The Parametric Oka Principle for Riemann Surfaces

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For Samantha Rose Griffiths.
The love of my life.
My heart and soul.
Now, forever, and always.
23/03/1995 – 19/09/2018

How lucky I am to have something that makes saying goodbye so hard.

– Winnie the Pooh.

To have been loved so deeply, even though the person who loved us is gone, will give us some protection forever.

– J.K. Rowling.

Contents

Abstract

In 1993, Winkelmann classified the pairs of Riemann surfaces which satisfy the basic Oka principle (BOP). We generalise Winkelmann's result to include the notion of the parametric Oka principle (POP). Using low-dimensional techniques from algebraic topology and Riemann surface theory, we provide accessible proofs of POP for all pairs of Riemann surfaces satisfying BOP, besides the case of an open Riemann surface mapping into the Riemann sphere. For this case, we provide partial results.

Winkelmann also provided a list of the pairs of Riemann surfaces which fail to satisfy BOP. To explore these pairs, we introduce the notion of the higher parametric Oka principle (hPOP). This is our own definition and is one of the main original contribution of this thesis. For Winkelmann's counterexamples (labelled (i)–(v)), we ask whether they satisfy hPOP. We provide a counterexample for case (i), showing hPOP fails. For cases (ii), (iv) and (v), we provide full proofs showing hPOP holds. For case (iii), we provide partial affirmative results of hPOP.

Signed Statement

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Chapter 1

Introduction

1.1 The context

Oka theory is a recent area of complex analysis that has bloomed since about 2000 that takes the rigidity of complex analysis and complex geometry, and combines them with the flexibility of topology. One of the basic questions in Oka theory is when can a continuous map from a Stein manifold — a complex manifold holomorphically embeddable into complex Euclidean space $\mathbb{C}^n$ for some n — into a complex manifold be deformed into a holomorphic map. For this reason we may think of Oka theory as the study of those complex spaces that have "*many*" holomorphic maps coming into them. Thus we may consider Oka theory as the dual to complex hyperbolic geometry, which is the study of spaces with "*few*" holomorphic maps coming into them.

A key concept in the study of Oka theory is the idea that analytical problems only have topological obstructions. One example of this is Grauert's classification of vector bundles over Stein manifolds [8]. Grauert showed that holomorphic vector bundles over Stein manifolds are holomorphically trivial if they are topologically trivial. Another example of this idea is the classical Cousin problems from the late 19th century, where the only obstruction to solutions to these problems on Stein manifolds are topological.

Modern Oka theory originates from Gromov's seminal paper in 1989 [9]. This paper introduced many of the fundamental concepts now central to Oka theory. Gromov asked for which complex manifolds Y can an arbitrary continuous map $X \to Y$ be deformed into a holomorphic map, where X is any Stein manifold. If a complex manifold Y satisfies this condition, then we say that Y satisfies the *basic Oka property* (BOP). He also explored the seemingly stronger *parametric Oka property* (POP) of a complex manifold Y, which says that, for an arbitrary Stein manifold X, the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ of the space of holomorphic maps $X \to Y$ into the space of continuous maps $X \to Y$ with the compact-open topology is a weak homotopy equivalence. Gromov established these results for *elliptic manifolds*, a class of complex manifolds he introduced.

If E is a holomorphic vector bundle over a complex manifold X, a *dominating spray* on X with spray bundle E is a holomorphic map $s : E \to X$ such that $s(0_x) = x$ and $s|_{E_x}$ is a submersion at 0_x for all $x \in X$, where 0_x is the origin of the fibre over x. A complex manifold is said to be *elliptic* if it has a dominating spray. This concept was introduced by Gromov to generalise the idea of the exponential map on complex Lie groups, which allow us to localise analytical problems in a particularly nice manner. Gromov proved the following.

Theorem (Gromov)**.** *Let X be a Stein manifold and Y be an elliptic manifold. Then every continuous map $X \to Y$ is homotopic to a holomorphic map. Furthermore, the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ is a weak homotopy equivalence.*

Gromov's work led to the introduction of a larger class of manifolds known as *Oka manifolds*. The notions of BOP and POP may be strengthened to include interpolation, approximation, or both, and doing this leads us to the various *Oka properties*. For instance, we may consider the *basic Oka property with interpolation* or the *parametric Oka property with approximation* to name a few. By a series of major theorems by Forstnerič, these Oka properties are non-trivially equivalent. We say a complex manifold Y is *Oka* if Y satisfies any of these properties. It is known that every elliptic manifold is Oka, but the converse is still unknown. That is, there are no current examples of Oka manifolds that fail to be elliptic. Forstnerič has given a thorough exposition of modern Oka theory in his monograph [3].

In 1993, Winkelmann explored BOP in the situation of Riemann surfaces [30]. Winkelmann completely classified the pairs of Riemann surfaces (X,Y) for which every continuous map $X \to Y$ is homotopic to a holomorphic map. This generalises the usual definition of BOP since he allows non-Stein sources. Winkelmann's main result is the following [30, Theorem 1].

Theorem (Winkelmann)**.** *Let X and Y be Riemann surfaces. Then the pair (X,Y) satisfies BOP in each of the following cases, and only these cases.*

(1) (*a*) *X is the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$ with any target Y.*
   (*b*) *Y is $\mathbb{C}$ or $\mathbb{D}$ with any source X.*
   (*c*) *X is the Riemann sphere $\mathbb{P}$ with any target $Y \neq \mathbb{P}$.*

(2) *X is open and Y is $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{P}$ or a one-dimensional complex torus.*

(3) *$X = \bar{X}\backslash\bigcup D_i$ and Y is $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$, where $\bar{X}$ is a compact Riemann surface and (D_i) is a non-empty finite collection of mutually disjoint closed discs in $\bar{X}$.*

We refer to (1) as the *topological pairs*, (2) as the *Gromov pairs* and (3) as the *non-Gromov pairs*. The topological pairs are so named because BOP holds purely for topological reasons. The Gromov pairs fall into Gromov's framework from [9], that is, $\mathbb{C}^*$,

$\mathbb{P}$ and one-dimensional complex tori are elliptic. The punctured plane and the tori are elliptic since they are complex Lie groups, thus the dominating spray on each of them is just their exponential map. The Riemann sphere is elliptic since it is a homogenous space of the complex Lie group $\mathrm{PGL}_2\mathbb{C}$ and thus inherits a dominating spray from the exponential map on $\mathrm{PGL}_2\mathbb{C}$. We acknowledge that $\mathbb{C}$ is also elliptic, but the proof of BOP is so topological that we include it among the topological pairs. Finally, there are the non-Gromov pairs, so named because they do not fall into Gromov's framework, since $\mathbb{D}^*$ is hyperbolic and we are only considering specific Stein sources.

We wish to improve Winkelmann's result to incorporate POP for all pairs of Riemann surfaces. The topological pairs will satisfy POP by the basic topological properties that make them satisfy BOP, whereas the Gromov pairs will satisfy POP by Gromov's work. The interesting question comes with the non-Gromov pairs, as these are an oddity in Oka theory. We aim to develop low-dimensional, accessible proofs of POP in all cases. That is, for a pair of Riemann surfaces (X, Y) that satisfies BOP, we aim to find accessible proofs that the map $\pi_k\mathscr{O}(X, Y) \to \pi_k\mathscr{C}(X, Y)$ induced by inclusion is an isomorphism for $k \geq 1$ and a bijection for $k = 0$, where π_k denotes the kth homotopy group.

Winkelmann also provided a complete list of pairs of Riemann surfaces for which BOP fails [30, Proposition 1].

Theorem (Winkelmann)**.** *Let X and Y be Riemann surfaces. Then in each of the following cases there exists a continuous map $X \to Y$ not homotopic to a holomorphic map.*

(i) *X is compact and $Y = \mathbb{P}$.*

(ii) *X is compact and $\pi_1(X), \pi_1(Y) \neq 0$.*

(iii) *X is open, $\pi_1(X) \neq 0$, and Y is not a torus, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^*$, $\mathbb{D}^*$ or $\mathbb{P}$.*

(iv) *$X = X'\backslash\{p\}$ for some Riemann surface X', and $Y = \mathbb{D}^*$.*

(v) *$\pi_1(X)$ is not finitely generated and $Y = \mathbb{D}^*$.*

These pairs are of interest as we wish to explore the higher properties of POP for each of them. It is a stimulating question to explore for which of the pairs of spaces that fail to satisfy BOP can we still get every other aspect of POP.

1.2 Research overview

The thesis will be broken into two main chapters. First will be the background chapter (Chapter 2), which will collate relevant background material needed to understand and complete the research in the second half of the thesis. The main areas we draw upon are from algebraic topology, differential geometry and complex analysis. Although we assume a base understanding of these areas, we attempt to provide all relevant results

from the theory, within reason, to make the thesis as self-contained as possible. Chapter 2 is concluded with a discussion of Winkelmann's main result on the basic Oka principle, where we provide a more thorough exposition to explore why the results hold.

Chapter 3 will focus on finding accessible proofs of the parametric Oka principle for all pairs of Riemann surfaces that satisfy the basic principle. Many results in the chapter rely heavily on Lemma 3.2, which is Lemma 2.67 from the background chapter (where we provide careful proof) restated in the context of the spaces of holomorphic and continuous maps between Riemann surfaces. For a given integer $k > 0$, this lemma gives a neatly packaged tool to show when the inclusion of the space of holomorphic maps into the space of continuous maps induces a surjection on π_k and an injection on π_{k-1}.

Section 3.1 looks at the topological pairs. We first use Lemma 3.2 to construct a proof of Theorem 3.3, which shows that the inclusion of the space of null-homotopic holomorphic maps (holomorphic maps null-homotopic through holomorphic maps) into the space of null-homotopic continuous maps induces an isomorphism on π_k for $k \geq 1$, and an injection for $k = 0$. This immediately gives POP for the topological pairs as highlighted in Corollaries 3.4 and 3.5. The rest of this section is dedicated to showing the stronger result that the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ is a homotopy equivalence for the topological pairs (X, Y). Theorems 3.6 and 3.7 do this when the source or target is contractible (that is, when X or Y is $\mathbb{C}$ or $\mathbb{D}$) by constructing an explicit homotopy inverse for the inclusion. We show that the inclusion $\mathscr{O}(\mathbb{P}, Y) \hookrightarrow \mathscr{C}(\mathbb{P}, Y)$ is a homotopy equivalence for $Y \neq \mathbb{P}$ by invoking a theorem of Milnor (Theorem 3.8), which tells us the space of continuous maps has the homotopy type of a CW-complex. Coupling this with Whitehead's theorem (Theorem 2.65) yields Theorem 3.9.

The next section (Section 3.2) explores the Gromov pairs. First we consider when the target Y is the punctured complex plane or a one-dimensional complex torus, as these cases are closely related. Lemma 3.10 shows us that the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ induces an injection on path components for an open Riemann surface X. Using the topological group structure (inherited from Y) on $\mathscr{O}(X, Y)$ and $\mathscr{C}(X, Y)$, we can then concentrate our efforts on the path component of the identity element. Our maps are then null-homotopic and POP follows from Theorem 3.3. We then consider the other Gromov pairs, an open Riemann surface X mapping into the Riemann sphere $\mathbb{P}$. This case proved difficult, and only partial results were obtained. The first step towards POP was achieved with Corollary 3.13, which shows the inclusion $\mathscr{O}(X, \mathbb{P}) \hookrightarrow \mathscr{C}(X, \mathbb{P})$ induces an injection of path components for any open Riemann surface as our source. We provide the partial result of Corollary 3.17 that shows the inclusion $\mathscr{O}(\mathbb{C}^*, \mathbb{P}) \hookrightarrow \mathscr{C}(\mathbb{C}^*, \mathbb{P})$ induces a surjection on fundamental groups. This is accomplished with the use of fibrations and degree theory. In particular, we construct a map of degree 1 from the torus $S^1 \times S^1$ to the Riemann sphere that can be holomorphically extended to $\mathbb{C}^*$ in one variable.

We then explore the non-Gromov pairs in Section 3.3. Theorem 3.19 shows that when our target $Y \neq \mathbb{P}$ has an abelian fundamental group, the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$

induces an isomorphism on π_k for $k \geq 1$ and an injection for $k = 0$ for any Riemann surface X. The proof once again uses Lemma 3.2. This result then gives an alternative proof of POP when our target is the punctured complex plane or a complex torus by Corollary 3.21, and proves POP holds for the non-Gromov pairs as shown in Corollary 3.22.

An interesting consequence of Theorem 3.19 is that it shows $\mathscr{O}(X, \mathbb{D}^*) \hookrightarrow \mathscr{C}(X, \mathbb{D}^*)$ induces an isomorphism on π_k for $k \geq 1$ and an injection on π_0 for *any* source X. Since BOP does not hold for an arbitrary Riemann surface mapping into $\mathbb{D}^*$, this shows that the higher properties of the parametric principle can hold even when the basic principle fails. This discovery is perhaps the main original contribution of this thesis, and spurred the work of Section 3.4. Definition 3.24 defines the *higher parametric Oka principle* (hPOP), which is our own definition not previously seen in the literature. A pair of Riemann surfaces (X, Y) is said to satisfy hPOP when the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ induces an an isomorphism on π_k for $k \geq 1$ and an injection on π_0. We explore the cases Winkelmann identified that do not satisfy BOP and see if they satisfy hPOP.

The first case concerns a compact Riemann surface X mapping into the Riemann sphere. We invoke theorems by Segal (Theorem 2.96) and Hansen (Theorem 2.90) to understand the topological structures of the spaces $\mathscr{O}(\mathbb{P}, \mathbb{P})$ and $\mathscr{C}(\mathbb{P}, \mathbb{P})$. Section 3.4.1 analyses these spaces and shows that the pair $(\mathbb{P}, \mathbb{P})$ fails to satisfy hPOP. The case of hPOP for an arbitrary compact Riemann surface mapping into the Riemann sphere remains open, but does not look promising.

We then explore the second case in Section 3.4.2, that is, the case of a compact source $X \neq \mathbb{P}$ mapping into any target Y with $\pi_1(Y) \neq 0$. With the use of fibrations and de Franchis' theorem (Theorem 2.17), we show that hPOP is satisfied with Theorem 3.30. This result is strengthened slightly with Corollary 3.32, which shows that the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ is a homotopy equivalence when we restrict $\mathscr{C}(X, Y)$ to the path components containing holomorphic maps. This is done with similar methods to Theorem 3.9.

Section 3.4.3 explores the case of an open Riemann surface X, $\pi_1(X) \neq 0$, mapping into any target besides $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^*$, $\mathbb{D}^*$, $\mathbb{P}$ or a one-dimensional complex torus. We use covering space theory, fibrations, and a careful adaptation of the five lemma from group theory to show that hPOP will hold whenever the source or the target has an abelian fundamental group (Corollary 3.36). This partial result is all we could achieve, with the general result looking much more difficult to approach.

We do not dedicate a section to the fourth and fifth cases where BOP fails in Winkelmann's classification. This is because hPOP holds in these cases by Theorem 3.19.

The thesis concludes with Section 3.5, which summarises our results in a table, highlighting the pairs for which we have been able to construct a proof of POP and hPOP. We also provide a brief discussion on the homotopy groups of the space $\mathscr{O}(X, Y)$ for Riemann surfaces X and Y.

1.3 Further research

The results of this thesis provide ample avenues for further exploration. First, a complete low-dimensional proof that POP holds for the Riemann sphere would be desirable. It would be of interest to know if the methods used to prove Corollary 3.13 could be generalised to show the inclusion $\mathscr{O}(X, \mathbb{P}) \to \mathscr{C}(X, \mathbb{P})$ induces an injection on higher homotopy groups for an open Riemann surface X. Our attempts at this proved fruitless due to the large zero sets of continuous families of holomorphic functions we encountered. It would also be of interest to see if the methods used for Corollary 3.17 could be generalised at all. No attempts were made at this due to time constraints.

A related topic is to look at dominating sprays on the Riemann sphere. Since $\mathbb{P}$ is a homogenous space of the 3-dimensional complex Lie group $\mathrm{PGL}_2\mathbb{C}$, it inherits a dominating spray of rank 3 (where the rank of a dominating spray is the rank of the vector bundle it is defined on). It would be interesting to see if one could construct a dominating spray of lower rank. This could be beneficial to providing a low-dimensional proof of POP for the Riemann sphere case. When the target is $\mathbb{C}^*$, the proof of POP relies on the exponential map, a dominating spray of rank 1 on $\mathbb{C}^*$. We have hope this idea may lend itself to the Riemann sphere case if we can find such a spray.

Generalising the idea of hPOP may also be of interest. It remains to provide a complete characterisation of Riemann surfaces satisfying hPOP as we only provide partial results towards this in Section 3.4. One could also consider hPOP in higher dimensions to see if we can find a larger class of spaces than just Oka manifolds satisfying this property.

One may also consider other Oka properties for Riemann surfaces, in particular one may consider the *parametric Oka principle with approximation and interpolation* (POPAI). This is, in a sense, the ultimate Oka property a manifold can satisfy, and it would be interesting to see a low-dimensional approach to classify the pairs of Riemann surfaces that satisfy POPAI. It is easily shown that the hyperbolic Riemann surfaces (those covered by the unit disc) will fail the approximation and interpolation properties for the basic principle, and so will not satisfy them for the parametric principle. Although this cuts out the oddity of the non-Gromov pairs, it is still of interest to construct low-dimensional proofs of POPAI for the Gromov pairs.

The fact that the various Oka properties are equivalent for Oka manifolds is highly non-trivial. It would be edifying to provide a low-dimensional proof of this equivalence for Riemann surfaces.

Finally, one could look at generalising the idea of the non-Gromov pairs. These are oddities in Oka theory in that we have a hyperbolic target that satisfies BOP for particular sources. Winkelmann generalised this idea to higher dimensions [30, Theorem 4], however he did not approach the idea of the parametric principle. It may be of interest to take Winkelmann's generalisation of these spaces and see if the parametric principle holds.

Chapter 2

Background

This chapter is dedicated to collecting the relevant material needed to understand and complete the work done in Chapter 3. Throughout, we assume a basic knowledge of algebraic topology, differential geometry, and complex analysis, and we strive to collect the results needed to provide a self-contained thesis. In Section 2.1, we discuss Riemann surface theory, presenting well-known results for both compact and non-compact Riemann surfaces. We also provide classification results for Riemann surfaces and completely identify the Riemann surfaces with abelian fundamental group to be used in Section 3.3. This section concludes with various results about centralisers and subgroups of the fundamental group of Riemann surfaces covered by the unit disc which are used throughout Section 3.4.

Section 2.2 presents a brief discussion on degree theory in which we introduce the notion of the degree of a smooth map and the famous Hopf degree theorem. We then move to Section 2.3, which looks at introducing the compact-open topology, which is the natural topology to place on the space of continuous maps between topological spaces. Together with the definition of this topology, we present basic notions on continuous maps in the compact-open topology. The next section (Section 2.4) gathers results from algebraic topology. We provide well-known results about covering maps and covering spaces, including the existence of universal covering spaces and necessary and sufficient conditions needed to lift continuous maps by covering maps. This section is concluded with Lemma 2.67, which is a powerful tool used throughout Chapter 3. Section 2.5 collects some results about topological groups and shows how this group structure is inherited by the space of continuous maps into a topological group.

The notion of a fibration is introduced in Section 2.6, where we collect results about induced fibrations and the long exact sequence of homotopy groups associated to a fibration. Section 2.7 is concerned with collecting a variety of results on the topological structure of the spaces of continuous and holomorphic maps in the compact-open topology. The results from these sections play a crucial role in Section 3.4.

Sections 2.8 and 2.9 first introduce the Whitney $\mathscr{C}^\infty$ topology on the space of smooth

maps between manifolds[1], and then Morse theory. The main goal is to prove that there is a strictly subharmonic Morse exhaustion function on an open Riemann surface, which in turn tells us that an open Riemann surface has the homotopy type of a bouquet of circles. This is a vital step in the proof of Theorem 2.122 in Section 2.10.

Finally, we introduce the notion of the basic Oka principle in Section 2.10. This section explores the driving force for this thesis, namely Winkelmann's classification of the pairs of Riemann surfaces that satisfy the basic Oka principle. We provide thorough, accessible exposition detailing the results of Winkelmann's paper, which provides the groundwork for Chapter 3.

2.1 Riemann surfaces

Complex analysis in one complex variable provides a rich, well-understood theory that has been developed for centuries. The natural extension from the complex plane is to generalise these ideas of complex analysis to one-dimensional complex manifolds, otherwise known as *Riemann surfaces*. A Riemann surface is an oriented topological surface – without loss of generality, we take it to be connected – equipped with a complex structure, that is, a surface equipped with a notion of how to measure angles between tangent vectors with orientation. Riemann surfaces may be categorised into compact Riemann surfaces and non-compact Riemann surfaces, also called *open* Riemann surfaces, both of which provide a deep theory to be investigated. In the following we collect well-known notation and results about Riemann surfaces.

To begin, we provide some fundamental examples of Riemann surfaces, some of which will be referred to throughout the text.

Example 2.1.

- The complex plane $\mathbb{C}$, the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, or the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ are all Riemann surfaces. Open domains in $\mathbb{C}$ are Riemann surfaces.

- The punctured plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ or the punctured disc $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$ are Riemann surfaces.

- The Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ is a Riemann surface. It is diffeomorphic to S^2 and complex projective space $\mathbb{CP}^1$.

- A one-dimensional complex torus[2] is a Riemann surface. This is obtained by identifying points of the complex plane by the action of the lattice $\mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$.

- $\mathbb{P}\backslash C$ where $C \subset \mathbb{P}$ is a Cantor set is a Riemann surface.

[1] We always mean smooth manifolds unless otherwise stated.
[2] Henceforth we shall simply say torus.

- The real projective plane $\mathbb{RP}^2$ does not have the structure of a Riemann surface. Although it is a closed topological surface, it fails to be orientable and hence cannot have a complex structure.

Riemann surfaces are well understood and those provided are just a handful of examples. By general surface theory we know the compact Riemann surfaces, which in particular are compact orientable smooth surfaces, are classified up to diffeomorphism by their genus g. That is, any compact Riemann surface is obtained by taking $\mathbb{P}$ and attaching g handles. Further, we have the following classification result for Riemann surfaces with finitely generated fundamental group.

Theorem 2.2. *Let X be a Riemann surface with $\pi_1(X)$ finitely generated. Then there is a compact Riemann surface $\bar{X}$, finitely many closed discs $D_i \subset \bar{X}$, and finitely many points $p_j \in \bar{X}$ such that X is biholomorphic to $\bar{X}\backslash(\bigcup_i D_i \cup \bigcup_j \{p_j\})$.*

Proof. This result goes back to Stout in 1965 [28, Theorem 8.1]. A proof can also be found in [30, Theorem 2]. □

This is a powerful result that provides insight into how Riemann surfaces are obtained. From our examples, we may write $\mathbb{C} = \mathbb{P}\backslash\{\infty\}$ or $\mathbb{D}^* = \mathbb{P}\backslash(\{z \in \mathbb{P} : |z| \geq 1\} \cup \{0\})$ for instance. The example $\mathbb{P}\backslash C$ for a Cantor set C provides an instance where this classification does not apply since $\pi_1(\mathbb{P}\backslash C)$ is not finitely generated.

Another key topic to discuss in conjunction with Riemann surfaces are the maps between them. That is, the notion of *holomorphic* maps between Riemann surfaces and *meromorphic* functions on Riemann surface are vital to the discussion of Riemann surface theory. A holomorphic map between Riemann surfaces X and Y is a continuous map $X \to Y$ that is locally holomorphic, that is, in charts on X and Y the map looks like a holomorphic map between open sets in the complex plane. A *biholomorphism* between Riemann surfaces is a holomorphic map with holomorphic inverse. A meromorphic function f on a Riemann surface X is a holomorphic map $f : X\backslash S \to \mathbb{C}$ (where $S \subset X$ is discrete), that is well behaved near S. By well behaved, we mean that for all $s \in S$,

$$\lim_{z \to s} |f(z)| = \infty\,.$$

We call S the set of *poles* of f.

Remark 2.3. A note on notation. Let X and Y be Riemann surfaces.

- The set of all holomorphic functions $X \to \mathbb{C}$ is denoted $\mathscr{O}(X)$ or $\mathscr{O}(X, \mathbb{C})$. The set of all non-zero holomorphic functions $X \to \mathbb{C}^*$ is denoted $\mathscr{O}^*(X)$ or $\mathscr{O}(X, \mathbb{C}^*)$. The set of all holomorphic maps $X \to Y$ is denoted $\mathscr{O}(X, Y)$.

- A meromorphic function $X \to \mathbb{C}$ extends to a holomorphic map $X \to \mathbb{P}$. The set of all meromorphic functions $X \to \mathbb{C}$ is denoted $\mathscr{M}(X) = \mathscr{O}(X, \mathbb{P})\backslash\{\infty\}$, where ∞ denotes the constant map with value ∞. The set of invertible meromorphic functions is denoted $\mathscr{M}^*(X)$.

We now move on to collating relevant results about open and compact Riemann surfaces, the first of which is the fundamental result that open Riemann surfaces are what we call *Stein manifolds*.

Definition 2.4. Let X be a complex manifold. Then X is said to be a *Stein manifold*, or simply *Stein*, if there is a proper holomorphic embedding $X \to \mathbb{C}^n$ for some $n \geq 1$.

Remark 2.5. There are many non-trivially equivalent definitions of a Stein manifold. Although the one given here is not the standard definition, it is the best suited for our needs.

Theorem 2.6. *A Riemann surface is Stein if and only if it is open.*

Proof. If X is not open, then every holomorphic map $X \to \mathbb{C}^n$ is constant for all n by Proposition 2.13, and hence not an embedding. Thus X is not Stein.

For a proof that every open Riemann surface is Stein, we refer the reader to [2, p. 205, Corollary 26.8]. □

Remark 2.7. Due to Theorem 2.6, we will synonymously refer to a non-compact Riemann surface as open or Stein depending on the situation.

Remark 2.8. A *divisor* D on a Riemann surface X is a map $D : X \to \mathbb{Z}$ such that for every compact $K \subset X$ there are only finitely many $x \in K$ such that $D(x) \neq 0$. For a meromorphic function $f \in \mathscr{M}(X)$, the *order of* f *at* $x \in X$ is

$$\operatorname{ord}_x(f) = \begin{cases} 0 & \text{if } f \text{ is holomorphic at } x \text{ and non-zero,} \\ k & \text{if } f \text{ has a zero of order } k \text{ at } x, \\ -k & \text{if } f \text{ has a pole of order } k \text{ at } x, \\ \infty & \text{if } f \text{ is identically 0 in a neighbourhood of } x. \end{cases}$$

If f is not identically zero on X, the map $x \mapsto \operatorname{ord}_x(f)$ is a divisor on X, denoted (f).

