

Metrics and Special Kähler Geometry on the Moduli Spaces of Higgs Bundles and Hitchin Systems

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Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

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Abstract

The notions of Hitchin systems and Higgs bundles (also called Higgs pairs) were introduced by N. Hitchin in 1987. They rapidly formed a subject lying on the crossroads of representation theory, symplectic geometry, and algebraic geometry. In this research area, the main objects that attract mathematicians' attention are the moduli spaces of Higgs bundles $\mathcal{M}_{n,d}$ (the moduli space of Higgs bundles is a space parameterizing the collection of all Higgs bundles). These moduli spaces have many good properties that make them interesting objects worthy of study. For example, they are symplectic manifolds, e.g. the moduli space of the Higgs bundles with rank one and degree zero is the cotangent bundle of the Jacobian variety of a Riemann surface. These moduli spaces are also equipped with Riemannian hyperkähler metrics g_{hk} which can not be written explicitly in general, but they can be approximated by another metric, g_{sf} which is called the semi-flat metric. Roughly, we can say that $g_{hk} = g_{sf} + \text{correction}$. In fact, Gaiotto, Moore and Neitzke conjectured what the "corrections" should be in 2011[GMN13]. To find the semi-flat metric of $\mathcal{M}_{n,d}$, we first investigate the integrable system (Hitchin system) $h : \mathcal{M}_{n,d} \rightarrow B$ where the map h is known as the Hitchin map. The Lagrangian fibers of the Hitchin systems are Jacobian (or Prym) varieties of spectral curves of a given compact Riemann surface. The semi-flat metric of $\mathcal{M}_{n,d}$ is induced by a special Kähler metric on B through the Lagrangian fibration.

In my thesis, I will compute these metrics as explicitly as possible and give close observations to them. In chapter 1, we will recall the basic notions of Hitchin systems [Hit86] [Hit87a] [Hit87b].

In chapter 2, 3 and 4, we will give formulas for $SL_n(\mathbb{C})$, $GL_n(\mathbb{C})$ Hitchin systems. To do that, we treat the regular part of the Hitchin systems as complex torus bundles with fibers the Jacobian varieties (or Prym varieties) of the spectral curves. These fibers are Lagrangian submanifolds and the base B^{reg} has a special Kähler structure. Then we apply the results of Freed [Fre99] for special Kähler manifolds and results of Hitchin [Hit99] for complex Lagrangian submanifolds to get two sets of complex affine coordinate systems on B^{reg} . By using these coordinates we get some formulas for these metrics. It seems harmless to have a peek here. For the $GL_n(\mathbb{C})$ -Hitchin system we have,

$$g_{sk} = \frac{1}{2} \sum_{i,j}^{g_s} (\text{Im}(\tau_{ij})(ds_1^i \otimes ds_1^j + ds_2^i \otimes ds_2^j)),$$

$$g_{sf} = \sum_{i,j,k} g_{ijk} ds_k^i \otimes ds_k^j + g_{ijk}^{-1} d\lambda_k^i \otimes d\lambda_k^j,$$

where s_m^i, λ_n^j are the real and imaginary parts of the affine coordinates, and τ_{ij} are the periods of the spectral curves. One may notice that the spectral curves play the key role in the geometry of Hitchin systems.

In chapter 5, we will study the variation of the complex structures of the spectral curves, and show a new residue formula for the Donagi-Markman cubic

$$\frac{\partial \tau_{jk}}{\partial \lambda_i} = 2\pi i \sum_{l=1}^m \operatorname{Res} \left(\frac{\omega_i \omega_j \omega_k \lambda \frac{\partial}{\partial \lambda}}{\theta}; \tilde{p}_l \right)$$

that measures the variation of the period of a spectral curve, and give a new description of the special Kähler geometry on B .

In chapter 6, we will discuss the relation between special Kähler geometry and topological recursion [EO07][Eyn14]. That will lead us to a conclusion that once we know the periods of one of the spectral curves, we can use it to derive the special Kähler geometry of the base everywhere else.

The paper "Special Kähler Geometry of the Hitchin System and Topological Recursion" [BH17] is joint work with David Baraglia. A copy of the paper is included at the end of the thesis.

Chapter 1

Stable and Semi-Stable Bundles

1.1 Stability of holomorphic vector bundles

In this chapter, we focus on the spaces of (semi)stable vector bundles over a Riemann surface. These spaces have some good geometric structures as we will see later. A short introduction to holomorphic vector bundles over a Riemann surface is given in Appendix A.

A remarkable property of a holomorphic line bundle $L \rightarrow \Sigma$ for Σ compact connected, is that it does not possess any non-zero holomorphic section if $\deg(L) < 0$ (see [For91], theorem 16.5). In other words, there is no non-zero homomorphism from L_1 to L_2 if $\deg(L_1) > \deg(L_2)$, since a homomorphism is a section of $L_1^* \otimes L_2$. In general the analogue of this does not hold for holomorphic vector bundles of higher rank. Because $\deg(E_1^* \otimes E_2) = \text{rank}(E_1)\deg(E_2) - \text{rank}(E_2)\deg(E_1)$, but

$$\frac{\deg(E_1)}{\text{rank}(E_1)} > \frac{\deg(E_2)}{\text{rank}(E_2)}$$

does not imply there does not exist a non-zero homomorphism from E_1 to E_2 . However, the rational number $\deg(E)/\text{rank}(E)$ called the slope is still a useful quantity to study the holomorphic structure of vector bundles.

Definition 1.1.1. (*Stable and semi-stable vector bundles*) A holomorphic vector bundle E ($\text{rank}(E) \neq 0$) over a compact connected Riemann surface is called

- **semi-stable** if for any non-trivial proper holomorphic sub-bundle F , we have

$$\frac{\deg(F)}{\text{rank}(F)} \leq \frac{\deg(E)}{\text{rank}(E)}.$$

- **stable** if for any non-trivial proper holomorphic sub-bundle F , we have

$$\frac{\deg(F)}{\text{rank}(F)} < \frac{\deg(E)}{\text{rank}(E)}.$$

Obviously, all stable bundles are semi-stable and holomorphic line bundles are stable. Also, a semi-stable bundle is stable when its rank and degree are relatively prime, because if r' and d' is the rank and degree of a non-zero sub-bundle, then

$$\frac{d}{r} = \frac{d'}{r'} \iff r = r', d = d'.$$

Proposition 1.1.2. *If E_1 and E_2 are stable bundles over Σ with the same rank and degree, then*

$$H^0(\Sigma; \text{Hom}(E_1, E_2)) \cong \begin{cases} \mathbb{C} & \text{if } E_1 \cong E_2, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we have $H^0(\Sigma; \text{End}(E)) = \mathbb{C}$.

Proof. Suppose $E_1 \not\cong E_2$, let $f : E_1 \rightarrow E_2$ be a non-zero homomorphism, since E_1 is stable, if $\text{Ker}(f)$ and $\text{rank}(E_1/\text{Ker}(f))$ are non-zero then,

$$\frac{\text{deg}(\text{Ker}(f))}{\text{rank}(\text{Ker}(f))} < \frac{\text{deg}(E_1)}{\text{rank}(E_1)}.$$

Using the degree formula in Appendix A, we have

$$\frac{\text{deg}(E_1/\text{Ker}(f))}{\text{rank}(E_1/\text{Ker}(f))} = \frac{\text{deg}(E_1) - \text{deg}(\text{Ker}(f))}{\text{rank}(E_1) - \text{rank}(\text{Ker}(f))}.$$

Then,

$$\frac{\text{deg}(\text{Im}(f))}{\text{rank}(\text{Im}(f))} = \frac{\text{deg}(E_1/\text{Ker}(f))}{\text{rank}(E_1/\text{Ker}(f))} > \frac{\text{deg}(E_1)}{\text{rank}(E_1)} = \frac{\text{deg}(E_2)}{\text{rank}(E_2)}$$

By stability of E_2 , $\text{Im}(f)$ must be E_2 . Therefore, f must be surjective, hence f is an isomorphism by rank-nullity theorem. Now if $E_1 \cong E_2$, W.L.O.G., we may take E_2 to be E_1 and let Id be the identity map, obviously, $c \cdot Id$ is also an isomorphism for all $c \in \mathbb{C}^*$. Suppose there exist another isomorphism $g \neq c \cdot Id$ for all $c \in \mathbb{C}$, pick an eigenvector v in one fiber with eigenvalue λ , then $g(v) = \lambda v$, hence $(g - \lambda Id)(v) = 0$, contradicting to the fact that $(g - \lambda Id)$ is also an isomorphism. Therefore, $H^0(\Sigma; \text{Hom}(E_1, E_2)) \cong \mathbb{C}$. \square

In this thesis we will focus on the moduli spaces of semi-stable vector bundles, although there are some more interesting properties of stable and semi-stable vector bundles that are worthy to know, one may find them in for example section 3 of [Sch13].

1.2 Moduli space of stable bundles

Given a compact connected Riemann surface, we define the **moduli space** $\mathcal{N}^{(n,d)}$ to be the space of S -equivalence classes [HN75] of rank n , degree d semi-stable holomorphic vector bundles. It can be given the structure of a complex algebraic variety. When (n, d) is known and fixed we use $\mathcal{N} := \mathcal{N}^{(n,d)}$, to simplify the notation.

The reason we only focus on semi-stable holomorphic vector bundle is that, the moduli space of all rank n , degree d holomorphic bundles is neither smooth nor Hausdorff. However if we restrict to those with semi-stability, we get a better moduli space which is a projective algebraic variety, and the stable points are smooth and form an open dense subset. This result is given by Mumford and Narasimhan-Seshadri [Mum65].

In the rest of the thesis, we may use the Dolbeault operator $\bar{\partial}_E$ to denote the holomorphic structure of the holomorphic vector bundle E . Let $(E, \bar{\partial}_E)$ be a (representative of a) smooth point of \mathcal{N} , the tangent space $T_E \mathcal{N}$ is obtained as follows. Firstly, note that $\text{Dol}(E)$, the space of $\bar{\partial}$ -operators on E , is an affine space of $\Omega^{0,1}(\Sigma; \text{End}(E))$, then $\bar{\partial}_E + t\varphi$ gives a path on \mathcal{N} where φ is a tangent vector of $\text{Dol}(E)$. If $\varphi = \bar{\partial}_E(g)$ for some $g \in \Omega^0(\Sigma; \text{End}(E))$, then there is a path $\alpha(t) := f_t^{-1} \bar{\partial}_E^1(f_t)$ on $\text{Dol}(E)$ where $f_t = e^{tg}$,

given by the group action of \mathcal{G}_E , where \mathcal{G}_E denotes the group of automorphisms of E , with $\alpha'(0) = \varphi$. Therefore φ is a tangent vector in the direction of the group action, and hence a zero vector of $T_E\mathcal{N}$. Now we have

$$\begin{aligned} T_E\mathcal{N} &= \frac{\Omega^{0,1}(\Sigma; \text{End}(E))}{\bar{\partial}_E(\Omega^0(\Sigma; \text{End}(E)))} \\ &= H^1(\Sigma; \text{End}(E)). \end{aligned}$$

Now we can use the Riemann-Roch theorem, and proposition 1.1.2 to compute the dimension of \mathcal{N} ,

$$\begin{aligned} &\dim H^0(\Sigma; \text{End}(E)) - \dim H^1(\Sigma; \text{End}(E)) \\ &= \text{rank}(\text{End}(E))(1 - g) + \text{deg}(\text{End}(E)) \\ &= \text{rank}(E)^2(1 - g), \end{aligned}$$

Therefore, $\dim(\mathcal{N}) = \dim H^1(\Sigma; \text{End}(E)) = \text{rank}(E)^2(g - 1) + 1$. Also, using Serre duality, we have $H^1(\Sigma, \text{End}(E))^* \simeq H^0(\Sigma, \text{End}(E) \otimes K)$, hence a pair (E, Φ) , where K is the canonical line bundle and $\Phi \in H^0(\Sigma, \text{End}(E) \otimes K)$ defines a point in $T^*\mathcal{N}$.

1.3 Moduli space of Higgs bundles

We have observed in the last section that a pair (E, Φ) is a point in $T^*\mathcal{N}$, and it gives us motivation to look at the space of all such pairs. In 1987, N. Hitchin showed in his paper [Hit87a] that the moduli space \mathcal{M} of such pairs is a complex symplectic manifold and an integrable system.

An element in $H^0(\Sigma, \text{End}(E) \otimes K)$ is called a **Higgs field** of E , and a pair (E, Φ) consisting of a holomorphic vector bundle and a Higgs field is called a **Higgs bundle**. Given a trivialization $E|_{U_i} \simeq U_i \times \mathbb{C}^n$ of E , a Higgs field Φ on U_i can be written as an $n \times n$ matrix

$$\Phi = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix},$$

where α_{ij} are holomorphic one forms.

Definition 1.3.1. (Stable Higgs bundle). Let (E, Φ) be a Higgs bundle, a sub-bundle $F \subseteq E$ is called Φ -invariant if $\Phi F \subseteq F \otimes K$.

- a Higgs bundle is **stable** if

$$\frac{\text{deg}(F)}{\text{rank}(F)} < \frac{\text{deg}(E)}{\text{rank}(E)}$$

for any proper Φ -invariant sub-bundle $F \subseteq E$.

- a Higgs bundle is **semi-stable** if

$$\frac{\text{deg}(F)}{\text{rank}(F)} \leq \frac{\text{deg}(E)}{\text{rank}(E)}.$$

for any proper Φ -invariant sub-bundle $F \subseteq E$

Now the group of isomorphisms of E acts on (E, Φ) in the way that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \otimes K \\ \downarrow g & & \downarrow g \\ E & \xrightarrow{g \cdot \Phi} & E \otimes K \end{array}$$

This means $g \cdot \Phi = g\Phi g^{-1}$, for $g \in \mathcal{G}_E$ where \mathcal{G}_E is the group of automorphisms of E . The **moduli space \mathcal{M} of stable Higgs bundles** is the quotient of the set of stable Higgs bundles by the group \mathcal{G} . There is also a moduli space of semistable Higgs bundles, but in this thesis it will suffice to consider only the stable moduli space when the genus $g \geq 2$. When $g = 1$, we consider semistable Higgs bundles. In this case we will work out the moduli space directly.

We first restrict attention to rank 2 bundles. In this case, a Higgs bundle (E, Φ) is stable if for each Φ -invariant holomorphic line bundle $L \subseteq E$, $\deg L < \frac{1}{2} \deg E$. Note that if the holomorphic bundle is stable, then the Higgs bundles are also stable for all Φ , but there exist stable Higgs bundles (E, Φ) with non-stable E .

According to N. Hitchin [Hit87a], [Hit87b], \mathcal{M} is a complex symplectic manifold and $T^*\mathcal{N}$ lies as an open dense subset in \mathcal{M} . The symplectic form on \mathcal{M} is defined as follows.

Let (E, Φ) be a (representative) element in \mathcal{M} , firstly, we need to find out the tangent space $T_{(E, \Phi)}\mathcal{M}$. Deform (E, Φ) ,

$$\begin{cases} \bar{\partial}_E + t\alpha & t \in \mathbb{R}, \quad \alpha \in \Omega^{0,1}(\Sigma; \text{End}(E)) \\ \Phi + t\varphi & \varphi \in \Omega^{1,0}(\Sigma; \text{End}(E)) \end{cases}$$

such that to first order in t , $(\bar{\partial}_E + t\alpha)(\Phi + t\varphi) = 0$. Then we have

$$[\alpha, \Phi] + \bar{\partial}_E \varphi = 0.$$

Therefore, the tangent space at (E, Φ) of the space of stable holomorphic Higgs bundles is

$$H := \{(\alpha, \varphi) \mid [\alpha, \Phi] + \bar{\partial}_E \varphi = 0\}.$$

Now if $\alpha = -\bar{\partial}_E(\lambda)$, $\varphi = [\lambda, \Phi]$ for some $\lambda \in \Omega^0(\Sigma; \text{End}(E))$, then $t\lambda$ defines a path given by the gauge group action

$$\begin{aligned} e^{t\lambda} \bar{\partial}_E(e^{-t\lambda}) &= \bar{\partial}_E - t\bar{\partial}_E(\lambda) + O(t^2) \\ e^{t\lambda} \Phi e^{-t\lambda} &= \Phi + t[\lambda, \Phi] + O(t^2) \end{aligned}$$

with tangent vector $(-\bar{\partial}_E(\lambda), [\lambda, \Phi])$. Therefore $(-\bar{\partial}_E(\lambda), [\lambda, \Phi])$ is a tangent vector of the group action direction, hence the tangent space of \mathcal{M} is

$$T_{(E, \Phi)}\mathcal{M} = \frac{H}{\{(\alpha, \varphi) \mid \alpha = -\bar{\partial}_E(\lambda), \varphi = [\lambda, \Phi]\}}.$$

Next, we define a two form $\Omega : T_{(E, \Phi)}\mathcal{M} \times T_{(E, \Phi)}\mathcal{M} \rightarrow \mathbb{C}$ on \mathcal{M} to be:

$$\Omega((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_{\Sigma} \text{tr}(\alpha_1 \wedge \varphi_2) - \text{tr}(\alpha_2 \wedge \varphi_1).$$

for representatives (α_1, φ_1) and (α_2, φ_2) . To see that Ω is a well defined two form, we need to check the definition doesn't depend on the choice of representative. In other words, we need to show

$$\int_{\Sigma} \text{tr}(-\bar{\partial}_E(\lambda) \wedge \varphi_2) - \text{tr}(\alpha_2 \wedge [\lambda, \Phi]) = 0.$$

for all $(\alpha_2, \varphi_2) \in H$ and $\lambda \in \Omega^0(\Sigma; \text{End}(E))$. Since

$$\begin{aligned} & \int_{\Sigma} \text{tr}(\bar{\partial}_E(\lambda) \wedge \varphi_2) + \text{tr}(\lambda \wedge (\bar{\partial}_E \varphi_2)) \\ &= \int_{\Sigma} \bar{\partial}(\text{tr}(\lambda \wedge \varphi_2)) \\ &= \int_{\Sigma} d(\text{tr}(\lambda \wedge \varphi_2)) \\ &= 0, \end{aligned}$$

where we use Stokes' theorem in the last equality, then we complete the proof by

$$\begin{aligned} \int_{\Sigma} \text{tr}(-\bar{\partial}_E(\lambda) \wedge \varphi_2) &= \int_{\Sigma} \text{tr}(\lambda \wedge (\bar{\partial}_E \varphi_2)) \\ &= - \int_{\Sigma} \text{tr}(\lambda \wedge [\Phi, \alpha_2]) \\ &= - \int_{\Sigma} \text{tr}([\lambda, \Phi] \wedge \alpha_2). \end{aligned}$$

Hitchin proved in [Hit87a] section 6,7 that Ω is a symplectic form. Note that $\Omega|_{T^*\mathcal{N}}$ is the canonical symplectic form on $T^*\mathcal{N}$.

1.4 Special Kähler manifolds

Let M be a complex manifold, J be a complex structure, g be a Riemannian metric on M , we say g is Hermitian if

$$g(Ju, Jv) = g(u, v).$$

In this case,

$$\omega(u, v) := g(Ju, v).$$

is skew symmetric, hence a 2-form. Note that ω is nondegenerate, we call it a Hermitian form. If ω is symplectic (i.e. if ω is closed), then (M, g, ω) is called a Kähler manifold and in this case, we say that ω is a Kähler form. In other words, a Kähler manifold is a manifold with a complex structure, a Riemannian structure, a symplectic structure, and these three structures are compatible.

Remark 1.4.1. *Let (M, g, ω) be a Kähler manifold, J be the complex structure of M , then*

1. g can be retrieved from ω by: $g(u, v) = \omega(u, Jv)$.
2. On the other hand, if g is a Riemannian metric, ω is the associated Kähler form, then $h := g + i\omega$ is a Hermitian metric.

Let ∇ be a connection on the tangent bundle of (M, g, ω) , and $d_\nabla : \Omega^k(TM) \rightarrow \Omega^{k+1}(TM)$ be exterior derivative determined by the Leibniz rule

$$d_\nabla(\alpha \otimes u) = d\alpha \otimes u + (-1)^{\deg(\alpha)} \alpha \wedge \nabla u,$$

where α is a k -form and u a vector field. Then if ∇ is flat, it provides a de Rham complex,

$$0 \longrightarrow \Omega^0(TM) \xrightarrow{d_\nabla} \Omega^1(TM) \xrightarrow{d_\nabla} \Omega^2(TM) \xrightarrow{d_\nabla} \dots$$

Note that for any vector bundle $E \rightarrow M$ and $\eta \in \Omega^k(E)$ (in our case here, E is TM), the definition of d_∇ is just:

$$\begin{aligned} d_\nabla \eta(X_0, \dots, X_k) &= \sum_{0 \leq i \leq k} (-1)^i \nabla_{X_i} \eta(X_0, \dots, \hat{X}_i, \dots, X_k) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \eta([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned} \quad (1.1)$$

where X_0, \dots, X_k are vector fields on M .

Definition 1.4.2. A Kähler manifold M is called special Kähler if it equipped with a connection ∇ on TM , such that

- ∇ is flat and torsion-free
- $\nabla \omega = 0$ and
- $d_\nabla J = 0 \in \Omega^2(TM)$.

The second condition means by definition:

$$\nabla_X \omega(Y, Z) := X\omega(Y, Z) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0,$$

for any vector fields X, Y, Z . A connection that satisfies this relation is called a symplectic connection. In the last condition, we think of the almost complex structure $J : TM \rightarrow TM$ to be an endomorphism of the tangent bundle of M , hence an element of $\Omega^1(TM)$.

Now for $J \in \Omega^1(TM)$ and the identity endomorphism $Id \in \Omega^1(TM)$, we use formula 1.1 to get:

$$\begin{aligned} d_\nabla J(X, Y) &= \nabla_X(J(Y)) - \nabla_Y(J(X)) - J([X, Y]); \\ d_\nabla Id(X, Y) &= \nabla_X(Y) - \nabla_Y(X) - [X, Y]. \end{aligned}$$

Since by definition the torsion tensor is $\nabla_X(Y) - \nabla_Y(X) - [X, Y]$, then torsion free condition is just $d_\nabla Id = 0$. The flatness and torsion free conditions of ∇ imply that we can choose coordinates $\{t_1^\alpha, \dots, t_{2n}^\alpha\}$ so that $\nabla(\frac{\partial}{\partial t_i^\alpha}) = 0$ and $\nabla(dt_i^\alpha) = 0$ ([Lee06] p.119, [Boo86] p.161). Then the condition $\nabla \omega = 0$ means the coefficients c_{ij} of $\omega = c_{ij} dt_i^\alpha \wedge dt_j^\alpha$ are constants. Since ω is a symplectic form, therefore we can choose Darboux coordinates $\{x_i, y_i\}$ of ω which are given by a linear constant transformation of t^α , hence $\nabla dx_i = \nabla dy_j = 0$ and

$$\omega = \sum_i^n dx_i \wedge dy_i.$$

By the condition $d_\nabla J = 0$, we have $d_\nabla(J \otimes dx_i) = d_\nabla J \otimes dx_i + J \otimes \nabla dx_i = 0$, which implies $d(J \cdot dx_i) = 0$. Similarly we have $d(J \cdot dy_i) = 0$. Therefore locally,

$$J \cdot dx_i = du_i \quad \text{and} \quad J \cdot dy_j = dv_j,$$

for some real coordinates $\{u_i, v_j\}$ of M . Now we let $z^i = x_i - iu_i$, and $w_j = y_j - iv_j$, then $\{z^i\}$ and $\{w_j\}$ are complex holomorphic coordinate systems of M . Now we have [BH17][Fre99]

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + \tau_{ij} \frac{\partial}{\partial y_j} \right), \quad \text{where } \tau_{ij} = \frac{\partial w_j}{\partial z^i} \text{ form the period matrix.}$$

Here are a few remarks about the affine coordinates:

- The connection ∇ is trivial in the coordinate system

$$\{\operatorname{Re}(z^1), \dots, \operatorname{Re}(z^{g_s}), \operatorname{Re}(w_1), \dots, \operatorname{Re}(w_{g_s})\};$$

- The Kähler form is given by $\omega = -\frac{1}{2} \operatorname{Im}(\tau_{ij}) dz^i \wedge d\bar{z}^j$; and

- $\tau_{ij} = \frac{\partial w_j}{\partial z^i}$.

N.Hitchin proved that the base of the Hitchin integrable system is in fact a special Kähler manifold. Both Hitchin[Hit99] and Freed[Fre99] provided us a way to study the special Kähler metrics on these bases by using two sets of affine coordinates. In the next few sections of the thesis, we will compute the coordinates for some explicit Hitchin systems, and derive the special Kähler metrics.

1.5 Integrable systems and the Hitchin map

In 1987, Hitchin proved in his paper [Hit87b](section 8), that the moduli space \mathcal{M} is equipped with an integrable system. Recall that an $2m$ -dimensional complex symplectic manifold is a (completely) integrable system if there exist holomorphic functions f_1, \dots, f_m which Poisson commute and $df_1 \wedge \dots \wedge df_m$ is non-zero in an open dense subset (see also Appendix B). In our case, the functions on \mathcal{M} are given by the **Hitchin map** for $GL_n(\mathbb{C})$ Higgs bundles, defined as follows.

Let $B := \bigoplus_{i=1}^n H^0(K^{\otimes i})$, and $p_a : T^*\Sigma \rightarrow (T^*\Sigma)^n$ be the polynomial which sends x to

$$p_a(x) = x^n + a_1 x^{n-1} + \dots + a_n,$$

for $a = a_1 \oplus \dots \oplus a_n \in B$. The polynomial p_a is called the characteristic polynomial. Now we define the Hitchin map to be:

$$\begin{aligned} h : \mathcal{M} &\rightarrow B \\ (E, \Phi) &\rightarrow a \end{aligned}$$

where $p_a(x) = \det(xI - \Phi)$. In other words, a is the direct sum of the coefficients of $\det(xI - \Phi)$. It is easy to see that this is a well defined map since we have $\det(AB) = \det(A)\det(B)$ and $\det(A) = 1/\det(A^{-1})$ for matrices A, B with A invertible. Since $T^*\mathcal{N}$ is open and dense in \mathcal{M} , to conclude that (\mathcal{M}, h) is an integrable system we shall at least show

Proposition 1.5.1.

$$\frac{1}{2} \dim(\mathcal{M}) = \dim(\mathcal{N}) = 1 + n^2(g-1) = \dim \bigoplus_{i=1}^n H^0(\Sigma; K^{\otimes i}).$$

Proof. Recall that $\dim \mathcal{N} = 1 + n^2(g - 1)$, and from Serre duality, we directly obtain $\dim H^0(\Sigma; K) = g$. And we claim that for $n > 1, g > 1$, $\dim H^0(\Sigma; K^n) = (2n - 1)(g - 1)$, since from the Gauss-Bonnet theorem, $\deg(K) = 2g - 2$, hence $\deg(K^n) = n(2g - 2)$, then $\dim H^1(\Sigma; K^n) = 0$, see [For91] Theorem 17.16. Therefore

$$\begin{aligned} \dim \bigoplus_{i=1}^n H^0(\Sigma; K^{\otimes i}) &= g + \sum_{i=2}^n (2n - 1)(g - 1) \\ &= 1 + n^2(g - 1). \end{aligned}$$

□

One can find a proof for the Poisson commutativity of components of h in [Hit87b]. When $g \geq 1, n = 1$, the moduli space \mathcal{M} is the cotangent bundle of the Jacobian variety $T^*Jac(\Sigma)$ with Hitchin map the projection onto the fibers (see [GX08]).

1.6 Hitchin systems for subgroups of $GL_n(\mathbb{C})$

In the study of Hitchin systems, we can also restrict the structure group of the vector bundle E to be a subgroup of $GL_n(\mathbb{C})$. Write the structure group to be G , let \mathfrak{g} be the Lie algebra, if $\{(u_\alpha, \phi_\alpha)\}$ is a trivialization of E , then we consider endomorphisms and automorphisms $\{h_\alpha : u_\alpha \rightarrow GL_n(\mathbb{C})\}$ taking values in \mathfrak{g} and G respectively. Denote the space of \mathfrak{g} -valued endomorphisms of E by $\text{End}_G(E)$, then the space of holomorphic structures of E is an affine space of $\Omega^{0,1}(\Sigma; \text{End}_G(E))$. Let \mathcal{N}_G be the space of isomorphism classes of semi-stable holomorphic vector bundles (the definition of stable/semistable Higgs bundles is different for different groups G) with G -structures, then

$$\begin{aligned} T_E \mathcal{N}_G &= \frac{\Omega^{0,1}(\Sigma; \text{End}_G(E))}{\bar{\partial}_E(\Omega^0(\Sigma; \text{End}_G(E)))} \\ &= H^1(\Sigma; \text{End}_G(E)). \end{aligned}$$

Similar to the computation in section 1.2, if G is reductive, then $\mathfrak{g} \cong \mathfrak{g}^*$ and $\text{End}_G(E) \cong \text{End}_G(E)^*$, hence $\deg(\text{End}_G(E)) = 0$. Therefore

$$\begin{aligned} &\dim H^0(\Sigma; \text{End}_G(E)) - \dim H^1(\Sigma; \text{End}_G(E)) \\ &= \text{rank}(\text{End}_G(E))(1 - g) + \deg(\text{End}_G(E)) \\ &= \dim(G)(1 - g) + \deg(\text{End}_G(E)) \\ &= \dim(G)(1 - g), \end{aligned}$$

Since by proposition 1.1.2, $H^0(\Sigma; \text{End}(E)) \cong \mathbb{C}$, therefore an endomorphism of E with the structure group embedded into $GL_n(\mathbb{C})$ is multiplication by a complex number (i.e. a map $\Sigma \rightarrow cId$, where c is a complex number and Id is the $n \times n$ identity matrix), hence when G is semisimple, e.g. $SL_n(\mathbb{C}), SO_n(\mathbb{C}), Sp_n(\mathbb{C})$, we have

$$H^0(\Sigma; \text{End}_G(E)) \cong 0,$$

because the centre of G is finite. Therefore

Proposition 1.6.1. *Let \mathcal{N}_G^S be the moduli space of stable holomorphic G -bundles, where G is semisimple, then*

$$\dim(\mathcal{N}_G^S) = \dim H^1(\Sigma; \text{End}(E)) = \dim(G)(g - 1).$$

Example 1.6.2. ($G = SL_n(\mathbb{C})$). An $SL_n(\mathbb{C})$ -Higgs bundle (E, Φ) consists of a rank n vector bundle E over Σ such that the determinant line bundle $\det(E)$ is trivial, and a Higgs field $\Phi \in H^0(\Sigma; \text{End}(E) \otimes K)$ such that $\text{tr}(\Phi) = 0$. Because $\dim(SL_n(\mathbb{C})) = n^2 - 1$, we have

$$\dim \mathcal{N}_{SL_n(\mathbb{C})}^S = (n^2 - 1)(g - 1).$$

When $g \geq 2$, The dimension of the moduli space of $SL_n(\mathbb{C})$ -Higgs bundles is $2(n^2 - 1)(g - 1)$.

1.7 Metrics on the moduli spaces

In previous sections, we saw that a tangent vector of the moduli space \mathcal{M} is given by a pair $(\alpha, \varphi) \in \Omega^{0,1}(\Sigma; \text{End}(E)) \times \Omega^{1,0}(\Sigma; \text{End}(E))$ [Hit87b].

Let \star be the Hodge star operator (on a Riemann surface with compatible metric, $\star dx = dy$, $\star dy = -dx$, if $z = x + iy$ is a local holomorphic coordinate then $\star dz = -idz$, $\star d\bar{z} = id\bar{z}$), recall that a hermitian inner product on $\Omega^1(\Sigma)$ is given by

$$\langle a, b \rangle := \int_{\Sigma} \bar{a} \wedge \star b.$$

The hermitian inner product gives a positive definite inner product on $\Omega^{0,1}(\Sigma) \oplus \Omega^{1,0}(\Sigma)$ by

$$\langle (a_1, b_1), (a_2, b_2) \rangle := \text{Re} \langle a_1, a_2 \rangle + \text{Re} \langle b_1, b_2 \rangle.$$

Similarly, note that $\Omega^{0,1}(\Sigma; \text{End}(E)) \otimes \Omega^{1,0}(\Sigma; \text{End}(E))$ is an infinite dimensional vector space and there is a Hermitian inner product on it given by:

$$\langle (\alpha_1, \varphi_1), (\alpha_2, \varphi_2) \rangle := \frac{i}{2} \int_{\Sigma} \text{tr}(\alpha_1^* \alpha_2) - \text{tr}(\varphi_1^* \varphi_2),$$

where $*$ is the conjugate transpose. The inner product is invariant under the action of the group of unitary gauge transformations $\mathcal{G}^{unitary}$. Let X be the subspace of Higgs bundles satisfying the Hitchin equations

$$\begin{cases} F_E + [\Phi, \Phi^*] = -\frac{2\pi id}{n} \text{vol}_{\Sigma} \otimes Id_E \\ \bar{\partial}_E \Phi = 0 \end{cases}$$

where F_E is the curvature of the Chern connection (the hermitian connection where the $(0, 1)$ -part equals $\bar{\partial}_E$), d is the degree of E . Let $\widetilde{\mathcal{M}} = X/\mathcal{G}^{unitary}$, then the inner product induces a metric on $\widetilde{\mathcal{M}}$. From the Hitchin-Simpson theorem [Hit87a] [Sim92], $\widetilde{\mathcal{M}}$ is diffeomorphic to \mathcal{M} therefore using the diffeomorphism, we get a metric g on \mathcal{M} .

In fact g is a **hyperKähler metric**. We can see it in the following way: firstly, we use \langle, \rangle on $\Omega^{0,1}(\Sigma; \text{End}(E)) \times \Omega^{1,0}(\Sigma; \text{End}(E))$ and the complex structure to get a Kähler form ω_1 , let $\omega_2 := \text{Re}\Omega$ and $\omega_3 := \text{Im}\Omega$. Then let J, K be the almost complex structures with respect to the Kähler forms ω_2, ω_3 . Now since

$$\Omega((\alpha_1, \varphi_1), i(\alpha_2, \varphi_2)) = i\Omega((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)),$$

so $\operatorname{Re}\Omega((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \operatorname{Im}\Omega((\alpha_1, \varphi_1), i(\alpha_2, \varphi_2))$. Therefore

$$\begin{aligned} & \omega_3((\alpha_1, \varphi_1), K(\alpha_2, \varphi_2)) \\ &= g((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) \\ &= \omega_2((\alpha_1, \varphi_1), J(\alpha_2, \varphi_2)) \\ &= \omega_3((\alpha_1, \varphi_1), IJ(\alpha_2, \varphi_2)), \end{aligned}$$

for any $(\alpha_1, \varphi_1), (\alpha_2, \varphi_2) \in \Omega^{0,1}(\Sigma; \operatorname{End}(E)) \times \Omega^{1,0}(\Sigma; \operatorname{End}(E))$. By non-degeneracy of ω_3 , we have $IJ = K$, hence \langle, \rangle is hyperkähler. The Hitchin equation can be interpreted as a hyperKähler moment map equation, hence \mathcal{M} can be interpreted as a hyperKähler quotient. Then according to Hitchin [Hit86], the natural metric g on the hyperKähler quotient of a hyperKähler manifold is again hyperkähler (see also [HKLR87]).

Studying these metrics is the key goal of this thesis. As you may see here that although we have the definitions of these metrics of Hitchin systems, one can hardly obtain any geometric information about the Hitchin systems by using these definitions. The metrics are given fiber wise on the tangent bundle, therefore we can not even tell the local geometry of the Hitchin system from the definitions. In later chapters, we will look closer at these metrics by using the affine coordinates. By doing that, we can indeed get an idea about what the metrics look like. For example, the special Kähler metric and the semi-flat metric on $GL_n(\mathbb{C})$ -Hitchin system are given by formulas 3.2, 3.3 in the affine coordinates, and the metrics of the $SL_n(\mathbb{C})$ -Hitchin systems are the restriction of the metrics on the general linear Hitchin systems.

Also, a more powerful result comes up when we start looking at the variation formulas about the metrics, which concludes that one can know the geometry of the base of a Hitchin system by looking at only one of its fibers.

1.8 Constructing special Kähler and semi-flat hyper-Kähler metrics

According to the theorem of Arnold-Liouville, the regular fibers of an integrable system are Lagrangian, hence we can think of the regular part of Hitchin systems as moduli spaces of complex Lagrangian submanifolds. We explain in this section how Hitchin constructs a special Kähler metric from the moduli space of Lagrangian submanifolds [Hit99].

Let $h : \mathcal{M}^{reg} \rightarrow B^{reg}$ be the regular part of a Hitchin system. Then a fiber $Y := h^{-1}(b)$ is a Lagrangian submanifold with respect to the symplectic form Ω . Let $\omega_1 = \operatorname{Re}\Omega$, $\omega_2 = \operatorname{Im}\Omega$, then ω_1, ω_2 are real symplectic forms and the fibers are Lagrangian with respect to ω_1 and ω_2 . Let U be a contractible open subset of B^{reg} containing $h(Y)$, let $\{x_1, \dots, x_{2m}, y_1, \dots, y_{2m}\}$ be real local coordinates on $h^{-1}(U) \subset \mathcal{M}^{reg}$ ($2m$ is the complex dimension of \mathcal{M}) where $\{y_1, \dots, y_{2m}\}$ are coordinates of U and $h(x_1, \dots, y_{2m}) = (y_1, \dots, y_{2m})$, then

$$\omega_1 = \sum a_{ij} dx_i \wedge dy_j + \sum b_{ij} dy_i \wedge dy_j.$$

Note that the $dx_i \wedge dx_j$ terms are missing because the fibers are Lagrangian. Since $d\omega_1 = 0$, for each j

$$\sum_{i,k} \frac{\partial a_{ij}}{\partial x_k} dx_k \wedge dx_i = 0,$$

which implies $\sum a_{ij}dx_i$ is closed on Y for each j . Therefore integrating $\sum a_{ij}dx_i$ along two homologous 1-cycles in a fiber of h gives the same result. Also, if Q is a tangent vector to B^{reg} then for any lift \tilde{Q} to a vector field along Y , the 1-form given by the contraction $\omega_1(\tilde{Q}, -)|_Y$ is closed and independent of the choice of the lifting. Because if we let $Q = c_i\partial_{y_i}$ be the tangent vector, where c_i are constants, and lift it to $\tilde{Q} = c_i\partial_{y_i} + f_i\partial_{x_i}$ for any functions f_i on the fiber, then

$$\omega_1(\tilde{Q}, -)|_Y \cdot (h_j \frac{\partial}{\partial x_j}) = -c_i a_{ij} h_j,$$

hence $\omega_1(\tilde{Q}, -)|_Y = -c_i a_{ij} dx_j$, which is independent of the f_i .

Next, let $A \in H_1(Y, \mathbb{Z})$ be a homology class, then A is also a homology class of $H_1(h^{-1}(U), \mathbb{Z})$. Choose a circle fibration over U , such that each circle is contained in the fiber of $h^{-1}(U)$ and represent the class A , define a 1-form $F_A \in \Omega^1(U)$ to be the integral of ω_1 over the fibers of h , i.e.

$$F_A = \sum_{i,j=1}^{2m} \left(\int_A a_{ij} dx_i \right) dy_j.$$

Since $d\omega_1 = 0$, writing it out explicitly we have that for any i, j, k ,

$$\frac{\partial a_{kj}}{\partial y_i} - \frac{\partial a_{ki}}{\partial y_j} = \frac{\partial b_{ij}}{\partial x_k} - \frac{\partial b_{ji}}{\partial x_k}.$$

Hence

$$\begin{aligned} dF_A &= \sum_{i,j,k=1}^{2m} \left(\int_A \frac{\partial a_{ij}}{\partial y_k} dx_i \right) dy_k \wedge dy_j \\ &= \sum_{j>k,i=1}^{2m} \left(\int_A \left(\frac{\partial a_{ij}}{\partial y_k} - \frac{\partial a_{ik}}{\partial y_j} \right) dx_i \right) dy_k \wedge dy_j \\ &= \sum_{j>k,i=1}^{2m} \left(\int_A \left(\frac{\partial b_{kj}}{\partial x_i} - \frac{\partial b_{jk}}{\partial x_i} \right) dx_i \right) dy_k \wedge dy_j \\ &= \sum_{j>k,i=1}^{2m} \left(\int_A d(b_{kj} - b_{jk}) \right) dy_k \wedge dy_j \\ &= 0, \end{aligned}$$

This says F_A is closed. Since U is assumed to be contractible, there is a function $u_A : U \rightarrow \mathbb{R}$ such that $du_A = F_A$ hence determines a map $u : U \rightarrow H^1(Y, \mathbb{R})$. We shall do the same to the real symplectic form ω_2 to get a map $v : U \rightarrow H^1(Y, \mathbb{R})$. If u', v' are two other solutions then $u' = u + \text{constant}$, $v' = v + \text{constant}$, hence u, v are not well defined on B^{reg} , but they can be patched together on the universal cover \tilde{B}^{reg} to give $u, v : \tilde{B}^{reg} \rightarrow H^1(Y, \mathbb{R})$

Proposition 1.8.1. (*[Hit99], section 3*) *The maps $u, v : \tilde{B}^{reg} \rightarrow H^1(Y, \mathbb{R})$ are local diffeomorphisms. Thus the image $w(\tilde{B}^{reg})$ of $w = (u, v) : \tilde{B}^{reg} \rightarrow H^1(Y, \mathbb{R}) \times H^1(Y, \mathbb{R})$ is a smooth submanifold and the projections onto each factor are local diffeomorphisms.*

Let k be the Kähler form of \mathcal{M}^{reg} , then there is a real constant symplectic form on $H^1(Y, \mathbb{R})$ defined by

$$\langle a, b \rangle = \int_Y \alpha \wedge \beta \wedge k^{n-1}$$

where α, β are representatives of a, b . If we let

$$\begin{aligned} \Omega_1((a_1, a_2), (b_1, b_2)) &= \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle \\ \Omega_2((a_1, a_2), (b_1, b_2)) &= \langle a_1, b_1 \rangle - \langle a_2, b_2 \rangle \end{aligned}$$

then they are clearly symplectic forms on $H^1(Y, \mathbb{R}) \times H^1(Y, \mathbb{R})$. Also Hitchin proved the following theorems:

Theorem 1.8.2. [Hit99] $w(\tilde{B}^{reg}) \subset H^1(Y, \mathbb{R}) \times H^1(Y, \mathbb{R})$ is Lagrangian with respect to Ω_1 and Ω_2 .

