Lifting Bundle Gerbes and Central Extensions of Gauge Groups

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Contents

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Abstract

The loop group $\mathrm{Map}(S^1, G)$ has a central extension by the circle called the Kac-Moody group. We use the techniques introduced in [29] to generalise this construction to higher mapping groups such as $\mathrm{Map}(\Sigma, G)$ and construct the central extension

$$1 \to \frac{\Omega^1(\Sigma, \mathbb{R})}{\Omega^1_{\mathbb{Z}}(\Sigma)} \to \widehat{\mathrm{Map}(\Sigma, G)} \to \mathrm{Map}(\Sigma, G) \to 1,$$

where $\Omega^1(\Sigma, \mathbb{R})$ is the space of all differentiable one-forms on a compact manifold Σ and $\Omega^1_{\mathbb{Z}}(\Sigma)$ is the subspace of all closed one-forms with integral periods. An alternative construction of $\widehat{\mathrm{Map}}(\Sigma, G)$ is given in [18].

In the case that $\mathrm{Map}(\Sigma, G)$ is 1-connected we can consider the topological problem of when a $\mathrm{Map}(\Sigma, G)$-bundle lifts to a $\widehat{\mathrm{Map}}(\Sigma, G)$-bundle. This was addressed in [27] for the case $\Sigma = S^1$. We have investigated the general case using those same methods. The geometric tool used to address this problem is the theory of bundle gerbes [25] and particularly the notion of the *lifting* bundle gerbe. In the case of $U(1)$, bundle gerbes on a manifold M provide a realisation of $H^3(M, \mathbb{Z})$. The lifting bundle gerbe is an important example of a bundle gerbe which is associated to a central extension such as the one above.

Chapter 1

Introduction

If a manifold M admits a spin structure then it is possible to define spinor fields and the Dirac operator on M, and thus to discuss the physics of fermions on M. To obtain a geometrical understanding of world-sheet anomalies in string theory we must replace the finite dimensional geometry associated with spin structures by a suitable infinite dimensional geometry involving loop spaces [6, 14].

We recall what it means for an orientable manifold to admit a spin structure. Let M be an orientable n-dimensional manifold and $P \to M$ the principal $SO(n)$-bundle of oriented frames on M. Recall that a manifold is orientable if its first Stiefel-Whitney class $w_1(M)$ is equal to zero. The group $SO(n)$ admits a central extension

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1. \tag{1.1}$$

The manifold M admits a spin structure if P lifts to a principal $Spin(n)$-bundle $\widehat{P} \to M$ via (1.1). There is a characteristic class of M which measures the obstruction to the existence of such a lift; the lift $\widehat{P}$ exists if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes [30].

We define the *loop space* of M to be the infinite dimensional manifold $LM = \mathrm{Map}(S^1, M)$ of all smooth maps $\gamma : S^1 \to M$. These are interesting objects as they are related to strings moving around on a manifold. Given a compact Lie group G the loop space LG forms a group under pointwise multiplication and is called the *loop group*. Given a principal G-bundle P over a manifold M we can take loops to obtain a principal LG-bundle LP over LM.

In the case of spin structures we had a principal $SO(n)$-bundle over M and we wished to lift this to a principal $Spin(n)$-bundle on M. In this case LG is the analogue of $SO(n)$ and so we need to find an analogue for $Spin(n)$. The loop group LG has a central extension $\widehat{LG}$ by the circle group S^1 called the Kac-Moody group. The obstruction to lifting the principal LG-bundle $LP \to LM$ to a $\widehat{LG}$-bundle $\widehat{LP} \to LM$ is a class in $H^3(LM, \mathbb{Z})$, called the *string class* [14].

The loop group LG and its extensions also appear in quantum field theory because an affine Lie algebra is the Lie algebra of non-abelian current densities in the one-dimensional compactified space S^1 [22]. Similarly, $\mathrm{Map}(S^3, G)$ is important in quantum field theory in a $3+1$ dimensional space-time. When going to dimension 3 or higher one has to deal with extensions by infinite-dimensional abelian groups. Mickelsson [22] shows that although $\widehat{LG}$ is a nontrivial S^1-bundle over LG ($\widehat{LG}$ is not the direct product of LG and S^1), the group DG consisting of the smooth maps of the unit disk D to the group G has a central extension $\widehat{DG}$ which is topologically (but not algebraically) trivial. It is then shown that $\widehat{LG}$ is given by the quotient of $\widehat{DG}$ by the group of maps which are equal to the identity on the boundary S^1 of the disk.

In [29] Pressley and Segal construct $\widehat{LG}$ in the following way. Let $\mathfrak{g}$ be the Lie algebra of G and $L\mathfrak{g}$ be the *loop algebra* of all loops in $\mathfrak{g}$. Then as a vector space $\widehat{L\mathfrak{g}}$ is given by $L\mathfrak{g} \oplus \mathbb{R}$, with the bracket given by

$$[(X, a), (Y, b)] = ([\alpha, \beta], \omega(\alpha, \beta))$$

for $\alpha, \beta \in L\mathfrak{g}$, $a, b \in \mathbb{R}$ and a Lie algebra 2-cocycle $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ given by

$$\omega(\alpha, \beta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \alpha(\theta), \beta'(\theta) \rangle d\theta.$$

It is shown in [29] that ω is a skew form on the tangent space to LG at the identity and therefore defines a left-invariant 2-form on LG. It follows that if G is simply connected the Lie algebra extension

$$1 \to \mathbb{R} \to \widehat{L\mathfrak{g}} \to L\mathfrak{g} \to 1$$

defined by ω corresponds to a group extension

$$1 \to S^1 \to \widehat{LG} \to LG \to 1$$

if and only if the differential form $\omega/2\pi$ represents an integral cohomology class on LG. It is also outlined in [29] how this construction can be generalised to give a central extension of the group $\mathrm{Map}(\Sigma, G)$ of smooth maps $f : \Sigma \to G$.

In [26] Murray and Stevenson used the theory of bundle gerbes to construct a central extension of the loop group and give an explicit realisation of the string class using differential forms. In this thesis we generalise these constructions to the case of groups of maps and gauge groups. Using our general constructions from chapter 3 (in particular, proposition 3.20) and proposition 4.5 we obtain the following theorem.

Theorem 1.1. *Suppose Σ is a compact manifold and G a Lie group and let Q be a G-bundle over Σ with a connection ∇. The gauge group*

$$\mathcal{G} = \mathrm{Aut}(Q) = \{f : Q \to G \mid f(qg) = g^{-1}f(q)g\}$$

of Q has a central extension by $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$ characterised by $\alpha \in \Omega^1(\mathcal{G}\times\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ defined by

$$\alpha((g_1, g_2); (g_1X_1, g_2X_2)) = \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\nabla_\Sigma g_2)g_2^{-1}))$$

and

$$R(g; gX, gY) = \frac{i}{4\pi}\Pi(\mathrm{tr}(X\nabla_\Sigma Y)),$$

where Π is the projection map $\Omega^1(\Sigma) \to \mathrm{Lie}(\mathcal{A}) = \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}$ and $\nabla_\Sigma Y \in \Omega^1(Q, \mathfrak{g})$ is defined by

$$\nabla_\Sigma Y(\xi) = dY(h\xi).$$

The Dixmier-Douady class of a lifting bundle gerbe is known to be the obstruction to a principal $\mathcal{G}$ bundle lifting to a principal $\hat{\mathcal{G}}$ bundle, where

$$1 \to \mathcal{A} \to \widehat{\mathcal{G}} \to \mathcal{G} \to 1$$

is a central extension. In the case where $\mathcal{A} = U(1)$ and $\mathcal{G}$ is the loop group, the Dixmier-Douady class of the lifting bundle gerbe is a class in $H^3(M, \mathbb{Z})$, the string class [27].

In this thesis we give an explicit formula for the de Rham representative of the Dixmier-Douady class of the lifting bundle gerbe in the more general case of groups of maps. Using our construction in chapter 5, in particular proposition 5.5 we obtain the following theorem.

Theorem 1.2. *Let $P \to M$ be a $\mathcal{G} = \mathrm{Map}(\Sigma, G)$ bundle with connection A and Higgs field $\Phi : P \to \Omega^1(\Sigma, \mathfrak{g})$. The Dixmier-Douady class of the lifting bundle gerbe which is the obstruction to lifting P to $\hat{\mathcal{G}}$ is represented in de Rham cohomology by*

$$\omega = -\Pi(\mathrm{tr}(FD_A\Phi)) \in H^3(M, \mathrm{Lie}(\mathcal{A})),$$

where $F = dA + [A, A]$ and

$$D_A\Phi = d\Phi + [A, \Phi] - d_\Sigma A \in \Omega^1(P, \Omega^1(\Sigma, \mathfrak{g})).$$

In [18] central extensions of $\mathrm{Map}(\Sigma, G)$ and $\mathrm{Aut}(Q)$ are constructed using slightly different techniques. Other more recent results in the area include the work of Maier and Neeb [19] and Vizman [33].

This thesis consists of six chapters; including this introduction. The second chapter presents background material on Lie groups, principal bundles and bundle gerbes. In the third chapter central extensions of groups are discussed and an explicit method for construction of central extension of groups is given using differential forms. In the fourth chapter a central extension of the loop group of a compact, simple, simply connected Lie group G is constructed; this construction is then generalised to a central extension of the group $\mathrm{Map}(\Sigma, G)$ of smooth maps between a compact manifold Σ and a simple Lie group G and also $\mathrm{Aut}(Q)$, the gauge group of a principal bundle Q over a compact manifold. The fifth chapter explains the connection between the lifting bundle gerbe and central extensions of groups. The final chapter serves as a conclusion for the preceding chapters and briefly explains some ideas for further research.

Chapter 2

Background

In this chapter we present some preliminary material about differential forms on manifolds, Lie groups, principal bundles and bundle gerbes as well as gauge groups. The following chapters will make use of these notions.

2.1 Differential forms

We recall some standard results on differential forms for later use. Readers can refer to [32] for further details.

Let V be a real vector space. We define a k-multilinear map to be a map

$$\omega : V^k \to \mathbb{R}$$

that is linear in each factor. The map ω is called antisymmetric if

$$\omega(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -\omega(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$ and all $1 \leq i < k$. The set of all k-multilinear antisymmetric maps on V forms a vector space under addition. This vector space is denoted $\Lambda^k(V^*)$ and its elements are called k-forms on V.

Definition 2.1. *If $\omega \in \Lambda^p(V^*)$ and $\rho \in \Lambda^q(V^*)$ we define $\omega \wedge \rho \in \Lambda^{p+q}(V^*)$ by*

$$(\omega \wedge \rho)(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, \ldots, v_{\pi(p)})\, \rho(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)}),$$

where S_{p+q} is the set of all permutations on $\{1, 2, \ldots, p+q\}$ and $\operatorname{sgn}(\pi)$ is the sign of the permutation π.

Definition 2.2. *Let M be a manifold. A k-form on M is a function ω that assigns to each point $p \in M$ a k-linear map $\omega_p \in \Lambda^k(T_p^*M)$.*

If ω is a k-form on a manifold M and $X_1, \ldots, X_k$ are vector fields on M, then $\omega(X_1, \ldots, X_k)$ is the function on M defined by

$$(\omega(X_1, \ldots, X_k))(p) = \omega_p((X_1)_p, \ldots, (X_k)_p).$$

Definition 2.3. *A k-form on a manifold M is smooth if its components with respect to the local coordinates on M are smooth. We denote the set of all smooth k-forms on M by $\Omega^k(M)$. We also define $\Omega^0(M)$ to be $C^\infty(M)$, the set of all smooth functions on M. Note that $\Omega^k(M)$ has a natural structure as a vector space over $\mathbb{R}$.*

The usual derivative on functions defines a linear differential operator

$$d : \Omega^0(M) \to \Omega^1(M)$$

satisfying the Leibniz rule

$$d(fg) = (df)g + f(dg)$$

for smooth functions $f, g : M \to \mathbb{R}$.

Proposition 2.4. [32, section 19] *Let M be an n dimensional manifold. Then there are unique linear maps*

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

for $p = 0, 1, \ldots, n-1$ satisfying

- *d is the usual derivative of smooth functions if $p = 0$,*
- *$d^2 = 0$,*
- *If $\omega \in \Omega^p(M)$ and $\rho \in \Omega^q(M)$ then $d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^p \omega \wedge (d\rho)$.*

This linear map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is called the exterior derivative on p-forms.

Definition 2.5 ([32], §18)**.** *Let $f : M \to N$ be a smooth map and $\omega \in \Omega^k(N)$. We define the pullback $f^*(\omega) \in \Omega^k(M)$ by*

$$f^*(\omega)(x)(X_1, \ldots, X_k) = \omega(f(x))(T_x(f)(X_1), \ldots, T_x(f)(X_k))$$

where $X_1, \ldots, X_k \in T_xM$. It can be shown that $f^*(\omega)$ is a smooth k-form on M.

Proposition 2.6. *If $f : M \to N$ is a smooth map and ω and ρ are differential forms on N then*

- $df^*(\omega) = f^*(d\omega)$,
- $f^*(\omega \wedge \rho) = f^*(\omega) \wedge f^*(\rho)$.

Later on we will be interested in computing the exterior derivative of differential forms. A useful formula for the exterior derivative of 1-forms and 2-forms is given by the following proposition.

Proposition 2.7 ([15], Proposition 3.11)**.** *Let M be a manifold and let ω be a 1-form and ρ a 2-form on M. Then*

$$(d\omega)(X, Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\}$$

and

$$(d\rho)(X, Y, Z) = \frac{1}{3}\{X(\rho(Y, Z)) + Y(\rho(Z, X)) + Z(\rho(X, Y)) - \rho([X, Y], Z) - \rho([Y, Z], X) - \rho([Z, X], Y)\},$$

where X, Y and Z are vector fields on M.

2.2 Lie groups

In this section we recall some basic definitions in the theory of Lie groups. Roughly speaking, a Lie group is a blend of the notions of group and manifold. We will give some examples of Lie groups which appear in later chapters.

Definition 2.8. *A* Lie group *is a manifold G with a group structure such that the multiplication map*

$$m : G \times G \to G, \quad m(g_1, g_2) = g_1 g_2$$

and the inverse map

$$i : G \to G, \quad i(g) = g^{-1}$$

are smooth.

Let n be a positive integer and $\mathrm{Mat}(n, \mathbb{C})$ be the set of n by n matrices with complex entries. We define the *general linear group* to be the set of all invertible n by n matrices over the complex numbers with matrix multiplication as the group operation. This group is denoted by

$$\mathrm{GL}(n, \mathbb{C}) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid \det(A) \neq 0\}.$$

The underlying set $\mathrm{GL}(n, \mathbb{C}) = \det^{-1}(\mathbb{C} \setminus \{0\})$ is open in $\mathrm{Mat}_n(\mathbb{C}) \simeq \mathbb{C}^{n^2} \simeq \mathbb{R}^{2n^2}$ since $\det : \mathrm{Mat}(n, \mathbb{C}) \to \mathbb{C}$ is continuous and $\mathbb{C} \setminus \{0\}$ is open in $\mathbb{C}$. Furthermore, matrix multiplication and inversion are smooth maps . So $\mathrm{GL}(n, \mathbb{C})$ is a Lie group.

We have the following result known as the *closed subgroup theorem.*

Theorem 2.9 ([32], Theorem 15.12)**.** *Let G be a Lie group and H a topologically closed subgroup of G. Then H is a Lie subgroup of G.*

Using the closed subgroup theorem we can deduce that the following are Lie groups.

- **Special linear group:** The set of matrices with determinant one with matrix multiplication denoted by

$$\mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{Mat}(n, \mathbb{C}) \mid \det(A) = 1\}.$$

- **Orthogonal group:** The group of orthogonal matrices

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\}.$$

- **Special orthogonal group:**

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{C}).$$

- **Unitary group:** The group of unitary matrices

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^* A = I\},$$

  where $A^* = \overline{A^T}$ is the conjugate transpose of A.

- **Special unitary group:**

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}).$$

All the Lie groups so far have been finite dimensional. Below we introduce the *loop group*, which is an infinite dimensional Lie group.

Let G be a compact Lie group. Then the loop group $LG = \mathrm{Map}(S^1, G)$ of all smooth maps from the circle to G forms a Lie group, with the group operation being given by point-wise multiplication of functions

$$(\gamma_1\gamma_2)(\theta) = \gamma_1(\theta)\gamma_2(\theta)$$

for all $\gamma_1, \gamma_2 \in LG$ and $\theta \in S^1$. See [29] for a detailed discussion of loop groups and Appendix A for infinite dimensional manifolds.

2.3 Principal bundles

Definition 2.10. *Let G be a group and X a manifold. A smooth right action of G on X is a smooth map $X \times G \to X$ denoted by $(x, g) \mapsto xg$ such that*

- *If $e \in G$ is the identity element then $xe = x$ for every $x \in X$.*
- *$(xg)h = x(gh)$ for all $g, h \in G$.*

We call a subset of the form $\{xg \mid g \in G\}$ an orbit of G on X.

Let G be a Lie group and M a manifold. A principal G-bundle P is a projection map $\pi : P \to M$ and a smooth right-action of G on P satisfying

- The action of G on P is free, meaning given $g \in G$ and $p \in P$, if $pg = p$ then g must be the identity element of G.
- The fibres of π are exactly the orbits of G.
- P is locally trivial, meaning every $m \in M$ has a neighbourhood $U \subset M$ for which there exists a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times G \\ & \searrow^{\pi} & \downarrow{\scriptstyle \mathrm{pr}_U} \\ & & U \end{array}$$

Here G acts on the right of $U \times G$ by $(x, h)g = (x, hg)$.

2.3.1 Connections

In the next two sections we introduce the notion of connection and curvature for a principal bundle and state some key results. A more detailed discussion along with the proofs can be found in [15] chapter II.

Let P be a principal G-bundle over a manifold M. For each $p \in P$ let T_pP be the tangent space of P at p and V_p the subspace of T_pP consisting of vectors tangent to the fibre through p. A *connection* Γ on P is an assignment of a subspace $H_p \subset T_pP$ to each $p \in P$ such that for all $p \in P$

- $T_pP = V_p \oplus H_p$,
- $H_{pg} = (R_g)_* H_p$ for all $g \in G$, where $R_g : P \to P$ is the map defined by $R_g(p) = pg$,
- If X is a smooth vector field on P, then the components of $X \in T_pP$ in V_p and H_p define smooth vector fields.

We call V_p the *vertical* subspace and H_p the *horizontal* subspace of T_pP. A vector field $X \in T_pP$ is called *vertical* if $X \in V_p$ and *horizontal* if $X \in H_p$. The components of X in V_p and H_p are called the *vertical* and *horizontal components* of X respectively.

Suppose M is a manifold and let G act smoothly on M on the right. Let $\mathfrak{g}$ be the Lie algebra of G and let $A \in \mathfrak{g}$. The action of the one-parameter subgroup $a_t = \exp tA$ on M induces a vector field on M, denoted by A^* and called the *fundamental vector field* corresponding to A. For each $p \in P$ we define $\omega_p : T_pP \to \mathfrak{g}$ which assigns to each $X \in T_pP$ the unique $A \in \mathfrak{g}$ such that $(A^*)_p$ is equal to the vertical component of X. This defines a linear map

$$\omega : \Gamma(TP) \to \mathfrak{g}$$

called *the connection 1-form* of the connection.

Proposition 2.11 ([15], Proposition 1.1)**.** *The connection form $\omega \in \Omega^1(P, \mathfrak{g})$ of a connection satisfies the following conditions.*

- $\omega(X) = 0$ *if and only if X is horizontal,*
- $\omega(A^*) = A$ *for every $A \in \mathfrak{g}$,*
- $(R_g)^*\omega = \mathrm{ad}(g^{-1})\omega$, *where* ad *denotes the adjoint representation of G in $\mathfrak{g}$.*

Note, in the case of $G = U(1)$ we have $\mathfrak{g} = \mathrm{Lie}(U(1)) = i\mathbb{R}$. So we can think of the connection of a $U(1)$-bundle as a purely-imaginary valued one-form.

The *horizontal lift* of a vector field X on M with respect to a connection is the unique vector field $\hat{X}$ on P which is horizontal and projects onto X, that is, $\pi_*(\hat{X}_p) = X_{\pi(p)}$ for all $p \in P$, where $\pi : P \to M$ is the principal bundle projection map. Given a connection on P and a vector field X on M, there is a unique horizontal lift $\hat{X}$ of X.

Proposition 2.12 ([15], Proposition 1.3)**.** *Let $\hat{X}$ and $\hat{Y}$ be the horizontal lifts of X and Y. Then*

- *$\hat{X} + \hat{Y}$ is the horizontal lift of $X + Y$,*
- *For every function f on M we have $f^*\hat{X} = \widehat{fX}$, where $f^* = f \circ \pi$,*
- *The horizontal component of $[\hat{X}, \hat{Y}]$ is the horizontal lift of $[X, Y]$.*

2.3.2 Curvature

In this section we introduce the notion of curvature of a principal bundle. We are following chapter II of [15].

Definition 2.13. *We define the exterior covariant derivative of a Lie algebra valued r-form φ to be the horizontal component of the exterior derivative of φ and denote it by $D\varphi$*

$$(D\varphi)(X_1, \ldots, X_r) = d\varphi(hX_1, \ldots, hX_r).$$

We can now define the notion of curvature for a principal bundle.

Definition 2.14. *Let P be a principal G-bundle with a connection form ω. We define the curvature form of ω to be the Lie algebra valued 2-form $D\omega$.*

We have the following important result, known as the "structure equation".

Theorem 2.15 ([15], Theorem 5.2)**.** *Let P be a principal G-bundle with connection form ω and curvature form $\Omega = D\omega$. Then for every $p \in P$ and $X, Y \in T_pP$ we have*

$$d\omega(X, Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y).$$

Corollary 2.16 (Bianchi's identity)**.** *Let P be a principal G-bundle with connection form ω and curvature form Ω. Then $D\Omega = 0$.*

2.4 Principal $U(1)$-bundles

In this section we consider principal $U(1)$-bundles, which are important as they provide a geometric realisation of $H^2(M, \mathbb{Z})$ via their chern class. That is, the isomorphism classes of $U(1)$ bundles are in a bijective correspondence with $H^2(M, \mathbb{Z})$. This suggests that the set of $U(1)$-bundles has a product structure: given two $U(1)$-bundles P_1 and P_2 we can form the tensor product $P_1 \otimes P_2$. We have the notion of trivial $U(1)$-bundle (identity). Also for each $U(1)$-bundle P there exists a dual bundle P^* (inverse).

Recall from section 2.3.1 that a connection on a principal $U(1)$-bundle P over a manifold M is an assignment of a subspace $H_p \subset T_pP$ to each $p \in P$ such that for all $p \in P$ we have $T_pP = \mathbb{R} \oplus H_p$. We can define an invariant 1-form A on P which assigns to each $X \in T_pP$ its vertical (real) component. Since dA vanishes on vertical vectors, there exists a unique closed 2-form F on M (the curvature) such that $dA = \pi^* F$.

Geometrically speaking, a connection lifts a path in M to a horizontal path in P with a fixed starting point. The lift of a closed path in M to a horizontal path in P is not closed in general. The gap between the end points of the lifted path is called the *holonomy* around the path. The curvature of the connection measures the holonomy around infinitesimally small closed paths [29].

