolio, it is not surprising that the plots in Figure 5 mirror the plots in Figures 3 and 4. The optimal portfolio achieves a very high ex post correlation with the market portfolio very quickly as more industries are added to it. Similarly, the ex post annualized tracking error recedes quickly as we add more industries to the optimal portfolio. The ex post median Sharpe ratio is relatively insensitive to the number of assets included in the portfolio and has a similarly wide possible range of values. The ex post median realized active return is quite close to zero but still positive, especially as we add more industries to the optimal portfolio. There is a very respectable ex post median α of between 1% and 2% depending on how many industries we add. Finally, the ex post median optimal portfolio β starts off a bit higher than the minimum variance portfolio at around 0.8 and reaches upwards of 0.9 (though still less than 1) when all 30 industries are included.

5.2 Individual Stocks

In this section, I present the findings regarding the performance of all three portfolios using a 1,000 individual US stocks. The stock return data is obtained from the Center for Research in Security Prices (CRSP) at the University of Chicago and covers the period between January 4, 1988 and December 30, 2011. The selection criterion employed selects the stocks with the largest market capitalization as of December 30, 2011. I repeat the same portfolio construction and evaluation process as in the previous subsection. The findings are largely consistent both quantitatively and qualitatively with the findings reported previously. Both the TG and the MV portfolios very quickly achieve returns that are very highly correlated with the US stock market return as the number of individual stocks increases. Similarly, both portfolios tracking errors are reduced dramatically as soon as the portfolio size increases beyond 50 to 100 stocks. The Sharpe ratio of both portfolio levels off early on and asymptotes to a value of around 1.25 on an annualized basis.

Figure 6 presents a few more variables of interest regarding the out-of-sample performance of the TG portfolio. The median annualized realized active return is quite stable across various portfolio sizes and around 1 to 2%. At the same time, the median annualized α is the neighbourhood of 3% to 5%. Finally, the portfolio median β increases slightly for $N_{securities} \approx 50$ and then levels off to a value of approximately 0.75. Nevertheless, the 80% interval between the top 10% β and the bottom 10% β across the entire sample period is quite wide at between 0.5 and 1.

Insert Figure 6 about here.

Next, Figure 7 plots the performance metrics and their distribution for the MV portfolio. The first notable difference between the behavior of the the MV portfolio and the TG portfolio is that the median annualized realized active return increases gradually as $N_{securities}$ increases reaching a value of about 3%. The second notable difference is that the median annualized α is not reduced for large portfolio sizes and, indeed, stabilizes at a value around 6% to 7%. Finally, and most interestingly, the median market β only increases up to a value of 0.6 when all 1,000 stocks are included in the MV portfolio. Nevertheless, the 80% range of possible outcomes is still wide and betwen 0.35 and 0.95.

These findings reinforce the reputation of the MV as a perhars even more desirable portfolio than the TG portfolio. Clearly, the MV portfolio tracks the stock market index very well and delivers a slightly higher realized active return, higher alpha and lower beta on average in comparison with the TG portfolio.

Insert Figure 7 about here.

Finally, turning to Figure 8 we can explore the performance of the optimal portfolio which is simply a combination of the MV and the TG portfolio. The evidence presented in the previous two figures will hint at the fact the optimal portfolio will share many of the features of the MV and the TG portfolio themselves. In fact, the performance of the optimal portfolio is a little better for very large $N_{securities}$ as can be seen from the last three panels in Figure 8. First, the median annualized realized active return is slightly higher at around 4% for $N_{securities} = 1000$. Secondly, the median annualized α peters off at around 8% for $N_{securities} = 1000$. Thirdly, the median β peaks when $N_{securities} \approx 800$ and comes down to around 0.6 for $N_{securities} = 1000$. Still, the range of actual βs is between around 0.35 and 0.95. The abnormal performance of all three portfolios appears to not be driven away on average as the universe of stocks expands. This phenomenon is largely consistent with the theoretical model in Dybvig (2005) where equity portfolio abnormal returns do not necessarily disappear as the number of risky assets in the portfolio increases.

Insert Figure 8 about here.

5.3 Notes on the Performance of New Investment Objective

In Figure 9 I present a summary of the performance of the TG, MV and the optimal portfolio relative to the new investment management objective as well as the values of the excess return per unit variance out-of-sample during the time period under consideration when 30 industry portfolios are used in constructing the TG, MV and optimal portfolios. Note that the returns I use are expressed in percent leading to a value for μ/σ^2 that will be off by two orders of magnitude (i.e., a factor of 100) relative to the case when returns are expressed as a decimal. In order to compare these values to investors' coefficient of absolute risk aversion the values reported in the Figure will need to be multiplied by a factor of 100. The upper-left panel of Figure 9 reports the top 10%, median, and bottom

10% values of realized out-of-sample excess return per unit variance for the TG portfolio. The upper-right panel plots those values for the MV portfolio while the lower-left panel plots the realized out-of-sample excess return per unit variance for the optimal portfolio proposed in the article. Finally, the lower-right panel of Figure 9 plots the median risk-return measure for all three portfolios. What is evident from the last panel is that the new measure can result in larger realized values of excess return per unit variance for the optimal portfolio than the ones for the TG and MV portfolios even though the optimal portfolio is a self-financed linear combination of the latter two. The way to interpret these findings is that an investor with absolute risk aversion of 6, for example, will invest fully in all three portfolios but only as long as $N_{securities} \geq 3$ since smaller portfolios deliver a lower risk-return trade-off. The same hypothetical investor will only allocate portion of her funds into portfolios with $N_{securities} < 3$.

Insert Figure 9 about here.

Finally, I present the distribution of the realized excess return per unit variance out-of-sample for all three portfolios when 1,000 individual stocks are used to construct all portfolios. Figure 10 is similar in spirit in the way the realized risk-return values are presented. Note the difference in out-of-sample performance with respect to the excess return per unit variance between the TG and the MV portfolio for up to 100 stocks. Around $N_{securities} \approx 100$ the risk-return measure stabilizes at roughly the same values for both portfolios. For larger portfolios the performance of both the TG and MV portfolios track each other very closely although the MV and the optimal portfolios offer consistently greater excess returns per unit of variance. Note that for certain value of the absolute risk aversion parameter, an investor with constant absolute risk aversion will prefer to invest her entire wealth into the MV or the optimal portfolio. At the same time she will prefer to invest only a fraction of her wealth into the TG portfolio.

Insert Figure 10 about here.