Theorem 1.8.3. [Hit99] The metric $g|_{B^{reg}}$ which is the restriction of

$$g((a, b), (a, b)) = \frac{1}{2} \langle a, b \rangle$$

to B^{reg} is a special Kähler metric.

In fact the above theorem was divided into two parts in [Hit99]. Firstly, Hitchin showed that if V is a vector space with skew form \langle, \rangle (in our case V is $H^1(Y, \mathbb{R})$), and M a Lagrangian submanifold in $V \times V$ with respect to Ω_1, Ω_2 , then $g|_M$ is a special pseudo-Kähler metric (not necessarily positive definite). This is Theorem 2 in [Hit99]. Secondly, the metric is in fact definite in our case (which is Theorem 3 in [Hit99]) because given a tangent vector Q of B^{reg} , $du(Q) = \omega_1(Q, -)|_Y \in H^1(Y, \mathbb{R})$, then

$$\begin{aligned} g(Q, Q) &= \langle du(Q), dv(Q) \rangle \\ &= \frac{i}{2} \langle (du + idv)(Q), (du - idv)(Q) \rangle \\ &= \frac{i}{2} \int_Y \Omega(Q, -) \wedge \bar{\Omega}(Q, -) \wedge h^{n-1}, \end{aligned}$$

which shows g is positive definite. In this way we are able to construct a special Kähler metric on B^{reg} explicitly.

Now starting from a special Kähler manifold (X, ω, I, ∇) , we can construct a hyper Kähler metric on the cotangent bundle T^*X [Fre99]. First of all, the tangent space of $T_{(x,p)}(T^*X)$ at $(x, p) \in T^*X$ is just $T_x X \oplus T_x^* X \cong T_x X \oplus \overline{T_x X}$. If g is the Kähler metric of the tangent space $T_x X$ such that $\omega(-, -) = g(I-, -)$, then g induces a metric on the cotangent space $T_x^* X$ as follows.

Let $\{e_1, \dots, e_{2m}\}$ be a basis of $T_x X$ such that $g_{ij} = g(e_i, e_j)$ and let e^1, \dots, e^{2m} be the dual basis, let $\xi = \sum a^i e_i$ be a tangent vector, then a cotangent vector $\alpha := g(\xi, -) = \sum b_i e^i$ is given by

$$\alpha = \sum_{j=1}^{2m} g(\xi, e_j) e^j = \sum_{j=1}^{2m} \left(\sum_{i=1}^{2m} a^i g(e_i, e_j) \right) e^j = \sum_{j=1}^{2m} \left(\sum_{i=1}^{2m} a^i g_{ij} \right) e^j.$$

Therefore, $b_j = \sum a^i g_{ij}$ (In matrices, $\alpha = ((g_{ij})\xi)^T = \xi^T (g_{ij})^T$). If the induced metric g^{-1} on $T_x^* X$ is defined by $g^{-1}(\alpha, \alpha) := g(\xi, \xi)$, then $g^{-1}(\alpha, \alpha) = \alpha (g_{ij})^{-1} \alpha^T$.

Now the hyper Kähler metric is given by

$$g_{hk}(\xi_1 \oplus \alpha_1, \xi_2 \oplus \alpha_2) = g(\xi_1, \xi_2) + g^{-1}(\alpha_1, \alpha_2), \quad (1.0)$$

where ξ_1, ξ_2 are real tangent vectors of X , α_1, α_2 are real cotangent vectors of X , g is the Kähler metric on the tangent spaces $T_x X$, and g^{-1} is the induced metric on the cotangent spaces $T_x^* X$. Let J be the complex structure such that $J(v_1 \oplus \bar{v}_2) = -v_2 \oplus \bar{v}_1$, define $K = IJ$ (i.e. $K(v_1 \oplus \bar{v}_2) = -iv_2 \oplus \bar{v}_1$), then the Kähler forms of the hyper Kähler structure are

$$\begin{aligned} \omega_I(-, -) &= g_{hk}(-, I-), \\ \omega_J(-, -) &= g_{hk}(-, J-), \\ \omega_K(-, -) &= g_{hk}(-, K-). \end{aligned}$$

The fibers of the Hitchin system are complex tori (e.g. $\text{Jac}(\Sigma)$, $\text{Prym}(\Sigma)$). They are the quotient of $\mathbb{C}^m \cong T_b^* B^{reg}$ with $2m$ -dimensional lattices. The metric g_{hk} is invariant under the translation action of this lattice. Therefore we can use this construction to get a hyper Kähler metric on \mathcal{M}^{reg} .

Chapter 2

$g = 1, \mathrm{SL}_n(\mathbb{C})$ Moduli Space

In this section, we let the Riemann surface Σ have $g = 1$, and it is well known that

$$\Sigma \cong \mathrm{Jac}(\Sigma) := \mathbb{C}/\{\mathbb{Z} + \tau\mathbb{Z}\}$$

for some $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. τ determines the complex structure of Σ . Let z be the coordinate of the complex plane \mathbb{C} given above. Now consider the rank n holomorphic vector bundles with trivial determinant. Note that the holomorphic 1-form dz on \mathbb{C} is translation invariant and so descends to $\Sigma \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. It therefore provides a non-vanishing holomorphic section of the canonical bundle K , therefore $K \cong \mathcal{O}$ is trivial. It follows that an $\mathrm{SL}_n(\mathbb{C})$ Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}_0(E) \otimes K) \cong \mathrm{End}_0(E)$ may be regarded as a trace-free endomorphism of E . The coefficients of the characteristic polynomial

$$\det(\lambda - \Phi) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

are globally defined holomorphic functions on Σ hence they are constant, hence the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of Φ are constant. In addition assume that all λ_i are different, since $\bar{\partial}_E(\Phi) = 0$, the eigen-subbundles of Φ are holomorphic sub-bundles of E , therefore $\bar{\partial}_E = \bar{\partial} + \{a_{ij}^{0,1}\}$ where $\{a_{ij}^{0,1}\}$ is also a diagonal matrix. Note that each $\bar{\partial} + a_{ij}^{0,1}$ gives a holomorphic structure of a line bundle, hence we proved the following

Theorem 2.0.1. *For any Higgs bundle (E, Φ) with distinct eigenvalues, there is a gauge transformation $(E, \Phi) \rightarrow (\tilde{E}, \tilde{\Phi})$ where $\tilde{\Phi}$ is diagonalised and \tilde{E} decomposes as the direct sum of line bundles*

$$\tilde{E} = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where L_i are eigen-subbundles of the eigenvalues λ_i of Φ .

Note that since $\det(E) = \mathcal{O}$ and $\mathrm{tr}(\Phi) = 0$, we have $\bigotimes_{i=1}^n L_i = \mathcal{O}$ and $\sum_{i=1}^n \lambda_i = 0$.

2.1 $n = 2$ Moduli space

Now we focus on the case when $n = 2$. First of all we need to find out what the moduli space and the regular fibers look like. Let (E, Φ) be a rank two Higgs bundle, since Φ is traceless and E is a vector bundle of trivial determinant, then

$$(E, \Phi) \sim (L \oplus L^*, (\lambda, -\lambda)),$$

for $\lambda \in H^0(\Sigma; K)$ and L the λ -eigen-subbundle of E . Note that when $g = 1$,

$$\dim(H^0(\Sigma; K)) = 1.$$

Let λ be the coordinate of $\dim(H^0(\Sigma; K))$, i.e., $\lambda \in \mathbb{C}$ corresponds to the 1-form λdz , then the Hitchin map is $h : [(E, \Phi)] \mapsto -\lambda^2$. The base B of the Hitchin map is also a complex plane \mathbb{C} . Let $(dz)^2$ be the basis, and β be the coordinate of B , by a gauge transformation of $SL_2(\mathbb{C})$, we have $(L \oplus L^*, (\lambda, -\lambda)) \sim (L^* \oplus L, (-\lambda, \lambda))$. Then for any $\beta \neq 0$, the fiber $h^{-1}(\beta)$ is

$$h^{-1}(\beta) = \{(L, \lambda) \in \text{Jac}(\Sigma) \times \mathbb{C} \mid -\lambda^2 = \beta\} / \sim,$$

where $(L, \lambda) \sim (L^*, -\lambda)$. Therefore

Theorem 2.1.1. *The regular fiber (i.e. $\beta \neq 0$) of the $g = 1$, $SL_2(\mathbb{C})$ -Hitchin system is the Jacobian variety of Σ ,*

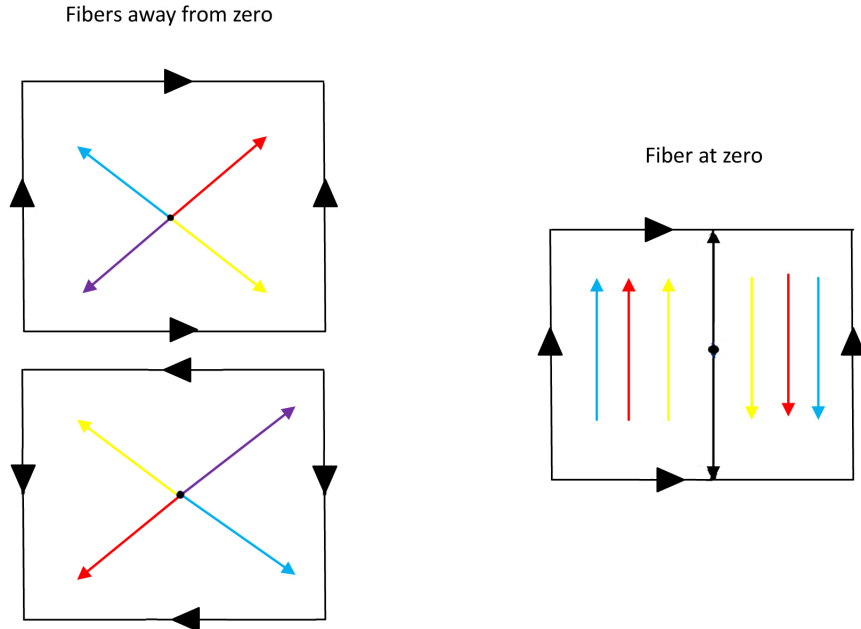
$$h^{-1}(\beta) \cong \text{Jac}(\Sigma)$$

When $\beta = 0$, the fiber is singular, given by

$$\begin{aligned} h^{-1}(0) &\cong \text{Jac}(\Sigma) / \sim \\ &\cong \mathbb{P}^1, \end{aligned}$$

where $L \sim L^*$.

The last \cong can be seen from the picture below, where the fiber at a point $\beta \neq 0$ is represented by the two tori in the left of the picture, and these two tori are identified completely by the relation \sim (for example, the lines with the same colour were attached). The fiber at $\beta = 0$ is represented by a torus, but half of the torus is identified with another half by \sim .



Now we study the regular part \mathcal{M}^{reg} of the Hitchin's fibration. Recall from section 1.3 that a tangent vector of \mathcal{M}^{reg} is a pair $(\alpha, \varphi) \in \Omega^{0,1}(\Sigma; \text{End}_0(E)) \times \Omega^{1,0}(\Sigma; \text{End}_0(E))$, and the symplectic form Ω on \mathcal{M}^{reg} is defined to be

$$\Omega((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_{\Sigma} \text{tr}(\alpha_1 \wedge \varphi_2) - \text{tr}(\alpha_2 \wedge \varphi_1). \quad (2.1)$$

In the case $g = 1$, we have $\Sigma \cong \text{Jac}(\Sigma)$. Let $\Sigma := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, and z be the coordinate of \mathbb{C} , $dz/(2i\text{Im}(\tau))$ be a basis of $H^0(\Sigma; K)$ with coordinate λ , then $\{z, \lambda\}$ are coordinates of $\text{Jac}(\Sigma) \times H^0(\Sigma; K)$. Note that \sim becomes $(z, \lambda) \sim (-z, -\lambda)$. A tangent vector on \mathcal{M}^{reg} is given by

$$p \frac{\partial}{\partial z} + q \frac{\partial}{\partial \lambda} = \left(\begin{bmatrix} p & 0 \\ 0 & -p \end{bmatrix} d\bar{z}, \begin{bmatrix} q & 0 \\ 0 & -q \end{bmatrix} \frac{dz}{2i\text{Im}(\tau)} \right)$$

where $\int_{\Sigma} dz \wedge d\bar{z} = -2i\text{Im}(\tau)$. Then formula 2.1 implies

$$\begin{aligned} \Omega \left(p \frac{\partial}{\partial z} + q \frac{\partial}{\partial \lambda}, m \frac{\partial}{\partial z} + n \frac{\partial}{\partial \lambda} \right) &= \frac{2pn - 2qm}{2i\text{Im}(\tau)} \int_{\Sigma} d\bar{z} \wedge dz \\ &= 2(pn - qm). \end{aligned}$$

Then we get

$$\Omega = 2dz \wedge d\lambda.$$

Note that Ω is invariant under $(z, \lambda) \sim (-z, -\lambda)$. Working locally on B , for any $\beta \in B$ and $\beta \neq 0$, there exists a locally defined holomorphic function λ on a neighbourhood U of β such that $\lambda^2 = -\beta$, then λ is local coordinate of B . Let

$$\begin{cases} z = z_1 + iz_2 \\ \lambda = x_1 + ix_2, \end{cases}$$

then

Theorem 2.1.2. *The special Kähler metric of B given by*

$$g_{sk} = 4\text{Im}(\tau)(dx_1 \otimes dx_1 + dx_2 \otimes dx_2).$$

Proof. We follow the procedure in [Hit99] also see section 1.8. First of all, define

$$\begin{aligned} \omega_1 &:= \text{Re}\Omega = 2(dz_1 \wedge dx_1 - dz_2 \wedge dx_2) \\ \omega_2 &:= \text{Im}\Omega = 2(dz_2 \wedge dx_1 + dz_1 \wedge dx_2). \end{aligned}$$

Let A, B be a symplectic basis of cycles for $H_1(\Sigma; \mathbb{Z})$, given by the path from z to $z + 1$ and z to $z + \tau$ respectively. Integrating ω_1, ω_2 along A, B we obtain four closed 1-forms of U :

$$\begin{cases} \left(\int_A 2dz_1 \right) dx_1 - \left(\int_A 2dz_2 \right) dx_2 = 2dx_1 \\ \left(\int_B 2dz_1 \right) dx_1 - \left(\int_B 2dz_2 \right) dx_2 = 2(\text{Re}(\tau)dx_1 - \text{Im}(\tau)dx_2) \end{cases}$$

$$\begin{cases} \left(\int_A 2dz_2 \right) dx_1 + \left(\int_A 2dz_1 \right) dx_2 = 2dx_2 \\ \left(\int_B 2dz_2 \right) dx_1 + \left(\int_B 2dz_1 \right) dx_2 = 2(\text{Im}(\tau)dx_1 + \text{Re}(\tau)dx_2). \end{cases}$$

They are differentials of the following functions on U :

$$\begin{cases} u_A(x_1, x_2) = 2\text{Re}(\lambda) \\ u_B(x_1, x_2) = 2\text{Re}(\tau\lambda) \end{cases}$$

$$\begin{cases} v_A(x_1, x_2) = 2\text{Im}(\lambda) \\ v_B(x_1, x_2) = 2\text{Im}(\tau\lambda) \end{cases}$$

According to Hitchin [Hit99], $u = (u_A, u_B), v = (v_A, v_B)$ are local diffeomorphisms onto the image. Then we get an embedding $w = (u, v) = (u_A, u_B, v_A, v_B) : U \rightarrow \mathbb{R}^4$. They are two systems of local coordinates on U , and we rewrite them to be $(u_A, u_B, v_A, v_B) := (s_1, y_1, s_2, y_2)$. Then we have

Theorem 2.1.3. *The image $w(U) \subset \mathbb{R}^4$ is a Lagrangian submanifold with respect to the symplectic form*

$$\Omega_1 = ds_1 \wedge dy_2 + ds_2 \wedge dy_1.$$

One can find the proof in [Hit99].

If we think of Ω_1 as the canonical form of \mathbb{R}^4 then $w(U)$ is the graph of a generating function

$$K : \mathbb{R}^2 \rightarrow \mathbb{R}$$

satisfying the Hamiltonian equations:

$$\begin{cases} \frac{\partial K}{\partial s_1} = y_2 \\ \frac{\partial K}{\partial y_1} = -s_2. \end{cases}$$

Since the graph $w(U)$ satisfies:

$$\begin{aligned} s_2 &= \frac{\text{Re}(\tau)s_1 - y_1}{\text{Im}(\tau)} \\ y_2 &= \frac{\text{Re}(\tau)^2 s_1 - \text{Re}(\tau)y_1}{\text{Im}(\tau)} + s_1 \text{Im}(\tau), \end{aligned}$$

solving the equations, we get

$$K(s_1, y_1) = \frac{\text{Re}(\tau)^2}{2\text{Im}(\tau)} s_1^2 - \frac{\text{Re}(\tau)}{\text{Im}(\tau)} y_1 s_1 + \frac{\text{Im}(\tau)}{2} s_1^2 + \frac{1}{2\text{Im}(\tau)} y_1^2.$$

Since $y_1 = \text{Re}(\tau)s_1 - \text{Im}(\tau)s_2$, we can rewrite K to be:

$$\begin{aligned} K(s_1, s_2) &= \frac{1}{2}(s_1^2 + s_2^2)\text{Im}(\tau) \\ &= 2(x_1^2 + x_2^2)\text{Im}(\tau) \\ &= 2|\lambda|^2\text{Im}(\tau), \end{aligned} \tag{2.2}$$

From Hitchin [Hit99] as well as Freed [Fre99], B has a special Kähler structure where K is the Kähler potential and the Kähler form is given locally by

$$\Omega_K = \sqrt{-1}\partial\bar{\partial}K = 2\sqrt{-1}\text{Im}(\tau)d\lambda \wedge d\bar{\lambda} = 4\text{Im}(\tau)dx_1 \wedge dx_2. \tag{2.3}$$

Now using the Kähler form of B , we get the special Kähler metric,

$$g_{sk} = 4\text{Im}(\tau)(dx_1 \otimes dx_1 + dx_2 \otimes dx_2). \tag{2.4}$$

□

Both Hitchin and Freed in their papers gave us a way to construct a hyperkähler metric on \mathcal{M}_{reg} (see section 1.8). The idea is that the special Kähler metric g_{sk} on the base B extends naturally to a hyperkähler metric on the cotangent bundle T^*B . Then we think of the fibers of the moduli space as the cotangent spaces quotiented out by the discrete action of $\{\mathbb{Z} + \tau\mathbb{Z}\}$, hence \mathcal{M}_{reg} inherits naturally a hyperkähler metric, written as g_{sf} . Using our convention of coordinates over U and formula 1.0, g_{sf} is given explicitly by

Theorem 2.1.4.

$$g_{sf}|_{\mathcal{M}_U} = 4\text{Im}(\tau)\left(\sum_{i=1}^2 dx_i \otimes dx_i\right) + \frac{1}{4\text{Im}(\tau)}\left(\sum_{i=1}^2 dz_i \otimes dz_i\right), \quad (2.5)$$

where the almost complex structure I, J, K are given as:

$$\begin{cases} I(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + b_1 \frac{\partial}{\partial z_1} + b_2 \frac{\partial}{\partial z_2}) = (-a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} - b_2 \frac{\partial}{\partial z_1} + b_1 \frac{\partial}{\partial z_2}) \\ J(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + b_1 \frac{\partial}{\partial z_1} + b_2 \frac{\partial}{\partial z_2}) = (-b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + a_1 \frac{\partial}{\partial z_1} - a_2 \frac{\partial}{\partial z_2}) \\ K = IJ. \end{cases}$$

We can also change our coordinates to $\beta = -\lambda^2 = \beta_1 + i\beta_2$ on the base B to see the global picture. Recall that β is a globally defined coordinate on B whereas λ is only locally defined away from $\beta = 0$. Note that $\beta = -\lambda^2 = -(x_1^2 - x_2^2) - i2x_1x_2$, then

$$\begin{cases} \frac{\partial}{\partial x_1} = -2x_1 \frac{\partial}{\partial \beta_1} - 2x_2 \frac{\partial}{\partial \beta_2} \\ \frac{\partial}{\partial x_2} = 2x_2 \frac{\partial}{\partial \beta_1} - 2x_1 \frac{\partial}{\partial \beta_2}, \end{cases}$$

hence

$$\begin{cases} \frac{\partial}{\partial \beta_1} = -\frac{1}{2(x_1^2 + x_2^2)}(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}) \\ \frac{\partial}{\partial \beta_2} = -\frac{1}{2(x_1^2 + x_2^2)}(x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}). \end{cases}$$

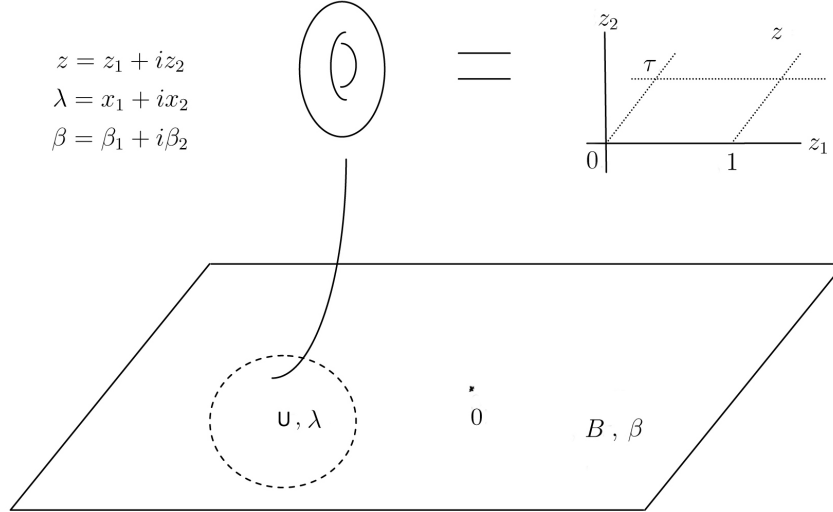
Then by direct computation, we get

$$\sum_{i=1}^2 dx_i \otimes dx_i = \frac{1}{4|\beta|} \left(\sum_{i=1}^2 d\beta_i \otimes d\beta_i \right)$$

Therefore,

$$\begin{aligned} g_{sk}|_U &= \text{Im}(\tau) \frac{\sum_{i=1}^2 (d\beta_i \otimes d\beta_i)}{|\beta|}, \\ g_{sf}|_{\mathcal{M}_U} &= \text{Im}(\tau) \frac{\sum_{i=1}^2 (d\beta_i \otimes d\beta_i)}{|\beta|} + \frac{1}{4\text{Im}(\tau)} \sum_{i=1}^2 (dz_i \otimes dz_i). \end{aligned} \quad (2.6)$$

We can see that when β goes to zero, the metric blows up. However the metric is flat when restricted to the fibers. The diagram shows the coordinates we are using for the moduli space.



2.2 $n > 2$ Moduli spaces

Now we study the $SL_n(\mathbb{C})$ -moduli spaces for $n > 2$, and use similar methods to derive the semi-flat metric. Let (E, Φ) be a rank n Higgs bundle, since Φ is traceless and E is a vector bundle with trivial determinant, then

$$(E, \Phi) \sim (L_1 \oplus L_2 \oplus \cdots \oplus L_n, (\lambda_1, \lambda_2, \cdots, \lambda_n)),$$

for $\lambda_i \in H^0(\Sigma; K)$ the eigen-values of Φ and L_i the λ_i -eigen-subbundle of E . Note that $\bigotimes_{i=1}^n L_i = \mathcal{O}$, $\sum_{i=1}^n \lambda_i = 0$, because $\det(E) = \mathcal{O}$, $\text{tr}\Phi = 0$. Then we have

Theorem 2.2.1. *For any $\beta \in B$ such that the eigen-values λ_i are different,*

$$h^{-1}(\beta) \cong \text{Jac}(\Sigma)^{n-1}.$$

Next we will compute the special Kähler metric on B^{reg} and the semi-flat metric of \mathcal{M}_{reg} . To do that, we need to write down the symplectic form of \mathcal{M}_{reg} explicitly. First of all, we rewrite the eigen-values λ_i to be $\lambda_i e, i = 1, \cdots, n-1$ where $\lambda_i \in \mathbb{C}$ and e is a basis of $H^0(\Sigma, K)$. Then analogous to the last section, $\{\lambda_i\}$ are the coordinates of a neighbourhood of β . Let z^i be the coordinates of the copy of $\text{Jac}(\Sigma) \cong \mathbb{C}/\{\mathbb{Z} + \tau\mathbb{Z}\}$ corresponding to λ_i .

Let $(\alpha, \varphi) \in \Omega^{0,1}(\Sigma; \text{End}_0(E)) \times \Omega^{1,0}(\Sigma; \text{End}_0(E))$ be a tangent vector of \mathcal{M}_{reg} , the symplectic form Ω on \mathcal{M}_{reg} is defined to be

$$\Omega((\alpha^1, \varphi^1), (\alpha^2, \varphi^2)) = \int_{\Sigma} \text{tr}(\alpha^1 \wedge \varphi^2) - \text{tr}(\alpha^2 \wedge \varphi^1).$$

More explicitly, if

$$\alpha^j = \begin{bmatrix} \alpha_1^j & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{n-1}^j & 0 \\ 0 & 0 & \cdots & 0 & -\sum_{i=1}^{n-1} \alpha_i^j \end{bmatrix} \quad \varphi^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1}^j & 0 \\ 0 & 0 & \cdots & 0 & -\sum_{i=1}^{n-1} \lambda_i^j \end{bmatrix}$$

for $j = 1, 2$, then choosing a suitable basis for $\Omega^{1,0}(\Sigma)$ and $\Omega^{0,1}(\Sigma)$, we have

$$\begin{aligned} \Omega((\alpha^1, \varphi^1), (\alpha^2, \varphi^2)) &= \int_{\Sigma} \text{tr}(\alpha^1 \wedge \varphi^2) - \text{tr}(\alpha^2 \wedge \varphi^1) \\ &= \sum_{i=1}^{n-1} (a_i^1 b_i^2) - \sum_{i=1}^{n-1} (a_i^2 b_i^1) + \sum_{i,j=1}^{n-1} (a_i^1 b_j^2 - a_i^2 b_j^1), \end{aligned}$$

where a_i^p, b_i^p are coordinates with respect to $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \lambda^i}$ for $p = 1, 2$. Therefore the symplectic form on the moduli space is

$$\Omega = \sum_{i=1}^{n-1} dz^i \wedge d\lambda_i + \sum_{i,j=1}^{n-1} dz^i \wedge d\lambda_j.$$

Let $A_1, B_1, \dots, A_{n-1}, B_{n-1}$ be a symplectic basis of cycles for $H_1(\Sigma^{n-1}; \mathbb{Z})$, and write

$$\begin{cases} \lambda_j = x_1^j + ix_2^j \\ z^j = z_1^j + iz_2^j. \end{cases}$$

Then the real and imaginary parts of the symplectic form are

$$\begin{cases} \omega_1 = \text{Re}\Omega = \sum_{i=1}^{n-1} (dz_1^i \wedge dx_1^i - dz_2^i \wedge dx_2^i) + \sum_{i,j=1}^{n-1} (dz_1^i \wedge dx_1^j - dz_2^i \wedge dx_2^j) \\ \omega_2 = \text{Im}\Omega = \sum_{i=1}^{n-1} (dz_2^i \wedge dx_1^i + dz_1^i \wedge dx_2^i) + \sum_{i,j=1}^{n-1} (dz_2^i \wedge dx_1^j + dz_1^i \wedge dx_2^j). \end{cases}$$

Analogous to the previous computation, we obtain two sets of functions,

$$\begin{cases} u_{A_k}(x_1^1, x_2^1, \dots, x_1^{n-1}, x_2^{n-1}) = \text{Re}(\lambda_k) + \sum_{j=1}^{n-1} \text{Re}(\lambda_j) \\ u_{B_k}(x_1^1, x_2^1, \dots, x_1^{n-1}, x_2^{n-1}) = \text{Re}(\tau\lambda_k) + \sum_{j=1}^{n-1} \text{Re}(\tau\lambda_j) \\ v_{A_k}(x_1^1, x_2^1, \dots, x_1^{n-1}, x_2^{n-1}) = \text{Im}(\lambda_k) + \sum_{j=1}^{n-1} \text{Im}(\lambda_j) \\ v_{B_k}(x_1^1, x_2^1, \dots, x_1^{n-1}, x_2^{n-1}) = \text{Im}(\tau\lambda_k) + \sum_{j=1}^{n-1} \text{Im}(\tau\lambda_j). \end{cases}$$

Each set of functions gives a local diffeomorphism of B to $\mathbb{R}^{2(n-1)}$. Let $w = (u, v)$, and $s_1^j, y_1^j, s_2^j, y_2^j$ be the coordinates corresponding to $u_{A_j}, u_{B_j}, v_{A_j}, v_{B_j}$, then the image $w(U) \subset \mathbb{R}^{4(n-1)}$ is a Lagrangian submanifold with respect to the symplectic form [Hit99]

$$\Omega_1 = \sum_{j=1}^{n-1} (ds_1^j \wedge dy_2^j + ds_2^j \wedge dy_1^j).$$

The image $w(U)$ is the graph of a generating function

$$K : \mathbb{R}^{2(n-1)} \rightarrow \mathbb{R}$$

satisfying the Hamiltonian equations:

$$\begin{cases} \frac{\partial K}{\partial s_1^j} = y_2^j \\ \frac{\partial K}{\partial y_1^j} = -s_2^j. \end{cases}$$

Solving the equations, we get

$$K(s_1^1, s_2^1, \dots, s_1^{n-1}, s_2^{n-1}, y_1^1, \dots, y_1^{n-1}) = \sum_{j=1}^{n-1} \left(\frac{\operatorname{Re}(\tau)^2}{2\operatorname{Im}(\tau)} s_1^{j2} - \frac{\operatorname{Re}(\tau)}{\operatorname{Im}(\tau)} y_1^j s_1^j + \frac{\operatorname{Im}(\tau)}{2} s_1^{j2} + \frac{1}{2\operatorname{Im}(\tau)} y_1^{j2} \right).$$

Since $y_1^j = \operatorname{Re}(\tau)s_1^j - \operatorname{Im}(\tau)s_2^j$, restricting K to the neighbourhood of the base B , we get

$$K(s_1^1, s_2^1, \dots, s_1^{n-1}, s_2^{n-1}) = \frac{1}{2} \sum_{j=1}^{n-1} (s_1^{j2} + s_2^{j2}) \operatorname{Im}(\tau) \quad (2.7)$$

hence by [Fre99], the Kähler form is

$$\begin{aligned} \Omega_K &= \sqrt{-1} \partial \bar{\partial} K \\ &= \frac{1}{2} \sqrt{-1} \operatorname{Im}(\tau) \sum_{j=1}^{n-1} d(s_1^j + i s_2^j) \wedge d(s_1^j - i s_2^j) \\ &= \operatorname{Im}(\tau) \sum_{j=1}^{n-1} ds_1^j \wedge ds_2^j. \end{aligned}$$

Therefore

Theorem 2.2.2. *The special Kähler metric is given by*

$$g_{sk} = \sum_{j=1}^{n-1} \operatorname{Im}(\tau) (ds_1^j \otimes ds_1^j + ds_2^j \otimes ds_2^j). \quad (2.8)$$

Since $ds_p^j = dx_p^j + \sum_{q=1}^{n-1} (dx_p^q)$ for $p = 1, 2$, then by using $\{x_1^k, x_2^k\}$, the real and imaginary part of the affine coordinates $\{\lambda_k\}$ the special Kähler metric can be rewritten as

$$g_{sk} = \operatorname{Im}(\tau) \sum_{i,j=1}^{n-1} g_{ij} (dx_1^i \otimes dx_1^j + dx_2^i \otimes dx_2^j). \quad (2.9)$$

with

$$g_{ij} = \begin{cases} n+1, & i \neq j, \\ n+2, & i = j. \end{cases}$$

By formula 1.0, the semi-flat metric of $\mathcal{M}|_U$ is given by

$$g_{sf} = \text{Im}(\tau) \sum_{i,j=1}^{n-1} g_{ij} (dx_1^i \otimes dx_1^j + dx_2^i \otimes dx_2^j) + \sum_{i,j=1}^{n-1} (\text{Im}(\tau) g_{ij})^{-1} (dz_1^i \otimes dz_1^j + dz_2^i \otimes dz_2^j). \quad (2.10)$$

Up to now, we have derived explicit formulas of semi-flat metrics for the smooth part of $SL_n(\mathbb{C})$ moduli spaces for all n when $g = 1$. One can check that the formulas for $n = 2$ coincide with the formulas we obtained in this section.

Chapter 3

The $GL_n(\mathbb{C})$ -moduli spaces for $g \geq 2$

3.1 The symplectic form and the canonical 1-form on the moduli spaces

In this section we will have a closer look at the symplectic form and the canonical 1-form on $T^*\mathcal{N}$. We will see later that the two sets of coordinates on the base given in the last chapter (also in [Fre99]) is the integration of the canonical 1-form along the basis of cycles of the fibers.

From Hitchin [Hit87a] and section 1.2, the cotangent space of the moduli space of stable bundles $T^*\mathcal{N}$ is an open dense subset of \mathcal{M} . If μ is the canonical one form on $T^*\mathcal{N}$, then the symplectic form

$$\Omega((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_{\Sigma} \text{tr}(\alpha_1 \wedge \varphi_2) - \text{tr}(\alpha_2 \wedge \varphi_1).$$

on \mathcal{M} equals to $-d\mu$ when restricting to $T^*\mathcal{N}$.

Proposition 3.1.1. *The canonical 1-form μ on $T^*\mathcal{N}$ is given by:*

$$\mu((\alpha, \varphi)) = \int_{\Sigma} \text{tr}(\alpha \wedge \Phi).$$

Proof. Let $\pi : T^*\mathcal{N} \rightarrow \mathcal{N}$ be the projection. Then for any point $(E, \Phi) \in T^*\mathcal{N}$ with $\pi(E, \Phi) = E$, the push forward of π gives a mapping:

$$T_{(E, \Phi)} T^*\mathcal{N} \xrightarrow{\pi_*} T_E \mathcal{N} \xrightarrow{\Phi} \mathbb{R}.$$

Recall from symplectic geometry that the canonical 1-form is defined by $\mu_{(E, \Phi)} = \Phi \circ \pi_*$. Therefore, for (α, φ) in the tangent space of (E, Φ) , the canonical 1-form applied to (α, φ) gives

$$\begin{aligned} \mu((\alpha, \varphi)) &= \Phi \circ \pi_*(\alpha, \varphi) \\ &= \Phi \cdot \alpha \\ &= \int_{\Sigma} \text{tr}(\alpha \wedge \Phi), \end{aligned}$$

where the last equality is given by Serre duality. □

In the cases of $g = 1$, using our construction of the $SL_n(\mathbb{C})$ moduli spaces in chapter 2, the canonical 1-form is $\mu = \sum_{i=1}^{n-1} \lambda_i dz^i + \sum_{i,j=1}^{n-1} \lambda_j dz^i$. Note that $\Omega = \sum_{i=1}^{n-1} dz^i \wedge d\lambda_i + \sum_{i,j=1}^{n-1} dz^i \wedge d\lambda_j$ and $\Omega = -d\mu$. Now we can see that if F is a fiber of the Hitchin map, and $C \in H_1(F, \mathbb{Z})$ is a cycle, then

$$\begin{aligned} d\left(\int_C \mu\right) &= d\left(\int_C \lambda_i dz^i\right) + \sum_{j=1}^{n-1} d\left(\int_C \lambda_j dz^i\right) \\ &= \left(\int_C dz^i\right) d\lambda_i + \sum_{j=1}^{n-1} \left(\int_C dz^i\right) d\lambda_j \\ &= \int_C \Omega. \end{aligned}$$

In fact, it is also true for $g \geq 2$.

Proposition 3.1.2. *Let M be a symplectic manifold, $\{x_i, y_i\}$ be local coordinates so that $\{y_i\}$ are coordinates on the base B , $\{x_i\}$ are coordinates on the fiber F , and let $M \rightarrow B$ be an integrable system given by (y_1, \dots, y_n) then*

$$\int_C \Omega = d\left(\int_C \mu\right),$$

where $C \in H_1(F, \mathbb{Z})$ is any cycle, and the left hand side of the formula is an integral over the fiber of the circle bundle defined locally over B with fibers representing the cycle C .

The proof is a direct computation. Since the fibers are Lagrangian, and $\{x_i\}$ are coordinates of the fibers, therefore we can write $\Omega = \sum_{i,j} a_{ij} dx_i \wedge dy_j + b_{ij} dy_i \wedge dy_j$, for some functions a_{ij}, b_{ij} . Write $\mu = \sum_i e_i dx_i + f_j dy_j$, then since $\Omega = -d\mu$, we have

$$\begin{aligned} a_{ij} &= \frac{\partial e_i}{\partial y_j} - \frac{\partial f_j}{\partial x_i} \\ b_{ij} &= -\frac{\partial f_j}{\partial y_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} d\left(\int_C \mu\right) &= \sum_i d\left(\int_C e_i dx_i\right) \\ &= \sum_{ij} \frac{\partial\left(\int_C e_i dx_i\right)}{\partial y_j} dy_j \\ &= \sum_{ij} \left(\int_C \frac{\partial e_i}{\partial y_j} dx_i - \int_C \frac{\partial f_j}{\partial x_i} dx_i\right) dy_j \\ &= \int_C \Omega, \end{aligned}$$

which completes the proof.

3.2 The fibers of the Hitchin systems

When the genus of the Riemann surface is greater than one, the Hitchin systems are more complicated because the Higgs fields are not diagonalizable in general, and vector bundles E do not decompose into direct sums of line bundles. Recall that $\dim \mathcal{N} = n^2(g-1) + 1$ for the moduli space of $GL_n(\mathbb{C})$ vector bundles, and $\dim \mathcal{N} = n^2(g-1)$ for the $SL_n(\mathbb{C})$ [Hit87b]. Therefore almost all vector bundles in \mathcal{N} can not decompose as the direct sum of line bundles because

$$\dim(\underbrace{\text{Jac}(\Sigma) \times \cdots \times \text{Jac}(\Sigma)}_n) = ng,$$

which is less than the dimension of \mathcal{N} . Hence we can not use the same method as in chapter 2.

To understand the $g \geq 2$ Hitchin systems, we first study the fibers. Let (E, Φ) be a point in the $GL_n(\mathbb{C})$ moduli space \mathcal{M} , and $\det(x - \Phi) = x^n + a_1x^{n-1} + \cdots + a_n$ be the characteristic polynomial, where $a_i \in H^0(\Sigma; K^i)$. Let $\beta = h((E, \Phi)) \in B$, and λ be the tautological section of $T^*(K)$, i.e. if $\pi : K \rightarrow \Sigma$ is the projection, then $\lambda \in H^0(K; \pi^*K)$ is a section such that $\lambda(p) = \pi^*(p)$ for $p \in K$. We call the zero set of $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n \in H^0(K, \pi^*K^n)$ the **spectral curve** S_β . For almost all β , S_β is non-singular, and we focus on the non-singular spectral curves.