Let P be a $U(1)$-bundle over a manifold M. We trivialise P using an open cover $\{U_\alpha\}$ on M with sections $s_\alpha : U_\alpha \to P$ and let $A_\alpha = s_\alpha^* A$. Then the curvature on U_α is given by $dA_\alpha = F$. The transition functions of the bundle are $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ and

$$A_\beta = A_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

Given a $U(1)$-bundle P over M with connection A and curvature F_A we have

$$\frac{1}{2\pi i} \int_\Sigma F_A \in \mathbb{Z},$$

for every closed surface Σ in M [29]. The de Rham class of $\frac{1}{2\pi i} F_A$ represents the image in real cohomology of the chern class of P, $c_1(P) \in H^2(M, \mathbb{Z})$. The chern class can also be represented in Čech cohomology [5] by the class of transition functions

$$[g_{\alpha\beta}] \in H^1(M, U(1)) \simeq H^2(M, \mathbb{Z}).$$

We define the *tensor product* $P \otimes P'$ of $U(1)$-bundles P and P' to be the $U(1)$-bundle whose fibre over $m \in M$ is $(P_m \otimes P'_m)/\mathbb{C}^\times$, where the action of $\mathbb{C}^\times$ is given by $(p, p')z = (pz, p'z^{-1})$. Note for every $p \otimes p' \in P \otimes P'$ and $z \in \mathbb{C}^\times$ we have $(pz) \otimes p' =$

$p \otimes (p'z) = (p \otimes p')z$. Suppose P and P' have connections A and A' with curvatures F_A and F'_A respectively. Then $P \otimes P'$ has an induced connection $A \otimes A'$ with curvature $F_A + F'_A$.

Given a $U(1)$-bundle P over M we define the dual bundle P^* to be the same space P but with the action $p^*g = (pg^{-1})^*$. It can be shown that $P \otimes P^*$ is trivial. Moreover, a connection A on P induces a connection A^* on P^* which has curvature $F_{A^*} = -F_A$.

If P is a $U(1)$-bundle over M and $f : N \to M$ is a smooth map then there is a $U(1)$-bundle $f^{-1}P$ over N defined by

$$f^{-1}P = \{(n, p) \in N \times P \mid f(n) = \pi(p)\}$$

with the projection map $\pi' : f^{-1}P \to N$ defined by $\pi'(n, p) = n$.

We have $f^{-1}(P \otimes P') = (f^{-1}P) \otimes (f^{-1}P')$ and $f^{-1}(P^*) = (f^{-1}(P))^*$.

Proposition 2.17. *Let P and P' be $U(1)$-bundles over M. Then*

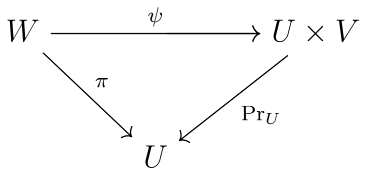
1. $c_1(P \otimes P') = c_1(P) + c_1(P')$,
2. $c_1(P^*) = -c_1(P)$,
3. $c_1(P) = 0$ *if and only if P is trivial,*
4. $c_1(f^{-1}P) = f^*(c_1(P))$.

An important fact about $U(1)$ bundles is that they are in a bijective correspondence with hermitian line bundles. That is, given a principal $U(1)$ bundle P over a manifold M, we can construct a line bundle L by $L = P \times_{U(1)} \mathbb{C}$, the quotient of the product $P \times \mathbb{C}$ by the diagonal action of $U(1)$. We call L an *associated line bundle* to P. Conversely, given a hermitian line bundle L over M we can construct a principal $U(1)$ bundle by restricting to the set of vectors of length one.

2.5 Bundle gerbes

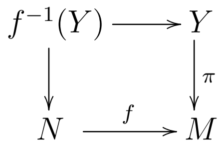
In this section we introduce the notion of bundle gerbes, initially defined in [25]. As discussed in the previous section, line bundles on a manifold M provide a geometric realisation of $H^2(M, \mathbb{Z})$ via their chern class; that is, there is an isomorphism between the isomorphism classes of line bundles on M and $H^2(M, \mathbb{Z})$. Analogously, bundle gerbes give a geometric realisation of $H^3(M, \mathbb{Z})$. Therefore bundle gerbes are like a higher dimensional analogue of line bundles. We start this section by introducing the notion of bundle gerbes as defined in [25], then generalise the notions of connection, curvature and their characteristic class.

Definition 2.18 ([10]). *Let M be a manifold. A surjective map $\pi : Y \to M$ is said to be a submersion if for all $y \in Y$ there exist open subsets $W \subset Y$, $U \subset M$ and V open in some Fréchet space with $y \in W$ and $\pi(W) \subset U$ and a diffeomorphism $\psi : W \to U \times V$ such that the following diagram commutes.*



Remark 2.19. *If Y is finite dimensional, then an equivalent definition for a surjective map $\pi : Y \to M$ to be a submersion is that $T_y\pi$ is onto for all $y \in Y$. As in the case of bundle gerbes we are generally interested in the infinite dimensional case [26], we use the alternative definition [10].*

Theorem 2.20. *Let $\pi : Y \to M$ be a surjective submersion and $f : N \to M$ a smooth map between manifolds. Then $f^{-1}(Y) = \{(n, y) \mid f(n) = \pi(y)\} \subset N \times Y$ is a submanifold of $N \times Y$ and the following diagram commutes.*



Moreover, $f^{-1}(Y) \to N$ is a surjective submersion.

Proof. For simplicity let us suppose $Y = M \times S$. So $\pi : M \times S \to M$ is the map $(m, s) \mapsto m$. Moreover we have

$$f^{-1}(Y) = \{(n, m, s) \in N \times M \times S \mid f(n) = m\}.$$

We define the map $\rho : N \times S \to f^{-1}(Y)$ by

$$\rho(n, s) = (n, f(n), s).$$

Its inverse is given by $\rho^{-1}(n, m, s) = (n, s)$. So $f^{-1}(Y) \simeq N \times S$. We will show that $f^{-1}(Y)$ is a manifold by proving it is a submanifold of $N \times Y$. We note that

$$f^{-1}(Y) = \mathrm{graph}(f) \times S \subset N \times M \times S$$

Since M and N are finite dimensional, graph(f) is a submanifold of $N \times M$. But graph(f) is isomorphic to N by $(n, f(n)) \mapsto n$. Therefore $f^{-1}(Y) = N \times S$ is a submanifold of $N \times M \times S$, and thus a manifold.

In general given $m \in M$ and $\pi(m) \in Y$ we can find neighbourhoods $V \subset M$ of m and $W \subset Y$ of $\pi(m)$ such that $W = V \times S$. Then apply the above constructions locally and then patch them together.

□

Let M be a manifold and $\pi : Y \to M$ a surjective submersion. We define $Y_m = \pi^{-1}(m)$ to be the *fibre* of Y over m. Let the fibre product of Y with itself be

$$Y^{[2]} = Y \times_\pi Y = \{(y_1, y_2) \in Y \times Y \mid \pi(y_1) = \pi(y_2)\}.$$

Similarly, we define the p-fold fibre product of Y to be

$$Y^{[p]} = \{(y_1, \ldots, y_p) \in Y^p \mid \pi(y_1) = \ldots = \pi(y_p)\}.$$

By theorem 2.20 we know $Y^{[p]}$ is a manifold. Note there is an obvious surjection $\hat{\pi} : Y^{[p]} \to M$ given by

$$\hat{\pi}(y_1, \ldots, y_p) = \pi(y_1).$$

We can now define what a bundle gerbe is.

Definition 2.21. *A bundle gerbe (P, Y, M) over a manifold M is a surjective submersion $\pi : Y \to M$ and a principal $U(1)$-bundle $P \to Y^{[2]}$ such that*

- *There is a bundle gerbe multiplication*

  $$m : \pi_3^{-1}(P) \otimes \pi_1^{-1}(P) \to \pi_2^{-1}(P)$$

  which is a smooth isomorphism of $U(1)$ bundles over $Y^{[3]}$. That is, for every $(y_1, y_2, y_3) \in Y^{[3]}$, the bundle gerbe multiplication defines an isomorphism

  $$m : P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \to P_{(y_1,y_3)}$$

  where $P_{(y_1,y_2)}$ is the fibre of P over (y_1, y_2).

- *The bundle gerbe multiplication is associative. That is, for every $(y_1, y_2, y_3, y_4) \in Y^{[4]}$ the following diagram commutes.*

$$\begin{CD}
P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} @>>> P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\
@VVV @VVV \\
P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} @>>> P_{(y_1,y_4)}
\end{CD}$$

Remark 2.22. *It can be shown that the bundle gerbe multiplication*

$$m : \pi_3^{-1}(P) \otimes \pi_1^{-1}(P) \to \pi_2^{-1}(P)$$

induces a trivial $U(1)$-bundle

$$m : \pi_1^{-1}(P) \otimes \pi_2^{-1}(P^*) \otimes \pi_3^{-1}(P) \to Y^{[3]}.$$

We can define a section of this $U(1)$-bundle as follows. Let $(y_1, y_2, y_3) \in Y^{[3]}$ and choose $u \in P_{(y_2,y_3)}$ and $v \in P_{(y_1,y_2)}$. Then $m(u \otimes v) \in P_{(y_1,y_3)}$ and

$$s(y_1, y_2, y_3) = u \otimes m(u \otimes v)^* \otimes v.$$

It turns out the associativity of the bundle gerbe multiplication is equivalent to the condition

$$\delta(s) := \pi_1^{-1}s \otimes \pi_2^{-1}s^* \otimes \pi_3^{-1}s \otimes \pi_4^{-1}s^* = 1.$$

Here by 1 we mean the canonical trivialisation of the bundle $\delta^2(P)$, with $\delta(P)$ defined in section 2.5.3.

Remark 2.23. *Given an abelian Lie group $\mathcal{A}$ we can define an $\mathcal{A}$-bundle gerbe (P, Y, M) similar to definition 2.21 but by taking $P \to Y^{[2]}$ to be a principal $\mathcal{A}$-bundle. The construction in this section all extend to $\mathcal{A}$-bundle gerbes.*

2.5.1 Pullback, dual and product

We extend the notions of pullback, dual and product of line bundles to bundle gerbes, following [25].

Let (P, Y, M) be a bundle gerbe and $f : N \to M$ be a map. We can pullback the surjective submersion $Y \to M$ to

$$f^{-1}Y = \{(n, y) \in N \times Y \mid f(n) = \pi(y)\} \to N$$

with a map $\hat{f} : f^{-1}Y \to Y$, defined by $\hat{f}(n, y) = f(y)$. This then induces the map $\hat{f}^{[2]} : (f^{-1}Y)^{[2]} \to Y^{[2]}$ defined by

$$\hat{f}^{[2]}((n_1, y_1), (n_2, y_2)) = (y_1, y_2)$$

where $((n_1, y_1), (n_2, y_2)) \in (f^{-1}Y)^{[2]}$. We define the *pullback* of the bundle gerbe (P, Y, M) by f to be

$$f^{-1}P = f^{-1}(P, Y, M) = ((\hat{f}^{[2]})^{-1}P, f^{-1}Y, N).$$

The *dual* P^* of a bundle gerbe (P, Y, M) is defined to be (P^*, Y, M), where $P^* \to Y^{[2]}$ is the dual of the $U(1)$-bundle $P \to Y^{[2]}$. We define the *product* of two bundle gerbes (P, Y, M) and (Q, X, M) by

$$P \otimes Q = (P \otimes Q, Y \times_M X, M)$$

where $P \otimes Q$ is taken as a $U(1)$-bundle over $(Y \times_M X)^{[2]}$ defined by

$$(P \otimes Q)_{((y_1,x_1),(y_2,x_2))} = P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)}.$$

2.5.2 The Dixmier-Douady class

Let M be a manifold of dimension n. An open cover $\{U_\alpha\}_{\alpha \in I}$ of M is called a *good cover* if all non-empty finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$.

Theorem 2.24 ([3], Theorem 5.1)**.** *Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.*

Consider a bundle gerbe (P, Y, M) over a compact manifold M. We can take a finite good cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M with sections $s_\alpha : U_\alpha \to Y$. Define $(s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$ by

$$(s_\alpha, s_\beta)(x) = (s_\alpha(x), s_\beta(x)),$$

for $x \in U_\alpha \cap U_\beta$. Let $P_{\alpha\beta} = (s_\alpha, s_\beta)^{-1}P$. As $U_\alpha \cap U_\beta$ is contractible, the bundle $P_{\alpha\beta}$ is trivial and hence we can choose sections $\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to P$ satisfying

$$\sigma_{\alpha\beta}(x) \in P_{(s_\alpha,s_\beta)(x)},$$

for $x \in U_\alpha \cap U_\beta$. For every $\alpha, \beta, \gamma \in I$ and $x \in U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$m(\sigma_{\alpha\beta}(x), \sigma_{\beta\gamma}(x)) = g_{\alpha\beta\gamma}(x)\sigma_{\alpha\gamma}(x)$$

for maps $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to U(1)$. By associativity of the bundle gerbe multiplication we have

$$g_{\beta\gamma\delta}(x)g^{-1}_{\alpha\gamma\delta}(x)g_{\alpha\beta\delta}(x)g^{-1}_{\alpha\beta\gamma}(x) = 1$$

for every $\alpha, \beta, \gamma, \delta \in I$ and $x \in U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$, showing that $g = \{g_{\alpha\beta\gamma}\}_{\alpha,\beta\gamma \in I}$ is a $U(1)$ valued Čech 2-cocycle relative to $\mathcal{U}$. We define the *Dixmier-Douady class* of a bundle gerbe (P, Y, M) to be

$$DD(P) = [g_{\alpha\beta\gamma}] \in H^2(M, U(1)) = H^3(M, \mathbb{Z}).$$

It can be shown that this class is independent of all the choices we made. For more details we refer to [25]. We have the following result about the Dixmier-Douady class.

Proposition 2.25 ([25]). *Let (P, Y, M) and (Q, X, M) be bundle gerbes and $f : N \to M$ be a map. Then*

- $DD(f^{-1}P) = f^*(DD(P))$,
- $DD(P^*) = -DD(P)$,
- $DD(P \otimes Q) = DD(P) + DD(Q)$.

2.5.3 Isomorphism, triviality and stable isomorphism

We define two bundle gerbes (P, Y, M) and (Q, X, M) to be *isomorphic* if there is an isomorphism $Y \to X$ covering the identity on M and a bundle isomorphism $P \to Q$ covering the induced map $Y^{[2]} \to X^{[2]}$ and commuting with the bundle gerbe multiplications.

Definition 2.26. *Let R be a $U(1)$-bundle over $Y^{[p-1]}$. We define $\pi_i : Y^{[p+1]} \to Y^{[p]}$ by*

$$\pi_i(y_1, \ldots, y_{p+1}) = (y_1 \ldots, \hat{y}_i, \ldots, y_{p+1})$$

and let $\delta R = {\pi_1}^{-1}R \otimes ({\pi_2}^{-1}R)^* \otimes {\pi_3}^{-1}R \otimes \ldots$

Definition 2.27. *Let $\pi : Y \to M$ be a surjective submersion and R a $U(1)$-bundle over Y. We define the trivial bundle gerbe to be*

$$\delta R = \pi_1^{-1}R \otimes (\pi_2^{-1}R)^*$$

with the bundle gerbe multiplication

$$(\delta R)_{(y_1,y_2)} \otimes (\delta R)_{(y_2,y_3)} \to (\delta R)_{(y_1,y_3)}$$

given by

$$R^*_{y_1} \otimes R_{y_2} \otimes R^*_{y_2} \otimes R_{y_3} \to R^*_{y_1} \otimes R_{y_3}$$

defined by contracting tensors and using associativity. A bundle gerbe (P, Y, M) is said to be trivial if there is a $U(1)$-bundle $R \to Y$ such that (P, Y, M) is isomorphic to $(\delta R, Y, M)$.

We have the following important result.

Theorem 2.28 ([25]). *A bundle gerbe (P, Y, M) is trivial if and only if $DD(P) = 0$.*

Proof. If P is trivial then without loss of generality we can assume

$$P = \delta R = \pi_1^{-1} R \otimes (\pi_2^{-1} R)^*$$

where R is a principal $U(1)$ bundle. As $\pi : Y \to M$ is a surjective submersion, we can find an open cover $\{U_\alpha\}_{\alpha \in I}$ of M with smooth sections $s_\alpha : U_\alpha \to Y$. Let $Y_\alpha = \pi^{-1}(U_\alpha)$. We define

$$(s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$$

which enables us to construct sections of P over $U_\alpha \cap U_\beta$ given by

$$\sigma_{\alpha\beta} \in \Gamma((s_\alpha, s_\beta)^{-1} P)$$

as follows. If we choose sections $r_\alpha : U_\alpha \to s_\alpha^* R$, then for every $x \in U_\alpha \cap U_\beta$ we define sections $\sigma_{\alpha\beta}$ of $P_{\alpha\beta}$ by

$$\sigma_{\alpha\beta}(x) = r_\beta(x) {r_\alpha}^*(x).$$

This is well defined since

$$\sigma_{\alpha\beta}(x) \in P_{(s_\alpha(x), s_\beta(x))} = R_{s_\beta(x)} \otimes R^*_{s_\alpha(x)}.$$

Therefore we have

$$\begin{aligned} \sigma_{\alpha\beta}\sigma_{\beta\gamma} &= r_\beta {r_\alpha}^* r_\gamma {r_\beta}^* \\ &= {r_\alpha}^* r_\gamma \\ &= \sigma_{\alpha\gamma}, \end{aligned}$$

implying $DD(P) = 0$. We now suppose that

$$DD(P) = 0$$

for a bundle gerbe (P, Y, M). This means the maps $\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \to P$ satisfy

$$\sigma_{\alpha\beta} {\sigma_{\alpha\gamma}}^* \sigma_{\beta\gamma} = 1. \tag{2.1}$$

Suppose $y \in Y$. Let $U_\alpha \subset M$ be an open subset containing $\pi(y)$. Then $y \in Y_\alpha = \pi^{-1}(U_\alpha)$. We define $f : Y_\alpha \to Y^{[2]}$ by $f(y) = (y, s_\alpha(\pi(y)))$. This gives the following commuting diagram.

$$\begin{array}{ccc} R_\alpha := f^{-1}P & \longrightarrow & P \\ \downarrow & & \downarrow \\ Y_\alpha & \longrightarrow & Y^{[2]} \end{array}$$

Therefore R_α is a principal $U(1)$-bundle $R_\alpha \to Y_\alpha$ with fibres defined by

$$(R_\alpha)_y = P_{(y, s_\alpha(\pi(y)))}.$$

We have

$$\begin{aligned}
\delta(R_\alpha)_{(y_1,y_2)} &= (R_\alpha)_{y_1} \otimes (R_\alpha)^*_{y_2} \\
&= P_{(y_1, s_\alpha(\pi(y)))} \otimes P^*_{(y_2, s_\alpha(\pi(y)))} \\
&= P_{(y_1, s_\alpha(\pi(y)))} \otimes P_{(s_\alpha(\pi(y)), y_2)} \\
&= P_{(y_1,y_2)}.
\end{aligned} \tag{2.2}$$

For $y \in Y_\alpha \cap Y_\beta$ we define the map $(\varphi_{\alpha\beta})_y : (R_\alpha)_y \to (R_\beta)_y$ by

$$(\varphi_{\alpha\beta})_y(u) = u \cdot \sigma_{\alpha\beta}(\pi(y))$$

This then defines a map $\varphi_{\alpha\beta} : R_\alpha \mid_{Y_\alpha \cap Y_\beta} \to R_\beta \mid_{Y_\alpha \cap Y_\beta}$. We define a relation $\sim$ by

$$(\alpha, r \in R_\alpha) \sim (\beta, \varphi_{\alpha\beta}(r) \in R_\beta).$$

Since (2.1) holds, $\sim$ is an equivalence relation. So we can define

$$R = \left(\bigcup_\alpha R_\alpha \right) / \sim .$$

It can be shown [25] that the isomorphism (2.2) induces a trivialisation $P = \delta R$. □

We now define a more useful notion of equivalence between bundle gerbes.

Definition 2.29 ([25])**.** *Two bundle gerbes (P, Y, M) and (Q, X, M) are said to be stably isomorphic if $P \otimes Q^*$ is trivial.*

Proposition 2.30 ([25])**.** *Two bundle gerbes (P, Y, M) and (Q, X, M) are stably isomorphic if and only if $DD(P) = DD(Q)$.*

Proof. We know that P and Q being stably isomorphic is equivalent to $P \otimes Q^*$ being trivial, meaning

$$DD(P \otimes Q^*) = DD(P) + DD(Q^*) = DD(P) - DD(Q) = 0.$$

Therefore $DD(P) = DD(Q)$. □

So the Dixmier-Douady class defines a bijection between stable isomorphism classes of bundle gerbes over M and $H^3(M, \mathbb{Z})$.

2.5.4 Connection, curving and curvature

Recall the p-fold fibre product of Y

$$Y^{[p]} = \{(y_1, \dots, y_p) \in Y^p \mid \pi(y_1) = \dots = \pi(y_p)\}.$$

We define the projection maps $\pi_i : Y^{[p]} \to Y^{[p-1]}$ for $i = 1, \dots, p$ by

$$\pi_i(y_1, \dots, y_p) = (y_1, \dots, \hat{y}_i, \dots, y_p).$$

We then define for all $p \geq 1$

$$\delta : \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]})$$

by

$$\delta(\omega) = \sum_{i=1}^{p} (-1)^i \pi_i^*(\omega)$$

for $\omega \in \Omega^q(Y^{[p-1]})$. It can be shown that $\delta^2 = 0$. Therefore there is a complex

$$\Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \cdots \tag{2.3}$$

It is shown in [25] that this complex has no cohomology.

A connection A on the $U(1)$-bundle $P \to Y^{[2]}$ is called a *bundle gerbe connection* if it respects the bundle gerbe product [25]. That is if the section s of $\delta(P) \to Y^{[3]}$ satisfies $s^*(\delta(A)) = 0$ (see remark 2.22). Given an arbitrary connection A we have $\delta(s^*(\delta(A))) = \delta(s)^*(\delta^2(A)) = 0$ as $\delta^2(A)$ is a flat connection on the trivial bundle $\delta^2(P)$. As the complex (2.3) has no cohomology, there is a 1-form ϵ on $Y^{[2]}$ such that $\delta(\epsilon) = s^*(\delta(A))$, implying $A - \epsilon$ is a bundle gerbe connection. Therefore bundle gerbe connections always exist. The curvature $F_A \in \Omega^2(Y^{[2]})$ of a bundle gerbe connection A satisfies $\delta(F_A) = 0$ and again since the complex (2.3) has no cohomology, the curvature satisfies $F_A = \delta(f)$ for some $f \in \Omega^2(Y)$. A choice of f is called a *curving* for the bundle gerbe connection. We have

$$\delta(df) = d(\delta(f)) = d(F_A) = 0.$$

Therefore $df = \pi^*(\omega)$ for some $\omega \in \Omega^3(M)$, called the *three-curvature* of (A, f). We have $\pi^*(d\omega) = d\pi^*(\omega) = d^2 f = 0$, hence ω is closed. It is shown in [25] that $\frac{1}{2\pi i}\omega$ is integral and represents the image of the Dixmier-Douady class $DD(P)$ in $H^3(M, \mathbb{R})$. We will compute the connection and curvature of a class of bundle gerbes in chapter 6.

2.6 Gauge groups

Suppose G is a compact Lie group and Q is a principal G-bundle over a compact manifold Σ. Let Q_m be a fibre of Q over a point $m \in \Sigma$. An *automorphism* of Q is a map $\varphi : Q \to Q$ that takes fibres Q_m to themselves such that $\varphi(qg) = \varphi(q)g$ for all $q \in Q$ and $g \in G$. We would like to realise automorphisms as sections of a bundle of groups called the *adjoint* bundle of Q. We define this by

$$\mathrm{Ad}(Q) = \frac{Q \times G}{G},$$

where the action of G is the adjoint one given by

$$(q, g)h = (qh, h^{-1}gh).$$

Note that the fibres of $\mathrm{Ad}(Q) \to M$ are groups isomorphic to G.

Given $\varphi \in \mathrm{Aut}(Q)$ we define $\tilde{\varphi} : \Sigma \to \mathrm{Ad}(Q)$ by $\tilde{\varphi}(m) = [q, g]$, where $\varphi(q) = qg$. If $\tilde{\varphi}(m) = [q', g']$ where $\varphi(q') = q'g'$ then $q' = qh$ for some $h \in G$ and we have

$$\varphi(q') = \varphi(q)h = qgh$$

and also

$$\varphi(q') = q'g' = qhg'.$$

Therefore $g' = h^{-1}gh$ and $[q', g'] = [qh, h^{-1}gh] = [q, g]$, showing $\tilde{\varphi}$ is well-defined.

