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GAP-LABELLING CONJECTURE WITH NONZERO MAGNETIC FIELD

MOULAY TAHAR BENAMEUR AND VARGHESE MATHAI

Abstract. Given a constant magnetic field on Euclidean space $\mathbb{R}^p$ determined by a skew-symmetric $(p \times p)$ matrix Θ, and a $\mathbb{Z}^p$-invariant probability measure μ on the disorder set Σ which is by hypothesis a Cantor set, where the action is assumed to be minimal, the corresponding Integrated Density of States of any self-adjoint operator affiliated to the twisted crossed product algebra $C(\Sigma) \rtimes_\sigma \mathbb{Z}^p$, where σ is the multiplier on $\mathbb{Z}^p$ associated to Θ, takes on values on spectral gaps in the *magnetic gap-labelling group*. The *magnetic frequency group* is defined as an explicit countable subgroup of $\mathbb{R}$ involving Pfaffians of Θ and its sub-matrices. We conjecture that the magnetic gap labelling group is a subgroup of the magnetic frequency group. We give evidence for the validity of our conjecture in 2D, 3D, the Jordan block diagonal case and the periodic case in all dimensions.

1. Introduction

The gap-labelling theorem was originally conjectured by Bellissard [5] in the late 1980s. It concerns the labelling of gaps in the spectrum of a Schrödinger operator (in the absence of a magnetic field) by the elements of a subgroup of $\mathbb{R}$ which results from pairing the K_0-group of the noncommutative analog for the Brillouin zone with the tracial state defined by the probability measure on the hull. The problem arises in a mathematical version of solid state physics in the context of aperiodic tilings. Its three proofs, discovered independently by the authors of [15, 29, 7] all concern the proof of a statement in K-theory. Earlier results include the proof of the gap-labelling conjecture in 1D [6], 2D [8, 50] and in 3D [11]. A more detailed account of the history of gap-labelling theorems can be found in Appendix B.

In the presence of a non-zero constant magnetic field in Euclidean space, the gap-labelling conjecture is much trickier to state, even though it was known to be the more interesting problem in spectral theory and in condensed matter physics since the 1980s, cf. [9]. Here, we manage to give, for the first time, a precise formulation of conjectures for the magnetic gap-labelling group in all dimensions which encompass all previously known results. More precisely, in this paper we initiate the study of the gap-labelling group in the case of the magnetic Schrödinger operator on Euclidean space $\mathbb{R}^p$ with disorder set a Cantor set Σ under a non-zero magnetic field $B = \frac{1}{2} dx^t \Theta dx$, where Θ is a $(p \times p)$ skew-symmetric matrix. We believe that proving (or disproving) our conjectures would constitute an important step in the understanding of aperiodic tilings under a constant magnetic field. Given a $\mathbb{Z}^p$-invariant

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probability measure μ on Σ, the corresponding Integrated Density of States of any self-adjoint operator affiliated to the twisted crossed product algebra $C(\Sigma) \rtimes_\sigma \mathbb{Z}^p$ takes values on spectral gaps in an explicit countable subgroup of $\mathbb{R}$ involving Pfaffians of Θ and its sub-matrices that we describe in Conjecture 1, where σ is the multiplier on $\mathbb{Z}^p$ associated to Θ. The physical interpretation of one side of our conjecture is a natural extension to the magnetic case of the notion of the group of frequencies studied in solid physics; see [5]. In 2D, the magnetic gap-labelling group applies to the magnetic Schrödinger operators that are the Hamiltonians which are pertinent to the study of the integer quantum Hall effect, cf. [10] and the bulk-boundary correspondence, cf. [30, 35].

Upon defining the *magnetic gap-labelling group* and the *magnetic frequency group* in Definition 1, our gap-labelling Conjecture 1 states that for minimal actions of $\mathbb{Z}^p$ on a Cantor set, the magnetic gap-labelling group is a subgroup of the magnetic frequency group. Our gap-labelling Conjecture 2 states that for *strongly* minimal actions of $\mathbb{Z}^p$ on a Cantor set, the magnetic gap-labelling group coincides with the magnetic frequency group. Our main achievements in this paper, besides the precise statement of the conjectures, are complete solutions to the conjectures in the 2D case and also in the 3D case. We also give other evidence that our conjectures should hold in higher dimensions such as in the periodic case and the Jordan block diagonal case.

The heart of our approach is a new index theorem, named the twisted index theorem for foliations, see subsection 3.3. We also use the Baum-Connes conjecture with coefficients, which is known to be true for the relevant free abelian discrete group $\mathbb{Z}^p$ (cf. [2]). In addition, the integrality of *all* the components of the Chern character is needed to complete the proof of our magnetic gap-labelling conjecture, and is the trickiest part of the proofs of our theorems. This is in contrast to the proof of Bellissard's gap-labelling conjecture, where only the integrality of the *top* dimensional component of the Chern character is needed, and it explains in a nutshell the difference in complexity of the two conjectures. Direct cohomological computations in the 3D case enabled us to prove Conjecture 2 (see Corollary 7.6) for *strongly minimal systems*, a notion that is introduced in Definition 2. We have included the complicated combinatorics that proves an independently interesting result in Theorem 8.1 and which makes possible a better understanding of our magnetic gap-labelling conjectures. The proof of this theorem is a *tour de force* computation, and although our method extends to all dimensions, it only allowed us to deduce Conjecture 2 under an extra hypothesis on the corresponding clopen subsets. The strategy of proving the results in Section 7 and Section 8 are outlined at the beginning of these sections.

In a forthcoming paper, we plan to weaken the hypotheses of Theorem 8.1 and to systematically study the magnetic gap-labelling group in all higher dimensions.

It is worth pointing out that the proofs mentioned earlier of the Bellissard gap-labelling conjecture, which is the special case of the zero magnetic field, use the integrality of the top degree component of the Chern character for *even* dimensions. Since no published proof of this Chern-integrality result is known for general minimal $\mathbb{Z}^p$-actions on Cantor sets, we *do not* use it in the present paper. Notice however that the Chern-integrality condition is fulfilled for low dimensions (2D and 3D) and it was also proved in all even dimensions whenever the relevant K-theory is torsion-free, see [15, 14, 16].

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Contents

2. Magnetic Schrödinger operators

We begin by reviewing the construction of magnetic Schrödinger operators. Consider Euclidean space $\mathbb{R}^p$ equipped with its usual metric $\sum_{j=1}^p dx_j^2$. Consider the uniform magnetic field $B = \frac{1}{2} dx^t \Theta dx = \frac{1}{2}\sum_{j,k} \Theta_{jk} dx_j \wedge dx_k$, where Θ is a constant $(p \times p)$ skew-symmetric matrix. The Euclidean group $G = \mathbb{R}^p \rtimes SO(p)$ acts transitively on $\mathbb{R}^p$ by affine transformations. The torus $\mathbb{T}^p$ can be realised as the quotient of $\mathbb{R}^p$ by the action of its fundamental group $\mathbb{Z}^p$.

Let us now pick a 1-form η such that $d\eta = B$. This is always possible since B is a closed 2-form and $\mathbb{R}^p$ is contractible. We may regard η as defining a connection $\nabla = d + i\eta$ on the trivial line bundle $\mathcal{L}$ over $\mathbb{R}^p$, whose curvature is iB. Physically we can think of η as an electromagnetic vector potential for the uniform magnetic field B normal to $\mathbb{R}^p$. Using the Riemannian metric, the Hamiltonian of an electron in this field is given in terms of suitable units by

$$H_\eta = \frac{1}{2}\nabla^\dagger \nabla = \frac{1}{2}(d+i\eta)^\dagger (d+i\eta),$$

where $\dagger$ denotes the adjoint. In a real material this Hamiltonian would be modified by the addition of a *real-valued* potential V, and called a magnetic Schrödinger operator $H_{\eta,V} = H_\eta + V$. The spectrum of the unperturbed Hamiltonian H_η for $\eta = \frac{1}{2}\sum \Theta_{jk} x_j dx_k$ has been computed by physicists. We record that it has discrete eigenvalues with infinite multiplicity. Any η is cohomologous to $\frac{1}{2}\sum \Theta_{jk} x_j dx_k$ since they both have B as differential, and forms differing by an exact form $d\phi$ give equivalent models: in fact, multiplying the wave functions by $\exp(i\phi)$ shows that the Hamiltonians for η and $\frac{1}{2}\sum \Theta_{jk} x_j dx_k$ are unitarily equivalent. This equivalence also intertwines the $\mathbb{Z}^p$-actions so that the spectral densities for the two models also coincide. However, it is the perturbed Hamiltonian $H_{\eta,V} = H_\eta + V$ which is the key, and the spectrum of this is unknown for general $\mathbb{Z}^p$-aperiodic V. Set $\eta = \frac{1}{2}\sum \Theta_{jk} x_j dx_k$ from now on. For $\gamma \in \mathbb{Z}^p$, consider the function on $\mathbb{R}^p$ given by $\psi_\gamma(x) = \frac{1}{2}\sum \Theta_{jk}\gamma_j x_k$. It satisfies $\gamma^*\eta - \eta = d\psi_\gamma$. Also, $\psi_\gamma(0) = 0$ for all $\gamma \in \mathbb{Z}^p$ and $\psi_\gamma(\gamma') = \frac{1}{2}\sum \Theta_{jk}\gamma_j\gamma'_k$ for $\gamma' \in \mathbb{Z}^p$. Define a projective unitary action T^σ of $\mathbb{Z}^p$ on $L^2(\mathbb{R}^p)$ as follows.

$$U_\gamma(f)(x) = f(x-\gamma), \tag{1}$$

$$S_\gamma(f)(x) = \exp(-2\pi i \psi_\gamma(x)) f(x), \tag{2}$$

$$T^\sigma_\gamma = U_\gamma \circ S_\gamma. \tag{3}$$

Then the operators T^σ_γ, also known as *magnetic translations*, satisfy $T^\sigma_e = \mathrm{Id}$, $\mathrm{T}^\sigma_{\gamma_1}\mathrm{T}^\sigma_{\gamma_2} = \sigma(\gamma_1,\gamma_2)\mathrm{T}^\sigma_{\gamma_1\gamma_2}$, where $\sigma(\gamma,\gamma') = \exp\left(-2\pi i \psi_\gamma(\gamma')\right)$ is a *multiplier* on $\mathbb{Z}^p$ satisfying,

(1) $\sigma(\gamma, e) = \sigma(e,\gamma) = 1$ for all $\gamma \in \mathbb{Z}^p$;

(2) $\sigma(\gamma_1,\gamma_2)\sigma(\gamma_1\gamma_2,\gamma_3) = \sigma(\gamma_1,\gamma_2\gamma_3)\sigma(\gamma_2,\gamma_3)$ for all $\gamma_j \in \mathbb{Z}^p$, $j = 1,2,3$.

Note that with the above choices, we also ensure the relation $\sigma(\gamma_1,\gamma_2) = \overline{\sigma(\gamma_2,\gamma_1)}$, and in particular $\sigma(\gamma,\gamma) = 1$, $\forall\, \gamma \in \mathbb{Z}^p$. An easy calculation shows that $T^\sigma_\gamma H_\eta = H_\eta T^\sigma_\gamma$. Also, we shall assume that V is aperiodic with hull equal to a Cantor set Σ and we conclude that the magnetic Schrödinger operator $H_{\eta,V}$ is also aperiodic with hull equal to Σ.

According to Bellissard's gap-labelling theorem, [5], under usual conditions on the aperiodic potential V, the C^*-algebra of observables is associated with a minimal dynamical system and is defined as follows. Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and consider the strong closure X of the space of all conjugates of the resolvent $(H - z\mathrm{I})^{-1}$ under the magnetic translations $(T_a^\sigma)_{a\in\mathbb{R}^p}$. Then X is independent of the choice of z up to homeomorphism, and it is a compact space with a minimal action of $\mathbb{R}^p$, through the same magnetic translations. The C^*-algebra of observables is then the twisted crossed product C^*-algebra $C(X) \rtimes_\sigma \mathbb{R}^p$. For the particular tilings we are interested in, and which include quasi-crystals [8], this latter C^*-algebra is Morita equivalent to a discrete twisted crossed product algebra $C(\Sigma) \rtimes_\sigma \mathbb{Z}^p$ for some Cantor space Σ.

If $\lambda \in \mathbb{R}$ is in a spectral gap of $H_{\eta,V}$, then the Riesz projection $\chi_{(-\infty,\lambda]}(H_{\eta,V})$ can be expressed as $p_\lambda(H_{\eta,V})$ where p_λ is a smooth compactly supported function which is identically equal to 1 in the interval $[\inf spec(H_{\eta,V}), \lambda]$, where $\inf spec(H_{\eta,V})$ denotes the bottom of the spectrum of the self-adjoint operator $H_{\eta,V}$ that is bounded below since V is bounded below by our hypotheses. Assume also that the support of p_λ is contained in the interval $[-\varepsilon + \inf spec(H_{\eta,V}), \lambda + \varepsilon]$ for some $\varepsilon > 0$. Then $p_\lambda(H_{\eta,V}) \in C(\Sigma) \rtimes_\sigma \mathbb{Z}^p \otimes \mathcal{K}$ and therefore one obtains an element,

$$[p_\lambda(H_{\eta,V})] \in K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p).$$

A standard assumption on the physical model is the *gap hypothesis*, which is that the Fermi level of the physical system described by $H_{\eta,V}$ lies in a spectral gap. We shall enunciate a precise conjecture generalising a famous conjecture of Bellissard in the absence of a magnetic field. We shall also give a complete proof of our conjecture in low dimensions, and give evidence for it in all dimensions.

3. The magnetic gap-labelling group

In this section, we introduce and start our study of what we shall call the magnetic gap-labelling group. Inspired by Bellissard's gap-labelling conjecture [5], we show that the magnetic gap-labelling conjecture can be completely stated in the language of minimal totally disconnected dynamical systems. More precisely, we assume that we are given p commuting homeomorphisms $T = (T_j)_{1\leq j\leq p}$ of a Cantor space Σ which preserve a Borel probability measure μ. So Σ is compact totally disconnected without isolated points and these homeomorphisms then generate a minimal action of the abelian free group $\mathbb{Z}^p$ so that T_j corresponds to the action of the canonical basis vector $\psi_j \in \mathbb{Z}^p$.

The subgroup of the real line $\mathbb{R}$ which is generated by μ-measures of clopen subspaces of Σ is denoted $\mathbb{Z}[\mu]$. This is known as the group of frequencies of the aperiodic potential associated with the quasi-crystal, i.e. appearing in the Fourier expansion of that potential. It can also be seen as the image under (the integral associated with) the probability measure μ of $C(\Sigma, \mathbb{Z})$, the group of continuous integer valued functions on Σ. That is,

$$\mathbb{Z}[\mu] = \left\{ \int_\Sigma f(z) d\mu(z) \middle| f \in C(\Sigma, \mathbb{Z}) \right\} = \mu(C(\Sigma, \mathbb{Z}))$$

Let I be an ordered subset of $\{1,\ldots,p\}$ with an even number of elements, and let $C(\Sigma,\mathbb{Z})_{\mathbb{Z}^{I^c}}$ denote the coinvariants of $C(\Sigma,\mathbb{Z})$ under the action of the subgroup $\mathbb{Z}^{I^c}$ of $\mathbb{Z}^p$, where I^c denotes the index set that is the complement to I. Let $\left(C(\Sigma,\mathbb{Z})_{\mathbb{Z}^{I^c}}\right)^{\mathbb{Z}^I}$ denote the subset of $C(\Sigma,\mathbb{Z})_{\mathbb{Z}^{I^c}}$ composed of those $\mathbb{Z}^{I^c}$-coinvariant classes in $C(\Sigma,\mathbb{Z})_{\mathbb{Z}^{I^c}}$ which are invariant under the induced action of the subgroup $\mathbb{Z}^I$. Define

$$\mathbb{Z}_I[\mu] = \mu\left(\left(C(\Sigma,\mathbb{Z})_{\mathbb{Z}^{I^c}}\right)^{\mathbb{Z}^I}\right).$$

Notice that

$$\mathbb{Z}_{\{1,\cdots,p\}}[\mu] = \mathbb{Z} \ \subset \ \mathbb{Z}_I[\mu] \ \subset \ \mathbb{Z}[\mu] = \mathbb{Z}_\emptyset[\mu].$$

3.1. **Labelling the gaps.** Let σ be a multiplier of $\mathbb{Z}^p$ which is associated with the skew symmetric matrix Θ. The $\mathbb{Z}^p$ invariant probability measure μ yields a regular trace τ^μ on the twisted crossed product C^*-algebra $C(\Sigma)\rtimes_\sigma \mathbb{Z}^p$, which is by fiat the operator norm completion of the $*$-algebra of compactly supported continuous functions $C_c(\mathbb{Z}^p\times\Sigma)$ acting via the left regular representation on the Hilbert space $L^2(\Sigma,d\mu)\otimes\ell^2(\mathbb{Z}^d)$. The trace τ^μ is defined on the dense subalgebra $C_c(\mathbb{Z}^p\times\Sigma)$ by the equality

$$\tau^\mu(f) = \langle\mu, f_0\rangle = \int_\Sigma f_0(z)d\mu(z), \qquad \text{where } f_0 : z\mapsto f(0,z), \tag{4}$$

with 0 the zero element of $\mathbb{Z}^p$. Hence τ^μ induces a trace

$$\tau^\mu(=\tau^\mu_*) \ : \ K_0(C(\Sigma)\rtimes_\sigma \mathbb{Z}^p) \longrightarrow \mathbb{R}.$$

The C^*-algebra $C(\Sigma)\rtimes_\sigma \mathbb{Z}^p$ (or rather a Morita equivalent one) is a receptacle for the spectral projections onto spectral gaps of the magnetic Schrödinger operator associated with our system, as explained earlier. Any gap in its spectrum may therefore be labelled by the trace of the corresponding projection.

Definition 1. The range of the trace τ^μ

$$\mathrm{Range}(\tau^\mu) \ := \ \tau^\mu\left(K_0(C(\Sigma)\rtimes_\sigma \mathbb{Z}^p)\right) \ \subset \ \mathbb{R}$$

is what is natural to call the *magnetic gap-labelling group.*

The countable subgroup of $\mathbb{R}$ given by:

$$\sum_{0\le|I|\le p} \mathrm{Pf}(\Theta_I)\mathbb{Z}_I[\mu],$$

is what is natural to call the *magnetic frequency group.*

It is easy to see the gap-labelling group (without magnetic field) is contained in the magnetic gap-labelling group.

We next formulate a conjecture that gives an explicit calculation of the magnetic gap-labelling group, and later give evidence for its validity.

Conjecture 1 (**Magnetic gap-labelling conjecture: minimal actions).** *Let Σ be a Cantor set with a minimal action of $\mathbb{Z}^p$ that preserves a Borel probability measure μ. Let σ be the multiplier on $\mathbb{Z}^p$ associated to a skew-symmetric $(p \times p)$ matrix Θ.*
Then the magnetic gap-labelling group is contained in the magnetic frequency group.

