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Heavy-quark axial charges to nonleading order

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We combine Witten’s renormalization group with the matching conditions of Bernreuther and Wetzel to calculate at next-to-leading order the complete heavy-quark contribution to the neutral-current axial-charge measurable in neutrino-proton elastic scattering. Our results are manifestly renormalization group invariant.

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This paper announces results for the next-to-leading-order (NLO) heavy-quark corrections to the axial charge \( g_A^{(Z)} \) for protons to couple to the weak neutral current

\[
J_{\mu}^{(Z)} = \frac{1}{2} \sum_{q=u,c,t} \left[ \sum_{a=r} q_{\gamma\mu}\gamma_5 q \right].
\]

The calculation is performed by decoupling heavy quarks \( h = t,b,c \) sequentially, i.e. one at a time. An extension to simultaneous decoupling of \( t,b,c \) quarks is foreshadowed in our concluding remarks.

The charge \( g_A^{(Z)} \) receives contributions from both light \( u,d,s \) and heavy \( c,b,t \) quarks,

\[
2g_A^{(Z)} = (\Delta u - \Delta d - \Delta s) + (\Delta c - \Delta b + \Delta t)
\]

where \( \Delta q \) refers to expectation value \( \langle p\bar{s}|q\gamma_{\mu}\gamma_5 q|p\rangle = 2m_p\gamma_\mu\Delta q \) for a proton of spin \( s_p \) and mass \( m_p \). It governs parity-violating effects due to \( Z \) exchange at low energies in elastic \( np \) and \( \bar{n}p \) scattering [1,2] or in light atoms [3,4]. A definitive measurement of \( np \) elastic scattering may be possible using the MiniBooNE setup at Fermilab [5].

Once heavy-quark corrections [2,6,7] have been taken into account, \( g_A^{(Z)} \) is related (modulo the issue of \( \delta \)-function terms at \( \chi = 0 \) [8]) to the flavor-singlet axial charge, defined scale invariantly and extracted from polarized deep inelastic scattering:

\[
g_A^{(0)} = 0.2 - 0.35.
\]

The small value of this quantity has inspired vast experimental and theoretical activity to understand the spin structure of the proton [9]. As a result, new experiments are being planned to map out the spin-flavor structure of the proton. These include polarized proton-proton collisions at the BNL Relativistic Heavy Ion Collider (RHIC) [10], semi-inclusive polarized deep inelastic scattering, and polarized \( ep \) collider studies [11]. Full NLO analyses are essential for a consistent interpretation of these experiments.

Many techniques for decoupling a single heavy quark are available. We rely on Witten’s method [12], where the renormalization scheme is mass independent and improved Callan-Symanzik equations [13] can be exploited. In such schemes, the decoupling of heavy particles required by the Appelquist-Carrazone theorem [14] is not manifest. However, correct decoupling is ensured by applying the matching conditions of Bernreuther and Wetzel [15]; these relate coupling constant, mass and operator normalizations before and after the decoupling of a heavy quark. The advantages of this approach are its rigor and the fact that the final results are expressed in terms of renormalization group (RG) invariants.

These invariants are Witten-style running couplings \( \bar{\alpha}_i \), one for each heavy quark \( h = t,b,c \), and axial charges for nucleons in the residual theory with three light flavors.

We find that, when first \( t \), then \( b \), and finally \( c \) are decoupled from Eq. (2), the full NLO result is

\[
2g_A^{(Z)}(\Delta u - \Delta d - \Delta s)_{\text{inv}} + P(\Delta u + \Delta d + \Delta s)_{\text{inv}} + O(m_{i,b,c}^{-1})
\]

where \( P \) is a polynomial in the running couplings \( \bar{\alpha}_i \),

\[
P = \frac{6}{23\pi}(\bar{\alpha}_b - \bar{\alpha}_t) \left[ 1 + \frac{125663}{82800\pi\alpha_t} + \frac{6167}{3312\pi\alpha_t} \right]
\]

\[
- \frac{22}{75\pi\alpha_t} \left( \frac{\bar{\alpha}_b}{\bar{\alpha}_t} - \frac{6}{27\pi\alpha_t} - \frac{181}{648\pi^2} \bar{\alpha}_b^2 + O(\bar{\alpha}_{t,b,c}^3) \right)
\]

and \( (\Delta q)_{\text{inv}} \) denotes the scale-invariant version of \( \Delta q \) defined in the following way.

