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## INVARIANT PROLONGATION OF THE KILLING TENSOR EQUATION

A. ROD GOVER AND THOMAS LEISTNER

ABSTRACT. The Killing tensor equation is a first order differential equation on symmetric covariant tensors that generalises to higher rank the usual Killing vector equation on Riemannian manifolds. We view this more generally as an equation on any manifold equipped with an affine connection, and in this setting derive its prolongation to a linear connection. This connection has the property that parallel sections are in 1-1 correspondence with solutions of the Killing equation. Moreover this connection is projectively invariant and is derived entirely using the projectively invariant tractor calculus which reveals also further invariant structures linked to the prolongation.

## 1. INTRODUCTION

On a Riemannian manifold  $(M, g)$  a tangent vector field  $k \in \mathfrak{X}(M)$  is an infinitesimal automorphism (or symmetry) if the Lie derivative of the metric  $g$  in direction of  $k$  vanishes. In terms of the Levi-Civita connection  $\nabla = \nabla^g$ , this may be written as

$$(1) \quad \nabla_{(a} k_{b)} = 0$$

where we use an obvious abstract index notation,  $k_a = g_{ab}k^b$ , and the  $(ab)$  indicates symmetrisation over the enclosed indices. This *Killing equation* is generalised to higher rank  $r \geq 1$  by the *Killing tensor equation*

$$(2) \quad \nabla_{(a} k_{b \dots c)} = 0$$

where  $k_{b \dots c}$  is a symmetric tensor, that is  $k \in \Gamma(S^r T^*M)$  and again  $(ab \dots c)$  indicates symmetrisation over the enclosed indices. Solutions of this, so-called *Killing tensors*, are important for treatment of separation of variables [2, 25, 30, 33], higher symmetries of the Laplacian and similar operators [1, 14, 16, 22, 28, 29], and for the theory of integrable systems, and superintegrability [11, 15, 13, 27, 26]. Partly these applications arise because a solution of (2) (for any  $r$ ) provides a first integral along geodesics: if  $\gamma : I \rightarrow M$  is a geodesic (where  $I \subset \mathbb{R}$  is an interval) and  $u := \dot{\gamma}$  is the velocity of this then  $\nabla_u u = 0$  and therefore by dint of (2) the function  $k_{b \dots c} u^b \dots u^c$  is constant along  $\gamma$ .

In dimensions  $n \geq 2$  (which we assume throughout) the equation (2) is an overdetermined finite type linear partial differential equation. This means, in particular, that it is equivalent to a linear connection on a system that involves the Killing tensor  $k$  but also additional variables, the *prolonged system* [4, 34]. For example for equation (1) above this prolonged system is very easily found to be

$$(3) \quad \bar{\nabla}_a \begin{pmatrix} k_c \\ \mu_{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a k_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a k_d \end{pmatrix}$$

where  $R_{bc}{}^d{}_a$  is the curvature of  $\nabla$  (see Section 4.2 below). In general such prolonged systems are not unique, but for any such connection its parallel sections correspond 1-1 with solutions of the original equation ((2) in this case). Thus, on connected manifolds, the rank of the prolonged systems gives an upper bound on the dimension of the space of solutions and curvature of the given connection can lead to obstructions to solving the equation, see e.g. [5, 20, 21].

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Two affine connections  $\nabla$  and  $\nabla'$  are said to be *projectively equivalent* if they share the same unparametrised geodesics. Connections differing only by torsion are projectively related, and we will lose no generality in our work here if we restrict to torsion free connections, which we do henceforth. An equivalence class of  $\mathbf{p} = [\nabla]$  of such projectively related torsion-free connections is called a *projective structure* and a manifold  $M^{n \geq 2}$  equipped with such a structure is called a *projective manifold*. An important but not fully exploited feature of the equation (2) is that it is *projectively invariant*. This will be explained fully in Section 2.2, but at this stage it will suffice to say the following. First when we introduced (2) above,  $\nabla$  denoted the Levi-Civita connection of a metric, but the equation makes sense and is important for any affine connection  $\nabla$ , and it is in this setting that we now study it. Next the projective invariance means that the equation (2) has a certain insensitivity and, in particular, descends to a well defined equation on a projective manifold  $(M, \mathbf{p})$ .

On a general projective manifold  $(M, \mathbf{p})$  there is no distinguished affine connection on  $TM$ . However there is a distinguished projectively invariant connection  $\nabla^{\mathcal{T}}$  on a vector bundle  $\mathcal{T}$  that extends (a density twisting of) the tangent bundle  $TM$ :

$$(4) \quad 0 \rightarrow \mathcal{E}(-1) \xrightarrow{X} \mathcal{T} \rightarrow TM \otimes \mathcal{E}(-1) \rightarrow 0$$

where  $\mathcal{E}(-1)$  is a natural real oriented line bundle defined in Section 2 below. This is the normal projective tractor connection and it (or the equivalent Cartan connection) provides the basic tool for invariant calculus on projective manifolds. An important feature of this connection is that it is on a low rank bundle (i.e.  $\dim(TM) + 1$ ) that is simply related to the tangent bundle. The tractor calculus is recalled in Section 2.2.

For most applications that one can imagine it makes sense then to seek a prolongation of (2) that is itself a projectively invariant connection. For example, if this can be found, then its curvature simultaneously constrains solutions for the entire class of projectively related connections. In fact such a connection exists. The equations (2) is an example of a first BGG equation and arises as a special case of the very general theory of Hammerl et al. in [24] (see also [23]). That theory describes an algorithm for producing an invariant connection giving the prolonged system for any of the large class of BGG equations (and we refer the reader to that source for the meaning of these terms) and in this sense is very powerful. Although the algorithm of [24] produces in the end an invariant connection it proceeds through stages that break the invariance of the given equation. For example in treating (2) the steps of the algorithm are not projectively invariant. Moreover beyond the case of rank 1 the explicit treatment of (2) using this algorithm seems practically intractable due to the number of steps involved. Finally although the construction of [24] is strongly linked to the calculus of the normal tractor connection (of [3, 6, 10]) the connection finally obtained is not easily linked to the normal tractor connection.

The aim of this article is to produce an alternative invariant prolongation procedure that is simple, conceptual, explicit, and that reflects the invariance properties of the original equations. It is well known that for the projective BGG equations the normal tractor connection easily recovers the required prolongation in the case that the structure is projectively flat (i.e., the projective tractor/Cartan connection is flat). A motivation is to be able to produce the explicit curvature correction terms that modify the normal tractor connection to deal with general solutions on a projectively curved manifold. An explicit knowledge of these terms will enable us to deduce properties of the prolongation and so properties of solutions in general. We develop here a projectively invariant prolongation of the equation (2) for each  $r \geq 1$ . This uses at all stages the calculus of the normal projective tractor connection  $\nabla^{\mathcal{T}}$  (as in [3]). The result is a connection on a certain projective tractor bundle (a tensor part of a power of the dual  $\mathcal{T}^*$  to  $\mathcal{T}$ ) that differs from the normal tractor connection by the algebraic action of a tractor field that is projectively invariant and produced in a simple way from the curvature of the normal tractor connection and iterations of a projectively invariant operator on this. An advantage is that the construction and calculation uses projectively invariant tools, and at all stages the link to the very simple normal

tractor connection is manifest. As an immediate application this approach typically simplifies the computation of integrability conditions, see Remark 18 and in particular equation (56).

A tensorial approach to prolonging the Killing equation has been developed for arbitrary rank in [35] (see also [12]). Concerning our results for the projectively flat case in Section 3.1 there are necessarily some strong links to the prolongation approach of [29]. However our route to the prolongation is very different and it is this that is important for the development of the curved theory.

In fact there is considerable information in some of the preliminary results along the way in our treatment. For example each Killing equation is captured in the very simple tractor equation of Proposition 6. This is part of a rather general picture which suggests that the theory here should generalise considerably. (In fact aspects of our treatment here were inspired by the conformally invariant prolongation of the conformal Killing equation via tractors in [19, Proposition 2.2].) This will be taken up in subsequent works. The Proposition 6 also may be interpreted as showing that solutions of the Killing tensor equation on  $(M, \mathbf{p})$  correspond in a simple way to Killing tensors for the canonical affine connection on the Thomas cone over  $(M, \mathbf{p})$ ; the Thomas cone is discussed in e.g. [7, 10].

Throughout we use Penrose's abstract index notation. As mentioned above  $(ab \cdots c)$  indicates symmetrisation over the enclosed indices, while  $[ab \cdots c]$  indicates skewing over the enclosed indices. Then  $\mathcal{E}$  is used to denote the trivial bundle, and for example  $\mathcal{E}_{(abc)}$  is the bundle of covariant symmetric 3-tensors  $S^3 T^* M$ .

## 2. BACKGROUND

**2.1. Conventions for affine geometry.** Let  $(M, \nabla)$  be an affine manifold (of dimension  $n \geq 2$ ), meaning that  $\nabla$  is a torsion-free affine connection. The curvature

$$R_{ab}{}^c{}_d \in \Gamma(\Lambda^2 T^* M \otimes TM \otimes T^* M)$$

of the connection  $\nabla$  is given by

$$[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d, \quad v \in \Gamma(TM).$$

The Ricci curvature is defined by  $R_{bd} = R_{cb}{}^c{}_d$ .

On an affine manifold the trace-free part  $W_{ab}{}^c{}_d$  of the curvature  $R_{ab}{}^c{}_d$  is called the *projective Weyl curvature* and we have

$$(5) \quad R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta_{[a}^c P_{b]d} + \beta_{ab}\delta_d^c,$$

where  $\beta_{ab}$  is skew and  $P_{ab}$  is called the *projective Schouten tensor*. That  $W_{ab}{}^c{}_d$  is trace-free means exactly that  $W_{ab}{}^a{}_d = 0$  and  $W_{ab}{}^d{}_d = 0$ . Since  $\nabla$  is torsion-free the Bianchi symmetry  $R_{[ab}{}^c{}_d] = 0$  holds, whence

$$\beta_{ab} = -2P_{[ab]} \quad \text{and} \quad (n-1)P_{ab} = R_{ab} + \beta_{ab}.$$

As we shall see below the curvature decomposition (5) is useful in projective differential geometry.

First some further notation. On a smooth  $n$ -manifold  $M$  the bundle  $\mathcal{K} := (\Lambda^n TM)^2$  is an oriented line bundle and thus we can take correspondingly oriented roots of this. For projective geometry a convenient notation for these is as follows: given  $w \in \mathbb{R}$  we write

$$(6) \quad \mathcal{E}(w) := \mathcal{K}^{\frac{w}{2n+2}}.$$

Of course the affine connection  $\nabla$  acts on  $\Lambda^n TM$  and hence on the *projective density bundles*  $\mathcal{E}(w)$ . As a point of notation, given a vector bundle  $\mathcal{B}$  we often write  $\mathcal{B}(w)$  as a shorthand for  $\mathcal{B} \otimes \mathcal{E}(w)$ .

**2.2. Projective geometry and tractor calculus.** Two affine torsion-free connections  $\nabla'$  and  $\nabla$  are projectively equivalent, that is they share the same unparametrised geodesics, if and only if there some  $\Upsilon \in \Gamma(T^*M)$  s.t.

$$(7) \quad \nabla'_a v^b = \nabla_a v^b + \Upsilon_a v^b + \Upsilon_c v^c \delta_a^b$$

for all  $v \in \Gamma(T^*M)$ . This implies that on sections of  $\mathcal{E}(w)$  we have

$$\nabla'_a \tau = \nabla_a \tau + w \Upsilon_a \tau,$$

while on sections of  $T^*M$ ,

$$\nabla'_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a$$

It follows at once that on  $k_{a_1 \dots a_k} \in S^k T^*M(2r)$  we have

$$\nabla'_{(a_0} k_{a_1 \dots a_k)} = \nabla_{(a_0} k_{a_1 \dots a_k)}.$$

Thus for  $k \in S^k T^*M(2r)$  the Killing equation (2) is *projectively invariant* and descends to a well defined equation on  $(M, \mathbf{p})$ , where  $\mathbf{p} = [\nabla] = [\nabla']$ , the projective equivalence class of  $\nabla$ .

On a general projective  $n$ -manifold  $(M, \mathbf{p})$  there is no distinguished connection on  $TM$ . However there is a projectively invariant connection on a related rank  $(n+1)$  bundle  $\mathcal{T}$ . This is the projective tractor connection that we now describe.

Consider the first jet prolongation  $J^1 \mathcal{E}(1) \rightarrow M$  of the density bundle  $\mathcal{E}(1)$ . (See for example [31] for a general development of jet bundles.) There is a canonical bundle map called the *jet projection map*  $J^1 \mathcal{E}(1) \rightarrow \mathcal{E}(1)$ , which at each point is determined by the map from 1-jets of densities to simply their evaluation at that point, and this map has kernel  $T^*M(1)$ . We write  $\mathcal{T}^*$ , or an in an abstract index notation  $\mathcal{E}_A$ , for  $J^1 \mathcal{E}(1)$  and  $\mathcal{T}$  or  $\mathcal{E}^A$  for the dual vector bundle. Then we can view the jet projection as a canonical section  $X^A$  of the bundle  $\mathcal{E}^A(1)$ . Likewise, the inclusion of the kernel of this projection can be viewed as a canonical bundle map  $\mathcal{E}_a(1) \rightarrow \mathcal{E}_A$ , which we denote by  $Z_A^a$ . Thus the jet exact sequence (at 1-jets) is written in this notation as

$$(8) \quad 0 \longrightarrow \mathcal{E}_a(1) \xrightarrow{Z_A^a} \mathcal{E}_A \xrightarrow{X^A} \mathcal{E}(1) \longrightarrow 0.$$

We write  $\mathcal{E}_A = \mathcal{E}(1) \uplus \mathcal{E}_a(1)$  to summarise the composition structure in (8) and  $X^A \in \Gamma(\mathcal{E}^A(1))$ , as defined in (8), is called the *canonical tractor* or *position tractor*. Note the sequence (4) is simply the dual to (8).

