AN APPLICATION OF FUNCTIONAL ANALYSIS TO A PROBLEM IN GEOPHYSICS

by

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Thesis submitted for the degree of

Master of Science

in the Department of Pure Mathematics,

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Revised Version January, 1992 .

CONTENTS

SUMMARY

The aim of this thesis was to provide a rigourous justification for certain aspects of a perturbation method used to calculate an approximation of transient electromagnetic field. This method used the quasi-static approximation to the solution of the vector wave equations ' in a region of two half-spaces of differing conductivities.
In the case in which neither of the half-spaces was insulating, it was shown, via a variational approach, that a unique solution of the time-domain problem exists. If one of the spaces was insulating it was only possible to establish existence in the scalar case. The solution to the scalar diffusion equation was shown to exist in a weighted Sobolev space.
It was shown that the elements of the fundamental matrix of the Laplace transformed (with respect to t) vector wave equation, tended spatially pointwise to their value at $\epsilon = 0$, as $\epsilon \to 0$. Formulae for the fundamental matrix obtained previously were verified. It was shown that the perturbation method gave a solution to the problem in the half-space of non-zero conductivity if the current source was considered to reside in this half-space. Further restrictions on the source were shown to be necessary if it was considered to reside in the insulating half-space. The spatial asymptotic behaviour of the field was determined.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and that, to the best of my knowledge and belief, the thesis contains no material previously published or written by another person other than where due reference is made in the text of the thesis.
I consent to the thesis being made available for photocopying and loan if applicable if accepted for the award of the degree.

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ACKNOWLEDGEMENTS

The author wishes to thank Dr. Alan Carey for his help and advice. During this research the author was supported by an ABSTUDY grant.

Chapter 1
Outline and Introduction

The transient electromagnetic (TEM) method is an important technique of geophysical surveying. The method, essentially, consists of placing two conducting loops on the surface of the Earth and passing an alternating current through one of the loops. This current creates induction currents in the other loop and the induced E.M.F., which depends upon the electrical properties of the Earth in the region of the loops, is measured in this receiving loop. The data gathered from the receiving loop is then compared with the data expected from various models of the substrate. This thesis investigates certain aspects of a method for calculating this model data.

Firstly, we recall that electromagnetic fields are governed by Maxwell's equations:

$$\begin{aligned}
\nabla \times \mathrm{E} &= -\,\partial_t \mathrm{B} \\
\nabla \times \mathrm{H} &= \partial_t \mathrm{D} + \mathrm{J} \\
\nabla \cdot \mathrm{B} &= 0 \\
\nabla \cdot \mathrm{D} &= \rho \\
\mathrm{J} &= \sigma \mathrm{E} + \mathrm{K}.
\end{aligned}$$

where E and H are the electric and magnetic field intensities, ρ is the density of electrical charge, J is the total current density, σE is the conduction current density, B is the magnetic induction field , D is the displacement current, K is a known current density maintained by an external energy source and σ is the conductivity.

If we assume that the following conditions hold:

(1) $\mathrm{D} = \epsilon \mathrm{E}$, $\mathrm{B} = \mu \mathrm{H}$ where, μ is the permeability and ϵ the permittivity,

(2) μ and ϵ are independent of the time variable,

(3) that K is switched on at t=0,

(4) for $t < 0$, $\mathrm{E}(\cdot,t) = 0$, $\mathrm{H}(\cdot,t) = 0$, $\rho(\cdot,t) = 0$,

then E is determined by the solution to the equation

$$\mu\epsilon\partial_t^2 \mathrm{E} + \mu\sigma\partial_t \mathrm{E} + \nabla \times \nabla \times \mathrm{E} = -\mu\partial_t \mathrm{K} \qquad (1 \cdot 1)$$

(Carey and O'Brien [1]).

Now, formally, (1· 1) can be Laplace transformed to

$$(\mu\epsilon s^2 + \mu\sigma s + \nabla \times \nabla \times)\mathrm{e} = -\mu s \mathrm{k} \qquad (1 \cdot 2)$$

(where $\mathcal{L}(\mathrm{E}) = \mathrm{e}, \mathcal{L}(\mathrm{K}) = \mathrm{k}$, and s is the Laplace transform variable) or Fourier transformed to

$$(-\mu\epsilon\omega^2 - \mu\sigma i\omega + \nabla \times \nabla \times)\tilde{\mathrm{e}} = \mu i\omega\tilde{\mathrm{k}} \qquad (1 \cdot 3)$$

(where $\mathcal{F}(\mathrm{E}) = \tilde{\mathrm{e}}, \mathcal{F}(\mathrm{K}) = \tilde{\mathrm{k}}$, and ω is the Fourier transform variable). Note that the fundamental matrices of the above transformed equations are determined by the fundamental matrix of the vector Helmholtz equation:

$$(\nabla \times \nabla \times -\kappa^2)u = f \qquad (1 \cdot 4)$$

with $\kappa^2 = -\mu\epsilon s^2 - \mu\sigma s$ or $\mu\epsilon\omega^2 + \mu\sigma i\omega$. The fundamental matrix or Green's tensor of (1· 4) can be calculated explicitly for certain $\sigma(x)$. However, even in these cases the form of the fundamental matrix is particularly inimical to both numerical calculation and inversion of the integral transforms.

A particularly useful simplification of the forms occurs when the quasi-static approximation is used, i.e. ϵ is taken to be zero. This asymptotic approximation corresponds to the physical situation when the electromagnetic wave-fronts are a large distance away from the observation point or when the source is of low-frequency. These two viewpoints are equivalent to asserting that the terms $-\mu\epsilon\omega^2$ and $\mu\epsilon s^2$ are negligible, and that the solution to (1· 2) is approximated in some sense by the inverse transforms of the solutions to

$$(\mu\sigma s + \nabla \times \nabla\times)\mathrm{e} = -\mu s\mathrm{k} \qquad (1 \cdot 5)$$

or

$$(-\mu\sigma i\omega + \nabla \times \nabla\times)\tilde{\mathrm{e}} = \mu i\omega\tilde{\mathrm{k}}. \qquad (1 \cdot 6)$$

Carey and O'Brien [1] have shown that this is indeed the case for the Laplace transformed version of (1· 2) with certain conditions on $\sigma, \mu,$ & ϵ. They required that μ be constant on R^3, while ϵ and σ were merely required to be constant, bounded functions on open sets $\Omega_i, i = 1, ..., n, \bigcup_{i=1}^{n} \overline{\Omega_i} = \mathrm{R}^3$, with smooth boundary, i.e.

$\epsilon(x) = \epsilon_i, \sigma(x) = \sigma_i, x \in \Omega_i$ and at least one of ϵ and σ is required to be non-vanishing.

It was proved in Carey and O'Brien [1] that the solution to (1· 3) has a bound:

$$||\mathrm{e}|| \leq 2\frac{1}{|s|}(\epsilon_*\Re s + \sigma_*)^{-1}||\mathrm{k}||, \qquad (1 \cdot 7)$$

where

$$\sigma_* = \inf_{x\in\mathrm{R}^3} \sigma(x), \epsilon_* = \inf_{x\in\mathrm{R}^3} \epsilon(x), \Re s = \text{real part of } s(\text{ and } \Im s = \text{imaginary part of } s)$$

and $||\cdot||$ is the norm of $\mathrm{L}^2(\mathrm{R}^3)$. This bound is well-behaved, for a model in which $\sigma_* > 0$, as $\epsilon \to 0$ and it is therefore possible to take the quasi-static limit. It was also shown by

Carey and O'Brien, via replacing ϵ by $\lambda\epsilon$ and considering the limit as λ vanished while ϵ is held fixed, that

$$||\mathrm{E}_\lambda(t) - \mathrm{E}_0(t)|| \leq \frac{2}{\pi}\lambda\epsilon^*\sigma_*^{-2}\mu \int_C |\exp(st)||s|||\mathrm{k}(s)||ds \qquad (1 \cdot 8)$$

and assuming that $\mathrm{K} \in \mathrm{C}^\infty(\mathrm{R}^3)$ this proved that $\mathrm{E}_\lambda \to \mathrm{E}_0$ as $\lambda \to 0$ i.e. the quasi-static limit is a good $\mathrm{L}^2(\mathrm{R}^3)$ approximation of the solution for small ϵ, if σ never vanishes.

It was also shown that for $\sigma_* > 0$:

(1) away from interfaces, all fields are smooth if the source K is smooth

(2) the transverse component, $\mathrm{e_T}$ lies in H^2 and is therefore continuous

(3) the longitudinal component, $\mathrm{e_L}$, lies in L^2, but cannot lie inH^1,

(4) $n \times \mathrm{e}$ and $n \cdot \sigma\mathrm{e}$ are continuous across any any interface Γ in the sense of distributions in $\mathrm{H}^{-\frac{1}{2}}(\Gamma)$.

(Note that result (4) corresponds to the classical boundary conditions that the tangential components of e and the normal components of σe should be continuous across any interface between two regions with different constitutive parameters.)

However, in the geophysical applications of interest to us the approximation $\sigma(x) = 0$ in a halfspace of R^3 is made. Thus most of Carey and O'Brien's results are inapplicable, though (1) and (4) are still true for the half-space in which $\sigma(x) > 0$ at interfaces between volumes of differing, non-zero, conductivities.

Nonetheless, the lack of a rigorous proof is not a true obstacle in numerical calculations, as it is always the case that intuition precedes rigour. Hohmann [1,2] has considered the case of an infinite flat-earth with an insulating upper half-space, i.e.

$$\sigma(\underline{x}) = \sigma(x,y,z) = \begin{cases} 0, & z \leq 0 \\ \sigma_{ground}, & z > 0 \end{cases}$$

where $\sigma_{ground} > 0$. From the vector diffusion equation,which is obtained from (1·1) by making the quasi-static approximation, viz.:

$$\mu\sigma\partial_t \mathrm{E} + \nabla \times \nabla \times \mathrm{E} = -\mu \mathrm{K} \qquad (1 \cdot 9)$$

Hohmann derives an integral equation which is asserted to be equivalent to the vector diffusion equation:

$$\mathrm{E} = G * \mathrm{K} + G * (\sigma_V \chi_V)\mathrm{E}$$

where an asterisk denotes convolution, σ_V is the difference between the conductivities of the ambient material and the ore-body, χ_V is the characteristic volume of the ore-body and G is the pointwise limit, as ϵ vanishes, of the kernel of the Greens' operator for the transmission problem with a source in the Earth in the presence of an insulating upper half-space with ϵ non-zero. Hohmann did not give a proof in either [1] or [2] that G is in fact a fundamental solution to the transmission problem with ϵ taken to be 0 and the conductivity of the air to be 0. The advantage of this integral equation approach is that it allows the electric field to be calculated as a perturbation of the field induced in the absence of the volume of differing conductivity. A finite element scheme is used to calculate this perturbation.

In this thesis, certain aspects of Hohmann's method are investigated. In chapter 2 :

(1) The existence and uniqueness of solutions to the vector and scalar diffusion equations are investigated using a modification of the proof in Treves[1, pp. 397-405] for the equation $\dot{u} + A(t, x, \partial_x)u = g$ where A is an elliptic operator.

(2) It is shown that in the scalar case, the solution lies in a weighted L^2 space.

In chapter 3 we:

(1) Outline Johnson et al.'s [1] derivation of the correction term for the spectral expansion of the vector Helmholtz equation's Green's dyadic for a flat Earth given in Tai [1].

(2) Reduce Tai's formulae to a more compact form.

(3) Take the quasi-static limit ($\epsilon \to 0$) in Tai's formulae after assuming that $\sigma_{air} \neq 0$. We then show that as $\sigma_{air} \to 0$, for z and z' non-zero, the terms of the Green's dyadic tend, pointwise, to the form obtained by Hohmann [1], even though we have taken the limits in the opposite order.

(4) Verify Hohmann's [2] inverse Fourier transforms of the scattering terms.

In chapter 4 we:

(1) Show that Hohmann's integral equation [2] and the weak form of the vector diffusion equation are equivalent.

(2) Show that the kernel used in Hohmann's integral equation is a bounded operator on $L^2(R^3_- \times [0, T])$ and that therefore there is,
under certain conditions, a solution to the integral equation.

(3) As a check on the validity of the solution, we show that the field obtained in the air by allowing the current source to approach the air/ground interface from below is the same as that obtained by allowing the current source to descend from the air, if the source is transverse.

(4) We examine the large negative z behaviour (i.e. in the air) of the electric field generated by a source in the ground.

(5) Finally, we examine the existence of surface charges and the boundary conditions at interfaces.

The notation is standard: $W^j(\Omega) = W^{j,2}(\Omega)$ is the Sobolev space of functions in $L^2(\Omega)$, $\Omega \subset R^3$, with distributional derivatives of order less than or equal to j in $L^2(\Omega)$. An asterisk denotes convolution, ∂_l denotes partial differentiation with respect to l, $< \cdot, \cdot >_H$ is the inner product in the Hilbert space H, where H may be $L^2(\Omega), (L^2(\Omega))^3$, etc., depending upon the context.

Chapter 2

Existence and Uniqueness of the Solution of the Generalised Diffusion Equation

2.1 Introduction

In this chapter the existence and uniqueness of solutions to the vector and scalar diffusion equations are investigated. We will denote by $\mathcal{O}$, depending on context, either the distributional $-\nabla^2$ (vector and scalar) or the distributional $\nabla \times \nabla \times$. That is, we study the equation $\sigma\dot{u} + \mathcal{O}u = g$. Note that σ must be bounded below by a positive constant in the case $\mathcal{O} = \nabla \times \nabla\times$ since the method we use to prove the existence and uniqueness of the solution fails when σ vanishes in a half-space. When $\mathcal{O} = -\nabla^2$ (scalar or vector) it is possible to allow σ to vanish in a half-space.

The proof of the existence and uniqueness of the solution is a modification of the proof in Treves[1, pp. 397-405] for the equation $\dot{u} + A(t, x, \partial_x)u = g$ where A is an elliptic operator. The essence of the proof is , after the definition of the spaces we seek the solution in (analogues of the Sobolev spaces used in, for example, Dirichlet problems on bounded domains), to

(1) Show that the bi-linear form corresponding to the generalised diffusion equation gives rise to a coercive , continuous map between certain spaces, which arise naturally from the variational approach to the problem using Lions' generalisation of the Lax - Milgram lemma.

(2) Show that this variational method gives a solution of the weak form of the generalised diffusion equation, which fulfills the initial conditions and is unique.

2.2 The Generalised Diffusion Equation

The vector/scalar generalised diffusion equation is

$$\sigma\dot{u}(\underline{x},t) + \mathcal{O}u(\underline{x},t) = g(\underline{x},t) \tag{2 ·1}$$

with initial condition $u(\underline{x},0) = 0, \forall \underline{x} \in \mathrm{R}^3$ where

$$g \in \mathrm{L}^2(\mathrm{R}^3 \times [0,T]; \mathrm{R}^n),$$

$$\sigma(x) = \begin{cases} \sigma_{air}, & x \in \mathrm{R}^3_+ \\ \sigma_{ground}, & x \in \mathrm{R}^3_- \end{cases},$$

where σ_{air} is a non-negative constant, σ_{ground} is an arbitrary positive, bounded function of x with lower bound $\sigma_* > 0$, the integer n denotes 1 when $\mathcal{O}$ is scalar and 3 otherwise, $\underline{x} = (x,y,z)$ and T is a finite, non-negative real number. Note that all functions are real-valued and that all spaces are over the reals. Firstly, we replace u by $\exp(-\tau t)u$, where τ is a positive real number. We will eventually restrict τ to be greater than some $\tau_0 \geq 0$, to ensure that an inequality of the form $\|h\|_{\mathrm{M}} \leq |A_{(\tau,\mathcal{O})}(h,h)|$ holds, where $A_{(\tau,\mathcal{O})}$ is defined below, M is a Banach space, which will be defined later, and $h \in \mathrm{M}$. In the case $\sigma_{air} \neq 0$ this substitution ensures that we obtain a contraction semi-group.

Thus, upon making the above substitution, (2· 1) becomes

$$\sigma\dot{u} + \tau\sigma u + \mathcal{O}u = \exp(-\tau t)g = \Gamma. \tag{2 ·2}$$

We define

$$B_{\mathcal{O}}(u,v) = < \mathcal{O}^{\frac{1}{2}}u, \mathcal{O}^{\frac{1}{2}}v >_{\mathrm{L}^2(\mathrm{R}^3)} .$$

where $< \cdot,\cdot >_{\mathrm{L}^2(\mathrm{R}^3)}$ indicates the inner product in $(\mathrm{L}^2(\mathrm{R}^3))^n$. Note that $\mathcal{O}^{\frac{1}{2}}$ exists since the operators we are dealing with are positive, self-adjoint and closed (For $-\nabla^2$ this is a standard result. For $\nabla \times \nabla\times$ see Carey and O'Brien [1].) For the cases of interest to us :

$$B_{-\nabla^2}(u,v) = \sum_{i=1}^{3} < \partial_{x_i} u, \partial_{x_i} v >_{L^2(R^3)}$$

($x_1 = x, x_2 = y, x_3 = z$) and

$$B_{\nabla\times\nabla\times}(u,v) = \sum_{i=1}^{3} < \partial_{x_i}(1-Q)u, \partial_{x_i}(1-Q)v >_{L^2(R^3)}$$

where Q is a pseudo- differential operator defined by (Carey and O'Brien [1])

$$\nabla \times \nabla\times = -\nabla^2(1-Q).$$

That is,

$$Q_{ij} = \frac{\partial_{x_i}\partial_{x_j}}{\nabla^2}.$$

We now define the spaces $\Phi_{\mathcal{O}}$ and $E_{\mathcal{O}}$ in which we seek the solution to (2·1).

$\Phi_{\mathcal{O}}$ is the completion of $(C_c^\infty(R^3))^n$ with respect to the topology induced by the norm corresponding to the following inner product:

$$\langle u,v\rangle_{\Phi_{\mathcal{O}}} =< \sigma u, v >_{L^2(R^3)} + B_{\mathcal{O}}(u,v). \qquad (2\cdot 3)$$

This norm is

$$||u||_{\Phi_{\mathcal{O}}} = \sqrt{< \sigma u, u >_{L^2(R^3)} + B_{\mathcal{O}}(u,u)}. \qquad (2\cdot 4)$$

This is a standard construction. It is easily seen that $< \cdot,\cdot >_{\Phi_{\mathcal{O}}}$ satisfies the Cauchy-Schwarz inequality since for $u, v \in \Phi_{\mathcal{O}}$

$$|B_{\mathcal{O}}(u,v)| \leq ||\mathcal{O}^{\frac{1}{2}}u||_{L^2(R^3)}||\mathcal{O}^{\frac{1}{2}}v||_{L^2(R^3)}$$

$$\leq ||u||_{\Phi_{\mathcal{O}}} ||v||_{\Phi_{\mathcal{O}}}$$

and

$$| < \sigma u, v >_{\mathrm{L}^2(\mathrm{R}^3)} | \leq ||\sigma^{\frac{1}{2}} u||_{\mathrm{L}^2(\mathrm{R}^3)} ||\sigma^{\frac{1}{2}} v||_{\mathrm{L}^2(\mathrm{R}^3)}$$

$$= < \sigma u, u >^{\frac{1}{2}}_{\mathrm{L}^2(\mathrm{R}^3)} < \sigma v, v >^{\frac{1}{2}}_{\mathrm{L}^2(\mathrm{R}^3)} .$$

Thus, $< \cdot, \cdot >_{\Phi_{\mathcal{O}}}$ is an inner product, since it is clearly bi-linear.

Note that when $\mathcal{O} = \nabla \times \nabla \times$ and $\sigma_{air} = 0$ a function u which has its support in R^3_+ and is longitudinal must satisfy $||u||_{\Phi_{\nabla\times\nabla\times}} = 0$, i.e. the inner product merely induces a semi-norm. We exclude this case from our consideration.

$\mathrm{E}_{\mathcal{O}}$ is defined as the completion of $\mathrm{L}^2([0,T]; (\mathrm{C}^\infty_c(\mathrm{R}^3))^3)$ in the norm

$$||u||_{\mathrm{E}_{\mathcal{O}}} = \sqrt{\int_0^T ||u||^2_{\Phi_{\mathcal{O}}} + ||\sigma \frac{du}{dt}||^2_{\Phi'_{\mathcal{O}}} dt.} \tag{2 ·5}$$

(If H is a Banach space ,H$'$ denotes its dual)

That is

$$\mathrm{E}_{\mathcal{O}} = \{u \in \mathrm{L}^2([0,T]; \Phi_{\mathcal{O}}) | \ \sigma \frac{du}{dt} \in \mathrm{L}^2([0,T]; \Phi'_{\mathcal{O}})\} \tag{2 ·6}$$

We now define the bi-linear form on $\mathrm{E}_{\mathcal{O}} \times \mathrm{E}_{\mathcal{O}}$ corresponding to the weak form of (2·1):

$$A_{(\tau,\mathcal{O})}(u,v) = \int_0^T - < u, \sigma \dot{v} > dt + \int_0^T < \tau \sigma u, v >_{\mathrm{L}^2(\mathrm{R}^3)} dt + \int_0^T B_{\mathcal{O}}(u,v) dt \tag{2 ·7}$$

where the notation $< \cdot, \cdot >$ without a subscript denotes the bracket of duality.

The following result Carey [1] shows that $\Phi_{\mathcal{O}}$ is a space of functions

Theorem

Let W be the space of functions u with

$$\|u\|_{\mathrm{W}} = \int_{\mathrm{R}^3} \frac{u^2}{(1+\theta(z)z)^2} dxdydz < \infty. \qquad (2 \cdot 8)$$

(θ is the Heaviside function.)Then, if $\sigma_{air} \neq 0$ or in the case where $\sigma_{air} = 0$, $\mathcal{O} = -\nabla^2$ (scalar), $\Phi_{\mathcal{O}} \subseteq \mathrm{W}$

Proof

Note that if $\sigma_{air} \neq 0$ then $\Phi_{\mathcal{O}} \subset \mathrm{L}^2(\mathrm{R}^3) \subset \mathrm{W}$, so we need only consider the case $\sigma_{air} = 0, \mathcal{O} = -\nabla^2$ (scalar). Let $u \in \mathrm{C}_c^\infty(\mathrm{R}^3)$ then

$$\partial_z \frac{u^2}{(1+\theta(z)z)} = \frac{-\theta(z)u^2}{(1+\theta(z)z)^2} + \frac{\partial_z u^2}{(1+\theta(z)z)}$$

therefore

$$\int_{-\infty}^{\infty} \frac{\theta(z)u^2}{(1+\theta(z)z)^2} dz = \int_{-\infty}^{\infty} \frac{\partial_z u^2}{(1+\theta(z)z)} dz$$

as u and thus u^2 is of compact support. Now

$$\int_{\mathrm{R}^3} \frac{\theta(z)u^2}{(1+\theta(z)z)^2} dx\ dy\ dz = \int_{\mathrm{R}^3} \frac{\partial_z u^2}{(1+\theta(z)z)} dx\ dy\ dz$$

$$\leq 2\left(\int_{\mathrm{R}^3} \frac{u^2}{(1+\theta(z)z)^2} dx\ dy\ dz\right)^{\frac{1}{2}} \left(\int_{\mathrm{R}^3} (\partial_z u)^2 dx\ dy\ dz\right)^{\frac{1}{2}}$$

that is,

$$\int_{\mathrm{R}^3_+} \frac{u^2}{(1+\theta(z)z)^2} dx\ dy\ dz \leq 2\|u\|_W \|\nabla u\|_{\mathrm{L}^2(\mathrm{R}^3)}. \qquad (2 \cdot 9)$$

(Since $\int_{\mathrm{R}^3} u^2 \geq \int_{\mathrm{R}^3_+} u^2$.)

