An Analytical Model for Two-layered Composite Beams with Partial Shear Interaction Based on a Higher Order Beam Theory

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Abstract

The application of composite structures is quite frequent in various structural engineering activities due to their super mechanical properties and structural performances. A composite beam consisting of two material layers, such as steel-concrete, steel-timber, and timber-timber is typically used in the construction industry to enhance the overall performance due to a proper utilization of two material layers in this structural system. In reality, the shear connectors such as bolts, nails or steel shear studs, commonly used to connect the two layers, are having a certain degree of deformability due to a finite stiffness of these shear connectors. This induces a shear slip at the interface between the two layers which is known as partial shear interaction. This is an important feature which needs to be considered in the modelling of composite beams. In the present study, the shear connectors are modelled as distributed shear springs along the length of composite beams in the present study.

A higher order beam theory (HBT) is used to consider the effect of transverse shear deformation accurately by taking a third order variation of the longitudinal displacement across the beam depth. Since HBT allows a true parabolic vibration of the shear stress that vanishes at the top and bottom fibres of the beam, no shear correction factor needs to be used. In addition to the prediction for the beam global response such as deflection or vibration frequency, HBT also predicts the local response such as distribution of stresses accurately, which cannot be achieved by the existing models based on Euler-Bernoulli beam theory (EBT) or Timoshenko beam theory (TBT).
In the present study, exact analytical models based on HBT are developed for the static bending response, flexural free and forced vibration response, and geometric nonlinear static flexural response of two-layered composite beams with partial shear interaction. The principle of virtual work and the Hamilton’s principle are applied to derive the governing equations for static and dynamic analysis, respectively, where the Navier type solution technique is used to solve these equations analytically. In order to assess the accuracy and efficiency of the proposed analytical models, the results produced by the models are compared with the results reported in literature by previous researchers and numerical results predicted by a one dimensional finite element model based on HBT as well as by a detailed two-dimensional finite element modelling of composite beams.
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Statement of Originality

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List of Manuscript


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Chapter 1: Introductory Background

1.1 Introduction

Composite beams consisting of two material layers, such as timber-timber, timber-concrete and steel-concrete, are widely used in bridges, buildings and many other civil engineering constructions. This structural configurations help to enhance the overall performance due to the proper utilisation of the two material layers. Taking steel-concrete composite beams as an example, this structural system is found to primarily utilise the concrete layer to carry the compressive stress whereas the steel layer carries the tensile stress. In terms of the connection between these two material layers, shear connectors such as bolts, nails and steel shear studs are commonly used to join these layers and transfer the shear force at the interface between them, which achieves the composite action of this structural system. Since shear connectors are having deformability with a finite stiffness in reality, the development of interfacial shear slip always occurs between the two layers. This scenario is defined as partial shear interaction (Oehlers and Bradford 1995), and its contribution on the structural behaviour is found to be so significant (e.g. Loh et al. 2004 and Uy and Nethercot 2005) that cannot be ignored in the modelling of these composite beams.

As the modelling of this structural system is a challenging task, a number of investigations have been conducted by various researchers considering the effect of partial interaction. It is interesting to note that most of the existing models (e.g. Adekola 1968, Faella et al. 2002, Jasim 1997, Ranzi et al. 2004 and 2006 and Wu et al. 2002) are developed for static response of composite beams based on Euler-Bernoulli beam theory (EBT) using numerical
techniques (e.g. direct stiffness method, finite difference method and finite element method). However, EBT does not consider the effect of shear deformation, which results in the underestimation of beam deflections.

Subsequently, it has been realised that the effect of shear deformation is considerable in many situations. Thus there has been a growing interest in recent years to incorporate this effect and Timoshenko’s beam theory (TBT) is typically used for this purpose. Ranzi and Zona (2007) developed a finite element model for steel-concrete composite beams where TBT was applied to model the steel girder while the concrete slab was modelled as EBT. On the other hand, a finite element formulation for two-layered composite beams has been developed by Schanbl et al. (2007a) who applied TBT in modelling both layers. It has been found that the effect of shear deformation is significant and cannot be ignored for beams with a small span-to-depth ratio, clamped boundary conditions, localized concentrated loads, and some other cases. In this approach (TBT), the actual parabolic variation of shear stress over the beam depth is replaced by an average shear stress of uniform distribution to simplify the problem. In order to minimise the effect of this simplification, an arbitrary shear correction factor is used to match the global response of the beam such as deflection well by scaling down the shear stiffness artificially, but this is not adequate to get a satisfactory local response of these beams such as distribution of stresses.

In order to address the abovementioned issue, a higher-order beam theory (HBT) has recently been developed by Sheikh and his co-workers for an accurate prediction of global as well as local responses of two-layered composite beams (Chakrabarti et al. 2012a and
2012b). The HBT takes a 3rd order variation of the longitudinal displacement of fibres over the beam depth to model the cross-sectional warping of the beam layers produced by the parabolic (nonlinear) variation of shear stress, where the basic concept of Reddy’s higher order shear deformation theory (Reddy 1984) developed for multi-layered laminated composite plates with no interfacial slip between the layers is used. However, these models (Chakrabarti et al. 2012a and 2012b) are restricted to numerical solution techniques using one dimensional (1D) finite element implementation where exact and unique results cannot be obtained as the numerical results are dependent on element sizes in a finite element analysis.

1.2 Analytical Solution

All the aforementioned models are developed using approximate numerical techniques, which results in excessive computational costs. Although some limitations exit in an analytical approach and the derivation procedure is always very complicated, an analytical solution can reduce the computational issues and its advantage of an exact solution attracted a number of researchers to developing analytical models for predicting the response of composite beams. Newmark et al. (1951), as one of the earliest researchers on the modelling of composite beams with partial shear interaction, developed an analytical model using EBT which is commonly treated as a benchmarking solution for the validation of a numerical model in the area of composite beams. A further development is done by Girhammar and Gopu (1993) who developed an analytical model based on EBT for first and second order analysis of composite beam columns considering partial interactions. Subsequently, Schanbl
et al. (2007b) developed an analytical model for two-layered composite beams using TBT.

1.3 Dynamic Response

In addition to the investigations on flexural response of composite beams under static loading, some studies have been conducted on dynamic response of composite structures subjected to time varying and moving loads but the number of such investigations is very limited. Girhammar and Pan (1993) have proposed an accurate analytical model with some approximations for dynamic response of composite beams subjected to time varying loads (an impulsive load and a suddenly applied step loading) using EBT. Huang and Su (2008) have developed an analytical model based on EBT for dynamic response of composite beams subject to a moving load. Recently, a finite element model for dynamic response of composite beams subjected to a step loading as well as a moving load has been developed by Chakrabarti et al. (2013) using HBT.

1.4 Geometric Nonlinear Analysis

As the deformation of composite beams cannot be restricted to a small range during their entire service life and it is quite common to have a moderately large deformation of these beams under service loads in reality, this effect should be considered in the modelling of these beams. This introduces a nonlinearity which is commonly known as geometric nonlinearity. Unfortunately, a very limited number of models have been developed considering the effect of geometrically nonlinearity and most of them are based on finite element approximation. Erkmen and Bradford (2009) developed a nonlinear finite element model for steel-concrete composite beams curved in-plan. A nonlinear model for straight
composite beams is developed by Ranzi et al. (2010) considering both horizontal shear slip and vertical separation the two material layers. Both these models are based on EBT which has its usual limitations in analysing these beams as mentioned before. Recently, Hjiaj et al. (2012) developed a model based on TBT for large deformation analysis of composite beams. All these models focused on investigating the global response (i.e. deflection) of composite beams considering the effect of large displacement but no one paid any attention on local response (i.e stresses and their distributions) of these beams under large deformation.

1.5 Research Gaps and Objectives

It is observed from the literature review that a number of research gap exists in the modelling of two-layered composite beams and it is attempted to fill few important gaps in the present study.

Most of the exiting models for two-layered composite beams with partial shear interaction are based on EBT and few are based on TBT. These models are not adequate in accurately predicting the local response (i.e. stresses) of composite beams. Even the global response (i.e. deflection) of these beams having a high degree of shear interaction, a small span-to-length ratio, a big elastic-to-shear ratio, a localised concentrated load and clamped boundary conditions cannot be estimated satisfactorily by these models. Although a higher order beam theory (HBT) has recently been proposed to overcome the abovementioned issues but it is restricted to its numerical solution through a finite element approximation and no one attempted to develop an exact analytical model for composite beams based on this beam theory.
✓ Objective 1: To develop an exact analytical model based on HBT for the flexural response of two-layered composite beams with interfacial shear slips under static loading utilising a Navier type solution technique.

Most of the exiting models for two-layered composite beams with interfacial shear slips are developed for flexural response of these beams under static loading. A very limited number of studies have been conducted on dynamic response of composite beams subjected to dynamic loading, where most of them are based on EBT. Although HBT has recently been applied to the dynamic analysis of two-layered composite beams but it is again based on finite element approximation and an exact analytical solution based on HBT for dynamic response of composite beams is yet to be developed. Moreover, no one has investigated the effect of shear deformation on the dynamic response of these beams.

✓ Objective 2: To develop an exact analytical model based on HBT for the dynamic response due to time varying loads as well as moving loads of two-layered composite beams with interfacial shear slips.

A limited number of models have been developed for geometric nonlinear analysis of composite beams with partial shear interaction. Most of these models are based on less refined beam theories (EBT and TBT) and their solutions are restricted to numerical techniques such as finite element approximation.

✓ Objective 3: To develop an analytical model based on HBT for accurately predicting the geometric nonlinear response of two-layered composite beams with partial shear interaction by taking into account the effect of large displacement.
1.6 Details of Manuscripts Included in this Thesis

This thesis contains three manuscripts which are submitted to internationally recognised journals. The following chapters (Chapter 2, 3 and 4) are presented in the form of journal papers which are individual and self-sufficient that do not need the information from previous chapters.

Chapter 2 presents an investigation on flexural static response of two-layered composite beams with partial shear interaction using an exact analytical model based on a higher order beam theory. The Navier type solution is used to solve the governing equations analytically which are derived based on the virtual work principle. The validation of the model is conducted by comparing the results predicted by this model with existing results published in literature and numerical results predicted by a one-dimensional finite element model as well as a detailed two-dimensional model using a reliable commercial finite element code. In order to assess the performance of the model, a number of numerical examples of composite beams having different beam dimensions, cross sectional configurations, loading types and degrees of shear interaction are analysed. Moreover, the relative merits of different beam theories (EBT, TBT and HBT) in predicting the global and local responses of two-layered composite beams are also investigated using this model.

Chapter 3 presents the development of an analytical model for the free and forced vibration response of two-layered composite beams with interfacial shear slips. Both time varying and moving loads are considered in predicting the dynamic response of composite beams considering the effect of damping. In addition to the higher order beam theory (HBT),
models are also developed using both EBT and TBT, which are used to solve some numerical examples of composite beams to investigate the relative performances of these beam theories in predicting vibration frequency, time history of deflection, velocity and stresses as well as the stress distribution over the depth of these beams. Moreover, composite beams with different damping ratios are studied to show the significance of the damping effect on flexural response of composite beams subjected dynamic loading. The model validation is conducted by comparing the results predicted by the model with experimental results, published results as well as numerical results generated by one-dimensional and two-dimensional finite element models.

Chapter 4 presents the development of an analytical model based on HBT for geometric nonlinear response of two-layered composite beams considering the effect of partial shear interaction. The Von-Karman large deflection theory is applied to capture the effect of geometric nonlinearity, which introduces a displacement dependent nonlinear component within the strain vector appeared in the governing equations. Therefore, the nonlinear equations are solved iteratively following the Newton-Raphson technique. The model is used to solve numerical examples of composite beams to assess its performance and range of applicability in predicting large deformation response of these beams. A detailed two-dimensional finite element model is also used for the verification of the proposed analytical model. The significance of geometric nonlinear in accurate prediction of both global and local responses of composite beams is investigated. Also the comparative studies between the results predicted by three beam theories (HBT, TBT and EBT) are conducted to show their relative merits.
Chapter 5 concludes the vital improvements and major findings of this research, and provides some possibilities for future studies in the area of two-layered composite beams.

Reference


Computers and Structure, vol. 80, no. 11, pp. 1001-1009.


Chapter 2: Flexural Response

2.1 Introduction

The manuscript of this chapter “An analytical model for flexural response of two-layered composite beams with interfacial shear slip using a higher order beam theory” presents the development of an exact analytical model for the flexural response of two-layered composite beams with partial shear interaction using a higher order beam theory (HBT). This study aims to investigate the effect of shear deformation on both global and local responses of these composite beams subjected to static loading. The principle of virtual work is used to derive the governing equations, and these equations are solved analytically using a Navier type solution technique satisfying specific boundary conditions. The performance and range of applicability of the proposed analytical model is tested by solving a number of numerical examples of composite beams having different cross sectional configurations (rectangular, T-section and flanged section), values of interfacial shear stiffness and loading types (concentrated load, uniformly distributed load and sinusoidal variation of distributed load). After conducting some comparative studies among three beam theories (EBT, TBT and HBT), it is observed that the effect of shear deformation is significant in predicting the flexural response of two-layered composite beams. Moreover, it is shown that the HBT exhibits an improvement in predicting both global and local response of composite beams, especially for the shear stress distribution. Due to the analytical nature of the proposed model, all the results predicted by the model are exact and unique that can be treated as a benchmarking solution for future studies in the area of
2.2 List of Manuscript


2.3 Statement of Authorship

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<tr>
<th>Title of Paper</th>
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<td>An analytical model for flexural response of two-layered composite beams with interfacial shear slip using a higher order beam theory</td>
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<td>Developed the analytical model, conducted numerical analysis and prepared manuscript</td>
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<td>Overall Percentage</td>
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This paper reports on original research I conducted during my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.

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3) the sum of all co-author contribution is equal to 100% less the candidate’s stated contribution

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2.4 Analytical model for flexural response of two-layered composite beams with interfacial shear slip using a higher order beam theory

Jie Wen, Abdul Hamid Sheikh, Md. Alhaz Uddin and Brian Uy

ABSTRACT

An exact analytical model based on a higher-order beam theory (HBT) is developed for an accurate prediction of the flexural response of two layered composite beams with partial shear interactions. This is achieved by taking a third order variation of the longitudinal displacement over the beam depth for the two layers separately. The deformable shear connectors joining the two different material layers are modelled as distributed shear springs along the beam length at their interface. The principle of virtual work is used to derive the governing equations which are solved analytically using a Navier type solution technique. To assess the performance of the proposed model, numerical examples of composite beams are solved using the model. The results predicted by the model are compared with published results and the numerical results produced by a one dimensional finite element model based on HBT as well as a detailed two-dimensional finite element modelling of composite beams.

Keywords: Composite beams; Partial shear interaction; Higher order beam theory; analytical solution; Flexural response
1. **Introduction**

Composite structures are widely used in many structural engineering activities for their superior mechanical properties and structural performance. A two layered composite beam such as timber-timber, timber-steel, timber-concrete and steel-concrete are typically used in the construction industry. In these structural configurations, the two material layers are properly utilised (e.g., in steel-concrete composite beams, the concrete layer is primarily used to carry the compressive stress whereas the steel layer carries the tensile stress) to enhance the performance of the overall structural system. Composite action of these beams is achieved by connecting the two different material layers with shear connectors such as bolts, nails or steel shear studs. Theoretically, if the connections are rigid with infinite stiffness, full composite action can be achieved. In such scenarios, the benefits of composite action can be fully achieved where no shear slip develops at the interface, which is defined as full shear interaction. However, these rigid and non-deformable connectors can hardly be realized in practise. In reality, shear connections are having deformability with a finite stiffness, which results in the development of interfacial shear slip between the two layers. That is defined as partial shear interaction [1] which always occurs due to the deformability of shear connectors. As the contribution of partial shear interaction on the structural behaviour is found to be significant [2, 3], this effect can’t be ignored in the analysis of these composite beams. This is an active area of research which is best demonstrated by the large number of studies on different aspects of composite beams. However, the main aim of the present study is to develop an efficient analytical model for accurately predicting the response of composite beams.
Newmark et al. [4] is one of the earliest researchers who developed an analytical model for composite beams considering the effect of partial interaction. This is a well-regarded model and due to the analytical nature of its solution, it is commonly treated as a benchmarking solution for the validation of a numerical model. It is obvious that an analytical model is applicable for a specific type of loading and boundary conditions used for deriving the analytical model, which motivated many researchers for developing numerical models (e.g. [5-9]). A further development is done by Girhammar and Gopu [10] who developed an analytical model for first and second order analysis of composite beam columns considering partial interactions. Girhammar and Pan [11] also developed an analytical model for dynamic analysis of composite beams. However, it should be noted that all the above mention studies are based on Euler-Bernoulli beam theory (EBT), which does not consider the effect of transverse shear deformation of the steel and concrete layers. The effect of this shear deformation is significant in some situations such as beams with a small span-to-depth ratio, clamped boundary conditions, localized concentrated loads, and some other cases.

Thus there has been a growing interest in recent years to incorporate the effect of shear deformation and the Timoshenko’s beam theory (TBT) is typically used for this purpose. Ranzi and Zona [12] developed a model for steel-concrete composite beams where TBT was applied to model the steel girder and the concrete slab was modelled as EBT. On the other hand, an analytical model for two-layered composite beams has been developed by Schnabl et al. [13] who applied TBT in modelling both layers. Schnabl et al. [14] also developed a finite element formulation of their analytical model [13]. In this context,
another representative study is due to Xu and Wu [15] who developed analytical models for bending, vibration and buckling response of composite beams considering shear deformation of both layers according to TBT. In a model based on TBT, the actual parabolic variation of shear stress over the beam depth is replaced by an average shear stress of uniform distribution over the beam depth in order to simplify the problem. In order to minimise the effect of this simplification, an arbitrary shear correction factor is used to shear stiffness of the beam. Unfortunately, the calculation of the exact value of this shear correction factor for a composite beam with partial shear interaction is cumbersome in comparison with that of a single layer homogeneous beam. Moreover, this factor helps to match the global response of the beam such as deflection well by scaling down the shear stiffness, but this is not adequate to get a satisfactory local response such as distribution of stresses or strains, which is shown in the section of numerical results.

In order to address the abovementioned issue, a higher-order beam theory (HBT) has recently been developed by Sheikh and his co-workers [16] for an accurate prediction of global as well as local responses of these composite beams. The cross-sectional warping of the beam layers produced by the parabolic (nonlinear) variation of shear stress is modelled by a higher order (3rd order) variation of the longitudinal displacement of fibres over the beam depth. This beam theory (HBT) utilised the basic concept of Reddy’s higher order shear deformation theory [17] developed for multi-layered laminated composite plates modelled as single layered plates with no interfacial slip. So far, the capability of HBT has been investigated numerically through a one dimensional (1D) finite element implementation of the model [16] which has exhibited an encouraging performance of the
model. This has encouraged to develop an exact analytical of two-layered composite beams with partial shear interaction based on HBT so that the analytical solution can be benchmarked for future research investigations in this area.

In order to ensure that the present formulation is consistent, a number of numerical examples are solved. The results predicted by the proposed analytical model are checked or validated with published results as well as numerical results produced by the 1D finite element model based on HBT developed by us in an earlier study [16]. Moreover, a detailed two-dimensional finite element modelling approach is employed to analyse a composite beam using a well-regarded finite element commercial code ABAQUS specifically for the validation of stresses. It is noted that the number of available results of shear deformable composite beams with partial shear interaction is very limited in the existing literature and no one reported results for the stresses induced within these beams. Therefore, the dataset reported in this paper will contribute to an important resource for future references.