Theorem 2.9 (Weierstrass' theorem)**.** *Let X be an open Riemann surface and D be a divisor on X. Then there is $f \in \mathscr{M}^*(X)$ with $(f) = D$.*

Proof. See [2, p. 203, Theorem 26.5]. □

Corollary 2.10. *Let X be an open Riemann surface and $f \in \mathscr{M}(X)$. Then there are f_1, $f_2 \in \mathscr{O}(X)$ such that $f = f_1/f_2$.*

Proof. Let D be the divisor that agrees with (f) at the poles of f and is 0 otherwise. Let $g \in \mathscr{O}(X)$ be such that $(g) = -D$ by Weierstrass' theorem. Then $fg \in \mathscr{O}(X)$, so taking $f_1 = fg$ and $f_2 = g$ yields the result. □

Remark 2.11. We let $\mathscr{E}(X)$ denote the set of smooth functions $X \to \mathbb{C}$ and $\mathscr{E}^{(0,1)}(X)$ denote the set of smooth $(0,1)$-forms on X, that is, 1-forms that locally look like $f\,d\bar{z}$ for a smooth function f. The differential operator $\bar{\partial}$ is the anti-holomorphic part of the exterior derivative d.

Lemma 2.12 (Dolbeault's lemma)**.** *Let X be an open Riemann surface. Given $\omega \in \mathscr{E}^{(0,1)}(X)$ there is $f \in \mathscr{E}(X)$ with $\bar{\partial} f = \omega$.*

Proof. See [2, p. 200, Theorem 25.6]. □

Dolbeault's lemma is a complex analogue of the well-known Poincaré lemma, which says a smooth differential form is closed if and only if it is locally exact. This lemma has far-reaching consequences in complex analysis and so it provides an interesting question of how it may be generalised. One such avenue explored for this thesis was the notion of parametrising the classical Dolbeault lemma, that is, if we have a family ω_t of $(0,1)$-forms parametrised by a space P, can we find a family f_t of smooth functions parametrised by P pointwise satisfying $\bar{\partial} f_t = \omega_t$. This was explored to try to find a parametrised version of the proof of Theorem 2.119, however was found to be unnecessary. This work is included in Appendix A.

Proposition 2.13. *Let X and Y be Riemann surfaces. Suppose X is compact and Y is open. Then every holomorphic map $X \to Y$ is constant.*

Proof. Suppose $f : X \to Y$ is a non-constant holomorphic map. Then $f(X)$ is open since non-constant holomorphic maps are open. However, $f(X)$ is closed and compact since X is compact. Then, by connectedness, $f(X) = Y$, but Y is not compact, a contradiction. □

Remark 2.14. A non-constant holomorphic map $p : X \to Y$ between compact Riemann surfaces is known as a *branched holomorphic covering map*. This means that $p|_{X \backslash p^{-1}(C)} : X \backslash p^{-1}(C) \to Y \backslash C$ is a covering map where $C \subset Y$ is the set of critical values of p. If $v(x,p)$ denotes the multiplicity with which p takes the value $p(x)$ at x, then the total branching order of p is

$$b = \sum_{x \in X} \left(v(x,p) - 1 \right) .$$

Theorem 2.15 (Riemann-Hurwitz formula)**.** *Suppose $p : X \to Y$ is an n-sheeted, branched holomorphic covering map between compact Riemann surfaces X and Y with total branching order b. Let g and g' denote the genera of X and Y respectively. Then*

$$g - 1 = \frac{b}{2} + n(g' - 1) .$$

Proof. See [2, p. 140]. □

Corollary 2.16. *Let X and Y be compact Riemann surfaces with genera g and g' respectively. Suppose $g < g'$. Then $\mathscr{O}(X, Y) = Y$, that is, there are no non-constant holomorphic maps $X \to Y$.*

Proof. Suppose $f : X \to Y$ is a non-constant, holomorphic map. Then f is an n-sheeted branched holomorphic covering map for some $n \geq 1$. By the Riemann-Hurwitz formula, we have

$$g - 1 = \frac{b}{2} + n(g' - 1)\,,$$

where b denotes the total branching order of f. Since $g < g'$, $g - 1 - n(g' - 1) < 0$ and so $b < 0$, which is absurd. □

Theorem 2.17 (De Franchis' theorem)**.** *Let X and Y be compact Riemann surfaces of genus at least 2. Then there are only finitely many non-constant holomorphic maps $X \to Y$.*

Remark 2.18. An accessible proof of de Franchis' theorem has turned out to be hard to find. An algebraic proof can be found in [23], however this is unenlightening for the work presented here. We present an outline of a method to prove de Franchis' theorem which follows the argument by Ebert in [1].

Discussion of the proof. Although the proof is omitted, it is edifying to discuss the techniques that go into proving de Franchis' theorem. The proof can be broken into two steps. The first is to show that the space $\mathscr{O}(X, Y)$ is compact in the compact-open topology. This is done using the hyperbolic geometry of X and Y. Since holomorphic maps $X \to Y$ cannot increase the hyperbolic distance between two points, we can deduce that any sequence in $\mathscr{O}(X, Y)$ is bounded and equicontinuous, so by the Arzelá-Ascoli theorem has a convergent subsequence and $\mathscr{O}(X, Y)$ is sequentially compact. From the hyperbolic metrics on X and Y, $\mathscr{O}(X, Y)$ is a metric space with the compact-open topology, and thus it is compact since it is sequentially compact.

The next step is to show that if $f : X \to Y$ is holomorphic and non-constant, then there are no other holomorphic maps homotopic to f. This is accomplished with intersection theory, the necessary elements of which can be found in [5]. Representing f by its graph defines a curve in the complex surface $X \times Y$. Intersection theory tells us that the self-intersection number of the graph of f must be negative since it is the degree of the normal bundle to the graph of f. The basic notions of intersection theory tell us that if $g : X \to Y$ is another holomorphic map distinct from f, then the graphs of f and g will have a non-negative intersection number in $X \times Y$. Thus, since the intersection number is a homotopy invariant, if $g : X \to Y$ is holomorphic and homotopic to f, then $g = f$. By deep theorems in complex analysis, $\mathscr{O}(X, Y)$ can be given the structure of a complex space with the compact-open topology. Thus, since the non-constant holomorphic maps are isolated, the set of non-constant holomorphic maps is discrete. We now have a compact

complex space made up from a copy of Y, representing the constant maps, and a discrete set of non-constant holomorphic maps. Hence there are only finitely many non-constant holomorphic maps. □

Remark 2.19. An interesting consequence of de Franchis' theorem and its proof is that it tells us a lot about the structure of $\mathscr{O}(X, Y)$. In particular, we see that

$$\pi_n(\mathscr{O}(X,Y), f) = \begin{cases} \pi_n(Y) & \text{if } f \text{ is constant,} \\ 0 & \text{if } f \text{ is non-constant,} \end{cases}$$

for all $n \geq 1$. The second point we take away is that no two distinct, non-constant holomorphic maps $X \to Y$ are homotopic. We can interpret this as saying that the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ induces an injection of path components. Thus, a path component of $\mathscr{C}(X, Y)$ contains at most one non-constant holomorphic map. This fact will be useful in Section 3.4.2.

Theorem 2.20 (Uniformisation theorem)**.** *Let X be a Riemann surface. Then the universal covering space of X is $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{P}$.*

Proof. See [2, p. 210, Theorem 27.9]. □

Remark 2.21. By Corollary 2.49 every Riemann surface has a universal covering, so the statement of the uniformisation theorem is very powerful, telling us there are only three possible choices of universal covering.

Lemma 2.22. *Let X and Y be Riemann surfaces, X compact and $Y \neq \mathbb{P}$. Suppose $f : X \to Y$ is holomorphic. Then if f is null-homotopic, f is constant.*

Proof. Let $\pi : \tilde{Y} \to Y$ denote the universal covering of Y. Then $\tilde{Y}$ is $\mathbb{C}$ or $\mathbb{D}$. Since f is null-homotopic, $f_*\pi_1(X) = 0$ and there is a holomorphic lifting $\tilde{f} : X \to \tilde{Y}$ of f by π (Proposition 2.60) such that

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "\pi"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. However, X is compact and $\tilde{Y}$ is open, so $\tilde{f}$ is constant by Proposition 2.13. Then $f = \pi \circ \tilde{f}$ is constant also. □

Remark 2.23. Let X be a Riemann surface. Then by $\operatorname{Aut} X$ we mean the group of biholomorphisms $X \to X$ with the group operation being composition.

Theorem 2.24. *Let X be a Riemann surface. Then $\pi_1(X)$ is abelian if and only if X is $\mathbb{P}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^*$, $\mathbb{D}^*$, an annulus or a torus.*

Proof. It is well known that if X is $\mathbb{P}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^*$, $\mathbb{D}^*$, an annulus or a torus, then $\pi_1(X) = 0$, $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ and is thus abelian.

Suppose $\pi_1(X)$ is abelian. By the uniformisation theorem (Theorem 2.20), the universal covering space $\tilde{X}$ of X is either $\mathbb{P}$, $\mathbb{C}$ or $\mathbb{D}$, and $X = \tilde{X}/\Gamma$ where $\Gamma \subset \operatorname{Aut} \tilde{X}$ is a discrete abelian subgroup acting without fixed points by Proposition 2.58. It is well known (for instance, [2, p. 212, Theorem 27.12]) that if $\tilde{X} = \mathbb{P}$ then $X = \mathbb{P}$ and if $\tilde{X} = \mathbb{C}$ then X is $\mathbb{C}$, $\mathbb{C}^*$ or a torus. So it remains to consider the case $\tilde{X} = \mathbb{D}$, or equivalently $\tilde{X} = \mathbb{H}$, and our question is reduced to exploring discrete abelian subgroups of $\operatorname{Aut} \mathbb{H}$.

Before we continue, let us recall that

$$\operatorname{Aut} \mathbb{H} = \left\{ \varphi : \mathbb{H} \to \mathbb{H} : \varphi(z) = \frac{az+b}{cz+d} \, , \, a, b, c, d \in \mathbb{R} \text{ satisfying } ad - bc = 1 \right\} .$$

This group is canonically isomorphic to $\mathrm{PSL}_2\mathbb{R}$ under the identification

$$\varphi \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.1}$$

and so we shall interchangeably refer to elements as maps and as matrices. By the proof of Theorem B.6 in the Appendix, the discrete abelian subgroups of $\operatorname{Aut} \mathbb{H}$ that act without fixed points in $\mathbb{H}$ are conjugate to subgroups generated by elements of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ,$$

where x and λ are positive real numbers, $\lambda \neq 1$. We shall consider subgroups generated by matrices of these types separately and show that they give rise to the punctured disc and an annulus respectively.

Fix $x > 0$ and let

$$\Gamma = \left\langle \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\rangle .$$

Define $\pi : \mathbb{H} \to \mathbb{D}^*$ by $\pi(z) = \exp(\frac{2\pi i}{x} z)$. Then π is a universal covering map with covering group

$$\{\varphi_n : \mathbb{H} \to \mathbb{H} : \varphi_n(z) = z + nx \, , \, n \in \mathbb{Z}\} .$$

This is Γ by making the identification defined by (2.1). Thus $\mathbb{D}^* = \mathbb{H}/\Gamma$ as required.

Now fix $\lambda \in (0, 1)$ (in the case $\lambda > 1$, consider λ^{-1}) and let

$$\Gamma = \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle .$$

For $s > 1$, let $A_s = \{z \in \mathbb{C} : \frac{1}{s} < |z| < s\}$. Set $r = \exp\left(\frac{-\pi^2}{\log \lambda^2}\right)$ and define the map $\pi : \mathbb{H} \to A_r$ by $\pi(z) = \exp(\frac{-2i \log r}{\pi} \operatorname{Log}(-iz))$, where $\operatorname{Log}(z)$ is the principal branch of the logarithm. Then π is a universal covering map with covering group

$$\{\varphi_n : \mathbb{H} \to \mathbb{H} : \varphi_n(z) = \lambda^{2n} z \, , \, n \in \mathbb{Z}\} = \Gamma \, ,$$

again making the identification defined by (2.1). Thus $A_r = \mathbb{H}/\Gamma$ as required. □

Remark 2.25. For a more complete discussion of discrete abelian subgroups of $\operatorname{Aut}\mathbb{H}$ we refer the reader to Appendix B.

Lemma 2.26. *Let Y be a Riemann surface with universal covering $\mathbb{H}$ (or equivalently $\mathbb{D}$). For $\gamma \in \pi_1(Y)$ denote by $C_\gamma = \{\alpha \in \pi_1(Y) : \alpha\gamma = \gamma\alpha\}$ the centraliser of γ in $\pi_1(Y)$. If γ is not the identity, then $C_\gamma < \pi_1(Y)$ is infinite cyclic.*

Proof. Let $p : \mathbb{H} \to Y$ denote the universal covering of Y. By considering $\pi_1(Y) = \operatorname{Aut} p < \operatorname{Aut}\mathbb{P}$ (Proposition 2.57), it follows that we may think of C_γ as a subgroup of $\operatorname{Aut}\mathbb{P}$. If $\alpha \in C_\gamma$, then $\alpha\gamma = \gamma\alpha$ and α and γ have the same fixed points by Lemma B.5. Since this is true for every $\alpha \in C_\gamma$, any two elements of C_γ have the same fixed points and hence commute by Lemma B.5, that is, if $\alpha_0, \alpha_1 \in C_\gamma$ then $\alpha_0\alpha_1 = \alpha_1\alpha_0$. Thus, C_γ is a discrete abelian subgroup of $\operatorname{Aut} p$ and hence infinite cyclic by Theorem B.6. □

Corollary 2.27. *Let Y be a Riemann surface with universal covering $\mathbb{H}$ (or equivalently $\mathbb{D}$). For $H < \pi_1(Y)$ denote by $C_H = \{\alpha \in \pi_1(Y) : \alpha\gamma = \gamma\alpha$ for all $\gamma \in H\}$ the centraliser of H in $\pi_1(Y)$. Suppose H is non-trivial. If $C_H < \pi_1(Y)$ is non-trivial, then C_H is infinite cyclic.*

Proof. We may write $C_H = \bigcap_{\gamma \in H} C_\gamma$. Since the intersection of infinite cyclic subgroups of a given group is either trivial or again infinite cyclic, the result follows from Lemma 2.26. □

Corollary 2.28. *Let Y be a Riemann surface with universal covering $\mathbb{H}$ (or equivalently $\mathbb{D}$). If $\pi_1(Y)$ is not abelian, then $\pi_1(Y)$ has trivial centre.*

Proof. Denote by $Z(\pi_1(Y)) = \{a \in \pi_1(Y) : ab = ba \text{ for all } b \in \pi_1(Y)\}$ the centre of $\pi_1(Y)$. If $\gamma \in Z(\pi_1(Y))$, then $C_\gamma = \pi_1(Y)$. If γ is not the identity, then C_γ is infinite cyclic by Lemma 2.26 and thus abelian. Since $\pi_1(Y)$ is not abelian by assumption, γ is the identity and $Z(\pi_1(Y)) = 0$. □

Lemma 2.29. *Let Y be a Riemann surface with universal covering $\mathbb{H}$ (or equivalently $\mathbb{D}$). Let $H < \pi_1(Y)$ be a subgroup such that the centraliser C_H of H in $\pi_1(Y)$ is infinite cyclic. Then H is abelian.*

Proof. Consider C_H and H as subgroups of $\operatorname{Aut}\mathbb{P}$ via the universal covering map (Proposition 2.57). Since C_H is abelian, every element in C_H has the same fixed points in $\mathbb{P}$. Further, given $\gamma \in C_H$ and $\alpha \in H$, then α has the same fixed points as γ since $\gamma\alpha = \alpha\gamma$, that is, every element of H has the same fixed points in $\mathbb{P}$. Thus the elements of H commute by Lemma B.5. □

2.2 Degree theory

One of the classical examples of a topological invariant of a map is the winding number. This is an invariant for maps $S^1 \to S^1$ (or more generally for maps $\mathbb{R}^2\backslash\{0\} \to \mathbb{R}^2\backslash\{0\}$) that counts how many times the unit circle wraps around the origin, where counter-clockwise wraps are counted positively and clockwise wraps are counted negatively. By this construction, the winding number is an integer that we can associate to a map, and it is a classical result that the winding number completely characterises the homotopy classes of maps $S^1 \to S^1$, that is, every integer is realisable by a map and two maps are homotopic if and only if they have the same winding number.

The winding number is an integer we can associate to a map between compact, oriented manifolds of real dimension one. The idea of the *degree* of a map is a generalisation of this notion to compact, oriented manifolds of the same real dimension n. Throughout this section we will introduce the idea of degree in the sense of smooth maps and the notion of local degree which provides a convenient method for calculating degrees. We will conclude this section with the classical Hopf degree theorem which shows the homotopy classes of maps $X \to S^n$, where X is a compact, oriented manifold of real dimension n, are classified by degree in much the same sense that maps $S^1 \to S^1$ are classified by the winding number.

Remark 2.30. Let Y be a manifold. Then the space of smooth k-forms on Y is denoted $\Lambda^k(Y)$.

Definition 2.31. Let $f : X \to Y$ be a smooth map between compact, oriented manifolds of real dimension n. Let $\omega \in \Lambda^n(Y)$ be a volume form on Y (that is, a nowhere zero n-form on Y). Then the *degree* of f, denoted $\deg f$, is given by the formula

$$\int_X f^*\omega = \deg f \int_Y \omega \,.$$

Remark 2.32. This definition is independent of the choice of $\omega \in \Lambda^n(Y)$. Since ω is a top degree form on Y, ω is closed and thus represents a class in the nth de Rham cohomology group of Y, denoted $H^n_{dR}(Y)$. It is well known that the map $H^n_{dR}(Y) \to \mathbb{R}$, $[\omega] \mapsto \int_Y \omega$, is an isomorphism since Y is compact, and we get a commuting diagram

$$\begin{tikzcd} H^n_{dR}(Y) \arrow[r, "f^*"] \arrow[d, "\int_Y(\cdot)"'] & H^n_{dR}(X) \arrow[d, "\int_X(\cdot)"] \\ \mathbb{R} \arrow[r, "\deg f"] & \mathbb{R} \end{tikzcd}$$

This shows that $\deg f$ is independent of choice of ω, and also that $\deg f$ is a homotopy invariant. We will not present the relevant material to explore this idea fully as it is not used elsewhere throughout the thesis, but for a full discussion of de Rham cohomology we refer the reader to [17, Chapter 17].

Remark 2.33. The notion of degree can be defined for a continuous map $f : X \to Y$ between compact, oriented manifolds of real dimension n. Let $H_n(X)$ and $H_n(Y)$ denote the nth singular homology groups of X and Y respectively. Fixing an orientation on X and Y (that is, choosing generators for $H_n(X)$ and $H_n(Y)$), we have $H_n(X) = H_n(Y) = \mathbb{Z}$ since X and Y are compact. Thus $f_* : H_n(X) \to H_n(Y)$ acts by multiplication by an integer $d \in \mathbb{Z}$, so we define $\deg f = d$. We will not work with this definition as a full discussion of the homology groups and the mathematics behind it will take us too far afield and will not benefit the rest of the thesis. Our definition suffices since both definitions of degree agree when f is smooth.

The definition we give of the degree of a map takes a global viewpoint, however there is a local description that lends itself nicely to calculations. If $f : X \to Y$ is a smooth map between compact, oriented manifolds of real dimension n and $p \in Y$ is a regular value of f (that is, if $q \in f^{-1}(p)$, the differential $d_q f : T_q X \to T_p Y$ is an isomorphism), then f is a local diffeomorphism about q by the inverse function theorem. This allows us to define a notion of degree locally about each point $q \in f^{-1}(p)$.

Definition 2.34. Let $f : X \to Y$ be a smooth map between compact, oriented manifolds of real dimension n and suppose $p \in Y$ is a regular value of f. Let $q \in f^{-1}(p)$. Then the *local degree* of f at q is

$$\deg_q f = \begin{cases} 1 & \text{if } d_q f : T_q X \to T_p Y \text{ is orientation preserving}\,, \\ -1 & \text{otherwise}\,. \end{cases}$$

Proposition 2.35. *Let $f : X \to Y$ be a smooth map between compact, oriented manifolds of real dimension n and suppose $p \in Y$ is a regular value of f. Then*

$$\deg f = \sum_{q \in f^{-1}(p)} \deg_q f\,.$$

Remark 2.36. The sum $\sum_{q \in f^{-1}(p)} \deg_q f$ is well defined since each point $q \in f^{-1}(p) \subset X$ is isolated by the inverse function theorem. Thus the closed set $f^{-1}(p)$ is discrete and hence finite since X is compact.

Proof. See [17, p. 457, Theorem 17.35]. □

Corollary 2.37. *Let $f : X \to Y$ be a smooth map between compact, oriented manifolds of real dimension n. If f is not surjective, then* $\deg f = 0$.

Proof. Let $p \notin f(X)$. Then by definition, p is a regular value for f. Since $f^{-1}(p) = \varnothing$, we have

$$\deg f = \sum_{q \in \varnothing} \deg_q f = 0\,.$$ □

Proposition 2.35 tells us that we can simply count the elements in the preimage of a regular value with orientation to obtain the degree of a given map, which in particular tells us that the degree of a smooth map is an integer. It is a simple exercise in the definitions of degrees and local degrees that diffeomorphisms have degree 1 if they are orientation preserving and degree -1 otherwise. We also find that homotopic maps have the same degree, and if $f : X \to Y$ and $g : Y \to Z$ are smooth maps between compact, oriented manifolds of real dimension n, then $\deg(g \circ f) = \deg g \deg f$. This leads us to want to think of the degree as a homomorphism from the homotopy classes of maps between compact, oriented manifolds of real dimension n to the integers. We see this is the case when $Y = S^n$ from the Hopf degree theorem, but first we show that every integer is obtainable.

Proposition 2.38. *Let X be a compact, oriented manifold of real dimension n and let $d \in \mathbb{Z}$. Then there is a smooth map $f : X \to S^n$ of degree d.*

Proof. If $d = 0$, then take $f : X \to S^n$ to be a constant map, which trivially has degree 0. Now suppose $d > 0$. Pick d distinct points $q_1, \dots, q_d \in X$ and let (U_i, φ_i) be pairwise disjoint coordinate charts centred at q_i for $i = 1, \dots, d$, that is, $U_i \cap U_j = \varnothing$ if $i \neq j$ and $\varphi_i : U_i \to \mathbb{R}^n$ is a diffeomorphism with $\varphi_i(q_i) = 0$. Let $p : \mathbb{R}^n \to S^n \backslash \{N\}$ be the inverse of the standard stereographic projection where $N \in S^n$ denotes the north pole. Define

$$\psi(x) = \begin{cases} p \circ \varphi_i(x) & \text{if } x \in U_i\,, \\ N & \text{if } x \in X \backslash \bigcup_i U_i\,. \end{cases}$$

Then it can be shown that $\psi : X \to S^n$ is smooth and that the south pole $S \in S^n$ is a regular value for ψ. By construction, $\psi^{-1}(S) = \{q_1, \dots, q_d\}$, so ψ has degree d if we choose φ_i to be orientation preserving for $i = 1, \dots, d$. If $d < 0$, we choose φ_i to be orientation reversing for $i = 1, \dots, d$. □

Remark 2.39. It is interesting to note that, when $n = 2$, we cannot obtain every degree for holomorphic maps. If X is a compact Riemann surface, the above construction cannot generally be applied to holomorphic maps since the diffeomorphisms $\varphi_i : U_i \to \mathbb{C}$ usually cannot be chosen to be biholomorphisms by Liouville's theorem. Further, since holomorphic maps are orientation preserving, we find that $\deg f \geq 0$. However, this does not mean every positive degree is obtainable. For instance, there is no holomorphic map of degree 1 from a torus to the Riemann sphere, as such a map would be a 1-sheeted holomorphic covering, and hence a biholomorphism.

Theorem 2.40 (The Hopf degree theorem)**.** *Let X be a compact, oriented manifold of real dimension n. Then the set $[X, S^n]$ of homotopy classes of maps $X \to S^n$ is isomorphic to $\mathbb{Z}$, where the isomorphism is given by degree.*

Proof. When considering the smooth homotopy classes of smooth maps, a proof can be found in [21, p. 50]. When considering the homotopy classes of continuous maps, a proof can be found in [12, p. 361, Corollary 4.25] when we take $X = S^n$. □

2.3 The compact-open topology

For topological spaces X and Y we may consider the set $\mathscr{C}(X,Y)$ of all continuous maps from X to Y. The first thing one may ask is how to place a topology on this set in a natural way, which leads us to the compact-open topology. In the following we will collect some key facts about the compact-open topology and see why it is the natural topology to place on the set of continuous maps.

Definition 2.41. Let X and Y be topological spaces. The *compact-open topology* on $\mathscr{C}(X,Y)$ is defined as follows. For $K \subset X$ compact and $U \subset Y$ open we declare

$$A(K,U) = \{f \in \mathscr{C}(X,Y) : f(K) \subset U\}$$

to be a subbasis element for the topology. Thus the open sets are arbitrary unions of finite intersections of sets of the form $A(K,U)$.

Remark 2.42. We say a topological space X is *locally compact* if every point in X has a compact neighbourhood. This is not necessarily what one might think of when talking about local compactness. Another definition of local compactness is that every point has a neighbourhood basis of compact sets, however these definitions are equivalent for Hausdorff spaces.

Proposition 2.43. *Let X, Y and Z be topological spaces. Suppose X is locally compact Hausdorff and Y is Hausdorff. Then in the compact-open topology we have the following:*

(a) *The evaluation map* $\mathrm{ev} : \mathscr{C}(X,Y) \times X \to Y$, $\mathrm{ev}(f,x) = f(x)$, *is continuous.*

(b) *A map $f : X \times Y \to Z$ is continuous if and only if the map $\hat{f} : Y \to \mathscr{C}(X,Z)$, $\hat{f}(y)(x) = f(x,y)$, is continuous.*

(c) *The map $\mathscr{C}(X \times Y, Z) \to \mathscr{C}(Y, \mathscr{C}(X,Z))$, $f \mapsto \hat{f}$, is a homeomorphism.*

Proof. This is standard. See [12, p. 530–531, Propositions A.14, A.16]. □

Proposition 2.43 (c) demonstrates what is known as the *universal property* of the compact-open topology. This gives a characteristic of continuous maps which we would expect to be true under our desired topology and shows that it holds under very general assumptions with the compact-open topology. Hence we view this as the most natural topology to put on the space of continuous maps. The following propositions are basic facts about continuous maps in the compact-open topology. We present them here for completeness and to demonstrate how to use this topology.

