

Improved Landau Gauge Fixing and Discretisation Errors ^{*}

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Lattice discretisation errors in the Landau gauge condition are examined. An improved gauge fixing algorithm in which $\mathcal{O}(a^2)$ errors are removed is presented. $\mathcal{O}(a^2)$ improvement of the gauge fixing condition displays the secondary benefit of reducing the size of higher-order errors. These results emphasise the importance of implementing an improved gauge fixing condition.

1. Introduction

Gauge fixing in lattice gauge theory simulations is crucial for many calculations e.g. the study of gauge dependent quantities such as the gluon propagator [1]. However, the standard lattice Landau gauge condition [2] is the same as the continuum condition, $\sum_{\mu} \partial_{\mu} A_{\mu} = 0$, only to leading order in the lattice spacing, a .

The focus of this talk is to use mean-field-improved perturbation theory [3] to compare different lattice definitions of the Landau gauge, and quantify the sizes of the discretisation errors. In particular, we derive a new $\mathcal{O}(a^2)$ improved Landau-gauge-fixing functional, and a method of generalising this to $\mathcal{O}(a^n)$.

2. Lattice Landau Gauge

Gauge fixing on the lattice is achieved by maximising a functional whose extremum implies the gauge fixing condition. The usual Landau gauge fixing functional is [2]

$$\mathcal{F}_1^G[\{U\}] = \sum_{\mu, x} \frac{1}{2} \text{Tr} \{U_{\mu}^G(x) + U_{\mu}^G(x)^{\dagger}\}, \quad (1)$$

where $U_{\mu}^G(x) = G(x)U_{\mu}(x)G(x + \hat{\mu})^{\dagger}$. By taking the functional derivative of (1), it can be shown

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that a maximum of that functional implies the continuum Landau gauge, with $\mathcal{O}(a^2)$ errors. It can further be shown that the gauge fixing condition implies that

$$\sum_{\mu} \partial_{\mu} A_{\mu}(x) = \sum_{\mu} \left\{ -\frac{a^2}{12} \partial_{\mu}^3 A_{\mu}(x) - \mathcal{H}_1 \right\}, \quad (2)$$

where \mathcal{H}_1 represents $\mathcal{O}(a^4)$ and higher-order terms. Naïvely one might hope that higher-order derivatives in the brackets are small, but it will be shown that the terms on the R.H.S. of (2) are large compared to the numerical accuracy possible in gauge fixing algorithms.

This “one-link” functional can be generalised to functionals using “n-link” terms:

$$\mathcal{F}_n^G[U] \equiv \sum_{x, \mu} \frac{1}{2n^2} \text{Tr} \{U_{n\mu}^G(x) + \text{h.c.}\} \quad (3)$$

where

$$U_{n\mu}^G(x) \equiv U_{\mu}^G(x)U_{\mu}^G(x + \hat{\mu}) \dots U_{\mu}^G(x + (n-1)\hat{\mu}).$$

Then

$$\begin{aligned} \frac{\delta \mathcal{F}_n^G}{\delta \omega^a(x)} &= \frac{1}{2n^2} i \sum_{\mu} \text{Tr} \{ [U_{n\mu}^G(x - \hat{\mu}) - U_{n\mu}^G(x) \\ &\quad \text{-h.c.}] T^a \} \\ &= ga^2 \left(\sum_{\mu} \text{Tr} \left\{ \left[\partial_{\mu} A_{\mu}(x) + \frac{2}{(na)^2} \left\{ \frac{(na)^4}{4!} \partial_{\mu}^3 A_{\mu}(x) \right. \right. \right. \right. \end{aligned} \quad (4)$$

$$\left. + \frac{(na)^6}{6!} \partial_\mu^5 A_\mu(x) + \dots \right\} T^a \Big\} + \mathcal{O}(g^2 a^2)$$

$\mathcal{O}(a^2)$ errors can be removed from the gauge fixing condition by taking a linear combination of the one-link and two-link functionals:

$$\mathcal{F}_{\text{Imp}}^G = \frac{4}{3} \mathcal{F}_1^G - \frac{1}{12u_0} \mathcal{F}_2^G \quad (5)$$

where we have included the plaquette-based, mean-field (tadpole) improvement parameter, u_0 [3].