The evidence regarding the out-of-sample performance of all three portfolios presented in Figures 9 and 10 concurs with the mathematical derivation of the properties of the three portfolios. Note that the excess return per unit variance measure for the TG portfolio closely follows the one for the MV portfolio. Nevertheless, their magnitudes are quite close to each other. Similarly, the excess return per unit variance measure for the optimal portfolio frequently exceeds the respective values for the TG and MV portfolio. Once again, this is not always the case and does not happen for portfolios with differing number of securities. Due to parameter drift, estimation errors related to the use of daily data with the single-factor model and the nature of the portfolio implementation (buy-and-hold) the two measures are not perfectly aligned out-of-sample.

6 Concluding Remarks

The standard mean-variance prescriptive focus on maximizing the Sharpe ratio and the associated tangent portfolio have long been the workhorse of finance theory and has had an important impact on practitioners' asset management policies. One particular feature of the Sharpe ratio is its lack of sensitivity the leverage while an important downside is that the Sharpe ratio is horizon-dependent.[9] At the same time, the risk premium or the excess return per unit variance, is sensitive to leverage but independent of the investment management horizon. This downside limits the applicability of this investment objective to all-equity portfolios only. Recognizing the fact that investors can improve on the risk premium offered by the tangent and the minimum variance portfolios, I have developed the optimal maximal risk premium portfolio which happens to be a linear combination of the two portfolios. In addition, I demonstrate how factor models of securities' returns can be used to improve on the calculation of the optimal portfolio weights of all three portfolios. Finally, I present evidence regarding the out-of-sample performance of the maximal risk premium portfolio using 30 industry portfolios as well as 1,000 individual US stocks and the market model as the choice of a single-factor model of securities' returns. The optimal portfolio shares many of the desirable features of the minimum variance portfolio while retaining the upside risk-return potential of the tangent portfolio with regards to tracking error, ex post active return (in the sense of Grinold and Kahn (1999)), risk-return trade-off as well as abnormal ex post return (α) and ex post level of systematic risk (β).

The implementation of the maximal risk premium portfolio can be extended to use-supplied forecasts of abnormal returns (α) as well as multiple factors, e.g., Fama and

[9] An annualized Sharpe ratio of 1 is equivalent to a Sharpe ratio of about 3.16 over the course of the next decade and a Sharpe ratio of 10 over the course of the next century. The general formula for the Shapre ratio over the next k years, s_k, can be obtained from the one-year Sharpe ratio, s_1, as $s_k = \sqrt{k}s_1$.

Fench (1992) or Carhart (1997) empirical asset pricing models. Furthermore, the important question of re-balancing and re-estimation frequency still remain largely unexplored. Future work in this area would not only be of great interest to the academic finance literature but would add tremendous value in terms of informing practitioners' decisions in actively managed equity portfolios.

Appendix

In order to derive Equation (1) it is helpful to solve the following:

$$\min_{w} \quad (w'Vw) \tag{21}$$

$$\text{s.t.} \quad w'\mu = \mu_0, \tag{22}$$

$$w'1_N = 1.$$

The Lagrangian for this program is given by:

$$\mathcal{L} = (w'Vw) + \lambda_1 (\mu_0 - w'\mu) + \lambda_2 \left(1 - w'_{tg}1_N\right), \tag{23}$$

with the following solution:

$$w^\star(\mu_0) = \left(\frac{\lambda_1}{2}\right) V^{-1}\mu + \left(\frac{\lambda_1}{2}\right) V^{-1}1_N, \tag{24}$$

where

$$\left(\frac{\lambda_1}{2}\right) = \frac{\mu_0(1'_N V^{-1}1_N) - (\mu' V^{-1}1_N)}{\Delta}, \tag{25}$$

$$\left(\frac{\lambda_2}{2}\right) = \frac{(\mu' V^{-1}\mu) - \mu_0(\mu' V^{-1}1)N)}{\Delta}, \tag{26}$$

$$\Delta = (\mu' V^{-1}\mu)(1'_N V^{-1}1_N) - (\mu' V^{-1}1_N)^2. \tag{27}$$

The last step involves finding the right value for μ_0 which maximizes the Sharpe ratio, $s^\star(\mu_0)$, associated with $w^\star(\mu_0)$. To ease the analytical derivation it is easier to minimize the inverse of the squared Sharpe ratio, $\frac{1}{(s^{\star}(\mu_0))^2}$, which is given as follows:

$$\frac{1}{(s^{\star}(\mu_0))^2} = \frac{1}{\Delta}\left[(1_N' V^{-1} 1_N) - \frac{2(1_N' V^{-1}\mu)}{\mu_0} + \frac{\mu' V^{-1}\mu}{\mu_0^2}\right], \tag{28}$$

which has as a solution

$$\mu_0^{\star} = \frac{\mu' V^{-1}\mu}{1_N' V^{-1}\mu}, \tag{29}$$

leading to the equation provided in the text, namely, $w_{tg} = \frac{V^{-1}\mu}{1_N' V^{-1}\mu}$.

Equation (2) is derived as the solution to the following problem:

$$\begin{aligned} \min_{w_{mv}} \quad & w_{mv}' V w_{mv} \\ \text{s.t.} \quad & w_{mv}' 1_N = 1. \end{aligned} \tag{30}$$

The Lagrangian for this program is given by:

$$\mathcal{L} = w_{mv}' V w_{mv} + \lambda_2 \left(1 - w_{mv}' 1_N\right), \tag{31}$$

with the associated first order condition:

$$2V w_{mv} - \lambda_2 1_N = 0_N, \tag{32}$$

which leads to:

$$w_{mv}^{\star} = \left(\frac{\lambda_2}{2}\right) V^{-1} 1_N. \tag{33}$$

The final step involved substituting the above equation into the budget constraint and solving for $\left(\frac{\lambda_2}{2}\right)$:

$$\left(\frac{\lambda_2}{2}\right) = \frac{1}{1_N' V^{-1} 1_N}, \tag{34}$$

leading to the stated result by plugging $\left(\frac{\lambda_2}{2}\right)$ into the prior expression for $w_{mv}^{\star}$.[10]

To obtain Equation (3), simply multiply the optimal tangent portfolio weight vector, $w_{tg}^{\star}$, by the vector of excess returns, μ. Similarly, Equation (4) obtains by multiplying the optimal minimum variance portfolio weight vector, $w_{mv}^{\star}$, by the vector of excess returns,

[10] The second order sufficient condition for this to be the objective-minimizing solution is $2V > 0$ which is fulfilled by any positive definite variance-covariance matrix V.