Now, we let $\pi^*\Phi$ be the pull back of Φ to S_β , then $\lambda \in H^0(S_\beta; \pi^*K)$ is an eigenvalue of $\pi^*\Phi$ and let L be the eigen-subbundle of $\pi^*(E)$ over S . If S_β is non-singular, then L is a line bundle on S . Suppose $g \in \Omega^0(\Sigma, \text{End}(E))$ is a gauge transformation, then for $x \in S$, $(x, \xi) \in L$,

$$g^{-1}\Phi_{\pi(x)}g \cdot g^{-1}\xi g = \lambda_x g^{-1}\xi g.$$

We can do this to Higgs bundles in other degrees. Therefore, we have constructed a map

$$\mathcal{F} : A_\beta \rightarrow \text{Pic}(S_\beta),$$

with $\mathcal{F}((E, \Phi)) = L$, where A_β is the set of isomorphism classes of rank n Higgs bundles with characteristic polynomial given by β .

Next, let's try to recover (E, Φ) from a line bundle $L \in \text{Pic}(S_\beta)$. Let \mathcal{O}_L be the sheaf of sections of L , since the projection $\pi : S \rightarrow \Sigma$ is continuous, we therefore obtain an image sheaf $\mathcal{O}_E := \pi_*\mathcal{O}_L$ of the direct image functor π_* , i.e. $\mathcal{O}_E(U) := \mathcal{O}_L(\pi^{-1}(U))$.

If $z_0 \in \Sigma$ is not a ramification point, we can choose a small neighbourhood U of z_0 such that $\pi^{-1}(U)$ is a disjoint union of connected open sets:

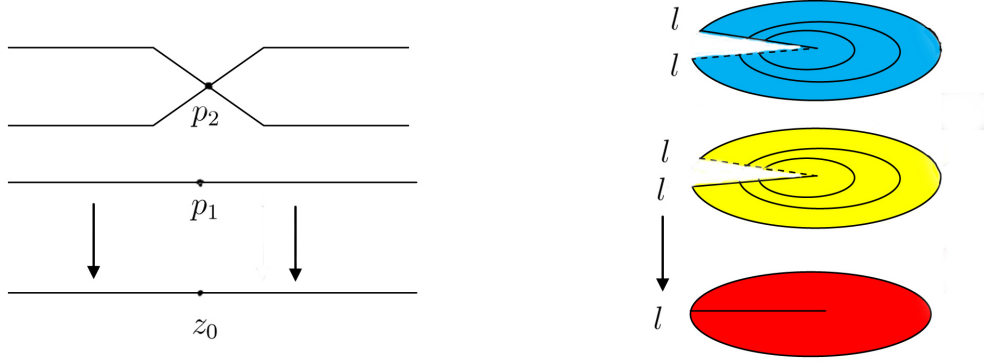
$$\pi^{-1}(U) = V_1 \sqcup \cdots \sqcup V_n.$$

Then

$$\mathcal{O}_E(U) = \bigoplus_{i=1}^n \mathcal{O}_L(V_i) \cong \bigoplus_{i=1}^n \mathcal{O}(U).$$

Therefore \mathcal{O}_E is locally free of rank n away from ramification points.

When z_0 is a ramification point, let $p_i \in \pi^{-1}(z_0)$, $i = 1, \dots, k$ be the branch points with ramification index d_i (To make things convenient, we allow d_i to be 1), then $\sum_{i=1}^k d_i = n$.



Choose a small neighbourhood U of z_0 , such that V_i are disjoint neighbourhoods of p_i . Let z be the coordinate of U , then we can choose coordinates w_i of V_i such that

$$z = w_i^{d_i}.$$

Locally, we have $\mathcal{O}_L(V_i) \cong \mathcal{O}(V_i)$, then

$$\mathcal{O}_E(U) = \mathcal{O}(\pi^{-1}(U)) \cong \bigoplus_{i=1}^k \mathcal{O}(V_i).$$

If $f(w_i)$ is a holomorphic function on V_i , then there exist unique holomorphic functions $f_1(z), \dots, f_{d_i}(z)$ on U such that $f(w_i) = f_1(z) + w_i f_2(z) + \dots + w_i^{d_i-1} f_{d_i}(z)$. Therefore,

$$\mathcal{O}_E(U) \cong \bigoplus_{i=1}^n \mathcal{O}(U) \quad (3.0)$$

Hence \mathcal{O}_E is locally free of rank n , and given $L \in \text{Jac}_d(S_\beta)$, E defined by $\mathcal{O}_E = \pi_*(\mathcal{O}_L)$ is a rank n holomorphic vector bundle.

Now we define the Higgs field Φ to be an element of $H^0(\Sigma; \text{End}(E) \otimes K)$ such that the diagram below commutes for any U .

$$\begin{array}{ccc} \mathcal{O}_E(U) & \xrightarrow{\Phi} & \mathcal{O}_{E \otimes K}(U) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_L(\pi^{-1}(U)) & \xrightarrow{\otimes \lambda} & \mathcal{O}_{L \otimes \pi^*(K)}(\pi^{-1}(U)) \end{array}$$

Then we obtain a map

$$\mathcal{J} : \text{Pic}(S_\beta) \rightarrow A_\beta,$$

that sends $L \rightarrow (E, \Phi)$.

Since λ is the eigenvalue of $\pi^*\Phi$, then by the definition of \mathcal{F} , we have $\mathcal{F} \circ \mathcal{J} = \text{Id}$. From [BNR89] proposition 3.6, the map \mathcal{J} is a bijection. Therefore

Proposition 3.2.1. \mathcal{F} is a homeomorphism.

From the Grothendieck-Riemann-Roch theorem, when $\mathcal{O}_E = \pi_*(\mathcal{O}_L)$, we have

$$\deg(E) = \deg(L) + (1 - g_s) - n(1 - g),$$

hence the degree of L is determined by the degree of E . If we restrict \mathcal{F} to $h^{-1}(\beta)$ (i.e. fixing the degree of Higgs bundles), then

Proposition 3.2.2. $h^{-1}(\beta) \cong \text{Jac}_d(S_\beta)$,

where $d = \deg(E) + n(n - 1)(g - 1)$.

3.3 The two sets of affine coordinates on the base

In this part we derive the two sets of affine coordinates on the base of the Hitchin systems. Recall from last section that the coordinates on the base are given by the integration of the canonical 1-form: $\int_C \mu$. Let us fix the degree of the line bundle L to be d or equivalently, fix the degree of the Higgs bundles to be $d - n(n - 1)(g - 1)$.

Let $F := \text{Jac}_d(S)$ be a regular fiber, $\{a_1, b_1, \dots, a_{g_s}, b_{g_s}\}$ be a basis of $H_1(S, \mathbb{Z})$. Then the Abel-Jacobi map gives a basis $\{A_1, B_1, \dots, A_{g_s}, B_{g_s}\}$ of $H_1(F, \mathbb{Z})$, where A_i, B_i correspond to a_i, b_i respectively, hence $H_1(S, \mathbb{Z}) \cong H_1(F, \mathbb{Z})$. From the universal coefficient theorems, we have $H^1(S, \mathbb{R}) \cong H^1(F, \mathbb{R})$.

Proposition 3.3.1. *If $\theta \in H^0(S, K_S)$ is the canonical 1-form on $T^*\Sigma|_S$ given by $\theta = d\pi(\lambda)$, where $d\pi \in H^0(S; \text{Hom}(K_S^{-1}, \pi^*K_\Sigma^{-1})) = H^0(S; \text{Hom}(\pi^*K_\Sigma, K_S))$ is the derivative of π , then*

$$\int_{A_i} \mu = \int_{a_i} \theta \quad \text{and} \quad \int_{B_i} \mu = \int_{b_i} \theta.$$

Proof. Since $F = \text{Jac}(S) \cong H^1(S; \mathcal{O})/H^1(S; \mathbb{Z})$, then $TF = H^1(S; \mathcal{O})$. Hence a $(0,1)$ -form $[\alpha] \in H^1(S; \mathcal{O})$ generates a 1-parameter family of line bundles $\{L_t\}$ in $\text{Jac}(S)$, given by their holomorphic structures $\bar{\partial}_L + t\alpha$.

Let $(\dot{A}, \dot{\Phi}) = d\mathcal{J}(\alpha)$, since the set of branch points has measure zero, they do not contribute to integrals over Σ or S . Recall that in an open neighbourhood U that does not contain a ramification point in Σ , $E|_U$ given by $\mathcal{O}_E(U)$ is the direct sum $\mathcal{O}_L(V_1) \oplus \dots \oplus \mathcal{O}_L(V_n)$. In other words,

$$\dot{A} = \begin{bmatrix} \alpha|_{V_1} & & & \\ & \alpha|_{V_2} & & \\ & & \ddots & \\ & & & \alpha|_{V_n} \end{bmatrix} \quad \dot{\Phi} = \begin{bmatrix} \lambda|_{V_1} & & & \\ & \lambda|_{V_2} & & \\ & & \ddots & \\ & & & \lambda|_{V_n} \end{bmatrix}.$$

Choose an open cover U^i of Σ (after removing the ramification points), let ϕ be the

partition of unity, then

$$\begin{aligned}
\mu((\dot{A}, \dot{\Phi})) &= \int_{\Sigma} \text{tr}(\dot{A}\dot{\Phi}) \\
&= \sum_i \phi_i \int_{U^i} \text{tr}(\dot{A}\dot{\Phi}) \\
&= \sum_i \phi_i \left(\sum_{j=1}^n \int_{U^i} \alpha|_{V_j^i} \cdot \lambda|_{V_j^i} \right) \\
&= \sum_i \phi_i \left(\sum_{j=1}^n \int_{V_j^i} \alpha(d\pi)(\lambda) \right) \\
&= \int_S \alpha\theta.
\end{aligned}$$

Therefore, $\mu((\dot{A}, \dot{\Phi})) = \theta(\alpha)$. This shows $\mathcal{J}^*(\mu) = \theta$ hence we complete the proof. \square

Therefore, the two sets of local coordinates of the base B are given by:

$$\left\{ \text{Re}\left(\int_{a_i} \theta\right), \text{Im}\left(\int_{a_i} \theta\right) \right\}, \left\{ \text{Re}\left(\int_{b_i} \theta\right), \text{Im}\left(\int_{b_i} \theta\right) \right\}$$

for $i = 1, \dots, g_s$, where g_s is the genus of S which can be computed by $g_s = \dim(\text{Jac}(S)) = \dim \mathcal{N} = n^2(g-1) + 1$.

3.4 The Metrics

We can compute the metrics by using a procedure similar to chapter 2. Let $\{s_1^i, s_2^i, y_1^i, y_2^i\}$ be the coordinates of \mathcal{R}^{4g_s} corresponding to $\left\{ \text{Re}\left(\int_{a_i} \theta\right), \text{Im}\left(\int_{a_i} \theta\right), \text{Re}\left(\int_{b_i} \theta\right), \text{Im}\left(\int_{b_i} \theta\right) \right\}$, let $\lambda_j = s_1^j + is_2^j$, $y^j = y_1^j + iy_2^j$. Locally, the graph in \mathcal{R}^{4g_s} is a Lagrangian submanifold with respect to the symplectic form

$$\Omega_1 = \sum_{j=1}^{g_s} (ds_1^j \wedge dy_2^j + ds_2^j \wedge dy_1^j).$$

Now choose a basis w_1, \dots, w_{g_s} of the space of holomorphic one forms $\Omega^1(S)$, such that

$$A = \begin{bmatrix} \int_{a_1} w_1 & \cdots & \int_{a_{g_s}} w_1 \\ \vdots & & \vdots \\ \int_{a_1} w_{g_s} & \cdots & \int_{a_{g_s}} w_{g_s} \end{bmatrix} = Id$$

Let the period matrix of S be

$$\tau = B = \begin{bmatrix} \int_{b_1} w_1 & \cdots & \int_{b_{g_s}} w_1 \\ \vdots & & \vdots \\ \int_{b_1} w_{g_s} & \cdots & \int_{b_{g_s}} w_{g_s} \end{bmatrix},$$

then we have $y^i = \sum_j \tau_{ji} \lambda_j$, therefore

$$\begin{cases} y_2^i = \sum_j \operatorname{Re}(\tau_{ji}\lambda_j) = \sum_j (\operatorname{Re}(\tau_{ji})s_1^j - \operatorname{Im}(\tau_{ji})s_2^j) \\ s_2^i = \sum_j \operatorname{Im}(\tau_{ji}\lambda_j) = \sum_j (\operatorname{Re}(\tau_{ji})y_2^j + \operatorname{Im}(\tau_{ji})y_1^j). \end{cases}$$

The Kähler potential K is a solution of

$$\begin{cases} \frac{\partial K}{\partial s_1^j} = y_2^j \\ \frac{\partial K}{\partial y_1^j} = -s_2^j. \end{cases}$$

Now we solve the equations to obtain the Kähler potential

$$\begin{aligned} K(s_1^1, s_1^2, \dots, s_1^{g_s}, s_2^1, \dots, s_2^{g_s}) &= \frac{1}{2} \sum_j^{g_s} \operatorname{Im}(y^j \bar{\lambda}_j) \\ &= \frac{1}{2} \sum_{i,j}^{g_s} ((s_1^j s_1^i + s_2^j s_2^i) \operatorname{Im}(\tau_{ji})) \\ &= \frac{1}{4} \sum_{i,j}^{g_s} \operatorname{Im}(\tau_{ji}) \left(\int_{a_i} \theta \int_{a_j} \bar{\theta} + \int_{a_i} \bar{\theta} \int_{a_j} \theta \right). \end{aligned}$$

Hence the Kähler form can be computed by

$$\begin{aligned} \Omega_K &= \sqrt{-1} \partial \bar{\partial} K \\ &= \frac{\sqrt{-1}}{4} \sum_{i,j}^{g_s} (\operatorname{Im}(\tau_{ji}) (d(s_1^i + is_2^i) \wedge d(s_1^j - is_2^j) + d(s_1^j + is_2^j) \wedge d(s_1^i - is_2^i))) \\ &= \frac{1}{2} \sum_{i,j}^{g_s} \operatorname{Im}(\tau_{ji}) (ds_1^i \wedge ds_2^j + ds_1^j \wedge ds_2^i) \\ &= \sum_{i,j}^{g_s} \operatorname{Im}(\tau_{ji}) ds_1^i \wedge ds_2^j. \end{aligned} \tag{3.1}$$

Theorem 3.4.1.

The special Kähler metric and semi-flat metrics are given by:

$$g_{sk} = \sum_{i,j}^{g_s} \operatorname{Im}(\tau_{ij}) (ds_1^i \otimes ds_1^j + ds_2^i \otimes ds_2^j), \tag{3.2}$$

$$g_{sf} = \sum_{i,j=1,\dots,g_s; k=1,2} g_{ij} ds_k^i \otimes ds_k^j + g_{ij}^{-1} d\lambda_k^i \otimes d\lambda_k^j, \tag{3.3}$$

where g_{ij} are the coefficients of $ds_1^i \otimes ds_1^j$ in g_{sk} , g_{ij}^{-1} is the inverse matrix of g_{ij} and λ_k^i are the coordinates correspond to ds_k^i .

3.5 Some results on the affine coordinates

There are a few good results concerning the affine coordinates $\{\lambda_i := \int_{a_i} \theta\}$, $\{y^i := \int_{b_i} \theta\}$ of the base B , and the periods of the fiber τ_{ij} . Recall that $y^j = \tau_{ij} \lambda_i$

Proposition 3.5.1.

$$\sum_{j=1}^{gs} \left(\frac{\partial \tau_{ij}}{\partial \lambda_k} - \frac{\partial \tau_{kj}}{\partial \lambda_i} \right) \lambda_j = 0. \quad (3.4)$$

Proof. Hitchin showed that the base $B \subset \mathbb{R}^{4gs}$ is Lagrangian with respect to Ω_1 and

$$\Omega_2 = \sum_{i=1}^{gs} ds_1^i \wedge dy_1^i - ds_2^i \wedge dy_2^i.$$

Note that $\Omega_2 + i\Omega_1 = \sum_{j=1}^{gs} d\lambda_j \wedge dy^j$, hence $\sum_{j=1}^{gs} d\lambda_j \wedge dy^j = 0$ on B . Since $y^i = \sum_{j=1}^{gs} \tau_{ji} \lambda_j$, then

$$dy^i = \sum_{k=1}^{gs} d\lambda_k \left(\tau_{ik} + \sum_{j=1}^{gs} \frac{\partial \tau_{ij}}{\partial \lambda_k} \lambda_j \right).$$

Therefore,

$$0 = \sum_{i=1}^{gs} d\lambda_i \wedge dy^i = \sum_{i,k=1}^{gs} \tau_{ik} d\lambda_i \wedge d\lambda_k + \sum_{i,k=1}^{gs} d\lambda_i \wedge d\lambda_k \left(\sum_{j=1}^{gs} \frac{\partial \tau_{ij}}{\partial \lambda_k} \lambda_j \right).$$

Hence we have

$$\sum_{j=1}^{gs} \left(\frac{\partial \tau_{ij}}{\partial \lambda_k} - \frac{\partial \tau_{kj}}{\partial \lambda_i} \right) \lambda_j = 0,$$

which completes the proof. \square

Proposition 3.5.2.

$$\sum_{j=1}^{gs} \frac{\partial \tau_{ij}}{\partial \lambda_k} \lambda_k = 0. \quad (3.5)$$

Proof. Define the \mathbb{C}^* -action on Higgs bundles given by

$$t \cdot (E, \Phi) = (E, t\Phi)$$

for $t \in \mathbb{C}^*$. Then the action induces a \mathbb{C}^* -action on the base $B = \bigoplus_{j=1}^n H^0(\Sigma; K^j)$ given by

$$t \cdot (a_1, a_2, \dots, a_n) = (ta_1, t^2 a_2, \dots, t^n a_n)$$

where a_1, \dots, a_n are the coefficients of the characteristic polynomial. Hence the \mathbb{C}^* -action sends a point (λ, p) in the spectral curve $S_b := \{\lambda|\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0\}$ to $(t\lambda, p)$ in the spectral curve $S_{tb} := \{\lambda|\lambda^n + ta_1\lambda^{n-1} + \dots + t^n a_n = 0\}$, for $b \in B$. By abuse of notation, also let λ be the tautological 1-form of $T^*\Sigma$, (i.e. $\lambda_{(\lambda,p)} = \lambda$), then given $t \in \mathbb{C}^*$, the pull back of the action by t is given by

$$(t^*\lambda)_{(\lambda,p)} = \lambda_{(t(\lambda,p))} = \lambda_{(t\lambda,p)} = t\lambda.$$

Hence, $t^*(\lambda) = t\lambda$. Since the canonical 1-form $\theta = d\pi\lambda$, therefore $t^*(\theta) = t\theta$. Now the coordinate of the base B is given by the integration $\lambda^i(b) = \int_{a_i} \theta$, then

$$t^*(\lambda^i)(b) = \lambda^i(tb) = \int_{a_i} t^*(\theta) = t\lambda^i.$$

Let ξ be the holomorphic vector field generated by the \mathbb{C}^* -action on B , then

$$\begin{aligned} \xi(\lambda^i)(b) &= \frac{d}{dt} \Big|_{t=0} \lambda^i(e^t b) \\ &= \frac{d}{dt} \Big|_{t=0} (e^t)^* \lambda^i(b) \\ &= \frac{d}{dt} \Big|_{t=0} (e^t \lambda^i(b)) \\ &= \lambda^i(b) \end{aligned}$$

If $\xi = \xi^i \frac{\partial}{\partial \lambda^i}$, then $\xi(\lambda^i) = \xi^j \frac{\partial \lambda^i}{\partial \lambda^j} = \xi^i$. Hence $\xi^i = \lambda^i$, and $\xi = \sum_i \lambda^i \frac{\partial}{\partial \lambda^i}$. Since the periods τ_{ij} are \mathbb{C}^* -invariant, then

$$0 = \xi(\tau_{ij}) = \sum_{k=1}^{gs} \lambda^k \frac{\partial \tau_{ij}}{\partial \lambda^k}.$$

□

Another main result of the periods τ_{ij} is that the partial derivative $\frac{\partial \tau_{ij}}{\partial \lambda^k}$ is symmetric with respect to i, j and k .

Theorem 3.5.3.

$$\frac{\partial \tau_{jk}}{\partial \lambda^i} \tag{3.6}$$

is symmetric in i, j, k for $1 \leq i, j, k \leq g_s$.

We prove the theorem as follows. Let $\{S_t\}$ be a family of spectral curves parameterised by a family $t \in \mathcal{T}$, as we have seen earlier, we shall think of a spectral curve S_t as the zero locus of a section

$$p_t \in H^0(T^*\Sigma; K^n)$$

of the form

$$p_t = \lambda^n + a_1(t)\lambda^{n-1} + a_2(t)\lambda^{n-2} + \dots + a_n(t).$$

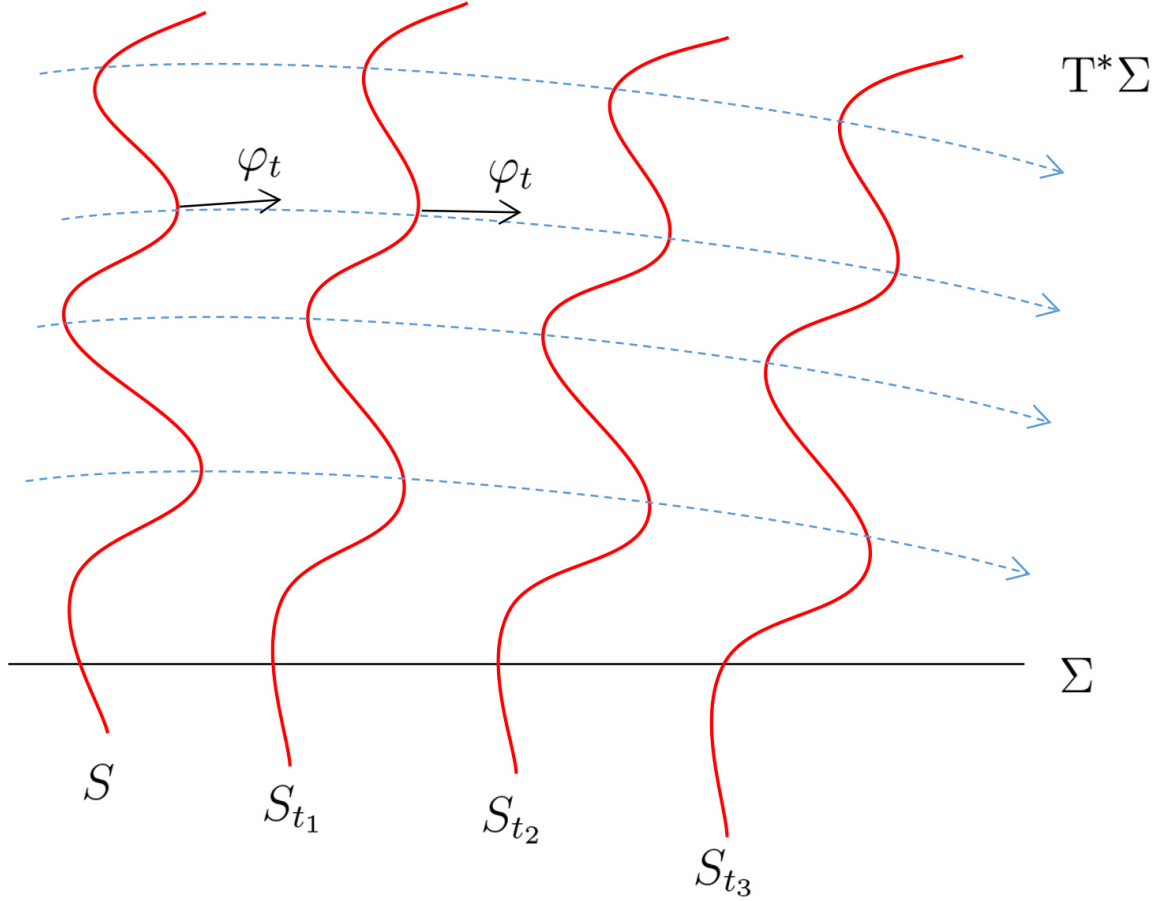
where $(a_1(t), \dots, a_n(t))$ can be considered as a path in B parameterised by t , and $\lambda \in H^0(T^*\Sigma; K)$ the tautological section. Let $S := S_{t_0}$ be a fixed spectral curve, note that for each t , the spectral curves S_t are differentiable manifolds, therefore there is a small enough neighbourhood of t_0 , so that the spectral curves S_t are diffeomorphic to each other (Ehresmann). Hence we obtain a family of diffeomorphisms

$$\varphi_t : S \rightarrow S_t$$

parameterised by t which are given by a normal vector field V . Let θ be the canonical 1-form of $T^*\Sigma$, and define

$$\theta_t = \varphi_t^* \theta.$$

Recall that the affine coordinates and periods are given by integration of θ . When a point in the base moves around, we can pull back the spectral curves to S by the diffeomorphisms φ_t . In this way, the deformation of the spectral curves S_t with a fixed canonical form θ can be considered as a deformation of θ with fixed S .



Then we have,

$$\dot{\theta} := \frac{d\theta_t}{dt} = \mathcal{L}_{\tilde{V}}(\theta_t),$$

where $\tilde{V} \in \Omega^0(S, T(T^*\Sigma))$ is a lift of V . By Cartan's magic formula,

$$\dot{\theta}|_S = \mathcal{L}_{\tilde{V}}\theta|_S = \mathbf{i}_{\tilde{V}}(d\theta)|_S + d(\mathbf{i}_{\tilde{V}}\theta)|_S.$$

Since the projection of \tilde{V} to the normal bundle is holomorphic, Therefore $\mathbf{i}_{\tilde{V}}(d\theta)|_S$ is a holomorphic 1-form on S . Moreover, $d(\mathbf{i}_{\tilde{V}}\theta)|_S$ is an exact 1-form. Therefore in cohomology, $\dot{\theta}$ can be written as the sum of holomorphic 1-forms

$$[\dot{\theta}] = \sum_{i=1}^{g_S} c_i(t)[\omega_i]$$

for some coefficients $c_i(t)$. Then by Stoke's theorem

$$\begin{aligned} \frac{\partial}{\partial t} \int_{a_i} \theta &= \int_{a_i} \dot{\theta} = \int_{a_i} \left(\sum_{j=1}^{gs} c_j \omega_j \right) \\ &= \sum_{j=1}^{gs} c_j \delta_{ij} \\ &= c_i. \end{aligned}$$

Now let $\frac{\partial}{\partial t} = \frac{\partial}{\partial \lambda_i}$, and rewrite $\dot{\theta}$ as $\dot{\theta}_i$. Since

$$\frac{\partial}{\partial \lambda_i} \int_{a_j} \theta = \frac{\partial}{\partial \lambda_i} (\lambda_j) = \delta_{ij},$$

then $c_j = \delta_{ij}$. Hence we have

Lemma 3.5.4. *In de Rham cohomology,*

$$\frac{\partial}{\partial \lambda_i} [\theta] = [\dot{\theta}_i] = [\omega_i],$$

therefore,

$$\frac{\partial}{\partial \lambda_i} [\omega_j] = \frac{\partial}{\partial \lambda_j} [\omega_i].$$

Since the periods are given by $\tau_{ij} = \int_{b_j} \omega_i$, then

$$\begin{aligned} \frac{\partial \tau_{jk}}{\partial \lambda_i} &= \frac{\partial}{\partial \lambda_i} \int_{b_k} \omega_j \\ &= \int_{b_k} \frac{\partial \omega_j}{\partial \lambda_i} \\ &= \int_{b_k} \frac{\partial \omega_i}{\partial \lambda_j} \\ &= \frac{\partial}{\partial \lambda_j} \int_{b_k} \omega_i \\ &= \frac{\partial \tau_{ik}}{\partial \lambda_j}. \end{aligned}$$

Because τ_{ij} is symmetric of i, j , therefore we completed the proof of theorem 3.5.3.

Corollary 3.5.5.

$$\tau_{ij} = \frac{\partial y^j}{\partial \lambda_i}.$$

Proof. Since $y^j = \tau_{ij} \lambda_i$, then

$$\frac{\partial y^j}{\partial \lambda_i} = \tau_{ij} + \sum_{l=1}^{gs} \frac{\partial \tau_{lj}}{\partial \lambda_i} \lambda_l.$$

By theorem 3.5.3 and proposition 3.5.2,

$$\sum_{l=1}^{gs} \frac{\partial \tau_{lj}}{\partial \lambda_i} \lambda_l = \sum_{l=1}^{gs} \frac{\partial \tau_{ij}}{\partial \lambda_l} \lambda_l = 0$$

□

Form the corollary, we can rewrite the Kähler form 3.1 as

$$\begin{aligned}\Omega_K &= \sum_{i,j}^{g_S} \text{Im}(\tau_{ji}) ds_1^i \wedge ds_2^j \\ &= \sum_i^{g_S} ds_1^i \wedge dy_1^i,\end{aligned}$$

because $dy^i = \sum_j \tau_{ij} d\lambda_j$ and $dy_1^i = \text{Re}(dy^i) = \sum_j (\text{Re}(\tau_{ij}) ds_1^j + \text{Im}(\tau_{ij}) ds_2^j)$.

If we define the holomorphic prepotential to be $F = \frac{1}{2} \sum_{j=1}^{g_S} \lambda_j y^j$, then by corollary 3.5 we have

$$\frac{\partial F}{\partial \lambda_j} = y^j,$$

hence

$$\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} = \tau_{ij}.$$

Proposition 3.5.6.

$$\frac{\partial \tau_{ij}}{\partial \lambda_k} = - \int_S \theta \wedge \partial_i \partial_j \partial_k \theta,$$

where $\partial_i = \frac{\partial}{\partial \lambda_i}$.

Proof.

$$\int_S \theta \wedge \partial_i \partial_j \partial_k \theta = \sum_{l=1}^{g_S} \left(\int_{A_l} \theta \int_{B_l} \partial_i \partial_j \partial_k \theta - \int_{B_l} \theta \int_{A_l} \partial_i \partial_j \partial_k \theta \right)$$

The second part of the right hand side is zero, because $\int_{a_l} \partial_i \partial_j \partial_k \theta = \partial_i \partial_j \partial_k \int_{a_l} \theta = \partial_i \partial_j \partial_k (\lambda_l) = 0$, and

$$\begin{aligned}\sum_{l=1}^{g_S} \left(\int_{A_l} \theta \int_{B_l} \partial_i \partial_j \partial_k \theta \right) &= \sum_{l=1}^{g_S} (\lambda_l \partial_i \partial_j \partial_k (\sum_m \lambda^m \tau_{lm})) \\ &= \sum_{l=1}^{g_S} \partial_i \partial_j \partial_k y^l \\ &= \sum_{l=1}^{g_S} (\lambda_l \partial_i \partial_j (\tau_{kl})) \\ &= \sum_{l=1}^{g_S} (\lambda_l \partial_i \partial_l (\tau_{kj})) \\ &= \sum_{l=1}^{g_S} \partial_i (\lambda_l \partial_l \tau_{kj}) - \partial_i \tau_{jk} \\ &= - \partial_i \tau_{jk}.\end{aligned}\tag{3.7}$$

The last equality is proposition 3.5.2. □

Analogous to the above, we also have

$$K = \frac{1}{4i} \int_S \theta \wedge \bar{\theta},$$

where K is the Kähler potential of section 3.4, since

$$\begin{aligned} \frac{1}{4i} \int_S \theta \wedge \bar{\theta} &= \frac{1}{4i} \sum_{l=1}^{gs} \left(\int_{A_l} \theta \int_{B_l} \bar{\theta} - \int_{B_l} \theta \int_{A_l} \bar{\theta} \right) \\ &= \frac{1}{4i} \sum_{l=1}^{gs} (\lambda_l \bar{y}^l - y^l \bar{\lambda}_l) \\ &= \frac{1}{2} \sum_{l=1}^{gs} \text{Im}(y^l \bar{\lambda}_l). \end{aligned}$$

Chapter 4

The $SL_n(\mathbb{C})$ -moduli spaces for $g \geq 2$

In this section, we let E be a degree 0, rank n $SL_n(\mathbb{C})$ -vector bundle over a Riemann surface Σ . Also let \mathcal{N} be the space of $SL_n(\mathbb{C})$ -stable bundles, and $\mathcal{M} := \mathcal{M}_{SL_n(\mathbb{C})}$ the moduli space of Higgs bundles. We will see in this section, that the fibers of the Hitchin integrable system in these cases are the Prym varieties of the spectral curves. By integrating the canonical one form of the spectral curve along the cycles corresponding to the Prym variety, we will obtain the affine coordinates of the base for the Hitchin system. Using these coordinates, we get the special Kähler metrics.

4.1 The Prym variety of the spectral curve

Let Σ be a compact Riemann surface, we have seen that a point β in the base B determines a spectral curve S which is an n -sheeted cover of Σ . Note that (see for example [For91] section 29; [Jos13] section 5.6 or Appendix A.2) the Picard group $\text{Pic}(S)$ of S is isomorphic to the group $\text{Div}(S)/\text{Div}_p(S)$, where $\text{Div}(S)$ is the group of divisors of S , and $\text{Div}_p(S)$ is the subgroup of principle divisors. Restricting the isomorphism to the same degree we get $\text{Jac}(S) \cong \text{Div}_0(S)/\text{Div}_p(S)$, where $\text{Div}_0(S)$ is the divisors of degree zero. Now given a line bundle $L \in \text{Jac}(S)$, we obtain a divisor $D = \sum_i n_i p_i$ such that $[D] = L$. The covering $\pi : S \rightarrow \Sigma$ gives a divisor $\tilde{D} = \sum_i n_i \pi(p_i)$ of Σ , which determines a line bundle over Σ .

This procedure gives a map

$$\text{Nm} : \{\text{line bundles over } S\} \rightarrow \{\text{line bundles over } \Sigma\},$$

which is called the norm map.

To show this, we check that Nm sends linearly equivalent divisors to linearly equivalent divisors. Let f be a meromorphic function of S and $[f]$ be the principle divisor generated by f . Let D, D' be divisors of S such that $D = D' + [f]$. Define a function \tilde{f} of Σ to be

$$\tilde{f}(p) = \prod_{p_i \in \pi^{-1}(p)} f(p_i),$$

counting points with multiplicity. Away from the ramification points, \tilde{f} is meromorphic. If p_0 is a branch point on S , then we can choose a local coordinate z in the neighbourhood of p_0 such that $\pi(z) = z^k$ where k is the multiplicity. Let $w = \pi(z)$ be the coordinate near $\pi(p_0)$, then $\pi^{-1}(w) = \{z, z\xi, z\xi^2, \dots\}$ where $\xi = e^{\frac{1}{k}i2\pi}$. Therefore \tilde{f} is a product of

terms of the form $f(z)f(z\xi)f(z\xi^2)\cdots f(z\xi^{k-1})$. Writing out the Laurent series of f , one can see that \tilde{f} is a meromorphic function of w . Since $\text{Nm}(D) = \text{Nm}D' + [\tilde{f}]$, therefore Nm is a homomorphism between the Jacobian varieties.

Definition 4.1.1. *The Prym variety $\text{Prym}(S; \Sigma)$ of the spectral curve is a subvariety of $\text{Jac}(S)$ defined by*

$$\text{Prym}(S; \Sigma) := \{L \in \text{Jac}(S) \mid \text{Nm}(L) = \mathcal{O}\}.$$

In other words, the Prym variety is the set of isomorphism classes of line bundles over S such that $\text{Nm}(L)$ is trivial.

4.2 The fibers of Hitchin's system

In the last section, we saw that a generic fiber of the $GL_n(\mathbb{C})$ -Hitchin system is the Jacobian variety of the spectral curve. By integrating the canonical one form along the cycles of $H_1(S; \mathbb{Z})$, we obtained two sets of affine coordinates, and used the coordinates to compute the special Kähler metrics of B . In the $SL_n(\mathbb{C})$ -moduli space, we will use similar methods to get the metrics. In these cases, the generic fibers of the Hitchin system are not the Jacobian varieties, but instead the Prym varieties of the spectral curves.

Let $h : \mathcal{M} := \mathcal{M}_{SL_n(\mathbb{C})} \rightarrow B$ be the Hitchin map, recall from section 3.2 that there is a map $\mathcal{F} : h^{-1}(\beta) \rightarrow \text{Jac}_d(S_\beta)$. Given a point $(E, \Phi) \in \mathcal{M}$, $\mathcal{F}((E, \Phi)) = L$ is a line bundle of the spectral curve S such that $E = \pi_*L$. Since $\det(E) = \mathcal{O}$ in $\mathcal{M}_{SL_n(\mathbb{C})}$, then we have

$$\mathcal{O} = \det(E) = \text{Nm}(L) \otimes K^{-n(n-1)/2},$$

(where the second equality is given in [BNR89] section 4), hence $\text{Nm}(L) = K^{n(n-1)/2}$.

Since the degree of K is even, choosing a square root $K^{1/2}$ of K , we may take $\tilde{L} = K^{-(n-1)/2}$, hence $\tilde{L}^n = K^{-n(n-1)/2}$. Let $L' = L \otimes \pi^*(\tilde{L})$, since $\text{Nm}(L \otimes \pi^*(\tilde{L})) = \text{Nm}(L) \otimes \tilde{L}^n$, we have $\text{Nm}(L') = \mathcal{O}$, hence $L' \in \text{Prym}(S; \Sigma)$.

Therefore, by shifting L with $\pi^*\tilde{L}$, the isomorphism \mathcal{F} in section 3.2 restricts to the $SL_n(\mathbb{C})$ -moduli space becomes an isomorphism

$$\mathcal{F} : h^{-1}(\beta) \rightarrow \text{Prym}(S; \Sigma)$$

sends (E, Φ) to $L' \in \text{Prym}(S; \Sigma)$.

Note that $\dim(\text{Prym}(S; \Sigma)) = \dim(\mathcal{N}) = (n^2-1)(g-1) := d$. Let $\{A_1, B_1, A_2, B_2, \dots, A_d, B_d\} \in H_1(\text{Prym}(S; \Sigma); \mathbb{Z}) \subset \text{Jac}(S)$ be a basis, and $\{a_1, b_1, a_2, b_2, \dots, a_d, b_d\}$ be the cycles of $\text{Ker}(\pi_* : H_1(S, \mathbb{R}) \rightarrow H_1(\Sigma, \mathbb{R}))$ respectively. Then the affine coordinates of B are given by

$$\left\{ \text{Re}\left(\int_{a_i} \theta\right), \text{Im}\left(\int_{a_i} \theta\right) \right\}, \left\{ \text{Re}\left(\int_{b_i} \theta\right), \text{Im}\left(\int_{b_i} \theta\right) \right\}.$$

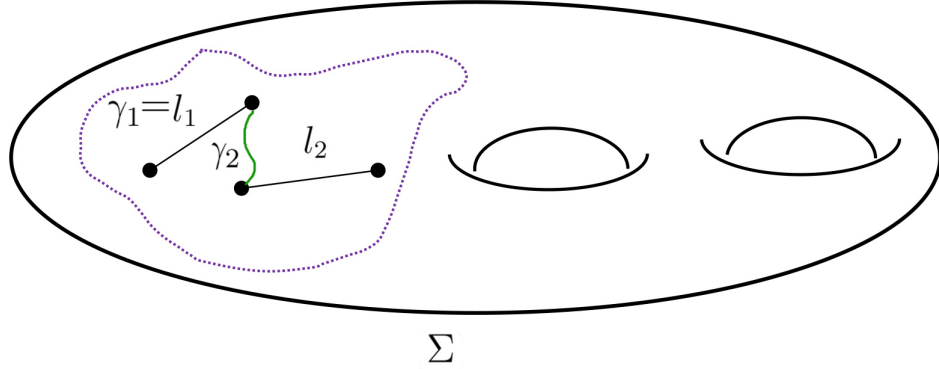
Let $\{s_1^i, s_2^i, y_1^i, y_2^i\}$ be the coordinates corresponds to $\left\{ \text{Re}\left(\int_{a_i} \theta\right), \text{Im}\left(\int_{a_i} \theta\right), \text{Re}\left(\int_{b_i} \theta\right), \text{Im}\left(\int_{b_i} \theta\right) \right\}$. The metrics of the $SL_n(\mathbb{C})$ -moduli space is computed by formulas (3.2), (3.3) given in section 3.4.