The map $\varphi \to \tilde{\varphi}$ is in fact an isomorphism from the group of automorphisms of Q, $\mathrm{Aut}(Q)$ and the group of sections of $\mathrm{Ad}(Q)$ [1].

Definition 2.31. *Let Q be a principal G-bundle over a compact manifold Σ. We define the gauge group of Q to be the Fréchet Lie group of smooth sections of $\mathrm{Ad}(Q)$ given by*

$$\mathcal{G} = \Gamma(\Sigma, \mathrm{Ad}(Q)) = \{f : Q \to G \mid f(qg) = g^{-1}f(q)g\},$$

with the group action being pointwise multiplication [1]. The Lie algebra of $\mathcal{G}$ is given by

$$\mathrm{Lie}(\mathcal{G}) = \Gamma\left(\Sigma, \frac{Q \times \mathrm{Lie}(G)}{G}\right) = \Gamma(\Sigma, \mathrm{ad}(Q)),$$

where $\mathrm{ad}(Q)$ is the bundle associated with Q via the adjoint action of G on its tangent space T_eG [1].

Remark 2.32. *If Q is taken to be the trivial bundle $\Sigma \times G$ then $\mathcal{G} = \mathrm{Map}(\Sigma, G)$.*

Let Q be a principal G bundle over a compact manifold Σ. An r-form ω on Q is said to be *invariant* if for all $g \in G$

$$R_g^* \omega = \omega.$$

Such a form is called *horizontal* if $\omega(X_1, \ldots, X_r) = 0$ whenever at least one of the tangent vectors X_i of P is vertical. An r-form ω on Q is said to be *ad-invariant* if for all $g \in G$

$$R_g^* \omega = \mathrm{ad} g^{-1} \omega.$$

Proposition 2.33 ([15], Proposition 5.1)**.** *If ω is an invariant and horizontal $(r+1)$-form it descends to X. Furthermore if ω is an ad-invariant r-form on Q, then*

1. *$d\omega$ is an ad-invariant $(r+1)$-form,*
2. *The $(r+1)$-form $D\omega$ (as defined in section 2.3.2) is ad-invariant and horizontal,*
3. *In the case of special unitary group $G = SU(n)$, $tr(\omega)$ is invariant.*

Chapter 3

Constructing central extensions of groups

In this chapter we introduce the notion of a central extension of a group. We then give an explicit construction of central extensions of groups using differential forms. Our discussion follows [27].

3.1 Central extensions of groups

Definition 3.1. *Let $\mathcal{G}$ and $\mathcal{N}$ be groups. Then $\widehat{\mathcal{G}}$ is an* extension *of $\mathcal{G}$ by $\mathcal{N}$ if there is a short exact sequence*

$$1 \to \mathcal{N} \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{p} \mathcal{G} \to 1.$$

This means that i and p are group homomorphisms such that

- $\ker(i) = 1$*, so i is injective*
- $\ker(p) = \mathrm{im}(i)$
- *$im(p) = \mathcal{G}$, so p is surjective.*

Note this means $\mathcal{N} \lhd \widehat{\mathcal{G}}$ and hence given $\mathcal{N}$ and $\widehat{\mathcal{G}}$, we simply have $\mathcal{G} = \widehat{\mathcal{G}}/\mathcal{N}$.

Definition 3.2. *An extension*

$$1 \to \mathcal{N} \to \widehat{\mathcal{G}} \to \mathcal{G} \to 1$$

is called a central extension *if $\mathcal{N}$ lies in $Z(\mathcal{G})$, the* centre *of $\mathcal{G}$ defined by*

$$Z(\mathcal{G}) = \{z \in \mathcal{G} \mid zg = gz \quad \forall g \in \mathcal{G}\}.$$

Remark 3.3. *In the case of central extensions, $\mathcal{N}$ is always abelian since $\mathcal{N} \subset Z(\mathcal{G})$. From now on we denote this abelian group by $\mathcal{A}$. We will often be interested in the case $\mathcal{A} = U(1)$.*

Example 3.4. *Let $\mathcal{G} = U(n)$. Then*

$$Z(\mathcal{G}) = \{\lambda I \mid \lambda \in U(1)\} \simeq U(1) = \{z \in \mathbb{C} \mid |z| = 1\}.$$

So we have the central extension

$$1 \to U(1) \to U(n) \to PU(n) \to 1,$$

where $PU(n)$ is the projective unitary group, defined to be the quotient $U(n)/U(1)$.

We define the notion of isomorphism of central extensions as follows.

Definition 3.5. *Let $\mathcal{G}$ be a group and let $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}'$ be central extensions of $\mathcal{G}$ by an abelian group $\mathcal{A}$. We call $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}'$ isomorphic central extensions if there exists a group isomorphism $\gamma : \hat{\mathcal{G}} \to \hat{\mathcal{G}}'$ and a commuting diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{A} & \longrightarrow & \hat{\mathcal{G}} & \longrightarrow & \mathcal{G} & \longrightarrow & 1 \\ \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \| \\ 1 & \longrightarrow & \mathcal{A} & \longrightarrow & \hat{\mathcal{G}}' & \longrightarrow & \mathcal{G} & \longrightarrow & 1 \end{array}$$

Note by using the Five Lemma, it suffices for γ to be a group homomorphism [3].

Let $\mathcal{A}$ be an abelian group and $\mathcal{G}$ be any group. Then

$$1 \to \mathcal{A} \xrightarrow{\iota} \mathcal{A} \times \mathcal{G} \xrightarrow{\pi} \mathcal{G} \to 1$$

is a central extension, where $\iota(z) = (z, e)$, $\pi(z, g) = g$ and the multiplication on $\mathcal{A} \times \mathcal{G}$ is defined by

$$(z_0, g_0) * (z_1, g_1) = (z_0 z_1, g_0 g_1).$$

We can deform the product on $\mathcal{A} \times \mathcal{G}$ by

$$(z_0, g_0) * (z_1, g_1) = (z_0 z_1 c(g_0, g_1), g_0 g_1)$$

for some function $c : \mathcal{G} \times \mathcal{G} \to \mathcal{A}$. For $*$ to be a product, we require some conditions on c. Primarily for $*$ to be associative we need

$$\begin{aligned} &((z_0, g_0) * (z_1, g_1)) * (z_2, g_2) = (z_0, g_0) * ((z_1, g_1) * (z_2, g_2)) \\ \therefore\ &(z_0 z_1 c(g_0, g_1), g_0 g_1) * (z_2, g_2) = (z_0, g_0) * (z_1 z_2 c(g_1, g_2), g_1 g_2) \\ \therefore\ &(z_0 z_1 c(g_0, g_1) z_2 c(g_0 g_1, g_2), g_0 g_1 g_2) = (z_0 z_1 z_2 c(g_1, g_2) c(g_0, g_1 g_2), g_0 g_1 g_2) \end{aligned}$$

Therefore

$$c(g_0, g_1) c(g_0 g_1, g_2) = c(g_1, g_2) c(g_0, g_1 g_2). \tag{3.1}$$

Similarly, to obtain an identity and inverse we require $c(e, g) = c(g, e) = 1$ for all $g \in \mathcal{G}$.

Definition 3.6. *A function $c : \mathcal{G} \times \mathcal{G} \to \mathcal{A}$ with*

$$c(e, g) = c(g, e) = 1\ , \quad \forall g \in \mathcal{G}$$

and satisfying (3.1) is called a group 2-cocycle on $\mathcal{G}$ with coefficients in $\mathcal{A}$.

We denote by $\mathcal{A} \times_c \mathcal{G}$, the set $\mathcal{A} \times \mathcal{G}$ equipped with the product arising from the cocycle c.

Consider the central extension

$$1 \to \mathcal{A} \to \mathcal{A} \times_c \mathcal{G} \xrightarrow{\pi} \mathcal{G} \to 1$$

arising from a cocycle c. Define $s : \mathcal{G} \to \mathcal{A} \times_c \mathcal{G}$ by $s(g) = (1, g)$, where 1 is the identity element in $\mathcal{A}$. Then s is a right inverse or a *section* of π as

$$(\pi \circ s)(g) = \pi(1, g) = g.$$

More generally we have the following result.

Proposition 3.7. *Let*

$$1 \to \mathcal{A} \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{p} \mathcal{G} \to 1$$

be a central extension admitting a (smooth) section $s : \mathcal{G} \to \widehat{\mathcal{G}}$. Then we have $\widehat{\mathcal{G}} \simeq \mathcal{A} \times_c \mathcal{G}$ for some cocycle $c : \mathcal{G} \times \mathcal{G} \to \mathcal{A}$. In this case we say $\widehat{\mathcal{G}}$ is a twisted product*.*

We denote by $Z^2(\mathcal{G}, \mathcal{A})$ the space of all cocycles arising from central extensions of $\mathcal{G}$ by $\mathcal{A}$, which forms a group under pointwise multiplication. Given a cocycle c, we can construct a new cocycle

$$c'(g_1, g_2) = c(g_1, g_2) \chi(g_1) \chi(g_2) \chi(g_1 g_2)^{-1} \tag{3.2}$$

for a function $f : \mathcal{G} \to \mathcal{A}$. We define $H^2(\mathcal{G}, \mathcal{A}) = Z^2(\mathcal{G}, \mathcal{A})/\sim$, where two cocycles are *cohomologous* if they satisfy (3.2). We have the following result.

Theorem 3.8 ([34], Theorem 6.6.3)**.** *There is a one-to-one correspondence between the isomorphism classes of central extensions of $\mathcal{G}$ by $\mathcal{A}$ and the elements in $H^2(\mathcal{G}, \mathcal{A})$.*

Proposition 3.9. *Given a central extension*

$$1 \to \mathcal{A} \xrightarrow{i} \widehat{\mathcal{G}} \xrightarrow{p} \mathcal{G} \to 1$$

arising from a cocycle c we have

$$\mathcal{A} \times_c \mathcal{G} \simeq \mathcal{A} \times \mathcal{G}$$

if and only if there exists a section $s : \mathcal{G} \to \widehat{\mathcal{G}}$ which is a homomorphism.

Proof. If c is cohomologous to the identity then by (3.2) we have

$$c(g_1, g_2) = \chi(g_1)\chi(g_2)\chi(g_1 g_2)^{-1}.$$

So if $s : \mathcal{G} \to \widehat{\mathcal{G}}$ is a section

$$\begin{aligned} s(g_1 g_2) &= s(g_1)s(g_2)c(g_1 g_2) \\ &= s(g_1)s(g_2)\chi(g_1)\chi(g_2)\chi(g_1 g_2)^{-1}. \end{aligned}$$

Therefore

$$s(g_1 g_2)\chi(g_1 g_2) = s(g_1)s(g_2)\chi(g_1)\chi(g_2)$$

showing $s\chi$ is a homomorphism. □

Note not every central extension admits a section. For example in the central extension

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1$$

if there was a smooth section $s : SO(3) \to SU(2)$, then we would have a homeomorphism between the connected group $SU(2)$ and the disconnected group $\mathbb{Z}_2 \times SO(3)$, a contradiction.

Remark 3.10. *Given a central extension*

$$1 \to \mathcal{A} \to \widehat{\mathcal{G}} \to \mathcal{G} \to 1$$

arising from a cocycle $c : \mathcal{G} \times \mathcal{G} \to \mathcal{A}$ and a homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ we can construct a central extension of $\mathcal{G}$ by $\mathcal{B}$ by the cocycle $\tilde{c} = \phi \circ c$.

3.2 Relation to group cohomology

Let $\mathcal{G}$ be a Lie group and $\mathcal{A}$ be an abelian Lie group. Denote by $\mathrm{Map}(\mathcal{G},\mathcal{A})$ the group of smooth maps between $\mathcal{G}$ and $\mathcal{A}$. For $n \geq 1$ define a homomorphism of abelian groups

$$\delta : \mathrm{Map}(\mathcal{G}^n,\mathcal{A}) \to \mathrm{Map}(\mathcal{G}^{n+1},\mathcal{A})$$

by

$$\delta(f)(g_0,\dots,g_n) = f(g_1,\dots,g_n)f(g_0g_1,g_2,\dots,g_n)^{-1}\dots f(g_0,\dots,g_{n-1})^{(-1)^{n+1}}.$$

A simple calculation shows that $\delta^2 = 1$. So we have a cochain complex

$$\mathrm{Map}(\mathcal{G},\mathcal{A}) \xrightarrow{\delta} \mathrm{Map}(\mathcal{G}^2,\mathcal{A}) \xrightarrow{\delta} \mathrm{Map}(\mathcal{G}^3,\mathcal{A}) \xrightarrow{\delta} \cdots$$

We define the cohomology groups

$$H^q(\mathcal{G},\mathcal{A}) = \frac{\ker(\delta : \mathrm{Map}(\mathcal{G}^q,\mathcal{A}) \to \mathrm{Map}(\mathcal{G}^{q+1},\mathcal{A}))}{\mathrm{im}(\delta : \mathrm{Map}(\mathcal{G}^{q-1},\mathcal{A}) \to \mathrm{Map}(\mathcal{G}^q,\mathcal{A}))}.$$

Note for $f \in \mathrm{Map}(\mathcal{G}^2,\mathcal{A})$ the condition $\delta(f) = 1$ is equivalent to the cocycle condition (3.1) and therefore as previously mentioned we can identify the isomorphism classes of central extensions of $\mathcal{G}$ by $\mathcal{A}$ by $H^2(\mathcal{G},\mathcal{A})$.

3.3 Central extensions of Lie groups

Let $\mathcal{G}$ be a Lie group and $\mathcal{A}$ an abelian Lie group. A Lie group $\hat{\mathcal{G}}$ is said to be *a central extension of* $\mathcal{G}$ *by* $\mathcal{A}$ *in the Lie group sense* if there is a smooth homomorphism $p : \hat{\mathcal{G}} \to \mathcal{G}$ such that

- $\ker(p) = \mathcal{A}$,
- $\hat{\mathcal{G}}$ has the structure of a principal $\mathcal{A}$ bundle over $\mathcal{G}$ with projection map $p : \hat{\mathcal{G}} \to \mathcal{G}$.

In this case, $\mathcal{A}$ is a normal subgroup of $\mathcal{G}$ and we have an underlying central extension

$$1 \to \mathcal{A} \to \widehat{\mathcal{G}} \to \mathcal{G} \to 1$$

of the abstract groups. Note here a smooth section $s : \mathcal{G} \to \hat{\mathcal{G}}$ exists if and only if the principal bundle $p : \hat{\mathcal{G}} \to \mathcal{G}$ is trivial. In the next chapter we look at central extensions of Lie groups which are not trivial as a principal bundle. Therefore a smooth section does not exist and the central extension cannot be identified by a cocycle. This leads us to the next section.

3.4 Contractible spaces

Definition 3.11. *A manifold M is* contractible *if it can be smoothly deformed to a point. That is, there exists a point $m_0 \in M$ and a smooth map $F : [0,1] \times M \to M$ such that for every $m \in M$ we have $F(0,m) = m$ and $F(1,m) = m_0$.*

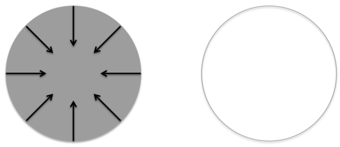


Figure 3.1: A disk is contractible but a circle is not

Theorem 3.12 ([23], page 15)**.** *Let M be a contractible manifold. Then a fibre bundle $P \xrightarrow{\pi} M$ over M is trivial.*

Proof. As M is contractible there exists a smooth map $F : [0,1] \times M \to M$ and some $m_0 \in M$ such that $F(0,m) = m$ and $F(1,m) = m_0$ for all $m \in M$. We have the following commutative diagram.

$$\begin{array}{ccc} F^{-1}P & \longrightarrow & P \\ \downarrow & & \downarrow \pi \\ [0,1] \times M & \xrightarrow{F} & M \end{array}$$

As F is smooth, we have the bundle isomorphism $F^{-1}P \mid_{\{0\}\times M} \simeq F^{-1}P \mid_{\{1\}\times M}$ ([13], part I, chapter 4). But $F^{-1}P \mid_{\{0\}\times M} = P$ and $F^{-1}P \mid_{\{1\}\times M} = P \mid_{m_0} \times M$ is trivial. Thus P is trivial. ☐

Corollary 3.13. *Let*

$$1 \to \mathcal{A} \to \widehat{\mathcal{G}} \xrightarrow{p} \mathcal{G} \to 1$$

be a central extension of a Lie group $\mathcal{G}$ for which $\hat{\mathcal{G}} \to \mathcal{G}$ is a principal $\mathcal{A}$-bundle. If $\mathcal{G}$ is contractible then p admits a section s and therefore $\hat{\mathcal{G}} \simeq \mathcal{A} \times_c \mathcal{G}$ for some cocycle c.

Definition 3.14. *Let $\mathcal{G}$ be a 1-connected Lie group; that is, $\mathcal{G}$ is connected and simply connected. We define $P\mathcal{G}$, the* path space *of $\mathcal{G}$, to be the group of all smooth functions $\gamma : [0,1] \to \mathcal{G}$ with $\gamma(0) = 1 \in \mathcal{G}$. We also define $\Omega\mathcal{G}$ to be the subgroup of $P\mathcal{G}$ containing all smooth functions $\gamma : [0,1] \to \mathcal{G}$ with $\gamma(0) = \gamma(1) = 1 \in \mathcal{G}$.*

Note $\Omega\mathcal{G}$ is not the same as the loop group $L\mathcal{G}$ since an element $\gamma \in \Omega\mathcal{G}$ does not need to be smooth at $\gamma(0) = \gamma(1)$.

We have the short exact sequence

$$1 \to \Omega\mathcal{G} \xrightarrow{\iota} P\mathcal{G} \xrightarrow{\text{ev}} \mathcal{G} \to 1$$

where ι is the inclusion map and ev denotes the evaluation map, that is, the map which sends a path $\gamma \in P\mathcal{G}$ to its endpoint $\gamma(1)$. In particular, it follows that $\Omega\mathcal{G}$ is a normal subgroup and

$$\mathcal{G} = \frac{P\mathcal{G}}{\Omega\mathcal{G}}.$$

Lemma 3.15. *The path space $P\mathcal{G}$ of a Lie group $\mathcal{G}$ is contractible.*

Proof. Define $F : [0,1] \times P\mathcal{G} \to P\mathcal{G}$ by

$$F(t,\gamma)(s) = \gamma((1-t)s + t).$$

Clearly F is smooth. Also $F(0,\gamma) = \gamma$ and $F(1,\gamma) = 1$.

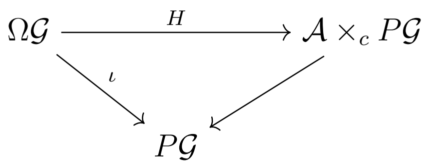
□

Let $\mathcal{G}$ be a 1-connected Lie group. As $P\mathcal{G}$ is contractible, by corollary 3.13 every central extension of $P\mathcal{G}$ is determined by a cocycle. For a cocycle $c : P\mathcal{G} \times P\mathcal{G} \to \mathcal{A}$, we can construct a central extension

$$1 \to \mathcal{A} \to \mathcal{A} \times_c P\mathcal{G} \to P\mathcal{G} \to 1,$$

using the method in section 3.1. If we could take the quotient of $\mathcal{A} \times_c P\mathcal{G}$ by $\Omega\mathcal{G}$ we would produce a central extension of $\mathcal{G} = P\mathcal{G}/\Omega\mathcal{G}$. So calculating $\widehat{\mathcal{G}}$ is reduced to finding a cocycle c and a homomorphism $H : \Omega\mathcal{G} \to \mathcal{A} \times_c P\mathcal{G}$ with image $H(\Omega\mathcal{G})$ a normal subgroup of $\mathcal{A} \times_c P\mathcal{G}$; enabling us to take the quotient of $\mathcal{A} \times_c P\mathcal{G}$ by $\Omega\mathcal{G}$.

The following calculations show that to define a homomorphism H, it is sufficient to find a map $h : \Omega\mathcal{G} \to \mathcal{A}$ satisfying $h(g_1 g_2) = h(g_1)h(g_2)c(g_1, g_2)$; setting $H(g) = (h(g), \iota(g))$ defines a homomorphism $H : \Omega\mathcal{G} \to \mathcal{A} \times_c P\mathcal{G}$ and the following diagram will commute.



Let $g_1, g_2 \in \Omega\mathcal{G}$. We have

$$H(g_1 g_2) = (h(g_1 g_2), \iota(g_1 g_2))$$

and

$$\begin{aligned} H(g_1)H(g_2) &= (h(g_1), \iota(g_1)) \times_c (h(g_2), \iota(g_2)) \\ &= (h(g_1)h(g_2)c(g_1, g_2), \iota(g_1 g_2)). \end{aligned}$$

So for H to be a homomorphism we need to show

$$h(g_1 g_2) = h(g_1)h(g_2)c(g_1, g_2).$$

The following proposition finds a criterion for $H(\Omega\mathcal{G})$ to be normal.

Proposition 3.16. *If $h : \Omega\mathcal{G} \to \mathcal{A}$ satisfies*

$$c(f, g)h(g)^{-1}c(fg, f^{-1})c(f, f^{-1})^{-1} = h(fgf^{-1})^{-1}, \quad \forall f \in P\mathcal{G},\ g \in \Omega\mathcal{G}$$

then $H : \Omega\mathcal{G} \to \mathcal{A} \times_c P\mathcal{G}$ defined by $H(g) = (h(g), \iota(g))$ has normal image in $\mathcal{A} \times_c P\mathcal{G}$.

Note here f^{-1} is defined by $f^{-1}(t) = f(t)^{-1}$.

Proof. A short calculation shows that if $(z, f) \in \mathcal{A} \times_c P\mathcal{G}$ then

$$(z, f)^{-1} = (z^{-1}c(f, f^{-1})^{-1}, f^{-1}).$$

For every $g \in \Omega\mathcal{G}$ we want there to be some $k \in \Omega\mathcal{G}$ such that

$$(z, f) * (h(g), g) * (z^{-1}c(f, f^{-1})^{-1}, f^{-1}) = (h(k), k). \tag{3.3}$$

Equating the second components of (3.3) yields $fgf^{-1} = k$ implying $h(k) = h(fgf^{-1})$. Equating the first components of (3.3) then shows

$$z\, h(g)c(f,g)z^{-1}c(f,f^{-1})^{-1}c(fg,f^{-1}) = h(fgf^{-1})$$

and hence

$$h(g)c(f,g)c(f,f^{-1})^{-1}c(fg,f^{-1}) = h(fgf^{-1}).$$

□

In summary, we need to construct a cocycle $c : P\mathcal{G} \times P\mathcal{G} \to \mathcal{A}$ satisfying

$$c(g_0,g_1)c(g_0g_1,g_2) = c(g_1,g_2)c(g_0,g_1g_2), \quad \forall g_0,g_1,g_2 \in P\mathcal{G} \tag{3.4}$$

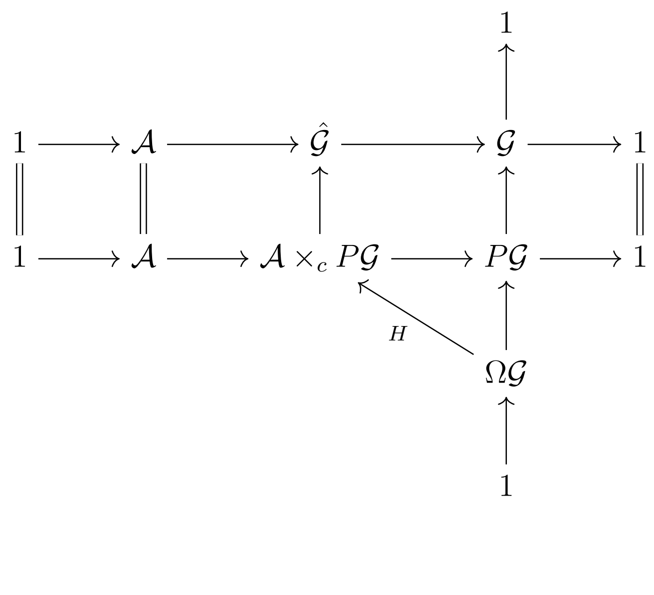
and a map $h : \Omega\mathcal{G} \to \mathcal{A}$ satisfying

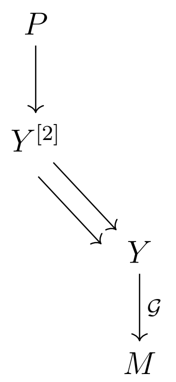
$$h(g_1g_2) = h(g_1)h(g_2)c(g_1,g_2), \quad \forall g_1,g_2 \in \Omega\mathcal{G} \tag{3.5}$$

and

$$c(f,g)h(g)c(fg,f^{-1})c(f,f^{-1})^{-1} = h(fgf^{-1}), \quad \forall f \in P\mathcal{G},\ g \in \Omega\mathcal{G}. \tag{3.6}$$

Then if $\mathcal{G}$ is 1-connected, the following diagram commutes.





and therefore the central extension of $\mathcal{G}$ is given by

$$\widehat{\mathcal{G}} = \frac{\mathcal{A} \times_c P\mathcal{G}}{H(\Omega\mathcal{G})}. \tag{3.7}$$

The following proposition shows how the isomorphism class of the central extension $\hat{\mathcal{G}}$ depends on the cocycle c and the homomorphism h.