Let

$$\alpha^2 = \int_{\mathrm{R}^3_+} \frac{u^2}{(1+\theta(z)z)^2} dx\ dy\ dz$$

$$\beta^2 = \int_{\mathrm{R}^3_-} u^2 dx\ dy\ dz$$

$$\gamma = \|\nabla u\|_{\mathrm{L}^2(\mathrm{R}^3)}$$

then (2· 9) implies

$$\alpha^2 \leq 2(\alpha^2 + \beta^2)^{\frac{1}{2}}\gamma \qquad (2\cdot 10)$$

which implies

$$\alpha^4 \leq 4(\alpha^2 + \beta^2)\gamma^2$$

and, considering the above equation as a quadratic in α^2 it is seen that:

$$\alpha^2 \leq 2\gamma^2(1 + \sqrt{\beta^2 + \gamma^2}) \qquad (2\cdot 11)$$

$$\leq 2\gamma^2 + 2\sqrt{\beta^4 + 2\beta^2\gamma^2 + \gamma^4}$$

$$= 4\gamma^2 + 2\beta^2$$

and thus

$$\alpha^2 + \beta^2 \leq 4(\beta^2 + \gamma^2). \qquad (2\cdot 12)$$

Now, we define σ_* as $\inf_{\underline{x}\in\mathrm{R}^3_-} \sigma(\underline{x})$. Thus, if $\sigma_* \leq 1$ then

$$\beta^2 + \gamma^2 \leq \frac{1}{\sigma_*}(\sigma_*\beta^2 + \gamma^2)$$

and if $\sigma_* > 1$

$$\beta^2 + \gamma^2 \leq \sigma_*\beta^2 + \gamma^2.$$

Noting that $||\cdot||_{\Phi_O}$ is equivalent, when $\sigma_{air} = 0$, to

$$\sqrt{\sigma_*||\cdot||^2_{\mathrm{L}^2(\mathrm{R}^3_-)} + B_O(\cdot,\cdot)}$$

it can be seen that there is a constant K_σ such that

$$||u||_W \leq K_\sigma ||u||_{\Phi_\mathcal{O}}, \tag{2 ·13}$$

that is, $\Phi_\mathcal{O} \subseteq$ W.

We now discuss the dual of $\Phi_\mathcal{O}$.

For $\mathcal{O} = -\nabla^2$ and $\sigma_{air} \neq 0$ we have that $\Phi_\mathcal{O}$ is just the Sobolev space $(W^{1,2}(\mathrm{R}^3))^n$ and therefore the dual of $\Phi_\mathcal{O}$ is merely $(W^{-1,2}(\mathrm{R}^3))^n$. However, if $\sigma_{air} = 0$, the situation is more complicated. In this case, $\sigma\dot{v}$ has support only in the lower half-space and therefore we are interested in the duality between $\Phi_\mathcal{O}$ and $\Phi'_\mathcal{O}$ only for the subspace of $\Phi_\mathcal{O}$ consisting of functions with support in R^3_-. However, this subspace is $(W^{1,2}(\mathrm{R}^3_-))^n$ and hence its dual is $(W^{-1,2}(\mathrm{R}^3_-))^n$. Thus, the space $\mathrm{E}_\mathcal{O}$ is the space of functions v such that

(1) $v \in \mathrm{L}^2([0,T];\Phi_\mathcal{O})$

(2) $\sigma\dot{v} \in \mathrm{L}^2([0,T];(W^{-1,2}(\mathrm{R}^3_-))^n)$.

For $\mathcal{O} = \nabla\times\nabla\times$ the situation is slightly more complicated. Since $\Phi_\mathcal{O} \in (\mathrm{L}^2(\mathrm{R}^3))^3$ if $\sigma_{air} \neq 0$ we can resolve $u \in \Phi_\mathcal{O}$ into its transverse and longitudinal components which we will denote by $u_\mathcal{T}$ and $u_\mathcal{L}$ respectively. Note that

$$\begin{aligned} ||u_\mathcal{T}||^2_{\Phi_\mathcal{O}} &= <\sigma u_\mathcal{T}, u_\mathcal{T}>_{\mathrm{L}^2(\mathrm{R}^3)} + <\nabla\times u_\mathcal{T}, \nabla\times u_\mathcal{T}>_{\mathrm{L}^2(\mathrm{R}^3)} \\ ||u_\mathcal{L}||^2_{\Phi_\mathcal{O}} &= <\sigma u_\mathcal{L}, u_\mathcal{L}>_{\mathrm{L}^2(\mathrm{R}^3)} . \end{aligned} \tag{2 ·14}$$

Now for $\sigma_{air} \neq 0$, (2· 14) implies that

$$u_\mathcal{T} \in (W^{1,2}(\mathrm{R}^3))^3$$

and

$$u_{\mathcal{L}} \in (\mathrm{L}^2(\mathrm{R}^3))^3.$$

That is, $\Phi_{\nabla\times\nabla\times} = (W_{\mathcal{T}}^{1,2}(\mathrm{R}^3))^3 \oplus (\mathrm{L}^2_{\mathcal{L}}(\mathrm{R}^3))^3$ ($\mathcal{T}, \mathcal{L}$ indicate the transverse and longitudinal subspaces respectively) and therefore $\Phi'_{\nabla\times\nabla\times} = ((W_{\mathcal{T}}^{1,2}(\mathrm{R}^3))^3)' \oplus ((\mathrm{L}^2_{\mathcal{L}}(\mathrm{R}^3))^3)'$.

It is also of interest to consider in exactly which space the solution to (2·1) lies. Now when $\mathcal{O} = \nabla \times \nabla\times$, (2·1) is

$$\sigma\dot{u}(\underline{x}, t) + \nabla \times \nabla \times u(\underline{x}, t) = g(\underline{x}, t)$$

where $g \in \mathrm{L}^2(\mathrm{R}^3 \times [0, T])$. (We ignore initial conditions for the moment and consider the case $\mathcal{O} = \nabla \times \nabla\times, \sigma_{air} = 0$, even though we have no existence proof in this case.) Now, for (2·1) to be meaningful it is necessary that $\nabla \times u$ &$\nabla \times \nabla \times u$ be elements of $(\mathrm{L}^2(\mathrm{R}^3))^3$. That is,

$$u_{\mathcal{T}} \in \begin{cases} (W^{1,2}(\mathrm{R}^3))^3, & \sigma_{air} \neq 0 \\ (W^{1,2}(\mathrm{R}^3_-))^3, & \sigma_{air} = 0 \end{cases}$$

which implies that

$$\nabla \times \nabla \times u_{\mathcal{T}} = -\nabla^2 u_{\mathcal{T}} \in \begin{cases} (W^{-1,2}(\mathrm{R}^3))^3, & \sigma_{air} \neq 0 \\ (W^{-1,2}(\mathrm{R}^3_-))^3, & \sigma_{air} = 0. \end{cases}$$

This,in view of (2·1), implies that

$$\sigma\dot{u}_{\mathcal{T}} \in \begin{cases} (W^{-1,2}(\mathrm{R}^3))^3, & \sigma_{air} \neq 0 \\ (W^{-1,2}(\mathrm{R}^3_-))^3, & \sigma_{air} = 0 \end{cases}$$

We also have from (2·1) that

$$\sigma\dot{u}_{\mathcal{L}} \in \begin{cases} (\mathrm{L}^2(\mathrm{R}^3))^3, & \sigma_{air} \neq 0 \\ (\mathrm{L}^2(\mathrm{R}^3_-))^3, & \sigma_{air} = 0. \end{cases}$$

That is

$$u \in (W_T^{1,2}(\Omega))^3 \oplus (\mathrm{L}^2_{\mathcal{L}}(\Omega))^3 \ \& \ \sigma\dot{u} \in ((W_T^{1,2}(\Omega))^3)' \oplus ((\mathrm{L}^2_{\mathcal{L}}(\Omega))^3)'$$

where $\Omega = \mathrm{R}^3_-$ (R^3) when $\sigma_{air} = 0$ $(\neq 0)$.

By a similar argument it can be seen that for $\mathcal{O} = -\nabla^2$ the solution $u \in (W^{1,2}(\Omega))^n$ and that $\sigma\dot{u} \in (W^{-1,2}(\Omega))^n$.

2.3 Existence of Solution

We now prove the existence of a solution to the generalised diffusion equation. To do this we consider a space of the form $\mathcal{G} \times \mathcal{I}$ where $\mathrm{E}_{\mathcal{O}} \subseteq \mathcal{G}$ and $\mathcal{I}$ contains the function corresponding to our choice of initial condition. A subset of $\mathcal{G} \times \mathcal{I}$ is chosen so that:

(1) the pair (v, v_0) corresponds to a choice of a function $v \in \mathrm{E}_{\mathcal{O}}$ with $\lim_{t \to 0} ||v - v_0||_{\mathcal{I}} = 0$

(2) $||(v, v_0)||_{\mathcal{G}\times\mathcal{I}} \leq |A_{(\tau,\mathcal{O})}(v, v)|$.

Lions' generalisation of the Lax-Milgram lemma is then used to prove that there exists a unique solution to the weak differential equation which satisfies the initial conditions.

Firstly we define $\mathcal{I}$ to be the completion of $(\mathrm{C}_c^\infty(\mathrm{R}^3))^n$ with respect to the norm induced by the inner product $< \sigma u, v >_{\mathrm{L}^2(\mathrm{R}^3)}$. We take $\mathcal{G} = \mathrm{L}^2([0,T]; \Phi_{\mathcal{O}})$ and thus $\mathcal{G} \times \mathcal{I} =M= \mathrm{L}^2([0,T]; \Phi_{\mathcal{O}}) \times \mathcal{I}$ with norm

$$||(u, u_0)||_{\mathrm{M}} = \sqrt{\int_0^T ||u||^2_{\Phi_{\mathcal{O}}} dt + < \sigma u_0, u_0 >_{\mathrm{L}^2(\mathrm{R}^3)}}.$$

We choose as the subset of M, $\hat{h} \subset$ M as

$$\{(v, v_0) \in \mathrm{M} | v \in \mathrm{E}_{\mathcal{O}}, \lim_{t \to T} ||v||_{\mathcal{I}} = 0 \& \lim_{t \to 0} ||v - v_0||_{\mathcal{I}} = 0\}.$$

If $\sigma_{air} = 0$ we make the additional restriction upon $\mathcal{I}$ that that its elements have support in R^3_-, i.e. it is the completion of $(\mathrm{C}_c^\infty(\mathrm{R}^3_-))^n$ with respect to $< \sigma\cdot, \cdot >_{\mathrm{L}^2(\mathrm{R}^3)}$. Now ,

$$2\int_0^T < \sigma\dot{u}, u > dt \tag{2 $\cdot$ 15}$$

$$= \int_0^T \frac{d}{dt} < \sigma u, u >_{\mathrm{L}^2(\mathrm{R}^3)} dt$$

$$= < \sigma u_T, u_T >_{\mathrm{L}^2(\mathrm{R}^3)} - < \sigma u_0, u_0 >_{\mathrm{L}^2(\mathrm{R}^3)}$$

& thus

$$2A_{(\tau,\mathcal{O})}(u,u) + < \sigma u_T, u_T >_{\mathrm{L}^2(\mathrm{R}^3)}$$

$$= < \sigma u_0, u_0 >_{\mathrm{L}^2(\mathrm{R}^3)} + 2(\int_0^T [< \tau\sigma u, u >_{\mathrm{L}^2(\mathrm{R}^3)} + B_{\mathcal{O}}(u,u)dt]).$$

Assume without loss of generality that

$$\tau \geq 1$$

thus we have the energy inequality :

$$2A_{(\tau,\mathcal{O})}(u,u) + < \sigma u_T, u_T >_{\mathrm{L}^2(\mathrm{R}^3)} \geq < \sigma u_0, u_0 >_{\mathrm{L}^2(\mathrm{R}^3)}$$

$$+2\int_0^T < \sigma u, u > + B_{\mathcal{O}}(u,u)dt.$$

Hence for $h \in \hat{h}$,

$$||h||^2_{\mathrm{M}} \leq |A_{(\tau,\mathcal{O})}(h,h)|.$$

Note that $A_{(\tau,\mathcal{O})}(w,h)$ is clearly a continuous linear functional on M for every fixed $(h, h_0) \in \hat{h}$. We now use Lions' generalisation of the Lax-Milgram Lemma (Treves [1],p. 403).

Lemma

Let $\mathbf{E}$ be a Hilbert space, $\tilde{h}$ a linear subspace of $\mathbf{E}$, $\mathbf{U}(w,h)$ a sesquilinear functional on $\mathbf{E}\times\tilde{h}$ having the following properties :

(a) for each fixed $h \in \tilde{h}, w \rightarrow \mathbf{U}(w,h)$ is a continuous linear functional on E.

(b) there is a $c_0 > 0$ such that , for every $h \in \tilde{h}$,

$$c_0||h||^2_{\mathbf{E}} \leq |\mathbf{U}(h,h)|$$

Conclusion:

There is a bounded linear map G of the antidual $\overline{\mathbf{E}}'$ of $\mathbf{E}$ into $\mathbf{E}$, with norm $\leq c_0^{-1}$, such that for every continuous linear functional λ on $\mathbf{E}$,

$$\mathbf{U}(G\lambda, h) = \lambda(h), \forall h \in \tilde{h}.$$

So let our continuous functional λ be

$$v \mapsto \int_0^T < \Gamma, v >_{L^2(R^3)} dt,$$

choose our $\tilde{h}$ to be $\hat{h}$ and $\mathbf{E}$ to be M. The energy inequality for $A_{(\tau,\mathcal{O})}$ on $\hat{h}$ and the fact that $(w,w_0) \mapsto A_{(\tau,\mathcal{O})}(w,h)$ is continuous on M for fixed $(h,h_0) \in \hat{h}, (w,w_0) \in$ M, shows that we can take $\mathbf{U}(u,v)$=$A_{(\tau,\mathcal{O})}(u,v)$ and apply the lemma to obtain:

$$\exists (V,V_0) \in \text{M such that } A_{(\tau,\mathcal{O})}(V,h) = \int_0^T < \Gamma, h >_{L^2(R^3)} dt \forall h \in \hat{h}$$

Choose

$$h \in C_c^\infty([0,T];\Phi_{\mathcal{O}})(\subset \hat{h} \subset M)$$

so

$$\int_0^T [< -\sigma V, \dot{h} >_{L^2(R^3)} + < \sigma\tau V, h >_{L^2(R^3)} + B_{\mathcal{O}}(V,h)]dt = \int_0^T < \Gamma, h >_{L^2(R^3)} dt$$

and therefore,

$$\sigma\dot{V} + (\tau\sigma V + \mathcal{O}V) = \Gamma \tag{2 $\cdot$ 16}$$

in sense of $\Phi'_{\mathcal{O}}$ valued distributions on$[0,T]$.

2.4 Initial Conditions

First we prove a technical result.

Lemma

The natural injection

$$C^\infty([0,T];\Phi_{\mathcal{O}}) \to C^0([0,T];\mathcal{I})$$

can be extended to a continuous map

$$E_{\mathcal{O}} \to C^0([0,T];\mathcal{I})$$

(We equip $C^0([0,T];\mathcal{I})$ with the natural norm

$$\sup_{0<t<T} \|u_t\|_{\mathcal{I}}$$

where $u_{t_0} = u(\underline{x},t)|_{t=t_0}$.)

Proof

Let $u \in E_{\mathcal{O}}$. We define $\tilde{u}$ on $(-T,T)$ by

$$\tilde{u} = \begin{cases} u_t, & t > 0 \\ u_{-t} & t < 0 \end{cases}.$$

It is clear that the map $u \mapsto \tilde{u}$ is a continuous injection , with norm 2 , from $\mathrm{E}_{\mathcal{O}} = \mathrm{E}_{\mathcal{O}}([0,T])$ to $\mathrm{E}_{\mathcal{O}}((-T,T))$.Consider now

$$\alpha \in \mathrm{C}^{\infty}(R), \alpha_t = \begin{cases} 0, & t < -T \\ 1, & t > 0. \end{cases}$$

Let $u \in \mathrm{C}^{\infty}([0,T]); \Phi_{\mathcal{O}})$.

Note that when $\sigma_{air} = 0$ we are really dealing with a truncation of u,

$$\mathrm{Trunc}(u)(\underline{x}) = \begin{cases} u(\underline{x}), & z \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

since we require the elements of $\mathcal{I}$ to have support in R^3_-. However, since $< \sigma u, u >_{\mathrm{L}^2(\mathrm{R}^3)} = < \sigma \mathrm{Trunc}(u), \mathrm{Trunc}(u) >_{\mathrm{L}^2(\mathrm{R}^3)}$ this is not a problem.

We now consider $< \sigma(\alpha\tilde{u}), (\alpha\tilde{u}) >_{\mathrm{L}^2(\mathrm{R}^3)}$

$$= 2\int_{-T}^{t} < (\alpha\tilde{u})_s, \sigma(\dot{\alpha\tilde{u}})_s > ds$$

$$\leq \int_{-T}^{T} ||(\alpha\tilde{u})_s||^2_{\Phi_{\mathcal{O}}} + ||\sigma(\dot{\alpha\tilde{u}})_s||^2_{\Phi'_{\mathcal{O}}} ds$$

$$\leq C_\alpha \int_{-T}^{T} ||\tilde{u}_s||^2_{\Phi_{\mathcal{O}}} + ||\sigma\dot{\tilde{u}}_s||^2_{\Phi'_{\mathcal{O}}} ds$$

$$\leq 2C_\alpha ||u||_{\mathrm{E}_{\mathcal{O}}}.$$

Since $\alpha\tilde{u}$ restricted to $[0,T]$ equals u , we see that the natural injection

$$\mathrm{C}^{\infty}([0,T]; \Phi_{\mathcal{O}}) \to \mathrm{C}^0([0,T]; \mathcal{I})$$

is continuous and thus has a unique continuous extension to all of $E_{\mathcal{O}}$, since $C^{\infty}([0,T];\Phi_{\mathcal{O}})$ is, clearly, dense in $E_{\mathcal{O}}$. (The density can be proved by using an approximate identity argument.)

Now (2· 16) implies

$$\sigma\dot{V} \in L^2([0,T];\Phi'_{\mathcal{O}})$$

i.e. $V \in E_{\mathcal{O}}$, hence we can consider V as a continuous function on $[0,T]$, valued in $\mathcal{I}$.

Now ,

$$\int_0^T < -\sigma V, \dot{h} >_{L^2(R^3)} dt =< \sigma V, h >_{L^2(R^3)} |_{t=0} + \int_0^T < \sigma\dot{V}, h >_{L^2(R^3)} dt$$

So by (2· 16)

$$< \sigma V, h >_{L^2(R^3)} |_{t=0} = 0 \forall h \in \hat{h}$$

i.e.

$$\sigma V_0 = 0 \text{ a.e. in } R^3$$

& therefore so doesV_0, since if σ has support only in R^3_- we have chosen $\mathcal{I}$ to ensure that the initial data V_0 is zero in R^3_+.

2.5 Uniqueness of the Solution

If we have two solutions $u, v \in E_{\mathcal{O}}$ to the above problem then $W = u - v$ is an element of $E_{\mathcal{O}}$ which satisfies

$$\sigma\dot{W} + (\tau\sigma + \mathcal{O})W = 0 \qquad (2 \cdot 17)$$

$$W_0 = 0 \text{ in } R^3. \qquad (2 \cdot 18)$$

We now show, using the the bilinear form $A'_{(\tau,\mathcal{O})}$ defined on $E_{\mathcal{O}} \times E_{\mathcal{O}}$ by

$$A'_{(\tau,\mathcal{O})}(u,v) = \int_0^T < \sigma\dot{u}, v > + < \tau\sigma u, v > + B_{\mathcal{O}}(u,v)dt \tag{2 $\cdot$ 19}$$

that $W \equiv 0$ in $L^2([0,T];\Phi_{\mathcal{O}})$.

Firstly, by a similar argument to that for $A_{(\tau,\mathcal{O})}$ above we have:

$$2A'_{(\tau,\mathcal{O})}(W,W) + < \sigma W_0, W_0 >_{L^2(R^3)} \geq < \sigma W_T, W_T >_{L^2(R^3)} + 2\|W\|^2_{L^2([0,T];\Phi_{\mathcal{O}})}. \tag{2 $\cdot$ 20}$$

Now

$$A'_{(\tau,\mathcal{O})}(W,W) = \int_0^T < \sigma\dot{W} + (\tau\sigma + \mathcal{O})W, W > dt = 0 \tag{2 $\cdot$ 21}$$

by (2· 17) and

$$< \sigma W_0, W_0 >_{L^2(R^3)} = 0 \tag{2 $\cdot$ 22}$$

by (2· 18). Therefore

$$0 \geq < \sigma W_T, W_T >_{L^2(R^3)} + 2\|W\|^2_{L^2([0,T];\Phi_{\mathcal{O}})} \geq 0 \tag{2 $\cdot$ 23}$$

and hence $W \equiv 0$ in $L^2([0,T];\Phi_{\mathcal{O}})$.

Chapter 3
Calculation of the Green's Dyadic for the Vector Helmholtz Equation.

3.1 Introduction

In this chapter we:

(1) Outline Johnson et al.'s [1] derivation of the correction term for the spectral expansion of the vector Helmholtz equation's Green's dyadic for a flat Earth given in Tai [1].

(2) Reduce Tai's formulae to a more compact form.

(3) Take the quasi-static limit ($\epsilon \to 0$) in Tai's formulae after assuming that $\sigma_{air} \neq 0$. (This is justified by the results of Carey and O'Brien[1].) We then show that as $\sigma_{air} \to 0$, for z and z' non-zero, the terms of the Green's dyadic tend, pointwise, to the form obtained by Hohmann [1], even though we have taken the limits in the opposite order.

(4) Verify Hohmann's [2] inverse Fourier transforms of the scattering terms.

3.2 Derivation of the Correction Term

In Tai[1] it was wrongly assumed that the eigenfunctions corresponding to the TE (Transverse Electric) and TM (Transverse Magnetic) modes formed a complete set. Johnson et al. [1] remark that this was probably inspired by Morse and Feshbach's [1,p. 1781]comments which implied that away from the source the transverse and longitudinal components of the unit delta dyadic ($I\delta(\underline{x})$) vanish identically. This error was further compounded by the use of a form for

$$\int_{-\infty}^{\infty} \frac{e^{ih(z-z')}}{h^2+\lambda^2-k_1^2}dh \left(= \pi\frac{e^{-\sqrt{\lambda^2-k_1^2}|z-z'|}}{\sqrt{\lambda^2-k_1^2}}\right) \tag{3 $\cdot$ 1}$$

which obscured the generation of a delta-function term by an application of $\partial_z\partial_{z'}$ to the above term. The existence of such a singular term reveals the incompleteness of the eigenfunctions corresponding to the TE and TM modes. The correction term, viz. $\frac{-1}{k_0^2}\hat{z}\hat{z}\delta(\underline{x}-\underline{x}')$ is not the total contribution of the longitudinal eigenfunctions. (This is obvious since $\nabla\times\delta(\underline{x}-\underline{x}')\hat{z}\hat{z}\neq 0$.) Rather, it is what remains of the contribution from the longitudinal eigenfunctions after partially integrating the spectral expansion with respect to one of the wave-numbers and a cancellation with part of the contribution from the eigenfunctions corresponding to the TM mode.

We now outline Johnson et al.'s [1] calculation of the correction term.

Firstly, the eigenfunctions of the vector Helmholtz equation in Cartesian co-ordinates are

$$\begin{aligned} \mathrm{L}(\underline{x}) &= \nabla\psi \\ \mathrm{M}(\underline{x}) &= \nabla\times(\hat{z}\psi) \\ \mathrm{N}(\underline{x}) &= \frac{\nabla\times\mathrm{M}(\underline{x})}{(k_x^2+k_y^2+k_z^2)^{\frac{1}{2}}} \\ \psi(\underline{x}) &= e^{-ik\cdot\underline{x}}, k = k_x\hat{x}+k_y\hat{y}+k_z\hat{z} \\ \underline{x} &= (x,y,z) \end{aligned} \tag{3 $\cdot$ 2}$$

and k_x, k_y, k_z are the wave numbers in the x, y, z directions respectively.

The free space Green's dyadic satisfies:

$$\nabla \times \nabla \times G(\underline{x}|\underline{x}') - k_0^2 G(\underline{x}|\underline{x}') = I\delta(\underline{x} - \underline{x}') \qquad (3 \cdot 3)$$

and the radiation condition at infinity:

$$\lim_{R\to\infty} R[\nabla \times G(\underline{x}|\underline{x}') - ik_0\hat{\mathrm{R}} \times G(\underline{x}|\underline{x}')] = 0 \qquad (3 \cdot 4)$$

where $R = |\underline{x} - \underline{x}'|$ and $\hat{\mathrm{R}}$ is the unit vector in the R direction in spherical co-ordinates.