Proposition 2.44. *Let X, Y and Z be topological spaces. Fix $g \in \mathscr{C}(X,Y)$. Then the pre-composition map $g^* : \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$ given by $g^*(f) = f \circ g$ is continuous in the compact-open topology.*

Proof. Let $K \subset X$ be compact and $U \subset Z$ be open. It suffices to show there are $K' \subset Y$ compact and $U' \subset Z$ open such that $(g^*)^{-1}(A(K,U)) = \{f \in \mathscr{C}(Y,Z) : f(K') \subset U'\}$.

We see that

$$\begin{aligned}(g^*)^{-1}(A(K,U)) &= \{f \in \mathscr{C}(Y,Z) : f \circ g \in A(K,U)\} \\ &= \{f \in \mathscr{C}(Y,Z) : f \circ g(K) \subset U\}\,.\end{aligned}$$

Setting $U' = U$ and $K' = g(K)$, which is compact since g is continuous, we see that

$$(g^*)^{-1}(A(K,U)) = \{f \in \mathscr{C}(Y,Z) : f(K') \subset U'\}\,,$$

and thus g^* is a continuous map. □

Proposition 2.45. *Let X, Y and Z be topological spaces. Fix $g \in \mathscr{C}(Y,Z)$. Then the post-composition map $g_* : \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$ given by $g_*(f) = g \circ f$ is continuous in the compact-open topology.*

Proof. Let $K \subset X$ be compact and $U \subset Z$ be open. It suffices to show there are $K' \subset X$ compact and $U' \subset Y$ open such that $(g_*)^{-1}(A(K,U)) = \{f \in \mathscr{C}(X,Y) : f(K') \subset U'\}$.

We see that

$$\begin{aligned}(g_*)^{-1}(A(K,U)) &= \{f \in \mathscr{C}(X,Y) : g \circ f \in A(K,U)\} \\ &= \{f \in \mathscr{C}(X,Y) : g \circ f(K) \subset U\} \\ &= \{f \in \mathscr{C}(X,Y) : f(K) \subset g^{-1}(U)\}\,.\end{aligned}$$

Setting $U' = g^{-1}(U)$, which is open by continuity, and $K' = K$ we see that

$$(g_*)^{-1}(A(K,U)) = \{f \in \mathscr{C}(X,Y) : f(K') \subset U'\}\,,$$

and thus g_* is a continuous map. □

Proposition 2.46. *Let X, Y and Z be locally compact Hausdorff spaces. Suppose $f : X \to Y$ is a homotopy equivalence. Then the induced maps $f_* : \mathscr{C}(Z,X) \to \mathscr{C}(Z,Y)$ and $f^* : \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$ are homotopy equivalences in the compact-open topology.*

Proof. Let $g : Y \to X$ be a homotopy inverse for f. We will first show that $g_* : \mathscr{C}(Z,Y) \to \mathscr{C}(Z,X)$ is a homotopy inverse for f_*. Let $H : X \times [0,1] \to X$ be a homotopy from $g \circ f$ to id_X and $G : Y \times [0,1] \to Y$ be a homotopy from $f \circ g$ to id_Y.

Define a map $\hat{H} : \mathscr{C}(Z,X) \times [0,1] \to \mathscr{C}(Z,X)$ by $\hat{H}(h,t) = H(\cdot,t)_*(h)$. By Proposition 2.43 (b), we consider this as a map $\mathscr{C}(X,Z) \times [0,1] \times Z \to X$ with $\hat{H}(h,t,z) = H(h(z),t)$. This is clearly continuous in z and t, and by Proposition 2.45 it is continuous in h. We see that $\hat{H}(\cdot,0) = (g \circ f)_* = g_* \circ f_*$ and $\hat{H}(\cdot,1) = \mathrm{id}_{\mathscr{C}(Z,X)}$. By the same argument, $\hat{G} : \mathscr{C}(Z,Y) \times [0,1] \to \mathscr{C}(Z,Y)$ defined by $\hat{G}(h,t) = G(\cdot,t)_*(h)$ is continuous with

$\hat{G}(\cdot,0) = f_* \circ g_*$ and $\hat{G}(\cdot,1) = \mathrm{id}_{\mathscr{C}(Z,Y)}$. This shows $f_* : \mathscr{C}(Z,X) \to \mathscr{C}(Z,Y)$ is a homotopy equivalence.

Similarly, $g^* : \mathscr{C}(X,Z) \to \mathscr{C}(Y,Z)$ is a homotopy inverse for f^*. Let H and G be as above. Define $\tilde{H} : \mathscr{C}(X,Z) \times [0,1] \to \mathscr{C}(X,Z)$ by $\tilde{H}(h,t) = H(\cdot,t)^*(h)$. Then $\tilde{H}$ is continuous by Propositions 2.43 and 2.44 in a similar manner to that above, and $\tilde{H}(\cdot,0) = (g \circ f)^* = f^* \circ g^*$ and $\tilde{H}(\cdot,1) = \mathrm{id}_{\mathscr{C}(X,Z)}$. In an identical manner, $\tilde{G} : \mathscr{C}(Y,Z) \times [0,1] \to \mathscr{C}(Y,Z)$ given by $\tilde{G}(h,t) = G(\cdot,t)^*(h)$ is continuous, $\tilde{G}(\cdot,0) = g^* \circ f^*$, and $\tilde{G}(\cdot,1) = \mathrm{id}_{\mathscr{C}(Y,Z)}$. This completes the proof. □

2.4 Algebraic topology

Algebraic topology is a thriving area of research where algebraic tools are used to solve topological problems. One of the key ideas in algebraic topology is to find algebraic invariants of topological spaces to identify whether two given spaces are not homotopy equivalent. An example of such invariants are the homotopy groups $\pi_n(X,x_0)$ of a topological space X, that is, homotopy classes of based maps $(S^n,s_0) \to (X,x_0)$. In this section we will collect some well-known and less-known results from algebraic topology to be used throughout the text. The vital results provided are Proposition 2.63, which explores an explicit homotopy equivalence between $\mathbb{C}^* \times \mathbb{C}^*$ and a torus, and Lemma 2.67, which provides a powerful tool to determine whether an inclusion between topological spaces induces an isomorphism of all homotopy groups and an injection of path components. To begin, we collect useful, well-known results about covering spaces and liftings of continuous maps for ease of reference.

Remark 2.47. Of particular interest to us is the study of homotopy groups, however it should be understood that these groups are in general hard to calculate. An example of this is that the homotopy groups of spheres have not all been calculated. To illustrate how complicated these groups can be, Table 2.1 gives some of the homotopy groups of spheres extracted from [12, p. 339].

Theorem 2.48. *Let X be a connected, locally path connected and semi-locally simply connected topological space. Then X has a universal covering space.*

Proof. See [12, p. 64]. □

Corollary 2.49. *Every Riemann surface has a universal covering.*

Proof. A Riemann surface is locally biholomorphic to an open subset of $\mathbb{C}$, so it follows that it is locally path connected and semi-locally simply connected. Since we take Riemann surfaces to be connected, the result follows. □

n \ k	1	2	3	4	5	6	7	8	9	10
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Table 2.1: The homotopy groups $\pi_k(S^n)$ of spheres for $k = 1, \ldots, 10$ and $n = 1, \ldots, 6$.

Example 2.50.

- The exponential map $\exp(2\pi i \cdot) : \mathbb{R} \to S^1$ is a universal covering map when realising $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.
- For $n \geq 2$, the projection map $S^n \to \mathbb{RP}^n$ defined by identifying antipodal points is a universal covering map of real projective space.
- The exponential map $\exp(2\pi i \cdot) : \mathbb{C} \to \mathbb{C}^*$ is a universal covering map for $\mathbb{C}^*$.
- The projection map $\mathbb{C} \to \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $z \mapsto z + \mathbb{Z} + \tau\mathbb{Z}$, for $\tau \in \mathbb{H}$ is a universal covering map for the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.
- The exponential map $\exp(2\pi i \cdot) : \mathbb{H} \to \mathbb{D}^*$ is a universal covering map of $\mathbb{D}^*$. Since $\mathbb{H}$ is biholomorphic to $\mathbb{D}$ we may realise this as a map $\mathbb{D} \to \mathbb{D}^*$.

Proposition 2.51. *Let X be a Riemann surface and $p : \tilde{X} \to X$ be a covering of X by a Hausdorff topological space $\tilde{X}$. Then there is a unique complex structure on $\tilde{X}$ such that p is holomorphic.*

Proof. Since $p : \tilde{X} \to X$ is a covering map, it is in particular a local homeomorphism. If (U, φ) is a chart on X, then there is an evenly covered neighbourhood $U' \subset U$ such that $(U', \varphi|_{U'})$ is a chart on X. If $V \subset p^{-1}(U')$ is a connected component, then $p|_V : V \to U'$ is a homeomorphism and $(V, \varphi|_{U'} \circ p)$ is a chart on $\tilde{X}$. The collection of all charts obtained in this manner is easily seen to define a complex structure on $\tilde{X}$. The fact that the structure is unique and that p is holomorphic both follow from this construction. □

Remark 2.52. Proposition 2.51 is an important result as it tells us that we can realise coverings of Riemann surfaces as complex spaces and holomorphic maps. If $p : \tilde{X} \to X$ is a covering map of a Riemann surface X, then p is a local biholomorphism since covering maps are local homeomorphisms. This fact tell us that, if we have a holomorphic map

$f : Y \to X$ between Riemann surfaces with a continuous lifting $\tilde{f} : Y \to \tilde{X}$ of f by p such that

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

commutes, then $\tilde{f}$ is holomorphic.

Lemma 2.53. *Let $p : Y \to X$ be a covering map of topological spaces. Then the induced map $p_* : \pi_1(Y, y_0) \to \pi_1(X, p(y_0))$ is injective for every choice of $y_0 \in Y$.*

Proof. See [12, p. 61, Proposition 1.31]. □

Theorem 2.54. *Let X be a connected, locally path connected and semi-locally simply connected topological space. Then for each subgroup $H < \pi_1(X)$ there is a covering space $p : \tilde{X} \to X$ with $\pi_1(\tilde{X}) \cong p_*\pi_1(\tilde{X}) = H$. Furthermore, there is a bijection between the fibre $p^{-1}(x)$ for any $x \in X$ and the set of cosets of H in $\pi_1(X)$.*

Proof. See [12, p. 66, Proposition 1.36]. □

Definition 2.55. Let $p : Y \to X$ be a covering map of topological spaces. An *automorphism* of p (also called a *deck transformation* or *covering transformation*) is a homeomorphism $h : Y \to Y$ such that

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. The automorphisms of p naturally make a group of homeomorphisms under composition, which we denote by $\operatorname{Aut} p$.

Remark 2.56. It is useful to note that $\operatorname{Aut} p$ is discrete. Since $\operatorname{Aut} p \subset \operatorname{Aut} Y$, we give it the compact open topology, so what we wish to do is find an open neighbourhood $A(K, U)$ of each point $h \in \operatorname{Aut} p$ such that $A(K, U) \cap \operatorname{Aut} p = \{h\}$. Fix $x \in X$ and $y \in p^{-1}(x)$. Then h is uniquely determined by $h(y) = y_0$ for some $y_0 \in p^{-1}(x)$. Since $p^{-1}(x)$ is discrete, there is an open neighbourhood U of y_0 such that $p^{-1}(x) \cap U = \{y_0\}$. Then $A(\{y\}, U) = \{g \in \mathscr{C}(Y, Y) : g(y) \in U\}$ is an open neighbourhood of h such that $A(\{y\}, U) \cap \operatorname{Aut} p = \{h\}$, showing $\operatorname{Aut} p$ is discrete.

Proposition 2.57. *Suppose X is a connected, locally path connected and semi-locally simply connected topological space. Let $p : \tilde{X} \to X$ denote the universal covering space of X. Then $\pi_1(X) \cong \operatorname{Aut} p$.*

Proof. See [12, p. 71, Proposition 1.39]. □

Proposition 2.58. *Suppose X is a connected, locally path connected and semi-locally simply connected topological space, and let $p : \tilde{X} \to X$ denote the universal covering space of X. Then there is a group Γ of homeomorphisms of $\tilde{X}$ that acts freely and properly discontinuously on $\tilde{X}$ such that $X = \tilde{X}/\Gamma$. Furthermore, $\Gamma = \operatorname{Aut} p \cong \pi_1(X)$.*

Proof. See [12, p. 72, Proposition 1.40]. □

Remark 2.59. Proposition 2.58 tells us the structure of a space X in terms of its universal covering space $\tilde{X}$. That is, X is a set of equivalence classes of points in $\tilde{X}$ where $\tilde{x}_0 \sim \tilde{x}_1$ if there is $h \in \Gamma$ with $h(\tilde{x}_0) = \tilde{x}_1$.

Proposition 2.60. *Let $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ be a based covering of topological spaces. Suppose $f : (X, x_0) \to (Y, y_0)$ is a based continuous map with X connected and locally path-connected. Then a continuous lifting $\tilde{f} : (X, x_0) \to (\tilde{Y}, \tilde{y}_0)$ of f by p exists if and only if $f_*\pi_1(X, x_0) \subset p_*\pi_1(\tilde{Y}, \tilde{y}_0)$. Furthermore, since X is connected, any such lift is unique.*

Proof. See [12, p. 61–62, Propositions 1.33–1.34]. □

Corollary 2.61. *Let $p : Y \to X$ be a covering map of topological spaces. Suppose Z is connected, simply connected and locally path connected and $f : Z \to X$ is a continuous map. Then for every choice of points $z_0 \in Z$ and $y_0 \in Y$ with $f(z_0) = p(y_0)$ there exists precisely one continuous lifting $\hat{f} : Z \to Y$ of f by p with $\hat{f}(z_0) = y_0$.*

Having collected the relevant results about covering spaces and liftings of maps, we move on to describing an important homotopy equivalence between $\mathbb{C}^* \times \mathbb{C}^*$ and a torus. From a topological perspective, the fact that $\mathbb{C}^* \times \mathbb{C}^*$ and a torus T are homotopy equivalent is almost trivial, since they are both homotopy equivalent to $S^1 \times S^1$. However, since both of these spaces are also complex manifolds, we would like to know if we can construct a homotopy equivalence between them that respects this complex structure. Obviously we cannot construct a holomorphic map $T \to \mathbb{C}^* \times \mathbb{C}^*$ that is a homotopy equivalence since any such map is necessarily constant and T is not contractible. To see this, let $f : T \to \mathbb{C}^* \times \mathbb{C}^*$ be a holomorphic map and $\mathrm{pr}_1 : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ be projection onto the first factor. Then $\mathrm{pr}_1 \circ f : T \to \mathbb{C}^*$ is holomorphic and so constant by Proposition 2.13. Similarly, projection onto the second factor of f is constant and thus f itself is constant. Despite this, Winkelmann constructed a holomorphic map $\mathbb{C}^* \times \mathbb{C}^* \to T$ that is a homotopy equivalence for his work in [30]. We give an explicit construction of this homotopy equivalence, but before we can do this we need the following definition.

Definition 2.62. Let Γ be a group and X and Y be sets such that Γ acts on X and on Y. A map $f : X \to Y$ is called *equivariant* if it preserves the action of Γ, that is,

$$f(\gamma \cdot x) = \gamma \cdot f(x)$$

for all $\gamma \in \Gamma$ and $x \in X$.

Proposition 2.63. *Let T be a torus. Then there is a holomorphic map $f : \mathbb{C}^* \times \mathbb{C}^* \to T$ that is a homotopy equivalence.*

Proof. Let $T = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be the complex torus defined by $\tau \in \mathbb{H}$. We have the universal covering maps $\mathrm{ex} = (\exp(2\pi i\cdot), \exp(2\pi i\cdot)) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$ and $\pi : \mathbb{C} \to T$, $z \mapsto z + \mathbb{Z} + \tau\mathbb{Z}$. The action of $\mathbb{Z}^2$ by covering transformations on $\mathbb{C}^2$ and $\mathbb{C}$ is given by

$$(m, n) \cdot (x, y) = (x + m, y + n)\,, \qquad (m, n) \cdot z = z + m + \tau n$$

for all $(m, n) \in \mathbb{Z}^2$, $(x, y) \in \mathbb{C}^2$ and $z \in \mathbb{C}$. Define a map $F : \mathbb{C}^2 \to \mathbb{C}$, equivariant with respect to the actions of $\mathbb{Z}^2$, by $F(x, y) = x + \tau y$. We wish to define $f : \mathbb{C}^* \times \mathbb{C}^* \to T$ such that we get a commuting diagram

$$\begin{tikzcd} \mathbb{C} \times \mathbb{C} \arrow[r, "F"] \arrow[d, "\mathrm{ex}"'] & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{C}^* \times \mathbb{C}^* \arrow[r, "f"] & T \end{tikzcd}$$

Our candidate map will be $f(x, y) = \frac{\log x}{2\pi i} + \tau\frac{\log y}{2\pi i} + \mathbb{Z} + \tau\mathbb{Z}$. This is the map pointwise obtained by tracing the above diagram, and by the equivariance of F we see it is well defined, that is, does not depend on the chosen branch of logarithm. Further we see that f is a holomorphic map since ex is a local biholomorphism and F and π are holomorphic.

It is interesting to note that we cannot find a holomorphic homotopy inverse to f. As discussed above, all holomorphic maps $T \to \mathbb{C}^* \times \mathbb{C}^*$ are constant, and since T is not contractible a homotopy inverse for f cannot be constant. Thus, to define a homotopy inverse for f we define an equivariant continuous map $G : \mathbb{C} \to \mathbb{C}^2$ by

$$G(z) = \left(\frac{\mathrm{Re}(-i\tau\bar{z})}{\mathrm{Im}(\tau)}, \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right).$$

To see G is equivariant, take $m, n \in \mathbb{Z}$ and note that

$$\begin{aligned} G(z + m + \tau n) &= \left(\frac{\mathrm{Re}(-i\tau(\overline{z + m + \tau n}))}{\mathrm{Im}(\tau)}, \frac{\mathrm{Im}(z + m + \tau n)}{\mathrm{Im}(\tau)}\right) \\ &= \left(\frac{\mathrm{Re}(-i\tau\bar{z} - i\tau m - i\tau\bar{\tau}n)}{\mathrm{Im}(\tau)}, \frac{\mathrm{Im}(z) + \mathrm{Im}(\tau)n}{\mathrm{Im}(\tau)}\right) \\ &= \left(\frac{\mathrm{Re}(-i\tau\bar{z}) + m\mathrm{Im}(\tau)}{\mathrm{Im}(\tau)}, \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} + n\right) \\ &= \left(\frac{\mathrm{Re}(-i\tau\bar{z})}{\mathrm{Im}(\tau)} + m, \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} + n\right), \end{aligned}$$

as required. This gives an induced map $g : T \to \mathbb{C}^* \times \mathbb{C}^*$ given by

$$g\,(z + \mathbb{Z} + \tau\mathbb{Z}) = \left(\exp\left(2\pi i\frac{\mathrm{Re}(-i\tau\bar{z})}{\mathrm{Im}(\tau)}\right), \exp\left(2\pi i\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right)\right)$$

for any representative z. We observe that $f \circ g = \mathrm{id}_T$, so consider $g \circ f : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$. For $x, y \in \mathbb{C}^*$, write $x = r_1 e^{i\theta_1}$, $y = r_2 e^{i\theta_2}$ for $r_1, r_2 > 0$ and $\theta_1, \theta_2 \in [0, 2\pi)$. We compute that

$$g \circ f(x,y) = \left(\exp\left(\frac{i\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} \log r_1 + \frac{i|\tau|^2}{\mathrm{Im}(\tau)} \log r_2 \right) \exp(i\theta_1) \ , \right.$$
$$\left. \exp\left(\frac{-i}{\mathrm{Im}(\tau)} \log r_1 + \frac{-i\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} \log r_2 \right) \exp(i\theta_2) \right) .$$

Define a homotopy $H : \mathbb{C}^* \times \mathbb{C}^* \times [0,1] \to \mathbb{C}^* \times \mathbb{C}^*$ by

$$H(x,y,t) = \left(\exp\left(\left(1 - t\left(1 - \frac{\mathrm{Im}(\tau)}{i\mathrm{Re}(\tau)} \right) \right) \frac{i\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} \log r_1 \right.\right.$$
$$\left. + (1-t)\frac{i|\tau|^2}{\mathrm{Im}(\tau)} \log r_2 \right) \exp(i\theta_1) \, , \exp\left((1-t)\frac{-i}{\mathrm{Im}(\tau)} \log r_1 \right.$$
$$\left.\left. + \left(1 - t\left(1 + \frac{\mathrm{Im}(\tau)}{i\mathrm{Re}(\tau)} \right) \right) \frac{-i\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} \log r_2 \right) \exp(i\theta_2) \right) ,$$

if $\mathrm{Re}(\tau) \neq 0$ and

$$H(x,y,t) = \left(\exp\left(t \log r_1 + (1-t)\frac{i|\tau|^2}{\mathrm{Im}(\tau)} \log r_2 \right) \exp(i\theta_1) \ , \right.$$
$$\left. \exp\left((1-t)\frac{-i}{\mathrm{Im}(\tau)} \log r_1 + t \log r_2 \right) \exp(i\theta_2) \right) ,$$

if $\mathrm{Re}(\tau) = 0$. In either case we see H is continuous with

$$H(x,y,0) = g \circ f(x,y)$$

and

$$H(x,y,1) = (\exp(\log r_1)\exp(i\theta_1), \exp(\log r_2)\exp(i\theta_2)) = (x,y) \, .$$

Thus, g is a homotopy inverse for f, proving the result. □

The notion of a homotopy equivalence is well known, however in some cases a full homotopy equivalence is in fact more than we are able to prove. This leads us to relax the definition somewhat to the notion of a weak homotopy equivalence.

Definition 2.64. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is a *weak homotopy equivalence* if the induced map $f_* : \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$ and every choice of $x \in X$, and the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.

A weak homotopy equivalence gives us a sense of when two spaces have the same "*rough shape*" in that they have the same number of connected components and the same homotopy groups. A homotopy equivalence is a weak homotopy equivalence, however it can be quite difficult to construct an explicit homotopy inverse for a given map. The notion of a weak equivalence[3] can therefore be quite powerful because a weak equivalence may be upgraded to a genuine homotopy equivalence for nice spaces. This is made precise in Theorem 2.65.

The "nice spaces" we refer to are CW-complexes. These are spaces built up from attaching cells together and they have good topological properties. Most spaces we encounter have the homotopy type of a CW-complex. For example, every manifold has the homotopy type of a CW-complex. The theory of CW-complexes is well known and expansive but would take us too far afield to delve into here. For a thorough discussion of these spaces, we refer the reader to [4].

Theorem 2.65 (Whitehead's theorem)**.** *Let X and Y be connected CW-complexes. If $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.*

Proof. See [12, p. 346, Theorem 4.5]. □

Example 2.66. We provide an example of where Whitehead's theorem can fail when our spaces do not have a CW-structure. Let $X = \{0, 1, 2, \dots\}$ have the discrete topology and $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ have the topology induced from the real line. Then X is a CW-complex and Y is not since it fails to be locally contractible, that is, the point $0 \in Y$ fails to have any neighbourhood which is contractible. Consider the map $f : X \to Y$ such that

$$f(n) = \begin{cases} 0 & \text{if } n = 0\,, \\ \frac{1}{n} & \text{otherwise}\,. \end{cases}$$

Then f is a weak equivalence since $\pi_k(X) = \pi_k(Y) = 0$ for all $k \geq 1$ and $f_* : \pi_0(X) \to \pi_0(Y)$ is clearly bijective. However, f fails to have a homotopy inverse. If such an inverse $g : Y \to X$ did exist, g would be locally constant since X is discrete. In particular, g would be constant on a neighbourhood of 0, U say. Then $g \circ f|_{f^{-1}(U)} : f^{-1}(U) \to X$ is constant and so not the identity on $f^{-1}(U)$. Since X is discrete, $\mathscr{C}(X, X)$ is a discrete space and all maps $[0, 1] \to \mathscr{C}(X, X)$ are constant. It follows that there is no homotopy from $g \circ f$ to id_X and f has no homotopy inverse.

We now provide a convenient method to show whether an inclusion of topological spaces is a weak homotopy equivalence. Lemma 2.67 is a concise method of showing an inclusion induces an isomorphism of all homotopy groups and an injection of path components, and so handles all requirements for a weak homotopy equivalence, except for a surjection of path components which must be done separately. This method is used heavily throughout Chapter 3.

[3] By weak equivalence we always mean a weak homotopy equivalence.

Lemma 2.67. *Let $X \subset Y$ be topological spaces. For $k \geq 1$, let B^k denote the closed unit ball in $\mathbb{R}^k$ and fix $b_0 \in \partial B^k = S^{k-1}$. Then the inclusion $X \hookrightarrow Y$ induces a surjection on π_k and an injection on π_{k-1} (for every choice of base point) if and only if for every map $\alpha_0 : B^k \to Y$ with $\alpha_0(\partial B^k) \subset X$, there is a continuous map $\alpha : B^k \times [0,1] \to Y$ with $\alpha(\cdot, 0) = \alpha_0$ such that, writing $\alpha_t = \alpha(\cdot, t)$,*

(1) *$\alpha_t(b_0) = \alpha_0(b_0)$ for all $t \in [0,1]$,*

(2) *$\alpha_t(\partial B^k) \subset X$ for all $t \in [0,1]$,*

(3) *$\alpha_1(B^k) \subset X$.*

Proof. Fix $x_0 \in X$. Suppose for every $\alpha_0 : B^k \to Y$ with $\alpha_0(\partial B^k) \subset X$, there is a deformation α as in the lemma. We first show that $\pi_k(X, x_0) \to \pi_k(Y, x_0)$ is a surjection, that is, given a representative $\gamma : (S^k, s_0) \to (Y, x_0)$ of a class in $\pi_k(Y, x_0)$, there is a map $\tilde{\gamma} : (S^k, s_0) \to (X, x_0)$ homotopic to γ relative to s_0. Considering $S^k = B^k/\partial B^k$, define $\alpha_0 : B^k \to Y$ such that

$$\begin{tikzcd} (B^k, \partial B^k) \arrow[d] \arrow[dr, "\alpha_0"] & \\ (S^k, s_0) \arrow[r, "\gamma"] & (Y, x_0) \end{tikzcd}$$

commutes, that is, $\alpha_0(b) = \gamma([b])$ where $[b]$ denotes the equivalence class of $b \in B^k$ when you collapse the boundary to a point. Then by assumption there is a continuous map $\alpha : B^k \times [0,1] \to Y$ with $\alpha(\cdot, 0) = \alpha_0$ satisfying (1)–(3). Define $\tilde{\alpha} : B^k \times [0,1] \to Y$ by

$$\tilde{\alpha}_t(b) = \begin{cases} \alpha_t((1+t)b) & \text{if } \|b\| \leq \frac{1}{1+t}\,, \\ \alpha_{(1+t)(1-\|b\|)}\left(\dfrac{b}{\|b\|}\right) & \text{if } \|b\| \geq \frac{1}{1+t}\,, \end{cases}$$

where $\tilde{\alpha}_t = \tilde{\alpha}(\cdot, t)$ and $\alpha_t = (\cdot, t)$. Then $\tilde{\alpha}_t$ is a deformation of α_0 with $\tilde{\alpha}_t(b) = \alpha_0(b_0)$ for all $b \in \partial B^k$ and $t \in [0,1]$. Thus $\tilde{\alpha}_t$ descends to a homotopy of maps $(S^k, s_0) \to (Y, x_0)$. Since $\tilde{\alpha}_1(B^k) \subset X$, $\tilde{\alpha}_1$ descends to a map $\tilde{\gamma} : (S^k, s_0) \to (X, x_0)$ representing the same class as γ, as required.