As mentioned above, any connection  $\nabla \in \mathbf{p}$  determines a connection on  $\mathcal{E}(1)$ . On the other hand, by definition, a connection on  $\mathcal{E}(1)$  is precisely a splitting of the 1-jet sequence (8). Thus given such a choice we have the direct sum decomposition  $\mathcal{E}_A \stackrel{\nabla}{=} \mathcal{E}(1) \oplus \mathcal{E}_a(1)$  and we write

$$(9) \quad Y_A : \mathcal{E}(1) \rightarrow \mathcal{E}_A \quad \text{and} \quad W^A_a : \mathcal{E}_A \rightarrow \mathcal{E}_a(1),$$

for the bundle maps giving this splitting of (8); so

$$X^A Y_A = 1, \quad Z_A^b W^A_a = \delta_a^b, \quad \text{and} \quad Y_A W^A_a = 0.$$

By definition  $X$  and  $Z$  are projectively invariant. The formulae for how  $Y_A$  and  $W^A_a$  transform when  $\nabla$  is replaced by  $\nabla'$ , is in (7), is easily deduced and can be found in [3].

With respect to a splitting (9) we define a connection on  $\mathcal{T}^*$  by

$$(10) \quad \nabla_a^{\mathcal{T}^*} \begin{pmatrix} \sigma \\ \mu_b \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab} \sigma \end{pmatrix}.$$

Here  $P_{ab}$  is the projective Schouten tensor of  $\nabla \in \mathbf{p}$ , as introduced earlier. It turns out that (10) is independent of the choice  $\nabla \in \mathbf{p}$ , and so  $\nabla^{\mathcal{T}^*}$  is determined canonically by the projective structure  $\mathbf{p}$ . We have followed the construction of [3, 9], but as mentioned in those sources this *ctractor connection* is due to T.Y. Thomas. Thus we shall also term  $\mathcal{T}^* = \mathcal{E}_A$  the *ctractor bundle*, and we note the dual *tractor bundle*  $\mathcal{T} = \mathcal{E}^A$  has canonically the dual *tractor connection*: in terms of a splitting dual to that above this is given by

$$(11) \quad \nabla_a^{\mathcal{T}} \begin{pmatrix} \nu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \nu^b + \rho \delta_a^b \\ \nabla_a \rho - P_{ab} \nu^b \end{pmatrix}.$$

Note that given a choice of  $\nabla \in \mathfrak{p}$ , by coupling with the tractor connection we can differentiate tensors taking values in tractor bundles and also weighted tractors. In particular we have

$$(12) \quad \nabla_a X^B = W^B{}_a, \quad \nabla_a W^B{}_b = -P_{ab} X^A, \quad \nabla_a Y_B = P_{ab} Z_B^b, \quad \text{and} \quad \nabla_a Z_B^b = -\delta_a^b Y_B.$$

The curvature of the tractor connection is given by

$$(13) \quad \kappa_{ab}{}^C{}_D = W_{ab}{}^c{}_d W^C{}_c Z_D^d - C_{abd} Z_D^d X^C,$$

where  $W_{ab}{}^c{}_d$  is the projective Weyl curvature, as above, and

$$(14) \quad C_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac}$$

is called the *projective Cotton tensor*.

The projective Thomas-D operator is a first order projectively invariant differential operator, or more accurately family of such operators. Given any tractor bundle  $\mathcal{V}$  (including the trivial bundle  $\mathcal{E}$ ) and any  $w \in \mathbb{R}$  it provides an operator on the weighted tractor bundle  $\mathcal{V}(w)$

$$\mathbb{D} : \mathcal{V}(w) \rightarrow \mathcal{T}^* \otimes \mathcal{V}(w-1)$$

given by

$$(15) \quad \mathbb{D}_A V = w Y_A V + Z_A^a \nabla_a V,$$

where  $\nabla_a$  is the connection induced on the weighted bundle  $\mathcal{V}$  from the tractor connection  $\nabla_a^{\mathcal{T}^*}$  and the connection on  $\mathcal{E}(1)$  coming from a representative in  $\mathfrak{p}$ . Note that from this definition and (12) follows

$$(16) \quad \mathbb{D}_A X^B = \delta_A^B, \quad \text{and} \quad X^A \mathbb{D}_A V = w V,$$

for  $V \in \Gamma(\mathcal{V}(w))$ . Also from the definition it follows that  $\mathbb{D}$  satisfies a Leibniz rule, in that if  $\mathcal{U}(w)$  and  $\mathcal{V}(w')$  are tractor (or density) bundles of weights  $w$  and  $w'$ , respectively then for sections  $U \in \Gamma(\mathcal{U}(w))$  and  $V \in \mathcal{V}(w')$  we have

$$\mathbb{D}(U \otimes V) = (\mathbb{D}U) \otimes V + U \otimes \mathbb{D}V.$$

Thus from (16), when commuting  $\mathbb{D}_A$  with the tensor product with  $X^B$ , we get the commutator identity

$$(17) \quad [\mathbb{D}_A, X^B] = \delta_A^B.$$

In view of the last property, as an operator on weighted tractor fields, the commutator  $[\mathbb{D}_A, \mathbb{D}_B]$  is a ‘‘curvature’’ in that it acts algebraically. We will treat it this way by writing,

$$(18) \quad [\mathbb{D}_A, \mathbb{D}_B] V^C = W_{AB}{}^C{}_D V^D$$

for its action on  $V \in \gamma(\mathcal{T}(w))$ . For this reason and for convenience we will refer to  $W_{AB}{}^C{}_D$  as the  $W$ -curvature. Investigating this, consider  $\mathbb{D}$  on projective densities  $\tau \in \Gamma(\mathcal{E}(w))$  to form  $\mathbb{D}_B \tau$ . Using (12) we have

$$\begin{aligned} \mathbb{D}_A \mathbb{D}_B \tau &= (w-1) Y_A \mathbb{D}_B \tau + Z_A^a \nabla_a \mathbb{D}_B \tau \\ &= w(w-1) Y_A Y_B \tau + 2(w-1) Y_{(A} Z_{B)}^b \nabla_b \tau + Z_A^a Z_B^b \nabla_a \nabla_b \tau, \end{aligned}$$

which we note is symmetric. Phrased alternatively, we have on sections of density bundles

$$(19) \quad [\mathbb{D}_A, \mathbb{D}_B] \tau = 0.$$

So  $\mathbb{D}$  is ‘‘torsion free’’ in this sense, and from the Jacobi identity we have at once the Bianchi identities

$$(20) \quad W_{[AB}{}^C{}_D] = 0 \quad \text{and} \quad \mathbb{D}_{[A} W_{BC]}{}^E{}_F = 0.$$

To compute  $W_{AB}{}^C{}_D$  it suffices to act on a section  $V \in \Gamma(\mathcal{T})$ . Note from (12)

$$\mathbb{D}_A \mathbb{D}_B V^C = -Y_A \mathbb{D}_B V^C - Y_B \mathbb{D}_A V^C + Z_A^a Z_B^b \nabla_a \nabla_b V^C.$$

Thus

$$(21) \quad W_{AB}{}^C{}_D = Z_A^a Z_B^b \kappa_{ab}{}^C{}_D,$$

where  $\kappa$  is the tractor curvature given above, and in particular

$$(22) \quad X^A W_{AB}{}^C{}_D = X^B W_{AB}{}^C{}_D = X^D W_{AB}{}^C{}_D = 0,$$

as well as

$$(23) \quad Z_C{}^c W_{AB}{}^C{}_D = Z_A{}^a Z_B{}^b Z_D{}^d W_{ab}{}^c{}_d, \quad Y_C W_{AB}{}^C{}_D = -Z_A{}^a Z_B{}^b Z_D{}^d C_{abd}$$

The action of the  $W$ -tractor, as on the right hand side of (18), extends to tensor products of  $\mathcal{T}$  and  $\mathcal{T}^*$  by the Leibniz rule and we use the shorthand  $W_{AB}{}^\sharp$  for this. For example, for any (possibly weighted) 2-cotractor field  $T_{CD}$  we have

$$W_{AB}{}^\sharp T_{CD} = -W_{AB}{}^E{}_C T_{ED} - W_{AB}{}^E{}_D T_{CE}.$$

*Remark 1.* The  $W$ -curvature  $W_{AB}{}^C{}_D$  satisfies, of course, stronger properties if the projective structure includes the Levi-Civita connection of a metric. An interesting case is when, in particular, the metric is Einstein but not scalar flat, as in this case there is a parallel (non-degenerate) metric on the projective tractor bundle. This can be used to raise and lower tractor indices [9] and it follows easily that the  $W$ -curvature  $W_{AB}{}^C{}_D$  has the same algebraic symmetries as a conformal Weyl tensor. This is potentially important for applications, but we will not exploit these observations in the current work.

**2.3. Young diagrams and some algebra.** For a real vector space  $\mathbb{V}$  of dimension  $N$  we consider irreducible representations of  $SL(\mathbb{V}) \cong SL(N, \mathbb{R})$  within  $\otimes^m \mathbb{V}^*$  for  $m \in \mathbb{Z}_{\geq 0}$ . Up to isomorphism, these are classified by Young diagrams [17, 18] and we assume an elementary familiarity with this notation. Each diagram is (equivalent to) a weight  $(a_1, a_2, \dots, a_N)$  where  $m \geq a_1 \geq \dots \geq a_N \geq 0$  with  $\sum_{i=1}^k a_i = m$ . We usually omit terminal strings of 0, strictly after  $a_1$ , that is for  $s \geq 2$  we usually omit  $a_s$  from the list if  $a_s = 0$ . In particular the trivial representation of  $SL(\mathbb{V})$  on  $\mathbb{R}$  (so  $m = 0$ ) will be denoted  $(0)$  rather than  $(0, \dots, 0)$  and the dual of the defining (or fundamental) representation of  $SL(\mathbb{V})$  on  $\mathbb{V}^*$  (so  $m = 1$ ) will be denoted  $(1)$  rather than  $(1, 0, \dots, 0)$ . Given this notation for weights the representation space for the representation  $(a_1, \dots, a_h)$  will usually be denoted  $\mathbb{V}_{(a_1, \dots, a_h)}$ , or by the weight  $(a_1, \dots, a_h)$ , simply, if  $\mathbb{V}$  is understood. We will term  $h$  the *height* of the diagram.

In fact for our current purposes we shall only need the Young diagrams of height at most 2, and  $\mathbb{V}$  will be  $\mathbb{R}^{n+1}$  with its standard representation of  $SL(n+1, \mathbb{R})$ . The symmetric representations  $S^m \mathbb{V}^*$  have the diagram  $(m)$ , while  $(k, \ell)$  with  $k + \ell = m \geq 1$ ,  $k \geq \ell \geq 1$ , can be realised by tensors  $T_{B_1 \dots B_k C_1 \dots C_\ell}$  on  $\mathbb{V}$  which are symmetric in the  $B_i$ 's, also symmetric in the  $C_i$ 's, and such that symmetrisation over the first (equivalently any)  $k+1$  indices vanishes:

$$(24) \quad T_{B_1 \dots B_k C_1 \dots C_\ell} = T_{(B_1 \dots B_k)(C_1 \dots C_\ell)} \quad \text{and} \quad T_{(B_1 \dots B_k C_1) C_2 \dots C_\ell} = 0.$$

In this article we will call these particular realisations *Young symmetries* and  $\mathbb{V}_{(k, \ell)}$  will mean the  $SL(\mathbb{V})$ -submodule of  $\otimes^m \mathbb{V}$  consisting of tensors on  $\mathbb{V}$  with these Young symmetries.

The key algebraic fact we need is then the following.

**Proposition 2.** *The map of  $SL(\mathbb{V})$  representations*

$$(25) \quad \mathbb{V}_{(r+1)} \otimes \mathbb{V}_{(r)} \rightarrow \mathbb{V}_{(r)} \otimes \mathbb{V}_{(r+1)}$$

given by

$$T_{B_1 \dots B_r B_{r+1} C_1 \dots C_r} \mapsto T_{B_1 \dots B_r (B_{r+1} C_1 \dots C_r)}$$

is an isomorphism.

*Proof.* This is an straightforward consequence of the well known Littlewood-Richardson rules for decomposing the tensor product  $U_{C_1 \dots C_r} \otimes V_{B_1 \dots B_{r+1}} \in \mathbb{V}_{(r)} \otimes \mathbb{V}_{(r+1)}$  into its direct sum of irreducible parts, and then the properties of these irreducibles in terms of Young symmetries as explained in [17, 18, 32]. Each of the summands is a representation equivalent to either  $\mathbb{V}_{(2k+1)}$  or  $\mathbb{V}_{(k, \ell)}$ , with  $\ell \geq 1$ ,  $k + \ell = 2r + 1$ , and each projection to such a component may be factored through the map (25).  $\square$

This yields the following consequence.

**Corollary 3.** For  $r \in \mathbb{Z}_{\geq 1}$  and  $k \geq \ell \geq 1$  with  $k + \ell = r + 1$ ,

$$(\mathbb{V}_{(r+1)} \otimes \mathbb{V}_{(r)}) \cap (\mathbb{V}_{(r)} \otimes \mathbb{V}_{(k,\ell)}) = \{0\}.$$

*Proof.* The irreducible components of  $\otimes^{r+1}\mathbb{V}^*$  isomorphic to  $\mathbb{V}_{(k,\ell)}$ , with  $k \geq \ell \geq 1$  and  $k + \ell = r + 1$  all lie in the kernel of the map

$$(26) \quad \otimes^{r+1}\mathbb{V}^* \rightarrow \mathbb{V}_{(r+1)}$$

However from the Proposition 2 the kernel of the map (25) is trivial.  $\square$

In fact the kernel of (2) is spanned by the irreducible components of  $\otimes^{r+1}\mathbb{V}^*$  isomorphic to  $\mathbb{V}_{(k,\ell)}$ , with  $k \geq \ell \geq 1$  and  $k + \ell = r + 1$ . Thus it is clear that in fact the Corollary 3 is equivalent to the Proposition 2. Thus it is interesting to prove this directly. We present this here, since for our later purposes this will be useful.

Another fact that will be useful is the following.