The longitudinal component of the free space Green's dyadic is given by

$$\frac{-1}{k_0^2}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{L}(\underline{x})\mathrm{L}^*(\underline{x}')}{k_x^2+k_y^2+k_z^2}dk_x dk_y dk_z \qquad (3 \cdot 5)$$

$$= \frac{-1}{k_0^2}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\nabla\nabla'\psi(\underline{x})\psi^*(\underline{x}')}{k_x^2+k_y^2+k_z^2}dk_x dk_y dk_z.$$

(* denotes complex conjugation and ∇', etc. indicates differentiation with respect to the primed co-ordinates.) We now integrate with respect to k_z, using (3· -1)

$$= \frac{-1}{k_0^2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla\nabla'\frac{e^{-ik_x(x-x')}e^{-ik_y(y-y')}e^{-k_c|z-z'|}}{2k_c}dk_x dk_y.$$

Differentiating, the following is obtained:

$$\frac{-1}{k_0^2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} k_x^2 & k_x k_y & -ik_x k_{cs} \\ k_x k_y & k_y^2 & -ik_y k_{cs} \\ -ik_x k_{cs} & -ik_y k_{cs} & -k_c^2 \end{pmatrix}$$

$$\cdot\frac{e^{-ik_x(x-x')}e^{-ik_y(y-y')}e^{-k_c|z-z'|}}{2k_c}dk_x dk_y - \frac{1}{k_0^2}\hat{z}\hat{z}\delta(\underline{x}-\underline{x}')$$

where

$$k_c = \sqrt{k_x^2+k_y^2},\ k_{cs} = k_c\mathrm{Sign}(z-z')$$

and we take, for convergence, the branch of the square root function with $\Im z^{1/2} \geq 0$. The transverse component of the free space Green's dyadic is given by:

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\nabla\times(\psi(\underline{x})\hat{z})\nabla'\times(\psi^*(\underline{x}')\hat{z})}{(k_x^2+k_y^2+k_z^2)(k_x^2+k_y^2+k_z^2-k_0^2)}+\frac{\nabla\times\nabla\times(\psi(\underline{x})\hat{z})\nabla'\times\nabla'\times(\psi^*(\underline{x}')\hat{z})}{(k_x^2+k_y^2+k_z^2)(k_x^2+k_y^2+k_z^2-k_0^2)}\right)dk_x dk_y dk_z. \tag{3 $\cdot$ 6}$$

The second term equals

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-\frac{1}{(k_x^2+k_y^2+k_z^2)k_0^2}+\frac{1}{k_0^2(k_x^2+k_y^2+k_z^2-k_0^2)}\right) \tag{3 $\cdot$ 7}$$

$$\begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'} & \partial_z\partial_{z'}\partial_x\partial_{y'} & -(\partial_{x'}^2+\partial_{y'}^2)\partial_z\partial_x \\ \partial_z\partial_{z'}\partial_y\partial_{x'} & \partial_z\partial_{z'}\partial_y\partial_{y'} & -(\partial_{x'}^2+\partial_{y'}^2)\partial_z\partial_y \\ -(\partial_x^2+\partial_y^2)\partial_{z'}\partial_{x'} & -(\partial_x^2+\partial_y^2)\partial_{z'}\partial_{y'} & (\partial_x^2+\partial_y^2)(\partial_{x'}^2+\partial_{y'}^2) \end{pmatrix}$$

$$e^{ik_x(x-x')}e^{ik_y(y-y')}e^{ik_z(z-z')}dk_x dk_y dk_z$$

Integrating with respect to k_z again, we obtain from the above expression

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-1}{k_c^2k_0^2} \tag{3 $\cdot$ 8}$$

$$\begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'} & \partial_z\partial_{z'}\partial_x\partial_{y'} & -(\partial_{x'}^2+\partial_{y'}^2)\partial_z\partial_x \\ \partial_z\partial_{z'}\partial_y\partial_{x'} & \partial_z\partial_{z'}\partial_y\partial_{y'} & -(\partial_{x'}^2+\partial_{y'}^2)\partial_z\partial_y \\ -(\partial_x^2+\partial_y^2)\partial_{z'}\partial_{x'} & -(\partial_x^2+\partial_y^2)\partial_{z'}\partial_{y'} & (\partial_x^2+\partial_y^2)(\partial_{x'}^2+\partial_{y'}^2) \end{pmatrix}$$

$$e^{-ik_x(x-x')}e^{-ik_y(y-y')}e^{-k_c|z-z'|}dk_x dk_y dk_z$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$\frac{1}{k_0^2 ik_g}\begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'} & \partial_z\partial_{z'}\partial_x\partial_{y'} & -(\partial_{x'}^2+\partial_{y'}^2)\partial_z\partial_x \\ \partial_z\partial_{z'}\partial_y\partial_{x'} & \partial_z\partial_{z'}\partial_y\partial_{y'} & -(\partial_{x'}^2+\partial_{y'}^2)\partial_z\partial_y \\ -(\partial_x^2+\partial_y^2)\partial_{z'}\partial_{x'} & -(\partial_x^2+\partial_y^2)\partial_{z'}\partial_{y'} & (\partial_x^2+\partial_y^2)(\partial_{x'}^2+\partial_{y'}^2) \end{pmatrix}$$

$$e^{-ik_x(x-x')}e^{-ik_y(y-y')}e^{ik_g|z-z'|}\hat{z}\hat{z}dk_x dk_y dk_z$$

where $k_g = \sqrt{k_0^2 - k_c^2}$.

Differentiating the above expression (3· 8) we obtain some singular terms which cancel out with each other, leaving

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} k_x^2 & k_x k_y & -ik_x k_{cs} \\ k_x k_y & k_y^2 & -ik_y k_{cs} \\ -ik_x k_{cs} & -ik_y k_{cs} & -k_c^2 \end{pmatrix}$$

$$\cdot\frac{e^{-ik_x(x-x')}e^{-ik_y(y-y')}e^{-k_c|z-z'|}}{2k_0^2 k_c}dk_x dk_y \qquad (3\cdot 9)$$

$$+\frac{1}{(2\pi^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} k_x^2 k_g^2 & k_x k_y k_g^2 & k_x k_c k_{cs} k_g \\ k_x k_y k_g^2 & k_y^2 k_g^2 & k_y k_c k_{cs} k_g \\ k_x k_c k_{cs} k_g & k_y k_c k_{cs} k_g & k_c^4 \end{pmatrix}$$

$$\cdot\frac{e^{-ik_x(x-x')}e^{-ik_y(y-y')}e^{-k_c|z-z'|}}{2k_0^2 k_c^2 k_g^2}dk_x dk_y.$$

Note that the first term of (3· 9) cancels out all but the delta-function term of the longitudinal contribution. Thus, this delta-function term is the correction term for the form obtained by Tai[1,p. 103].

3.3 Cylindrical Vector Wave Functions in Cartesian Co-ordinates

Tai's spectral expansion of the Green's dyadic is in terms of the transverse cylindrical vector wave functions. These functions, $\mathrm{M}_{\substack{e\\o}n\lambda}(h)$ & $\mathrm{N}_{\substack{e\\o}n\lambda}(h)$, are eigenfunctions of the vector Helmholtz equation and are related symmetrically:

$$\nabla \times \mathrm{M}_{\substack{e\\o}n\lambda}(h) = \kappa \mathrm{N}_{\substack{e\\o}n\lambda}(h)$$

$$\nabla \times \mathrm{N}_{\substack{e\\o}n\lambda}(h) = \kappa \mathrm{M}_{\substack{e\\o}n\lambda}(h)$$

The notation $\mathrm{F}_{\substack{e\\o}}$ indicates a choice of even or odd function of ϕ. For example, $\mathrm{M}_{e\ n\lambda}(h) = \nabla \times \mathrm{J}_n(\lambda\rho)\cos(n\phi)\exp(ihz)\hat{z}$. The dyad $\cdot_{\substack{e\\o}n\lambda}(h)\cdot'_{\substack{e\\o}n\lambda}(h)$ is defined as $\cdot_{e\ n\lambda}(h)\cdot'_{e\ n\lambda}(h) + \cdot_{o\ n\lambda}(h)\cdot'_{o\ n\lambda}(h)$ where $\cdot$ may be replaced by M or N.

We present these functions in a form slightly different from that used by Tai, in that rather than expressing them in terms of a cylindrical co-ordinate system, we use Cartesian co-ordinates. We now list the cylindrical vector wave functions:

$$\begin{aligned}\mathrm{M}_{\substack{e\\o}n\lambda}(h) =& \nabla \times (\psi_n \hat{z}) \\ =& \partial_y \psi_n \hat{x} - \partial_x \psi_n \hat{y}\end{aligned} \tag{3 · 10}$$

$$\begin{aligned}[\text{ Where } r =& \sqrt{x^2+y^2} \\ \phi =& \arctan\frac{y}{x} \\ \psi_n =& \mathrm{J}_n(\lambda r)^{\cos}_{\sin}(n\phi)\exp(ihz) \\ \text{and } \nabla^2\psi_n + \kappa^2\psi_n =& 0]\end{aligned} \tag{3 · 11}$$

$$\begin{aligned}\mathrm{N}_{\substack{e\\o}n\lambda}(h) =& \frac{1}{\kappa}\nabla \times \nabla \times (\psi_n \hat{z}) \\ =& \frac{1}{\kappa}(\partial_z\partial_x\psi_n\hat{x} + \partial_z\partial_y\psi_n\hat{y} - (\partial_x^2 + \partial_y^2)\psi_n\hat{z}) \\ =& \frac{1}{\kappa}(\partial_z\partial_x\psi_n\hat{x} + \partial_z\partial_y\psi_n\hat{y} + (\partial_z^2 + \kappa^2)\psi_n\hat{z}) \\ =& \frac{1}{\kappa}(\nabla(\partial_z\psi_n) + \kappa^2\psi_n\hat{z}) \\ =& \frac{1}{\kappa}(\partial_z\partial_x\psi_n\hat{x} + \partial_z\partial_y\psi_n\hat{y} + \lambda^2\psi_n\hat{z}) \\ & \text{where } \kappa^2 = \lambda^2 + h^2\end{aligned} \tag{3 · 12}$$

3.4 Green's Dyadic for the Vector Helmholtz Equation

Our model in this section is of an upper half-space (i.e. $z \geq 0$) with parameters k_2, h_2, corresponding to the ground and a lower half-space (i.e. $z \leq 0$) with parameters k_1, h_1, corresponding to the air. There is no need to substantially alter Tai's [1] form for the Green's dyadic for a flat earth, as the expression for the free-space dyadic is easily corrected by the addition of the delta-function term derived previously and an appropriate (in fact naive) substitution for the derivatives with respect to z and z'. Also, Tai's reasoning about the form of the scattering terms remains valid. The anterior elements must still be eigenfunctions of the vector wave equation in their respective media, i.e. $\mathrm{M}_{\substack{e\\o}n\lambda}(h_1), \mathrm{N}_{\substack{e\\o}n\lambda}(h_1)$ in the ground and $\mathrm{M}_{\substack{e\\o}n\lambda}(-h_2), \mathrm{N}_{\substack{e\\o}n\lambda}(-h_2)$ in the air (h_1 and h_2 are defined below.) These choices also ensure that the radiation condition is satisfied at $z \to \infty$ and $z \to -\infty$ respectively. The posterior elements of the scattering terms in both media must be the same as that for the free-space Green's dyadic to ensure that the boundary conditions at the interface can be satisfied. These boundary conditions, which follow from the classical boundary conditions for an electromagnetic field, are that at $z = 0$ the components of the Green's dyadic and the curl of the Green's dyadic which are tangential to the plane $z = 0$ are continuous across this plane.

From Tai[1], the Green's dyadic for the source in ground, receiver in ground, after the addition of the correction term is :

$$\mathrm{G}(\underline{x}|\underline{x}') =$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{\lambda h_2}\sum_{n=0}^{\infty}(2-\delta_0)\mathrm{M}_{\substack{e\\o}n\lambda}(h_2)[\mathrm{M}'_{\substack{e\\o}n\lambda}(-h_2) + a\mathrm{M}'_{\substack{e\\o}n\lambda}(h_2)] \qquad (3\cdot 13)$$

$$+\mathrm{N}_{\substack{e\\o}n\lambda}(h_2)[\mathrm{N}'_{\substack{e\\o}n\lambda}(-h_2) + b\mathrm{N}'_{\substack{e\\o}n\lambda}(h_2)]d\lambda$$

$$-\frac{1}{k_2^2}\hat{z}\hat{z}\delta(\underline{x} - \underline{x}')$$

$$z \geq z' \geq 0$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{\lambda h_2}\sum_{n=0}^{\infty}(2-\delta_0)[\mathrm{M}_{\substack{e\\o}n\lambda}(-h_2) + a\mathrm{M}_{\substack{e\\o}n\lambda}(h_2)]\mathrm{M}'_{\substack{e\\o}n\lambda}(h_2) \tag{3 · 14}$$

$$+[\mathrm{N}_{\substack{e\\o}n\lambda}(-h_2) + b\mathrm{N}_{\substack{e\\o}n\lambda}(h_2)]\mathrm{N}'_{\substack{e\\o}n\lambda}(h_2)d\lambda$$

$$-\frac{1}{k_2^2}\hat{z}\hat{z}\delta(\underline{x} - \underline{x}')$$

$$z' \geq z \geq 0$$

where the primes denote dependence on $\underline{x}'$, δ_0 is the Kronecker symbol with respect to n:

$$\delta_0 = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0, \end{cases}$$

$$a = \frac{h_2 - h_1}{h_2 + h_1},$$
$$b = \frac{k_1^2 h_2 - k_2^2 h_1}{k_1^2 h_2 + k_2^2 h_1},$$
$$k_i^2 = \lambda^2 + h_i^2, (i = 1, 2),$$

$$k_1 = \sqrt{-\mu\sigma_{air}s} \text{ (taking} \epsilon = 0),$$

and $k_2 = \sqrt{-\mu\sigma_{ground}s}$.

Note that for $\lambda \neq 0$:

$$\lim_{\sigma_{air}\to 0} k_1 = 0,$$
$$\lim_{\sigma_{air}\to 0} k_2 = \sqrt{-\mu\sigma_{ground}s} = k_2,$$
$$\lim_{\sigma_{air}\to 0} h_1 = i\lambda,$$
$$\lim_{\sigma_{air}\to 0} h_2 = \lim_{\sigma_{air}\to 0} i\sqrt{\lambda^2 - k_2^2} = i\sqrt{\lambda^2 - k_2^2} = iK,$$
$$\lim_{\sigma_{air}\to 0} a = \frac{iK - i\lambda}{iK + i\lambda} = \frac{K-\lambda}{K+\lambda},$$
$$\lim_{\sigma_{air}\to 0} b = -1.$$

It can be seen from the formulae for the Green's dyadic that the form of the dyadic for $z \geq z'$ is merely the form for $z \leq z'$ with z and z' interchanged. Thus, we need only use the form for $z \geq z'$ in our calculations, though the discontinuity at $z = z'$ must still be taken into account.

Now, the dyadic operator $\mathrm{M}_{\substack{e\\o}n\lambda}(h_2)\mathrm{M}'_{\substack{e\\o}n\lambda}(\pm h_2)$ in matrix form is

$$\begin{pmatrix} \partial_y\partial_{y'}\psi_n\psi'_n & -\partial_y\partial_{x'}\psi_n\psi'_n & 0 \\ -\partial_x\partial_{y'}\psi_n\psi'_n & \partial_x\partial_{x'}\psi_n\psi'_n & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (3\cdot 15)$$

and, similarly, the dyadic operator $\mathrm{N}_{\substack{e\\o}n\lambda}(h_2)\mathrm{N}'_{\substack{e\\o}n\lambda}(\pm h_2)$

in matrix form is

$$\frac{1}{k_2^2}\begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'}\psi_n\psi'_n & \partial_z\partial_{z'}\partial_x\partial_{y'}\psi_n\psi'_n & \lambda^2\partial_z\partial_x\psi_n\psi'_n \\ \partial_z\partial_{z'}\partial_y\partial_{x'}\psi_n\psi'_n & \partial_z\partial_{z'}\partial_y\partial_{y'}\psi_n\psi'_n & \lambda^2\partial_z\partial_y\psi_n\psi'_n \\ \lambda^2\partial_{z'}\partial_{x'}\psi_n\psi'_n & \lambda^2\partial_{z'}\partial_{y'}\psi_n\psi'_n & \lambda^4\psi_n\psi'_n \end{pmatrix}. \qquad (3\cdot 16)$$

(Here we have interpreted the derivatives with respect to z and z' naively, for reasons explained in section 3.2.) Now ,

$$\begin{aligned} \psi_n \psi_n' &= \mathrm{J}_n(\lambda r)\mathrm{J}_n(\lambda r')\exp(ih_2(z \pm z'))(\cos n\phi \cos n\phi' + \sin n\phi \sin n\phi') \\ &= \mathrm{J}_n(\lambda r)\mathrm{J}_n(\lambda r')\exp(ih_2(z \pm z'))\cos(n(\phi - \phi')). \end{aligned} \tag{3 $\cdot$ 17}$$

Noting that

$$\sum_{n=0}^{\infty} \mathrm{J}_n(\lambda r)\mathrm{J}_n(\lambda r')\cos(n(\phi - \phi'))(2 - \delta_0) = \mathrm{J}_0(\lambda \rho) \tag{3 $\cdot$ 18}$$

(Graff's addition formula, Erdelyi, et. al [1])

where

$$\rho = (r^2 + r'^2 - 2rr' \cos(\phi - \phi'))^{\frac{1}{2}} \tag{3 $\cdot$ 19}$$

so interchanging summation and differentiation in (1.4), and defining

$$\psi_{\pm} = \mathrm{J}_0(\lambda\rho) \ \exp(ih_2(z \pm z')) \tag{3 $\cdot$ 20}$$

gives

$$\mathrm{G}(\underline{x}|\underline{x}') =$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{\lambda h_2}\left\{\begin{pmatrix} \partial_y\partial_{y'}(\psi_- + a\psi_+) & -\partial_y\partial_{x'}(\psi_- + a\psi_+) & 0 \\ -\partial_x\partial_{y'}(\psi_- + a\psi_+) & \partial_x\partial_{x'}(\psi_- + a\psi_+) & 0 \\ 0 & 0 & 0 \end{pmatrix}\right. \tag{3 $\cdot$ 21}$$

$$\left. + \frac{1}{k_2^2}\begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'}(\psi_- + b\psi_+) & \partial_z\partial_{z'}\partial_x\partial_{y'}(\psi_- + b\psi_+) & \lambda^2\partial_z\partial_x(\psi_- + b\psi_+) \\ \partial_z\partial_{z'}\partial_y\partial_{x'}(\psi_- + b\psi_+) & \partial_z\partial_{z'}\partial_y\partial_{y'}(\psi_- + b\psi_+) & \lambda^2\partial_z\partial_y(\psi_- + b\psi_+) \\ \lambda^2\partial_{z'}\partial_{x'}(\psi_- + b\psi_+) & \lambda^2\partial_{z'}\partial_{y'}(\psi_- + b\psi_+) & \lambda^4(\psi_- + b\psi_+) \end{pmatrix}\right\} d\lambda$$

$$-\frac{1}{k_2^2}\delta(\underline{x}-\underline{x}')\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

Note that here we interpret the derivatives of ψ_- with respect to z and z' naively, i.e. $\partial_z\partial_{z'}\psi_- = -h_2^2\psi_-$.

3.5 Convergence of the Green's Dyadic

In this section it will be shown in what sense the Green's dyadic evaluated at $\sigma_{air} \neq 0$ tends to the dyadic evaluated at $\sigma_{air} = 0$. Firstly, note that the free-space Green's dyadic, which from Tai[1,p. 55] is :

$$(I + \frac{1}{k_2^2}\nabla\nabla)\frac{\exp(-ik_2R)}{R}$$

does not change as $\sigma_{air} \rightarrow 0$. We now show that for $z, z' \neq 0$, the integrands of the scattering terms are dominated by a λ-integrable function and thus prove that pointwise the scattering terms tend to the expressions obtained by Hohmann[1]. We now calculate some elementary estimates, needed here and in the next chapter.

First, we calculate a bound for $\Re K = \Re\sqrt{\lambda^2 + s\mu\sigma_{ground}}$ $= \Re\sqrt{\lambda^2 + (\alpha + i\beta)\mu\sigma_{ground}}$. For convergence we pick the branch of $\sqrt{z}$ with $\Re\sqrt{z} > 0$ for $\Re z > 0$. Since we wish only to take an inverse Laplace transform it suffices to consider s in the right half plane. Consider a vertical line in the $\alpha - \beta$ plane: $\alpha = c, c \geq 0$. This corresponds to the set of s with $\Re s = c$ i.e.

$$|s|\cos(\arg s) = c.$$

This implies that

$$|s^{\frac{1}{2}}|^2\cos(2\arg s^{\frac{1}{2}}) = c. \qquad (3\cdot 22)$$

However,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

and thus (3· 22) is

$$|s^{\frac{1}{2}}|^2(\cos^2(\arg s^{\frac{1}{2}}) - \sin^2(\arg s^{\frac{1}{2}})) = c$$

i.e.

$$(\Re s^{\frac{1}{2}})^2 - (\Im s^{\frac{1}{2}})^2 = c.$$

Thus, if we let $s^{\frac{1}{2}} = x + iy$, the point corresponding to $s^{\frac{1}{2}}$ in the $x - y$ plane moves along a hyperbola with vertex $(\sqrt{\Re s}, 0)$ and asymptotes $y = \mp x$ as the point corresponding to $s = \alpha + i\beta$ in the $\alpha - \beta$ plane moves along $\alpha = c$. We have immediately from this that:

$$\Re\sqrt{(\lambda^2 + s\mu\sigma_{ground})} \geq \sqrt{(\lambda^2 + \Re s\mu\sigma_{ground})} \geq \sqrt{\Re s\mu\sigma_{ground}} \qquad (3 \cdot 23)$$

Note also that $|\lambda + K| \geq |\Re(\lambda + K)| \geq \lambda$ and $|K| \geq \lambda$.

We now consider the transmission coefficients a and b. First, note that since $\Re s \geq 0$, $h_2 = \sqrt{\lambda^2 + s\mu\sigma_{ground}}$ and $h_1 = \sqrt{\lambda^2 + s\mu\sigma_{air}}$ must lie in the same quadrant of the complex plane. Thus, by considering a parallelogram with sides h_1 and h_2 in the complex plane, it can be seen that $|h_1 - h_2| \leq |h_1 + h_2|$. Thus,

$$\left|\frac{h_1 - h_2}{h_2 + h_1}\right| \leq 1.$$

Similarly, assuming w.o.l.g. $s \neq 0$,

$$\begin{aligned}|b| &= \left|\frac{k_2^2 h_1 - k_1^2 h_2}{k_2^2 h_1 + k_1^2 h_2}\right| \\ &= \frac{|s\mu|}{|s\mu|}\left|\frac{\sigma_{ground}h_1 - \sigma_{air}h_2}{\sigma_{air}h_2 + \sigma_{ground}h_1}\right| \leq 1\end{aligned}$$

since $\sigma_{ground} h_1$ and $\sigma_{air} h_2$ must also lie in the same quadrant. We now derive some bounds for the derivatives of Bessel functions which will be used throughout both this chapter and the next.

Note first that:

$$\begin{aligned}\partial_{x_i}\partial_{x'_j} \mathrm{J}_0(\lambda\rho) =&\delta_{ij}\lambda^2 \frac{\mathrm{J}_1(\lambda\rho)}{\lambda\rho} + \frac{\lambda^2}{\rho^2}(x_i - x'_i)(x_j - x'_j)\mathrm{J}_0(\lambda\rho)\\ &-2\lambda^2 \frac{\mathrm{J}_1(\lambda\rho)}{\lambda\rho} \frac{(x_i - x'_i)(x_j - x'_j)}{\rho^2}\end{aligned} \tag{3 $\cdot$ 24}$$

where i, j are either 1 or 2, $x_1 = x, x_2 = y$ and δ_{ij} is the Kronecker delta function.