μ.

Equations (5) and (6) obtain using the following relations along with $w^\star_{tg}$ and $w^\star_{mv}$:

$$\sigma^2_{tg} = w^{\star'}_{tg} V w^\star_{tg}, \tag{35}$$

$$\sigma^2_{mv} = w^{\star'}_{mv} V w^\star_{mv}. \tag{36}$$

Finally, to obtain Equations (10) it is best to make use of the supplementary problem solved previously in this appendix in Equation (24) but this time we will focus on minimizing the ratio of $(\sigma^\star(\mu_0))^2$ to μ_0[11] which leads to:

$$\mu_0^\star = \sqrt{\frac{(\mu' V^{-1}\mu)}{1'_N V^{-1} 1_N}}. \tag{37}$$

Substituting the above in (24) above produces the following:

$$\begin{aligned} w^\star = \; & \frac{1}{\Delta}\left\{\left(\sqrt{(\mu' V^{-1}\mu)}\sqrt{(1'_N V^{-1} 1_N)} - (\mu' V^{-1} 1_N)^2\right) w_{tg} + \right. \\ & \left. \left((\mu' V^{-1}\mu)(1'_N V^{-1} 1_N) - \sqrt{(\mu' V^{-1}\mu)}\sqrt{(1'_N V^{-1} 1_N)}(1'_N V^{-1}\mu)\right) w_{mv}\right\}. \end{aligned}$$

Recognizing that the two terms multiplying w_{tg} and w_{mv} add up to one we can use that simpler representation in the paper:

$$w^\star = a w_{tg} + (1-a) w_{mv}, \tag{38}$$

where

$$a = \frac{(\mu' V^{-1} 1_N)\left[\sqrt{(\mu' V^{-1}\mu)}\sqrt{(1'_N V^{-1} 1_N)} - (\mu' V^{-1} 1_N)\right]}{(\mu' V^{-1}\mu)(1'_N V^{-1} 1_N) - (\mu' V^{-1} 1_N)^2}. \tag{39}$$

The latter simplifies to the expression supplied in the article after realizing that $(\mu' V^{-1}\mu)(1'_N V^{-1} 1_N) - (\mu' V^{-1} 1_N)^2$ can represented as:

$$\left[\sqrt{(\mu' V^{-1}\mu)}\sqrt{(1'_N V^{-1} 1_N)} - (\mu' V^{-1} 1_N)\right]\left[\sqrt{(\mu' V^{-1}\mu)}\sqrt{(1'_N V^{-1} 1_N)} + (\mu' V^{-1} 1_N)\right]. \tag{40}$$

[11]This is mathematically equivalent to maximizing the ratio of excess return to return variance and less unwieldy to work with analytically.

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Figure 1. MV Example in $\mu - \sigma$ Space

This figure presents the tangent portfolio TG (square), the minimum variance portfolio MV (diamond) and the optimal risk-reward portfolio (circle) in mean-standard deviation space.

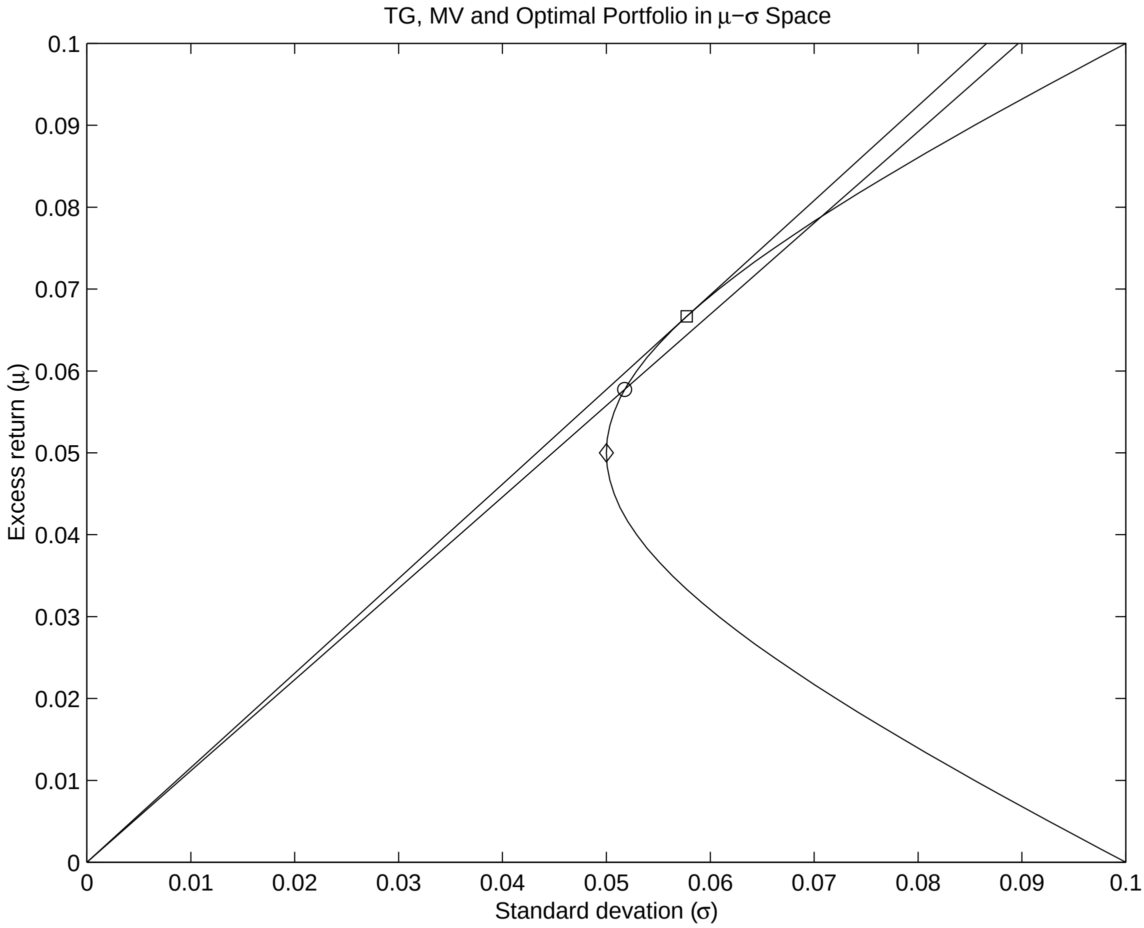

Figure 2. MV Example in $\mu - \sigma^2$ Space

This figure presents the tangent portfolio TG (square), the minimum variance portfolio MV (diamond) and the optimal risk-reward portfolio (circle) in mean-variance space.