Let \( \alpha_f = g_f^2/4\pi \) and \( \beta_f(\alpha_f) \) be the gluon coupling and beta function for \( \overline{\text{MS}} \) renormalized quantum chromodynamics (QCD) with \( f \) flavors and \( N_c = 3 \) colors, and let \( \gamma_f(\alpha_f) \) be the gamma function for the singlet current

\[
(u_{\gamma_f} \gamma_{5} u + \bar{d} \gamma_f \gamma_{5} d + \cdots)_{f} = \sum_{k=1}^{f} (\bar{q}_k \gamma_f \gamma_{5} q_k f)_{f}.
\]

A scale-invariant current \( (S_{f})_{5} \) is obtained when Eq. (6) is multiplied by

\[
E_f(\alpha_f) = \exp \int_{0}^{\alpha_f} dx \frac{\gamma_f(x)}{\beta_f(x)}.
\]
Flavor-dependent, scale-invariant axial charges \( \Delta q_{\text{inv}} \) such as
\[
\Delta q_{\text{inv}}^{(s)} = \frac{1}{\pi} \left( g_{A}^{(s)} - g_{A}^{(8)} \right)
\]
can then be obtained from linear combinations of Eq. (8) and
\[
g_{A}^{(3)} = \Delta u - \Delta d = (\Delta u - \Delta d)_{\text{inv}}
\]
\[
g_{A}^{(8)} = \Delta u + \Delta d - 2 \Delta s = (\Delta u + \Delta d - 2 \Delta s)_{\text{inv}}.
\]
Here \( g_{A}^{(3)} \approx 1.267 \pm 0.004 \) is the isotriplet axial charge measured in neutron beta-decay, and \( g_{A}^{(8)} = 0.58 \pm 0.03 \) is the octet charge measured independently in hyperon beta decay. Taking \( \bar{\alpha}_{3} = 0.1, \bar{\alpha}_{8} = 0.2 \) and \( \bar{\alpha}_{c} = 0.35 \) in Eq. (5), we find a small heavy-quark correction factor \( \mathcal{P} = -0.02 \), with LO terms dominant.

Our results extend and make more precise the well known work of Collins, Wilczek and Zee [6] and Kaplan and Manohar [2], where heavy-quark effective theory was used to estimate \( g_{A}^{(2)} \) in leading order (LO) for sequential decoupling of \( t, b \) and \( t, b, c \) respectively. This analysis is also influenced by a discussion of [6] by Chetyrkin and Kühn [16], who considered some aspects of NLO decoupling of the \( t \) quark from the neutral current and in particular, the requirement that the result be scale invariant. Related work has been done on heavy-quark production in polarized deep inelastic scattering using the QCD parton model [17] and in high-energy polarized \( \gamma p \) and \( pp \) at NLO [18].

The plan of this paper is as follows. First is a brief review of Witten's application of improved Callan-Symanzik equations [13] to the decoupling of a heavy quark in mass-independent renormalization schemes. Next, we combine it with matching conditions [15] to deal with next-to-leading-order (NLO) calculations involving axial-vector currents. Following is then a direct derivation of Eq. (5) from Eq. (1) for the neutral current. Our concluding remarks indicate the result of extending Eq. (5) to simultaneous decoupling of \( t, b, c \)—done not only for numerical reasons, but also to check that the \( t, b \) contributions cancel for \( m_{t} = m_{b} \).

We begin by considering mass-independent schemes, such as the modified minimal subtraction scheme (MS), where renormalized masses behave like coupling constants. This key property is exploited in Witten's method.

Let \( \mu \) be the scale used to define dimensional regularization and renormalization. Then the MS scale is
\[
\bar{\mu} = \mu \sqrt{4 \pi e^{-\gamma/2}}, \quad \gamma = 0.5772 \ldots
\]
We choose the same scale \( \bar{\mu} \) irrespective of the number of flavors \( f \) being considered, and so hold \( \bar{\mu} \) fixed as the heavy quarks (masses \( m_{b} \)) decouple.
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\[ \delta_f(\alpha_f) = D_f \ln m_f \]  
(21)

\[ \gamma_f(\alpha_f) = Z^{-1}[\gamma_f(\alpha_F) + D_F] \]  
(22)

depend on \( \alpha_f \) alone. The lack of \( m_f \) dependence of the renormalization factors in Eqs. (14) and (15) ensures mass-independent renormalization for the residual theory.