2. Mathematical formulation

2.1 Higher order beam theory

Fig.1 shows a typical two layered composite beam with a deformable interface in shear. According to the HBT [16], the variation of longitudinal displacements of the two layers over their depths can be expressed as

\[ u_a = u_{a0} - y_a \theta_a + y_a^2 \alpha_a + y_a^3 \beta_a \]  

(1)

\[ u_b = u_{b0} - y_b \theta_b + y_b^2 \alpha_b + y_b^3 \beta_b \]  

(2)
where \( u_{a0} \) and \( u_{b0} \) are longitudinal displacements of the two layers at their reference axis 
\((y_a = 0 \text{ or } y_b = 0)\), \( \theta_a \) and \( \theta_b \) are bending rotations of these layers, and \( \alpha \) and \( \beta \) are higher order terms. As the vertical separation between the layers is not common under static loading for a straight beam, its effect is not considered in this study. Thus the vertical displacement is assumed to be the same for both the layers and it can be expressed as

\[
w_a = w_b = w(x) \tag{3}
\]

**Fig. 1.** Typical two layer composite beam with displacement variations throughout the beam

The partial shear interaction between the two layers is modelled by a distributed spring layer along the interface between these layers where the shear slip can be defined in the form of longitudinal displacement jump at their interface as

\[
s = \bar{u}_b - \bar{u}_a \tag{4}
\]

where \( \bar{u}_a \) is the longitudinal displacement at the bottom fibre of the upper layer and \( \bar{u}_b \) is that at the top fibre of the lower layer.
Using the shear stress free condition at the exterior surfaces \((y_a = h_a/2\) and \(y_b = -h_b/2\)), and taking \(\bar{u}_a\) and \(\bar{u}_b\) as independent unknowns, the higher order terms having no physical meaning appeared in Eqs. (1) and (2) can be expressed in terms of other unknowns. Using Eq. (3) and the above information, Eqs. (1) and (2) can be rewritten in terms of all the physical parameters as

\[
\begin{align*}
    u_a &= A_u u_{a0} + B_u \bar{u}_a + C_u \theta_a + D_u \frac{dw}{dx} \\
    u_b &= A_u u_{b0} + B_u \bar{u}_b + C_u \theta_b + D_u \frac{dw}{dx}
\end{align*}
\]

where \(A, B, C\) and \(D\) are functions of \(y\) and cross-sectional properties of the two layers which are expressed as follows, and the derivation process is given in Appendix 1.

\[
\begin{align*}
    A_a &= 1 - \frac{12}{5h_a^2} y_a^2 + \frac{16}{5h_a^2} y_a^3, \\
    B_a &= \frac{12}{5h_a^2} y_a^2 - \frac{16}{5h_a^2} y_a^3, \\
    C_a &= -y_a - \frac{4}{5h_a} y_a^2 + \frac{12}{5h_a^2} y_a^3, \\
    D_a &= -\frac{2}{5h_a} y_a^2 - \frac{4}{5h_a^2} y_a^3, \\
    A_b &= 1 - \frac{12}{5h_b^2} y_b^2 - \frac{16}{5h_b^2} y_b^3, \\
    B_b &= \frac{12}{5h_b^2} y_b^2 + \frac{16}{5h_b^2} y_b^3, \\
    C_b &= -y_b + \frac{4}{5h_b} y_b^2 + \frac{12}{5h_b^2} y_b^3, \\
    D_b &= \frac{2}{5h_b} y_b^2 - \frac{4}{5h_b^2} y_b^3.
\end{align*}
\]

\section*{2.2 Variational formulation}

The governing equation can be derived using the principle of virtual work and it can be expressed as

\[
\int_a^b \int_a^b \left( \delta \sigma_a \sigma_a + \delta \gamma_a \tau_a \right) dA dx + \int_a^b \int_a^b \left( \delta \sigma_b \sigma_b + \delta \gamma_b \tau_b \right) dA dx + \int_a^b \delta \sigma \tau_a dx - \int_a^b \delta w q dx = 0
\]

where \(\delta\) is an operate to show the variation of any parameter, \(\sigma_a\) and \(\sigma_b\) are normal stresses while \(\tau_a\) and \(\tau_b\) are shear stresses of the upper and lower layers, \(\varepsilon_a\) and \(\varepsilon_b\) are...
normal strains while \( \gamma_a \) and \( \gamma_b \) are shear strains of the layers, \( \tau_{sh} \) is the distributed shear force (per unit length) at their interface, \( q \) is the distributed external load (per unit length) acting on the beam and \( A \) represents the cross-sectional area. Eq. (5) and (6) can be used to expresses the strain components appeared in Eq. (7) as follows.

\[
\varepsilon_a = \frac{du_a}{dx} = A_d \frac{du_{a0}}{dx} + B_d \frac{du_a}{dx} + C_d \frac{d\theta_a}{dx} + D_d \frac{d^2w}{dx^2}
\]

\[
\varepsilon_b = \frac{du_b}{dx} = A_d \frac{du_{b0}}{dx} + B_d \frac{du_b}{dx} + C_d \frac{d\theta_b}{dx} + D_d \frac{d^2w}{dx^2}
\]

\[
\gamma_a = \frac{\partial u_a}{\partial y_a} + \frac{\partial w_a}{\partial x} = \frac{dA_a}{dy_a} u_{a0} + \frac{dB_a}{dy_a} \bar{u}_a + \frac{dC_a}{dy_a} \theta_a + \frac{dD_a}{dy_a} \frac{dw}{dx} + \frac{dw}{dx}
\]

\[
\gamma_b = \frac{\partial u_b}{\partial y_b} + \frac{\partial w_b}{\partial x} = \frac{dA_b}{dy_b} u_{b0} + \frac{dB_b}{dy_b} \bar{u}_b + \frac{dC_b}{dy_b} \theta_b + \frac{dD_b}{dy_b} \frac{dw}{dx} + \frac{dw}{dx}
\]

(8)

2.3 Differential equations and their solution

After substitution of the above strain components and the expression of shear slip as in Eq. (4) along with its constitutive relationship in Eq. (7), the integration over the cross-sections (upper and lower material layers) can be carried out conveniently which will lead to express Eq. (7) in the following form.
\[
\begin{align*}
\int_a^b & \left[ \left( \frac{N_a}{h_a} - \frac{12}{5h_a^2} P_a + \frac{16}{5h_a^3} S_a \right) \frac{d\delta u_a}{dx} + \left( \frac{12}{5h_a^2} P_a - \frac{16}{5h_a^3} S_a \right) \frac{d\delta \bar{u}_a}{dx} + \left( -M_a - \frac{4}{5h_a} P_a + \frac{12}{5h_a^2} S_a \right) \frac{d\delta \theta_a}{dx} \\
&+ \left( -\frac{2}{5h_a^2} P_a - \frac{4}{5h_a^3} S_a \right) \frac{d^2 \delta w}{dx^2} + \left( -\frac{12}{5h_a^2} R_a + \frac{3}{5h_a^3} T_a \right) \delta u_a + \left( \frac{12}{5h_a^2} R_a - \frac{3}{5h_a^3} T_a \right) \delta \bar{u}_a \\
&+ \left( -Q_a - \frac{2}{5h_a^2} R_a + \frac{12}{5h_a^3} T_a \right) \delta \theta_a + \left( -\frac{2}{5h_a^2} R_a - \frac{4}{5h_a^3} T_a \right) \frac{d\delta w}{dx} \right] dx \\
&+ \int_a^b \left[ \left( \frac{N_b}{h_b} - \frac{12}{5h_b^2} P_b + \frac{16}{5h_b^3} S_b \right) \frac{d\delta u_{b0}}{dx} + \left( \frac{12}{5h_b^2} P_b + \frac{16}{5h_b^3} S_b \right) \frac{d\delta \bar{u}_{b0}}{dx} + \left( -M_b - \frac{4}{5h_b} P_b + \frac{12}{5h_b^2} S_b \right) \frac{d\delta \theta_{b0}}{dx} \\
&+ \left( \frac{2}{5h_b^2} P_b - \frac{4}{5h_b^3} S_b \right) \frac{d^2 \delta w}{dx^2} + \left( -\frac{12}{5h_b^2} R_b - \frac{3}{5h_b^3} T_b \right) \delta u_{b0} + \left( \frac{12}{5h_b^2} R_b + \frac{3}{5h_b^3} T_b \right) \delta \bar{u}_{b0} \\
&+ \left( -Q_b + \frac{2}{5h_b^2} R_b + \frac{12}{5h_b^3} T_b \right) \delta \theta_{b0} + \left( \frac{2}{5h_b^2} R_b - \frac{4}{5h_b^3} T_b \right) \frac{d\delta w}{dx} \right] dx + \int_a^b \delta (\bar{\tau}_a - \bar{\tau}_b) k_s (\bar{\sigma}_b - \bar{\sigma}_a) \\
- \int_a^b \delta w q dx &= 0
\end{align*}
\]

where $k_s$ is the stiffness of the interfacial distributed springs and the stress resultants ($N$, $M$, $S$, $Q$, $R$ and $T$) used in the above equation can be defined as

\[
(N_a, M_a, P_a, S_a) = \int_{A_a} \sigma_a (1, y_a, y_a^2, y_a^3) dA_a
\]

\[
(Q_a, R_a, T_a) = \int_{A_a} \tau_a (1, y_a^2, y_a^3) dA_a
\]

\[
(N_b, M_b, P_b, S_b) = \int_{A_b} \sigma_b (1, y_b, y_b^2, y_b^3) dA_b
\]

\[
(Q_b, R_b, T_b) = \int_{A_b} \tau_b (1, y_b^2, y_b^3) dA_b
\]

(10)

In order to develop the proposed model analytically without major complications, the present study is restricted to beams with simply supported ends. Therefore, $w_a = w_b = 0$ and $\sigma_a = \sigma_b = 0$ at the two ends ($x = 0$ and $x = L$) of a beam having a length of $L$. As the normal stresses are zero ($\sigma_a = \sigma_b = 0$) at the two ends, Eq. (10) can be utilised to get the following boundary conditions.
\[ N_a(x = 0) = N_a(x = L) = N_b(x = 0) = N_b(x = L) = 0 \]
\[ M_a(x = 0) = M_a(x = L) = M_b(x = 0) = M_b(x = L) = 0 \]
\[ P_a(x = 0) = P_a(x = L) = P_b(x = 0) = P_b(x = L) = 0 \]
\[ S_a(x = 0) = S_a(x = L) = S_b(x = 0) = S_b(x = L) = 0 \] (11)

Eq. (9) contains seven one-dimensional displacement components \((u_{a0}, \bar{u}_a, \theta_a, w, u_{b0}, \bar{u}_b, \theta_b)\) but they appear in terms of their variations (e.g., \(\delta u_{a0}\)) as well as derivatives of these terms (e.g., \(d\delta u_{a0}/dx\)). In order to eliminate these derivatives, Eq. (9) is integrated by parts and the boundary conditions given in Eq. (11) are used to remove some terms. This will help to collect all the terms associated with the individual displacement components in their variational forms (e.g., \(\delta u_{a0}\)) that will lead to get seven differential equations. As the integration of the entire equation is too long, a portion of Eq. (9) associated with \(u_{a0}\) (one of these displacement components) is shown below as a representative example.

\[
\int_a^b \left[ N_a \left( \frac{12}{5h_a^2} P_a + \frac{16}{5h_a^3} S_a \right) \frac{d\delta u_{a0}}{dx} + \left( \frac{-2}{5h_a^2} R_a + \frac{16}{5h_a^3} T_a \right) \delta u_{a0} \right] dx
\]
\[ = \left[ N_a \left( \frac{12}{5h_a^2} P_a + \frac{16}{5h_a^3} S_a \right) \right] _0^L - \int_a^b \delta u_{a0} \left( \frac{dN_a}{dx} \frac{12}{5h_a^2} \frac{dP_a}{dx} + \frac{16}{5h_a^3} \frac{dS_a}{dx} \right) dx + \int_a^b \left( \frac{-2}{5h_a^2} R_a + \frac{16}{5h_a^3} T_a \right) \delta u_{a0} dx
\]
\[ = \int_a^b \left( -\frac{dn_a}{dx} + \frac{12}{5h_a^2} \frac{dp_a}{dx} - \frac{16}{5h_a^3} \frac{ds_a}{dx} - \frac{2}{5h_a^2} R_a + \frac{16}{5h_a^3} T_a \right) \delta u_{a0} dx = \int_a^b C_{u_{a0}} \delta u_{a0} dx \] (12)

In the above equation, the terms associated with \(\delta u_{a0}\) is expressed in a compact form \(C_{u_{a0}}\) as the coefficient of \(\delta u_{a0}\). Similarly, the coefficients of \(\delta \bar{u}_a\), \(\delta \theta_a\), \(\delta u_{b0}\), \(\delta \bar{u}_b\), \(\delta \theta_b\) and \(\delta w\) are derived which can be used to express the whole equation as

\[
\int_a^b \left( C_{u_{a0}} \delta u_{a0} + C_{\bar{u}_a} \delta \bar{u}_a + C_{\theta_a} \delta \theta_a + C_{u_{b0}} \delta u_{b0} + C_{\bar{u}_b} \delta \bar{u}_b + C_{\theta_b} \delta \theta_b + C_w \delta w \right) dx = 0 \] (13)
For any non-zero values of the displacement components, the above equation will be satisfied if the individual coefficients are zero \( C_{u_0} = 0, \ C_{\alpha} = 0, \ldots \) which will give the following equations.

\[
\delta u_{a_0} : -\frac{dN_a}{dx} + \frac{12}{5h_a^2} \frac{dP_a}{dx} - \frac{16}{5h_a^2} \frac{dS_a}{dx} - 2 \frac{12}{5h_a} R_a + 3 \frac{16}{5h_a^2} T_a = 0
\]

\[
\delta u_{a} : \frac{12}{5h_a^2} \frac{dP_a}{dx} + \frac{16}{5h_a^2} \frac{dS_a}{dx} + 2 \frac{12}{5h_a} R_a - \frac{3}{5h_a} T_a + k_a (\overline{u}_b - \overline{u}_a) = 0
\]

\[
\delta \theta_a : \frac{dM_a}{dx} + \frac{4}{5h_a^2} \frac{dP_a}{dx} - \frac{12}{5h_a^2} \frac{dS_a}{dx} - Q_a - 2 \frac{4}{5h_a} R_a + 3 \frac{12}{5h_a^2} T_a = 0
\]

\[
\delta u_{b_0} : -\frac{dN_b}{dx} + \frac{12}{5h_b^2} \frac{dP_b}{dx} + \frac{16}{5h_b^2} \frac{dS_b}{dx} - 2 \frac{12}{5h_b} R_b - 3 \frac{16}{5h_b^2} T_b = 0
\]

\[
\delta u_{b} : \frac{12}{5h_b^2} \frac{dP_b}{dx} - \frac{16}{5h_b^2} \frac{dS_b}{dx} + 2 \frac{12}{5h_b} R_b + \frac{3}{5h_b} T_b + k_b (\overline{u}_a - \overline{u}_b) = 0
\]

\[
\delta \theta_b : \frac{dM_b}{dx} - \frac{4}{5h_b^2} \frac{dP_b}{dx} - \frac{12}{5h_b^2} \frac{dS_b}{dx} - Q_b + 2 \frac{4}{5h_b} R_b + 3 \frac{12}{5h_b^2} T_b = 0
\]

\[
\delta w : \frac{2}{5h_a} \frac{d^2P_a}{dx^2} - \frac{4}{5h_a^2} \frac{d^2S_a}{dx^2} - \frac{dQ_a}{dx} - \frac{4}{5h_a} \frac{dR_a}{dx} + \frac{12}{5h_a^2} \frac{dT_a}{dx}
\]

\[
+ \frac{2}{5h_b} \frac{d^2P_b}{dx^2} - \frac{4}{5h_b^2} \frac{d^2S_b}{dx^2} - \frac{dQ_b}{dx} - \frac{4}{5h_b} \frac{dR_b}{dx} + \frac{12}{5h_b^2} \frac{dT_b}{dx} - q = 0
\]

Now the stresses at any point of the two material layers can be expressed in terms of their strains as follows:

\[
\sigma_a = E_a \varepsilon_a, \quad \tau_a = G_a \gamma_a, \quad \sigma_b = E_b \varepsilon_b, \quad \tau_b = G_b \gamma_b
\]

where \( E_a \) and \( E_b \) are elastic modulus while \( G_a \) and \( G_b \) are shear modulus of these layers. The above constitutive relationships are substituted in Eq. (10) which leads to express the stress
resultants in terms of the four strain components (15) and a further substitution of Eq. (8)
for these strain components will give the final expression of the stress resultants in terms of
the 1D displacement components \( \Delta = \{u_{a0}, \bar{u}_a, \theta_a, u_{b0}, \bar{u}_b, \theta_b, w\} \) as follows:

\[
N_a = A_{11} \frac{du_{a0}}{dx} + A_{13} \left( -12 \frac{du_{a0}}{5h_u^2} + 12 \frac{du_u}{5h_u^4} - 4 \frac{d\theta_a}{5h_u^5} - 2 \frac{d^2 w}{5h_u^6} \right)
\]

\[
M_a = A_{22} \frac{d\theta_a}{dx} + A_{24} \left( 16 \frac{du_{a0}}{5h_u^2} - 16 \frac{du_u}{5h_u^4} + 12 \frac{d\theta_a}{5h_u^5} - 4 \frac{d^2 w}{5h_u^6} \right)
\]

\[
P_a = A_{33} \frac{du_{a0}}{dx} + A_{33} \left( -12 \frac{du_{a0}}{5h_u^2} + 12 \frac{du_u}{5h_u^4} - 4 \frac{d\theta_a}{5h_u^5} - 2 \frac{d^2 w}{5h_u^6} \right)
\]

\[
S_a = A_{24} \frac{d\theta_u}{dx} + A_{44} \left( 16 \frac{du_{a0}}{5h_u^2} - 16 \frac{du_u}{5h_u^4} + 12 \frac{d\theta_u}{5h_u^5} - 4 \frac{d^2 w}{5h_u^6} \right)
\]

\[
Q_a = C_{11} \left( -\theta_a + \frac{dw}{dx} \right) + 3C_{13} \left( 16 \frac{du_{a0}}{5h_u^2} - 16 \frac{du_u}{5h_u^4} + 12 \frac{d\theta_u}{5h_u^5} - 4 \frac{d^2 w}{5h_u^6} \right)
\]

\[
R_a = 2C_{22} \left( -\frac{12}{5h_u} u_{a0} + \frac{12}{5h_u} \bar{u}_a - 4 \frac{d\theta_u}{5h_u} - \frac{2}{5h_u} \frac{dw}{dx} \right)
\]

\[
T_a = C_{13} \left( -\theta_a + \frac{dw}{dx} \right) + 3C_{33} \left( 16 \frac{du_{a0}}{5h_u^2} - 16 \frac{du_u}{5h_u^4} + 12 \frac{d\theta_u}{5h_u^5} - 4 \frac{d^2 w}{5h_u^6} \right)
\]

\[
N_b = B_{11} \frac{du_{b0}}{dx} + B_{13} \left( -12 \frac{du_{b0}}{5h_b^2} + 12 \frac{du_u}{5h_b^4} + 4 \frac{d\theta_b}{5h_b^5} + 2 \frac{d^2 w}{5h_b^6} \right)
\]

\[
M_b = B_{22} \frac{d\theta_b}{dx} + B_{24} \left( -16 \frac{du_{b0}}{5h_b^2} + 16 \frac{du_u}{5h_b^4} + 12 \frac{d\theta_b}{5h_b^5} - 4 \frac{d^2 w}{5h_b^6} \right)
\]

\[
P_b = B_{33} \frac{du_{b0}}{dx} + B_{33} \left( -12 \frac{du_{b0}}{5h_b^2} + 12 \frac{du_u}{5h_b^4} + 4 \frac{d\theta_b}{5h_b^5} + 2 \frac{d^2 w}{5h_b^6} \right)
\]
\[ S_b = B_{24} \left( -\frac{d\theta_b}{dx} \right) + B_{44} \left( -\frac{16}{5h_b^3} \frac{du_{b0}}{dx} + \frac{16}{5h_b^3} \frac{d\theta_b}{dx} \right) + \frac{16}{5h_b^3} \frac{d\alpha_{b0}}{dx} + \frac{12}{5h_b^3} \frac{d\theta_b}{dx} - \frac{4}{5h_b^3} \frac{d^2 w}{dx^2} \]  

\[ Q_b = D_{14} \left( -\frac{d\theta_b}{dx} \right) + D_{13} \left( -\frac{16}{5h_b^3} \frac{u_{b0}}{dx} + \frac{16}{5h_b^3} \frac{d\alpha_{b0}}{dx} + \frac{12}{5h_b^3} \frac{d\theta_b}{dx} - \frac{4}{5h_b^3} \frac{d^2 w}{dx^2} \right) \]  

\[ R_b = 2D_{22} \left( -\frac{12}{5h_b^3} \frac{u_{b0}}{dx} + \frac{12}{5h_b^3} \frac{d\alpha_{b0}}{dx} + \frac{4}{5h_b^3} \frac{d\theta_b}{dx} + \frac{2}{5h_b^3} \frac{d^2 w}{dx^2} \right) \]  

\[ T_b = D_{13} \left( -\frac{d\theta_b}{dx} \right) + 3D_{13} \left( -\frac{16}{5h_b^3} \frac{u_{b0}}{dx} + \frac{16}{5h_b^3} \frac{d\alpha_{b0}}{dx} + \frac{12}{5h_b^3} \frac{d\theta_b}{dx} - \frac{4}{5h_b^3} \frac{d^2 w}{dx^2} \right) \]  

(16)

where  \((A_{11}, A_{13}, A_{22}, A_{24}, A_{33}, A_{44}) = \int_{A_b} \frac{E_a (1, y^2, y^4, y^6)}{dA_a}, \]

\((B_{11}, B_{13}, B_{22}, B_{24}, B_{33}, B_{44}) = \int_{B_b} \frac{E_b (1, y^2, y^4, y^6)}{dA_a}, \]

\((C_{11}, C_{13}, C_{22}, C_{33}) = \int_{C_b} \frac{G_a (1, y^2, y^4, y^6)}{dA_a} \text{ and} \]

\((D_{11}, D_{13}, D_{22}, D_{33}) = \int_{D_b} \frac{G_b (1, y^2, y^4, y^6)}{dA_a}. \]

For a Navier type solution of the present problem, the 1D displacement components are expressed in the form of Fourier series as follows:

\[ u_{a0} = \sum_{m=1}^{\infty} u_{a0m} \cos \frac{m\pi}{L} x, \]

\[ u_a = \sum_{m=1}^{\infty} u_{am} \cos \frac{m\pi}{L} x, \]

\[ \theta_a = \sum_{m=1}^{\infty} \theta_{am} \cos \frac{m\pi}{L} x, \]

\[ u_{b0} = \sum_{m=1}^{\infty} u_{b0m} \cos \frac{m\pi}{L} x, \]

\[ \overline{u}_b = \sum_{m=1}^{\infty} \overline{u}_{bm} \cos \frac{m\pi}{L} x, \]

\[ \theta_b = \sum_{m=1}^{\infty} \theta_{bm} \cos \frac{m\pi}{L} x, \]

\[ w = \sum_{m=1}^{\infty} w_m \sin \frac{m\pi}{L} x \]  

(17)

Similarly, the transverse load acting on the beam can be expressed as
\[ q = \sum_{m=1}^{\infty} q_m \sin \frac{m\pi}{L} x \]  

(18)

where the coefficients \( q_m \) can be obtained for a given load \( q \) following the usual steps [18].