So, more explicitly, we conjecture the following:

(1) If p is even, then the magnetic gap-labelling group is contained in the countable subgroup of $\mathbb{R}$ given by:

$$\mathbb{Z}[\mu] + \sum_{0<|I|<p} \mathrm{Pf}(\Theta_I)\mathbb{Z}_I[\mu] + \mathrm{Pf}(\Theta)\mathbb{Z}.$$

(2) If p is odd, then the magnetic gap-labelling group is contained in the countable subgroup of $\mathbb{R}$ given by:

$$\mathbb{Z}[\mu] + \sum_{0<|I|\leq p} \mathrm{Pf}(\Theta_I)\mathbb{Z}_I[\mu].$$

In both cases, I is an ordered subset of $\{1, \ldots, p\}$ with an even number of elements, Θ_I denotes the skew-symmetric submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$, and $\mathrm{Pf}(\Theta_I)$ denotes the Pfaffian of Θ_I.

We mention here the Pfaffian $\mathrm{Pf}(\Theta)$ was recognised first as the top degree coefficient in the range of the trace of the noncommutative torus associated to Θ in [24], whereas the other terms were only given in terms of the coefficients of Θ. The complete set of coefficients in terms of the Pfaffians of submatrices of Θ is given over here for the first time in Proposition 4.1, and is also due to [34] in another context.

Remark 3.1. When p is even, we note that $C(\Sigma, \mathbb{Z})^{\mathbb{Z}^p} = \mathbb{Z}$ since the $\mathbb{Z}^p$ action on Σ is minimal, which accounts for why the last term in part (1) above is a multiple of $\mathbb{Z}$. More precisely, when p is even, by the Baum-Connes map μ_θ and the measured foliated twisted L^2-index theorem (see Section 3.3),

$$\tau_\mu(\mu_\theta(E)) = \sum_I \mathrm{Pf}(\Theta_I) \int_X dx_I \wedge ch(E)$$

where I is an ordered subset of $\{1, \ldots, p\}$ with an even number of elements, dx_I is the differential form of degree equal to $|I|$ on the torus $\mathbb{T}^p$ but lifted to $X = \Sigma \times_{\mathbb{Z}^p} \mathbb{R}^p$, which is the total space of a fibre bundle over the torus $\mathbb{T}^p$ with fibre the Cantor set Σ and E is a vector bundle over X whose Chern character is defined as in [37]. Consider the top degree term,

$$\mathrm{Pf}(\Theta) \int_X dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p \, \mathrm{rank}(E). \tag{5}$$

Since the action of $\mathbb{Z}^p$ on the Cantor set Σ is minimal, X is therefore a *connected* space cf. Lemma 3 [16]. So the rank of the vector bundle E on X is constant and (5) now becomes

$$\mathrm{Pf}(\Theta)\mathrm{rank}(E)\mathrm{vol}(\mathbb{T}^\mathrm{p})\mu(\Sigma) = \mathrm{Pf}(\Theta)\mathrm{rank}(\mathrm{E}), \tag{6}$$

since μ is a probability measure and the volume of the torus is normalised to equal 1. This implies that the last term in Conjecture 1 is always of the form $\mathrm{Pf}(\Theta)\mathbb{Z}$.

Remark 3.2. We mention that for the dynamical systems arising from the cut-and-project quasi-crystals, the action is moreover free. One might thus tackle our conjecture under the extra freeness assumption but we do not make this assumption in the present paper, see again [16].

In order to ensure the equality in the previous conjecture, it seems necessary to strengthen the minimality assumption. In particular, we mention the following interesting question:

Problem 1. *Under which (dynamical) condition, can one replace containment in Conjecture 1 by equality?*

Of particular interest is the case of strongly minimal actions which are defined as follows.

Definition 2. When p is odd, an action of $\mathbb{Z}^p$ is said to be *strongly minimal* when for every generator T_j, the infinite cyclic group generated by it $\langle T_j \rangle$ acts minimally. When p is even, an action of $\mathbb{Z}^p$ is said to be *strongly minimal* when for every pair of generators $(T_i, T_j)_{i\neq j}$, the $\mathbb{Z}^2$ group generated by the pair $\langle T_i, T_j \rangle$ acts minimally.

Remark 3.3. Strongly minimal actions are clearly minimal. In the 2D case, minimal actions are strongly minimal.

For $p \leq 3$, we prove that if the action of $\mathbb{Z}^3$ on Σ is strongly minimal, then the answer is yes. The strong minimality condition might though be too strong for 3D (see Section 8), or even too weak for higher dimensions. It is surprising that despite the case of the zero magnetic field, where the missing containment relation always holds without any extra-condition, when the magnetic field is non-zero, the answer to the above problem is not clear. As a starting point, we then state:

Conjecture 2 (**Magnetic Gap Labelling conjecture, strongly minimal actions).** *If the action if strongly minimal, then the magnetic gap labelling group coincides with the magnetic frequency group.*

Remark 3.4. As mentioned before, in the 2D case, minimal actions are strongly minimal and we prove Conjecture 2 in this case in Section 5, in the 3D case in Section 7 and also in the periodic case in Section 4.

Remark 3.5. There are several motivations for the magnetic gap-labelling conjecture, starting with Bellissard's gap-labelling conjecture [5], the formula (see section 1 in [34])

$$e^{\frac{1}{2}dx^t\Theta dx} = \sum_I \mathrm{Pf}(\Theta_I)dx^I$$

in the notation above, the magnetic gap-labelling group when the potential of the magnetic Schrödinger operator is purely periodic which reduces to the computation of the range of the trace on the noncommutative torus A_Θ using the formula above (see Section 4), and finally the proof of both the magnetic gap-labelling conjectures in 2D and in 3D, that we give in this paper.

3.2. **Some reductions of Conjectures 1 and 2.** Let us first mention that it suffices to prove Conjecture 1 and Conjecture 2 for p even.

Lemma 3.6. *If Conjecture 1 is true for some $p_0 \in \mathbb{N}$, then it is also true for all $p \in \mathbb{N}$ such that $p \leq p_0$. In particular, if Conjecture 1 is true for all $p \in 2\mathbb{N}$, then it is true for all $p \in \mathbb{N}$. If Conjecture 1 is true for all $p \in 2\mathbb{N}+1$, then it is true for all $p \in \mathbb{N}$.*

Proof. By an easy trick, see for instance [16], given a minimal Cantor system of dimension p as above we may embed $\mathbb{Z}^p$ in the group $\mathbb{Z}^{p+1}$ by using $\iota(n) = (n, 0)$ and notice that $\mathbb{Z}^{p+1}$ acts minimally on Σ by using the $\mathbb{Z}^p$-action of the projection onto the p-first factors. We get in this way a new minimal Cantor system of dimension $p+1$. The multiplier σ on $\mathbb{Z}^p$ then gives rise to the multiplier $\iota_*\sigma$ on $\mathbb{Z}^{p+1}$ which is given by

$$\iota_*\sigma((n, n_{p+1}), (m, m_{p+1})) = \sigma(n, m).$$

Then $\iota_*\sigma$ is associated with the skew symmetric matrix

$$\iota_*\Theta = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, the above inclusion ι yields an inclusion $i : C(\Sigma) \rtimes_\sigma \mathbb{Z}^p \hookrightarrow C(\Sigma) \rtimes_{\iota_*\sigma} \mathbb{Z}^{p+1}$ and it is easy to check that the following diagram commutes

$$\begin{array}{ccc} K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p) & \xrightarrow{i_*} & K_0(C(\Sigma) \rtimes_{\iota_*\sigma} \mathbb{Z}^{p+1}) \\ {\scriptstyle \tau_*^\mu}\downarrow & & \downarrow{\scriptstyle \tau_*^\mu} \\ \mathbb{R} & \xrightarrow{=} & \mathbb{R} \end{array}$$

Hence if we assume that Conjecture 1 is true for $p+1$, then we get

$$\tau_*^\mu \left(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p)\right) = (\tau_*^\mu \circ i_*) \left(K_0(C(\Sigma) \rtimes_{\iota_*\sigma} \mathbb{Z}^{p+1})\right) \subset \sum_{0 \leq |I| \leq p+1} \mathrm{Pf}((\iota_*\Theta)_I)\mathbb{Z}_I[\mu]. \tag{7}$$

But from the expression of $\iota_*\Theta$, we see that if I contains $p+1$ then $\mathrm{Pf}((\iota_*\Theta)_I) = 0$ while for $I \subset \{1, \cdots, p\}$, we have $\mathrm{Pf}((\iota_*\Theta)_I) = \mathrm{Pf}(\Theta_I)$. □

The similar proof shows:

Lemma 3.7. *If Conjecture 2 is true for some $p_0 \in 2\mathbb{N}$, then it is also true for $p_0 - 1$. In particular, if Conjecture 2 is true for all $p \in 2\mathbb{N}$, then it is true for all $p \in \mathbb{N}$.*

Proof. We apply the same proof as for the previous lemma but replace the containment in (7) with equality and notice that $p = p_0 - 1$ is odd and hence we consider a strongly minimal system in the odd case. This means that every generator T_j for $j = 1, \cdots, p_0 - 1$ acts minimally. But then adding a trivial action of $\mathbb{Z}$ as above, we get again a strongly minimal action for $p = p_0$ even. □

We make some observations now to help investigate the conjectures. In comparison with the non-magnetic case, we expect the inclusion of the magnetic gap-labelling group into the magnetic frequency group to be the easy half of the conjecture. Recall that $\mathbb{Z}^J$ denotes the subgroup of $\mathbb{Z}^p$ which corresponds to a given ordered subset $J = \{j_1 < \cdots < j_{|J|}\}$ of $\{1, \cdots, p\}$ where we put zero components for the entries corresponding to the complement set J^c. So the action of $\mathbb{Z}^J$ on Σ is generated by the homeomorphisms $(T_j)_{j\in J}$. We denote for $\alpha = (\alpha_1, \cdots, \alpha_{|J|}) \in \mathbb{Z}^{|J|}$ by T^α the homeomorphism of Σ which is given by

$$T^\alpha := \Pi_{1\leq \ell \leq |J|}\ T_{j_\ell}^{\alpha_\ell} = T_{j_1}^{\alpha_1} \circ \cdots \circ T_{j_{|J|}}^{\alpha_{|J|}}.$$

Notice that the $\mathbb{Z}$-module $C(\Sigma, \mathbb{Z})$ is generated by the characteristic functions of the clopen subsets of Σ, however an identification of $C(\Sigma, \mathbb{Z})$ with a $\mathbb{Z}$-module ($\mathbb{Z}$-measures) constructed out of the Boolean algebra $\mathcal{S}$ of clopen subsets of Σ is not helpful. For $J = I^c$ with I an ordered multi-index set of even length as in the statement of the conjecture, one expects to give a simple description of the class in $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}$ of any clopen set Λ so that the $\mathbb{Z}^I$ invariance can be exploited. So, in order to show for instance that the magnetic frequency group (RHS) is contained in the magnetic gap-labelling group, one needs to show that for any such Λ, the real number $\mathrm{Pf}(\Theta_I) \times \mu(\Lambda)$ belongs to the magnetic gap-labelling group (LHS).

This inclusion will be given explicitly in Section 8 when Λ has a convenient presentation. We shall only give this construction in the $3D$ case where we succeeded to prove both our conjectures. Although the construction is already combinatorially involved, it turns out to be feasible by direct inspection. We expect that our proof will help to deduce an explicit construction of the easy half of the conjecture for general algebraic combinations of clopens satisfying the invariance property in the coinvariants, that is whose class belongs to $(C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}})^{\mathbb{Z}^I}$.

Another important observation is that the K-theory group of the twisted crossed product algebra is isomorphic to that of the untwisted one. More precisely, the following is probably known to experts, but we give the short proof. Let $X = \Sigma \times_{\mathbb{Z}^p} \mathbb{R}^p$ be the suspension of Σ, that is the quotient of the cartesian product $\Sigma \times \mathbb{R}^p$ under the diagonal action of $\mathbb{Z}^p$. The additive group $\mathbb{R}^p$ acts on X and this action yields a lamination of X and the action groupoid $X \rtimes \mathbb{R}^p$. Recall that the groupoid $X \rtimes \mathbb{R}^p$ is strongly Morita equivalent to the groupoid $\Sigma \rtimes \mathbb{Z}^p$, see [46, 27], see also [29].

Theorem 3.1 (Twisted Connes-Thom isomorphism)**.** *Let Σ be a Cantor set with an action of $\mathbb{Z}^p$ and let $X = \Sigma \times_{\mathbb{Z}^p} \mathbb{R}^p$. Let σ be the multiplier on $\mathbb{Z}^p$ associated to a skew-symmetric $(p \times p)$ matrix Θ. Then*

$$K^p(X) \cong K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p).$$

Proof. By the strong Morita equivalence of the groupoids $X \rtimes \mathbb{R}^p$ and $\Sigma \rtimes \mathbb{Z}^p$, we see that the crossed product C^*-algebra $C(X) \rtimes \mathbb{R}^p$, is strongly Morita equivalent to the crossed product C^*-algebra $C(\Sigma) \rtimes \mathbb{Z}^p$. In particular, using Connes-Thom isomorphism [20, 25] , one has

$$K^p(X) \cong K_0(C(X) \rtimes \mathbb{R}^p) \cong K_0(C(\Sigma) \rtimes \mathbb{Z}^p) \tag{8}$$

For $t \in [0,1]$, let σ_t denote the multiplier corresponding to the $(p \times p)$ skew symmetric matrix $t\Theta$. More precisely,

$$\sigma_t(\gamma, \gamma') = \exp\left(2\pi\sqrt{-1}t \sum_{j<k} \Theta_{jk}\gamma_j\gamma'_k\right), \qquad \text{where} \quad \gamma, \gamma' \in \mathbb{Z}^p.$$

Now $\{C(\Sigma) \rtimes_{\sigma_t} \mathbb{Z}^p : t \in [0,1]\}$ is a homotopy of twisted crossed products in the sense of section 4, [40], where $\sigma_0 = 1$ and $\sigma_1 = \sigma$. By Theorem 4.2 in [40], we deduce that

$$K_0(C(\Sigma) \rtimes \mathbb{Z}^p) \cong K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p). \tag{9}$$

The Theorem follows from (8) and (9). □

3.3. **The measured twisted foliated index theorem.** Here we sketch a proof of a special case of the measured twisted index theorem that we need in this paper. It is a twisted analogue of a special case of the index theorem in Benameur-Piazza [17]. The general case will be treated in [13].

Let $\rho : \mathbb{Z}^p \longrightarrow \mathrm{Homeo}(\Sigma)$ denote the minimal action of $\mathbb{Z}^p$ on Σ. We suppose that μ is an invariant measure on Σ and that p is even. Then the suspension $X = \mathbb{R}^p \times_{\mathbb{Z}^p} \Sigma$ is a compact foliated space with transversal the Cantor set Σ, and with invariant transverse measure induced from μ (cf. Proposition 2.2 [29]). The monodromy groupoid is:

$$\mathcal{G} = (\mathbb{R}^p \times \mathbb{R}^p \times \Sigma)/\mathbb{Z}^p. \tag{10}$$

Then the twisted monodromy groupoid C^*-algebra, $C^*(X, \mathcal{F}, \sigma)$ consists of the operator norm closure of continuous functions k satisfying the following conditions:

(1) $k \in C(\mathbb{R}^p \times \mathbb{R}^p \times \Sigma)$;
(2) $k(x.\gamma, y.\gamma, \vartheta.\gamma) = e^{i\varphi_\gamma(x)} k(x, y, \vartheta) e^{-i\varphi_\gamma(y)}$ for all $x, y \in \mathbb{R}^p$, $\vartheta \in \Sigma$, $\gamma \in \mathbb{Z}^p$;

Define the transverse trace as,

$$\tau_\mu(k) = \int_X k(x, x, \vartheta) d\mu(\vartheta) dx. \tag{11}$$

It easily extends to matrix valued kernel functions, by composing with the pointwise matrix trace. Next define the continuous functions $\varphi_\gamma(x, \gamma)$. Let $\Theta(\vartheta)$ be a skew-symmetric $p \times p$ matrix that is a continuous function of $\vartheta \in \Sigma$ and such that $\Theta(\vartheta.\gamma) = \Theta(\vartheta)$ for all $\vartheta \in \Sigma$, $\gamma \in \mathbb{Z}^p$. Since the action of $\mathbb{Z}^p$ on Σ is assumed to be minimal, it follows that $\Theta = \Theta(\vartheta)$ is independent of $\vartheta \in \Sigma$.

Set $B = \frac{1}{2} dx^t \Theta dx$, which is a closed 2-form on $\mathbb{R}^p \times \Sigma$ (which is independent of $\vartheta \in \Sigma$) satisfying $\gamma^* B = B$ for all $\gamma \in \mathbb{Z}^p$. Since $B = d\eta$ where for instance $\eta = \sum_{j<k} \Theta_{jk} x_j dx_k$, we get $0 = d(\gamma^*\eta - \eta)$. Since $\mathbb{R}^p$ is simply-connected, we see that $\gamma^*\eta - \eta = d\phi_\gamma$, where ϕ_γ is a smooth function on $\mathbb{R}^p \times \Sigma$ (which is independent of $\vartheta \in \Sigma$). We normalise it so that $\phi_\gamma(0) = 0$ for all $\gamma \in \mathbb{Z}^p$.

Consider functions $f \in L^2(\mathbb{R}^p \times \Sigma; dx d\mu)$ and bounded operators on it defined as follows,

(1) $S_\gamma f(x, \vartheta) = e^{i\varphi_\gamma(x)} f(x, \vartheta)$;
(2) $U_\gamma f(x, \vartheta) = f(x.\gamma, \vartheta.\gamma)$.

Then for all $\gamma \in \mathbb{Z}^p$, the bounded operators $T_\gamma = U_\gamma \circ S_\gamma$ satisfy the relation

$$T_{\gamma_1} T_{\gamma_1} = \sigma(\gamma_1, \gamma_2)\, T_{\gamma_1 \gamma_2} \tag{12}$$

where $\sigma(\gamma_1, \gamma_2) = \phi_{\gamma_1}(\gamma_2)$ is a multiplier on $\mathbb{Z}^p$.

Let $\partial\!\!\!/$ denote the Dirac operator on $\mathbb{R}^p$ and $\nabla = d + i\eta$ the connection on the trivial line bundle on $\mathbb{R}^p$, ∇^E the lift to $\mathbb{R}^p \times \Sigma$ of a connection on a vector bundle $E \to X$. Consider the twisted Dirac operator along the leaves of the lifted foliation,

$$D = \partial\!\!\!/ \otimes \nabla \otimes \nabla^E : L^2(\mathbb{R}^p \times \Sigma, \mathcal{S}^+ \otimes E) \longrightarrow L^2(\mathbb{R}^p \times \Sigma, \mathcal{S}^- \otimes E). \tag{13}$$

Then one computes that $T_\gamma \circ D = D \circ T_\gamma$ for all $\gamma \in \mathbb{Z}^p$. The heat kernel $k(t, x, y, \vartheta)$ of D, although not compactly supported, can be shown as usual to belong to the C^*-algebra of the foliation. More precisely, $k(t, x, y, \vartheta) \in C^*(X, \mathcal{F}, \sigma) \otimes \mathcal{K}$. For $t > 0$, define the idempotent

$$e_t(D) \in M_2(C^*(X, \mathcal{F}, \sigma) \otimes \mathcal{K})$$

as follows:

$$e_t(D) = \begin{pmatrix} e^{-tD^- D^+} & e^{-\frac{t}{2} D^- D^+} \dfrac{(1 - e^{-tD^- D^+})}{D^- D^+} D^+ \\ e^{-\frac{t}{2} D^+ D^-} D^+ & 1 - e^{-tD^+ D^-} \end{pmatrix},$$

It is the analogue of the Wassermann idempotent, see e.g. [22]. Then the $C^*(X, \mathcal{F}, \sigma)$-*twisted analytic index* is defined as

$$\mathrm{Index}_{C^*(X,\mathcal{F},\sigma)}(D^+) = [e_t(D)] - [E_0] \in K_0(C^*(X, \mathcal{F}, \sigma)), \tag{14}$$

where $t > 0$ and E_0 is the idempotent

$$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, using the fact that $C^*(X, \mathcal{F}, \sigma)$ and $C(X) \rtimes_\sigma \mathbb{R}^p$ are isomorphic, this constructs a *twisted foliated index map*, generalizing Section 2, [32] and also [21], [37]

$$\mathrm{Index}_{C^*(X,\mathcal{F},\sigma)} : K^0(X) \longrightarrow K_0(C(X) \rtimes_\sigma \mathbb{R}^p) \text{ as } [E] \longmapsto \mathrm{Index}_{C^*(X,\mathcal{F},\sigma)}(D^+).$$

The twisted measured index of D^+ is then by definition the τ^μ-trace of the index class, a real number.