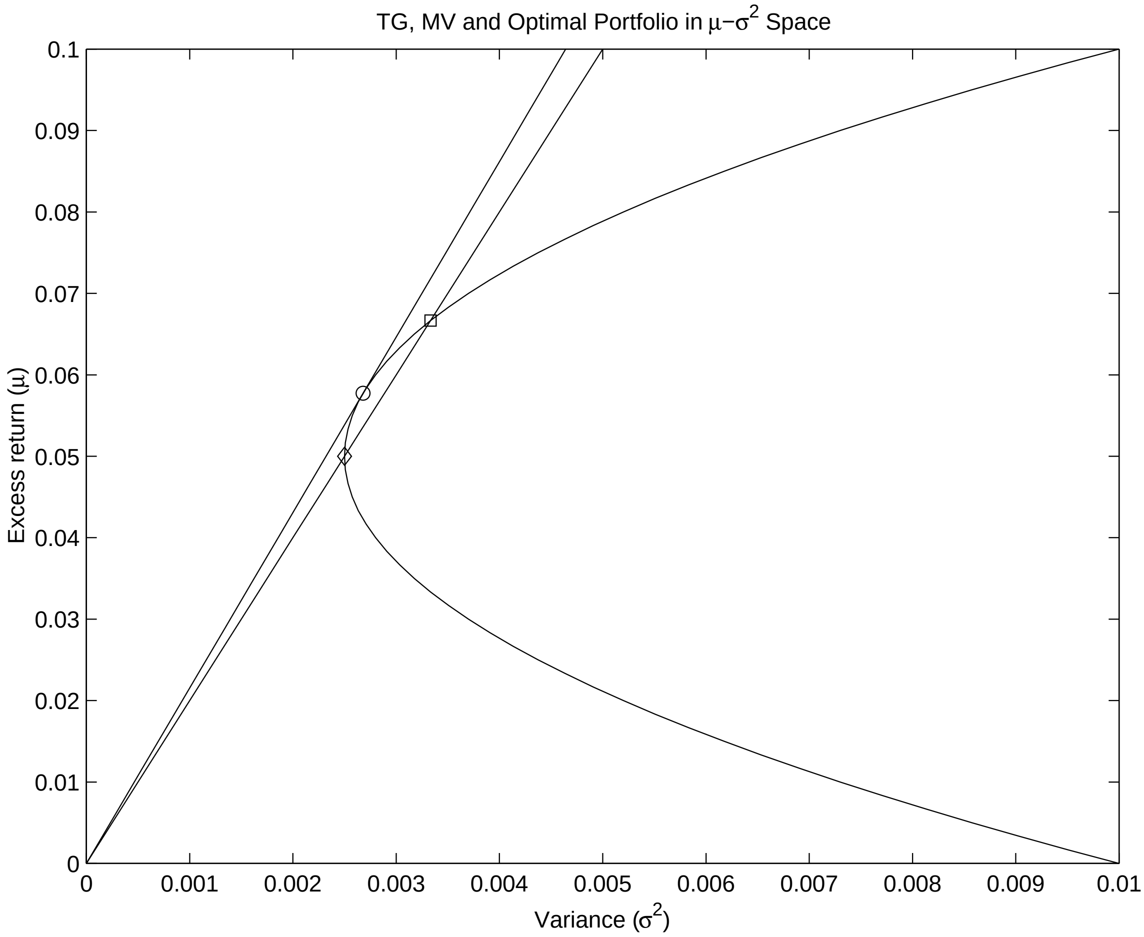

Figure 3. Out-of-sample performance of the Tangent Portfolio: 30 Industry Portfolios

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of the TG portfolio return correlation with the VWRETD market return, the annualized tracking error in percent, the annualized Sharpe ratio, the annualized realized active return, $RAR = \Pi_{t=1}^{t=T}(1 + r_{p,t}) - \Pi_{t=1}^{t=T}(1 + r_{b,t})$, as well as the annualized alpha, $\alpha = \overline{r}_p - \beta_p \overline{r}_b$, and the market beta, β, as a function of the number of industry portfolios, $N_{securities}$, included in the TG portfolio.