**Lemma 4.** Suppose that  $T_{B_1 \dots B_r C_1 \dots C_r} = T_{(B_1 \dots B_r)(C_1 \dots C_r)} \in \mathbb{V}_{(r,r)}$ . Then

$$(27) \quad T_{B_1 \dots B_r C_1 \dots C_r} = (-1)^r T_{C_1 \dots C_r B_1 \dots B_r}.$$

*Proof.* The projector  $P_{(r,r)} : \otimes^{2r}\mathbb{V}^* \rightarrow \mathbb{V}_{(r,r)}$  is given by

$$(28) \quad P_{(r,r)} T = S_{(1, \dots, r)} \circ S_{(r+1, \dots, 2r)} \circ S_{[1, r+1]} \circ \dots \circ S_{[r, 2r]}(T),$$

where  $S_{(1, \dots, r)}$  denotes symmetrisation over the first  $r$  indices,  $S_{(r+1, \dots, 2r)}$  denotes symmetrisation over the last  $r$  indices,  $S_{[i, j]}$  denotes anti-symmetrisation over the two indices in, respectively, the  $i^{\text{th}}$  and  $j^{\text{th}}$  positions.

The claim in the Lemma is an immediate consequence.  $\square$

In the following we extend these conventions, notations, and definitions to vector bundles (with fibre  $\mathbb{V}$ ) in the obvious way.

### 3. KILLING EQUATIONS: PROLONGATION VIA THE TRACTOR CONNECTION

Here we treat the Killing type equations

$$(29) \quad \nabla_{(a_0} k_{a_1 \dots a_r)} = 0,$$

on an affine manifold with an affine connection  $\nabla$ . For simplicity we assume this is torsion free, but this plays almost no role. There is such an equation for each  $r \in \mathbb{Z}_{>0}$  and as discussed above the equations are each projectively invariant if we take the symmetric rank  $r$  tensor to have projective weight  $2r$ , i.e.  $k_{b \dots c} \in \Gamma(\mathcal{E}_{(b \dots c)}(2r))$ . In the following, we denote by  $\mathcal{T}_{(k,\ell)}$  the tractor bundle with fibre  $\mathbb{V}_{(k,\ell)}$  where  $\mathbb{V} = \mathbb{R}^{n+1} = \mathcal{T}|_p$ . Moreover we include the weight  $w$  in the notation as  $\mathcal{T}_{(k,\ell)}(w)$ .

Via the cotractor filtration sequence (8) we evidently have the following.

**Lemma 5.** There is a projectively invariant bundle inclusion

$$S^r T^* M(2r) \rightarrow S^r \mathcal{T}^*(r) = \mathcal{T}_{(r)}(r)$$

given by

$$(30) \quad S^r T^* M(2r) \ni k_{b \dots c} \mapsto K_{B \dots C} := Z_B^b \dots Z_C^c k_{b \dots c} \in \mathcal{T}_{(r)}(r).$$

Note that for  $K$  as here we have

$$(31) \quad X^B K_{B \dots C} = 0.$$

Moreover if  $K \in \mathcal{T}_{(r)}(r)$  satisfies (31) then it is in the image of (30).

This enables a tractor interpretation of the Killing type equations, as follows.

**Proposition 6.** For each rank  $r$  the equation (29) is equivalent to the tractor equation

$$(32) \quad \mathbb{D}_{(A} K_{B \dots C)} = 0,$$

where  $K_{B \dots C}$  is given by (30).

*Proof.* From the tractor formulae (12) and (15) we have

$$\begin{aligned} \mathbb{D}_{A_0} K_{A_1 \dots A_r} &= r Y_{A_0} K_{A_1 A_2 \dots A_r} - Y_{A_1} K_{A_0 A_2 \dots A_r} - \dots - Y_{A_r} K_{A_1 A_2 \dots A_{r-1} A_0} \\ &\quad + Z_{A_0}{}^{a_0} Z_{A_1}{}^{a_1} \dots Z_{A_r}{}^{a_r} \nabla_{a_0} k_{a_1 \dots a_r}, \end{aligned}$$

which implies

$$\mathbb{D}_{(A_0} K_{A_1 \dots A_r)} = Z_{(A_0}{}^{a_0} Z_{A_1}{}^{a_1} \dots Z_{A_r)}{}^{a_r} \nabla_{a_0} k_{a_1 \dots a_r},$$

from which the result follows immediately.  $\square$

In the following  $K_{A_1 \dots A_r}$  will always refer to a weight  $r$  symmetric tractor as given by (30). We now define a projectively invariant operator

$$(33) \quad \mathcal{L} : S^r T^* M(2r) \rightarrow P_{(r,r)}(\otimes^{2r} \mathcal{T}^*) = \mathcal{T}_{(r,r)},$$

where  $P_{(r,r)}$  is the  $(r,r)$  Young symmetry as described in expression (28), by applying the Young projection  $P_{(r,r)}$  to  $\mathbb{D}^r K$ , as follows

$$k_{c_1 \dots c_r} \mapsto P_{(r,r)}(\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r}),$$

with  $K_{C_1 \dots C_r} = Z_{C_1}{}^{c_1} \dots Z_{C_r}{}^{c_r} k_{c_1 \dots c_r}$ .

**Proposition 7.** *The operator  $\mathcal{L} : S^r T^* M(2r) \rightarrow \mathcal{T}_{(r,r)}$  of (33) is a differential splitting operator.*

*Proof.* We claim that

$$(34) \quad X^{B_1} \dots X^{B_r} W^{C_1}{}_{c_1} \dots W^{C_r}{}_{c_r} P_{(r,r)}(\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r}) = c k_{c_1 \dots c_r},$$

where  $c$  is a non-zero constant. It clearly suffices to show that

$$(35) \quad X^{B_1} \dots X^{B_r} P_{(r,r)}(\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r}) = c K_{C_1 \dots C_r}.$$

Contract  $X^{B_1} \dots X^{B_r}$  into the explicit expansion of  $P_{(r,r)}(\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r})$ . Use (i)  $[\mathbb{D}_A, X^B] = \delta_A^B$ , (ii)  $X^A \mathbb{D}_A f = w f$ , for any tractor field  $V$  of weight  $w$  (see (16)), and that (iii)  $X^A K_{A \dots C} = 0$ , to eliminate all occurrences of  $X$ . It follows easily that the result is  $c K_{C_1 \dots C_r}$  for some constant  $c$ , since there is no way to include a term involving  $\mathbb{D}$ s that has the correct valence (i.e. the tractor rank  $r$ ). That  $c \neq 0$  is found by explicit computation or more simply the fact that it is not zero in the case that the affine connection  $\nabla$  is projectively flat, as we shall see below.  $\square$

The above definition is motivated by the projectively flat case where the situation is particularly elegant. (It is easily verified that the operator  $\mathcal{L}$  above is a co-called first BGG splitting operator, as discussed in e.g. [8], and see references therein. We will not use this fact however.)

We conclude this section with an observation. It shows, in particular, that sections of  $\mathcal{T}_{(r,r)}$  that are parallel for the usual tractor connection determine solutions of (29). These are the so-called *normal solutions* (see e.g. [8]):

**Proposition 8.** *Let  $(M, \mathfrak{p})$  be a projective manifold (not necessarily flat) and let  $L \in \Gamma(\mathcal{T}_{(r,r)})$  such that*

$$(36) \quad 0 = X^{B_1} \dots X^{B_r} \mathbb{D}_A L_{B_1 \dots B_r C_1 \dots C_r}.$$

*Then  $K_{C_1 \dots C_r} \in \Gamma(\mathcal{T}_{(r)})$  defined by  $K_{C_1 \dots C_r} = X^{B_1} \dots X^{B_r} L_{B_1 \dots B_r C_1 \dots C_r}$  satisfies equation (32). If we assume in addition that*

$$(37) \quad 0 = \mathbb{D}_A L_{B_1 B_2 \dots B_r C_1 \dots C_r},$$

*then  $L$  defines a rank  $r$  Killing tensor via (34) such that  $L$  is a constant multiple of  $\mathcal{L}(k)$ .*

*Proof.* The proof is a direct rewriting of (36),

$$(38) \quad \begin{aligned} 0 &= X^{B_1} \dots X^{B_r} \mathbb{D}_A L_{B_1 \dots B_r C_1 \dots C_r} \\ &= X^{B_2} \dots X^{B_r} (\mathbb{D}_{A_1} (X^{B_1} L_{B_1 \dots B_r C_1 \dots C_r}) - L_{A_1 B_2 \dots B_r C_1 \dots C_r}) \\ &= -r X^{B_2} \dots X^{B_r} L_{A_1 B_2 \dots B_r C_1 \dots C_r} + \mathbb{D}_{A_1} K_{C_1 \dots C_r}, \end{aligned}$$

where we successively apply  $[\mathbb{D}_A, X^B] = \delta_A^B$  to commute and eliminate  $X$ 's and  $\mathbb{D}$ 's and use the symmetries of  $L$ . Note that this computation does not require any mutual commutations of

$\mathbb{D}_A$ 's. Now since  $L_{B_2 \dots B_r (A C_1 \dots C_r)} = 0$  this equation implies equation (32). Moreover, because of the symmetries of  $L$ , we also have that

$$X^{C_i} K_{C_1 \dots C_r} = X^{C_i} X^{B_1} \dots X^{B_r} L_{B_1 \dots B_r C_1 \dots C_r} = 0,$$

for each  $i = 1, \dots, r$ . This implies that  $K$  is given by a  $k$  as in relation (30).

Applying  $\mathbb{D}_{A_r}, \dots, \mathbb{D}_{A_2}$  successively to equation (38), commuting with the  $X$ 's successively by  $[\mathbb{D}_A, X^B] = \delta_A^B$  and finally using the additional hypothesis (37), shows that  $\mathbb{D}_{A_r} \dots \mathbb{D}_{A_1} K_{C_1 \dots C_r}$  is a nonzero constant multiple of  $L_{A_r \dots A_1 C_1 \dots C_r}$ . Hence,  $L$  is a constant multiple of  $\mathcal{L}(k)$ .  $\square$

**3.1. Projectively flat structures.** In this subsection we restrict to affine (or projective) manifolds that are projectively flat, i.e. where the projective tractor curvature vanishes. According to equation (21) this also means that the Thomas- $\mathbb{D}$  operators mutually commute when acting on weighted tractor sections.

In the projectively flat setting we obtain a nice characterisation of Killing tensors.

**Proposition 9.** *Let  $(M, \mathfrak{p})$  be a projectively flat manifold. Let  $k_{c_1 \dots c_r} \in \Gamma(S^r T^* M(2r))$  and define  $K_{C_1 \dots C_r} := Z_{C_1}^{c_1} \dots Z_{C_r}^{c_r} k_{c_1 \dots c_r}$ , as in (5). Then  $k$  satisfies the Killing equation (29) if and only if*

$$(39) \quad \mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r} \in \Gamma(\mathcal{T}_{(r,r)}).$$

*In particular, on a projectively flat manifold there is a non-zero constant  $c$  so that*

$$\mathcal{L}(k) = c \mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r},$$

*if and only if  $k$  solves (29).*

*Proof.* ( $\Rightarrow$ ) Since we work in the projectively flat setting the Thomas- $\mathbb{D}$  operators commute. So

$$\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r} \in \Gamma(\mathcal{T}_{(r)} \otimes \mathcal{T}_{(r)})$$

Suppose that (29) holds. Then (32) holds, so symmetrising the left hand side of the display over any  $r + 1$  indices that include  $C_1 \dots C_r$  results in annihilation and so we conclude (39) from the definition of  $\mathbb{V}_{(r,r)}$  and hence of  $\mathcal{T}_{(r,r)}$  in (24).

( $\Leftarrow$ ) If (39) holds then

$$\mathbb{D}_{B_1} \dots \mathbb{D}_{B_{r-1}} \mathbb{D}_{(B_r} K_{C_1 \dots C_r)} = 0$$

so

$$X^{B_1} \dots X^{B_{r-1}} \mathbb{D}_{B_1} \dots \mathbb{D}_{B_{r-1}} \mathbb{D}_{(B_r} K_{C_1 \dots C_r)} = (r-1)! \mathbb{D}_{(B_r} K_{C_1 \dots C_r)} = 0,$$

from (16), thus we obtain the result from Proposition 6.  $\square$

Here and throughout, as above,  $K \in \Gamma(\mathcal{T}_{(r)}(r))$  is the image of some  $k \in \Gamma(S^r T^* M(2r))$  as in formula (30).

**Proposition 10.** *The constant  $c$  in equation (34) is not 0.*

*Proof.* In the case that the structure is projectively flat this is immediate from the Proposition 9, since  $X^{B_1} \dots X^{B_r}$  contracted into  $\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r}$  gives  $r! K_{C_1 \dots C_r}$ . But it is clear from the argument in the proof of Proposition 7 that  $c$  does not depend on curvature, as no commutation of  $\mathbb{D}$ s is involved.  $\square$

**Theorem 11.** *Let  $(M, \mathfrak{p})$  be projectively flat manifold. Then the splitting operator  $\mathcal{L}$  gives an isomorphism between Killing tensors of rank  $r$  and sections of  $\mathcal{T}_{(r,r)}$  that are parallel for the projective tractor connection.*

*Proof.* Since  $\mathcal{L}$  is a splitting operator, it does not have a kernel. Moreover, using that  $\nabla_a L = 0$  is equivalent to  $\mathbb{D}_A L = 0$ , Proposition 8 shows that every parallel section of  $\mathcal{T}_{(r,r)}$  arises as  $\mathcal{L}(k)$  for a Killing tensor  $k$ . So it remains to show that  $\mathcal{L}(k)$  is a parallel section of the projective tractor connection whenever  $k$  is a Killing tensor: Suppose that (29) holds. Then by Proposition 9,

$$\mathbb{D}_{B_1} \dots \mathbb{D}_{B_r} K_{C_1 \dots C_r} = \mathcal{L}(k),$$

and  $\mathcal{L}(k)$  has weight 0 so

$$\mathbb{D}_A \mathcal{L}(k) = Z_A^a \nabla_a \mathcal{L}(k).$$

Thus it suffices to show that  $\mathbb{D}_A \mathcal{L}(k) = 0$ . But

$$\mathbb{D}_A \mathcal{L}(k) = \mathbb{D}_A \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r} = 0,$$

because of the identity  $[\mathbb{V}_{(r+1)} \otimes \mathbb{V}_{(r)}] \cap [\mathbb{V}_{(r)} \otimes \mathbb{V}_{(r,1)}] = \{0\}$  from Corollary 3 (where we have used (32) which implies that  $\mathbb{D}K$  is a section of  $\mathcal{T}_{(r,1)}(2r-1)$ ).  $\square$

As a final note in this section we observe that it is easy to “discover” the projectively invariant Killing equation using the tractor machinery, as follows. Consider a symmetric rank  $r$  covariant tensor field  $k_{c_1 \cdots c_r}$  of projective weight  $2r$ . Form

$$K_{C_1 \cdots C_r} \in S^r \mathcal{T}^*(r)$$

by Lemma 5. We wish to prolong this to a parallel tractor. This requires a tractor field of weight 0. Thus we apply the  $r$ -fold composition of  $\mathbb{D}$ . Altogether we have the projectively invariant operator

$$k \mapsto \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} Z_{C_1}^{c_1} \cdots Z_{C_r}^{c_r} k_{c_1 \cdots c_r} = \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r},$$

and the image has weight zero. Thus we can form

$$\nabla_a \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} Z_{C_1}^{c_1} \cdots Z_{C_r}^{c_r} k_{c_1 \cdots c_r},$$

by construction it is projectively invariant and we can ask what it means for this to be zero. Equivalently we seek the condition on  $k$  determined by

$$\mathbb{D}_A \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r} = 0.$$

But this implies  $X^{C_1} \cdots X^{C_r} \mathbb{D}_A \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r} = 0$  and from equation (36) in the proof of Theorem 11 it follows that

$$\mathbb{D}_{(A} K_{B_1 \cdots B_r)} = 0, \quad \text{implies} \quad \nabla_{(a} k_{b_1 \cdots b_r)} = 0$$

where we again used Proposition 6.