Eq. (17) and (18) are substituted in Eq. (16) which is subsequently substituted in Eq. (13), the seven governing equations are finally be derived. As an representative example, the last (seventh) governing equations corresponding to \( \delta \nu \) can be expressed as

\[
\sum_{m=1}^{\infty} H_m \sin \frac{m\pi}{L} x = 0
\]  

(19)

where

\[
H_m = -\frac{2}{5h_a} \left[ A_{ij} \mu_{a0m} + A_{ik} \left( -12 \frac{L}{h_a^3} \theta_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} - 4 \frac{L}{h_a} \phi_{am} - 2 \frac{L}{h_a} w_m \right) \right] \left( \frac{m\pi}{L} \right)^3 \\
- \frac{4}{5h_a} \left[ A_{ij}(-\theta_{am}) + A_{ij} \left( -16 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \right] \left( \frac{m\pi}{L} \right)^3 \\
- 3C_6 \left( - \theta_{am} + w_m \right) + 3C_7 \left( - \theta_{am} + w_m \right) \left( \frac{m\pi}{L} \right)^3 \\
- \frac{12}{5h_a} \left[ C_{ij}(-\theta_{am}) + A_{ij} \left( -16 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \right] \left( \frac{m\pi}{L} \right)^3 \\
+ \frac{2}{5h_a} \left[ B_{1j} \theta_{a0m} + B_{3j} \left( -12 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \right] \left( \frac{m\pi}{L} \right)^3 \\
- \frac{4}{5h_a} \left[ B_{2j}(-\theta_{am}) + A_{ij} \left( -16 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \right] \left( \frac{m\pi}{L} \right)^3 \\
+ \left[ D_{1j}(-\theta_{am}) + w_m \right] + 3D_{3j} \left( -16 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \left( \frac{m\pi}{L} \right)^3 \\
+ \frac{2}{5h_a} \left[ D_{2j} \theta_{a0m} + A_{ij} \left( -12 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \right] \left( \frac{m\pi}{L} \right)^3 \\
- \frac{12}{5h_a} \left[ D_{3j}(-\theta_{am}) + w_m \right] + 3D_{3j} \left( -16 \frac{L}{h_a^3} u_{a0m} + 12 \frac{L}{h_a^2} \pi_{am} + 4 \frac{L}{h_a} \phi_{am} + 2 \frac{L}{h_a} w_m \right) \left( \frac{m\pi}{L} \right)^3 \\
- q_m
\]
In order to satisfy Eq. (19), the individual terms should be zero for any value of \( m \) i.e.,

\[ H_m = 0 \]

which can be expressed in terms of Fourier components of the displacements

\[ \{ \Delta_m \} = \{ u_{a0m}, \bar{u}_{am}, \theta_{am}, u_{b0m}, \bar{u}_{bm}, \theta_{bm}, w_m \} \]

corresponding to \( m \) as

\[ K_{17}u_{a0m} + K_{27}\bar{u}_{am} + K_{37}\theta_{am} + K_{47}u_{b0m} + K_{57}\bar{u}_{bm} + K_{67}\theta_{bm} + K_{77}w_m - q_m = 0 \]  

(20)

Similarly, the other six governing equations are derived and the entire system of equations for any value of \( m \) is expressed in a matrix form as

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & 0 & 0 & 0 & K_{17} \\
K_{12} & K_{22} & K_{23} & 0 & K_{25} & 0 & K_{27} \\
K_{13} & K_{23} & K_{33} & 0 & 0 & 0 & K_{37} \\
0 & 0 & 0 & K_{44} & K_{45} & K_{46} & K_{47} \\
0 & K_{25} & 0 & K_{45} & K_{55} & K_{56} & K_{57} \\
0 & 0 & 0 & K_{46} & K_{56} & K_{66} & K_{67} \\
K_{17} & K_{27} & K_{37} & K_{47} & K_{57} & K_{67} & K_{78}
\end{bmatrix}
\begin{bmatrix}
u_{a0m} \\
p_{a} \\
\theta_{a} \\
u_{b0m} \\
p_{b} \\
\theta_{b} \\
w_{m}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]  

(21)

where the explicit expressions for the stiffness parameters \( K_{ij} \) are given in Appendix 2.

The above equation can easily be solved to get \( \Delta_m \) and it can be substituted in Eq. (17) to get the 1D displacement components which can be utilised to get stresses using Eqs. (5), (6), (8) and (15).

3. Numerical results

The proposed analytical model based on HBT is used to solve numerical examples of composite beams to show the performance and range of applicability of the model. For the validation of the proposed analytical model, the 1D finite element model based on this beam theory developed by Sheikh and his co-workers in an earlier study [16] is also used as the number of suitable results available in literature is very limited. As no one reported results
for the stresses induced within these beams, an attempt is made to pay a proper attention on this aspect. Specifically, for the validation of the stress results, a well-regarded finite element software is also used to analyse a composite beam based on a detailed 2D finite element model of the beam. Though the proposed model is derived on the basis of HBT, the present formulation can easily be modified to accommodate TBT and EBT by eliminating some terms from the expression given in Eqs. (1) and (2). In many cases, the results based on these three beam theories are presented together in order to assess their relative merits. As TBT needs a shear correction factor for getting an acceptable deflection result, an approximate value of 5/6 is taken for the shear correction factor in the present study.

3.1 A composite beam having a rectangular section under a point load

In this section, a simply supported two-layered composite beam subjected to a point load \( P = 100\text{kN} \) at the mid-span (see Fig. 2) is considered to investigate the performance of the proposed model based on HBT in comparison with that on the basis of TBT and EBT. For this purpose, the values of some parameters such as interfacial shear stiffness \( (k_s) \), span-to-depth ratio \( (L/H) \) and elastic-to-shear modulus ratio \( (E/G) \) are varied and the deflections predicted by HBT, TBT and EBT at the mid-span are presented in Table 1. The analysis is carried out taking \( h_a = 0.2 \text{ m}, h_b = 0.3 \text{ m}, E_a = 12 \text{ GPa} \) and \( E_b = 8 \text{ GPa} \). The table shows that the deviation between the results predicted by HBT and EBT (H-E) is significant (ranging from 82.8\% to 3.8\%) and it increases with the increasing values of \( k_s \) and \( E/G \), and decreasing values of \( L/H \). A similar trend is observed between the results obtained by HBT and TBT (H-T) but their deviations are not that high (ranging from 14.9\% to 0.15\%). Now the effect of these parameters \( (L/H, k_s \text{ and } E/G) \) on the interfacial shear slip at one of the
beam ends is studied. First, the variation of the slip, in terms of deviations between its values predicted by HBT and EBT (H-E) as well as HBT and TBT (H-T), with respect to $L/H$ ratio is plotted in Fig. 3a for a specific value of the other parameters ($k_s$ and $E/G$). It also shows that the deviation between the results obtained by HBT and EBT i.e. H-E (ranging from 7.1% to 0.29%) is more than that of H-T (ranging from 5.2% to 0.16%) but the difference between them (H-T and H-E) is not that wide as observed in the case of deflection. Similarly, the variations of slip with respect to $k_s$ and $E/G$ are plotted in Fig. 3b (H-E: ranging from 7.1% to 5.2%, H-T: ranging from 5.2% to 3.4%) and Fig. 3c (H-E: ranging from 13.3% to 1.8%, H-T: ranging from 10.1% to 1.2%), respectively. It should be noted that the shear correction factor used in TBT helps to predict the deflection well but the performance of TBT is deteriorated in the case of interfacial slip and it is affected more for predicting the local response (i.e. stresses) which will now be shown. For this purpose, the bending and shear stresses are calculated for a specific case ($k_s = 50$ MPa, $E/G = 50$ and $L/H = 5$) of the composite beam. The variations of bending stress over the entire beam depth predicted by HBT and TBT at the mid-span section of the beam are plotted in Fig. 4a. Similarly the shear stress variations at the quarter span of the beam are plotted in Fig. 4b. These figures show that the performance of TBT is not encouraging particularly for the shear stress as shown in Fig. 4b where TBT could not even predict the trend of the stress variation apart from big differences of its values.
Fig. 2. A simply supported two-layer composite beam under a point load

Table 1. Mid-span deflections of the composite beam (Fig. 2) for different values of $k_s$, $L/H$ and $E/G$

<table>
<thead>
<tr>
<th>$k_s$</th>
<th>$E/G$</th>
<th>Ref.</th>
<th>$L/H=2.5$</th>
<th>$L/H=5$</th>
<th>$L/H=10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>100 50 10</td>
<td>100 50 10</td>
<td>100 50 10</td>
</tr>
<tr>
<td>100</td>
<td>HBT</td>
<td></td>
<td>27.112 16.591 7.2951</td>
<td>77.632 54.367 34.585</td>
<td>255.75 205.91 164.82</td>
</tr>
<tr>
<td></td>
<td>TBT</td>
<td></td>
<td>31.159 18.113 7.4631</td>
<td>83.552 56.838 34.949</td>
<td>263.77 209.42 165.40</td>
</tr>
<tr>
<td></td>
<td>% Deviation (H-T)</td>
<td></td>
<td>14.926 9.1696 2.3034</td>
<td>7.6257 4.5452 1.0519</td>
<td>3.1375 1.7066 0.3510</td>
</tr>
<tr>
<td></td>
<td>% Deviation (H-E)</td>
<td></td>
<td>82.817 71.921 36.139</td>
<td>62.249 46.094 15.261</td>
<td>39.694 25.097 6.4234</td>
</tr>
<tr>
<td>50</td>
<td>HBT</td>
<td></td>
<td>27.671 17.014 7.5923</td>
<td>83.155 59.507 39.338</td>
<td>294.04 244.15 202.97</td>
</tr>
<tr>
<td></td>
<td>TBT</td>
<td></td>
<td>31.449 18.397 7.7310</td>
<td>88.509 61.683 39.639</td>
<td>302.80 248.05 203.63</td>
</tr>
<tr>
<td></td>
<td>EBT</td>
<td></td>
<td>4.9158 4.9158 4.9158</td>
<td>33.941 33.941 33.941</td>
<td>192.35 192.35 192.35</td>
</tr>
<tr>
<td></td>
<td>% Deviation (H-T)</td>
<td></td>
<td>13.654 8.1239 1.8259</td>
<td>6.4383 3.6565 0.7648</td>
<td>2.9773 1.5950 0.3232</td>
</tr>
<tr>
<td></td>
<td>% Deviation (H-E)</td>
<td></td>
<td>82.235 71.108 35.253</td>
<td>59.183 42.962 13.719</td>
<td>34.583 21.216 5.2332</td>
</tr>
<tr>
<td>10</td>
<td>HBT</td>
<td></td>
<td>28.244 17.432 7.871</td>
<td>90.844 66.353 45.399</td>
<td>387.30 335.41 292.44</td>
</tr>
<tr>
<td></td>
<td>TBT</td>
<td></td>
<td>31.717 18.659 7.9778</td>
<td>94.781 67.801 45.549</td>
<td>394.09 338.30 292.88</td>
</tr>
<tr>
<td></td>
<td>EBT</td>
<td></td>
<td>5.1525 5.1525 5.1525</td>
<td>39.777 39.777 39.777</td>
<td>281.33 281.33 281.33</td>
</tr>
<tr>
<td></td>
<td>% Deviation (H-T)</td>
<td></td>
<td>12.296 7.040 1.3573</td>
<td>4.336 2.1824 0.3320</td>
<td>1.7530 0.8609 0.1537</td>
</tr>
<tr>
<td></td>
<td>% Deviation (H-E)</td>
<td></td>
<td>81.757 70.442 34.539</td>
<td>56.214 40.053 12.383</td>
<td>27.363 16.126 3.7991</td>
</tr>
</tbody>
</table>

Note: ¹(TBT-HBT)/HBT*100%
²(HBT-EBT)/HBT*100%
a. Effect of \( L/H \) ratio \((k_s = 10 \text{ MPa}, E/G = 10)\)

b. Effect of \( k_s \) \((L/H = 2.5, E/G = 10)\)

c. Effect of \( E/G \) ratio \((k_s = 10 \text{ MPa}, L/H = 5)\)

Fig. 3. Effect of different parameters on the interfacial slip of a composite beam
3.2 A timber-timber composite beam under uniformly distributed load

An example of a 5 m long simply supported timber-timber composite beam under a uniformly distributed load of 50 kN/m studied by Schnabl et al. [13] is considered in this section for the validation of the proposed model. The beam has a rectangular cross-section (300 mm wide and 500 mm deep) which is consisting of two layers (200 mm deep upper layer and 300 mm deep lower layer) having same material properties ($E_a = E_b = 12000$ MPa, $G_a = G_b = 750$MPa). Schnabl et al. [13] developed an analytical model based on TBT which they used to solve this problem and presented mid-span deflection results for different values of stiffness of the shear connectors ($k_s$). They have also presented results based on EBT. The problem is solved by the proposed analytical model (HBT, TBT and EBT) as well as the 1D finite element model (HBT and TBT) developed by us, and the results obtained by all these models are presented in Table 2 along with those reported by Schnabl et al. [13]. Table 2 conformed the accuracy of the proposed model in its different variants.

This example is also used for the convergence study of the proposed analytical model.
with respect to the number of terms \( m \) of the Fourier series (Eq. 17) taking a specific value of \( k_s \) (10 MPa). The convergence study is carried for mid-span deflection, interfacial slip at the left end, bending stress at the bottom fibre of mid-span section, and shear stress at mid-depth of the lower layer at the left end using the three beam theories. However, results obtained by HBT (representative case) with increasing values of \( m \) are presented in Table 3 which shows the model needs only 15 terms for the displacement to converge whereas a minimum of 30 terms are required for the shear stress. Based on the observation, \( m \) is taken as 30 for solving all examples in this paper in order to avoid any convergence issue.

Table 2. Mid-span deflections from different beam models for different \( k_s \)

<table>
<thead>
<tr>
<th>( k_s ) (MPa)</th>
<th>Schnabl et al. [13]</th>
<th>Proposed Analytical Model</th>
<th>Proposed FE Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EBT</td>
<td>TBT</td>
<td>EBT</td>
</tr>
<tr>
<td>0.01</td>
<td>38.75</td>
<td>40.62</td>
<td>38.674</td>
</tr>
<tr>
<td>0.1</td>
<td>38.69</td>
<td>40.57</td>
<td>38.621</td>
</tr>
<tr>
<td>1</td>
<td>38.18</td>
<td>40.05</td>
<td>38.105</td>
</tr>
<tr>
<td>10</td>
<td>33.91</td>
<td>35.73</td>
<td>33.833</td>
</tr>
</tbody>
</table>

Table 3. Convergence study with respect to the number of terms \( m \) used in the series solution of the proposed analytical model

<table>
<thead>
<tr>
<th>( m )</th>
<th>Deflection (mm)</th>
<th>Slip (mm)</th>
<th>Bending stress (MPa)</th>
<th>Shear stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35.867</td>
<td>5.1004</td>
<td>25.168</td>
<td>1.2149</td>
</tr>
<tr>
<td>2</td>
<td>35.641</td>
<td>5.1814</td>
<td>24.125</td>
<td>1.3503</td>
</tr>
<tr>
<td>5</td>
<td>35.664</td>
<td>5.1985</td>
<td>24.325</td>
<td>1.4311</td>
</tr>
<tr>
<td>10</td>
<td>35.663</td>
<td>5.2003</td>
<td>24.297</td>
<td>1.4546</td>
</tr>
<tr>
<td>15</td>
<td>35.663</td>
<td>5.2006</td>
<td>24.306</td>
<td>1.4611</td>
</tr>
<tr>
<td>20</td>
<td>35.663</td>
<td>5.2007</td>
<td>24.301</td>
<td>1.4638</td>
</tr>
<tr>
<td>30</td>
<td>35.663</td>
<td>5.2007</td>
<td>24.302</td>
<td>1.4661</td>
</tr>
<tr>
<td>50</td>
<td>35.663</td>
<td>5.2007</td>
<td>24.302</td>
<td>1.4661</td>
</tr>
</tbody>
</table>
3.3 A timber-concrete composite beam having a T-section

A 4 m long simply supported timber-concrete composite beam subjected to a point load of 10 kN at its mid-span is considered in this section to assess the prediction capability of the proposed analytical model specifically for the evaluation of stresses. As there is no results available in literature for the validation of stresses, numerical results are generated by analysing the beam with a well-regarded finite element software ABAQUS where a detailed 2D model of the beam is used to produce realistic stress results. Fig. 5 shows the cross-section of the beam which consists of a concrete slab and a timber girder, and the material properties used for these two components are: $E_{\text{concrete}} = 24 \text{ GPa}$, $G_{\text{concrete}} = 10 \text{ GPa}$, $E_{\text{wood}} = 15 \text{ GPa}$ and $G_{\text{wood}} = 5 \text{ GPa}$. The stiffness of the shear connectors, idealised as interfacial distributed springs, is taken as 1 MPa, 10 MPa and 100 MPa.

![Fig. 5. Cross-section of a timber-concrete composite beam](image)

From the analysis of the beam using ABAQUS, the individual material layers are modelled with four node plane stress rectangular elements (CPS4R) placed in a vertical plane, and the cohesive contact model is used between these two sets of elements used for these two material layers to simulate the partial shear interaction at their interface. The
variation of bending stress over the beam depth obtained by the 1D analytical model based on HBT and the 2D detailed FE model at the mid-span of the beam using $k_s = 10$ MPa is plotted in Fig. 6a. Similarly, the variation of shear stress obtained at the quarter span is presented in Fig. 6b. These figures show a very good performance of the proposed analytical model in predicting the local response of the beam. In addition to the stresses, the mid-span deflection and the interfacial shear slip at the left end of the beam predicted by both these models are also presented in Table 4 which shows a very good agreement between the results produced by these models.

Table 4. Deflection and interfacial slip of a simply-supported timber-concrete composite beam

<table>
<thead>
<tr>
<th>$k_s$ (MPa)</th>
<th>Deflection (mm)</th>
<th>Slip *10 (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical HBT</td>
<td>ABAQUS</td>
</tr>
<tr>
<td>1</td>
<td>22.189</td>
<td>22.194</td>
</tr>
<tr>
<td>10</td>
<td>15.963</td>
<td>15.956</td>
</tr>
<tr>
<td>100</td>
<td>8.0801</td>
<td>8.0693</td>
</tr>
</tbody>
</table>

Fig. 6. Variation of stresses over the beam depth ($k_s = 5$ MPa)
3.4 A steel-concrete composite beam having a flanged section

Fig. 7 shows the cross-section of a 20 m long simply supported composite beam, consists of a concrete slab and a steel I girder connected by steel shear studs, which has been investigated by Ranzi et al. [9] using different techniques based on EBT where the beam is subjected to a uniformly distributed load of 35 kN/m. This beam problem is used in this example and the response of the beam under a point load of 100kN at the mid-span, a distributed load having a sinusoidal variation (\(10\sin \frac{\pi x}{L}\)) with a maximum value of 10kN/m at the mid-span \((x = L/2)\) and zero at the two ends \((x = 0, x =L)\), as well as a uniformly distributed load of 35kN/m is studied by the proposed analytical model based on HBT as well as our 1D finite element model (HBT). The material properties of the concrete slab and the steel girder are: \(E_c = 34,200 \text{ MPa}, G_c = 14,250 \text{ MPa}, E_s = 210,000 \text{ MPa}\) and \(G_s = 84,000 \text{ MPa}\). The stiffness of shear connectors as expressed by Ranzi et al. [9] in terms of non-dimensional parameter \((\alpha L)\) is followed in this example and it is varied from 0.1 to 50.

![Cross-section of a simply supported steel-concrete composite beam](image)

The mid-span deflection, interfacial shear slip at the left end and bending stress at the bottom fibre of the mid-span section predicted by the proposed analytical and FE models
are presented in Table 5. It also includes the closed form solutions reported by Ranzi et al. [9] for the mid-span deflection of the beam under the uniformly distributed load which served up to a certain range of validation. Also, the variations of deflection and interfacial shear slip along the beam length obtained by the proposed analytical model (HBT) for the load case having a sinusoidal variation are presented in Fig. 8, for three different values of the shear connectors \(aL\) as 1, 5 and 10. The results show a consistent trend of variation as expected.

![Deflection Graph](image)

**a. Deflection**

![Interfacial Shear Slip Graph](image)

**b. Interfacial shear slip**

*Fig. 8. Variation of deflection and interfacial shear slip along the beam length*
Table 5. Deflection, slip and bending stress of a simply-supported flanged composite beam

<table>
<thead>
<tr>
<th></th>
<th>Uniformly distributed load</th>
<th>Point Load</th>
<th>Sinusoidal load</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical - HBT</td>
<td>Analytical - HBT</td>
<td>FEM - HBT</td>
</tr>
<tr>
<td><strong>αL</strong>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>23.025</td>
<td>5.3110</td>
<td>5.1927</td>
</tr>
<tr>
<td>1</td>
<td>21.939</td>
<td>5.0667</td>
<td>4.9489</td>
</tr>
<tr>
<td>5</td>
<td>14.493</td>
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<td>3.2752</td>
</tr>
<tr>
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<td>12.205</td>
<td>2.8545</td>
<td>2.7583</td>
</tr>
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<td>11.435</td>
<td>2.6650</td>
<td>2.5824</td>
</tr>
<tr>
<td>50</td>
<td>11.202</td>
<td>2.6022</td>
<td>2.5286</td>
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<table>
<thead>
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<th>Point Load</th>
<th>Sinusoidal load</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Analytical - HBT</td>
<td>Analytical - HBT</td>
<td>FEM - HBT</td>
</tr>
<tr>
<td><strong>αL</strong>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>2549.8</td>
<td>544.69</td>
<td>563.03</td>
</tr>
<tr>
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<td>2320.6</td>
<td>493.09</td>
<td>511.70</td>
</tr>
<tr>
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<td>743.18</td>
<td>145.41</td>
<td>159.39</td>
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<tr>
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<td>246.77</td>
<td>42.739</td>
<td>50.576</td>
</tr>
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<td>20</td>
<td>70.054</td>
<td>10.821</td>
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<tr>
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<td>1.7311</td>
<td>2.2138</td>
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<table>
<thead>
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<th>Point Load</th>
<th>Sinusoidal load</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical - HBT</td>
<td>Analytical - HBT</td>
<td>FEM - HBT</td>
</tr>
<tr>
<td><strong>αL</strong>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>66.875</td>
<td>19.384</td>
<td>15.492</td>
</tr>
<tr>
<td>1</td>
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<td>19.195</td>
<td>15.215</td>
</tr>
<tr>
<td>5</td>
<td>57.395</td>
<td>17.201</td>
<td>13.318</td>
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<tr>
<td>10</td>
<td>54.828</td>
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<td>12.732</td>
</tr>
<tr>
<td>20</td>
<td>54.061</td>
<td>16.034</td>
<td>12.533</td>
</tr>
<tr>
<td>50</td>
<td>53.842</td>
<td>15.729</td>
<td>12.472</td>
</tr>
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</table>
considered and modelled as distributed shear springs along the entire length of the beam.

The HBT provides a true parabolic variation of shear stress over the beam depth, and therefore, the use of a potentially arbitrary shear correction factor is not required for the correct prediction of the global response such as deflection. Moreover, HBT has helped to predict the local responses such as distribution of stresses satisfactorily in addition to the global responses such as deflection.

The principle of virtual work is applied to derive the governing equations which are solved analytically by using a Navier type solution technique. The expressions of individual terms/parameters necessary for solving a problem are provided explicitly which will help to use this model for solving a new problem easily.

The relative performances of three beam theories (HBT, TBT and EBT) implemented within the proposed analytical model are investigated. It is observed that variations between the results predicted by HBT and EBT are very high as expected. On the other hand, TBT is found to predict the deflection well with the help of the shear correction factor but its performance is deteriorated in the case of interfacial slip and it is significantly affected for predicting the local responses (i.e. stresses).

The proposed analytical model is used to solve numerical examples of composite beams having different dimensions, cross sectional configurations, loading types and degrees of shear interaction. The ability of the proposed model in accurately predicting the stress distribution, deflection and interfacial shear slip of composite beams is tested by comparing the results produced by the model with published results and numerical results
produced by a 1D finite element model based on HBT as well as a detailed two-dimensional finite element modelling of composite beams. In this process, a large number new results are produced analytically which can be used as benchmarking solution for future research studies in the area of composite structures.

Acknowledgement

The financial support provide by the University of Adelaide in the form of an Adelaide Graduate Research Scholarship (AGRS) award to the first author for her doctoral studies is gratefully acknowledged.

