Holomorphic Flexibility Properties
of Spaces of Elliptic Functions

David Bowman

Thesis submitted for the degree of
Doctor of Philosophy
in
Pure Mathematics
at
The University of Adelaide
Faculty of Engineering, Computer
and Mathematical Sciences

School of Mathematical Sciences

July 13, 2016
# Contents

Signed Statement ........................................ iii  
Acknowledgements ....................................... iv  
Dedication .................................................... v  
Abstract ....................................................... vi  

## 1 Introduction .......................................... 1  
1.1 Background and Context ............................. 1  
1.2 Results of the Thesis ............................... 4  
1.3 Further Directions ................................ 9  

## 2 Proof of Main Theorem ............................. 11  
2.1 Topology ............................................... 11  
2.2 Complex Structure ................................ 13  
2.2.1 The Universal Complex Structure .............. 13  
2.2.2 Symmetric Products ............................. 17  
2.2.3 The Jacobi Map ................................ 19  
2.2.4 The Theta Function ............................ 22  
2.2.5 The Divisor Map ................................ 25  
2.2.6 Nonsingularity of $R_n$ ....................... 26  
2.2.7 The Space $R_n/M$ ............................... 27
2.3 Elliptic Functions of Degree 2 ........................................ 29
  2.3.1 Elliptic Curves .................................................... 29
  2.3.2 The Weierstrass $\wp$-function ................................. 31
  2.3.3 Degree 2 Case ...................................................... 32

2.4 A 9-sheeted Covering of $R_3/M$ .................................... 35

2.5 Oka Branched Covering Space of $\mathbb{P}_2 \setminus C$ ............... 43

2.6 The Main Theorem ..................................................... 48

3 Further Results ................................................................ 53
  3.1 Strong Dominability .................................................... 53
  3.2 The Remmert Reduction of $R_3$ ..................................... 58
  3.3 An Alternative Proof of the Main Theorem ......................... 60

Bibliography ...................................................................... 69
Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

I also give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library Search and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

SIGNED: 

DATE: 10/11/16
Acknowledgements

I would like to thank the following for their contributions to this thesis.

Finnur Lárusson, my principal supervisor, for his unending support, for nurturing the growth of this thesis and for many, many helpful discussions.

Nicholas Buchdahl, my secondary supervisor, for exposing me to many exciting ideas in geometry and analysis.

Franc Forstnerič, Philipp Naumann, Richard Lärkäng and Tuyen Truong for many interesting mathematical discussions.

The staff at Kathleen Lumley College for making my stay in Adelaide an enjoyable one.

And finally, my friends and family for their love and support.
Dedication

To Grandad, for teaching me to count cars.
Abstract

Let $X$ be an elliptic curve and $\mathbb{P}$ the Riemann sphere. Since $X$ is compact, it is a deep theorem of Douady that the set $\mathcal{O}(X, \mathbb{P})$ consisting of holomorphic maps $X \to \mathbb{P}$ admits a complex structure. If $R_n$ denotes the set of maps of degree $n$, then Namba has shown for $n \geq 2$ that $R_n$ is a $2n$-dimensional complex manifold.

We study holomorphic flexibility properties of the spaces $R_2$ and $R_3$. Firstly, we show that $R_2$ is homogeneous and hence an Oka manifold. Secondly, we present our main theorem, that there is a 6-sheeted branched covering space of $R_3$ that is an Oka manifold. It follows that $R_3$ is $\mathbb{C}$-connected and dominable. We show that $R_3$ is Oka if and only if $\mathbb{P}_2 \setminus C$ is Oka, where $C$ is a cubic curve that is the image of a certain embedding of $X$ into $\mathbb{P}_2$.

We investigate the strong dominability of $R_3$ and show that if $X$ is not biholomorphic to $\mathbb{C}/\Gamma_0$, where $\Gamma_0$ is the hexagonal lattice, then $R_3$ is strongly dominable.

As a Lie group, $X$ acts freely on $R_3$ by precomposition by translations. We show that $R_3$ is holomorphically convex and that the quotient space $R_3/X$ is a Stein manifold.

We construct an alternative 6-sheeted Oka branched covering space of $R_3$ and prove that it is isomorphic to our first construction in a natural way. This alternative construction gives us an easier way of interpreting the fibres of the branched covering map.