**3.2. Restoring curvature.** We return now to the general curved case and seek the generalisations of the results in the previous subsection. First we observe the following first generalisation of Proposition 9:

**Proposition 12.** *Let  $k \in \Gamma(S^r T^* M(2r))$  on a general affine manifold  $(M, \nabla)$  (or projective manifold  $(M, \mathfrak{p})$ ) and  $K = K(k) \in \Gamma(\mathcal{T}_{(r)}(r))$ , as in (30). Then  $k$  is a Killing tensor, i.e., a solution of (29), if and only if we have*

$$(40) \quad \mathcal{L}(k) = \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r} + \mathbf{Kurv}(K),$$

where  $\mathbf{Kurv}$  is a specific projectively invariant linear differential operator on  $\Gamma(\mathcal{T}_{(r)}(r))$ , of order at most  $(r-2)$ , constructed with the  $W$ -curvature and the Thomas- $\mathbb{D}$  operators and such that the  $W$ -curvature and its  $\mathbb{D}$ -derivatives appear in the coefficients of every term.

*Proof.* ( $\Rightarrow$ ) Suppose that  $k$  solves (29). We have

$$\mathcal{L}(k) = P_{(r,r)}(\mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r}).$$

We expand out this expression on the right hand side using the definition of the operator  $P_{(r,r)}$  in (28). We would like to show that the resulting terms can be combined and rearranged to yield (40). We have the identity (32) available. In the projectively flat case we also have the identity  $[\mathbb{D}_A, \mathbb{D}_B] = 0$  as an operator on (weighted) tractors. In the flat case the two identities are enough to conclude (40) (with  $\mathbf{Kurv}(K) = 0$ ), according to the proof of Proposition 9. In the curved case we perform the same formal computation but keep track of the curvature, i.e., replace each  $[\mathbb{D}_A, \mathbb{D}_B]$  with  $W_{AB}\sharp$  (instead of 0). The order statement follows by construction (or elementary weight arguments), so this proves the result in this direction and generates a specific formula for  $\mathbf{Kurv}(K)$ .

( $\Leftarrow$ ) Now we suppose that  $k \in \Gamma(S^r T^* M(2r))$  is any section such that

$$\mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r} + \mathbf{Kurv}(K)_{B_1 \cdots B_r C_1 \cdots C_r}$$

is a section of  $\mathcal{T}_{(r,r)}(r)$ . Then in particular

$$\mathbb{D}_{B_1} \cdots \mathbb{D}_{B_{r-1}} \mathbb{D}_{(B_r} K_{C_1 \cdots C_r)} + \mathbf{Kurv}(K)_{B_1 \cdots B_{r-1} (B_r C_1 \cdots C_r)} = 0,$$

according to (24). As in the proof of Proposition 9, we contract now with  $X^{B_1} \cdots X^{B_{r-1}}$ . This contraction annihilates the second term in the display as follows. Each of the  $X^{B_i}$ 's is contracted into either a  $\mathbb{D}_{B_i}$ , into  $K$ , or into the curvature  $W$ . Thus every  $X^{B_i}$  can be eliminated using the identities (16), that  $X^B K_{B \cdots C} = 0$ , and that similarly  $X^B$  contracted into any of the lower indices of the curvature  $W$  is zero. But, by the construction of the operator  $\mathbf{Kurv}$ , in any term there are at most  $(r-2)$   $\mathbb{D}$  operators (either applied to the curvature or directly to the argument) and so the identities (16) remove only  $(r-2)$  of the  $(r-1)$   $X$ 's. This means that in every term produced we have a contraction of the form  $X^B K_{B \cdots C} = 0$ , so that term vanishes, or  $X$  into  $W$  so also that term vanishes. Thus we are left with

$$0 = X^{B_1} \cdots X^{B_{r-1}} \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_{r-1}} \mathbb{D}_{(B_r} K_{C_1 \cdots C_r)} = (r-1)! \mathbb{D}_{(B_r} K_{C_1 \cdots C_r)},$$

as in the proof of Proposition 9.  $\square$

**Proposition 13.** *Let  $k \in \Gamma(S^r T^* M(2r))$  on a general affine manifold  $(M, \nabla)$  (or projective manifold  $(M, \mathbf{p})$ ) and  $K = K(k) \in \Gamma(\mathcal{T}_{(r)}(r))$ , as in (30). Then  $k$  is a solution of (29) if and only if we have*

$$(41) \quad \mathbb{D}\mathcal{L}(k) = \mathbf{Curv}(K),$$

where  $\mathbf{Curv}$  is a projectively invariant linear differential operator, of order at most  $(r-1)$ , on  $\Gamma(\mathcal{T}_{(r,r)}(r))$  given by a specific formula constructed with the  $W$ -curvature, and the Thomas- $\mathbb{D}$  operator such that the  $W$ -curvature and its derivatives appear in the coefficients of every term. Moreover, if  $\mathcal{L}(k)$  satisfies equation (41), then

$$(42) \quad X^{B_1} \cdots X^{B_r} \mathbb{D}_A \mathcal{L}(k)_{B_1 \cdots B_r C_1 \cdots C_r} = 0.$$

*Proof.* ( $\Rightarrow$ ) Suppose that  $k$  solves (29). We apply  $\mathbb{D}_A$  to both sides of (40). This yields

$$(43) \quad \mathbb{D}_A \mathcal{L}(k) = \mathbb{D}_A \mathbb{D}_{B_1} \cdots \mathbb{D}_{B_r} K_{C_1 \cdots C_r} + \mathbb{D}_A \mathbf{Kurv}(K)_{B_1 \cdots B_r C_1 \cdots C_r}.$$

In the case when  $\nabla$  is projectively flat the first term on the right can be shown to be zero by a formal calculation using just the identities  $[\mathbb{D}_A, \mathbb{D}_B] = 0$  and  $\mathbb{D}_{(A_0} K_{A_1 \cdots A_r)} = 0$ . This follows from the proof of Theorem 11. Performing the same formal calculation, but now instead replacing the commutator of  $\mathbb{D}$ 's with  $[\mathbb{D}_A, \mathbb{D}_B] = W_{AB} \sharp$  and combining the result with the second term on the right hand side yields the result:  $\mathbb{D}\mathcal{L}(k)$  is equal to a specific formula for a linear differential operator  $\mathbf{Curv}$  on  $K$  that is constructed polynomially, and with usual tensor operations, involving just the  $W$ -curvature, and the Thomas- $\mathbb{D}$  operator. Thus by construction it is projectively invariant, and also by construction (or weight arguments) the order claim follows.

( $\Leftarrow$ ) We suppose now that (41) holds with  $k \in \Gamma(S^r T^* M(2r))$ ,  $K$  as in (30) and with the operator  $\mathbf{Curv}$  given by the formula found the first part of the proof. So we have

$$\mathbb{D}_A \mathcal{L}(k)_{B_1 \cdots B_r C_1 \cdots C_r} = \mathbf{Curv}(K)_{AB_1 \cdots B_r C_1 \cdots C_r}.$$

Note that contraction of  $X^{C_1} \cdots X^{C_r}$  annihilates the right hand side by an easy analogue of the argument used in the second part of the proof of the Proposition 12 above: in this case there are at most  $(r-1)$  many  $\mathbb{D}$  operators in any term but we are contracting in  $\otimes^r X$ , so in each term an  $X$  is contracted directly into and undifferentiated  $K$  or  $W$ . The result now follows by the argument used in second part of the proof of Theorem 11 for the projectively flat case. Thus we have just shown that we have the equation (42). Then the result follows from the first part of Proposition 8.  $\square$

For the proof of the main theorem we recall the following fact, which follows from the theory of overdetermined systems of PDE.

**Lemma 14.** *For every  $T \in \mathcal{T}_{(r,r)}|_x$ , where  $x \in M$ , there is a local section  $k \in \Gamma(S^r T^* M|_U)$ , such that  $T = \mathcal{L}(k)|_x$ .*

*Proof.* In the case of (projectively) flat  $(M, \mathbf{p})$  this follows at once from the fact that in the flat case for  $L \in \Gamma(T_{(r,r)})$  we have shown that  $\nabla L = 0$  implies  $L = \mathcal{L}(k)$ .

For the general case the result then follows as the formula for the operator  $\mathcal{L}(k)$  generalises that from the flat case by the simply the addition (at each order) of lower order curvature terms.  $\square$

Now we state and prove the main results of the paper.

**Theorem 15.** *Let  $(M, \mathbf{p})$  be a projective manifold. Then there is a specific section  $\mathcal{R}_A \sharp \in \mathcal{T}^* M \otimes \text{End}(\mathcal{T}_{(r,r)})$  (where we suppress the endomorphism indices) such that  $X^A \mathcal{R}_A \sharp = 0$  and such that the differential splitting operator  $\mathcal{L} : \Gamma(S^r T^* M(2r)) \rightarrow \Gamma(\mathcal{T}_{(r,r)})$  gives an isomorphism between Killing tensors of rank  $r$  and sections  $L$  of the bundle  $\mathcal{T}_{(r,r)}$  that satisfy the equation*

$$(44) \quad \mathbb{D}_A L = \mathcal{R}_A \sharp L.$$

*Proof.* Again, the splitting operator  $\mathcal{L}$  is injective. Hence, we have to show the following:

- (A) For every Killing tensor  $k$  the image  $\mathcal{L}(k)$  satisfies equation (44) with a specific  $\mathcal{R}_A \sharp \in \mathcal{T}^* M \otimes \text{End}(\mathcal{T}_{(r,r)})$  that will be determined;
- (B)  $\mathcal{L}$  restricted to Killing tensors (i.e. the solutions of (29)) is surjective onto the sections  $L$  that satisfy equation (44), where the right hand side is as determined in (A).

We prove (A): Assume that  $k$  solves (29). Then we have equation (41),

$$\mathbb{D}\mathcal{L}(k) = \mathbf{Curv}(K).$$

from Proposition 13. The operator  $\mathbf{Curv}$  is given by a formula polynomial in the  $W$ -curvature, its  $\mathbb{D}$  derivatives, and the Thomas- $\mathbb{D}$  operators up to order  $(r-1)$ . Now observe that each term of the form  $\mathbb{D}_{B_1} \cdots \mathbb{D}_{B_s} K_{C_1 \cdots C_r}$ , for  $0 \leq s < r$  can be replaced using (40) from Proposition 12,

$$\mathbb{D}_{B_1} \cdots \mathbb{D}_{B_s} K_{C_1 \cdots C_r} = c X^{s+1} \cdots X^r \mathcal{L}(k)_{B_1 \cdots B_r C_1 \cdots C_r} + \mathbf{Curv}^{(s)}(K),$$

where  $\mathbf{Curv}^{(s)}$  is a differential operator is given by a formula polynomial in the  $W$ -curvature, its  $\mathbb{D}$  derivatives, and the Thomas- $\mathbb{D}$  operators up to order  $(s-2)$ . In this way we can successively eliminate all applications of  $\mathbb{D}$  to  $K$  by terms algebraic in  $\mathcal{L}(k)$  arriving at an equation of the form

$$(45) \quad \mathbb{D}_A \mathcal{L}(k) = \mathcal{R}_A \sharp \mathcal{L}(k), \quad \text{with } \mathcal{R}_A \in \Gamma(\mathcal{T}^* \otimes \text{End}(\otimes^{2r} \mathcal{T}^*)),$$

given by a polynomial in the  $W$ -curvature and its  $\mathbb{D}$ -derivatives. Now we have to verify:

- (i) that  $\mathcal{R}_A \sharp$  is indeed a section of  $\mathcal{T}^* \otimes \text{End}(\mathcal{T}_{(r,r)})$ , and
- (ii) that for every  $L \in \mathcal{T}_{(r,r)}$ , the contraction of  $\mathcal{R}_A \sharp L$  with  $X^A$  is equal to zero.

In order to verify (i) and (ii) we have to make a key observation: Although we phrased the discussion above in a naive way that supposes there is a solution to (29), in fact to derive (45) we do not actually require that there exist solutions, even locally, to the equation (29). Equation (45) simply expresses relations on the jets, of a section  $k \in \Gamma(S^r T^* M(2r))$  that are formally determined by a finite jet prolongation of the Killing equation (29). It is clear that we can derive (45) at any point  $x \in M$  by working with just the  $r+1$ -jet,  $j_x^{r+1} k$ , of  $k$  at  $x$ . Following the argument as above, but working formally with such jets and assuming (29) holds to order  $r$  at  $x$ , we come to

$$(46) \quad \mathbb{D}_A L|_x = \mathcal{R}_A \sharp \mathcal{L}(k)|_x$$

where all curvatures and their derivatives are evaluated at  $x$ . From the results in the projectively flat case we know that this is exactly the point where the prolongation of the finite type PDE (29) has closed: The prolongation up to order  $r$  may be viewed as simply the introduction of new variables labelling the part of the jet that is not constrained by the equation, and these are exactly parametrised by the elements in the fibre  $\mathcal{T}_{(r,r)}(x)$ . At the next order the derivative of

these variables is expressed algebraically in terms of the variables from  $\mathcal{T}_{(r,r)}(x)$ . That is (a key part of) the content of (46). Viewing this as a computation in slots (via a choice of  $\nabla \in \mathfrak{p}$ ) the computation is the same in the curved case as in the projectively flat case except that additional curvature terms may enter when derivatives are commuted. It follows that  $\mathcal{L}(k)(x)$  may be an arbitrary element  $L$  of  $\mathcal{T}_{(r,r)}(x)$ . Using this, and since contraction with  $X^A$  annihilates the left hand side of (46) it follows that it annihilates the right hand side for any  $L \in \mathcal{T}_{(r,r)}(x)$ . Similarly since the left hand side of (46) is a section of  $(\mathcal{T}^* \otimes \mathcal{T}_{(r,r)})(x)$  so is the right hand side, for arbitrary  $L = \mathcal{L}(k)(x)$  and thus (ii) also follows.

