Graphs With Transitive Automorphism Groups

Walter David Neumann B.A.

Department of Mathematics of

The University of Adelaide

and

Mathematisches Institut der

Universität Bonn.

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Summary

Graph theory to date tends to be mostly of a combinatorial or topological nature. More algebraic aspects of graph theory have been studied, however they deal almost exclusively with the problem of determining graphs with given combinatorial properties whose automorphism group is isomorphic to a given abstract group. A bibliography to the literature on this subject is given in [11] p.263.

This thesis discusses a different algebraic side of graph theory, namely the theory of graphs with transitive automorphism groups. A large class of such graphs is given by the Cayley graphs, and the problem immediately arises as to how the class γ of graphs with transitive automorphism groups is related to the class $\mathcal{L}$ of graphs which are isomorphic to Cayley graphs. This problem is discussed in the first four chapters of the thesis.

In chapter I the basic definitions and terminology are given, and it is shown by examples that γ properly contains $\mathcal{L}$.

In chapter II it is shown that γ and $\mathcal{L}$ are both closed under cartesian products, but not under the reverse operation of factorising a graph with respect to cartesian products. It is also shown that to each simple graph G in γ, there exists a complete graph whose cartesian product with G is in $\mathcal{L}$.

Two natural generalizations of Cayley graphs are discussed in chapter III. The first of these is shown to give arbitrary simple graphs in γ, generalizing a theorem of Sabidussi which

characterises the graphs in $\mathcal{L}$ by means of their automorphism groups. This is used to deduce a theorem on homomorphisms, which states in Reidemeister's language that any simple graph in γ may be covered by a graph in $\mathcal{L}$. Similar results hold for the second generalization of Cayley graphs, and are used to deduce a further characterisation of the graphs in $\mathcal{L}$.

The problem of finding usable sufficient conditions for a graph to be in $\mathcal{L}$ is discussed in §7 using the results of chapter III, and in §10 using Petersen's alternating path method. It is for example shown that if a regular graph of degree 2 with p^2 vertices (p prime) is in γ, then it is already in $\mathcal{L}$. This is deduced from a rather stronger result involving the alternate composition graph of a graph.

Some further applications of the alternating path method are also considered in §9 and §10, and the strong practical applications of this method are demonstrated in §11 in the construction of an infinite set of regular graphs of degree 2 which are in $\mathcal{L}$ but not in γ.

In chapter V Hamiltonian arcs in Cayley graphs are discussed. It is shown for instance that a connected Cayley graph of a finite abelian group always has a Hamiltonian arc, and the problem of existence and classification of Hamiltonian arcs in Cayley graphs is solved or partially solved in a number of other special cases.

Declaration

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except when due reference is made in the text of this thesis.

Signed

Bonn, 15th March 1967.

Notes on Terminology

The basic graph theoretical terminology follows as far as possible Oystein Ore's "Theory of Graphs" (A.M.S. Colloquium publications vol.38).

The terminology for permutation groups is that of H. Wielandt "Finite Permutation Groups" (Academic press 1964). This terminology differs from the classical terminology in a few instances. In particular "regular" is used instead of "regular transitive" to describe a transitive permutation group whose stabilizer subgroups are trivial, and the term "block" is used for "set of imprimitivity".

In the first three chapters the arithmetic used is cardinal arithmetic, though it is often restricted to the usual finite arithmetic.