Proposition 3.17. *A choice of cohomologous cocycles c and c' related by*

$$c'(g_1, g_2) = c(g_1, g_2)\chi(g_1)\chi(g_2)\chi(g_1 g_2)^{-1}$$

and homomorphisms h and h' related by

$$h'(g) = h(g)\chi(g)^{-1}$$

where $\chi : P\mathcal{G} \to \mathcal{A}$ give rise to isomorphic central extensions.

Proof. Define $\phi : \mathcal{A} \times P\mathcal{G} \to \mathcal{A} \times P\mathcal{G}$ by

$$\phi(z, g) = (z\chi(g)^{-1}, g).$$

Then

$$\begin{aligned}
\phi((z_1, g_1) *_c (z_2, g_2)) &= \phi(z_1 z_2 c(g_1, g_2), g_1 g_2) \\
&= (z_1 z_2 c(g_1, g_2)\chi(g_1 g_2)^{-1}, g_1 g_2) \\
&= (z_1 z_2 c'(g_1, g_2)\chi(g_1)^{-1}\chi(g_2)^{-1}, g_1 g_2) \\
&= (z_1\chi(g_1)^{-1}, g_1) *_{c'} (z_2\chi(g_2)^{-1}, g_2)) \\
&= \phi(z_1, g_1) *_{c'} \phi(z_2, g_2).
\end{aligned}$$

Also

$$\begin{aligned}\phi(H(g)) &= \phi(h(g), g)\\ &= (h(g)\chi(g)^{-1}, g)\\ &= (h'(g), g)\\ &= H'(g).\end{aligned}$$

Therefore ϕ induces an isomorphism

$$\tilde{\phi} : \frac{\mathcal{A} \times_c P\mathcal{G}}{H(\Omega\mathcal{G})} \to \frac{\mathcal{A} \times_{c'} P\mathcal{G}}{H'(\Omega\mathcal{G})}$$

of central extensions.

□

3.5 A cocycle and a homomorphism

In this section we show how to construct a cocycle c and a homomorphism h using differential forms. We are following [26] and [27].

Define $d_i : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ for $i = 0, 1, 2, 3$ by

$$\begin{aligned}d_0(g_0, g_1, g_2) &= (g_1, g_2)\\ d_1(g_0, g_1, g_2) &= (g_0 g_1, g_2)\\ d_2(g_0, g_1, g_2) &= (g_0, g_1 g_2)\\ d_3(g_0, g_1, g_2) &= (g_0, g_1).\end{aligned}$$

Similarly we define $d_i : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ for $i = 0, 1, 2$ by

$$\begin{aligned}d_0(g_0, g_1) &= g_1\\ d_1(g_0, g_1) &= g_0 g_1\\ d_2(g_0, g_1) &= g_0.\end{aligned}$$

Let μ be a 1-form on $\mathcal{G} \times \mathcal{G}$ and η be a 2-form on $\mathcal{G}$. We define

$$\delta(\mu) = d_0^*(\mu) - d_1^*(\mu) + d_2^*(\mu) - d_3^*(\mu),$$

a 1-form on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ and

$$\delta(\eta) = d_0^*(\eta) - d_1^*(\eta) + d_2^*(\eta),$$

a 2-form on $\mathcal{G} \times \mathcal{G}$.

Remark 3.18. *We can use the maps $d_i : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ for $i = 0, 1, 2$ and $d_i : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ for $i = 0, 1, 2, 3$ to pullback bundles over $\mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$ respectively. In particular if P is a $U(1)$-bundle over $\mathcal{G}$ we can form the bundle $\delta(P)$ over $\mathcal{G} \times \mathcal{G}$ defined by*

$$\delta(P) = d_0^{-1}P \otimes d_1^{-1}P^* \otimes d_2^{-1}P$$

analogous to the definition of $\delta(R)$ in definition 2.26. We can also define the bundle $\delta\delta(P)$ and similarly to the case for bundle gerbes the bundle $\delta\delta(P)$ has a canonical trivialisation.

Definition 3.19. *Let $\mathcal{G}$ be a Lie group and $\mathcal{A}$ be an abelian Lie group. We define $\Lambda = \ker(\exp)$ where*

$$\exp : \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}$$

is the exponential map ([15]) and call Λ the integer lattice of $\mathcal{A}$. A 2-form $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ is said to be Λ-integral if for every $f : S^2 \to \mathcal{G}$ we have

$$\int_{S^2} f^* R \in \Lambda.$$

We denote the set of Lie($\mathcal{A}$) valued Λ-integral 2-forms on $\mathcal{G}$ by $\Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$.

Proposition 3.20. *Let $\mathcal{G}$ be a 1-connected Lie group and $\mathcal{A}$ be an abelian Lie group. Let $R \in \Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $\alpha \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$. We define*

$$c(g_1, g_2) = \exp\left[\int_0^1 (g_1, g_2)^* \alpha\right], \quad g_1, g_2 \in P\mathcal{G} \tag{3.8}$$

and

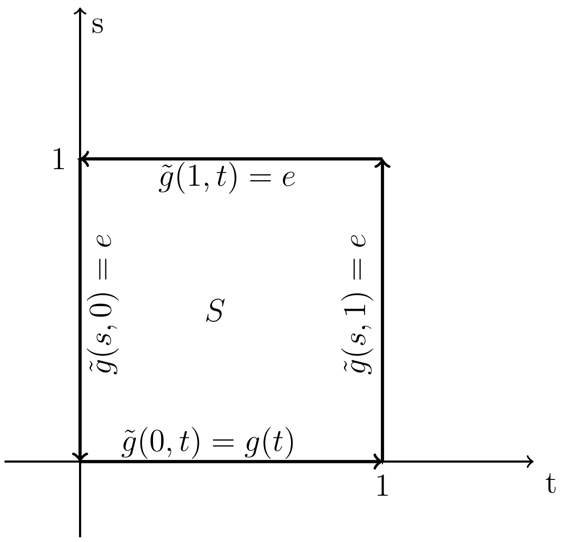
$$h(g) = \exp\left[-\int_S \tilde{g}^* R\right], \tag{3.9}$$

where $\tilde{g} : S \to \mathcal{G}$ is an extension of $g \in \Omega(\mathcal{G})$ to the unit square $S = [0, 1] \times [0, 1]$ such that

$$\tilde{g}(0, t) = g(t), \quad \tilde{g}(s, 1) = e, \quad \tilde{g}(1, t) = e \quad \text{and } \tilde{g}(s, 0) = e.$$

This definition is independent of the choice of extension $\tilde{g}$ as R is Λ-integral. Note, if $\mathcal{A} = U(1)$, then $\exp$ is the usual exponentiation and $\Lambda = 2\pi i\mathbb{Z}$. Therefore the condition we need is $\frac{1}{2\pi i}R$ to be integral.

Suppose $\delta(\alpha) = 0$ and $\delta(R) = d\alpha$. Then c and h defined by (3.8) and (3.9) will satisfy the requirements (3.4), (3.5) and (3.6).



Proof. We have

$$\delta(\alpha)=d_0^*(\alpha)-d_1^*(\alpha)+d_2^*(\alpha)-d_3^*(\alpha)=0$$

in $\Omega^1(\mathcal{G}^3)$. Therefore, pulling back via the map $(g_1,g_2,g_3):I\to\mathcal{G}^3$ we have

$$(g_2,g_3)^*(\alpha)-(g_1g_2,g_3)^*(\alpha)+(g_1,g_2g_3)^*(\alpha)-(g_1,g_2)^*(\alpha)=0.$$

Rearranging this we have

$$(g_1,g_2)^*(\alpha)+(g_1g_2,g_3)^*(\alpha)=(g_2,g_3)^*(\alpha)+(g_1,g_2g_3)^*(\alpha).$$

Integrating both sides over [0,1] yields

$$\int_0^1(g_1,g_2)^*(\alpha)+\int_0^1(g_1g_2,g_3)^*(\alpha)=\int_0^1(g_2,g_3)^*(\alpha)+\int_0^1(g_1,g_2g_3)^*(\alpha).$$

Finally by taking the exponential of both sides we have

$$c(g_1,g_2)c(g_1g_2,g_3)=c(g_2,g_3)c(g_1,g_2g_3).$$

To show (3.5) we use the identity $\delta(R)=d\alpha$ and Stokes' theorem:

$$
\begin{aligned}
h(g_1)h(g_2)c(g_1,g_2) &= \exp\left[-\int_S \tilde{g_1}^*R\right]\exp\left[-\int_S \tilde{g_2}^*R\right]\exp\left[\int_0^1 (g_1,g_2)^*\alpha\right]\\
&= \exp\left[-\int_S \tilde{g_1}^*R - \int_S \tilde{g_2}^*R + \int_0^1 (g_1,g_2)^*\alpha\right]\\
&= \exp\left[-\int_S (\tilde{g}_1,\tilde{g}_2)^*d_2^*R - \int_S (\tilde{g}_1,\tilde{g}_2)^*d_0^*R + \int_0^1 (g_1,g_2)^*\alpha\right]\\
&= \exp\left[-\int_S (\tilde{g}_1,\tilde{g}_2)^*d_1^*R - \int_S (\tilde{g}_1,\tilde{g}_2)^*\mathrm{d}\alpha + \int_0^1 (g_1,g_2)^*\alpha\right]\\
&= \exp\left[-\int_S (\tilde{g}_1\tilde{g}_2)^*R - \int_{\partial S} (\tilde{g}_1,\tilde{g}_2)^*\alpha + \int_0^1 (g_1,g_2)^*\alpha\right]\\
&= \exp\left[-\int_S \widetilde{g_1g_2}^*R - \int_0^1 (g_1,g_2)^*\alpha + \int_0^1 (g_1,g_2)^*\alpha\right]\\
&= \exp\left[-\int_S \widetilde{g_1g_2}^*R\right]\\
&= h(g_1g_2),
\end{aligned}
$$

noting that

$$
\int_{\partial S} (\tilde{g}_1,\tilde{g}_2)^*\alpha = \int_0^1 (g_1,g_2)^*\alpha,
$$

which is illustrated by the following figure.

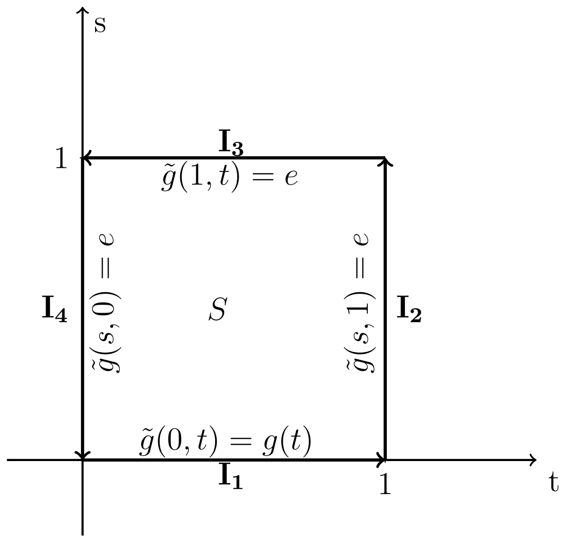


Figure 3.2: An extension $\tilde{g}$ of g to the unit square S

To prove (3.6) we note

$$d_1(d_1(f, g, f^{-1})) = d_1(fg, f^{-1}) = fgf^{-1}.$$

So using the relation $\delta R = d\alpha$ we have

$$\begin{aligned}
h(fgf^{-1}) &= h(d_1(d_1(f, g, f^{-1}))) \\
&= \exp\left[-\int_S (f, \tilde{g}, f^{-1})^* d_1^* d_1^* R\right] \\
&= \exp\left[-\int_S (f, \tilde{g}, f^{-1})^* d_1^* (d_0^* R + d_2^* R - d\alpha)\right] \\
&= \exp\left[-\int_S (f, \tilde{g}, f^{-1})^* d_1^* d_0^* R - \int_S (f, \tilde{g}, f^{-1})^* d_1^* d_2^* R + \int_S (f, \tilde{g}, f^{-1})^* d_1^* d\alpha\right].
\end{aligned}$$

For the first integral note

$$d_0 d_1(f, g, f^{-1}) = d_0(fg, f^{-1}) = f^{-1},$$

a function of t only. Therefore the pullback 2-form $(f^{-1})^* R$ vanishes, implying

$$\exp\left(-\int_S (f, \tilde{g}, f^{-1})^* d_1^* d_0^* R\right) = 1. \tag{3.10}$$

For the second integral we have

$$d_2 d_1(f, g, f^{-1}) = d_2(fg, f^{-1}) = fg.$$

So using $\delta(R) = d\alpha$ again we have

$$\begin{aligned}\int_S (f,\tilde g,f^{-1})^* d_1^* d_2^* R &= \int_S (f\tilde g)^* R \\ &= \int_S (f,\tilde g)^* d_1^* R \\ &= \int_S (f,\tilde g)^*(d_0^* R + d_2^* R - d\alpha) \\ &= \int_S (f,\tilde g)^* d_0^* R + \int_S (f,\tilde g)^* d_2^* R - \int_S (f,\tilde g)^* d\alpha \\ &= \int_S \tilde g^* R + \int_S f^* R - \int_S (f,\tilde g)^* d\alpha \\ &= \int_S \tilde g^* R - \int_S (f,\tilde g)^* d\alpha,\end{aligned}$$

as f^*R vanishes. Hence

$$\exp\left(-\int_S (f,\tilde g,f^{-1})^* d_1^* d_2^* R\right) = h(\tilde g)\exp\left[\int_S (f,\tilde g)^* d\alpha\right].$$

But by Stoke's theorem

$$\begin{aligned}\int_S (f,\tilde g)^* d\alpha &= \int_{\partial S} (f,\tilde g)^*\alpha \\ &= \int_{I_1} (f,\tilde g)^*\alpha + \int_{I_2} (f,\tilde g)^*\alpha + \int_{I_3} (f,\tilde g)^*\alpha + \int_{I_4} (f,\tilde g)^*\alpha \\ &= \int_0^1 (f,g)^*\alpha + \int_0^1 (e,g)^*\alpha - \int_0^1 (f,e)^*\alpha - \int_0^1 (e,g)^*\alpha.\end{aligned}$$

Here $I_1,\dots,I_4$ are oriented intervals whose union is ∂S, as illustrated in figure 3.2. So

$$\begin{aligned}\exp\int_S (f,g)^* d\alpha &= \exp\left[\int_0^1 (f,g)^*\alpha + \int_0^1 (e,g)^*\alpha - \int_0^1 (f,e)^*\alpha - \int_0^1 (e,g)^*\alpha\right] \\ &= c(f,g)c(e,g)c(f,e)^{-1}c(e,g)^{-1} \\ &= c(f,g),\end{aligned}$$

since $c(e,g)=c(f,e)=1$ by definition (see (3.8)). So we have

$$\exp\left(-\int_S (f,\tilde g,f^{-1})^* d_1^* d_2^* R\right) = h(g)c(f,g). \tag{3.11}$$

For the final integral we note that by Stoke's theorem

$$\int_S (f, \tilde{g}, f^{-1})^* d_1^* d\alpha = \int_{\partial S} (f, \tilde{g}, f^{-1})^* d_1^* \alpha = \int_{\partial S} (f\tilde{g}, f^{-1})^* \alpha.$$

Therefore, using the decomposition of ∂S as the union of $I_1, \ldots, I_4$ (see figure 3.2) we have

$$\begin{aligned}
\int_{\partial S} (f\tilde{g}, f^{-1})^* \alpha &= \int_{I_1} (f\tilde{g}, f^{-1})^* \alpha + \int_{I_2} (f\tilde{g}, f^{-1})^* \alpha \\
&\quad - \int_{I_3} (f\tilde{g}, f^{-1})^* \alpha - \int_{I_4} (f\tilde{g}, f^{-1})^* \alpha \\
&= \int_0^1 (fg, f^{-1})^* \alpha + \int_0^1 (g, e)^* \alpha \\
&\quad - \int_0^1 (f, f^{-1})^* \alpha - \int_0^1 (g, e)^* \alpha.
\end{aligned}$$

Therefore

$$\begin{aligned}
\exp & \int_S (f, \tilde{g}, f^{-1})^* d_1^* d\alpha \\
&= \exp \left[\int_0^1 (fg, f^{-1})^* \alpha + \int_0^1 (g, e)^* \alpha - \int_0^1 (f, f^{-1})^* \alpha - \int_0^1 (g, e)^* \alpha \right] \\
&= c(fg, f^{-1}) c(g, e) c(f, f^{-1})^{-1} c(g, e)^{-1} \\
&= c(fg, f^{-1}) c(f, f^{-1})^{-1}. \qquad (3.12)
\end{aligned}$$

Combining (3.10), (3.11) and (3.12) shows

$$h(fgf^{-1}) = h(g) c(f, g) c(fg, f^{-1}) c(f, f^{-1})^{-1}.$$

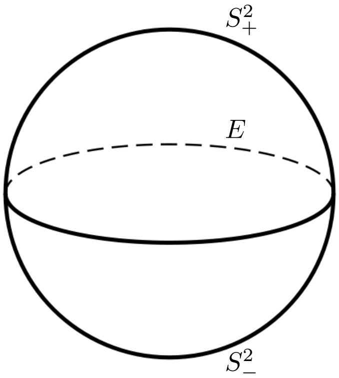
□

Let P be a principal bundle over $\mathcal{G}$ with connection μ and curvature R. If s is a section of P then there is a section $\delta(s)$ of $\delta(P)$ and an induced connection $\delta(\mu)$ on $\delta(P)$ with curvature denoted by $\delta(R)$.

Consider a (Lie group) central extension $\hat{\mathcal{G}}$ of a Lie group $\mathcal{G}$ by an abelian Lie group $\mathcal{A}$. Let $\mu \in \Omega^1(\hat{\mathcal{G}}, \mathrm{Lie}(\mathcal{A}))$ be a connection on $\hat{\mathcal{G}}$ thought as an $\mathcal{A}$-bundle over $\mathcal{G}$ with curvature $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$.

Lemma 3.21. *The curvature R as defined above is Λ-integral.*

Proof. We prove this for the case of an $\mathcal{A}$-bundle over S^2. Write $S^2 = S^2_- \cup S^2_+$ where S^2_- and S^2_+ are the lower and the upper hemispheres separated by the equator E.



Then

$$\begin{aligned}\exp\left(-\int_{S^2} R\right) &= \exp\left(-\int_{S^2_-} R\right)\exp\left(-\int_{S^2_+} R\right)\\ &= \mathrm{hol}(E,\mu)\,\mathrm{hol}(-E,\mu)\\ &= 1\end{aligned}$$

showing R is Λ-integral. □

Recall the central extension

$$1 \to \mathcal{A} \to \widehat{\mathcal{G}} \xrightarrow{p} \mathcal{G} \to 1.$$

There is a canonical trivialisation of $\delta(\hat{\mathcal{G}})$ over $\mathcal{G} \times \mathcal{G}$ induced by the product in $\hat{\mathcal{G}}$ as follows. Let $g_1, g_2 \in \mathcal{G}$ and choose $\hat{g}_1, \hat{g}_2 \in \hat{\mathcal{G}}$ such that $p(\hat{g}_1) = g_1$ and $p(\hat{g}_2) = g_2$. Then there is a section $s : \mathcal{G} \times \mathcal{G} \to \hat{\mathcal{G}}$ defined by $s(g_1, g_2) = \hat{g}_1 \otimes (\hat{g}_1\hat{g}_2)^* \otimes \hat{g}_2$ which is independent of the choices of $\hat{g}_1, \hat{g}_2$. This trivialisation s of $\delta(\hat{\mathcal{G}})$ induces a trivialisation $\delta(s)$ of $\delta\delta(\hat{\mathcal{G}})$ which is equal to the canonical trivialisation of $\delta\delta(\hat{\mathcal{G}})$. See remark 3.18.

By letting $\alpha = s^*(\delta(\mu)) \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ we have

$$\delta(\alpha) = (\delta(s^*))(\delta(\mu)) = (1)^*(\delta(\delta(\mu))) = 0$$

and

$$d(\alpha) = d(s^*(\delta(\mu))) = s^*(d\delta(\mu)) = \delta(R).$$

Let $\Psi(\mathcal{G},\mathcal{A})$ be the collection of all pairs (α, R) where $\alpha \in \Omega^1(\mathcal{G}\times\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$, $R \in \Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and

$$\delta(\alpha) = 0, \quad dR = 0, \quad \text{and } \delta(R) = d(\alpha).$$

The computations above indicate a map from the equivalence classes of central extensions onto $\Psi(\mathcal{G},\mathcal{A})$. This map is constructed rigorously in [26] and [27]. We aim to construct an inverse map and conclude that central extensions (up to isomorphism) of a group $\mathcal{G}$ by $\mathcal{A}$ are precisely determined by pairs (α, R) in $\Psi(\mathcal{G},\mathcal{A})$.

3.6 Isomorphism of central extensions

Recall the definition of isomorphism of central extensions from section 3.1. In this section we investigate the relationship between isomorphism of central extensions and the associated R and α.

Lemma 3.22. *Suppose $\mathcal{G}$ is a Lie group and suppose there exist $R \in \Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $\alpha \in \Omega^1(\mathcal{G}\times\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ which satisfy*

- *dR=0,*
- *$\delta(\alpha) = 0$ and*
- *$\delta(R) = d\alpha$.*

Let $\eta \in \Omega^1(\mathcal{G})$. If we define $R' = R + d\eta$ and $\alpha' = \alpha + \delta(\eta)$ then we have

$$dR' = 0, \quad \delta(\alpha') = 0 \quad \textit{and} \quad \delta(R') = d\alpha'.$$

Moreover R' is Λ-integral.

