Nontopological Finite Temperature Induced Fermion Number

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We show that while the zero temperature induced fermion number in a chiral sigma model background depends only on the asymptotic values of the chiral field, at finite temperature the induced fermion number depends also on the detailed shape of the chiral background. We resum the leading low temperature terms to all orders in the derivative expansion, producing a simple result that can be interpreted physically as the different effect of the chiral background on virtual pairs of the Dirac sea and on the real particles of the thermal plasma. By contrast, for a kink background, not of sigma model form, the finite T induced fermion number is temperature dependent but topological.

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The phenomenon of induced fermion number due to the interaction of fermions with topological backgrounds (e.g., solitons, vortices, monopoles, Skyrmions) has many applications ranging from polymer physics to particle physics [1–8]. The original fractional fermion number result of Jackiw-Rebbi [1] has a deep connection with the existence of spinless charged excitations in polymers [2]. The adiabatic analysis of Goldstone-Wilczek [3] in systems without conjugation symmetry has important implications for bag models [5], monopoles, and sigma models, which provide effective field theory descriptions of systems ranging from condensed matter, to AMO, to particle and nuclear physics [9]. The induced fermion number is related to the spectral asymmetry of the relevant Dirac operator, and mathematical results concerning index theorems [8] relate the fermion number to asymptotic topological properties of the background. At finite temperature, the situation is less clear. In several examples [10–14], the fermion number is known to be temperature dependent, but is still topological in the sense that the only dependence on the background field is through its asymptotic properties. In this Letter, we present a simple physical case for which this is not true: in a $1 + 1$ dimensional chiral sigma model, the finite temperature induced fermion number depends on the detailed structure of the background. This contradicts a previous analysis [15] and claim [16] that the finite T fermion number is in general a topological quantity. We give a simple physical explanation of the origin of the nontopological dependence. Our analysis has been motivated in part by the results of [17] concerning the T dependence of anomalous amplitudes in nuclear decays.

Consider an abelian model in $1 + 1$ dimensions with fermions interacting via scalar and pseudoscalar couplings to two boson fields ϕ_1 and ϕ_2:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}(\phi_1 + i\gamma_5\phi_2)\psi\,. \tag{1}$$

There are two important physical cases: (i) kink case [1]:

$$\phi_1 = m \quad \text{and} \quad \phi_2(\pm\infty) = \pm\hat{\phi}_2\,, \tag{2}$$

(ii) sigma model case [3]:

$$\phi_1^2 + \phi_2^2 = m^2. \tag{3}$$

In the sigma model case (3), the interaction term in the Lagrangian (1) can be expressed as

$$m\bar{\psi}e^{i\gamma_5\theta}\psi = m\bar{\psi}(\cos\theta + i\gamma_5\sin\theta)\psi\,. \tag{4}$$

At $T = 0$, both these cases have an induced topological current $J^\mu \equiv \langle\bar{\psi}\gamma^\mu\psi\rangle$ given by [3]

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\theta + \cdots, \tag{5}$$

where the angular field θ is defined by $\theta \equiv \arctan(\phi_2/\phi_1)$. The dots in (5) refer to higher derivative terms, which are all of the form of a total derivative of θ and its derivatives [4]. The induced fermion number $N \equiv \int dx\, J^0$ is

$$N = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \theta' = \frac{1}{\pi}\hat{\theta}\,, \tag{6}$$

where $\pm\hat{\theta}$ are the asymptotic values of $\theta(x)$ at $x = \pm\infty$. The fermion number N is topological as it depends only on $\hat{\theta}$, not on the detailed shape of $\theta(x)$. The conjugation symmetric case of Jackiw and Rebbi [1] is with $m \to 0$ in the kink case (2), in which case $N \to \pm\frac{1}{2}$.