We now show that $\pi_{k-1}(X, x_0) \to \pi_{k-1}(Y, x_0)$ is an injection. Suppose $\hat{\alpha}_0 : (S^{k-1}, s_0) \to (X, x_0)$ represents a null-homotopic class in $\pi_{k-1}(Y, x_0)$. Then $\hat{\alpha}_0$ extends to a continuous map $\alpha_0 : B^k \to Y$ such that $\alpha_0|_{\partial B^k} = \hat{\alpha}_0$. We wish to show $\alpha_0|_{\partial B^k}$ represents a null-homotopic class in $\pi_{k-1}(X, x_0)$, that is, we can extend $\alpha_0|_{\partial B^k}$ to a map $\alpha_0' : B^k \to X$. By assumption, there is a continuous map $\alpha : B^k \times [0,1] \to Y$ with $\alpha(\cdot, 0) = \alpha_0$ satisfying (1)–(3). Writing $\alpha(\cdot, t) = \alpha_t$, define

$$\alpha_0'(b) = \begin{cases} \alpha_1(2b) & \text{if } \|b\| \leq \frac{1}{2}\,, \\ \alpha_{(2-2\|b\|)}\left(\dfrac{b}{\|b\|}\right) & \text{if } \|b\| \geq \frac{1}{2}\,. \end{cases}$$

Then $\alpha_0'(B^k) \subset X$ and $\alpha_0'|_{\partial B^k} = \alpha_0|_{\partial B^k}$. Thus $\alpha_0|_{\partial B^k}$ represents a null-homotopic class in $\pi_{k-1}(X, x_0)$ and $\pi_{k-1}(X, x_0) \to \pi_{k-1}(Y, x_0)$ is injective.

Now suppose that

$$\pi_k(X, x_0) \to \pi_k(Y, x_0)$$

is surjective and

$$\pi_{k-1}(X, x_0) \to \pi_{k-1}(Y, x_0)$$

is injective. Let $\alpha_0 : (B^k, b_0) \to (Y, x_0)$ be a continuous map with $\alpha_0(\partial B^k) \subset X$. Then $\alpha_0|_{\partial B_k}$ represents a null-homotopic class in $\pi_{k-1}(Y, x_0)$, and since $\pi_{k-1}(X, x_0) \to \pi_{k-1}(Y, x_0)$ is injective, $\alpha_0|_{\partial B_k}$ is null-homotopic in X. Thus there is a map $\tilde{\alpha}_0 : (B^k, x_0) \to (X, x_0)$ with $\tilde{\alpha}_0|_{\partial B_k} = \alpha_0|_{\partial B_k}$. Writing $\alpha_t = \alpha(\cdot, t)$, define a continuous map $\alpha : B^k \times [0, 1] \to Y$ by

$$\alpha_t(b) = \begin{cases} \alpha_0((1+t)b) & \text{if } \|b\| \leq \frac{1}{1+t}\,, \\ \tilde{\alpha}_0(s(b,t)) & \text{if } \|b\| \geq \frac{1}{1+t}\,, \end{cases}$$

where

$$s(b,t) = (1-t)\frac{b}{\|b\|} + ((1+t)\,\|b\| - 1)b_0 + (1+t)(1 - \|b\|)\frac{b}{\|b\|}\,.$$

Then $\alpha_t(\partial B^k) \subset X$, $\alpha_t(b_0) = \alpha_0(b_0) = x_0$ for all $t \in [0, 1]$, and $\alpha_1(b) = \alpha_0(b_0)$ for all $b \in \partial B^k$. Let $\alpha_0' = \alpha_1$. Then α_0' descends to a map $S^k \to Y$ making

$$\begin{tikzcd} (B^k, \partial B^k) \arrow[d] \arrow[dr, "\alpha_0'"] & \\ (S^k, s_0) \arrow[r] & (Y, x_0) \end{tikzcd}$$

commute. Since $\pi_k(X, x_0) \to \pi_k(Y, x_0)$ is surjective, α_0' is then homotopic as a map $(S^k, s_0) \to (Y, x_0)$ to a map $\alpha_1' : (S^k, s_0) \to (X, x_0)$. This homotopy then lifts to give a homotopy $\alpha' : B^k \times [0, 1] \to Y$ from α_0' to α_1' such that (writing $\alpha_t' = \alpha'(\cdot, t)$) $\alpha_t'(b_0) = \alpha_0'(b_0) = \alpha_1(b_0) = \alpha_0(b_0)$, $\alpha_t'(\partial B^k) \subset X$ and $\alpha_1'(B^k) \subset X$. Thus performing first α_t and then α_t' gives the desired deformation of α_0. $\square$

2.5 Topological groups

A *topological group* is a topological space together with a group action that interacts nicely with the topology, that is, the group multiplication and inverse maps are continuous with respect to the topology. When considering a topological group, we can also consider the space of continuous maps into the group. We would like the space of continuous maps with the compact open topology to inherit the topological group structure from our target with pointwise multiplication as the group action. Lemma 2.69 shows that this is in fact

the case. Moreover, Corollary 2.70 shows that the topological group structure is preserved when considering maps between Riemann surfaces and restricting to holomorphic maps, and Corollary 2.71 shows that if our target is $\mathbb{C}$ then the space of continuous functions will inherit a topological vector space structure. To begin, we have a result that helps us describe the topology of a topological group.

Proposition 2.68. *Let X be a topological group and $1 \in X$ be the identity element. Let X_{x_0} denote the path component of $x_0 \in X$. Then X_{x_0} is homeomorphic to X_1 for all $x_0 \in X$.*

Proof. The map $X_1 \to X_{x_0}$, $x \mapsto x \cdot x_0$, with inverse $X_{x_0} \to X_1$, $x \mapsto x \cdot x_0^{-1}$, provides the required homeomorphism. □

Lemma 2.69. *Let X be a locally compact Hausdorff space and Y be a Hausdorff topological group. Then the space $\mathscr{C}(X, Y)$ is a topological group with the compact-open topology where the group action is pointwise multiplication.*

Proof. Since Y is a topological group there are continuous maps $m : Y \times Y \to Y$, $(x, y) \mapsto xy$, and $i : Y \to Y$, $y \mapsto y^{-1}$, and an identity element $1 \in Y$ such that for all $x, y, z \in Y$ we have:

(1) $(xy)z = x(yz)$,

(2) $y \cdot 1 = y = 1 \cdot y$,

(3) $yy^{-1} = 1 = y^{-1}y$.

Define induced maps

$$\mu : \mathscr{C}(X, Y) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Y) \quad \text{and} \quad \iota : \mathscr{C}(X, Y) \to \mathscr{C}(X, Y)$$

by

$$\mu(f, g)(x) = m(f(x), g(x)) = f(x)g(x)\,, \qquad \iota(f)(x) = i(f(x)) = f(x)^{-1}\,.$$

It is clear that μ and ι satisfy requirements (1)–(3) with the identity element the constant map $1 : X \to Y$, and so it remains to show that μ and ι are continuous.

The map $\mu : \mathscr{C}(X, Y) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Y)$ is continuous if and only if the corresponding map $\hat{\mu} : \mathscr{C}(X, Y) \times \mathscr{C}(X, Y) \times X \to Y$ given by $\hat{\mu}(f, g, x) = \mu(f, g)(x)$ is continuous by Proposition 2.43 (b). However we see that

$$\hat{\mu}(f, g, x) = \mu(f, g)(x) = m(f(x), g(x)) = m(\mathrm{ev}(f, x), \mathrm{ev}(g, x))\,,$$

which is the composition of the evaluation map $\mathrm{ev} : \mathscr{C}(X, Y) \times X \to Y$, which is continuous by Proposition 2.43 (a), and $m : Y \times Y \to Y$, which is continuous since Y is a topological group. Thus μ is continuous.

Similarly, $\iota : \mathscr{C}(X,Y) \to \mathscr{C}(X,Y)$ is continuous if and only if the corresponding map $\hat{\iota} : \mathscr{C}(X,Y) \times X \to Y$ given by $\hat{\iota}(f,x) = \iota(f)(x)$ is continuous. We see that

$$\hat{\iota}(f,x) = \iota(f)(x) = i(f(x)) = i \circ \mathrm{ev}(f,x)\,,$$

which again is continuous. Thus $\mathscr{C}(X,Y)$ is a topological group. □

Corollary 2.70. *Let X and Y be Riemann surfaces such that Y is a complex Lie group (that is, Y is $\mathbb{C}$, $\mathbb{C}^*$ or a torus). Then the space $\mathscr{O}(X,Y)$ is a topological group in the compact-open topology.*

Proof. Since $\mathscr{O}(X,Y) \subset \mathscr{C}(X,Y)$ is both a subgroup and a subspace, the result follows from Lemma 2.69. □

Corollary 2.71. *Let X be a locally compact Hausdorff space. Then $\mathscr{C}(X,\mathbb{C})$ is a topological vector space over $\mathbb{C}$ with the compact-open topology.*

Proof. By Lemma 2.69, $\mathscr{C}(X,\mathbb{C})$ obtains the structure of an abelian group from pointwise addition in $\mathbb{C}$, so what remains to be shown is that the scalar multiplication $\mathscr{C}(X,\mathbb{C}) \times \mathbb{C} \to \mathscr{C}(X,\mathbb{C})$, $(f,z) \mapsto zf$, is continuous and satisfies the necessary distributive laws. Again by Lemma 2.69, $\mathscr{C}(X,\mathbb{C})$ obtains the structure of an abelian group with pointwise multiplication, so in particular the multiplication map $\mathscr{C}(X,\mathbb{C}) \times \mathscr{C}(X,\mathbb{C}) \to \mathscr{C}(X,\mathbb{C})$, $(f,g) \mapsto fg$, is continuous. Then scalar multiplication is continuous since it is simply restriction to the constant maps in the second factor of the multiplication map. We also require that, for $f,g \in \mathscr{C}(X,\mathbb{C})$ and $z,w \in \mathbb{C}$,

1. $z(f+g) = zf + zg$,
2. $(z+w)f = zf + wf$,
3. $z(wf) = (zw)f$,
4. $1 \cdot f = f$.

Clearly all of these are satisfied by the scalar multiplication map, proving the result. □

2.6 Fibrations

The notion of a fibration in algebraic topology is a generalisation of a covering map between topological spaces, that is, a fibration generalises the path lifting property of covering maps. We show that a fibration $p : Y \to X$ between topological spaces X and Y induces a fibration $p_* : \mathscr{C}(Z,Y) \to \mathscr{C}(Z,X)$ by post-composition for any space Z. This can be refined to holomorphic maps when the source and target are Riemann surfaces and the fibration is holomorphic. The notion of a fibre bundle is briefly introduced to

show that the evaluation map is a fibration from the space of continuous and holomorphic maps to the Riemann sphere. We also provide the long exact sequence of homotopy groups induced by a fibration for use in Sections 3.2 and 3.4. To begin, we provide the definition of a fibration.

Definition 2.72. Let $p : Y \to X$ be a continuous map between topological spaces. Then p is said to satisfy the *homotopy lifting property* for a topological space Z if, given a continuous map $f : Z \times [0,1] \to X$ and a continuous lifting $\tilde{f}_0 : Z \times \{0\} \to Y$ of $f|_{Z\times\{0\}}$ by p, there is a continuous lifting $\tilde{f} : Z \times [0,1] \to Y$ of f by p making

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, hook] & Y \arrow[d, "p"] \\ Z \times [0,1] \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & X \end{tikzcd}$$

commute.

Definition 2.73. Let $p : Y \to X$ be a continuous map between topological spaces. Then p is said to be a *fibration* if p satisfies the homotopy lifting property for all spaces.

Remark 2.74. Fibrations thus defined are known as *Hurewicz fibrations*. The other common type of fibration, a *Serre fibration*, is not used in this thesis and so is not defined here.

Proposition 2.75. *Let $p : Y \to X$ be a fibration over a path connected space X, and let $x_0, x_1 \in X$. Then $p^{-1}(x_0)$ is homotopy equivalent to $p^{-1}(x_1)$.*

Sketch of proof. Let $\gamma : [0,1] \to X$ be a path from x_0 to x_1. For each choice $y \in p^{-1}(x_0)$, let $\gamma_y : [0,1] \to Y$ be a continuous lift of γ by p, which exists since p is a fibration, such that $\gamma_y(0) = y$. Then the map $y \mapsto \gamma_y(1)$ provides the desired homotopy equivalence $p^{-1}(x_0) \to p^{-1}(x_1)$. □

Remark 2.76. By Proposition 2.75, we tend to talk about *the fibre* of p instead of any particular fibre since the fibres are determined up to homotopy type.

Proposition 2.77. *Let $p : Y \to X$ be a covering map between topological spaces. Then p is a fibration.*

Proof. See [26, p. 67]. □

Proposition 2.77 is part of the reason why we think of fibrations as generalisations of covering maps. A covering map is a special type of fibration with discrete fibres, a property that makes covering maps so convenient to work with. We now collect some results about fibrations.

Proposition 2.78. *Let $p : Y \to X$ be a fibration between locally compact Hausdorff spaces. Let Z be a locally compact Hausdorff space. Then the induced map $p_* : \mathscr{C}(Z, Y) \to \mathscr{C}(Z, X)$ given by post-composition is a fibration.*

Proof. Let W be a topological space and suppose $f : W \times [0,1] \to \mathscr{C}(Z, X)$ is a continuous map such that we have a continuous lifting $\tilde{f}_0 : W \times \{0\} \to \mathscr{C}(Z, Y)$ of $f|_{W\times\{0\}}$ by p. By Proposition 2.43 (b), we consider f as a map $f : W \times [0,1] \times Z \to X$ and $\tilde{f}_0$ as a map $\tilde{f}_0 : W \times \{0\} \times Z \to Y$. Then we have a diagram

$$\begin{tikzcd}
W \times \{0\} \times Z \arrow[r, "\tilde{f}_0"] \arrow[d, hook] & Y \arrow[d, "p"] \\
W \times [0,1] \times Z \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & X
\end{tikzcd}$$

and since $p : Y \to X$ is a fibration, there exists a continuous lifting $\tilde{f} : W \times [0,1] \times Z \to Y$ of f by p with $\tilde{f}|_{W\times\{0\}\times Z} = \tilde{f}_0$, making this diagram commute. Again by Proposition 2.43 (b), we consider $\tilde{f}$ as a map $\tilde{f} : W \times [0,1] \to \mathscr{C}(Z, Y)$ to get our desired lifting of f by p_*. □

Corollary 2.79. *Let X and Y be Riemann surfaces. Suppose $p : \tilde{Y} \to Y$ is a holomorphic covering map. Then the induced map $p_* : \mathscr{O}(X, \tilde{Y}) \to \mathscr{O}(X, Y)$ is a fibration.*

Proof. By Proposition 2.77, p is a fibration and so $p_* : \mathscr{C}(X, \tilde{Y}) \to \mathscr{C}(X, Y)$ is a fibration by Proposition 2.78. Since $\mathscr{O}(X, Y)$ is a subspace of $\mathscr{C}(X, Y)$ and $p_*^{-1}(\mathscr{O}(X, Y)) = \mathscr{O}(X, \tilde{Y})$, the restriction of p_* to $\mathscr{O}(X, \tilde{Y})$ is then a fibration. □

One useful tool in topology is the notion of a *fibre bundle*, which is a generalisation of a vector bundle. Fibre bundles are surjective maps between topological spaces that locally look like projections, which is made precise with the next definition. The concept of a fibre bundle is introduced here because, under very general assumptions, fibre bundles are fibrations. This provides a powerful tool since it is comparatively easier to show a map is a fibre bundle than to show it is a fibration in general.

Definition 2.80. Let $p : Y \to X$ be a continuous surjection between topological spaces. Then p is said to be a *fibre bundle* if for every $x \in X$ there is a neighbourhood U of x and a homeomorphism $\varphi : p^{-1}(x) \times U \to p^{-1}(U)$ making

$$\begin{tikzcd}
p^{-1}(x) \times U \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_2"'] & & p^{-1}(U) \arrow[dl, "p"] \\
& U &
\end{tikzcd}$$

commute.

Remark 2.81. The usual definition of a fibre bundle deals with a fixed fibre F over each point. However, on each connected component of X, the above definition implies the fibres over each point are homeomorphic. Since we always deal with a connected base space X, our definition is equivalent to the usual definition.

Remark 2.82. A topological space X is said to be *paracompact* if every open cover of X has a locally finite refinement. Recall that every manifold is paracompact.

Proposition 2.83. *Let $p : Y \to X$ be a fibre bundle. Suppose X is paracompact. Then p is a fibration.*

Proof. See [18, p. 49]. □

We can now use Proposition 2.83 to obtain a useful fibration to be used in Section 3.2.

Proposition 2.84. *Let X be a Riemann surface and fix $x_0 \in X$. Then the evaluation maps $\mathrm{ev}_{x_0} : \mathscr{C}(X, \mathbb{P}) \to \mathbb{P}$ and $\mathrm{ev}_{x_0} : \mathscr{O}(X, \mathbb{P}) \to \mathbb{P}$ defined by $f \mapsto f(x_0)$ are fibrations.*

Proof. Since the proof for holomorphic maps is identical, we just consider the case for continuous maps. We show $\mathrm{ev}_{x_0} : \mathscr{C}(X, \mathbb{P}) \to \mathbb{P}$ is a fibre bundle and hence a fibration by Proposition 2.83.

Let $p \in \mathbb{P}\backslash\{\infty\}$ and let $D \subset \mathbb{P}$ be a disc neighbourhood of p not containing ∞. Define $\varphi : \mathrm{ev}_{x_0}^{-1}(p) \times D \to \mathrm{ev}_{x_0}^{-1}(D)$ by $(f, q) \mapsto f - p + q$. Then φ is continuous with continuous inverse $\mathrm{ev}_{x_0}^{-1}(D) \to \mathrm{ev}_{x_0}^{-1}(p) \times D$, $g \mapsto (g - g(x_0) + p, g(x_0))$, and is such that

$$\begin{tikzcd} \mathrm{ev}_{x_0}^{-1}(p) \times D \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_2"'] & & \mathrm{ev}_{x_0}^{-1}(D) \arrow[dl, "\mathrm{ev}_{x_0}"] \\ & D & \end{tikzcd}$$

commutes.

Now suppose $p = \infty$ and let D be a disc neighbourhood of p. Define $\varphi : \mathrm{ev}_{x_0}^{-1}(p) \times D \to \mathrm{ev}_{x_0}^{-1}(D)$ by

$$\varphi(f, q) = \frac{1}{\frac{1}{f} + \frac{1}{q}} \, .$$

Then φ is continuous with continuous inverse $\varphi^{-1} : \mathrm{ev}_{x_0}^{-1}(D) \to \mathrm{ev}_{x_0}^{-1}(p) \times D$ given by

$$\varphi^{-1}(g) = \left(\frac{1}{\frac{1}{g} - \frac{1}{g(x_0)}}, \, g(x_0) \right) \, .$$

Further, we see that

$$\begin{tikzcd} \mathrm{ev}_{x_0}^{-1}(p) \times D \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_2"'] & & \mathrm{ev}_{x_0}^{-1}(D) \arrow[dl, "\mathrm{ev}_{x_0}"] \\ & D & \end{tikzcd}$$

commutes. This completes the proof. □

We now move on to introduce how fibrations can be used to calculate homotopy groups through long exact sequences. The notion of a long exact sequence of groups provides a powerful tool for analysing groups. For a collection of groups (G_n) with maps $\varphi_n : G_n \to G_{n+1}$, the sequence

$$\cdots \longrightarrow G_{n-1} \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} G_{n+1} \longrightarrow \cdots$$

is said to be *exact* if $\text{image}(\varphi_{n-1}) = \ker(\varphi_n)$ for all n. The following results show that we can assign a long exact sequence of homotopy groups to a pair of topological spaces $A \subset X$, and more importantly we can assign one to a fibration $p : Y \to X$ between topological spaces X and Y.

Remark 2.85. Recall that for a pair of topological spaces $A \subset X$ with $a_0 \in A$, we may define the nth *relative homotopy group* $\pi_n(X, A, a_0)$ to be $\pi_{n-1}P(X; a_0, A)$, where $P(X; a_0, A)$ is the space of paths starting at a_0 and ending in A. This is done as in [18, p. 63].

Theorem 2.86. *Let $A \subset X$ be a pair of topological spaces and $a_0 \in A$. Then there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_n(A, a_0) \longrightarrow \pi_n(X, a_0) \longrightarrow \pi_n(X, A, a_0) \xrightarrow{\partial} \pi_{n-1}(A, a_0) \longrightarrow \cdots$$

where ∂ is restriction of maps $(D^n, S^{n-1}, s_0) \to (X, A, a_0)$ to $(S^{n-1}, s_0) \to (A, a_0)$.

Proof. See [18, p. 63]. □

Lemma 2.87. *Let $p : (Y, y_0) \to (X, x_0)$ be a fibration with fibre $Y_0 = p^{-1}(x_0)$. Then there is a natural isomorphism*

$$\pi_n(Y, Y_0, y_0) \to \pi_n(X, x_0)$$

for $n \geq 1$.

Proof. See [18, p. 64]. □

Corollary 2.88. *Let $p : (Y, y_0) \to (X, x_0)$ be a fibration with fibre $Y_0 = p^{-1}(x_0)$. Then there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_n(Y_0, y_0) \longrightarrow \pi_n(Y, y_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_{n-1}(Y_0, y_0) \longrightarrow \cdots$$

Proof. Applying Theorem 2.86 to the pair $Y_0 \subset Y$ and combining this with Lemma 2.87 provides the result. □

Remark 2.89. We have to be careful when talking about the long exact sequence of homotopy groups because at the tail end of the sequence, when $n = 0$, we are no longer dealing with groups but pointed sets. Despite this, we can still make sense of what it means for a sequence to be exact. For a map $f : (A, a_0) \to (B, b_0)$ between pointed sets, we define the kernel of f to be $\ker(f) = \{a \in A : f(a) = b_0\}$. Thus a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of pointed sets is exact if $\mathrm{im}(f) = \ker(g)$.

To unravel this in terms of homotopy groups, consider the tail end of the sequence

$$\cdots \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_0(Y_0) \longrightarrow \pi_0(Y) \longrightarrow \pi_0(X)\,.$$

For these sets it is understood that the base point of $\pi_0(Y_0)$ is the path component of y_0 in Y_0, the base point of $\pi_0(Y)$ is the path component of y_0 in Y, and the base point of $\pi_0(X)$ is the path component of x_0 in X.

Now, this sequence is exact at $\pi_0(Y_0)$ if $\mathrm{im}(\pi_1(X, x_0) \to \pi_0(Y_0))$ consists precisely of the elements mapped to the path component of y_0 in Y by $\pi_0(Y_0) \to \pi_0(Y)$. Similarly, it is exact at $\pi_0(Y)$ if the image of the map $\pi_0(Y_0) \to \pi_0(Y)$ consists precisely of the elements mapped to the path component of x_0 in X by $\pi_0(Y) \to \pi_0(X)$.

2.7 Homotopy groups of mapping spaces

In the following we collate useful results in regards to the homotopy groups of mapping spaces. The mapping spaces we are interested in are the space of continuous maps between topological spaces and the space of holomorphic maps between Riemann surfaces. A variety of work has been done in this area (see [25] for a thorough survey), but we will focus on results by Hansen [11] and Segal [24]. Hansen focuses on calculating the homotopy groups of the space of continuous maps from a compact surface into the two-sphere. Hansen also presents a nice exposition on theorems by Thom and Gottlieb [10], which we will include here. Segal looks at the homotopy groups of the space of holomorphic maps into the Riemann sphere of a particular degree and how these homotopy groups relate to the corresponding continuous maps. This work in particular is in a similar vein to Oka theory, in the sense that Segal shows the inclusion of the space of holomorphic maps into the space of continuous maps of degree d from the Riemann sphere to itself is a d-equivalence. The results collected here are vital to the work done in Section 3.4.

The first is a result due to Hansen, describing the homotopy groups of the space of continuous maps from a compact Riemann surface into the Riemann sphere. Hansen calculates these groups in terms of the homotopy groups of spheres.

Theorem 2.90. *Let Σ_g be a compact Riemann surface of genus g, $d \neq 0$ and $k \geq 2$. Then*

$$\pi_k(\mathscr{C}_d(\Sigma_g, \mathbb{P})) = \pi_k(S^3) \oplus \left(\pi_{k+1}(S^2)\right)^{2g} \oplus \pi_{k+2}(S^2)\,,$$

where $\mathscr{C}_d(\Sigma_g, \mathbb{P})$ is the space of continuous maps $\Sigma_g \to \mathbb{P}$ of degree d.

Proof. See [11, Theorem 4.2]. □

The next result presented is due to Thom [29, Theorem 10] in the abelian case and Gottlieb [7, Lemma 2] in the non-abelian case. Although the results are due to these authors, Hansen provides an accessible exposition of them in [10], which is the main reference used. These results pertain to the homotopy groups of the space of continuous maps into an Eilenberg-MacLane space.

Definition 2.91. Let Y be a path connected topological space. Then Y is said to be an *Eilenberg-MacLane space of type* (π, k) if

$$\pi_n(Y) = \begin{cases} \pi & \text{if } n = k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Example 2.92.