Although these equations hold for any \( f < F \), their practical application is straightforward only when heavy quarks are decoupled one at a time. So we set \( F = f + 1 \), where just one quark \( h \) is heavy. Then it is convenient to introduce a running coupling [12]

\[ \tilde{\alpha}_h = \tilde{\alpha}_h(\alpha_F, \ln(m_h/\mu)) \]  
(23)

associated with the \( \Sigma_F \) renormalized mass \( m_h \):

\[ \ln(m_h/\mu) = \int_{\alpha_f}^{\alpha_h} \frac{d\alpha_f}{\gamma_f(\alpha_f)} \]  
(24)

It satisfies the constraints

\[ \tilde{\alpha}_h(\alpha_F, 0) = \alpha_F. \tilde{\alpha}_h(\alpha_F, \infty) = 0 \]  
(25)

the latter being a consequence of the asymptotic freedom of the \( F \)-flavor theory (\( F \leq 16 \)). Also, Eqs. (16), (20) and (24) imply that \( \tilde{\alpha}_h \) is renormalization group (RG) invariant:

\[ D_F \tilde{\alpha}_h = 0. \]  
(26)

Witten’s solution of Eq. (22) for the matrix \( Z \) is

\[ Z(\alpha_F, m_h/\mu) = \exp\left\{ \int_{\alpha_f}^{\alpha_h} \frac{d\alpha_f}{\beta_f(\alpha_f)} Z(\tilde{\alpha}_h, 1) \right\} \]  
(27)

where “ord” indicates \( x \)-ordering of matrix integrands in the exponentials. Note that it is the relative scaling between the initial and residual theories which matters.

For our NLO calculation, we need the formulas

\[ \beta_f(x) = -\frac{x^2}{3\pi} \left( \frac{33}{2} - f \right) - \frac{x^3}{12\pi^2} (153 - 19f) + O(x^4) \]  

\[ \gamma_f(x) = \frac{x^2}{\pi^2} f + \frac{x^3}{36\pi^2} (177 - 2f) + O(x^4) \]  

\[ \delta_f(x) = -\frac{2x}{\pi} + O(x^2) \]  
(28)

where \( \gamma_f \) refers to the \( f \)-flavor singlet current (6) and includes the three-loop term found by Larin [19] and Chetyrkin and Kühn [16].

Our matching procedure amounts to evaluating to NLO accuracy the quantities \( \tilde{\alpha}_h, \alpha_f(\tilde{\alpha}_h, 1) \) and \( Z(\tilde{\alpha}_h, 1) \) in Eq. (27), such that the answers depend on \( \alpha_f \) and not \( \alpha_F \).

Bernreuther and Wetzel [15] applied the Appelquist-Carrazzone decoupling theorem [14] to the gluon coupling constant \( \alpha^M_Q \) renormalized at space-like momentum \( Q \),

\[ \alpha^{M_Q}_{\text{with } h} = \alpha^{M_Q}_{\text{no } h} + O(m_h^{-1}) \]  
(29)

and compared calculations of \( \alpha^{M_Q}_Q \) in the \( F = f + 1 \) and \( f \)-flavor MS theories. This reduces to a determination of the leading power of the one-\( h \)-loop MSF gluon self-energy. The result is a matching condition

\[ \alpha^M_{f+1} = \alpha^M_f = C_{\text{LO}} \ln \left( \frac{m_h}{\mu} \right) + C_{\text{NLO}} + O(\alpha_f, m_h^{-1}) \]  
(30)

with \( \alpha_f \)-independent LO and NLO coefficients given by

\[ C_{\text{LO}} = \frac{1}{3\pi}, \quad C_{\text{NLO}} = 0. \]  
(31)

As a result, we find

\[ \alpha_f(\tilde{\alpha}_h, 0) = \tilde{\alpha}_h + O(\tilde{\alpha}_h^{3/2}) = \tilde{\alpha}_h. \]  
(32)