Theorem 3.2. *Under the previous assumptions, the measured index of D^+ is given by the formula*

$$\tau_\mu(\mathrm{Index}(D^+)) = \sum_I \mathrm{Pf}(\Theta_I) \int_X dx_I \wedge \mathrm{ch}(F_E) d\mu(\vartheta).$$

Here I runs over subsets of $\{1, \ldots, p\}$ with an even number of elements, dx_I is the differential form of degree equal to $|I|$ on the torus $\mathbb{T}^p$ which is lifted to $X = \Sigma \times_{\mathbb{Z}^p} \mathbb{R}^p$, and Θ_I is the skew-symmetric submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$. Finally, F_E denotes the curvature of the connection ∇^E on the vector bundle E over X.

When p is odd, a similar construction gives the odd measured index formula.

Proof. A standard McKean-Singer type argument shows that the supertrace

$$\tau_\mu(\mathrm{tr}_s(k(t,\cdots))) = \tau_\mu(e^{-tD^-D^+}) - \tau_\mu(e^{-tD^+D^-}) = \tau_\mu(\mathrm{Index}_{C^*(X,\mathcal{F},\sigma)}(D^+)) \tag{15}$$

is independent of $t > 0$ and represents the measured twisted foliated index. To be self-contained, we outline the argument. First we show that equation (15) is independent of $t > 0$. The heat operator e^{-tD^2} can be differentiated with respect to t, since $\frac{d}{dt}e^{-tD^2}$ is a smoothing operator equal to $-D^2e^{-tD^2}$, and therefore

$$\frac{d}{dt}\tau^s_\mu(e^{-tD^2}) = -\tau^s_\mu(D^2e^{-tD^2}) = -\frac{1}{2}\tau^s_\mu([D, De^{-tD^2}]) = 0$$

where the last equality holds since D is an odd operator. Here τ^s_μ denotes the graded version of the trace τ_μ. This shows that $\tau^s_\mu(e^{-tD^2})$ is independent of $t > 0$.

To complete the proof, we need to show that the smoothing kernel $k(t,\cdots)$ of e^{-tD^2} converges to the smoothing kernel of the projection P to the nullspace of D, uniformly on compact subsets as $t \to \infty$. But this is identical to the argument given in the proof of Proposition 15.11 in [47].

Now, using the expression of the trace as an integral over the fundamental domain $\Sigma \times (0,1)^p$ with respect to the product measure $\mu \otimes dvol$ on $\Sigma \times \mathbb{R}^p$ and applying the standard local index method, see [18], we obtain

$$\lim_{t\downarrow 0}\tau_\mu(\mathrm{tr}_s(k(t,\cdots))) = \frac{1}{(2\pi)^p}\int_{\Sigma\times(0,1)^p}\exp\left(\frac{1}{2}dx^t\Theta dx\right)\wedge \mathrm{ch}(F_E)d\mu(\vartheta), \tag{16}$$

$$= \frac{1}{(2\pi)^p}\sum_I \mathrm{Pf}(\Theta_I)\int_X dx_I\wedge\mathrm{ch}(F_E)d\mu(\vartheta). \tag{17}$$

Here I runs over subsets of $\{1,\dots,p\}$ with an even number of elements, and Θ_I denotes the skew-symmetric submatrix of $\Theta = (\Theta_{ij})$ with $i,j \in I$. Observe that $\frac{i}{2}dx^t\Theta dx$ is the curvature of the connection ∇, and that $\exp\left(\frac{1}{2}dx^t\Theta dx\right)$ is the Chern character of ∇. □

4. Magnetic gap-labelling group for periodic potentials

Let $\Lambda[dx] = \Lambda[dx_1,\dots,dx_p]$ denote the exterior algebra with generators $dx_1,\dots,dx_p$. It has basis the monomials $dx_I = dx_{i_1},\dots,dx_{i_p},\ I = \{i_1,\dots,i_p\},\ i_1 < \cdots < i_p$. Given a skew-symmetric matrix Θ, we can associate a quadratic element $\frac{1}{2}dx^t\Theta dx$ in $\Lambda[dx]$. Here dx is the column vector with entries dx_j and dx^t is the row vector with the same entries. Then the Gaussian $e^{\frac{1}{2}dx^t\Theta dx}$ can be expressed in terms of the Pfaffians, see section 1 in the paper of Mathai-Quillen [34],

$$e^{\frac{1}{2}dx^t\Theta dx} = \sum_I \mathrm{Pf}(\Theta_I)dx_I \tag{18}$$

where I runs over subsets of $\{1,\dots,p\}$ with an even number of elements, and Θ_I denotes the submatrix of $\Theta = (\Theta_{ij})$ with $i,j\in I$, which is clearly also skew-symmetric.

Let $\tau : A_\Theta \to \mathbb{C}$ denote the von Neumann trace. Then the magnetic gap-labelling group for magnetic Schrödinger operators $H_{\eta,V}$ where V is periodic is given by Proposition 4.1, which is originally due to Elliott in [24], although in that paper, only the Pfaffian $\mathrm{Pf}(\Theta)$ was recognized, whereas the other terms were only given in terms of the coefficients of Θ, but were not recognized as the the Pfaffians of submatrices of Θ as neatly given over here, and was originally due to [34] in another context. Moreover our proof is different, being based on index theory and algebraic topology.

Proposition 4.1 (Magnetic gap-labelling group for periodic potentials)**.** *The range of the trace on the K-theory of A_Θ is:*

(1) If p is even, then

$$\tau(K_0(A_\Theta)) = \mathbb{Z} + \sum_{0<|I|<p} \mathrm{Pf}(\Theta_I)\mathbb{Z} + \mathrm{Pf}(\Theta)\mathbb{Z},$$

(2) If p is odd, then

$$\tau(K_0(A_\Theta)) = \mathbb{Z} + \sum_{0<|I|<p} \mathrm{Pf}(\Theta_I)\mathbb{Z},$$

where I runs over subsets of $\{1,\dots,p\}$ with an even number of elements, and Θ_I denotes the submatrix of $\Theta = (\Theta_{ij})$ with $i, j \in I$.

Proof. Since the Baum-Connes conjecture with coefficients is true for $\mathbb{Z}^p$, it follows that the twisted Baum-Connes conjecture is true for $\mathbb{Z}^p$.

$$\mu_\Theta : K^p(\mathbb{T}^p) \xrightarrow{\sim} K_0(A_\Theta)$$

is an isomorphism. Then by Appendix A and the twisted L^2-index theorem [33] and equation (18), one has

$$\begin{aligned}\tau(\mu_\Theta(\xi)) &= \int_{\mathbb{T}^p} e^{\frac{1}{2}dx^t\Theta dx} \wedge \mathrm{Ch}(\xi)\\ &= \sum_I \mathrm{Pf}(\Theta_I) \int_{\mathbb{T}^p} dx_I \wedge \mathrm{Ch}(\xi)_{I^c}\end{aligned}$$

where I^c is the index that is complement to I and $\mathrm{Ch}(\xi)_{I^c}$ denotes the component of the Chern character $\mathrm{Ch}(\xi)$ of the vector bundle ξ containing dx_{I^c}. Since the Chern character is integral on the torus $\mathbb{T}^p$, the result follows by varying ξ over all K-theory classes.

□

Remark 4.2. In fact, Elliott in [24], proved more in that he computed the whole Connes-Chern character in terms of the canonical action of the p-dimensional torus. The special case when $p = 2$ was partially due to Rieffel, [45] and partially to Pimsner-Voiculescu [43, 44].

5. Computation of the 2D magnetic gap-labelling group

We now compute the magnetic gap-labelling group in the easiest case of $p = 2$. Let again $\mathbb{Z}^2 \curvearrowright \Sigma$ be a minimal action with invariant probability measure μ. Let σ be a multiplier on $\mathbb{Z}^2$. Then the group cohomology class of $[\sigma] \in H^2(\mathbb{Z}^2; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$ can be identified with a real number θ, $0 \leq \theta < 1$. More precisely, we take $\sigma = e^{2\pi i\theta\omega}$ where ω is the standard symplectic form on $\mathbb{Z}^2$.

Notice that an easy inspection shows that the natural inclusion $A_\theta = C_r^*(\mathbb{Z}^2, \sigma) \overset{\iota}{\hookrightarrow} C(\Sigma) \rtimes_\sigma \mathbb{Z}^2$, takes the Rieffel projection $\mathcal{P}_\theta$ [45] to the projection $\iota(\mathcal{P}_\theta)$ in $C(\Sigma) \rtimes_\sigma \mathbb{Z}^2$ which generates a $\mathbb{Z}$ factor in $K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2)$. Indeed let $\mu : C(\Sigma) \to \mathbb{C}$ and $\tau^\mu : C(\Sigma) \rtimes_\sigma \mathbb{Z}^2 \to \mathbb{C}$ be the traces induced by μ, see equation (4), and by the same notation the maps induced on K-theory. Then we have,

Lemma 5.1. *Let $\tau : A_\theta \to \mathbb{C}$ denote the von Neumann trace. Then there is a commutative diagram,*

$$\begin{tikzcd} A_\theta \arrow[r, hook, "\iota"] \arrow[d, "\tau"] & C(\Sigma) \rtimes_\sigma \mathbb{Z}^2 \arrow[d, "\tau_\mu"] \\ \mathbb{C} \arrow[r, "="] & \mathbb{C} \end{tikzcd}$$

In particular, we see that $\theta = \tau(\mathcal{P}_\theta) = \tau_\mu(\iota(\mathcal{P}_\theta))$.

Proof. Let $f(\gamma) \in A_\theta$. Then by equation (4), we see that

$$\tau(f) = f(0), \quad \tau^\mu(\iota(f)) = \tau^\mu(f) = \int_\Sigma f(0)\, d\mu(z) = f(0),$$

since μ is a probability measure on Σ. □

Theorem 5.1.

$$K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2) \simeq C(\Sigma, \mathbb{Z})_{\mathbb{Z}^2} \oplus \mathbb{Z}[\iota(\mathcal{P}_\theta)].$$

where $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^2}$ denotes the space of coinvariants.

Proof. The computation of $K_0(C(\Sigma) \rtimes \mathbb{Z}^2)$ in the untwisted case was carried out in [8],

$$K_0(C(\Sigma) \rtimes \mathbb{Z}^2) \simeq C(\Sigma, \mathbb{Z})_{\mathbb{Z}^2} \oplus \mathbb{Z}.$$

By Theorem 3.1 above, we have

$$K^0(X) \cong K_0(C(\Sigma) \rtimes \mathbb{Z}^2) \cong K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2).$$

Therefore the result follows.

□

Remark 5.2. The computation in this 2D case is thus identical to the untwisted case and we only replace the Bott projection by the Rieffel projection.

Corollary 5.3 (Magnetic gap-labelling in 2 dimensions)**.**

$$\tau^\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2)) = \mathbb{Z}[\mu] + \mathbb{Z}\theta.$$

Proof. We use Theorem 5.1. Now $\tau^\mu(C(\Sigma, \mathbb{Z})_{\mathbb{Z}^2}) = \mu(C(\Sigma, \mathbb{Z})) = \mathbb{Z}[\mu]$ since the measure is invariant. By Lemma 5.1, $\tau_\mu(\iota(\mathcal{P}_\theta)) = \theta$.

□

We next give another proof of this result, using index theory, in order to help investigate the general case.

2nd Proof. By Theorem 3.1,

$$\mu_\theta : K^0(X) \xrightarrow{\sim} K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2)$$

is an isomorphism, where $X = \Sigma \times_{\mathbb{Z}^2} \mathbb{R}^2$.

By the measured foliated twisted L^2-index theorem (see Section 3.3),

$$\tau_\mu(\mu_\theta(\xi)) = \int_X e^{\theta dx_1 \wedge dx_2} \wedge \mathrm{Ch}(\xi)$$

X is connected since the $\mathbb{Z}^2$-action is minimal cf. Lemma 3 [16], so every vector bundle ξ on X has constant rank.

$$\begin{aligned} &= \theta \int_{\mathbb{T}^2} dx_1 \wedge dx_2 \, \mu(\Sigma)\mathrm{rank}(\xi) + \int_X c_1(\xi) \\ &= \theta \, \mathrm{rank}(\xi) + \int_X c_1(\xi) \end{aligned}$$

Varying over all vector bundles ξ on X, and using Bellissard's gap-labelling theorem in 2D [8] (when the magnetic field vanishes i.e. $\theta = 0$), the result follows. □

Remark 5.4. In section 5, [26], they compute a useful example which we now recall and that does not use their Theorem 4.2 in [26]. Suppose that $0 < \alpha_1 < \alpha_2 < 1$ are two rationally independent irrational numbers. Then $T_j x = x + \alpha_j \pmod 1$, $j = 1, 2$ defines a minimal $\mathbb{Z}^2$-action on the circle $\mathbb{R}/\mathbb{Z}$. Define the Cantor set Σ to be the circle disconnected along the orbit of $\mathbb{Z}^2$ through the origin. Then by fiat, $\mathbb{Z}^2$ also acts minimally on Σ and this example has a unique invariant probability measure μ. What is shown on page 623 in [26] is that in this case, one has,

$$\int_X c_1(\xi) \in \mathbb{Z}[\mu] = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2.$$

Therefore the magnetic gap-labelling theorem, Corollary 5.3, in this particular 2D case is:

$$\tau^\mu(K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^2)) = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta.$$

Remark 5.5. Corollary 5.3 was known to Kellendonk, in [30], and we thank him for pointing it out to us. He uses the bulk-boundary correspondence method to reduce to the 1D case. Recently another proof using groupoids has been given in [31].

Remark 5.6. In [36], they consider the 2D magnetic Schrödinger operator,

$$H = -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - \theta x\right)^2 + V(x).$$

Here $B = \theta \, dx \wedge dy$, $\theta \neq 0$ is a constant magnetic field, and V is a non-constant smooth τ-periodic electric potential that is independent of the y variable. The self-adjoint operator H on $L^2(\mathbb{R}^2)$ is proved in [36] to generically have *infinitely* many open spectral gaps. This is in stark contrast to the Bethe-Sommerfeld conjecture, which says that there are only a *finite* number of gaps in the spectrum of any Schrödinger operator with smooth periodic potential V on Euclidean space, in the case when the magnetic field vanishes, i.e. $\theta = 0$, whenever the dimension is greater than or equal to 2. The Bethe-Sommerfeld conjecture was proved by

L. Parnovski in [41]. In fact in [36], they also study the Hamiltonian $H_\pm = H \pm W$, where $W \in L^\infty(\mathbb{R}^2)$ is non-negative and decays at infinity and $\theta \neq 0$, so that $H_\pm$ is the sort of Hamiltonians that we consider in our paper. They find in [36] that there are *infinitely* many discrete eigenvalues of $H_\pm$ in any open gap in the spectrum of $\mathrm{spec}(H)$, and the convergence of these eigenvalues to the corresponding endpoint of the spectral gap is asymptotically Gaussian. This shows that the spectral gaps of magnetic Schrödinger operators (of the type considered in this paper) can be rather interesting even in higher dimensions.

In fact, let Θ be a skew-symmetric $(2n \times 2n)$ matrix. Putting Θ in Jordan normal form, we can assume without loss of generality that the associated magnetic field $B = \frac{1}{2} dx^t \Theta dx = \sum_{i=1}^n \Theta_{2i-1,2i} dx_{2i-1} \wedge dx_{2i}$. Choosing the vector potential $A = \sum_{i=1}^n \Theta_{2i-1,2i} x_{2i-1} \wedge dx_{2i}$, we see that with $H_A = (d+iA)^\dagger(d+iA)$ that $H_{A,V} = H_A + V$, where

$$H_{A,V} = -\sum_{i=1}^n \left(\left(\frac{\partial}{\partial x_{2i-1}} \right)^2 + \sum_{i=1}^n \left(-i\frac{\partial}{\partial x_{2i}} - \Theta_{2i-1,2i}\, x_{2i-1} \right)^2 + V_i(x_{2i-1}) \right)$$

Arguing exactly as in the 2D case, we see that H_A has discrete spectrum with infinite multiplicity, where V_i is a smooth periodic function similar to the 2D case. The argument of [36] easily extends to the higher (even) dimensional case in this way, giving examples of magnetic Schrödinger operators (of the type considered in this paper) that have infinitely many open spectral gaps that are interesting.

6. Proof of the conjecture in the Jordan block diagonal case

In this section, we establish the magnetic gap labelling conjecture 1 in the case when the skew symmetric matrix associated to the constant magnetic field on Euclidean space is in Jordan block diagonal form. Although we consider only even dimensional systems here, the odd dimensional case is similar.

Given a constant magnetic field on Euclidean space $\mathbb{R}^2$ determined by $\theta \in \mathbb{R}$, and a $\mathbb{Z}^2$-invariant probability measure μ on the disorder set Σ that is a Cantor set, where the action is minimal, we have seen in Corollary 5.3 that the range of the trace on K-theory is,

$$\tau^\mu(K_0(C(\Sigma) \rtimes_\theta \mathbb{Z}^2)) = \mathbb{Z}[\mu] + \theta\mathbb{Z}.$$

Now consider the special 4D situation of the same phenomenon, where the skew-symmetric matrix Θ is in Jordan block diagonal form,

$$\Theta = \begin{pmatrix} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{pmatrix}$$

Consider $\mathbb{Z}^2$-invariant probability measures μ_i on the disorder sets Σ_i that are Cantor sets for $i = 1, 2$, where the actions are minimal.