Now we prove (B): Suppose that  $L \in \Gamma(\mathcal{T}_{(r,r)})$  satisfies (44) for the specific  $\mathcal{R}_A \in \Gamma(\mathcal{T}^* \otimes \mathcal{T}_{(r,r)})$  obtained from the argument above. We now claim that

$$(47) \quad X^{B_1} \dots X^{B_r} (\mathcal{R}_A \# L)_{B_1 \dots B_r C_1 \dots C_r} = 0.$$

Indeed, in the case that  $L = \mathcal{L}(k)$  for a tensor  $k$  that solves (29), we know from Proposition 13 that  $X^{C_1} \dots X^{C_r}$  annihilates the right hand side of equation (44) for  $\mathcal{L}(k)$ , because then it is simply a rewriting of the right hand side of (41). However, as mentioned above, at a point  $x \in M$  and for  $k$  satisfying (29) to order  $r$  at  $x$ , any element of  $\mathcal{T}_{(r,r)}|_x$  can arise as  $\mathcal{L}(k)|_x$  because this is the full prolonged system for the overdetermined PDE (29). Thus it follows that  $X^{C_1} \dots X^{C_r}$  must annihilate the right hand side of (44) for  $L$  even if  $L$  is not  $\mathcal{L}(k)$  for a  $k \in \Gamma(S^r T^* M(2r))$  satisfying (29).

Having established equation (47), we can apply the first part of Proposition 8 to ensure that  $L$  determines a Killing tensor  $k$ . Then we have that  $L = \mathcal{L}(k)$  unless the map

$$L_{B_1 \dots B_r C_1 \dots C_r} \mapsto K_{B_1 \dots B_r} = X^{C_1} \dots X^{C_r} L_{B_1 \dots B_r C_1 \dots C_r} \in \mathcal{T}_{(r)}(r).$$

has a kernel. To exclude this possibility, assume there is a section  $L$  of  $\mathcal{T}_{(r,r)}$  that satisfies (44) and such that

$$(48) \quad X^{C_1} \dots X^{C_r} L_{B_1 \dots B_r C_1 \dots C_r} = 0.$$

The following lemma shows that this implies the vanishing of  $L$ .

**Lemma 16.** *Let  $L_{B_1 \dots B_r C_1 \dots C_r}$  be a section of  $\mathcal{T}_{(r,r)}$  that satisfies equation (44) for the specific  $\mathcal{R}_A \# \in \Gamma(\mathcal{T}^* \otimes \mathcal{T}_{(r,r)})$ . Then we have the following implication: if*

$$(49) \quad X^{B_1} \dots X^{B_k} L_{B_1 \dots B_k \dots B_r C_1 \dots C_r} = 0 \quad \text{for } a \ k \in \{1, \dots, r\},$$

then

$$X^{B_1} \dots X^{B_{k-1}} L_{B_1 \dots B_{k-1} \dots B_r C_1 \dots C_r} = 0,$$

and hence  $L_{B_1 \dots B_r C_1 \dots C_r} = 0$ .

*Proof.* Assume that equation (49) holds. Applying  $\mathbb{D}_A$ , the Leibniz rule for  $\mathbb{D}_A$  gives

$$(50) \quad 0 = c X^{B_1} \dots X^{B_{k-1}} L_{B_1 \dots B_{k-1} A B_{k+1} \dots B_r C_1 \dots C_r} + X^{B_1} \dots X^{B_k} \mathbb{D}_A L_{B_1 \dots B_r C_1 \dots C_r},$$

with a nonzero constant  $c$ . Hence, we have to show that equation (49) implies

$$(51) \quad X^{B_1} \dots X^{B_k} \mathbb{D}_A L_{B_1 \dots B_k \dots B_r C_1 \dots C_r} = 0,$$

by using equation (44) and the specific form of  $\mathcal{R}_A \#$ . The proofs of the previous propositions and of (A) provide us with the following information about  $\mathcal{R}_A \#$ : In Proposition 13 we have seen that the expression  $\mathbf{Curv}(K)$  was of order at most  $(r-1)$  in  $\mathbb{D}$  and is a linear combination of in terms of the form  $\mathcal{A}^{(s-1)} \otimes \mathbb{D}^{r-s} K$  for  $1 \leq s \leq r$  and where  $\mathcal{A}^{(s-1)}$  is a tractor of valence  $s$  containing at most  $s-1$  applications of  $\mathbb{D}$  to the tractor curvature  $W$ . Then in (A) of the present proof we have expressed the terms  $\mathbb{D}^{r-s} K$  by an  $s$ -fold contraction of  $\mathcal{L}(k)$  with  $X$ . Hence  $\mathcal{R}_A \# L$  is a linear combination of terms of the form

$$(52) \quad \mathcal{A}^{(s-1)} \otimes \mathcal{B}^{(s)},$$

where  $\mathcal{B}^{(s)}$  is of the form  $X^{E_1} \dots X^{E_s} L_{E_1 \dots E_s E_{s+1} \dots E_r C_1 \dots C_r}$ . Because of (49), the only terms that are nonzero in  $\mathcal{R}_A \# L$  are those of the form (52) with  $s < k$ . Hence the terms  $\mathcal{A}^{(s-1)}$  contain

at most  $(k - 2)$   $\mathbb{D}$ -derivatives of the tractor curvature. Now since  $X^A W_{AB} = 0$  and therefore  $X^A \mathbb{D}_B W_{AC} = -W_{BC}$ , each of the  $\mathcal{A}^{(s-1)}$  is annihilated by  $s$  contractions with  $X$ . Hence the only terms of the form (52) that are non zero when contracted with  $k$  many  $X$ 's must have at least  $(k + 1 - s)$  contractions with  $X$  at  $B^{(s)}$ , which already is obtained by  $s$  contractions with  $X$ . Hence the only terms  $\mathcal{B}^{(s)}$  that may remain nonzero when contracted with  $(k + 1 - s)$  many  $X$ 's are of the form

$$(53) \quad X^{B_1} \dots X^{B_s} X^{C_1} \dots X^{C_{k+1-s}} L_{B_1 \dots B_s \dots B_r C_1 \dots C_{k+1-s} \dots C_r}.$$

Now an induction over  $s$  shows that these terms are actually zero. In fact, for  $s = 1$  this follows from the assumption (49). If  $s > 1$  we use that  $L \in \mathcal{T}_{(r,r)}$  to get

$$\begin{aligned} X^{B_1} \dots X^{B_s} X^{C_1} \dots X^{C_{k+1-s}} L_{B_1 \dots B_r C_1 \dots C_r} &= \\ &= - \sum_{i=1}^r X^{B_1} \dots X^{B_s} X^{C_1} \dots X^{C_{k+1-s}} L_{B_1 \dots B_{s-1} C_i B_{s+1} \dots B_r C_1 \dots C_{i-1} B_s C_{i-1} \dots C_r} \\ &= -(k + 1 - s) X^{B_1} \dots X^{B_s} X^{C_1} \dots X^{C_{k+1-s}} L_{B_1 \dots B_r C_1 \dots C_r} \end{aligned}$$

by the induction hypothesis. This shows that the terms in (53) are indeed zero and finishes the proof of the lemma.  $\square$

This shows that every  $L \in \Gamma(\mathcal{T}_{(r,r)})$  that satisfies equation (44) is the image of a Killing tensor under the splitting operator  $\mathcal{L}$ . This finishes the proof of (B) and hence of the theorem.  $\square$

Rewriting the result of this theorem in terms of the tractor connection gives:

**Corollary 17.** *Let  $(M, \mathfrak{p})$  be a projective manifold. Then there is a projectively invariant section  $\mathcal{Q}_a \# \in \Gamma(T^*M \otimes \text{End}(\mathcal{T}_{(r,r)}))$  such that the splitting operator  $\mathcal{L}$  gives an isomorphism between weighted Killing tensors of rank  $r$  and sections  $L \in \Gamma(\mathcal{T}_{(r,r)})$  that satisfy satisfies the equation*

$$(54) \quad \nabla_a^T L = \mathcal{Q}_a \# L,$$

or equivalently, sections  $L$  that are parallel for connection

$$(55) \quad \nabla_a^T - \mathcal{Q}_a \#.$$

*Proof.* This follows by contracting equation (44) with  $W^A_a$  yielding equation (54) with some  $\mathcal{Q}_a \# \in \Gamma(T^*M \otimes \text{End}(\mathcal{T}_{(r,r)}))$ . Moreover, since  $X^A \mathcal{R}_A \# = 0$ , the resulting  $\mathcal{Q}_a$  is projectively invariant.  $\square$

*Remark 18.* As a final remark we note that there is a considerable gain in understanding the prolongation of (29) in the form (54) (or equivalently (55)), rather than simply as some (possible invariant) connection  $\tilde{\nabla}$  on  $\mathcal{T}_{r,r}$  without the structure (55) (or some equivalent) made explicit. An obvious example of such a gain is for the explicit computation of integrability conditions. Given such a connection the standard way to compute integrability conditions is via the curvature of  $\tilde{\nabla}$ , since this must annihilate any section of  $\mathcal{T}_{(r,r)}$  that corresponds to a solution of (29). However, because the bundle  $\mathcal{T}_{(r,r)}$  has very high rank (e.g. for  $r = 2$  it has rank  $n^2(n^2 - 1)/12$ ) and the prolongation connection is necessarily very complicated, computing such curvature is typically out of reach without the development of specialised software. However given (54) we obtain integrability conditions immediately from the curvature  $\kappa$  (see (13)) of the normal tractor connection: Differentiating (54) with the latter and skewing in the obvious way we obtain

$$(56) \quad 2\nabla_{[b}^T \nabla_{a]}^T L = \kappa_{ba} \# L = \nabla_{[b}(\mathcal{Q}_{a]} \# L).$$

Then using similar ideas to the treatments above, we can expand the (far) right hand side by replacing any instance of  $\nabla_b^T L$  with  $\mathcal{Q}_b \# L$  and thus, by subtracting  $\kappa_{ba} \# L$ , obtain at once a projectively invariant 2-form with values in  $\text{End}(T_{r,r})$ , that must annihilate any  $L(k)$  for  $k$  solving (29). Thus the existence of solutions 29 constrains the rank of this natural projective invariant constructed from the tractor curvature and its derivatives. From there one can compute

invariants that must vanish following standard ideas, as in e.g. [21, Section 3] (applied there to a different problem).

#### 4. EXPLICIT RESULTS FOR LOW RANK

**4.1. The curved rank  $r = 1$  case.** The rank one case is well known and here we compare it to our approach. We construct the connection corresponding to the equation

$$(57) \quad \nabla_{(a}k_b) = 0 \quad \nabla \in \mathfrak{p}$$

on  $k_b \in \Gamma(T^*M(2))$  on a projective manifold  $(M, \mathfrak{p})$ . Following Lemma 5 we form  $K_C = Z_C^c k_c \in \mathcal{T}^*(1)$ , where  $k_c$  is a solution of (29), and then according to the definition (33), set

$$\mathcal{L}(k)_{BC} := \mathbb{D}_{[B}K_{C]}.$$

Consider the case that  $k$  is a solution of (57). Then from Proposition 6,

$$\mathbb{D}_B K_C \in \Gamma(\Lambda^2 \mathcal{T}^*),$$

and because the  $W$ -tractor satisfies the algebraic Bianchi identity  $W_{AB}{}^E{}_C + W_{BC}{}^E{}_A + W_{CA}{}^E{}_B = 0$  we have  $\mathbb{D}_{[A}\mathbb{D}_B K_{C]} = 0$ , that is

$$\mathbb{D}_A \mathbb{D}_B K_C = [\mathbb{D}_C, \mathbb{D}_B] K_A = -W_{CB}{}^E{}_A K_E.$$

So for solutions  $k$  we have

$$\mathbb{D}_A \mathbb{D}_{[B} K_{C]} - W_{BC}{}^E{}_A X^F \mathbb{D}_{[F} K_{E]} = 0.$$

So  $\nabla_a \mathcal{L}(k)_{BC} + W_{BC}{}^E{}_A W_a^A X^F \mathcal{L}(k)_{EF} = 0$ . But for any  $k \in \Gamma(T^*M(2))$

$$X^F \mathbb{D}_{[F} K_{E]} = X^F \mathcal{L}(k)_{FE} = K_E.$$

Thus the projectively invariant connection on  $\Lambda^2 \mathcal{T}^*$  is given by

$$\nabla_a V_{BC} + W_{BC}{}^E{}_A W_a^A X^F V_{EF}.$$

It is easily checked that this agrees with the formula (3) from the introduction (and so that connection  $\bar{\nabla}$  is projectively invariant).

**4.2. The curved rank  $r = 2$  case.** Here we consider the case  $r = 2$ . We will make the computations in Section 3.2 explicit and in particular provide explicit formulae for the curvature tractor fields  $\mathcal{R}_A \sharp$  and  $\mathcal{Q}_a \sharp$ .