At nonzero temperature, the induced fermion number for a static background is [8,14]

$$N = -\frac{1}{2}\int_C \frac{dz}{2\pi i}\,\mathrm{tr}\left(\frac{1}{H - z}\right)\tanh\left(\frac{\beta z}{2}\right). \tag{7}$$

Here $\beta = 1/T$ is the inverse temperature, and $\mathrm{tr}(\frac{1}{H-z})$ is the resolvent of the Dirac Hamiltonian H. The contour C is $(-\infty + i\epsilon, +\infty + i\epsilon)$ and $(+\infty - i\epsilon, -\infty - i\epsilon)$. The technical part of the calculation of the induced fermion number (7) is computing the resolvent of H. Then the induced fermion number has an integral representation (7), or a sum by deforming the contour in (7) around the simple poles of the tanh function. For static backgrounds $\phi_1(x)$ and $\phi_2(x)$ in (1), the Dirac Hamiltonian is

$$H = -i\gamma^0\gamma^1\frac{d}{dx} + \gamma^0\phi_1(x) + i\gamma^0\gamma_5\phi_2(x)\,. \tag{8}$$

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We use Dirac matrices $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, $\gamma^5 = -\sigma_1$. Only the *even* part (in the argument z) of the resolvent $\mathrm{tr}(\frac{1}{H-z})$ contributes to the fermion number N in (7).

Consider first the kink case in (2). The even part of the resolvent can be computed exactly using an index theorem trace identity [8,10,18]:

$$\left[\mathrm{tr}\left(\frac{1}{H-z}\right)\right]_{\mathrm{even}} = \frac{-m\hat{\phi}_2}{(m^2-z^2)\sqrt{m^2+\hat{\phi}_2^2-z^2}}. \quad (9)$$

Then the fermion number (7) for the kink case (2) is

$$N = \sum_{n=0}^{\infty} \frac{\frac{2}{\pi}(\frac{m\beta}{\pi})^2 \sin\hat{\theta}}{[(2n+1)^2+(\frac{m\beta}{\pi})^2]\sqrt{(2n+1)^2\cos^2\hat{\theta}+(\frac{m\beta}{\pi})^2}}, \quad (10)$$

where $\hat{\theta} \equiv \arctan(\hat{\phi}_2/m)$. The fermion number (10) is plotted in Fig. 1 as a function of $\hat{\theta}$ for various values of T. As $T \to 0$, this result reduces smoothly to the zero T result (6). Despite its complicated form, the nonzero T result (10) is still topological, as it refers only to the background through $\hat{\theta}$.

In the sigma model case (3), the trace identity formula (9) does not apply. Another approach is needed to evaluate the resolvent. One such approach is the derivative expansion [19], in which (at $T = 0$) we assume that the spatial derivatives of the background fields are small compared to the fermion mass scale m. At finite T, however, there is an additional scale, namely T, and one might expect a new condition that the derivatives of the background fields are small compared to T. But in that case, a smooth $T \to 0$ limit would be precluded. As we shall see, this condition does not in fact arise, and we recover a smooth $T \to 0$ limit by making an exact resummation of the leading low temperature term at each order in the derivative expansion.

Returning to the general Hamiltonian (8), the derivative expansion can be obtained by separating H^2 as

$$H^2 = (-\nabla^2+\phi_1^2+\phi_2^2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\phi_2' & i\phi_1'\\ -i\phi_1' & -\phi_2'\end{pmatrix} \quad (11)$$

and expanding in powers of derivatives. To first order:

$$\left[\mathrm{tr}\left(\frac{1}{H-z}\right)\right]_{\mathrm{even}} = -\frac{1}{2}\int_{-\infty}^{\infty} dx\, \frac{(\phi_1\phi_2'-\phi_2\phi_1')}{(\phi_1^2+\phi_2^2-z^2)^{3/2}} + \dots,$$

where dots refer to terms with three or more derivatives.

In the kink case (2), where $\phi_1 = m$ is constant, this first order calculation actually reproduces the *exact* trace identity result (9). But in the sigma model case (3), (4), where $\phi_1^2+\phi_2^2 = m^2$ is a constant, this first order derivative expansion result leads to

$$N^{(1)} = \left[\sum_{n=0}^{\infty} \frac{\frac{1}{\pi}(\frac{m\beta}{\pi})^2}{[(2n+1)^2+(\frac{m\beta}{\pi})^2]^{3/2}}\right]\int_{-\infty}^{\infty} dx\, \theta', \quad (12)$$

which is simply the zero temperature answer (6) multiplied by a smooth function of T. As $T \to 0$, this prefactor reduces to $\frac{1}{2\pi}$, so the *full* $T = 0$ result (6) is regained unproblematically. But at finite temperature, the first order (in the derivative expansion) formula (12) for the sigma model case differs from the kink case formula (10), even though each of (12) and (10) reduces to (6) at $T = 0$.