References


Appendix 1

The derivation process of $A$, $B$, $C$ and $D$ in Eqs. (5) and (6):

-Using Eqs. (1) - (3), the shear stress at any point of the upper and lower material layers can be expressed as:

\[
\tau_a = G_a y_a = G_a \left( \frac{\partial u_a}{\partial y_a} + \frac{\partial w}{\partial x} \right) = G_a \left( -\theta_a + 2y_a \alpha_a + 3y_a^2 \beta_a + \frac{dw}{dx} \right)
\] (21)

\[
\tau_b = G_b y_b = G_b \left( \frac{\partial u_b}{\partial y_b} + \frac{\partial w}{\partial x} \right) = G_b \left( -\theta_b + 2y_b \alpha_b + 3y_b^2 \beta_b + \frac{dw}{dx} \right)
\] (22)

The shear stress is equal to zero at the beam top and bottom surface (i.e. $y_a = \frac{h_a}{2}$, $\tau_a = 0$; $y_b = -\frac{h_b}{2}$, $\tau_b = 0$):

\[
\tau_a \left( y_a = \frac{h_a}{2} \right) = G_a \left( -\theta_a + h_a \alpha_a + \frac{3}{4} h_a^2 \beta_a + \frac{dw}{dx} \right) = 0
\] (23)

\[
\tau_b \left( y_b = -\frac{h_b}{2} \right) = G_b \left( -\theta_b - h_b \alpha_b + \frac{3}{4} h_b^2 \beta_b + \frac{dw}{dx} \right) = 0
\] (24)

-Substituting $y_a = \frac{-h_a}{2}$ and $y_b = \frac{h_b}{2}$ into Eqs. (1) and (2) respectively, $\bar{u}_a$ and $\bar{u}_b$ can be expressed as:

\[
\bar{u}_a = u_{a0} + \frac{h_a}{2} \theta_a + \frac{h_a^2}{4} \alpha_a - \frac{h_a^3}{8} \beta_a
\] (25)

\[
\bar{u}_b = u_{b0} - \frac{h_b}{2} \theta_b + \frac{h_b^2}{4} \alpha_b + \frac{h_b^3}{8} \beta_b
\] (26)
From Eq. (25), \( \alpha_a \) can be expressed in terms of \( \beta_a \) and the displacement components \( u_{a_0}, \bar{u}_a \) and \( \theta_a \) as follows:

\[
\alpha_a = \frac{4}{h_a} \left( u_{a_0} - \bar{u}_a + \frac{h_a}{2} \theta_a - \frac{h_a^3}{8} \beta_a \right) \quad (27)
\]

Substituting the above equation into Eq. (23), \( \alpha_a \) and \( \beta_a \) can be expressed in terms of \( u_{a_0}, \bar{u}_a \), \( \theta_a \) and \( w \) as follows:

\[
\alpha_a = \frac{12}{5h_a^2} (\bar{u}_a - u_{a_0}) - \frac{6}{5h_a} \theta_a - \frac{2}{5h_a} \left( \frac{dw}{dx} - \theta_a \right) \quad (28)
\]

\[
\beta_a = \frac{16}{5h_a^3} (\bar{u}_a - u_{a_0}) - \frac{8}{5h_a^2} \theta_a - \frac{4}{5h_a} \left( \frac{dw}{dx} - \theta_a \right) \quad (29)
\]

Similarly, \( \alpha_b \) and \( \beta_b \) also can be obtained:

\[
\alpha_b = \frac{12}{5h_b^2} (\bar{u}_b - u_{b_0}) + \frac{6}{5h_b} \theta_b + \frac{2}{5h_b} \left( \frac{dw}{dx} - \theta_b \right) \quad (30)
\]

\[
\beta_b = \frac{16}{5h_b^3} (\bar{u}_b - u_{b_0}) - \frac{8}{5h_b^2} \theta_b - \frac{4}{5h_b} \left( \frac{dw}{dx} - \theta_b \right) \quad (31)
\]

Substituting Eqs. (28) – (31) into Eqs. (1) and (2), the expressions of \( A, B, C \) and \( D \) can be obtained.
Appendix 2

The expressions of individual terms for the stiffness matrix in Eq. (21) are:

\[
K_{11} = \left[ A_{13} - 2A_{13} \frac{12}{5h_y^2} + A_{33} \left( \frac{12}{5h_y^2} \right)^2 + A_{44} \left( \frac{16}{5h_y^2} \right)^2 \left( \frac{m\pi}{L} \right)^2 + \left[ C_{22} \left( \frac{24}{5h_y^2} \right)^2 + C_{33} \left( \frac{48}{5h_y^2} \right)^2 \right] \right]
\]

\[
K_{12} = \left[ A_{33} \frac{12}{5h_y^2} - A_{33} \left( \frac{12}{5h_y^2} \right)^2 - A_{44} \left( \frac{16}{5h_y^2} \right)^2 \left( \frac{m\pi}{L} \right)^2 - \left[ C_{22} \left( \frac{24}{5h_y^2} \right)^2 + C_{33} \left( \frac{48}{5h_y^2} \right)^2 \right] \right]
\]

\[
K_{13} = -\left( A_{13} \frac{4}{5h_y^2} - A_{13} \frac{12}{5h_y^2} \frac{12}{5h_y^2} + A_{33} \frac{16}{5h_y^2} \frac{12}{5h_y^2} - A_{44} \frac{16}{5h_y^2} \frac{12}{5h_y^2} \left( \frac{m\pi}{L} \right) \right)^2 + \left( C_{22} - 4 \frac{24}{5h_y^2} \frac{8}{5h_y^2} - C_{33} \frac{48}{5h_y^2} + C_{13} \frac{48}{5h_y^2} \frac{36}{5h_y^2} \right)
\]

\[
K_{17} = -\left( A_{13} \frac{2}{5h_y^2} - A_{13} \frac{12}{5h_y^2} \frac{2}{5h_y^2} + A_{44} \frac{16}{5h_y^2} \frac{4}{5h_y^2} \left( \frac{m\pi}{L} \right) \right)^2 + \left( C_{22} - 4 \frac{4}{5h_y^2} \frac{24}{5h_y^2} \frac{8}{5h_y^2} \frac{48}{5h_y^2} \frac{48}{5h_y^2} \frac{36}{5h_y^2} \right)
\]

\[
K_{22} = -\left[ -A_{13} \left( \frac{12}{5h_y^2} \right)^2 - A_{44} \left( \frac{16}{5h_y^2} \right)^2 \left( \frac{m\pi}{L} \right)^2 + \left[ C_{22} \left( \frac{24}{5h_y^2} \right)^2 + C_{33} \left( \frac{48}{5h_y^2} \right)^2 \right] \right]
\]

\[
K_{23} = -\left( A_{33} \frac{12}{5h_y^2} \frac{4}{5h_y^2} - A_{33} \frac{16}{5h_y^2} \frac{12}{5h_y^2} \left( \frac{m\pi}{L} \right)^2 + \left( -C_{22} \frac{24}{5h_y^2} \frac{4}{5h_y^2} \frac{8}{5h_y^2} \frac{4}{5h_y^2} \frac{24}{5h_y^2} \frac{48}{5h_y^2} \right) \right)
\]

\[
K_{25} = -k_s
\]

\[
K_{27} = -\left( A_{33} \frac{12}{5h_y^2} \frac{2}{5h_y^2} - A_{44} \frac{16}{5h_y^2} \frac{4}{5h_y^2} \left( \frac{m\pi}{L} \right)^2 + \left( -C_{22} \frac{24}{5h_y^2} \frac{4}{5h_y^2} \frac{8}{5h_y^2} \frac{4}{5h_y^2} \frac{48}{5h_y^2} \frac{36}{5h_y^2} \right) \right)
\]

\[
K_{33} = -\left( -A_{22} + 2A_{24} \frac{12}{5h_y^2} - A_{44} \frac{4}{5h_y^2} - A_{44} \left( \frac{12}{5h_y^2} \right)^2 \left( \frac{m\pi}{L} \right)^2 + \left[ C_{13} - 2C_{13} \frac{36}{5h_y^2} + C_{22} \frac{8}{5h_y^2} \frac{36}{5h_y^2} \right] \right)
\]

\[
K_{35} = -\left( -A_{34} \frac{4}{5h_y^2} - A_{13} \frac{2}{5h_y^2} + A_{44} \frac{12}{5h_y^2} \frac{4}{5h_y^2} \left( \frac{m\pi}{L} \right)^2 + \left( -C_{14} + C_{13} \frac{12}{5h_y^2} + C_{24} \frac{8}{5h_y^2} \frac{4}{5h_y^2} + C_{13} \frac{36}{5h_y^2} \frac{12}{5h_y^2} \right) \right)
\]

\[
K_{45} = \left[ B_{13} \frac{12}{5h_y^2} - B_{33} \left( \frac{12}{5h_y^2} \right)^2 - B_{44} \left( \frac{16}{5h_y^2} \right)^2 \left( \frac{m\pi}{L} \right)^2 - D_{22} \left( \frac{24}{5h_y^2} \right)^2 + D_{33} \left( \frac{48}{5h_y^2} \right)^2 \right]
\]
\[
K_{46} = \left( B_{13} \frac{4}{5 h_b} - B_{13} \frac{12}{5 h_b^2} + B_{24} \frac{16}{5 h_b^3} - B_{44} \frac{16}{5 h_b^4} \right) \left( \frac{m \pi}{L} \right)^2 - \left( D_{22} \frac{24}{5 h_b^2} - D_{13} \frac{48}{5 h_b^3} + D_{33} \frac{48}{5 h_b^4} \right) \left( \frac{m \pi}{L} \right)
\]
\[
K_{47} = \left( B_{13} \frac{2}{5 h_b} - B_{13} \frac{12}{5 h_b^2} + B_{24} \frac{4}{5 h_b^3} + B_{44} \frac{12}{5 h_b^4} \right) \left( \frac{m \pi}{L} \right)^3 - \left( D_{22} \frac{24}{5 h_b^2} + D_{13} \frac{48}{5 h_b^3} - D_{33} \frac{48}{5 h_b^4} \right) \left( \frac{m \pi}{L} \right)
\]
\[
K_{55} = -\left[ -B_{33} \left( \frac{12}{5 h_b^2} \right)^2 - B_{44} \left( \frac{16}{5 h_b^3} \right)^2 \right] \left( \frac{m \pi}{L} \right)^2 + \left[ D_{22} \left( \frac{24}{5 h_b^2} \right)^2 + D_{33} \left( \frac{48}{5 h_b^3} \right)^2 \right] + k_i
\]
\[
K_{56} = \left( B_{33} \frac{12}{5 h_b^2} - B_{33} \frac{12}{5 h_b^3} + B_{44} \frac{16}{5 h_b^3} + B_{44} \frac{12}{5 h_b^4} \right) \left( \frac{m \pi}{L} \right)^2 + \left( D_{22} \frac{24}{5 h_b^2} + D_{13} \frac{48}{5 h_b^3} + D_{33} \frac{48}{5 h_b^4} \right) \left( \frac{m \pi}{L} \right)
\]
\[
K_{57} = -\left[ -B_{22} + 2 B_{44} \frac{12}{5 h_b^2} - B_{33} \left( \frac{4}{5 h_b} \right)^2 - B_{44} \left( \frac{12}{5 h_b^2} \right)^2 \right] \left( \frac{m \pi}{L} \right)^2 \left[ D_{11} - 2 D_{13} \frac{36}{5 h_b^3} + D_{22} \left( \frac{8}{5 h_b^2} \right)^2 + D_{33} \left( \frac{36}{5 h_b^3} \right)^2 \right]
\]
\[
K_{77} = A_{33} \left( \frac{2}{5 h_b} \right)^2 + A_{44} \left( \frac{4}{5 h_b} \right)^2 + B_{33} \left( \frac{2}{5 h_b} \right)^2 + B_{44} \left( \frac{4}{5 h_b} \right)^2 \left( \frac{m \pi}{L} \right)^4
\]
\[
- C_{11} + 2 C_{13} \frac{12}{5 h_b^2} - C_{22} \left( \frac{4}{5 h_b} \right)^2 - C_{33} \left( \frac{12}{5 h_b^2} \right)^2 - D_{11} + 2 D_{13} \frac{12}{5 h_b^2} - D_{22} \left( \frac{4}{5 h_b} \right)^2 - D_{33} \left( \frac{12}{5 h_b^2} \right)^2 \left( \frac{m \pi}{L} \right)^2
\]
\[
K_{67} = \left[ -B_{24} \frac{4}{5 h_b^2} - B_{33} \frac{12}{5 h_b^3} + B_{44} \frac{12}{5 h_b^4} \right] \left( \frac{m \pi}{L} \right)^3 + \left[ -D_{11} + D_{13} \frac{12}{5 h_b^2} + D_{22} \left( \frac{4}{5 h_b} \right)^2 + D_{33} \frac{36}{5 h_b^3} - D_{33} \frac{36}{5 h_b^4} \right] \left( \frac{m \pi}{L} \right)
\]

(32)
Chapter 3: Dynamic Response

3.1 Introduction

The title of the manuscript presented in this chapter is “An analytical model based on a higher order beam theory for the dynamic response of two-layered composite beams with partial shear interaction”, which proposes an analytical model for free and forced vibration response of two-layered composite beams with the interfacial shear slip based on a higher order beam theory (HBT). The main objective of this study is to investigate the flexural response of composite beams subjected to various dynamic loads. In addition, the free vibration analysis of these beams is also conducted for an initial assessment of the proposed analytical model. Different types of dynamic loading, such as stationary loads (distributed and point loads) having different time variations (harmonic excitation, triangular pulse and step loading), and moving point loads of unaltered magnitude are considered in the forced vibration analysis of composite beams considering the effect of damping. The capability of the proposed model in accurately predicting the time history for deflection, velocity and stress is demonstrated with numerical examples. Furthermore, it is shown that the performance of the proposed HBT model is better than existing models based on EBT and TBT in both free and forced vibration analysis of composite beams, especially for the local response (i.e. stresses and their distributions).

3.2 List of Manuscript

partial shear interaction.” *Construction and building materials* (Elsevier).

### 3.3 Statement of Authorship

<table>
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<th>An analytical model based on a higher order beam theory for the dynamic response of two-layered composite beams with partial shear interaction</th>
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<tr>
<td>Contribution to the Paper</td>
<td>Developed the analytical model, conducted numerical analysis and prepared manuscript</td>
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<td>Overall Percentage</td>
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**Certification**

This paper reports on original research I conducted during my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.

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**Date**

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<td>Supervised in developing the model, helped in data interpretation, provided critical manuscript evaluation and acted as the corresponding author</td>
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3.4 An analytical model based on a higher order beam theory for the dynamic response of two-layered composite beams with partial shear interaction

Jie Wen, Abdul Hamid Sheikh, Md. Alhaz Uddin and Brian Uy

ABSTRACT

In this paper, an analytical solution is derived for accurately predicting the free and forced vibration responses due to time varying as well as moving loads of two layered composite beams. For this purpose, a higher order beam theory (HBT) is used which allows to take a parabolic (third order) variation of the longitudinal displacement over the beam depth for the two material layers. The partial shear interaction between the two material layers produced by the deformability of shear connectors joining these layers in the form of shear slip at their interface is modelled using distributed shear springs along the beam length. Hamilton’s principle is applied to derive the governing equations of the dynamic system which are solved analytically using a Navier type solution technique. In order to assess the performance of the proposed model, numerical examples of composite beams are solved and the results predicted by the model are compared with published results as well as numerical results generated by one dimensional (1D) and two dimensional (2D) finite element models.

Keywords: Composite beams; Partial shear interaction; Higher order beam theory; Analytical solution; Dynamic response
1. Introduction

The application of composite structures is quite common in various structural engineering activities due to their superior performance. A two layered-composite beam such as steel-concrete, steel-timber, and timber-timber are widely used in the construction industry. The advantage of these structural configurations is that the two material layers are properly utilised (e.g., in steel-concrete composite beams, the concrete layer is primarily used to carry the compressive stress whereas the steel layer carries the tensile stress) to enhance the performance of the overall structural system. Shear connectors such as bolts, nails or steel shear studs are commonly used to connect the two layers to achieve composite action. Theoretically, if shear connection is considered to have an infinite stiffness, the benefits of composite action can be fully achieved where no shear slip develops at the interface, which is defined as full shear interaction. However, shear connections are having some level of deformability with a finite stiffness in reality, which results in shear slip at the interface between the two layers and this scenario is popularly known as partial shear interaction [1]. As the effect of partial shear interaction on the structural response has been proven to be significant [2, 3], this should be considered in the analysis of composite beams. In general, this is an active area of research which is best demonstrated by the large number of studies on different aspects of composite beams. However, the main focus of the present investigation is to develop an efficient analytical model for accurately predicting the response of composite beams under dynamic actions.

One of the earliest studies on the modelling of composite beams was completed by Newmark et al. [4] who developed an analytical model based on Euler-Bernoulli beam
theory (EBT) for static response of composite beams considering the effect of partial interaction. This is a well-regarded model which is commonly used as one of the benchmarking solutions for the validation of many numerical models. Due to the analytical nature of this model, it is obvious that this model is applicable for a specific type of loading and boundary conditions, which motivated many researchers for developing a number of numerical models (e.g. [5-9]). In the model developed by Ranzi et al. [9], the effect of vertical separation of the two material layers has also been incorporated in addition to their horizontal shear slip. A numerical model can have a good degree of generality but it needs a computer having adequate capabilities so as to use the model for analysing a structure. Moreover, the numerical results are not unique as they are dependent on element sizes. In terms of analytical solutions, a further development is done by Girhammar and Gopu [10] who developed an analytical model for first and second order analysis of composite beam columns considering partial shear interaction.

All the aforementioned models are applicable to flexural response of composite beams under static loads. On the other hand, a very limited number of studies have been conducted on the dynamic response of composite structures subjected to time varying loads such as harmonic, triangular or some other type of pulse, and moving loads commonly encountered in bridge and other structures. Girhammar and Pan [11] have proposed an accurate as well as an approximate analytical model for dynamic response of composite beams subjected to time varying loads which can be an impulsive load or a suddenly applied step loading. Recently, Huang and Su [12] have developed an analytical model for the dynamic response of composite beams subjected to moving loads. However, it should be noted that all the
above mentioned studies related to static as well as dynamic response of composite beams are based on EBT, which does not consider the effect of transverse shear deformation of the two layers. The effect of shear deformation is found to be significant and cannot be ignored for beams with a small span-to-depth ratio, clamped boundary conditions, localized concentrated loads, and some other scenarios.

The above fact has motivated some researchers and there is a growing interest in recent years to incorporate the effects of shear deformation where the Timoshenko’s beam theory (TBT) is typically used for this. Ranzi and Zona [13] have developed a numerical model for steel-concrete composite beams where TBT is used to model the steel girder whereas the concrete slab is modelled as EBT. On the other hand, an analytical model for two-layered composite beams has been developed by Schnabl et al. [14] who used TBT for modelling both material layers. Schnabl et al. [15] have also developed a finite element formulation following similar idealisation used in their analytical model [14]. It is observed that all these studies [13-15] are restricted to static analysis of composite beams. Nguyen et al. [16] has made an attempt to develop an analytical model using TBT for both material layers of composite beams but they have studied the free vibration of these beams only and did not paid any attention on their dynamic response. Authors could not find any papers on the dynamic response of composite beams where TBT is used to consider shear deformation of the beam layers.

It should be noted that the actual parabolic variation of shear stress over the beam depth is replaced by an average shear stress of uniform distribution over the beam depth in TBT to simplify the problem. In order to minimise the effect of this simplification, an arbitrary shear
correction factor is used to represent the shear stiffness of the beam. Unfortunately, it is quite cumbersome to calculate the exact value of this shear correction factor for a composite beam with partial shear interaction in comparison with that of a single layer homogeneous beam. Moreover, this factor helps to match the global response of the beam (e.g. deflection or natural frequency) well by scaling down the shear stiffness, but this is not adequate to get a satisfactory local response such as the distribution of stresses or strains, which is shown in the section 3.

In order to address the abovementioned issues, a higher-order beam theory (HBT) has recently been developed by Sheikh and his co-workers [17] for an accurate prediction of global as well as local responses of these composite beams. The HBT takes a 3rd order variation of the longitudinal displacement of fibres over the beam depth to model the cross-sectional warping of the beam layers produced by the parabolic (nonlinear) variation of shear stress, where the basic concept of Reddy’s higher order shear deformation theory [18] developed for multi-layered laminated composite plates with no interfacial slip between the layers is used. The model has also been extended to dynamic response [19] and vertical separation [20] of composite beams. So far, the capability of HBT has been investigated numerically through a one dimensional (1D) finite element implementation of the model [17, 19-20] which has exhibited an encouraging performance. This has encouraged to develop an exact analytical model of two-layered composite beams with partial shear interaction based on HBT so that the analytical solution can be benchmarked for future research investigations in this area.

In this paper, a number of numerical examples of two-layered composite beams with partial
shear interaction are solved by the proposed analytical model to assess the performance of
the model. The validation of the proposed analytical model is conducted by comparing the
results predicted by the model with published results as well as numerical results generated
by the 1D finite element model based on HBT [19]. Moreover, a detailed two-dimensional
finite element modelling approach is employed to analyse a composite beam using a well-
regarded finite element commercial code ABAQUS specifically for the validation of
stresses. It should be noted that the number of available results of dynamic response of
composite beams even with a model based on EBT is very limited in the existing literature
and no one reported results for the stresses induced within these beams. The dynamic
response of composite beams is conducted for moving loads as well as time varying loads
consisting of harmonic excitation, triangular pulse and suddenly applied step load. The free
vibration analysis of these beams is also conducted at the beginning to provide initial
confidence of the proposed model. A large number of new results are produced taking
different cross-sectional geometries, interfacial shear stiffness values, loading types and
damping ratios, and these results reported in this paper will contribute to an important
resource for future references.