The first observation was established as part of a more involved argument in the second part of the proof of Proposition 13:

**Lemma 19.** *If  $K_{DE} \in \Gamma(\mathcal{T}_{(2)}(2))$ , then*

$$X^D X^E \mathbb{D}_A \mathbb{D}_B \mathbb{D}_C K_{DE} = 6 \mathbb{D}_{(A} K_{BC)}.$$

*In particular,  $X^E X^D \mathbb{D}_A \mathbb{D}_B \mathbb{D}_C K_{DE}$  is totally symmetric.*

*Proof.* A direct computation using the relation (17) implies

$$(58) \quad X^C \mathbb{D}_A V_{CB\dots} = [X^C, \mathbb{D}_A] V_{CB\dots} + \mathbb{D}_A (X^C V_{CB\dots}) = -V_{AB\dots} + \mathbb{D}_A (X^C V_{CB\dots}).$$

This can be used to commute  $X^E$  and  $X^D$  past the  $\mathbb{D}$ 's until  $X^E K_{EA} = 0$  can be applied.  $\square$

Now we study the projection  $P := P_{(2,2)}$  from  $\otimes^4 \mathcal{T}^*$  to  $\mathcal{T}_{(2,2)}$  defined in (28). If  $S_{BCDE}$  is an element in  $\otimes^4 \mathcal{T}^*$  that is symmetric in  $D$  and  $E$ , i.e.,  $S_{BCDE} = S_{BC(DE)}$ , then a straightforward computation shows that for  $S_{BCDE} \in \otimes^2 \mathcal{T}^* \otimes \mathcal{T}_{(2)}$  we have

$$(59) \quad (PS)_{BCDE} = \frac{1}{4} (S_{(BC)DE} + S_{(DE)BC}) - \frac{1}{8} (S_{(DC)BE} + S_{(EB)CD} + S_{(DB)CE} + S_{(EC)BD}).$$

This implies indeed that

$$(S_{(ijk)} PS)_{BCDE} = 0,$$

i.e., the symmetrisation of  $PS$  over any three indices  $1 \leq i < j < k \leq 4$  vanishes.

Next, for a section  $K_{DE} \in \Gamma(\mathcal{T}_{(2)}(2))$  we set  $S_{BCDE} := \mathbb{D}_B \mathbb{D}_C K_{DE}$ . Note that the differential splittig operator  $\mathcal{L}$  is given by  $\mathcal{L}(k)_{bd} = (P\mathbb{D}^2 K)_{BCDE}$ . We obtain the following statement, which was already observed in the proof Theorem 11 and Proposition 15 for general rank:

**Lemma 20.** *If  $K_{DE} \in \Gamma(\mathcal{T}_{(2)}(2))$ , then*

$$X^E X^D \mathbb{D}_A (P\mathbb{D}^2 K)_{BCDE} = \frac{1}{4} X^E X^D \mathbb{D}_A \mathbb{D}_B \mathbb{D}_C K_{DE}$$

*Proof.* We use the formula (59) for  $S_{BCDE} := \mathbb{D}_B \mathbb{D}_C K_{DE}$  and apply  $\mathbb{D}_A$  to it. Using relation (58) as well as  $X^D K_{DB} = 0$  and equations (16), a direct computation shows that each of the last eight terms in the right hand side of (59) vanishes when contracted with  $X^D$  and  $X^E$ . For example,

$$\begin{aligned} X^E X^D \mathbb{D}_A \mathbb{D}_D \mathbb{D}_C K_{BE} &= -\mathbb{D}_C K_{BA} + X^D \mathbb{D}_A \mathbb{D}_C K_{BD} - X^D \mathbb{D}_A \mathbb{D}_C K_{BE} \\ &= -\mathbb{D}_A X^D \mathbb{D}_C K_{BD} - X^D \mathbb{D}_A \mathbb{D}_D K_{BC} \\ &= 2\mathbb{D}_A K_{BC} - \mathbb{D}_A X^D \mathbb{D}_D K_{BC} \\ &= 0. \end{aligned}$$

A similar computation shows that

$$X^E X^D \mathbb{D}_A \mathbb{D}_D \mathbb{D}_E K_{BC} = \mathbb{D}_A K_{BC} + [X^E, \mathbb{D}_A] \mathbb{D}_E K_{BC} = 0.$$

Hence, equation (59) implies that

$$X^E X^D \mathbb{D}_A (P\mathbb{D}^2 K)_{BCDE} = \frac{1}{4} X^E X^D \mathbb{D}_A \mathbb{D}_B \mathbb{D}_C K_{DE} = \frac{1}{4} X^E X^D \mathbb{D}_A \mathbb{D}_B \mathbb{D}_C K_{DE},$$

where the second equality follows from Lemma 19.  $\square$

The following lemma will give a formula for the projection  $P$ , when restricted to  $\mathcal{T} \otimes \mathcal{T}_{(2,1)}$ , i.e., applied to  $S_{BCDC} \in \mathcal{T}^* \otimes \mathcal{T}_{(2,1)}$ .

**Lemma 21.** *Let  $P := P_{(2,2)}$  be the projection of  $\otimes^4 \mathcal{T}^*$  onto  $\mathcal{T}_{(2,2)}$  defined above and  $S_{BCDC} \in \mathcal{T}^* \otimes \mathcal{T}_{(2,1)}$ . Then*

$$(60) \quad (PS)_{BCDE} = \frac{3}{4} (S_{BCDE} - S_{[BC]DE}) - \frac{3}{8} (S_{[DC]BE} + S_{[EB]CD} + S_{[DB]CE} + S_{[EC]BD}).$$

*Proof.* We use equation (59) under the additional assumption that  $S_{BCDC} \in \mathbb{V}^* \otimes \mathbb{V}_{(2,1)}$ , i.e.,

$$(61) \quad S_{B(CDE)} = 0.$$

For the the third term on the right-hand-side in (59) we compute

$$S_{(DC)BE} = S_{CDBE} + S_{[DC]BE} = -S_{CBDE} - S_{CEBD} + S_{[DC]BE},$$

where the last equation uses equation (61). This allows to compute the sum of the last four terms in (59) as

$$\begin{aligned} (62) \quad & S_{(DC)BE} + S_{(EB)CD} + S_{(DB)CE} + S_{(EC)BD} = \\ & = -4S_{(CB)DE} - S_{CEDB} - S_{CDBE} - S_{BEDC} - S_{BDEC} \\ & \quad + S_{[DC]BE} + S_{[EB]CD} + S_{[DB]CE} + S_{[EC]BD} \\ & = -2S_{(CB)DE} + S_{[DC]BE} + S_{[EB]CD} + S_{[DB]CE} + S_{[EC]BD}, \end{aligned}$$

where the last equation again follows from (61).

Now we look at the second term on the right-hand-side of (59): using (61) we get that

$$\begin{aligned} S_{(DE)BC} &= -\frac{1}{2} (S_{DBCE} + S_{DCEB} + S_{EBCD} + S_{ECDB}) \\ &= -\frac{1}{2} (S_{BDCE} + S_{CDEB} + S_{BECD} + S_{CEDB}) \\ & \quad - (S_{[DB]CE} + S_{[DC]EB} + S_{[EB]CD} + S_{[EC]DB}) \\ &= S_{(BC)DE} - (S_{[DB]CE} + S_{[DC]EB} + S_{[EB]CD} + S_{[EC]DB}). \end{aligned}$$

Hence, equation (27) from the flat case generalises to

$$(63) \quad S_{(BC)DE} = S_{(DE)BC} + (S_{[DB]CE} + S_{[DC]EB} + S_{[EB]CD} + S_{[EC]DB}).$$

Then putting (62) and (63) together, for  $S_{BCDE} \in \mathbb{V}^* \otimes \mathbb{V}_{(2,1)}$ , finishes the proof.  $\square$

Now assume that  $\mathbb{D}_C$  is the Thomas  $\mathbb{D}$ -operator and  $K_{DE}$  is symmetric such that

$$(64) \quad \mathbb{D}_{(C}K_{DE)} = 0.$$

Then set  $S_{BCDE} := \mathbb{D}_B\mathbb{D}_C K_{DE}$  in the above equations. Observe that

$$\begin{aligned} S_{[BC]DE} &= \mathbb{D}_{[B}\mathbb{D}_{C]}K_{DE} = \frac{1}{2}(\mathbb{D}_B\mathbb{D}_C K_{DE} - \mathbb{D}_C\mathbb{D}_B K_{DE}) \\ &= \frac{1}{2}W_{BC} \# K_{DE} = -W_{BC}{}^F{}_{(D}K_{E)F}. \end{aligned}$$

Then, from Lemma 21 we get an explicit version of the curvature terms in Proposition 12:

**Proposition 22.** *Let  $\mathbb{D}$  be the Thomas  $\mathbb{D}$ -operator for a projective structure with curvature  $W_{AB}{}^C{}_D$  and let  $P$  be the projection from  $\mathcal{T}^* \otimes \mathcal{T}_{(2,1)}$  to  $\mathcal{T}_{(2,2)}$ . Then  $K \in \Gamma(\mathcal{T}_{(2)}^*)$  satisfies  $\mathbb{D}_{(A}K_{BC)} = 0$ , i.e.,  $\mathbb{D}_A K_{BC} \in \mathcal{T}_{(2,1)}$ , if and only if*

$$(65) \quad (P\mathbb{D}^2 K)_{BCDE} = \frac{3}{4}\mathbb{D}_B\mathbb{D}_C K_{DE} - \frac{3}{8}(W_{BC}\#K_{DE} + W_{D(B}\#K_{C)E} + W_{E(B}\#K_{C)D})$$

that is

$$\mathbb{D}_B\mathbb{D}_C K_{DE} + \frac{1}{2}(W_{BC}\#K_{DE} + W_{D(B}\#K_{C)E} + W_{E(B}\#K_{C)D}) \in V_{(2,2)}.$$

*Proof.* One direction immediately follows from Lemma 21 applied to  $S_{BCDE} := \mathbb{D}_B\mathbb{D}_C K_{DE}$ .

For the other direction assume that equation (65) holds. Contracting with  $X^B$  and noting that  $X^B W_{B\dots} = 0$  as well as  $X^B K_{BC} = 0$  implies that

$$(66) \quad X^B(P\mathbb{D}^2 K)_{BCDE} = \frac{3}{4}X^B\mathbb{D}_B\mathbb{D}_C K_{DE} = \frac{3}{4}\mathbb{D}_C K_{DE}$$

from the definition of  $\mathbb{D}_B$ . Hence, since  $P\mathbb{D}^2 K \in \Gamma(\mathcal{T}_{(2,2)})$ , the symmetrisation over  $CDE$  vanishes.  $\square$

Note that, from equation (65) we obtain that

$$(67) \quad X^B X^C(P\mathbb{D}^2 K)_{BCDE} = \frac{3}{4}X^B X^C \mathbb{D}_B\mathbb{D}_C K_{DE} = \frac{3}{4}X^C \mathbb{D}_C K_{DE} = \frac{3}{2}K_{DE},$$

because of (16) and (22).

Next we determine the connection for which  $(P\mathbb{D}^2 K)_{BCDE}$  is going to be parallel, i.e., we determine explicitly the curvature terms in Proposition 13, Theorem 15 and Corollary 17. To get a formula for its covariant derivative with respect to the projective tractor connection, we apply  $\mathbb{D}$  to the equality in Proposition 22 to get

$$(68) \quad 4\mathbb{D}_C(P\mathbb{D}^2 K)_{DEAB} = 3\mathbb{D}_C\mathbb{D}_D\mathbb{D}_E K_{AB} - \frac{3}{2}\mathbb{D}_C(W_{DE}\#K_{AB} + W_{A(D}\#K_{E)B} + W_{B(D}\#K_{E)A}).$$

We are now going to obtain a formula for  $T_{CDEAB} = \mathbb{D}_C\mathbb{D}_D\mathbb{D}_E K_{AB} \in \otimes^5 \mathcal{V}^*$ . This is achieved by the following lemmas.

**Lemma 23.** *For every  $T \in \otimes^5 \mathcal{T}^*$  it holds*

$$\begin{aligned} &T_{C(DE)AB} + T_{D(EC)AB} + T_{E(CD)AB} \\ &= 3T_{CDEAB} + 3T_{C[ED]AB} + T_{D[EC]AB} + T_{E[DC]AB} + 2T_{[EC]DAB} + 2T_{[DC]EAB}. \end{aligned}$$

*Proof.* The proof is by inspection.  $\square$

**Lemma 24.** *Let  $T_{ABCDE} \in \otimes^2 \mathcal{T}^* \otimes \mathcal{T}_{(2,1)}$ , i.e.,  $T_{AB(CDE)} = 0$ . Then*

$$\begin{aligned} -3T_{CDEAB} &= 2T_{[EC]DAB} + 2T_{[DC]EAB} + 2T_{[AC]BDE} + 2T_{[AD]BEC} + 2T_{[AE]BCD} \\ &\quad + 2T_{A[BC]DE} + 2T_{A[BD]EC} + 2T_{A[BE]CD} \\ &\quad + 3T_{C[ED]AB} + T_{C[DA]BE} + T_{C[DB]EA} + T_{C[EA]BD} + T_{C[EB]DA} + T_{C[AB]DE} \\ &\quad + T_{D[EC]AB} + T_{D[EA]BC} + T_{D[EB]CA} + T_{D[CA]BE} + T_{D[CB]EA} + T_{D[AB]EC} \\ &\quad + T_{E[DC]AB} + T_{E[CA]BD} + T_{E[CB]DA} + T_{E[DA]BC} + T_{E[DB]CA} + T_{E[AB]CD}. \end{aligned}$$

*Proof.* First we can swap the pair  $AB$  with  $DE$  by using (63) for the second equality in

$$\begin{aligned} T_{ABCDE} &= T_{C(AB)DE} + T_{C[AB]DE} + 2T_{[AC]BDE} + 2T_{A[BC]DE} \\ &= T_{C(DE)AB} + T_{C[DA]BE} + T_{C[DB]EA} + T_{C[EA]BD} + T_{C[EB]DA} \\ &\quad + T_{C[AB]DE} + 2T_{[AC]BDE} + 2T_{A[BC]DE}. \end{aligned}$$

In an analogous computation as in the flat case, this can be used to evaluate

$$\begin{aligned} 0 &= 3T_{AB(CDE)} \\ &= T_{C(DE)AB} + T_{D(EC)AB} + T_{E(CD)AB} \\ &\quad + T_{C[DA]BE} + T_{C[DB]EA} + T_{C[EA]BD} + T_{C[EB]DA} + T_{C[AB]DE} + 2T_{[AC]BDE} + 2T_{A[BC]DE} \\ &\quad + T_{D[EA]BC} + T_{D[EB]CA} + T_{D[CA]BE} + T_{D[CB]EA} + T_{D[AB]EC} + 2T_{[AD]BEC} + 2T_{A[BD]EC} \\ &\quad + T_{E[CA]BD} + T_{E[CB]DA} + T_{E[DA]BC} + T_{E[DB]CA} + T_{E[AB]CD} + 2T_{[AE]BCD} + 2T_{A[BE]CD} \end{aligned}$$