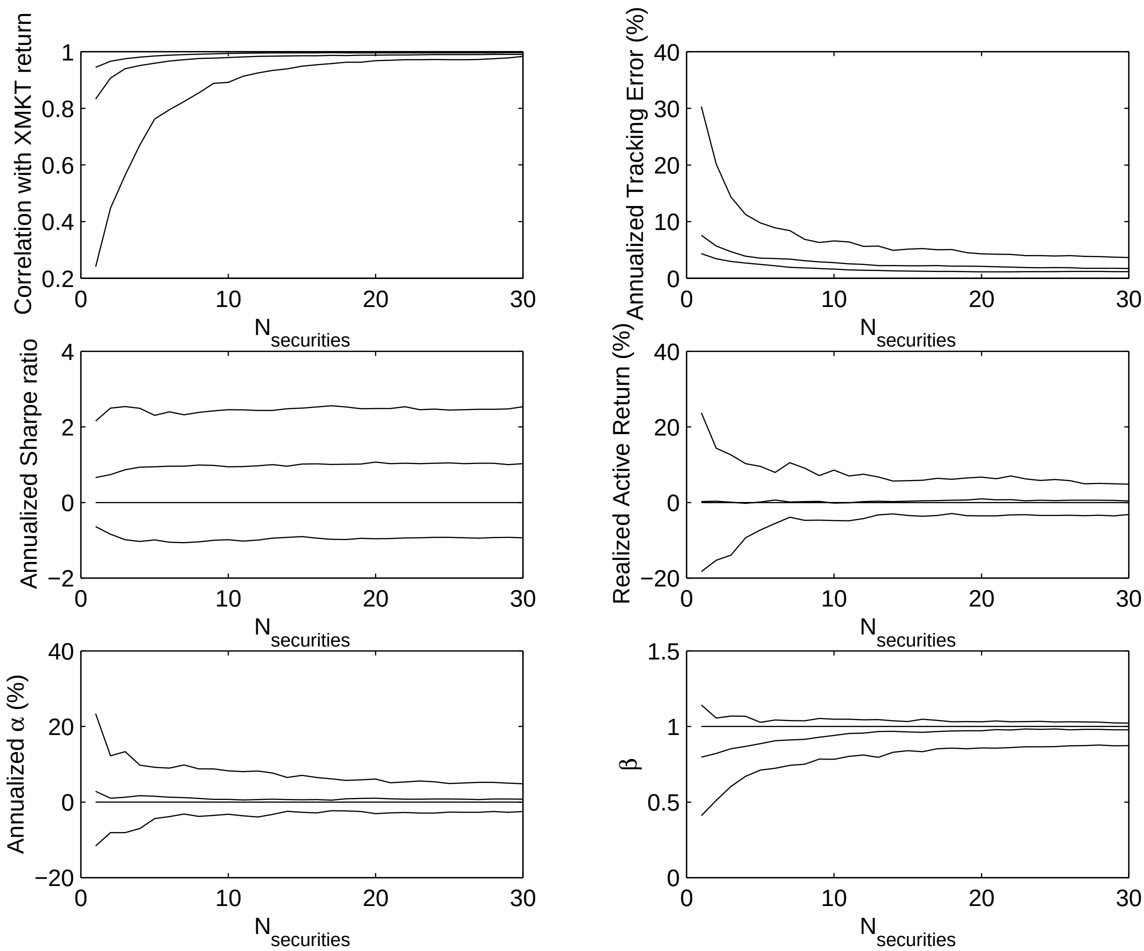

Figure 4. Out-of-sample performance of the Minimum Variance Portfolio: 30 Industry Portfolios

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of the MV portfolio return correlation with the VWRETD market return, the annualized tracking error in percent, the annualized Sharpe ratio, the annualized realized active return, $RAR = \Pi_{t=1}^{t=T}(1 + r_{p,t}) - \Pi_{t=1}^{t=T}(1 + r_{b,t})$, as well as the annualized alpha, $\alpha = \overline{r}_p - \beta_p \overline{r}_b$, and the market beta, β, as a function of the number of industry portfolios, $N_{securities}$, included in the MV portfolio.

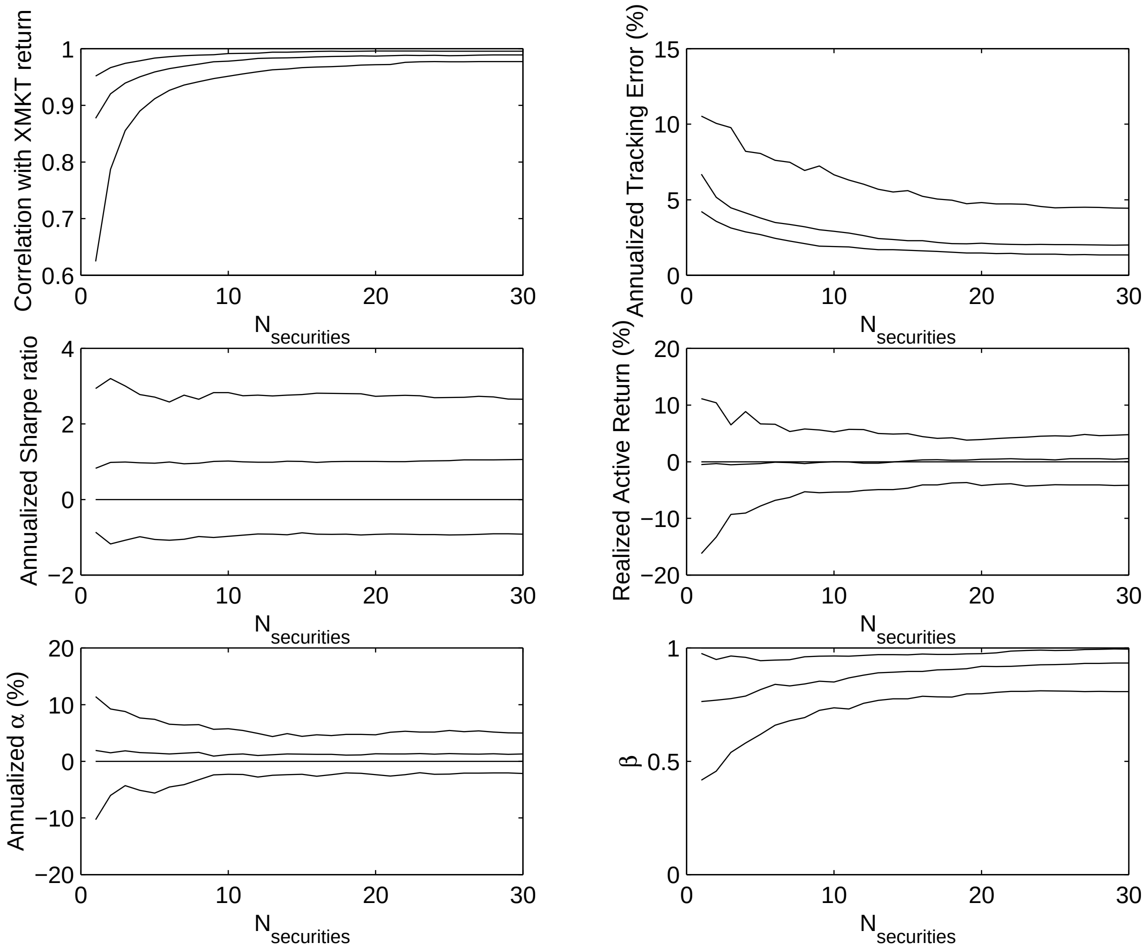

Figure 5. Out-of-sample performance of the Optimal Portfolio: 30 Industry Portfolios

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of the optimal portfolio return correlation with the VWRETD market return, the annualized tracking error in percent, the annualized Sharpe ratio, the annualized realized active return, $RAR = \Pi_{t=1}^{t=T}(1 + r_{p,t}) - \Pi_{t=1}^{t=T}(1 + r_{b,t})$, as well as the annualized alpha, $\alpha = \overline{r}_p - \beta_p \overline{r}_b$, and the market beta, β, as a function of the number of industry portfolios, $N_{securities}$, included in the optimal portfolio.