- Suppose $X \neq \mathbb{P}$ is a Riemann surface. Then X is an Eilenberg-MacLane space of type $(\pi_1(X), 1)$ since the universal covering is contractible. For instance, a torus T is an Eilenberg-MacLane space of type $(\mathbb{Z} \oplus \mathbb{Z}, 1)$ and $\mathbb{C}^*$ is an Eilenberg-MacLane space of type $(\mathbb{Z}, 1)$.
- S^1 is an Eilenberg-MacLane space of type $(\mathbb{Z}, 1)$.
- Consider the lattice $\mathbb{Z}^{2n}$ in $\mathbb{C}^n$. Then the torus $\mathbb{C}^n/\mathbb{Z}^{2n}$ is an Eilenberg-MacLane space of type $(\mathbb{Z}^{2n}, 1)$.
- $\mathbb{P}$ fails to be an Eilenberg-MacLane space since $\pi_2(\mathbb{P}) = \pi_3(\mathbb{P}) = \mathbb{Z}$.

Theorem 2.93. *Let Y be an Eilenberg-MacLane space of type (π, n), $n \geq 1$. Let X be a finite-dimensional CW-complex and $f : X \to Y$ a based map. Then*

$$\pi_k(\mathscr{C}(X, Y), f) = \begin{cases} H^{n-k}(X; \pi) & \textit{for } 0 \leq k \leq n\,, \\ 0 & \textit{if } k > n\,, \end{cases}$$

if π is abelian, and

$$\pi_k(\mathscr{C}(X, Y), f) = \begin{cases} C_\pi(f_*\pi_1(X)) & \textit{if } k = 1 \\ 0 & \textit{if } k > 1 \end{cases}$$

if π is non-abelian, where $C_\pi(f_*\pi_1(X))$ denotes the centraliser of $f_*\pi_1(X)$ in π.

Proof. As mentioned above, a nice exposition of this is found in [10]. □

We now move on to Segal's work. Presented here are three of the main results from [24] in regards to the inclusion of the space of holomorphic maps from a compact Riemann surface into the Riemann sphere into the space of continuous maps between the same spaces. The methods Segal uses to obtain these results are quite advanced and rely heavily on what he calls the *scanning map*. This map looks at the zeros and poles of a holomorphic map into the Riemann sphere as if through a very fine microscope and provides great geometric and topological insight into the workings of the space of holomorphic maps of a given degree. We will not go into further details, but this can be found in Segal's paper [24]. The proofs of these results make up the majority of Segal's paper.

Remark 2.94. Let $m \geq 1$ and X and Y be topological spaces. A continuous map $f : X \to Y$ is an *m-equivalence* if the induced map $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is a bijection for $k < m$ and a surjection for $k = m$ for every choice of $x \in X$.

Theorem 2.95. *Let $\mathscr{O}_d^{(*)}$ (resp. $\mathscr{C}_d^{(*)}$) denote the space of based holomorphic (resp. continuous) maps $\mathbb{P} \to \mathbb{P}$ of degree d. Then the inclusion*

$$\mathscr{O}_d^{(*)} \hookrightarrow \mathscr{C}_d^{(*)}$$

is a d-equivalence.

Theorem 2.96. *Let $\mathscr{O}_d$ (resp. $\mathscr{C}_d$) denote the space of holomorphic (resp. continuous) maps $\mathbb{P} \to \mathbb{P}$ of degree d. Then the inclusion*

$$\mathscr{O}_d \hookrightarrow \mathscr{C}_d$$

is a d-equivalence.

Remark 2.97. A *homology m-equivalence* is defined as in Remark 2.94, with homology with integer coefficients replacing homotopy groups.

Theorem 2.98. *Let X be a compact Riemann surface of genus g. Let $\mathscr{O}_d(X, \mathbb{P})$ (resp. $\mathscr{C}_d(X, \mathbb{P})$) denote the space of holomorphic (resp. continuous) maps $X \to \mathbb{P}$ of topological degree d. Then the inclusion*

$$\mathscr{O}_d(X, \mathbb{P}) \hookrightarrow \mathscr{C}_d(X, \mathbb{P})$$

is a homology $(d - 2g)$-equivalence.

2.8 Jet spaces and the Whitney $\mathscr{C}^\infty$ topology

The goal of the next two sections is to show the existence of a strictly subharmonic Morse exhaustion function on an open Riemann surface. We do this by first introducing the Whitney $\mathscr{C}^\infty$ topology in this section and then explaining how this interacts with Morse functions on a manifold in the following section. The aim is to show that an open Riemann surface has the homotopy type of a bouquet of circles, to be used in the proof of Theorem 2.122 in Section 2.10. Throughout, we let $\mathscr{C}^\infty(X,Y)$ denote the space of smooth maps from X to Y. Before we can define the Whitney $\mathscr{C}^\infty$ topology, we need to first look at jet spaces.

Definition 2.99. Let X and Y be smooth manifolds and $f, g : X \to Y$ be smooth maps. Let $p \in X$ and suppose $f(p) = g(p) = q$.

(1) The maps f and g are said to agree to *first order at* p if $d_pf = d_pg$ as linear maps $T_pX \to T_qY$.

(2) The maps f and g are said to agree to k^{th} *order at* p if $df : TX \to TY$ and $dg : TX \to TY$ agree to $(k-1)^{st}$ order at every point of T_pX.

(3) Let $(p,q) \in X \times Y$. The kth *jet space at* (p,q), denoted $J^k_{(p,q)}(X,Y)$, is the set of equivalence classes of smooth maps $f : X \to Y$ with $f(p) = q$, where two smooth maps $f, g : X \to Y$ are equivalent if they agree to k^{th} order at p.

(4) The kth *jet space* from X to Y is

$$J^k(X,Y) = \bigsqcup_{(p,q)\in X\times Y} J^k_{(p,q)}(X,Y)\,.$$

(5) The map $j^kf : X \to J^k(X,Y)$ that sends x to the equivalence class of f in $J^k_{(x,f(x))}(X,Y)$ is called the k-*jet* of f.

We can think of the k-jet of a smooth map $X \to Y$ as a generalised notion of a Taylor polynomial. In fact when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, it is easily seen that the k-jet of a map can be identified with the kth degree Taylor polynomial. The following theorem provides useful information about jet spaces.

Theorem 2.100. *Let X and Y be smooth manifolds. Then for each $k \in \mathbb{N}$,*

(1) *$J^k(X,Y)$ inherits the structure of a smooth manifold from charts on X and Y,*

(2) *for every $f \in \mathscr{C}^\infty(X,Y)$, the map $j^kf : X \to J^k(X,Y)$ is smooth.*

Proof. See [6, p. 40]. □

We can now introduce the Whitney $\mathscr{C}^\infty$ topology on $\mathscr{C}^\infty(X,Y)$.

Definition 2.101. Let X and Y be smooth manifolds.

(1) Let $k \in \mathbb{N}$. Then a basis for the *Whitney $\mathscr{C}^k$ topology* on $\mathscr{C}^\infty(X,Y)$ is given by sets of the form
$$A_k(U) = \{f \in \mathscr{C}^\infty(X,Y) : j^k f(X) \subset U\},$$
where $U \subset J^k(X,Y)$ is open. Let A_k denote the set of basis elements for the Whitney $\mathscr{C}^k$ topology.

(2) The *Whitney $\mathscr{C}^\infty$ topology* on $\mathscr{C}^\infty(X,Y)$ is defined by the basis $\bigcup_{k=1}^{\infty} A_k$.

Remark 2.102. The Whitney $\mathscr{C}^\infty$ topology is much finer than the usual Fréchet topology on the space of smooth maps between smooth manifolds when the source is non-compact. For example, in the Fréchet topology, for a sequence of maps (f_n) to converge to a map f we require the maps f_n and all their partial derivatives to converge uniformly on compact subsets to f and its partial derivatives. Proposition 2.103 gives a much stronger condition on what it means for a sequence to converge in the Whitney $\mathscr{C}^\infty$ topology, which tells us that convergent sequences are in some sense too rare to be useful. We should note that to obtain such a fine topology we must sacrifice metrisability, as the Whitney $\mathscr{C}^\infty$ topology fails to be first countable for a non-compact source.

Proposition 2.103. *Let X and Y be smooth manifolds and endow $\mathscr{C}^\infty(X,Y)$ with the Whitney $\mathscr{C}^\infty$ topology. Then a sequence (f_n) in $\mathscr{C}^\infty(X,Y)$ converges to $f \in \mathscr{C}^\infty(X,Y)$ if and only if there is a compact subset $K \subset X$ with $j^k f_n \to j^k f$ uniformly on K for all $k \in \mathbb{N}$, and only finitely many $f_n \neq f$ on $X \backslash K$.*

Proof. See [6, p. 43]. □

2.9 Morse theory

Morse theory is an interesting area of differential topology that provides a strong connection between topology and analysis. One of the main results in Morse theory tells us that all smooth manifolds, and in particular Riemann surfaces, have the homotopy type of a CW-complex. In this section we will present a few well-known facts about Morse theory and give an idea of how we can find a strictly subharmonic Morse exhaustion function on an open Riemann surface. This is used to show that every open Riemann surface has the homotopy type of a bouquet of circles. We start with the definition of a Morse function.

Definition 2.104. Let X be a smooth manifold and $f : X \to \mathbb{R}$ be smooth. A point $p \in X$ is said to be a *critical point* of f if $d_p f = 0$. A critical point p is said to be

non-degenerate if, in any coordinate system (U, x) (or equivalently, in every coordinate system) centred at p, the Hessian matrix

$$\mathrm{Hess}_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

is non-degenerate, that is, the eigenvalues of $\mathrm{Hess}_p(f)$ are all non-zero. The function f is said to be a *Morse function*, or simply *Morse*, if it has only non-degenerate critical points.

Definition 2.105. Let $f : X \to \mathbb{R}$ be a Morse function on a smooth manifold X, and $p \in X$ a critical point of f. Then the *index* of f at p, written $\mathrm{Index}_p(f)$, is defined to be the number of negative eigenvalues of the Hessian matrix $\mathrm{Hess}_p(f)$.

Remark 2.106. We can relate the index of a Morse function $f : X \to \mathbb{R}$ at a critical point $p \in X$ of f to the behaviour of f. For instance, when $\dim_{\mathbb{R}} X = 2$, we have

- $\mathrm{Index}_p(f) = 0$ if f has a minimum at p,
- $\mathrm{Index}_p(f) = 1$ if f has a saddle point at p,
- $\mathrm{Index}_p(f) = 2$ if f has a maximum at p.

In the following we will let $\mathscr{C}^\infty(X)$ denote the set of smooth functions $X \to \mathbb{R}$, where X is a smooth manifold.

Theorem 2.107. *Let X be a smooth manifold. Then the set of Morse functions is open and dense in $\mathscr{C}^\infty(X)$ with respect to the Whitney $\mathscr{C}^\infty$ topology.*

Proof. See [6, p. 63]. □

We now state the fundamental theorems of Morse theory to give an understanding of how we can reconstruct the topology of a manifold given a Morse function. The proofs of these can be found in [20, Chapter 3].

Theorem 2.108. *Let X be a smooth manifold and $f : X \to \mathbb{R}$ be Morse. Let $X^a = f^{-1}(-\infty, a]$. If $a < b$ is such that $f^{-1}[a, b]$ is compact and contains no critical points of f, then X^a is diffeomorphic to X^b. Further, X^a is a deformation retract of X^b.*

Theorem 2.109. *Let X be a smooth manifold and $f : X \to \mathbb{R}$ be Morse. Let $p \in X$ be a critical point of f of index λ and set $c = f(p)$. Suppose there is $\epsilon > 0$ such that the set $f^{-1}[c - \epsilon, c + \epsilon]$ is compact and contains no other critical points of f. Then $X^{c+\epsilon}$ is diffeomorphic to $X^{c-\epsilon}$ with a λ-cell attached.*

In simplistic terms, these theorems tell us that we can construct X by attaching a λ-cell for each critical point of index λ, and so every smooth manifold has the homotopy type of a CW-complex. We now explore the existence of a strictly subharmonic Morse exhaustion function on an open Riemann surface X.

Remark 2.110. Recall the definition of a strictly subharmonic exhaustion function on a Riemann surface X. A smooth function $f : X \to \mathbb{R}$ is said to be *subharmonic*[4] if $\Delta f(x) \geq 0$ for all $x \in X$, where Δ is any Laplacian determined by the complex structure on X. It is said to be *strictly subharmonic* if $\Delta f(x) > 0$ for all $x \in X$. A smooth function $f : X \to \mathbb{R}$ is said to be an *exhaustion function* if $f^{-1}(-\infty, \epsilon]$ is compact in X for all $\epsilon \in \mathbb{R}$.

Theorem 2.111. *Let X be an open Riemann surface. Then there exists a strictly subharmonic smooth exhaustion function $\rho : X \to \mathbb{R}$.*

Proof. A construction of such a function can be found in [22, p. 82]. □

Remark 2.112. This is not the only way to find a strictly subharmonic exhaustion function on an open Riemann surface. Alternatively, using the fact that an open Riemann surface is Stein (Theorem 2.6), we may holomorphically embed it into $\mathbb{C}^3$. Taking the usual norm squared on $\mathbb{C}^3$ and pulling it back to our Riemann surface via the embedding will then give a strictly subharmonic exhaustion function.

Theorem 2.113. *Let X be an open Riemann surface. Then there exists a strictly subharmonic Morse exhaustion function $\psi : X \to \mathbb{R}$.*

Proof. In the Whitney $\mathscr{C}^\infty$ topology, we can define an open neighbourhood of a map $f \in \mathscr{C}^\infty(X)$ by

$$U_k = \{g \in \mathscr{C}^\infty(X) : \left\| j^k f(x) - j^k g(x) \right\| < \varphi(x) \text{ for all } x \in X\},$$

where $\varphi : X \to (0, \infty)$ is continuous. Let $\rho : X \to \mathbb{R}$ be as in Theorem 2.111 and let U_2 be an open neighbourhood of ρ. For a suitably chosen $\varphi : X \to (0, \infty)$, every $g \in U_2$ is both an exhaustion function and strictly subharmonic. Indeed, if φ is chosen to be bounded by a constant, then the 0-jet of $g \in U_2$ is close enough to the 0-jet of ρ to be an exhaustion function. Further if ρ is chosen to decay fast enough at infinity we ensure that the 2-jets are close enough such that $\Delta g > \Delta \rho > 0$ for all $g \in U_2$. Now, by Lemma 2.107 Morse functions are dense, so there is a Morse function $\psi \in U_2$ satisfying the conditions of the theorem. □

Corollary 2.114. *Let X be an open Riemann surface. Then X has the homotopy type of a bouquet of circles.*

Proof. By Theorem 2.113, we have a strictly subharmonic Morse exhaustion function $\psi : X \to \mathbb{R}$. By the maximum principle for subharmonic functions[5], ψ never attains a maximum, and so the index of each critical point is 0 or 1. Thus by Theorem 2.109, X is made (up to homotopy) by attaching a sufficient number of 0-cells and 1-cells together, giving a bouquet of circles. □

[4] The notion of subharmonicity is generally defined for upper semi-continuous functions, however, we will only work with smooth subharmonic functions.

[5] If a subharmonic function attains its maximum, then it is constant. For a precise statement see [2, p. 179, Theorem 22.9].

2.10 The basic Oka principle

In 1993, Winkelmann completely classified the pairs of Riemann surfaces for which the basic Oka principle is satisfied [30, Theorem 1], identifying examples previously not accounted for by Gromov [9]. This classification is the driving force for this thesis and this section will be dedicated to presenting a thorough understanding of Winkelmann's result. We first provide the definition we work with for the basic Oka principle.

Definition 2.115. Let X and Y be Riemann surfaces. The pair (X, Y) is said to satisfy the *basic Oka principle* (BOP) if every continuous map $X \to Y$ is homotopic to a holomorphic map. Equivalently, the pair (X, Y) satisfies BOP if the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ induces a surjection of path components.

In a loose sense, the basic Oka principle is telling us that the only obstructions to solving analytical problems are topological. This is a common theme in Oka theory dating back to the Cousin problems in 1895. Another example of this comes from Grauert's classification of holomorphic vector bundles over Stein manifolds [8], where he shows that two such vector bundles are holomorphically equivalent if and only if they are topologically equivalent. Winkelmann's classification is the following.

Theorem 2.116 (Winkelmann, 1993 [30])**.** *Let X and Y be Riemann surfaces. Then the pair (X, Y) satisfies BOP in each of the following cases, and only these cases.*

1. (a) *X is $\mathbb{C}$ or $\mathbb{D}$ with any target Y.*
   (b) *Y is $\mathbb{C}$ or $\mathbb{D}$ with any source X.*
   (c) *$X = \mathbb{P}$ with any target $Y \neq \mathbb{P}$.*
2. *X is open and Y is $\mathbb{C}^*$, $\mathbb{P}$ or a torus.*
3. *$X = \bar{X} \setminus \bigcup D_i$ and Y is $\mathbb{D}^*$, where $\bar{X}$ is a compact Riemann surface and (D_i) is a non-empty finite collection of mutually disjoint closed discs in $\bar{X}$.*

This is seen to be a complete classification of the pairs that satisfy BOP when combined with Winkelmann's list of counterexamples in Theorem 3.23 and the classification result Theorem 2.2. The rest of this section is dedicated to breaking down Theorem 2.116 into subcases and providing thorough, understandable proofs.

Theorem 2.117. *Let X and Y be Riemann surfaces such that one of the following is true:*

1. *Y is $\mathbb{C}$ or $\mathbb{D}$.*
2. *X is $\mathbb{C}$ or $\mathbb{D}$.*

Then every continuous map $X \to Y$ is homotopic to a holomorphic map.

Proof. First suppose Y is $\mathbb{C}$. Note that $\mathbb{C}$ is contractible. To see this, take the homotopy $H : \mathbb{C} \times [0,1] \to \mathbb{C}$ given by

$$H(z,t) = tz\,.$$

Then $H(\cdot,0) = 0$ and $H(\cdot,1) = \mathrm{id}_{\mathbb{C}}$. Let $f : X \to \mathbb{C}$ be continuous. Then the map $F : X \times [0,1] \to \mathbb{C}$ given by

$$F(x,t) = H(f(x),t)$$

is a homotopy such that $F(x,0) = 0$ and $F(x,1) = f(x)$. Thus f is homotopic to the zero map, which is holomorphic. An identical argument shows that any continuous map $X \to \mathbb{D}$ is homotopic to a holomorphic map.

Now suppose X is $\mathbb{C}$. Let H be the homotopy as above and $f : \mathbb{C} \to Y$ be continuous. Let $G : \mathbb{C} \times [0,1] \to Y$ be given by

$$G(z,t) = f \circ H(z,t)\,.$$

Then G is continuous as it is the composition of continuous maps and $G(z,0) = f(0)$ and $G(z,1) = f(z)$. Thus G is a homotopy from f to the constant map $f(0)$ and hence f is homotopic to a holomorphic map. Again an identical argument handles the case $X = \mathbb{D}$. □

Theorem 2.118. *Let Y be a Riemann surface such that $Y \neq \mathbb{P}$. Then every continuous map $\mathbb{P} \to Y$ is homotopic to a holomorphic map.*

Proof. Let $\pi : \tilde{Y} \to Y$ denote the universal covering map of Y. By the uniformisation theorem (Theorem 2.20), $\tilde{Y}$ is biholomorphic to $\mathbb{C}$ or $\mathbb{D}$. Let $f : \mathbb{P} \to Y$ be continuous. As $\mathbb{P}$ is simply connected, there is a holomorphic lifting $\tilde{f}$ of f by π such that

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "\pi"] \\ \mathbb{P} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes by Corollary 2.61. By Theorem 2.117 we know that $\tilde{f}$ is homotopic to a constant map (in fact, $\tilde{f}$ is homotopic to the zero map). Let $H : \mathbb{P} \times [0,1] \to \tilde{Y}$ be such a homotopy from $\tilde{f}$ to the zero map. Define $F : \mathbb{P} \times [0,1] \to Y$ by

$$F(z,t) = \pi \circ H(z,t)\,.$$

Now F is continuous, being the composition of continuous maps, and $F(z,0) = \pi(0)$ and $F(z,1) = \pi \circ \tilde{f}(z) = f(z)$. Thus F defines a homotopy from f to a constant map and hence a holomorphic map. □

Theorem 2.119. *Let X be an open Riemann surface. Then every continuous map $X \to \mathbb{C}^*$ is homotopic to a holomorphic map.*

Remark 2.120. The following proof is as presented by Lárusson in [16] . This is a detailed explanation of why the basic principle holds in this situation. These techniques were adapted and used in Chapter 3.

Proof. Let $f : X \to \mathbb{C}^*$ be continuous. For each $x \in X$, there is a simply connected open neighbourhood U of x (for example, a coordinate disc centred at x) such that $f|_U : U \to \mathbb{C}^*$ has a continuous logarithm, that is, there is a continuous function $\lambda : U \to \mathbb{C}$ such that $f|_U = \exp(2\pi i\lambda)$. Cover X by such open neighbourhoods U_α. We thus get a family of continuous functions $\lambda_\alpha : U_\alpha \to \mathbb{C}$ such that $f|_{U_\alpha} = \exp(2\pi i\lambda_\alpha)$. As $\exp(2\pi i(\lambda_\alpha - \lambda_\beta)) = 1$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we see that $n_{\alpha\beta} = \lambda_\alpha - \lambda_\beta : U_{\alpha\beta} \to \mathbb{Z}$ is a locally constant, integer-valued function. It is easy to see that $n_{\alpha\beta} + n_{\beta\gamma} = n_{\alpha\gamma}$, so $n_{\alpha\beta}$ satisfies what is known as the *cocycle condition*. Note that the cocycle condition immediately implies that $n_{\alpha\alpha} = 0$ and $n_{\alpha\beta} = n_{\beta\alpha}$.

We wish to construct a holomorphic splitting of $n_{\alpha\beta}$, that is, find holomorphic functions $\mu_\alpha : U_\alpha \to \mathbb{C}$ with $n_{\alpha\beta} = \mu_\alpha - \mu_\beta$. Let (ρ_α) be a smooth partition of unity subordinate to the cover (U_α). This means that:

1. $\operatorname{supp}(\rho_\alpha) \subset U_\alpha$.
2. Each point of X has a neighbourhood V such that $\operatorname{supp}(\rho_\alpha) \cap V = \varnothing$ for all but finitely many α.
3. $\sum_\alpha \rho_\alpha(x) = 1$ for every $x \in X$.

Using this partition of unity we can first define a smooth splitting of $n_{\alpha\beta}$. Note that $\rho_\gamma n_{\alpha\gamma}$ may be extended to a smooth function on U_α by assigning it the value 0 outside $\operatorname{supp}(\rho_\gamma)$. Define $\nu_\alpha = \sum_\gamma \rho_\gamma n_{\alpha\gamma}$, which is a well-defined smooth function on U_α. On $U_{\alpha\beta}$,

$$\begin{aligned}
\nu_a - \nu_\beta &= \sum_\gamma \rho_\gamma n_{\alpha\gamma} - \sum_\gamma \rho_\gamma n_{\beta\gamma} \\
&= \sum_\gamma \rho_\gamma (n_{\alpha\gamma} + n_{\gamma\beta}) \\
&= \sum_\gamma \rho_\gamma n_{\alpha\beta} \quad \text{(the cocycle condition)} \\
&= n_{\alpha\beta}\,.
\end{aligned}$$

Thus we have a smooth splitting of $n_{\alpha\beta}$.

Now, as $n_{\alpha\beta}$ is locally constant, we have $0 = \bar\partial n_{\alpha\beta} = \bar\partial\nu_\alpha - \bar\partial\nu_\beta$ on $U_{\alpha\beta}$. This gives us that $\bar\partial\nu_\alpha = \bar\partial\nu_\beta$ on $U_{\alpha\beta}$, so there is a well-defined $(0,1)$-form ω on X with $\omega|_{U_\alpha} = \bar\partial\nu_\alpha$.

As X is open, there is a smooth function $u : X \to \mathbb{C}$ with $\bar{\partial} u = \omega$ by Dolbeault's lemma (Lemma 2.12). Define $\mu_\alpha : U_\alpha \to \mathbb{C}$ by $\mu_\alpha = \nu_\alpha - u|_{U_\alpha}$. Then μ_α is holomorphic since $\bar{\partial}\mu_\alpha = \bar{\partial}\nu_\alpha - \bar{\partial}u|_{U_\alpha} = \omega|_{U_\alpha} - \omega|_{U_\alpha} = 0$. Further, $\mu_\alpha - \mu_\beta = \nu_\alpha - u|_{U_\alpha} - \nu_\beta + u|_{U_\beta} = n_{\alpha\beta}$ on $U_{\alpha\beta}$, so (μ_α) is a splitting of $(n_{\alpha\beta})$.