Now the recurrence relation

$$2\frac{\mathrm{J}_1(x)}{x} = \mathrm{J}_0(x) + \mathrm{J}_2(x)$$

implies that

$$|\frac{\mathrm{J}_1(\lambda\rho)}{\lambda\rho}| \leq 1.$$

Thus,

$$\begin{aligned}\left|\partial_{x_i}\partial_{x'_j} \mathrm{J}_0(\lambda\rho)\right| &\leq (1 + \delta_{ij})\lambda^2 + 2\lambda^2 \frac{|x_i - x'_i||x_j - x'_j|}{\rho^2}\\ &\leq 2\lambda^2 + 2\lambda^2 \frac{|x_i - x'_i||x_j - x'_j|}{\rho^2}\\ &= F(\underline{x})\lambda^2.\end{aligned} \tag{3 $\cdot$ 25}$$

We can now estimate the scattering terms of the dyadic. We have that

$$|h_2| = |K| = \sqrt{|\lambda^2 - k_2^2|} \geq \sqrt{\mu\sigma\Re s}, \tag{3 $\cdot$ 26}$$

$|a| \leq 1$ and that $\Re K \geq \lambda$. Thus, recalling (3$\cdot$ 25), we see that

$$\partial_{x_i}\partial_{x'_j} a\psi_+ \frac{1}{\lambda h_2} = \partial_{x_i}\partial_{x'_j} a\mathrm{J}_0(\lambda\rho)\ \exp(-K(z + z'))\frac{1}{\lambda h_2}$$

is dominated by

$$\frac{1}{\sqrt{\mu\sigma\Re s}}F(\underline{x})\lambda\exp(-\lambda(z+z')).$$

There are similar bounds for terms involving $\partial_z\partial_{z'}\partial_{x_i}\partial_{x'_j}, \partial_z\partial_{x_i}, b\psi_+$ etc. since the derivatives with respect to z and z' only multiply terms by $-K$ or K^2, and

$$|K| = |\sqrt{\lambda^2 - k_2^2}| \leq \lambda + |k_2|$$

and $|b| \leq 1$. Since F is a bounded, continuous function of $\underline{x}$ we need only consider the following integral:

$$\frac{1}{4\pi}\int_0^\infty \lambda\exp(-\lambda(z+z'))d\lambda \qquad (3\cdot 27)$$

$$= (-1)\frac{1}{4\pi}\frac{d}{dz}\int_0^\infty \exp(-\lambda(z+z'))d\lambda$$

$$= (-1)\frac{1}{4\pi}\frac{d}{dz}\frac{1}{(z+z')}.$$

Thus, for z and z' non-zero we have that the scattering terms of the dyadic tend, as $\sigma_{air} \to 0$, to the elements of the dyadic evaluated at $\sigma_{air} = 0$ pointwise in space.

3.6 Calculation of the Terms of the Dyadic.

We now consider the elements of the dyadic, putting $\sigma_{air} = 0$ but not expressing a and b in terms of their limits until this is needed. Note that part of the following is merely a reversal of Tai's[1] spectral expansion of the free-space Green's dyadic. We include this as a check on the rest of our calculation. By the symmetry of the expression for G in section 3.4 we need only consider four elements: $G_{13}, G_{21}, G_{11}, G_{33}$.

Consider, firstly G_{13}, viz.

$$\frac{1}{k_2^2}\frac{i}{4\pi}\int_0^\infty \frac{d\lambda}{\lambda h_2}\ \lambda^2\partial_z\partial_x(\psi_- + b\psi_+). \tag{3 ·28}$$

Noting that as σ_{air} tends to 0, h_2 tends to iK, where $K^2 = \lambda^2 - k_2^2$

$$\begin{aligned}
&= \frac{1}{4\pi}\frac{1}{k_2^2}\int_0^\infty d\lambda\ \partial_z\partial_x\frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z-z'))\\
&\quad - \frac{1}{4\pi}\frac{1}{k_2^2}\int_0^\infty d\lambda\ \partial_z\partial_x\frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z+z'))\\
&= \frac{1}{4\pi}\frac{1}{k_2^2}\partial_z\partial_x\int_0^\infty d\lambda\ \frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z-z')) - \mathrm{J}_0(\lambda\rho)\ \exp(-K(z+z'))\\
&= \frac{1}{4\pi}\frac{1}{k_2^2}\partial_z\partial_x(\frac{\exp(-ik_2R)}{R} - \frac{\exp(-ik_2R_S)}{R_S})\\
&= \frac{1}{4\pi}\frac{1}{k_2^2}\partial_{z'}\partial_x(-\frac{\exp(-ik_2R)}{R} - \frac{\exp(-ik_2R_S)}{R_S})
\end{aligned}$$

where $R = (\rho^2 + (z-z')^2)^{\frac{1}{2}}, R_S = (\rho^2 + (z+z')^2)^{\frac{1}{2}}$ since from Ward[1] :

$$\frac{\exp(-ik_2r)}{r} = \int_0^\infty \frac{\lambda}{(\lambda^2 - k_2^2)^{\frac{1}{2}}}\exp((\lambda^2 - k_2^2)^{\frac{1}{2}}z)\mathrm{J}_0(\lambda\rho)\ d\lambda \tag{3 ·29}$$

where

$$r = \sqrt{(\rho^2 + z^2)}.$$

Consider secondly, G_{21},

$$\frac{i}{4\pi}\int_0^\infty \frac{d\lambda}{\lambda h_2}\left(-\partial_y\partial_{x'}(\psi_- + a\psi_+) + \frac{-K^2}{k_2^2}\partial_{y'}\partial_x\psi_- + \frac{1}{k_2^2}\partial_{y'}\partial_x b\psi_+\right). \tag{3 ·30}$$

Since $\partial_y\partial_{x'}\mathrm{J}_0(\lambda\rho)\ = \partial_x\partial_{y'}\mathrm{J}_0(\lambda\rho)$ equation (3· 30) becomes

$$\frac{1}{4\pi}\int_0^\infty \partial_y\partial_{x'}(-1 + \frac{-K^2}{k_2^2})\psi_-\ + \partial_y\partial_{x'}(-a + \frac{b}{k_2^2}\partial_{z'}\partial_z)\psi_+\frac{d\lambda}{\lambda K}. \tag{3 ·31}$$

Now,

$$
\begin{aligned}
&\frac{1}{4\pi}\int_0^\infty \left((-1+\frac{-K^2}{k_2^2})\psi_-\right)\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty (-1-\frac{K^2}{k_2^2})\mathrm{J}_0(\lambda\rho)\ \exp(-K(z-z'))\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty (\frac{-k_2^2-\lambda^2+k_2^2}{k_2^2})\mathrm{J}_0(\lambda\rho)\ \exp(-K(z-z'))\frac{d\lambda}{\lambda K} \qquad (3\cdot 32) \\
=&\frac{-1}{k_2^2 4\pi}\int_0^\infty \frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z-z'))d\lambda \\
=&\frac{-1}{k_2^2 4\pi}\frac{\exp(-ik_2R)}{R}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{4\pi}\int_0^\infty \partial_y\partial_{x'}(-a+\frac{b}{k_2^2})\partial_z\partial_{z'}\psi_+\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty \partial_y\partial_{x'}(-a+\frac{b}{k_2^2}K^2)\psi_+\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty -\partial_y\partial_{x'}(\frac{K-\lambda}{K+\lambda}+\frac{K^2}{k_2^2})\psi_+\frac{d\lambda}{\lambda K} \qquad (3\cdot 33) \\
=&\frac{1}{k_2^2 4\pi}\int_0^\infty -\partial_y\partial_{x'}(-\lambda^2+2\lambda K)\psi_+\frac{d\lambda}{\lambda K} \\
=&\frac{1}{k_2^2 4\pi}\int_0^\infty -\partial_y\partial_{x'}(2-\frac{\lambda}{K})\psi_+ d\lambda.
\end{aligned}
$$

Consider, now, G_{11}

$$\frac{1}{4\pi}\int_0^\infty \left(\partial_y\partial_{y'}(\psi_- + a\psi_+) + \frac{1}{k_2^2}\partial_z\partial_{z'}\partial_x\partial_{x'}(\psi_- + b\psi_+)\right)\frac{d\lambda}{\lambda K}$$
$$=\frac{1}{4\pi}\int_0^\infty \left(\partial_y\partial_{y'}\psi_- + \frac{-K^2}{k_2^2}\partial_x\partial_{x'}\psi_- + (a\partial_y\partial_{y'} + \frac{b}{k_2^2}\partial_z\partial_{z'}\partial_x\partial_{x'})\psi_+\right)\frac{d\lambda}{\lambda K}. \tag{3 $\cdot$ 34}$$

We now look at the first term, viz.:

$$\frac{1}{4\pi}\int_0^\infty (\partial_y\partial_{y'} - \frac{K^2}{k_2^2}\partial_x\partial_{x'})\psi_-\frac{d\lambda}{\lambda K}. \tag{3 $\cdot$ 35}$$

Noting that

$$\partial_y\partial_{y'}\mathrm{J}_0(\lambda\rho) = \lambda^2\mathrm{J}_0(\lambda\rho)\,\frac{(y-y')^2}{\rho^2} - \mathrm{J}_1(\lambda\rho)\lambda\frac{(y-y')^2}{\rho^3} + \mathrm{J}_1(\lambda\rho)\lambda\frac{(x-x')^2}{\rho^3} \tag{3 $\cdot$ 36}$$

and

$$\partial_x\partial_{x'}\mathrm{J}_0(\lambda\rho) = \lambda^2\mathrm{J}_0(\lambda\rho)\,\frac{(x-x')^2}{\rho^2} - \mathrm{J}_1(\lambda\rho)\lambda\frac{(x-x')^2}{\rho^3} + \mathrm{J}_1(\lambda\rho)\lambda\frac{(y-y')^2}{\rho^3} \tag{3 $\cdot$ 37}$$

we see that (3· 35) is

$$\frac{1}{4\pi}\int_0^\infty (\partial_y\partial_{y'} - \frac{(\lambda^2-k_2^2)}{k_2^2}\partial_x\partial_{x'})\psi_-\,\frac{d\lambda}{\lambda K}$$
$$=\frac{1}{4\pi}\int_0^\infty \frac{1}{\lambda K}(\partial_y\partial_{y'} + \partial_x\partial_{x'})\psi_- - \partial_x\partial_{x'}\frac{1}{k_2^2}\frac{\lambda}{K}\psi_- d\lambda$$
$$=\frac{1}{4\pi}\int_0^\infty \frac{1}{\lambda K}\lambda^2\mathrm{J}_0(\lambda\rho)\,\exp(-K(z-z'))d\lambda \tag{3 $\cdot$ 38}$$
$$-\frac{1}{4\pi}\partial_x\partial_{x'}\int_0^\infty \frac{1}{k_2^2}\frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\,\exp(-K(z-z'))d\lambda$$
$$=\frac{1}{4\pi}(\frac{\exp(-ik_2R)}{R} - \frac{1}{k_2^2}\partial_x\partial_{x'}\frac{\exp(-ik_2R)}{R}).$$

The second term is

$$
\begin{aligned}
&\frac{1}{4\pi}\int_0^\infty (a\partial_y\partial_{y'} + \frac{b}{k_2^2}\partial_{z'}\partial_z\partial_x\partial_{x'})\psi_+\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty (\frac{K-\lambda}{K+\lambda}\partial_y\partial_{y'} - \frac{K^2}{k_2^2}\partial_x\partial_{x'})\psi_+\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty \frac{K-\lambda}{K+\lambda}(\partial_y\partial_{y'} + \partial_x\partial_{x'})\psi_+\frac{d\lambda}{\lambda K} - \frac{1}{4\pi}\int_0^\infty (\frac{K-\lambda}{K+\lambda} + \frac{K^2}{k_2^2})\partial_x\partial_{x'}\psi_+\frac{d\lambda}{\lambda K} \\
=&\frac{1}{4\pi}\int_0^\infty (\frac{K-\lambda}{K+\lambda})\frac{\lambda}{K}\psi_+ d\lambda - \frac{1}{k_2^2 4\pi}\partial_x\partial_{x'}\int_0^\infty (2-\frac{\lambda}{K})\psi_+ d\lambda.
\end{aligned}
\tag{3 $\cdot$ 39}
$$

We now consider G_{33} :

$$
\frac{1}{4\pi}\frac{1}{k_2^2}\int_0^\infty \lambda^4(\psi_- + b\psi_+)\frac{d\lambda}{\lambda K} - \frac{1}{k_2^2}\delta(\underline{x}-\underline{x}') \tag{3 $\cdot$ 40}
$$

$$
= \frac{1}{4\pi}\frac{1}{k_2^2}\int_0^\infty \frac{\lambda^3}{K}(\psi_- + b\psi_+)d\lambda - \frac{1}{k_2^2}\delta(\underline{x}-\underline{x}')
$$

$$
= \frac{1}{4\pi}\frac{1}{k_2^2}\int_0^\infty \frac{\lambda}{K}(k_2^2 + K^2)(\psi_- + b\psi_+)d\lambda - \frac{1}{k_2^2}\delta(\underline{x}-\underline{x}')
$$

$$
= \frac{1}{4\pi}\int_0^\infty \frac{\lambda}{K}(\psi_- - \psi_+) + \frac{\lambda}{Kk_2^2}\partial_z\partial_{z'}(-\psi_- - \psi_+) + \frac{1}{k_2^2}\lambda \mathrm{J}_0(\lambda\rho)\, 2\delta(z-z')d\lambda - \frac{1}{k_2^2}\delta(\underline{x}-\underline{x}').
$$

Note that

$$
\frac{\delta(r-r')}{r} = \int_0^\infty \lambda \mathrm{J}_n(\lambda r)\mathrm{J}_n(\lambda r')d\lambda
$$

(Tai[1], p. 9)

$$
\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{n=0}^\infty \cos(n(\phi-\phi'))(2-\delta_0)
$$

(Fourier series for the delta-function) and

$$\sum_{n=0}^{\infty} \mathrm{J}_n(\lambda r)\mathrm{J}_n(\lambda r') \cos(n(\phi - \phi'))(2 - \delta_0) = \mathrm{J}_0(\lambda \rho)$$

(Graff's addition formula). Thus,

$$\frac{1}{k_2^2}\delta(z - z')\left(\frac{1}{2\pi}\int_0^{\infty} \lambda \mathrm{J}_1(\lambda\rho)\, d\lambda\right) = \frac{1}{k_2^2}\frac{\delta(r - r')}{r}\delta(\phi - \phi')\delta(z - z')$$

$$= \frac{1}{k_2^2}\delta(\underline{x} - \underline{x}')(\text{ in cylindrical co-ordinates})$$

and therefore (3· 40)

$$= \frac{1}{4\pi}(\frac{\exp(-ik_2R)}{R} - \frac{\exp(-ik_2R_s)}{R_s} - \frac{1}{k_2^2}\partial_z\partial_{z'}\frac{\exp(-ik_2R)}{R} - \frac{1}{k_2^2}\partial_z\partial_{z'}\frac{\exp(-ik_2R_s)}{R_s}).$$

Thus in matrix form, $G(\underline{x}|\underline{x}') =$

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{1}{k_2^2}\partial_x\partial_{x'}\alpha_1 & -\frac{1}{k_2^2}\partial_x\partial_{y'}\alpha_1 & -\frac{1}{k_2^2}\partial_x\partial_{z'}\alpha_1 \\ -\frac{1}{k_2^2}\partial_y\partial_{x'}\alpha_1 & -\frac{1}{k_2^2}\partial_y\partial_{y'}\alpha_1 & -\frac{1}{k_2^2}\partial_y\partial_{z'}\alpha_1 \\ -\frac{1}{k_2^2}\partial_z\partial_{x'}\alpha_1 & -\frac{1}{k_2^2}\partial_z\partial_{y'}\alpha_1 & -\frac{1}{k_2^2}\partial_{z'}\partial_z\alpha_1 \end{pmatrix}$$
$$+ \begin{pmatrix} \alpha_3 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ 0 & 0 & -\alpha_2 \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{1}{k_2^2}\partial_x\partial_{x'}\alpha_4 & -\frac{1}{k_2^2}\partial_x\partial_{y'}\alpha_4 & -\frac{1}{k_2^2}\partial_x\partial_{z'}\alpha_2 \\ -\frac{1}{k_2^2}\partial_y\partial_{x'}\alpha_4 & -\frac{1}{k_2^2}\partial_y\partial_{y'}\alpha_4 & -\frac{1}{k_2^2}\partial_y\partial_{z'}\alpha_2 \\ -\frac{1}{k_2^2}\partial_z\partial_{x'}\alpha_2 & -\frac{1}{k_2^2}\partial_z\partial_{y'}\alpha_2 & -\frac{1}{k_2^2}\partial_{z'}\partial_z\alpha_2 \end{pmatrix} \quad (3 \cdot 41)$$

where,

$$\begin{aligned}
\alpha_1 &= \frac{1}{4\pi}\frac{\exp(-ik_2R)}{R} \\
\alpha_2 &= \frac{1}{4\pi}\frac{\exp(-ik_2R_s)}{R_s} \\
\alpha_3 &= \frac{1}{4\pi}\int_0^\infty (\frac{K-\lambda}{K+\lambda})\frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z+z'))d\lambda \\
\alpha_4 &= \frac{1}{4\pi}\int_0^\infty (2-\frac{\lambda}{K})\mathrm{J}_0(\lambda\rho)\ \exp(-K(z+z'))d\lambda.
\end{aligned} \tag{3 $\cdot$ 42}$$

We now perform an integration by parts on the second and third matrices above using $\int_V \nabla G\ dV = \int_S G\ dS$, obtaining:

$$\begin{pmatrix} -\frac{1}{k_2^2}\partial_x\alpha_1 & -\frac{1}{k_2^2}\partial_x\alpha_1 & -\frac{1}{k_2^2}\partial_x\alpha_1 \\ -\frac{1}{k_2^2}\partial_y\alpha_1 & -\frac{1}{k_2^2}\partial_y\alpha_1 & -\frac{1}{k_2^2}\partial_y\alpha_1 \\ -\frac{1}{k_2^2}\partial_z\alpha_1 & -\frac{1}{k_2^2}\partial_z\alpha_1 & -\frac{1}{k_2^2}\partial_z\alpha_1 \end{pmatrix}$$

$$= -\frac{1}{k_2^2}\frac{d\alpha_1}{dR}\begin{pmatrix} (x-x')\alpha_1 & (x-x')\alpha_1 & (x-x')\alpha_1 \\ (y-y')\alpha_1 & (y-y')\alpha_1 & (y-y')\alpha_1 \\ (z-z')\alpha_1 & (z-z')\alpha_1 & (z-z')\alpha_1 \end{pmatrix}$$

(where

$$\frac{d\alpha_1}{dR} = -(ik_2R+1)\frac{\exp(-ik_2R)}{4\pi R^3})$$

and

$$\begin{pmatrix} -\frac{1}{k_2^2}\partial_x\alpha_4 & -\frac{1}{k_2^2}\partial_x\alpha_4 & -\frac{1}{k_2^2}\partial_x\alpha_2 \\ -\frac{1}{k_2^2}\partial_y\alpha_4 & -\frac{1}{k_2^2}\partial_y\alpha_4 & -\frac{1}{k_2^2}\partial_y\alpha_2 \\ -\frac{1}{k_2^2}\partial_z\alpha_2 & -\frac{1}{k_2^2}\partial_z\alpha_2 & -\frac{1}{k_2^2}\partial_z\alpha_2 \end{pmatrix}$$

$$= \frac{-1}{k_2^2} \begin{pmatrix} (x-x')\frac{1}{\rho}\frac{d\alpha_4}{d\rho} & (x-x')\frac{1}{\rho}\frac{d\alpha_4}{d\rho} & (z+z')\frac{d\alpha_2}{dR} \\ (y-y')\frac{1}{\rho}\frac{d\alpha_4}{d\rho} & (y-y')\frac{1}{\rho}\frac{d\alpha_4}{d\rho} & (z+z')\frac{d\alpha_2}{dR} \\ (z+z')\frac{d\alpha_2}{dR} & (z+z')\frac{d\alpha_2}{dR} & (z+z')\frac{d\alpha_2}{dR} \end{pmatrix}$$

(where

$$\frac{d\alpha_4}{d\rho} = -\frac{1}{4\pi}\int_0^\infty (2 - \frac{\lambda}{K})\mathrm{J}_1(\lambda\rho)\ \exp(-K(z+z'))d\lambda$$

and

$$\frac{d\alpha_2}{dR} = -(ik_2R+1)\frac{\exp(-ik_2R)}{4\pi R^3}).$$

Now the formula for the Green's tensor appearing in Hohmann[1] is $\frac{k_2^2}{\sigma_{ground}}$ times the formula obtained above after replacing the Fourier variable $i\omega$ in Hohmann's formula by the Laplace transform variable. Note that while our original dyadic and Hohmann's [1] original dyadic both satisfy the same boundary conditions, Hohmann's dyadic which we shall denote by $\mathcal{G}$, satisfies

$$(\nabla \times \nabla \times -k^2)\mathcal{G} = -\mu\delta sI$$

while our dyadic satisfies

$$(\nabla \times \nabla \times -k^2)G = \delta I.$$

Thus,

$$\mathcal{G} = -\mu sG = \frac{k_2^2}{\sigma}G$$

in the ground, where $\sigma(\underline{x})$ does not vanish. This relationship will hold in the pointwise limit as $\sigma_{air} \rightarrow 0$ and thus our results agree with those obtained by Hohmann.