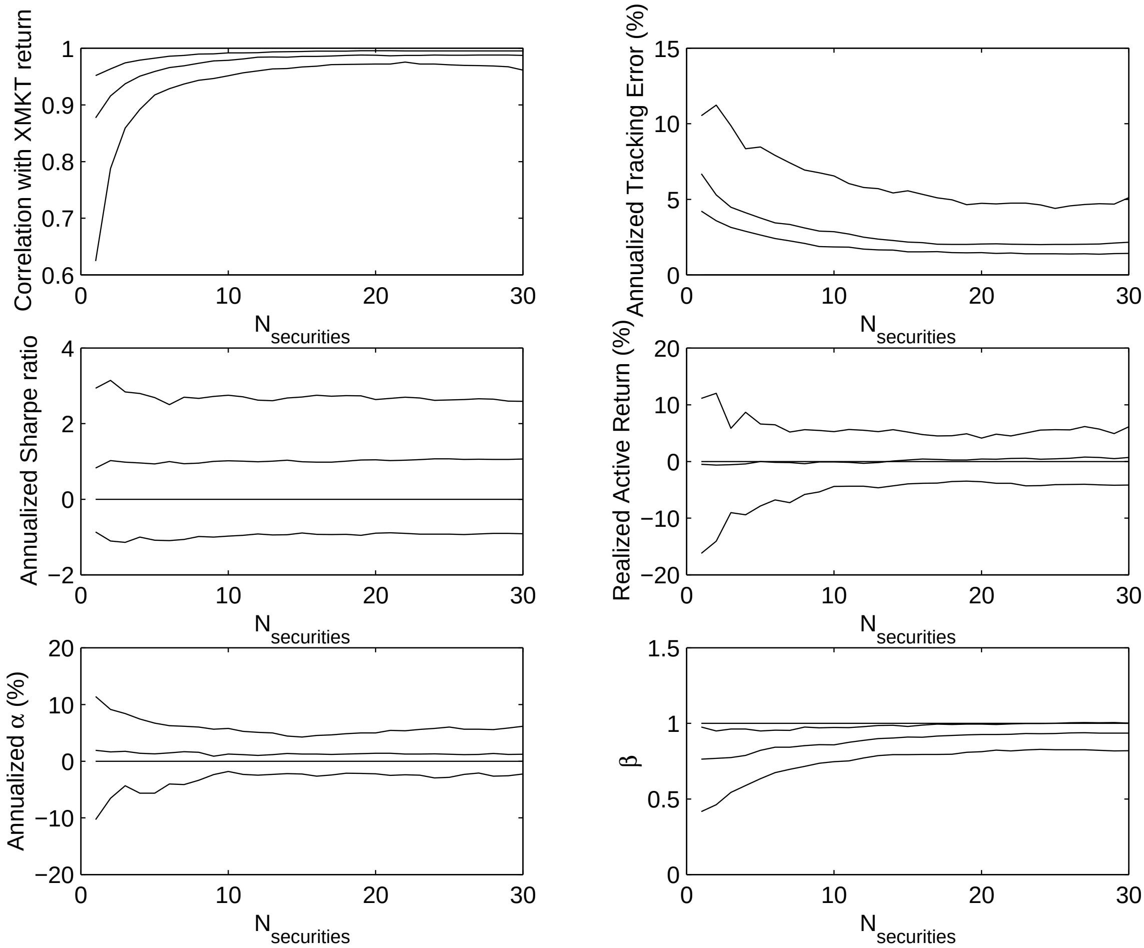

Figure 6. Out-of-sample performance of the Tangent Portfolio: 1000 Individual Stocks

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of TG portfolio return correlation with the VWRETD market return, the annualized tracking error in percent, the annualized Sharpe ratio, the annualized realized active return, $RAR = \Pi_{t=1}^{t=T}(1 + r_{p,t}) - \Pi_{t=1}^{t=T}(1 + r_{b,t})$, as well as the annualized alpha, $\alpha = \overline{r}_p - \beta_p \overline{r}_b$, and the market beta, β, as a function of the number of individual stocks, $N_{securities}$, included in the TG portfolio.

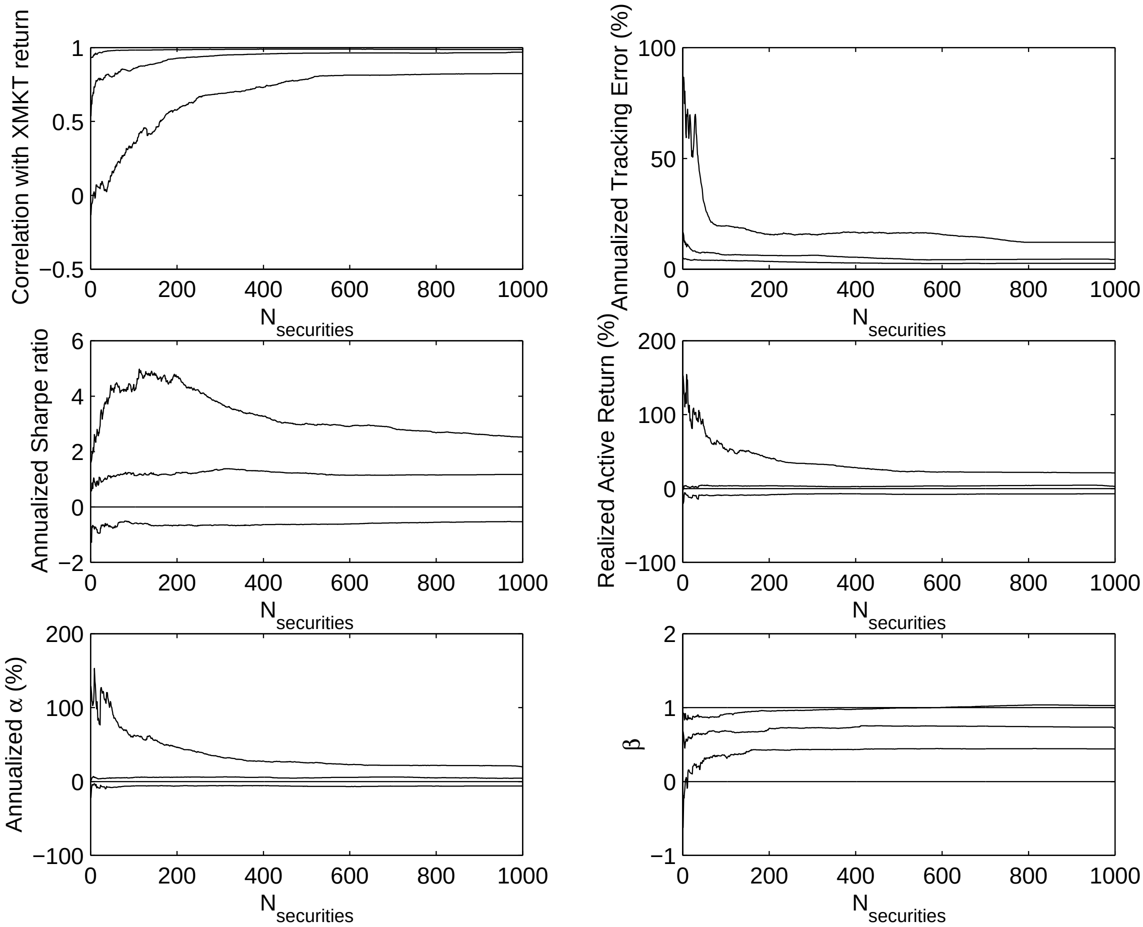

Figure 7. Out-of-sample performance of the Minimum Variance Portfolio: 1000 Individual Stocks