Proof. We have

$$dR' = d(R + d\eta) = dR = 0,$$

as $d^2\eta = 0$ and

$$\delta(\alpha') = \delta(\alpha + \delta(\eta)) = \delta(\alpha) = 0,$$

since $\delta^2(\eta) = 0$. Moreover from $R' = R + d\eta$ we have $\delta(R') = \delta(R) + \delta(d\eta)$ and hence using $\delta(R) = d\alpha$

$$\delta(R') = d\alpha + d(\delta(\eta)) = d(\alpha + \delta(\eta)) = d\alpha'.$$

Finally using Stokes' theorem, for every $f : S^2 \to \mathcal{G}$ we have

$$\int_{S^2} f^* R' = \int_{S^2} f^*(R + d\eta) = \int_{S^2} f^* R \in \Lambda.$$

$\square$

Proposition 3.23. *Suppose* $\alpha, \alpha' \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ *and* $R, R' \in \Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ *satisfy*

$$dR = 0, \quad \delta(\alpha) = 0, \quad \delta(R) = d\alpha$$

and

$$dR' = 0, \quad \delta(\alpha') = 0, \quad \delta(R') = d\alpha'.$$

Let $\hat{\mathcal{G}}$ *and* $\hat{\mathcal{G}}'$ *and be central extensions of* $\mathcal{G}$ *defined by* (α, R) *and* (α', R') *respectively. That is,*

$$\widehat{\mathcal{G}} = \frac{\mathcal{A} \times_c P\mathcal{G}}{H(\Omega\mathcal{G})} \quad \textit{and} \quad \widehat{\mathcal{G}}' = \frac{\mathcal{A} \times_{c'} P\mathcal{G}}{H'(\Omega\mathcal{G})},$$

where c *and* H *(respectively* c' *and* H'*) are defined using 3.8 and 3.9 by* α *and* R *(respectively* α' *and* R'*).*

Suppose moreover that there exists a 1-form $\eta \in \Omega^1(\mathcal{G})$ *such that* $R' = R + d\eta$ *and* $\alpha' = \alpha + \delta(\eta)$. *Then* $\hat{\mathcal{G}} \simeq \hat{\mathcal{G}}'$

Proof. By using proposition 3.17 we need to find some $\chi : P\mathcal{G} \to \mathcal{A}$ such that

$$c'(g_1, g_2) = c(g_1, g_2)\chi(g_1)\chi(g_2)\chi(g_1 g_2)^{-1}$$

and

$$h'(g) = h(g)\chi(g)^{-1}.$$

As in the proof of proposition 3.20 we define the cocycle $c : P\mathcal{G} \times P\mathcal{G} \to \mathcal{A}$ for a central extension of $P\mathcal{G}$ by

$$c(g_1, g_2) = \exp\left[\int_0^1 (g_1, g_2)^* \alpha\right]$$

and similarly define c' for $\hat{\mathcal{G}}'$. We have

$$\begin{aligned} c'(g_1, g_2) &= \exp\left[\int_0^1 (g_1, g_2)^* \alpha'\right] \\ &= \exp\left[\int_0^1 (g_1, g_2)^* (\alpha + \delta(\eta))\right] \\ &= \exp\left[\int_0^1 (g_1, g_2)^* \alpha\right] \exp\left[\int_0^1 (g_1, g_2)^* \delta(\eta)\right] \\ &= c(g_1, g_2) \exp\left[\int_0^1 (g_1, g_2)^* \delta(\eta)\right] \\ &= c(g_1, g_2) \exp\left[\int_0^1 g_1^* \eta\right] \exp\left[\int_0^1 g_2^* \eta\right] \exp\left[\int_0^1 (g_1 g_2)^* \eta\right]^{-1}. \end{aligned}$$

Let $\chi(g) = \exp\left[\int_0^1 g^* \eta\right]$. Then

$$c'(g_1, g_2) = c(g_1, g_2)\chi(g_1)\chi(g_2)\chi(g_1 g_2)^{-1}. \tag{3.13}$$

As in the proof of proposition 3.20 we can define $h : \Omega\mathcal{G} \to \mathcal{A}$ by

$$h(g) = \exp\left[-\int_S \tilde{g}^* R\right],$$

where $\tilde{g} : S \to \mathcal{G}$ is some extension of g to the unit square $S = [0, 1] \times [0, 1]$ which agrees with g on the boundary and similarly define h'. Using Stokes' theorem we have

$$\begin{aligned} h'(g) &= \exp\left[-\int_S \tilde{g}^* R'\right] \\ &= \exp\left[-\int_S \tilde{g}^* (R + d\eta)\right] \\ &= \exp\left[-\int_S \tilde{g}^* R\right] \exp\left[-\int_S \tilde{g}^* d\eta\right] \\ &= h(g) \exp\left[-\int_S d(\tilde{g}^* \eta)\right] \\ &= h(g) \exp\left[-\int_{\partial S} \tilde{g}^* \eta\right] \\ &= h(g) \exp\left[-\int_0^1 g^* \eta\right] \\ &= h(g)\chi(g)^{-1}. \end{aligned}$$

□

Chapter 4

Examples of central extensions

We begin this chapter by constructing a central extension of the loop group of a compact, simple, simply connected group G by an abelian Lie group $\mathcal{A}$ [27]. We then generalise this construction to a central extension of the group $\text{Map}(\Sigma, G)$ of smooth maps from a compact manifold Σ to a Lie group G. We conclude this chapter by constructing a central extension of the gauge group $\text{Aut}(Q)$ where Q is a principal G bundle over a compact manifold Σ. These central extensions are all constructed by the techniques we developed in chapter 3.

For the case of loop groups we set $\mathcal{A} = U(1)$. For other cases we set $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$. We explain in section 4.2 and 4.3 why this is a good choice for $\mathcal{A}$.

4.1 Loop groups

The central extension of the loop group by the circle, known as the Kac-Moody group was first constructed in [29, 21] and later in [22]. In this section we give an explicit construction of the Kac-Moody group in terms of differential forms using the results of chapter 3 and following [24].

Let G be a compact, simple, simply connected Lie group and let

$$\mathcal{G} = LG = C^\infty(S^1, G).$$

Our aim in this section is to construct a central extension

$$1 \to U(1) \to \widehat{\mathcal{G}} \to \mathcal{G} \to 1.$$

It will turn out that the Chern class of the bundle $\hat{\mathcal{G}} \to \mathcal{G}$ is non-vanishing and hence the bundle $\hat{\mathcal{G}} \to \mathcal{G}$ is non-trivial. So we apply the results from the previous chapter and construct a central extension of the loop group using an R and α.

Proposition 4.1. *Let $\mathcal{G} = C^\infty(S^1, \mathrm{SU}(n))$ and define $R \in \Omega^2(\mathcal{G})$ and $\alpha \in \Omega^1(\mathcal{G} \times \mathcal{G})$ by*

$$\alpha((g_1, g_2); (g_1X_1, g_2X_2)) = \frac{i}{2\pi} \int_{S^1} \mathrm{tr}\left(X_1 \frac{\partial g_2}{\partial \theta} g_2^{-1}\right) d\theta,$$

where $g_1, g_2 \in \mathcal{G}$ and X_1, X_2 are vector fields on $\mathcal{G}$ and

$$R(g; gX, gY) = \frac{i}{4\pi} \int_{S^1} \mathrm{tr}\left(X \frac{\partial Y}{\partial \theta}\right) d\theta,$$

where $g \in \mathcal{G}$ and X, Y are vector fields on $\mathcal{G}$.

Then we have (1) $\delta(\alpha) = 0$ and (2) $\delta(R) = d\alpha$. Moreover R is $2\pi i$-integral.

Proof. For (1), by defining

$$\mu(g, X) = \frac{i}{2\pi} \int_{S^1} \mathrm{tr}\left(X \frac{\partial g}{\partial \theta} g^{-1}\right) d\theta, \tag{4.1}$$

we can write

$$\alpha((g_1, g_2); (g_1X_1, g_2X_2)) = \mu(g_2, X_1). \tag{4.2}$$

Using the definition of the maps $d_i : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ (see section 3.5) we find that

$$(\delta\alpha)((g_1, g_2, g_3); (g_1X_1, g_2X_2, g_3X_3))$$

is equal to

$$\begin{aligned}&\alpha((g_2, g_3); (g_2X_2, g_3X_3)) - \alpha((g_1g_2, g_3); (g_1X_1g_2 + g_1g_2X_2, g_3X_3))\\ &+\alpha((g_1, g_2g_3); (g_1X_1, g_2X_2g_3 + g_2g_3X_3)) - \alpha((g_1, g_2); (g_1X_1, g_2X_2))\end{aligned}$$

which by using (4.2) is equal to

$$-\mu(g_3, g_2^{-1}X_1g_2) + \mu(g_2g_3, X_1) - \mu(g_2, X_1).$$

Therefore by using (4.1) and invariance of trace

$$(\delta\alpha)((g_1, g_2, g_3); (g_1X_1, g_2X_2, g_3X_3))$$

is equal to

$$\frac{i}{2\pi}\int_{S^1}\left[-\mathrm{tr}\left(g_2^{-1}X_1g_2\frac{\partial g_3}{\partial\theta}g_3^{-1}\right)+\mathrm{tr}\left(X_1\frac{\partial(g_2g_3)}{\partial\theta}(g_2g_3)^{-1}\right)-\mathrm{tr}\left(X_1\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)\right]d\theta$$

$$=\frac{i}{2\pi}\int_{S^1}\left[-\mathrm{tr}\left(g_2^{-1}X_1g_2\frac{\partial g_3}{\partial\theta}g_3^{-1}\right)+\mathrm{tr}\left(X_1g_2\frac{\partial g_3}{\partial\theta}g_3^{-1}g_2^{-1}\right)+\mathrm{tr}\left(X_1\frac{\partial g_2}{\partial\theta}g_3g_3^{-1}g_2^{-1}\right)\right.$$

$$\left.-\mathrm{tr}\left(X_1,\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)\right]d\theta$$

$$=\frac{i}{2\pi}\int_{S^1}\left[-\mathrm{tr}\left(g_2^{-1}X_1g_2\frac{\partial g_3}{\partial\theta}g_3^{-1}\right)+\mathrm{tr}\left(X_1g_2\frac{\partial g_3}{\partial\theta}g_3^{-1}g_2^{-1}\right)\right]d\theta$$

$$=0.$$

For (2), by using the definition of the maps $d_i : \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ (see section 3.5) and left invariance of R we find that

$$(\delta R)((g_1,g_2);(g_1X_1,g_2X_2),(g_1Y_1,g_2Y_2))$$

is equal to

$$R(e;X_2,Y_2)-R(e;(g_2^{-1}X_1g_2+X_2),(g_2^{-1}Y_1g_2+Y_2))+R(e;X_1,Y_1)$$
$$=\frac{i}{4\pi}\int_{S^1}\left[\mathrm{tr}\left(X_2\frac{\partial Y_2}{\partial\theta}\right)-\mathrm{tr}\left((g_2^{-1}X_1g_2+X_2)\frac{\partial(g_2^{-1}Y_1g_2+Y_2)}{\partial\theta}\right)+\mathrm{tr}\left(X_1\frac{\partial Y_1}{\partial\theta}\right)\right]d\theta.$$

Differentiating both sides of $g^{-1}g=1$, using the product rule yields

$$\frac{\partial(g^{-1})}{\partial\theta}g+g^{-1}\frac{\partial g}{\partial\theta}=0 \implies \frac{\partial(g^{-1})}{\partial\theta}=-g^{-1}\frac{\partial g}{\partial\theta}g^{-1}.$$

So we have

$$\begin{aligned}
&\operatorname{tr}\left((g_2^{-1}X_1g_2+X_2)\frac{\partial(g_2^{-1}Y_1g_2+Y_2)}{\partial\theta}\right)\\
&= \operatorname{tr}\left((g_2^{-1}X_1g_2+X_2)\left(\frac{\partial(g_2^{-1})}{\partial\theta}Y_1g_2+g_2^{-1}\frac{\partial Y_1}{\partial\theta}g_2+g_2^{-1}Y_1\frac{\partial g_2}{\partial\theta}+\frac{\partial Y_2}{\partial\theta}\right)\right)\\
&= \operatorname{tr}\left((g_2^{-1}X_1g_2+X_2)\left(-g_2^{-1}\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1g_2+g_2^{-1}\frac{\partial Y_1}{\partial\theta}g_2+g_2^{-1}Y_1\frac{\partial g_2}{\partial\theta}+\frac{\partial Y_2}{\partial\theta}\right)\right)\\
&= \operatorname{tr}\left(-g_2^{-1}X_1\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1g_2\right)+\operatorname{tr}\left(g_2^{-1}X_1\frac{\partial Y_1}{\partial\theta}g_2\right)\\
&\quad+\operatorname{tr}\left(g_2^{-1}X_1Y_1\frac{\partial g_2}{\partial\theta}\right)+\operatorname{tr}\left(g_2^{-1}X_1g_2\frac{\partial Y_2}{\partial\theta}\right)\\
&\quad+\operatorname{tr}\left(-X_2g_2^{-1}\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1g_2\right)+\operatorname{tr}\left(X_2g_2^{-1}\frac{\partial Y_1}{\partial\theta}g_2\right)\\
&\quad+\operatorname{tr}\left(X_2g_2^{-1}Y_1\frac{\partial g_2}{\partial\theta}\right)+\operatorname{tr}\left(X_2\frac{\partial Y_2}{\partial\theta}\right)\\
&= \operatorname{tr}\left(-X_1\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1\right)+\operatorname{tr}\left(X_1\frac{\partial Y_1}{\partial\theta}\right)+\operatorname{tr}\left(g_2^{-1}X_1Y_1\frac{\partial g_2}{\partial\theta}\right)+\operatorname{tr}\left(g_2^{-1}X_1g_2\frac{\partial Y_2}{\partial\theta}\right)\\
&\quad+\operatorname{tr}\left(-X_2g_2^{-1}\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1g_2\right)+\operatorname{tr}\left(X_2g_2^{-1}\frac{\partial Y_1}{\partial\theta}g_2\right)+\operatorname{tr}\left(X_2g_2^{-1}Y_1\frac{\partial g_2}{\partial\theta}\right)+\operatorname{tr}\left(X_2\frac{\partial Y_2}{\partial\theta}\right)
\end{aligned}$$

So

$$\begin{aligned}
\delta(R) &= \frac{i}{4\pi}\int_{S^1}\left[\operatorname{tr}\left(X_1\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1\right)-\operatorname{tr}\left(g_2^{-1}X_1Y_1\frac{\partial g_2}{\partial\theta}\right)-\operatorname{tr}\left(g_2^{-1}X_1g_2\frac{\partial Y_2}{\partial\theta}\right)\right.\\
&\qquad\left.+\operatorname{tr}\left(X_2g_2^{-1}\frac{\partial g_2}{\partial\theta}g_2^{-1}Y_1g_2\right)-\operatorname{tr}\left(X_2g_2^{-1}\frac{\partial Y_1}{\partial\theta}g_2\right)-\operatorname{tr}\left(X_2g_2^{-1}Y_1\frac{\partial g_2}{\partial\theta}\right)\right]d\theta\\
&= \frac{i}{4\pi}\int_{S^1}\left[\operatorname{tr}\left(-[X_1,Y_1]\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)-\operatorname{tr}\left(X_2\frac{\partial}{\partial\theta}\left\{g_2^{-1}Y_1g_2\right\}\right)\right.\\
&\qquad\left.-\operatorname{tr}\left(g_2^{-1}X_1g_2\frac{\partial Y_2}{\partial\theta}\right)\right]d\theta
\end{aligned}$$

By integrating the second term in this expression by parts we have

$$\delta(R) = \frac{i}{4\pi}\int_{S^1}\left[\operatorname{tr}\left(-[X_1,Y_1]\frac{\partial g_2}{\partial\theta}g_2^{-1}\right) + \operatorname{tr}\left(\frac{\partial X_2}{\partial\theta}g_2^{-1}Y_1g_2\right)\right.$$
$$\left. - \operatorname{tr}\left(g_2^{-1}X_1g_2\frac{\partial Y_2}{\partial\theta}\right)\right]d\theta.$$

On the other hand, using the formula for the exterior derivative of a 1-form (see theorem 2.7) we have

$$\begin{aligned}
d\alpha((g_1,g_2);(g_1X_1,g_2X_2),&(g_1Y_1,g_2Y_2)) \\
=&\frac{1}{2}\Big\{(g_1X_1,g_2X_2)\alpha((g_1,g_2);(g_1Y_1,g_2Y_2)) \\
&- (g_1Y_1,g_2Y_2)\alpha((g_1,g_2);(g_1X_1,g_2X_2)) \\
&- \alpha((g_1,g_2),(g_1[X_1,Y_1],g_2[X_2,Y_2]))\Big\} \\
=&\frac{1}{2}\Big\{(g_1X_1,g_2X_2)\frac{i}{2\pi}\int_{S^1}\operatorname{tr}\left(Y_1\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)d\theta \\
&- (g_1Y_1,g_2Y_2)\frac{i}{2\pi}\int_{S^1}\operatorname{tr}\left(X_1\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)d\theta \\
&- \frac{i}{2\pi}\int_{S^1}\operatorname{tr}\left([X_1,Y_1]\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)d\theta\Big\}.
\end{aligned}$$

A calculation gives the following expressions for the Lie derivatives

$$\begin{aligned}
(g_1X_1,g_2X_2)&\int_{S^1}\operatorname{tr}\left(Y_1\frac{\partial g_2}{\partial\theta}g_2^{-1}\right)d\theta \\
&= \left.\frac{d}{ds}\right|_{s=0}\int_{S^1}\operatorname{tr}\left(Y_1\frac{\partial(g_2(1+sX_2))}{\partial\theta}(1-sX_2)g_2^{-1}\right)d\theta \\
&= \int_{S^1}\operatorname{tr}\left(Y_1g_2\frac{\partial X_2}{\partial\theta}g_2^{-1}\right)d\theta
\end{aligned}$$

and

$$
\begin{aligned}
(g_1Y_1, g_2Y_2) \int_{S^1} \operatorname{tr}\left(X_1 \frac{\partial g_2}{\partial\theta} g_2^{-1}\right) d\theta
&= \frac{d}{ds}\bigg|_{s=0} \int_{S^1} \operatorname{tr}\left(X_1 \frac{\partial(g_2(1+sY_2))}{\partial\theta}(1-sX_2)g_2^{-1}\right) d\theta \\
&= \int_{S^1} \operatorname{tr}\left(X_1 g_2 \frac{\partial Y_2}{\partial\theta} g_2^{-1}\right) d\theta.
\end{aligned}
$$

Therefore $d\alpha((g_1, g_2); (g_1X_1, g_2X_2), (g_1Y_1, g_2Y_2))$ is equal to

$$
\frac{i}{4\pi}\left\{ \int_{S^1} \left[\operatorname{tr}\left(Y_1 g_2 \frac{\partial X_2}{\partial\theta} g_2^{-1}\right) - \operatorname{tr}\left(X_1 g_2 \frac{\partial Y_2}{\partial\theta} g_2^{-1}\right) - \operatorname{tr}\left([X_1, Y_1] \frac{\partial g_2}{\partial\theta} g_2^{-1}\right)\right] d\theta \right\},
$$

which is the expression we had for $\delta(R)$. For the integrality condition on R we refer to section 4.4 of [29]. □

Now that we have established R and α satisfy the required properties we can define the cocycle c and the homomorphism h by (3.8) and (3.9) (see section 3.5). Therefore the central extension of the loop group by $U(1)$ is given by (3.7) (see section 3.4).

4.2 Group of maps

In this section we aim to construct a central extension of the group $\mathcal{G} = \operatorname{Map}(\Sigma, G)$ of smooth maps from a compact manifold Σ to a Lie group G. A method to construct such central extensions was introduced in [29] by thinking of $\operatorname{Map}(\Sigma, G)$ as a generalisation of the loop group $\operatorname{Map}(S^1, G)$. An alternative construction can be found in [18]. Throughout this section we assume the group $\mathcal{G}$ is 1-connected. If $\mathcal{G}$ is not 1-connected we believe our approach extends to give a central extension of $\widetilde{\mathcal{G}}_0$, the universal cover of the identity component of $\mathcal{G}$. We intend to pursue this question in future research.

Before beginning to construct a central extension of $\mathrm{Map}(\Sigma, G)$ by the abelian group $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$ we would like to discuss the topological properties of $\mathcal{A}$. Recall the de Rham complex of Σ

$$\Omega^0(\Sigma) \xrightarrow{d} \Omega^1(\Sigma) \xrightarrow{d} \Omega^2(\Sigma) \xrightarrow{d} \cdots$$

We define the *codifferentials* to be the linear maps $\delta : \Omega^p(\Sigma) \to \Omega^{p-1}(\Sigma)$ defined by

$$\delta = (-1)^{n(p+1)+1} \star d \star,$$

where $\star$ is the Hodge star operator $\star : \Omega^p(\Sigma) \to \Omega^{n-p}(\Sigma)$ [7] and $n = \dim(\Sigma)$. The Hodge decomposition theorem states that for every $\omega \in \Omega^p(\Sigma)$ there are unique forms $\alpha \in \Omega^{p-1}(\Sigma)$, $\beta \in \Omega^{p+1}(\Sigma)$ and a harmonic form $\gamma \in \Omega^p(\Sigma)$ such that

$$\omega = d\alpha + \delta\beta + \gamma,$$

where by γ being a harmonic form we mean $(d\delta - \delta d)\gamma = 0$. In particular, we have

$$\Omega^1(\Sigma) = \ker(d) \oplus \mathrm{im}(\delta).$$

Therefore

$$\mathcal{A} = \frac{\Omega^1(\Sigma)}{\Omega^1_{\mathbb{Z}}(\Sigma)} = \frac{\ker(d)}{\Omega^1_{\mathbb{Z}}(\Sigma)} \oplus \mathrm{im}(\delta).$$

As $\mathrm{im}(\delta)$ is a vector space, it makes no contribution to the cohomology of $\mathcal{A}$. To compute the cohomology of $\ker(d)/\Omega^1_{\mathbb{Z}}(\Sigma)$ we note the inclusion

$$\mathbb{Z}^{b_1} = \frac{\Omega^1_{\mathbb{Z}}(\Sigma)}{d\Omega^0(\Sigma)} \subset \frac{\ker d}{d\Omega^0(\Sigma)} = \mathbb{R}^{b_1},$$

where b_1 is the first Betti number of Σ [31]. Hence

$$\frac{\ker(d)}{\Omega^1_{\mathbb{Z}}(\Sigma)} \simeq \frac{\mathbb{R}^{b_1}}{\mathbb{Z}^{b_1}} = (S^1)^{b_1}.$$

That is, $\mathcal{A}$ is the sum of a vector space and b_1 copies of S^1. This shows that $\mathcal{A}$ has the structure of an abelian Lie group. Note, if $b_1 = 0$ then $\mathcal{A}$-bundles over Σ are trivial and there is a smooth section $s : \mathcal{G} \to \hat{\mathcal{G}}$. Therefore the central extension is topologically trivial. However for the central extension to be algebraically trivial, s needs to be a homomorphism (see proposition 3.9). Whether $b_1 = 0$ implies this is a question we wish to address in future research.

Now we use the results from section 4.1 to construct a central extension of $\mathrm{Map}(\Sigma, G)$ by $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$.

Let Σ_1 and Σ_2 be compact manifolds and G a Lie group. Let $\mathcal{A}_{\Sigma_i} = \Omega^1(\Sigma_i)/\Omega^1_{\mathbb{Z}}(\Sigma_i)$ and $\mathcal{G}_{\Sigma_i} = \mathrm{Map}(\Sigma_i, G)$ for $i = 1, 2$. Given a map $f : \Sigma_2 \to \Sigma_1$ we can pull back with f to get homomorphisms $f^*_{\mathcal{G}} : \mathcal{G}_{\Sigma_1} \to \mathcal{G}_{\Sigma_2}$ defined by

$$(f^*_{\mathcal{G}}\gamma)(x) = \gamma(f(x))$$

for $\gamma \in \mathcal{G}_{\Sigma_1}$ and $x \in \Sigma_2$ and $f^*_{\mathcal{A}} : \mathcal{A}_{\Sigma_1} \to \mathcal{A}_{\Sigma_2}$ defined by

$$(f^*_{\mathcal{A}}\omega(x))(X) = \omega(f(x))(T_x f(X)),$$

for $\omega \in \mathcal{A}_{\Sigma_1}$, $x \in \Sigma_2$ and $X \in T_x\Sigma_2$.

This gives us the following commuting diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{A}_{\Sigma_1} & \longrightarrow & \widehat{\mathcal{G}}_{\Sigma_1} & \longrightarrow & \mathcal{G}_{\Sigma_1} & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & \mathcal{A}_{\Sigma_2} & \longrightarrow & \widehat{\mathcal{G}}_{\Sigma_2} & \longrightarrow & \mathcal{G}_{\Sigma_2} & \longrightarrow & 1
\end{array}$$

where $\widehat{\mathcal{G}}_{\Sigma_i}$ is a central extension of $\mathcal{G}_{\Sigma_i}$ by $\mathcal{A}_{\Sigma_i}$ for $i = 1, 2$.