3.7 Inverse Laplace Transform of the Green's Dyadic

First we note that (Oberhettinger and Badii [1])

$$\mathcal{L}^{-1}\left(\frac{\exp(-ik_2R_s)}{R_s}\right) \tag{3 ·43}$$

$$= \mathcal{L}^{-1}\left(\frac{\exp(-\sqrt{(s\mu\sigma)}R_s)}{R_s}\right)$$

$$= \frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp(-\frac{\mu\sigma R_s^2}{4t})$$

where $\sigma = \sigma_{ground}$. Secondly,

$$\mathcal{L}^{-1}\left(s^{-\frac{1}{2}}\exp\left(-\sqrt{s\mu\sigma}R_s\right)\right) \tag{3 ·44}$$

$$= \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\mu\sigma R_s^2}{4t}\right).$$

We now verify Hohmann's [2] time domain expressions, using our notation. We first consider:

$$\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma\rho}\int_0^\infty\left(2-\frac{\lambda}{K}\right)\exp\left(-K(z+z')\right)\mathrm{J}_1(\lambda\rho)\ \lambda d\lambda\right)$$

$$=\mathcal{L}^{-1}\left(2\frac{1}{4\pi\sigma\rho}\int_0^\infty\exp\left(-K(z+z')\right)\mathrm{J}_1(\lambda\rho)\ \lambda d\lambda\right) \tag{3 ·45}$$

$$-\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma\rho}\int_0^\infty\frac{\lambda}{K}\exp\left(-K(z+z')\right)\mathrm{J}_1(\lambda\rho)\ d\lambda\right)$$

Now,

$$\mathcal{L}^{-1}\left(2\frac{1}{4\pi\sigma\rho}\int_0^\infty\exp\left(-K(z+z')\right)\mathrm{J}_1(\lambda\rho)\ \lambda d\lambda\right) \tag{3 ·46}$$

becomes using the substitution $s = s' - \frac{\lambda^2}{\mu\sigma}$,

$$2\frac{1}{4\pi\sigma\rho}\int_0^\infty(z+z')\mathrm{J}_1(\lambda\rho)\ \lambda\frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp(-\frac{\mu\sigma(z+z')^2}{4t})\exp(-\frac{\lambda^2t}{\mu\sigma})d\lambda$$

$$=\frac{1}{2\pi^{3/2}\sigma\rho}(z+z')\frac{\sqrt{\mu\sigma}}{2t^{\frac{3}{2}}}\exp(-\frac{\mu\sigma(z+z')^2}{4t})\int_0^\infty\mathrm{J}_1(\lambda\rho)\ \lambda\exp(-\frac{\lambda^2t}{\mu\sigma})d\lambda \tag{3 ·47}$$

We now define as in Hohmann[2]:

$$\theta = \sqrt{\frac{\mu\sigma}{4t}}, W(R_s) = \frac{\exp(-\theta^2 R_s^2)}{t}, \beta = \frac{\lambda}{\theta}, r = \theta\rho \tag{3 ·48}$$

so (3· 45) becomes

$$\frac{1}{2\pi^{3/2}\sigma}\frac{(z+z')}{r}\theta W(z+z')\int_0^\infty \beta\theta \exp\left(\frac{-\beta^2}{4}\right) \mathrm{J}_1(\beta r)\theta d\beta \tag{3 ·49}$$

$$-\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma\rho}\int_0^\infty \frac{\lambda}{K}\exp\left(-K(z+z')\right)\mathrm{J}_1(\lambda\rho)\, d\lambda\right)$$

$$= \frac{1}{2\pi^{3/2}\sigma}\frac{(z+z')}{r}\theta^3 W(z+z')\int_0^\infty \beta \exp\left(\frac{-\beta^2}{4}\right) \mathrm{J}_1(\beta r) d\beta \tag{3 ·50}$$

$$-\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma\rho}\int_0^\infty \frac{\lambda}{K}\exp\left(-K(z+z')\right)\mathrm{J}_1(\lambda\rho)\, d\lambda\right). \tag{3 ·51}$$

We now assume that $x_i - x_i' \neq 0$ for at least one $i (= 1, 2, 3)$, then

$$\mathcal{L}^{-1}\left(-\frac{1}{4\pi\sigma}(x_i - x_i')^{-1}\partial_{x_i'}\int_0^\infty \frac{\lambda}{K}\mathrm{J}_1(\lambda\rho)\ \exp(-K(z+z')d\lambda\right)$$

$$= \mathcal{L}^{-1}\left(-\frac{1}{4\pi\sigma}(x_i - x_i')^{-1}\partial_{x_i'}\frac{\exp(-ik_2R_s)}{R_s}\right)$$

$$= -\frac{1}{4\pi\sigma}(x_i - x_i')^{-1}\partial_{x_i'}\frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp\left(-\frac{\mu\sigma R_s^2}{4t}\right)$$

$$= \frac{1}{4\pi\sigma}\frac{-(\mu\sigma)^{3/2}}{2\sqrt{\pi}4t^{\frac{5}{2}}}2\exp\left(-\frac{\mu\sigma R_s^2}{4t}\right)$$

$$= -\frac{1}{2\pi^{3/2}\sigma}\theta^3 W(R_s).$$

That is (3· 45) is

$$\frac{1}{2\pi^{3/2}\sigma}\frac{(z+z')}{r}\theta^3 W(z+z')\int_0^\infty \beta \exp\left(\frac{-\beta^2}{4}\right) \mathrm{J}_1(\beta r) d\beta - \frac{1}{2\pi^{3/2}\sigma}\theta^3 W(R_s) \tag{3 ·52}$$

which agrees with Hohmann [2].

We next consider

$$\mathcal{L}^{-1}\left(\frac{k_2^2}{4\pi\sigma}\int_0^\infty \frac{K-\lambda}{K+\lambda}\frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z+z'))d\lambda\right) \tag{3 $\cdot$ 53}$$

(Recall that $K^2 = \lambda^2 - k_2^2$)

$$=\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\int_0^\infty -(\lambda-K)^2\frac{\lambda}{K}\mathrm{J}_0(\lambda\rho)\ \exp(-K(z+z'))d\lambda\right)$$
$$=\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\int_0^\infty \frac{-\lambda^3}{K}\exp(-K(z+z'))\mathrm{J}_0(\lambda\rho)\ d\lambda\right)$$
$$+\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\int_0^\infty -\lambda K\exp(-K(z+z'))\mathrm{J}_0(\lambda\rho)\ d\lambda\right)$$
$$+\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\int_0^\infty 2\lambda^2\exp(-K(z+z'))\mathrm{J}_0(\lambda\rho)\ d\lambda\right)$$

Now, put $s = s' - \frac{\lambda^2}{\mu\sigma}$, so $k_2 = \sqrt{-s\mu\sigma} = \sqrt{\lambda^2 + s'\mu\sigma}$ and $K = \sqrt{\lambda^2 - k_2^2} = \sqrt{s'\mu\sigma}$.

Thus the first term above becomes

$$\frac{1}{4\pi\sigma}\int_0^\infty -\lambda^3\mathcal{L}^{-1}\left(\frac{\exp\left(-\sqrt{s'\mu\sigma(z+z')}\right)}{\sqrt{s'\mu\sigma}}\right)\exp\left(-\frac{\lambda^2 t}{\mu\sigma}\right)\mathrm{J}_0(\lambda\rho)\ d\lambda \tag{3 $\cdot$ 54}$$

$$=-\frac{1}{4\pi\sigma}\int_0^\infty -\lambda^3\frac{1}{\sqrt{\pi t\mu\sigma}}\exp\left(-\frac{\mu\sigma(z+z')^2}{4t}\right)\exp\left(-\frac{\lambda^2 t}{\mu\sigma}\right)\mathrm{J}_0(\lambda\rho)\ d\lambda$$
$$=-\frac{1}{4\pi^{3/2}\sigma}\frac{1}{2}W(z+z')\theta^{-1}\int_0^\infty \lambda^3\exp\left(-\left(\frac{\lambda}{\theta}\right)^2\right)\mathrm{J}_0(\lambda\rho)\ d\lambda$$
$$=-\frac{1}{2\pi^{3/2}\sigma}\frac{1}{4}W(z+z')\theta^{-1}\int_0^\infty \beta^3\theta^4\exp\left(-\frac{\beta^2}{4}\right)\mathrm{J}_0(\beta r)d\beta$$
$$=-\frac{1}{2\pi^{3/2}\sigma}\frac{1}{4}W(z+z')\theta^3\int_0^\infty \beta^3\exp\left(-\frac{\beta^2}{4}\right)\mathrm{J}_0(\beta r)d\beta.$$

Also, the third term becomes

$$\frac{1}{4\pi\sigma}2\int_0^\infty \lambda^2\mathcal{L}^{-1}\left(\exp(-\sqrt{s'\mu\sigma}(z+z'))\right)\exp\left(-\frac{\lambda^2 t}{\mu\sigma}\right)\mathrm{J}_0(\lambda\rho)\,d\lambda \qquad (3\cdot 55)$$

$$=\frac{1}{2\pi\sigma}\int_0^\infty \lambda^2(z+z')\frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp(-\frac{\mu\sigma(z+z')^2}{4t})\exp\left(-\frac{\lambda^2 t}{\mu\sigma}\right)\mathrm{J}_0(\lambda\rho)\,d\lambda$$

$$=\frac{1}{2\pi^{3/2}\sigma}W(z+z')(z+z')\theta t\int_0^\infty \beta^2\theta^2\exp\left(-\frac{\beta^2}{4}\right)\mathrm{J}_0(\beta r)\theta d\beta$$

$$=\frac{1}{2\pi^{3/2}\sigma}W(z+z')(z+z')\theta^4\int_0^\infty \beta^2\exp\left(-\frac{\beta^2}{4}\right)\mathrm{J}_0(\beta r)d\beta.$$

The second term is

$$\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\int_0^\infty -\lambda K\exp(-K(z+z'))\mathrm{J}_0(\lambda\rho)\,d\lambda\right)$$

$$=\partial_{z'}^2\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\int_0^\infty \frac{-\lambda}{K}\exp(-K(z+z'))\mathrm{J}_0(\lambda\rho)\,d\lambda\right)$$

$$=\partial_{z'}^2\frac{1}{4\pi\sigma}\frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp(-\frac{\mu\sigma R_s^2}{4t})$$

$$=-\frac{1}{4\pi\sigma}\frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\left(-\frac{\mu\sigma 2R_s}{R_s 4t}\right)\partial_{z'}2(z+z')\exp(-\frac{\mu\sigma R_s^2}{4t}) \qquad (3\cdot 56)$$

$$=\frac{1}{2\pi^{3/2}\sigma}\theta^3\partial_{z'}(z+z')\exp(-\frac{\mu\sigma R_s^2}{4t})$$

$$=\frac{1}{2\pi^{3/2}\sigma}\theta^3(1+(-\frac{\mu\sigma}{4t}(z+z')^2)\exp(-\frac{\mu\sigma R_s^2}{4t})$$

$$=\frac{1}{2\pi^{3/2}\sigma}\theta^3 W(R_s)(1-2(z+z')^2\theta^2)$$

That is, (3· 53) equals

$$-\frac{1}{2\pi^{3/2}\sigma}\frac{1}{4}W(z+z')\theta^3\int_0^\infty \beta^3\exp\left(-\frac{\beta^2}{4}\right)\mathrm{J}_0(\beta r)d\beta$$

$$+\frac{1}{2\pi^{3/2}\sigma}\theta^3 W(R_s)(1-2(z+z')^2\theta^2) \qquad (3\cdot 57)$$

$$+\frac{1}{2\pi^{3/2}\sigma}W(z+z')(z+z')\theta^4\int_0^\infty \beta^2\exp\left(-\frac{\beta^2}{4}\right)\mathrm{J}_0(\beta r)d\beta$$

which agrees with Hohmann [2].

We now consider

$$
\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{k_2^2}{4\pi\sigma}\frac{\exp(-ik_2R_s)}{R_s}\right)\\
=&\partial_t\mathcal{L}^{-1}\left(\frac{-\mu_o\sigma}{4\pi\sigma}\frac{\exp(-ik_2R_s)}{R_s}\right)\\
=&-\frac{\mu}{4}\partial_t\frac{\sqrt{\mu\sigma}}{2(\pi t)^{\frac{3}{2}}}\exp(-\frac{\mu\sigma R_s^2}{4t})\\
=&-\frac{\mu}{4\pi^{3/2}}\partial_t\theta W(R_s)
\end{aligned}
\qquad (3\cdot 58)
$$

which agrees with Hohmann's [2] result.

Finally, we consider

$$
\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}(ik_2R_s+1)\frac{\exp(-ik_2R_s)}{R_s^3}\right) \qquad (3\cdot 59)
$$

assume $x_i - x_i' \neq 0$ for at least one $i(=1,2,3)$, then

$$
\begin{aligned}
=&-(x_i-x_i')^{-1}\partial_{x_i}\mathcal{L}^{-1}\left(\frac{1}{4\pi\sigma}\frac{\exp(-ik_2R_s)}{R_s}\right)\\
=&-(x_i-x_i')^{-1}\partial_{x_i}\frac{1}{4\pi\sigma}\frac{\sqrt{\mu\sigma}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp(-\frac{\mu\sigma R_s}{4t})\\
=&-\frac{1}{2\pi^{3/2}\sigma}\frac{\theta}{2}2\frac{-\mu\sigma}{4t}\exp(-\frac{\mu\sigma R_s^2}{4t})\\
=&\frac{1}{2\pi^{3/2}\sigma}\theta^3W(R_s)
\end{aligned}
$$

which does not agree with Hohmann's [2] results. This difference is due to a misprint in Hohmann [2], where the 3/2 power of π in the last line does not appear.

Chapter 4
Equivalence of the Weak Form of the Vector Diffusion Equation and Hohmann's Integral Equation.

4.1 Introduction

In this chapter we investigate the validity of Hohmann's [2] method of approximating the electric field in the case of an insulating upper half-space. The main problem in this case is that there is no Green's tensor since the operator $\nabla \times \nabla \times + \sigma(x)\partial_t$ has a non-zero kernel. Specifically, if u is a transverse infinitely differentiable function with compact support in the upper half-space then $(\nabla \times \nabla \times + \sigma(x)\partial_t)u = 0$. Thus, the inverse of the operator cannot exist without further restrictions. Hohmann's [2] method implicitly places a restriction on the solution, viz. that it is the limit as $\epsilon \to 0$ in some sense of the solutions to $(\nabla \times \nabla \times + \mu\sigma(x)\partial_t + \epsilon\partial_t^2)u = f$, $\sigma_{air} = 0$, which satisfy a radiation condition at infinity. However, we can derive a fairly weak result for a transverse source. We recall the definition of the space in which we sought solutions,

$$\mathrm{M} = \mathrm{L}^2([0,T];\Phi_O) \times \mathcal{I}$$

(where $\mathcal{I}$ was the space of initial conditions) with norm

$$||(u,u_0)||_{\mathrm{M}} = \sqrt{\int_0^T ||u||^2_{\Phi_O}\,dt + < \sigma u_0, u_0 >_{\mathrm{L}^2(\mathrm{R}^3)}}.$$

Our first result shows that the solutions to the vector diffusion equation with $\sigma_{air} \neq 0$ are bounded in $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$ independently of the conductivity of the upper half-space, if the source can be written as

$$J(t,\underline{x}) = \nabla \times Y(t,\underline{x})$$

where

$$Y(t,\underline{x}) \in \mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3), \nabla \times Y(t,\underline{x}) \in \mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3).$$

We note that

$$< \nabla \times Y(t), h(t) >_{\mathrm{L}^2(\mathrm{R}^3)} = - < Y(t), \nabla \times h(t) >_{\mathrm{L}^2(\mathrm{R}^3)}$$

,for

$$h(t,x) \in \mathrm{C}_c^\infty([0,T];\mathrm{C}_c^\infty(\mathrm{R}^3)^3).$$

Thus $||J||_{\overline{\mathrm{M}}'} \leq ||Y||_{\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3)}$. Now , our map, G, from the antidual $\overline{\mathrm{M}}'$ to M has norm less than or equal to 1 for fixed σ , by our previous estimates , so

$$||G(\nabla \times Y)||_{\mathrm{M}} \leq ||Y||_{\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3)}.$$

We now note that

$$||Y||_{\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3)} \geq ||G(\nabla \times Y)||_{\mathrm{M}} \geq \sigma_{ground}||G(\nabla \times Y)||_{\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)}$$

i.e. the solution is bounded in the ground independently of the conductivity in the air.

If the source has support entirely in the ground , then since J acts on

$$h \in \mathrm{C}_c^\infty([0,T];\mathrm{C}_c^\infty(\mathrm{R}^3)^3).$$

via the inner product in $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$ we have $||J||_{\overline{\mathrm{M}}'} \leq ||J||_{\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)}$ and as before this leads to a bound for the $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$ norm of the solution,$G(J)$, independent of σ_{air}.

Thus, there is a subsequence of the sequence of solutions, ordered by the value of the conductivity of the upper half-space, which converges weakly in $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$.

However, it is unclear as to how to prove that this limit satisfies the weak differential equation.

In view of these difficulties, we propose in this chapter to

(1) Show that Hohmann's integral equation and the weak form of the vector diffusion equation are equivalent.

(2) Show that the kernel used in Hohmann's integral equation is a bounded operator on $L^2([0,T];(L^2(R^3_-))^3)$ and that therefore there is a solution, under certain conditions, to the integral equation.

(3) As a check on the validity of the solution, we show that the field obtained in the air by allowing the current source to approach the air/ground interface from the ground is the same as that obtained by allowing the current source to approach the interface from the air, if the source is transverse.

(4) We examine the large negative z behaviour (i.e. in the air) of the electric field generated by a source in the ground.

(5) Finally, we examine the existence of surface charges and the boundary conditions at interfaces.

4.2 Equivalence of the Weak Form of the Vector Diffusion Equation and Hohmann's Integral Equation

In this section we show that the weak form of the vector diffusion equation

$$(\nabla \times \nabla \times +\mu\sigma(x)\partial_t)u = -\mu\partial_t J \qquad (4 \cdot 1)$$

is equivalent to an integral equation involving the field $E'(\underline{x}, t)$ generated by the source J in the absence of an ore body, the fundamental solution G to the above equation and the conductivity contrast of the ore-body $\sigma_V = (\sigma_{body} - \sigma_{Host})\chi_{body}$ where χ_{body} is the characteristic function of the ore-body. We first consider the case in which the σ is non-vanishing.

Consider $E'(\underline{x}, t) \in L^2([0,T]; D(\nabla \times \nabla \times))$ such that for

$$J(\underline{x}, t) \in \{S \in C_c^\infty(R^3 \times [0,T]) | \nabla \cdot S(\underline{x}, t) = 0, \forall t \in [0,T]\}$$

the equation

$$\nabla \times \nabla \times E'(\underline{x}, t) + \mu \sigma_{Host} \partial_t E'(\underline{x}, t) = -\mu \partial_t J(\underline{x}, t) \quad (4 \cdot 2)$$

with initial condition

$$E'(\underline{x}, t) = 0 \; \forall t \le 0 \text{ and } \forall x \in R^3$$

holds weakly, where

$$\sigma_{Host}(x) = \begin{cases} \sigma_+, & x \in R^3_+ \\ \sigma_-, & x \in R^3_- \end{cases}$$

with σ_+, σ_- both positive constants. (That is, $E'(\underline{x}, t)$ is the solution to the vector diffusion equation in the absence of an ore-body.)

Let $G(\underline{x}, \underline{x}', t)$ be the solution of, in the sense of distributions

$$\nabla \times \nabla \times G(\underline{x}, \underline{x}', t) + \mu \sigma_{Host} \partial_t G(\underline{x}, \underline{x}', t) = -\mu \partial_t I \delta(x, t) \quad (4 \cdot 3)$$

$$G(\underline{x}, \underline{x}', t) = 0, t \le 0 \quad (4 \cdot 4)$$

(I is the identity tensor.) G is a distribution on the set $\mathcal{D}_+$ of $C_c^\infty(R \times R^3)$ functions $\phi(t, x)$ with support in $t > 0$, valued in the space of linear maps from $L^2([0,T]; (L^2(R^3))^3)$ to $D(\nabla \times \nabla \times)$, i.e.

$$G \in \mathcal{D}'_+(L(L^2([0,T]; (L^2(R^3))^3); D(\nabla \times \nabla \times))$$

i.e. G is the causal solution to (4· 3) (Lions and Magnes [1]). Hohmann[2] uses this Green's dyadic, though this is not immediately apparent because of a misprint in the paper.

Let $\mathrm{E} \in \mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3)$ be the solution of

$$\mathrm{E}(\underline{x},t) = \mathrm{E}'(\underline{x},t) + \mathrm{G} * \sigma_{\mathrm{V}}\mathrm{E}(\underline{x},t) \qquad (4 \cdot 5)$$

with

$$\mathrm{E}(\underline{x},t) = 0, t \leq 0, \qquad (4 \cdot 6)$$

where $\sigma_{\mathrm{V}} = (\sigma_{Body} - \sigma_{Host})\chi_{Body}$ and χ_{Body} is the characteristic function of the body. The convolution $\mathrm{G} * \sigma_{\mathrm{V}}\mathrm{E}$ is well defined since by the results of chapter 2 we have that G is a bounded operator on $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3)$.

Now if,

$$\mathrm{E}(\underline{x},t) = \mathrm{E}'(\underline{x},t) + (\mathrm{G} * \sigma_{\mathrm{V}}\mathrm{E})(\underline{x},t) \qquad (4 \cdot 7)$$

holds then

$$\mu\sigma_{Host}\partial_t\mathrm{E}(\underline{x},t) = \mu\sigma_{Host}\partial_t\mathrm{E}'(\underline{x},t) + \mu\sigma_{Host}\partial_t\mathrm{G} * \sigma_{\mathrm{V}}\mathrm{E}(\underline{x},t) \qquad (4 \cdot 8)$$

weakly, i.e. in $\mathcal{D}'(\mathrm{R}^3)$. Noting that,

$$\partial_t \int_0^\infty \mathrm{G} * \mathrm{E}\; dt' = \int_0^\infty (\partial_t\mathrm{G}) * \mathrm{E}\; dt' + \mathrm{G} * \mathrm{E}(0).$$

(Since $\mathrm{G}(\underline{x},\underline{x}',t) = 0, t \leq 0$.) we see that (4· 8) equals

$$\begin{aligned} &- \nabla \times \nabla \times \mathrm{E}'(\underline{x},t) - \mu\partial_t\mathrm{J}(\underline{x},t) \\ &+ \int_0^t \int_{\mathrm{V}} (-\nabla \times \nabla \times \mathrm{G}(\underline{x},\underline{x}',t-t') - \mu\delta(\underline{x}-\underline{x}',t-t'))\sigma_{\mathrm{V}}\mathrm{E}(x',t')dv'dt' \\ &+\mu\sigma_{Host}\int_{\mathrm{V}} \mathrm{G}(\underline{x},\underline{x}',0)\sigma_{\mathrm{V}}\mathrm{E}(x',t)dv' \qquad (4 \cdot 9) \\ &= - \nabla \times \nabla \times (\mathrm{E}'(\underline{x},t) + \int_0^t \int_{\mathrm{V}} \mathrm{G}(\underline{x},\underline{x}',t,t')\sigma_{\mathrm{V}}\mathrm{E}(x',t')dv'dt') \\ &- \mu\partial_t\mathrm{J}(\underline{x},t) - \sigma_{\mathrm{V}}\partial_t\mathrm{E}(\underline{x},t) \end{aligned}$$

Therefore,

$$\nabla \times \nabla \times \mathrm{E}(\underline{x},t) + \mu(\sigma_{Host} + \sigma_{\mathrm{V}})\partial_t \mathrm{E}(\underline{x},t) = -\mu\partial_t \mathrm{J}(\underline{x},t) \tag{4 $\cdot$ 10}$$

but $\sigma_{Host} + \sigma_{\mathrm{V}} = \sigma$, thus

$$\nabla \times \nabla \times \mathrm{E}(\underline{x},t) + \mu\sigma\partial_t \mathrm{E}(\underline{x},t) = -\mu\partial_t \mathrm{J}(\underline{x},t) \tag{4 $\cdot$ 11}$$

in the weak sense. That is $\mathrm{E}'(\underline{x},t)$ satisfies the weak form of the vector diffusion equation.

Conversely, if $\mathrm{E}(\underline{x},t)$ satisfies the weak form of the vector diffusion equation,

$$\nabla \times \nabla \times \mathrm{E}(\underline{x},t) + \mu\sigma\partial_t \mathrm{E}(\underline{x},t) = -\mu\partial_t \mathrm{J}(\underline{x},t) \tag{4 $\cdot$ 12}$$

then

$$\nabla \times \nabla \times \mathrm{E}(\underline{x},t) + \mu\sigma_{Host}\partial_t \mathrm{E}(\underline{x},t) = -\mu\sigma_{\mathrm{V}}\partial_t \mathrm{E}(\underline{x},t) - \mu\partial_t \mathrm{J}(\underline{x},t) \tag{4 $\cdot$ 13}$$

i.e. $\mathrm{E}(\underline{x},t)$ is response to source $\sigma_{\mathrm{V}}\mathrm{E}(\underline{x},t) + \mathrm{J}(\underline{x},t)$ i.e

$$\begin{aligned} \mathrm{E}(\underline{x},t) =& \mathrm{G} * (\sigma_{\mathrm{V}}\mathrm{E} + \mathrm{J})(\underline{x},t) \\ =& \mathrm{G} * \mathrm{J}(\underline{x},t) + \mathrm{G} * \sigma_{\mathrm{V}}\mathrm{E}(\underline{x},t) \\ =& \mathrm{E}'(\underline{x},t) + \mathrm{G} * \sigma_{\mathrm{V}}\mathrm{E}(\underline{x},t). \end{aligned} \tag{4 $\cdot$ 14}$$

That is, $\mathrm{E}(\underline{x},t)$ satisfies the integral equation.

The case $\sigma_+ = 0$ is similar. Although we cannot show *a priori* that $\mathrm{E}'(\underline{x},t) \in \mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$ or that there is a fundamental solution of the vector diffusion equation which is a bounded integral operator on $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$, we show explicitly in the following section that this is the case. Thus, considering the weak form of the vector diffusion equation in the ground (in $\mathcal{D}'(\mathrm{R}^3_-)$) the argument above from equation (4·10) onwards holds in this sense. Thus, in the case of an insulating upper half-space, the integral equation and the weak form of the vector diffusion equation are equivalent in the ground.

4.3 Existence of the Solution to Hohmann's Integral Equation.

In this section we show that the operator $L(f) = \mathrm{G} * \sigma_{\mathrm{V}} f$, where G is defined as in the previous section, with $\sigma_{+} = 0$, is a continuous operator on $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$. We then show that for an appropriate source in the ground that the field $\mathrm{E}'(\underline{x},t)$ in the absence of an ore body is in $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$ and that therefore, for $|\sigma_{body} - \sigma_{Host}|$ sufficiently small there is a solution to $\mathrm{E}(\underline{x},t) = \mathrm{E}'(\underline{x},t) + L(\mathrm{E})$ in $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$.