Now define the holomorphic function $g : X \to \mathbb{C}^*$ by $g|_{U_\alpha} = \exp(2\pi i \mu_\alpha)$. It is well defined because $\mu_\alpha - \mu_\beta = n_{\alpha\beta}$ is locally constant. Define a map $H : X \times [0,1] \to \mathbb{C}^*$ by

$$H(\cdot, t)|_{U_\alpha} = \exp(2\pi i((1-t)\lambda_\alpha + t\mu_\alpha)) .$$

Then H is a well-defined continuous map with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$, that is, H is a homotopy from f to a holomorphic map as desired. □

Theorem 2.121. *Let X be an open Riemann surface and let T be a torus. Then every continuous map $X \to T$ is homotopic to a holomorphic map.*

Proof. Recall there is a holomorphic map $f : \mathbb{C}^* \times \mathbb{C}^* \to T$ that is a homotopy equivalence by Proposition 2.63. Then f induces maps on the spaces of holomorphic and continuous maps such that

$$\begin{tikzcd} \mathscr{O}(X, \mathbb{C}^* \times \mathbb{C}^*) \arrow[r, "f_*"] \arrow[d, hook, "i"'] & \mathscr{O}(X, T) \arrow[d, hook, "i"] \\ \mathscr{C}(X, \mathbb{C}^* \times \mathbb{C}^*) \arrow[r, "f_*"] & \mathscr{C}(X, T) \end{tikzcd}$$

commutes. By Proposition 2.46, $f_* : \mathscr{C}(X, \mathbb{C}^* \times \mathbb{C}^*) \to \mathscr{C}(X, T)$ is a homotopy equivalence, and by Theorem 2.119, $i_* : \pi_0\mathscr{O}(X, \mathbb{C}^* \times \mathbb{C}^*) \to \pi_0\mathscr{C}(X, \mathbb{C}^* \times \mathbb{C}^*)$ is a surjection. Thus the commuting diagram

$$\begin{tikzcd} \pi_0\mathscr{O}(X, \mathbb{C}^* \times \mathbb{C}^*) \arrow[r, "f_{**}"] \arrow[d, hook, "i_*"'] & \pi_0\mathscr{O}(X, T) \arrow[d, hook, "i_*"] \\ \pi_0\mathscr{C}(X, \mathbb{C}^* \times \mathbb{C}^*) \arrow[r, "f_{**}"] & \pi_0\mathscr{C}(X, T) \end{tikzcd}$$

gives that $f_{**} \circ i_* = i_* \circ f_{**}$ is a surjection. Thus $i_* : \pi_0\mathscr{O}(X, T) \to \pi_0\mathscr{C}(X, T)$ is a surjection and every continuous map $X \to T$ is homotopic to a holomorphic map. □

Theorem 2.122. *Let X be an open Riemann surface. Then every continuous map $X \to \mathbb{P}$ is homotopic to a holomorphic map.*

Proof. By Corollary 2.114 we know X has the homotopy type of a bouquet of circles, $\bigvee_\alpha S^1_\alpha$ say. Then there is a bijection $[X, \mathbb{P}] \to [\bigvee_\alpha S^1_\alpha, S^2]$, where $[X, Y]$ denotes the set of homotopy classes of continuous maps $X \to Y$. Since S^2 is simply connected it follows that $[\bigvee_\alpha S^1_\alpha, S^2]$ is a singleton. Hence every continuous map $X \to \mathbb{P}$ is null-homotopic and so homotopic to a holomorphic map. □

Theorem 2.123. *Let $\bar{X}$ be a compact Riemann surface and (D_i) a non-empty finite collection of mutually disjoint closed discs in $\bar{X}$. Let $X = \bar{X} \setminus \bigcup_i D_i$. Then every continuous map $X \to \mathbb{D}^*$ is homotopic to a holomorphic map.*

Proof. Let $f : X \to \mathbb{D}^*$ be a continuous map. Let (D_i') be a collection of non-empty closed discs in $\bar{X}$ with $D_i' \subsetneqq \mathring{D}_i$ for each i. Then $X' = \bar{X} \setminus \bigcup_i D_i'$ is such that $X \Subset X'$ and there exists a continuous map $r : X' \to X$ such that the composition

$$X \hookrightarrow X' \xrightarrow{r} X$$

is homotopic to the identity on X. To see this, consider non-empty *open* discs $\tilde{D}_i$ in $\bar{X}$ with $D_i' \subset D_i \subset \tilde{D}_i$ and $\tilde{X} = \bar{X} \setminus \bigcup \tilde{D}_i$. It is easy to see that $\tilde{X}$ is a strong deformation retract of X' and so there is a continuous map $r : X' \to \tilde{X}$ such that $r \circ \tilde{\iota} = \mathrm{id}_{\tilde{X}}$ and $\tilde{\iota} \circ r$ is homotopic to $\mathrm{id}_{X'}$, where $\tilde{\iota}$ denotes the inclusion $\tilde{X} \hookrightarrow X'$. Let $\iota : X \hookrightarrow X'$ denote the inclusion map and extend r as a continuous map to X by post-composing by the inclusion $\tilde{X} \hookrightarrow X$. Then $r \circ \iota$ is homotopic to $\mathrm{id}_{X'} \circ \iota = \mathrm{id}_X$.

Now, $f \circ r : X' \to \mathbb{D}^*$ is continuous and we may consider it as a map into $\mathbb{C}^*$. Then by Theorem 2.119, $f \circ r$ is homotopic to a holomorphic map. Let $H : X' \times [0,1] \to \mathbb{C}^*$ be such a homotopy. Then $H(X \times [0,1]) \Subset \mathbb{C}^*$ since $X \Subset X'$. Thus there is $k \in \mathbb{C}^*$ with $kH(X \times [0,1]) \Subset \mathbb{D}^*$ where $kH(\cdot, 0) = kf \circ r$ and $kH(\cdot, 1) : X' \to \mathbb{C}^*$ is holomorphic. It follows then that f is homotopic to $kf \circ r|_X$ which in turn is homotopic to $kH(\cdot, 1)|_X$, proving the result. □

We have now given complete, detailed proofs that BOP holds in each of the cases identified in Theorem 2.116. Although we have not shown that these are the only pairs for which BOP holds, Winkelmann addresses this with Proposition 1 from his paper [30]. Winkelmann provides counterexamples for each pair identified in Proposition 1 (we present this list in Theorem 3.23), but we will not go into a full discussion here. We now have all the required background material to proceed to Chapter 3.

Chapter 3

The parametric Oka principle

In his 1993 paper "The Oka-principle for mappings between Riemann surfaces" [30], Winkelmann presented a full classification of the pairs of Riemann surfaces which satisfy the basic Oka principle (BOP). The parametric Oka principle is a stronger notion than BOP that provides a deeper insight into the topological and analytical structure of our Riemann surfaces, so we wish to classify the spaces that satisfy this property. As with all properties and definitions in Oka theory, the definition of the parametric Oka principle is easily stated in the context of arbitrary complex manifolds. However, due to our low-dimensional approach, we provide a definition restricted to Riemann surfaces.

Definition 3.1. Let X and Y be Riemann surfaces. Then the pair (X, Y) is said to satisfy the *parametric Oka principle* (POP) if the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ of the space of holomorphic maps into the space of continuous maps is a weak homotopy equivalence in the compact-open topology.

This definition may seem abstract at first glance, but has a very down-to-earth interpretation. We see that a pair of Riemann surfaces (X, Y) satisfies POP if, given a continuous family of continuous maps $f : X \times S^n \to Y$ such that $f(\cdot, s_0)$ is holomorphic for a fixed $s_0 \in S^n$, we can continuously deform it into a continuous family of holomorphic maps $g : X \times S^n \to Y$ (that is, g is continuous and $g(\cdot, s)$ is holomorphic for all $s \in S^n$) preserving $g(\cdot, s_0) = f(\cdot, s_0)$. Moreover, POP tells us this deformation is somewhat unique in the sense that if two continuous families of holomorphic maps are homotopic through continuous families of continuous maps, they are homotopic through continuous families of holomorphic maps.

One easily sees that if a pair of Riemann surfaces (X, Y) satisfies POP then it automatically satisfies BOP as well. If $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ is a weak homotopy equivalence, then, in particular, the induced map $\pi_0\mathscr{O}(X, Y) \to \pi_0\mathscr{C}(X, Y)$ is a surjection. This says precisely that every path component of the space of continuous maps contains a holomorphic map, or, otherwise said, that every continuous map $X \to Y$ is homotopic to a

holomorphic map. From this, we see if a pair (X, Y) fails to satisfy BOP then it cannot satisfy POP since $\pi_0(X) \to \pi_0(Y)$ will not be a surjection. This fact reduces our study of POP from all pairs of Riemann surfaces to those identified to satisfy BOP by Winkelmann.

These pairs of spaces can be divided into three types: the topological pairs, the Gromov pairs and the non-Gromov pairs. The *topological pairs* consist of the source being the disc or the plane, the target being the disc or the plane, or the source being the Riemann sphere and the target being anything but the Riemann sphere. We call these the topological pairs since the basic principle holds for purely topological reasons.

The *Gromov pairs* consist of an open source mapping into either the Riemann sphere, the punctured plane or a complex torus. These are so called because they fit into Gromov's framework as presented in his 1989 seminal paper [9]. That is to say that an open Riemann surface is Stein and the Riemann sphere, punctured plane and complex tori are elliptic as defined by Gromov. We acknowledge that the complex plane is also elliptic, but the fact that BOP holds is purely topological, so is fits more naturally with the topological pairs.

Finally, the *non-Gromov pairs* are oddities identified by Winkelmann. Suppose $\bar{X}$ is a compact Riemann surface and (D_i) is a non-empty finite collection of mutually disjoint closed discs in $\bar{X}$. Then the pair $(X, \mathbb{D}^*)$ satisfies BOP where $X = \bar{X} \setminus \bigcup D_i$. These are called the non-Gromov pairs because they do not fit into Gromov's framework. We have a very specific type of open Riemann surface X mapping into a hyperbolic manifold $\mathbb{D}^*$, which truly is an oddity in Oka theory.

We wish to construct low-dimensional, accessible proofs of POP in each of these situations. The topological pairs and the Gromov pairs are known to satisfy POP: the topological pairs due to the basic topological facts that cause BOP to hold, and the Gromov pairs due to Gromov's work [9]. Accessible proofs for these are presented for all but the Riemann sphere in Sections 3.1 and 3.2. When the target is the Riemann sphere, the problem becomes increasingly more difficult and only partial results are obtained. The non-Gromov pairs provide an interesting open problem as to whether POP will hold or not. This is solved in Section 3.3.

Finally, Section 3.4 discusses ideas about the higher parametric Oka principle. We look through the pairs of Riemann surfaces for which BOP fails and explore the other parts of the parametric principle. In other words, we take the Riemann surfaces X and Y for which $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ does not induce a surjection of path components and explore whether we have an injection of path components and isomorphisms of all homotopy groups when we choose a holomorphic map as our base point. This work was fuelled by the proof of Theorem 3.19 in Section 3.3, and is one of the main original contribution of this thesis.

3.1 The topological pairs

We begin by recalling Lemma 2.67, worded more precisely for our situation, which provides a useful tool in proving the parametric Oka principle for spaces for which BOP holds.

Lemma 3.2. *Let X and Y be Riemann surfaces, B^k denote the closed unit ball in $\mathbb{R}^k$, $k \geq 1$, and let $b_0 \in \partial B^k$. Then the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces a surjection of π_k and an injection of π_{k-1} if and only if for every continuous map $\alpha_0 : B^k \to \mathscr{C}(X,Y)$ with $\alpha_0(\partial B^k) \subset \mathscr{O}(X,Y)$, there is a deformation $\alpha : B^k \times [0,1] \to \mathscr{C}(X,Y)$ of α_0 such that, writing $\alpha_t = \alpha(\cdot, t)$,*

(1) $\alpha_t(b_0) = \alpha_0(b_0)$ *for all* $t \in [0,1]$,

(2) $\alpha_t(\partial B^k) \subset \mathscr{O}(X,Y)$ *for all* $t \in [0,1]$,

(3) $\alpha_1(B^k) \subset \mathscr{O}(X,Y)$.

This is just Lemma 2.67 taking $X = \mathscr{O}(X,Y)$ and $Y = \mathscr{C}(X,Y)$. Thus if we show this holds for all $k \geq 1$, and we know that BOP holds for the pair (X,Y), then we see that the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces a weak homotopy equivalence. We now turn our attention to the topological cases.

Theorem 3.3. *Let X and $Y \neq \mathbb{P}$ be Riemann surfaces. Denote by $\mathscr{C}_0$ (resp. $\mathscr{O}_0$) the space of null-homotopic continuous (resp. holomorphic) maps $X \to Y$. Then the map $\pi_k\mathscr{O}_0 \to \pi_k\mathscr{C}_0$ induced by inclusion is an isomorphism for $k \geq 1$ and an injection for $k = 0$.*

Proof. Let B^k denote the closed unit ball in $\mathbb{R}^k$, $k \geq 1$, and let $p : \tilde{Y} \to Y$ be the universal covering space of Y. Suppose we have a continuous map $\alpha_0 : B^k \to \mathscr{C}_0$ such that $\alpha_0(\partial B^k) \subset \mathscr{O}_0$. Fix a base point $b_0 \in \partial B^k$. Consider α_0 as a continuous map $\alpha_0 : B^k \times X \to Y$ by Proposition 2.43 (b). Since B^k is contractible, $\{b_0\} \times X \hookrightarrow B^k \times X$ is a homotopy equivalence and we get a commuting diagram of fundamental groups

$$\begin{tikzcd}
\pi_1(\{b_0\} \times X) \arrow[d] \arrow[dr] & \\
\pi_1(B^k \times X) \arrow[r] & \pi_1(Y)
\end{tikzcd}$$

where $\pi_1(\{b_0\} \times X) \to \pi_1(Y)$ is induced by the restriction $\alpha_0|_{\{b_0\}\times X}$, and $\pi_1(\{b_0\} \times X) \to \pi_1(B^k \times X)$ is an isomorphism. Since $\alpha_0|_{\{b_0\}\times X}$ is null-homotopic, $\pi_1(\{b_0\} \times X) \to \pi_1(Y)$ is the trivial map and so $\pi_1(B^k \times X) \to \pi_1(Y)$ is trivial also. Thus $(\alpha_0)_*\pi_1(B^k \times X) = 0$ and we can lift α_0 by the universal covering map of Y by Proposition 2.60, that is, we can find a continuous map $\tilde{\alpha}_0 : B^k \times X \to \tilde{Y}$ with $\tilde{\alpha}_0(b, \cdot) : X \to \tilde{Y}$ holomorphic for each

$b \in \partial B^k$, such that $\alpha_0 = p \circ \tilde{\alpha}_0$. Realising $\tilde{Y}$ as $\mathbb{C}$ or $\mathbb{D}$ by the uniformisation theorem (Theorem 2.20), define $\tilde{\alpha} : B^k \times X \times [0,1] \to \tilde{Y}$ by

$$\tilde{\alpha}_t = (1-t)\tilde{\alpha}_0 + t\tilde{\alpha}_0(b_0, \cdot),$$

where $\tilde{\alpha}_t = \tilde{\alpha}(\cdot, t)$. Then $\tilde{\alpha}_t$ satisfies the following criteria:

- $\tilde{\alpha}_t(b, \cdot) \in \mathscr{O}(X, \tilde{Y})$ for every $b \in \partial B^k$,
- $\tilde{\alpha}_t(b_0, \cdot) = \tilde{\alpha}_0(b_0, \cdot)$ for all $t \in [0,1]$,
- $\tilde{\alpha}_1(b, \cdot) \in \mathscr{O}(X, \tilde{Y})$ for every $b \in B^k$.

Now, defining $\alpha_t = p \circ \tilde{\alpha}_t$, we get a deformation of α_0 to α_1 satisfying the criteria of Lemma 3.2, proving the result. □

Corollary 3.4. *Let X and Y be Riemann surfaces such that X or Y is contractible. Then $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ is a weak homotopy equivalence.*

Proof. If X or Y is contractible, then it is $\mathbb{D}$ or $\mathbb{C}$, so by Theorem 2.117 the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces a surjection of path components. If $f \in \mathscr{C}(X,Y)$, then f is null-homotopic as we see from the homotopy $X \times [0,1] \to Y$, $(z,t) \mapsto (1-t)f(z)$, when Y is contractible and the homotopy $X \times [0,1] \to Y$, $(z,t) \mapsto f((1-t)z)$, when X is contractible. So $\mathscr{C}_0 = \mathscr{C}(X,Y)$. If f is holomorphic, these homotopies are through holomorphic maps, so we have $\mathscr{O}_0 = \mathscr{O}(X,Y)$ also. The result then follows from Theorem 3.3. □

Corollary 3.5. *Let $Y \neq \mathbb{P}$ be a Riemann surface. Then $\mathscr{O}(\mathbb{P},Y) \hookrightarrow \mathscr{C}(\mathbb{P},Y)$ is a weak homotopy equivalence.*

Proof. By Theorem 2.118 we know $\mathscr{O}(\mathbb{P},Y) \hookrightarrow \mathscr{C}(\mathbb{P},Y)$ induces a surjection of path components. If Y is open, then every holomorphic map $\mathbb{P} \to Y$ is constant (Proposition 2.13), and if Y is compact, then every holomorphic map is constant by Corollary 2.16 since $Y \neq \mathbb{P}$. Thus $\mathscr{O}(\mathbb{P},Y) = Y$, and since every continuous map $\mathbb{P} \to Y$ is null-homotopic (as seen in the proof of Theorem 2.118), it follows that $\mathscr{O}_0 = \mathscr{O}(\mathbb{P},Y)$ and $\mathscr{C}_0 = \mathscr{C}(\mathbb{P},Y)$. Thus the result follows from Theorem 3.3. □

When working with CW-complexes, a weak homotopy equivalence can always be upgraded to a genuine homotopy equivalence by Whitehead's theorem (Theorem 2.65). However, in our situation these mapping spaces do not in general have a CW-structure: they are in some sense too large for this to occur and a weak homotopy equivalence may be the best we can hope for. In saying this, we can provide some deeper insight into these mapping spaces for the topological pairs, and in particular for a contractible source or target.

Theorem 3.6. *Let X and Y be Riemann surfaces such that Y is $\mathbb{C}$ or $\mathbb{D}$. Then the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ is a homotopy equivalence.*

Proof. We prove the case $Y = \mathbb{C}$ as the argument is the same for $Y = \mathbb{D}$. To show $\iota : \mathscr{O}(X,\mathbb{C}) \hookrightarrow \mathscr{C}(X,\mathbb{C})$ is a homotopy equivalence we define a homotopy inverse $\kappa : \mathscr{C}(X,\mathbb{C}) \to \mathscr{O}(X,\mathbb{C})$ such that $\kappa(f) = 0$, the zero map, for every $f \in \mathscr{C}(X,\mathbb{C})$.

Showing that the composition $\iota \circ \kappa : \mathscr{C}(X,\mathbb{C}) \to \mathscr{C}(X,\mathbb{C})$ is homotopic to the identity on $\mathscr{C}(X,\mathbb{C})$ is tantamount to showing that $\mathscr{C}(X,\mathbb{C})$ is contractible. Define $H : \mathscr{C}(X,\mathbb{C}) \times [0,1] \to \mathscr{C}(X,\mathbb{C})$ by $H(f,t) = tf$. Since H acts by scalar multiplication in the topological vector space $\mathscr{C}(X,\mathbb{C})$ (Corollary 2.71), it is continuous. Further $H(f,0) = 0$ and $H(f,1) = f$, so H is a homotopy from the identity to the zero map, showing that $\mathscr{C}(X,\mathbb{C})$ is contractible.

To show that the constant map $\kappa \circ \iota : \mathscr{O}(X,\mathbb{C}) \to \mathscr{O}(X,\mathbb{C})$ is homotopic to the identity is to show that $\mathscr{O}(X,\mathbb{C})$ is contractible, that is, there is a homotopy $G : \mathscr{O}(X,\mathbb{C}) \times [0,1] \to \mathscr{O}(X,\mathbb{C})$ from the identity to a constant map. However, if we take $G = H|_{\mathscr{O}(X,\mathbb{C})\times[0,1]}$ we are done as $H(f,t)$ is holomorphic for all $t \in [0,1]$ precisely when $f \in \mathscr{O}(X,\mathbb{C})$. $\square$

Theorem 3.7. *Let X and Y be Riemann surfaces such that X is $\mathbb{C}$ or $\mathbb{D}$. Then the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ is a homotopy equivalence.*

Proof. We take $X = \mathbb{C}$ as the case $X = \mathbb{D}$ is similar. To show $\mathscr{O}(\mathbb{C},Y) \hookrightarrow \mathscr{C}(\mathbb{C},Y)$ is a homotopy equivalence we shall first show that each space is in fact homotopy equivalent to Y via the evaluation maps at 0. Consider first the map $\mathrm{ev}_1 : \mathscr{C}(\mathbb{C},Y) \to Y$, $f \mapsto f(0)$. We claim this is a homotopy equivalence with homotopy inverse $\iota_1 : Y \to \mathscr{C}(\mathbb{C},Y)$, such that $\iota_1(y)$ is the constant map $z \mapsto y$. We see immediately that $\mathrm{ev}_1 \circ \iota_1 = \mathrm{id}_Y$. Now consider the map $H : \mathscr{C}(\mathbb{C},Y) \times [0,1] \to \mathscr{C}(\mathbb{C},Y)$ defined by $H(f,t)(x) = f(tx)$. It is continuous by Proposition 2.43 (b), and as $H(f,0) = f(0)$ and $H(f,1) = f$, it is a homotopy from $\iota_1 \circ \mathrm{ev}_1$ to $\mathrm{id}_{\mathscr{C}(\mathbb{C},Y)}$, showing $\mathscr{C}(\mathbb{C},Y)$ is homotopy equivalent to Y.

A homotopy equivalence between $\mathscr{O}(\mathbb{C},Y)$ and Y is obtained similarly. As constant maps are holomorphic, the maps $\mathrm{ev}_2 : \mathscr{O}(\mathbb{C},Y) \to Y$ and $\iota_2 : Y \to \mathscr{O}(\mathbb{C},Y)$ will again provide the equivalence. Immediately we have $\mathrm{ev}_2 \circ \iota_2 = \mathrm{id}_Y$ as before. Further if we take $G = H|_{\mathscr{O}(\mathbb{C},Y)\times[0,1]}$ we have a homotopy from $\iota_2 \circ \mathrm{ev}_2$ to the identity $\mathrm{id}_{\mathscr{O}(\mathbb{C},Y)}$. Indeed $G(\cdot,t) \in \mathscr{O}(\mathbb{C},Y)$ for each t as $z \mapsto tz$ is holomorphic and G is the composition of this map with a holomorphic map. Thus $\mathscr{O}(\mathbb{C},Y)$ is homotopy equivalent to Y.

We can now prove that $\iota : \mathscr{O}(\mathbb{C},Y) \hookrightarrow \mathscr{C}(\mathbb{C},Y)$ is a homotopy equivalence with homotopy inverse $\mathrm{ev} : \mathscr{C}(\mathbb{C},Y) \to \mathscr{O}(\mathbb{C},Y)$ that maps f to the constant map $f(0)$. We have the commutative diagram

$$\begin{tikzcd}
\mathscr{O}(\mathbb{C},Y) \arrow[rr, hook, "\iota"] \arrow[dr, shift left, "\mathrm{ev}_2"] & & \mathscr{C}(\mathbb{C},Y) \arrow[ll, bend right, "\mathrm{ev}"'] \arrow[dl, shift left, "\mathrm{ev}_1"] \\
& Y \arrow[ul, bend left, "\iota_2"] \arrow[ur, bend right, "\iota_1"'] &
\end{tikzcd}$$

Now we see that $\iota \circ \mathrm{ev} = \iota_2 \circ \mathrm{ev}_2$ which is homotopic to $\mathrm{id}_{\mathscr{O}(\mathbb{C},Y)}$ and that $\mathrm{ev} \circ \iota = \iota_1 \circ \mathrm{ev}_1$ which is homotopic to $\mathrm{id}_{\mathscr{C}(\mathbb{C},Y)}$. This shows that $\mathscr{O}(\mathbb{C}, Y) \hookrightarrow \mathscr{C}(\mathbb{C}, Y)$ is a homotopy equivalence. □

Now consider the final topological pairs $(\mathbb{P}, Y \neq \mathbb{P})$. My attempts to find an explicit homotopy inverse for the inclusion $\mathscr{O}(\mathbb{P}, Y) \hookrightarrow \mathscr{C}(\mathbb{P}, Y)$ have been fruitless, however we know this inclusion is a homotopy equivalence due to a theorem of Milnor.

Theorem 3.8. *Let X have the homotopy type of a compact CW-complex and Y have the homotopy type of a countable CW-complex. Then $\mathscr{C}(X, Y)$ has the homotopy type of a CW-complex.*

Proof. See [19, Corollary 2]. □

Theorem 3.9. *Let $Y \neq \mathbb{P}$ be a Riemann surface. Then the inclusion $\mathscr{O}(\mathbb{P}, Y) \hookrightarrow \mathscr{C}(\mathbb{P}, Y)$ is a homotopy equivalence.*

Proof. By Corollary 3.5, the inclusion $\mathscr{O}(\mathbb{P}, Y) \hookrightarrow \mathscr{C}(\mathbb{P}, Y)$ is a weak homotopy equivalence. As discussed in the proof of Corollary 3.5, $\mathscr{O}(\mathbb{P}, Y) = Y$ and thus is a CW-complex. By Theorem 3.8, $\mathscr{C}(\mathbb{P}, Y)$ has the homotopy type of a CW-complex. Since Whitehead's theorem (Theorem 2.65) can easily be seen to apply to spaces with the homotopy type of a CW-complex, the result then follows. □

3.2 The Gromov pairs

Next we look at the Gromov pairs. These pairs consist of an arbitrary Stein source mapping into an elliptic target. In the case of Riemann surfaces, the Stein manifolds are the open Riemann surfaces (Theorem 2.6), and the elliptic targets in the sense of Gromov are the plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, the complex tori, and the Riemann sphere $\mathbb{P}$. Since the case of target $\mathbb{C}$ has been covered with the topological pairs, we focus our attention on target $\mathbb{C}^*$, a torus, or $\mathbb{P}$. Due to Gromov [9], POP will hold for these pairs. We now construct more accessible proofs of this fact when the target is the punctured plane or a torus, and prove partial results when the target is the Riemann sphere.

Lemma 3.10. *Let X be an open Riemann surface, Y be either $\mathbb{C}^*$ or a torus, and $f, g : X \to Y$ be holomorphic maps. Then if f and g are homotopic through continuous maps, they are homotopic through holomorphic maps.*

Proof. Let $\pi : \mathbb{C} \to Y$ be the universal covering map. Let $H : X \times [0, 1] \to Y$ be a homotopy of continuous maps from f to g, that is, $H(\cdot, 0) = f$, $H(\cdot, 1) = g$ and $H(\cdot, t) \in \mathscr{C}(X, Y)$ for all $t \in [0, 1]$. Let (U_α) be an open cover of X by simply connected sets. We

may lift the homotopy H when restricted to $U_\alpha \times [0,1]$ to obtain maps $\mu_a : U_\alpha \times [0,1] \to \mathbb{C}$ such that

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\pi"] \\ U_\alpha \times [0,1] \arrow[ur, "\mu_\alpha"] \arrow[r, "H|_{U_\alpha \times [0,1]}"'] & Y \end{tikzcd}$$

commutes for each α. This gives us holomorphic lifts $\mu_\alpha(\cdot, 0)$ and $\mu_\alpha(\cdot, 1)$ on U_α of f and g respectively. Since $H|_{U_\alpha \times [0,1]} = \pi \circ \mu_\alpha$, we have that $\mu_\alpha - \mu_\beta = n_{\alpha\beta}$ is an $\mathrm{Aut}(\pi)$-valued map on $U_\alpha \cap U_\beta \times [0,1]$, and is thus locally constant since $\mathrm{Aut}\,\pi$ is discrete. In particular $n_{\alpha\beta}$ is independent of t since $[0,1]$ is connected, so $\mu_\alpha(\cdot, 0) - \mu_\beta(\cdot, 0) = n_{\alpha\beta} = \mu_\alpha(\cdot, 1) - \mu_\beta(\cdot, 1)$. It follows that we have a well-defined homotopy $G : X \times [0,1] \to Y$ given by

$$G(z,t) = \pi((1-t)\mu_\alpha(z,0) + t\mu_\alpha(z,1))$$

on $U_\alpha \times [0,1]$. Since $\mu_\alpha(\cdot, 0)$ and $\mu_\alpha(\cdot, 1)$ are holomorphic for each α, this gives a homotopy from f to g through holomorphic maps. □

Theorem 3.11. *Let X be an open Riemann surface and Y be $\mathbb{C}^*$ or a torus. Then $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ is a weak homotopy equivalence.*

Proof. By Theorems 2.119 and 2.121, we know that $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces a surjection of path components. By Lemma 3.10, the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces an injection of path components also. Since Y is a topological group (in fact, it is a complex Lie group), $\mathscr{C}(X,Y)$ and $\mathscr{O}(X,Y)$ are topological groups by Lemma 2.69 and Corollary 2.70. Let $\mathscr{O}_0(X,Y)$ and $\mathscr{C}_0(X,Y)$ denote the path components of the identity group element, which consist of the null-homotopic maps. By Theorem 3.3, it follows that $\mathscr{O}_0(X,Y) \hookrightarrow \mathscr{C}_0(X,Y)$ is a weak homotopy equivalence. Since $\mathscr{O}(X,Y)$ and $\mathscr{C}(X,Y)$ are topological groups, each path component is homeomorphic to the path component of the identity (Proposition 2.68) and the result then follows. □

This proof relies on the topological group structure of the spaces $\mathscr{O}(X,Y)$ and $\mathscr{C}(X,Y)$ inherited from the Lie group structure of the target, coupled with the results about null-homotopic maps obtained in Theorem 3.3. We also provide an alternative proof of this result in Section 3.3 that relies more on the topology of the target rather than the induced group structure.