Proposition 4.2. *The map $\widehat{\mathcal{G}}_{\Sigma_1} \to \widehat{\mathcal{G}}_{\Sigma_2}$ is a homomorphism.*

Proof. Given $f^*_{\mathcal{G}} : \mathcal{G}_{\Sigma_1} \to \mathcal{G}_{\Sigma_2}$ and $f^*_{\mathcal{A}} : \mathcal{A}_{\Sigma_1} \to \mathcal{A}_{\Sigma_2}$ we can construct the maps

$$Pf^*_{\mathcal{G}} : P\mathcal{G}_{\Sigma_1} \to P\mathcal{G}_{\Sigma_2}$$

and

$$\Omega f^*_{\mathcal{G}} : \Omega\mathcal{G}_{\Sigma_1} \to \Omega\mathcal{G}_{\Sigma_2}.$$

Then given cocycles c_i and homomorphisms h_i corresponding to the central extensions $\widehat{\mathcal{G}}_{\Sigma_i}$ we have

$$c_2 \circ (Pf^*_{\mathcal{G}}, Pf^*_{\mathcal{G}}) = f^*_{\mathcal{G}} \circ c_1$$

and

$$h_2 \circ \Omega f^*_{\mathcal{G}} = f^*_{\mathcal{G}} \circ h_1.$$

Therefore $\widehat{\mathcal{G}}_{\Sigma_1} \to \widehat{\mathcal{G}}_{\Sigma_2}$ is a homomorphism.

□

We now let $\Sigma_1 = S^1$ and use the results from 4.1 to construct a central extension for an arbitrary $\Sigma_2 = \Sigma$. By integrating 1-forms on S^1 we have

$$\mathcal{A}_{S^1} = \Omega^1(S^1)/\Omega^1_{\mathbb{Z}}(S^1) = \mathbb{R}/\mathbb{Z} = U(1)$$

and

$$\mathcal{G}_{S^1} = \mathrm{Map}(S^1, G) = L\mathcal{G}.$$

Let $\widehat{\mathcal{G}}_\Sigma$ be the central extension of $\mathcal{G}_\Sigma$ by $\mathcal{A}_\Sigma$ arising from the cocycle c and homomorphism h as outlined in proposition 4.2. Then the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & \widehat{L\mathcal{G}} & \longrightarrow & L\mathcal{G} & \longrightarrow & 1 \\ \| & & \downarrow & & \downarrow & & \downarrow & & \| \\ 1 & \longrightarrow & \mathcal{A}_\Sigma & \longrightarrow & \widehat{\mathcal{G}}_\Sigma & \longrightarrow & \mathcal{G}_\Sigma & \longrightarrow & 1 \end{array}$$

In the loop group case, α and R were given by integrals. To generalise this to the case $\mathrm{Map}(\Sigma, G)$ we consider the commuting diagram

$$\begin{array}{ccc} \Omega^1(S^1) & \xrightarrow{\int} & \mathbb{R} \\ \| & & \downarrow \\ \Omega^1(S^1) & \xrightarrow{\Pi} & \frac{\Omega^1(S^1)}{d\Omega^0(S^1)} \end{array}$$

where $\Pi : \Omega^1(S^1) \to \Omega^1(S^1)/d\Omega^0(S^1)$ is the projection map. To construct a central extension of $\mathcal{G} = \mathrm{Map}(\Sigma, G)$ by $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$ we need to define $\alpha \in \Omega^1(\mathcal{G}\times\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ satisfying $dR = 0$, $\delta(\alpha) = 0$ and $\delta(R) = d\alpha$.

Proposition 4.3. *Let Σ be a compact manifold and G a Lie group and define $\mathcal{G} = \mathrm{Map}(\Sigma, G)$ and $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$. Let Π be the projection map $\Omega^1(\Sigma) \to \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}$. Define $\alpha \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ by*

$$\alpha((g_1, g_2); (g_1X_1, g_2X_2)) = \frac{i}{2\pi}\Pi(tr(X_1 dg_2 g_2^{-1}))$$

and

$$R(g; gX, gY) = \frac{i}{4\pi}\Pi(tr(XdY)).$$

Then

(1) $dR = 0$,

(2) $\delta(\alpha) = 0$,

(3) $\delta(R) = d\alpha$.

Note R is a 2-form as

$$\begin{aligned} 0 &= \Pi(d(\mathrm{tr}(XY))) \\ &= \Pi(\mathrm{tr}(dXY)) - \Pi(\mathrm{tr}(XdY)) \\ &= \Pi(\mathrm{tr}(YdX)) - \Pi(\mathrm{tr}(XdY)). \end{aligned}$$

Proof. For (1) let X, Y and Z be vector fields. Then

$$\begin{aligned} dR(X,Y,Z) &= X.R(Y,Z) + Y.R(Z,X) + Z.R(X,Y) \\ &\quad - R([X,Y],Z) - R([Y,Z],X) - R([Z,X],Y) \\ &= -R([X,Y],Z) - R([Y,Z],X) - R([Z,X],Y) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}([X,Y]dZ)) - \frac{i}{4\pi}\Pi(\mathrm{tr}([Y,Z]dX)) - \frac{i}{4\pi}\Pi(\mathrm{tr}([Z,X]dY)) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}([X,Y]dZ + [Y,Z]dX + [Z,X]dY)) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}(XYdZ - YXdZ + YZdX - ZYdX + ZXdY - XZdY)) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}(XYdZ + YZdX + ZXdY)) \\ &\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(-YXdZ - ZYdX - XZdY)) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}(XY(dZ) + (dX)YZ + X(dY)Z)) \\ &\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(-(dZ)YX - ZY(dX) - Z(dY)X)) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}(d(XYZ))) - \frac{i}{4\pi}\Pi(\mathrm{tr}(-d(ZYX))) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}(d(XYZ - ZYX))) \\ &= -\frac{i}{4\pi}\Pi(d(\mathrm{tr}(X(YZ - ZY)))) \\ &= 0. \end{aligned}$$

For (2), we have

$$\begin{aligned}
d_0^*(\alpha) &= (d_0^*\alpha)((g_1, g_2, g_3); (g_1X_1, g_2X_2, g_3X_3)) \\
&= \alpha((g_2, g_3); (g_2X_2, g_3X_3)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2 dg_3 g_3^{-1}))
\end{aligned}$$

$$\begin{aligned}
d_1^*(\alpha) &= (d_1^*\alpha)((g_1, g_2, g_3); (g_1X_1, g_2X_2, g_3X_3)) \\
&= \alpha((g_1g_2, g_3); (g_1X_1g_2 + g_1g_2X_2, g_3X_3)) \\
&= \alpha((g_1g_2, g_3); (g_2^{-1}X_1g_2 + X_2, g_3X_3)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1g_2 + X_2)dg_3g_3^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dg_3g_3^{-1})) + \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2dg_3g_3^{-1}))
\end{aligned}$$

$$\begin{aligned}
d_2^*(\alpha) &= (d_2^*\alpha)((g_1, g_2, g_3); (g_1X_1, g_2X_2, g_3X_3)) \\
&= \alpha((g_1, g_2g_3); (g_1X_1, g_2X_2g_3 + g_2g_3X_3)) \\
&= \alpha((g_1, g_2g_3); (g_1X_1, g_3^{-1}X_2g_3 + X_3)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1d(g_2g_3)(g_2g_3)^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(dg_2g_3 + g_2dg_3)g_3^{-1}g_2^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1dg_2g_2^{-1})) + \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1g_2dg_3g_3^{-1}g_2^{-1}))
\end{aligned}$$

$$\begin{aligned}
d_3^*(\alpha) &= (d_3^*\alpha)((g_1, g_2, g_3); (g_1X_1, g_2X_2, g_3X_3)) \\
&= \alpha((g_1, g_2); (g_1X_1, g_2X_2)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1dg_2g_2^{-1})).
\end{aligned}$$

Therefore

$$\begin{aligned}
\delta(\alpha) &= d_0^*(\alpha) - d_1^*(\alpha) + d_2^*(\alpha) - d_3^*(\alpha) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2 dg_3 g_3^{-1})) - \frac{i}{2\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1 g_2 dg_3 g_3^{-1})) - \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2 dg_3 g_3^{-1})) \\
&+ \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1 dg_2 g_2^{-1})) + \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1 g_2 dg_3 g_3^{-1} g_2^{-1})) - \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1 dg_2 g_2^{-1})) \\
&= 0.
\end{aligned}$$

For (3), we have

$$\begin{aligned}
\delta(R) &= d_0^*(R) - d_1^*(R) + d_2^*(R) \\
&= R(g_2; X_2, Y_2) - R(g_1 g_2; g_1 X_1 g_2 + g_1 g_2 X_2, g_1 Y_1 g_2 + g_1 g_2 Y_2) + R(g_1; X_1, Y_1) \\
&= \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 dY_2)) - \frac{i}{4\pi}R(e; g_2^{-1}X_1 g_2 + X_2, g_2^{-1}Y_1 g_2 + Y_2) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1 dY_1)) \\
&= \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 dY_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1 g_2 + X_2) d(g_2^{-1}Y_1 g_2 + Y_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1 dY_1)) \\
&= \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 dY_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1 g_2) d(g_2^{-1}Y_1 g_2))) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1 g_2 dY_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 d(g_2^{-1}Y_1 g_2))) - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 dY_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1 dY_1)) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1 g_2) d(g_2^{-1}Y_1 g_2))) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1 g_2 dY_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 d(g_2^{-1}Y_1 g_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1 dY_1)) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1 g_2)[d(g_2^{-1})Y_1 g_2 + g_2^{-1}d(Y_1)g_2 + g_2^{-1}Y_1 d(g_2)])) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1 g_2 dY_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2 d(g_2^{-1}Y_1 g_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1 dY_1))
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2d(g_2^{-1})Y_1g_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2g_2^{-1}d(Y_1)g_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2g_2^{-1}Y_1d(g_2))) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dY_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2d(g_2^{-1}Y_1g_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1dY_1)) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(X_1g_2d(g_2^{-1})Y_1)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1d(Y_1))) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1Y_1d(g_2))) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dY_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2d(g_2^{-1}Y_1g_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1dY_1)) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(X_1g_2d(g_2^{-1})Y_1)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1Y_1dg_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dY_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2d(g_2^{-1}Y_1g_2))) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(X_1g_2d(g_2^{-1})Y_1)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1Y_1dg_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dY_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(d(X_2)g_2^{-1}Y_1g_2)) \\
&= \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1g_2g_2^{-1}d(g_2)g_2^{-1}Y_1)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1Y_1dg_2)) \\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dY_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(d(X_2)g_2^{-1}Y_1g_2)) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}([X_1,Y_1]d(g_2)g_2^{-1})) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2dY_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(d(X_2)g_2^{-1}Y_1g_2))
\end{aligned}
$$

The exterior derivative of a 1-form ω is given by proposition 2.7. So we have

$$\begin{aligned}
d\alpha((g_1,g_2);&(g_1X_1,g_2X_2),(g_1Y_1,g_2Y_2))\\
&=\frac{1}{2}\{(g_1X_1,g_2X_2)(\alpha((g_1,g_2);(g_1Y_1,g_2Y_2)))\\
&\quad-(g_1Y_1,g_2Y_2)(\alpha((g_1,g_2);(g_1X_1,g_2X_2)))\\
&\quad-\alpha((g_1,g_2);(g_1[X_1,Y_1],g_2[X_2,Y_2]))\}\\
&=\frac{i}{4\pi}(g_1X_1,g_2X_2)(\Pi(\mathrm{tr}(Y_1dg_2g_2^{-1})))\\
&\quad-\frac{i}{4\pi}(g_1Y_1,g_2Y_2)(\Pi(\mathrm{tr}(X_1dg_2g_2^{-1})))\\
&\quad-\frac{i}{4\pi}(\Pi(\mathrm{tr}([X_1,Y_1]dg_2g_2^{-1})))\\
&=\frac{i}{4\pi}\frac{d}{dt}(\Pi(\mathrm{tr}(Y_1d(g_2(1+tX_2))(g_2(1+tX_2))^{-1})))\mid_{t=0}\\
&\quad-\frac{i}{4\pi}\frac{d}{dt}(\Pi(\mathrm{tr}(X_1d(g_2(1+tY_2))(g_2(1+tY_2))^{-1})))\mid_{t=0}\\
&\quad-\frac{i}{4\pi}(\Pi(\mathrm{tr}([X_1,Y_1]dg_2g_2^{-1})))\\
&=\frac{i}{4\pi}\Pi(\mathrm{tr}(Y_1d(g_2)g_2^{-1}+tY_1g_2d(X_2)g_2^{-1}))\mid_{t=0}\\
&\quad-\frac{i}{4\pi}(\Pi(\mathrm{tr}(X_1d(g_2)g_2^{-1}+tX_1g_2d(Y_2)g_2^{-1})))\mid_{t=0}\\
&\quad-\frac{i}{4\pi}(\Pi(\mathrm{tr}([X_1,Y_1]dg_2g_2^{-1})))\\
&=\frac{i}{4\pi}(\Pi(\mathrm{tr}(Y_1g_2d(X_2)g_2^{-1})))\\
&\quad-\frac{i}{4\pi}(\Pi(\mathrm{tr}(X_1g_2d(Y_2)g_2^{-1})))\\
&\quad-\frac{i}{4\pi}(\Pi(\mathrm{tr}([X_1,Y_1]dg_2g_2^{-1})))\\
&=\delta(R).
\end{aligned}$$

□

We would also need R to be a Λ-integral 2-form in the sense that if we integrate over a 2-surface in $\mathcal{G}$, the result is in

$$\frac{\Omega^1_{\mathbb{Z}}(\Sigma)}{d\Omega^0(\Sigma)} \subset \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}.$$

Proposition 4.4. *Given a map $\gamma : S^1 \to G$ with the pullback $\gamma^* : \Omega^1(\Sigma) \to \Omega^1(S^1)$ we have*

$$\int_{S^1} \gamma^*(R) \in \Omega^2_\Lambda(\mathcal{G}, \mathbb{Z}).$$

Proof. Let (u, v) be coordinates on $S^2 - \{(0,0,1)\}$ with domain D. For a map $f : S^2 \to \mathcal{G}$ we have

$$\begin{aligned}\int_{S^1} \gamma^* \int_{S^2} f(u,v)^*(R) &= \int_{S^1} \gamma^* \int_D (f^*R)(\partial u, \partial v) du dv \\ &= \frac{i}{4\pi} \int_{S^1} \gamma^* \int_D \mathrm{tr}[d_\Sigma (f^{-1}\partial_u f)(f^{-1}\partial_v f)] du dv \\ &= \frac{i}{4\pi} \int_{S^1} \int_D \mathrm{tr}[d_{S^1}((f^{-1}\partial_u f) \circ \gamma)((f^{-1}\partial_v f)\circ\gamma)] du dv.\end{aligned}$$

Defining $\tilde{f} : S^2 \times \Sigma \to G$ by $\tilde{f}((u,v),x) = f(u,v)(x)$ this is equal to

$$\begin{aligned}&\frac{i}{4\pi} \int_{S^1} \int_D \mathrm{tr}[d_{S^1}(\tilde{f}^{-1} \circ (1,\gamma)\partial_u(\tilde{f} \circ (1,\gamma)))(\tilde{f}^{-1} \circ (1,\gamma)\partial_v(\tilde{f} \circ (1,\gamma)))] du dv \\ =&\frac{i}{4\pi} \int_D \int_{S^1} \mathrm{tr}[d_{S^1}(\tilde{f}^{-1} \circ (1,\gamma)\partial_u(\tilde{f} \circ (1,\gamma)))(\tilde{f}^{-1} \circ (1,\gamma)\partial_v(\tilde{f} \circ (1,\gamma)))] du dv \\ =& \int_{S^2} (\tilde{f} \circ (1,\gamma))^*(R).\end{aligned}$$

Therefore

$$\int_{S^1} \gamma^* \int_{S^2} f^*(R) = \int_{S^2} (\tilde{f} \circ (1,\gamma))^*(R).$$

Since R as defined in section 4.1 for the loop group case is $2\pi i$-integral, we can deduce that f^*R gives us a Λ-integral 2-form for the case of $\mathrm{Map}(\Sigma, G)$.

□

4.3 Gauge group

Suppose Σ is a compact manifold and G a Lie group and let Q be a G-bundle over Σ. As previously introduced in section 2.6, the gauge group of Q is given by

$$\mathcal{G} = \{f : Q \to G \mid f(qg) = g^{-1}f(q)g\}$$

with Lie algebra

$$\mathrm{Lie}(\mathcal{G}) = \{X : Q \to \mathrm{Lie}(G) \mid X(qg) = g^{-1}X(q)g\}.$$

We would like to construct a central extension of $\mathcal{G}$ by $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$. Let Π be the projection map $\Omega^1(\Sigma) \to \mathrm{Lie}(\mathcal{A}) = \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}$. We define $\alpha \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ by

$$\alpha((g_1, g_2); (g_1X_1, g_2X_2)) = \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\nabla_\Sigma g_2)g_2^{-1}))$$

and

$$R(g; gX, gY) = \frac{i}{4\pi}\Pi(\mathrm{tr}(X\nabla_\Sigma Y)),$$

where ∇_Σ is a chosen connection on the G-bundle Q over Σ and $\nabla_\Sigma Y \in \Omega^1(Q, \mathfrak{g})$ is defined by

$$\nabla_\Sigma Y(\xi) = dY(h\xi).$$

Therefore

$$\begin{aligned}
(R_g^*\nabla_\Sigma Y)_q(\xi) &= (\nabla_\Sigma Y)_{qg}((R_g)_*\xi) \\
&= (dY)_{qg}(h(R_g)_*\xi) \\
&= (dY)_{qg}((R_g)_*h\xi) \\
&= (R_g^* dY)_q(h\xi) \\
&= (dR_g^*Y)_q(h\xi) \\
&= (d\,\mathrm{ad}(g^{-1})Y)_q(h\xi) \\
&= \mathrm{ad}(g^{-1})(dY)_q(h\xi) \\
&= \mathrm{ad}(g^{-1})(\nabla_\Sigma Y)_q(\xi).
\end{aligned}$$

So $\nabla_\Sigma Y$ is also ad-invariant and horizontal and hence by theorem 2.33, $\mathrm{tr}(X\nabla_\Sigma Y)$ is pullback of an element in $\Omega^1(\Sigma)$. So Π maps it to an element of $\mathrm{Lie}(\mathcal{A})$.

Proposition 4.5. *For α, R and δ as defined above, we have*

(1) $dR = 0$,

(2) $\delta(\alpha) = 0$,

(3) $\delta(R) = d\alpha$,

(4) R is Λ-integral.

Proof. For (1) let X, Y and Z be vector fields. Then

$$\begin{aligned} dR(X,Y,Z) &= \{X.R(Y,Z) + Y.R(Z,X) + Z.R(X,Y) \\ &\quad - R([X,Y],Z) - R([Y,Z],X) - R([Z,X],Y)\} \\ &= -R([X,Y],Z) - R([Y,Z],X) - R([Z,X],Y) \end{aligned}$$

which up to a constant is equal to

$$\begin{aligned} &\Pi(\mathrm{tr}([X,Y]\nabla_\Sigma Z)) + \Pi(\mathrm{tr}([Y,Z]\nabla_\Sigma X)) + \Pi(\mathrm{tr}([Z,X]\nabla_\Sigma Y)) \\ =&\Pi(\mathrm{tr}([X,Y]\nabla_\Sigma Z + [Y,Z]\nabla_\Sigma X + [Z,X]\nabla_\Sigma Y)) \\ =&\Pi(\mathrm{tr}(XY\nabla_\Sigma Z - YX\nabla_\Sigma Z + YZ\nabla_\Sigma X - ZY\nabla_\Sigma X + ZX\nabla_\Sigma Y - XZ\nabla_\Sigma Y)) \\ =&\Pi(\mathrm{tr}(XY\nabla_\Sigma Z + YZ\nabla_\Sigma X + ZX\nabla_\Sigma Y)) \\ &\qquad + \Pi(\mathrm{tr}(-YX\nabla_\Sigma Z - ZY\nabla_\Sigma X - XZ\nabla_\Sigma Y)) \\ =&\Pi(\mathrm{tr}(XY(\nabla_\Sigma Z) + (\nabla_\Sigma X)YZ + X(\nabla_\Sigma Y)Z)) \\ &\qquad + \Pi(\mathrm{tr}(-(\nabla_\Sigma Z)YX - ZY(\nabla_\Sigma X) - Z(\nabla_\Sigma Y)X)) \\ =&\Pi(\mathrm{tr}(\nabla_\Sigma(XYZ))) - \Pi(\mathrm{tr}(-\nabla_\Sigma(ZYX))) \\ =&\Pi(\mathrm{tr}(\nabla_\Sigma(XYZ - ZYX))) \\ =&\Pi(\nabla_\Sigma(\mathrm{tr}(X(YZ - ZY)))) \\ =&\Pi(\nabla_\Sigma(\mathrm{tr}(X[Y,Z]))) \\ =&0. \end{aligned}$$

For (2), we have

$$\begin{aligned}
d_0^*(\alpha) &= (d_0^*\alpha)((g_1,g_2,g_3);(g_1X_1,g_2X_2,g_3X_3)) \\
&= \alpha((g_2,g_3);(g_2X_2,g_3X_3)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma g_3g_3^{-1})), \\
d_1^*(\alpha) &= (d_1^*\alpha)((g_1,g_2,g_3);(g_1X_1,g_2X_2,g_3X_3)) \\
&= \alpha((g_1g_2,g_3);(g_1X_1g_2+g_1g_2X_2,g_3X_3)) \\
&= \alpha((g_1g_2,g_3);(g_2^{-1}X_1g_2+X_2,g_3X_3)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1g_2+X_2)\nabla_\Sigma g_3g_3^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma g_3g_3^{-1}))+\frac{i}{2\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma g_3g_3^{-1})), \\
d_2^*(\alpha) &= (d_2^*\alpha)((g_1,g_2,g_3);(g_1X_1,g_2X_2,g_3X_3)) \\
&= \alpha((g_1,g_2g_3);(g_1X_1,g_2X_2g_3+g_2g_3X_3)) \\
&= \alpha((g_1,g_2g_3);(g_1X_1,g_3^{-1}X_2g_3+X_3)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma(g_2g_3)(g_2g_3)^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\nabla_\Sigma g_2g_3+g_2\nabla_\Sigma g_3)g_3^{-1}g_2^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma g_2g_2^{-1}))+\frac{i}{2\pi}\Pi(\mathrm{tr}(X_1g_2\nabla_\Sigma g_3g_3^{-1}g_2^{-1})), \\
d_3^*(\alpha) &= (d_3^*\alpha)((g_1,g_2,g_3);(g_1X_1,g_2X_2,g_3X_3)) \\
&= \alpha((g_1,g_2);(g_1X_1,g_2X_2)) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma g_2g_2^{-1})).
\end{aligned}$$

Therefore

$$\begin{aligned}
\delta(\alpha) &= d_0^*(\alpha)-d_1^*(\alpha)+d_2^*(\alpha)-d_3^*(\alpha) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma g_3g_3^{-1}))-\frac{i}{2\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma g_3g_3^{-1}))-\frac{i}{2\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma g_3g_3^{-1})) \\
&+\frac{i}{2\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma g_2g_2^{-1}))+\frac{i}{2\pi}\Pi(\mathrm{tr}(X_1g_2\nabla_\Sigma g_3g_3^{-1}g_2^{-1}))-\frac{i}{2\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma g_2g_2^{-1})) \\
&= 0.
\end{aligned}$$

For (3), we have

$$\begin{aligned}
\delta(R) &= d_0^*(R) - d_1^*(R) + d_2^*(R)\\
&= R(g_2; X_2, Y_2) - R(g_1g_2; g_1X_1g_2 + g_1g_2X_2, g_1Y_1g_2 + g_1g_2Y_2) + R(g_1; X_1, Y_1)\\
&= \Pi(\mathrm{tr}(X_2\nabla_\Sigma Y_2)) - R(e; g_2^{-1}X_1g_2 + X_2, g_2^{-1}Y_1g_2 + Y_2) + \Pi(\mathrm{tr}(X_1\nabla_\Sigma Y_1))\\
&= \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma Y_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1g_2 + X_2)\nabla_\Sigma(g_2^{-1}Y_1g_2 + Y_2)))\\
&\quad + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma Y_1)).
\end{aligned}$$

