THE COVERING OF SETS OF CONSTANT WIDTH

by

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CONTENTS

Summary (ii)
Statement (iv)

CHAPTER 1. INTRODUCTION 1
1.1 Lebesgue tile problem 1
1.2 Extension of the Lebesgue tile problem to \( E^3 \) 6
1.3 Further generalizations of Lebesgue tile problem 8
1.4 Application to Borsuk's problem 9

CHAPTER 2. UNIVERSAL COVERS 15
2.1 Definition and statement of problems 15
2.2 Simple examples of \( k \)-covers 19
2.3 Formal results on \( k \)-covers 28

CHAPTER 3. EXISTENCE OF AN OPTIMAL \( k \)-COVER 40

CHAPTER 4. FURTHER EXAMPLES OF \( k \)-COVERS 63

CHAPTER 5. APPLICATION TO THE MODIFIED BORSUK'S PROBLEM 83
5.1 The principle involved 83
5.2 Partitioning the sets of the 4-cover into four sets 84
5.3 Dissection of a 1-cover 112
5.4 Use of a different \( k \)-cover 119
5.5 Conclusion 121

Bibliography 123
SUMMARY

In 1914, Lebesgue posed the problem of determining a closed set of least area which covers every set of diameter 1 in the plane. Although such a set has not been found, various sets with the required covering property have been discovered. Also the problem has been generalized to higher dimensions n and to measures other than n-dimensional volume.

In this thesis, the problem has been generalized further. A universal cover of order k is a collection of k bounded closed sets such that every n-dimensional set of diameter 1 can be covered by at least one of the k sets. Various examples are considered, and a number of formal results produced. In particular, it is shown that for fixed positive integers n and k there exists a universal cover of order k whose sets are optimal with respect to any one of the measures: volume, diameter or surface area.

The covering sets of Lebesgue's problem have been used successfully to solve the two and three dimensional cases of Borsuk's problem: "Can every set of diameter 1 in n-dimensional space be partitioned into n+1 sets of diameter less than 1?" Now, suppose \( d_n(n+1) \) is the infimum of all numbers d such that each set of diameter 1
in n-dimensional space can be partitioned into $n+1$ sets of diameter at most $d$. Universal covers can be used to obtain upper bounds on the value of $d_n(n+1)$ for each integer $n$. The method is illustrated in the case $n=3$. 

(iii)
STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge, contains no material previously published or written by another person, except when due reference is made.

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