We first note that the free-space Green's function is a bounded operator on $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3))^3)$ and therefore also on $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$, *a priori*, by the theory of parabolic partial differential equations with constant coefficients. Thus we need only show that the part of $L(f)$ involving the scattering component of the Green's function is a continuous operator on $\mathrm{L}^2([0,T];(\mathrm{L}^2(\mathrm{R}^3_-))^3)$.

Note that throughout this section, unlike the previous, we follow the geophysical convention of the ground being in the region $z \geq 0$. Thus, to prove our contention it suffices to show that the elements of the scattering term of Hohmann's [2] tensor define bounded operators on $\mathrm{L}^2\left(\mathrm{R}^3_+ \times [0,T]\right)$. We first need a few lemmas.

Lemma 1.

$$\max x^m \exp(-\alpha x) = \begin{cases} \left(\frac{(\exp(-1)m)}{\alpha}\right)^m, & m > 0; \\ 1, & m = 0. \end{cases}$$

Proof

If $m = 0$, then the maximum of $\exp(-\alpha x)$ occurs at $x = 0$. If $m > 0$,

$$\partial_x x^m \exp(-\alpha x) = x^{m-1} \exp(-\alpha x)(m - \alpha x).$$

Thus, for $x \neq 0$, the above has a root at $x = \frac{m}{\alpha}$ and, by inspection of the sign of the derivative near this point, the maximum value of $x^m \exp(-\alpha x)$ is

$$\left(\frac{(\exp(-1)m)}{\alpha}\right)^m .$$

Let

$$\begin{aligned} x_1 &= \inf_{\underline{x} \in \mathrm{V}} x & x_2 &= \sup_{\underline{x} \in \mathrm{V}} x \\ y_1 &= \inf_{\underline{x} \in \mathrm{V}} y & y_2 &= \sup_{\underline{x} \in \mathrm{V}} y \\ z_1 &= \inf_{\underline{x} \in \mathrm{V}} z & z_2 &= \sup_{\underline{x} \in \mathrm{V}} z \end{aligned}$$

where V is the volume modelling the ore body in the ground. Also, let

$$\mathcal{C} = [x_1 - \epsilon, x_2 + \epsilon] \times [y_1 - \epsilon, y_2 + \epsilon] \times [z_1 - \epsilon, z_2 + \epsilon]$$

where $\epsilon > 0$ is such that $\mathcal{C}$ is entirely below ground.

Lemma 2

If $F(\underline{x}, t) \in \mathrm{L}^2([0,T]) \times \mathrm{L}^2(\mathrm{R}^2) \times \mathrm{L}^2(\mathrm{R}_+)$ with compact support in $\mathcal{C}$ and

$$g(\underline{x}, \underline{x}', t) = \frac{(z + z')^m}{t^l} \exp\left(-\frac{(z + z')^2}{\phi t}\right) g_1(x, x', y, y', t)$$

and

$$\|\int_{\mathrm{R}^2} g_1(x, x', y, y', t) f(x', y')\|_{\mathrm{L}^2(\mathrm{R}^2)} \leq \frac{K}{t^k} \|f(x', y')\|_{\mathrm{L}^2(\mathrm{R}^2)}$$

for some constant K independent of $f(x, y)$ and $2(l + k) - m > \frac{1}{2}$. Then,

$$\|\int_{\mathrm{R}^3_+} g(\underline{x}, \underline{x}', t - t') F(\underline{x}', t') d\underline{x}'\|_{\mathrm{L}^2(\mathrm{R}^3_+)} \leq H \|F(\underline{x}, t')\|_{\mathrm{L}^2(\mathrm{R}^3_+)}$$

where H is independent of F and $t - t' > 0$.

Proof

Since

$$F(\underline{x}, t) \in \mathrm{L}^2([0,T]) \times \mathrm{L}^2(\mathrm{R}^2) \times \mathrm{L}^2(\mathrm{R}_+),$$

$F(\underline{x}, t) = F_1(x,y)F_2(z)F_3(t)$ where

$$F_1(x,y) \in \mathrm{L}^2(\mathrm{R}^2),$$

$$F_2(z) \in \mathrm{L}^2(\mathrm{R}_+),$$

$$F_3(t) \in \mathrm{L}^2([0,T]).$$

Therefore,

$$\begin{aligned}
&|| \int_{\mathrm{R}^3_+} g(\underline{x}, \underline{x}', t-t')F(\underline{x}', t')||_{\mathrm{L}^2(\mathrm{R}^3_+)} \\
&= || \int_{\mathrm{R}^2} g_1(x, x', y, y', t-t')F_1(x', y')dx'dy'||_{\mathrm{L}^2(\mathrm{R}^2)} \\
&\times || \int_{\mathrm{R}_+} \frac{(z+z')^m}{(t-t')^l} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) F_2(z')dz'||_{\mathrm{L}^2(\mathrm{R}_+)} \\
&\qquad \times |F_3(t')| \\
&\leq \frac{K}{(t-t')^k}||F_1(x', y')||_{\mathrm{L}^2(\mathrm{R}^2)} \\
&\times || \int_{\mathrm{R}_+} \frac{(z+z')^m}{(t-t')^l} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) F_2(z')dz'||_{\mathrm{L}^2(\mathrm{R}_+)} \\
&\qquad \times |F_3(t')| \\
&= K||F_1(x', y')||_{\mathrm{L}^2(\mathrm{R}^2)} \\
&\times || \int_{\mathrm{R}_+} \frac{(z+z')^m}{(t-t')^{(l+k)}} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) F_2(z')dz'||_{\mathrm{L}^2(\mathrm{R}_+)} \\
&\qquad \times |F_3(t')|
\end{aligned}$$

Now,

$$\left|\int_{\mathrm{R}_+} \frac{(z+z')^m}{(t-t')^{(l+k)}} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) F_2(z')dz'\right|$$
$$\leq \int_{[z_1-\epsilon, z_2+\epsilon]} \frac{(z+z')^m}{(t-t')^{(l+k)}} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) |F_2(z')|dz'.$$

(Since F has compact support in $\mathcal{C}$.) Now, by lemma 1, placing $x = \frac{1}{t-t'}$, $\alpha = \frac{(z+z')^2}{\phi}$ and $m = l + k$, we have

$$\frac{(z+z')^m}{(t-t')^{(l+k)}} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right)$$
$$\leq \frac{(z+z')^m}{(z+z')^{2(l+k)}}(\exp(-1)(l+k)\phi)^{l+k}$$
$$=(z+z')^{-2(l+k)+m} J$$
$$\leq J(z+(z_1-\epsilon))^{-2(l+k)+m}$$

(where $J = (\exp(-1)(l+k)\phi)^{l+k}$) since $z' \in [z_1-\epsilon, z_2+\epsilon]$. Thus,

$$\int_{[z_1-\epsilon, z_2+\epsilon]} \frac{(z+z')^m}{(t-t')^{(l+k)}} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) |F_2(z')|dz'$$
$$\leq J(z+(z_1-\epsilon))^{-2(l+k)+m} \int_{[z_1-\epsilon, z_2+\epsilon]} |F_2(z')|dz'$$
$$\leq J(z+(z_1-\epsilon))^{-2(l+k)+m} \sqrt{\int_{[z_1-\epsilon, z_2+\epsilon]} 1^2 dz'} ||F_2||_{\mathrm{L}^2(\mathrm{R}_+)}.$$

Letting $L = \sqrt{\int_{[z_1-\epsilon, z_2+\epsilon]} 1^2 dz'} J$, the last inequality implies that

$$|| \int_{\mathrm{R}_+} \frac{(z+z')^m}{(t-t')^{(l+k)}} \exp\left(-\frac{(z+z')^2}{\phi(t-t')}\right) F_2(z')dz' ||_{\mathrm{L}^2(\mathrm{R}_+)}$$
$$\leq L||(z+(z_1-\epsilon))^{-2(l+k)+m}||_{\mathrm{L}^2(\mathrm{R}_+)} ||F_2||_{\mathrm{L}^2(\mathrm{R}_+)}$$

and therefore,

$$
\begin{aligned}
||\int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')d\underline{x}'||_{\mathrm{L}^2(\mathrm{R}^3_+)} & \\
&\leq KL||(z+(z_1-\epsilon))^{-2(l+k)+m}||_{\mathrm{L}^2(\mathrm{R}_+)} \\
&\quad \times ||F_2||_{\mathrm{L}^2(\mathrm{R}_+)}|F_3(t')|\,||F_1(x',y')||_{\mathrm{L}^2(\mathrm{R}^3)} \\
&= H||F(\underline{x},t')||_{\mathrm{L}^2(\mathrm{R}^3_+)}
\end{aligned}
$$

where

$$H = KL||(z+(z_1-\epsilon))^{-2(l+k)+m}||_{\mathrm{L}^2(\mathrm{R}_+)}$$

Corollary 1

Let F, g be as in Lemma 2, then

$$||\int_0^t \int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,\mathrm{T}])} \leq C||F||_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,\mathrm{T}])}.$$

Proof

Note that

$$
\begin{aligned}
||\int_0^t \int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+)} & \\
&\leq \int_0^t ||\int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+)}\, dt' \\
&\leq H\int_0^t ||F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+)}dt' \\
&\leq H\int_0^T ||F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+)}dt' \\
&\leq H\sqrt{\int_0^T 1dt'}\, ||F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,T])} \\
&= H\sqrt{T}||F(\underline{x}',t')||_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,T])}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&||\int_0^t \int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')||^2_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,\mathrm{T}])} \\
&= \int_0^T ||\int_0^t \int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')||^2_{\mathrm{L}^2(\mathrm{R}^3_+)} dt \\
&\leq \int_0^T H^2T||F(\underline{x}',t')||^2_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,T])} dt \\
&= H^2T^2||F(\underline{x}',t')||^2_{\mathrm{L}^2(\mathrm{R}^3_+\times[0,T])}.
\end{aligned}
$$

Now, since the tensor space is dense in $\mathrm{L}^2(\mathcal{C}\times[0,T])$ we see that the operation

$$F(\underline{x},t) \mapsto \int_0^t \int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')F(\underline{x}',t')d\underline{x}'dt'$$

defines a bounded linear operator from $\mathrm{L}^2(\mathcal{C}\times[0,T])$ to $\mathrm{L}^2\left(\mathrm{R}^3_+\times[0,T]\right)$. It is easy to see that

$$F(\underline{x},t) \mapsto \chi_{\mathrm{V}}(\underline{x})F(\underline{x},t)$$

(where χ_{V} is the characteristic function of the body V) is a bounded linear operator from $\mathrm{L}^2\left(\mathrm{R}^3_+\times[0,T]\right)$ to $\mathrm{L}^2(\mathcal{C}\times[0,T])$. Thus,

$$F(\underline{x},t) \mapsto \int_0^t \int_{\mathrm{R}^3_+} g(\underline{x},\underline{x}',t-t')\chi_{\mathrm{V}}(\underline{x}')F(\underline{x}',t')d\underline{x}'dt'$$

is a bounded linear operator on $\mathrm{L}^2\left(\mathrm{R}^3_+\times[0,T]\right)$. We now estimate some operator norms.

Lemma 3

As convolution operators on $\mathrm{L}^2\left(\mathrm{R}^2\right), t>0$

$$||\partial_{x_i}\partial_{x_j}\int_0^\infty \lambda^n \mathrm{J}_0(\lambda\rho)\ \exp(-\lambda^2\frac{t}{K})d\lambda|| \leq (\frac{(n+1)}{2}\exp(-1))^{\frac{n+1}{2}}\left(\frac{K}{t}\right)^{\frac{n+1}{2}} \qquad (1)$$

$$||\int_0^\infty \lambda^n \mathrm{J}_0(\lambda\rho)\ \exp(-\lambda^2\frac{t}{K})d\lambda|| \leq (\frac{(n-1)}{2}\exp(-1))^{\frac{n-1}{2}} \left(\frac{K}{t}\right)^{\frac{n-1}{2}} \quad (2)$$

$$||\exp(-\frac{(x^2+y^2)K}{t})|| \leq \frac{t}{2K} \quad (3)$$

$$||\partial_{x_i}\exp(-\frac{(x^2+y^2)K}{t})|| \leq \sqrt{\exp(-1)\frac{t}{2K}} \quad (4)$$

$$||(x^2+y^2)\exp(-\frac{(x^2+y^2)K}{t})|| \leq \frac{t^2}{2K^2} \quad (5)$$

$$||\partial_{x_i}\partial_{x_j}\exp\left(-r^2\frac{K}{t}\right)|| \leq 2\exp(-1) \quad (6)$$

Proof

We prove the inequalities by showing that corresponding inequalities are true for the L^∞ norms of the Fourier transforms. Recalling that for a radially symmetric function of two variables,

$$\mathcal{F}(f(x,y);u,v) = \mathcal{H}_0(f(r);\rho)$$

where $\mathcal{F}(f)$ denotes the Fourier transform:

$$\frac{1}{2\pi}\int_0^\infty\int_0^\infty f(x,y)\exp(-i(xu+yv))dx$$

and $\mathcal{H}_\mu(f)$ is the Hankel transform of order μ:

$$\int_0^\infty r\mathrm{J}_\mu(\rho r)f(r)dr$$

with $r^2 = x^2+y^2, \rho^2 = u^2+v^2$, we see that

$$\mathcal{F}(\partial_{x_i}\partial_{x_j}\int_0^\infty \lambda^n \mathrm{J}_0(\lambda\rho)\ \exp(-\lambda^2\frac{t}{K})d\lambda)(u,v)$$
$$= -iu(-iu)^{\delta_{ij}}(-iv)^{1-\delta_{ij}}\sqrt{u^2+v^2}^{n-1}\exp(-(u^2+v^2)\frac{t}{K}).$$

Changing to radial co-ordinates in transform space,

$$u = \rho\cos(\theta), v = \rho\sin(\theta)$$

and the absolute value of the above transform is

$$\begin{aligned}
&|-i(\rho\cos(\theta))(-i(\rho\cos(\theta)))^{\delta_{ij}}(-i(\rho\sin(\theta)))^{1-\delta_{ij}}\rho^{n-1}\exp(-\rho^2\frac{t}{K})| \\
&\leq (\rho^2)^{\frac{n+1}{2}}\exp(-\rho^2\frac{t}{K}) \\
&\leq (\frac{n+1}{2}\exp(-1))^{\frac{n+1}{2}}\left(\frac{K}{t}\right)^{\frac{n+1}{2}}
\end{aligned}$$

by lemma 1. A similar argument gives the corresponding bound for the other Hankel transform. Noting that

$$\mathcal{F}\left(\exp\left(-\frac{(x^2+y^2)K}{t}\right)\right) = \frac{1}{2}\frac{t}{K}\exp\left(-(u^2+v^2)\frac{t}{4K}\right)$$

and thus

$$\left|\mathcal{F}\left(\exp\left(-\frac{(x^2+y^2)K}{t}\right)\right)\right| \leq \frac{t}{2K}$$

giving inequality (3).

Similarly,

$$\mathcal{F}\left(\partial_x\exp\left(-\frac{(x^2+y^2)K}{t}\right)\right) = -iu\frac{1}{2}\frac{t}{K}\exp\left(-(u^2+v^2)\frac{t}{4K}\right)$$

and thus

$$\left|\mathcal{F}\left(\partial_x\exp\left(-\frac{(x^2+y^2)K}{t}\right)\right)\right| = \frac{1}{2}(\rho^2)^{\frac{1}{2}}\frac{t}{K}\exp\left(-\rho^2\frac{t}{K}\right)$$

and by Lemma 1, with $x = \rho^2, m = \frac{1}{2}$

$$\leq \frac{1}{2}\frac{t}{K}\left(\frac{\exp(-1)\frac{1}{2}}{\frac{t}{4K}}\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\frac{t}{K}\left(\exp(-1)\frac{1}{2}\right)^{\frac{1}{2}} 2\sqrt{\frac{t}{K}}$$

$$= \left(\exp(-1)\frac{1}{2}\right)^{\frac{1}{2}} \sqrt{\frac{t}{K}}$$

giving inequality (4) (By symmetry, the argument also holds for ∂_y). Inequality (6) is proved in a similar manner, by an application of Lemma 1, with $x = \rho^2, m = 1$.

Noting that

$$\left|\mathcal{F}\left((x^2+y^2)\exp\left(-\frac{(x^2+y^2)K}{t}\right)\right)\right|$$

$$= \left|\frac{1}{2\pi}\int_0^\infty\int_0^\infty (x^2+y^2)\exp\left(-\frac{(x^2+y^2)K}{t}\right)\exp(-i(ux+vy))\,dxdy\right|$$

$$\leq \frac{1}{2\pi}\int_0^\infty\int_0^\infty \left|(x^2+y^2)\exp\left(-\frac{(x^2+y^2)K}{t}\right)\right| dxdy$$

$$= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} r^2\exp\left(-\frac{r^2K}{t}\right) rdrd\theta$$

$$= \int_0^\infty r^3\exp\left(-\frac{r^2K}{t}\right) dr$$

$$= \frac{1}{2}\int_0^\infty r\exp\left(-\frac{rK}{t}\right) dr$$

$$= \frac{1}{2}\left(\frac{t}{K}\right)^2$$

$$= \frac{t^2}{2K^2}$$

giving inequality (5). We now turn to Hohmann's form of the scattering terms of the tensor and show that the elements of this part of the tensor satisfy the conditions of lemma 2 and thus by corollary 1 are bounded operators from $L^2(\mathcal{C}\times[0,T])$ to $L^2(R^3_+\times[0,T])$.

In Hohmann [2] the scattering component of the tensor is written in terms of

$$\begin{aligned}
\theta =& \sqrt{\left(\frac{\mu\sigma_{ground}}{4t}\right)} \\
R =& \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \\
R_I =& \sqrt{(x-x')^2+(y-y')^2+(z+z')^2} \\
W(x) =& \frac{\exp\left(-\theta^2 x^2\right)}{t} \\
g(\underline{x},\underline{x}',t) =& \frac{1}{4\pi^{3/2}}\theta W(R) \\
r =& \sqrt{(x-x')^2+(y-y')^2} \\
\rho =& \theta r \\
\beta =& \frac{\lambda}{\theta}.
\end{aligned}$$

Hohmann's [2] form of the scattering component of the tensor is:

$$\begin{pmatrix}
\partial_{x'}(x-x')\alpha_1+\alpha_2 & \partial_{y'}(x-x')\alpha_1 & \partial_{z'}(x-x')\alpha_3 \\
\partial_{x'}(y-y')\alpha_1 & \partial_{y'}(y-y')\alpha_1+\alpha_2 & \partial_{z'}(y-y')\alpha_3 \\
\partial_{x'}(z+z')\alpha_3 & \partial_{y'}(z+z')\alpha_3 & \partial_{z'}(z+z')\alpha_3+\alpha_4
\end{pmatrix}$$

where

$$
\begin{aligned}
\alpha_1 =& \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\Big\{\frac{(z+z')}{r}\theta^3 W(z+z') \\
&\times \int_0^\infty \beta \exp\left(-\frac{\beta^2}{4}\right) \mathrm{J}_1(\beta\rho)\, d\beta \\
&- \theta^3 W(R_I)\Big\} \\
\alpha_2 =& \frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\Big\{\theta(z+z')W(z+z') \\
&\times \int_0^\infty \beta^2 \exp(-\frac{\beta^2}{4})\mathrm{J}_0(\beta\rho)\, d\beta \\
&- \frac{1}{4}W(z+z')\int_0^\infty \beta^3 \exp(-\frac{\beta^2}{4})\mathrm{J}_0(\beta\rho)\, d\beta \\
&+ \left(1 - 2\theta^2(z+z')^2\right) W(R_I)\Big\} \\
\alpha_3 =& \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta^3 W(R_I) \\
\alpha_4 =& -\mu\partial_t g(R_I, t)
\end{aligned}
$$

4.3.1 Boundedness of terms involving α_1

Expanding the expression for the first term of α_1 we have

$$
\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left\{\frac{(z+z')}{r}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}}\frac{1}{t}\exp(-\frac{\theta^2(z+z')}{4t})\int_{\frac{\lambda}{\theta}=0}^{\infty}\frac{\lambda}{\theta}\exp(-\frac{\lambda^2}{\theta^2 4})\mathrm{J}_0(\lambda r)\ d\beta\right\}
$$

Noting that

$$
\frac{d\beta}{d\lambda} = \frac{1}{\theta}
$$

and therefore

$$
d\beta = \frac{1}{\theta}d\lambda
$$

we have that the first term of α_1 is

$$
\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\frac{(z+z')}{r}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{1}{2}}\frac{\exp(-\frac{\theta^2(z+z')}{4t})}{t}\int_0^\infty \lambda\exp(-\frac{\lambda^2}{\theta^2 4})\mathrm{J}_0(\lambda r)\ d\lambda
$$

The second term of α_1 is

$$-\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}} \exp\left(-(z+z')^2\frac{\mu\sigma_{ground}}{4t}\right)\frac{\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t}$$

Consider, the terms of the tensor of the form

$$\partial_{x'}(x-x')\alpha_1 = -\partial_x(x-x')\alpha_1.$$

Noting that

$$\begin{aligned}
&\partial_x \int_0^\infty \mathrm{J}_0(\lambda r)\exp\left(-\lambda^2\frac{t}{\mu\sigma_{ground}}\right)d\lambda \\
&= -\frac{(x-x')}{r}\int_0^\infty \lambda^1 \mathrm{J}_1(\lambda r)\exp\left(-\lambda^2\frac{t}{\mu\sigma_{ground}}\right)d\lambda
\end{aligned}$$

it can be seen that the first term of $-\partial_x(x-x')\alpha_1$ is

$$\begin{aligned}
&\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}(z+z')\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{1}{2}}\exp\left(-\theta^2(z+z')\right) \\
&\partial_x^2\int_0^\infty \mathrm{J}_0(\lambda r)\exp\left(-\lambda^2\frac{t}{\mu\sigma_{ground}}\right)d\lambda \\
&= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\frac{\sqrt{\mu\sigma_{ground}}}{2}\frac{(z+z')\exp\left(-\theta^2(z+z')^2\right)}{t^{\frac{3}{2}}} \\
&\partial_x^2\int_0^\infty \mathrm{J}_0(\lambda r)\exp\left(-\lambda^2\frac{t}{\mu\sigma_{ground}}\right)d\lambda.
\end{aligned}$$

Now,

$$\|\partial_x^2\int_0^\infty \mathrm{J}_0(\lambda r)\exp\left(-\lambda^2\frac{t}{\mu\sigma_{ground}}\right)d\lambda\| \le \sqrt{\frac{1}{2e}}\sqrt{\frac{4\mu\sigma_{ground}}{t}}$$

from inequality (1) in Lemma 3 and thus by Corollary 1 with

$$\begin{aligned}
m &= 1 \\
l &= \frac{3}{2} \\
k &= \frac{1}{2}
\end{aligned}$$

we have that the first term of $-\partial_x(x-x')\alpha_1$ is a bounded operator from $L^2(\mathcal{C}\times[0,T])$ to $L^2\left(R^3_+\times[0,T]\right)$.

Now, consider the second term of $-\partial_x(x-x')\alpha_1$ i.e.

$$
\begin{aligned}
&-\partial_x(x-x')\left(\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-(z+z')^2\frac{\mu\sigma_{ground}}{4t}\right)\frac{\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t}\\
=&\frac{1}{2}\partial_x\frac{1}{t}\left(\sqrt{\frac{\mu\sigma_{ground}}{4t}}\exp\left(-(z+z')^2\frac{\mu\sigma_{ground}}{4t}\right)\partial_x\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\right)\\
=&\frac{\sqrt{\mu\sigma_{ground}}}{2}\frac{1}{t^{\frac{3}{2}}}\exp\left(-(z+z')^2\frac{\mu\sigma_{ground}}{4t}\right)\partial_x^2\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)
\end{aligned}
$$

Now, by Lemma 3, inequality (6)

$$\|\partial_x^2\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\|\le\frac{2}{e}$$

and we apply Lemma 2 and Corollary 1 with

$$
\begin{aligned}
k=&0\\
m=&0\\
l=&\frac{3}{2}.
\end{aligned}
$$

Thus $-\partial_x(x-x')\alpha_1$ defines a bounded operator from $L^2(\mathcal{C}\times[0,T])$ to $L^2\left(R^3_+\times[0,T]\right)$. The symmetry of α_1 respect to x and y means that a similar argument holds for all other terms of the tensor involving α_1's derivatives.