To perform the gauge fixing we adopt a ‘‘steepest descents’’ approach [2]. The gauge transformation is $G(x) = \exp\{-i\alpha \sum_\mu \partial_\mu A_\mu(x)\}$. To maximise, for example, \mathcal{F}_1^G , we use (4) to derive the gauge transformation

$$G_1(x) = \exp\left\{\frac{\alpha}{2} \Delta_1(x)\right\}, \quad (6)$$

where

$$\Delta_1(x) = \sum_\mu \{U_\mu^G(x - \hat{\mu}) - U_\mu^G(x) - \text{h.c.}\}_{\text{traceless}}.$$

Similarly, Δ_2 and Δ_{Imp} are obtained from the functional derivatives of \mathcal{F}_2 and \mathcal{F}_{Imp} respectively. For a given functional, \mathcal{F}_i^G , the gauge fixing algorithm proceeds by calculating the relevant Δ_i , applying the associated gauge transformation to the gauge field, and iterating until the lattice Landau gauge condition is satisfied, to within some numerical accuracy. The approach to Landau gauge is measured by

$$\theta_i = \frac{1}{VN_c} \sum_x \text{Tr} \{ \Delta_i(x) \Delta_i(x)^\dagger \}. \quad (7)$$

A configuration fixed using $\Delta_1(x)$ will satisfy (2). It will also satisfy

$$\Delta_2(x) = -2iga^2 \sum_\mu \left\{ \frac{a^2}{4} \partial_\mu^3 A_\mu(x) - \mathcal{H}_1 + \mathcal{H}_2 \right\}$$

and similarly,

$$\begin{aligned} \Delta_{\text{Imp}}(x) = & -2iga^2 \sum_\mu \left\{ -\frac{a^2}{12} \partial_\mu^3 A_\mu(x) \right. \\ & \left. - \mathcal{H}_1 + \mathcal{H}_{\text{Imp}} \right\}. \end{aligned} \quad (8)$$

Since the improved measure has no $\mathcal{O}(a^2)$ error of its own, (8) provides an estimate of the absolute size of these discretisation errors.

Table 1

Values of the gauge-fixing measures obtained using the improved gluon action on 6^4 lattices at three values of the lattice spacing, fixed to Landau gauge with the one-link, two-link and improved functionals respectively.

| β | u_0 | \mathcal{F} | θ_{Imp} | θ_2 | $\frac{\theta_{\text{Imp}}}{\theta_2}$ |
|---------|-------|---------------|-----------------------|------------|--|
| 3.92 | 0.837 | 1 | 0.102 | 0.921 | 0.111 |
| 4.38 | 0.880 | 1 | 0.0585 | 0.526 | 0.111 |
| 5.00 | 0.904 | 1 | 0.0410 | 0.369 | 0.111 |
| β | u_0 | \mathcal{F} | θ_{Imp} | θ_1 | $\frac{\theta_1}{\theta_{\text{Imp}}}$ |
| 3.92 | 0.837 | 2 | 57.5 | 32.3 | 0.563 |
| 4.38 | 0.880 | 2 | 53.4 | 30.0 | 0.563 |
| 5.00 | 0.904 | 2 | 52.2 | 29.4 | 0.563 |
| β | u_0 | \mathcal{F} | θ_1 | θ_2 | $\frac{\theta_1}{\theta_2}$ |
| 3.92 | 0.837 | Imp | 0.0638 | 1.02 | 0.0625 |
| 4.38 | 0.880 | Imp | 0.0366 | 0.586 | 0.0625 |
| 5.00 | 0.904 | Imp | 0.0261 | 0.417 | 0.0625 |

3. Calculations on the Lattice

We use an $\mathcal{O}(a^2)$ tadpole-improved action. For the exploration of gauge fixing errors we consider 6^4 lattices at $\beta = 3.92, 4.38$, and 5.00 , corresponding to lattice spacings of approximately $0.35, 0.17$, and 0.1 fm respectively.

The configurations are gauge fixed, using Conjugate Gradient Fourier Acceleration [4] until $\theta_1 < 10^{-12}$. θ_{Imp} and θ_2 are then measured, to see the size of the residual higher order terms. The evolution of the gauge fixing measures is shown for one of the lattices in Fig. 1. This procedure is then repeated, fixing with each of the other two functionals, and the results are shown in Table 1. Results from additional lattices, as well as a more detailed discussion, are in [5].