What about higher order corrections to the derivative expansion? In the kink case (2), there are *no* higher order corrections to the even part of the resolvent. This is due to the special form of the Hamiltonian in the kink background [8,18]. There can, of course, be higher order corrections to the fermion number *density*, but these are all total (spatial) derivatives, and do not contribute to the integrated fermion number, even at $T > 0$.

In the sigma model case (3), (4), where the trace identity does not apply, the situation is very different. Going to the next order in the derivative expansion, we find

$$\left[\mathrm{tr}\left(\frac{1}{H-z}\right)\right]_{\mathrm{even}} = -\frac{m^2}{2(m^2-z^2)^{3/2}}\int\theta' - \frac{m^2}{8(m^2-z^2)^{5/2}}\int\theta''' - \frac{m^2(4z^2+m^2)}{16(m^2-z^2)^{7/2}}\int(\theta')^3 + \cdots, \quad (13)$$

where the dots refer to terms with five or more derivatives. For a chiral background with $\theta(x)$ approaching its asymptotic values exponentially fast, $\int dx\,\theta'''$ vanishes. But $\int dx\,(\theta')^3$ does not vanish. Thus, the first order fermion number (12) acquires a third order correction:

$$N^{(3)} = \frac{m^2\beta^4}{8\pi^5}\sum_{n=0}^{\infty}\frac{[-4(2n+1)^2+(\frac{m\beta}{\pi})^2]}{[(2n+1)^2+(\frac{m\beta}{\pi})^2]^{7/2}}\int(\theta')^3. \quad (14)$$

This is *not* just a function of the asymptotic value $\hat{\theta}$ of the chiral field $\theta(x)$; it also depends on the actual *shape* of $\theta(x)$. Thus, the induced fermion number is no longer topological. This contradicts [15], where it is stated that the first order derivative expansion contribution (12) is the full answer. However, the energy trace prefactor in (14) vanishes at $T = 0$, so the nontopological third order contribution (14) vanishes at $T = 0$. Thus, the nontopological nature of the finite temperature induced fermion number is still consistent (at this order) with the topological nature of the $T = 0$ induced fermion number (6).

We now turn to a physical explanation of why, in the sigma model case, the finite temperature induced charge is more sensitive to the background field than at zero temperature. Note first of all that the chiral background acts like a static but spatially inhomogeneous electric field, as

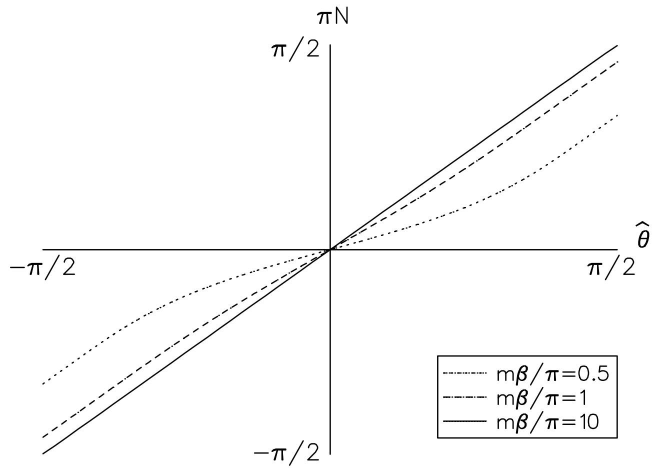

FIG. 1. Plots of πN, where N is the finite temperature fermion number (10) for the kink case (2), as a function of $\hat{\theta}$. These plots are for $m\beta/\pi$ taking values 0.5, 1, and 10, as labeled. As $T \to 0$, note that $\pi N \to \hat{\theta}$, as in (6).