Example 4.2.1. *The regular fibers of the $SL_2(\mathbb{C})$ -moduli space for $g = 2$.*

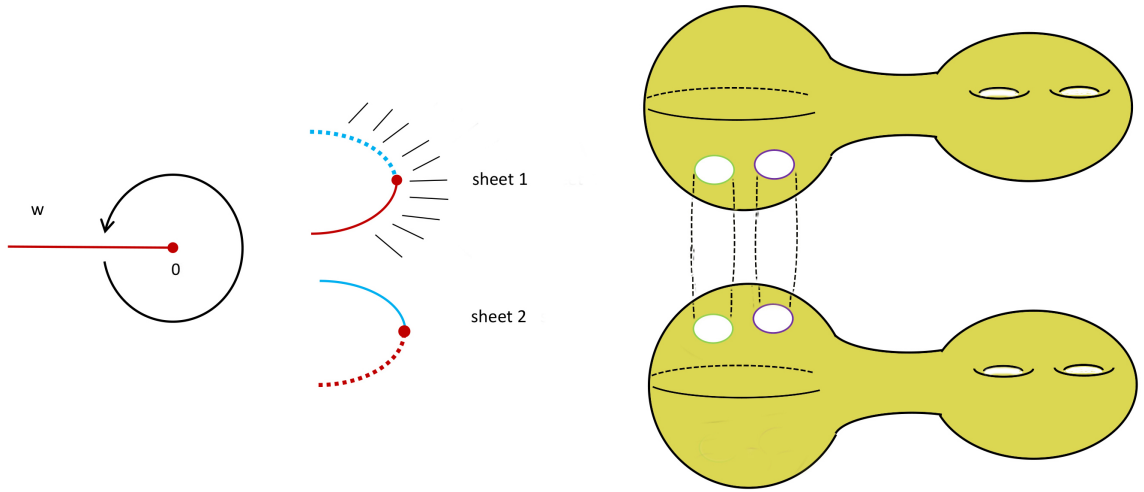
When $g = 2, n = 2$, the spectral curve is the set of solutions of the equation

$$\lambda^2 + a_2 = 0,$$

where $a_2 \in H^0(\Sigma; K^2)$. Since $\deg(K^2) = 2(2g - 2) = 4$, then there are four branch points with order two of the spectral cover $S \xrightarrow{\pi} \Sigma$. According to Baraglia ([Bar18] section 3), we can find an open disc D on Σ which contains the four ramification points such that S restrict to $\Sigma \setminus D$ is a trivial covering space (two disjoint copies of $\Sigma \setminus D$).



Let z be the coordinate of the neighbourhood of a branch point on S , and w be the coordinate of the corresponding neighbourhood on Σ such that $w = \pi(z) = z^2$, let l_1, l_2 be two branch cuts for the functions $z = \sqrt{w}$. Then the spectral curve is obtained by identifying the points along the branch cut of the two sheets (see the picture below).



We can also see from the picture that the genus g_s of the spectral curve S is $n^2(g-1) + 1 = 5$. Let $\{a_1, a_2, b_1, b_2\}$ be a symplectic basis of $H_1(\Sigma; \mathbb{Z})$, and $\{a_1^1, b_1^1, a_2^1, b_2^1, a_1^2, b_1^2, a_2^2, b_2^2, c, d\}$ be a symplectic basis of $H_1(S; \mathbb{Z})$, where $\pi(a_j^k) = a_j$, $\pi(b_j^k) = b_j$, for $k = 1, 2$ and c, d are cycles represented by closed curves in $\pi^{-1}(D)$ with intersection number 1 (i.e. $c = \pi^{-1}(\gamma_1)$, $d = \pi^{-1}(\gamma_2)$ where γ_1, γ_2 are the shown paths. They are the new cycles induced by the "new hole").

Then $H_1(\text{Prym}(S; \Sigma), \mathbb{Z})$ is generated by $\{A_i^1 - A_i^2, B_i^1 - B_i^2, C, D\}$ where $i = 1, 2$ and A_i^j, B_i^j, C, D are the cycles correspond to a_i^j, b_i^j, c, d of the Abel-Jacobi isomorphism.

In fact, the affine coordinates of B can be computed by integrating the canonical 1-form λ on Σ along the paths corresponds to a_i^j, b_i^j, c, d . Recall from section 3.3 that $\int_{a_i^j} \theta = \int_{a_i^j} d\pi(\lambda)$, note that θ changes sign when restricted to another branch, then $\int_{a_i^1} \theta = \int_{a_i} \lambda$ (where we view λ as the 1-form $\sqrt{-a_2}$, which is well defined on a neighborhood of the

cycle a_i in Σ) and $\int_{a_i^2} \theta = \int_{a_i} -\lambda$, hence

$$\int_{a_i^1 - a_i^2} = 2 \int_{a_i} \lambda,$$

for $i = 1, 2$. Similarly, for the two paths γ_1, γ_2 on Σ corresponding to c, d , we have $\int_c \theta = 2 \int_{\gamma_1} \lambda$ and $\int_d \theta = 2 \int_{\gamma_2} \lambda$. Note that this example can be generalised to any values of (g, n) .

4.3 The metrics

From the above discussions, the base $B_{SL} := \bigoplus_{i=2}^n H^0(K^{\otimes i})$ of the special linear moduli space is a subspace of B_{GL} , the base of the $GL_n(\mathbb{C})$ moduli space, with inclusion $(a_2, \dots, a_n) \rightarrow (a_1 = 0, a_2, \dots, a_n)$ where a_i 's are coefficients of the characteristic polynomials. We can construct a special Kähler metric on B_{SL} by using Hitchin's construction (see section 1.8). It turns out that this special Kähler metric is the restriction of g_{sk}^{GL} , the special Kähler metric on the base B_{GL} of general linear Hitchin system, to the base B_{SL} .

To show this, note that since $\text{Prym}(S; \Sigma) = \text{Ker}(\text{Nm})$, there is a short exact sequence

$$0 \longrightarrow \text{Prym}(S; \Sigma) \longrightarrow \text{Jac}(S) \xrightarrow{\text{Nm}} \text{Jac}(\Sigma) \longrightarrow 0.$$

It gives a long exact sequence of homotopy groups:

$$\pi_2(\text{Jac}(\Sigma)) \longrightarrow \pi_1(\text{Prym}(S; \Sigma)) \longrightarrow \pi_1(\text{Jac}(S)) \xrightarrow{\pi_*} \pi_1(\text{Jac}(\Sigma)) \longrightarrow 0.$$

Since $\pi_2(\text{Jac}(\Sigma)) = 0$ and $\pi_1(\text{Prym}(S; \Sigma)) = H_1(\text{Prym}(S; \Sigma), \mathbb{Z})$, $\pi_1(\text{Jac}(S)) = H_1(\text{Jac}(S), \mathbb{Z})$, $\pi_1(\text{Jac}(\Sigma)) = H_1(\text{Jac}(\Sigma))$. Then $H_1(\text{Prym}(S; \Sigma), \mathbb{Z})$ is the kernel of π_* and tensoring by \mathbb{R} , we get

$$H_1(\text{Jac}(S), \mathbb{R}) = \ker(\pi_*) \oplus \pi^*(H_1(\text{Jac}(\Sigma), \mathbb{R})). \quad (4.1)$$

If we let the coefficients of the homology groups to be \mathbb{R} then

$$0 \longrightarrow H_1(\text{Prym}(S; \Sigma); \mathbb{R}) \longrightarrow H_1(\text{Jac}(S); \mathbb{R}) \xrightarrow{\pi_*} H_1(\text{Jac}(\Sigma); \mathbb{R}) \longrightarrow 0.$$

By Poincaré duality,

$$0 \longrightarrow H^1(\text{Prym}(S; \Sigma); \mathbb{R}) \longrightarrow H^1(\text{Jac}(S); \mathbb{R}) \xrightarrow{\pi_*} H^1(\text{Jac}(\Sigma); \mathbb{R}) \longrightarrow 0.$$

(To avoid too many notations, we use π_* for a few different mappings above, which should not cause confusions).

Let $V_{GL} = H^1(\text{Jac}(S); \mathbb{R})$, $V_{SL} = H^1(\text{Prym}(S; \Sigma); \mathbb{R}) = \ker(\pi_*) = \text{im}(\pi^*)^\perp$ and $W = H^1(\text{Jac}(\Sigma); \mathbb{R})$, then we also have

$$V_{GL} = V_{SL} \oplus \pi^*W.$$

Recall from section 1.8 that to find the special Kähler metric we defined a pair of maps $u, v : B_{GL} \rightarrow V_{GL}$ such that for $b \in B_{GL}$, $u(b)$ is determined by sending cycles $A \in H_1(\text{Jac}(S))$ to $\text{Re}(\int_A \mu)$ and $v(b)$ is determined by sending A to $\text{Im}(\int_A \mu)$. From the decomposition 4.1, we can write $A = \alpha + \beta$ where $\alpha \in \ker(\pi_*)$ and $\beta \in \pi^*(H_1(\text{Jac}(\Sigma)))$. Then $\int_A \mu = \int_{\alpha+\beta} \mu$. Let $b \in B_{SL} \subset B_{GL}$, then for any cycle $\gamma \in H_1(\text{Jac}(S))$,

$$\int_{\pi^*(\gamma)} \mu = \int_{\pi^*(r)} \theta = \sum_{i=1}^n \int_r \lambda_i = \int_r (-a_1) = 0$$

where the first equality is proposition 3.3.1, $r \in H_1(\Sigma)$ is a cycle in Σ representing γ , θ is the canonical 1-form of S_b , λ_i is, as in proposition 3.3.1, the restriction $\lambda|_{V_i}$ of λ to the i^{th} sheet of the trivial covering (by removing a closed disc containing all branched points) $S/\pi^*(\bar{D}) \rightarrow \Sigma/\bar{D}$.

Therefore when $b \in B_{SL}$, $\int_A \mu = \int_\alpha \mu$. A similar argument applies to the map v . This shows the map u sends b to the V_{SL} component and the following diagrams commute,

$$\begin{array}{ccc} V_{SL} & \xrightarrow{i} & V_{GL} \\ \uparrow u & & \uparrow u \\ B_{SL} & \xrightarrow{i} & B_{GL} \end{array} \quad \begin{array}{ccc} V_{SL} \times V_{SL} & \xrightarrow{i} & V_{GL} \times V_{GL} \\ \uparrow (u,v) & & \uparrow (u,v) \\ B_{SL} & \xrightarrow{i} & B_{GL} \end{array}$$

where i is the inclusions and $u, (u, v)$ are the maps given in section 1.8. More concretely, let $\{e_1, \dots, e_{2(g_s-g)}, f_1, \dots, f_{2g}\}$ be a basis of the decomposition $\ker(\pi_*) \oplus \pi^*(H_1(\text{Jac}(\Sigma)))$ given in 4.1, and $b \in B_{SL}$, then

$$\begin{aligned} u \circ i(b) &= (\text{Re}(\int_{e_1} \mu), \dots, \text{Re}(\int_{e_{2(g_s-g)}} \mu), \text{Re}(\int_{f_1} \mu), \dots, \text{Re}(\int_{f_{2g}} \mu)) \\ &= (\text{Re}(\int_{e_1} \mu), \dots, \text{Re}(\int_{e_{2(g_s-g)}} \mu), 0, \dots, 0) \end{aligned}$$

and

$$\begin{aligned} i \circ u(b) &= i(\text{Re}(\int_{e_1} \mu), \dots, \text{Re}(\int_{e_{2(g_s-g)}} \mu)) \\ &= (\text{Re}(\int_{e_1} \mu), \dots, \text{Re}(\int_{e_{2(g_s-g)}} \mu), 0, \dots, 0). \end{aligned}$$

The computation is the same for the mapping v in the second diagram. We can also check the necessary condition that $\dim_{\mathbb{C}}(B_{GL}) - \dim_{\mathbb{C}}(B_{SL}) = n^2(g-1) + 1 - (n^2-1)(g-1) = g$. Since the special Kähler metrics g_{sk}^{GL} and g_{sk}^{SL} are induced from the inner products $\langle \cdot, \cdot \rangle_{V_{GL} \times V_{GL}}$ and $\langle \cdot, \cdot \rangle_{V_{SL} \times V_{SL}}$, where $\langle \cdot, \cdot \rangle_{V_{SL} \times V_{SL}} = \langle \cdot, \cdot \rangle_{V_{GL} \times V_{GL}}|_{V_{SL} \times V_{SL}}$, therefore

Theorem 4.3.1. *The special Kähler metric of the $SL_n(\mathbb{C})$ -Hitchin system is the restriction of the special Kähler metric of the $GL_n(\mathbb{C})$ -Hitchin system. i.e. $g_{sk}^{SL} = g_{sk}^{GL}|_{B_{SL}}$.*

Chapter 5

Donagi-Markman Cubic

In this section, we introduce the Donagi-Markman Cubic [DM96] for Lagrangian fibrations, which is useful in the computation of metrics in Hitchin systems. We also provide a formula of the Donagi-Markman Cubic in terms of the affine coordinates of B .

Let $h : \mathcal{M} \rightarrow B$ be a Hitchin system, $\pi : \mathcal{M}^{reg} \rightarrow B^{reg}$ be the regular fibration, then the regular fibers are complex tori with dimension g_s (In the $GL_n(\mathbb{C})$ -Hitchin system, g_s is the genus of the spectral curves). Let \mathcal{V} be the vector bundle over B^{reg} whose sections are constant vector fields along the fibers of h , then we call $\mathcal{V} \rightarrow B^{reg}$ the vertical bundle on B^{reg} . Since each constant vector field on a torus can be represented by a vector in the tangent space at the origin of the torus, therefore \mathcal{V} may be considered as the bundle of tangent spaces at zero of each fiber, at least, once a local section of $\mathcal{M}^{reg} \rightarrow B^{reg}$ has been chosen. This means \mathcal{V} is the disjoint union $\bigsqcup_{\beta \in B^{reg}} T_0 F(\beta)$, where F_β is the fiber at β . Define a map

$$i : TB^{reg} \rightarrow \mathcal{V}^*$$

sending $X \in T_\beta B^{reg}$ to $\Omega(X, -)|_{F_\beta}$. Note that $\Omega(X, -)|_{F_\beta}$ is a holomorphic 1-form on the torus F_β , because the fibers are Lagrangian hence $\Omega(X, -)|_{F_\beta}$ does not depend on the choice of vectors that project to X . Every such holomorphic 1-form is translation invariant, hence $\Omega(X, -)|_{F_\beta}$ defines an element of \mathcal{V}_β^* . Obviously, i is a homomorphism. Since the fibers are Lagrangian and Ω is non-degenerate, hence \mathcal{V}^* has the same dimension as TB^{reg} , then i is an isomorphism.

Now let F be a fiber, $\gamma_1, \dots, \gamma_{2g_s}$ be a symplectic basis of $H_1(F, \mathbb{Z})$, then the change of basis is given by a symplectic transformation. Let $\alpha_1, \dots, \alpha_{g_s}$ be a basis of holomorphic 1-forms, such that for $1 \leq i, j \leq g_s$,

$$\int_{\gamma_i} \alpha_j = \delta_{ij}.$$

Then the period matrix $\tau = \{\tau_{ij}\}$ is given by $\tau_{ij} = \int_{\gamma_{i+g_s}} \alpha_j$.

By Riemann's bilinear relations (see [GX08]), τ is a $g_s \times g_s$ symmetric matrix depending on the complex structure of F with positive definite imaginary part. Let a point vary in the base B , then τ is locally a matrix of complex functions of B^{reg} (τ depends on the local choice of basis $\gamma_1, \dots, \gamma_{2g_s}$). Let

$$\mathcal{A}_{g_s} := \frac{\{\text{symmetric matrices with positive definite imaginary part}\}}{Sp(2g_s; \mathbb{Z})},$$

then we obtain a map

$$p : B^{reg} \rightarrow \mathcal{A}_{g_s}.$$

Note that F is determined by the period matrix, hence F can be regarded as a point in \mathcal{A}_{g_s} , and p is the map which sends a point $b \in B$ to a matrix $(\tau_{ij}) \in \mathcal{A}_{g_s}$. An element in $\mathrm{T}\mathcal{A}_{g_s}$ is a symmetric matrix, which can be considered as a symmetric tensor of \mathcal{V} , then the differential of p gives

$$dp : \mathrm{T}B^{reg} \rightarrow \mathrm{T}\mathcal{A}_{g_s} \cong \mathrm{Sym}^2\mathcal{V}.$$

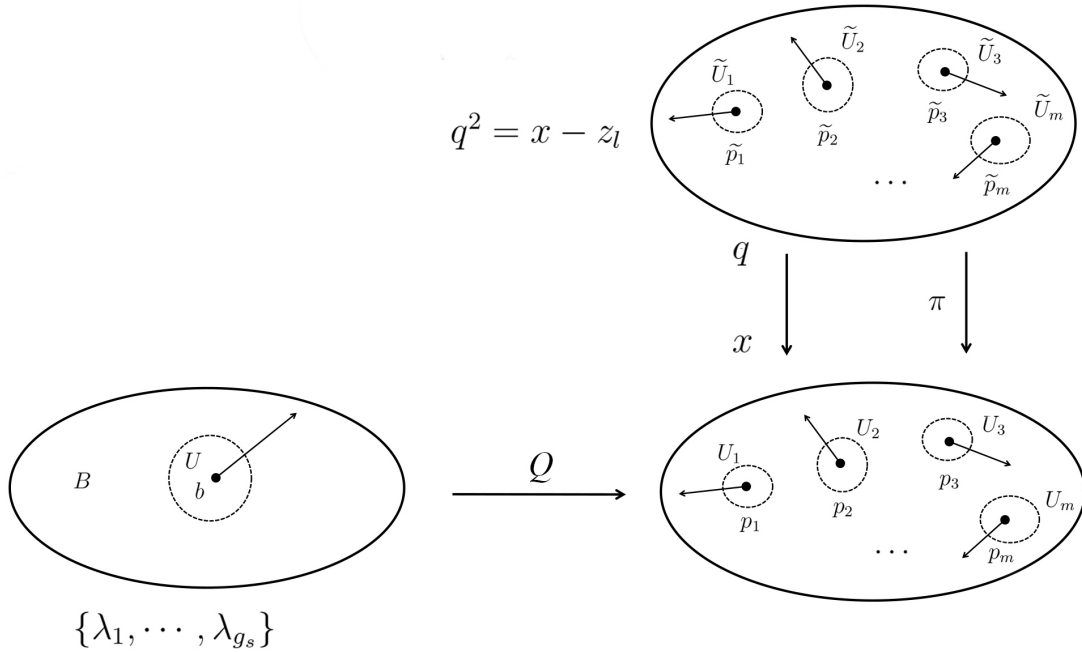
Donagi and Markman showed (in [DM96]) that the map $dp \circ i^{-1} : \mathcal{V}^* \rightarrow \mathrm{Sym}^2\mathcal{V}$ is a cubic in $\mathrm{H}^0(B^{reg}; \mathrm{Sym}^3\mathcal{V})$.

Let $\{\lambda_1, \dots, \lambda_{g_s}\}$ be the affine coordinates of B given as in section 3.3, 3.4, then

$$dp \circ i^{-1} \in \mathrm{H}^0(B^{reg}; \mathrm{Sym}^3\mathcal{V})$$

is equivalent to the condition that $\partial_k \tau_{ij} := \frac{\partial \tau_{ij}}{\partial \lambda_k}$ is symmetric in i, j, k which we have proven in theorem 3.5.3. From now on, we call $\partial_k \tau_{ij}$ the Donagi-Markman cubic.

Given a point $b \in B^{reg}$, with local coordinates $\lambda_1, \dots, \lambda_{g_s}$, since τ_{ij} is the periods of the spectral curve S_b , hence the Donagi-Markman cubic $\partial_k \tau_{ij}$ is given by the deformation of a family of spectral curves and by the theorem of Ehresmann, locally on B^{reg} , the spectral curves S_b are diffeomorphic ([Huy06] section 6.2). Therefore we can think of the deformation of spectral curves as the deformation of complex structures on S_b . It is known that the deformation of complex structure is given by the Kodaira-Spencer class in $\mathrm{H}^1(S_b; \mathrm{T}S_b)$, which can be computed as follows.



Let B^{sim} be the open and dense subset of B^{reg} so that the spectral curves S_b have only ramification points with ramification order one (i.e. the spectral curves S_b only have simple ramification points), for simplicity, we will work on B^{sim} instead of B^{reg} . Let m be the number of ramification points on S_b , $\{p_1, \dots, p_m\}$, $\{\tilde{p}_1, \dots, \tilde{p}_m\}$ be the branch points on Σ and ramification points on S_b , $\{U_l\}$, $\{\tilde{U}_l\}$ be the neighbourhoods of the branch points and ramification points respectively as the picture showed, then for each l , we can find coordinates x_l of U_l such that $x_l(p_l) = z_l$, and the local coordinate of S_b in a

neighbourhood of the ramification point \tilde{p}_l is given by $q^2 = x - z_l$. Then we have a short exact sequence,

$$0 \longrightarrow \mathcal{O}(TS) \xrightarrow{\pi_*} \mathcal{O}(\pi^*T\Sigma) \xrightarrow{\text{restriction}} \mathcal{O}_{br}(T\Sigma) \longrightarrow 0,$$

where $\mathcal{O}_{br}(T\Sigma)$ is the skyscraper sheaf of the branch points with values in $T_{p_l}\Sigma$, because π_* vanishes to first order of each ramification point. Let $\delta : H^0(\Sigma; \mathcal{O}_{br}(T\Sigma)) \rightarrow H^1(S; TS)$ be the coboundary map of the induced long exact sequence of Čech cohomology. Since Σ and S_b are complex manifolds, we prefer to think of the cohomologies of the first two terms in the short exact sequence to be Dolbeault cohomologies (though there is no Dolbeault cohomology of the third term).

Explicitly, when a point $b \in B^{sim}$ varies smoothly, let y be a local coordinate of the fiber of $T^*\Sigma$ (i.e. (x, y) corresponds to ydx). The spectral curve given by the characteristic polynomial $y^n + a_1y^{n-1} + \cdots + a_n = 0$ varies in $T^*\Sigma$, hence the branch points on Σ also vary smoothly. Therefore locally we obtain a map (which sends a point $b \in B^{sim}$ to the m -tuple of ramification points of S_b)

$$Q : B^{sim} \rightarrow U_1 \times \cdots \times U_m.$$

Let $\frac{\partial}{\partial \lambda_i} \in T_b B^{sim}$ be the tangent vector at b in the λ_i direction, then $dQ(\frac{\partial}{\partial \lambda_i})$ is an element in $H^0(\Sigma; \mathcal{O}_{br}(T\Sigma))$. The corresponding Kodaira-Spencer class κ_i is given by $\delta \circ dQ(\frac{\partial}{\partial \lambda_i})$.

In the local coordinates we have chosen on Σ , let λ be the local coordinate of S_b around ramification points, write

$$dQ(\frac{\partial}{\partial \lambda_i}) = (\dot{z}_1^i \frac{\partial}{\partial z_1}, \dots, \dot{z}_m^i \frac{\partial}{\partial z_m}),$$

where z_l is a local coordinate on U_l . Then we choose a smooth extension of the local vector fields $\{\dot{z}_l^i \frac{\partial}{\partial z_l}\}$ to Σ , and call the extended vector field F_i . In addition, we can require F_i to be holomorphic in a small neighbourhood of each branch point (e.g. we can assume $F_i = \dot{z}_l^i \frac{\partial}{\partial z_l}$ in a neighbourhood of p_l). Let W_i be the singular vector field on S such that

$$\pi_* W_i = F_i$$

At ramification points, we have

$$W_i = \frac{\dot{z}_l^i}{2q} \frac{\partial}{\partial q}, \quad (5.1)$$

where q is a local coordinate on S at \tilde{p}_l satisfying $q^2 = z_l - z_l(p_l)$, hence $2q dq = dz_l$ and

$$\frac{\partial}{\partial z_l} = \frac{1}{2q} \frac{\partial}{\partial q}.$$

Therefore the Kodaira-Spencer class is given by $\kappa_i = \bar{\partial} W_i$ (note that $\bar{\partial} W_i$ has no pole since W_i is meromorphic near ramification points).

Now by Riemann's bilinear relation,

$$\begin{aligned} \int_{S_b} \frac{\partial \omega_j}{\partial \lambda_i} \wedge \omega_k &= \sum_{l=1}^{g_s} \left(\int_{a_l} \frac{\partial \omega_j}{\partial \lambda_i} \int_{b_l} \omega_k - \int_{b_l} \frac{\partial \omega_j}{\partial \lambda_i} \int_{a_l} \omega_k \right) \\ &= - \int_{b_k} \frac{\partial \omega_j}{\partial \lambda_i}. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial \tau_{jk}}{\partial \lambda_i} &= \int_{b_k} \frac{\partial \omega_j}{\partial \lambda_i} \\ &= - \int_{S_b} \frac{\partial \omega_j}{\partial \lambda_i} \wedge \omega_k \\ &= - \int_{S_b} \frac{\partial \omega_j^{(0,1)}}{\partial \lambda_i} \wedge \omega_k\end{aligned}$$

where $\frac{\partial \omega_j^{(0,1)}}{\partial \lambda_i}$ is the $(0,1)$ -part of $\frac{\partial \omega_j}{\partial \lambda_i}$. From Griffiths transversality,

$$\left(\frac{\partial}{\partial \lambda_i} [\omega_j]\right)^{(0,1)} = [\kappa_i] \smile [\omega_j].$$

Therefore, we have

$$\begin{aligned}\frac{\partial \tau_{jk}}{\partial \lambda_i} &= - \int_{S_b} \kappa_i \omega_j \wedge \omega_k \\ &= - \int_{S_b} (\bar{\partial} W_i) \omega_j \wedge \omega_k \\ &= - \int_{S_b} \bar{\partial} (W_i \cdot \omega_j \wedge \omega_k).\end{aligned}$$

Now by Stokes' theorem and Cauchy's residue theorem, we have

Theorem 5.0.1.

$$\begin{aligned}\frac{\partial \tau_{jk}}{\partial \lambda_i} &= 2\pi i \sum_{l=1}^m \text{Res}(W_i \omega_j \omega_k; \tilde{p}_l) \\ &= -2\pi i \sum_{l=1}^m \text{Res}\left(\frac{\omega_i \omega_j \omega_k y \frac{\partial}{\partial y}}{\theta}; \tilde{p}_l\right).\end{aligned}$$

The second equality is due to the lemma below.

Lemma 5.0.2. $\text{Res}(W_i \omega_j \omega_k; \tilde{p}_l) = -\text{Res}\left(\frac{\omega_i \omega_j \omega_k y \frac{\partial}{\partial y}}{\theta}; \tilde{p}_l\right)$.

Proof. The proof is in fact Lemma 6.2.4 which shows

$$\hat{V} = \frac{\delta_i \theta}{\theta|_S} y \frac{\partial}{\partial y} = \frac{\omega_i}{\theta|_S} y \frac{\partial}{\partial y},$$

where \hat{V} is a lift of the normal vector field with $\text{Res}(\hat{V}) = -\text{Res}(W)$. The second equality is given by Lemma 3.5.4 and the fact that $\delta_i \theta$ is holomorphic (see section 6.2 and Proposition 6.2.2 for details), because if df is an exact holomorphic 1-form, then $\bar{\partial}(df) = \bar{\partial} \partial f = 0$, which implies f is a harmonic function hence constant by the maximum principle. All these together give a proof of the Lemma. \square

Chapter 6

Topological Recursion

We have seen in the previous sections that the special Kähler metric on the base B^{reg} , and the semi-flat metric of the moduli space \mathcal{M} depend on the variation of periods of the spectral curves. Under this circumstance, it allow us to use a tool called topological recursion [EO07] [Eyn14], introduced by Eynard and Orantin, to describe the variation of the periods τ_{ij} . We will also give an overview of the relation between topological recursion and the Bergmann tau-function τ_B [KK04], which is defined locally as a flat section of the Bergmann projective connection near ramification points. Although the Bergmann tau-function is only defined locally under some chosen coordinates, it turns out that the 12th power of the function is a global section of a line bundle of the Hurwitz space, and we have worked out the transition functions for the line bundle. We begin by giving the definitions of topological recursion.

6.1 Definitions of topological recursion

Let τ be the period matrix of a compact Riemann surface with genus g_s , and \vec{a}, \vec{b} be fixed vectors in \mathbb{Z}^{g_s} , and

$$t = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix},$$

we define:

1. **The Riemann theta-function** with characteristic t [Fay73], $\theta_t : \mathbb{C}^{g_s} \rightarrow \mathbb{C}$ to be

$$\theta_t(v) = \sum_{\vec{n} \in \mathbb{Z}^{g_s}} e^{i\pi(\vec{n} + \vec{a}/2)^T \tau (\vec{n} + \vec{a}/2) + 2i\pi(\vec{v} + \vec{b}/2)^T (\vec{n} + \vec{a}/2)}.$$

The fixed vector t is called the characteristic of the theta function and we say that it is odd if $\sum_{i=1}^{g_s} a_i b_i$ is an odd number where $\vec{a} = (a_i)^T, \vec{b} = (b_i)^T$;

2. **The prime form** of $S \times S$ to be

$$E(p, q) = \frac{\theta_t(u(p) - u(q))}{\sqrt{dh_t(p)dh_t(q)}},$$

where u is the Abel-Jacobi map, t is an odd characteristic and

$$dh_t(p) = \sum_{i=1}^{g_s} d\omega_i(p) \cdot \frac{\partial \theta_t(v)}{\partial v_i} \Big|_{v=0}.$$

Note that E is independent of the choice of t ;

3. The Bergmann kernel to be

$$B(p, q) = d_p d_q \ln(\theta_t(u(p) - u(q))),$$

for any odd characteristic t . We will use the following properties of the Bergmann kernel in this section. All of them can be found in [EO07] [Fay73].

The Bergmann kernel is a section of $K_S \boxtimes K_S$. It has double poles with no residue on the diagonal of $S \times S$ and is holomorphic elsewhere, choose a coordinate $\{p, q\}$ in the neighbourhood of a point $(a, a) \in S \times S$, such that $p \circ \pi_1 = q \circ \pi_2$ on the diagonal of $S \times S$ where π_1, π_2 are the projections of $S \times S$ onto the first and second component, then $B(p, q)$ has the form

$$B(p, q) = \left(\frac{1}{(p-q)^2} + \frac{1}{6} S_B(a) + \dots \right) dp dq,$$

where S_B is called the Bergmann projective connection [Fay73] that depends on the choice of local coordinates, such that if \hat{q} is another coordinate near a , and \hat{S}_B is the Bergmann projective connection corresponding to \hat{q} , assume that $q(a) = \hat{q}(a) = 0$ then

$$S_B(a) = (f')^2 \hat{S}_B(a) + \left(\frac{f'''}{f'} - \frac{3f''^2}{2f'^2} \right),$$

where f is the change of coordinate, i.e. $\hat{q} = f(q) = f'q + \frac{1}{2}f''q^2 + \frac{1}{6}f'''q^3 + \dots$.

Also $B(p, q)$ is the unique symmetric form of the type described above satisfying,

$$\frac{1}{2\pi i} \int_{p \in a_i} B(p, q) = 0, \quad \frac{1}{2\pi i} \int_{p \in b_i} B(p, q) = \omega_i(q). \quad (6.1)$$

Now given a ramification point \tilde{b} of the spectral curve S , and let b be the corresponding branch point on Σ , x be the coordinate around b such that $x(b) = z_0$, then we can find a local coordinate q on the spectral curve near \tilde{b} , such that $q^2 = x - z_0$. Let

$$dE_q(p) := \frac{1}{2} \int_{\xi=q}^{-q} B(\xi, p),$$

$$\omega(q) = (y(q) - y(-q)) dx(q),$$

where y is defined by $\theta = y dx$, then we define the topological recursion for meromorphic sections $W_n^{(g)}$ (they are called Eynard-Orantin invariants) of the n -th exterior tensor product $K_s^{\boxtimes n} = K_s \boxtimes K_s \boxtimes \dots \boxtimes K_s$ for $n \geq 1, g \geq 0$ as follows.

$$W_1^{(0)}(p) = 0,$$

$$W_2^{(0)}(p, q) = B(p, q),$$

and by recursion for $g \geq 0, n \geq 3$,

$$W_{n+1}^{(g)}(p, P_N) = \sum_a \operatorname{Res}_{q \rightarrow a} \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^g \sum_{K \subset N} W_{|K|+1}^{(m)}(q, P_K) W_{n-|K|+1}^{(g-m)}(-q, P_{N \setminus K}) + W_{n+2}^{(g-1)}(q, -q, P_N) \right),$$

where a are ramification points, $P_N = \{p_1, \dots, p_n\}$, $P_K = \{p_{i_1}, p_{i_2}, \dots, p_{i_k}\}$ for $K = \{i_1, \dots, i_k\}$ a subset of $\{1, 2, \dots, n\}$. One can check that the recursion formula is well defined when $W_1^{(0)}$ and $W_2^{(0)}$ are given. Using the recursion formula we can compute

$$W_3^{(0)}(p, p_1, p_2) = \sum_a \operatorname{Res}_{q \rightarrow a} \frac{B(q, p)B(q, p_1)B(q, p_2)}{dx(q)dy(q)}, \quad (6.2)$$

$$W_1^{(1)}(p) = \sum_a \operatorname{Res}_{q \rightarrow a} \frac{dE_q(p)}{\omega(q)} B(q, -q). \quad (6.3)$$

Note that $W_n^{(g)}$ is symmetric in its n variables [EO07].

6.2 Variation formulas and the Donagi-Markman cubic

Recall that given a point $b \in U$ the spectral curve S_b lies in $T^*\Sigma$ and the normal bundle N_{S_b} is defined by the short exact sequence

$$0 \rightarrow TS_b \rightarrow T(T^*\Sigma) \rightarrow N_{S_b} \rightarrow 0.$$

When b is moving along an integral curve of a vector field on B^{reg} , the deformation of S_b is described by a section of the normal bundle N_{S_b} . This is called the characteristic map

$$\chi : T_a B^{reg} \cong B \rightarrow H^0(S_b, N_{S_b}),$$

which is given as follows.

Let S be a spectral curve which is the zero divisor of a $p \in H^0(T^*\Sigma, \pi^*(K^n))$, V be a section in N_S , by the adjunction formula we have an isomorphism $\phi : N_S \cong \pi^*(K^n)|_S$. If we choose local trivialisations of π^*K^n with transition functions $g_{ij} : U_i \cap U_j \rightarrow \mathbb{C}^*$, then p is just a collection of functions $s_i : U_i \rightarrow \mathbb{C}$ with $s_i = g_{ij}s_j$ on $U_i \cap U_j$ and $s_i|_S = 0$. That implies $\partial_V(s_i)|_S = g_{ij}\partial_V(s_j)|_S$, where $\partial_V(s_i)$ is the derivative of s_i in the direction V . Hence $\{\partial_V(s_i)|_S\}$ is a section of $\pi^*(K^n)|_S$ denoted as $\partial_V p|_S$. Then ϕ is given by

$$\phi(V) = \partial_V p|_S.$$

Let $b \in B^{reg}$ and

$$\rho : B = \bigoplus_{j=1}^n H^0(\Sigma, K^j) \rightarrow H^0(S_b, \pi^*(K^n))$$

be the map such that

$$\rho(b_1, \dots, b_n) = \pi^*(b_1)\lambda^{n-1} + \pi^*(b_2)\lambda^{n-2} + \dots + \pi^*(b_n).$$

Then we have,

Proposition 6.2.1. *The characteristic map is given by*

$$\chi = -\phi^{-1} \circ \rho.$$

Proof. Let $b(t)$ be a 1-parameter family in B^{reg} with $b(0) = b$, $b'(0) \in T_b B^{reg}$ a tangent vector, and x a point in S_b , $x(t)$ a 1-parameter family of points in $T^*(\Sigma)$ such that $x(t) \in S_{b(t)}$. Then by the definition of χ , the projection of $x'(0)$ to the normal bundle is $\chi(b'(0)) := V$. Since $p_{b(t)}(x(t)) = 0$, then

$$\left. \frac{dp_{b(t)}(x(t))}{dt} \right|_{t=0} = b'_1(0)x^{n-1}(0) + \cdots + b'_n(0) + \left. \frac{p_{b(0)}(x(t))}{dt} \right|_{t=0} = 0.$$

This says $\rho(b'(0))(x) + \partial_V p_{b(0)} = 0$ or $\phi(V(x)) = -\rho(b'(0))(x)$. Therefore

$$\chi(b'(0)) = -\phi^{-1}\rho b'(0).$$

□

In fact, ρ is an isomorphism since by [BNR89],

$$\pi_* \mathcal{O}_S = \mathcal{O}_\Sigma \oplus K^{-1} \oplus \cdots \oplus K^{-(n-1)},$$

and then multiplying both sides by K^n and taking global sections gives the result. That means χ is an isomorphism, so B^{reg} can be identified (at least locally) with the moduli space of deformations of S_b . Since in a local neighborhood U of b , the topology of all the spectral curves $S_{b(t)}$ are the same for $\forall t \in U$. We can think of the moduli space B^{reg} as the space of a fixed curve with different complex structures and each curve $S_{b(t)}$ is equipped with a 1-form θ_t , which is the pull back of the canonical 1-form of $S_{b(t)}$ to $S_{b(0)}$. In concrete terms, let Z^{reg} be the universal moduli spaces of spectral curves:

$$Z^{reg} = \{(x, b) \in T^*\Sigma \times B^{reg} \mid p_b(x) = 0\},$$

then for any $b \in B^{reg}$ we can find an open neighborhood $U \subset B^{reg}$ such that

$$Z^{reg}|_U \cong U \times S_b.$$

Here we think of $Z^{reg}|_U$ as the product of U with a fixed topological surface S_b , but there is a family of complex structures $I(t)$ of S_b that is varying with $t \in U$. In other words, all spectral curves $S_t \subset T^*\Sigma$ for $t \in U$ were identified with S_b by a diffeomorphism $\varphi_t : S_b \rightarrow S_t$. Then

$$\pi_t : S_b \xrightarrow{\varphi_t} S_t \xrightarrow{\pi} \Sigma$$

is holomorphic with respect to $I(t)$. Given a tangent vector $\partial \in T_b U$, let $Y = \dot{\pi} \in \Omega^0(S, \pi^*(T\Sigma))$ be the differentiation of π by ∂ , then $\bar{\partial}Y = \pi_*(\kappa(\partial))$, since π_* is holomorphic and $\kappa(\partial) = \bar{\partial}W$ where W is the vector field as defined in Chapter 5. If we view π_* as a section of $K_S \otimes \pi^*(T\Sigma)$, then

$$W = \frac{Y}{\pi_*}$$

is a vector field on S with poles at the ramification points and satisfies $\bar{\partial}W = \kappa$. Let:

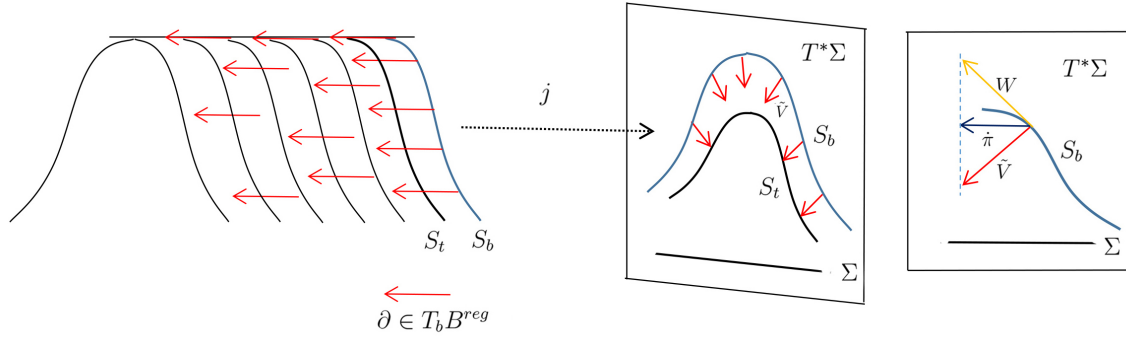
- $j : Z^{reg} \rightarrow T^*\Sigma$ the map sending a spectral curve to its image in $T^*\Sigma$;
- $\tilde{\pi} = \pi \circ j$;
- ∂ a $(1,0)$ -vector field on B^{reg} ;
- $\delta = \partial - W(\partial)$, where $W(\partial)$ means the vector field W determined by ∂ ,

then δ is the unique lift of ∂ to $Z^{reg}|_U$ such that $\tilde{\pi}(\delta) = 0$ (i.e., $\tilde{\pi}$ is constant along the integral curves of δ). Since W has poles at ramification points, δ is a vector field with poles of $Z^{reg}|_U$. Let $S := S_b$, the variation of θ is given by the next proposition. [BH17]

Proposition 6.2.2. *Let V be the section of the normal bundle N_S given by $\chi(\partial)$, then $\delta\theta = \mathbf{i}_V d\theta|_S$ (where in the right hand side, to simplify the notation, we also use θ as the canonical 1-form of $T^*\Sigma$, and $\mathbf{i}_V d\theta|_S$ means restricting $\mathbf{i}_V d\theta$ to S , i.e. we only apply the 1-form to tangent vectors in TS).*

Proof. Let $D = \sum_a (r(a) - 1)a$ be the ramification divisor of π , where a are the ramification points and $r(a)$ the ramification index, let $\tilde{V} = j_*(\partial) \in \Omega^0(S, T(T^*\Sigma))$ and $\hat{V} = j_*(\delta) \in \Omega^0(S, T(T^*\Sigma))(D)$ (see the picture below). Note that \hat{V} is meromorphic, since W has poles.