The middle term is equal to

$$\begin{aligned}
&\frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma(g_2^{-1}Y_1g_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma Y_2))\\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(\nabla_\Sigma(X_2)(g_2^{-1}Y_1g_2))) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma Y_2))\\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2g_2^{-1}\nabla_\Sigma(g_2)g_2^{-1}Y_1g_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2g_2^{-1}\nabla_\Sigma(Y_1)g_2))\\
&\quad + \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2g_2^{-1}Y_1\nabla_\Sigma g_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma Y_2))\\
&\quad - \frac{i}{4\pi}\Pi(\mathrm{tr}(\nabla_\Sigma(X_2)g_2^{-1}Y_1g_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma Y_2))\\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma(g_2)g_2^{-1}Y_1)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma Y_1)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1Y_1\nabla_\Sigma g_2))\\
&\quad + \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma Y_2)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(\nabla_\Sigma(X_2)g_2^{-1}Y_1g_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(X_2\nabla_\Sigma Y_2)).
\end{aligned}$$

Therefore

$$\begin{aligned}\delta(R) &= \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1\nabla_\Sigma(g_2)g_2^{-1}Y_1)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1Y_1\nabla_\Sigma g_2)) \\ &- \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma Y_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(\nabla_\Sigma(X_2)g_2^{-1}Y_1g_2)) \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}([X_1,Y_1]\nabla_\Sigma(g_2)g_2^{-1})) \\ &- \frac{i}{4\pi}\Pi(\mathrm{tr}(g_2^{-1}X_1g_2\nabla_\Sigma Y_2)) + \frac{i}{4\pi}\Pi(\mathrm{tr}(\nabla_\Sigma(X_2)g_2^{-1}Y_1g_2)).\end{aligned}$$

The exterior derivative of a 1-form ω is given by proposition 2.7. Therefore

$$\begin{aligned}&d\alpha((g_1,g_2);(g_1X_1,g_2X_2),(g_1Y_1,g_2Y_2)) \\ &\quad= \frac{1}{2}\{(g_1X_1,g_2X_2)(\alpha((g_1,g_2);(g_1Y_1,g_2Y_2))) \\ &\quad\quad - (g_1Y_1,g_2Y_2)(\alpha((g_1,g_2);(g_1X_1,g_2X_2))) \\ &\quad\quad - \alpha((g_1,g_2);(g_1[X_1,Y_1],g_2[X_2,Y_2]))\} \\ &\quad= \frac{i}{4\pi}(g_1X_1,g_2X_2)(\Pi(\mathrm{tr}(Y_1\nabla_\Sigma g_2g_2^{-1}))) \\ &\quad\quad - \frac{i}{4\pi}(g_1Y_1,g_2Y_2)(\Pi(\mathrm{tr}(X_1\nabla_\Sigma g_2g_2^{-1}))) \\ &\quad\quad - \frac{i}{4\pi}(\Pi(\mathrm{tr}([X_1,Y_1]\nabla_\Sigma g_2g_2^{-1})))\end{aligned}$$

We have

$$\begin{aligned}&(g_1X_1,g_2X_2)(\Pi(\mathrm{tr}(Y_1\nabla_\Sigma g_2g_2^{-1}))) \\ &\quad= \frac{d}{dt}\{\Pi(\mathrm{tr}(Y_1\nabla_\Sigma(g_2(1+tX_2))(g_2(1+tX_2))^{-1}))\}\mid_{t=0} \\ &\quad= \frac{d}{dt}\{\Pi(\mathrm{tr}(Y_1\nabla_\Sigma(g_2)g_2^{-1}+tY_1g_2\nabla_\Sigma(X_2)g_2^{-1}))\}\mid_{t=0} \\ &\quad= \Pi(\mathrm{tr}(Y_1g_2\nabla_\Sigma(X_2)g_2^{-1})).\end{aligned}$$

Similarly

$$(g_1Y_1, g_2Y_2)(\Pi(\mathrm{tr}(X_1\nabla_\Sigma g_2 g_2^{-1}))) = \Pi(\mathrm{tr}(X_1 g_2 \nabla_\Sigma(Y_2) g_2^{-1})).$$

Therefore

$$\begin{aligned} d\alpha((g_1, g_2); (g_1X_1, g_2X_2), (g_1Y_1, g_2Y_2)) & \\ &= \frac{i}{4\pi}\Pi(\mathrm{tr}(Y_1 g_2 \nabla_\Sigma(X_2) g_2^{-1})) \\ &- \frac{i}{4\pi}\Pi(\mathrm{tr}(X_1 g_2 \nabla_\Sigma(Y_2) g_2^{-1})) \\ &- \frac{i}{4\pi}(\Pi(\mathrm{tr}([X_1, Y_1]\nabla_\Sigma g_2 g_2^{-1}))) \\ &= \delta(R). \end{aligned}$$

To show the Λ-integrality of R we recall that given a smooth map $\gamma : \Sigma_1 \to \Sigma_2$ between manifolds and a G-bundle P over Σ_2 with connection ∇_Σ there is a pullback bundle $\gamma^{-1}P$ over Σ_1 with connection $\gamma^*\nabla_\Sigma$. This induces the maps

$$\gamma^* : \Omega^1(\Sigma_2) \to \Omega^1(\Sigma_1)$$

and

$$\tilde{\gamma} : \mathcal{G}_{\Sigma_2} = \Gamma(\Sigma_2, \mathrm{Ad}P) \to \mathcal{G}_{\Sigma_1} = \Gamma(\Sigma_1, \mathrm{Ad}(\gamma^* P)).$$

Given $R_{\Sigma_2}^{\nabla_\Sigma} \in \mathcal{G}_{\Sigma_2}$ and $R_{\Sigma_1}^{\gamma^*\nabla_\Sigma} \in \mathcal{G}_{\Sigma_1}$ we have

$$\tilde{\gamma}^*(R_{\Sigma_1}^{\gamma^*\nabla_{\Sigma_1}}) = \gamma^*(R_{\Sigma_2}^{\nabla_{\Sigma_2}}).$$

The construction in proposition 4.4 then applies for $\gamma : S^1 \to \Sigma$ except that we have a $\mathcal{G}$-bundle over S^1 with a possibly non-trivial connection A. We can trivialise the bundle but the connection may not be trivial. The 2-form R then takes the form

$$R_A(X, Y) = \frac{i}{4\pi}\int_{S_1} \mathrm{tr}(X d_A Y),$$

where $d_A Y = dY + [A, Y]$. A calculation shows that this is cohomologous to R_0 given by

$$\frac{i}{4\pi}\int_{S_1} \mathrm{tr}(X dY)$$

and is therefore integral.

□

Remark 4.6. *Notice that this demonstrates theorem 1.1 in the introduction.*

Proposition 4.7. *Let ∇_Σ and ∇'_Σ be connections on Q. Then the pairs (α, R) and (α', R') defined by ∇_Σ and ∇'_Σ respectively construct isomorphic central extensions.*

Proof. As ∇_Σ and ∇'_Σ are connections on Q, their difference is given by

$$\nabla_\Sigma - \nabla'_\Sigma = \pi^*\mu$$

for a one-form $\mu \in \Omega^1(\Sigma, \mathrm{ad}Q)$. Let $q \in Q$ and $\xi \in T_qQ$. We have

$$(\nabla_\Sigma Y)(\xi) = dY(h\xi) = dY(\xi - \iota_q(\omega(\xi))).$$

So

$$\begin{aligned}
(\nabla_\Sigma Y - \nabla'_\Sigma Y)(\xi) &= dY(\xi - \iota_q(\omega(\xi))) - dY(\xi - \iota_q(\omega'(\xi))) \\
&= dY(\iota_q(\omega'(\xi)) - \iota_q(\omega(\xi))) \\
&= \frac{d}{dt} Y(q \exp t\pi^*\mu) \mid_{t=0} \\
&= \frac{d}{dt}((1 - t\pi^*\mu)Y(q)(1 + t\pi^*\mu)) \mid_{t=0} \\
&= -[Y, \pi^*\mu].
\end{aligned}$$

Therefore

$$\begin{aligned}
R - R' &= \frac{i}{4\pi}\Pi(\mathrm{tr}(X\nabla_\Sigma Y)) - \frac{i}{4\pi}\Pi(\mathrm{tr}(X\nabla'_\Sigma Y)) \\
&= \frac{i}{4\pi}\Pi(\mathrm{tr}(X(\nabla_\Sigma Y - \nabla'_\Sigma Y))) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}(X[Y, \pi^*\mu])) \\
&= -\frac{i}{4\pi}\Pi(\mathrm{tr}([X, Y]\pi^*\mu)).
\end{aligned}$$

We also have

$$\begin{aligned}
\alpha - \alpha' &= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\nabla_\Sigma g_2)g_2^{-1})) - \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\nabla'_\Sigma g_2)g_2^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\nabla_\Sigma g_2 - \nabla'_\Sigma g_2)g_2^{-1})) \\
&= \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\pi^*\mu - g_2\pi^*\mu g_2^{-1}))).
\end{aligned}$$

So by Proposition 3.23 we need $\eta \in \Omega^1\left(\mathcal{G}, \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}\right)$ such that

$$d\eta = -\frac{i}{4\pi}\Pi(\mathrm{tr}([X,Y]\pi^*\mu)) \tag{4.3}$$

and

$$\delta\eta = \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\pi^*\mu - g_2\pi^*\mu g_2^{-1}))). \tag{4.4}$$

We show that $\eta(g,X) = \frac{i}{4\pi}\Pi(\mathrm{tr}(X\pi^*\mu))$ satisfies conditions (4.3) and (4.4). As η does not depend on the first entry, we have

$$\begin{aligned} d\eta(g,X,Y) &= \frac{1}{2}\{X(\eta(g,Y)) - Y(\eta(g,X)) - \eta(g,[X,Y])\} \\ &= -\frac{i}{4\pi}\Pi(\mathrm{tr}([X,Y]\pi^*\mu)). \end{aligned}$$

We also have

$$\begin{aligned} &\delta\eta((g_1,g_2);(X_1,X_2)) \\ &\quad = \eta(g_2,X_2) - \eta(g_1g_2, g_2^{-1}X_1g_2 + X_2) + \eta(g_1,X_1) \\ &\quad = \frac{i}{2\pi}\Pi(\mathrm{tr}(X_2\pi^*\mu)) - \frac{i}{2\pi}\Pi(\mathrm{tr}((g_2^{-1}X_1g_2 + X_2)\pi^*\mu)) + \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1\pi^*\mu)) \\ &\quad = \frac{i}{2\pi}\Pi(\mathrm{tr}(-g_2^{-1}X_1g_2\pi^*\mu + X_1\pi^*\mu)) \\ &\quad = \frac{i}{2\pi}\Pi(\mathrm{tr}(-X_1g_2\pi^*\mu g_2^{-1} + X_1\pi^*\mu)) \\ &\quad = \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1(\pi^*\mu - g_2\pi^*\mu g_2^{-1}))). \end{aligned}$$

□

Chapter 5

The lifting bundle gerbe

This chapter introduces the notion of lift for principal bundles and bundle gerbes and explains how this relates to the central extensions of groups [25] and [27]. The important notion of the lifting bundle gerbe is used to construct a de Rham three-form obstructing the lift of a $\mathcal{G}$-bundle to a central extension $\hat{\mathcal{G}}$. The relation to reduced splittings [9] is also discussed.

5.1 Bundle lifting

Let

$$1 \to \mathcal{A} \xrightarrow{\iota} \hat{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \to 1 \tag{5.1}$$

be a central extension of Lie groups and $P \to M$ be a principal $\mathcal{G}$ bundle over M. Then P is said to *lift to a $\hat{\mathcal{G}}$ bundle* if there exists a principal $\hat{\mathcal{G}}$ bundle $\widehat{P}$ over M and a bundle map $\widehat{P} \to P$ which commutes with $\pi : \hat{\mathcal{G}} \to \mathcal{G}$. We can trivialise P by taking a good cover $\{U_\alpha\}_{\alpha \in I}$ over M, choose sections $s_\alpha : U_\alpha \to \mathcal{G}$ and present P by the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{G}$ defined by $s_\alpha = g_{\alpha\beta} s_\beta$. We can then lift $g_{\alpha\beta}$ to maps $\hat{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \hat{\mathcal{G}}$ such that $\pi(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. These maps are transition functions for a lift $\widehat{P}$ of P if they satisfy the cocycle condition

$$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = 1.$$

We define the maps $\epsilon_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathcal{A}$ by

$$\iota(\epsilon_{\alpha\beta\gamma}) = \hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha}.$$

Let $\alpha, \beta, \gamma, \delta \in I$. As $\mathcal{A}$ is central in $\hat{\mathcal{G}}$ we have

$$\begin{aligned}
\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\delta} &= \hat{g}_{\alpha\gamma}\epsilon_{\alpha\beta\gamma}\hat{g}_{\gamma\delta} \\
&= \hat{g}_{\alpha\gamma}\epsilon_{\alpha\beta\gamma}\hat{g}_{\gamma\delta} \\
&= \hat{g}_{\alpha\gamma}\hat{g}_{\gamma\delta}\epsilon_{\alpha\beta\gamma} \\
&= \hat{g}_{\alpha\delta}\epsilon_{\alpha\gamma\delta}\epsilon_{\alpha\beta\gamma}
\end{aligned}$$

and

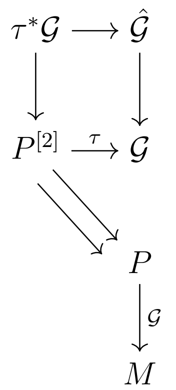
$$\begin{aligned}
\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\delta} &= \hat{g}_{\alpha\gamma}\epsilon_{\alpha\beta\gamma}\hat{g}_{\gamma\delta} \\
&= \hat{g}_{\alpha\beta}\hat{g}_{\beta\delta}\epsilon_{\beta\gamma\delta} \\
&= \hat{g}_{\alpha\delta}\epsilon_{\alpha\beta\delta}\epsilon_{\beta\gamma\delta}.
\end{aligned}$$

Therefore $\epsilon_{\alpha\gamma\delta}\,\epsilon_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\delta}\,\epsilon_{\beta\gamma\delta}$ which rearranges to give the cocycle condition $(\delta\epsilon)_{\alpha\beta\gamma\delta} = 1$. Therefore ϵ is a cocycle and defines a class $[\epsilon]$ in $H^2_{\check{\text{C}}\text{ech}}(M, \mathcal{A})$ which one can show is independent of the choices made. This class measures the obstruction to lifting P to a $\hat{\mathcal{G}}$ bundle. That is, $[\epsilon] = 0$ if and only if there is a lift of P.

We now describe an important example of a bundle gerbe associated to a principal $\mathcal{G}$-bundle $P \to M$ and a central extension such as (5.1). There is a map $\tau : P^{[2]} \to \mathcal{G}$ defined by $p_1\tau(p_1, p_2) = p_2$ satisfying

$$\tau(p_1, p_2)\tau(p_2, p_3) = \tau(p_1, p_3). \tag{5.2}$$

It can be shown via (5.2) that $\tau^*\hat{\mathcal{G}} \to P^{[2]}$ has a natural product making $(\tau^*\hat{\mathcal{G}}, P, M)$ a bundle gerbe. We depict this as



We have the following important result.

Proposition 5.1 ([27]). *The lifting bundle gerbe has Dixmier-Douady class precisely the obstruction to lifting the bundle $P \to M$. So the lifting bundle gerbe is trivial if and only if the $\mathcal{G}$ bundle $P \to M$ lifts to $\hat{\mathcal{G}}$.*

Example 5.2. *Let Σ be a compact manifold and G be a Lie group. Let $\mathcal{G} = \mathrm{Map}(\Sigma, G)$. If $b_1(\Sigma) = 0$ then by the discussion in section 4.2 the central extension $\hat{\mathcal{G}}$ is topologically trivial. Therefore the lifting bundle gerbe is trivial and the $\mathcal{G}$-bundle lifts to a $\hat{\mathcal{G}}$-bundle.*

5.2 Connection and curvature

Let Σ be a compact manifold and G a Lie group and define $\mathcal{G} = \mathrm{Map}(\Sigma, G)$. Consider the central extension

$$1 \to \mathcal{A} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

of $\mathcal{G}$ by $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$, as constructed in section 4.2.

Let Π be the projection map $\Omega^1(\Sigma) \to \mathrm{Lie}(\mathcal{A}) = \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}$. Recall we defined $\alpha \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2_\Lambda(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ in section 4.2 to be

$$\alpha((g_1, g_2); (g_1 X_1, g_2 X_2)) = \frac{i}{2\pi}\Pi(\mathrm{tr}(X_1 dg_2 g_2^{-1}))$$

and

$$R(g; gX, gY) = \frac{i}{4\pi}\Pi(\mathrm{tr}(X dY)).$$

We think of $\hat{\mathcal{G}}$ as a principal $\mathcal{A}$ bundle over $\mathcal{G}$ and we let μ be a left invariant connection. It turns out the pullback connection $\tau^*\mu$ is not a bundle gerbe connection as it does not preserve the bundle gerbe multiplication. Consider the following commutative diagram

$$\begin{array}{ccc}
\delta(\tau^*(\widehat{\mathcal{G}})) & \longrightarrow & \delta(\hat{\mathcal{G}}) \\
\downarrow\uparrow s & & \downarrow\uparrow s \\
P^{[3]} & \xrightarrow{\bar{\tau}} & \mathcal{G}^2
\end{array}$$

with $s : \mathcal{G}^2 \to \delta(\hat{\mathcal{G}})$ defined in section 3.5. Let $\beta = \tau^*\alpha$. We have

$$\begin{aligned}
s^*\delta(\tau^*\mu) &= s^*\tau^*(\delta\mu) \\
&= \tau^* s^*(\delta\mu) \\
&= \tau^*(\alpha) \\
&= \beta,
\end{aligned}$$

where $\bar{\tau} : P^{[3]} \to \mathcal{G}^2$ is defined by

$$\bar{\tau}(p_1, p_2, p_3) = (\tau_{12}(p_1, p_2, p_3), \tau_{23}(p_1, p_2, p_3)) = (\tau(p_1, p_2), \tau(p_2, p_3)).$$

So we have

$$\beta = \frac{i}{2\pi}\Pi(\mathrm{tr}(\tau_{13}^* A - \mathrm{ad}(\tau_{12}^*)\tau_{23}^* A \, \mathrm{d}\tau_{23} \, \tau_{23}^{-1})),$$

with A being the connection 1-form on $P \to M$. By letting $g_{12} = \tau(p_1, p_2)$, it turns out

$$\begin{aligned}\epsilon((p_1, p_2), (X_1, X_2)) &= \alpha((1, g_{12}), (A(X_1, 0))) \\ &= \frac{i}{2\pi}\Pi(\mathrm{tr}(A(X_1) d_\Sigma(g_{12}) g_{12}^{-1})) \\ &= \frac{i}{2\pi}\Pi(\mathrm{tr}(\pi_1^*(A) d_\Sigma(g_{12}) g_{12}^{-1}))\end{aligned}$$

solves $\delta(\epsilon) = \beta$. Therefore $\tau^*\mu - \epsilon$ is a bundle gerbe connection with curvature

$$R(1; A(X_2), A(X_2')) - R(1; A(X_1), A(X_1')) + \alpha((1, g_{12}), (F(X_1, X_1'), 0))$$

$$= \delta(R)((1, A); (X_1, Y_1), (X_2, Y_2)) + \alpha((1, g_{12}), (F(X_1, X_1'), 0)).$$

We would like to write $\alpha((1, g_{12}), (F(X_1, X_1'), 0)) = \delta(f)$ for some $f \in \Omega^2(P, \Omega^1(\Sigma)/d\Omega^0(\Sigma))$.

For this we would need to generalise the notion of a Higgs field, which was used for the case of loop groups in [27] to the case of $\mathrm{Map}(\Sigma, G)$.

Definition 5.3. *Let P be a principal $\mathcal{G}$ bundle over a manifold M. A map $\Phi : P \to \Omega^1(\Sigma, \mathfrak{g})$ is said to be a Higgs field if*

$$\Phi(pg) = ad(g^{-1})\Phi(p) + g^{-1} d_\Sigma g$$

for all $p \in P$ and $g \in \mathcal{G}$.