can be seen by making a local chiral rotation [4]: $\psi \to \tilde{\psi} = e^{i\theta\gamma_5/2}\psi$. In terms of these chirally rotated fields the Lagrangian (1), with interaction (4), becomes

$$\mathcal{L} = i\bar{\tilde{\psi}}\partial\!\!\!/\tilde{\psi} - m\bar{\tilde{\psi}}\tilde{\psi} - \bar{\tilde{\psi}}\gamma^0 \frac{\theta'}{2}\tilde{\psi}. \qquad (15)$$

Thus, the chiral field leads to an inhomogeneous electric field $E(x) = -\frac{1}{2}\theta''(x)$. Given that $\theta(x)$ itself has a kinklike spatial profile, the electric field is such that it changes sign as a function of x, as shown in Fig. 2 (we choose $\theta' > 0$). This electric field acts on the Dirac sea to polarize the vacuum by aligning the virtual vacuum dipoles of the Dirac sea, producing a localized buildup of charge near the kink center. But at nonzero T, the electric field also has an effect on the thermal plasma, as we show below.

First, consider the *full* derivative expansion (13) of the even part of the resolvent, at low but nonzero temperature. At fifth order, there are three independent terms involving θ''''', $\theta'''(\theta')^2$, and $(\theta')^5$. The θ''''' term vanishes when integrated over x, but the other two terms are generally nonzero. However, as $T \to 0$ the $(\theta')^5$ term dominates the $\theta'''(\theta')^2$ term. Indeed, for low temperature, the dominant term with $(2l - 1)$ derivatives in the derivative expansion (13) involves $(\theta')^{2l-1}$. Using the chirally rotated form (15), the dominant term at $(2l - 1)$th order is simply

$$N^{(2l-1)}_{\rm dom} = 2T\sum_{n=0}^{\infty}\int \frac{dk}{2\pi}\,\frac{{\rm tr}([\gamma^0(p\!\!\!/ + m)]^{2l})}{(p^2 + m^2)^{2l}} \times \int dx\left(\frac{\theta'}{2}\right)^{2l-1} \qquad (16)$$

with Euclidean $p = ((2n + 1)\pi T, k)$.

At zero temperature, all these terms $N^{(2l-1)}$ vanish, except for $l = 1$. This fact is not obvious; it involves highly nontrivial cancellations between terms in the expansion of the trace. But at nonzero temperature, all the terms in (16) are nonvanishing. Moreover, they have a remarkably simple low temperature ($T \ll m$) limit:

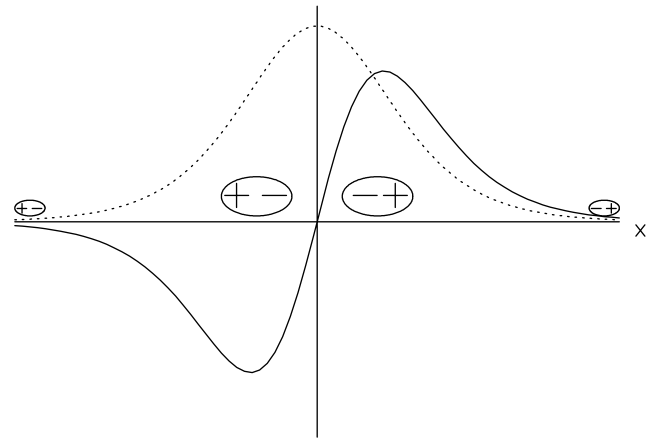

FIG. 2. For a kinklike chiral field $\theta(x)$, the electric field $E = -\theta''/2$ has the form shown in the solid line, producing a vacuum polarization charge distribution localized near the kink center, roughly following the dotted line $\theta'/2$.

$$N^{(2l-1)} = \delta_{l,1}\int dx\,\frac{\theta'}{2\pi} - \sqrt{\frac{2mT}{\pi}}\,e^{-m/T}\int \frac{(\frac{\theta'}{2T})^{2l-1}}{(2l - 1)!} + \cdots. \qquad (17)$$

Thus, in the low temperature limit, we can resum the *entire* derivative expansion, to obtain the induced fermion number in the sigma model case (3), (4):

$$N = \int dx\,\frac{\theta'}{2\pi} - \sqrt{\frac{2mT}{\pi}}\int dx\,e^{-m/T}\sinh\!\left(\frac{\theta'}{2T}\right) + \cdots \qquad (18)$$

(the dots are power-law subleading terms for $T \ll m$).