Now we apply Lemma 23 to the terms  $T_{C(DE)AB} + T_{D(EC)AB} + T_{E(CD)AB}$  in this equation to get to get

$$\begin{aligned} 0 &= 3T_{CDEAB} + 3T_{C[ED]AB} + T_{D[EC]AB} + T_{E[DC]AB} + 2T_{[EC]DAB} + 2T_{[DC]EAB} \\ &\quad + T_{C[DA]BE} + T_{C[DB]EA} + T_{C[EA]BD} + T_{C[EB]DA} + T_{C[AB]DE} + 2T_{[AC]BDE} + 2T_{A[BC]DE} \\ &\quad + T_{D[EA]BC} + T_{D[EB]CA} + T_{D[CA]BE} + T_{D[CB]EA} + T_{D[AB]EC} + 2T_{[AD]BEC} + 2T_{A[BD]EC} \\ &\quad + T_{E[CA]BD} + T_{E[CB]DA} + T_{E[DA]BC} + T_{E[DB]CA} + T_{E[AB]CD} + 2T_{[AE]BCD} + 2T_{A[BE]CD}, \end{aligned}$$

which implies the formula in the lemma.  $\square$

By applying this lemma to  $T_{CDEAB} = \mathbb{D}_C \mathbb{D}_D \mathbb{D}_E K_{AB} \in \Gamma(\otimes^2 \mathcal{T}^* \otimes \mathcal{T}_{(2,1)})$  for  $K_{AB} \in \Gamma(\mathcal{T}_{(2)})$  and by replacing skew-symmetrisations by curvature, for example,

$$T_{[EC]DAB} = \frac{1}{2} (\mathbb{D}_E \mathbb{D}_C \mathbb{D}_D K_{AB} - \mathbb{D}_C \mathbb{D}_E \mathbb{D}_D K_{AB}) = \frac{1}{2} W_{EC} \sharp \mathbb{D}_D K_{AB}$$

and

$$T_{A[BC]DE} = \frac{1}{2} (\mathbb{D}_A \mathbb{D}_B \mathbb{D}_C K_{DE} - \mathbb{D}_A \mathbb{D}_C \mathbb{D}_B K_{DE}) = \frac{1}{2} \mathbb{D}_A (W_{BC} \sharp K_{DE}),$$

we obtain the following result. Here and henceforth we use the following convention: the notation  $|B|$  or  $|A \cdots B|$  means that the index  $B$ , or the indices  $A \cdots B$ , are excluded from any surrounding symmetrisation.

**Proposition 25.** *Let  $\mathbb{D}$  be the Thomas  $\mathbb{D}$ -operator for a projective structure with curvature  $W_{AB}{}^C{}_D$  and let  $P$  be the map from  $\mathcal{T}^* \otimes \mathcal{T}_{(2,1)}$  to  $\mathcal{T}_{(2,2)}$  defined in (28). Then  $K \in \mathcal{T}_{(2)}^*$  satisfies  $\mathbb{D}_{(A} K_{BC)} = 0$ , i.e.,  $\mathbb{D}_A K_{BC} \in \mathcal{T}_{(2,1)}$ , if and only if,*

$$\begin{aligned} (69) \quad \mathbb{D}_C (P \mathbb{D}^2 K)_{DEAB} &= \frac{1}{2} W_{C(D} \sharp \mathbb{D}_E) K_{AB} - \frac{3}{4} W_{A(C} \sharp \mathbb{D}_{|B|} K_{DE}) - \frac{3}{4} \mathbb{D}_A (W_{B(C} \sharp K_{DE})) \\ &\quad - \frac{1}{8} \mathbb{D}_C (W_{AB} \sharp K_{DE} - W_{E(A} \sharp K_{B)D} - W_{D(A} \sharp K_{B)E}) \\ &\quad - \frac{1}{8} \mathbb{D}_D (W_{AB} \sharp K_{EC} + W_{EC} \sharp K_{AB} + 2W_{E(A} \sharp K_{B)C} + 2W_{C(A} \sharp K_{B)E}) \\ &\quad - \frac{1}{8} \mathbb{D}_E (W_{AB} \sharp K_{DC} + W_{DC} \sharp K_{AB} + 2W_{C(A} \sharp K_{B)D} + 2W_{D(A} \sharp K_{B)C}). \end{aligned}$$

*Proof.* First assume that equation (69) holds. We contract this equation with  $X^A$  and  $X^B$ . It is a direct computation to see the then the right hand side is zero: to see this, recall that  $X^A W_{A\dots} = 0$  and  $X^A K_{AC} = 0$  and that equation (58) applied to  $V_{C\dots}$  with  $X^C V_{C\dots} = 0$  gives

$$(70) \quad X^C \mathbb{D}_A V_{CB\dots} = [X^C, \mathbb{D}_A] V_{CB\dots} + \mathbb{D}_A (X^C V_{CB\dots}) = -V_{AB\dots}$$

Then, from the obtained  $X^A X^B \mathbb{D}_C (P \mathbb{D}^2 K)_{DEAB} = 0$  and from Lemmas 19 and 20 we obtain the required symmetry of  $\mathbb{D}_C K_{ED}$ .

For the other direction we apply Lemma 24 to  $T_{CDEAB} = \mathbb{D}_C \mathbb{D}_D \mathbb{D}_E K_{AB} \in \otimes^2 \mathcal{T}^* \otimes \mathcal{T}_{(2,1)}$ . Equation in Lemma 24 then becomes

$$\begin{aligned} -3\mathbb{D}_C \mathbb{D}_D \mathbb{D}_E K_{AB} &= -2W_{C(E\#\mathbb{D}_D)} K_{AB} + 3W_{A(C\#\mathbb{D}_B)} K_{DE} + 3\mathbb{D}_A (W_{B(C\#K_{DE})}) \\ &\quad + \frac{1}{2}\mathbb{D}_C (3W_{ED\#K_{AB}} + W_{AB\#K_{DE}} - 2W_{A(E\#K_D)B} - 2W_{B(D\#K_E)A}) \\ &\quad + \frac{1}{2}\mathbb{D}_D (W_{AB\#K_{EC}} + W_{EC\#K_{AB}} + 2W_{E(A\#K_B)C} + 2W_{C(A\#K_B)E}) \\ &\quad + \frac{1}{2}\mathbb{D}_E (W_{AB\#K_{DC}} + W_{DC\#K_{AB}} + 2W_{C(A\#K_B)D} + 2W_{D(A\#K_B)C}). \end{aligned}$$

Now we plug this in for the term  $\mathbb{D}_C \mathbb{D}_D \mathbb{D}_E K_{AB}$  in (69) that was obtained by differentiating the equality in 22:

$$\begin{aligned} 4\mathbb{D}_C (P\mathbb{D}^2 K)_{BCDE} &= 3\mathbb{D}_C \mathbb{D}_D \mathbb{D}_E K_{AB} - \frac{3}{2}\mathbb{D}_C (W_{DE\#K_{AB}} + W_{A(D\#K_E)B} + W_{B(D\#K_E)A}) \\ &= 2W_{C(E\#\mathbb{D}_D)} K_{AB} - 3W_{A(C\#\mathbb{D}_B)} K_{DE} - 3\mathbb{D}_A (W_{B(C\#K_{DE})}) \\ &\quad - \frac{1}{2}\mathbb{D}_C (W_{AB\#K_{DE}} + W_{A(E\#K_D)B} + W_{B(D\#K_E)A}) \\ &\quad - \frac{1}{2}\mathbb{D}_D (W_{AB\#K_{EC}} + W_{EC\#K_{AB}} + 2W_{E(A\#K_B)C} + 2W_{C(A\#K_B)E}) \\ &\quad - \frac{1}{2}\mathbb{D}_E (W_{AB\#K_{DC}} + W_{DC\#K_{AB}} + 2W_{C(A\#K_B)D} + 2W_{D(A\#K_B)C}). \end{aligned}$$

This finishes the proof.  $\square$

Now are going to expand the terms in (69) using the Leibniz rule

$$(71) \quad \mathbb{D}_A (W_{BC\#K_{DE}}) = (\mathbb{D}_A W_{BC})\#K_{DE} + W_{BC}\#(\mathbb{D}_A K_{DE}) + W_{BC}{}^H{}_A \mathbb{D}_H K_{DE},$$

and then substituting  $K_{DE}$  and  $\mathbb{D}_A K_{DE}$  terms by contractions of  $X^F$  with  $L_{FADE} = (P\mathbb{D}^2 K)_{FADE}$  using relations (67) and (66):

$$K_{DE} = \frac{2}{3}X^F X^G L_{FGDE}, \quad \mathbb{D}_A K_{DE} = \frac{4}{3}X^F L_{FADE}.$$

To this end, first one checks that  $X^F W_{FBCD} = 0$  and  $\mathbb{D}_A X^F = \delta_A^F$  imply that

$$W_{BC}\#(X^F Q_{F\dots}) = X^F W_{BC}\#Q_{F\dots},$$

and

$$\mathbb{D}_A W_{BC}\#(X^F Q_{F\dots}) = X^F \mathbb{D}_A W_{BC}\#Q_{F\dots} - W_{BC}{}^H{}_A Q_{H\dots},$$

for any tensor  $Q_{F\dots}$ . For  $Q = L$  and  $Q = X^F L_{F\dots}$  this implies

$$W_{BC}\#(\mathbb{D}_A K_{DE}) = \frac{4}{3}W_{BC}\#(X^F L_{FADE}) = \frac{4}{3}X^F W_{BC}\#L_{FADE}$$

and

$$\begin{aligned} (\mathbb{D}_A W_{BC})\#K_{DE} &= \frac{2}{3}(\mathbb{D}_A W_{BC})\#(X^F X^G L_{FGDE}) \\ &= \frac{2}{3}X^F X^G \mathbb{D}_A W_{BC}\#L_{FGDE} - \frac{4}{3}X^F W_{BC}{}^H{}_A L_{FHDE}. \end{aligned}$$

Substituting this into equation (71), the terms  $W_{BC}{}^H{}_A \mathbb{D}_H K_{DE}$  are cancelled and we get

$$(72) \quad \mathbb{D}_A (W_{BC}\#K_{DE}) = \frac{2}{3}X^F X^G \mathbb{D}_A W_{BC}\#L_{FGDE} + \frac{4}{3}X^F W_{BC}\#L_{FADE}.$$

Then we compute step by step the terms in the right-hand-side of (69):

$$\begin{aligned} W_{C(D\#\mathbb{D}_E)} K_{AB} - \frac{1}{4}(\mathbb{D}_D (W_{EC\#K_{AB}}) + \mathbb{D}_E (W_{DC\#K_{AB}})) &= \\ = 2X^F W_{C(D\#L_E)FAB} - \frac{1}{3}X^F X^G \mathbb{D}_{(D} W_{E)C}\#L_{FGAB}. \end{aligned}$$

Next we consider the terms that are not evidently symmetric in  $A$  and  $B$ : using  $L_{A(CDE)} = 0$  as well as the second Bianchi identity for the Weyl tensor we compute

$$\begin{aligned} -\frac{3}{4}(\mathbb{D}_A (W_{B(C\#K_{DE})}) + W_{A(C\#\mathbb{D}_B)} K_{DE}) &= \\ -\frac{1}{8}(\mathbb{D}_C (W_{AB\#K_{DE}}) + \mathbb{D}_D (W_{AB\#K_{EC}}) + \mathbb{D}_E (W_{AB\#K_{CD}})) &= \\ = -2X^F W_{(B|(C\#L_{DE})|A)F} - \frac{1}{2}X^F X^G \mathbb{D}_{(A} W_{B)(C\#L_{DE})FG} &= \\ = \frac{2}{3}X^F (W_{C(A\#L_B)FED} - 2W_{A|(D\#L_E)C|B)F}) &= \\ -\frac{1}{6}X^F X^G (\mathbb{D}_{(A} W_{B)C}\#L_{DEFG} + 2\mathbb{D}_{(A} W_{B)(D\#L_E)CFG}) &= \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2}\mathbb{D}_C (W_{E(A\sharp K_B)D} + W_{D(A\sharp K_B)E}) \\ & - \mathbb{D}_D (W_{E(A\sharp K_B)C} + W_{C(A\sharp K_B)E}) - \mathbb{D}_E (W_{C(A\sharp K_B)D} + W_{D(A\sharp K_B)C}) = \\ & = \frac{4}{3}X^F (W_{C(A\sharp L_B)FED} + 2W_{(D|(A\sharp L_B)F|E)C} + 3W_{(D|(A\sharp L_B)|E)FC}) \\ & - \frac{2}{3}X^F X^G (2\mathbb{D}_{(D}W_{E)(A\sharp L_B)CFG} + \mathbb{D}_{(A}W_{|C(D\sharp L_E)|B)FG} + \mathbb{D}_{(D}W_{|C(A\sharp L_B)|E)FG}). \end{aligned}$$

Now note that because of the pairwise symmetry of  $L$  and the skew symmetry of  $W$ , we have

$$W_{(A|(D\sharp L_E)C|B)F} = -W_{(D|(A\sharp L_B)F|E)C}.$$

This allows to collect some of the terms above as

$$\begin{aligned} & \frac{2}{3}W_{(D|(A\sharp L_B)F|E)C} + W_{(D|(A\sharp L_B)|E)FC} - \frac{4}{3}W_{(A|(D\sharp L_E)C|B)F} = \\ & = 2W_{(D|(A\sharp L_B)F|E)C} + W_{(D|(A\sharp L_B)|E)FC} \\ & = W_{(D|(A\sharp L_B)F|E)C} + W_{(A|(D\sharp L_E)F|B)C}, \end{aligned}$$

where the last equality follows from  $L_{ECBF} = L_{BFEC}$  and  $L_{B(FEC)} = 0$ . Hence, we get the following formula for  $\mathbb{D}_C L_{DEAB}$  for  $L := P(\mathbb{D}^2 K)$ :

$$\begin{aligned} (73) \quad \mathbb{D}_C L_{DEAB} & = X^F (W_{C(D\sharp L_E)FAB} + W_{C(A\sharp L_B)FED}) \\ & + X^F (W_{(D|(A\sharp L_B)F|E)C} + W_{(A|(D\sharp L_E)F|B)C}) \\ & - \frac{1}{6}X^F X^G (\mathbb{D}_{(D}W_{E)C\sharp L_{ABFG}} + \mathbb{D}_{(A}W_{B)C\sharp L_{DEFG}}) \\ & - \frac{1}{3}X^F X^G (\mathbb{D}_{(D}W_{E)(A\sharp L_B)CFG} + \mathbb{D}_{(A}W_{B)(D\sharp L_E)CFG}) \\ & - \frac{1}{6}X^F X^G (\mathbb{D}_{(A}W_{|C(D\sharp L_E)|B)FG} + \mathbb{D}_{(D}W_{|C(A\sharp L_B)|E)FG}). \end{aligned}$$

Having this formula, we can formulate the following result:

**Theorem 26.** *Let  $(M, \mathfrak{p})$  be an arbitrary projective manifold. Then the splitting operator  $\mathcal{L} : S^2 T^* M(4) \rightarrow \mathcal{T}_{(2,2)}$  gives an isomorphism between weighted Killing tensors of rank 2 and sections  $L_{DEAB}$  of the tractor bundle  $\mathcal{T}_{(2,2)}$  of weight zero that satisfy equation (73).*

*Proof.* Given a rank 2 tensor  $k_{ab}$  we define  $L_{DEAB} = \mathbb{D}_D \mathbb{D}_E K_{AB}$  and  $L_{DEAB} := (P\mathbb{D}^2 K)_{DEAB}$ . Then, if  $k_{ab}$  is Killing, it follows from Proposition 25 and the above computations that  $L_{DEAB}$  satisfies equation (73).