4.3.2 Boundedness of terms involving α_3

We now consider the terms involving α_3

$$
\begin{aligned}
\alpha_3=&\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta^3W(R_I)\\
=&\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}}\frac{\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)}{t}\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)
\end{aligned}
$$

Thus,

$$\partial_{z'}(x-x')\alpha_3$$

$$= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}} \frac{\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t}(x-x')\partial_{z'}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)$$

$$= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}} \frac{\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t}(x-x')$$

$$\left(-2(z+z')\frac{\mu\sigma_{ground}}{4t}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\right)$$

$$= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}} \frac{(z+z')}{t}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)$$

$$\left(-2(x-x')\frac{\mu\sigma_{ground}}{4t}\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\right)$$

$$= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}} \frac{(z+z')}{4t}\partial_x\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right).$$

Now, by inequality (4) of Lemma 3,

$$\|\partial_x \exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\| \le \sqrt{\frac{t}{e\frac{\mu\sigma_{ground}}{4}}}$$

and thus by Corollary 1 with

$$m = 1$$

$$l = \frac{5}{2}$$

$$k = -\frac{1}{2}$$

this is a bounded operator from $L^2(\mathcal{C}\times[0,T])$ to $L^2\left(R^3_+\times[0,T]\right)$. (By the symmetry of α_3 with respect to x and y, a similar argument holds for the $\partial_{z'}(y-y')\alpha_3$ term.)

Similarly,

$$\partial_{x'}(z+z')\alpha_3$$

$$= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{3}{2}}\partial_{x'}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)$$

$$= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4}\right)^{\frac{3}{2}}\frac{(z+z')}{t^{\frac{5}{2}}}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\partial_{x'}\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)$$

and as before

$$m = 1$$
$$l = \frac{5}{2}$$
$$k = -\frac{1}{2}$$

and thus defines a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R_+^3 \times [0,T])$. (The symmetry of α_3 with respect to x and y means that a similar arguments hold for the $\partial_{y'}(z+z')\alpha_3$ term.)

We now consider

$$\partial_{z'}(z+z')\alpha_3$$
$$= \alpha_3 + (z+z')\partial_{z'}\alpha_3$$
$$= \frac{1}{16\pi^{\frac{3}{2}}\sigma_{ground}}(\mu\sigma_{ground})^{\frac{3}{2}}\Big(\frac{\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t^{\frac{5}{2}}}$$
$$+(z+z')\frac{\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t^{\frac{5}{2}}}\partial_{z'}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\Big)$$
$$= \frac{1}{16\pi^{\frac{3}{2}}\sigma_{ground}}(\mu\sigma_{ground})^{\frac{3}{2}}\left(\frac{1}{t^{\frac{5}{2}}}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\right.$$
$$\left.+(z+z')\frac{\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)}{t^{\frac{5}{2}}}\left(-2(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\right)$$
$$= \frac{1}{16\pi^{\frac{3}{2}}\sigma_{ground}}(\mu\sigma_{ground})^{\frac{3}{2}}\frac{1}{t^{\frac{5}{2}}}\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)$$
$$-\frac{2}{4}\frac{1}{16\pi^{\frac{3}{2}}\sigma_{ground}}(\mu\sigma_{ground})^{\frac{7}{2}}\frac{(z+z')^2}{t^{\frac{7}{2}}}\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)$$

Now, by inequality (3) of Lemma 3,

$$\|\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\| \le \frac{t}{2\frac{\mu\sigma_{ground}}{4}}$$

and thus by Corollary 1 with

$$k = -1$$
$$l = \frac{5}{2}$$
$$m = 1$$

for the first term and

$$k = -1$$
$$l = \frac{7}{2}$$
$$m = 2$$

for the second term, this term of the tensor is also a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$.

4.3.3 Boundedness of terms involving α_4

$$\begin{aligned}
&\alpha_4 \\
&= -\mu \partial_t \frac{1}{4\pi^{\frac{3}{2}}} \sqrt{\mu\sigma_{ground}} \frac{1}{t^{\frac{3}{2}}} \exp\left(-{R_I}^2 \frac{\mu\sigma_{ground}}{4t}\right) \\
&= -\mu \frac{1}{4\pi^{\frac{3}{2}}} \Big\{ -\frac{3}{2} \sqrt{\mu\sigma_{ground}} \frac{1}{t^{\frac{5}{2}}} \exp\left(-{R_I}^2 \frac{\mu\sigma_{ground}}{4t}\right) \\
&\quad + \frac{1}{4} (\mu\sigma_{ground})^{\frac{3}{2}} \frac{1}{t^{\frac{7}{2}}} R_I^2 \exp\left(-{R_I}^2 \frac{\mu\sigma_{ground}}{4t}\right) \Big\} \\
&= -\mu \frac{1}{4\pi^{\frac{3}{2}}} \Big\{ \\
&\quad -\frac{3}{2} \sqrt{\mu\sigma_{ground}} \frac{1}{t^{\frac{5}{2}}} \exp\left(-(z+z')^2 \frac{\mu\sigma_{ground}}{4t}\right) \exp\left(-r^2 \frac{\mu\sigma_{ground}}{4t}\right) \\
&\quad + \frac{1}{4} (\mu\sigma_{ground})^{\frac{3}{2}} \frac{1}{t^{\frac{7}{2}}} (z+z')^2 \exp\left(-(z+z')^2 \frac{\mu\sigma_{ground}}{4t}\right) \exp\left(-r^2 \frac{\mu\sigma_{ground}}{4t}\right) \\
&\quad + \frac{1}{4} (\mu\sigma_{ground})^{\frac{3}{2}} \frac{1}{t^{\frac{7}{2}}} r^2 \exp\left(-(z+z')^2 \frac{\mu\sigma_{ground}}{4t}\right) \exp\left(-r^2 \frac{\mu\sigma_{ground}}{4t}\right) \Big\}.
\end{aligned}$$

Now,

$$\| \exp\left(-r^2 \frac{\mu\sigma_{ground}}{4t}\right) \| \leq \frac{t}{2\frac{\mu\sigma_{ground}}{4}}$$

and

$$\|r^2 \exp\left(-r^2 \frac{\mu\sigma_{ground}}{4t}\right) \| \leq \frac{t^2}{(2\frac{\mu\sigma_{ground}}{4})^2}.$$

Thus, in the first term, $k = -1, l = \frac{5}{2}, m = 0$, in the second term, $k = -1, l = \frac{7}{2}, m = 2$, and in the third term, $k = -2, l = \frac{7}{2}, m = 0$, and by Corollary 1, these terms and thus α_4 determine a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$.

4.3.4 Boundedness of terms involving α_2

Turning to the remaining element of the scattering component of the tensor,

$$\begin{aligned}
&\alpha_2 \\
&= \frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\Big\{\theta(z+z')W(z+z')\int_0^\infty \beta^2 \exp(-\frac{\beta^2}{4})J_0(\beta\rho)d\beta \\
&\quad - \frac{1}{4}W(z+z')\int_0^\infty \beta^3 \exp(-\frac{\beta^2}{4})J_0(\beta\rho)d\beta \\
&\quad + \left(1 - 2\theta^2(z+z')^2\right)W(R_I)\Big\}
\end{aligned}$$

Now, the first term of this is

$$\frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta(z+z')W(z+z')\int_0^\infty \beta^2 \exp(-\frac{\beta^2}{4})J_0(\beta\rho)d\beta$$

and since $\beta = \frac{\lambda}{\theta}$, the expression becomes

$$\begin{aligned}
&\frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta(z+z')W(z+z')\int_0^\infty \beta^2 \exp(-\frac{\beta^2}{4})J_0(\beta\rho)d\beta \\
&\frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta(z+z')W(z+z')\int_0^\infty \frac{\lambda^2}{\theta^2} \exp(-\frac{\lambda^2}{4\theta^2})J_0(\lambda r)\frac{1}{\theta}d\lambda \\
&= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta(z+z')W(z+z')\int_0^\infty \lambda^2 \exp(-\frac{\lambda^2}{4\theta^2})J_0(\lambda r)d\lambda \\
&= \frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{1}{2}}(z+z')\frac{\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)}{t} \\
&\int_0^\infty \lambda^2 \exp(-\frac{\lambda^2 t}{\mu\sigma_{ground}})J_0(\lambda r)d\lambda.
\end{aligned}$$

Now, by Lemma 3, inequality (2)

$$||\int_0^\infty \lambda^2 J_0(\lambda r)\ \exp(-\lambda^2 \frac{t}{\mu\sigma_{ground}})d\lambda|| \leq (\frac{1}{2}\exp(-1))^{\frac{1}{2}} \left(\frac{\mu\sigma_{ground}}{t}\right)^{\frac{1}{2}}.$$

Thus, by Corollary 1 with

$$k = \frac{1}{2}$$
$$l = \frac{3}{2}$$
$$m = 1$$

this term is a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$.

The second term of α_2 is

$$-\frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\frac{1}{4}W(z+z')\int_0^\infty \beta^3 \exp(-\frac{\beta^2}{4})J_0(\beta\rho)d\beta$$
$$= -\frac{1}{4}\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\frac{1}{\theta}W(z+z')$$
$$\int_0^\infty \lambda^3 \exp(-\frac{\lambda^2 t}{\mu\sigma_{ground}})J_0(\lambda r)d\lambda$$
$$= -\frac{1}{4}\frac{1}{2\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4t}\right)^{\frac{-1}{2}} \frac{\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)}{t}$$
$$\int_0^\infty \lambda^3 \exp(-\frac{\lambda^2 t}{\mu\sigma_{ground}})J_0(\lambda r)d\lambda.$$

Now, by Lemma 3 inequality (2),

$$||\int_0^\infty \lambda^3 J_0(\lambda r)\ \exp(-\lambda^2 \frac{t}{\mu\sigma_{ground}})d\lambda|| \leq \exp(-1)\frac{\mu\sigma_{ground}}{t}.$$

Thus, by Corollary 1 with

$$k = 1$$
$$l = \frac{1}{2}$$
$$m = 0$$

this term is a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$.

The third term of α_2 is

$$\frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}} W(R_I)$$
$$= \frac{\left(\frac{\mu\sigma_{ground}}{4}\right)^{\frac{3}{2}}}{2\pi^{\frac{3}{2}}\sigma_{ground}} \frac{1}{t^{\frac{5}{2}}} \exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right) \exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right).$$

Now by Lemma 3, inequality 3,

$$\|\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\| \leq \frac{t}{2\left(\frac{\mu\sigma_{ground}}{4}\right)}$$

and thus by Corollary 1, with

$$k = -1$$
$$l = \frac{5}{2}$$
$$m = 0$$

this term also defines a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$.

The fourth term of α_2 is

$$-2\frac{\theta^3}{2\pi^{\frac{3}{2}}\sigma_{ground}}\theta^2(z+z')^2 W(R_I)$$
$$= -\frac{1}{\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4}\right)^{\frac{5}{2}}\frac{1}{t^{\frac{5}{2}}}$$
$$(z+z')^2\frac{\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)}{t}\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)$$
$$= -\frac{1}{\pi^{\frac{3}{2}}\sigma_{ground}}\left(\frac{\mu\sigma_{ground}}{4}\right)^{\frac{5}{2}}\frac{1}{t^{\frac{7}{2}}}$$
$$(z+z')^2\exp\left(-(z+z')\frac{\mu\sigma_{ground}}{4t}\right)\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)$$

and again by Lemma 3, inequality 3 we have

$$\|\exp\left(-r^2\frac{\mu\sigma_{ground}}{4t}\right)\| \leq \frac{t}{2\left(\frac{\mu\sigma_{ground}}{4}\right)}$$

and so by Corollary 1, with

$$k = -1$$

$$m = 2$$

$$l = \frac{7}{2}$$

this term also defines a bounded operator from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$.

Since each of the terms of the tensor are bounded operators from $L^2(\mathcal{C} \times [0,T])$ to $L^2(R^3_+ \times [0,T])$, it can be seen from the discussion following the proof of Corollary 1 that the map

$$F(\underline{x},t) \in L^2([0,T];(L^2(R^3_+))^3) \qquad \mapsto \qquad G * \sigma_V F(\underline{x},t)$$

defines a bounded linear operator on $L^2([0,T];(L^2(R^3_+))^3)$.

Recall that in the geophysical situation, the source is typically a current loop on the ground surface, thus the initial field is in $L^2([0,T];(L^2(R^3_+))^3)$ and by a Picard type argument, it can be seen that a solution for Hohmann's integral equation exists in $L^2([0,T];(L^2(R^3_+))^3)$, for $|\sigma_{body} - \sigma_{Host}|$ sufficiently small.

4.4 Recipocrity of Solution

In this section we wish to check the consistency of Hohmann's method by investigating whether the Green's tensor (and hence solution) we obtain by allowing a specific type of source to approach the ground/air interface from the ground is the same as that obtained by allowing the source to approach the interface from the air. Our model is that of an upper half-space (the air) with parameters k_1, h_1 and a lower half-space (the ground) with parameters k_2, h_2. Since we are only interested in the case where the source eventually lies on the air/ground interface, we shall consider only sources with support in a plane parallel to the plane $z = 0$. We will observe below that except for special sources, the convolution G $*$ J diverges as $\sigma_{air} \to 0$. We investigate the reasons for this in more detail later. We now give the corrected forms of the Green's tensors for the cases and regions of interest.

The corrected form of the Green's tensor for the source in air, response in air case from Tai[1] is :

$$G^{(11)}(\underline{x}|\underline{x}') =$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{\lambda h_1}\left\{\begin{pmatrix} \partial_y\partial_{y'}(\psi_- + a\psi_+) & -\partial_y\partial_{x'}(\psi_- + a\psi_+) & 0 \\ -\partial_x\partial_{y'}(\psi_- + a\psi_+) & \partial_x\partial_{x'}(\psi_- + a\psi_+) & 0 \\ 0 & 0 & 0 \end{pmatrix}\right. \qquad (4\cdot 15)$$

$$\left. + \frac{1}{k_1^2}\begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'}(\psi_- + b\psi_+) & \partial_z\partial_{z'}\partial_x\partial_{y'}(\psi_- + b\psi_+) & \lambda^2\partial_z\partial_x(\psi_- + b\psi_+) \\ \partial_z\partial_{z'}\partial_y\partial_{x'}(\psi_- + b\psi_+) & \partial_z\partial_{z'}\partial_y\partial_{y'}(\psi_- + b\psi_+) & \lambda^2\partial_z\partial_y(\psi_- + b\psi_+) \\ \lambda^2\partial_{z'}\partial_{x'}(\psi_- + b\psi_+) & \lambda^2\partial_{z'}\partial_{y'}(\psi_- + b\psi_+) & \lambda^4(\psi_- + b\psi_+) \end{pmatrix}\right\} d\lambda$$

$$-\frac{1}{k_1^2}\delta(R - R')\hat{z}\hat{z}$$

where

$$k_i^2 = \lambda^2 + h_i^2, i = 1, 2$$

$$k_1 = \sqrt{-\mu\sigma_{air}s} \quad (\text{taking}\epsilon = 0)$$

$$k_2 = \sqrt{-\mu\sigma_{ground}s}$$

$$a = \frac{h_1 - h_2}{h_2 + h_1}$$

$$b = \frac{k_2^2 h_1 - k_1^2 h_2}{k_1^2 h_2 + k_2^2 h_1}$$

$$\psi_{\mp} = \mathrm{J}_0(\lambda r)\ \exp(ih_1(z \mp z'))$$

and $z \geq z' \geq 0$.(We interpret the derivatives with respect to z and z' naively, ignoring the discontinuity at $z = z'$ for the reasons given in chapter 3.) For the case $z' \geq z \geq 0$ we merely interchange z and z' in the above formulae. The corrected form of the Green's tensor for the source in ground, response in air case in Tai[1] is :

$$\mathrm{G}^{(12)}(\underline{x}|\underline{x}') =$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{\lambda h_2}\left\{ c\begin{pmatrix} \partial_y\partial_{y'}(\psi) & -\partial_y\partial_{x'}(\psi) & 0 \\ -\partial_x\partial_{y'}(\psi) & \partial_x\partial_{x'}(\psi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \tag{4 $\cdot$ 16}$$

$$\left. + \frac{d}{k_2 k_1}\begin{pmatrix} \partial_z\partial_x\partial_{z'}\partial_{x'}(\psi) & \partial_z\partial_x\partial_{z'}\partial_{y'}(\psi) & \lambda^2\partial_z\partial_x(\psi) \\ \partial_z\partial_y\partial_{z'}\partial_{x'}(\psi) & \partial_z\partial_y\partial_{z'}\partial_{y'}(\psi) & \lambda^2\partial_z\partial_y(\psi) \\ \lambda^2\partial_{z'}\partial_{x'}(\psi) & \lambda^2\partial_z\partial_{z'}(\psi) & \lambda^4(\psi) \end{pmatrix}\right\} d\lambda$$

where $c = \frac{2h_2}{h_1+h_2}$, $d = \frac{2k_1k_2h_2}{k_1^2h_2+k_2^2h_1}$ and $\psi = \exp(i(-h_2z' + h_1z))\mathrm{J}_0(\lambda r)$.

Now (4· 15) solves

$$\nabla \times \nabla \times \mathrm{G}^{(11)}(x,t) + \mu\sigma_{Host}\partial_t\mathrm{G}^{(11)}(x,t) = -\mu\partial_t I\delta(x,t), \quad ,z \geq z' \geq 0 \tag{4 $\cdot$ 17}$$

$\mathrm{G}^{(11)}$will clearly also solve the above equation for $0 \geq z' \geq z$, if z and z' are interchanged. We note that as $\sigma_{air} \to 0$, the terms of the tensor $\mathrm{G}^{(11)}$ become singular if $z' \neq 0$, that is the source does not have support contained in the air/ground interface, for a general source due to a factor of $\frac{1}{\sigma_{air}}$. We also note that even if the source resides on the interface , $\frac{\lambda^4}{k_1^2} \& \frac{\lambda^2}{k_1^2} \longrightarrow \infty$, so to take the limit as σ_{air} vanishes we need to restrict our source functions to those which have zero z-component. Note also that since $\psi_- \neq \psi_+$ unless $z' = 0$, and $\frac{b}{k_1^2} \longrightarrow \infty$ as $\epsilon \longrightarrow 0$, we need to allow the source current to reside on the air/earth interface and then allow σ_{air} to vanish, but this is precisely the type of source we are concerned with and so is not really a restriction. These restrictions are somewhat unphysical in as much as they indicate that a source cannot be approximated, in general, by that part of it on the air/ground interface. It will be shown in a later section that the requirement that the source lie on the interface in order for the limit as $\sigma_{air} \longrightarrow 0$ to exist is unnecessary for a transverse source.

In order to investigate the continuity of the field we first note that

$$\partial_{z'} \psi_- |_{z'=0} = -\partial_{z'} \psi_+ |_{z'=0}$$

and therefore

$$\frac{1}{k_1^2} \partial_{z'} \left(\psi_- + b\psi_+ \right) |_{z'=0}$$

$$= \partial_{z'} \left(\frac{b-1}{k_1^2} \psi_+ \right) |_{z'=0}.$$

Now

$$\frac{b-1}{k_1^2} = \frac{1}{k_1^2} \frac{-2k_1^2 h_2}{k_2^2 h_1 + k_1^2 h_2} \tag{4 $\cdot$ 18}$$

and

$$\lim_{\sigma_{air} \to 0} \frac{b-1}{k_1^2} = \frac{-2}{k_0^2} \frac{K}{i\lambda} = b_0 \tag{4 $\cdot$ 19}$$

where $K = \sqrt{\lambda^2 - k_2^2}$. Note that

$$\lim_{\sigma_{air}\to 0} a = \lim_{\sigma_{air}\to 0} \frac{h_1 - h_2}{h_1 + h_2} = \frac{-i\lambda + iK}{i\lambda - iK} = a_0 \tag{4 ·20}$$

$$\lim_{\sigma_{air}\to 0} c = \lim_{\sigma_{air}\to 0} \frac{2h_2}{h_1 + h_2} = \frac{2K}{\lambda + K} = c_0 \tag{4 ·21}$$

$$\lim_{\sigma_{air}\to 0} \frac{d}{k_1 k_2} = \lim_{\sigma_{air}\to 0} \frac{1}{k_1 k_2} \frac{2k_1 k_2 h_2}{k_2^2 h_1 + k_1^2 h_2} = \frac{2iK}{i\lambda k_0} = -b_0 \tag{4 ·22}$$

and

$$\psi_+|_{z'=0} = \psi_-|_{z'=0} = \psi|_{z'=0}. \tag{4 ·23}$$

We denote the non-diverging part of (4· 17), after taking the pointwise limit $\sigma_{air} \to 0$, by

$$\tilde{\mathrm{G}}^{(11)}(\underline{x}|\underline{x}') =$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{i\lambda^2}\left\{(1+a_0)\begin{pmatrix} \partial_y\partial_{y'}\psi_+ & -\partial_y\partial_{x'}\psi_+ & 0 \\ -\partial_x\partial_{y'}\psi_+ & \partial_x\partial_{x'}\psi_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}\right. \tag{4 ·24}$$

$$\left. + b_0 \begin{pmatrix} \partial_z\partial_{z'}\partial_x\partial_{x'}\psi_+ & \partial_z\partial_{z'}\partial_x\partial_{y'}\psi_+ & 0 \\ \partial_z\partial_{z'}\partial_y\partial_{x'}\psi_+ & \partial_z\partial_{z'}\partial_y\partial_{y'}\psi_+ & 0 \\ \lambda^2\partial_{z'}\partial_{x'}\psi_+ & \lambda^2\partial_z\partial_{z'}\psi_+ & 0 \end{pmatrix}\right\} d\lambda$$

and the corresponding part of (4· 19)

$$\tilde{\mathrm{G}}^{(12)}(\underline{x}|\underline{x}') =$$

$$\frac{i}{4\pi}\int_0^\infty \frac{1}{i\lambda K}c_0\left\{\begin{pmatrix} \partial_y\partial_{y'}\psi_0 & -\partial_y\partial_{x'}\psi_0 & 0 \\ -\partial_x\partial_{y'}\psi_0 & \partial_x\partial_{x'}\psi_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right. \tag{4 ·25}$$

$$\left. + -b_0\begin{pmatrix} \partial_z\partial_x\partial_{z'}\partial_{x'}\psi_0 & \partial_z\partial_x\partial_{z'}\partial_{y'}\psi_0 & \lambda^2\partial_z\partial_x\psi_0 \\ \partial_z\partial_y\partial_{z'}\partial_{x'}\psi_0 & \partial_z\partial_y\partial_{z'}\partial_{y'}\psi_0 & \lambda^2\partial_z\partial_y\psi_0 \\ \lambda^2\partial_{z'}\partial_{x'}\psi_0 & \lambda^2\partial_z\partial_{z'}\psi_0 & \lambda^4\psi_0 \end{pmatrix}\right\} d\lambda$$

where $\psi_0 = \lim_{\epsilon\to 0}\psi = \exp i(iKz' - i\lambda z)\mathrm{J}_0(\lambda r)$ and at $z' = 0, \psi_0 = \exp(\lambda z)\mathrm{J}_0(\lambda r) = \psi_+$ Note that $\frac{1+a_0}{\lambda} = \frac{c_0}{K} = \frac{2}{\lambda+K}$, $\partial_z\psi_0|_{z'=0} = \partial_z\psi_+|_{z'=0}$ and $-\frac{1}{K}\partial_{z'}\psi_0 = \frac{1}{\lambda}\partial_{z'}\psi_+$. Thus, for a general source residing on the air/ground interface the fields obtained by the limiting process we have used are the same.