Bernreuther and Wetzel showed that it is possible to deduce all LO and NLO terms in Eq. (30) from Eq. (31) and \( \beta_f \) and \( \delta_f \) in Eq. (28). We have done the calculation explicitly:

\[ \alpha_{f+1}^{NLO} = \alpha_f^{-1} + \frac{3}{3\pi} \ln \frac{m_h}{\mu} + \frac{1}{3\pi} \ln \left( \frac{m_h}{\mu} \right) \]  

\[ + \frac{d_f}{1 + \frac{3\pi}{3\pi} \left( \frac{1}{2} - f \right) \ln \frac{m_h}{\mu}} \]  

\[ c_f = \frac{142 - 19f}{2\pi(31 - 2f)}, \quad d_f = \frac{57 + 16f}{2\pi(33 - 2f)(31 - 2f)}. \]  
(33)

From Eq. (24), we have also found \( \tilde{\alpha}_h \) in NLO,

\[ \tilde{\alpha}_h^{-1} = \alpha_f^{-1} + \frac{3\pi}{3\pi} \left( \frac{1}{2} - f \right) \ln \frac{m_h}{\mu} + \frac{153 - 19f}{2\pi(33 - 2f)} \]  

\[ \times \ln \left( 1 + \frac{3\pi}{3\pi} \left( \frac{1}{2} - f \right) \ln \frac{m_h}{\mu} \right) \]  
(34)

where \( \tilde{m}_h \) is Witten’s RG invariant mass:

\[ \tilde{m}_h = m_h \exp \int_{\alpha_f}^{\tilde{\alpha}_h} \frac{d\alpha_f}{\beta_f(\alpha_f)} \tilde{\delta}_f(\alpha_f)/\beta_f(\alpha_f). \]  
(35)

If desired, \( \ln(\tilde{m}_h/\mu) \) can be eliminated by substituting

\[ \ln \frac{\tilde{m}_h}{\mu} = \ln \frac{m_h}{\mu} - \frac{12}{31 - 2f} \left[ 1 + \frac{3\pi}{3\pi} \left( \frac{1}{2} - f \right) \ln \frac{m_h}{\mu} \right]. \]  
(36)

Therefore the asymptotic formula for \( \tilde{\alpha}_h \) as \( m_h \to \infty \) is
\[ \tilde{\alpha}_b \sim 3 \pi \left\{ \left( \frac{33}{2} - f \right) \ln \frac{m_b}{\mu} + k_f \ln \frac{m_b}{\mu} + O(1) \right\} \]

\[ k_f = \frac{3(153 - 19f)}{2(33 - 2f)} \cdot \frac{6(33 - 2f)}{31 - 2f}. \] (37)

To find the matrix \( \mathcal{Z}(\tilde{\alpha}_b, 1) \) in NLO, we need a matching condition for the MS amplitude \( \Gamma_{\mu 5} \) for \( \bar{t} g_{\mu} \gamma_5 h \) to couple to a light quark \( l \). We have calculated the leading power due to the two-loop diagram \( \mathcal{O}_{(\alpha)} \):

\[ \Gamma_{\mu 5} = \left( \frac{\alpha_F}{\pi} \right)^2 \gamma_{\mu} \gamma_5 \left( \ln \frac{m_b}{\mu} + \frac{1}{8} \right) + O(\alpha_F^3, m_b^{-1}). \] (38)

Consequently, there is a NLO term \( \tilde{\alpha}_b^2/8\pi^2 \) in \( \mathcal{Z}(\tilde{\alpha}_b, 1) \) for \( \bar{t} g_{\mu} \gamma_5 h \) to produce \( \bar{t} \gamma_{\mu} \gamma_5 l \) as \( m_b \to \infty \).