On the other hand, let  $L_{DEAB}$  be a section of  $\mathcal{T}_{(2,2)}$  of weight zero that satisfies equation (73). Contracting (73) with  $X^D$  and  $X^E$ , one can easily check, using the same arguments as before and that  $L_{(DEF)B} = 0$ , that the right-hand-side vanishes and thus

$$0 = X^D X^E \mathbb{D}_C L_{DEAB}$$

Then from Proposition 8 it follows that  $L_{DEAB}$  defines a Killing tensor  $k_{ab}$ . Moreover we see that  $L_{DEAB} = \mathcal{L}(k)_{DEAB}$  unless the map

$$L_{DEAB} \mapsto K_{AB} = X^D X^E L_{DEAB} \in \mathcal{T}_{(2)}(2).$$

has a kernel. So let's assume there is a section  $L_{DEAB}$  of  $\mathcal{T}_{(2,2)}$  that satisfies (73) and such that

$$(74) \quad X^D X^E L_{DEAB} = 0.$$

Applying  $\mathbb{D}_C$  to this and using  $0 = X^D X^E \mathbb{D}_C L_{DEAB}$  implies that  $0 = X^D L_{DEAB}$ . Applying  $\mathbb{D}_C$  to this gives

$$0 = L_{CEAB} + X^D \mathbb{D}_C L_{DEAB} = L_{CEAB}.$$

Here the second equality uses (73), which allows us to compute

$$X^D \mathbb{D}_C L_{DEAB} = X^D X^F (W_{C(A\sharp L_B)FED} + \frac{1}{2}W_{E(A\sharp L_B)FDC}).$$

But now  $L_{B(FED)} = 0$  and (74) imply that

$$X^D X^F W_{C(A\sharp L_B)FED} = -X^D X^F W_{C(A\sharp L_B)DEF} = 0,$$

which proves that  $X^D \mathbb{D}_C L_{DEAB} = 0$  and finishes the proof.  $\square$

Note that the right hand side of (73) indeed defines a section  $\mathcal{R}_C \sharp$  of  $\mathcal{T}^* \otimes \mathcal{T}_{(2,2)}$  as claimed in the proof of Theorem 15.

In order to extract a covariant derivative from this, we have to contract it with  $W^C_c$ . In general this contraction is not projectively invariant. However, since  $L_{DEAB}$  has weight zero, applying  $\mathbb{D}_C$  to it and contracting with  $X^C$  gives zero,  $X^C \mathbb{D}_C L_{DEAB} = 0$ . Hence, the contraction  $W^C_c \mathbb{D}_C L_{DEAB}$  is also projectively invariant for sections  $L_{DEAB}$  that satisfy equation (73). However we need that the curvature term in right hand side of (73) is projectively invariant as claimed in the proof of Theorem 15, i.e., that the right hand side of (73) is projectively invariant for *any*  $L_{DEAB} \in \mathcal{T}_{(2,2)}$  not only for solutions of (73). This is the statement of the following lemma.

**Lemma 27.** *For any  $L_{ABDE} \in \mathcal{T}_{(2,2)}$  the right hand side in equation (73) gives zero when contracted with  $X^C$ . In particular, the section of  $\mathcal{T}^* \otimes \text{End}(\mathcal{T}_{(2,2)})$  defined by the right hand side in (73) is projectively invariant.*

*Proof.* Clearly both of the terms of the form  $X^C W_{C(D \sharp L_E)FAB}$  in the first line of (73) vanish separately because  $X^C W_{CABC} = 0$ . Also both terms of the form  $X^C X^F X^G \mathbb{D}_{(D} W_{E)(A \sharp L_B)CFG}$  in the fourth line of (73) vanish separately because  $L_{B(CFG)} = 0$ . Similarly both terms of the form  $X^C X^F X^G \mathbb{D}_{(D} W_{E)C \sharp L_{ABFG}}$  in the third line of (73) vanish separately because  $X^C W_{CABC} = 0$  and

$$X^C \mathbb{D}_{(D} W_{E)C \sharp L_{ABFG}} = -\delta^C_{(D} W_{E)C \sharp L_{ABFG}} = W_{(EC) \sharp L_{ABFG}} = 0.$$

All the other terms in the second and fifth line of (73) do not vanish separately but cancel against each other when contracted with  $X^C$ . In fact we have

$$X^C (\mathbb{D}_{(A} W_{|C(D \sharp L_E)|B)FG} + \mathbb{D}_{(D} W_{|C(A \sharp L_B)|E)FG}) = -W_{(A|(D \sharp L_E)|B)FG} - W_{(D|(A \sharp L_B)|E)FG} = 0,$$

and for the terms in the second line

$$X^C X^F (W_{(D|(A \sharp L_B)F|E)C} + W_{(A|(D \sharp L_E)F|B)C}) = 0,$$

because of the skew-symmetry of  $W_{DA}$ .  $\square$

In order to obtain from equation (73) an equation involving the tractor derivative  $\nabla_c$ , we have to contract it with  $W^C_c$ . First we look at terms that for which the contracted index  $C$  is at the curvature (or its derivative)  $W_{AC}$ . These will turn out to be manifestly invariant as we can eliminate  $W^C_c$ : First we observe that

$$W^C_c W_{CA \sharp L_{BFED}} = Z_A^a \kappa_{ca} \sharp L_{BFED},$$

where  $\kappa_{ca}^H_G$  is the tractor curvature defined in (13). Hence, for the terms in the first line in equation (73) we get

$$X^F (W_{C(D \sharp L_E)FAB} + W_{C(A \sharp L_B)FED}) = X^F (Z_{(A}^a \kappa_{|ca|} \sharp L_{B)FED} + Z_{(D}^a \kappa_{|ca|} \sharp L_{E)FAB}),$$

which is manifestly invariant. Next we compute, using formulae (12) and that the weight of  $W_{CD}^H_F$  is  $-2$ , that

$$\begin{aligned} W^C_c \mathbb{D}_A W_{BC} &= -2Y_A Z_B^b \kappa_{bc} + Z_A^a W^C_c \nabla_a W_{BC} \\ &= -2Y_A Z_B^b \kappa_{bc} + Z_A^a (\nabla_a (W^C_c W_{BC}) - \nabla_a W^C_c W_{BC}) \\ &= -2Y_A Z_B^b \kappa_{bc} + Z_A^a \nabla_a (Z_B^b \kappa_{bc}) \\ &= -(2Y_A Z_B^b + Y_B Z_A^b) \kappa_{bc} + Z_A^a Z_B^b \nabla_a \kappa_{bc}, \end{aligned}$$

because  $\nabla_a W^C_c W_{BC} = -P_{ac} X^C W_{BC} = 0$  and  $\nabla_a Z_B^b = -\delta_a^b Y_B$ . Hence, for the expressions in the third line of (73) we get,

$$X^F X^G \mathbb{D}_{(A} W_{B)C \sharp L_{DEFG}} = X^F X^G (-3Y_{(A} Z_{B)}^b \kappa_{bc} + Z_{(A}^a Z_{B)}^b \nabla_a \kappa_{bc}) \sharp L_{DEFG}$$

and

$$X^F X^G \mathbb{D}_{(D)W_E C} \# L_{ABFG} = X^F X^G (-3Y_{(D)Z_E}{}^b \kappa_{bc} + Z_{(D^a Z_E)}{}^b \nabla_a \kappa_{bc}) \# L_{ABFG}.$$

Similarly we get for the expressions in the fifth line of (73),

$$\begin{aligned} & (\mathbb{D}_{(A)W_{|C(D)\#L_E|B}FG} + \mathbb{D}_{(D)W_{|C(A)\#L_B|E}FG}) = \\ &= -\frac{1}{2} (\mathbb{D}_{(A)W_D C} \# L_{BEFG} + \mathbb{D}_{(A)W_E C} \# L_{DBFG} + \mathbb{D}_{(B)W_D C} \# L_{AEFG} + \mathbb{D}_{(B)W_E C} \# L_{DAFG}) \\ &= \frac{1}{2} (3Y_{(A)Z_D}{}^b \kappa_{bc} - Z_{(A^a Z_D)}{}^b \nabla_a \kappa_{bc}) \# L_{BEFG} \\ &+ \frac{1}{2} (3Y_{(A)Z_E}{}^b \kappa_{bc} - Z_{(A^a Z_E)}{}^b \nabla_a \kappa_{bc}) \# L_{BDFG} \\ &+ \frac{1}{2} (3Y_{(B)Z_D}{}^b \kappa_{bc} - Z_{(B^a Z_D)}{}^b \nabla_a \kappa_{bc}) \# L_{AEFG} \\ &+ \frac{1}{2} (3Y_{(B)Z_E}{}^b \kappa_{bc} - Z_{(B^a Z_E)}{}^b \nabla_a \kappa_{bc}) \# L_{ADFG}. \end{aligned}$$

Finally, we compute

$$W_{(D|A)\#L_B F|E)C} + W_{(A|(D)\#L_E F|B)C} = Z_{(A^a Z_{|(D^d \kappa_{|ad|}\#(L_E)F|B)C} - L_E)C|B)F}$$

and

$$\mathbb{D}_{(A)W_B(D)\#L_E C)FG} = - (3Y_{(A)Z_B}{}^b Z_{(D^d \kappa_{|bd|} - Z_{(A^a Z_B)}{}^b Z_{(D^d \nabla_{|a \kappa_{bd|})} \# L_E)CFG},$$

to rewrite equation (73) in terms of the tractor connection as

$$\begin{aligned} (75) \quad \nabla_c L_{DEAB} &= X^F (Z_{(A^a \kappa_{|ca|}\#L_B)FED} + Z_{(D^a \kappa_{|ca|}\#L_E)FAB}) \\ &+ X^F W^C{}_c Z_{(A^a Z_{|(D^d \kappa_{|ad|}\#(L_E)F|B)C} - L_E)C|B)F} \\ &- \frac{1}{12} X^F X^G (3Y_{(A)Z_D}{}^b \kappa_{bc} - Z_{(A^a Z_D)}{}^b \nabla_a \kappa_{bc}) \# L_{BEFG} \\ &- \frac{1}{12} X^F X^G (3Y_{(A)Z_E}{}^b \kappa_{bc} - Z_{(A^a Z_E)}{}^b \nabla_a \kappa_{bc}) \# L_{BDFG} \\ &- \frac{1}{12} X^F X^G (3Y_{(B)Z_D}{}^b \kappa_{bc} - Z_{(B^a Z_D)}{}^b \nabla_a \kappa_{bc}) \# L_{AEFG} \\ &- \frac{1}{12} X^F X^G (3Y_{(B)Z_E}{}^b \kappa_{bc} - Z_{(B^a Z_E)}{}^b \nabla_a \kappa_{bc}) \# L_{ADFG} \\ &+ \frac{1}{6} X^F X^G (3Y_{(A)Z_B}{}^b \kappa_{bc} - Z_{(A^a Z_B)}{}^b \nabla_a \kappa_{bc}) \# L_{DEFG} \\ &+ \frac{1}{6} X^F X^G (3Y_{(D)Z_E}{}^b \kappa_{bc} - Z_{(D^a Z_E)}{}^b \nabla_a \kappa_{bc}) \# L_{ABFG} \\ &- \frac{1}{3} X^F X^G W^C{}_c (3Y_{(A)Z_B}{}^b Z_{(D^d \kappa_{|bd|} - Z_{(A^a Z_B)}{}^b Z_{(D^d \nabla_{|a \kappa_{bd|})} \# L_E)CFG} \\ &- \frac{1}{3} X^F X^G W^C{}_c (3Y_{(D)Z_E}{}^b Z_{(A^d \kappa_{|bd|} - Z_{(D^a Z_E)}{}^b Z_{(A^d \nabla_{|a \kappa_{bd|})} \# L_B)CFG}, \end{aligned}$$

where  $\nabla_c$  is the projective tractor connection and  $\kappa_{bc}$  its curvature. The right hand side of this equation defines the section  $\mathcal{Q}_a \# \in \Gamma(T^*M \otimes \mathcal{T}_{(2,2)})$  in Corollary 17. Hence we arrive at:

**Theorem 28.** *Let  $(M, \mathfrak{p})$  be an arbitrary projective manifold. Then the splitting operator  $\mathcal{L} : S^2 T^* M(4) \rightarrow \mathcal{T}_{(2,2)}$  gives an isomorphism between weighted Killing tensors of rank 2 and sections  $L_{DEAB}$  of the tractor bundle  $\mathcal{T}_{(2,2)}$  of weight zero that satisfy equation (75) for the projective tractor connection  $\nabla_a$ , or equivalently, parallel sections of the connection  $\nabla_a - \mathcal{Q}_a \#$ . Moreover, the right hand side of (75) is projectively invariant.*

*Proof.* The proof follows immediately from Theorem 26 and Lemma 27 and from the computations above.  $\square$

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