2. Formulation

2.1. Higher Order Beam Theory

A typical two layered composite beam with an interfacial shear slip due to shear connector
deformations is shown in Fig. 1. The HBT [17] is utilised to express the variation of
longitudinal displacements \( u_1 \) and \( u_2 \) (function of \( x \) and \( y_1 \) or \( y_2 \) in space apart from time \( t \))
of the two layers over their depths as

\begin{align}
  u_1 &= u_{10} - y_1 \theta_1 + y_1^2 \alpha_1 + y_1^3 \beta_1 \\
  u_2 &= u_{20} - y_2 \theta_2 + y_2^2 \alpha_2 + y_2^3 \beta_2
\end{align}

(1) \quad (2)

where \( u_{10} \) and \( u_{20} \) are longitudinal displacements of the two layers (function of \( x \) in space) at their reference axis (\( y_1 = 0 \) or \( y_2 = 0 \)), \( \theta_1 \) and \( \theta_2 \) are bending rotations (function of \( x \) in space) of these layers at their RA, and \( \alpha \) and \( \beta \) are higher order terms (function of \( x \) in space). As the vertical separation between the layers is not common for a straight beam under normal loading, its effect is not considered in this study. Thus the vertical displacement is assumed to be the same for both the layers and it can be expressed as

\[ w_1 = w_2 = w(x) \]

(3)

The partial shear interaction between the two layers is modelled by a distributed spring layer along the interface between these layers where the shear slip can be defined in the form of longitudinal displacement jump at their interface as

\[ s = \bar{u}_2 - \bar{u}_1 \]

(4)

where \( \bar{u}_1 \) is the longitudinal displacement at the bottom fibre of the upper layer and \( \bar{u}_2 \) is that at the top fibre of the lower layer.
Fig. 1. Two-layered composite beam with the variation of longitudinal displacement over the beam depth

Using the shear stress free condition at the exterior surfaces ( \( y_1 = h_1 / 2 \) and \( y_2 = -h_2 / 2 \)), and taking \( \pi_1 \) and \( \pi_2 \) as independent unknowns, the higher order terms appeared in Eqs. (1) and (2), which are having no physical meaning, can be replaced in terms of other unknowns having a physical meaning. Using Eqs. (1) - (3) and the above information/conditions, Eqs. (1) and (2) can be rewritten in terms of all the physical parameters as

\[
u_1 = A_1 u_{10} + B_1 \bar{u}_1 + C_1 \theta_1 + D_1 \frac{dw}{dx}
\]

(5)

\[
u_2 = A_2 u_{20} + B_2 \bar{u}_2 + C_2 \theta_2 + D_2 \frac{dw}{dx}
\]

(6)

where \( A, B, C \) and \( D \) are functions of \( y \) (\( y_1, y_2 \)) and cross-sectional properties of the two layers which are expressed as follows:
\begin{align*}
A_i &= 1 - \frac{12}{5h_i^2} y_1^2 + \frac{16}{5h_i^2} y_1^3, \quad B_i = \frac{12}{5h_i^2} y_1^2 - \frac{16}{5h_i^2} y_1^3, \quad C_i = -y_1 - \frac{4}{5h_i^2} y_1^2 + \frac{12}{5h_i^2} y_1^3, \\
D_i &= -\frac{2}{5h_i^2} y_1^2 - \frac{4}{5h_i^2} y_1^3, \quad A_2 = 1 - \frac{12}{5h_2^2} y_2^2 - \frac{16}{5h_2^2} y_2^3, \quad B_2 = \frac{12}{5h_2^2} y_2^2 + \frac{16}{5h_2^2} y_2^3, \\
C_2 &= -y_2 + \frac{4}{5h_2^2} y_2^2 + \frac{12}{5h_2^2} y_2^3, \quad D_2 = \frac{2}{5h_2^2} y_2^2 - \frac{4}{5h_2^2} y_2^3.
\end{align*}

### 2.2. Variational Formulation

The governing equation can be derived using the virtual work principle in its dynamic form, which can be extracted from the Hamilton’s principle, and it can be expressed as

\[\sum_{j=1}^{2} \left[ \int_{x_{j}}^{x_{j+1}} \left( \delta \varepsilon \sigma_{j} + \delta \gamma \tau_{j} \right) dA dx - \int_{x_{j}}^{x_{j+1}} \delta \tau_{s_{j}} dA dx - \sum_{j=1}^{2} \left[ \int_{x_{j}}^{x_{j+1}} \left( \delta u \rho_{j} \dot{u}_{j} + \delta \omega \left( \rho_{j} \ddot{u}_{j} \right) \right) dA dx \right] \right] = \int_{x}^{\theta} \delta w q dx \quad (7)\]

where \( \delta \) is an operate used to show the variation of any parameter, \( \sigma_{j}, \tau_{j}, \varepsilon_{j}, \gamma_{j} \) and \( \rho_{j} \) are the normal stress, shear stress, normal strain, shear strain and mass density of the \( j \)th layers (\( j = 1 \) for the upper layer and \( j = 2 \) for the lower layer as shown in Fig. 1), \( \tau_{s_{j}} \) is the distributed shear force (per unit length) at their interface, \( (\cdot') \) denotes the second derivative with respect to time \( t \) and \( A_{j} \) represents the cross-sectional area of the two layers.

Eqs. (5) and (6) can be used to expresses the strain components appeared in Eq. (7) as follows.

\begin{align*}
\varepsilon_{j} &= \frac{d u_{j}}{d x} = A_{j} \frac{d u_{j}}{d x} + B_{j} \frac{d \ddot{u}_{j}}{d x} + C_{j} \frac{d \theta_{j}}{d x} + D_{j} \frac{d^{2} w}{d x^{2}} \\
\gamma_{j} &= \frac{d u_{j}}{d y} + \frac{\ddot{u}_{j}}{d x} = A_{j} \frac{d u_{j}}{d y} + B_{j} \frac{d \ddot{u}_{j}}{d y} + C_{j} \frac{d \theta_{j}}{d y} + D_{j} \frac{d w}{d y} + dw 
\end{align*}

\[\left(8\right)\]

### 2.3. Equations of Motion and their Solution

The governing equation as given in Eq. (7) may be rewritten in a compact form as
\[
\sum_{j=1}^{2} \int_{A_{j}} \left\{ \delta \varepsilon_{j} \right\}^{T} [\sigma_{j}] dAdx + \int_{z} \delta \tau_{sh} dz - \sum_{j=1}^{2} \int_{A_{j}} \left\{ \delta \bar{u}_{j} \right\}^{T} [\rho_{j}] [\bar{u}_{j}] dAdx - \int_{z} \delta w dz = 0
\]  \tag{9}

where \( \left\{ \delta \varepsilon_{j} \right\}^{T} = \left[ \delta \varepsilon_{j} \right. \ \delta \gamma_{j} \left. \right] \), \( \left\{ \sigma_{j} \right\} = \left[ \sigma_{j} \right. \ \tau_{j} \left. \right] \), \( \left\{ \delta \bar{u}_{j} \right\}^{T} = \left[ \delta \bar{u}_{j} \right. \ \delta w \left. \right] \) and \( \left\{ \bar{u}_{j} \right\}^{T} = [u_{j} \ w] \)

The stresses at any point of these two material layers can be expressed in terms of their strains and material properties (elastic modulus \( E_{i} \) and shear modulus \( G_{i} \)) as follows:

\[
\sigma_{j} = E_{j} \varepsilon_{j} \quad \text{and} \quad \tau_{j} = G_{j} \gamma_{j}
\]  \tag{10}

Using the constitutive relationships for the two layers as given above and the shear slip as in Eq. (4) along with its constitutive relationship \( \tau_{sh} = k_{s} \varepsilon_{s} \) (\( k_{s} \) is the stiffness of distributed interfacial spring layer), Eq. (9) is expressed as

\[
\sum_{j=1}^{2} \int_{A_{j}} \left\{ \delta \varepsilon_{j} \right\}^{T} [D_{j}] [\varepsilon_{j}] dAdx + \int_{z} \delta (\bar{u}_{2} - \bar{u}_{1}) k_{s} (\bar{u}_{2} - \bar{u}_{1}) dz - \sum_{j=1}^{2} \int_{A_{j}} \left\{ \delta \bar{u}_{j} \right\}^{T} [\rho_{j}] [\bar{u}_{j}] dAdx - \int_{z} \delta w dz = 0 \quad \tag{11}
\]

where \( [D_{j}] = \begin{bmatrix} E_{j} & 0 \\ 0 & G_{j} \end{bmatrix} \)

Using Eqs. (5), (6) and (8), the 2D strain and displacement vectors (function of \( x \) and \( y \) in space) in the above equation can be expressed in a decoupled form as

\[
\left\{ \varepsilon_{j} \right\} = [H_{j}] \left\{ \varepsilon_{j} \right\} \quad \text{and} \quad \left\{ \bar{u}_{j} \right\} = [I_{j}] \left\{ \Delta_{j} \right\} \quad \tag{12}
\]

where \([H_{j}]\) and \([I_{j}]\) are functions of \( y_{j} \) while \( \left\{ \varepsilon_{j} \right\} \) and \( \left\{ \Delta_{j} \right\} \) are 1D strain vector and 1D displacement vector as they are functions of \( x \). The expressions of these quantities are:

\[
[H_{j}] = \begin{bmatrix} A_{j} & B_{j} & C_{j} & D_{j} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dA_{j} / dy_{j} & dB_{j} / dy_{j} & dC_{j} / dy_{j} & 1 + dD_{j} / dy_{j} & & \end{bmatrix}
\]

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\[ \{ \hat{\varepsilon}_j \}^T = \left[ du_j / dx \quad d\bar{u}_j / dx \quad d\theta_j / dx \quad d^2 w / dx^2 \quad u_{j0} \quad \bar{u}_j \quad \theta_j \quad dw / dx \right], \]

\[ [I_j] = \begin{bmatrix} A_j & B_j & C_j & D_j & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \{ \Delta_j \}^T = \left[ u_{j0} \quad \bar{u}_j \quad \theta_j \quad dw / dx \quad w \right] \]

Substituting equations (12) in Eq. (11) and performing integration over the cross-sectional areas of the two material layers \( \int A (...) dA \) for the first and third terms, Eq. (11) is expressed as

\[ \sum_{j=1}^{2} \int \{ \delta \hat{\varepsilon}_j \}^T [D_j] \{ \hat{\varepsilon}_j \} dx + \int_{x} \sigma (\bar{u}_2 - \bar{u}_1) k (\bar{u}_2 - \bar{u}_1) dx - \sum_{j=1}^{2} \int \{ \delta \Delta_j \}^T [R_j] \{ \hat{\Delta}_j \} dx \\
- \int \sigma w q dx = 0 \tag{13} \]

where \([D_j] = \int A_j [H_j]^T [D_j] [H_j] dA\) and \([R_j] = \int A_j [I_j]^T \rho [I_j] dA\).

For a Navier type solution of the present problem, the 1D displacement components are expressed in the following forms:

\[ u_{j0} = \sum_{m=1}^{\infty} u_{jm0} e^{i \omega_m t} \cos \frac{m \pi x}{L}, \quad \bar{u}_j = \sum_{m=1}^{\infty} \bar{u}_{jm} e^{i \omega_m t} \cos \frac{m \pi x}{L}, \quad \theta_j = \sum_{m=1}^{\infty} \theta_{jm} e^{i \omega_m t} \cos \frac{m \pi x}{L}, \]

\[ w = \sum_{m=1}^{\infty} w_m e^{i \omega_m t} \sin \frac{m \pi x}{L} \tag{14} \]

where \( i = \sqrt{-1} \) and \( \omega_m \) is the frequency for the \( m \)-th mode of vibration.

Similarly, the transverse load \( q \) (may be a function in terms of \( x \) and \( t \)) acting on the beam can be expressed as
\[ q = \sum_{m=1}^{\infty} q_m \sin \frac{m \pi x}{L} \quad (15) \]

where the coefficients \( q_m \) (function of \( t \)) can be obtained for a given load \( q \) as

\[ q_m = \int_0^L q \sin \frac{m \pi x}{L} \, dx \]

Using Eq. (14), the 1D strain vector \( \{ \varepsilon \} \) and the displacement vector \( \{ \Delta \} \) appeared in Eq. (13) can be expressed as

\[ \{ \varepsilon \} = \sum_{m=1}^{\infty} [F_{jm}] \{ \Delta_m \} \quad \text{and} \quad \{ \Delta \} = \sum_{m=1}^{\infty} [S_{jm}] \{ \Delta_m \} \quad (16) \]

where \( \{ \Delta_m \} = \begin{bmatrix} u_{10m} \quad \bar{u}_{1m} \quad \theta_{1m} \quad u_{20m} \quad \bar{u}_{2m} \quad \theta_{2m} \quad w_m \end{bmatrix}^T e^{i \omega \tau} = \{ \Delta_{m0} \} e^{i \omega \tau} \]

\[
[F_{1m}] = \begin{bmatrix}
-\frac{m \pi}{L} \sin \frac{m \pi x}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{m \pi}{L} \sin \frac{m \pi x}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{m \pi}{L} \sin \frac{m \pi x}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cos \frac{m \pi x}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \frac{m \pi x}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \frac{m \pi x}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \frac{m \pi x}{L} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{m \pi}{L} \cos \frac{m \pi x}{L} & \frac{m \pi}{L} \cos \frac{m \pi x}{L} & \frac{m \pi}{L} \cos \frac{m \pi x}{L}
\end{bmatrix}
\]
Similarly, the interfacial slip $s$ as in Eq. (4) and the transverse displacement $w$ can be expressed in terms of $\{\Delta_m\}$ using Eq. (14) as

$$s = \bar{u}_2 - \bar{u}_1 = \sum_{m=1}^{\infty} [F_{2m}] \{\Delta_m\}, \quad w = \sum_{m=1}^{\infty} [F_{wm}] \{\Delta_m\}$$

(17)

where $[F_{2m}] = \begin{bmatrix} 0 - \cos \frac{m \pi x}{L} & 0 & 0 & \cos \frac{m \pi x}{L} & 0 \end{bmatrix}$ and
\[ [F_{\text{w.m}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sin \frac{m \pi x}{L} \end{bmatrix}. \]

Substituting Eqs. (15), (16) and Eq. (17) in Eq. (13), it can be expressed as

\[
\begin{align*}
\sum_{j=1}^{2} \left[ \left( \sum_{m=1}^{\infty} \{\Delta_m\}^T F_{jm} \right) \right] D_j \sum_{n=1}^{\infty} [F_{nm}] \{\Delta_n\} dx + \int \left[ \sum_{m=1}^{\infty} \{\Delta_m\}^T F_{wm} \right] k_j \sum_{n=1}^{\infty} [F_{nm}] \{\Delta_n\} dx \\
- \sum_{j=1}^{2} \left[ \left( \sum_{m=1}^{\infty} \{\Delta_m\}^T S_{jm} \right) \right] R_j \sum_{n=1}^{\infty} [S_{jn}] \{\Delta_n\} dx - \int \left[ \sum_{m=1}^{\infty} \{\Delta_m\}^T F_{wm} \right] \sum_{n=1}^{\infty} q_n \sin \frac{n \pi x}{L} dx = 0
\end{align*}
\]

As \( \int_{x=0}^{L} \sin \frac{m \pi}{L} \sin \frac{n \pi}{L} \cdot dx = \int_{x=0}^{L} \cos \frac{m \pi}{L} \cos \frac{n \pi}{L} \cdot dx = 0 \) if \( m \neq n \) and it is equal to \( \frac{L}{2} \) if \( m = n \),

the above equation can be rewritten as

\[
\begin{align*}
\sum_{j=1}^{2} \left[ \left( \sum_{m=1}^{\infty} \{\Delta_m\}^T F_{jm} \right) \right] D_j \sum_{n=1}^{\infty} [F_{nm}] \{\Delta_n\} dx + \int \left[ \sum_{m=1}^{\infty} \{\Delta_m\}^T F_{wm} \right] k_j \sum_{n=1}^{\infty} [F_{nm}] \{\Delta_n\} dx \\
- \sum_{j=1}^{2} \left[ \left( \sum_{m=1}^{\infty} \{\Delta_m\}^T S_{jm} \right) \right] R_j \sum_{n=1}^{\infty} [S_{jn}] \{\Delta_n\} dx - \int \left[ \sum_{m=1}^{\infty} \{\Delta_m\}^T F_{wm} \right] \sum_{n=1}^{\infty} q_m \sin \frac{m \pi x}{L} dx = 0
\end{align*}
\]

In order to satisfy the above equation, its individual terms must be zero for any value of \( m \) and it leads to the final form of the equation for a specific value of \( m \) as follows:

\[
\begin{align*}
\left[ \sum_{j=1}^{2} \left[ F_{jm} \right] D_j \sum_{n=1}^{\infty} [F_{nm}] \{\Delta_n\} dx + \int \left[ F_{wm} \right] k_j \sum_{n=1}^{\infty} [F_{nm}] \{\Delta_n\} dx \right] \{\Delta_m\} = -\sum_{j=1}^{2} \left[ \sum_{n=1}^{\infty} \left[ S_{jm} \right] R_j \sum_{n=1}^{\infty} [S_{jn}] \{\Delta_n\} dx \right] \{\Delta_m\} \\
= \int \left[ F_{wm} \right] q_m \sin \frac{m \pi x}{L} dx
\end{align*}
\]

or \( [K_m] \{\Delta_m\} - [M_m] \{\Delta_m\} = \{Q_m\} \)

For the free vibration response of composite beams where \( q = q_m = Q_m = 0 \) and \( \{\Delta_m\} = -\omega_m^2 \{\Delta_m\} \) due to a harmonic response of these structures, the above equation leads to:

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\[
[K_m][\Delta_{m0}] + \omega_m^2[M_m][\Delta_{m0}] = 0
\]  
(22)

The above equation is an eigenvalue problem which is solved using the simultaneous iteration technique of Corr and Jennings [21] to obtain the eigenvalues in the form of natural frequencies of vibration \( \omega_m \) and their corresponding mode shapes as eigenvectors \( \{\Delta_{m0}\} \).

In order to consider the effect of damping, the concept of Rayleigh damping [22] is followed which helped to express the damping matrix in terms of stiffness and mass matrix as

\[
[C_m] = \alpha[K_m] + \beta[M_m]
\]  
(23)

where \( \alpha \) and \( \beta \) are Rayleigh’s proportionality constants and can be expressed in terms of frequencies \( (\omega_m \text{ and } \omega_n) \) and damping ratios \( (\zeta_m \text{ and } \zeta_n) \) corresponding to \( m \)-th and \( n \)-th modes of vibration as

\[
\begin{align*}
\{\alpha\} &= 2 \frac{\omega_m \omega_n}{\omega_n^2 - \omega_m^2} \left[ \frac{\omega_n}{\omega_m} - \frac{\omega_m}{\omega_n} \right] \{\xi_m\} \\
\{\beta\} &= \frac{1}{\omega_m^2 - \omega_n^2} \left[ \frac{\omega_n}{\omega_m} - \frac{\omega_m}{\omega_n} \right] \{\xi_n\}
\end{align*}
\]  
(24)

In the presence of damping in the structural system, the forced vibration equation i.e. the equation motion for the undamped system as appeared in Eq. (21) will be modified as

\[
[K_m][\Delta_m] + [C_m][\dot{\Delta}_m] - [M_m][\ddot{\Delta}_m] = [Q_m]
\]  
(25)

The above equation is solved using Newmark’s time integration method [23].

3. Numerical examples

A number of numerical examples of composite beams are solved by the proposed analytical model based on HBT to show its performance and range of applicability. As the number of
suitable results available in literature is very limited for the validation of the proposed model, the results produced by a 1D finite element (FE) model based on HBT are also utilised in many cases. Moreover, since no one reported results for the stresses induced within these beams, results for the stresses are produced from a detailed 2D finite element model of a composite beam developed by a well-regarded finite element software for the validation of the stress results. Though the proposed model is derived on the basis of HBT, the present formulation can easily be modified to accommodate TBT and EBT by eliminating some terms from the expression given in Eqs. (1) and (2). In many cases, the results based on these three beam theories are presented together in order to assess their relative merits. As TBT needs a shear correction factor for getting an acceptable result of deflection and natural frequency, an approximate value of 5/6 is taken for this factor in the present study. Though the primary focus of this paper is on dynamic response of composite beams, the free vibration analysis of these structures is also conducted to have an initial assessment of the proposed model. In addition to time varying stationary loads, a moving load having a time independent magnitude is used to conduct the forced vibration analysis. The effect of damping is also considered in a few cases.

3.1. A steel-concrete composite beam having a flanged section

A 6.0 m long steel-concrete composite beam tested by Henderson et al. [24] for its free vibration response is used in this example for the experimental validation of the proposed analytical model. The beam is simply supported at its two ends and it is consisting of a concrete slab and a steel I girder as shown in Fig. 2 where the conventional welded steel shear studs - cast-in type [24] are used to connect the steel girder with the concrete slab.
This is a sample case amongst the six similar beam specimens having six different types of shear connectors tested by Henderson et al. [24]. The material properties used for the two material layers are: \( E_1 = E_{\text{concrete}} = 37519 \text{ MPa}, \) \( v_1 = 0.2, \) \( \rho_1 = 2356 \text{ kg/m}^3, \) \( E_2 = E_{\text{steel}} = 210000 \text{ MPa}, \) \( v_2 = 0.265 \) and \( \rho_2 = 7800 \text{ kg/m}^3. \) The stiffness parameter \( k_s \) used for the shear connectors is 205.9875 MPa which is obtained from the measured data in a push test conducted by Henderson et al. [24].

![Cross-section of a simply supported steel-concrete composite beam](image)

**Fig. 2. Cross-section of a simply supported steel-concrete composite beam**

The beam is analysed with the proposed analytical model as well as a detailed finite element model using a reliable software (ABAQUS) for a cross checking of the analytical results. From the analysis of the beam using ABAQUS, the individual material layers are modelled with four node plane stress rectangular elements (CPS4R) placed in the vertical plane, and the cohesive contact model is used between these two sets of elements used for these two material layers to simulate the partial shear interaction at their interface. The first six natural frequencies of vibration predicted by the proposed analytical model are presented in Table 1 along with the experimental results [24] as well as the numerical results generated by analysing the beam with ABAQUS where a detailed 2D model of the beam is used to produce realistic results. The table shows that the results predicted by the proposed
analytical model have in general a reasonable agreement with the experimental results. However, a visible deviation between them is observed for the second mode which may be due to a measurement problem since this deviation is also observed with the numerical model.