If we fix a configuration to Landau gauge by using the basic, one-link functional, then the improved measure, θ_{Imp} , will consist entirely of the discretisation errors. Looking at Table 1, we see that at $\beta = 4.38$, $\theta_{\text{Imp}} = 0.059$, a substantial deviation from the continuum Landau gauge compared to the tolerance of the gauge fixing. We note that the relationship between the functionals in (5) provides a constraint on the gauge fixing

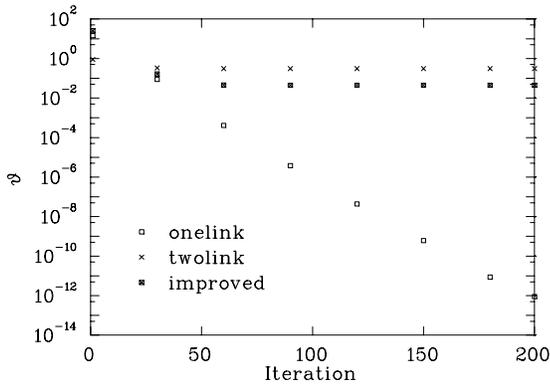


Figure 1. The gauge fixing measures for a 6^4 lattice with Wilson action at $\beta = 6.0$. This lattice was gauge fixed with Δ_1 , so θ_1 drops steadily whilst θ_2 and θ_{Imp} plateau at much higher values.

measures. For example, when fixing with Δ_1

$$\frac{\theta_{\text{Imp}}}{\theta_2} = \frac{\left(-\frac{1}{12}\right)^2}{\left(-\frac{1}{12} + \frac{1}{3}\right)^2} = \frac{1}{9} \simeq 0.111. \quad (9)$$

A configuration fixed using $\Delta_{\text{Imp}}(x)$ will satisfy

$$\sum_{\mu} \partial_{\mu} A_{\mu}(x) = \sum_{\mu} \{-\mathcal{H}_{\text{Imp}}\}. \quad (10)$$

With the help of (2) we see that

$$\Delta_1(x) = -2iga^2 \sum_{\mu} \left\{ \frac{a^2}{12} \partial_{\mu}^3 A_{\mu}(x) + \mathcal{H}_1 - \mathcal{H}_{\text{Imp}} \right\}$$

and (8) are identical to within a sign. If the three different methods presented all fixed in exactly the same way, then the θ_{Imp} of a configuration fixed with Δ_1 , would be equal to θ_1 when the configuration is fixed with Δ_{Imp} . It is clear from the table that they are not, signaling the higher-order derivative terms $\partial_{\mu}^n A_{\mu}(x)$, contained in the the \mathcal{H}_i , take different values depending on the gauge fixing functional used.

Examining the values in Table 1 reveals that in every case θ_1 is smaller when we have fixed with \mathcal{F}_{Imp} than θ_{Imp} under the \mathcal{F}_1 . This suggests that the additional long range information used by the improved functional is producing a gauge fixed configuration with smaller, higher-order derivatives; a secondary effect of improvement.

Equally, one can compare the value of θ_2 when fixed using \mathcal{F}_1 , and θ_1 when fixed using the \mathcal{F}_2 . In this case, their differences are rather large and are once again attributed to differences in the size of higher-order derivatives of the gauge field. \mathcal{F}_2 is coarser, knows little about short range fluctuations, and fails to constrain higher-order derivatives. Similar conclusions are drawn from a comparison of θ_2 fixed with the \mathcal{F}_{Imp} and θ_{Imp} fixed with \mathcal{F}_2 .

We also find that in terms of the absolute errors, the Wilson action at $\beta = 6.0$ is comparable to the improved lattice at $\beta = 4.38$, where the lattice spacing is three times larger.

4. Conclusions

We have fixed gluon field configurations to Landau gauge by three different functionals: one-link and two-link functionals, both with $\mathcal{O}(a^2)$ errors, and an improved functional, with $\mathcal{O}(a^4)$ errors. Using these functionals we have devised a method for estimating the discretisation errors involved. Lattice Landau gauge, in its standard implementation, deviates from its continuum counterpart by one part in 20, despite fixing the Lattice gauge condition to one part in 10^{12} . Our results indicate that order $\mathcal{O}(a^2)$ improvement of the gauge fixing condition improves comparison with the continuum Landau gauge through: 1) the elimination of $\mathcal{O}(a^2)$ errors and 2) reducing the size of higher-order errors.

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