Now consider the final Gromov pair: the case of an open Riemann surface X mapping into the Riemann sphere $\mathbb{P}$. Theorem 2.122 tells us that the inclusion $\mathscr{O}(X,\mathbb{P}) \hookrightarrow \mathscr{C}(X,\mathbb{P})$ induces a surjection of path components since $\mathscr{C}(X,\mathbb{P})$ is path connected. We will provide the partial results that $\mathscr{O}(X,\mathbb{P}) \hookrightarrow \mathscr{C}(X,\mathbb{P})$ induces an injection of path components for any open Riemann surface X, and also that $\mathscr{O}(\mathbb{C}^*,\mathbb{P}) \hookrightarrow \mathscr{C}(\mathbb{C}^*,\mathbb{P})$ induces a surjection on fundamental groups.

First, we show that the inclusion induces an injection of path components by showing that $\mathscr{O}(X,\mathbb{P})$ is path connected for any open source X. To do this we use the following lemma.

Lemma 3.12. *Let X be an open Riemann surface. Then $\mathscr{M}(X)$ is path connected in the compact open topology (considering $\mathscr{M}(X)$ as a subspace of $\mathscr{O}(X,\mathbb{P})$).*

Proof. Let $h \in \mathscr{M}(X)$. Since X is an open Riemann surface we may write $h = \frac{f}{g}$ for $f, g \in \mathscr{O}(X)$ with no common zeros by Corollary 2.10. If either f or g is constant, then h maps into either $\mathbb{P}\backslash\{0\}$ or $\mathbb{C}$, which are contractible, so h is null-homotopic. Thus we assume that both f and g are non-constant. Let $Z_f \subset X$ denote the zero set of f. Then Z_f is discrete and countable. This implies $g(Z_f) \subset \mathbb{C}$ is countable and $0 \notin g(Z_f)$. For each $z \in g(Z_f)$, let ℓ_z denote the line in $\mathbb{C}$ through z and the origin. Let $\theta_z \in [0,\pi)$ denote the anti-clockwise angle from the positive real axis to ℓ_z. Then $\{\theta_z : z \in g(Z_f)\}$ is countable, so choose $\theta \in [0,\pi)\backslash\{\theta_z : z \in g(Z_f)\}$. Let $H_1 : X \times [0,1] \to \mathbb{C}$ be given by

$$H_1(\cdot,t) = (1 - t + te^{-i\theta})g\,.$$

Then H_1 is a homotopy from g to $e^{-i\theta}g$ through holomorphic maps such that $H_1(\cdot,t)$ has no common zeros with f for all $t \in [0,1]$. To see this, suppose $z_t \in X$ is such that $H_1(z_t,t) = 0$ for some $t \in [0,1]$. Then z_t is a zero of g since $1 - t + te^{-i\theta} \neq 0$ for all $t \in [0,1]$, thus $z_t \notin Z_f$. Note that $e^{-i\theta}g(Z_f) \cap \mathbb{R} = \varnothing$. Define $H_2 : X \times [0,1] \to \mathbb{C}$ by

$$H_2(\cdot,t) = (1-t)e^{-i\theta}g + t\,.$$

Again, H_2 gives us a homotopy from $e^{-i\theta}g$ to 1 through holomorphic maps with no common zeros with f. This is clear for $t = 0,1$, and for $t \in (0,1)$ we see $H(z,t) = 0$ if $e^{-i\theta}g(z) = \frac{t}{t-1} \in \mathbb{R}$. Since $e^{-i\theta}g(Z_f) \cap \mathbb{R} = \varnothing$ we get $z \notin Z_f$. Thus $H : X \times [0,1] \to \mathbb{C}$ given by

$$H(\cdot,t) = \begin{cases} H_1(\cdot,2t) & \text{for } t \in [0,\frac{1}{2}]\,, \\ H_2(\cdot,2t-1) & \text{for } t \in [\frac{1}{2},1]\,, \end{cases}$$

is a homotopy through holomorphic maps from g to 1 such that $H(\cdot,t)$ has no common zeros with f for all t. Now $G : X \times [0,1] \to \mathbb{C}$ given by

$$G = \frac{f}{H}$$

is continuous since $H(\cdot,t)$ and f have no common zeros for all t. Then G is a homotopy from h to f through meromorphic maps. Since $f \in \mathscr{O}(X)$, f is null-homotopic, and so h is null-homotopic also. Thus $\mathscr{M}(X)$ is path connected. □

Corollary 3.13. *Let X be an open Riemann surface. Then the inclusion $\mathscr{O}(X,\mathbb{P}) \hookrightarrow \mathscr{C}(X,\mathbb{P})$ induces an injection of path components.*

Proof. Since $\mathscr{C}(X,\mathbb{P})$ is path connected, this is tantamount to showing $\mathscr{O}(X,\mathbb{P})$ is path connected. We know $\mathscr{O}(X,\mathbb{P}) = \mathscr{M}(X) \cup \{\infty\}$, where ∞ denotes the constant map with value ∞. By Lemma 3.12, $\mathscr{M}(X) = \mathscr{O}(X,\mathbb{P})\backslash\{\infty\}$ is path connected. Since the continuous map $X \times [0,1] \to \mathbb{P}$ given by $(x,t) \mapsto \frac{1}{1-t}$ defines a path from the constant map 1 to the constant map ∞, this shows that $\mathscr{O}(X,\mathbb{P})$ is path connected also. □

The next step in showing the inclusion is a weak homotopy equivalence is to look at the fundamental groups. For this, we provide the partial result that the inclusion $\mathscr{O}(\mathbb{C}^*,\mathbb{P}) \hookrightarrow \mathscr{C}(\mathbb{C}^*,\mathbb{P})$ induces a surjection on fundamental groups with the use of degree theory and fibrations.

Lemma 3.14. *For $x_0 \in S^1 \subset \mathbb{C}$, let $\mathrm{ev}_{x_0} : \mathscr{C}(\mathbb{C}^*,\mathbb{P}) \to \mathbb{P}$ be the evaluation map at x_0. Fix $p \in \mathbb{P}$ and set $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P}) = \mathrm{ev}_{x_0}^{-1}(p)$. Then $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ is path connected.*

Proof. Recall that $\mathrm{ev}_{x_0} : \mathscr{C}(\mathbb{C}^*,\mathbb{P}) \to \mathbb{P}$ is a fibration with fibre $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ by Proposition 2.84. The long exact sequence of homotopy groups associated to this fibration (Corollary 2.88) is

$$\cdots \longrightarrow \pi_n\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P}) \longrightarrow \pi_n\mathscr{C}(\mathbb{C}^*,\mathbb{P}) \longrightarrow \pi_n(\mathbb{P}) \longrightarrow \cdots$$

Since $\pi_1(\mathbb{P}) = 0$, at $n = 0$ we see

$$0 \longrightarrow \pi_0\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P}) \longrightarrow \pi_0\mathscr{C}(\mathbb{C}^*,\mathbb{P}) \longrightarrow \pi_0(\mathbb{P})$$

Now, $\pi_0\mathscr{C}(\mathbb{C}^*,\mathbb{P})$ is a singleton since $\mathscr{C}(\mathbb{C}^*,\mathbb{P})$ is path connected as a consequence of Theorem 2.122, thus $\pi_0\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ is a singleton and $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ is path connected. □

Lemma 3.15. *For $x_0 \in S^1 \subset \mathbb{C}$, let $\mathrm{ev}_{x_0} : \mathscr{C}(\mathbb{C}^*,\mathbb{P}) \to \mathbb{P}$ be the evaluation map at x_0. Fix $p \in \mathbb{P}$ and set $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P}) = \mathrm{ev}_{x_0}^{-1}(p)$. Then $\pi_1\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P}) = \mathbb{Z}$ and the homotopy classes are classified by degree.*

Proof. We ignore base points since $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ is path connected by Lemma 3.14. The spaces $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ and $\mathscr{C}_{x_0}(S^1,S^2)$ are homotopy equivalent through pre-composition by the inclusion $S^1 \hookrightarrow \mathbb{C}^*$ and the usual identification $\mathbb{P} = S^2$. Thus $\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$ is homotopy equivalent to the loop space[1] of S^2, which we denote by ΩS^2. A standard identification in algebraic topology is that, for a topological space X, $\pi_n(\Omega X) = \pi_{n+1}(X)$ (see [12, p. 395] for instance). Thus it follows that $\pi_1\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P}) = \pi_1(\Omega S^2) = \pi_2(S^2) = \mathbb{Z}$ by the Hopf degree theorem (Theorem 2.40). Further, the Hopf degree theorem tells us that the homotopy classes in $\pi_2(S^2)$ (and hence in $\pi_1\mathscr{C}_{x_0}(\mathbb{C}^*,\mathbb{P})$) are classified by degree. □

[1]The loop space of a path-connected topological space X is the space of based loops $S^1 \to X$ with the compact-open topology.

Proposition 3.16. *Set $x_0 = -i$. Let $\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P}) = \{f \in \mathscr{C}(\mathbb{C}^*, \mathbb{P}) : f(x_0) = 0\}$ and let $\mathscr{O}_{x_0}(\mathbb{C}^*, \mathbb{P}) \subset \mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P})$ be the subspace of holomorphic maps. Then the inclusion $\mathscr{O}_{x_0}(\mathbb{C}^*, \mathbb{P}) \hookrightarrow \mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P})$ induces a surjection on fundamental groups.*

Proof. By Lemma 3.15, it suffices to find a holomorphic representative for each degree in $\pi_1\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P})$. This means for each degree $d \in \mathbb{Z}$ we wish to find a continuous map $f : S^1 \to \mathscr{O}_{x_0}(\mathbb{C}^*, \mathbb{P})$, which we may consider as a map $f : \mathbb{C}^* \times S^1 \to \mathbb{P}$ that is holomorphic in the first variable, whose restriction $f|_{S^1 \times S^1} \to \mathbb{P}$ has degree d. We do this by first defining a map of degree 1.

Define $f : \mathbb{C}^* \times [0, 1] \to \mathbb{P}$ by

$$f(z,t) = \begin{cases} \dfrac{2t(\alpha z + \bar{\alpha})}{2t\bar{\alpha}z + \alpha} & \text{for } t \in [0, \frac{1}{2}]\,, \\[2ex] \dfrac{2(1-t)(\alpha z + \bar{\alpha})}{\bar{\alpha}z + 2(1-t)\alpha} & \text{for } t \in [\frac{1}{2}, 1]\,, \end{cases}$$

where $\alpha = 1 - i$. This is a continuous map such that $f(\cdot, 0) = f(\cdot, 1) = 0$ and $f(x_0, t) = 0$ for all $t \in [0, 1]$, and so represents a class in $\pi_1\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P})$.

For each $t \in (0, 1)$, $f(\cdot, t)$ is a Möbius transformation and so maps S^1 to a circle in $\mathbb{P}$, with $f(\cdot, \frac{1}{2})$ being the Möbius transformation that takes S^1 to the great circle $\mathbb{R} \cup \{\infty\}$. We wish to show that $f|_{S^1 \times [0,1]} : S^1 \times [0, 1] \to \mathbb{P}$ has degree 1, but first we show that it is surjective (otherwise $\deg f = 0$ by Corollary 2.37). To do this, we will show that the image of S^1 by $f(\cdot, t)$ is different for each t, which is best argued geometrically. This shows $f|_{S^1 \times [0,1]}$ is surjective as any $z \in \mathbb{C}$ will sit on a circle tangent to the real axis and perpendicular to the imaginary axis (we only need to consider $z \in \mathbb{C}$ since it is easily checked that $f(i, \frac{1}{2}) = \infty$).

By conformality, the images of S^1 and the imaginary axis $i\mathbb{R}$ under $f(\cdot, t)$ will be perpendicular for all $t \in (0, 1)$. For $t \in (0, \frac{1}{2}]$, $f(i\mathbb{R}\backslash\{0\}, t) = i\mathbb{R}\backslash\{2ti\}$, and for $t \in (\frac{1}{2}, 1)$, $f(i\mathbb{R}\backslash\{0\}, t) = i\mathbb{R}\backslash\{i\}$. Thus the image of S^1 under $f(\cdot, t)$ will be perpendicular to $i\mathbb{R}$. We have $f(x_0, t) = 0$ for all $t \in [0, 1]$ and

$$f(i,t) = \begin{cases} \dfrac{4t}{1-2t} i & \text{for } t \in [0, \frac{1}{2}]\,, \\[2ex] \dfrac{4(1-t)}{1-2t} i & \text{for } t \in [\frac{1}{2}, 1]\,, \end{cases}$$

so $f(i, t) \neq f(i, t')$ if $t \neq t'$ for all $t, t' \in (0, 1)$. Thus the image of S^1 under $f(\cdot, t)$ is distinct for every $t \in (0, 1)$. Geometrically this tells us that the image of S^1 maps out the upper-half plane for $t \in [0, \frac{1}{2})$, peaks at the great circle described by the extended real axis at $t = \frac{1}{2}$, and then drags back down to 0 through the lower-half plane when $t \in (\frac{1}{2}, 1]$.

Under this construction we see that $f|_{(S^1\times[0,1])\backslash f^{-1}(0)} : (S^1 \times [0,1])\backslash f^{-1}(0) \to \mathbb{P}\backslash\{0\}$ is in fact a homeomorphism, so $f|_{S^1\times[0,1]}$ is (in a sense) the closest map we can get to a homeomorphism from the torus to the 2-sphere.

Now, we may re-parametrise $[0,1]$ so that f is smooth near $t = \frac{1}{2}$, which gives a smooth map from the torus to the 2-sphere. Since $f|_{(S^1\times[0,1])\backslash f^{-1}(0)} : (S^1 \times [0,1])\backslash f^{-1}(0) \to \mathbb{P}\backslash\{0\}$ is bijective, the preimage of a regular value is a singleton, and so $\deg f = 1$ since f is clearly orientation preserving. Note that $\tilde{f} : \mathbb{C}^* \times [0,1] \to \mathbb{P}$ defined by $\tilde{f}(\cdot, t) = f(\cdot, 1-t)$ gives a map of degree -1.

Picking a holomorphic map $p : \mathbb{P} \to \mathbb{P}$ of degree $d > 0$ (for instance, a polynomial of degree d), we get $p \circ f : \mathbb{C}^* \times [0,1] \to \mathbb{P}$ is a continuous family of based holomorphic maps with degree d and $p \circ \tilde{f} : \mathbb{C}^* \times [0,1] \to \mathbb{P}$ is a continuous family of based holomorphic maps with degree $-d$. Since families of constant maps trivially have degree 0, this proves the result. □

Corollary 3.17. *The inclusion $\mathscr{O}(\mathbb{C}^*, \mathbb{P}) \hookrightarrow \mathscr{C}(\mathbb{C}^*, \mathbb{P})$ induces a surjection on fundamental groups.*

Proof. In the following we ignore base points since all the spaces are path connected. Set $x_0 = -i$. For the fibrations $\mathrm{ev}_{x_0} : \mathscr{C}(\mathbb{C}^*, \mathbb{P}) \to \mathbb{P}$ and $\mathrm{ev}_{x_0} : \mathscr{O}(\mathbb{C}^*, \mathbb{P}) \to \mathbb{P}$ (Proposition 2.84) defined by evaluation at $x_0 \in \mathbb{C}^*$, consider the long exact sequence of homotopy groups

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n\mathscr{O}_{x_0}(\mathbb{C}^*, \mathbb{P}) \arrow[r] \arrow[d] & \pi_n\mathscr{O}(\mathbb{C}^*, \mathbb{P}) \arrow[r] \arrow[d] & \pi_n(\mathbb{P}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_n\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P}) \arrow[r] & \pi_n\mathscr{C}(\mathbb{C}^*, \mathbb{P}) \arrow[r] & \pi_n(\mathbb{P}) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical arrows are those induced by inclusion and $\mathscr{C}_{x_0}(\mathscr{C}^*, \mathbb{P}) = \mathrm{ev}_{x_0}^{-1}(0)$ and $\mathscr{O}_{x_0}(\mathscr{C}^*, \mathbb{P}) = \mathrm{ev}_{x_0}^{-1}(0)$. For $n = 1$, since $\pi_1(\mathbb{P}) = 0$, the commuting diagram reduces to

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1\mathscr{O}_{x_0}(\mathbb{C}^*, \mathbb{P}) \arrow[r] \arrow[d] & \pi_1\mathscr{O}(\mathbb{C}^*, \mathbb{P}) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \pi_1\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P}) \arrow[r] & \pi_1\mathscr{C}(\mathbb{C}^*, \mathbb{P}) \arrow[r] & 0
\end{tikzcd}$$

This shows that $\pi_1\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P}) \to \pi_1\mathscr{C}(\mathbb{C}^*, \mathbb{P})$ is surjective. By Proposition 3.16, we have $\pi_1\mathscr{O}_{x_0}(\mathbb{C}^*, \mathbb{P}) \to \pi_1\mathscr{C}_{x_0}(\mathbb{C}^*, \mathbb{P})$ is a surjection, and so $\pi_1\mathscr{O}(\mathbb{C}^*, \mathbb{P}) \to \pi_1\mathscr{C}(\mathbb{C}^*, \mathbb{P})$ is a surjection also. □

This method looks promising to generalise to the complex plane with finitely many punctures, however in this case we no longer have the Hopf degree theorem to call upon to classify our maps. The extension of this method to an arbitrary open Riemann surface

X, as well as finding accessible proofs that the map $\pi_n\mathscr{O}(X,\mathbb{P}) \to \pi_n\mathscr{C}(X,\mathbb{P})$ induced by inclusion is an isomorphism for $n \geq 1$, provide a stimulating and interesting question for further research.

3.3 The non-Gromov pairs

We now consider the strange case of the non-Gromov pairs. In this situation we consider maps from a particular kind of Stein manifold, a Riemann surface of specific finite topological type, to a hyperbolic target, the punctured disc $\mathbb{D}^*$. This is an oddity that does not fit into Gromov's framework as hyperbolic targets are generally considered as spaces with few holomorphic maps coming into them. However, in this special situation we find there are in some sense as many holomorphic maps as there are continuous maps. We will show that POP holds for these non-Gromov pairs.

Lemma 3.18. *Let $Y \neq \mathbb{P}$ be a Riemann surface, and $\pi : \tilde{Y} \to Y$ be the universal covering space of Y. If $\pi_1(Y)$ is abelian, then the covering transformations of π respect convex linear combinations when realising $\tilde{Y}$ as $\mathbb{C}$ or $\mathbb{H}$, that is, if $h \in \operatorname{Aut}\pi$ and $x, y \in \tilde{Y}$, then $h((1-t)x + ty) = (1-t)h(x) + th(y)$ for all $t \in [0,1]$.*

Proof. Using the standard identification $\pi_1(Y) = \operatorname{Aut}\pi$ (Proposition 2.57), we find that the group of covering transformations of π is abelian. If $\tilde{Y} = \mathbb{C}$, then Y is $\mathbb{C}$, $\mathbb{C}^*$ or a torus, and elements of $\operatorname{Aut}\pi$ are translations $z \mapsto z + \alpha$, $\alpha \in \mathbb{C}$, which clearly respect convex linear combinations. If $\tilde{Y} = \mathbb{H}$, then Y is $\mathbb{D}$, $\mathbb{D}^*$ or an annulus, and elements of $\operatorname{Aut}\pi$ are either translations $z \mapsto z + a$, $a \in \mathbb{R}$, or scalings $z \mapsto \lambda z$, $\lambda > 0$. Again, these transformations clearly respect convex linear combinations. □

Theorem 3.19. *Let X and Y be Riemann surfaces. Suppose $Y \neq \mathbb{P}$ and $\pi_1(Y)$ is abelian. Then the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces an isomorphism of π_k for $k \geq 1$ and an injection of π_0.*

Remark 3.20. We emphasise that in the setting of the theorem, the inclusion can easily fail to induce a surjection of π_0: take for example $X = \mathbb{C}^*$, $Y = \mathbb{D}^*$. The discovery that, even if BOP fails, the rest of POP may still hold is perhaps the main original contribution of this thesis.

Proof. Let B^k denote the closed unit ball in $\mathbb{R}^k$, $k \geq 1$, and fix $b_0 \in \partial B^k$. We show for any continuous map $\alpha_0 : B^k \to \mathscr{C}(X,Y)$ with $\alpha_0(\partial B^k) \subset \mathscr{O}(X,Y)$, there is a deformation $\alpha : B^k \times [0,1] \to \mathscr{C}(X,Y)$ such that, writing $\alpha_t = \alpha(\cdot, t)$,

(1) $\alpha_t(b_0) = \alpha_0(b_0)$ for all $t \in [0,1]$,

(2) $\alpha_t(\partial B^k) \subset \mathscr{O}(X,Y)$ for all $t \in [0,1]$,

(3) $\alpha_1(B^k) \subset \mathscr{O}(X,Y)$,

as in Lemma 3.2. Let (U_γ) be an open cover of X by simply connected sets and $\pi : \tilde{Y} \to Y$ be the universal covering map. Considering α_0 as a continuous map $X \times B^k \to Y$ by Proposition 2.43 (b), we may lift $\alpha_0|_{U_\gamma \times B^k}$ to the universal covering space $\tilde{Y}$ by π since $U_\gamma \times B^k$ is simply connected (Corollary 2.61), that is, there are continuous maps $\lambda_\gamma : U_\gamma \times B^k \to \tilde{Y}$ such that

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "\pi"] \\ U_\gamma \times B^k \arrow[ur, "\lambda_\gamma"] \arrow[r, "\alpha_0"] & Y \end{tikzcd}$$

commutes for each γ. On $(U_\gamma \cap U_\delta) \times B^k$ we have $\pi(\lambda_\gamma) = \pi(\lambda_\delta)$, so for each $(x,b) \in (U_\gamma \cap U_\delta) \times B^k$ there is $h_{(x,b)} \in \operatorname{Aut}\pi$ with $\lambda_\gamma(x,b) = h_{(x,b)} \circ \lambda_\delta(x,b)$. This defines a continuous map $h : (U_\gamma \cap U_\delta) \times B^k \to \operatorname{Aut}\pi$, $(x,b) \mapsto h_{(x,b)}$, which is locally constant since $\operatorname{Aut}\pi$ is discrete. Since B^k is connected, h is independent of the parameter value in B^k, that is, $h(x,b) = h(x,b')$ for all $b, b' \in B^k$. In particular this gives that $h(x,b) = h(x,b_0)$ for all $b \in B^k$. For this reason, we shall write $h(x,b) = h(x,b_0) = h_x$.

Realising $\tilde{Y}$ as $\mathbb{C}$ or $\mathbb{H}$, define $\alpha : X \times B^k \times [0,1] \to Y$ by

$$\alpha(x,b,t) = \alpha_t(x,b) = \pi((1-t)\lambda_\gamma(x,b) + t\lambda_\gamma(x,b_0))$$

on $U_\gamma \times B^k \times [0,1]$. Since $\pi_1(Y)$ is abelian, elements $h \in \operatorname{Aut}\pi$ respect convex linear combinations by Lemma 3.18, so for $(x,b) \in (U_\gamma \cap U_\delta) \times B^k$ we have

$$\begin{aligned} \pi((1-t)\lambda_\gamma(x,b) + t\lambda_\gamma(x,b_0)) &= \pi((1-t)h_x(\lambda_\delta(x,b)) + th_x(\lambda_\delta(x,b_0))) \\ &= \pi(h_x((1-t)\lambda_\delta(x,b) + t\lambda_\delta(x,b_0))) \\ &= \pi((1-t)\lambda_\delta(x,b) + t\lambda_\delta(x,b_0)), \end{aligned}$$

for all $t \in [0,1]$, so α_t is well defined. Thus α_t defines a deformation from α_0 to α_1 satisfying criteria (1)–(3). This proves the result by Lemma 3.2. □

Next is our alternative proof of Theorem 3.11, and POP for the non-Gromov pairs.

Corollary 3.21. *Let X be an open Riemann surface and Y be $\mathbb{C}^*$ or a torus. Then the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ is a weak homotopy equivalence.*

Proof. By Theorems 2.119 and 2.121 we know the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ induces a surjection of path components. Since $\pi_1(Y) = \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ if Y is $\mathbb{C}^*$ or a torus, the result follows by Theorem 3.19. □

Corollary 3.22. *Let $X = \bar{X} \setminus \bigcup_i D_i$, where $\bar{X}$ is a compact Riemann surface and (D_i) is a non-empty finite collection of mutually disjoint closed discs in $\bar{X}$. Then the inclusion $\mathscr{O}(X,\mathbb{D}^*) \hookrightarrow \mathscr{C}(X,\mathbb{D}^*)$ is a weak homotopy equivalence.*

Proof. By Theorem 2.123, the inclusion $\mathscr{O}(X,\mathbb{D}^*) \hookrightarrow \mathscr{C}(X,\mathbb{D}^*)$ induces a surjection of path components. Since $\pi_1(\mathbb{D}^*) = \mathbb{Z}$, the result follows from Theorem 3.19. □

3.4 hPOP

In Winkelmann's classification of Riemann surfaces that satisfy BOP [30] he provides a list of the pairs of Riemann surfaces for which BOP fails. Winkelmann's result is the following [30, Proposition 1]:

Theorem 3.23. *Let X and Y be Riemann surfaces. Then in each of the following cases there exists a continuous map $X \to Y$ not homotopic to a holomorphic map.*

(i) *X is compact and $Y = \mathbb{P}$.*

(ii) *X is compact and $\pi_1(X), \pi_1(Y) \neq 0$.*

(iii) *X is open, $\pi_1(X) \neq 0$ and Y is not a torus, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^*$, $\mathbb{D}^*$ or $\mathbb{P}$.*

(iv) *$X = X' \backslash \{p\}$ for some Riemann surface X', and $Y = \mathbb{D}^*$.*

(v) *$\pi_1(X)$ is not finitely generated and $Y = \mathbb{D}^*$.*

An interesting consequence of Theorem 3.19 is that it provides examples of spaces for which BOP fails, but the other aspects of POP hold. We make this precise with the next definition.