∂ direction of the fibers (spectral curves) in $Z^{reg}|_U \cong U \times S_b$



Note that \tilde{V} and \hat{V} are lifts of V because the flow of them gives spectral curves for each t , but they depend on the trivialisation of $Z^{reg}|_U \cong U \times S_b$. Also note that $\pi_* \tilde{V} = \pi_* W$. Since δ preserves $\pi \circ j$, $\pi_* j_* \delta = 0$. Hence $\pi_* \hat{V} = 0$. Since $j_*|_{TS} = Id$, we have $\hat{V} = j_*(\delta) = j^*(\partial - W) = \tilde{V} - W$. Applying $\bar{\partial}$ to $\pi_* \tilde{V} = \pi_* W$, we have $\pi_*(\bar{\partial} \tilde{V}) = \pi_*(\bar{\partial} W) = \pi_* \kappa$. Since $\pi_* : TS \rightarrow \pi^* T\Sigma$ is generically an isomorphism, so $\kappa = \bar{\partial} W = \bar{\partial} \tilde{V}$, hence \hat{V} is meromorphic. The identity $j_*(\partial) = \tilde{V}$ means the diffeomorphisms φ_t are given by integral curves of \tilde{V} , hence the change in θ is $\delta\theta = \mathcal{L}_{\tilde{V}} \theta|_S$. Then we have

$$\delta\theta = \partial\theta - \mathcal{L}_W \theta = \mathcal{L}_{\tilde{V}-W} \theta|_S = \mathcal{L}_{\tilde{V}} \theta|_S = \mathbf{i}_{\tilde{V}} d\theta|_S + d(\mathbf{i}_{\tilde{V}} \theta)|_S,$$

where the last equality is the Cartan's formula. Since θ vanishes at the ramification points with order at least that of π_* , hence $\mathbf{i}_{\tilde{V}} \theta$ has no pole on S , i.e. it is a holomorphic function on S , hence a constant. Therefore $\delta\theta = \mathbf{i}_{\tilde{V}} d\theta|_S = \mathbf{i}_V d\theta|_S$, since \hat{V} is a lift of V (note that a tangent line of S in $T^*\Sigma$ is always Lagrangian with respect to $d\theta$). \square

Example 6.2.3. ($g = 1, SL_2(\mathbb{C})$ moduli space) Let \mathbb{C} be a complex plane with coordinate x , think of Σ as $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, then a holomorphic one form on Σ has the form λdx . In particular, λ gives the coordinate of the base B . When $\lambda \neq 0$, the spectral curve of Σ is $\{(y, p) \mid y^2 = -\lambda, p \in \Sigma\}$. In these settings, we have

$$V = \frac{\partial}{\partial \lambda} = -\frac{1}{2y} \frac{\partial}{\partial y},$$

hence

$$\mathbf{i}_V d\theta = -\frac{1}{2y} dx.$$

On the other hand,

$$\delta_\lambda \theta = \left(\frac{\partial \sqrt{-\lambda}}{\partial \lambda} \right) dx = -\frac{1}{2y} dx.$$

Now let x be a local coordinate of Σ , y be the coordinate of the fiber of $T^*\Sigma$ such that (x, y) corresponds to $(x, ydx) \in T^*\Sigma$, then $\theta = ydx$. Let $\xi = y\frac{\partial}{\partial y}$, and $\alpha = \mathbf{i}_V d\theta|_S$ then we have

Lemma 6.2.4.

$$\hat{V} = \frac{\alpha}{\theta|_S} \xi. \quad (6.4)$$

Proof. Since \hat{V} is a lift of V satisfying $\pi_*(\hat{V}) = 0$, therefore \hat{V} is uniquely determined (because for a tangent vector in $X \in TS$, $\pi_*(X)$ is non zero generically). Therefore all we need to prove is that the right hand side of 6.4, denoted as V^* , is a section of $T(T^*\Sigma)(D)$ which is a lift of V and $\pi_*(V^*) = 0$. The condition $\pi_*(V^*) = 0$ is automatically satisfied by the definition of V^* since $\pi_*(\xi) = 0$. Now we write $\alpha = \alpha(q)dq$ for some local coordinate of S , then

$$\begin{aligned} V^* &= \frac{\alpha(q)dq}{ydx} y \frac{\partial}{\partial y} \\ &= \frac{\alpha(q)}{x'(q)} \frac{\partial}{\partial y}, \end{aligned}$$

which says the polar divisor of V^* is D . Since

$$\mathbf{i}_{V^*} d\theta|_S = \frac{\alpha}{\theta|_S} \theta|_S = \alpha,$$

thus V^* is a lift of V . Hence $V^* = \hat{V}$. \square

Also we can vary the Bergmann kernel and other Eynard-Orantin invariants by δ [EO07] [BH17].

Theorem 6.2.5. (*Rauch variational formula*)

$$\delta B(p, q) = - \sum_a \operatorname{Res}_{u \rightarrow a} \frac{\delta \theta(u) B(u, p) B(u, q)}{dx(u) dy(u)}$$

where p, q are distinct and are not ramification points.

Proof. Note that the vector field W such that $\pi_* W = \dot{\pi}$ depends on the choice of the local trivialisation of $Z^{\text{reg}}|_U \cong U \times S$ (the diffeomorphisms φ_t). We can choose a local trivialisation such that W vanishes in a neighbourhood of p and q (i.e. we choose a trivialisation so that p, q deforms inside the fibers $T^*\Sigma$ they belong to, we can do that because p, q are not ramification points). Then

$$\delta B(p, q) = \partial B(p, q) - \mathcal{L}_{W(p)} B(p, q) - \mathcal{L}_{W(q)} B(p, q) = \partial B(p, q).$$

Next we have the variational formula for $B(p, q)$ ([Fay92], page 57),

$$\partial B(p, q) = \kappa(p) B(p, q) + \kappa(q) B(q, p) - \frac{1}{2\pi i} P.V. \int_S \kappa(-) B(-, p) \wedge B(-, q),$$

where $P.V.$ is the Cauchy principal value of the integral. Since $\kappa = \bar{\partial} W$, then $\kappa(p) = \kappa(q) = 0$ and

$$\partial B(p, q) = \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} W(u) B(u, p) B(u, q),$$

where \sum_a is the sum over all poles of $W(u)B(u, p)B(u, q)$ (i.e. ramification points and p and q , but since W is zero near p and q , we only need to take the sum over the ramification points), γ_a is a contour around a . Note that if $a = q(0)$ is a ramification point, $Y(a) = c \frac{\partial}{\partial x} \in T_{\pi(a)}\Sigma$ is a tangent vector describing the motion of the branch point $\pi(a)$ correspond to ∂ , then under these coordinates,

$$W(a) = c \frac{\partial}{\partial x} + d \frac{\partial}{\partial y} = \frac{c}{x'(0)} \frac{\partial}{\partial q} + \frac{d}{y'(0)} \frac{\partial}{\partial q}$$

where the second term has no pole. Let $\zeta = 1/(dx dy)$, V in these coordinates is $V(a) = Y(a)$, and $-(i_V d\theta|_S)(a)\zeta(a) = c/(x'(0)dq)$ has the same pole as $W(a)$. Therefore by proposition 6.2.2 we can write W as $-(\delta\theta)\zeta + W'$ where W' has no pole. Then

$$\begin{aligned} \partial B(p, q) &= \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} W(u)B(u, p)B(u, q) \\ &= - \sum_a \operatorname{Res}_{u \rightarrow a} (\delta\theta)(u)\zeta(u)B(u, p)B(u, q) \\ &= - \sum_a \operatorname{Res}_{u \rightarrow a} \frac{\delta\theta(u)B(u, p)B(u, q)}{dx(u)dy(u)}. \end{aligned} \quad (6.5)$$

Hence we finish the proof. \square

If $\{\lambda_1, \dots, \lambda_{g_S}\}$ are (the affine) coordinates of B^{reg} as before, δ_i is the lift of ∂_i as in proposition 6.2.2, then from theorem 6.2.5 we have

$$\delta_i B(p, r) = - \sum_a \frac{\omega_i(a)}{y'(a)} \frac{B(a, p)B(a, r)}{2dqdq}, \quad (6.6)$$

where the sum is over all simple ramification points, q the local coordinate of a . Because if p, r are not ramification points then $B(u, p)B(u, r)$ have no pole when $u \rightarrow a$. Also $\delta\theta$ is ω_i in this case from Lemma 3.5.4, and note that $\delta\theta$ is holomorphic from the proof of Proposition 6.2.2, and $dx = 2q dq$.

Theorem 6.2.6. (Variational formula) (See [EO07] Theorem 5.7, [BH17] Theorem 7.2) For $g + k > 1$,

$$\delta_i W_k^{(g)}(p_1, \dots, p_k) = - \frac{1}{2i\pi} \int_{p \in b_i} W_{k+1}^{(g)}(p, p_1, \dots, p_k),$$

where δ_i is the δ corresponding to the i -th affine coordinate of B^{reg} , and the cycle b_i is chosen so that it contains no ramification points.

Now using the variational formulas, we are able to recompute the Donagi-Markman cubic. Recall that $\frac{1}{2\pi i} \int_{p_1 \in b_j} B(p_1, p_2) = \omega_j(p_2)$ and $\int_{p_2 \in b_k} \omega_j = \tau_{jk}$. Therefore the cubic c_{ijk} can be computed by

$$\begin{aligned} c_{ijk} = \partial_i \tau_{jk} &= \frac{1}{2\pi i} \partial_i \int_{p_1 \in b_j} \int_{p_2 \in b_k} B(p_1, p_2) \\ &= \frac{1}{2\pi i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} \delta_i B(p_1, p_2) \\ &= - \left(\frac{1}{2\pi i} \right)^2 \int_{p \in b_i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} W_3^{(0)}(p, p_1, p_2). \end{aligned}$$

From formula 6.2, we have

$$c_{ijk} = -2\pi i \sum_a \operatorname{Res}_{q \rightarrow a} \left(\frac{\omega_i(q)\omega_j(q)\omega_k(q)}{dx(q)dy(q)} \right),$$

which is the same as the formula of c_{ijk} we obtained in the last section. Using the recursion and variation formulas, we are also able to compute [BH17],

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-2}} \tau_{i_{m-1} i_m} = - \left(\frac{i}{2\pi} \right)^{m-1} \int_{p_1 \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_m^{(0)}(p_1, \dots, p_m).$$

In this way, we have a power series expansion for the period matrix τ_{ij} about any point in B^{reg} by using $W_k^{(0)}$. From Chapters 2,3,4, we have seen that the special Kähler metrics are given by the periods of the spectral curves, hence the Kähler geometry of B^{reg} around a point $p \in B^{reg}$ is determined by the spectral curve corresponds to p . Next, we will study what $W_k^{(1)}$ computes.

6.3 Bergmann tau-function, $F^{(1)}$ and $W_k^{(1)}$

Start with $\pi : S \rightarrow \Sigma$, a holomorphic n -fold covering with only simple ramification points. Let p_1, \dots, p_m be the branch points on Σ and a_1, \dots, a_m be the ramification points on S , x_i be local coordinates of a small neighbourhood of p_i with $x_i(p_i) = 0$. Now fix Σ and vary S in B^{reg} then both the branch points and ramification points are varying.

Let $\mathcal{H}_{n,m}$ be the set of all n -fold coverings with m simple branch points, let x_i be a local coordinate of an open disc on Σ containing the branch point p_i , then the coordinates x_1, \dots, x_m of branch points on Σ can be considered as local coordinates of a neighbourhood of a point $\pi : S \rightarrow \Sigma$ in $\mathcal{H}_{n,m}$. This gives $\mathcal{H}_{n,m}$ the structure of a complex manifold. We shall rewrite it to be $\{z_1, \dots, z_m\}$ to distinguish the coordinate on Σ , then $z_i = x_i = x(a_i)$. Let $B^{sim} \subset B^{reg}$ be the dense subset consisting of all points whose spectral curves only have simple ramification points, then there is a natural map $B^{sim} \rightarrow \mathcal{H}_{n,m}$, where $m = \deg(K^{n(n-1)})$, then by the same reason in the proof of Theorem 6.2.5, formula 6.5 extends to any vector field ∂ in $\mathcal{H}_{n,m}$ with $\delta\theta$ describing the variation of θ with respect to ∂ . The Rauch variational formula of the Bergmann kernel in the coordinates $\{z_1, \dots, z_m\}$ is given by (see e.g. [EO07] formula 5-3, 5-4 or Theorem 6.2.5, where it is clear that $\delta\theta = (dy/dx)dx = dy$ in this case)

$$\frac{\partial B(P, Q)}{\partial z_i} = \operatorname{Res}_{R \rightarrow a_i} \frac{B(R, P)B(R, Q)}{dx(R)}. \quad (6.7)$$

Let $q_k = \sqrt{x - z_k}$ for $1 \leq k \leq m$, then q_k is a local coordinate of S in the neighbourhood of the ramification point a_k , and

$$q_k(a_k) = 0.$$

Define

$$\mathcal{B}_k := -\frac{1}{12} S_B(0).$$

then \mathcal{B}_k is a function of z_k . Note that \mathcal{B}_k depends on the choice of local coordinate x_k . We will examine the dependence of \mathcal{B}_k on the choice of coordinate later. We can do this

for all $k = 1, \dots, m$, and define a connection on $\mathcal{H}_{n,m}$ in a neighbourhood of $\pi : S \rightarrow \Sigma$ to be

$$d_B = d - \sum_{k=1}^m \mathcal{B}_k dz_k. \quad (6.8)$$

This is a locally defined connection in the coordinates $\{z_1, \dots, z_m\}$.

Theorem 6.3.1. *For $i, j = 1, \dots, m$ we have*

$$\frac{\partial \mathcal{B}_i}{\partial z_j} = \frac{\partial \mathcal{B}_j}{\partial z_i}.$$

Proof. When $i = j$, the formula is correct automatically. Assume $i \neq j$, let a_i be a ramification point, U_i be an open disc of a_i , U_j an open disc of a_j , let $q : U_i \rightarrow \mathbb{C}$, $r : U_j \rightarrow \mathbb{C}$ be coordinates, such that $q = \sqrt{x - z_i}$ near a_i , and $r = \sqrt{x - z_j}$ near a_j . Then we have

$$B(q, -q) = - \left(\frac{1}{4q^2} + \frac{1}{6} S_B(q(a_i)) + \mathfrak{o}(1) + \dots \right) dqdq. \quad (6.9)$$

Therefore

$$\mathcal{B}_i = \operatorname{Res}_{q \rightarrow a_i} \frac{B(q, -q)}{2q dq} = \operatorname{Res}_{q \rightarrow a_i} \frac{B(q, -q)}{dx}.$$

From the Rauch variational formula 6.7 (which is a formula for x fixed), we have

$$\begin{aligned} \frac{\partial \mathcal{B}_i}{\partial z_j} &= \operatorname{Res}_{q \rightarrow a_i} \frac{\partial B(q, -q)}{\partial z_j} \frac{1}{dx} \\ &= \operatorname{Res}_{q \rightarrow a_i} \left(\operatorname{Res}_{r \rightarrow a_j} \frac{B(r, q) B(r, -q)}{2r dr} \right) \frac{1}{2q dq} \\ &= \operatorname{Res}_{q \rightarrow a_i} \operatorname{Res}_{r \rightarrow a_j} \frac{B(r, q) B(r, -q)}{4q r dq dr} \\ &= \operatorname{Res}_{r \rightarrow a_j} \operatorname{Res}_{q \rightarrow a_i} \frac{B(r, q) B(r, -q)}{4q r dq dr}. \end{aligned} \quad (6.10)$$

Since we need to consider the situation when $r \rightarrow a_j$, $q \rightarrow a_i$ and $i \neq j$, $B(r, q)$ and $B(r, -q)$ are holomorphic in a small neighbourhood of $(a_j, a_i) \in S \times S$. The series expansions of $B(r, q)$ and $B(r, -q)$ near $q = 0$ (note that $q \rightarrow a_i$) have the form

$$\begin{aligned} B(r, q) &= (c_0(r) + c_1(r)q + \dots) drdq \\ B(r, -q) &= (c_0(r) - c_1(r)q + \dots) drdq. \end{aligned}$$

Therefore the first residue of the last equality in 6.10 is just

$$\operatorname{Res}_{q \rightarrow a_i} \frac{B(r, q) B(r, -q)}{4q r dq dr} = \operatorname{Res}_{q \rightarrow a_i} \frac{B(r, q) B(r, q)}{4q r dq dr} = \frac{c_0(r) c_0(r)}{4r} dr.$$

This means,

$$\frac{\partial \mathcal{B}_i}{\partial z_j} = \operatorname{Res}_{r \rightarrow a_j} \operatorname{Res}_{q \rightarrow a_i} \frac{B(r, q) B(r, q)}{4q r dq dr}$$

which is symmetric in i, j . □

Definition 6.3.2. The Bergmann tau-function τ_B is the solution of the equations

$$\frac{\partial \ln(\tau_B)}{\partial z_k} = \mathcal{B}_k = -\frac{1}{12}S_B(a_k) \quad (6.11)$$

for all k with $1 \leq k \leq m$.

We will soon see that τ_B can be interpreted as a section of a line bundle on the universal cover $\tilde{\mathcal{H}}_{n,m}$ of $\mathcal{H}_{n,m}$.

Let's fix all z_i except z_k at the moment, and omit the subscript k when it is not necessary to be mentioned. Consider the coordinate change $z \rightarrow \hat{z}$ of $\mathcal{H}_{n,m}$. The coordinate change of z corresponds to a coordinate change $x \rightarrow \hat{x} = h(x)$.

For each z , expand $h(x)$ in a neighbourhood of $x = z$, we have

$$\hat{x} = h(x) = h(z) + h'(z)(x - z) + \frac{1}{2}h''(z)(x - z)^2 + \frac{1}{6}h'''(z)(x - z)^3 + \dots$$

Define, $q = \sqrt{x - z}$, and $\hat{q} = \sqrt{\hat{x} - \hat{z}} = f(q)$ as coordinate change on S near the ramification point a_k . Note that $q(a_k) = \hat{q}(a_k) = 0$, then $f(0) = 0$. Expand f at $q = 0$,

$$\hat{q} = f(q) = f'q + \frac{1}{2}f''q^2 + \frac{1}{6}f'''q^3 + \dots$$

From the relation $q = \sqrt{x - z}$, and $\hat{q} = \sqrt{\hat{x} - \hat{z}} = f(q)$, we have

$$\begin{aligned} 0 &= f'' \\ h'(z) &= f'^2 \\ h''(z) &= \frac{1}{2}f''^2 + \frac{2}{3}f'f''' = \frac{2}{3}f'f''' \end{aligned} \quad (6.12)$$

Note that f' , f''' depend on z (when necessary, write $f' := f'_z$ etc.), because we get different f for each z . Recall that

$$S_B(a) = (f')^2 \hat{S}_B(a) + \left(\frac{f'''}{f'} - \frac{3f''^2}{2f'^2} \right),$$

then we have

$$\frac{1}{12}S_B(a)dz - \frac{1}{12}\hat{S}_B(a)d\hat{z} = \frac{3}{24}d\ln(h'(z)) = \frac{1}{24}d\ln(h'(z)^3). \quad (6.13)$$

Note that if $\hat{z}_i = h_i(z_i)$ is a coordinate change on V_i , where V_i is a neighbourhood of the branch point on Σ corresponds to a_i , then $h'_i(z_i)$ is the transition function of $K_\Sigma|_{V_i}$. Let

$$\mathcal{H}_{n,m} \supset V_1 \times V_2 \times \dots \times V_m \xrightarrow{r_i} V_i$$

be the projection, then $\{\sum_{i=1}^m 2S_B(a_i)dz_i\}$ is a set of connection 1-forms of the line bundle $\bigotimes_{i=1}^k r_i^* K_\Sigma^3|_{V_i}$ on $\mathcal{H}_{n,m}$ because of 6.13.

Let $d_B = d + \sum_{i=1}^m 2S_B(a_i)dz_i$ be the connection, from 6.11, we have

$$\begin{aligned} d_B(\tau_B^{24}) &= d\tau_B^{24} + 2 \sum_{i=1}^m S_B(a_i)\tau_B^{24}dz_i \\ &= d\tau_B^{24} + 2 \sum_{i=1}^m -12\tau_B^{23} \frac{\partial \tau_B}{\partial z_i} dz_i \\ &= 0. \end{aligned}$$

Hence τ_B^{24} is a flat section of the line bundle $\mathcal{L} = \bigotimes_{i=1}^m r_i^* K_\Sigma^3|_{V_i}$ (pulled back to the universal cover $\tilde{\mathcal{H}}_{n,m}$) with connection d_B .

Now choose a point $[S] \in V_1 \times \cdots \times V_m$, then locally $K_\Sigma|_{V_i}$ is isomorphic to $K_S^2|_{U_i}$ by identifying $dx_i \sim dqdq$. If $\hat{x}_i = h(x_i)$ and $\hat{q} = f(q)$ are coordinate changes, then the isomorphism gives $f'(q)^2 = h'(x_i)$. At the point $x_i = z_i$, the formula 6.12 is satisfied. Note that the isomorphism can be given by the jet bundle $J^1(d\pi) \in \text{Hom}(K_\Sigma|_{V_i}, K_S^2|_{U_i})$, where simply branching of the covering implies $J^1(d\pi)(x_i) \neq 0$. Therefore, the line bundle \mathcal{L} is isomorphic to $\bigotimes_{i=1}^m K_S^6|_{U_i}$. Then τ_B^{12} is a section of $\bigotimes_{i=1}^m K_S^3|_{U_i}$ (pulled back to the universal cover). In the z coordinate, the transition function is given by $g(z) = (f')^3$ when we change the coordinate $\hat{z} = h(z)$ (note that the value f' depends on z).

Definition 6.3.3. *The free energy $F^{(1)}$ is defined to be*

$$F^{(1)} = -\frac{1}{2} \ln(\tau_B) - \frac{1}{24} \ln\left(\prod_a y'(a)\right),$$

where a 's are the ramification points and y is the coordinate of dx_i , and consider it as a function of q when a branch was chosen, and $y'(a) = y'(q)|_{q=0}$.

Proposition 6.3.4. $e^{24F^{(1)}}$ is a well-defined function on the universal cover \tilde{B}^{sim} of B^{sim} .

Proof. From the definition above,

$$e^{24F^{(1)}} = \frac{\tau_B^{-12}}{\prod y'(a)}.$$

We have seen that τ_B^{-12} is a section of $\mathcal{L}^{-\frac{1}{2}} \cong \bigotimes_{i=1}^m K_S^{-3}|_{U_i}$. We can restrict the section to $\tilde{B}^{sim} \rightarrow \tilde{\mathcal{H}}_{n,m}$. Since

$$\theta = y(q)dx = \hat{y}(\hat{q})d\hat{x},$$

apply $\frac{d}{dq}$ to the middle and the right hand side, we have $y'(0) = \hat{y}'(0)(f')^3$. So $\prod y'(a)$ is also a section of $\bigotimes_{i=1}^m K_S^{-3}|_{U_i}$. Therefore $e^{24F^{(1)}}$ is the ratio of two sections of the same line bundle, so is a function on \tilde{B}^{reg} . \square

Next we will give the relation between $F^{(1)}$ and $W_1^{(1)}$ [EO07]. From formula 6.3 we have,

$$\begin{aligned} \frac{1}{2\pi i} \int_{p \in b_i} W_1^{(1)}(p) &= \sum_a \text{Res}_{q \rightarrow a} \frac{\frac{1}{2\pi i} \int_{p \in b_i} dE_q(p)}{\omega(q)} B(q, -q) \\ &= \sum_a \frac{1}{24y'(a)} \left(\omega_i(a) S_B(a) + \frac{\omega_i''(a)}{4} - \frac{y'''(a)\omega_i(a)}{4y'(a)} \right), \end{aligned}$$

where the second equality is computed by the expansions of $\frac{1}{\omega(q)}$, $B(q, -q)$ (see formula 6.9) and the fact that ([BH17], section 7.3)

$$\frac{1}{2\pi i} \int_{p \in b_i} dE_q(p) = -q(\omega_i(0) + \frac{1}{6}\omega_i''(0)q^2 + \cdots).$$

Proposition 6.3.5. *The partial derivative of z_j with respect to λ_i of the map $B^{sim} \rightarrow \mathcal{H}_{n,m}$ is given by*

$$\frac{\partial z_j}{\partial \lambda_i} = -\frac{\omega_i(a_j)}{y'(a_j)}. \quad (6.14)$$

Proof. Near the ramification point a_j , from the definition of W (or formula 5.1)

$$W = \frac{\partial z_j}{\partial \lambda_i} \frac{\partial}{\partial x} = \frac{\partial z_j}{\partial \lambda_i} \frac{1}{2q} \frac{\partial}{\partial q}.$$

Then

$$\begin{aligned} \frac{\partial z_j}{\partial \lambda_i} &= 2\text{Res}(W) \\ &= -2\text{Res}(\hat{V}). \end{aligned}$$

From (the proof of) Proposition 6.2.2, $\delta_i \theta$ is holomorphic. Together with Lemma 3.5.4, we have $\omega_i = \delta_i \theta$ (because if df is an exact holomorphic 1-form, then $\bar{\partial}(df) = \bar{\partial} \partial f = 0$, which implies f is a harmonic function. Since the spectral curve is compact, hence f is constant by the maximum principle). By Lemma 6.2.4

$$\text{Res}(\hat{V}) = \frac{\omega_i(a_j)}{2y'(a_j)},$$

which finishes the proof. \square

Proposition 6.3.6. [EO07]

$$-\frac{1}{2\pi i} \int_{p \in b_i} W_1^{(1)}(p) = \delta_i F^{(1)}.$$

Proof. Let $\lambda_1, \dots, \lambda_{g_S}, x_1, \dots, x_m, z_1, \dots, z_m$ as defined before, and choose a ramification point a_j , let $q = \sqrt{x_j - z_j}$. Recall that $\frac{\partial z_j}{\partial \lambda_i} = -\frac{\omega_i(0)}{y'(0)}$. By the definition of τ_B , we have

$$\sum_a \frac{1}{24y'(a)} \omega_i(a) S_B(a) = \delta_i \left(\frac{1}{2} \ln(\tau_B) \right).$$

Since δ_i is a lift of $\frac{\partial}{\partial \lambda_i}$ such that $\delta_i x = 0$, then $\delta_i = \frac{\partial}{\partial \lambda_i} + g \frac{\partial}{\partial q}$ for some function g of $q, \lambda_1, \dots, \lambda_{g_S}$. Near the ramification point a_j , we have $\frac{\partial x}{\partial \lambda_i} = \frac{\partial z_j}{\partial \lambda_i}$, hence $g = \frac{\omega_i(0)}{2qy'(0)}$.

Now since $\delta_i(y) dx = \omega_i(q) dq$, we have

$$2q \frac{\partial y(q)}{\partial \lambda_i} + \frac{\omega_i(0)y'(q)}{y'(0)} = \omega_i(q).$$

Expanding it to order q^2 , we have

$$\begin{aligned} & 2q \left(\frac{\partial y(0)}{\partial \lambda_i} + q \frac{\partial y'(0)}{\partial \lambda_i} + \dots \right) + \frac{\omega_i(0)}{y'(0)} \left(y'(0) + qy''(0) + \frac{1}{2}q^2y'''(0) + \dots \right) \\ &= \omega_i(0) + q\omega_i'(0) + \frac{1}{2}q^2\omega_i''(0) + \dots \end{aligned}$$

Comparing the coefficients of q^2 , we get

$$\frac{\partial y'(0)}{\partial \lambda_i} = \frac{1}{4}(\omega_i''(0) - \frac{\omega_i(0)y'''(0)}{y'(0)}), \quad (6.15)$$

and finishes the proof. \square

Proposition 6.3.7.

$$\frac{\partial \omega_i(a_j)}{\partial \lambda_k} = - \sum_{a \neq a_j} \frac{\omega_k(a) \omega_i(a) B(r(a), q)}{2y'(a) dr} - \frac{\omega_k(a) \omega_i(a) S_B(a_j)}{12y'(a)}.$$

Proof. Let q be a local coordinate in a neighbourhood of a_j . In this neighbourhood, let $\omega_i = \omega_i(q) dq$. Then by formulas 6.1 and 6.6, we have

$$\begin{aligned} \delta_k(\omega_i(q) dq) &= \frac{1}{2\pi i} \int_{p \in b_i} \delta_k B(p, q) \\ &= - \sum_a \frac{\omega_k(a)}{y'(a)} \frac{\int_{p \in b_i} B(r(a), p)}{2dr dr} B(a, q) \\ &= - \sum_a \frac{\omega_k(a) \omega_i(a) B(a, q)}{2y'(a) dr}, \end{aligned}$$

where r is the local coordinate of a . Note that $2q\delta_i(q) = -\delta_i z = -\partial_i z$. Then

$$\delta_i(q) = \frac{1}{2q} \frac{\omega_i(a)}{y'(a)},$$

and

$$\delta_i(dq) = -\frac{1}{2q^2} \frac{\omega_i(a)}{y'(a)} dq.$$

Therefore, we have

$$\delta_k \omega_i(q) = - \sum_a \frac{\omega_k(a) \omega_i(a) B(r(a), q)}{2y'(a) dr} + \frac{1}{2q^2} \frac{\omega_i(q) \omega_k(a_j)}{y'(a_j)}. \quad (6.16)$$

Hence

$$\partial_k \omega_i(a_j) = \delta_k \omega_i(a_j) = - \sum_{a \neq a_j} \frac{\omega_k(a) \omega_i(a) B(a, q)}{2y'(a) dr} - \frac{\omega_k(a_j) \omega_i(a_j) S_B(a_j)}{12y'(a_j)}.$$

□

Now combine with formulas 6.14, 6.15, we have

$$\frac{\partial^2 z_j}{\partial \lambda_i \partial \lambda_k} = \sum_{a \neq a_j} \frac{\omega_k(a) \omega_i(a) B(a, q)}{2y'(a_j) y'(a) dr} + \frac{\omega_k(a_j) \omega_i(a_j) S_B(a_j)}{12y'(a_j)^2} + \frac{\omega_i(a_j)}{4y'(a_j)^2} \left(\omega_k''(a_j) - \frac{\omega_k(a_j) y'''(a_j)}{y'(a_j)} \right).$$

One can see that the second order of the expansion of $B^{sim} \rightarrow \mathcal{H}_{n,m}$ depends on the one forms ω_i and the Bergmann kernel of the spectral curves.

Appendix A

Vector Bundles on Compact Riemann Surfaces

A.1 Holomorphic vector bundles and Dolbeault operators

I will briefly give a few facts about complex vector bundles on (compact connected) Riemann surfaces that are needed in my thesis, more detail about holomorphic vector bundles should be found in text books such as [For91] [Jos13].

Let Σ be a compact connected Riemann surface, and $p : E \rightarrow \Sigma$ a complex vector bundle on Σ , the **rank of the vector bundle** E is defined to be the complex dimension of $p^{-1}(x)$ for any $x \in \Sigma$. It is denoted $rank(E)$. A **line bundle** is a rank one vector bundle.

The **degree of the vector bundle** E is defined to be the integral of the first Chern class $c_1(E) \in H^2(\Sigma; \mathbb{Z})$ of E :

$$deg(E) := \int_{\Sigma} c_1(E) \in \mathbb{Z}.$$

The degree of vector bundles satisfies the following properties, which are useful in computations:

$$\begin{cases} deg(E^*) = -deg(E), & \text{where } E^* \text{ is the dual of } E, \\ deg(E_1 \otimes E_2) = deg(E_1)rank(E_2) + rank(E_1)deg(E_2). \end{cases}$$

Theorem A.1.1. *The topology of a vector bundle E over Σ only depends on $deg(E)$ and $rank(E)$.*

In other words, if E_1, E_2 are two complex vector bundles on Σ , if $deg(E_1) = deg(E_2)$, $rank(E_1) = rank(E_2)$, then E_1 and E_2 are isomorphic as topological complex vector bundles. Moreover, each $(r, d) \in \mathbb{Z}_+ \times \mathbb{Z}$ gives an isomorphism class. A proof of the theorem can be found in [Tha96].

Let $p : E \rightarrow \Sigma$ be a complex vector bundle, a **holomorphic structure** on E is by definition a (maximal) holomorphic atlas on it (i.e. the transition maps of the local trivialisations are holomorphic). Sometimes one just write E as a complex vector bundle with a holomorphic structure. A holomorphic structure of E is a global structure that can not be easily described in general. When studying the space of holomorphic structures, one may find it more convenient to work with differential operators that are attached

to the holomorphic structures instead. These differential operators are called Dolbeault operators of E , and we have a bijection between the space of Dolbeault operators and space of holomorphic structures of E .

Let's define a **connection** ∇ on $E \rightarrow \Sigma$ to be a \mathbb{C} -linear map

$$\nabla : \Omega^0(\Sigma; E) \rightarrow \Omega^1(\Sigma; E)$$

where $\Omega^p(\Sigma; E)$ is the space of E valued k -forms, satisfies the Leibniz rule

$$\nabla(fs) = (df)s + f(\nabla s),$$

for all $f \in C^\infty(\Sigma; \mathbb{C})$ and $s \in \Omega^0(\Sigma; E)$. Note that as a consequence of the existence of partitions of unity, any smooth vector bundle admits a connection. Also,

$$\begin{aligned} (\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs) \\ &= (df)s + f(\nabla_1 s) - (df)s - f(\nabla_2 s) \\ &= f(\nabla_1 - \nabla_2)s. \end{aligned}$$

Therefore, $\nabla_1 - \nabla_2$ is a $C^\infty(\Sigma; \mathbb{C})$ -linear map from $\Omega^0(\Sigma; E) \rightarrow \Omega^1(\Sigma; E)$, hence an $\text{End}(E)$ -valued one form on Σ .

Proposition A.1.2. *The space of connections on Σ is an affine space with translation vector space $\Omega^1(\Sigma; \text{End}(E))$.*

Now, since on a Riemann surface $\Omega^1(\Sigma; E) = \Omega^{1,0}(\Sigma; E) \oplus \Omega^{0,1}(\Sigma; E)$, then

$$\nabla = \nabla^{1,0} \oplus \nabla^{0,1} : \Omega^0(\Sigma; E) \rightarrow \Omega^{1,0}(\Sigma; E) \oplus \Omega^{0,1}(\Sigma; E)$$

Note that the operator $\nabla^{0,1}$ is also \mathbb{C} linear and satisfies the Leibniz rule.

Definition A.1.3. (*Dolbeault operator*). *A Dolbeault operator on a vector bundle $E \rightarrow \Sigma$ is a \mathbb{C} -linear map*

$$\bar{\partial}_E : \Omega^0(\Sigma; E) \rightarrow \Omega^{0,1}(\Sigma; E),$$

satisfying the following rule: $\bar{\partial}_E(fs) = (\bar{\partial}f)s + f(\bar{\partial}_E s)$, for any $f \in C^\infty(\Sigma; \mathbb{C})$ and $s \in \Omega^0(\Sigma; E)$.

Similar to how connections extend to tensor product of vector bundles, a Dolbeault operator extends uniquely to $E \otimes E^* = \text{End}(E)$, given by $(\bar{\partial}_E \phi)(s) := \bar{\partial}_E(\phi(s)) - \phi(\bar{\partial}_E s)$, for $\phi \in \text{End}(E)$. Now, let $\text{Dol}(E)$ be the space of all Dolbeault operators on E , and \mathcal{G}_E be the group of automorphisms of E , $g \in \mathcal{G}_E$ acts on $\bar{\partial}_E$ by $(g \cdot \bar{\partial}_E)(s) := g(\bar{\partial}_E(g^{-1}s))$. And one can easily show that $(gh) \cdot \bar{\partial}_E = g \cdot (h \cdot \bar{\partial}_E)$, so the action is a group action.

Theorem A.1.4. *Let $\text{Hol}(E)$ be the space of holomorphic structures on E , then there are bijections:*

$$\begin{aligned} \text{Hol}(E) &\longleftrightarrow \text{Dol}(E), \\ \text{Hol}(E)/\text{isomorphism} &\longleftrightarrow \text{Dol}(E)/\mathcal{G}_E. \end{aligned}$$

And the mappings above send a holomorphic structure \mathcal{E} on E to a Dolbeault operator $\bar{\partial}_E$ such that $\bar{\partial}_E(s) = 0$ for all holomorphic sections $s \in \mathcal{E}$.

A proof of the theorem can be found in [Wel07]. The idea of the proof is to construct a Dolbeault operator for each holomorphic structure using the fact that the clutching functions of E are holomorphic, and use the Newlander-Nirenberg theorem to prove the converse.

A.2 Holomorphic line bundles

Definition A.2.1. Let Σ be a Riemann surface, a divisor on Σ is a locally finite linear combination

$$D = \sum z_i p_i$$

where $z_i \in \mathbb{Z}$ and $p_i \in \Sigma$.

Let f be a meromorphic function on Σ , if f has a zero of order k at a point $z \in \Sigma$, we say that the order of f at p is k , denote as $\text{ord}_p f = k$; if f has a pole of order k at a point $p \in \Sigma$, we say that the order of f at p is $-k$, denote as $\text{ord}_p f = -k$; otherwise we say that the order of f at p is 0, denote as $\text{ord}_p f = 0$.

Definition A.2.2. The divisor of a meromorphic function f is defined by

$$(f) = \sum (\text{ord}_{p_i} f) p_i.$$

If a divisor D on Σ is given by a meromorphic function f , i.e. $D = (f)$, then we called D a principal divisor. Two divisors D_1 and D_2 are called linearly equivalent if $D_1 - D_2 = (f)$ for some meromorphic function f .

Let $\text{Div}(\Sigma)$ be the group of divisors on Σ (with group operation "+") and \sim be the linearly equivalent relation. Let $\text{Pic}(\Sigma)$ be Picard group i.e. the group of isomorphism classes of line bundles

Theorem A.2.3. Let Σ be a compact Riemann surface, then

$$\text{Pic}(\Sigma) \cong \text{Div}(\Sigma) / \sim .$$

Proof. Let D be a divisor on Σ , since D is locally finite (hence finite on Σ), we can find an open cover $\Sigma = \cup U_i$ and meromorphic functions f_i so that the order of f_i coincide with D . On $U_i \cap U_j$ set

$$g_{ij} = \frac{f_i}{f_j},$$

then it is clear that $\{g_{ij}\}$ are non-vanishing meromorphic functions satisfying $g_{ij}g_{ji} = 1$ and the cocycle condition. Therefore $\{g_{ij}\}$ determines a line bundle, we denote it as $[D]_{g_{ij}}$. Now if we have another set of meromorphic functions \tilde{g}_{ij} on $U_i \cap U_j$ and a line bundle $[D]_{\tilde{g}_{ij}}$ also given by D as above, then there are non-vanishing holomorphic functions ϕ_i on U_i such that

$$\tilde{g}_{ij} = \frac{\phi_i}{\phi_j} g_{ij}.$$

That means $[D]_{g_{ij}} \cong [D]_{\tilde{g}_{ij}}$, hence D determines an isomorphism class of line bundle, which is written as $[D]$. Therefore we obtained a map

$$[\] : D \mapsto [D].$$

Note that by definition of $[\]$, we have $[D_1 - D_2] = [D_1] \otimes [D_2]^{-1}$. Also note that $[(g)] = \mathcal{O}$ for any meromorphic function g , since the transition functions given by (g) are identity functions. Therefore

$$[\] : \text{Div}(\Sigma) / \sim \rightarrow \text{Pic}(\Sigma)$$

is an one to one homomorphism.

The surjectivity of $[\]$ comes from the well known fact that every line bundle L on Σ has a meromorphic section. Let D_L be the divisor determined by the meromorphic section then by definition, $[D_L] = L$. This finishes the proof. \square

Let L be a line bundle over Σ , the degree of L is defined as

$$\deg(L) := \int_{\Sigma} c_1(L).$$

Remark A.2.4. Given a divisor D , if we define the degree of D to be $\deg(D) := \sum z_i$, it is clear that $\deg(D)$ is finite. Then we have (see, for example [Jos13] Theorem 5.6.3)

$$\deg(D) = \int_{\Sigma} c_1([D]).$$

This means $\deg([D]) = \deg(D)$.