Let $\Sigma = \Sigma_1 \times \Sigma_2$. Then $\mathbb{Z}^4$ acts minimally on Σ with invariant measure $\mu = \mu_1 \times \mu_2$ and one can form the twisted crossed product algebra,

$$C(\Sigma) \rtimes_\Theta \mathbb{Z}^4 \cong (C(\Sigma_1) \rtimes_{\theta_1} \mathbb{Z}^2) \otimes (C(\Sigma_2) \rtimes_{\theta_2} \mathbb{Z}^2)$$

Recall that the Kunneth Theorem for Tensor Products [49] asserts that if A and B are C^*-algebras, with A being nuclear and $K_\bullet(A)$ being torsion-free, then there is a natural isomorphism,

$$K_0(A \otimes B) \cong K_0(A) \otimes K_0(B) \oplus K_1(A) \otimes K_1(B)$$

Note that $A = C(\Sigma_1) \rtimes_{\theta_1} \mathbb{Z}^2$ and $B = C(\Sigma_2) \rtimes_{\theta_2} \mathbb{Z}^2$ satisfy the hypotheses of the Kunneth theorem, and that

$$K_1(C(\Sigma) \rtimes_\theta \mathbb{Z}^2)) = \mathbb{Z}[u_1] + \mathbb{Z}[u_2],$$

where u_1, u_2 are the unitaries generating $K_1(C^*(\mathbb{Z}^2))$. Then the tensor product of the K_1 groups is,

$$(\mathbb{Z}[u_1] + \mathbb{Z}[u_2])(\mathbb{Z}[u_1] + \mathbb{Z}[u_2]) = \mathbb{Z}[u_1 \cup u_1] + \mathbb{Z}[u_2 \cup u_2] + \mathbb{Z}[u_1 \cup u_2]$$

Applying the trace, we see that

$$\tau^\mu(K_1(C(\Sigma) \rtimes_\theta \mathbb{Z}^2)) \otimes K_1(C(\Sigma) \rtimes_\theta \mathbb{Z}^2))) \subset \mathbb{Z}.$$

Therefore the range of the trace on K-theory is,

$$\begin{aligned}
\tau^\mu(K_0(C(\Sigma) \rtimes_\Theta \mathbb{Z}^4)) &= (\mathbb{Z}[\mu_1] + \theta_1\mathbb{Z})(\mathbb{Z}[\mu_2] + \theta_2\mathbb{Z}) \\
&= \mathbb{Z}[\mu_1]\mathbb{Z}[\mu_2] + \theta_2\mathbb{Z}[\mu_1] + \theta_1\mathbb{Z}[\mu_2] + \theta_1\theta_2\mathbb{Z} \\
&\subset \mathbb{Z}[\mu_1 \times \mu_2] + \theta_2\mathbb{Z}[\mu_1] + \theta_1\mathbb{Z}[\mu_2] + \theta_1\theta_2\mathbb{Z}
\end{aligned}$$

since it is not hard to see that $\mathbb{Z}[\mu_1]\mathbb{Z}[\mu_2] \subset \mathbb{Z}[\mu_1 \times \mu_2]$. This proves another special case of our magnetic gap-labelling conjecture 1 and also motivates our conjecture.

By elementary induction, this works for all even dimensional systems of this sort. More precisely, consider the constant magnetic field on Euclidean space $\mathbb{R}^{2n}$ given by the skew-symmetric matrix in Jordan diagonal form,

$$\Theta = \bigoplus_{j=1}^n \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix}$$

Consider $\mathbb{Z}^2$-invariant probability measures μ_i on the disorder sets Σ_i that are Cantor sets for $i = 1, 2, \ldots, n$, where the actions are all minimal. Let $\Sigma = \Sigma_1 \times \Sigma_2 \ldots \times \Sigma_n$. Then $\mathbb{Z}^{2n}$ acts on Σ and one can form the twisted crossed product algebra,

$$C(\Sigma) \rtimes_\Theta \mathbb{Z}^{2n} \cong \bigotimes_{j=1}^n (C(\Sigma_j) \rtimes_{\theta_j} \mathbb{Z}^2)$$

Assume the induction hypothesis that

$$\tau^\mu(K_0(C(\Sigma) \rtimes_\Theta \mathbb{Z}^{2n})) \subset \sum_I \theta_I \mathbb{Z}[\mu_{I^c}]$$

where $\theta_I = \prod_{j\in I}\theta_j, \quad \mu_J = \prod_{j\in J}\mu_j, \quad \mu = \prod_{j=1}^n \mu_j$, I^c denotes the index set that is complement to I and $|I| \le n$. This is exactly the statement of our conjecture in this special case.

Now let $\Sigma' = \Sigma \times \Sigma_{n+1}$, with $\mathbb{Z}^2$-invariant probability measure μ_{n+1} on the disorder set Σ_{n+1} that is a Cantor set, where the actions is minimal. Consider the constant magnetic field on Euclidean space $\mathbb{R}^{2n+2}$ given by the skew-symmetric matrix in Jordan diagonal form,

$$\Theta' = \Theta \oplus \begin{pmatrix} 0 & -\theta_{n+1} \\ \theta_{n+1} & 0 \end{pmatrix}$$

Then $\mathbb{Z}^{2n+2}$ acts on Σ' and one can form the twisted crossed product algebra,

$$C(\Sigma') \rtimes_{\Theta'} \mathbb{Z}^{2n+2} \cong C(\Sigma) \rtimes_{\Theta} \mathbb{Z}^{2n} \otimes C(\Sigma_{n+1}) \rtimes_{\theta_{n+1}} \mathbb{Z}^2.$$

Using the Kunneth Theorem for Tensor Products [49] and arguing as before, we see that the range of the trace on K-theory is,

$$\begin{aligned}\tau^{\mu}(K_0(C(\Sigma') \rtimes_{\Theta'} \mathbb{Z}^{2n+2})) &\subset (\sum_I \theta_I \mathbb{Z}[\mu_{I^c}])(\mathbb{Z}[\mu_{n+1}] + \theta_{n+1}\mathbb{Z}) \\ &\subset \sum_I \left(\theta_I \mathbb{Z}[\mu_{I^c \cup (n+1)}] + \sum_I \theta_{I\cup(n+1)}\mathbb{Z}[\mu_{I^c}] \right),\end{aligned}$$

completing the induction step for $n+1$, thereby establishing our magnetic gap-labelling conjecture 1 in the Jordan block diagonal case.

7. The 3D case

We restrict ourselves in this section to the 3D case where we have succeeded to prove Conjecture 1 and Conjecture 2.

7.1. **Proof of Conjecture 1.** We now proceed to prove Conjecture 1 in the 3D case. We prove more precisely that the magnetic gap-labelling group of an aperiodic tiling which corresponds to an action of $\mathbb{Z}^3$ on the Cantor space Σ, is contained in the magnetic frequency group defined in Section 3. We assume that there are no nontrivial globally invariant open subspaces in Σ. The Boolean algebra of clopen subspaces of Σ is denoted $\mathcal{S}$ and it is also endowed with the induced action of $\mathbb{Z}^3$. The generators of the $\mathbb{Z}^3$ action on Σ but also the induced one on $\mathcal{S}$ are denoted generically $(T_i)_{1\le i\le 3}$. The free subgroup of $\mathbb{Z}^3$ generated by T_j is again denoted $\langle T_j\rangle$.

For any $1 \le i < j \le 3$, we denote as before by $\mathbb{Z}_{ij}[\mu]$ the subgroup of the real line generated by μ-integrals of $\mathbb{Z}$-valued functions on Σ whose image in the coinvariants under $\mathbb{Z}^{\{i,j\}^c}$ is $\mathbb{Z}^{\{i,j\}}$-invariant. A multiplier $\sigma \in H^2(\mathbb{Z}^3, \mathbb{R}/\mathbb{Z}) \simeq \Lambda^2(\mathbb{R}/\mathbb{Z})^3$ as before is associated with a skew matrix $\Theta \in M_3(\mathbb{R})$ and hence with the three real numbers Θ_{12}, Θ_{13} and Θ_{23}, which are the entries of the matrix Θ. We are now in position to state the main result of this section.

Theorem 7.1. *With the previous notations,*

$$\mathrm{Range}(\tau^{\mu}_*) \subset \mathbb{Z}[\mu] + \Theta_{12}\,\mathbb{Z}_{12}[\mu] + \Theta_{13}\,\mathbb{Z}_{13}[\mu] + \Theta_{23}\,\mathbb{Z}_{23}[\mu].$$

Before embarking on the long proof of Theorem 7.1, we first outline the strategy and the main steps of this proof for the convenience of the reader. Lemma 7.1 simplifies the topological side of the measured twisted foliated index theorem (see subsection 3.3) in the 3D case, using in particular the Baum-Connes map [2]. Using Lemma 7.1 and some homological algebra of group cohomology with coefficients in a module, Corollary 7.2 identifies the magnetic gap labelling group in these terms. Further homological properties are exploited in both Lemma 7.3 and Lemma 7.4, together with integrality of the Chern character in the 3D case, to reduce the proof of Theorem 7.1 to the computation of the range of the *integral* group cohomology with coefficients, under the cup-product morphism with respect to the magnetic field. The computation of this latter range yields to the proof of Theorem 7.1 and verifies Conjecture 1 in the 3D case. In subsection 7.2, upon assuming that the action is strongly minimal, Lemma 7.5 studies the boundary map in group cohomology with coefficients in a module and reduces the proof of Conjecture 2 to an injectivity condition, see Theorem 7.2. Finally the injectivity condition of Theorem 7.2 is established for strongly minimal actions in Corollary 7.6, proving Conjecture 2 in the 3D case.

The rest of the section will thus be devoted to this long direct computation that we have split into lemmas for the sake of clarity as described above. Recall that the mapping torus is the space $X := (\Sigma \times \mathbb{R}^3)/\mathbb{Z}^3$ where we have divided out by the diagonal action. This is a transversely Cantor foliated space which fibres over the three torus $\mathbb{T}^3$, in particular pulling back cohomology classes on $\mathbb{T}^3$ we get cohomology classes on X. There is a well defined "Poincaré duality" isomorphism which was explicitly described in [16] using leafwise cohomology:

$$\Psi_{\mathbb{Z}^3} : H^3(X,\mathbb{R}) \longrightarrow C(\Sigma,\mathbb{R})_{\mathbb{Z}^3} \text{ and also the Cech version } \Psi_{\mathbb{Z}^3} : H^3(X,\mathbb{Z}) \longrightarrow C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3}.$$

Here and as in the other sections, for any subgroup Γ, the subscript $(\ )_\Gamma$ refers to coinvariants while the superscript $(\ )^\Gamma$ refers to invariants. Recall the 2-cohomology class B on the torus $\mathbb{T}^3$ which is associated with σ.

Lemma 7.1. *The magnetic gap-labelling group is given by*

$$\mathrm{Range}(\tau^\mu_*) \;=\; \mathbb{Z}[\mu] \;+\; \left\{ \left\langle \mu\circ\Psi_{\mathbb{Z}^3}, \mathrm{tr}\left(\frac{u^{-1}\,du}{2i\pi}\right) \cup B \right\rangle, u\in K^1(X) \right\}.$$

Proof. Denote by $\partial\!\!\!/_\sigma$ the leafwise σ-twisted Dirac operator (see subsection 3.3) along the leaves of our foliated space X. By classical arguments about the Baum-Connes map for $\mathbb{Z}^3$ with coefficients in the $\mathbb{Z}^3$-algebra $C(\Sigma)$, it is easy to check that the twisted Connes' Thom isomorphism of Theorem 3.1, $K^1(X)\to K_0(C(X)\rtimes_\sigma \mathbb{R}^3)$ coincides with the twisted foliated index map, see subsection 3.3 and also [16]. More precisely, the K-theory of X can be trivially identified with the K-theory of the algebra $C^{\infty,0}(X)$ of continuous leafwise smooth functions on X. Given a unitary u in a matrix algebra of $C^{\infty,0}(X)$, we may consider the Toeplitz operator $T_{\sigma,u}$ associated with $\partial\!\!\!/_\sigma$ and with symbol u. This is a leafwise elliptic 0-th order pseudodifferential operator on the foliated space X and it has a well defined index class $\mathrm{Ind}(T_{\sigma,u})$ in $K_0(C^{\infty,0}(X)\rtimes_\sigma \mathbb{R}^3)$ which is defined in [13] following the lines of [37]. So the Connes' Thom isomorphism is described as the map $[u] \longmapsto \mathrm{Ind}(T_{\sigma,u})$. On the twisted foliation algebra $C^{\infty,0}(X) \rtimes_\sigma \mathbb{R}^3$ we have the semi-finite normal trace, denoted equally τ^μ, associated with the monodromy invariant measure defined by μ, and we know that this trace and the original trace τ^μ on $C(\Sigma) \rtimes_\sigma \mathbb{Z}^3$ agree in K-theory with respect to the Morita isomorphism (see [17])

$$K_0(C(X) \rtimes_\sigma \mathbb{R}^3) \longrightarrow K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^3).$$

As a corollary of this discussion, we see that the magnetic gap-labelling group coincides with the range of the map

$$K^1(X) \longrightarrow \mathbb{R} \text{ given by } [u] \longmapsto \tau^\mu_* \left(\mathrm{Ind}(T_{\sigma,u})\right).$$

We now apply the twisted foliated index theorem from subsection 3.3 and [13] which gives exactly the statement of the lemma. Recall indeed that $\mathrm{ch}(u) = \mathrm{ch}_1(u) + \mathrm{ch}_3(u)$ where $\mathrm{ch}_1(u) = \mathrm{tr}\left(\frac{u^{-1}\,du}{2i\pi}\right)$. On the other hand the gap-labelling theorem for $p = 3$ implies that the group

$$\left\{ \left\langle \mu, \int_{(0,1)^3} \mathrm{ch}_3(u) \right\rangle, u \in K^1(X) \right\} \text{ coincides with } \mathbb{Z}[\mu].$$

$\square$

It remains thus to identify the second additive subgroup appearing in the previous lemma. Notice that the range of $\mathrm{ch}_1 : K^1(X) \to H^1(X, \mathbb{Q})$ is exactly given by $H^1(X, \mathbb{Z})$ and is hence isomorphic through $\Psi_{\mathbb{Z}^3}$ to $H^1(\mathbb{Z}^3, C(\Sigma, \mathbb{Z}))$. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
H^*(X,\mathbb{Z}) \otimes H^*(\mathbb{T}^3,\mathbb{R}) & \xrightarrow{\cup} & H^*(X,\mathbb{R}) \\
\Big\downarrow{\scriptstyle \nu\otimes P} & & \Big\downarrow{\scriptstyle \nu_{\mathbb{R}}} \\
H^*(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \otimes H^*(\mathbb{Z}^3,\mathbb{R}) & \xrightarrow{\cup} & H^*(\mathbb{Z}^3, C(\Sigma,\mathbb{R}))
\end{array}$$

where ν is the isomorphism $H^*(X, \mathbb{Z}) \to H^*(\mathbb{Z}^3, C(\Sigma, \mathbb{Z}))$ and $\nu_{\mathbb{R}}$ its version with real coefficients. Here P is the Pontryagin isomorphism $H^*(\mathbb{T}^3, \mathbb{R}) \to H^*(\mathbb{Z}^3, \mathbb{R})$.

We denote as before by $(\psi_j)_{1\le j\le 3}$ the generators of $H^1(\mathbb{Z}^3, \mathbb{Z})$. So, Θ corresponds to the element of $H^2(\mathbb{Z}^3, \mathbb{R})$, still denoted by Θ, given by

$$\Theta := \Theta_{12}\psi_1 \cup \psi_2 + \Theta_{13}\psi_1 \cup \psi_3 + \Theta_{23}\psi_2 \cup \psi_3,$$

whose cohomology class corresponds through the Pontryagin duality to the de Rham cohomology class of the form B on $\mathbb{T}^3$. In view of the previous lemma, we need to compute the range of the composite map

$$H^1(X, \mathbb{Z}) \xrightarrow{\cup B} H^3(X, \mathbb{R}) \xrightarrow{\Psi_{\mathbb{Z}^3}} C(\Sigma, \mathbb{R})_{\mathbb{Z}^3} \xrightarrow{\mu} \mathbb{R}.$$

Corollary 7.2. *The magnetic gap-labelling group coincides with the subgroup of* $\mathbb{R}$ *which is* $\mathbb{Z}[\mu]$ *plus the range of the map*

$$H^1(\mathbb{Z}^3, C(\Sigma, \mathbb{Z})) \xrightarrow{\cup\Theta} H^3(\mathbb{Z}^3, C(\Sigma, \mathbb{R})) \xrightarrow{\Psi} C(\Sigma, \mathbb{R})_{\mathbb{Z}^3} \xrightarrow{\mu} \mathbb{R}.$$

where Ψ *is the usual Poincaré duality isomorphism for the group* $\mathbb{Z}^3$.

Proof. From the previous considerations, we deduce the following commutative diagram

$$\begin{array}{ccccc}
H^1(X,\mathbb{Z}) & \xrightarrow{\cup B} & H^3(X,\mathbb{R}) & \xrightarrow{\Psi_{\mathbb{Z}^3}} & C(\Sigma,\mathbb{R})_{\mathbb{Z}^3} \\
{\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\nu_{\mathbb{R}}} & & \big\downarrow{\scriptstyle =} \\
H^1(\mathbb{Z}^3,C(\Sigma,\mathbb{Z})) & \xrightarrow{\cup\Theta} & H^3(\mathbb{Z}^3,C(\Sigma,\mathbb{R})) & \xrightarrow[\Psi]{} & C(\Sigma,\mathbb{R})_{\mathbb{Z}^3}
\end{array}$$

This completes the proof of the corollary, upon using Lemma 7.1. $\square$

In the sequel, we shall for simplicity no more denote the isomorphism $\nu : H^*(X,\mathbb{Z}) \simeq H^*(\mathbb{Z}^3, C(\Sigma,\mathbb{Z}))$ and hence denote by $\Psi_{\mathbb{Z}^3}$ the Poincaré duality isomorphism

$$\Psi_{\mathbb{Z}^3} : H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \longrightarrow H_0(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) = C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3}.$$

This discussion allows to deduce that the magnetic gap-labelling group coincides with $\mathbb{Z}[\mu]$ plus the sum of the ranges of the three maps corresponding to $1 \le i < j \le 3$ which are

$$H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \overset{\cup(\Theta_{ij}\psi_i\cup\psi_j)}{\longrightarrow} H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{R}) \xrightarrow{\Psi_{\mathbb{Z}^3}} C(\Sigma,\mathbb{R})_{\mathbb{Z}^3} \xrightarrow{\mu} \mathbb{R}.$$

It thus suffices to give the proof for $\Theta_{12}\psi_1 \cup \psi_2$ and the two other ranges will be obtained similarly. Since Θ_{12} is constant, we only need to deal with the element $\psi_1 \cup \psi_2$ in $H^2(\mathbb{Z}^3,\mathbb{Z})$. We shall denote by

$$\Psi_{\langle T_i,T_j\rangle} : H^2(\langle T_i,T_j\rangle, C(\Sigma,\mathbb{Z})) \longrightarrow C(\Sigma,\mathbb{Z})_{\langle T_i,T_j\rangle},$$

the similar isomorphism to $\Psi_{\mathbb{Z}^3}$ but corresponding to the Poincaré duality for the mapping torus associated with the $\mathbb{Z}^2$ action on Σ corresponding to the generators T_i and T_j. Notice that there is an induced action of T_2 on $H^*(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z}))$ and on $C(\Sigma,\mathbb{Z})_{\langle T_1 T_3\rangle}$ and that $\Psi_{\langle T_1,T_3\rangle}$ is $\langle T_2\rangle$-equivariant [16]. In particular, the invariants

$$H^2(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle}$$

are sent under the map $\Psi_{\langle T_1,T_3\rangle}$ into the invariants $\left(C(\Sigma,\mathbb{Z})_{\langle T_1,T_3\rangle}\right)^{\langle T_2\rangle}$.