Now we consider the special case where heavy quarks are decoupled from the weak neutral axial current. Let us adopt the shorthand notation \( q_f \) for MS currents \( (\bar{q}_f) \gamma_5 q_f \) in the \( f \)-flavor theory, e.g. the neutral current \( \bar{q}_f^2 \) and the scale-invariant singlet current \( S_{\mu 5} \):

\[ J_f^5 = \frac{1}{2} (t - b + c - s + u - d)_6. \] (39)

\[ S_f = E_f(\alpha_f)(u + d + s + \cdots)_f. \] (40)

We begin by decoupling the \( t \) quark. Because of

\[ (c - s + u - d)_6 = (c - s + u - d)_5 + O(1/m_t), \] (41)

we see that Eq. (27) is nontrivial only for

\[ (t - b)_6 = Z_{\mu 5}(u + d + s + c + b)_5 + \frac{1}{2} (u + d + s + c - 4b)_5 + O(1/m_t). \] (42)

Since \( (t - b)_6 \) is scale invariant, we have \( \gamma_F = 0 \) in Eq. (27):

\[ Z_{\mu 5}(\alpha_6, m_t/\mu) = Z_{\mu 5}(\tilde{\alpha}_b, 1) \exp[- \int_{\alpha_5}^{\alpha_6} \frac{\gamma_5(x)}{\beta_5(x)}]. \] (43)

The operator matching condition (38) corresponds to

\[ t_b = \frac{\alpha_2^2}{\pi^2} \left( \ln \frac{m_t}{\mu} + \frac{1}{8} \right) (u + d + s + c + b)_5 + O(\alpha_6^3, m_t^{-1}). \] (44)

and so we conclude:

\[ Z_{\mu 5}(\tilde{\alpha}_b, 1) = - \frac{1}{6} + (8 \pi^2)^{-1} \tilde{\alpha}_7^2 + O(\tilde{\alpha}_7^3). \] (45)

Equation (43) is to be expanded about \( \tilde{\alpha}_7 \sim 0 \) with \( \alpha_7 \) held fixed. In that limit, the exponential tends to the constant factor \( E_5(\alpha_3) \) of Eq. (7). This factor combines with the singlet current in Eq. (42) to form the scale-invariant operator \( S_5 \), as required by RG\( f_{\mu 5} \) invariance. The full NLO result is then obtained by writing

\[ (t - b)_6 = Z_{\mu 5}(\tilde{\alpha}_b, 1) \exp[- \int_{\alpha_5}^{\alpha_6} \frac{\gamma_5(x)}{\beta_5(x)}] S_5 \]

\[ + \frac{1}{2} (u + d + s + c - 4b)_5 \] (46)

and expanding in \( \tilde{\alpha}_7 \), keeping all quadratic terms:

\[ (t - b)_6 = \left\{ \frac{1}{5} - \frac{6}{23} \tilde{\alpha}_7 + \frac{6167 \tilde{\alpha}_7}{3312} \right\} S_5 + O(\tilde{\alpha}_7^2). \] (47)

Next we decouple the \( b \) quark. Here, it is natural to define five-flavor quantities \( \tilde{\alpha}_{b_5} \) and \( \tilde{m}_{b_5} \) analogous to the six-flavor running coupling \( \tilde{\alpha}_7 \) and mass \( \tilde{m}_7 \), for the top quark:

\[ \ln \frac{m_5}{\mu} = \int_{\alpha_5}^{\alpha_6} \frac{1 - \delta_5(x)}{\beta_5(x)} dx, \]

\[ \ln \frac{\tilde{m}_{b_5}}{\mu} = \int_{\alpha_5}^{\alpha_6} \frac{\delta_5(x)}{\beta_5(x)} dx. \] (48)

Equations (20) and (21) imply that \( \tilde{\alpha}_{b_5} \) and \( \tilde{m}_{b_5} \) are both RG\( f_{\mu 5} \) and RG\( f_{\mu 6} \) invariant

\[ D_\mu \tilde{\alpha}_{b_5} = 0 = D_\mu \tilde{m}_{b_5}, \quad D_\mu \tilde{m}_{b_5} = 0 = D_\mu \tilde{m}_{b_5} \] (49)

and hence physically significant in the original six-flavor theory. So we write \( \tilde{\alpha}_b \) and \( \tilde{m}_b \) for \( \tilde{\alpha}_{b_5} \) and \( \tilde{m}_{b_5} \).