Table 1. The first 6 natural frequencies of a simply-supported steel-concrete composite beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment [24]</th>
<th>Proposed analytical model</th>
<th>Detailed 2D finite element model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>24.8</td>
<td>22.952 (8.05%)*</td>
<td>23.476 (5.34%)*</td>
</tr>
<tr>
<td>2</td>
<td>98.7</td>
<td>75.121 (31.4%)</td>
<td>75.579 (23.4%)</td>
</tr>
<tr>
<td>3</td>
<td>146.0</td>
<td>148.97 (2.00%)</td>
<td>149.10 (2.12%)</td>
</tr>
<tr>
<td>4</td>
<td>224.0</td>
<td>237.86 (5.83%)</td>
<td>236.38 (5.53%)</td>
</tr>
<tr>
<td>5</td>
<td>297.0</td>
<td>337.31 (12.0%)</td>
<td>332.89 (12.1%)</td>
</tr>
<tr>
<td>6</td>
<td>383.0</td>
<td>445.12 (14.0%)</td>
<td>435.36 (13.7%)</td>
</tr>
</tbody>
</table>

* Absolute percentage error with respect to experimental results

3.2. A timber-concrete composite beam having a T-section

An example of a 4 m long timber-concrete composite beam simply supported at its two ends is considered in this section for the validation of the proposed model. The free vibration response of this beam was studied by Xu and Wu [25] analytically using both EBT and TBT but they used same bending rotation for both layers. The same problem was used by Girhammar and Pan [11] for the investigation of the dynamic response of the beam subjected to a suddenly applied uniformly distributed step load of 1.0 kN/m analytically using EBT. Fig. 3 shows the cross-section of the beam which is consisting of a concrete slab and a timber girder, and the material properties used for these two components are: $E_1 = E_{\text{concrete}} = 12$ GPa, $G_1 = 5$ GPa, $E_2 = E_{\text{timber}} = 8$ GPa, $G_2 = 3$ GPa, $\rho_1 = 2400$ kg/m$^3$, $\rho_2 = 500$ kg/m$^3$, while the value of interfacial shear stiffness ($k_s$) is taken as 50 MPa. In addition to this shear stiffness ($k_s = 50$ MPa) giving a partial interaction case, Girhammar and Pan [11] also studied
this problem taking two extreme values of \( k_s \) which are no interaction \((k_s = 0)\) and full interaction \((k_s = \infty)\) cases. For the present study, the value of \( k_s \) is taken as \(10^{10}\) MPa to simulate the full interaction case.

![Cross-sectional view of the timber-concrete composite beam](image)

**Fig. 3. Cross-sectional view of the timber-concrete composite beam**

Both free and forced vibration responses of the composite beam are predicted by the proposed analytical model having the capabilities of HBT, TBT and EBT as well as an in-house computer code based on the 1D finite element model (HBT) developed by Sheikh and his group [19]. The first eight natural frequencies of vibration obtained in the present analysis along with those reported by Xu and Wu [25] are presented in Table 2 which shows a good agreement between the results and relative performances of the different beam models. The maximum values of the time varying deflections predicted by the proposed analytical model (HBT, TBT and EBT) and the 1D-FE model (HBT) at the mid-span section of the beam having three different values of shear stiffness are presented in Table 3 along with the results of Ghammar and Pan [11]. Table 3 also shows a good correlation between the results. Due to different consideration of the shear deformation effect, results obtained from different beam theories are a little bit different in both free and forced vibration response of composite beams, but the differences are in an acceptable range.
Table 2. The first 8 frequencies of a simply-supported timber-concrete composite beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Xu and Wu [25]</th>
<th>Proposed analytical model</th>
<th>1D-FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EBT</td>
<td>TBT</td>
<td>EBT</td>
</tr>
<tr>
<td>2</td>
<td>33.526</td>
<td>33.357</td>
<td>33.503</td>
</tr>
<tr>
<td>3</td>
<td>66.483</td>
<td>65.881</td>
<td>66.397</td>
</tr>
<tr>
<td>4</td>
<td>110.17</td>
<td>108.61</td>
<td>109.93</td>
</tr>
<tr>
<td>5</td>
<td>165.27</td>
<td>161.91</td>
<td>164.72</td>
</tr>
<tr>
<td>6</td>
<td>232.11</td>
<td>225.67</td>
<td>231.01</td>
</tr>
<tr>
<td>7</td>
<td>310.83</td>
<td>299.59</td>
<td>308.83</td>
</tr>
<tr>
<td>8</td>
<td>401.51</td>
<td>383.21</td>
<td>398.15</td>
</tr>
</tbody>
</table>

Table 3. Maximum mid-span deflections of a simply-supported timber-concrete composite beam subjected to a suddenly applied step load for different shear interaction

<table>
<thead>
<tr>
<th>No interaction</th>
<th>Partial interaction</th>
<th>Full interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girhammar and Pan [11]</td>
<td>44.400</td>
<td>15.200</td>
</tr>
<tr>
<td>Proposed analytical model (EBT)</td>
<td>44.529</td>
<td>15.215</td>
</tr>
<tr>
<td>Proposed analytical model (TBT)</td>
<td>44.755</td>
<td>15.352</td>
</tr>
<tr>
<td>Proposed analytical model (HBT)</td>
<td>44.755</td>
<td>15.357</td>
</tr>
<tr>
<td>1D-FEM (HBT)</td>
<td>44.761</td>
<td>15.364</td>
</tr>
</tbody>
</table>

3.3. A composite beam subjected to a uniformly distributed load with sinusoidal time variation

An example of a 5 m long simply supported timber-concrete composite beam subjected to a time varying uniformly distributed load of $10 \sin(45t)$ kN/m ($t$ will be in sec) is fabricated in this section to assess the prediction capability of the proposed analytical model with an emphasis for the stresses. As there is no results available in literature for the validation of stresses, numerical results are generated by analysing the beam with a detailed 2D FE model of the beam. For this purpose, a rectangular cross-section (300 mm wide and 500 mm deep) of the beam consisting of two material layers (200 mm deep upper layer and
300 mm deep lower layer) is used. The material properties used for these two layers are: $E_1 = 24$ GPa, $G_1 = 10$ GPa, $\rho_1 = 2400$ kg/m$^3$, $E_2 = 15$ GPa, $G_2 = 5$ GPa and $\rho_2 = 700$ kg/m$^3$. The stiffness of the shear connectors ($k_s$), idealised as interfacial distributed springs, is taken as 5MPa, 50 MPa and 500MPa.

The time history for the bending stress at the top fibre of the upper layer and the bottom fibre of the lower layer predicted by the proposed analytical model (1D) based on HBT and the 2D detailed FE model at the mid-span section of the beam with $k_s = 50$ MPa is plotted in Fig. 4. Moreover, the variation of stress over the beam depth at critical beam sections obtained by the 1D analytical model based on HBT and the 2D detailed FE is shown in Fig. 5. The figures show a very good agreement between the results produced by these two models. In addition to the stresses, the natural frequency for the fundamental mode and the maximum value of the time varying deflection at the mid-span section of the beam predicted by both these models are also presented in Table 4. Also the time histories for the mid-span deflection and the mid-span velocity obtained from these analyses are plotted in Fig. 6 for $k_s = 50$ MPa. These results show a very good performance of the proposed analytical model in predicting the dynamic behaviour of the composite beam.
Fig. 4. Time history for the bending stress at the mid-span section of the composite beam subjected to a time varying uniformly distributed load ($k_s = 50$ MPa)

Fig. 5. Variation of stresses over the beam depth at a specific time ($t = 0.03$ sec)
Table 4. Fundamental frequency and maximum mid-span deflection of a simply-supported timber-concrete composite beam for different shear interaction

<table>
<thead>
<tr>
<th>$k_s$ (MPa)</th>
<th>Proposed analytical model (HBT)</th>
<th>2D-FEM</th>
<th>Proposed analytical model (HBT)</th>
<th>2D-FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>17.215</td>
<td>17.111</td>
<td>8.9380</td>
<td>8.9658</td>
</tr>
<tr>
<td>50</td>
<td>20.177</td>
<td>20.056</td>
<td>5.9026</td>
<td>5.8993</td>
</tr>
<tr>
<td>500</td>
<td>28.277</td>
<td>27.986</td>
<td>2.5921</td>
<td>2.3169</td>
</tr>
</tbody>
</table>

Fig. 6. Time history for the mid-span deflection and velocity of a composite beam subjected to a time varying uniformly distributed load ($k_s = 50$ MPa)

3.4. A composite beam subjected to a harmonic point load

In this section, a simply supported two-layered composite beam having a rectangular section subjected to a stationary point load having a harmonic time variation ($P = 300 \sin(45t)$ kN) at the beam mid-span is considered to investigate the relative performance of the proposed analytical model based on the three beam theories: HBT, TBT and EBT. For this purpose, the values of some key parameters such as interfacial shear stiffness ($k_s$) for the shear connectors, span-to-depth ratio ($L/h$) of the beam and elastic-to-shear modulus ratio ($E/G$)
of the two layers are varied. The analysis is carried out taking \( h_1 = 0.2 \text{ m} \), \( h_2 = 0.3 \text{ m} \), \( h = h_1 + h_2 = 0.5 \text{ m} \), \( E_1 = 12 \text{ GPa} \), \( E_2 = 8 \text{ GPa} \), \( \rho_1 = 700 \text{ kg/m}^3 \), \( \rho_2 = 500 \text{ kg/m}^3 \) and the breadth of the beam as 0.3 m. The natural frequency for the fundamental mode of vibration \( (\omega_1) \) and the maximum value of the time varying deflection at the beam mid-span \( (w_m) \) are first used to undertake this investigation.

First of all, the variation of \( \omega_1 \) and \( w_m \), in terms of deviations between their values predicted by HBT and EBT (H-E) (e.g. \( (\omega_{1\text{-HBT}} - \omega_{1\text{-EBT}})/\omega_{1\text{-HBT}} \)) as well as HBT and TBT (H-T), with respect to \( k_s \) is plotted in Fig. 7 for a specific value of other two parameters \((L/h \text{ and } E/G)\). Fig 7 shows that the deviations between HBT and EBT i.e. H-E (ranging from 177.5% to 121.7% for \( \omega_1 \) and 88.0% to 80.9% for \( w_m \)) are more than those of H-T (ranging from 4.8% to 2.0% for \( \omega_1 \) and 0.7% to 4.4% for \( w_m \)). Similarly, the variations of \( \omega_1 \) and \( w_m \) with respect to \( L/h \) and \( E/G \) are plotted in Fig. 8 (H-E: ranging from 177.5% to 38.7% for \( \omega_1 \) and 88.0% to 52.5% for \( w_m \), H-T: ranging from 4.8% to 0.5% for \( \omega_1 \) and 10.7% to 1.3% for \( w_m \)) and Fig. 9 (H-E: ranging from 177.5% to 31.1% for \( \omega_1 \) and 88.0% to 43.8% for \( w_m \), H-T: ranging from 4.8% to 1.4% for \( \omega_1 \) and 10.7% to 3.1% for \( w_m \)), respectively. A similar trend is found in Figs. 8 and 9 as well. It should be noted that the shear correction factor used in TBT helps to predict the global response such as natural frequencies and deflections well but the use of this factor may not adequate in predicting the local response (i.e. stresses and their distributions) which will now be studied.

For this purpose, the maximum values of the bending stress at the top fibre of the upper layer predicted by HBT, TBT and EBT at the mid-span are presented in Table 5 for different values of \( k_s \), \( L/h \) and \( E/G \). The table shows that the deviation between the results predicted
by H-E is quite significant (ranging from 61.0% to 2.6%) and it increases with the increase of $k_s$ and $E/G$ values, and decrease of $L/h$. A similar trend is observed for the case of H-T where the deviations between the results predicted by these two models is relatively less (ranging from 39.8% to 0.2%) but it is still significant in many situations. Furthermore, the variation of bending stress over the entire beam depth predicted by HBT and TBT at the mid-span section of the beam for a specific values of $k_s$ (100 MPa), $L/h$ (5) and $E/G$ (50) at a time $t = 0.04$ s is plotted in Fig. 10a. Similarly the shear stress variation at the quarter span of the beam is plotted in Fig. 10b. The figure shows that TBT is not at all capable in predicting the shear stress. In general the results indicates that TBT is acceptable for predicting the global response but it is not recommended for local response of composite beams subjected to dynamic loads.

![Graph](image-url)

a. Fundamental frequency  

b. Maximum mid-span deflection

Fig. 7. Effect of $k_s$ on the deviation of the fundamental frequency and the maximum mid-span deflection of a composite beam subjected to a harmonic point load
a. Fundamental frequency  

b. Maximum mid-span deflection  

Fig. 8. Effect of $L/h$ ratio on the deviation of the fundamental frequency and the maximum mid-span deflection of a composite beam subjected to a harmonic point load  

Fig. 9. Effect of $E/G$ ratio on the deviation of the fundamental frequency and the maximum mid-span deflection of a composite beam subjected to a harmonic point load
Table 5. Bending stress at the top fibre of the upper layer at the mid-span section of the composite beam (Fig. 2) for different values of $k_s$, $L/H$ and $E/G$

<table>
<thead>
<tr>
<th>$k_s$</th>
<th>$E/G$ Ref.</th>
<th>$L/H=2.5$</th>
<th>$L/H=5$</th>
<th>$L/H=10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100 50 10</td>
<td>100 50 10</td>
<td>100 50 10</td>
<td>100 50 10</td>
</tr>
<tr>
<td>HBT</td>
<td>22.388 16.869 11.027</td>
<td>27.330 22.185 17.155</td>
<td>46.222 38.840 35.887</td>
<td></td>
</tr>
<tr>
<td>EBT</td>
<td>8.7390 8.7390 8.7390</td>
<td>15.663 15.663 15.663</td>
<td>34.827 34.827 34.827</td>
<td></td>
</tr>
<tr>
<td>% Daviation (H-T)$^1$</td>
<td>39.832 25.912 7.1286</td>
<td>20.802 12.397 2.9117</td>
<td>7.8218 4.1820 0.8776</td>
<td></td>
</tr>
<tr>
<td>% Daviation (H-E)$^2$</td>
<td>60.965 48.196 20.746</td>
<td>42.691 29.401 8.7000</td>
<td>24.653 10.330 2.9541</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100 % Daviation (H-T)</td>
<td>30.227 17.671 4.2531</td>
<td>11.916 6.4609 1.4503</td>
<td>5.0822 3.2113 0.4775</td>
</tr>
<tr>
<td></td>
<td>% Daviation (H-E)</td>
<td>54.396 42.239 17.969</td>
<td>35.429 23.818 6.8336</td>
<td>22.671 12.726 2.9556</td>
</tr>
<tr>
<td></td>
<td>10 % Daviation (H-T)</td>
<td>28.615 16.702 3.9804</td>
<td>9.3588 4.8212 1.0725</td>
<td>2.6319 1.3283 0.1720</td>
</tr>
<tr>
<td></td>
<td>% Daviation (H-E)</td>
<td>53.404 41.381 17.761</td>
<td>33.113 22.177 5.3155</td>
<td>17.607 10.207 2.5526</td>
</tr>
</tbody>
</table>

$^1$($TBT$-$HBT$)$/HBT$*100%; $^2$($HBT$-$EBT$)$/HBT$*100%

Fig. 10. Variation of stresses over the beam depth at a specific time ($t = 0.04$ s)
3.5. A steel-concrete composite beam having a flanged section subjected to a moving load

Fig. 11 shows the cross-section of a 20 m long simply supported composite beam consisting of a concrete slab and a steel I girder connected by steel shear studs. This problem is used in this example to study the dynamic response of the beam due to a moving point load of 100 kN travelling from one end of the beam to its other end with a speed of 25 m/sec. The analysis is carried out using the proposed analytical model (HBT) as well as the 1D FE model based on HBT taking three values of the interfacial shear stiffness: $k_s = 0$ (no shear interaction), $k_s = 200$ MPa (partial shear interaction) and $k_s = 10^{10}$ MPa (full shear interaction). The material properties used for the concrete slab and the steel girder are: $E_c = 34,200$ MPa, $G_c = 14,250$ MPa, $\rho_c = 2400$ kg/m$^3$, $E_s = 210,000$ MPa and $G_s = 84,000$ MPa and $\rho_s = 8000$ kg/m$^3$. The effect of damping is considered taking three different values of damping ratios (0%, 5% and 10%). The free vibration response of the beam is first predicted, and natural frequencies for the first 3 vibration modes are presented in Table 6 for different degrees of shear interaction. For the dynamic response of the beam due to the moving load, the time history for the deflection and velocity at the mid-span of the beam is plotted in Fig. 12. As the results in Figs. 12 are produced with a damping ratio of $\xi = 0$ (i.e. no damping), the time history for the mid-span deflection corresponding to three damping ratios is presented in Fig. 13 for the case of partial shear interaction ($k_s = 200$ MPa) of the beam. Also the maximum values of the mid-span deflection of the beam are presented in Table 7 for three values of $k_s$ and $\xi$. The results show an increase of the damping ratio caused a reduction of the deflection as expected.
Fig. 11. Cross-section of a simply supported steel-concrete composite beam

Table 6. The first 3 frequencies of a simply supported steel concrete composite beam for different shear interaction

<table>
<thead>
<tr>
<th>No interaction</th>
<th>Partial interaction</th>
<th>Full interaction</th>
</tr>
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<tr>
<td>Mode</td>
<td>Proposed analytical model</td>
<td>1D – FEM</td>
</tr>
<tr>
<td>1</td>
<td>5.2997</td>
<td>5.2997</td>
</tr>
<tr>
<td>2</td>
<td>19.948</td>
<td>19.948</td>
</tr>
<tr>
<td>3</td>
<td>41.188</td>
<td>41.187</td>
</tr>
</tbody>
</table>

a. Mid-span deflection
b. Mid-span velocity

Fig. 12. Time history for the mid-span deflection and velocity of a steel-concrete composite beam subjected to a moving point load for different shear interaction ($\xi = 0$)

Table 7. Maximum mid-span deflections of a steel concrete composite beam subjected to a moving point load for different damping ratios and shear interaction

<table>
<thead>
<tr>
<th>Damping ratio (%)</th>
<th>Proposed analytical model</th>
<th>1D-FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No interaction</td>
<td>Partial interaction</td>
</tr>
<tr>
<td>0</td>
<td>5.7140</td>
<td>3.6513</td>
</tr>
<tr>
<td>5</td>
<td>5.4579</td>
<td>3.4695</td>
</tr>
<tr>
<td>10</td>
<td>5.3237</td>
<td>3.3924</td>
</tr>
</tbody>
</table>

Fig. 13. Time history for the mid-span deflection of a steel-concrete composite beam subjected to a moving point load for different damping ratios ($k_s = 200$ MPa)
3.6. A steel-concrete composite beam subjected to a triangular pulse load

The behaviour of the same beam used in the previous example is studied in this section for a uniformly distributed stationary load having a triangular time variation. The magnitude of the load is 5.0 kN/m at the beginning \((t = 0)\) which (load) is reduced to zero at a time of \(t = 0.5\) sec. The beam is analysed by the proposed analytical model (HBT) in a similar manner and the time history for the mid-span deflection obtained with different degrees of shear interaction \((k_s = 0, 200\, \text{MPa and } 10^{10}\, \text{MPa})\) is plotted in Fig. 14 for the undamped case \((\zeta = 0)\). In order to show the damping effect on the dynamic response of the beam, Fig. 15 presents the time history for the mid-span deflection for three damping ratios considering the case of partial shear interaction \((k_s = 200\, \text{MPa})\) of the beam, and the maximum values of the mid-span deflection of the beam with three values of \(\zeta\) as well as \(k_s\) are presented in Table 8.

Fig. 14. Time history for the mid-span deflection of a steel composite beam subjected to a triangular pulse loading for different shear interaction \((\zeta = 0)\)
Fig. 15. Time history for the mid-span deflection of a steel-concrete composite beam subjected to a triangular pulse loading for different damping ratios \((k_s = 200 \text{ MPa})\)

Table 8. Maximum mid-span deflections of a steel concrete composite beam subjected to a triangular pulse loading for different damping ratios and shear interaction

<table>
<thead>
<tr>
<th>Damping ratio (%)</th>
<th>Maximum mid-span deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No interaction</td>
<td>Partial interaction</td>
</tr>
<tr>
<td>0</td>
<td>6.0075</td>
</tr>
<tr>
<td>5</td>
<td>5.5630</td>
</tr>
<tr>
<td>10</td>
<td>5.1792</td>
</tr>
</tbody>
</table>

4. Conclusions

An efficient analytical model is developed in this paper for accurately predicting the dynamic response of two layered composite beams having deformable shear connectors.

The partial shear interaction caused by the shear slip or longitudinal separation between the two layers at their interface due to the deformability of shear connectors is considered and this is modelled in the form of distributed shear springs along the length of the beam. The higher order beam theory, which allows the true parabolic variation of shear stress over the beam depth, is used to develop this one dimensional efficient analytical model.

The forced vibration analysis of the composite beams is conducted for types of dynamic
loads: a) stationary loads (distributed and point loads) having different time variations (harmonic excitation, triangular pulse and step loading) and b) moving point loads of unaltered magnitude. The free vibration analysis of these beams is also carried out to have an initial assessment of the proposed model in the dynamic range. The proposed analytical model is used to solve numerical examples of composite beams having different geometries and material properties including damping ratios, loading types and degrees of shear interaction. The capability of the model in accurately predicting the time history for deflection, velocity and stress of these beams under different dynamic excitations as well as their natural frequencies of vibration is tested by comparing the results predicted by the proposed model with the published results and numerical results produced by a one dimensional FE model based on HBT as well as a detailed two dimensional FE model. The results confirmed that the model can predict both global and local responses of these beams satisfactorily.

The relative performances of three beam theories (HBT, TBT and EBT) implemented within the proposed analytical model are also investigated. It is observed that differences between the results predicted by HBT and EBT are significant as expected. On the other hand, TBT is found to predict the global response such as natural frequency and deflection well with the shear correction factor but its performance is not satisfactory in predicting the local responses such as stresses and their distributions. The numerical analysis produced a large number of results which will serve as valuable benchmarking solutions in future due to the analytical nature of the model.
5. References


Chapter 4: Geometric Nonlinear Response

4.1 Introduction

This chapter contains a manuscript entitled “An analytical model for geometric nonlinear response of two-layered composite beams with partial shear interaction based on a higher order beam theory”, where the development of an exact analytical model based on a higher order beam theory is presented for large deformation response of two-layered composite beams. The Von-Karman large deflection theory is applied to capture the effect of geometric nonlinearity in the governing equations of the proposed model. Due to the utilization of Navier type solution technique in the present formulation, the results generated by the proposed model are exact and unique, which cannot be achieved with a numerical model. A number of numerical examples are solved by the proposed analytical model to investigate the effect of large displacement on the flexural response of composite beams with interfacial shear slips, and the significance of geometric nonlinearity is confirmed. Moreover, the relative performances of three beam theories (EBT, TBT and HBT) are conducted by using the proposed model. It is observed that the results predicted by EBT are significantly different from the results produced by HBT in all cases whereas TBT has deficiency in accurate predicting the stress within these composite beams. As the proposed analytical model based on HBT exhibited a better performance in predicting both global and local response, it is recommended for the geometric nonlinear analysis of composite beams.

4.2 List of Manuscript

response of two-layered composite beams with partial shear interaction based on a higher order beam theory.” *Journal of Engineering Mechanics (ASCE)*.