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of MV portfolio return correlation with the VWRETD market return, the annualized tracking error in percent, the annualized Sharpe ratio, the annualized realized active return, $RAR = \Pi_{t=1}^{t=T}(1 + r_{p,t}) - \Pi_{t=1}^{t=T}(1 + r_{b,t})$, as well as the annualized alpha, $\alpha = \overline{r}_p - \beta_p \overline{r}_b$, and the market beta, β, as a function of the number of individual stocks, $N_{securities}$, included in the MV portfolio.

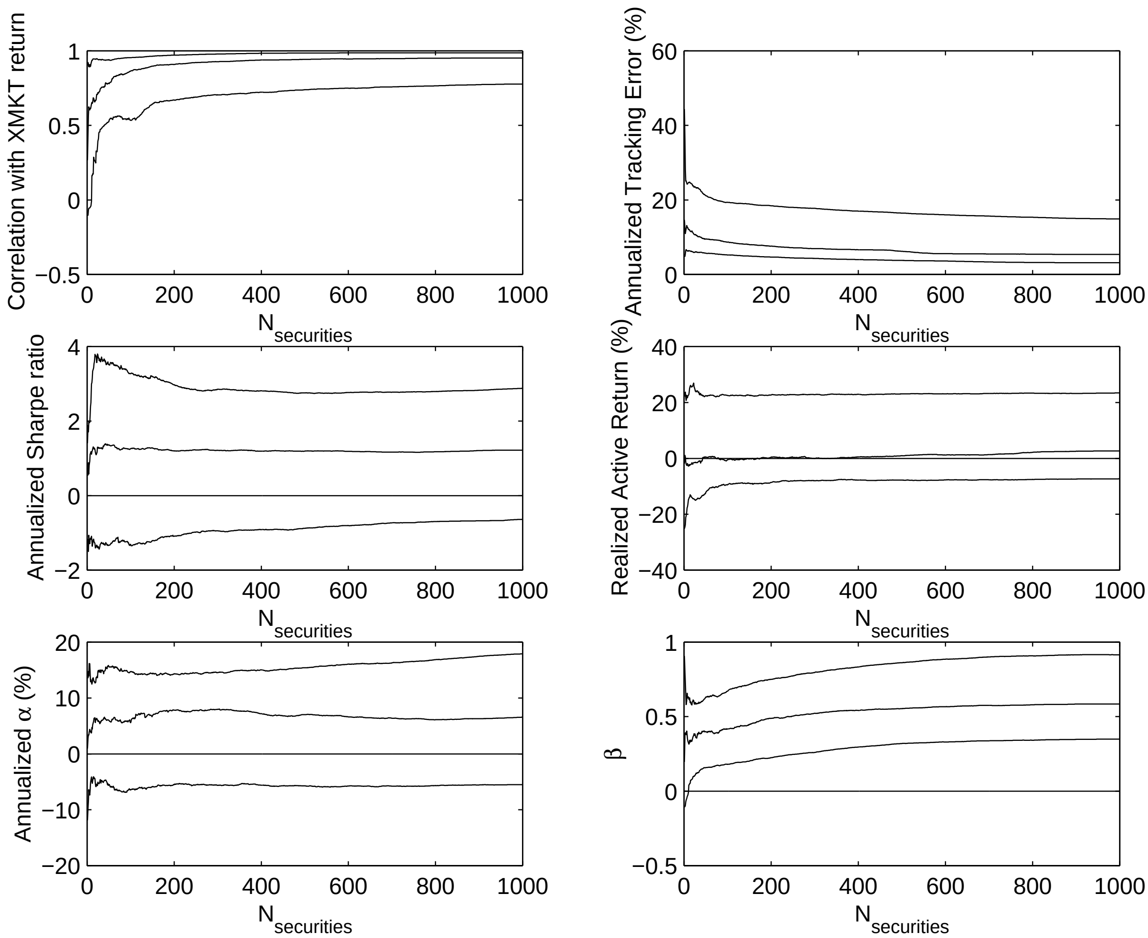

Figure 8. Out-of-sample performance of the Optimal Portfolio: 1000 Individual Stocks

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of optimal portfolio return correlation with the VWRETD market return, the annualized tracking error in percent, the annualized Sharpe ratio, the annualized realized active return, $RAR = \Pi_{t=1}^{t=T}(1 + r_{p,t}) - \Pi_{t=1}^{t=T}(1 + r_{b,t})$, as well as the annualized alpha, $\alpha = \overline{r}_p - \beta_p \overline{r}_b$, and the market beta, β, as a function of the number of individual stocks, $N_{securities}$, included in the optimal portfolio.