Several features of this result (18) deserve comment. First, at zero temperature, only the first term survives, producing the familiar result (6) that the induced fermion number depends on the chiral field $\theta(x)$ only through its asymptotic value $\hat{\theta} \equiv \theta(\infty)$. Second, although (17) contains powers of the "dangerous" ratio θ'/T, the resummed expression (18) has a smooth $T \to 0$ limit: the resummed exponential factors $e^{-(m\mp\theta'/2)/T}$ in (18) require only the derivative expansion condition $\theta' \ll m$, precisely as at $T = 0$. This is consistent with the fact that for a static background, T does not enter the computation of the single particle fermion spectrum. Third, at zero temperature, one can invoke Lorentz invariance to constrain the form of higher order corrections to (5) to be total derivatives [4], but these arguments do not apply at finite temperature. We see this in (18): the temperature dependent corrections are not total derivatives of terms made from θ and its derivatives. At nonzero temperature this shows clearly that the induced fermion number is nontopological—it depends also on the detailed shape of $\theta(x)$. Finally, the form of the exponential factors in (18) suggests an interpretation of this result as an adiabatic change of the local Fermi level with a local chemical potential $\mu(x) = -\theta'/2$, which once again is sensible only in the derivative expansion regime where $\theta' \ll m$.

To make this physical picture more precise, we can interpret the result (18) as follows. The first, topological, term refers to the induced charge coming from the polarization of the Dirac sea. This is temperature independent as the short-lived virtual "electron-positron dipoles" of the Dirac sea do not come to thermal equilibrium. The next term in (18) corresponds to the induced charge arising from the response of the real charges in the thermal plasma to the spatially inhomogeneous electric field. Indeed, the linear response [20] of the plasma at low temperature to such an electric field yields an induced fermion number density $\rho(x) = \int \frac{dk}{2\pi} f(x,k)$ where the static distribution function $f(x,k)$ satisfies the Boltzmann equation: $v\frac{\partial}{\partial x} f(x,k) = -E(x)\frac{\partial}{\partial k} f(x,k)$, with $v = k/\sqrt{k^2+m^2}$. Regarding $\mu(x) = -\theta'/2$ as a local chemical potential, this is satisfied by local Fermi particle and antiparticle distribution functions

$$f_\pm(x,k) = \frac{1}{e^{\beta[\sqrt{k^2+m^2}\mp\mu(x)]}+1}. \qquad (19)$$

Taking $f = f_+ - f_-$, we obtain precisely the second, nontopological, term in (18) in the low temperature limit.

At $T = 0$, the fermion number may be defined as a sharp observable [21]; but at $T > 0$, thermal fluctuations introduce an rms deviation. Thus, the finite T fermion number in (7) and (18) is a thermal expectation value $\langle N\rangle$, as in the monopole cases [12–14]. We have estimated $\langle N^2\rangle - \langle N\rangle^2$, in the derivative expansion regime, in an analogous manner to the computation presented here for $\langle N\rangle$. We find that the rms deviation vanishes at $T = 0$, but at nonzero T can be significant compared to the thermal shift in (18). Details of this will be reported elsewhere.