From Theorem 8 in [16], it is easy to deduce the following exact sequences

$$0 \to \left(C(\Sigma,\mathbb{Z})^{\langle T_1,T_3\rangle}\right)_{\langle T_2\rangle} \xrightarrow{i} H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \xrightarrow{\pi} H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \to 0 \tag{19}$$

and

$$0 \to \left(C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}\right)_{\langle T_1\rangle} \xrightarrow{i'} H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z})) \xrightarrow{\pi'} \left(C(\Sigma,\mathbb{Z})_{\langle T_3\rangle}\right)^{\langle T_1\rangle} \to 0. \tag{20}$$

For the convenience of the reader, let us briefly describe the maps appearing in these exact sequences. The maps i and i' are pull-back maps corresponding to projections onto $\langle T_2\rangle$ and $\langle T_1\rangle$ respectively, composed with inclusions of coefficients. For instance, in the first exact sequence (19), $\left(C(\Sigma,\mathbb{Z})^{\langle T_1,T_3\rangle}\right)_{\langle T_2\rangle}$ is first identified with $H^1(\langle T_2\rangle, C(\Sigma,\mathbb{Z})^{\langle T_1,T_3\rangle})$. Then i is the composite map

$$\left(C(\Sigma,\mathbb{Z})^{\langle T_1,T_3\rangle}\right)_{\langle T_2\rangle} \simeq H^1(\langle T_2\rangle, C(\Sigma,\mathbb{Z})^{\langle T_1,T_3\rangle}) \xrightarrow{\pi_2^*} H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})^{\langle T_1,T_3\rangle}) \to H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})).$$

Here we have denoted by $\pi_j : \mathbb{Z}^3 \to \langle T_j\rangle$ the projection. The similar description holds for the map i' in the second exact sequence.

In the same way, the maps π and π' are just pull-back maps. More precisely, again in the first exact sequence, the natural inclusion $\iota : \langle T_1, T_3\rangle \hookrightarrow \mathbb{Z}^3$ obtained by crossing with zero for the missing $\langle T_2\rangle$, induces

$$\iota^* : H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \longrightarrow H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z})),$$

and it is easy to see that the range of this map is exactly $H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle}$. For instance, if $\iota_2 : \langle T_2\rangle \hookrightarrow \mathbb{Z}^3$ is the similar inclusion then for any $g \in \langle T_1, T_3\rangle$ and any $g_2 \in \langle T_2\rangle$, one checks the following relation for any 1-cocycle $c \in H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z}))$

$$(g_2\iota^* c - c)(g) = (g\iota_2^* c - \iota_2^* c)(g_2).$$

Again the map π' is defined similarly.

It is then clear by compatibility of cup products with pull-backs that the map

$$\cup\psi_2 : H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \longrightarrow H^2(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})),$$

vanishes on the image of i, say on the subgroup $H^1(\langle T_2\rangle, C(\Sigma,\mathbb{Z})^{\langle T_1, T_3\rangle})$, and hence we deduce that cup product with $\psi_1 \cup \psi_2$ induces a well defined map

$$\alpha : H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \longrightarrow H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})).$$

Lemma 7.3. *The composite map $\Psi_{\mathbb{Z}^3} \circ \alpha$ coincides, up to sign, with the expected map*

$$H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \xrightarrow{\cup\psi_1} H^2(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle}$$
$$\xrightarrow{\Psi_{\langle T_1, T_3\rangle}} \left(C(\Sigma,\mathbb{Z})_{\langle T_1, T_3\rangle}\right)^{\langle T_2\rangle} \longrightarrow C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3}.$$

Proof. The proposed composite map is denoted $\Psi_{\mathbb{Z}^3} \circ \alpha'$ and it is clearly well defined. The map α is also well defined and it is by definition induced by

$$\cup(\psi_1 \cup \psi_2) : H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \longrightarrow H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})).$$

Since the map $H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \to H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle}$ of (19) is an epimorphism, it remain to show that α' fits in a commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) & \longrightarrow & H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \\ {\scriptstyle \cup(\psi_1\cup\psi_2)}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) & \xrightarrow{=} & H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \end{array}$$

As recalled above, the epimorphism $H^*(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \longrightarrow H^*(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle}$ is given by restriction using the inclusion ι of the subgroup $\langle T_1, T_3\rangle$ in $\mathbb{Z}^3$, and using that the restricted cocycles are automatically $\langle T_2\rangle$ invariant. Therefore, compatibility of cup products with pullbacks yields the commutativity of the following diagram

$$\begin{array}{ccc} H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) & \xrightarrow{\iota^*} & H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \\ {\scriptstyle \cup\psi_1}\downarrow & & \downarrow{\scriptstyle \cup\psi_1} \\ H^2(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) & \xrightarrow{\iota^*} & H^2(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \end{array}$$

Now, if $[\mathbb{Z}^3] \in H_3(\mathbb{Z}^3,\mathbb{Z})$ is the fundamental class which embodies, through cap product, the Poincaré duality map $\Psi_{\mathbb{Z}^3}$, and if similarly $[\langle T_1,T_3\rangle] \in H_2(\langle T_1,T_3\rangle,\mathbb{Z})$ is the corresponding fundamental class for the subgroup $\langle T_1,T_3\rangle$, then the following relation holds for any $c \in H^2(\mathbb{Z}^3, C(\Sigma,\mathbb{Z}))$:

$$(c\cup\psi_2)\cap[\mathbb{Z}_3] = \pm J_2\left(\iota^* c\cap[\langle T_1,T_3\rangle]\right) \quad \in C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3},$$

where $J_2 : C(\Sigma,\mathbb{Z})_{\langle T_1,T_3\rangle} \to C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3}$. The reason this relation holds is simply that $\psi_2\cap[\mathbb{Z}^3]$ is the 2-homology class which is dual to $\pm\psi_1\cup\psi_3$. The proof is now complete. $\square$

We denote in the following lemma by A the image of $H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle}$ in $\left(C(\Sigma,\mathbb{Z})_{\langle T_3\rangle}\right)^{\langle T_1\rangle}$ under the epimorphism of the above second exact sequence (20).

Lemma 7.4. *The map α induces a well defined morphism $\beta : A \to H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z}))$ such that the composite map $\Psi_{\mathbb{Z}^3}\circ\beta$ is given by the natural map*

$$\left(C(\Sigma,\mathbb{Z})_{\langle T_3\rangle}\right)^{\langle T_1\rangle} \longrightarrow C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3}.$$

Proof. From the previous lemma 7.3, we see that α is a composite map $\theta\circ(\cup\psi_1)$ with θ some morphism. Clearly, the map $\cup\psi_1$ vanishes on $H^1(\langle T_1\rangle, C(\Sigma,\mathbb{Z})^{\langle T_3\rangle})$, hence α vanishes on the kernel of the epimorphism $H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z})) \longrightarrow H^1(\langle T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_1\rangle}$ and finally also on its invariants under the group $\langle T_2\rangle$. We deduce that the morphism β is well defined.

The proposed composite map can be written as $\Psi_{\mathbb{Z}^3}\circ\beta'$ and it is clearly well defined and can then be restricted to the subgroup A. Now, the restriction of π' yields the epimorphism

$$H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_2\rangle} \xrightarrow{\pi'} A.$$

Hence, it remains as in the proof of the previous lemma to show that β' fits in the following commutative diagram

$$\begin{array}{ccc}
H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z})) & \xrightarrow{\iota'^*} & (H^1(\langle T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_1\rangle} \\
{\scriptstyle\cup\psi_1}\big\downarrow & & \big\downarrow{\scriptstyle J_1\circ\Psi_{\langle T_3\rangle}} \\
H^2(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z})) & \xrightarrow{\Psi_{\langle T_1,T_3\rangle}} & C(\Sigma,\mathbb{Z})_{\langle T_1,T_3\rangle}
\end{array}$$

where $\iota' : \langle T_3\rangle \hookrightarrow \langle T_1,T_3\rangle$ is the inclusion as before and $J_1 : C(\Sigma,\mathbb{Z})_{\langle T_3\rangle} \to C(\Sigma,\mathbb{Z})_{\langle T_1,T_3\rangle}$ is again the natural quotient map. The Poincaré maps $\Psi_{\langle T_3\rangle}$ and $\Psi_{\langle T_1,T_3\rangle}$ are cap products by fundamental classes which are denoted respectively $[\langle T_3\rangle]$ and $[\langle T_1,T_3\rangle]$ and the homology class $\psi_1\cap[\langle T_1,T_3\rangle]$ clearly coincides with the Poincaré dual to ψ_3. Hence, for any $c \in H^1(\langle T_1,T_3\rangle, C(\Sigma,\mathbb{Z}))$, we can write

$$(c\cup\psi_1)\cap[\langle T_1,T_3\rangle] = \pm J_1\left(\iota'^* c\cap[\langle T_3\rangle]\right).$$

Therefore, the proof is complete. $\square$

Proof. (of Theorem 7.1)

From Lemma 7.3, we deduce that the range of the map

$$H^1(\mathbb{Z}^3, C(\Sigma,\mathbb{Z})) \overset{\cup(\psi_1\cup\psi_2)}{\longrightarrow} H^3(\mathbb{Z}^3, C(\Sigma,\mathbb{Z}) \xrightarrow{\Psi_{\mathbb{Z}^3}} C(\Sigma,\mathbb{Z})_{\mathbb{Z}^3} \xrightarrow{\mu} \mathbb{R}.$$

coincides with the range of the map

$$\langle \mu, \bullet \rangle \circ \Psi_{\langle T_1, T_3 \rangle} \circ (\cup \psi_1) : H^1(\langle T_1, T_3 \rangle, C(\Sigma, \mathbb{Z}))^{\langle T_2 \rangle} \longrightarrow \mathbb{R}.$$

From Lemma 7.4, we further deduce that the range of this latter map is equal to the range of A under the map $\langle \mu, \bullet \rangle \circ \Psi_{\langle T_3 \rangle}$. Since A is contained in $H^1(\langle T_3 \rangle, C(\Sigma, \mathbb{Z}))^{\langle T_1, T_2 \rangle}$ we deduce from the $\langle T_2 \rangle$-equivariance of $\Psi_{\langle T_3 \rangle}$ that the magnetic gap-labelling group is contained in $\mathbb{Z}[\mu]$ plus the image under $\langle \mu, \bullet \rangle$ of $\left(C(\Sigma, \mathbb{Z})_{\langle T_3 \rangle}\right)^{\langle T_1, T_2 \rangle}$. Since this latter is by definition $\mathbb{Z}_{12}[\mu]$ the computation is complete for the pairing with $\psi_1 \cup \psi_2$.

Reproducing the same proof for $\psi_1 \cup \psi_3$ and $\psi_2 \cup \psi_3$ respectively, we deduce that the corresponding ranges are contained respectively in the range under μ of

$$\left(C(\Sigma, \mathbb{Z})_{\langle T_2 \rangle}\right)^{\langle T_1, T_3 \rangle} \quad \text{and} \quad \left(C(\Sigma, \mathbb{Z})_{\langle T_1 \rangle}\right)^{\langle T_2, T_3 \rangle},$$

that is in $\mathbb{Z}_{13}[\mu]$ and $\mathbb{Z}_{23}[\mu]$. If we sum up using again that Θ is constant, we see that the proof is complete. $\square$

7.2. **Proof of Conjecture 2.** We now prove Conjecture 2 in the 3D case. So, we assume that T_1, T_2 and T_3 all act minimally, this is the strong minimality condition. Let us show more precisely that if the action of the subgroup $\langle T_3 \rangle$ is minimal then $\mathbb{Z}_{12}[\mu]$ is contained in our magnetic gap-labelling group. Here and as before we have denoted by $\mathbb{Z}_{12}[\mu] = \mu([C(\Sigma, \mathbb{Z})_{\langle T_3 \rangle}]^{\langle T_1, T_2 \rangle})$. So, this result will only use the condition that T_3 acts minimally and in fact a priori a weaker assumption, see Theorem 7.2 below. Then the same statement can be proved for T_1 and T_2 yielding to our proof of Conjecture 2.

We notice that the exact sequence (20) is $\langle T_2 \rangle$-equivariant and we thus deduce the cohomology long sequence

$$0 \to H^0(< T_2 >, [C(\Sigma, \mathbb{Z})^{\langle T_3 \rangle}]_{\langle T_1 \rangle}) \longrightarrow H^0(< T_2 >, H^1(< T_1, T_3 >, C(\Sigma, \mathbb{Z}))) \longrightarrow$$
$$H^0(< T_2 >, [C(\Sigma, \mathbb{Z})_{\langle T_3 \rangle}]^{\langle T_1 \rangle}) \xrightarrow{\partial} H^1(< T_2 >, [C(\Sigma, \mathbb{Z})^{\langle T_3 \rangle}]_{\langle T_1 \rangle}) \simeq [C(\Sigma, \mathbb{Z})^{\langle T_3 \rangle}]_{\langle T_1, T_2 \rangle} \longrightarrow \cdots$$

We thus need to describe the boundary map ∂, and more precisely the map

$$\hat{\partial} : \left(H^1(\langle T_3 \rangle, C(\Sigma, \mathbb{Z}))\right)^{\langle T_1, T_2 \rangle} \longrightarrow [C(\Sigma, \mathbb{Z})^{\langle T_3 \rangle}]_{\langle T_1, T_2 \rangle},$$

obtained out of ∂ using the following two isomorphisms

$$\left(H^1(\langle T_3 \rangle, C(\Sigma, \mathbb{Z}))\right)^{\langle T_1, T_2 \rangle} \simeq H^0(< T_2 >, [C(\Sigma, \mathbb{Z})_{\langle T_3 \rangle}]^{\langle T_1 \rangle})$$

and

$$H^1(< T_2 >, [C(\Sigma, \mathbb{Z})^{\langle T_3 \rangle}]_{\langle T_1 \rangle}) \simeq [C(\Sigma, \mathbb{Z})^{\langle T_3 \rangle}]_{\langle T_1, T_2 \rangle}.$$

We can state the following general result for our 3D dynamical systems:

Lemma 7.5. *(1) An element of $(H^1(\langle T_3 \rangle, C(\Sigma, \mathbb{Z}))^{\langle T_1, T_2 \rangle}$ is a class $[\psi]$ of a 1-cocycle $\psi \in Z^1(\langle T_3 \rangle, C(\Sigma, \mathbb{Z})$ such that there exists $f, f' \in C(\Sigma, \mathbb{Z})$ with*

$$T_1\psi(1) - \psi(1) = T_3 f - f \text{ and } T_2\psi(1) - \psi(1) = T_3 f' - f', \quad \text{where } 1 \text{ is here viewed in } \langle T_3 \rangle.$$

(2) *In the notations of the first item, the element $\hat{\partial}[\psi]$ is the class in $[C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}]_{\langle T_1,T_2\rangle}$ of the $\langle T_3\rangle$-invariant element*

$$T_2f - f - T_1f' + f'.$$

Proof. For the first item, we notice that by definition, any element of $(H^1(\langle T_3\rangle, C(\Sigma,\mathbb{Z}))^{\langle T_1,T_2\rangle}$ is a class $[\psi]$ of a 1-cocycle $\psi \in Z^1(\langle T_3\rangle, C(\Sigma,\mathbb{Z})$ such that $T_1\psi - \psi$ and $T_2\psi - \psi$ are coboundaries for the $\langle T_1\rangle$-action. Now, such 1-cocycle ψ is totally determined by its value at $1 \in \langle T_3\rangle$ and it is then easy to check that the above conditions coincide exactly with the assumption of existence of f and f' satisfying

$$T_1\psi(1) - \psi(1) = T_3f - f \text{ and } T_2\psi(1) - \psi(1) = T_3f' - f', \quad \text{where 1 is here viewed in } \langle T_3\rangle.$$

Notice first that $T_2f - f - T_1f' + f'$ is $\langle T_3\rangle$-invariant, for setting $g := \psi(1)$, we have using the commutation of the actions

$$\begin{aligned} T_3\left(T_2f - f - T_1f' + f'\right) &= (T_2 - I)(f + T_1g - g) - (T_1 - I)(f' + T_2g - g) \\ &= T_2f - f - T_1f' + f'. \end{aligned}$$

To describe the boundary map, we introduce the 1-cocycle $\varphi \in Z^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z}))$ which satisfies:

$$\varphi(1,0) = T_3f \text{ and } \varphi(0,1) = T_3g = T_3\psi(1).$$

The explicit formula for φ is then obvious and we have for instance when $n_1, n_3 \geq 1$:

$$\varphi(n_1,n_3) = T_1^{n_1}\sum_{k=1}^{n_3} T_3^k g + T_3\sum_{k=0}^{n_1-1} T_1^k f,$$

and a similar explicit formula for any $(n_1,n_3) \in \langle T_1, T_3\rangle$. It is then straightforward to show that the class of φ is a preimage of $[\psi]$. Now, the map

$$\langle T_2\rangle \ni n_2 \longmapsto [T_2^{n_2}\varphi - \varphi] \in H^1(\langle T_1, T_3\rangle, C(\Sigma,\mathbb{Z})),$$

is clearly valued in the range of the monomorphism i' of the exact sequence (20). Hence, we get in this way a representative for a class in

$$H^1(\langle T_2\rangle, H^1(\langle T_1\rangle, C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}) \simeq [C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}]_{\langle T_1,T_2\rangle}.$$

The last isomorphism is evaluation at 1 in $\langle T_2\rangle$ followed by evaluation at 1 in $\langle T_1\rangle$. To conclude the proof we need a $\langle T_3\rangle$-invariant representative of

$$T_2\varphi(1,0) - \varphi(1,0) = T_2T_3f - T_3f$$

But,

$$\begin{aligned} T_2T_3f - T_3f - [T_1(T_3f') - T_3f'] &= (T_2 - I)(f + T_1g - g) - (T_1 - I)(f' + T_2g - g) \\ &= T_2f - T_1f' - f + f' \end{aligned}$$

Hence the $\langle T_3\rangle$ -invariant element $T_2f - T_1f' - f + f'$ is such a representative and represents the class $\hat{\partial}[\psi]$ as announced. ∎

Theorem 7.2. *Assume that the natural map*

$$\left[C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}\right]_{\langle T_1,T_2\rangle} \to C(\Sigma,\mathbb{Z})_{\langle T_1,T_2\rangle}$$

induced by the inclusion $C(\Sigma,\mathbb{Z})^{\langle T_3\rangle} \hookrightarrow C(\Sigma,\mathbb{Z})$, is injective, then the group $\mathbb{Z}_{12}[\mu]$ is contained in the magnetic gap-labelling group.

Proof. This is an easy corollary of the previous lemma. It is clear from the previous lemma that the composition of the boundary map $\hat{\partial}$ with the map

$$\left[C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}\right]_{\langle T_1,T_2\rangle} \to C(\Sigma,\mathbb{Z})_{\langle T_1,T_2\rangle}$$

is the zero map. Therefore, under the assumption of Theorem 7.2, we deduce that the boundary map must be the zero map. Applying the exactness of the cohomology exact sequence, we deduce that the restriction of the epimorphism π' in the exact sequence (20) to the $\langle T_2\rangle$-invariants is hence still an epimorphism onto the $\langle T_2\rangle$-invariants. Therefore, the subgroup A of Lemma 7.4 coincides with the whole group $[C(\Sigma,\mathbb{Z})_{\langle T_3\rangle}]^{\langle T_1,T_2\rangle}$ and this finishes the proof. □

We are now in position to deduce the proof of Conjecture 2 in the 3D case.