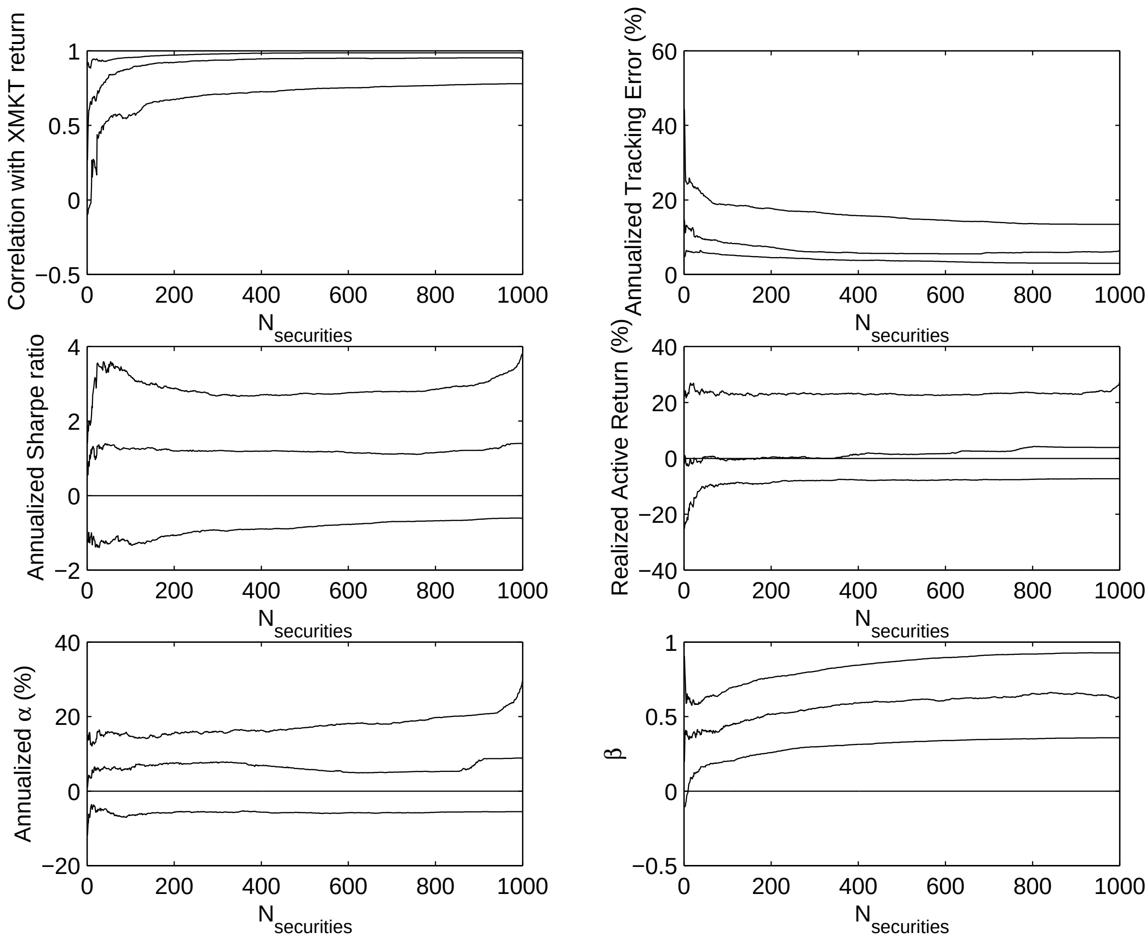

Figure 9. Comparison of realized excess returns per unit variance: 30 Industry Portfolios

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of realized out-of-sample excess return per unit variance for the TG (upper left panel), MV (upper right panel), and the optimal portfolio (lower left panel) as well as a superimposed plot of the median values for all three portfolios as a function of the number of industry portfolios, $N_{securities}$.

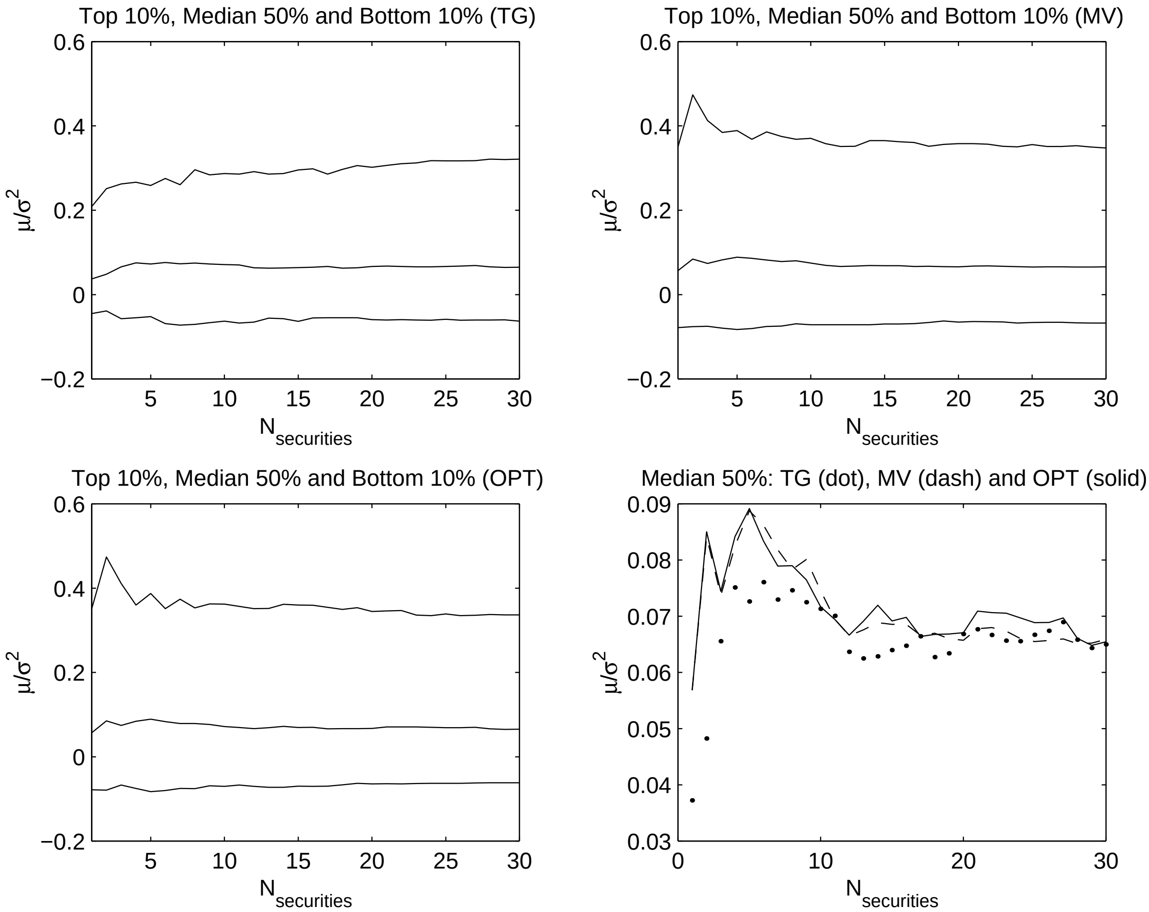

Figure 10. Comparison of realized excess returns per unit variance: 1000 Individual Stocks

This figure plots the summary of the of the top 10% (top line), median (middle line), and bottom 10% (bottom line) of realized out-of-sample excess return per unit variance for the TG (upper left panel), MV (upper right panel), and the optimal portfolio (lower left panel) as well as a superimposed plot of the median values for all three portfolios as a function of the number of individual stocks, $N_{securities}$.