We comment briefly on possible implications of these results for models in other dimensions. In $2+1$ dim, fermions in a static magnetic background acquire an induced charge that is T dependent but topological [11]. In $3+1$ dim, fermions in a static Dirac monopole background acquire an induced charge that is T dependent, but still depends only on the background through the total magnetic charge and the self-adjoint extension parameter [12–14]. For a static SU(2) 't Hooft-Polyakov monopole background (A_μ, ϕ), with coupling $\mathcal{L}_{\text{int}} = \bar\psi(\not{A} + \phi + i\gamma_5 m)\psi$, with ψ an isodoublet fermion, we have computed the finite T induced fermion number, using the $3+1$ trace identity used in the $T = 0$ case [22], and we find *precisely the same expression* (10) as in the $1+1$ kink case, with the identification $\hat\theta = \arctan(\hat\phi/m)$, where $\hat\phi$ is the asymptotic value of the magnitude $|\phi| = \sqrt{\phi^a\phi^a}$ of the Higgs field. Given that (10) reduces to $\frac{1}{\pi}\hat\theta$ at $T = 0$, this monopole result is consistent with the familiar $T = 0$ result [3,22,23]. The $3+1$ dim analog of the $1+1$ sigma model case (3), (4) is the sigma model with coupling $\mathcal{L}_{\text{int}} = m\bar\psi(\pi_0 + i\gamma_5\vec\pi\cdot\vec\tau)\psi$, where $\vec\tau$ are SU(2) generators, the fields π_0 and $\vec\pi$ are constrained by $\pi_0^2 + \vec\pi^2 = 1$. At $T = 0$ [3,4,6–8], there is an induced topological charge density $J^0 = \frac{1}{24\pi^2}\epsilon^{ijk}\text{tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g)$, where g is defined by $g = \pi_0 + i\vec\pi\cdot\vec\tau$. The corresponding $T = 0$ integrated charge is given by the winding number of the background field g at zero temperature. We conjecture that at finite temperature this induced charge will acquire additional nontopological contributions similar to those found here for the $1+1$ sigma model case.

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[1] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[2] W. Su, J. R. Schrieffer, and A. Heeger, Phys. Rev. Lett. **42**, 1698 (1979); Phys. Rev. B **22**, 2099 (1980); M. Rice, Phys. Lett. A **71**, 152 (1979).
[3] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[4] R. MacKenzie and F. Wilczek, Phys. Rev. D **30**, 2194 (1984); **30**, 2260 (1984); F. Wilczek, in *Proceedings of TASI 84,* edited by D. Williams (TASI Publications, Ann Arbor, MI, 1984).
[5] J. Goldstone and R. Jaffe, Phys. Rev. Lett. **51**, 1518 (1983).
[6] E. D'Hoker and E. Farhi, Nucl. Phys. **B248**, 59 (1984); **248**, 77 (1984); E. D'Hoker and J. Goldstone, Phys. Lett. B **158**, 429 (1985).
[7] D. Diakonov, V. Petrov, and P. Pobylitsa, Nucl. Phys. **B306**, 809 (1988); R. D. Ball, Phys. Rep. **182**, 1 (1989).
[8] A. Niemi and G. Semenoff, Phys. Rep. **135**, 99 (1986).
[9] S. Weinberg, *Quantum Theory of Fields II* (Cambridge University Press, Cambridge, England, 1996).
[10] A. Niemi and G. Semenoff, Phys. Lett. B **135**, 121 (1984); V. Soni and G. Baskaran, Phys. Rev. Lett. **53**, 523 (1984); W. Keil, Phys. Lett. A **106**, 89 (1984).
[11] K. Babu, A. Das, and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987); G. Dunne, in *Topological Aspects of Low Dimensional Systems,* Proceedings of the Les Houches Summer School, Session LXIX, edited by A. Comtet *et al.* (Springer, Berlin, 1999).
[12] C. Coriano and R. Parwani, Phys. Lett. B **363**, 71 (1995).
[13] A. Goldhaber, R. Parwani, and H. Singh, Phys. Lett. B **386**, 207 (1996).
[14] G. Dunne and J. Feinberg, Phys. Lett. B **477**, 474 (2000).
[15] S. Midorikawa, Prog. Theor. Phys. **69**, 1831 (1983).
[16] A. Niemi, Nucl. Phys. B **251**, 155 (1985).
[17] R. Pisarski, L. Trueman, and M. Tytgat, Prog. Theor. Phys. Suppl. **131**, 427 (1998).
[18] C. Callias, Commun. Math. Phys. **62**, 213 (1978).
[19] I. Aitchison and C. Fraser, Phys. Rev. D **31**, 2605 (1985).
[20] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[21] A. Goldhaber and S. Kivelson, Phys. Lett. B **255**, 445 (1991).
[22] M. Paranjape and G. Semenoff, Phys. Lett. **132B**, 369 (1983).
[23] This is similar to, but slightly different from, the point-monopole case in [13] where the theta parameter is related to the boundary conditions used in the point limit.