Corollary 7.6. *(1) Assume that the group $\langle T_3\rangle$ acts minimally, then the group $\Theta_{12}\mathbb{Z}_{12}[\mu]$ is contained in the magnetic gap-labelling group.*

(2) Assume that our action of $\mathbb{Z}^3$ on Σ is strongly minimal, then the magnetic gap-labelling group coincides with the magnetic frequency group, i.e. with

$$\mathbb{Z}[\mu] \;+\; \Theta_{12}\mathbb{Z}_{12}[\mu] \;+\; \Theta_{13}\mathbb{Z}_{13}[\mu] \;+\; \Theta_{23}\mathbb{Z}_{23}[\mu].$$

Proof. We need to show that when the group $\langle T_3\rangle$ acts minimally on Σ, the natural map

$$\left[C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}\right]_{\langle T_1,T_2\rangle} \to C(\Sigma,\mathbb{Z})_{\langle T_1,T_2\rangle}$$

is a monomorphism and apply Theorem 7.2. But notice that if the action of $\langle T_3\rangle$ is minimal, all elements of $C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}$ are constant integer valued functions given by $n\times\chi$ where $n\in\mathbb{Z}$ and χ is the constant function with value 1 on Σ. Assume then that the image of the class $[n\times\chi]\in\left[C(\Sigma,\mathbb{Z})^{\langle T_3\rangle}\right]_{\langle T_1,T_2\rangle}$ under the above map is zero in $C(\Sigma,\mathbb{Z})_{\langle T_1,T_2\rangle}$, then its integral against μ must be trivial. Since μ is a probability measure, this implies in turn that $n=0$.

The second item is clear since we can permute the roles of the generators T_1, T_2, T_3. □

8. An explicit construction for the "easy-half"

The main result of this section is Theorem 8.1 which allows in particular to deduce equality in Conjecture 1 under a technical hypothesis on the given tiling, see Proposition 8.3 and Definition 3. As explained previously, the inclusion of the magnetic frequency group in the magnetic gap-labelling group is expected to hold under suitable dynamical conditions on the given aperiodic tiling. In the present section, given a coinvariant class $f\in C(\Sigma,\mathbb{Z})_{\mathbb{Z}^{I^c}}$ which is $\mathbb{Z}^I$-invariant, we prove under a suitable combinatorial assumption on representatives of f, that its integral against the probability measure μ, multiplied by the Pfaffian of Θ_I, belongs to the magnetic gap-labelling group. The proof relies on the existence, for any multiplier σ of the subgroup $\mathbb{Z}^I$ of $\mathbb{Z}^p$, of a commutative diagram (see Theorem 8.1 again):

$$\begin{array}{ccc} K_0(C^*\mathbb{Z}^I, \sigma) & \xrightarrow{\Phi_{f,*}} & K_0(C(\Sigma) \rtimes_{i_*\sigma} \mathbb{Z}^p) \\ {\scriptstyle \tau_*}\downarrow & & \downarrow{\scriptstyle \tau_*^\mu} \\ \mathbb{R} & \xrightarrow{\mu(f)\times \bullet} & \mathbb{R} \end{array}$$

Such commutative diagram can be interpreted as a twisted (non-smooth) version of the classical Morita extension map associated with a given transversal in a foliation [12, 23]. For the clarity of the exposition, we have restricted ourselves to the 3D case, where the construction is already technically involved. The assumption on the class f allows us to reduce the problem to classes represented by characteristic classes of clopen subspaces which live in the magnetic frequency group and the statement of Theorem 8.1 corresponds to such clopen subspaces. The main step in the proof of Theorem 8.1 is Proposition 8.2 which allows one to construct an explicit Morita morphism between the relevant C^*-algebras, and the corresponding commutative diagram then follows immediately.

Fix a clopen subspace Λ of the Cantor space Σ such that the image of the characteristic function of Λ in the $\langle T_3\rangle$-coinvariants, is a $\langle T_1, T_2\rangle$-invariant class. In the following definition and in the major part of this section, we have given a specific role to the third generator T_3, but the similar constructions and proofs work if we operate any permutation of the generators T_1, T_2 and T_3.

Definition 3. The clopen subspace Λ satisfies Hypothesis (H) if we can decompose Λ into clopen subsets $(K_i)_{1\leq i\leq q}$

$$\Lambda = (K_1 \amalg \cdots \amalg K_r) \amalg (K_{r+1} \amalg \cdots \amalg K_q) = K \amalg (K_{r+1} \amalg \cdots \amalg K_q),$$

such that

- For $r+1 \leq i \leq q$, there exists $j_3(i) \in \{1, \cdots, r\}$ such that $K_i = T_3^{\beta_i} K_{j_3(i)}$ for some $\beta_i \in \mathbb{Z}$. In particular, $\langle T_3\rangle\Lambda = \langle T_3\rangle K$.
- $\langle T_3\rangle K_i \cap \langle T_3\rangle K_j = \emptyset$ for $1 \leq i \neq j \leq r$.
- For any $1 \leq i \leq q$, $\exists (j_1(i), j_2(i)) \in \{1, \cdots, r\}^2$ and $(k_1(i), k_2(i)) \in \mathbb{Z}^2$ such that

$$T_1(K_i) = T_3^{k_1(i)}\left(K_{j_1(i)}\right) \text{ and } T_2(K_i) = T_3^{k_2(i)}\left(K_{j_2(i)}\right), \quad 1 \leq i \leq q.$$

So, this means more specifically that there are (unique) surjections $j_1, j_2 : \{1, \cdots, q\} \to \{1, \cdots, r\}$ whose restrictions to $\{1, \cdots, r\}$ are permutations. Notice that we get a well defined map $j_3 : \{r+1, \cdots, q\} \to \{1, \cdots, r\}$ that we shall extend to $\{1, \cdots, q\}$ by setting $j_3(i) := i$ and $\beta_i = 0$ if $1 \leq i \leq r$. Moreover, the values of $j_1(i)$ and $j_2(i)$ for $i = r+1, \cdots, q$ are prescribed by the values on $\{1, \cdots, r\}$ since we must have

$$j_1 = j_1 \circ j_3 \text{ and } j_2 = j_2 \circ j_3 \text{ on } \{r+1, \cdots, q\}.$$

Since the projection of the characteristic function of Λ in the coinvariants modulo $\langle T_3\rangle$ is $\langle T_1, T_2\rangle$-invariant, the cardinal φ_j of $j_3^{-1}(j)$ is automatically constant on each orbit under $\langle j_1, j_2\rangle$.

We shall concentrate on the restricted permutations to $\{1, \cdots, r\}$ and then extend the constructions to $\{1, \cdots, q\}$. It is important in the sequel that we can exploit the relative freeness in the choice of the integer valued maps k_1 and k_2. An easy consequence of the definitions is that the two permutations j_1 and j_2 of $\{1, \cdots, r\}$ commute. More precisely, notice that since $T_1T_2 = T_2T_1$, we have by definition of j_1 and j_2 that

$$\langle T_3 \rangle K_{j_1 j_2(i)} = \langle T_3 \rangle K_{j_2 j_1(i)}.$$

But for $1 \leq i \leq r$ we know that the orbits under $\langle T_3 \rangle$ of the clopen sets K_j are disjoint. Hence necessarily $j_2 j_1(i) = j_1 j_2(i)$. In fact, we easily see that this commutation relation holds on $\{1, \cdots, q\}$.

The goal of this section is to prove the following theorem and to explain its relation with the easy-half of the conjecture.

Theorem 8.1. *For any clopen Λ in Σ as above which further satisfies Hypothesis (H) and any multiplier σ of the group $\langle T_1, T_2 \rangle$, there exists a C^*-algebra homomorphism*

$$\Phi_\Lambda : C^*(\langle T_1, T_2 \rangle, \sigma) \longrightarrow M_\infty(C(\Sigma) \rtimes_{i_*\sigma} \mathbb{Z}^3),$$

such that the following diagram commutes

$$\begin{array}{ccc} K_0(C^*(\langle T_1, T_2 \rangle, \sigma) & \xrightarrow{\Phi_{\Lambda,*}} & K_0(C(\Sigma) \rtimes_{i_*\sigma} \mathbb{Z}^3) \\ {\scriptstyle \tau_*} \downarrow & & \downarrow {\scriptstyle \tau_*^\mu} \\ \mathbb{R} & \xrightarrow{\mu(\Lambda) \times \bullet} & \mathbb{R} \end{array}$$

The same statement holds after any permutation of the generators T_1, T_2 and T_3.

The proof of this theorem will occupy the rest of this section and will be split into many lemmas. Since the generators play a perfect symmetric role, once the proof is given with the special role of T_3, it will hold immediately for all permutations of the generators.

Lemma 8.1. *With the previous notations, there exist integer valued maps $k_1, k_2 : \{1, \cdots, q\} \to \mathbb{Z}$ such that for any $i = 1, \cdots, q$:*

- *$T_1 T_3^{-k_1(i)}(K_i) = K_{j_1(i)}$ and $T_2 T_3^{-k_2(i)}(K_i) = K_{j_2(i)}$.*
- *The following relations hold:*

$$k_2(j_1(i)) - k_2(i) = k_1(j_2(i)) - k_1(i). \tag{21}$$

Proof. We first concentrate on $\{1, \cdots, r\}$ and will extend k_1 and k_2 later on. The relation (21) must be satisfied independently on every orbit of j_1 and j_2 of the form

$$A := \{j_1^{l_1} j_2^{l_2}(\lambda), l_1, l_2 \in \mathbb{Z}\}, \text{ for a given } \lambda \in \{1, \cdots, r\}.$$

So we only need to give the construction for one such orbit. We denote by $p_1 \geq 1$ and $p_2 \geq 1$ the respective orders of λ with respect to j_1 and j_2 and we set

$$A_{l_2} := \{j_2^{l_2}(\lambda), j_1 j_2^{l_2}(\lambda), \cdots, j_1^{p_1-1} j_2^{l_2}(\lambda)\} \text{ so that } A = \cup_{0 \leq l_2 \leq p_2-1} A_{l_2}$$

We point out that since j_1 and j_2 commute, the order of all elements of A_0 under j_2 is equal to p_2. Let ϱ_2 be the global order of A_0, that is the least integer $\varrho \geq 1$ such that $\varrho(A_0) = A_0$.

Then $\varrho_2 \leq p_2$ and for $0 \leq l_1 \leq p_1 - 1$ and $1 \leq l_2 \leq \varrho_2 - 1$, the integers $j_1^{l_1} j_2^{l_2}(\lambda)$ are all distinct from each other so that the first item of the lemma would be easy to satisfy in the sequel, and there is a unique integer $0 \leq \varrho_1 \leq p_1 - 1$ such that $j_2^{\varrho_2}(\lambda) = j_1^{\varrho_1}(\lambda)$. Notice also that if we write each A_{l_2} in the following order

$$A_{l_2} = \{j_2^{l_2}(\lambda), \cdots, j_2^{p_2-1}(\lambda), \lambda, \cdots, j_2^{l_2-1}(\lambda)\},$$

then $j_2^{\varrho_2}$ is nothing but the ϱ_1 power of the cyclic permutation of p_1 variables. We now construct $k_1(l_1, l_2) := k_1(j_1^{l_1} j_2^{l_2}(\lambda))$ on $[0, p_1 - 1] \times [0, \varrho_2 - 1]$. Assume that k_1 is given arbitrarily on A_0 and on any $A_{l_2} \setminus \{j_1^{p_1-1} j_2^{l_2}(\lambda)\}$ satisfying the first item of the lemma. More precisely, we assume that $k_1(l_1, l_2)$ are given integers for $l_2 = 0$ and $0 \leq l_1 \leq p_1 - 1$ on the one hand and for $1 \leq l_2 \leq \varrho_2 - 1$ and $0 \leq l_1 \leq p_1 - 2$ on the other hand, so that they satisfy

$$T_1 T_3^{-k_1(l_1,l_2)} \left(K_{j_1^{l_1} j_2^{l_2}(\lambda)}\right) = K_{j_1^{l_1+1} j_2^{l_2}(\lambda)} \text{ and } T_2 T_3^{-k_2(l_1,l_2)} \left(K_{j_1^{l_1} j_2^{l_2}(\lambda)}\right) = K_{j_1^{l_1} j_2^{l_2+1}(\lambda)}.$$

The second item actually imposes the missing values $k_1(p_1 - 1, l_2)$ of k_1. More precisely, the sum $C_{l_2} := \sum_{0 \leq l_1 \leq p_1 - 1} k_1(l_1, l_2)$ is then necessarily constant in l_2 and thus equal to C_0, for by (21)

$$\begin{aligned} \sum_{0 \leq l_1 \leq p_1 - 1} [k_1(l_1, l_2 + 1) - k_1(l_1, l_2)] &= \sum_{0 \leq l_1 \leq p_1 - 1} [k_2(l_1 + 1, l_2) - k_2(l_1, l_2)] \\ &= k_2(p_1, l_2) - k_2(0, l_2) = 0. \end{aligned}$$

We thus set for $1 \leq l_2 \leq \varrho_2 - 1$

$$k_1(p_1 - 1, l_2) := C_0 - \sum_{0 \leq l_1 \leq p_1 - 2} [k_2(l_1 + 1, l_2).$$

An easy verification shows that $k_1(p_1 - 1, l_2)$ then satisfies the first item. Indeed, notice that

$$T_1^{-1} T_3^{k_1(p_1-2,l_2)}(K_{j_1^{p_1-1} j_2^{l_2}(\lambda)}) = K_{j_1^{p_1-2} j_2^{l_2}(\lambda)}$$

and similarly for $(p - 3, l_2)$ etc. Therefore,

$$\begin{aligned} T_1 T_3^{-k_1(p_1-1,l_2)}(K_{j_1^{p_1-2} j_2^{l_2}(\lambda)}) &= T_1^{-(p_1-1)} T_3^{k_1(0,l_2)+\cdots+k_1(p_1-2,l_2)} (K_{j_1^{p_1-2} j_2^{l_2}(\lambda)}) \\ &= \left[T_1^{-1} T_3^{k_1(0,l_2)}\right] \cdots \left[T_1^{-1} T_3^{k_1(p_2-2,l_2)}\right] \left(K_{j_1^{p_1-1} j_2^{l_2}(\lambda)}\right) \\ &= K_{j_2^{l_2}(\lambda)} \end{aligned}$$

It is easy to check that no other condition is imposed on the values of k_1 on $[0, p_1 - 1] \times [0, \varrho_2 - 1]$ by the compatibility condition (21). Hence k_1 is now well defined on $[0, p_1 - 1] \times [0, \varrho_2 - 1]$ and satisfies the first item of the lemma. It is then extended to $[0, p_1 - 1] \times [0, p_2 - 1]$ by using that ϱ_2 is a divisor of p_2 and that k_1 (and also k_2) must satisfy that for any integer ν

$$k_1(l_1, l_2 + \nu\varrho_2) := k_1(l_1 + \nu\varrho_1, l_2).$$

That such extension of $k_1(l_1, l_2 + \nu\varrho_2)$ still satisfies the first item of the lemma is again straightforward since $k_1(l_1 + \nu\varrho_1, l_2)$ does for $l_2 \leq \varrho_2 - 1$ and since

$$j_1^{l_1+\nu\varrho_1} j_2^{l_2}(\lambda) = j_1^{l_1} j_2^{l_2+\nu\varrho_2}(\lambda).$$

We shall now impose the compatibility condition (21) to deduce k_2. Again, we need first to choose the values $k_2(0, l_2)$ for any $l_2 \in [0, \varrho_2 - 1]$ which turn out to be arbitrary as far as they satisfy the first item, and we now show that all the values of $k_2(l_1, l_2)$ are prescribed on $[0, p_1 - 1] \times [0, p_2 - 1]$ and satisfy the lemma. This is done on $[0, p_1 - 1] \times [0, \varrho_2 - 1]$ and then deduced again by the relation

$$k_2(l_1, l_2 + \nu\varrho_2) := k_2(l_1 + \nu\varrho_1, l_2).$$

We proceed, for $0 \leq l_2 \leq \varrho_2 - 1$, inductively on l_1. We set for instance,

$$k_2(1, l_2) := k_2(0, l_2) + k_1(0, l_2 + 1) - k_1(0, l_2).$$

Again such expression automatically satisfies the first item of the lemma. We then repeat the process for the induction in l_1 and deduce the values of $k_2(l_1, l_2)$ by using

$$k_2(l_1 + 1, l_2) - k_2(l_1, l_2) = k_1(l_1, l_2 + 1) - k_1(l_1, l_2).$$

We notice that as for k_1, the values of the integers $k_2(l_1, \varrho_2 - 1)$ could as well be deduced from the relation:

$$\sum_{0 \leq l_2 \leq \varrho_2 - 1} k_2(l_1 + 1, l_2) - \sum_{0 \leq l_2 \leq \varrho_2 - 1} k_2(l_1, l_2) = k_1(l_1 + \varrho_1, 0) - k_1(l_1, 0).$$

This is however compatible with the previous definition and is redundant.

We conclude by explaining how to extend the maps k_1 and k_2 defined so far on $\{1, \cdots, r\}$ only, to $\{1, \cdots, q\}$. We set for $r + 1 \leq i \leq q$:

$$k_1(i) := k_1(j_3(i)) + \beta_i \text{ and } k_2(i) := k_2(j_3(i)) + \beta_i$$

Then an easy verification shows that the extended maps also satisfy the relations. More specifically, we can write for $r + 1 \leq i \leq q$:

$$k_2(j_1(i)) - k_2(i) = [k_2(j_1(j_3(i))) - k_2(j_3(i))] - \beta_i = [k_1(j_2(j_3(i))) - k_1(j_3(i))] - \beta_i = k_1(j_2(i)) - k_1(i).$$

Moreover,

$$T_1 K_i = T_1 T_3^{\beta_i}(K_{j_3(i)}) = T_3^{\beta_i} T_3^{k_1(j_3(i))} K_{j_1 j_3(i)} = T_3^{k_1(j_3(i)) + \beta_i} K_{j_1(i)} = T_3^{k_1(i)} K_{j_1(i)},$$

and similarly for T_2. $\square$

We denote for $r + 1 \leq i \leq q$ by $j_1^{-1}(i)$ the integer $j_1^{-1}(j_3(i))$ and similarly for j_2. So notice that

$$j_1^{-1} \circ j_1 = j_1 \circ j_1^{-1} = j_3 \text{ and } j_2^{-1} \circ j_2 = j_2 \circ j_2^{-1} = j_3,$$

where j_3 has been extended to $\{1, \cdots, r\}$ by the identity map.