4.5 Behaviour of the Solution with Source in Ground as $z \to -\infty$

In this section our model is of a conductive upper half-space (the ground) and an insulating lower half-space (the air). The behaviour of the solution in this case for large negative z (in the air) depends upon the behaviour of the Green's tensor for large negative z. Here we prove that for large negative z, the Laplace transform of the Green's tensor behaves like z^{-2}. First, however, at this point it is of interest to note that the pointwise limit of the electric field in the case of a flat earth satisfies the radiation condition. The radiation condition which is used in the case of a flat earth is that the following pointwise relation holds (Tai [1]):

$$\lim_{z\to\mp\infty} z[\nabla\times \mathrm{E}(\underline{x},t) \mp s\hat{z}\mathrm{E}(\underline{x},t)] = 0.$$

However, $E(\underline{x}, t) = E'(\underline{x}, t) + G * \sigma_V E(\underline{x}, t)$ and $E'(\underline{x}, t) = G * J$. Thus, it is sufficient to prove that the anterior elements of G (which contain its z dependence) satisfy the radiation condition, which they do since they were constructed to do so (Tai [1, p. 104]).

From the previous section it can be seen for the case of a source in the ground ($z' \geq 0$) that the terms of the Laplace transform of the Green's tensor, denoted in this section by G ,after taking the quasi-static limit and putting $\sigma_{air} = 0$ are of the form:

$$\frac{1}{4\pi} \int_0^\infty \frac{\lambda^n}{iK\lambda} \partial^\beta J_0(\lambda r) \ \exp(i(h_2 z' - h_1 z)) d\lambda$$

where n is an integer and β is an appropriate multi-index. Now,

$$|\exp(i(h_2 z' - h_1 z))| \tag{4 $\cdot$ 26}$$

$$\leq \exp(\Re(i(h_2 z' - h_1 z)))$$

(taking the quasi-static limit and $\sigma_{air} = 0$), the above expression becomes

$$\exp(\Re(i(iKz' - i\lambda z))$$

$$\leq \exp(-\lambda z' + \lambda z)$$

$$= \exp(-\lambda(z' + |z|)) \text{ (since } z < 0 \text{).}$$

We also have from $3 \cdot 5$ the inequalities:

$$|\partial_{x_i} \partial_{x_j} J_0(\lambda r) \ | \leq F(\rho)\lambda^2, \tag{4 $\cdot$ 27}$$

and

$$|\partial_{x_i} J_0(\lambda r) \ | = | - J_1(\lambda r) \ \lambda \frac{(x_i - x_i')}{\rho} | \leq \lambda, \tag{4 $\cdot$ 28}$$

where F is a continuous function. Also, note that applications of ∂_z and $\partial_{z'}$ to a term only result in multiplication of the term by $-K$ and λ, and as we noted in chapter 3 $|K| \leq \lambda + |k_0|$, which ensures that these operations do not lower the powers of λ occurring in any term. Noting that $|K| = \sqrt{|\lambda^2 + s\mu\sigma_{ground}|} \geq |\sqrt{s\mu\sigma_{ground}}|$ since $\Re s \geq 0$, we have from the above that the terms of the tensor are bounded by terms of the form:

$$\left|\frac{1}{4\pi}\int_0^\infty \frac{\lambda^n}{K\lambda}\partial^\beta \mathrm{J}_0(\lambda r)\ \exp(i(h_2 z' - h_1 z))d\lambda\right|$$

$$\leq F(\rho)f(|s|)\int_0^\infty \lambda^{n-1+g(\beta)}\exp(-\lambda(z' + |z|))d\lambda$$

(where F is a continuous function, $f(x)$ is, depending upon the value of the multi-index β, $\sqrt{x}$ or 1 and $g(\beta)$ is, depending upon the value of β either zero,1 or 2)

$$= F(\rho)f(|s|)(-1)^{n-1+g(\beta)}\frac{d^{n-1+g(\beta)}}{dz^{n-1+g(\beta)}}\frac{1}{(z' + |z|)}$$

Now, the smallest possible power of $n+g(\beta)$ occurring in the original terms is 3. Thus, for $|z|$ sufficiently large the terms are dominated by a continuous function of ρ multiplied by $|s|$, if $\Re s > 1$ then $\left|\frac{1}{K}\right| \leq \frac{1}{\sqrt{\mu\sigma_{ground}}}$, multiplied by $|z|^{-2}$.($z' \geq 0 \Rightarrow (|z|+z')^{-2} \leq (|z|)^{-2}$.)
Thus, as a matrix $\|\mathrm{G}(s, \underline{x}, \underline{x}')\| \leq |s|3F(\rho)(|z|^{-2})$.

Lemma

Let $\mathrm{F}(s, x, y, z) \in \mathrm{L}^2(\mathrm{R}^3_-)$ with compact support then, pointwise,
$\mathrm{G} * \mathrm{F}(s, x, y, z) \to 0$ as $z \to -\infty$.

Proof

Note first that $\mathrm{G} * \mathrm{F}$ is differentiable away from the plane $z = 0$ since $\mathrm{G}(s, x, y, z)$ is differentiable in this region.

We have from above the estimate

$$|G(s,x,y,z,x',y',z')| \quad (4\cdot 29)$$

$$\leq 3|s|F(\rho)(|z|^{-2})$$

so G is locally L^2 away from the ground/air interface. Now

$$|\int_{R^3} G(s,x,y,z,x',y',z')F(s,x',y',z')dV'| \quad (4\cdot 30)$$

$$= |\int_{\text{supp } F} G(s,x,y,z,x',y',z')F(s,x',y',z')dV'|$$

$$\leq \int_{\text{supp } F} |G(s,x,y,z,x',y',z')|\ |F(s,x',y',z')|dV'$$

$$\leq \int_{\text{supp } F} 3|s|F(\rho)(|z|^{-2})|F(s,x',y',z')|dV' \quad (4\cdot 31)$$

Let

$$C_F = 3|s| \max_{x',y' \in \text{supp} F} F(\rho)$$

thus (4· 31)

$$= C_F|z|^{-2} \int_{\text{supp } F} |F(s,x',y',z')|dV'$$

$$\leq C_F|z|^{-2} \sqrt{\int_{\text{supp } F} 1^2 dV'} \|F\|_{L^2(R^3_-)}$$

Now,

$$\lim_{z \to \infty} C_F(|z|^{-2}) \sqrt{\int_{\text{supp } F} 1^2 dV'} \|F\|_{L^2(R^3_-)} = 0 \quad (4\cdot 32)$$

and thus the lemma follows.

We have the following theorem from Treves [1] pp. 420-421:

Theorem

Let $\mathbf{h}(p)$ denote a holomorphic function in the half-plane $\Re p > \sigma_0$, valued in the Banach space $\mathbf{E}$. The two following conditions are equivalent:

(1) $\exists \mathbf{T} \in \mathcal{D}'_+(\mathbf{E})$ such that $\mathcal{L}(\mathbf{T}) = \mathrm{h}(p)$;

(2) $\exists \sigma_1 \in \mathrm{R}, \sigma_0 \leq \sigma_1 < \infty$, a constant $C > 0$ and an integer $k \geq 0$ such that $\forall p, p$ complex, $\Re p > \sigma_1$,

$$\|\mathbf{h}(p)\|_{\mathbf{E}} \leq C(1 + |p|)^k.$$

In the proof this theorem it is shown that T$= \frac{d^{k+2}}{dt^{k+2}} f$ where f is a continuous, $\mathbf{E}$ valued function of t, in fact $f = \mathcal{L}^{-1}(\frac{1}{s^{k+2}}\mathbf{h}(p))$. The estimate we obtained in the proof the above lemma allows us to apply Treves' theorem with:

$$p = s, \Re s > 1$$

$$\mathbf{h} = \mathrm{G} * \mathrm{F}$$

$$\mathbf{E} = (\mathrm{C}^1(\mathrm{R}^3/\{z \leq l\}))^9 = \mathrm{E}_l$$

Thus, as $l \to \infty$, $\mathcal{L}^{-1}(\mathrm{G} * \mathrm{F}) \to 0$ in $\mathcal{D}'_+(\mathrm{E}_l)$.

4.6 Behaviour of the Quasi-static Limit as $\sigma_{air} \to 0$

It was noted in section 4.3 that for a general source with $\epsilon =$ 0the Green's tensor for the source in air, response in air case becomes meaningless in the limit $\sigma_{air} \to 0$. In this section it is shown why this is the case.

Firstly, consider the free-space Green's tensor:

$$\mathrm{G}_f(R) = (I - \frac{1}{k_1^2}\nabla\nabla)\frac{\exp(-ik_1 R)}{R} \tag{4 ·33}$$

For a non-transverse source, we have that this 'blows up ' as $k_1 \to 0$ since

$$\lim_{k_1 \to 0} \nabla\nabla\frac{\exp(-ik_1 R)}{R} = \nabla\nabla\frac{1}{R}$$

as a distribution and this is not the zero distribution. Of course, for a transverse source we have that the $-\frac{1}{k_1^2}\nabla\nabla$ terms disappear and thus the solution does not 'blow up' as $k_1 \to 0$.

Now we turn to the case of a flat Earth. In this case a study of the behaviour of the eigenfunctions in the limit as $\sigma_{air} \to 0$ reveals the conditions the source must satisfy for the a meaningful pointwise limit to exist.

Note that no problem arises in the eigenfunctions corresponding to the TE mode. However a problem does arise in the eigenfunctions corresponding to the TM mode. These functions are

$$\begin{aligned}
\mathrm{N}_{\substack{e\\o}n\lambda}(h) =& \frac{1}{\kappa}\nabla \times \nabla \times (\psi_n) \\
=& \frac{1}{\kappa}(\nabla(\partial_z\psi_n(h)) + \kappa^2\psi_n(h)\hat{z}) \\
& \text{where } \kappa^2 = \lambda^2 + h^2 \\
\psi_n(h) =& \mathrm{J}_n(\lambda r)^{\cos}_{\sin}(n\phi)\exp(ihz).
\end{aligned}$$

and thus as $\kappa \to 0$ we find that these expressions 'blow up'. However, if we consider an expression of the form

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{N}_{\substack{e\\o}n\lambda}(h)\mathrm{N}'_{\substack{e\\o}n\lambda}(h')(2-\delta_0) \\
=&\sum_{n=0}^{\infty} \frac{1}{\kappa}(\nabla(\partial_z\psi_n(h)) + \kappa^2\psi_n(h)\hat{z})\frac{1}{\kappa'}(\nabla'(\partial_{z'}\psi'_n(h')) + \ \kappa^{2\prime}\psi'_n(h')\hat{z})(2-\delta_0) \\
=&\frac{1}{\kappa\kappa'}\nabla\partial_z\nabla'\partial_{z'}\phi(h,h') + \frac{\kappa}{\kappa'}\hat{z}\nabla'\partial_{z'}\phi(h,h') + \frac{\kappa'}{\kappa}\nabla\partial_z\phi(h,h')\hat{z} + \kappa\kappa'\phi(h,h')
\end{aligned}
$$

where

$$
\sum_{n=0}^{\infty} \psi_n(h)\psi_n(h')(2-\delta_0) = \mathrm{J}_0(\lambda r)\ \exp(i(ih - ih')) = \phi(h,h')
$$

as in chapter 3.

For a transverse, infinitely differentiable source with compact support, the terms containing ∇' vanish and only terms of the form

$$
\frac{\kappa'}{\kappa}\nabla\partial_z\phi(h,h')\hat{z} + \kappa\kappa'\hat{z}\hat{z}
$$

contribute to the field. In the particular cases where $h = h_1, h' = \mp h_1$, as $k_1 \to 0$ the above terms tend, as distributions, to

$$
\nabla\partial_z\phi(h,h')\hat{z}
$$

which has both zero curl and divergence. These terms will give rise to a part of the field which is not square-integrable in the air, since it will 'inherit ' the property of having both zero-divergence and curl. This part of the field, which is in the kernel of the distributional $\nabla\times\nabla\times$, is 'invisible ' to the semi-norm constructed in Chapter 2 and reflects the difficulty in finding a variational proof of the existence of a solution to this problem.

Now, the TM contribution to the field in the ground is given by an integral with respect to λ of

$$\frac{1}{\lambda h_1} d \sum_{n=0}^{\infty} \mathrm{N}_{\substack{e\\o}n\lambda}(-h_2)\mathrm{N}'_{\substack{e\\o}n\lambda}(h_1)(2-\delta_0), z \leq 0$$

where $d = \frac{k_1 k_2 h_1}{k_2^2 h_1 + k_1^2 h_2}$. From the previous discussion, it can be seen that the expression for the TM contribution to the field in the ground, for a tranverse test function source in the air, involves only terms of the form

$$\frac{1}{\lambda h_1} \frac{k_1 k_2 h_1}{k_2^2 h_1 + k_1^2 h_2} \left(\frac{k_1}{k_2} \nabla \partial_z \phi(-h_2, h_1)\hat{z} + k_1 k_2 \phi(-h_2, h_1)\hat{z}\hat{z} \right).$$

Now, as σ_{air} vanishes, so does k_1 and d. Thus for a transverse source in the air the TM modes do not contribute and the field in the ground is parallel to the air/ground interface, irrespective of the source ! Note that by switching k_1 with k_2, etc. we obtain an expression for the field in the air due to a source in the ground. In this case, the k_1 in the expression for d cancels with the $\frac{1}{k_1}$ in the expression for the TM contributions and we obtain a term of the form

$$f(\lambda) \nabla \partial_z \phi(-h_1, h_2)\hat{z}.$$

Thus, if the source in the ground has a non-zero z component, so will the field in the air . Note that this field cannot be in $\mathrm{L}^2\left(\mathrm{R}^3\right)$, for reasons noted earlier.

We now return to an investigation of the recipocrity of the solution. Recalling that the TE modes are well behaved in the quasi-static limit as $\sigma_{air} \to 0$ for any source, we concentrate on the behaviour of the TM modes in the quasi-static limit as $\sigma_{air} \to 0$ for a transverse test function source. The above discussion shows that the restriction that the source must be on the interface and then σ_{air} allowed to vanish is not required in this case.

The asymmetric behaviour of the field may reflect the fact that the eigenfunctions of the vector Laplacian:

$$\nabla\nabla \cdot F - \nabla \times \nabla \times F = 0$$

are not the limit as $\kappa \to 0$ of eigenfunctions of the vector Helmholtz equation:

$$\nabla\nabla \cdot F - \nabla \times \nabla \times F + \kappa F$$

(cf. Morse and Feschbach [1] pp. 1784-1789).

We now show that a general source cannot be approximated by that part of the it in the ground by considering the scattering terms of $\mathrm{G}_{i3}^{(11)}, i = 1, 2, 3$ which from Tai, after taking the quasi-static limit and assuming $\sigma_{air} \neq 0$, are

$$\mathrm{G}_{13}^{(11)} = -\frac{1}{4\pi}\int_0^\infty \frac{1}{-s\mu\sigma_{air}} \frac{\lambda^2}{\lambda\sqrt{\lambda^2 + s\mu\sigma_{air}}} \partial_z\partial_x\left(b\psi_+\right)d\lambda$$

$$\mathrm{G}_{23}^{(11)} = -\frac{1}{4\pi}\int_0^\infty \frac{1}{-s\mu\sigma_{air}} \frac{\lambda^2}{\lambda\sqrt{\lambda^2 + s\mu\sigma_{air}}} \partial_z\partial_y\left(b\psi_+\right)d\lambda$$

$$\mathrm{G}_{33}^{(11)} = -\frac{1}{4\pi}\int_0^\infty \frac{1}{-s\mu\sigma_{air}} \frac{\lambda^4}{\lambda\sqrt{\lambda^2 + s\mu\sigma_{air}}} \partial_z\partial_z\left(b\psi_+\right)d\lambda$$

with

$$b = \frac{(s\mu\sigma_{ground})\sqrt{\lambda^2 + s\mu\sigma_{air}} - (s\mu\sigma_{air})\sqrt{\lambda^2 + s\mu\sigma_{ground}}}{(s\mu\sigma_{ground})\sqrt{\lambda^2 + s\mu\sigma_{air}} + (s\mu\sigma_{air})\sqrt{\lambda^2 + s\mu\sigma_{ground}}}$$

$$= \frac{\sigma_{ground}\sqrt{\lambda^2 + s\mu\sigma_{air}} - \sigma_{air}\sqrt{\lambda^2 + s\mu\sigma_{ground}}}{\sigma_{ground}\sqrt{\lambda^2 + s\mu\sigma_{air}} + \sigma_{air}\sqrt{\lambda^2 + s\mu\sigma_{ground}}}$$

Now, let $\mathrm{G}_{i3}^{(11)\prime} = -s\mu\sigma_{air}\mathrm{G}_{i3}^{(11)}$

By the same argument as used in chapter 3 to prove the spatial pointwise convergence of the tensor in the source in ground, receiver in ground case we obtain that, for

$J_z \in C_c^\infty(\mathrm{R}^3)$, which is zero in a neighbourhood of the air/Earth interface, ($z = 0$), and $\hat{x} \cdot J_z = \hat{y} \cdot J_z = 0$ in all R^3 and denoting the scattering part of our modified Green's tensor by $\mathrm{G}_s^{(11)\prime}$

$$\mathrm{G}^{(11)'}_{s_{\sigma_{air} \neq 0}} * J_z \rightarrow \mathrm{G}^{(11)'}_{s_{\sigma_{air} = 0}} * J_z \text{ pointwise as } \sigma_{air} \rightarrow 0 \qquad (4 \cdot 34)$$

i.e. $\frac{1}{\sigma_{air}\mu}\mathrm{G}^{(11)\prime}_{\sigma_{air}\neq 0} * J_z$ diverges as $\sigma_{air} \rightarrow 0$.

4.7 Boundary Conditions

In the quasi-static limit there are no true total surface charges. Since we assume that $\mathrm{D} = \epsilon \mathrm{E}$ we have from Maxwell's equations that if $\epsilon = 0$ then

$$\nabla \cdot \mathrm{D} = \nabla \cdot \epsilon \mathrm{E} = 0.$$

However, Hohmann's integral equation contains a perturbation term corresponding to the response to a scattering current j_s which has associated with it a rate of change of surface charge $\nabla \cdot j_s = \nabla \cdot \sigma_{\mathrm{V}} \mathrm{E}$. Now, since the surface charge is always zero this implies that

$$\nabla \cdot \sigma(x) \mathrm{E} = 0$$

and writing $\sigma(x) = \sigma_{\mathrm{V}} + \sigma_{Host}$ we have that

$$\nabla \cdot \sigma_V \mathrm{E} = -\nabla \cdot \sigma_{Host} \mathrm{E},$$

which reflects the physically obvious fact that the field in the ground gains its longitudinal component as a consequence of the presence and geometry of the ore-body.

We now turn to the behaviour of the field at interfaces. In Carey and O'Brien [1] it was shown that for Ω, a region spanning the boundary Γ between two regions Ω_-, Ω_+ that

Lemma

Suppose $f \in L^2(\Omega)$ and $\nabla \cdot f = 0$ in Ω. Let $n \cdot f_{\mp}$ denote the normal trace of $n \cdot f$ on Γ from $\Omega_{\mp}$ then

$$n \cdot f_- = n \cdot f_+$$

in the sense of equality of distributions in $H^{-\frac{1}{2}}(\Gamma)$.

Now for a source in the ground, the field E in the absence of an ore-body is in $L^2(R^3_-)$ and when an ore-body is present, the field, as given by the solution to Hohmann's integral equation, is also in $L^2(R^3_-)$. Thus, we have that,irrespective of the presence or absence of an ore-body, $\sigma E = 0$ in the air (E is pointwise finite) , $E \in L^2([0,T];(L^2(R^3_-))^3)$ and $\nabla \cdot \sigma E = 0$. Thus, the conditions of Carey and O'Brien's lemma are satisfied and we have that the normal component of σE is continuous across all interfaces, both air/ground and ore-body/host, in the sense of traces on the interfaces.

Chapter 5
Conclusion

To summarise our results:

(1) We have proved the existence of the solution to a generalised diffusion equation under certain conditions and shown that the non-existence of the solution in the case of a vanishing conductivity is a purely vector phenomena.

(2) We have verified that Hohmann's [1] & [2] integral equation method gives a consistent solution for the case of an insulating upper half-space when we adopt the simple geometry of an ore-body of finite size in a uniform conducting lower half-space.

(3) We have shown that, in general, the electric field, in the quasi-static approximation, ceases to be in $L^2(R^3)$ in the presence of an insulating half-space and that this behaviour is due to the nature of the eigenfunctions of the vector Laplacian.

It is apparently paradoxical that the large time (that is, near T)behaviour of a function valued in $L^2(R^3)$ for $t \in [0, T]$ should not be in $L^2(R^3)$. This paradox suggests that it is worthwhile recalling that the quasi-static limit is an asymptotic approximation made by investigating the behaviour of the inverse Laplace transform of the Green's function for the scalar Helmholtz equation, i.e.

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\exp(-\sqrt{\epsilon\mu s^2 + \mu\sigma s})}{R} \exp(st)ds$$

for large t.

Now, in an insulator ($\sigma \equiv 0$) we have that the integrand reduces to

$$\frac{\exp((t - \sqrt{\mu\epsilon}R)s)}{R}.$$

Now, $\frac{1}{\sqrt{\mu\epsilon}} = c$ where c is the speed of light *in vacuo*, which is approximately 2×10^5 km/s. Thus, since R in geophysical applications will be of order 10^2 km, we have that for t of order 10^{-3} seconds that the integral is well approximated by taking $\epsilon = 0$. However, in the corresponding vector case, we are interested in

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \left(I + \frac{1}{\mu\epsilon s^2}\nabla\nabla\right) \frac{\exp(-\sqrt{\mu\epsilon s}R)}{R} \exp(st)ds$$

and therefore we are concerned with terms like:

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{\sqrt{\mu\epsilon s}} \frac{\exp(-\sqrt{\epsilon\mu s}R)}{R^3} \exp(st)ds.$$

Again, we may ignore the $\sqrt{\mu\epsilon}R$ term in the exponential, but this is clearly not equivalent to assuming $\epsilon = 0$. This suggests that the quasi-static limit does not give the correct asymptotic behaviour in the air and that the above paradox may be resolved by an explicit asymptotic analysis of the Green's tensor in the air.

Bibliography

Carey, A.L. [1] Private communication.

Carey, A.L. and O'Brien, D.M. [1] Existence and Regularity results for Maxwell's equations in the quasi-static limit, J. Aus. Math. Soc. (Ser. B), **27**, pp. 402-415,1986.

Cheng, D.H.S.[1] On the fomulation of the dyadic Green's function in a layered medium, Electromagnetics, **6** pp. 171 - 182, 1986

Colins, R.E.[1] Dyadic Green's function expansions in spherical co-ordinates, Electromagnetics, **6** pp. 183 - 207, 1986.

Gradshteyn and Ryzhik, Table of integrals, series, and products, Academic Press, New York, 1987.

Hohmann, G.W.[1] Three-dimensional induced polarisation and electromagnetic modeling, Geophysics, **40**, 309-324., 1975.

[2] (with San Filipo, W.A.), Integral equation solution for the transient electromagnetic response of a three-dimensional body in a conductive half-space, Geophysics, **50**, pp. 798 - 809, 1985.

Johnson, W.A., Howard, A. Q. and Dudley, D. G. [1] On the irrotational component of the electric Green's Dyadic, Radio Science, **14**, pp. 961-967, 1979.

Lions, J.L. and Magnes, E. [1] Non-Homogeneous Boundary Value Problems and Applications, Vol. 2,Springer-Verlag, Berlin, 1972.

Morse, P. and Feshbach, H.[1] Methods of theoretical physics, McGraw-Hill, New York, 1953.

Oberhettinger, F. and Badii, L. [1] Tables of Laplace Transforms, Springer-Verlag, 1973.

Olver, F. W. J., [1] Asymptotics and special functions, Academic Press, New York, 1974.

Tai, Chen-To, [1] Dyadic Green's functions in electromagnetic theory, Intext, Scranton, Penns., 1971.

Treves, Francois, [1] Basic linear partial differential equations, Academic Press, New York, 1975.

Ward, S.H. [1] Electromagnetic theory for Geophysical Applications,in Mining Geophysics (Vol. 2), Society of Exploration Geophysicists, Mining Geophysics Volume Editorial Committee, 1971.