Proposition 5.4. *Higgs fields exist.*

Proof. Let $P = \Sigma \times \mathcal{G}$ be a trivial $\mathcal{G}$ bundle over Σ. We can define $\Phi : P \to \Omega^1(\Sigma, \mathrm{ad}(P))$ by

$$\Phi(m, g) = g^{-1} d_\Sigma g.$$

Then for $p = (m, g) \in P$ we have

$$\begin{aligned}\Phi(ph) &= \Phi(mh, gh) \\ &= (gh)^{-1} d_\Sigma(gh) \\ &= h^{-1} g^{-1}(g d_\Sigma(h) + d_\Sigma(g) h) \\ &= h^{-1} d_\Sigma(h) + h^{-1} g^{-1} d_\Sigma(g) h \\ &= h^{-1} d_\Sigma h + \mathrm{ad} h^{-1}(g^{-1} d_\Sigma g) \\ &= h^{-1} d_\Sigma h + \mathrm{ad} h^{-1} \Phi(m, g) \\ &= \mathrm{ad} h^{-1} \Phi(p) + h^{-1} d_\Sigma h.\end{aligned}$$

Therefore Φ is a Higgs field. If P is nontrivial, we can choose an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Σ which trivialises P. So by the previous calculations there exists a Higgs field Φ_i on U_i for every $i \in I$. By choosing a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\mathcal{U}$ and defining

$$\Phi(p) = \sum_{i \in I} \rho_i \Phi_i(p)$$

we have

$$\begin{aligned}\Phi(pg) &= \sum_{i \in I} \rho_i \Phi_i(pg) \\ &= \sum_{i \in I} \rho_i (\mathrm{ad} g^{-1} \Phi_i(p) + g^{-1} d_\Sigma g) \\ &= \sum_{i \in I} \mathrm{ad} g^{-1}(\rho_i \Phi_i(p)) + \sum_{i \in I} (\rho_i g^{-1} d_\Sigma g) \\ &= \mathrm{ad} g^{-1} \sum_{i \in I} (\rho_i \Phi_i(p)) + g^{-1} d_\Sigma g \sum_{i \in I} \rho_i \\ &= \mathrm{ad} g^{-1} \Phi(p) + g^{-1} d_\Sigma g.\end{aligned}$$

Therefore Φ is a Higgs field on Σ. □

We now choose a Higgs field $\Phi : P \to \Omega^1(\Sigma, \mathfrak{g})$ such that

$$\Phi(pg) = \mathrm{ad}(g^{-1}) \Phi(p) + g^{-1} d_\Sigma g.$$

Proposition 5.5. *If*
$$f = \frac{1}{2}\Pi(\mathrm{tr}(Ad_\Sigma A)) - \Pi(\mathrm{tr}(F\Phi)),$$
then the 3-curvature is given by $df = -\Pi(\mathrm{tr}(FD_A\Phi))$ *for*
$$D_A\Phi = d\Phi + [A, \Phi] - d_\Sigma A \in \Omega^1(P, \Omega^1(\Sigma, \mathfrak{g})).$$

Proof. Using the identities $F = dA + [A, A]$ and $dF = [F, A]$ we have

$$\begin{aligned}
df &= d\left(\frac{1}{2}\Pi(\mathrm{tr}(Ad_\Sigma A)) - \Pi(\mathrm{tr}(F\Phi))\right) \\
&= \Pi\left(\mathrm{tr}\left(\frac{1}{2}dAd_\Sigma A - \frac{1}{2}Add_\Sigma A - dF\Phi - Fd\Phi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(\frac{1}{2}dAd_\Sigma A - \frac{1}{2}Add_\Sigma A - [F, A]\Phi - Fd\Phi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(\frac{1}{2}dAd_\Sigma A + \frac{1}{2}dAd_\Sigma A - [F, A]\Phi - Fd\Phi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(dAd_\Sigma A - F[A, \Phi] - Fd\Phi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(Fd_\Sigma A - [A, A]d_\Sigma A - F[A, \Phi] - Fd\Phi\right)\right)
\end{aligned}$$

But
$$\begin{aligned}
\Pi(\mathrm{tr}([A, A]d_\Sigma A)) &= \frac{1}{3}\Pi\Big(\mathrm{tr}\Big([A(X), A(Y)]d_\Sigma(A(Z)) \\
&\qquad + [A(Y), A(Z)]d_\Sigma(A(X)) \\
&\qquad + [A(Z), A(X)]d_\Sigma(A(Y))\Big)\Big) \\
&= \frac{1}{3}\Pi\Big(\mathrm{tr}\Big(A(X)A(Y)d_\Sigma(A(Z)) - A(Y)A(X)d_\Sigma(A(Z)) \\
&\qquad + A(Y)A(Z)d_\Sigma(A(X)) - A(Z)A(Y)d_\Sigma(A(X)) \\
&\qquad + A(Z)A(X)d_\Sigma(A(Y)) - A(X)A(Z)d_\Sigma(A(Y))\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\Pi\Big(\mathrm{tr}\Big(- d_\Sigma(A(X))A(Y)A(Z) - A(X)d_\Sigma(A(Y))A(Z) \\
&\qquad\qquad + d_\Sigma(A(Y))A(X)A(Z) + A(Y)d_\Sigma(A(X))A(Z) \\
&\qquad\qquad - d_\Sigma(A(Y))A(Z)A(X) - A(Y)d_\Sigma(A(Z))A(X) \\
&\qquad\qquad + d_\Sigma(A(Z))A(Y)A(X) + A(Z)d_\Sigma(A(Y))A(X) \\
&\qquad\qquad - d_\Sigma(A(Z))A(X)A(Y) - A(Z)d_\Sigma(A(X))A(Y) \\
&\qquad\qquad + d_\Sigma(A(X))A(Z)A(Y) + A(X)d_\Sigma(A(Z))A(Y)\Big)\Big) \\
&= \frac{1}{3}\Pi\Big(\mathrm{tr}\Big(- d_\Sigma(A(X))[A(Y), A(Z)] - d_\Sigma(A(Y))[A(Z), A(X)] \\
&\qquad\qquad - d_\Sigma(A(Y))[A(Z), A(X)] - d_\Sigma(A(Z))[A(X), A(Y)] \\
&\qquad\qquad - d_\Sigma(A(Z))[A(X), A(Y)] - d_\Sigma(A(X))[A(Y), A(Z)]\Big)\Big) \\
&= -\frac{2}{3}\Pi\big(\mathrm{tr}\big(d_\Sigma(A(X))[A(Y), A(Z)] + d_\Sigma(A(Y))[A(Z), A(X)] + d_\Sigma(A(Z))[A(X), A(Y)]\big)\big) \\
&= -2\mathrm{tr}([A, A]d_\Sigma A).
\end{aligned}
$$

Therefore $\mathrm{tr}([A, A]d_\Sigma A) = 0$, implying the 3-curvature is given by

$$
\begin{aligned}
df &= \Pi\left(\mathrm{tr}\left(F d_\Sigma A - [A, A]d_\Sigma A - F[A, \Phi] - F d\Phi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(F d_\Sigma A - F[A, \Phi] - F d\Phi\right)\right) \\
&= -\Pi(\mathrm{tr}(F D_A \Phi)).
\end{aligned}
$$

□

Remark 5.6. *The calculations above establish theorem 1.2 in the introduction. Furthermore, it generalises to the $\mathcal{G} = \mathrm{Aut}(Q)$ case by defining a Higgs field $\Phi : P \to \Omega^1(\Sigma, \mathrm{ad}\, Q)$ and*

$$D_A\Phi = d\Phi + [A, \Phi] - \nabla_\Sigma A.$$

5.3 The loop group case

Suppose G is a compact, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathcal{G} = LG = C^\infty(S^1, G)$. Let P be a principal $\mathcal{G}$ bundle over a manifold M with connection $A \in \Omega^1(P, \mathrm{Lie}(\mathcal{G}))$ and curvature $F \in \Omega^2(P, \mathrm{Lie}(\mathcal{G}))$. We have $F = dA + \frac{1}{2}[A, A]$. It can be shown [27] that out the curvature of the bundle gerbe connection is $\tau^* R - d\epsilon \in \Omega^2(P^{[2]})$, where $R \in \Omega^2(\mathcal{G})$ and $\epsilon \in \Omega^1(P^{[2]})$. This can be expressed as

$$\delta\left(\frac{i}{4\pi}\int_{S^1}\langle A, \partial_\theta A\rangle d\theta\right) - \frac{i}{2\pi}\int_{S^1}\langle \pi_2^*(F), \tau^*(z)\rangle d\theta$$

where $z : \mathcal{G} \to \mathrm{Lie}(\mathcal{G})$ is defined by $z(k) = (\partial_\theta k)k^{-1}$. We seek f satisfying

$$\delta(f) = \frac{i}{2\pi}\int_{S^1}\langle \pi_2^*(F), \tau^*(z)\rangle d\theta.$$

Let $\Phi : P \to P\mathfrak{g} = C^\infty([0, 2\pi], \mathfrak{g})$ and suppose

$$\Phi(pg) = \mathrm{ad}g^{-1}\Phi(p) + g^{-1}\frac{\partial g}{\partial \theta}.$$

We have $\mathrm{ad}(\tau)(\pi_1{}^*\Phi) = \pi_2{}^*(\Phi) + \tau^*(z)$. So

$$\begin{aligned}
\pi_2{}^*(\Phi)(p_1, p_2) &= \Phi(p_1) \\
&= \Phi(p_2\tau^{-1}) \\
&= \mathrm{ad}\tau\Phi(p_2) + \tau\frac{\partial \tau^{-1}}{\partial \theta} \\
&= \mathrm{ad}(\tau)\pi_1{}^*\Phi + \tau\tau^{-1}\frac{\partial \tau}{\partial \theta}\tau^{-1} \\
&= \mathrm{ad}(\tau)\pi_1{}^*\Phi + \frac{\partial \tau}{\partial \theta}\tau^{-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{i}{2\pi}\int_{S^1}\langle {\pi_2}^*(F),\tau^*(z)\rangle d\theta &= \frac{i}{2\pi}\int_{S^1}\langle \mathrm{ad}(\tau^{-1})\pi_2^*(F),\pi_1^*\Phi\rangle d\theta - \frac{i}{2\pi}\int_{S^1}\langle {\pi_2}^*(F),{\pi_2}^*(\Phi)\rangle d\theta \\ &= \frac{i}{2\pi}\int_{S^1}\langle \pi_1^*(F),\pi_1^*\Phi\rangle d\theta - \frac{i}{2\pi}\int_{S^1}\langle {\pi_2}^*(F),{\pi_2}^*(\Phi)\rangle d\theta \\ &= \pi_1^*\left(\frac{i}{2\pi}\int_{S^1}\langle F,\Phi\rangle d\theta\right) - \pi_2^*\left(\frac{i}{2\pi}\int_{S^1}\langle F,\Phi\rangle d\theta\right) \\ &= \delta\left(\frac{i}{2\pi}\int_{S^1}\langle F,\Phi\rangle d\theta\right).\end{aligned}$$

So the curvature of the lifting bundle gerbe connection for the loop group case is

$$\delta\left(\frac{i}{2\pi}\int_{S^1}\left[\frac{1}{2}\langle A,\partial_\theta A\rangle - \langle F,\Phi\rangle\right]d\theta\right).$$

5.4 Reduced splitting

In this section we recall the notion of reduced splitting due to Gomi [9] and use it to find an expression for the curvature of a bundle gerbe connection.

Let Σ be a compact manifold and G a Lie group and define $\mathcal{G} = \mathrm{Map}(\Sigma, G)$. Consider the central extension

$$1 \to \mathcal{A} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

of $\mathcal{G}$ by $\mathcal{A} = \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma)$ as constructed in section 4.2. Let Π be the projection map $\Omega^1(\Sigma) \to \frac{\Omega^1(\Sigma)}{d\Omega^0(\Sigma)}$. Recall from section 4.2, $\alpha \in \Omega^1(\mathcal{G}\times\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ defined by

$$\alpha((g_1,g_2);(g_1X_1,g_2X_2)) = \Pi(\mathrm{tr}(X_1 d_\Sigma g_2 {g_2}^{-1}))$$

and

$$R(g; gX, gY) = \Pi(\mathrm{tr}(X d_\Sigma Y)).$$

Let $Y \to M$ be a principal bundle with an associated lifting bundle gerbe $P \to Y^{[2]}$. Let μ be a left invariant connection on $\hat{\mathcal{G}}$, thought as a principal $\mathcal{A}$ bundle over $\mathcal{G}$. From section 5.2 we know $\tau^*\mu - \epsilon$ is a bundle gerbe connection with curvature

$$(1; A(X_2), A(X_2')) - R(1; A(X_1), A(X_1')) + \alpha((1,g_{12}),(F(X_1,X_1'),0))$$

$$= \delta(R)((1, A); (X_1, Y_1), (X_2, Y_2)) + \alpha((1, g_{12}), (F(X_1, X_1'), 0)).$$

We would like to write $\alpha((1, g_{12}), (F(X_1, X_1'), 0)) = \delta(f)$ for some $f \in \Omega^2(P, \Omega^1(\Sigma)/d\Omega^0(\Sigma))$.

Definition 5.7 ([9], Definition 3.10). *Let Σ be a compact manifold and G a Lie group and define $\mathcal{G} = \mathrm{Map}(\Sigma, G)$. We define the* group cocycle $Z : \mathcal{G} \times \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{A})$ *by*

$$Z(g, X) = \mathrm{Ad}\hat{g}(\hat{X}) - \widehat{\mathrm{Ad}g(X)}.$$

Proposition 5.8 (cf. [27]). *The group cocycle $Z : \mathcal{G} \times \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{A})$ satisfies*

$$Z(g^{-1}, X) = -\alpha((1, g); (X, 0)) = -\Pi(\mathrm{tr}(X d_\Sigma g g^{-1})).$$

Definition 5.9 ([9], Definition 3.12). *A* reduced splitting *for a principal $\mathcal{G}$ bundle $P \to M$ is a map $\ell : P \times \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{A})$ which is linear in the second factor and satisfies*

$$\ell(p, X) = \ell(pg, Ad(g^{-1})(X)) + Z(g^{-1}, X).$$

Recall from section 4.3 that for a compact manifold Σ a Lie group G and a G-bundle Q over Σ, the gauge group of Q is given by

$$\mathcal{G} = \{f : Q \to G \mid f(qg) = g^{-1} f(q) g\}.$$

Then the central extension of $\mathcal{G}$ by $\mathcal{A} = \Omega^1_{\mathbb{Z}}(\Sigma)$ is constructed by $\alpha \in \Omega^1(\mathcal{G} \times \mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2(\mathcal{G}, \mathrm{Lie}(\mathcal{A}))$ given by

$$\alpha((g_1, g_2); (g_1 X_1, g_2 X_2)) = \Pi(\mathrm{tr}(X_1 (\nabla_\Sigma g_2) g_2^{-1}))$$

and

$$R(g; gX, gY) = \Pi(\mathrm{tr}(X \nabla_\Sigma Y)),$$

where ∇_Σ is a chosen connection on the G-bundle Q over Σ and $\nabla_\Sigma Y \in \Omega^1(Q, \mathfrak{g})$ is defined by

$$\nabla_\Sigma Y(\xi) = dY(h\xi).$$

Here we give an explicit formula for the reduced splitting associated with this central extension via the Higgs field.

Proposition 5.10. *Let $\Phi : Y \to \Omega^1(\Sigma, \mathfrak{g})$ be a Higgs field such that*

$$\Phi(pg) = Ad(g^{-1})\Phi(p) + g^{-1}\nabla_\Sigma g.$$

Then the map $\ell : P \times \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{A})$ defined by

$$\ell(p, \xi) = \Pi(\mathrm{tr}(\Phi(p)\xi))$$

is a reduced splitting.

Proof. We have

$$\begin{aligned}
\ell(pg, Ad(g^{-1})(\xi)) &= \Pi\left(\mathrm{tr}\left(\Phi(pg)\mathrm{Ad}(g^{-1})\xi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left((\mathrm{Ad}(g^{-1})\Phi(p) + g^{-1}\nabla_\Sigma g)\mathrm{Ad}(g^{-1})\xi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(\mathrm{Ad}(g^{-1})\Phi(p)\mathrm{Ad}(g^{-1})\xi\right)\right) + \Pi\left(\mathrm{tr}\left(g^{-1}\nabla_\Sigma g\mathrm{Ad}(g^{-1})\xi\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(g^{-1}\Phi(p)gg^{-1}\xi g\right)\right) + \Pi\left(\mathrm{tr}\left(g^{-1}(\nabla_\Sigma g)g^{-1}\xi g\right)\right) \\
&= \Pi\left(\mathrm{tr}\left(\Phi(p)\xi\right)\right) + \Pi\left(\mathrm{tr}\left((\nabla_\Sigma g)g^{-1}\xi\right)\right) \\
&= \ell(p,\xi) - Z(g^{-1},\xi).
\end{aligned}$$

□

Chapter 6

Conclusion

6.1 The topology of Σ

As mentioned earlier if $b_1(\Sigma) = 0$ then there must be a section $s : \mathcal{G} = \mathrm{Map}(\Sigma, G) \to \hat{\mathcal{G}}$ and therefore the central extension $\hat{\mathcal{G}}$ is topologically trivial. To know whether $\hat{\mathcal{G}}$ is algebraically trivial a possible approach would be to seek a 1-form ρ such that $R = d\rho$ and $\alpha = \delta(\rho)$, where R and α are the differential forms used to construct $\hat{\mathcal{G}}$ (see section 3.5). Then theorem 3.23 implies that $\hat{\mathcal{G}}$ is algebraically trivial.

6.2 The topology of $\mathcal{G}$

Throughout this thesis we assumed $\mathcal{G}$ to be 1-connected and constructed its central extension based on this assumption. If $\mathcal{G}$ is not connected then we can take the identity component $\mathcal{G}_0$ of $\mathcal{G}$. As $\mathcal{G}_0$ is connected it has a universal covering space $\widetilde{\mathcal{G}_0}$ and we have the following short exact sequence

$$1 \to \pi_1(\mathcal{G}_0) \to \widetilde{\mathcal{G}_0} \xrightarrow{\pi} \mathcal{G}_0 \to 1,$$

where $\pi_1(\mathcal{G}_0)$ is the first fundamental group of $\mathcal{G}_0$. Suppose $\alpha \in \Omega^1(\mathcal{G}_0 \times \mathcal{G}_0, \mathrm{Lie}(\mathcal{A}))$ and $R \in \Omega^2_\Lambda(\mathcal{G}_0, \mathrm{Lie}(\mathcal{A}))$ satisfy

$$\delta(\alpha) = 0, \quad dR = 0, \quad \text{and } \delta(R) = d(\alpha).$$

As $\widetilde{\mathcal{G}_0}$ is 1-connected, we believe our constructions apply to give a central extension

$$1 \to \mathcal{A} \to \widehat{\widetilde{\mathcal{G}_0}} \to \widetilde{\mathcal{G}_0} \to 1$$

determined by the differential forms $\tilde{R} = \pi^*(R)$ and $\tilde{\alpha} = \pi^*(\alpha)$. This is an interesting problem for further research.

6.3 The caloron correspondence

In [8], [11] and [27] a correspondence between LG-bundles over a manifold M and G-bundles over $M \times S^1$ was introduced.

$$\begin{array}{ccc} P & & Q \\ \pi_1 \downarrow LG & & \pi_2 \downarrow G \\ M & & M \times S^1 \end{array}$$

We review this result here and explain how it relates to our results. Suppose G is a 1-connected, compact Lie group and M a compact manifold and consider a principal LG-bundle $P \xrightarrow{\pi_1} M$. Let $\tilde{Q} = P \times G \times S^1$. We can define a map

$$LG \times \tilde{Q} \to \tilde{Q}$$

by

$$h(p, g, \theta) \mapsto (ph^{-1}, h(\theta)g, \theta).$$

For $h_1, h_2 \in LG$ we have

$$\begin{aligned} h_1(h_2(p, g, \theta)) &\mapsto h_1(ph_2^{-1}, h_2(\theta)g, \theta) \\ &= (ph_2^{-1}h_1^{-1}, h_1(\theta)h_2(\theta)g, \theta) \\ &= (p(h_1h_2)^{-1}, (h_1h_2)(\theta)g, \theta) \\ &= (h_1h_2)(p, g, \theta). \end{aligned}$$

Therefore this defines a left action on $\tilde{Q}$ by LG. We define $Q = \tilde{Q}/LG$ and denote the equivalence class of $(p, g, \theta) \in \tilde{Q}$ by $[p, g, \theta] \in Q$. We define a right action

$$Q \times G \to Q$$

by

$$([p, g, \theta], h) \mapsto [p, gh, \theta].$$

Define $\pi_2 : Q \to M \times S^1$ by

$$\pi_2([p, g, \theta]) = (\pi_1(p), \theta).$$

The fibres of this map are given by

$$\begin{aligned}\pi_2^{-1}(m, \theta) &= \{[p, g, \theta] \mid \pi_1(p) = m\}\\ &= \{[ph, g, \theta] \mid h \in G\}\\ &= \{[p, hg, \theta] \mid h \in G\}\\ &= \{[p, g, \theta]h \mid h \in G\}\\ &\simeq G.\end{aligned}$$

Therefore Q is a G-bundle over $M \times S^1$. We now start with a principal G-bundle Q over $M \times S^1$. For every $m \in M$, define

$$\begin{aligned}P_m &:= \Gamma(Q\mid_{\{m\}\times S^1})\\ &= \{s : \{m\} \times S^1 \to Q \mid \pi_2 \circ s = \mathrm{Id}\}\\ &= \{s : S^1 \to Q \mid p_2 \circ \pi_2 \circ s = \mathrm{Id}\},\end{aligned}$$

where $p_2 : M \times S^1 \to S^1$ is the projection map $p_2(m, \theta) = \theta$. This is acted on by LG, meaning we have constructed an LG-bundle $P \to M$.

By similar computations [11] a more general correspondence can be found; let Q be a principal G-bundle over a manifold Σ and define $\mathcal{G} = \mathrm{Aut}(Q)$. Assuming $\mathcal{G}$ is 1-connected we have the following caloron correspondence.

$$\begin{array}{ccc} P & \qquad & \tilde{P} = (P \times Q)/\mathcal{G} \\ {\scriptstyle \pi_1}\big\downarrow{\scriptstyle \mathcal{G}} & & {\scriptstyle \pi_2}\big\downarrow{\scriptstyle G} \\ M & & M \times \Sigma \end{array}$$

The $\mathcal{G}$-bundle P corresponds to a bundle gerbe over M with an associated Dixmier-Douady class in $H^2(M, \Omega^1(\Sigma)/\Omega^1_{\mathbb{Z}}(\Sigma))$. Considerations similar to those in section 4.2 suggest this can be mapped to $H^3(M, \mathbb{Z}) \otimes H^1(\Sigma, \mathbb{Z})$. The G bundle $\tilde{P}$ on the other hand gives rise to a class in $H^4(M \times \Sigma, \mathbb{Z})$. By the Kunneth formula for cohomology we have

$$\begin{aligned}H^4(M \times \Sigma, \mathbb{Z}) = {} & H^0(M, \mathbb{Z}) \otimes H^4(\Sigma, \mathbb{Z}) + H^1(M, \mathbb{Z}) \otimes H^3(\Sigma, \mathbb{Z})\\ &+ H^2(M, \mathbb{Z}) \otimes H^2(\Sigma, \mathbb{Z}) + H^3(M, \mathbb{Z}) \otimes H^1(\Sigma, \mathbb{Z})\\ &+ H^4(M, \mathbb{Z}) \otimes H^0(\Sigma, \mathbb{Z}).\end{aligned}$$

The first term on the right hand side corresponds to the Pontryagin class of Q. The second last term should arise from the Dixmier-Douady class of P. The last term is the Pontryagin class of the fibres $\tilde{P}\mid_{M\times\{x\}}$ of $\tilde{P}$ over $M \times \{x\} \simeq M$. Further research is required to determine a geometric interpretation of the other two terms.

Appendix A

Infinite-dimensional manifolds

Most manifolds studied throughout this thesis (loop group, smooth maps from a manifold to a Lie group, etc) are infinite-dimensional. These manifolds belong to a larger family of spaces, namely Fréchet manifolds. Here we briefly discuss these manifolds as introduced in [10].

A.1 Fréchet spaces

Definition A.1. *A seminorm on a real vector space V is a map $\|\cdot\| : V \to \mathbb{R}$ such that*

- $\|v\| \geq 0$ *for all* $v \in V$,
- $\|v + w\| \leq \|v\| + \|w\|$ *for all* $v, w \in V$,
- $\|av\| = |a| \cdot \|v\|$ *for all* $v \in V$ *and* $a \in \mathbb{R}$.

Definition A.2. *A locally convex topological vector space is a vector space with a topology which arises from some collection of seminorms* $\{\|\cdot\|_i\}_{i \in I}$.

A locally convex topological vector space is Hausdorff if and only if $v = 0$ is equivalent to $\|v\|_i = 0$ for all $i \in I$. The topology is metrisable if and only if it may be defined by a countable collection of seminorms. The space V is *complete* if every Cauchy sequence v_i converges.

Definition A.3. *A Fréchet space is a complete Hausdorff metrisable locally convex topological vector space.*

Example A.4. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the space* $\mathrm{Map}([a,b],\mathbb{R})$ *of smooth maps on $[a,b]$ equipped with seminorms*

$$\|f\|_n = \sum_{i=0}^{n} \sup_{x\in[a,b]} \left|\frac{d^i}{dx^i} f(x)\right|$$

is a Fréchet space.

Let F be a Fréchet space and $G \subset F$ a closed subset of F. Then G and F/G are both Fréchet spaces. The direct sum $F_1 \oplus F_2$ of two Fréchet spaces F_1 and F_2 is also a Fréchet space.

A.2 Differentiation

Definition A.5. *Let F and G be Fréchet spaces, $U \subset F$ an open subset of F and $P : U \to G$ a continuous nonlinear map. The derivative of P at $f \in U$ in the direction $h \in F$ is defined by*

$$DP(f)h = \lim_{t\to 0} \frac{P(f+th) - P(f)}{t}.$$

Furthermore, P is said to be continuously differentiable (or C^1) on U if the above limit exists for all $(f,h) \in U \times F$ and if $DP : U \times F \to G$ is continuous.

The directional derivative on Fréchet spaces behaves in the usual way with the sum, product or composition of functions.

We can define the higher derivatives recursively: for F and G be Fréchet spaces, $U \subset F$ open and $P : U \to G$ a continuous nonlinear map, the nth derivative of P at $f \in U$ in the direction $\{h_1, \ldots, h_n\} \in F^n$ is denoted by $D^n P(f)\{h_1, \ldots, h_n\}$ and is defined to be

$$\lim_{t\to 0} \frac{D^{n-1}P(f+th_n)\{h_1,\ldots,h_{n-1}\} - D^{n-1}P(f)\{h_1,\ldots,h_{n-1}\}}{t}.$$

Similar to the previous case P is said to be C^n on U if the $D^n P : U \times F^n \to G$ exists and is continuous. We say a map is smooth or C^∞ if it is C^n for all n.

A.3 Fréchet manifolds

Definition A.6. *A Fréchet manifold is a Hausdorff topological space with an atlas of coordinate charts taking their value in Fréchet spaces, such that the coordinate transition* *functions are all smooth maps between Fréchet spaces.*

Example A.7. *Let X and Y be manifolds with X compact. The space* $\mathrm{Map}(X, Y)$ *of all smooth maps of X into Y forms a Fréchet manifold.*

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