### 4.3 Statement of Authorship

<table>
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<th>An analytical model for geometric nonlinear response of two-layered composite beams with partial shear interaction based on a higher order beam theory</th>
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<td>Publication Status</td>
<td>Submitted for publication</td>
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**Principle Author**

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<thead>
<tr>
<th>Name of Principle Author (Candidate)</th>
<th>Jie Wen</th>
</tr>
</thead>
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<tr>
<td>Contribution to the Paper</td>
<td>Developed the analytical model, conducted numerical analysis and prepared manuscript</td>
</tr>
<tr>
<td>Overall Percentage</td>
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**Certification**

This paper reports on original research I conducted during my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.

**Signature**

**Date**

**Co-Author Contribution**

By singing the Statement of Authorship, each author certifies that:

1) the candidate’s stated contribution to the publication is accurate (as detailed above);

2) permission is granted for the candidate including the publication in the thesis; and

3) the sum of all co-author contribution is equal to 100% less the candidate’s stated contribution
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<th>Contribution to the Paper</th>
<th>Signature</th>
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<tr>
<td>Abdul Hamid Sheikh</td>
<td>Supervised in developing the model, helped in data interpretation, provided critical manuscript evaluation and acted as the corresponding author</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brian Uy</td>
<td>Helped in experimental validation of the proposed model and provided some manuscript evaluation</td>
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</table>
4.4 Analytical Model for Geometric Nonlinear Response of Two-layered Composite Beams with Partial Shear Interaction Based on a Higher Order Beam Theory

Jie Wen, Abdul Hamid Sheikh and Brian Uy

ABSTRACT

An efficient analytical model is developed for accurately predicting the geometric nonlinear response of two-layered composite beams with interfacial shear slips by taking into account the effect of large displacement. This is achieved by using a higher order beam theory applied to the two material layers so as to represent a third order variation of the longitudinal displacement over the beam depth in each layer. The partial shear interaction due to the deformable shear connectors between the two different material layers is considered which is modelled as distributed shear springs along the beam length at their interface. The Von-Karman large deflection theory is applied to capture the effect of geometric nonlinearity in the governing equations, and these equations are solved analytically by using a Navier type solution technique. Numerical examples are solved to assess the performance of the proposed model where the results predicted by the model are compared with published results as well as numerical results produced by a detailed finite element (FE) model.

Keywords: Composite beam; Partial shear interaction; Higher-order beam theory; analytical solution; Geometric nonlinearity.
INTRODUCTION

Composite beams consisting of two interconnected material layers, such as timber-timber, timber-steel, timber-concrete and steel-concrete, are commonly used in many structural engineering applications for their superior mechanical properties and structural performance. The performance of the overall structural system is enhanced in these structural configurations due to a proper utilisation of the two layers (e.g., in steel-concrete composite beams, the concrete layer is primarily used to carry the compressive stress whereas the steel layer carries the tensile stress). The two different material layers are commonly connected by shear connectors such as bolts, nails or steel shear studs, which provides a composite action of the structural system. Theoretically, if shear connectors are rigid with infinite stiffness, the benefits of composite action can be fully achieved where no shear slip develops at the interface between two material layers and this scenario is defined as full shear interaction. However, shear connectors have some degree of deformability with a finite stiffness in reality. Therefore, the development of interfacial shear slip always occurs due to the deformability of shear connectors and this is defined as partial shear interaction (Oehlers and Bradford 1995). As the contribution of partial shear interaction on the structural behaviour of composite beams has been proven to be significant (e.g. Loh et al. 2004 and Uy and Nethercot 2005), this effect should be considered in the analysis of these beams. This is an active area of research which is best demonstrated by a large number of studies on different aspects of composite beams. However, the main objective of the present study is to develop an efficient analytical model for accurately predicting the response of composite beams considering the effect of large deformations.
One of the initial pioneering studies on modelling of composite beams was carried out by Newmark et al. (1951) who developed an analytical model based on Euler-Bernoulli beam theory (EBT) considering the effect of partial interaction. Due to its analytical nature, the results produced by this well-regarded model is commonly used as benchmarking solutions for the validation of many numerical models. It is obvious that an analytical model is applicable for a specific type of loading and boundary conditions. Thus some researchers attempted to develop numerical models (e.g. Adekola 1968, Faella et al. 2002, Jasim 1997 and Ranzi et al. 2004, 2006) for addressing different boundary and loading conditions. Such a numerical model needs computers with adequate capabilities so as to utilise the numerical model for analysing these structures though this approach can have a good degree of generality. Moreover, a unique solution can be obtained from an analytical model which is not so in the case of numerical results as they are dependent on element sizes used in the analysis. In terms of analytical solutions, a further development has been carried out by Girhammar and Gopu (1993) who developed an analytical model for first and second order analysis of composite beam columns considering partial shear interaction. Girhammar and Pan (1993) also developed an analytical model for the dynamic analysis of composite beams.

All the above mentioned studies except the second order model of Girhammar and Gopu (1993) are based on the assumption of small displacement theory where a simple linear model is adequate for analysing these composite beams. However, the deformation of these composite beams cannot be restricted to a small range during their entire service life and it is quite common to have a moderately large deformation of these beams under service loads in reality. In order to predict the response of these beams accurately, the effect of large
deformations should be considered but unfortunately this effect introduces a specific type of nonlinearity commonly known as geometric nonlinearity in the analysis. It is interesting to note that a very limited number of studies have been conducted on the geometric nonlinear analysis of composite beams. Erkmen and Bradford (2009) developed a finite element (FE) model for steel-concrete composite beams curved in-plan where the effects of geometric nonlinearity has been considered. Ranzi et al. (2010) presented a finite element model for geometric nonlinear analysis of straight composite beams with the consideration of both horizontal shear slip and vertical separation between the two material layers. However, it should be noted that all the aforementioned models related to linear as well as nonlinear analysis of composite beams are based on EBT which does not consider the effect of transverse shear deformation of the two material layers. It has been found that the effect of shear deformation is significant in beams with a small span-to-depth ratio, clamped boundary conditions, localized concentrated loads, and some other situations.

Due to the abovementioned fact, there is a growing trend of incorporating the effect of shear deformation in recent years and the Timoshenko’s beam theory (TBT) is typically applied for this purpose. Ranzi and Zona (2007) have developed a numerical model for steel-concrete composite beams where TBT is used to model the steel girder whereas the concrete slab is modelled as EBT. On the other hand, an analytical model for two-layered composite beams has been developed by Schnabl et al. (2007a) who used TBT for modelling both material layers. Schnabl et al. (2007b) have also developed a finite element formulation following similar idealisation used in their analytical model (Schnabl et al., 2007a). It is observed that all these studies (Ranzi and Zone 2007; Schnabl et al. 2007a; Schnabl et al.
2007b) are restricted to linear analysis of composite beams based on small deformation theory. Recently, Hjiaj et al. (2012) developed a geometric nonlinear finite element model for two-layered composite beams with interfacial shear slip using TBT.

In a model based on TBT, the actual parabolic variation of shear stress over the beam depth is replaced by an average shear stress of uniform distribution over the beam depth to simplify the problem. In order to minimise the effect of this simplification, an arbitrary shear correction factor is required to adjust the shear stiffness of beams. With this correction factor, a model based on TBT can match the global response (e.g., deflection) of beams well by scaling down the shear stiffness, but it is not adequate for satisfactorily predicting their local response (e.g., distribution of stresses). Moreover, the calculation of the exact value of this shear correction factor is quite cumbersome for a composite beam with partial shear interaction in comparison with that of a single layer homogeneous beam.

In order to address the abovementioned issues, a higher-order beam theory (HBT) has recently been developed by Sheikh and his co-workers (Chakrabarti et al. 2012a) for accurately predicting global as well as local responses of these composite beams with interfacial shear slip. This beam theory (HBT) utilised the basic concept of Reddy’s higher order shear deformation theory (Reddy 1984) developed for multi-layered laminated composite plates with no interfacial slip between the layers. In a model based on HBT, the cross-sectional warping of the beam layers produced by the parabolic variation of shear stress is modelled by a higher order (3rd order) variation of the longitudinal displacement of fibres over the beam depth. The model (Chakrabarti et al. 2012a) has also been extended to capture the effect of vertical separation of the two material layers (Chakrabarti et al. 2012a).
2012b) as well as dynamic analysis (Chakrabarti et al. 2013) of composite beams. So far, the capability of HBT has been investigated numerically through a one dimensional (1D) finite element implementation of the model (Chakrabarti et al. 2012a, 2012b and 2013) which has exhibited an encouraging performance. Moreover, all these investigations are based on linear analysis considering small deformation of the composite beams. These facts have encouraged to develop an exact analytical model for large deformation response of two-layered composite beams with interfacial shear slips using HBT in this paper with the aim that the model will provide benchmarked solution which should be useful to future research investigations in this area. In this study, the Von-Karman’s large deformation theory is used to capture the effect of geometric nonlinearity within the composite beams.

A number of numerical examples of two-layered composite beams with partial shear interaction are solved by the proposed analytical model to assess its performance. For the validation of the proposed model, a detailed two-dimensional finite element modelling approach is employed to analyse composite beams to generate some realistic results using a reliable finite element commercial code ABAQUS. Since the number of available results of composite beams, considering geometric nonlinearity, even with a model based on EBT is very limited in the existing literature, a number of new results reported in this paper will contribute to an important resource for future references.

MATHEMATICAL FORMULATION

Higher Order Beam Theory

Fig. 1 shows a typical two layered composite beam with an interfacial shear slip due to
deformable shear connectors along with the longitudinal displacement profile over the beam depth. Based on HBT (Chakrabarti et al. 2012a), the variation of longitudinal displacements $u_1$ (function of $x$ and $y_1$ for the upper/first layer) and $u_2$ ($x$ and $y_2$ for the lower/second layer) of the two layers over their depths can be expressed as

$$u_1 = u_{10} - y_1 \theta_1 + y_1^2 \alpha_1 + y_1^3 \beta_1$$  \hspace{1cm} (1)

$$u_2 = u_{20} - y_2 \theta_2 + y_2^2 \alpha_2 + y_2^3 \beta_2$$  \hspace{1cm} (2)

where $u_{10}$ and $u_{20}$ are longitudinal displacements (function of $x$) of the two layers at their reference axes (RA: $y_1 = 0$ or $y_2 = 0$), $\theta_1$ and $\theta_2$ are bending rotations (function of $x$) of these layers at their RA, and $\alpha$ and $\beta$ are higher order terms (function of $x$). The effect of vertical separation between the two layers is not considered in this study since it is not significant for a straight beam under usual static loading. Thus the vertical displacement is assumed to be the same for both layers and it can be expressed as

$$w_1 = w_2 = w(x)$$  \hspace{1cm} (3)

Fig. 1. Two-layered composite beam with the longitudinal displacement variation over the beam depth
The deformability of shear studs connecting the two layers produces partial shear interaction which is modelled by a distributed spring layer along the interface between these layers. The interfacial shear slip is defined as a longitudinal displacement jump at the interface between the two layers and it can be expressed as

\[ s = \bar{u}_2 - \bar{u}_1 \]  \hspace{1cm} (4)

where \( \bar{u}_1 \) is the longitudinal displacement at the bottom fibre of the upper layer and \( \bar{u}_2 \) is that at the top fibre of the lower layer.

Using the shear stress free condition at the exterior surfaces (\( y_1 = h_1 / 2 \) and \( y_2 = -h_2 / 2 \)) and taking \( \bar{u}_1 \) and \( \bar{u}_2 \) as independent unknowns or displacement fields, the higher order non-physical terms appeared in Eqs. (1) and (2) can be replaced in terms of other unknowns having physical meanings. Using Eqs. (1) - (3) along with the above definitions and stress free boundary conditions, Eqs. (1) and (2) can be rewritten in terms of all the physical parameters as

\[ u_1 = A_1 u_{10} + B_1 \bar{u}_1 + C_1 \beta_1 + D_1 \frac{dw}{dx} \]  \hspace{1cm} (5)

\[ u_2 = A_2 u_{20} + B_2 \bar{u}_2 + C_2 \beta_2 + D_2 \frac{dw}{dx} \]  \hspace{1cm} (6)

where \( A, B, C \) and \( D \) are functions of \( y (y_1, y_2) \) and cross-sectional properties of the two layers which are expressed as follows:
Variational Formulation

The governing equations can be derived using the virtual work principle and it can be expressed as

\[
\sum_{i=1}^{2} \int_{x} \left( \delta \varepsilon_i \sigma_i + \delta \gamma_i \tau_i \right) dA + \int_{x} \delta \tau_{si} dx - \int_{x} \delta w q dx = 0
\]  

(7)

where \( \delta \) is an operate used to show the variation of any parameter, \( \sigma_i, \tau_i, \varepsilon_i \) and \( \gamma_i \) are normal stress, shear stress, normal strain and shear strain of the \( i \)-th layer (\( i = 1 \) and 2, see Fig. 1), \( \tau_{si} \) is the distributed shear force (per unit length) at the interface between two layers and \( A_i \) represents their cross-sectional area.

Based on the Von-Karman’s large deformation theory, the strain components at any point within the beam can be written in terms of displacement components at that point which can further be expressed in terms of reference plane displacements using Eqs. (5) and (6) as follows

\[
\varepsilon_i = \frac{d u_i}{d x} + \frac{1}{2} \left( \frac{d w}{d x} \right)^2 = A_i \frac{d u_{i0}}{d x} + B_i \frac{d u_i}{d x} + C_i \frac{d \theta_i}{d x} + D_i \frac{d^2 w}{d x^2} + \frac{1}{2} \left( \frac{d w}{d x} \right)^2
\]

\[
\gamma_i = \frac{\partial u_i}{\partial y_i} + \frac{\partial w_i}{\partial x} = \frac{d A_i}{d y_i} u_{i0} + \frac{d B_i}{d y_i} u_i + \frac{d C_i}{d y_i} \theta_i + \frac{d D_i}{d y_i} w_i + w_i
\]  

(8)
**Equilibrium Equation**

The equation as given in Eq. (7) may be rewritten in a compact form as

\[
\sum_{i=1}^{2} \int \{\delta \varepsilon_i\}^T \{\sigma_i\} dAdx + \int \delta s_{st} dx - \int \delta w q dx = 0
\]

(9)

where \(\{\delta \varepsilon_i\} = [\delta \varepsilon_i \ \delta \gamma_i]^T\) and \(\{\sigma_i\} = [\sigma_i \ \tau_i]^T\)

Substituting Eqs. (5), (6) into Eq. (8), the 2D strain vector \(\{\varepsilon_i\} = [\varepsilon_i \ \gamma_i]^T\) (function of \(x\) and \(y\)) at a point within the beam and its variation \(\{\delta \varepsilon_i\}\) used in the above equation can be expressed in a decoupled form as

\[
\{\varepsilon_i\} = \left( [H_L] + [N][H_N] \right) \{\varepsilon_i\}
\]

and

\[
\{\delta \varepsilon_i\} = \left( [H_L] + [N][H_N] \right) \{\delta \varepsilon_i\}
\]

(10)

where \([H_N] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\)

and

\([N] = \begin{bmatrix} dw/dx \\ 0 \end{bmatrix}\).

It should be noted that the variation of the strain vector has also changed the \([H_N]\) matrix to \([H_N]\) apart from the usual change of the 1D strain vector from \(\{\varepsilon_i\}\) to \(\{\delta \varepsilon_i\}\) (Zienkiewicz and Taylor 1989) \([H_L]\) is a function of \(y_i\) while \(\{\varepsilon_i\}\) is the 1D strain vector (function of \(x\)). The expressions of these two quantities (\([H_L]\) and \(\{\varepsilon_i\}\)) are:

\[
[H_L] = \begin{bmatrix} A_i & B_i & C_i & D_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dA_i/ dy_i & dB_i/ dy_i & dC_i/ dy_i & 1 + dD_i/ dy_i \end{bmatrix}
\]

\[
\{\varepsilon_i\}^T = \begin{bmatrix} du_{i0} / dx & d\bar{u}_{i} / dx & d\theta_{i} / dx & d^2 w / dx^2 & u_{i0} & \bar{u}_i & \theta_i & dw / dx \end{bmatrix}
\]
Using the above equation (10) and Eq. (4) for the shear slip, Eq. (9) can be expressed as

\[
\sum_{i=1}^{2} \int \left( \{\delta \hat{\varepsilon}_i\}^T \left[ H_L \right] + \left[ N \right] \left[ H_N \right]^T \right) \{\sigma_i\} dAdx
\]

\[\quad + \int_\delta (\bar{u}_2 - \bar{u}_1) \tau_{sh} dx - \int_\delta \delta w q dx = 0\] (11)

For a Navier type solution of the present problem, the 1D displacement components for a composite beam having simply supported boundary conditions at its two ends are expressed in the following forms:

\[u_{i0} = \sum_{m=1}^{\infty} u_{i0m} \cos \frac{m\pi x}{L}, \quad \bar{u}_i = \sum_{m=1}^{\infty} \bar{u}_{im} \cos \frac{m\pi x}{L}, \quad \theta_i = \sum_{m=1}^{\infty} \theta_{im} \cos \frac{m\pi x}{L},\]

\[w = \sum_{m=1}^{\infty} w_m \sin \frac{m\pi x}{L}\] (12)

where \(L\) is the length of the beam. Similarly, the transverse load \(q\) (may be a function in terms of \(x\)) acting on the beam can be expressed as

\[q = \sum_{m=1}^{\infty} q_m \sin \frac{m\pi x}{L}\] (13)

where the coefficients \(q_m\) can be obtained for a given load \(q\) as

\[q_m = \int_0^L q \sin \frac{m\pi x}{L} dx\]

Using Eq. (12), the 1D strain vector \(\{\hat{\varepsilon}_i\}\) appeared in Eq. (10) can be expressed as

\[\{\hat{\varepsilon}_i\} = \sum_{m=1}^{\infty} \left[ F_{im} \right] \{\Delta_m\}\] (14)

where \(\{\Delta_m\} = [u_{i0m} \quad \bar{u}_{im} \quad \theta_{im} \quad u_{20m} \quad \bar{u}_{2m} \quad \theta_{2m} \quad w_m]^T\)
Similarly, the interfacial slip \( s \) as expressed in Eq. (4) and transverse displacement \( w \) can be expressed in terms of \( \{ \Delta_m \} \) using Eq. (12) as

\[
\begin{aligned}
\Delta_m = \Delta_{m-1} \quad \text{and} \quad \Delta_0 &= 0.
\end{aligned}
\]
\[
\sum_{i=1}^{2} \int_{A} \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \left[ H_L \right] + \left[ N \right] \left[ \dot{H}_N \right] \right) \{\sigma_i\} dAdx \\
+ \int \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \right) \tau_{sh} dx - \int \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \right) \sum_{n=1}^{\infty} q_n \sin \frac{n \pi x}{L} dx = 0
\]  

(16)

**Incremental Equilibrium Equation and its solution**

In the above equation (16), the stresses \( \{\sigma_i\} \) can be expressed in terms of strains \( \{\varepsilon_i\} \) or \( \{\ddot{\varepsilon}_i\} \) (Eq. 10) using constitutive relationships and these strains can subsequently be expressed in terms of the 1D displacement fields \( \{\Delta_m\} \) as in Eq. (14). However, due to the presence of displacement \( (w) \) dependent nonlinear components within the strain vector (Eq. 10), the resulting equation cannot be solved directly. Thus Eq. (16) is solved iteratively until the residual force (difference between external and internal forces in an iteration) as expressed in the following equation becomes less than a user defined small tolerance to ensure the convergence of the solution.

\[
\{\psi\} = \int \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \right) \sum_{n=1}^{\infty} q_n \sin \frac{n \pi x}{L} dx \\
- \sum_{i=1}^{2} \int_{A} \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \left[ H_L \right] + \left[ N \right] \left[ \dot{H}_N \right] \right) \{\sigma_i\} dAdx - \int \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \right) \tau_{sh} dx
\]  

(17)

The Newton-Raphson method is used to solve the nonlinear governing equation (16) iteratively, and it needs an incremental form of this governing equation which can be obtained by taking a variation of the equation with respect to displacements as

\[
\sum_{i=1}^{2} \int_{A} \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \delta \left( H_L \right) + \left[ N \right] \left[ \delta \dot{H}_N \right] \right) \{\sigma_i\} dAdx \\
+ \int \left( \sum_{m=1}^{n} (\partial \Delta_m)^T [F_m]^T \right) \delta \tau_{sh} dx = \{\psi\}
\]  

(18)
where \( \delta\left([H_L] + [N][\vec{H}_N]\right) = [\delta N][\vec{H}_N] \)

The stresses at any point within a material layer can be expressed in terms of strains at that point using the material properties of that material layer as

\[
\{\sigma_i\} = \begin{bmatrix} E_i & 0 \\ 0 & G_i \end{bmatrix} \begin{bmatrix} \varepsilon_i \\ \gamma_{ij} \end{bmatrix} = [D_i] \{\varepsilon_i\}
\]

(19)

where \( E_i \) is the elastic modulus and \( G_i \) is the shear modulus of the layer. Using Eqs. (10) and (14), the stress vector as expressed in the above equation can be rewritten as

\[
\{\sigma_i\} = [D_i] \sum_{m=1}^{\infty} ([H_L] + [N][H_N]) [F_{im}] [\Delta_m] = [D_i] \sum_{m=1}^{\infty} \{\bar{\sigma}_{im}\} = \sum_{m=1}^{\infty} [D_i] \{\bar{\sigma}_{im}\} = \sum_{m=1}^{\infty} \{\bar{\sigma}_{im}\}
\]

(20)

where \( \{\varepsilon_{im}\} \) and \( \{\bar{\sigma}_{im}\} \) are components of strain and stress vectors, respectively, corresponding to the \( m \)-th mode of deformation. In a similar manner, the variation of the stress vector can be written as

\[
\{\delta\sigma_i\} = [D_i] \{\delta\bar{\sigma}_i\} = \sum_{m=1}^{\infty} [D_i] ([H_L] + [N][\vec{H}_N]) [F_{im}] [\delta\Delta_m]
\]

(21)

Using the expression for the interfacial shear slip (\( s \)) as in Eq. (15) along with its constitutive relationship \( \tau_{sh} = k_s s \) (\( k_s \) is the stiffness of the shear connectors in the form of distributed interfacial spring layer), the incremental interfacial shear force can be expressed as

\[
\delta\tau_{sh} = k_s \delta s = k_s \left( \sum_{m=1}^{\infty} [F_{im}] [\delta\Delta_m] \right)
\]

(22)