The theorem also implies that

$$\text{Jac}(\Sigma) \cong \text{Div}_0(\Sigma) / \sim,$$

where $\text{Div}_0(\Sigma)$ is the space of divisors with degree zero.

Theorem A.2.5. (Riemann-Roch) Let g be the genus of Σ , E be a vector bundle over Σ with rank n , then

$$\dim H^0(\Sigma, E) - \dim H^1(\Sigma, E) = n(1 - g) + \deg(E).$$

Theorem A.2.6. (Riemann-Hurwitz formula) Let S, Σ be two compact Riemann surfaces and $f : S \rightarrow \Sigma$ be an m -fold holomorphic covering with ramification points $r_i \in S$. Let d_i be the ramification order of r_i (i.e. there are coordinates in the neighbourhoods of r_i and $f(r_i)$, such that f is given by $f(z) = z^{d_i}$), let $d_f = \sum (d_i - 1)$. Then

$$2 - 2g_s = m(2 - 2g) - d_f,$$

where g_s and g are the genera of S and Σ respectively.

By using these theorems, we are able to compute the dimensions of the space of stable bundles, moduli space of Higgs bundles etc. in this thesis.

Appendix B

Symplectic Geometry

B.1 Symplectic manifolds

Symplectic geometry studies a special kind of manifolds called symplectic manifolds, which are smooth manifolds equipped with closed nondegenerate differential 2-forms. The symplectic form gives certain topological properties of a symplectic manifold. For example, all symplectic manifolds are orientable and even-dimensional. It is crucial to point out that the local structure of a symplectic manifold is, by Moser's and Darboux's theorems, always equivalent to Euclidean space with the canonical symplectic form. Some symplectic manifolds can be treated as Lagrange fibrations, such as the moduli space of Higgs bundles. They are known as integrable systems, to which I will give an introduction in [B.2](#).

Definition B.1.1. *Let M be a real manifold, a symplectic manifold is a pair (M, ω) where ω is a closed and non-degenerate differential 2-form of M .*

The 2-form ω is called a symplectic form, and note that the non-degeneracy of ω forces M to be even-dimensional.

Definition B.1.2. *A submanifold Y of a $2n$ -dimensional symplectic manifold (M, ω) is called isotropic if for $\forall y \in Y$,*

$$\omega_y|_{T_y Y} = 0.$$

And if in addition $\dim Y = \frac{1}{2} \dim M$, we call Y a Lagrangian submanifold.

Theorem B.1.3. *(Moser's Theorem) Let ω_0 and ω_1 be two symplectic forms on a manifold M . If ω_0 and ω_1 coincide on a closed submanifold V , then there exist a neighborhood U_0 of V and a map*

$$\varphi : U_0 \rightarrow M$$

such that

$$\varphi|_V = Id, \varphi^* \omega_1 = \omega_0.$$

As a corollary of Moser's theorem, we have

Theorem B.1.4. *(Darboux's Theorem) Let (M, ω) be a $2n$ -dimensional symplectic manifold, then for $\forall p \in M$, there exist a neighborhood U centered at p and a local coordinate chart $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

The coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ are known as Darboux coordinates.

There is a complex version of symplectic geometry. If we assume M is a complex manifold with complex dimension $2n$, moreover if there is a closed and non-degenerate holomorphic 2-form ω^c attached to M , then we say that M is a complex symplectic manifold. Similarly, a complex submanifold of complex dimension n is complex Lagrangian if $\omega^c|_Y = 0$. In the complex version, we will still have complex Darboux coordinates $(w_1, \dots, w_n, z_1, \dots, z_n)$ of M such that locally $\omega^c = \sum_{i=1}^n dw_i \wedge dz_i$. Now, one can write ω^c in terms of the real and imaginary parts,

$$\omega^c = \omega_1 + i\omega_2.$$

Then one can easily prove that ω_1 and ω_2 are real symplectic forms.

Proposition B.1.5. [*Hit99*] *A real $2n$ -dimensional submanifold $Y \subset M$ is complex Lagrangian if and only if $\omega^c|_Y = 0$.*

Therefore, in Hitchin's words, we lose nothing by focussing our attention on the symplectic aspects of complex Lagrangian submanifolds. When Y is compact, there is a good local moduli space of deformations of the compact complex Lagrangian submanifolds $Y \subset M$. One may see more about this in chapters 3, 5.

B.2 Integrable systems

Let (M, ω) be a real symplectic manifold with dimension $2n$, $H : M \rightarrow \mathbb{R}$ be any function, then the Hamiltonian vector field associated with H is a vector field X_H such that $i_{X_H}\omega = dH$, i.e. $\omega(X_H, -) = dH(-)$.

If $(x_1, \dots, x_n, y_1, \dots, y_n)$ is a local coordinate chart such that $\omega = \sum_{i=1}^n dx_i \wedge dy_i$, then

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i}$$

Note that, by non-degeneracy of ω , X_H is unique. In physics, if we think of M as the phase space, H the total energy, then the Hamiltonian vector field determines the motion of particles.

Definition B.2.1. *A $2n$ -dimensional symplectic manifold (M, ω) is an integrable system if there exist n (maximal number of) independent Poisson commuting functions f_1, f_2, \dots, f_n on M .*

Poisson commute means $\{f_i, f_j\} = 0$, for all i, j where $\{f, g\} := \omega(X_f, X_g)$ is the Poisson bracket, and the functions f_1, f_2, \dots, f_n on M are said to be independent if $(df_1)_x, \dots, (df_n)_x$ are linearly independent at all points x in an open dense subset of M .

If a symplectic manifold M^{2n} is an integrable system, then we can think of it as the foliation of n -dimensional Lagrange submanifolds (integral manifolds of the Hamiltonian vector fields). The advantage of being an integrable system is given by the Arnold-Liouville theorem.

Theorem B.2.2. (*Arnold-Liouville*) *Let a be a regular value of the map $F = (f_1, \dots, f_n)$, then $F^{-1}(a)$ is a Lagrangian submanifold of M . Also,*

- if the flows X_{f_1}, \dots, X_{f_n} starting at $p \in f^{-1}(a)$ are complete, then $f^{-1}(a)$ is a homogeneous space for \mathbb{R}^n . The components have angle coordinates $\theta_1, \dots, \theta_n$, in which the flows of X_{f_1}, \dots, X_{f_n} are linear.

- There exist coordinates $\varphi_1, \dots, \varphi_n$, called action coordinates, such that $\{\varphi_i, f_j\} = 0$ for $\forall i, j = 1, \dots, n$, and $\{\theta_i, \varphi_i\}$ is a Darboux coordinate.

If in addition, the map $F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$ is proper, this implies the compact connected components of $F^{-1}(a)$ for a regular value a is topologically a torus \mathbb{T}^n , and there exist special coordinates called angle-action coordinates on each neighborhood of regular fibers of F .

There is a complex version if M is a complex symplectic manifold. In this case the Darboux coordinates are complex and the functions f_i become complex functions $f_i : M \rightarrow \mathbb{C}$. We call the integrable system a complex integrable system (The Hitchin systems are complex integrable systems).

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SPECIAL KÄHLER GEOMETRY OF THE HITCHIN SYSTEM AND TOPOLOGICAL RECURSION

DAVID BARAGLIA AND ZHENXI HUANG

ABSTRACT. We investigate the special Kähler geometry of the base of the Hitchin integrable system in terms of spectral curves and topological recursion. The Taylor expansion of the special Kähler metric about any point in the base may be computed by integrating the $g = 0$ Eynard-Orantin invariants of the corresponding spectral curve over cycles. In particular, we show that the Donagi-Markman cubic is computed by the invariant $W_3^{(0)}$. We use topological recursion to go one step beyond this and compute the symmetric quartic of second derivatives of the period matrix.

1. INTRODUCTION

The Hitchin integrable system [14, 15] ties together many seemingly different branches of geometry and physics, including twistor theory, integrable systems, mirror symmetry and supersymmetric Yang-Mills theory to name just a few. The goal of this paper is to elucidate one particular aspect of this rich story, namely the relation between the special Kähler geometry of the base of the Hitchin system with the theory of Eynard-Orantin topological recursion of the spectral curves.

Suppose that $f : \mathcal{M} \rightarrow B$ is a complex integrable system, so \mathcal{M} is a complex manifold with a holomorphic symplectic form and f is a complex Lagrangian fibration, whose generic fibres are complex tori. In general f will have singular fibres so let $f : \mathcal{M}^{\text{reg}} \rightarrow B^{\text{reg}}$ denote the regular locus consisting of the non-singular torus fibres of f . As we recall in §3, under mild assumptions, there is a naturally defined metric g^{sk} on the base B^{reg} , known as a *special Kähler metric* [4, 10, 16]. We recall the fundamentals of special Kähler geometry in §2. In particular, B^{reg} has a Kähler metric of the form

$$(1.1) \quad \omega = -\frac{i}{2} \text{Im}(\tau_{ij}) dz^i \wedge d\bar{z}^j$$

where τ_{ij} is a matrix of functions, the periods of the torus fibres of the integrable system. It is well known that the special Kähler metric on B^{reg} can be combined with a metric along the fibres to produce a hyperkähler metric g^{sf} on \mathcal{M}^{reg} , known as the *semi-flat hyperkähler metric* [3, 10, 16]. This metric is called “semi-flat” because its restriction to the fibres of $f : \mathcal{M}^{\text{reg}} \rightarrow B^{\text{reg}}$ is flat.

In this paper we concentrate on the case that $f : \mathcal{M} \rightarrow B$ is the Hitchin system. Then $\mathcal{M} = \mathcal{M}_{n,d}$ is the *moduli space of Higgs bundles* of given rank n and degree d . In this case \mathcal{M} is known to admit a complete hyperkähler metric g [14]. The

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semi-flat metric on \mathcal{M}^{reg} may be thought of as an approximation of the complete hyperkähler metric. The semi-flat metric fails to extend over the singular fibres, however it is expected that g can be recovered from g^{sf} by incorporating instanton corrections [11].

To define the Hitchin system, one takes a compact Riemann surface Σ of genus greater than 1 and \mathcal{M} to be the moduli space of semistable Higgs bundles of fixed rank n and degree d (see §4). In this case, the period matrices $\tau_{ij}(b)$ for $b \in B^{\text{reg}}$ are not arbitrary, in fact they are the periods of a Riemann surface S_b , the *spectral curve* associated to the point $b \in B^{\text{reg}}$. The spectral curve S_b is a smooth compact Riemann surface $S_b \subset T^*\Sigma$ embedded in the cotangent bundle of Σ , such that the projection $\pi : S_b \rightarrow \Sigma$ is a branched covering of degree n . We now give a brief summary of the special Kähler geometry on B^{reg} in terms of spectral curves. Let θ be the canonical 1-form on $T^*\Sigma$. A pair of local holomorphic coordinate systems (z^1, \dots, z^{g_S}) and (w_1, \dots, w_{g_S}) for B^{reg} are given by integrating the canonical 1-form over a basis of 1-cycles in S_b :

$$z^i = \int_{a_i} \theta, \quad w_i = \int_{b_i} \theta,$$

where $a_1, \dots, a_{g_S}, b_1, \dots, b_{g_S}$ is a symplectic basis of 1-cycles, which may be defined over any sufficiently small neighbourhood in B^{reg} . The monodromy of the Hitchin system prevents us from choosing such a basis globally on B^{reg} . Having chosen such a basis of 1-cycles, we have the normalised basis of holomorphic 1-forms $\omega_1, \dots, \omega_{g_S}$ characterised by $\int_{a_i} \omega_j = \delta_{ij}$. The period matrix of S_b is then given by $\tau_{ij} = \int_{b_i} \omega_j$. With respect to the coordinate system (z^1, \dots, z^{g_S}) , the special Kähler metric is given by Equation (1.1). Alternatively, we have a real coordinate system $(x^1, \dots, x^{g_S}, y_1, \dots, y_{g_S})$ given by:

$$x^i = \text{Re}(z^i) = \text{Re}\left(\int_{a_i} \theta\right), \quad y_i = \text{Re}(w_i) = \text{Re}\left(\int_{b_i} \theta\right).$$

These coordinates are globally defined up to monodromy, which acts by linear transformations, hence they define an affine structure on B^{reg} . Moreover they are Darboux coordinates for the Kähler form:

$$\omega = dx^1 \wedge dy_1 + \dots + dx^{g_S} \wedge dy_{g_S}.$$

Recall the *prepotential* (see §2) is a locally defined holomorphic function \mathcal{F} on B^{reg} such that $w_i = \frac{\partial \mathcal{F}}{\partial z^i}$. We will recall that the period matrix is given by $\tau_{ij} = \frac{\partial w_i}{\partial z^j}$ and thus

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial z^i \partial z^j}.$$

From the prepotential \mathcal{F} , one obtains a Kähler potential K by:

$$K = -\frac{1}{2} \text{Im} \left(\frac{\partial \mathcal{F}}{\partial z^i} \bar{z}^i \right).$$

An important quantity in special Kähler geometry is the symmetric cubic $c \in H^0(\mathcal{M}, \text{Sym}^3(T_{\mathcal{M}}))$ which measures the variation of the period matrix τ_{ij} :

$$c = c_{ijk} dz^i \otimes dz^j \otimes dz^k, \quad c_{ijk} = \frac{\partial \tau_{jk}}{\partial z^i} = \frac{\partial^3 \mathcal{F}}{\partial z^i \partial z^j \partial z^k}.$$

We call c the *Donagi-Markman cubic*, since in the case where \mathcal{M} is the base of a complex integrable system, c is the cubic studied by Donagi and Markman [5]. One

can for instance express the Riemann, Ricci and scalar curvatures in terms of τ_{ij} and c_{ijk} [10]. The following proposition (cf. [13] for similar results) summarises the relation between the special Kähler geometry on B^{reg} and the spectral curves:

Proposition 1.1. *We have the following relations:*

- (1) *A (local) prepotential for the special Kähler structure on B^{reg} is given by:*

$$\mathcal{F} = \frac{1}{2} z^i w_i = \frac{1}{2} \tau_{ij} z^i z^j.$$

- (2) *A (global) Kähler potential is given by:*

$$K = -\frac{i}{4} \int_S \theta \wedge \bar{\theta}.$$

- (3) *Let $\mathcal{V}_{\mathbb{C}}$ be the local system on B^{reg} whose fibre over b is $H^1(S_b, \mathbb{C})$. We may think of θ as a section of $\mathcal{V}_{\mathbb{C}}$. The Donagi-Markman cubic is given by the “Yukawa couplings”:*

$$c(X, Y, Z) = \int_S \nabla_X \nabla_Y \nabla_Z \theta \wedge \theta,$$

where ∇ is the Gauss-Manin connection.

The above formula for the Donagi-Markman cubic involves differentiation with respect to the Gauss-Manin connection. This requires knowledge of the family of spectral curves. In §6 we give another formula for the c in terms of the geometry of a single spectral curve. Our formula holds for any smooth spectral curve, but in this section we will for simplicity consider the case where $\pi : S \rightarrow \Sigma$ has only simple branching. Around each ramification point $a \in S$ we can thus find local coordinates x on Σ and q on S such that $\pi(q) = x = q^2$. Near a we have $\theta = y dx$ for some function $y(q)$ with $y'(a) \neq 0$. We have:

Theorem 1.2. *The Donagi-Markman cubic is given by:*

$$c_{ijk} = -2\pi i \sum_a \text{Res}_a \left(\frac{\omega_i \omega_j \omega_k}{dx dy} \right),$$

where the sum is over the ramification points of π .

Similar-looking formulas for the Donagi-Markman cubic have appeared in [1, 13], however these formulas use cameral curves instead of spectral curves and involve quadratic residues instead of ordinary residues. One advantage of our formula (in the form of Theorem 6.6) is that it applies even when π has higher order ramification. Moreover, our formula looks very similar to formulas appearing in the theory of Eynard-Orantin topological recursion. This is not a coincidence, as we show in Section §7.

1.1. Relation to topological recursion. Eynard-Orantin topological recursion [7] is a recursive formula which takes as input a Riemann surface S (which we assume is compact) with a pair of meromorphic functions x, y and produces a series of symmetric multidifferentials $W_n^{(g)}$, for $g \geq 0$, $n \geq 1$, the *Eynard-Orantin invariants*. More precisely, $W_n^{(g)}$ is a meromorphic section of the n -th exterior tensor product $K_S^{\boxtimes n} = K_S \boxtimes K_S \boxtimes \cdots \boxtimes K_S$ on S^n , where K_S denotes the canonical bundle of S , which is symmetric under interchange of factors. The function x , viewed as a map $x : S \rightarrow \mathbb{P}^1$ is assumed to be a branched covering with only simple branching.

The topological recursion formula has been extended to Hitchin spectral curves $S \subset T^*\Sigma$ in [6], which gives an interpretation of the Eynard-Orantin invariants in terms of quantisation of spectral curves. Our paper gives another interpretation of these invariants, at least in the case of $g = 0$. To make sense of the Eynard-Orantin invariants in this setting, first note that the projection $\pi : S \rightarrow \Sigma$ plays the role of x . As we will recall in §7, the topological recursion formula continues to make sense as long as π has only simple branching. The key point is that the recursion formula does not directly involve the functions x, y but only the 1-form ydx . In the case of a spectral curve $S \subset T^*\Sigma$, the canonical 1-form θ plays the role of ydx . We show that the variational formula for Eynard-Orantin invariants in [7] holds in our setting and leads to the following formula for derivatives of the period matrix τ_{ij} about any point $b \in B^{\text{reg}}$.

Theorem 1.3. *For any $b \in B^{\text{reg}}$, we have:*

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-2}} \tau_{i_{m-1} i_m}(b) = - \left(\frac{i}{2\pi} \right)^{m-1} \int_{p_1 \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_m^{(0)}(p_1, \dots, p_m)$$

where on the right hand side, $W_m^{(0)}(p_1, \dots, p_m)$ is the $W_m^{(0)}$ Eynard-Orantin invariant of the corresponding spectral curve S_b . One can show that $W_m^{(0)}$ in each variable only has poles with zero residue, so the above expression does not depend on the choice of paths representing the given cycles.

This formula shows that the $g = 0$ Eynard-Orantin invariants $W_k^{(0)}$ for a spectral curve S_b compute the Taylor series expansion of the period matrix τ_{ij} about $b \in B^{\text{reg}}$. Since the special Kähler metric on B^{reg} is given in terms of the period matrix, the invariants $W_k^{(0)}$ also compute the power series expansion of the special Kähler metric. For instance, we consider the $m = 1$ case and show that it recovers our formula (Theorem 1.2) for the Donagi-Markman cubic. Theorem 1.3 is remarkable in that the left hand side of the equation is related to geometry of the *family of spectral curves*, while the right hand side is given by invariants of a *single spectral curve* $S_b \subset T^*\Sigma$. In a sense, any single spectral curve S_b “knows” about the geometry of the entire family $\{S_b\}_{b \in B^{\text{reg}}}$.

Let us remark that Theorem 1.3 is certainly not surprising, as it is essentially a consequence of the well known variational formula for Eynard-Orantin invariants given in [7]. The main point we would like to emphasise is that this formula is applicable to Hitchin spectral curves. This is not immediately obvious as one needs to check that the proof of the variational formula in [7] holds for Hitchin spectral curves. Ultimately, this boils down to proving a version of the so-called “Rauch variational formula”, which we prove in Proposition 7.1:

Proposition 1.4. *Let $\partial \in T_b B^{\text{reg}}$. Assume p, r are distinct and are not ramification points. Then:*

$$\delta B(p, r) = - \sum_a \text{Res}_{u \rightarrow a} \frac{\delta \theta(u) B(u, p) B(u, r)}{dx(u) dy(u)},$$

where the sum is over the ramification points of π and for each ramification point $a \in S_b$, we choose coordinate functions x on Σ and q on S with $x = q^2$ and write $\theta = ydx$.

In this proposition, B is the *Bergman kernel* (see §7). For the meaning of the variational operator δ , see Definition 6.1. Versions of this formula are well known in the literature, but we could not find a proof that holds in the setting of Hitchin spectral curves, so we give a proof in §7.

We finish the paper by using topological recursion to go beyond the Donagi-Markman cubic and compute the symmetric quartic of second derivatives of the periods in terms of geometry of the spectral curve. Around any ramification point $a \in S_b$, we may write the normalised 1-forms as $\omega_i = \omega_i(q)dq$ for some functions $\omega_i(q)$. Our result is:

Theorem 1.5. *The second derivatives of the period matrix are given by:*

$$\begin{aligned} \partial_i \partial_j \tau_{kl} &= 2\pi i \sum_{a \neq b} B(a, b) \left(\frac{\omega_i(a)\omega_j(a)}{2y'(a)} \right) \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + cyc_{j,k,l} \\ &\quad + 2\pi i \sum_a \frac{1}{8y'(a)^2} \left(S_B(a) - \frac{y'''(a)}{y'(a)} \right) \omega_i(a)\omega_j(a)\omega_k(a)\omega_l(a) \\ &\quad + 2\pi i \sum_a \frac{1}{8y'(a)^2} (\omega_i''(a)\omega_j(a)\omega_k(a)\omega_l(a)) + cyc_{i,j,k,l}, \end{aligned}$$

where S_B is the Bergman projective connection (see §7), the sum $\sum_{a \neq b}$ is over distinct pairs of ramification points and cyc means to sum over cyclic permutations of the specified indices.

1.2. On the $g > 0$ invariants. In this paper, we have established the relation between the $g = 0$ Eynard-Orantin invariants of spectral curves and the special Kähler geometry of B^{reg} . An interesting problem would be to find a similar geometric interpretation for the $g > 0$ invariants. In particular one may ask whether there is a relation between the $g > 0$ invariants and the instanton corrections relating the semi-flat hyperkähler metric on \mathcal{M}^{reg} to the complete hyperkähler metric on \mathcal{M} .

1.3. Summary of paper. A brief summary of the contents of this paper is as follows. In §2 we give a review of special Kähler geometry. In §3 we recall a result of Hitchin showing that the moduli space of deformations of a compact complex Lagrangian in a complex symplectic manifold M has a natural special Kähler metric (assuming M admits a Kähler metric). In §4 we briefly review the relevant aspects of the moduli space of Higgs bundles and the Hitchin system $f : \mathcal{M}_{n,d} \rightarrow B$, in particular the spectral curve construction. There are two possible ways in which B^{reg} inherits a special Kähler geometry: (i) view B^{reg} as a family of complex Lagrangians in $\mathcal{M}_{n,d}$ (the fibres of the map f), or (ii) view B^{reg} as a family of spectral curves $S_b \subset T^*\Sigma$ (clearly S_b is a complex Lagrangian submanifold of $T^*\Sigma$). We show in §5 that these two points of view give rise to the same special Kähler geometry, and we describe this geometry in terms of the family of spectral curves. In §6, we give a residue formula for the Donagi-Markman cubic, essentially by a computation of Kodaira-Spencer classes. In §7, we consider the Eynard-Orantin invariants of Hitchin spectral curves and relate the $g = 0$ invariants to the special Kähler geometry on B^{reg} . We show that in this way, we recover our formula from §6 for the Donagi-Markman cubic and then proceed to compute the quartic of second derivatives of the periods by topological recursion.

2. SPECIAL KÄHLER GEOMETRY

2.1. Review of special Kähler geometry.

Definition 2.1 ([10, 16]). A special Kähler manifold is a Kähler manifold (M, g, I, ω) together with a torsion free, flat affine connection ∇ such that

- $\nabla\omega = 0$, and
- $d_\nabla I = 0$

Here $I \in \Omega^1(M, TM)$ is viewed as a TM -valued 1-form and $d_\nabla : \Omega^1(M, TM) \rightarrow \Omega^2(M, TM)$ is the differential induced by ∇ .

Let us examine what the special Kähler condition implies in terms of local coordinates, following Freed [10]. Since ∇ is flat and torsion free, we can find local coordinates in which ∇ becomes the trivial connection. Moreover, since $\nabla\omega = 0$, we can choose these coordinates to be Darboux, that is, M has local flat coordinates $(x^1, \dots, x^n, y_1, \dots, y_n)$ for which

$$\omega = dx^1 \wedge dy_1 + \dots + dx^n \wedge dy_n.$$

Next, we observe that the 1-forms $I dx^i$ are closed, because

$$d(I dx^i) = d_\nabla(I dx^i) = (d_\nabla I) \wedge dx^i + I d_\nabla(dx^i) = 0,$$

where in the first equality we used that ∇ is torsion free. It follows that locally there exist functions u^1, \dots, u^n such that $I dx^i = du^i$. Let $z^i = x^i - iu^i$. Then $dz^i = dx^i - i du^i = dx^i - i I dx^i$ is a $(1, 0)$ -form. This together with the fact that $Re(dz^i) = dx^i$ implies that (z^1, \dots, z^n) is a local holomorphic coordinate system for M . Similarly, one can find functions v_1, \dots, v_n such that $I dy_j = dv_j$ and setting $w_j = y_j - i v_j$ gives another holomorphic coordinate system (w_1, \dots, w_n) ¹. A simple computation gives

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} + \tau_{ij} \frac{\partial}{\partial y_j} \right), \quad \text{where } \tau_{ij} = \frac{\partial w_j}{\partial z^i}.$$

Compatibility of ω and I gives, after a short computation, the condition $\tau_{ij} = \tau_{ji}$. So there is a local holomorphic function \mathcal{F} , called the *prepotential* such that

$$w_i = \frac{\partial \mathcal{F}}{\partial z^i}, \quad \tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial z^i \partial z^j}.$$

From symmetry of τ_{ij} , we also deduce that

$$\omega = -\frac{i}{2} Im(\tau_{ij}) dz^i \wedge d\bar{z}^j.$$

If we use the convention that g and ω are related by $g(X, Y) = \omega(IX, Y)$, this means that τ_{ij} is a symmetric, complex $n \times n$ matrix with $Im(\tau_{ij})$ positive definite. That is, τ_{ij} is a period matrix, a point in the Siegel upper half-space. We note that a Kähler potential for ω is given by:

$$(2.1) \quad K = -\frac{1}{2} Im(w_i \bar{z}^i) = -\frac{1}{2} Im \left(\frac{\partial \mathcal{F}}{\partial z^i} \bar{z}^i \right).$$

As in the introduction, we have the Donagi-Markman cubic $c \in H^0(M, Sym^3(TM))$ which measures the variation of the period matrix τ_{ij} :

$$c = c_{ijk} dz^i \otimes dz^j \otimes dz^k, \quad c_{ijk} = \frac{\partial \tau_{jk}}{\partial z^i} = \frac{\partial^3 \mathcal{F}}{\partial z^i \partial z^j \partial z^k}.$$

¹Note that Freed uses a slightly different convention in which $Re(w_i) = -y_i$.

2.2. Special Kähler manifolds as “bi-Lagrangians”. In [16], Hitchin establishes a close relation between special Kähler manifolds and submanifolds of a complex symplectic vector space satisfying a “bi-Lagrangian” condition. We recall the result. Let V be a real symplectic vector space with symplectic form ω . Define $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V$ with complex structure $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Define a complex symplectic form $\omega^c = \omega_1 + i\omega_2$ on $V_{\mathbb{C}}$ as the \mathbb{C} -bilinear extension of ω , that is:

$$\omega^c((x, y), (x', y')) = \omega(x+iy, x'+iy') = \underbrace{(\omega(x, x') - \omega(y, y'))}_{\omega_1((x, y), (x', y'))} + i \underbrace{(\omega(x, y') - \omega(x', y))}_{\omega_2((x, y), (x', y'))}.$$

Define in addition an (indefinite signature) inner product g on $V_{\mathbb{C}}$ by

$$g((x, y), (x', y')) = \frac{1}{2} (\omega(x, y') + \omega(x', y)).$$

Note that $g(\alpha, \beta) = \operatorname{Re}(\frac{i}{2}\omega^c(\alpha, \bar{\beta}))$, where we define $\overline{(x, y)} = (x, -y)$.

Theorem 2.2 (Hitchin, [16]). *Let $M \subset V_{\mathbb{C}}$ be a submanifold which is Lagrangian with respect to ω_1 and ω_2 and such that $g|_M$ is positive definite. Then $(M, g|_M, I|_M)$ is special Kähler. The projection of M to the first factor $V \subset V_{\mathbb{C}}$ defines a system of local coordinates, and the flat affine connection ∇ is the trivial connection on TM with respect to these coordinates. In a similar manner, the symplectic form ω on M is obtained by pullback of the symplectic form ω on V . Conversely, any special Kähler metric is locally of this form.*

The relation between the local embedding $M \subset V_{\mathbb{C}}$ and the holomorphic coordinates z^i, w_i is as follows: choose a symplectic bases $a^1, \dots, a^n, b^1, \dots, b^n$ for V , giving an explicit isomorphism $V \cong \mathbb{R}^{2n}$. Then the map $s : M \rightarrow V_{\mathbb{C}} = V \oplus V \cong (\mathbb{R}^n)^4$ is given by

$$(Re(z), Re(w), Im(z), Im(w)),$$

where we think of $z = (z^1, \dots, z^n)$, $w = (w_1, \dots, w_n)$ as vectors in \mathbb{C}^n . Let $a_1, \dots, a_n, b_1, \dots, b_n \in V^*$ be the dual basis and $\langle \cdot, \cdot \rangle : V^* \otimes V \rightarrow \mathbb{R}$ the dual pairing, which we extend to a pairing $V_{\mathbb{C}}^* \otimes V_{\mathbb{C}} \rightarrow \mathbb{C}$ by \mathbb{C} -linearity. Then the coordinate systems z^i, w_i can be recovered as:

$$z^i(m) = \langle a_i, s(m) \rangle, \quad w_i(m) = \langle b_i, s(m) \rangle.$$

A slight extension of Theorem 2.2 is to consider the following situation: suppose M is an n -manifold and let $(\mathcal{V}, \omega, \nabla)$ be a real symplectic vector bundle of rank $2n$ equipped with a flat symplectic connection ∇ . Let $\mathcal{V}_{\mathbb{C}} = \mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification and as above, define a complex symplectic form $\omega^c = \omega_1 + i\omega_2$ and an inner product g . Let $s : M \rightarrow \mathcal{V}_{\mathbb{C}}$ be a section of $\mathcal{V}_{\mathbb{C}}$ satisfying the following conditions:

- (SK1) The bundle map $\rho : TM \rightarrow \mathcal{V}_{\mathbb{C}}$ given by $\rho(X) = \nabla_X s$ is injective.
- (SK2) The image of ρ is Lagrangian with respect to ω_1 and ω_2 .
- (SK3) The image of ρ is positive definite with respect to g .

Then M inherits a special Kähler geometry. Indeed, locally on M we choose a flat trivialisation $\mathcal{V} \cong M \times V$. Then s defines an immersion $s : M \rightarrow V_{\mathbb{C}}$ and we are back to the setting of Theorem 2.2.

3. RELATION TO MODULI SPACES OF COMPLEX LAGRANGIANS

We recall the relationship between moduli spaces of complex Lagrangians and special Kähler geometry [16].

3.1. Deformations of complex Lagrangians. Let \mathcal{M} be a complex manifold of complex dimension $2n$ and let Ω be a holomorphic symplectic form, by which we mean a closed $(2,0)$ -form such that $\wedge^n \Omega$ is non-vanishing. A *complex Lagrangian* in \mathcal{M} is a complex submanifold $Y \subset \mathcal{M}$ which is Lagrangian with respect to Ω , i.e. Y has complex dimension n and $\Omega|_Y = 0$. A real submanifold $Y \subset \mathcal{M}$ of real dimension $2n$ such that $\Omega|_Y = 0$ is in fact automatically a complex Lagrangian [16, Proposition 1]. If $Y \subset \mathcal{M}$ is a complex Lagrangian, then Ω yields an isomorphism $N_Y \rightarrow T_Y^*$ between the normal bundle of Y and the cotangent bundle, which sends a normal vector field X to $i_X \Omega|_Y$. In particular, this gives an isomorphism $H^0(Y, N_Y) \cong H^1(Y, T_Y^*)$ between normal vector fields and holomorphic 1-forms. Recall that $H^0(Y, N_Y)$ describes the space of infinitesimal deformations of Y as a complex submanifold of \mathcal{M} . The infinitesimal deformations as a complex Lagrangian are those for which the corresponding holomorphic 1-form is closed.

Following Hitchin, we make the following two assumptions: (i) Y is compact and (ii) \mathcal{M} has a Kähler 2-form h . In this case, Y is also Kähler and since Y is compact Kähler, it follows that all holomorphic 1-forms are closed. Thus every infinitesimal deformation of Y respects the Lagrangian condition. Furthermore, it follows from [20] that in the Lagrangian case, all deformations are unobstructed. Hence there exists a local moduli space B of complex Lagrangian submanifolds parametrising (sufficiently small) deformations of a given complex Lagrangian $Y_0 \subset \mathcal{M}$. A point $[Y] \in B$ is a complex Lagrangian $Y \subset \mathcal{M}$ which is deformation equivalent to Y_0 . Moreover there is a natural isomorphism $T_{[Y]}B \cong H^0(Y, T_Y^*)$.

We recall from [16] how B inherits a naturally defined special Kähler structure. Let Z be a local universal family of deformations of the complex Lagrangian submanifold $Y_0 \subset \mathcal{M}$, so Z is a complex manifold with a proper holomorphic surjective submersion $f : Z \rightarrow B$ and a holomorphic map $j : Z \rightarrow \mathcal{M}$ such that the restriction of j to each fibre $L_b = f^{-1}(b)$ of f gives an embedding $j : L_b \rightarrow \mathcal{M}$ whose image is the complex Lagrangian corresponding to the point $b \in B$. Let $\mathcal{V} = R^1 f_* \mathbb{R}$ be the vector bundle on B whose fibre over $b \in B$ is given by the first cohomology $H^1(L_b, \mathbb{R})$ of the fibre L_b . The bundle \mathcal{V} is equipped with a natural flat connection ∇ , the Gauss-Manin connection. The Kähler form h on \mathcal{M} yields a symplectic structure ω on \mathcal{V} by setting

$$\omega_b(\alpha, \beta) = \int_{L_b} \alpha \wedge \beta \wedge h^{n-1},$$

for all $\alpha, \beta \in \mathcal{V}_b = H^1(L_b, \mathbb{R})$. Clearly ω is preserved by the Gauss-Manin connection.

The natural isomorphism $T_b B \cong H^0(L_b, T_{L_b}^*)$ together with the inclusion $H^0(L_b, T_{L_b}^*) \subset H^1(L_b, \mathbb{C})$ (recall L_b is compact Kähler) yields a bundle map $\phi : TB \rightarrow \mathcal{V}_{\mathbb{C}}$:

$$T_b B \cong H^0(L_b, T_{L_b}^*) \subset H^1(L_b, \mathbb{C}) \cong (\mathcal{V}_{\mathbb{C}})_b$$

ϕ_b

More explicitly, let $X \in T_b B$ be a tangent vector at b . Let \tilde{X} be a lift of X to L_b , that is, \tilde{X} is a section of $TZ|_{L_b}$ such that $f_*(\tilde{X}(y)) = X$ for all $y \in L_b$. Then $\phi(X)$ is given by:

$$(3.1) \quad \phi(X) = i_{\tilde{X}} j^* \Omega|_{L_b},$$

which is a closed 1-form on L_b representing a class in $H^1(L_b, \mathbb{C})$. The map ϕ can be viewed as a $\mathcal{V}_{\mathbb{C}}$ -valued 1-form on B . We then have:

Theorem 3.1 (Hitchin, [16]). *The $\mathcal{V}_{\mathbb{C}}$ -valued 1-form ϕ is d_{∇} -closed, so locally we can write $\phi = \nabla s$ for a section s of $\mathcal{V}_{\mathbb{C}}$. Then s satisfies the conditions (SK1)-(SK3) and hence locally defines a special Kähler structure on B . The special Kähler structure is independent of the choice of s , hence this construction gives rise to a globally defined special Kähler structure on B .*

Proof. We give only a sketch of the proof here and refer the reader to [16] for further details. We have that $j^*\Omega$ is a closed 2-form on Z . The complex Lagrangian condition means that $j^*\Omega|_{L_b} = 0$ for any fibre L_b . If we consider the Leray-Serre spectral sequence for $f : Z \rightarrow B$, we see that $j^*\Omega$ yields a class in $E_2^{1,1} = H^1(B, R^1 f_* \mathbb{C}) = H^1(B, \mathcal{V}_{\mathbb{C}})$, which is represented by the $\mathcal{V}_{\mathbb{C}}$ -valued 1-form ϕ . This explains why ϕ is closed. Condition (SK1) follows from the isomorphism $T_b B \cong H^0(L_b, T_{L_b}^*)$ and the inclusion $H^0(L_b, T_{L_b}^*) \subset H^1(L_b, \mathbb{C})$. Condition (SK2) follows from the fact that $H^0(L_b, T_{L_b}^*) \subset H^1(L_b, \mathbb{C})$ is Lagrangian with respect to ω . Condition (SK3) follows from

$$g_b(\alpha, \alpha) = \operatorname{Re} \left(\frac{i}{2} \int_{L_b} \alpha \wedge \bar{\alpha} \wedge h^{n-1} \right),$$

which is positive definite, since h is a Kähler form. \square

We give another way of understanding the $\mathcal{V}_{\mathbb{C}}$ -valued 1-form ϕ . The fibre of the dual local system $\mathcal{V}_{\mathbb{C}}^*$ over $b \in B$ is given by the homology group $(\mathcal{V}_{\mathbb{C}}^*)_b = H_1(L_b, \mathbb{C})$. Let $\langle \cdot, \cdot \rangle : \mathcal{V}_{\mathbb{C}}^* \otimes \mathcal{V}_{\mathbb{C}} \rightarrow \mathbb{C}$ be the dual pairing:

$$\langle \gamma, \alpha \rangle = \int_{\gamma} \alpha.$$

Then ϕ is determined by

$$\langle \gamma, \phi \rangle = \int_{\gamma} j^* \Omega,$$

for all locally defined covariantly constant sections γ of $\mathcal{V}_{\mathbb{C}}^*$, where the integral on the right hand side is fibrewise integration. More precisely, for any $b \in B$, choose an open neighborhood U of b over which $f : Z \rightarrow B$ is trivialisable: $f^{-1}(U) \cong L_b \times U$. Let $\hat{\gamma} \in H^{n-1}(L_b, \mathbb{C})$ be the Poincaré dual of γ . Then $\int_{\gamma} j^* \Omega$ is given by the integration over the fibres of $L_b \times U \rightarrow U$ of $j^* \Omega \wedge \hat{\gamma}$.

Proposition 3.2. *Suppose that $\Omega = d\mu$, where μ is a holomorphic 1-form on \mathcal{M} . Then $\phi = d_{\nabla} s$, where $s : B \rightarrow \mathcal{V}_{\mathbb{C}}$ is the section defined by*

$$s(b) = [\mu|_{L_b}] \in H^1(L_b, \mathbb{C}).$$

Proof. First of all, note that the restriction $\mu|_{L_b}$ is a closed 1-form because $d\mu = \Omega$ and $\Omega|_{L_b} = 0$, so the section s is well-defined. Let γ be a local constant section of $\mathcal{V}_{\mathbb{C}}$. Then

$$\begin{aligned} \langle \gamma, d_{\nabla} s \rangle &= d\langle \gamma, s \rangle - \langle d_{\nabla} \gamma, s \rangle \\ &= d\langle \gamma, s \rangle \quad (\text{as } \gamma \text{ is constant}) \\ &= d \left(\int_{\gamma} \mu \right) \\ &= \int_{\gamma} d\mu = \int_{\gamma} \Omega = \langle \gamma, \Omega \rangle, \end{aligned}$$

where we have used the fact that exterior differentiation commutes with fibre integration. This shows $\phi = d_{\nabla} s$. \square

Suppose we are in the situation of Proposition 3.2. For any $b \in B$, choose a symplectic basis $a_1, \dots, a_n, b_1, \dots, b_n$ of $H_1(L_b, \mathbb{C})$. If U is any simply connected neighborhood of b , we can extend a_i, b_i to be covariantly constant sections of \mathcal{V}^* over U . Then the local holomorphic coordinates z^i and w_i of the special Kähler structure are given by

$$z^i = \int_{a_i} \mu, \quad w_i = \int_{b_i} \mu.$$

Example 3.3 (Holomorphic symplectic surfaces). Let \mathcal{M} be a holomorphic symplectic surface, i.e. a complex surface with trivial canonical bundle. Let Ω be the symplectic form on \mathcal{M} . Then any complex 1-dimensional submanifold $S \subset \mathcal{M}$ is automatically Lagrangian, because there are no $(2, 0)$ -forms on S . The moduli space B of deformations of S then carries a natural special Kähler structure. We note that in this case since $n = 1$, it is not necessary to assume the existence of a Kähler form h on \mathcal{M} . Indeed, the symplectic structure on $\mathcal{V}_{\mathbb{C}}$ is given by

$$\omega_b(\alpha, \beta) = \int_{L_b} \alpha \wedge \beta,$$

which is defined without assuming the existence of h . Moreover, since S is a Riemann surface it is automatically Kähler. As we will see, this example is closely related to the Hitchin system, where $\mathcal{M} = T^*\Sigma$ is the cotangent bundle of a Riemann surface Σ and $S \subset T^*\Sigma$ is a spectral curve.