Proposition 8.2. *Given k_1 and k_2 as in the previous lemma (so satisfying (21)), the inductive relations*

$$k_2^{n_2+1} k_1^{n_1}(i) = k_2^{n_2} k_1^{n_1}(j_2(i)) + k_2(i) \quad \textit{and} \quad k_2^{n_2} k_1^{n_1+1}(i) = k_2^{n_2} k_1^{n_1}(j_1(i)) + k_1(i). \tag{22}$$

together with $k_2^1 k_1^0 = k_2$ and $k_2^0 k_1^1 = k_1$ allow to well define for any $(n_1, n_2) \in \mathbb{Z}^2$, integer valued functions $k_2^{n_2} k_1^{n_1} : \{1, \cdots, q\} \to \mathbb{Z}$ satisfying

(1) $\left(T_1^{n_1}T_2^{n_2}T_3^{-k_2^{n_2}k_1^{n_1}(i)}\right)(K_i) = K_{j_2^{n_2}j_1^{n_1}(i)}$.

(2) $k_2^{m_2}k_1^{m_1}(j_2^{n_2-m_2}j_1^{n_1-m_1}(i))+k_2^{n_2-m_2}k_1^{n_1-m_1}(i) = k_2^{n_2}k_1^{n_1}(i)$, *for any integers* n_1, n_2, m_1, m_2 *and any* $i \in \{1, \cdots, q\}$.

Notice that necessarily $k_2^0k_1^0 = 0$ and that the last item is trivially satisfied when $(m_1, m_2) = (n_1, n_2)$ or $(m_1, m_2) = (0, 0)$. Once such sequence $k_2^{n_2}k_1^{n_1}$ is proved to be well defined, one can deduce many expressions for it. For instance, for $n_1, n_2 > 0$, we have the following two expressions which then must fit by the previous proposition:

$$k_2^{n_2}k_1^{n_1}(i) = \left[k_2(j_2^{n_2-1}j_1^{n_1}(i)) + \cdots + k_2(j_1^{n_1}(i))\right] + \left[k_1(j_1^{n_1-1}(i)) + \cdots + k_1(i)\right]$$

and

$$k_2^{n_2}k_1^{n_1}(i) = \left[k_1(j_1^{n_1-1}j_2^{n_2}(i)) + \cdots + k_2(j_2^{n_2}(i))\right] + \left[k_2(j_2^{n_2-1}(i)) + \cdots + k_2(i)\right]$$

It is also a straightforward exercise to check directly the coincidence of these two expressions. There are similar expressions for negative n_i's, and for any $n_i's$. When $n_1 < 0$ and $n_2 < 0$, one gets for instance an expression

$$-k_2^{n_2}k_1^{n_1}(i) = \left[k_1(j_1^{n_1}j_2^{n_2}(i)) + \cdots + k_1(j_1^{-1}j_2^{n_2}(i))\right] + \left[k_2(j_2^{n_2}(i) + \cdots + k_2(j_2^{-1}(i))\right]$$

We shall first prove the well-definiteness of these maps and then use any of the above two expressions to deduce for instance the second item.

Proof. We again concentrate on $\{1, \cdots, r\}$ where j_1 and j_2 are permutations and explain later the different expressions on $[r + 1, q]$. The inductive definition allows to define many candidates for $k_2^{n_2}k_1^{n_1}$ obtained by using different polygonal paths from $(0, 0)$ to (n_1, n_2) in $\mathbb{Z}^2$, i.e. paths composed of horizontal and vertical segments with endpoints in $\mathbb{Z}^2$ and which start at $(0, 0)$ and end at (n_1, n_2). Now, consider for any $(m_1, m_2) \in \mathbb{Z}^2$, the two paths joining (m_1, m_2) to $(m_1 + 1, m_2 + 1)$ given by

$$\mathcal{C}_1 := [(m_1, m_2), (m_1 + 1, m_2)] \cup [(m_1 + 1, m_2), (m_1 + 1, m_2 + 1)] \quad \text{and}$$
$$\mathcal{C}_2 := [(m_1, m_2), (m_1, m_2 + 1)] \cup [(m_1, m_2 + 1), (m_1 + 1, m_2 + 1)].$$

Applying the inductive argument with $\mathcal{C}_1$ we get for any $i \in [1, q]$:

$$k_2^{m_2+1}k_1^{m_1+1}(i) = k_2^{m_2}k_1^{m_1}(j_1j_2(i)) + k_1(j_2(i)) + k_2(i).$$

Now using $\mathcal{C}_2$, we get

$$k_2^{m_2+1}k_1^{m_1+1}(i) = k_2^{m_2}k_1^{m_1}(j_2j_1(i)) + k_2(j_1(i)) + k_1(i).$$

Therefore, and since we have chosen k_1 and k_2 so that $k_1(j_2(i)) + k_2(i) = k_2(j_1(i)) + k_1(i)$ and also since $j_1j_2 = j_2j_1$, we see that we obtain the same result by using $\mathcal{C}_1$ or $\mathcal{C}_2$. We could as well use the inverse paths $-\mathcal{C}_1$ and $-\mathcal{C}_2$ and compute $k_2^{m_2}k_1^{m_1}(i)$ in terms of $k_2^{m_2+1}k_1^{m_1+1}$ and see that we also get the same result. Now if $\mathcal{C}$ is any polygonal path from the base point $(0, 0)$, where $k_2^0k_1^0 = 0$, to (n_1, n_2), composed of horizontal and vertical segments, we can use the inductive formulae to deduce an expression of $k_2^{n_2}k_1^{n_1}(i)$ which by repeating the previous argument as many times as necessary, will not depend on the chosen path.

So, if we choose the simple path with one horizontal segment and one vertical segment $[(0,0),(n_1,0)]\cup[(n_1,0),(n_1,n_2)]$ then we get the expression

$$k_2^{n_2}k_1^{n_1}(i)=\left[k_1(j_1^{n_1-1}j_2^{n_2}(i))+\cdots+k_2(j_2^{n_2}(i))\right]+\left[k_2(j_2^{n_2-1}(i))+\cdots+k_2(i)\right].$$

while the other simple path $[(0,0),(0,n_2)]\cup[(0,n_2),(n_1,n_2)]$ yields the expression

$$k_2^{n_2}k_1^{n_1}(i)=\left[k_2(j_2^{n_2-1}j_1^{n_1}(i))+\cdots+k_2(j_1^{n_1}(i))\right]+\left[k_1(j_1^{n_1-1}(i))+\cdots+k_1(i)\right].$$

Let us now prove the first item of the proposition. If $n_1=n_2=0$ then the item is satisfied by obvious observation. We assume now that $k_2^{n_2}k_1^{n_1}(i)$ satisfies the first item for any $i=1,\cdots,r$. Then

$$\begin{aligned}
T_1^{n_1}T_2^{n_2+1}T_3^{-k_2^{n_2+1}k_1^{n_1}(i)}(K_i) &= T_1^{n_1}T_2^{n_2}T_3^{-k_2^{n_2}k_1^{n_1}(j_2(i))}\left(T_2T_3^{-k_2(i)}(K_i)\right)\\
&= T_1^{n_1}T_2^{n_2}T_3^{-k_2^{n_2}k_1^{n_1}(j_2(i))}\left(K_{j_2(i)}\right)\\
&= K_{j_1^{n_1}j_2^{n_2+1}(i)}
\end{aligned}$$

In a similar way we prove that

$$T_1^{n_1+1}T_2^{n_2}T_3^{-k_2^{n_2}k_1^{n_1+1}(i)}(K_i)=K_{j_1^{n_1+1}j_2^{n_2}(i)}.$$

We obtain as well

$$\begin{aligned}
T_1^{n_1}T_2^{n_2-1}T_3^{-k_2^{n_2-1}k_1^{n_1}(i)}(K_i) &= T_1^{n_1}T_2^{n_2}T_3^{-k_2^{n_2}k_1^{n_1}(j_2^{-1}(i))}\left(T_2^{-1}T_3^{k_2(j_2^{-1}(i))}(K_i)\right)\\
&= T_1^{n_1}T_2^{n_2}T_3^{-k_2^{n_2}k_1^{n_1}(j_2^{-1}(i))}\left(K_{j_2^{-1}(i)}\right)\\
&= K_{j_1^{n_1}j_2^{n_2-1}(i)}
\end{aligned}$$

and again similarly we get

$$T_1^{n_1-1}T_2^{n_2}T_3^{-k_2^{n_2-1}k_1^{n_1}(i)}(K_i)=K_{j_1^{n_1-1}j_2^{n_2}(i)}.$$

We hence get the first item for any $(n_1,n_2)\in\mathbb{Z}^2$.

We now prove the second item. We prove first the following relation (which corresponds to the second item for $(n_1,n_2)=(0,0)$ and where we changed the notation):

$$k_2^{n_2}k_1^{n_1}(j_2^{-n_2}j_1^{-n_1}(i))+k_2^{-n_2}k_1^{-n_1}(i)=0,\quad (n_1,n_2)\in\mathbb{Z}^2.$$

For $n_1,n_2\geq 0$ we just use the expression

$$\begin{aligned}
k_2^{n_2}k_1^{n_1}(j_2^{-n_2}j_1^{-n_1}(i)) &= k_2(j_2^{n_2-1}j_1^{n_1}(j_2^{-n_2}j_1^{-n_1}(i)))+\cdots+k_2(j_1^{n_1}j_2^{-n_2}j_1^{-n_1}(i))\\
&\quad +k_1(j_1^{n_1-1}j_2^{-n_2}j_1^{-n_1}(i))+\cdots+k_1(j_2^{-n_2}j_1^{-n_1}(i))\\
&= k_2(j_2^{-1}(i)+\cdots+k_2(j_2^{-n_2}(i))+k_1(j_1^{-1}j_2^{-n_2}(i))+\cdots+k_1(j_1^{-n_1}j_2^{-n_2}(i))
\end{aligned}$$

But this is precisely the expression that we already got for $k_2^{-n_2}k_1^{-n_1}(i)$ in this case. If $n_1<0$ and $n_2\geq 0$ then a direct computation gives

$$\begin{aligned}
k_2^{n_2}k_1^{n_1}(j_2^{-n_2}j_1^{-n_1}(i)) &= \left[k_2(j_2^{-n_2}j_1^{-n_1}(i))+\cdots+k_2(j_2^{-1}j_1^{-n_1}(i))\right]\\
&\qquad -\left[k_1(i)+\cdots+k_1(j_1^{-n_1-1}(i))\right]
\end{aligned}$$

Computing $k_2^{-n_2}k_1^{-n_1}(i)$ by applying first the induction to the positive integer $-n_1$, we get exactly the opposite to this expression. Notice now that by applying the previous results to $(-n_1, -n_2)$ and to $i' = j_2^{n_2}j_1^{n_1}(i)$ we see that the formula for (n_1, n_2) is equivalent to the formula for $(-n_1, -n_2)$. Hence we have proved the formula of the second item when $n_1 = n_2 = 0$. Assume that this formula is satisfied for a given $(n_1, n_2) \in \mathbb{Z}^2$ and for any $i \in \{1, \cdots, r\}$ and any $(m_1, m_2) \in \mathbb{Z}^2$, then

$$\begin{aligned} k_2^{n_2+1}k_1^{n_1}(i) &= k_2^{n_2}k_1^{n_1}(j_2(i)) + k_2(i) \\ &= k_2^{m_2}k_1^{m_1}(j_2^{n_2-m_2}j_1^{n_1-m_1}(j_2(i))) + \left(k_2^{n_2-m_2}k_1^{n_1-m_1}(j_2(i)) + k_2(i)\right) \\ &= k_2^{m_2}k_1^{m_1}(j_2^{n_2+1-m_2}j_1^{n_1-m_1}(i)) + k_2^{n_2-m_2+1}k_1^{n_1-m_1}(i). \end{aligned}$$

Hence the formula is satisfied for $(n_1, n_2 + 1)$. We leave it to the interested reader to check that the formula is then also satisfied for $(n_1 + 1, n_2)$, $(n_1 - 1, n_2)$ as well as for $(n_1, n_2 - 1)$ by applying each time the induction formula.

So far we have concentrated on $i \in \{1, \cdots, r\}$ and we now give the expressions of the maps $k_2^{n_2}k_1^{n_1}$ on $\{r + 1, \cdots, q\}$ which extend the proposition in a straightforward manner. This is easy since we just set for $r + 1 \le i \le q$:

$$k_2^{n_2}k_1^{n_1}(i) := k_2^{n_2}k_1^{n_1}(j_3(i)) + \beta_i.$$

This expression satisfies again the inductive relations since

$$k_2^{n_2+1}k_1^{n_1}(i) = k_2^{n_2+1}k_1^{n_1}(j_3(i)) + \beta_i = k_2^{n_2}k_1^{n_1}(j_2j_3(i)) + k_2(j_3(i)) + \beta_i = k_2^{n_2}k_1^{n_1}(j_2(i)) + k_2(i),$$

and similarly for $k_2^{n_2}k_1^{n_1+1}(i)$. Also, we have $k_2^1k_1^0(i) = k_2(j_3(i)) + \beta_i = k_2(i)$ and $k_2^0k_1^1(i) = k_1(j_3(i)) + \beta_i = k_1(i)$. Moreover, if $(m_1, m_2) \ne (n_1, n_2)$ and $r + 1 \le i \le q$ then

$$\begin{aligned} k_2^{m_2}k_1^{m_1}(j_2^{n_2-m_2}j_1^{n_1-m_1}(i)) + k_2^{n_2-m_2}k_1^{n_1-m_1}(i) &= k_2^{m_2}k_1^{m_1}(j_2^{n_2-m_2}j_1^{n_1-m_1}(j_3(i))) + k_2^{m_2}k_1^{m_1}(j_3(i)) + \beta_i \\ &= k_2^{n_2}k_1^{n_1}(j_3(i)) + \beta_i \\ &= k_2^{n_2}k_1^{n_1}(i) \end{aligned}$$

Notice that we have used as before the convention $j_2^0 = j_1^0 = j_3$. $\square$

We are now in a position to prove our main theorem.

Proof. (of Theorem 8.1)

We know that $\Lambda = \amalg_{1\le i\le q}K_i$ as before, so where the first r indices represent all the disjoint orbits of Λ under T_3 with the maps j_1, j_2, j_3 as well as the family of maps $k_2^{n_2}k_1^{n_1}$ constructed in the previous proposition. We denote for any $j \in \{1, \cdots, q\}$ by $\varphi_j \in \{1, \cdots, q-r+1\}$ the cardinal of the set $j_3^{-1}\{j_3(j)\}$ and we set $\varphi_0 = 0$ and $\hat\varphi_j := \sum_{0\le i\le j}\varphi_i$. We then define the diagonal projection matrix $\hat\chi_\Lambda$ in $M_{\hat\varphi_q}(C(\Sigma) \rtimes_{i_*\sigma} \mathbb{Z}^3)$ by setting for $\hat\varphi_{j-1} + 1 \le h \le \hat\varphi_j$:

$$(\hat\chi^\Lambda)_{hh'}(n_1, n_2, n_3) := \delta_{h,h'}\delta_{(n_1,n_2,n_3),(0,0,0)} \cdot \chi_{K_j}.$$

where χ_{K_j} is the characteristic function of the minimal clopen K_j and δ stands as usual for the Kronecker symbol. If $F \in \mathbb{C}[\langle T_1, T_2\rangle, \sigma]$ is a finitely supported function and if $1 \le j, j' \le q$, then we set

$$\widetilde{F}^\Lambda_{jj'}(n_1, n_2, n_3) := \delta_{j_3(j), j_2^{n_2}j_1^{n_1}(j')}\delta_{n_3, -k_2^{n_2}k_1^{n_1}(j')+\beta_j} \times F(n_1, n_2)$$

where β_j was defined so that $T_3^{\beta_j}(K_{j_3(j)}) = K_j$ with the convention that for $1 \le j \le r$, $\beta_j = 0$. Now, we can define the matrix $\hat{F}^\Lambda \in M_{\hat{\varphi}_q}(\mathbb{C}[\mathbb{Z}^3, i_*\sigma])$ by setting

$$\hat{F}^\Lambda_{hh'} := \frac{1}{\sqrt{\varphi_{j'}\varphi_j}} \delta_{h-\hat{\varphi}_{j-1}, h'-\hat{\varphi}_{j'-1}} \widetilde{F}^\Lambda_{jj'} \text{ if } \hat{\varphi}_{j-1} + 1 \le h \le \hat{\varphi}_j \text{ and } \hat{\varphi}_{j'-1} + 1 \le h' \le \hat{\varphi}_{j'}.$$

Notice that this is possibly non zero only when $j_3(j) = j_2^{n_2} j_1^{n_1}(j')$ and in this case $\varphi_j = \varphi_{j'}$. We now set

$$\Phi_\Lambda(F) := \hat{F}^\Lambda \star \hat{\chi}^\Lambda \quad \in M_{\hat{\varphi}_q}(C(\Sigma) \rtimes_{i_*\sigma} \mathbb{Z}^3),$$

where $\star$ is the product in $M_{\hat{\varphi}_q}(C(\Sigma) \rtimes_{i_*\sigma} \mathbb{Z}^3)$. Notice that each $\hat{F}^\Lambda_{hh'}$ is clearly finitely supported in $\mathbb{Z}^3$. In order to show that Φ_Λ extends to a $*$-homomorphism and using that $\hat{\chi}^\Lambda$ is a self-adjoint idempotent, we will check the following relations on $\mathbb{C}[\langle T_1, T_2\rangle, \sigma]$:

$$\hat{F}^\Lambda \star \hat{\chi}_\Lambda = \hat{\chi}_\Lambda \star \hat{F}^\Lambda, \; (\hat{F}^\Lambda)^* = \widehat{F^*}^\Lambda \text{ and } \hat{F}^\Lambda \star \hat{G}^\Lambda = \widehat{FG}^\Lambda.$$

Computing for $\hat{\varphi}_{j-1}+1 \le h \le \hat{\varphi}_j$ and $\hat{\varphi}_{j'-1}+1 \le h' \le \hat{\varphi}_{j'}$, we get using that $\sigma((0,0),(n_1,n_2)) = 1$:

$$\begin{aligned}
(\hat{F}^\Lambda \star \hat{\chi}^\Lambda)_{hh'}(n_1, n_2, n_3) &= \frac{1}{\sqrt{\varphi_{j'}\varphi_j}} \delta_{h-\hat{\varphi}_{j-1}, h'-\hat{\varphi}_{j'-1}} \times \widetilde{F}^\Lambda_{jj'}(n_1, n_2, n_3) T_1^{n_1} T_2^{n_2} T_3^{n_3}(\chi_{K_{j'}}) \\
&= \frac{1}{\sqrt{\varphi_{j'}\varphi_j}} \delta_{h-\hat{\varphi}_{j-1}, h'-\hat{\varphi}_{j'-1}} F(n_1, n_2) \delta_{j_3(j), j_2^{n_2} j_1^{n_1}(j')} \delta_{n_3, -k_2^{n_2} k_1^{n_1}(j') + \beta_j} \chi_{K_j}
\end{aligned}$$

The last equality is a consequence of the relations

$$\begin{aligned}
\delta_{j_3(j), j_2^{n_2} j_1^{n_1}(j')} T_1^{n_1} T_2^{n_2} T_3^{-k_2^{n_2} k_1^{n_1}(j') + \beta_j}(\chi_{K_{j'}}) &= \delta_{j_3(j), j_2^{n_2} j_1^{n_1}(j')} T_3^{\beta_j} \chi_{K_{j_2^{n_2} j_1^{n_1}(j')}} \\
&= \delta_{j_3(j), j_2^{n_2} j_1^{n_1}(j')} T_3^{\beta_j}(\chi_{K_{j_3(j)}}) = \delta_{j_3(j), j_2^{n_2} j_1^{n_1}(j')}(\chi_{K_j})
\end{aligned}$$