Consider decoupling the \( b \) quark from Eq. (47). The NLO matching condition (38) becomes

\[ b_5 = \frac{\alpha_2^2}{\pi^2} \left( \ln \frac{m_{b_5}}{\mu} + \frac{1}{8} \right) (u + d + s + c)_4 + O(\alpha_6^3, m_{b_5}^{-1}). \] (50)

so the nonsinglet current in Eq. (47) can be written

\[ (u + d + s + c - 4b)_5 = \left( 1 - \frac{\alpha_2^2/2}{\pi^2} \right) E_4^{-1}(\tilde{\alpha}_b) S_4 \]

\[ + O(\alpha_6^3, m_{b_5}^{-1}). \] (51)

For the singlet current \( S_5 \) in Eq. (47), we find

\[ S_5 = E_5(\tilde{\alpha}_b) \left\{ 1 + \frac{\tilde{\alpha}_b^2}{8\pi^2} \right\} E_4^{-1}(\tilde{\alpha}_b) S_4 + O(\alpha_6^3, m_{b_5}^{-1}). \] (52)

taking into account the definitions (7) and (40). Then we expand Eqs. (51) and (52) in \( \tilde{\alpha}_b \), keeping quadratic terms:

\[ (t - b)_6 = \frac{6}{23\pi} (\tilde{\alpha}_b - \tilde{\alpha}_7) \left\{ 1 + \frac{125663}{82800} \tilde{\alpha}_b \right\} \]

\[ + \frac{6167}{3312} S_4 + O(\alpha_6^3, m_{b_5}^{-1}). \] (53)
The same technique can be applied to decouple the c quark from $S_4$ in Eq. (53) and $(c-s+u-d)_4$ [the result of decoupling b from Eq. (41)]. That yields the final results (4) and (5) given in the introduction.

Notice that our results depend on two key features:

(i) Like previous workers in this area, we decouple heavy quarks sequentially, i.e. one at a time.

(ii) Our running couplings $\bar{\alpha}_i$, $\bar{\alpha}_b$ and $\bar{\alpha}_c$, which correspond to Witten's prescription [12], are all renormalization group invariant.

The restriction to sequential decoupling is numerically reasonable for the t quark, but dubious for the b and c quarks, because it amounts to an assumption that $\ln(m_t/\mu)$ is negligible compared with $\ln(m_b/\mu)$. This inhibits detailed comparison of NLO results with data, which ought to be carried out with NLO accuracy [20].

There is also a theoretical issue here: one would like to check that, in the limit $m_t=m_b$, the t and b contributions cancel. However, that is outside the region of validity $\ln(m_t/\mu)\gg \ln(m_b/\mu)$ for sequential decoupling.

For these reasons, we have extended our analysis to the case of simultaneous decoupling, where the mass logarithms are allowed to grow large together: $\ln(m_t/\mu)\sim \ln(m_b/\mu)\sim \ln(m_c/\mu)$—large. This requires a considerable theoretical development of matching conditions and the renormalization group, which we will present separately. It involves the construction of running couplings $\alpha_i$, $\alpha_b$, $\alpha_c$ with the following properties: (i) They are renormalization group invariant; (ii) they are defined for $m_i\geq m_j\geq m_k$, and can have a nontrivial dependence on more than one heavy-quark mass; (iii) in the special case of sequential decoupling, they agree with $\bar{\alpha}_i$, $\bar{\alpha}_b$ and $\bar{\alpha}_c$ to NLO; and (iv) for the case of equal masses, they coincide, e.g.

$$\alpha_i = \alpha_b \quad \text{for} \quad m_i = m_b. \quad (54)$$

Then we find that the result for the simultaneous decoupling of the $t,b,c$ quarks from the neutral current is of the same form (4) as the sequential answer, but with the sequential running couplings in Eq. (5) replaced by our simultaneous couplings $\alpha_i$, $\alpha_b$, and $\alpha_c$:

$$\mathcal{P} = \frac{6}{23\pi} \left( \alpha_b - \alpha_t \right) \left( 1 + \frac{125663}{82800\pi} \alpha_b + \frac{6167}{3312\pi} \alpha_t - \frac{22}{75\pi} \alpha_c \right)$$

$$- \frac{6}{27\pi} \alpha_c - \frac{181}{648\pi} \alpha^2 + O(\alpha^3_{t,b,c}). \quad (55)$$

Notice the factorization of the terms depending on $\alpha_i$ and $\alpha_b$. Given Eq. (54), the factor $\alpha_b - \alpha_t$ ensures that all contributions from b and t quarks cancel (as they should) for $m_i = m_b$.

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[20] This includes matching conditions for the b and c masses, to be discussed elsewhere.