Example 3.4 (Complex integrable systems). Suppose that \mathcal{M} admits a proper holomorphic Lagrangian fibration $f : \mathcal{M} \rightarrow B$, by which we mean B is a complex n -manifold and $f : \mathcal{M} \rightarrow B$ is a proper, surjective, holomorphic map whose fibres are Lagrangian with respect to Ω . Assume further that the fibres of f are connected. Then Liouville's theorem implies that the fibres of f are in fact complex tori. We shall refer to the data $(\mathcal{M}, \Omega, B, f)$ as a complex integrable system. Since the normal bundle to the fibres is given by $f^*(TB)$, it follows that the deformations of any given fibre $L_b \subset \mathcal{M}$ are precisely the other fibres of f . Hence B is the moduli space of deformations of any given fibre. If we make the additional assumption that \mathcal{M} admits a Kähler 2-form h , then B inherits a natural special Kähler structure (which depends on the choice of h).

4. HIGGS BUNDLES AND THE HITCHIN SYSTEM

4.1. Review of Higgs bundles. Let Σ be a compact Riemann surface of genus $g > 1$ and let K denote the canonical bundle of Σ .

Definition 4.1. A Higgs bundle on Σ of rank n and degree d is a pair (E, Φ) , where E is a holomorphic vector bundle on Σ of rank n , degree d and Φ is a holomorphic bundle map $\Phi : E \rightarrow E \otimes K$, called the Higgs field.

Recall that the slope $\mu(E)$ of a holomorphic vector bundle E on Σ is defined to be the number $\mu(E) = \deg(E)/\text{rank}(E)$.

Definition 4.2. A Higgs bundle (E, Φ) is called semistable if for all proper, non-zero subbundles $F \subset E$ such that $\Phi(F) \subseteq F \otimes K$, we have $\mu(F) \leq \mu(E)$.

Using geometric invariant theory [18], one constructs a moduli space $\mathcal{M}_{n,d}$ of rank n , degree d semistable Higgs bundles up to a suitable notion of equivalence (called S -equivalence). Here we will recall only the details of $\mathcal{M}_{n,d}$ which are relevant to understanding the special Kähler geometry of the Hitchin system. The moduli space $\mathcal{M}_{n,d}$ is a quasi-projective, complex algebraic variety of dimension $2n^2(g-1)+2$. The moduli space is in general singular, but the smooth locus $\mathcal{M}_{n,d}^{\text{sm}}$ has a naturally defined holomorphic symplectic 2-form Ω . The smooth locus $\mathcal{M}_{n,d}^{\text{sm}}$ is a hyperkähler manifold with a triple of complex structures I, J, K and corresponding Kähler 2-forms $\omega_I, \omega_J, \omega_K$. The complex structure which arises from viewing $\mathcal{M}_{n,d}$ as the moduli space of Higgs bundles is customarily taken to be I and it is the only complex structure of relevance to us here. So we will regard $\mathcal{M}_{n,d}^{\text{sm}}$ as a Kähler manifold $(\mathcal{M}_{n,d}^{\text{sm}}, I, \omega_I)$ equipped with a holomorphic symplectic form $\Omega = \omega_J + i\omega_K$. One can also consider moduli spaces of Higgs bundles (E, Φ) , where Φ is trace-free and E has fixed determinant. This gives a subvariety of $\mathcal{M}_{n,d}$. As the corresponding special Kähler geometry is obtained by restriction, it is sufficient for our purposes to consider just the case of $\mathcal{M}_{n,d}$.

Associated to $\mathcal{M}_{n,d}$ is a complex integrable system, known as the *Hitchin system* [15], which is defined as follows. If A is a complex $n \times n$ matrix, write the characteristic polynomial of A as

$$\det(\lambda - A) = \lambda^n + p_1(A)\lambda^{n-1} + \cdots + p_n(A).$$

The coefficients p_1, \dots, p_n of the characteristic polynomial can be viewed as maps $\mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathbb{C}$ which are well known to be a basis for the ring of conjugation-invariant polynomial functions on $\mathfrak{gl}(n, \mathbb{C})$. The Hitchin system is the analogue of this where the matrix A is replaced by a Higgs field. If (E, Φ) is a rank n Higgs bundle, then by conjugation invariance, $p_j(\Phi)$ is a well-defined section of K^j . Define

$$B = \bigoplus_{j=1}^n H^0(\Sigma, K^j).$$

Then we have a natural map $f : \mathcal{M}_{n,d} \rightarrow B$, called the *Hitchin map*, which sends a Higgs bundle (E, Φ) to $(p_1(\Phi), p_2(\Phi), \dots, p_n(\Phi))$. It is known [14, 19] that f is a proper, surjective, holomorphic map whose non-singular fibres are Lagrangian submanifolds with respect to Ω . Let $B^{\text{reg}} \subset B$ denote the *regular locus*, i.e. the locus of points $b \in B$ over which f is a submersion. Let $\mathcal{M}_{n,d}^{\text{reg}} \subset \mathcal{M}_{n,d}$ denote the locus of points in $\mathcal{M}_{n,d}$ lying over B^{reg} , so that $f : \mathcal{M}_{n,d}^{\text{reg}} \rightarrow B^{\text{reg}}$ is a proper, holomorphic surjective submersion of complex manifolds (since B is smooth and $f : \mathcal{M}_{n,d}^{\text{reg}} \rightarrow B^{\text{reg}}$ is a submersion, it follows that $\mathcal{M}_{n,d}^{\text{reg}}$ is contained in the smooth locus $\mathcal{M}_{n,d}^{\text{sm}}$). As seen in Theorem 4.3 below, the fibres of f over B^{reg} are connected and hence, as in Example 3.4, the fibres are complex tori. We then have that B^{reg} can be identified with the moduli space of deformations of any given fibre in $\mathcal{M}_{n,d}^{\text{reg}}$ and thus B^{reg} carries a natural special Kähler geometry. The complement $\mathcal{D} = B \setminus B^{\text{reg}}$, called the *discriminant locus*, is the locus of all singular fibres of $f : \mathcal{M}_{n,d} \rightarrow B$. It is known that \mathcal{D} is an irreducible hypersurface of B [17].

4.2. Spectral curves. The fibres of the Hitchin system can be described using the notion of *spectral curves* [2, 15]. Let $\pi : T^*\Sigma \rightarrow \Sigma$ be the projection from $T^*\Sigma$ to Σ . Observe that $T^*\Sigma$ is the total space of the canonical bundle $K \rightarrow \Sigma$. Therefore the pullback $\pi^*(K)$ has a natural section λ , the *tautological section*, defined by the

property that if $p \in K_x$, then

$$\lambda(p) = p \in (\pi^*(K))_p = K_x.$$

Let $b = (p_1, p_2, \dots, p_n) \in B$, so $p_j \in H^0(\Sigma, K^j)$. To the point b , we associate a section $p = p_b \in H^0(T^*\Sigma, \pi^*(K^n))$, given by

$$p_b = \lambda^n + \pi^*(p_1)\lambda^{n-1} + \dots + \pi^*(p_n).$$

The spectral curve S_b associated to $b \in B$ is defined as the zero divisor of p_b in $T^*\Sigma$. Thus S_b is defined as a subscheme of $T^*\Sigma$. Bertini's theorem implies that for generic points $b \in B$, the spectral curve $S_b \subset T^*\Sigma$ is smooth. In fact, it can be shown that S_b is smooth if and only if $b \in B^{\text{reg}}$ [17]. Thus, the discriminant locus \mathcal{D} is also the locus of singular spectral curves.

If S_b is a spectral curve, we denote by $\pi : S_b \rightarrow \Sigma$ the restriction of π to S_b . Let $\tilde{d} = d + (g-1)n(n-1)$. For $b \in B^{\text{reg}}$, let $Jac_{\tilde{d}}(S_b)$ denote the degree \tilde{d} component of the Picard variety of S_b , which is a torsor for the Jacobian of S_b . If $L \in Jac_{\tilde{d}}(S_b)$, then by Grothendieck-Riemann-Roch, one sees that the push-forward $E = \pi_*L$ is a rank n , degree d vector bundle on Σ . The tautological section λ , viewed as a map $\lambda : L \rightarrow L \otimes \pi^*K$ pushes down to a map $\Phi : E \rightarrow E \otimes K$ and hence the pair (E, Φ) is a Higgs bundle. Since $p_b = 0$ on S_b , it follows that:

$$\Phi^n + \pi^*(p_1)\Phi^{n-1} + \dots + \pi^*(p_n) = 0.$$

Since $b \in B^{\text{reg}}$, S_b is smooth and it follows that p_b is irreducible. Thus p_b must be the characteristic polynomial of Φ , in other words (E, Φ) belongs to the fibre of $\mathcal{M}_{n,d}$ over $b \in B$. In this way, we obtain all Higgs bundles in the fibre over b :

Theorem 4.3 ([15, 2]). *Let $b \in B^{\text{reg}}$. The map sending $L \in Jac_{\tilde{d}}(S_b)$ to (E, Φ) described above gives an isomorphism between $Jac_{\tilde{d}}(S_b)$ and the fibre of $\mathcal{M}_{n,d}$ over $b \in B$.*

4.3. On deformations of spectral curves. Let $\theta \in \Omega^{1,0}(T^*\Sigma)$ denote the canonical 1-form on $T^*\Sigma$. Then $d\theta$ is the canonical holomorphic symplectic 2-form on $T^*\Sigma$ and gives a trivialisation of the canonical bundle of $T^*\Sigma$. Now let $S \subset T^*\Sigma$ be a non-singular spectral curve. We denote by K_S the canonical bundle of S and N_S the normal bundle. By definition of N_S , we have a short exact sequence

$$0 \rightarrow K_S^{-1} \rightarrow T_{(T^*\Sigma)}|_S \rightarrow N_S \rightarrow 0.$$

Taking determinants and using $d\theta$, we get an isomorphism $N_S \cong K_S$. Explicitly, the isomorphism is given by

$$\varphi : N_S \rightarrow K_S, \quad \varphi(V) = i_V d\theta|_S$$

which sends a normal vector field V to the contraction of $d\theta$ with V . Next, since S is by definition a divisor of the linear system of sections of $\pi^*(K^n)$, we have by the adjunction formula that $N_S \cong \pi^*(K^n)|_S$. We will make this isomorphism explicit as follows. Suppose that S is the zero divisor of $p \in H^0(T^*\Sigma, \pi^*(K^n))$. Choose an open covering $\{U_i\}$ of $T^*\Sigma$ over which $\pi^*(K^n)$ is trivial. Let $g_{ij} : U_i \cap U_j \rightarrow \mathbb{C}^*$ be the transition functions, so p corresponds to a collection of functions $s_i : U_i \rightarrow \mathbb{C}$ such that $s_i = g_{ij}s_j$ on $U_i \cap U_j$ and $s_i|_S = 0$ for all i . Now let V be a normal vector field along S . Denote by $\partial_V s_i$ the derivative of s_i in the direction V . Then $\partial_V(s_i)|_S = g_{ij}\partial_V(s_j)|_S$, because $s_j|_S = 0$. Therefore, $\{\partial_V(s_i)|_S\}_i$ is a well-defined

section of $\pi^*(K^n)|_S$, which we denote simply as $\partial_V p|_S$. The desired isomorphism is

$$\varphi' : N_S \rightarrow \pi^*(K^n), \quad \varphi'(V) = \partial_V p|_S.$$

Lemma 4.4. *Let $b \in B^{\text{reg}}$. The map*

$$\rho : B = \bigoplus_{j=1}^n H^0(\Sigma, K^j) \rightarrow H^0(S_b, \pi^*(K^n))$$

given by

$$\rho(b_1, b_2, \dots, b_n) = \pi^*(b_1)\lambda^{n-1} + \pi^*(b_2)\lambda^{n-2} + \dots + \pi^*(b_n)$$

is an isomorphism.

Proof. From [2], we have

$$\pi_* \mathcal{O}_S = \mathcal{O}_\Sigma \oplus K^{-1} \oplus \dots \oplus K^{-(n-1)}.$$

Multiplying both sides by K^n and taking global sections gives the result. \square

Let $b \in B^{\text{reg}}$ and $p_b \in H^0(T^*\Sigma, \pi^*(K^n))$ the corresponding section of $\pi^*(K^n)$. A tangent vector $X \in T_b B^{\text{reg}} \cong B$ gives rise to a deformation of p_b and hence to a deformation of the divisor $S_b \subset T^*\Sigma$ of p_b . Such a deformation is described by a section of N_{S_b} , hence we get a map, known as the *characteristic map* (cf., [12]):

$$(4.1) \quad \chi : T_b B^{\text{reg}} \cong B = \bigoplus_{j=1}^n H^0(\Sigma, K^j) \rightarrow H^0(S_b, N_{S_b}).$$

Proposition 4.5. *The characteristic map is given by $\chi = -(\varphi')^{-1} \circ \rho$.*

Proof. Let $b(t)$ be a 1-parameter family in B^{reg} with $b(0) = b$, and $b'(0) \in T_b B^{\text{reg}} \cong B$ a tangent vector at $t = 0$. The derivative of $p_{b(t)}$ at $t = 0$ is clearly given by $\rho(b'(0))$. Let $x(t)$ denote a 1-parameter family of points in $T^*\Sigma$ such that $x(t)$ lies on $S_{b(t)}$ for all times t , i.e. $p_{b(t)}(x(t)) = 0$. Note that the projection of $x'(0)$ to the normal bundle is given by $V(x)$, where $V = \chi(b'(0)) \in H^0(S_b, N_{S_b})$ is the normal vector field describing the deformation of S_b in $T^*\Sigma$. Expanding $p_{b(t)}(x(t)) = 0$ to first order at $t = 0$, we get $\rho(b'(0))(x) + \partial_{V(x)} p_b = 0$, or $\varphi'(V(x)) = -\rho(b'(0))(x)$. So $V = \chi(b'(0)) = -(\varphi')^{-1} \rho(b'(0))$, as required. \square

Note that $-(\varphi')^{-1} \circ \rho : T_b B^{\text{reg}} \rightarrow H^0(S_b, N_{S_b})$ is an isomorphism. Thus as a consequence of Proposition 4.5, we see that B^{reg} can also be identified with the moduli space of deformations of $S_b \subset T^*\Sigma$.

5. SPECIAL KÄHLER GEOMETRY OF THE HITCHIN SYSTEM

5.1. Two special Kähler geometries. We now have two possible ways of obtaining a special Kähler geometry on B^{reg} :

- (1) View B^{reg} as parametrising a family of spectral curves $S \subset T^*\Sigma$, which are complex Lagrangians, or
- (2) View B^{reg} as parametrising a family of complex Lagrangians in $\mathcal{M}_{n,d}^{\text{reg}}$, the fibres of the Hitchin system.

We will show that both of these give rise to the same special Kähler geometry on B^{reg} .

Definition 5.1. Let $(\mathcal{V}, \omega, \nabla)$ and $(\mathcal{V}', \omega', \nabla')$ denote the flat symplectic bundles on B^{reg} corresponding to (1) and (2) above. Let $\phi : TB^{\text{reg}} \rightarrow \mathcal{V}_{\mathbb{C}}$ and $\phi' : TB^{\text{reg}} \rightarrow \mathcal{V}'_{\mathbb{C}}$ denote the $\mathcal{V}_{\mathbb{C}}$ and $\mathcal{V}'_{\mathbb{C}}$ -valued 1-forms corresponding to (1) and (2).

Consider first the special Kähler geometry on B^{reg} given by $(\mathcal{V}, \omega, \nabla, \phi)$. Recall that θ is the canonical 1-form on $T^*\Sigma$ and $d\theta$ is the symplectic form on $T^*\Sigma$. The map $B^{\text{reg}} \ni b \mapsto [\theta|_{S_b}] \in H^1(S_b, \mathbb{C})$ defines a section of $\mathcal{V}_{\mathbb{C}}$ which by abuse of notation we will denote by θ . Then Proposition 3.2 gives $\phi = d_{\nabla}\theta$, and hence the section θ determines a special Kähler geometry on B^{reg} .

Next consider the special Kähler geometry on B^{reg} by $(\mathcal{V}', \omega', \nabla', \phi')$. For this we need to introduce the canonical 1-form on $\mathcal{M}_{n,d}$. Let (E, Φ) be a semistable Higgs bundle in the non-singular locus of $\mathcal{M}_{n,d}$. The tangent space to (E, Φ) is given by the hypercohomology group $\mathbb{H}^1(\Sigma, \text{End}(E) \xrightarrow{ad_{\Phi}} \text{End}(E) \otimes K)$. Thus tangent vectors to (E, Φ) are represented by pairs $(\dot{A}, \dot{\Phi}) \in \Omega^{0,1}(\Sigma, \text{End}(E)) \oplus \Omega^{0,0}(\Sigma, \text{End}(E) \otimes K)$ satisfying $\bar{\partial}_E \dot{\Phi} + [\dot{A}, \Phi] = 0$. Here \dot{A} represents a deformation of holomorphic structure of E and $\dot{\Phi}$ represents a deformation of the Higgs field Φ . The natural map $\mathbb{H}^1(\Sigma, \text{End}(E) \xrightarrow{ad_{\Phi}} \text{End}(E) \otimes K) \rightarrow H^1(\Sigma, \text{End}(E))$, sending a deformation to (E, Φ) to a deformation of E alone is given in terms of Dolbeault representatives by $(\dot{A}, \dot{\Phi}) \mapsto \dot{A}$.

The holomorphic symplectic form Ω on $\mathcal{M}_{n,d}$ is of the form $\Omega = d\mu$, where μ is a holomorphic $(1,0)$ -form, the *canonical 1-form*. Up to an overall scale factor, which is not important for us, the canonical 1-form is given by

$$\mu_{(E, \Phi)}(\dot{A}, \dot{\Phi}) = \int_{\Sigma} \text{Tr}(\Phi \dot{A}).$$

By abuse of notation, let μ denote the section of $\mathcal{V}'_{\mathbb{C}}$ on B^{reg} given by $B^{\text{reg}} \ni b \mapsto [\mu|_{L_b}] \in H^1(L_b, \mathbb{C})$, where $L_b = f^{-1}(b) \cong \text{Jac}_{\bar{d}}(S_b)$ is the fibre of the Hitchin map over b . Then by Proposition 3.2, $\phi' = d_{\nabla'}\mu$. Hence the section μ defines a special Kähler geometry on B^{reg} .

Proposition 5.2. *There is a natural isomorphism of local systems $u : (\mathcal{V}, \nabla) \rightarrow (\mathcal{V}', \nabla')$. Under this isomorphism, ω and ω' agree up to a positive constant factor.*

Proof. Let Z^{reg} denote the universal moduli space of spectral curves, which may be defined as

$$Z^{\text{reg}} = \{(x, b) \in T^*\Sigma \times B^{\text{reg}} \mid p_b(x) = 0\}.$$

Thus Z^{reg} is a fibre bundle $q : Z^{\text{reg}} \rightarrow B^{\text{reg}}$ over B^{reg} whose fibre over $b \in B^{\text{reg}}$ is the spectral curve S_b . Let $j : Z^{\text{reg}} \rightarrow T^*\Sigma$ be given by $j(x, b) = x$. Then $\mathcal{V} = R^1 q_* \mathbb{R}$ is the local system $\mathcal{V}_b = H^1(S_b, \mathbb{R})$ equipped with the Gauss-Manin connection and ω is given by

$$\omega_b(\alpha, \beta) = \int_{S_b} \alpha \wedge \beta.$$

Recall the Hitchin map $f : \mathcal{M}_{n,d}^{\text{reg}} \rightarrow B^{\text{reg}}$ whose fibre over $b \in B^{\text{reg}}$ is $\text{Jac}_{\bar{d}}(S_b)$. Then $\mathcal{V}' = R^1 f_* \mathbb{R}$ is the local system $\mathcal{V}'_b = H^1(\text{Jac}_{\bar{d}}(S_b), \mathbb{R})$ with the Gauss-Manin connection and ω' given by

$$\omega'_b(\alpha', \beta') = \int_{\text{Jac}_{\bar{d}}(S_b)} \alpha' \wedge \beta' \wedge h^{n-1}.$$

Recall that $\text{Jac}_{\bar{d}}(S_b)$ is a torsor over $\text{Jac}(S_b)$. Hence there is a canonical isomorphism $H^1(\text{Jac}_{\bar{d}}(S_b), \mathbb{R}) \cong H^1(\text{Jac}(S_b), \mathbb{R})$. We may identify the Jacobian $\text{Jac}(S_b)$

with $H_1(S_b, \mathbb{R})/H_1(S_b, \mathbb{Z})$, hence we have a canonical isomorphism $H^1(\text{Jac}(S_b), \mathbb{R}) \cong H^1(S_b, \mathbb{R})$. By composing we get a canonical isomorphism $H^1(S_b, \mathbb{R}) \cong H^1(\text{Jac}_{\bar{d}}(S_b), \mathbb{R})$. Clearly this isomorphism can be carried out fibrewise over B^{reg} to give an isomorphism of flat vector bundles $\mathcal{V} \cong \mathcal{V}'$. Furthermore, the natural Kähler form $h = \omega_I$ on $\mathcal{M}_{n,d}$ restricted to the fibre $\text{Jac}_{\bar{d}}(S_b)$ gives a multiple of the usual principal polarisation on $\text{Jac}_{\bar{d}}(S_b)$. It follows that ω and ω' agree up to a positive constant factor (which is independent of b). \square

Proposition 5.3. *Let $b \in B^{\text{reg}}$. Under the canonical isomorphism $H^1(\text{Jac}_{\bar{d}}(S_b), \mathbb{C}) \cong H^1(S_b, \mathbb{C})$, we have $\mu|_{\text{Jac}_{\bar{d}}(S_b)} \cong \theta|_{S_b}$.*

Proof. The restriction of μ to the fibre $\text{Jac}_{\bar{d}}(S_b)$ over b is a holomorphic 1-form, which is necessarily translation invariant, since this is true of all holomorphic 1-forms on a complex torus. The tangent space to $\text{Jac}_{\bar{d}}(S_b)$ at a point $[L] \in \text{Jac}_{\bar{d}}(S_b)$ is canonically isomorphic to $H^1(S_b, \mathcal{O})$. Under the isomorphism $H^1(\text{Jac}_{\bar{d}}(S_b), \mathbb{C}) \cong H^1(S_b, \mathbb{C})$, the pairing of a tangent vector with a holomorphic 1-form coincides with the Serre duality pairing $H^0(S_b, K_{S_b}) \otimes H^1(S_b, \mathcal{O}) \rightarrow \mathbb{C}$. In other words, let $X \in T_{[L]}\text{Jac}_{\bar{d}}(S_b)$ and let α_X be the corresponding element of $H^1(S_b, \mathcal{O})$. Then the statement of the proposition is equivalent to showing:

$$\mu(X) = \int_{S_b} \theta \wedge \alpha_X.$$

The point $[L] \in \text{Jac}_{\bar{d}}(S_b)$ corresponds to a line bundle $L \rightarrow S_b$. Under the spectral data construction, the Higgs bundle (E, Φ) corresponding to L is given by $E = \pi_*(L)$ and Φ by pushing forward the map $\lambda : L \rightarrow L \otimes \pi^*K$. We view the holomorphic line bundle L as a \mathcal{C}^∞ -line bundle together with a $\bar{\partial}$ -operator $\bar{\partial}_L$. Then the tangent vector X corresponds to the tangent at $t = 0$ of the 1-parameter family of deformations of L given by $\bar{\partial}_{L_t} = \bar{\partial}_L + t\alpha_X$. Similarly view E as a \mathcal{C}^∞ -vector bundle with a $\bar{\partial}$ -operator $\bar{\partial}_E$. Let $E_t = \pi_*(L_t)$. We will construct an explicit family of $\bar{\partial}$ -operators $\bar{\partial}_{E_t}$ on the fixed \mathcal{C}^∞ -vector bundle E such that $(E, \bar{\partial}_{E_t})$ is isomorphic to E_t .

By the Dolbeault Lemma, adding a $\bar{\partial}$ -exact term to α_X if necessary, we can assume that α_X vanishes identically in a neighborhood of each point of S_b lying over a branch point. Now let $U \subset \Sigma$ be an open, simply-connected subset containing no branch points. Then the pre-image $\pi^{-1}(U) = U_1 \cup U_2 \cup \dots \cup U_n$ is the disjoint union of n open subsets of S_b , and the restriction $\pi : U_j \rightarrow U$ of π to each U_j is a diffeomorphism. Over U , we have a canonical isomorphism $E|_U \cong L|_{U_1} \oplus L|_{U_2} \oplus \dots \oplus L|_{U_n}$. Define an $\text{End}(E)$ -valued $(0,1)$ -form $\dot{A}|_U$ on U by

$$\dot{A}|_U = \text{diag}((\pi^*)^{-1}(\alpha_X|_{U_1}), \dots, (\pi^*)^{-1}(\alpha_X|_{U_n})).$$

Here $\pi^* : \Omega^{0,1}(U) \rightarrow \Omega^{0,1}(U_j)$ denotes the pullback of $(0,1)$ -forms and $(\pi^*)^{-1} : \Omega^{0,1}(U_j) \rightarrow \Omega^{0,1}(U)$ the inverse map. The $\dot{A}|_U$ defined in this way patch together to give an $\text{End}(E)$ -valued $(0,1)$ -form on Σ minus the branch points. But since α_X vanishes in a neighborhood of each point of S_b lying over a branch point, we have that \dot{A} vanishes in a punctured neighborhood of each branch point. We extend \dot{A} by zero over each branch point to get a well-defined $\dot{A} \in \Omega^{0,1}(\Sigma, \text{End}(E))$. Set $\bar{\partial}_{E_t} = \bar{\partial}_E + t\dot{A}$. By construction of \dot{A} , it is clear that $(E, \bar{\partial}_{E_t})$ is isomorphic to E_t . Observe that $\lambda : L_t \rightarrow L_t \otimes \pi^*K$ pushes down to Φ independent of t and note that Φ is holomorphic with respect to $\bar{\partial}_{E_t}$ for all t (because \dot{A} and Φ are simultaneously diagonalisable away from the branch points). So (E_t, Φ) is the 1-parameter family

of Higgs bundles corresponding to L_t . In particular, differentiating at $t = 0$, we have that $(\dot{A}, \dot{\Phi}) = (\dot{A}, 0) \in \mathbb{H}^1(\Sigma, \text{End}(E) \xrightarrow{\text{ad}_{\Phi}} \text{End}(E) \otimes K)$ is the tangent vector corresponding to $X \in T_{[L]} \text{Jac}_{\bar{d}}(S_b)$. Let $U \subset \Sigma$ be a simply-connected open subset of Σ containing no branch points, as above, and let ψ be a smooth compactly supported function on U . Over U , we have

$$\Phi = \text{diag}(\lambda|_{U_1} \circ \pi^{-1}, \dots, \lambda|_{U_n} \circ \pi^{-1})$$

and so

$$\begin{aligned} \int_U \psi \text{Tr}(\Phi \dot{A}) &= \sum_{j=1}^n \int_U \psi (\lambda|_{U_j} \circ \pi^{-1}) \wedge ((\pi^*)^{-1}(\alpha_X|_{U_j})) \\ &= \sum_{j=1}^n \int_{U_j} \pi^*(\psi) \theta \wedge \alpha_X \\ &= \int_{\pi^{-1}(U)} \pi^*(\psi) \theta \wedge \alpha_X. \end{aligned}$$

Combining this with a partition of unity argument, we get

$$\mu(X) = \int_{\Sigma} \text{Tr}(\Phi \dot{A}) = \int_S \theta \wedge \alpha_X$$

as required. \square

Corollary 5.4. *The special Kähler geometries on B^{reg} given by (1) and (2) coincide (up to a constant rescaling of the metric g and symplectic form ω).*

5.2. Special Kähler geometry of B^{reg} . We summarise our findings so far concerning the special Kähler geometry of the Hitchin system and make some further observations. Recall that B^{reg} is the regular locus of the Hitchin base, that is $B^{\text{reg}} = B \setminus \mathcal{D}$, where \mathcal{D} is the locus of singular spectral curves. On B^{reg} we have the flat symplectic bundle $(\mathcal{V}, \omega, \nabla)$ whose fibre over b is $\mathcal{V}_b = H^1(S_b, \mathbb{R})$. The flat connection ∇ is the Gauss-Manin connection and the symplectic structure ω is the intersection form $\omega(\alpha, \beta) = \int_{S_b} \alpha \wedge \beta$. Let θ denote the canonical 1-form on $T^*\Sigma$. We think of θ as a section $\theta : B^{\text{reg}} \rightarrow \mathcal{V}_{\mathbb{C}}$ which sends b to $[\theta|_{S_b}] \in H^1(S_b, \mathbb{C})$. For any $b \in B^{\text{reg}}$, let $a_1, \dots, a_{g_S}, b_1, \dots, b_{g_S}$ be a symplectic basis for $H_1(S_b, \mathbb{R})$ (where g_S denotes the genus of the spectral curves). We can extend a_1, \dots, b_{g_S} to covariantly constant sections of \mathcal{V}^* in any simply connected open neighborhood $U \subset B^{\text{reg}}$ of b . Then the holomorphic coordinate systems (z^1, \dots, z^{g_S}) and (w_1, \dots, w_{g_S}) are given by:

$$(5.1) \quad z^i(u) = \int_{a_i} \theta, \quad w_i(u) = \int_{b_i} \theta.$$

For each $u \in U$, let $\omega_1(u), \dots, \omega_{g_S}(u)$ be the corresponding normalised basis of holomorphic 1-forms on S_u , which are characterised by:

$$(5.2) \quad \int_{a_i} \omega_j(u) = \delta_{ij}.$$

The period matrix of S_u with respect to the symplectic basis a_1, \dots, b_{g_S} is

$$\tau_{ij}(u) = \int_{b_i} \omega_j(u).$$

In these coordinates the special Kähler metric is given in terms of its Kähler form by

$$\omega = -\frac{i}{2} \operatorname{Im}(\tau_{ij}) dz^i \wedge d\bar{z}^j.$$

Since θ is a holomorphic 1-form, it can be written as a linear combination of the ω_i . From Equations (5.1) and (5.2), we immediately get

$$\theta = z^i \omega_i.$$

This equation holds not just at the level of cohomology classes, but as 1-forms. Combining this with (5.1), we also find

$$w_i = \int_{b_i} \theta = \int_{b_i} z^j \omega_j = z^j \tau_{ij}.$$

Such a relation between the z and w -coordinates does not hold for special Kähler manifolds in general. We now deduce a simple formula for the Kähler potential:

Proposition 5.5. *The Kähler potential K in Equation (2.1) is given by:*

$$K = -\frac{i}{4} \int_S \theta \wedge \bar{\theta}.$$

Proof. This is a straightforward computation:

$$\begin{aligned} -\frac{i}{4} \int_S \theta \wedge \bar{\theta} &= \frac{1}{4i} \sum_{i=1}^{g_S} \left(\int_{a_i} \theta \int_{b_i} \bar{\theta} - \int_{b_i} \theta \int_{a_i} \bar{\theta} \right) \\ &= \frac{1}{4i} \sum_{i=1}^{g_S} \left(z^i \bar{w}_i - w_i \bar{z}^i \right) \\ &= \frac{1}{2} \sum_{i=1}^{g_S} \operatorname{Im}(z^i \bar{w}_i) \\ &= -\frac{1}{2} \sum_{i=1}^{g_S} \operatorname{Im}(w_i \bar{z}^i) = K. \end{aligned}$$

□

Remark 5.6. An interesting feature of this formula is that it does not depend on a choice of symplectic basis and K is thus a *globally defined* Kähler potential on B^{reg} .

Recall that the moduli space of Higgs bundles has a natural \mathbb{C}^* -action given by rescaling the Higgs field $(E, \Phi) \mapsto (E, c\Phi)$, $c \in \mathbb{C}^*$. The \mathbb{C}^* -action on $\mathcal{M}_{n,d}$ is compatible with a \mathbb{C}^* -action on the base given by

$$c(a_1, a_2, \dots, a_n) = (ca_1, c^2 a_2, \dots, c^n a_n).$$

Let ξ be the vector field on B generating this action.

Proposition 5.7. *In terms of local special Kähler coordinates (z^1, \dots, z^{g_S}) or (w_1, \dots, w_{g_S}) on B^{reg} , we have:*

$$\xi = z^i \frac{\partial}{\partial z^i} = w_i \frac{\partial}{\partial w_i}.$$

Proof. In terms of spectral data, the \mathbb{C}^* -action on Higgs bundles corresponds to the natural \mathbb{C}^* scaling action on the fibres of $T^*\Sigma \rightarrow \Sigma$. It follows that the \mathbb{C}^* -action scales θ linearly. On the other hand, for any $c \in \mathbb{C}^*$, the action of c on $T^*\Sigma$ induces an isomorphism of spectral curves $c : S_b \rightarrow S_{cb}$. Suppose that U is an open simply connected subset of B^{reg} containing b . Then for all $c \in \mathbb{C}^*$ sufficiently close to 1, we have $cb \in U$ and the isomorphism $H_1(S_b, \mathbb{R}) \cong H_1(S_{cb}, \mathbb{R})$ induced by multiplication by c clearly agrees with parallel translation by the Gauss-Manin connection. In particular, since $z^i = \int_{a_i} \theta$, each of the coordinate functions z^i must scale linearly with the \mathbb{C}^* -action and hence $\xi(z^i) = z^i$ for each i . This gives $\xi = z^i \frac{\partial}{\partial z^i}$ as required. The same argument applied to B -cycles gives $\xi(w_i) = w_i$, hence $\xi = w_i \frac{\partial}{\partial w_i}$. \square

From now on, whenever special Kähler coordinates (z^1, \dots, z^{g_S}) are being used we will let ∂_i denote $\frac{\partial}{\partial z^i}$.

Lemma 5.8. *Let (z^1, \dots, z^{g_S}) be local special Kähler coordinates on B^{reg} . Then (as cohomology classes) we have:*

$$\nabla_{\partial_i} \theta = \omega_i.$$

Proof. Recall that for any vector field X on B^{reg} we have $\phi(X) = \nabla_X \theta$ where $\phi(X)$ is a $(1,0)$ -form. We can determine the 1-form by integrating against a -cycles:

$$\int_{a_j} \nabla_{\partial_i} \theta = \partial_i \int_{a_j} \theta = \partial_i(z^j) = \delta_{ij}.$$

Hence $\nabla_{\partial_i} \theta = \omega_i$ as claimed. \square

Proposition 5.9. *Let X, Y, Z be vector fields on B^{reg} . We have:*

$$(5.3) \quad \int_S \nabla_X \theta \wedge \theta = 0,$$

$$(5.4) \quad \int_S \nabla_X \nabla_Y \theta \wedge \theta = 0,$$

$$(5.5) \quad \int_S \nabla_X \nabla_Y \nabla_Z \theta \wedge \theta = c(X, Y, Z),$$

where $c(X, Y, Z)$ is the Donagi-Markman cubic.

Remark 5.10. The expression $\int_S \nabla_X \nabla_Y \nabla_Z \theta \wedge \theta$ is the analogue of Yukawa couplings for moduli spaces of Calabi-Yau 3-folds.

Proof. Recall as above that $\phi(X) = \nabla_X \theta$ is a $(1,0)$ -form. Thus $\nabla_X \theta \wedge \theta = 0$, since it is a $(2,0)$ -form on S . This proves (5.3). Applying ∇_Y to (5.3), we get:

$$0 = Y \left(\int_S \nabla_X \theta \wedge \theta \right) = \int_S \nabla_Y \nabla_X \theta \wedge \theta + \int_S \nabla_X \theta \wedge \nabla_Y \theta.$$

But $\int_S \nabla_X \theta \wedge \nabla_Y \theta = 0$, as $\nabla_X \theta$ and $\nabla_Y \theta$ are both $(1,0)$ -forms, so we get (5.4). Observe now that the left hand side of (5.5) is a symmetric cubic tensor because of (5.4). Therefore it suffices to consider the case where $X = \partial_i, Y = \partial_j, Z = \partial_k$. In

this case we get

$$\begin{aligned}
\int_S \nabla_{\partial_i} \nabla_{\partial_j} \nabla_{\partial_k} \theta \wedge \theta &= \sum_{l=1}^{g_S} \left(\int_{a_l} \nabla_{\partial_i} \nabla_{\partial_j} \nabla_{\partial_k} \theta \int_{b_l} \theta - \int_{b_l} \nabla_{\partial_i} \nabla_{\partial_j} \nabla_{\partial_k} \theta \int_{a_l} \theta \right) \\
&= \sum_{l=1}^{g_S} \left(w_l \partial_i \partial_j \partial_k \int_{a_l} \theta - z^l \partial_i \partial_j \partial_k \int_{b_l} \theta \right) \\
&= \sum_{l=1}^{g_S} (w_l \partial_i \partial_j \partial_k z^l - z^l \partial_i \partial_j \partial_k w_l) \\
&= - \sum_{l=1}^{g_S} z^l \partial_i \partial_j \tau_{kl} \\
&= - \sum_{l=1}^{g_S} z^l \partial_l \partial_j \tau_{ki} \quad (\text{by symmetry of } \partial_j \tau_{kl}) \\
&= -\xi \partial_j \partial_k w_i.
\end{aligned}$$

Then using the commutation relation $[\xi, \partial_i] = -\partial_i$ and the fact that $\xi(w_i) = w_i$, we see that $-\xi \partial_j \partial_k w_i = \partial_j \partial_k w_i = \partial_j \tau_{ki} = c_{jki} = c_{ijk}$. \square

Proposition 5.11. *A prepotential for the special Kähler structure on B^{reg} is given by*

$$\mathcal{F} = \frac{1}{2} z^i w_i = \frac{1}{2} \tau_{ij} z^i z^j.$$

Proof. This is a simple calculation:

$$\partial_i \mathcal{F} = \frac{1}{2} w_i + \frac{1}{2} z^j \partial_i w_j = \frac{1}{2} w_i + \frac{1}{2} z^j \tau_{ij} = \frac{1}{2} w_i + \frac{1}{2} w_i = w_i.$$

\square

6. DONAGI-MARKMAN CUBIC

The Donagi-Markman cubic measures the change in periods of the spectral curve S_b as b varies. Since the periods are essentially given by the Hodge structure on $H^1(S_b, \mathbb{C})$, this amounts to computing the variation of Hodge structure, which is controlled by the Kodaira-Spencer class of the deformation of S_b . Therefore, we need to compute the Kodaira-Spencer class $\kappa(\partial) \in H^1(S_b, T_{S_b})$ associated to a tangent vector $\partial \in T_b B^{\text{reg}}$.

6.1. Kodaira-Spencer class computation. Recall the characteristic map $\chi : T_b B^{\text{reg}} \rightarrow H^0(S_b, N_{S_b})$ in (4.1) which sends a tangent vector in $T_b B^{\text{reg}}$ to the corresponding normal vector field on S_b . Let us write $V = \chi(\partial)$ for the normal vector field corresponding to ∂ . Next, recall the universal moduli spaces of spectral curves:

$$Z^{\text{reg}} = \{(x, b) \in T^* \Sigma \times B^{\text{reg}} \mid p_b(x) = 0\}.$$

Let $q : Z^{\text{reg}} \rightarrow B^{\text{reg}}$ be the natural projection and $j : Z^{\text{reg}} \rightarrow T^* \Sigma$ the map sending a spectral curve to its image in $T^* \Sigma$. We denote the composition of j with the natural projection $T^* \Sigma \rightarrow \Sigma$ simply as $\pi : Z^{\text{reg}} \rightarrow \Sigma$. For any $b \in B$ we can find an open neighbourhood $U \subset B^{\text{reg}}$ over which the family Z^{reg} can be differentiably trivialised:

$$Z^{\text{reg}}|_U = q^{-1}(U) \cong U \times S,$$

where $S = S_b$. With respect to this trivialisation, the family $Z^{\text{reg}}|_U$ consists of a fixed topological surface S with a complex structure $I(t)$ that varies with $t \in U$. The map $\pi : Z^{\text{reg}}|_U \rightarrow \Sigma$ corresponds to a family of maps $\pi_t : S \rightarrow \Sigma$ such that π_t is holomorphic with respect to $I(t)$. Given a tangent vector $\partial \in T_b U$, we use a dot to denote differentiation by ∂ . Differentiating the condition that π_t is holomorphic with respect to $I(t)$, we get

$$(6.1) \quad \pi_*(\kappa(\partial)) = \bar{\partial}Y,$$

where $\kappa(\partial) = -\frac{i}{2}\dot{I}$ is the Kodaira-Spencer class of the deformation of I in the direction ∂ and $Y = \dot{\pi} \in \Omega^0(S, \pi^*(T_\Sigma))$ is the corresponding deformation of the map π . If $a \in S$ is a ramification point of π then $Y(a) \in (T_\Sigma)_{\pi(a)}$ is a tangent vector describing the motion of the branch point $\pi(a) \in \Sigma$. Let D be the ramification divisor of π , that is $D = \sum_a (r(a) - 1)a$ where the sum is over ramification points of π and $r(a)$ is the ramification degree of π at a . View π_* as a section of $T_S^{-1} \otimes \pi^*(T_\Sigma)$ and define

$$W = \frac{Y}{\pi_*} \in \Omega^0(S, T_S(D)),$$

which is a vector field on S having poles at the ramification points. Then Equation (6.1) says

$$\kappa(\partial) = \bar{\partial}W.$$

Note that W depends on ∂ and we will write $W(\partial)$ if we wish to show this dependence. Note also that W depends on the choice of local differentiable trivialisation of the family of spectral curves.