Computing $(\hat{\chi}^\Lambda \star \hat{F}^\Lambda)_{hh'}(n_1, n_2, n_3)$, we get the same expression, so that

$$\hat{\chi}^\Lambda \star \hat{F}^\Lambda = \hat{F}^\Lambda \star \hat{\chi}^\Lambda.$$

On the other hand, for a given extra element $G \in \mathbb{C}[\langle T_1, T_2\rangle, \sigma]$, we compute similarly using the same notations

$$\begin{aligned}
(\hat{F}^\Lambda \star \hat{G}^\Lambda)_{hh'}(n_1,n_2,n_3) &= \frac{1}{\sqrt{\varphi_{j'}\varphi_j}} \sum_{j''=1}^{q} \sum_{h''=\hat{\varphi}_{j''-1}+1}^{\hat{\varphi}_{j''}} \delta_{h-\hat{\varphi}_{j-1},h''-\hat{\varphi}_{j''-1}} \delta_{h'-\hat{\varphi}_{j'-1},h''-\hat{\varphi}_{j''-1}} \\
&\quad \sum_{m_1,m_2} \frac{1}{\varphi_{j''}} \delta_{j_3(j),j_2^{m_2} j_1^{m_1}(j'')} \delta_{j_3(j''),j_2^{n_2-m_2} j_1^{n_1-m_1}(j')} \\
&\quad \delta_{m_3,-k_2^{m_2}k_1^{m_1}(j'')+\beta_j} \delta_{n_3-m_3,-k_2^{n_2-m_2}k_1^{n_1-m_1}(j')+\beta_{j''}} \\
&\quad F(m_1,m_2)G(n_1-m_1,n_2-m_2)\sigma((m_1,m_2),(n_1-m_1,n_2-m_2)) \\
&= \frac{1}{\sqrt{\varphi_{j'}\varphi_j}} \delta_{h-\hat{\varphi}_{j-1},h'-\hat{\varphi}_{j'-1}} \delta_{j,j_2^{n_2}j_1^{n_1}(j')} \delta_{n_3,-k_2^{n_2}k_1^{n_1}(j')+\beta_j} \\
&\quad \sum_{m_1,m_2} F(m_1,m_2)G(n_1-m_1,n_2-m_2) \\
&\quad \sigma((m_1,m_2),(n_1-m_1,n_2-m_2)) \sum_{j''=1}^{q} \frac{1}{\varphi_{j''}} \delta_{j_3(j''),j_2^{n_2-m_2}j_1^{n_1-m_1}(j')}.
\end{aligned}$$

Indeed, recall that by the $\langle T_1, T_2\rangle$-invariance of the class of the characteristic function of Λ in the coinvariants with respect to $\langle T_3\rangle$, we know that for any n_1, n_2, m_1 and m_2, and for any $j'' \in j_3^{-1}(j_2^{n_2-m_2} j_1^{n_1-m_1}(j'))$, we have $\varphi_{j''} = \varphi_{j_3(j'')} = \varphi_{j'} (= \varphi_j)$. But for any fixed $(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4$ we have

$$\sum_{j''=1}^{q} \frac{1}{\varphi_{j''}} \delta_{j_3(j''),j_2^{n_2-m_2}j_1^{n_1-m_1}(j')} = \sum_{j_3(j'')=j_2^{n_2-m_2}j_1^{n_1-m_1}(j')} (\varphi_{j_2^{n_2-m_2}j_1^{n_1-m_1}(j')})^{-1} = 1.$$

Therefore, we get for any h, h':

$$(\hat{F}^\Lambda \star \hat{G}^\Lambda)_{hh'} = \widehat{FG}^{\Lambda}_{hh'}.$$

using the previous results we deduce that $\Phi_\Lambda(FG) = \Phi_\Lambda(F) \star \Phi_\Lambda(G)$.

In the same way, we compute

$$\begin{aligned}
\left[\hat{F}^\Lambda_{h'h}\right]^*(n_1,n_2,n_3) &= \overline{\hat{F}^\Lambda_{h'h}(-n_1,-n_2,-n_3)} \\
&= \frac{1}{\sqrt{\varphi_{j'}\varphi_j}} \delta_{h'-\hat{\varphi}_{j'-1},h-\hat{\varphi}_j} \overline{\widetilde{F}^\Lambda_{j'j}(-n_1,-n_2,-n_3)}
\end{aligned}$$

So forgetting the cocycle σ which doesn't perturb the following computation, we can write

$$\begin{aligned}
\widetilde{F}^{\Lambda}_{j'j}(-n_1,-n_2,-n_3) &= \delta_{j_3(j'),j_2^{-n_2}j_1^{-n_1}(j_3(j))}\delta_{-n_3,-k_2^{-n_2}k_1^{-n_1}(j)+\beta_{j'}}F(-n_1,-n_2)\\
&= \delta_{j_3(j),j_2^{n_2}j_1^{n_1}(j')}\delta_{n_3,k_2^{-n_2}k_1^{-n_1}(j)-\beta_{j'}}F(-n_1,-n_2)\\
&= \delta_{j_3(j),j_2^{n_2}j_1^{n_1}(j')}\delta_{n_3,k_2^{-n_2}k_1^{-n_1}(j_3(j))+\beta_j-\beta_{j'}}F(-n_1,-n_2)\\
&= \delta_{j_3(j),j_2^{n_2}j_1^{n_1}(j')}\delta_{n_3,k_2^{-n_2}k_1^{-n_1}(j_2^{n_2}j_1^{n_1}(j_3(j')))+\beta_j-\beta_{j'}}F(-n_1,-n_2)\\
&= \delta_{j_3(j),j_2^{n_2}j_1^{n_1}(j')}\delta_{n_3,-k_2^{n_2}k_1^{n_1}(j_3(j'))+\beta_j-\beta_{j'}}F(-n_1,-n_2)\\
&= \delta_{j_3(j),j_2^{n_2}j_1^{n_1}(j')}\delta_{n_3,-k_2^{n_2}k_1^{n_1}(j')+\beta_j}F(-n_1,-n_2)
\end{aligned}$$

Hence we see that

$$[\Phi_\Lambda(F)]^* = \Phi_\Lambda(F^*),$$

proving finally that Φ_Λ is a $*$-morphism from the involutive algebra $\mathbb{C}[\langle T_1,T_2\rangle,\sigma]$ to $M_{\hat{\varphi}_q}(C(\Sigma)\rtimes_{i_*\sigma}\mathbb{Z}^3)$ which is valued in the algebraic crossed product. It is easy to check that Φ_Λ extends to a C^*-algebra morphism and hence induces a group morphism

$$\Phi_{\Lambda,*} : K_*(C^*(\langle T_1,T_2\rangle,\sigma)) \longrightarrow K_*(C(\Sigma)\rtimes_{i_*\sigma}\mathbb{Z}^3).$$

Moreover, for any $F\in\mathbb{C}[\langle T_1,T_2\rangle,\sigma]$, we have

$$\begin{aligned}
(\tau^\mu\sharp\,\mathrm{tr})\Phi_\Lambda(F) &= \sum_{j-1}^{q}\sum_{h=\hat{\varphi}_{j-1}+1}^{\hat{\varphi}_j}\tau^\mu\Phi_\Lambda(F)_{hh}\\
&= \sum_{j=1}^{q}\mu(K_j)\sum_{h=\hat{\varphi}_{j-1}+1}^{\hat{\varphi}_j}\hat{F}^{\Lambda}_{hh}(0,0,0)
\end{aligned}$$

But $\hat{F}^{\Lambda}_{hh}(0,0,0)=\frac{1}{\varphi_j}\times F(0,0)$, therefore

$$(\tau^\mu\sharp\,\mathrm{tr})\Phi_\Lambda(F) = \sum_{j=1}^{q}\mu(K_j)F(0,0)\sum_{h=\hat{\varphi}_{j-1}+1}^{\hat{\varphi}_j}\frac{1}{\varphi_j} = \mu(\Lambda)\times\tau(F).$$

□

We end this section with an explanation of the relation between Theorem 8.1 and the easy-half of our conjecture in the 3D case.

Proposition 8.3. *Assume that $p=3$ so that $\mathbb{Z}^3$ acts minimally on Σ. Assume further that $\Theta_{13}=\Theta_{23}=0$ and that any $\mathbb{Z}$-valued function λ on Σ which represents a class in $\left(C(\Sigma,\mathbb{Z})_{\langle T_3\rangle}\right)^{\mathbb{Z}^3}$ can be decomposed as a finite algebraic sum of characteristic functions of clopen subspaces satisfying the hypothesis (H). Then the magnetic frequency group $\mathbb{Z}[\mu]+\Theta_{12}\mathbb{Z}_{12}[\mu]$ is contained in the magnetic gap-labelling group associated with the multiplier σ which corresponds to Θ.*

Proof. We only need to prove that $\Theta_{12}\mathbb{Z}_{12}[\mu]$ is contained in the magnetic gap-labelling group.

We denote by σ the multiplier of $\langle T_1, T_2\rangle$ which is associated with the 2×2-matrix

$$\begin{pmatrix} 0 & \Theta_{12} \\ -\Theta_{12} & 0 \end{pmatrix}.$$

Then using the inclusion $i : \langle T_1, T_2\rangle \hookrightarrow \mathbb{Z}^3$ and applying Theorem 8.1, we deduce that for any clopen subspace Λ which satisfies (H) for T_3,

$$\{\mu(\Lambda)\times\tau_*(x), x\in K_0(C^*(\langle T_1, T_2\rangle, \sigma), \Lambda \text{ as before}\} \subset \tau_*^{\mu}(K_0(C(\Sigma)\rtimes_{i_*\sigma_{12}}\mathbb{Z}^3)),$$

where $i_*\sigma$ is associated with the skew matrix

$$\begin{pmatrix} 0 & \Theta_{12} & 0 \\ -\Theta_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\Theta_{12}\mathbb{Z}_{12}[\mu]$ is contained in the magnetic gap-labelling group.

□

Appendix A. The coinvariants as a direct summand in K-theory

Here we give a direct proof that the coinvariants are always direct summands in the K-theory of the twisted crossed product C^*-algebra, without using the Packer-Raeburn trick [39]. The method is an important step towards a complete solution to our easy-half conjecture for all dimensions.

Let Σ be a Cantor space on which $\mathbb{Z}^p$ acts by homeomorphisms. We do not assume that the action is minimal in this section, because we want to use induction on p. Consider the twisted crossed product C^*-algebra $C(\Sigma)\rtimes_\sigma\mathbb{Z}^p$, with Θ a skew-symmetric $(p\times p)$ matrix determining the multiplier σ. We can write $C(\Sigma)\rtimes_\sigma\mathbb{Z}^p = \left(C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}\right)\rtimes_\phi\mathbb{Z}^{(p)}$ where $\sigma|$ is the restriction of the multiplier σ to $\mathbb{Z}^{p-1}\times\mathbb{Z}^{p-1}$ and $\mathbb{Z}^{(p)}$ is the p-th copy of $\mathbb{Z}$ and ϕ is an automorphism of $C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}$ given by

$$\phi(U_j) = \exp(2\pi\sqrt{-1}\Theta_{jp})U_j, \qquad j = 1,\dots(p-1).$$

where $U_1,\dots U_{(p-1)}$ are unitary automorphisms of $C(\Sigma)$ generating the twisted crossed product, $C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}$. More precisely, the map ϕ is an automorphism of $C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}$ which is given, on elementary elements $gU_{k_1,\cdots,k_{p-1}}$ of $C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}$, by the formula

$$\phi(gU_{k_1,\cdots,k_{p-1}}) := T_p(g)\exp(2\pi\sqrt{-1}[k_1\Theta_{p1}+\cdots+k_{p-1}\Theta_{p\,p-1}])U_{k_1,\cdots,k_{p-1}}.$$

Then an easy inspection shows that we recover in this way $C(\Sigma)\rtimes_\sigma\mathbb{Z}^p$ from $C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}$ and the $\mathbb{Z}$ action generated by ϕ. Indeed, the map

$$\left(gU_{k_1,\cdots,k_{p-1}}\right)\delta_{k_p} \longmapsto \exp(\pi\sqrt{-1}[k_1\Theta_{1p}+\cdots+k_{p-1}\Theta_{p-1\,p}])gU_{k_1,\cdots,k_p},$$

extends to the allowed C^*-algebra isomorphism

$$\left[C(\Sigma)\rtimes_{\sigma|}\mathbb{Z}^{p-1}\right]\rtimes\mathbb{Z}\longrightarrow C(\Sigma)\rtimes_\sigma\mathbb{Z}^p.$$

Clearly ϕ is homotopic to the identity, since the exponential defining it can be scaled. This is essentially an argument in [42].

The Pimsner-Voiculescu (PV) sequence [43] gives the exact sequences for $i = 0, 1$,

$$0 \longrightarrow K_i(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^{p-1})_{\mathbb{Z}^{(p)}} \longrightarrow K_i(C(\Sigma) \rtimes_{\Theta} \mathbb{Z}^p) \xrightarrow{\partial_p} K_{i-1}(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^{p-1})^{\mathbb{Z}^{(p)}} \longrightarrow 0, \tag{23}$$

where $(\cdot)^{\mathbb{Z}^{(p)}}$ and $(\cdot)_{\mathbb{Z}^{(p)}}$ denote, respectively, the invariants and coinvariants of $(\cdot)$ under the induced map in K-theory of the $\mathbb{Z}^{(p)}$ action. We will abbreviate these by $(\cdot)^p$ and $(\cdot)_p$ respectively. We can calculate these K-theory groups iteratively. Set $\mathcal{A} = C(\Sigma, \mathbb{Z})$. Then the following is a consequence of the general spectral sequence which computes K-theory if one uses the Packer-Raeburn trick [39]. We give it an independent treatment here.

Proposition A.1. *In the notation above, $\mathcal{A}_{123\ldots p}$ is a direct summand in $K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p)$.*

Proof. It is known that

$$K_0(C(\Sigma)) = \mathcal{A}, \qquad K_1(C(\Sigma)) = 0. \tag{24}$$

For the first $\mathbb{Z}$ action (for which $\Theta|$ vanishes trivially), we have

$$K_0(C(\Sigma) \rtimes \mathbb{Z}) \;\cong\; \mathcal{A}_1 \tag{25}$$

$$K_1(C(\Sigma) \rtimes \mathbb{Z}) \;\cong\; \mathcal{A}^1. \tag{26}$$

For the first $\mathbb{Z}^2$ action, using the PV sequence we have

$$K_0(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^2) \cong \mathcal{A}_{12} \oplus \mathcal{A}^{12} \tag{27}$$

because $\mathcal{A}^{12} \subset \mathcal{A}$ is free abelian. So, $\mathcal{A}_{12}$ is a direct summand in $K_0(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^2)$.

For the first $\mathbb{Z}^3$ action, we have

$$0 \longrightarrow (\mathcal{A}_{12} \oplus \mathcal{A}^{12})_3 \longrightarrow K_0(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^3) \longrightarrow K_1(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^2)^3 \longrightarrow 0, \tag{28}$$

that is

$$0 \longrightarrow \mathcal{A}_{123} \oplus (\mathcal{A}^{12})_3 \longrightarrow K_0(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^3) \longrightarrow K_1(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^2)^3 \longrightarrow 0, \tag{29}$$

Therefore, $\mathcal{A}_{123}$ is a direct summand in $K_0(C(\Sigma) \rtimes_{\Theta|} \mathbb{Z}^3)$.

Suppose now that $\mathcal{A}_{123\ldots(p-1)}$ is a direct summand in $K_0(C(\Sigma) \rtimes_{\sigma|} \mathbb{Z}^{p-1})$. That is,

$$K_0(C(\Sigma) \rtimes_{\sigma|} \mathbb{Z}^{p-1}) = \mathcal{A}_{123\ldots(d-1)} \oplus \mathcal{M}.$$

Then the PV sequence implies that

$$0 \longrightarrow \left(\mathcal{A}_{12\ldots(p-1)} \oplus \mathcal{M}\right)_p \longrightarrow K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p) \longrightarrow K_1(C(\Sigma) \rtimes_{\sigma|} \mathbb{Z}^{p-1})^p \longrightarrow 0, \tag{30}$$

that is,

$$0 \longrightarrow \mathcal{A}_{12\ldots p} \oplus \mathcal{M}_p \longrightarrow K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p) \longrightarrow K_1(C(\Sigma) \rtimes_{\sigma|} \mathbb{Z}^{p-1})^p \longrightarrow 0. \tag{31}$$

In particular, $\mathcal{A}_{123\ldots p}$ is a direct summand in $K_0(C(\Sigma) \rtimes_\sigma \mathbb{Z}^p)$, thereby proving the Proposition.

□

Appendix B. A more detailed history of gap-labelling theorems

We give here a brief overview of the history of the gap-labelling theorems and conjectures for the last 35 years. We thank Jean Bellissard for his invaluable help concerning this section.

The first mention of a *gap-labelling theorem* probably goes back to a paper by J. Moser in 1981 [38], concerning the Schrödinger operator in 1D with an almost periodic potential: he proved that the gaps are labeled by the *frequency module* of the potential, namely the $\mathbb{Z}$-module generated by the frequencies of the Bohr-Fourier decomposition of the almost periodic potential. These ideas were further developed by Johnson and Moser [28].

In higher dimensions, it turns out that the frequency module doesn't label spectral gaps, as seen by a counter-example [4]. Bellissard's version of the *gap-labelling theorem* says that the spectral gaps of any self-adjoint operator H (bounded or not) are labeled by the K_0-group of a C*-algebra this operator is affiliated with. In the case of an operator which is *homogeneous* (see [9]) with respect to some translations group $G = \mathbb{R}^p$ or $G = \mathbb{Z}^p$, this C*-algebra can be chosen to be, up to Morita equivalence, the crossed product algebra $C(\Omega) \rtimes G$ where Ω is a compact space in the strong topology. In addition, using the Shubin formula (see [9]) the labels are expressed as the image of K_0 by a canonical trace and the label of a given gap is also given by the value of the Integrated Density of States, see [3, 4, 9].

The general proof was finally given by [15, 29, 7] under some integrality assumption on the associated Čech Chern character. We recall that a quasicrystal is a distribution of points in the space obtained by the so-called *cut-and-project method* (also called *model sets* following the PhD work of Yves Meyer in 1972). The result concerning a $\mathbb{Z}^p$-action on a Cantor set Σ is more general and applies also to aperiodic systems which are *not* cut-and-project, such as the *Thue-Morse sequence* or the *chair tiling*, to cite only a few (for other examples, see [48]). The modern way of computing the gap labels involves the Čech cohomology of the hull, initiated by [1].

Proposition 6.2.1 in [5] has a formula involving determinants that superficially resembles our Conjecture 1. We emphasize that it does *not* include the magnetic field. The relevant matrix in that formula is the *frequency matrix* of the aperiodic potential (actually square submatrices of it). Another assumption made is that Ω is a manifold, in which case one is able to get more precise results. In our case, Ω is never a manifold as it is either a Cantor set Σ when $G = \mathbb{Z}^p$, or a fibre bundle X over the torus with fibre a Cantor set Σ when $G = \mathbb{R}^p$.

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