After substituting Eqs. (20), (21) and (22) into Eq. (18), the incremental equilibrium equation can be expressed as
\[ \sum_{i=1}^{3} \int_{x_0}^{x} \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \left[ \tilde{H}_N^{v} \right] \left[ \delta \tilde{N}^{v} \right] \left( \sum_{n=1}^{\infty} \{ \sigma_{in} \} \right) \right) dAdx \]

\[ + \sum_{i=1}^{2} \int_{x_0}^{x} \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \left( [H_{I}] + [N] \left[ \tilde{H}_N \right] \right) \right) D_i \left( \sum_{n=1}^{\infty} \left( [H_{I}] + [N] \left[ \tilde{H}_N \right] \right) [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dAdx \]

\[ + \int \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \right) k_i \left( \sum_{n=1}^{\infty} [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dx = \{ \psi \} \]

As \[ \int_{x=0}^{L} \sin \left( \frac{m \pi}{L} \right) \sin \left( \frac{n \pi}{L} \right) dx = \frac{L}{2} \] and \[ \int_{x=0}^{L} \cos \left( \frac{m \pi}{L} \right) \cos \left( \frac{n \pi}{L} \right) dx = \frac{L}{2} \] when \( m = n \) and both these expressions are zero when \( m \neq n \), the above equation can be reduced to

\[ \sum_{i=1}^{3} \int_{x_0}^{x} \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \left[ \tilde{H}_N^{v} \right] \left[ \delta \tilde{N}^{v} \right] \{ \sigma_{in} \} \right) dAdx \]

\[ + \sum_{i=1}^{2} \int_{x_0}^{x} \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \left( [H_{I}] + [N] \left[ \tilde{H}_N \right] \right) \right) D_i \left( \sum_{n=1}^{\infty} \left( [H_{I}] + [N] \left[ \tilde{H}_N \right] \right) [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dAdx \]

\[ + \int \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \right) k_i \left( \sum_{n=1}^{\infty} [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dx = \{ \psi \} \]

After rearranging the placement of stress components appeared in first term of the above equation according to the usual procedure (Zienkiewicz and Taylor 1989) and expanding the residual force (Eq. 17), Eq. (22) can be rewritten as

\[ \sum_{i=1}^{3} \int_{x_0}^{x} \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \left[ \tilde{H}_N^{v} \right] \sigma_{in} \left[ \tilde{H}_N \right] [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dAdx \]

\[ + \sum_{i=1}^{2} \int_{x_0}^{x} \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \left( [H_{I}] + [N] \left[ \tilde{H}_N \right] \right) \right) D_i \left( \sum_{n=1}^{\infty} \left( [H_{I}] + [N] \left[ \tilde{H}_N \right] \right) [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dAdx \]

\[ + \int \left( \sum_{m=1}^{\infty} \left\langle \delta \Delta_m \right\rangle^{v} [F_{m}]^{v} \right) k_i \left( \sum_{n=1}^{\infty} [F_{in}] \left\langle \delta \Delta_m \right\rangle \right) dx = \sum_{m=1}^{\infty} \{ \psi_m \} \]

In order to satisfy the above equation, its individual terms must be satisfied individually for any value of \( m \) and this will lead to the final form of the incremental equilibrium equation for a specific value of \( m \) as follows:

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\[
\sum_{i=1}^{2} \int_{A} \left[ F_{m} \right]^{T} \int_{A} \left[ \hat{H}_{i} \right]_{V} \sigma_{m} \left[ \hat{H}_{i} \right]_{V} dA \left[ F_{m} \right] dx
\]

\[
+ \sum_{i=1}^{2} \int_{A} \left[ \left( H_{i} \right)_{L} + [N] \left[ \hat{H}_{i} \right]_{V} \right] \left[ D_{i} \right] \left[ \hat{H}_{i} \right]_{V} dA \left[ F_{m} \right] dx + \int_{A} \left[F_{m} \right]^{T} \left[k_{s} \left[F_{m} \right] \right] dx \right\{\Delta_{m} \right\} = \left\{\psi_{m} \right\}
\]

or

\[
\left[K_{m}\right] \left\{\Delta_{m} \right\} = \left\{\psi_{m} \right\}
\]

NUMERICAL RESULTS

The performance and range of applicability of the proposed analytical model is tested in this section by solving a number of numerical examples of composite beams considering the effect of large displacement. Since the number of suitable results available in literature is very limited for the validation of the proposed model, numerical results are produced using detailed 2D FE models of composite beams which are mostly used for comparing the results predicted by the proposed analytical model. Moreover, no one reported results for the stresses induced within these composite beams. Thus the present paper has attempted to report a reasonable amount of stress results produced by the proposed analytical model which are again cross checked by the abovementioned 2D FE model in one sample case. Though the proposed model is derived on the basis of HBT, the present formulation can easily be modified to accommodate TBT and EBT by eliminating some terms from the expression given in Eqs. (1) and (2). In one case, the results predicted by the analytical model based on these three beam theories are presented together to show the relative merits of HBT, TBT and EBT. TBT needs a shear correction factor for achieving an acceptable result of deflection, which is taken as 5/6 (an approximate value) in the present study.
A Two-layered Composite Beam with a Deformable Interface

An example of a 14.0 m long composite beam, studied by Ranzi et al. (2010) using EBT, is used in this section for the validation of the proposed analytical model. The beam is simply supported at its two ends and consists of two identical material layers, each 200 mm wide and 150 mm deep. The material properties used for these layers are: $E_1 = E_2 = 30000\text{MPa}$ and $v_1 = v_2 = 0.2$. Ranzi et al. (2010) analysed the beam under uniformly distributed load using EBT taking five different values of shear connector stiffness, which is expressed in a non-dimensional parameter ($\alpha L$). Using one of these values of this non-dimensional stiffness parameter ($\alpha L = 1$, i.e. $k_s = 0.0574 \text{MPa}$) as a sample case, the beam is analysed with the proposed analytical model based on HBT as well as EBT by varying the applied load from zero to 5.0 kN/m incrementally.

This beam is also analysed with a reliable finite element software ABAQUS where a detailed 2D finite element modelling of the beam is used to produce realistic numerical results. From this 2D FE model, the individual material layers are modelled with four node plane stress rectangular elements (CPS4R) placed in the vertical plane, and the partial shear interaction at the interface between the two layers is simulated with a cohesive contact model.

The variation of the mid-span deflection with respect to the load intensity predicted by the proposed analytical model based on HBT and EBT as well as the 2D finite element method is plotted in Fig. 2 along with the corresponding results reported by Ranzi et al. (2010). The figure shows very good agreement with the proposed analytical model. It also shows that...
the performance of EBT is degraded with an increase of deformation.

![Graph showing variation of mid-span deflection with respect to load of the composite beam]

**Fig. 2. Variation of mid-span deflection with respect to the load of the composite beam**

*A steel-Concrete Composite Beam Having a T-section*

A 5.5 m long steel-concrete composite beam as shown in Fig. 3 is considered in this example to assess the prediction capability of the proposed analytical model specifically for the evaluation of stresses. The beam is consisting of a concrete slab and a steel girder (Fig. 2) and the material properties used for these two components are: \( E_1 = E_{\text{concrete}} = 32920 \text{ MPa}, \) \( v_1 = 0.15, \) \( E_2 = E_{\text{steel}} = 205000 \text{ MPa} \) and \( v_2 = 0.3, \) while the stiffness value of interfacial distributed springs for the shear connectors \( (k_s) \) is taken as 243 MPa. The beam is simply supported at the two ends and subjected to a uniformly distributed load \( (q) \) over its entire length, which is analysed with the proposed analytical model as well as a detailed 2D finite element model of the beam where the magnitude of the distributed load is increased incrementally from zero to a maximum value of 3800 kN/m.

The variation of mid-span deflection of the composite beam with respect to the load
intensity (q) predicted by the proposed analytical model and the 2D FE model is plotted in Fig. 4. Similarly, the variation of interfacial shear slip at one of the ends of the composite beam is presented in Fig. 5. The variation of deflection along the beam length is also shown in Fig. 6 for a load intensity of 1900 kN/m and 3800 kN/m. Moreover, the variation of bending stress over the beam depth at the mid-span section of the beam predicted by both these modelling approaches for a load intensity of 1900 kN/m is plotted in Fig. 7. In a similar manner, the variation of shear stress over the beam depth is presented in Fig. 8 where the shear stress is captured at quarter span of the beam as it is zero at the mid-span. The agreement between the results produced by the two approaches is found to be very good in all cases except for the case of shear stress. In principle, the shear stress must be zero at the top and bottom fibres of the beam which is unfortunately deviated with a large extent at the bottom fibre by the 2D FE model and leading to an overall mismatch. It is to be noted that this deviation is not significant in the linear range (Wen et al. 2017) but it gets magnified as the effect of nonlinearity increases.

![Cross-section of a 5.5 m long simply supported steel-concrete composite beam](image)

**Fig. 3.** Cross-section of a 5.5 m long simply supported steel-concrete composite beam
Fig. 4. Variation of mid-span deflection with respect to the load of the composite beam

Fig. 5. Variation of shear slip at one end of the beam with respect to the load

Fig. 6. Variation of deflection along the length of the beam for two load intensities
Fig. 7. Variation of bending stress over the depth of the beam for $q = 1900 \text{kN/m}$

Fig. 8. Variation of shear stress over the depth of the beam for $q = 1900 \text{kN/m}$

*A Timber-Concrete Composite Beam under a Point Load*

An example of a timber-concrete composite beam having a T-section (Fig. 9), simply supported at its two ends and subjected to a point load ($P$) action at the mid-span section of the beam, is studied in this section. In order to show the significance of the large displacement effect, the response of the beam in both linear and nonlinear ranges is analysed by the proposed analytical model using HBT. For this purpose, values of some key parameters such as interfacial shear stiffness ($k_o$) for shear connectors and span-to-depth
ratio \((L/H)\) of the beam are varied. The material properties of these two layers are taken as:

\[ E_1 = E_{\text{concrete}} = 24000 \text{ MPa}, \quad G_1 = 10000 \text{ MPa}, \quad E_2 = E_{\text{timber}} = 15000 \text{ MPa} \text{ and } G_2 = 5000 \text{ MPa}. \]

Fig. 9. Cross-section of a 5.0 m long simply supported steel-concrete composite beam

![Fig. 9](image_url)

Fig. 10. Variation of mid-span deflection with respect to the load of the beam

![Fig. 10](image_url)

a. \(L/H = 15\)  

b. \(L/H = 10\)
The variations of mid-span deflection of the beam with respect to the applied load ($P$) ranging from zero to 1000 kN obtained in the linear and nonlinear analysis are presented in Fig. 10 for different values of $k_s$ and $L/H$ ratio. Similarly, the variations of shear slip at one end of the beam are plotted in Fig. 11. Also, the numerical values of mid-span deflection and shear slip at the beam end corresponding to the maximum value of the load ($P = 1000$ kN) are presented in Table 1 for all cases. These results show that the effect of geometric nonlinearity is significant in the form of reductions of both deflection and slip with the increase of load. The table also shows that the effect of nonlinearity increases with an increase of shear connector deformability and span-to-depth ratio. In other words, the effect of geometric nonlinearity cannot be ignored for a composite beam subjected to a large load or having small values of interfacial shear stiffness and span-to-depth ratio.
Table 1. Mid-span deflection and shear slip at one end of a simply-supported timber-concrete composite beam \((P = 1000 \text{ kN})\).

<table>
<thead>
<tr>
<th>(k_s) (MPa)</th>
<th>Linear model ((w_L))</th>
<th>Nonlinear model ((w_N))</th>
<th>(w_L/ w_N)</th>
<th>Linear model ((s_L))</th>
<th>Nonlinear model ((s_N))</th>
<th>(s_L/ s_N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L/H = 15)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>363.15</td>
<td>180.95</td>
<td>2.0069</td>
<td>22.145</td>
<td>9.8661</td>
<td>2.2445</td>
</tr>
<tr>
<td>500</td>
<td>220.52</td>
<td>152.24</td>
<td>1.4485</td>
<td>2.8564</td>
<td>1.7543</td>
<td>1.6282</td>
</tr>
<tr>
<td>(L/H = 10)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>136.38</td>
<td>97.681</td>
<td>1.3962</td>
<td>15.819</td>
<td>10.980</td>
<td>1.4408</td>
</tr>
<tr>
<td>500</td>
<td>74.654</td>
<td>67.703</td>
<td>1.1027</td>
<td>2.7995</td>
<td>2.4866</td>
<td>1.1258</td>
</tr>
</tbody>
</table>

For further investigations, the variations of bending stress over the beam depth at the mid-span section of the beam corresponding to \(P = 1000 \text{ kN}\) obtained all the above mentioned cases are plotted in Figure. 12. The figure shows significant reductions of bending stress values at critical points due to the nonlinear effect, and this is amplified with the decrease of the interfacial shear stiffness and the increase of beam length. Similarly, the variations of shear stress over the beam depth at the quarter span of the beam is presented in Fig. 13 which shows a similar behaviour as observed for the bending stress.

![Graph](image-url)

\[ F_{12}. \text{Variation of bending stress over the beam depth at mid-span of the beam for } P = 1000 \text{ kN} \]

a. \(L/H = 15\)  
b. \(L/H = 10\)
Fig. 13. Variation of shear stress over the beam depth at quarter span of the beam for $P = 1000$ kN

A Timber-Timber Composite Beam under Uniformly Distributed Load

In this section, a 1.5 m long timber-timber composite beam simply supported at its two ends and subjected to a uniformly distributed load ($q$) is considered to investigate the relative performance of the proposed analytical model based on different beam theories (HBT, TBT and EBT). The analysis is carried out taking $h_1 = h_2 = 200$ mm (Fig. 1), 200 mm breadth for both layers, $E_1 = 12000$ MPa, $E_2 = 8000$ MPa, $G_1 = E_1 / 15$, $G_2 = E_2 / 15$, and two different values of interfacial shear stiffness ($k_s = 100$ MPa and $10^{10}$ MPa) to simulate a partial interaction and a full interaction scenarios. The variation of mid-span deflection with respected to the load intensity ($q$) ranging from zero to 5000 kN/m predicted by the proposed analytical model based these three beam theories (HBT, TBT and EBT) is plotted in Fig. 14. The figure shows that the differences between the results obtained by HBT and EBT is more than those by HBT and TBT.
As the performance of TBT is found to be very similar to HBT in predicting the global response (deflection) of the composite beams, their relative performance in predicting the local response of these beams is investigated. First, the variation of bending stress over the entire beam depth corresponding to the maximum value of the load \((q = 5000 \text{ kN/m})\) predicted by HBT and TBT at the mid-span section of the beam is plotted in Fig. 15. Similarly, the shear stress variation over the beam depth at the quarter span of the beam is plotted in Fig. 16. The performance of TBT is acceptable for the prediction of bending stress but it is not recommended for shear stress.

Fig. 14. Variation of mid-span deflection with respect to the load of the beam

Fig. 15. Variation of bending stress over the beam depth at mid-span of the beam for \(q = 5000 \text{ kN/m}\)
A Steel-Concrete Composite Beam Having a Flanged Section

The large deformation response of a 20 m long steel-concrete composite beam consisting of a concrete slab and a steel I girder connected by steel shear studs as shown in Fig. 17 is studied in this section. The investigation is carried out for two different load cases taking simply supported boundaries at its two ends of the beam. A point load \((P)\) is applied at the mid-span section of the beam in one case where the load \((P)\) is varied from zero to 20000 kN incrementally. For the other case, a distributed load having a sinusoidal variation \((q_0 \sin \pi x / L)\) is applied over the entire length of the beam where the peak load intensity \((q_0)\) at the mid-span \((x = L/2)\) is increased similarly from zero to 2000 kN/m. The proposed analytical model (HBT) is used to analyse the nonlinear response of the composite beam taking the value of interfacial shear stiffness \((k_s)\) as 100 MPa. The material properties used for the concrete slab and the steel girder are: \(E_c = 34200\) MPa, \(G_c = 14250\) MPa, \(E_s = 210000\) MPa and \(G_s = 84000\) MPa. The numerical values of mid-span deflection and shear slip at one
end of the beam predicted by the proposed model for different load values are presented in Table 2.

![Diagram of a steel-concrete composite beam](image)

**Fig. 17. Cross-sectional details of a steel-concrete composite beam**

Table 2. Mid-span deflection and shear slip at one end of the steel-concrete composite beam

<table>
<thead>
<tr>
<th>Point load</th>
<th>Sinusoidal load</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P ) (kN)</td>
<td>Deflection (mm)</td>
</tr>
<tr>
<td>2500</td>
<td>94.825</td>
</tr>
<tr>
<td>5000</td>
<td>181.94</td>
</tr>
<tr>
<td>7500</td>
<td>258.37</td>
</tr>
<tr>
<td>10000</td>
<td>324.81</td>
</tr>
<tr>
<td>12500</td>
<td>383.03</td>
</tr>
<tr>
<td>15000</td>
<td>434.70</td>
</tr>
<tr>
<td>17500</td>
<td>481.20</td>
</tr>
<tr>
<td>20000</td>
<td>523.52</td>
</tr>
</tbody>
</table>

**CONCLUSIONS**

In this study, an efficient analytical model based on a higher order beam theory is developed for accurately predicting the geometric nonlinear response of two layered composite beams simply supported at two ends considering partial shear interaction with deformable shear connectors. Based on the results obtained from the numerical analysis carried out using the proposed analytical model, the following conclusions can be drawn:
1. The proposed analytical solution based on HBT can accurately predict both global (i.e. deflection) and local (i.e. stresses and their distributions) responses of composite beams in the large deflection range, as verified by published results as well as numerical results generated by a detailed finite element analysis.

2. The effect of large displacement is found to have significantly influence on bending response of composite beams, which results in the reduction of deflection, interfacial shear slip and stress values compared to those predicted by the linear model. The parametric study carried out with different values of interfacial shear stiffness and beam length shows that the effect of geometric nonlinear is more significant for beams having bigger span-to-depth ratio and higher degree of shear connector deformability.

3. A comparative study was conducted between the results predicted by different beam theories (EBT, TBT and HBT) for different values of interfacial shear stiffness. It is observed that the difference between the results predicted by HBT and EBT is significant in all cases. In term of TBT, it can predict similar deflection and normal stress results as those by HBT for lower interfacial shear stiffness values, but the performance of HBT is deteriorated with the increase of interfacial shear stiffness. Moreover, TBT is not adequate in accurately predicting the shear stress distribution of composite beams in any situation.

4. The proposed analytical model has a good degree of generality that helped to solve numerical examples of composite beams having different dimensions, cross sectional configurations, loading types and degrees of shear interactions.
5. Due to analytical nature of the proposed model, the results reported in this paper will serve as benchmarking solutions for validation of any numerical model in the area of composite beams in future.

REFERENCES


Chapter 5: Conclusions and Recommendations for Future Study

5.1 Conclusions

Exact analytical models are developed in this thesis for static bending response, flexural free and forced vibration response and geometric nonlinear bending response of two-layered composite beams with interfacial shear slips using a higher order beam theory (HBT). The partial shear interaction caused by the shear slip or longitudinal separation between the two layers at their interface due to the deformability of shear connectors is considered and this is modelled in the form of distributed shear springs along the length of the beam. Since HBT allows the true parabolic variation of shear stress over the beam depth, the use of an arbitrary shear correction factor used in Timoshenko beam theory is not required. Moreover, all these proposed models based on HBT can be used for satisfactory prediction of global response (i.e. deflection) as well as local response (i.e. stresses and their distributions) of composite beams.

As the number of results available in literature suitable for validation of the proposed analytical models is very limited especially for dynamic and geometric nonlinear analysis of composite beams, detailed two-dimensional finite element analysis of a number of beams are conducted using a reliable commercial finite element software. In order to investigate the relative merits of EBT, TBT and HBT, the present formulations based on HBT are modified to accommodate TBT and EBT by eliminating the higher order terms. Based on the results obtained from the numerical analysis using the proposed analytical models, the following conclusions can be drawn:
The proposed analytical models are extremely efficient, which can be used to generate an accurate solution of static bending response, free and forced vibration response, and geometric nonlinear response of two-layered composite beams.

The results predicted by EBT show significant differences from those produced by HBT in all cases, and the deviations between the results obtained by HBT and EBT are more than those between HBT and TBT.

The use of shear correction factor helps TBT to predict the global response (i.e. deflection) well for some cases but its performance is deteriorated with the increase of interfacial shear stiffness \((k_s)\) and elastic-to-shear modulus ratio \((E/G)\), and the decrease of span-to-depth ratio \((L/H)\). However, TBT is not adequate in accurately predicting local response of composite beams especially for the shear stress distribution in any situation.

The performance of the proposed models is found to be superior when the results for deflection, interlayer slip and stress predicted by the proposed models are compared with experimental results, published results as well as numerical results generated by finite element analysis in all cases.

From the convergence study of the proposed models with respect to the number of terms \((m)\) used in the series solution, it is observed that 30 terms \((m = 30)\) are adequate to obtain converged results for both global and local responses in all cases.

The effect of damping on the response of composite beams subjected to both time varying and moving load is investigated using the proposed model, where an increase of the damping ratio is found to cause a reduction of the deflection.
The effect of geometric nonlinearity is found to be significant in the bending response of composite beams where the reduction of deflection, interfacial shear slip and stress values compared to those predicted by the linear model is increased steadily with the increase of load. These reductions are amplified with the increase of shear connector deformability.

The proposed analytical models can be used to investigate linear static and dynamic response as well as geometric nonlinear response of composite beams having different cross sectional configurations (rectangular, T-section and flanged section), degrees of shear interaction (i.e. no interaction, partial interaction and full interaction), and types of static loading (e.g. uniformly distributed load, point load and sinusoidal variation of distributed load) and dynamic loading (e.g. stationary distributed or point loads having different time variations and moving point loads having unaltered magnitude).

A large number of new results including both global and local responses of these beams generated from the proposed analytical models can well serve as valuable benchmarking solutions in future due to their analytical nature.

5.2 Recommendations for Future Study

Based on the literature review and present investigation, a number of possible scopes exists which could be taken into consideration for further improvement of the current model:

✓ The current nonlinear model considered geometric nonlinearity to capture the effect of large deformation; therefore, the model can be extended to capture the effect of inelastic material behaviour.
The current nonlinear model is restricted to static response of composite beams, where the dynamic response of two-layered composite beams considering the effect of large displacement with/without inelastic material behaviour can be incorporated in future studies.

Buckling and post buckling analysis of two-layered composite beams is another possible task for future studies.

The current analytical models are developed in one dimension, and an extension to two dimension or even three dimension can be considered for future studies in the modelling of composite beams.

In the current study, only straight beams are considered to conduct the analysis for their response, which can be extended to beams curved in plane or out of plane.

The long term effects, such as creep, shrinkage and temperature can be incorporated in future studies in the area of composite beams.