Definition 6.1. Let ∂ be a $(1,0)$ -vector field on B^{reg} . We denote by δ the unique vector field on $Z^{\text{reg}} \setminus D$ which is a lift of ∂ (that is, $q_*(\delta) = \partial$), satisfying $\pi_*(\delta) = 0$ (i.e., π is constant along the integral curves of δ). In a local differentiable trivialisation $Z^{\text{reg}}|_U \cong U \times S$, δ is given by:

$$(6.2) \quad \delta = \partial - W(\partial) \in \Omega^0(Z^{\text{reg}}, TZ^{\text{reg}}(D)).$$

Note that δ acquires poles along the ramification divisor D and so it may be regarded as a section of $TZ^{\text{reg}}(D)$.

We use δ to differentiate objects on the family Z^{reg} in a trivialisation-independent manner. The only drawback of this is that differentiation with respect to δ produces poles at the ramification points.

Lemma 6.2. *We have $\delta\theta = i_V d\theta|_S$, where $V = \chi(\partial)$ is the normal vector field corresponding to $\partial \in TB^{\text{reg}}$.*

Proof. Choose a local differentiable trivialisation of the family of spectral curves: $Z^{\text{reg}}|_U \cong U \times S$. Define $\tilde{V} = j_*(\partial) \in \Omega^0(S, T_{T^*\Sigma})$ and $\hat{V} = j_*(\delta) \in \Omega^0(S, T_{T^*\Sigma}(D))$. By this definition, \tilde{V} and \hat{V} are lifts of $V = \chi(\partial)$. Next, we note that since $\delta = \partial - W$ preserves $\pi \circ j : Z^{\text{reg}}|_U \rightarrow \Sigma$, we have that $j_*(\delta)$ preserves π in the sense that $\pi_* j_* \delta = 0$. Thus, $\pi_* \hat{V} = 0$. Observing that $j_* : T(U \times S) \rightarrow T_{T^*\Sigma}$ restricted to T_S acts as the identity $j_* : T_S \rightarrow T_S$, we find $\hat{V} = j_*(\delta) = j_*(\partial - W) = \tilde{V} - W$. We see that \hat{V} is a lift of V to a section of $T_{T^*\Sigma}(D)$ which is required to satisfy $\pi_* \hat{V} = 0$. Applying $\bar{\partial}$ to $\pi_* \tilde{V} = \pi_* W$, we get $\pi_*(\bar{\partial}\tilde{V}) = \pi_*(\bar{\partial}W) = \pi_* \kappa$, but $\pi_* : T_S \rightarrow \pi^* T_\Sigma$ is generically an isomorphism, so we deduce that $\kappa = \bar{\partial}W = \bar{\partial}\tilde{V}$ and \hat{V} is meromorphic. The identity $j_*(\partial) = \tilde{V}$ means that our local differentiable

trivialisation of the family of spectral curves is given by integrating the flow lines of \tilde{V} . This means that in our given trivialisation, the change in θ is $\partial\theta = \mathcal{L}_{\tilde{V}}\theta|_S$. Then:

$$\delta\theta = \partial\theta - \mathcal{L}_W\theta = \mathcal{L}_{\tilde{V}-W}(\theta)|_S = \mathcal{L}_{\tilde{V}}\theta|_S = i_{\tilde{V}}d\theta + d(i_{\tilde{V}}\theta)|_S.$$

But θ vanishes at the ramification points with order at least that of π_* , so $i_{\tilde{V}}\theta$ is a holomorphic function, hence constant on S . Therefore $d(i_{\tilde{V}}\theta)|_S = 0$ and $\delta\theta = i_{\tilde{V}}d\theta|_S$. But \hat{V} is a lift of V , so $i_{\tilde{V}}d\theta|_S = i_{\hat{V}}d\theta|_S$. \square

Consider the natural \mathbb{C}^* -action on $T^*\Sigma$ given by scalar multiplication in the fibres. Let $\xi \in H^0(T^*\Sigma, T_{T^*\Sigma})$ be the vector field generating this action. In local coordinates (x, y) , where x is a local coordinate on Σ and (x, y) corresponds to the point $(x, ydx) \in T^*\Sigma$, we have $\xi = y\frac{\partial}{\partial y}$. Note that in these coordinates the canonical 1-form is given by $\theta = ydx$, hence $\theta(\xi) = 0$, $i_\xi d\theta = \theta$.

Lemma 6.3. *Let V, \hat{V} be as in Lemma 6.2. Then \hat{V} is given by:*

$$\hat{V} = \frac{\alpha}{\theta}\xi,$$

where $\alpha = i_V d\theta|_{S_b}$.

Proof. Recall that \hat{V} is a lift of V satisfying $\pi_*(\hat{V}) = 0$. These conditions uniquely determine \hat{V} , because $\text{Ker}(\pi_*) \cap T_S$ is generically zero. Let $V^* = \frac{\alpha}{\theta}\xi$. By uniqueness, it is enough to check that V^* satisfies the necessary requirements, i.e. V^* is a section of $T_{T^*\Sigma}(D)$ which is a lift of V and satisfies $\pi_*V^* = 0$. Clearly $\pi_*V^* = 0$, because $\xi \in \text{Ker}(\pi_*)$. We need to check that V^* is a section of $T_{T^*\Sigma}(D)$. Let (x, y) be local coordinates on $T^*\Sigma$ as above and let q be a local coordinate on S_b . Then π is the map $\pi(q) = x = x(q)$ and with respect to these coordinates $d\pi = \frac{dx}{dq}$. The 1-form α has the form $\alpha(q)dq$ for some holomorphic function $\alpha(q)$. Then:

$$\begin{aligned} V^* &= \frac{\alpha}{\theta}\xi = \frac{\alpha(q)dq}{ydx}y\frac{\partial}{\partial y} \\ &= \frac{\alpha(q)dq}{dx}\frac{\partial}{\partial y} \\ &= \frac{\alpha(q)}{\left(\frac{dx}{dq}\right)}\frac{\partial}{\partial y} = \frac{\alpha(q)}{d\pi}\frac{\partial}{\partial y}. \end{aligned}$$

So the polar divisor of V^* is at most D . To show that V^* projects to V , we just need to show that

$$i_{V^*}d\theta = i_V d\theta = \alpha,$$

where the second equality is the definition of α . This follows easily from $i_\xi d\theta = \theta$, indeed:

$$i_{V^*}d\theta = \frac{\alpha}{\theta}i_\xi d\theta = \frac{\alpha}{\theta}\theta = \alpha.$$

Hence $V^* = \hat{V}$. \square

The following result relates the characteristic map to the Gauss-Manin connection:

Lemma 6.4. *For any $\partial \in T_b B^{\text{reg}}$, we have:*

$$\nabla_{\partial}\theta = \delta\theta = i_{\chi(\partial)}d\theta|_{S_b}.$$

More precisely, the right hand side is the unique holomorphic 1-form representing the cohomology class $\nabla_{\partial}\theta \in H^1(S_b, \mathbb{C})$.

Proof. The second equality is Lemma 6.2. Moreover the right hand side is clearly holomorphic. Thus it remains only to show that $\nabla_{\partial}\theta = i_{\chi(\partial)}d\theta|_{S_b}$, at the level of cohomology classes. For this we view B^{reg} as a moduli space of spectral curves, which are complex Lagrangians in the symplectic surface $(T^*\Sigma, d\theta)$. The $\mathcal{V}_{\mathbb{C}}$ -valued 1-form ϕ is essentially by definition given by $\phi(X) = i_{\chi(X)}d\theta|_{S_b}$ (this is just a special case of Equation (3.1)). On the other hand $\phi = d_{\nabla}\theta$, and the lemma follows. \square

Remark 6.5. By Lemma 6.4, we have $\alpha = \nabla_{\partial}\theta$ and so \hat{V} can be written as

$$\hat{V} = \frac{\nabla_{\partial}\theta}{\theta}\xi = \frac{\delta\theta}{\theta}\xi,$$

where it is understood that by $\nabla_{\partial}\theta$, we mean the holomorphic 1-form representing $\nabla_{\partial}\theta$ in cohomology.

6.2. Residue formula for the Donagi-Markman cubic.

Theorem 6.6. *The Donagi-Markman cubic is given by the following residue formula:*

$$c(X, Y, Z) = -2\pi i \sum_a \text{Res}_a \left(\frac{(\nabla_X\theta)(\nabla_Y\theta)(\nabla_Z\theta)\xi}{\theta} \right),$$

where the sum is over the ramification points of π . With respect to local special Kähler coordinates (z^1, \dots, z^{g_S}) this expression takes the form

$$(6.3) \quad c_{ijk} = -2\pi i \sum_a \text{Res}_a \left(\frac{\omega_i \omega_j \omega_k \xi}{\theta} \right).$$

Remark 6.7. Before giving the proof we should comment on how to interpret the right hand side of this formula. The vector field ξ is a section of $T_{T^*\Sigma}|_{S_b}$ which in general is not tangent to S_b . However, it is precisely at the ramification points where we have that ξ is tangent to S_b . Let ζ be a meromorphic section of T_S^2 defined in a neighbourhood of a and such that, when viewed as a section of $T_S \otimes T_{T^*\Sigma}$, the difference $\frac{\xi}{\theta} - \zeta$ is holomorphic. Then the right hand side of the formula can be read as:

$$-2\pi i \sum_a \text{Res}_a ((\nabla_X\theta)(\nabla_Y\theta)(\nabla_Z\theta)\zeta).$$

Note that such a ζ always exists. Indeed, in local coordinates (x, y) on $T^*\Sigma$, $\frac{\xi}{\theta} = \frac{y\partial_y}{ydx} = \frac{\partial_y}{dx}$. If a is a ramification point of degree $r(a)$ then $dx|_S$ vanishes to order $r(a) - 1$. But $y(a)$ is a zero of the characteristic polynomial $p(y, x)$ of order $r(a)$, so $(\partial_y^j p)(a) = 0$ for $j = 0, \dots, r(a) - 1$. In other words, ∂_y is tangent to S at a to order $r(a) - 1$, hence the polar part of $\frac{\xi}{\theta}$ is tangent to S . We will give a more explicit formula for this expression below, in the case where π has simple branching only (Remark 6.8).

Proof. We start with Equation (5.4):

$$\int_S \nabla_Y \nabla_Z \theta \wedge \theta = 0.$$

Applying ∇_X then gives

$$\int_S \nabla_X \nabla_Y \nabla_Z \theta \wedge \theta + \int_S \nabla_Y \nabla_Z \theta \wedge \nabla_X \theta = 0.$$

Combined with Equation (5.5), we get:

$$c(X, Y, Z) = - \int_S \nabla_Y \nabla_Z \theta \wedge \nabla_X \theta.$$

Since $\nabla_X \theta$ is a $(1, 0)$ -form, this integral only depends on the $(0, 1)$ -part of $\nabla_Y \nabla_Z \theta$. By the Griffiths transversality theorem for variations of Hodge structure we know that:

$$(\nabla_Y \nabla_Z \theta)^{(0,1)} = \kappa(Y) \cup \nabla_Z \theta \in H^1(S, \mathcal{O}),$$

where \cup denotes the cup product $\cup : H^1(S, T_S) \otimes H^0(S, K_S) \rightarrow H^1(S, \mathcal{O})$. Thus

$$c(X, Y, Z) = - \int_S (\kappa(Y) \cup \nabla_Z \theta) \wedge \nabla_X \theta.$$

Let $V = \chi(Y)$, and let \tilde{V} be a lift of V to a smooth section of $T_{T^*\Sigma}|_{S_b}$. Let \hat{V} be as in Lemma 6.3, which by the Remark 6.5 is given by

$$\hat{V} = \frac{\nabla_Y \theta}{\theta} \xi.$$

Recall that $W = \tilde{V} - \hat{V}$ is a smooth section of $T_{S_b}(D)$ and that $\bar{\partial}(W) = \bar{\partial}(\tilde{V})$ is a representative for the Kodaira-Spencer class $\kappa(Y)$. Let ζ be as in Remark 6.7. It follows that $W = -(\nabla_Y \theta)\zeta + W'$, where W' is smooth. We then have

$$\begin{aligned} c(X, Y, Z) &= - \int_S (\bar{\partial}W \cup \nabla_Z \theta) \wedge \nabla_X \theta \\ &= - \int_S \bar{\partial}(i_W(\nabla_Z \theta) \wedge \nabla_X \theta) \\ &= -2\pi i \sum_a \text{Res}_a (i_{(\nabla_Y \theta)\zeta}(\nabla_Z \theta) \wedge \nabla_X \theta) \\ &= -2\pi i \sum_a \text{Res}_a ((\nabla_X \theta)(\nabla_Y \theta)(\nabla_Z \theta)\zeta), \\ &= -2\pi i \sum_a \text{Res}_a \left(\frac{(\nabla_X \theta)(\nabla_Y \theta)(\nabla_Z \theta)\xi}{\theta} \right), \end{aligned}$$

where the sum is over the ramification points of S . □

Remark 6.8. We may re-write Equation 6.3 in terms of local coordinates as follows. For this, we will assume π has only simple ramification points. Let $a \in S$ be a ramification point. Let x be a local coordinate on Σ centered at $\pi(a)$, q a local coordinate on S centered at a chosen so that $\pi(q) = x = q^2$. Lastly, let (x, y) be local coordinates on $T^*\Sigma$ with $\theta = ydx$ the canonical 1-form and $\xi = y\partial_y$. At the point a , we have $\partial_y = \frac{dq}{dy}\partial_q$. Therefore we may choose ζ in Remark 6.7 to be

$$(6.4) \quad \zeta = \frac{y \frac{dq}{dy} \partial_q}{ydx} = \frac{1}{dxdy}.$$

Then the residue contribution to c_{ijk} at a is:

$$\text{Res}_{q \rightarrow 0} (\omega_i \omega_j \omega_k \zeta) = \text{Res}_{q \rightarrow 0} \left(\frac{\omega_i \omega_j \omega_k}{dxdy} \right).$$

We will assume that coordinates x, q have been chosen around each ramification point and so we will write Equation (6.3) as:

$$c_{ijk} = -2\pi i \sum_a \operatorname{Res}_a \left(\frac{\omega_i \omega_j \omega_k}{dx dy} \right).$$

7. RELATION TO TOPOLOGICAL RECURSION

7.1. Topological recursion for Hitchin spectral curves. Topological recursion, as introduced in [7], is a recursive procedure which takes a Riemann surface S (we assume S is compact) with meromorphic functions x, y such that dx has only simple zeros, and produces a collection of symmetric multidifferentials $W_n^{(g)}$, for $g \geq 0$, $n \geq 1$, known as the *Eynard-Orantin invariants*. By “multidifferential” we mean that $W_n^{(g)}$ is a meromorphic section of the n -th exterior tensor product $K_S^{\boxtimes n} = K_S \boxtimes K_S \boxtimes \cdots \boxtimes K_S$ on S^n . By symmetric we mean invariant under the action of the permutation group on S^n . The recursion formula requires a choice of symplectic basis $a_1, \dots, a_{g_S}, b_1, \dots, b_{g_S}$. The two base cases² are:

$$\begin{aligned} W_1^{(0)}(p) &= 0, \\ W_2^{(0)}(p_1, p_2) &= B(p_1, p_2), \end{aligned}$$

where $B(p_1, p_2)$ is the Bergman kernel³ on $S \times S$. Recall (eg, [8]) that this is the unique meromorphic symmetric bi-differential which is holomorphic away from the diagonal and which, in a local coordinate $q(p)$, has an expansion around the diagonal of the form:

$$B(q(p_1), q(p_2)) = \frac{dq(p_1) dq(p_2)}{(q(p_1) - q(p_2))^2} + O(1) dq(p_1) dq(p_2)$$

and such that B is normalised with respect to the symplectic basis in the sense that $\int_{p_1 \in a_i} B(p_1, p_2) = 0$ for $i = 1, \dots, g_S$. To define the recursion formula we need some further notation. By assumption the map $x : S \rightarrow \mathbb{P}^1$ is simply branched. Thus, for any ramification point $a \in S$, we can find a neighbourhood $a \in U \subset S$ and a non-trivial involution $\sigma : U \rightarrow U$ such that $x \circ \sigma = x$. If $q \in U$ one writes $\bar{q} = \sigma(q)$. Thus $x(q) = x(\bar{q})$. Let $\omega(q)$ denote the 1-form on U given by

$$\omega(q) = (y(q) - y(\bar{q})) dx(q) = (\theta - \sigma^* \theta)(q),$$

where $\theta = y dx$. For $p \in S$ and $q \in U$, we define $dE_q(p)$ as follows. We assume the neighbourhood U is chosen to be simply-connected and set

$$dE_q(p) = \frac{1}{2} \int_{\xi \in q}^{\bar{q}} B(\xi, p),$$

where the path of integration is taken to lie in U . Clearly this is independent of the choice of path of integration.

Now we are ready to state the recursion formula. Let p_1, \dots, p_n be points on S . If $K = \{i_1, \dots, i_k\}$ is a subset of $\{1, 2, \dots, n\}$, we let p_K be the k -tuple $p_K =$

²An alternative convention is to define $W_1^{(0)} = y dx$, but all other $W_n^{(g)}$'s are unchanged.

³One can extend topological recursion by replacing B with a modified version of the Bergman kernel, however we will not make use of this generalisation.

$(p_{i_1}, p_{i_2}, \dots, p_{i_k})$. Then for $2g - 2 + k > 0$ we define

$$(7.1) \quad W_{k+1}^{(g)}(p, p_K) = \operatorname{Res}_{q \rightarrow a} \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^g \sum_{J \subseteq K} W_{|J|+1}^{(m)}(q, p_J) W_{k-|J|+1}^{(g-m)}(\bar{q}, p_{K \setminus J}) + W_{k+2}^{(g-1)}(q, \bar{q}, p_K) \right)$$

where the sum $\sum_{J \subseteq K}$ is over all subsets $J \subseteq K$. Notice that all non-zero terms on the right hand side involve only terms $W_{k'}^{(g')}$ with $2g' - 2 + k' < 2g - 2 + k$. Therefore, this gives a recursive definition of the $W_k^{(g)}$.

The topological recursion formula was adapted to the case of Hitchin spectral curves in [6]. Here the map $x : S \rightarrow \mathbb{P}^1$ is replaced by $\pi : S \rightarrow \Sigma$. To make sense of the recursion formula (7.1) on a spectral curve $S \subset T^*\Sigma$, note that the formula does not directly involve the functions x, y only the 1-form $\theta = ydx$. For the recursive formula, we only need the Bergman kernel B , the local involutions σ about each ramification point, and the 1-forms $\omega = \theta - \sigma^*\theta$, defined around each ramification point. The local involutions σ are well-defined in a neighbourhood of each ramification point *provided* $\pi : S \rightarrow \Sigma$ has only simple branching. We will assume for the rest of this paper that this is the case.

7.2. Variational formulas. Our goal in this section is to relate the $g = 0$ Eynard-Orantin invariants of spectral curves to the special Kähler geometry of B^{reg} . The key result which ties these together is the following variational formula for the Bergman kernel:

Proposition 7.1 (Rauch variational formula). *Let $\partial \in T_b B^{\text{reg}}$. Assume p, r are distinct and are not ramification points. Then:*

$$\delta B(p, q) = - \sum_a \operatorname{Res}_{u \rightarrow a} \frac{\delta \theta(u) B(u, p) B(u, r)}{dx(u) dy(u)},$$

where the sum is over the ramification points of π and for each ramification point $a \in S_b$, we choose coordinate functions x, q as in Remark 6.8.

Proof. Although this formula is well known, we were unable to find a satisfactory proof in the literature that applies to our setting, so we provide a proof here. Choose a local differentiable trivialisation in a neighborhood U of b : $Z^{\text{reg}}|_U \cong U \times S$, so that $\delta = \partial - W$. Since δ is independent of the choice of local trivialisation, we are free to choose such a trivialisation at our convenience. Changing a given local trivialisation by a suitably chosen diffeomorphism yields a change in W of the form $W \mapsto W + X$, where X is an arbitrary smooth vector field on S (X has no poles). From this it is clear that, for a fixed choice of points p, r distinct from the ramification points, we can assume W vanishes in a neighbourhood of p and r . Then

$$\delta B(p, r) = \partial B(p, r) - \mathcal{L}_{W(p)} B(p, r) - \mathcal{L}_{W(r)} B(p, r) = \partial B(p, r),$$

because W vanishes around p and r . Next, we have the following variational formula for $B(p, r)$ [9, page 57]:

$$\begin{aligned} \partial B(p, r) &= \kappa(p) B(p, r) + \kappa(r) B(r, p) \\ &\quad - \frac{1}{2\pi i} \text{p.v.} \int_S (\kappa(\cdot) B(\cdot, p)) \wedge B(\cdot, r), \end{aligned}$$

where p.v. denotes the Cauchy principal value of the integral. Using $\kappa = \overline{\partial}W$, we obtain

$$\begin{aligned} \partial B(p, r) &= \kappa(p)B(p, r) + \kappa(q)B(r, p) \\ &\quad + \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} W(u)B(u, p)B(u, r), \\ &= \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} W(u)B(u, p)B(u, r), \end{aligned}$$

where we obtain the last line because $\kappa = 0$ at p and r by our assumption on W . The sum \sum_a is taken over all poles of $W(u)B(u, p)B(u, r)$, namely a is a ramification point, $a = p$ or $a = r$ and γ_a is a contour around the point a . However, we again have by our assumptions on W that there are no residue contributions from the points p, r and therefore we can take the sum to just be over the ramification points of π . Now, as in the proof of Theorem 6.6, we may write $W = -(\delta\theta)\zeta + W'$, where W' is smooth and ζ is given by (6.4). Then

$$\begin{aligned} \delta B(p, r) &= \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} (-(\delta\theta)(u)\zeta(u) + W'(u))B(u, p)B(u, r), \\ &= - \sum_a \operatorname{Res}_{u \rightarrow a} (\delta\theta)(u)\zeta(u)B(u, p)B(u, r) \\ &= - \sum_a \operatorname{Res}_{u \rightarrow a} \frac{\delta\theta(u)B(u, p)B(u, r)}{dx(u)dy(u)}. \end{aligned}$$

□

Let (z^1, \dots, z^{g_s}) be local special Kähler coordinates on B^{reg} . Let ∂_i denote the vector field $\frac{\partial}{\partial z^i}$ and let δ_i denote $\delta(\partial_i)$. We have:

Theorem 7.2 (Variational formula [7]). *For $g + k > 1$,*

$$\delta_i W_k^{(g)}(p_1, \dots, p_k) = -\frac{1}{2\pi i} \int_{p \in b_i} W_{k+1}^{(g)}(p, p_1, \dots, p_k).$$

Proof. This is essentially Theorem 5.1 of [7]. Since we are working in a setting where θ is not globally of the form $\theta = ydx$, one needs to check the proof of Theorem 5.1 in [7] holds in this setting. In fact the proof in [7, pages 32-34] essentially only relies on the Rauch variational formula (which we have proven in Proposition 7.1) and the diagrammatic representation of $W_k^{(g)}$ [7, Theorem 4.8], the proof of which only uses the recursive definition of $W_k^{(g)}$ and does not involve any global properties of the spectral curve. □

Remark 7.3. Note that the poles of $W_k^{(g)}$, in any one of its variables, have zero residues. This can easily be deduced from symmetry and the diagrammatic representation of $W_k^{(g)}$ [7, Theorem 4.8]. Therefore the integration of $W_k^{(g)}$ over a cycle γ (chosen so as to avoid the poles) depends only on the homology class of γ in S . Similarly one can also show that the integration of $W_k^{(g)}$ over an a -cycle is zero.

Consider the case $(g, k) = (0, 2)$, where $W_2^{(0)}(p_1, p_2) = B(p_1, p_2)$ is the Bergman kernel. Applying the variational formula, we obtain

$$\delta_i B(p_1, p_2) = -\frac{1}{2\pi i} \int_{p \in b_i} W_3^{(0)}(p, p_1, p_2).$$

Recall that $\int_{p_1 \in b_j} B(p_1, p_2) = 2\pi i \omega_j(p_2)$ [8] and thus $\int_{p_1 \in b_j} \int_{p_2 \in b_k} B(p_1, p_2) = 2\pi i \tau_{jk}$. The variational formula then gives

$$\begin{aligned} c_{ijk} &= \partial_i \tau_{jk} = \frac{1}{2\pi i} \partial_i \int_{p_1 \in b_j} \int_{p_2 \in b_k} B(p_1, p_2) \\ &= \frac{1}{2\pi i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} \delta_i B(p_1, p_2) \\ &= - \left(\frac{1}{2\pi i} \right)^2 \int_{p \in b_i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} W_3^{(0)}(p, p_1, p_2). \end{aligned}$$

From [7, Theorem 4.1], we have:

$$W_3^{(0)}(p, p_1, p_2) = \sum_a \operatorname{Res}_{q \rightarrow a} \left(\frac{B(p, q) B(p_1, q) B(p_2, q)}{dx(q) dy(q)} \right).$$

Therefore, we obtain

$$\begin{aligned} c_{ijk} &= - \left(\frac{1}{2\pi i} \right)^2 \sum_a \int_{p \in b_i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} \operatorname{Res}_{q \rightarrow a} \left(\frac{B(p, q) B(p_1, q) B(p_2, q)}{dx(q) dy(q)} \right) \\ &= -2\pi i \sum_a \operatorname{Res}_{q \rightarrow a} \left(\frac{\omega_i(q) \omega_j(q) \omega_k(q)}{dx(q) dy(q)} \right), \end{aligned}$$

which agrees with Theorem 6.6 (see Remark 6.8). In a similar manner, starting with $W_2^{(0)}(p_1, p_2) = B(p_1, p_2)$ and applying the variational formula multiple times, we obtain:

Theorem 7.4. *We have:*

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-2}} \tau_{i_{m-1} i_m} = - \left(\frac{i}{2\pi} \right)^{m-1} \int_{p_1 \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_m^{(0)}(p_1, \dots, p_m).$$

Therefore, the $g = 0$ Eynard-Orantin invariants $W_k^{(0)}$ for a spectral curve S_b compute the power series expansion of the period matrix τ_{ij} about $b \in B^{\text{reg}}$. Since the special Kähler metric on B^{reg} is given in terms of the period matrix, the invariants $W_k^{(0)}$ also compute the power series expansion of the special Kähler metric.

7.3. Second derivatives of the period matrix by topological recursion.

We will use Theorem 7.4 to compute the symmetric quartic $\partial_i \partial_j \tau_{kl}$ of second derivatives of the period matrix. From the diagrammatic representation of Eynard-Orantin invariants, one finds ([7, Equation (4-46)]):

$$\begin{aligned} W_4^{(0)}(p, p_1, p_2, p_3) &= \sum_{a,b} \operatorname{Res}_{q \rightarrow a} \operatorname{Res}_{r \rightarrow b} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(q)}{\omega(r)} [B(\bar{q}, p_1) B(r, p_2) B(\bar{r}, p_3) + \text{perm}_{1,2,3}] \\ &\quad + \sum_{a,b} \operatorname{Res}_{q \rightarrow a} \operatorname{Res}_{r \rightarrow b} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(\bar{q})}{\omega(r)} [B(q, p_1) B(r, p_2) B(\bar{r}, p_3) + \text{perm}_{1,2,3}], \end{aligned}$$

where $\text{perm}_{1,2,3}$ means we sum over all permutations of p_1, p_2, p_3 . To this, we apply $(\frac{1}{2\pi i})^4 \int_{p \in b_i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} \int_{p_3 \in b_l}$, giving

$$(7.2) \quad \begin{aligned} & \frac{1}{2\pi i} \int_{p \in b_i} \sum_{a,b} \text{Res}_{q \rightarrow a} \text{Res}_{r \rightarrow b} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(q)}{\omega(r)} [\omega_j(\bar{q})\omega_k(r)\omega_l(\bar{r}) + \text{perm}_{j,k,l}] \\ & + \frac{1}{2\pi i} \int_{p \in b_i} \sum_{a,b} \text{Res}_{q \rightarrow a} \text{Res}_{r \rightarrow b} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(\bar{q})}{\omega(r)} [\omega_j(q)\omega_k(r)\omega_l(\bar{r}) + \text{perm}_{j,k,l}]. \end{aligned}$$

Around each ramification point, choose coordinates x, q as usual. Then $dx(q) = 2q dq$, $dy = y'(q) dq$ and we have:

$$\text{Res}_{q \rightarrow b} \frac{dE_q(p)f(q)}{\omega(q)} = -\frac{1}{2} \text{Res}_{q \rightarrow b} \frac{B(q,p)f(q)}{dx(q)dy(q)} = -\frac{B(b,p)}{4y'(0)} f(b),$$

where $f(q)$ is any local section of K_S^2 , holomorphic in a neighbourhood of b . Using this, (7.2) simplifies to:

$$(7.3) \quad \begin{aligned} & \frac{1}{4\pi i} \int_{p \in b_i} \sum_{a,b} \text{Res}_{q \rightarrow a} B(q,b) \frac{dE_q(p)}{\omega(q)} \omega_j(\bar{q}) \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + \text{perm}_{j,k,l} \\ & + \frac{1}{4\pi i} \int_{p \in b_i} \sum_{a,b} \text{Res}_{q \rightarrow a} B(\bar{q},b) \frac{dE_q(p)}{\omega(q)} \omega_j(q) \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + \text{perm}_{j,k,l}. \end{aligned}$$

Consider first the terms where $a \neq b$. Then $B(q,b)$ and $B(\bar{q},b)$ have no pole at $q = a$, so these terms give, on further simplification:

$$\begin{aligned} & \frac{1}{4\pi i} \int_{p \in b_i} \sum_{a \neq b} B(a,b) \frac{B(a,p)\omega_j(a)}{2y'(a)} \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + \text{perm}_{j,k,l} \\ & = \frac{1}{2} \sum_{a \neq b} B(a,b) \left(\frac{\omega_i(a)\omega_j(a)}{2y'(a)} \right) \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + \text{perm}_{j,k,l} \\ & = \sum_{a \neq b} B(a,b) \left(\frac{\omega_i(a)\omega_j(a)}{2y'(a)} \right) \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + \text{cyc}_{j,k,l} \end{aligned}$$

where $\text{cyc}_{j,k,l}$ means a sum over cyclic permutations of j, k, l . Now let us consider the terms where $a = b$. In this case we must compute the residue as $q \rightarrow a$ of $\frac{1}{2\pi i} \int_{p \in b_i} B(q,a) \frac{dE_q(p)}{\omega(q)} \omega_j(\bar{q})$ and of $\frac{1}{2\pi i} \int_{p \in b_i} B(\bar{q},a) \frac{dE_q(p)}{\omega(q)} \omega_j(q)$. Both of these have poles of third order. We have the following expansions near $q = 0$:

$$\begin{aligned} y(q) &= y(0) + y'(0)q + \frac{1}{2}y''(0)q^2 + \frac{1}{6}y'''(0)q^3 + \dots \\ \omega(q) &= (y(q) - y(-q))2q dq = 4q^2(y'(0) + \frac{1}{6}y'''(0)q^2 + \dots) dq \\ \omega_j(q) &= \left(\omega_j(0) + \omega'_j(0)q + \frac{1}{2}\omega''_j(0)q^2 + \frac{1}{6}\omega'''_j(0)q^3 + \dots \right) dq \\ B(q,a) &= \left(\frac{1}{q^2} + \frac{1}{6}S_B(a) + \dots \right) dq da \end{aligned}$$

where $S_B(a)$ is the Bergman projective connection [8]. Also, we may compute the expansion of $\frac{1}{2\pi i} \int_{p \in b_i} dE_q(p)$:

$$\begin{aligned} \frac{1}{2\pi i} \int_{p \in b_i} dE_q(p) &= \frac{1}{2\pi i} \int_{p \in b_i} \frac{1}{2} \int_{\xi=q}^{\bar{q}} B(\xi, p) \\ &= \frac{1}{2} \int_{\xi=q}^{\bar{q}} \omega_i(\xi) \\ &= \frac{1}{2} \int_{\xi=q}^{-q} \left(\omega_i(0) + \omega_i'(0)q + \frac{1}{2}\omega_i''(0)q^2 + \dots \right) \\ &= -q \left(\omega_i(0) + \frac{1}{6}\omega_i''(0)q^2 + \dots \right). \end{aligned}$$

Therefore, the expansion of $\frac{1}{2\pi i} \int_{p \in b_i} B(q, a) \frac{dE_q(p)}{\omega(q)} \omega_j(\bar{q})$ has the form

$$\begin{aligned} &\frac{1}{2\pi i} \int_{p \in b_i} B(q, a) \frac{dE_q(p)}{\omega(q)} \omega_j(\bar{q}) \\ &= \left(\frac{1}{q^2} + \frac{1}{6}S_B(a) + \dots \right) \frac{q \left(\omega_i(0) + \frac{1}{6}\omega_i''(0)q^2 + \dots \right)}{4q^2(y'(0) + \frac{1}{6}y'''(0)q^2 + \dots)} \left(\omega_j(0) - \omega_j'(0)q + \frac{1}{2}\omega_j''(0)q^2 + \dots \right) dqda \\ &= \left(\frac{1}{q^2} + \frac{1}{6}S_B(a) + \dots \right) \frac{\left(\omega_i(0) + \frac{1}{6}\omega_i''(0)q^2 + \dots \right)}{4q(y'(0) + \frac{1}{6}y'''(0)q^2 + \dots)} \left(\omega_j(0) - \omega_j'(0)q + \frac{1}{2}\omega_j''(0)q^2 + \dots \right) dqda. \end{aligned}$$

Adding this to the corresponding expansion for $\frac{1}{2\pi i} \int_{p \in b_i} B(\bar{q}, a) \frac{dE_q(p)}{\omega(q)} \omega_j(q)$, we get

$$\begin{aligned} &\frac{1}{2\pi i} \int_{p \in b_i} B(q, a) \frac{dE_q(p)}{\omega(q)} \omega_j(\bar{q}) + \frac{1}{2\pi i} \int_{p \in b_i} B(\bar{q}, a) \frac{dE_q(p)}{\omega(q)} \omega_j(q) \\ &= \left(\frac{1}{q^2} + \frac{1}{6}S_B(a) + \dots \right) \frac{\left(\omega_i(0) + \frac{1}{6}\omega_i''(0)q^2 + \dots \right)}{2q(y'(0) + \frac{1}{6}y'''(0)q^2 + \dots)} \left(\omega_j(0) + \frac{1}{2}\omega_j''(0)q^2 + \dots \right) dqda. \end{aligned}$$

The coefficient of $\frac{dq}{q}$ in this is:

$$\frac{1}{12y'(0)} \left(S_B(a) - \frac{y'''(0)}{y'(0)} \right) \omega_i(0)\omega_j(0) + \frac{1}{2y'(0)} \left(\frac{1}{2}\omega_i(0)\omega_j''(0) + \frac{1}{6}\omega_i''(0)\omega_j(0) \right).$$

Therefore, the $a = b$ terms in (7.3) are given by:

$$\begin{aligned} &\frac{1}{2} \sum_a \frac{1}{24y'(a)^2} \left(S_B(a) - \frac{y'''(a)}{y'(a)} \right) \omega_i(a)\omega_j(a)\omega_k(a)\omega_l(a) + \text{perm}_{j,k,l} \\ &+ \frac{1}{2} \sum_a \frac{1}{4y'(a)^2} \left(\frac{1}{2}\omega_i(a)\omega_j''(a)\omega_k(a)\omega_l(a) + \frac{1}{6}\omega_i''(a)\omega_j(a)\omega_k(a)\omega_l(a) \right) + \text{perm}_{j,k,l} \\ &= \sum_a \frac{1}{8y'(a)^2} \left(S_B(a) - \frac{y'''(a)}{y'(a)} \right) \omega_i(a)\omega_j(a)\omega_k(a)\omega_l(a) \\ &\quad + \sum_a \frac{1}{8y'(a)^2} \left(\omega_i''(a)\omega_j(a)\omega_k(a)\omega_l(a) \right) + \text{cyc}_{i,j,k,l}. \end{aligned}$$

Putting this all together we have shown:

Theorem 7.5. *The second derivatives of the period matrix are given by:*

$$\begin{aligned}
\partial_i \partial_j \tau_{kl} &= \left(\frac{1}{2\pi i} \right)^3 \int_{p \in b_i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} \int_{p_3 \in b_l} W_4^{(0)}(p, p_1, p_2, p_3) \\
&= 2\pi i \sum_{a \neq b} B(a, b) \left(\frac{\omega_i(a)\omega_j(a)}{2y'(a)} \right) \left(\frac{\omega_k(b)\omega_l(b)}{2y'(b)} \right) + cy c_{j,k,l} \\
(7.4) \quad &+ 2\pi i \sum_a \frac{1}{8y'(a)^2} \left(S_B(a) - \frac{y'''(a)}{y'(a)} \right) \omega_i(a)\omega_j(a)\omega_k(a)\omega_l(a) \\
&+ 2\pi i \sum_a \frac{1}{8y'(a)^2} (\omega_i''(a)\omega_j(a)\omega_k(a)\omega_l(a)) + cy c_{i,j,k,l}.
\end{aligned}$$

Remark 7.6. Let us verify that the right hand side of (7.4) is independent of the choice of local coordinates q, x satisfying $x = q^2$, as it must be, since $\partial_i \partial_j \tau_{kl}$ is independent of such choices. Consider a change of variables $\hat{x} = h(x)$, $\hat{q} = f(q)$, where $x = q^2$ and $\hat{x} = \hat{q}^2$. If $h(x) = h'x + \frac{1}{2}h''x^2 + \frac{1}{6}h'''x^3 + \dots$ and $f(q) = f'q + \frac{1}{2}f''q^2 + \frac{1}{6}f'''q^3 + \dots$ then the relation $\hat{x} = \hat{q}^2$ implies $f''(0) = 0$. Let $S_B(a), \hat{S}_B(a)$ denote the Bergman projective connection in the q and \hat{q} -coordinates. The property of being a projective connection means

$$S_B(a) = (f')^2 \hat{S}_B(a) + \left(\frac{f'''}{f'} - \frac{3}{2} \frac{f''^2}{f'^2} \right),$$

where $\left(\frac{f'''}{f'} - \frac{3}{2} \frac{f''^2}{f'^2} \right) = S(f)$ is the Schwarzian derivative of f at $q = 0$. From $\theta = y(q)dx = \hat{y}(\hat{q})d\hat{x}$, one finds

$$(7.5) \quad y'(0) = \hat{y}'(0)(f')^3$$

and

$$\frac{y'''(0)}{y'(0)} = \frac{\hat{y}'''(0)}{\hat{y}'(0)}(f')^2 + 5 \frac{f'''}{f'}.$$

Then since $f'' = 0$, we get

$$(7.6) \quad \left(S_B(a) - \frac{y'''(0)}{y'(0)} \right) = (f')^2 \left(\hat{S}_B(a) - \frac{\hat{y}'''(0)}{\hat{y}'(0)} \right) - 4 \frac{f'''}{f'}.$$

But using $\omega_i(q)dq = \hat{\omega}_i(\hat{q})d\hat{q}$, we also find

$$(7.7) \quad \omega_i(a) = (f')\hat{\omega}_i(a)$$

$$(7.8) \quad \omega_i''(a) = (f')^3 \hat{\omega}_i''(a) + \hat{\omega}_i(a)f''.$$

Substituting (7.5)-(7.8) into (7.4), we see that the result is coordinate independent.

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