Definition 3.24. Let X and Y be Riemann surfaces. Then we say the pair (X, Y) satisfies the *higher parametric Oka principle* (hPOP) if the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ induces an isomorphism on π_k for $k \geq 1$ and an injection on π_0.

Clearly spaces that satisfy POP satisfy hPOP, but how about the spaces for which BOP fails? That is, what can we say about the pairs identified in Theorem 3.23? We explore this question by working through these pairs (X, Y) and providing some insight into the homotopy groups of the spaces $\mathscr{O}(X, Y)$ and $\mathscr{C}(X, Y)$. As a direct consequence of Theorem 3.19, $(X, \mathbb{D}^*)$ satisfies hPOP for every Riemann surface X since $\pi_1(\mathbb{D}^*)$ is abelian. This covers hPOP completely for cases (iv)–(v), and as to whether hPOP holds in cases (i)–(iii) provides an interesting open question explored in this section.

Throughout, unless otherwise stated, we shall be using the following notation. Let X and Y be Riemann surfaces and let $f : X \to Y$ be a continuous (resp. holomorphic) map. Denote by $\mathscr{C}_f(X, Y)$ (resp. $\mathscr{O}_f(X, Y)$) the path component of f in $\mathscr{C}(X, Y)$ (resp. in $\mathscr{O}(X, Y)$): when X and Y are understood, we will just write $\mathscr{C}_f$ (resp. $\mathscr{O}_f$). We shall also denote by

$$\mathscr{C}_{\mathscr{O}}(X, Y) = \bigcup_{f \in \mathscr{O}(X,Y)} \mathscr{C}_f(X, Y),$$

the union of the path components in $\mathscr{C}(X, Y)$ containing holomorphic maps. Again, if X and Y are understood we shall just write $\mathscr{C}_{\mathscr{O}}$. In this notation, to show a pair (X, Y) satisfies hPOP is to show the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X, Y)$ is a weak homotopy equivalence since this inclusion induces a surjection of path components by definition.

3.4.1 Case (i)

We begin by looking at case (i), that is, when the source is compact and the target is $\mathbb{P}$. This was explored in 1979 by Segal [24]. Segal presents results regarding the topology of the space of holomorphic maps $\mathbb{P} \to \mathbb{P}$. Let $n \geq 0$ be an integer and denote by $\mathscr{O}_n$ and $\mathscr{C}_n$ the spaces of holomorphic and continuous maps of degree n from the Riemann sphere to itself. The following is one of Segal's results [24, Proposition 1.1'], which is Theorem 2.96 from the background chapter, restated here for convenience.

Theorem 3.25. *The inclusion $\mathscr{O}_n \hookrightarrow \mathscr{C}_n$ is an n-equivalence.*

Remark 3.26. Recall the definition of an n-equivalence from Remark 2.94.

Segal's work does not address whether the inclusion $\mathscr{O}_n \hookrightarrow \mathscr{C}_n$ is a weak equivalence, that is, whether the inclusion is an m-equivalence for all m. We shall show here that the inclusion $\mathscr{O}_n \hookrightarrow \mathscr{C}_n$ is not a weak equivalence for all n: in particular we show $\mathscr{O}_1 \hookrightarrow \mathscr{C}_1$, the inclusion of maps of degree 1, is not a weak equivalence.

First consider $\mathscr{O}_1$, the space of holomorphic maps $\mathbb{P} \to \mathbb{P}$ of degree one. Since these maps have degree one, they consist of one-sheeted holomorphic covering maps $\mathbb{P} \to \mathbb{P}$, that is, the maps in $\mathscr{O}_1$ are precisely the automorphisms of $\mathbb{P}$. Thus $\mathscr{O}_1 = \operatorname{Aut}\mathbb{P}$, which is well known to be the complex Lie group $\mathrm{PGL}_2\mathbb{C}$. Since it is a complex Lie group, $\mathrm{PGL}_2\mathbb{C}$ deformation retracts onto its maximal compact subgroup by the manifold splitting theorem [13, p. 544, Theorem 14.3.11], which is well known to be $\mathrm{SU}(2)/\mathbb{Z}_2$. Since $\mathrm{SU}(2)$ is diffeomorphic to S^3, it follows that $\mathscr{O}_1$ is homotopy equivalent to $S^3/\mathbb{Z}_2 = \mathbb{RP}^3$. Hence $\pi_k\mathscr{O}_1 = \pi_k\mathbb{RP}^3$ for all $k \geq 0$.

Now consider $\mathscr{C}_1$. The homotopy groups for $\mathscr{C}_n$ were calculated for $k \geq 2$ by Hansen [11, Theorem 4.2]. The following is Hansen's result (Theorem 2.90) in the special case of genus 0.

Theorem 3.27. *For $n \neq 0$ and $k \geq 2$, $\pi_k\mathscr{C}_n = \pi_k S^3 \oplus \pi_{k+2}S^2$.*

Thus we have $\pi_2\mathscr{O}_1 = \pi_2\mathbb{RP}^3 = 0$ and $\pi_2\mathscr{C}_1 = \pi_4 S^2 = \mathbb{Z}_2$ (see Table 2.1), so the inclusion cannot induce an isomorphism of all homotopy groups. It is interesting to note however that by Hu [14, Theorem 5.3], $\pi_1\mathscr{C}_1 = \mathbb{Z}_2$. This means the map $\pi_1\mathscr{O}_1 \to \pi_1\mathscr{C}_1$ induced by inclusion is in fact an isomorphism. By Theorem 3.25, we know that $\pi_1\mathscr{O}_1 \to \pi_1\mathscr{C}_1$ is surjective, and since $\pi_1\mathscr{O}_1 = \pi_1\mathscr{C}_1 = \mathbb{Z}_2$, it must have trivial kernel. This slightly extends Segal's result, however does not generalise easily.

From this we can conclude that the pair $(\mathbb{P}, \mathbb{P})$ does not satisfy hPOP and thus case (i) does not satisfy hPOP in general. For a general compact Riemann surface X, Segal presents some insight into the topology of $\mathscr{O}_n(X, \mathbb{P})$, the space of holomorphic maps $X \to \mathbb{P}$ of degree n, showing that the inclusion $\mathscr{O}_n(X, \mathbb{P}) \hookrightarrow \mathscr{C}_n(X, \mathbb{P})$ is a homology $(n-2g)$-equivalence, where g is the genus of X (Theorem 2.98). Despite this, Segal does not determine whether it is a homotopy $(n-2g)$-equivalence.

3.4.2 Case (ii)

Next, we move on to case (ii). This case concerns maps between a compact source and a target that are not simply connected. By Theorem 3.19, it follows that $(X, \mathbb{C}^*)$ and (X, T) satisfy hPOP for every Riemann surface X, where T is a torus. This shows at least a partial positive result in case (ii). We now explore the other pairs that occur.

Lemma 3.28. *Let X and Y be Riemann surfaces such that every holomorphic map $X \to Y$ is constant. Then the pair (X, Y) satisfies hPOP.*

Proof. To show the pair (X, Y) satisfies hPOP is to show the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X, Y)$ is a weak homotopy equivalence. By definition of $\mathscr{C}_{\mathscr{O}}(X, Y)$ we have that the inclusion induces a surjection of path components. Further, $\mathscr{O}(X, Y) = Y$ by assumption and so $\mathscr{C}_{\mathscr{O}}(X, Y)$ is precisely the space of null-homotopic continuous maps. Hence the result follows by Theorem 3.3. □

Lemma 3.29. *Let X and Y be compact Riemann surfaces of genus at least 2. Suppose there is a non-constant holomorphic map $f : X \to Y$. Then $\pi_1(\mathscr{C}(X, Y), f) = 0$.*

We provide two proofs of Lemma 3.29 demonstrating two different techniques. The first relies more on fibrations and the associated long exact sequence. We use this tool and covering space theory to analyse $\pi_1(\mathscr{C}(X, Y), f)$. The second proof uses a direct relation between $\pi_1(\mathscr{C}(X, Y), f)$ and subgroups of $\pi_1(Y)$ and $\operatorname{Aut} \mathbb{H}$ to provide a more geometric analysis of what is occurring.

Proof (1). By Theorem 2.93, we know

$$\pi_1(\mathscr{C}(X, Y), f) = \{\gamma \in \pi_1(Y) : \gamma\alpha = \alpha\gamma \text{ for all } \alpha \in f_*\pi_1(X)\}.$$

Let $\pi_1(Y) = G$ and $f_*\pi_1(X) = H$ for brevity. Let $p : Y_H \to Y$ be the covering map associated to $H < G$ (Theorem 2.54) and $\tilde{f} : X \to Y_H$ be a holomorphic lifting of f by p (Proposition 2.60). We first show that H cannot have infinite index in G. If H has infinite index, then p has infinitely many sheets. Thus Y_H must be non-compact. Now, since f and p are holomorphic, $\tilde{f}$ is holomorphic also. Since X is compact, $\tilde{f}$ is constant (Proposition 2.13) and hence f is constant, a contradiction.

Thus H has finite index in G and corresponds to a compact covering space. Note that the genus of Y_H is at least 2, otherwise p would be constant by Corollary 2.16. Considering $\tilde{f}$ as above, we see that $\tilde{f}_* : \pi_1(X) \to \pi_1(Y_H)$ is surjective. Indeed, we have a commuting diagram of groups

$$\begin{tikzcd} & \pi_1(Y_H) \arrow[d, "p_*"] \\ \pi_1(X) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & H \end{tikzcd}$$

such that p_* is injective since p is a covering map (Lemma 2.53), and f_* is surjective by definition of H. Taking $\gamma \in \pi_1(Y_H)$, there is $\alpha \in \pi_1(X)$ such that $f_*\alpha = p_*\gamma$. Then $f_*\alpha = \left(p_* \circ \tilde{f}_*\right)\alpha = p_*\gamma$, so $\tilde{f}_*\alpha = \gamma$ by injectivity of p_* and $\tilde{f}_*$ is surjective.

Thus the centraliser of $\tilde{f}_*\pi_1(X)$ in $\pi_1(Y_H)$ is the centre of $\pi_1(Y_H)$, so

$$\pi_1(\mathscr{C}(X, Y_H), \tilde{f}) = Z(\pi_1(Y_H))^2$$

by Theorem 2.93, and $Z(\pi_1(Y_H)) = 0$ by Corollary 2.28, that is, $\pi_1(\mathscr{C}(X, Y_H), \tilde{f}) = 0$. Since p is a covering map, it is a fibration by Proposition 2.77. Then the induced map $p_* : \mathscr{C}(X, Y_H) \to \mathscr{C}(X, Y)$ is a fibration by Proposition 2.78 and there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(\mathscr{C}(X, Y_H), \tilde{f}) \longrightarrow \pi_n(\mathscr{C}(X, Y), \tilde{f}) \longrightarrow \pi_{n-1}(p_*^{-1}(f), \tilde{f}) \longrightarrow \cdots$$

associated to p_* (Corollary 2.88). Since $\pi_1(\mathscr{C}(X, Y_H), \tilde{f}) = 0$, the end of the sequence reduces to

$$0 \longrightarrow \pi_1(\mathscr{C}(X, Y), \tilde{f}) \longrightarrow \pi_0(p_*^{-1}(f), \tilde{f}) \longrightarrow \pi_0(\mathscr{C}(X, Y_H), \tilde{f}) \longrightarrow \cdots$$

Now, $\pi_1(\mathscr{C}(X, Y), f) = C_H$, so $\pi_1(\mathscr{C}(X, Y), \tilde{f}) = 0$ or $\mathbb{Z}$ by Corollary 2.27.

Suppose $\pi_1(\mathscr{C}(X, Y), \tilde{f}) = \mathbb{Z}$. Exactness of our long exact sequence tells us that

$$\pi_1(\mathscr{C}(X, Y), \tilde{f}) \to \pi_0(p_*^{-1}(f), \tilde{f})$$

is injective, so if $\pi_1(\mathscr{C}(X, Y), \tilde{f}) = \mathbb{Z}$, then the discrete set $\pi_0(p_*^{-1}(f), \tilde{f})$ is infinite. Since the elements of $\pi_0(p_*^{-1}(f), \tilde{f})$ correspond to the liftings of f by p, it follows that p must have infinitely many sheets, which is absurd. Thus $\pi_1(\mathscr{C}(X, Y), f) = 0$, as required. $\square$

Proof (2). Let $\pi_1(Y) = G$ and $f_*\pi_1(X) = H$ for brevity. By Theorem 2.93, we know that

$$\pi_1(\mathscr{C}(X, Y), f) = \{\gamma \in \pi_1(Y) : \gamma\alpha = \alpha\gamma \text{ for all } \alpha \in f_*\pi_1(X)\} = C_H\,,$$

and by Corollary 2.27 we know that $C_H = \mathbb{Z}$, 0 or G. Since f is non-constant, f is not null-homotopic by Lemma 2.22 and so $H \neq 0$, that is, $C_H \neq G$.

If $C_H = \mathbb{Z}$, then H is abelian by Lemma 2.29. Let $p : Y_H \to H$ denote the covering of Y associated to $H < G$. Then since $\pi_1(Y_H) = \mathbb{Z}$, Y_H is either $\mathbb{D}^*$ or an annulus (Y_H cannot be $\mathbb{C}^*$ since this would imply Y is covered by $\mathbb{C}$, which is absurd). Further, since $f_*\pi_1(X) = H = p_*\pi_1(Y_H)$, there exists a lifting of f by p, that is, there is a holomorphic map $\tilde{f} : X \to Y_H$ such that

$$\begin{tikzcd} & Y_H \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

[2]Here, $Z(\pi_1(Y_H))$ denotes the centre of $\pi_1(Y_H)$ as in the proof of Corollary 2.28.

commutes. However, since X is compact and Y_H is open, $\tilde{f}$ (and hence f) must be constant by Proposition 2.13, a contradiction. Thus we have $\pi_1(\mathscr{C}(X,Y), f) = C_H = 0$. □

Theorem 3.30. *Let X and Y be Riemann surfaces. Suppose X is compact and both X and Y are not simply connected. Then the pair (X,Y) satisfies h*POP*.*

Proof. We can separate this into three cases:

- Y is open,
- Y is compact with the genus g of X smaller than the genus g' of Y,
- Y is compact with the genus g of X larger than or equal to the genus g' of Y.

The first and second cases are covered by Lemma 3.28 since $\mathscr{O}(X,Y) = Y$ when Y is open and when $g < g'$. So it remains to consider the case $g \geq g'$.

Let $f : X \to Y$ be our holomorphic base point map. By de Franchis' theorem (Theorem 2.17), there are only finitely many, non-constant holomorphic maps $X \to Y$, so we must consider when f is constant and when f is non-constant. If f is constant, then the path component $\mathscr{O}_f$ in $\mathscr{O}(X,Y)$ consists of the constant maps by Lemma 2.22. Then $\mathscr{O}_f \hookrightarrow \mathscr{C}_0(X,Y)$ is a weak homotopy equivalence by Theorem 3.3, where $\mathscr{C}_0(X,Y)$ denotes the null-homotopic continuous maps.

If f is non-constant then $\mathscr{O}_f = \{f\}$ by de Franchis' theorem. Thus $\pi_n(\mathscr{O}(X,Y), f) = 0$ for $n \geq 1$. By Lemma 3.29 and Theorem 2.93, $\pi_n(\mathscr{C}(X,Y), f) = 0$ for $n \geq 1$, so $\mathscr{O}_f \hookrightarrow \mathscr{C}_f$ is trivially a weak homotopy equivalence.

Finally we need $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}(X,Y)$ to induce an injection of path components. As mentioned in Remark 2.19, this follows from the proof of de Franchis' theorem with the use of intersection theory. Thus $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X,Y)$ is a weak homotopy equivalence and (X,Y) satisfies hPOP. □

We can now use this result to show that the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X,Y)$ is a genuine homotopy equivalence when X is a compact Riemann surface and $Y \neq \mathbb{P}$. To do this however, we need one more result about $\mathscr{C}_{\mathscr{O}}(X,Y)$.

Lemma 3.31. *Let X and Y be Riemann surfaces. Suppose X is compact. Then $\mathscr{C}_{\mathscr{O}}(X,Y)$ has the homotopy type of a CW-complex.*

Proof. By Theorem 3.8, $\mathscr{C}(X,Y)$ has the homotopy type of a CW-complex. Thus there is a CW-complex W and continuous maps $f : \mathscr{C}(X,Y) \to W$ and $g : W \to \mathscr{C}(X,Y)$ such that $f \circ g$ is homotopic to id_W and $g \circ f$ is homotopic to $\mathrm{id}_{\mathscr{C}(X,Y)}$. For each $h \in \mathscr{C}(X,Y)$, $f(\mathscr{C}_h(X,Y))$ is contained in a unique path component of W by continuity of f. Call this path component W_h. Defining

$$W' = \bigcup_{h \in \mathscr{O}(X,Y)} W_h$$

gives a union of path components W' such that $f(\mathscr{C}_{\mathscr{O}}(X,Y)) \subset W'$. Further, by continuity of g and the fact that $g \circ f$ is homotopic to $\mathrm{id}_{\mathscr{C}_{\mathscr{O}}(X,Y)}$, we have $g(W_h) \subset \mathscr{C}_h(X,Y)$ for all $h \in \mathscr{C}(X,Y)$. In particular this gives $g(W') \subset \mathscr{C}_{\mathscr{O}}(X,Y)$. Note that $W' \subset W$ is a subcomplex of W since it is a union of path components (see [4, p. 38, Proposition 1.4.11] for more details).

We claim that $f' = f|_{\mathscr{C}_{\mathscr{O}}(X,Y)} : \mathscr{C}_{\mathscr{O}}(X,Y) \to W'$ is a homotopy equivalence with homotopy inverse $g' = g|_{W'} : W' \to \mathscr{C}_{\mathscr{O}}(X,Y)$. Indeed, suppose $H : W \times [0,1] \to W$ is a homotopy from $f \circ g$ to id_W. Then $\mathrm{image}(H|_{W' \times [0,1]}) \subset W'$ and $H|_{W' \times \{0\}} = (f \circ g)|_{W'} = f' \circ g'$ and $H|_{W' \times \{1\}} = \mathrm{id}_W|_{W'} = \mathrm{id}_{W'}$, that is, $H|_{W' \times [0,1]}$ is a homotopy from $f' \circ g'$ to $\mathrm{id}_{W'}$. In an identical manner we find that $g' \circ f'$ is homotopic to $\mathrm{id}_{\mathscr{C}_{\mathscr{O}}(X,Y)}$. This proves the result. □

Corollary 3.32. *Let X and Y be Riemann surfaces. Suppose X is compact and both X and Y are not simply connected. Then the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X,Y)$ is a homotopy equivalence.*

Proof. Under the assumptions on X and Y, we know that either $\mathscr{O}(X,Y) = Y$ when Y is open or when the genus of X is less than the genus of Y by Proposition 2.13 and Corollary 2.16, or $\mathscr{O}(X,Y) = Y \cup \{f_1, \ldots, f_n\}$, for finitely many non-constant holomorphic maps $f_1, \ldots, f_n : X \to Y$, when the genus of X is greater than or equal to the genus of Y by de Franchis' theorem (Theorem 2.17). In either case, $\mathscr{O}(X,Y)$ is a CW-complex. By Lemma 3.31, $\mathscr{C}_{\mathscr{O}}(X,Y)$ has the homotopy type of a CW-complex. Thus the inclusion $\mathscr{O}(X,Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X,Y)$ is a homotopy equivalence by Whitehead's theorem (Theorem 2.65) since it is a weak equivalence by Theorem 3.30. □

3.4.3 Case (iii)

Now consider the final case: an open source X, not simply connected, with target different from $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^*$, $\mathbb{D}^*$, $\mathbb{P}$ or a torus. By Theorem 3.19, hPOP holds whenever the target is not $\mathbb{P}$ and has abelian fundamental group, and in particular when the target is an annulus. We extend this idea by considering Riemann surfaces with abelian fundamental groups to provide a partial answer for case (iii).

Theorem 3.33. *Let X and Y be Riemann surfaces, $Y \neq \mathbb{P}$. Let $f : X \to Y$ be a holomorphic map with $f_*\pi_1(X)$ abelian. Then $\mathscr{O}_f \hookrightarrow \mathscr{C}_f$ is a weak homotopy equivalence.*

Proof. By definition, $\mathscr{O}_f \hookrightarrow \mathscr{C}_f$ induces a bijection on path components, so we need to show that $\mathscr{O}_f \hookrightarrow \mathscr{C}_f$ induces an isomorphism of all homotopy groups. Let $p : A \to Y$ be the covering space associated to $f_*\pi_1(X) < \pi_1(Y)$. Then $\pi_1(A)$ is abelian since $p_*\pi_1(A) = f_*\pi_1(X)$. Fix a holomorphic lifting $\tilde{f} : X \to A$ of f by p.

Since the covering map $p : A \to Y$ is a fibration (Proposition 2.77), by Proposition 2.78 and Corollary 2.79 we get induced fibrations

$$p_*^{-1}(f) \longrightarrow \mathscr{C}(X,A) \xrightarrow{p_*} \mathscr{C}(X,Y)$$

and

$$p_*^{-1}(f) \longrightarrow \mathscr{O}(X,A) \xrightarrow{p_*} \mathscr{O}(X,Y)$$

from post-composition by p, with the same fibre $p^{-1}(f)$. Restricting the image to the path components $\mathscr{C}_f(X,Y)$ and $\mathscr{O}_f(X,Y)$, and the source to $p_*^{-1}\mathscr{C}_f(X,Y) \subset \mathscr{C}(X,A)$ and $p_*^{-1}\mathscr{O}_f(X,Y) \subset \mathscr{O}(X,A)$, gives the fibrations

$$p_*^{-1}(f) \longrightarrow p_*^{-1}\mathscr{C}_f(X,Y) \xrightarrow{p_*} \mathscr{C}_f(X,Y)$$

and

$$p_*^{-1}(f) \longrightarrow p_*^{-1}\mathscr{O}_f(X,Y) \xrightarrow{p_*} \mathscr{O}_f(X,Y)\,.$$

There are natural maps between these fibration sequences induced by inclusion, so from the long exact sequence of homotopy groups (Corollary 2.88, considering $\tilde{f}$ as a base point in $p_*^{-1}\mathscr{O}_f(X,Y)$ and $p_*^{-1}\mathscr{C}_f(X,Y)$) we get a commuting diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_n(p_*^{-1}(f),\tilde{f}) & \longrightarrow & \pi_n(\mathscr{O}(X,A),\tilde{f}) & \longrightarrow & \pi_n(\mathscr{O}(X,Y),f) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \pi_n(p_*^{-1}(f),\tilde{f}) & \longrightarrow & \pi_n(\mathscr{C}(X,A),\tilde{f}) & \longrightarrow & \pi_n(\mathscr{C}(X,Y),f) & \longrightarrow & \cdots
\end{array}$$

Note we have replaced $p_*^{-1}\mathscr{O}(X,Y)$ and $p_*^{-1}\mathscr{C}(X,Y)$ with the full spaces $\mathscr{O}(X,A)$ and $\mathscr{C}(X,A)$ in the long exact sequence since, for $n \geq 1$,

$$\pi_n(p_*^{-1}\mathscr{O}(X,Y),\tilde{f}) = \pi_n(\mathscr{O}(X,A),\tilde{f})$$

and

$$\pi_n(p_*^{-1}\mathscr{C}(X,Y),\tilde{f}) = \pi_n(\mathscr{C}(X,A),\tilde{f})\,,$$

simply from the specification of the base point $\tilde{f}$.

We know the inclusion $\mathscr{O}(X,A) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X,A)$ is a weak homotopy equivalence by Theorem 3.19 since $\pi_1(A)$ is abelian, so $\pi_n(\mathscr{O}(X,A),\tilde{f}) \to \pi_n(\mathscr{C}_{\mathscr{O}}(X,A),\tilde{f})$ is an isomorphism for $n \geq 1$ and a bijection for $n = 0$. By Theorem 2.93, since A is an Eilenberg-MacLane space of type $(\pi_1(A),1)$, we know

$$\pi_n(\mathscr{O}(X,A),\tilde{f}) = \pi_n(\mathscr{C}(X,A),\tilde{f}) = 0 \text{ for } n \geq 2.$$

Also, since $p_*^{-1}(f)$ is discrete, $\pi_n(p_*^{-1}(f),\tilde{f}) = 0$ for $n \geq 1$. Thus for $n > 1$ the long exact sequence of homotopy groups gives us

$$\begin{array}{ccccc}
0 & \longrightarrow & \pi_n(\mathscr{C}(X,Y),f) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_n(\mathscr{O}(X,Y),f) & \longrightarrow & 0
\end{array}$$

that is, $\pi_n(\mathscr{O}(X,Y),f) = \pi_n(\mathscr{C}(X,Y),f) = 0$ for $n \geq 1$. This, together with Lemma 3.34 which covers the case $n = 1$, proves the result. $\square$

What remains is the proof of Lemma 3.34, handling the case $n = 1$ for Theorem 3.33. This proof is an adaptation of the five lemma in which we have to deal with the pointed sets of path components for our spaces. A careful consideration of the proof of the five lemma shows that it holds in this situation.

Lemma 3.34. *Let X and Y be Riemann surfaces, $Y \neq \mathbb{P}$. Let $f : X \to Y$ be a holomorphic map with $f_*\pi_1(X)$ abelian. Then $\mathscr{O}_f \hookrightarrow \mathscr{C}_f$ induces an isomorphism of fundamental groups.*

Proof. Throughout this proof, we continue with the notion, spaces and maps as introduced in the proof of Theorem 3.33. For $n = 1$, the long exact sequence of homotopy groups reduces to

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_1(\mathscr{O}(X,A),\tilde{f}) \arrow[r] \arrow[d] & \pi_1(\mathscr{O}(X,Y),f) \arrow[r] \arrow[d] & \pi_0(p_*^{-1}(f),\tilde{f}) \arrow[r] \arrow[d] & \pi_0(p_*^{-1}\mathscr{O}_f(X,Y),\tilde{f}) \arrow[d] \\
0 \arrow[r] & \pi_1(\mathscr{C}(X,A),\tilde{f}) \arrow[r] & \pi_1(\mathscr{C}(X,Y),f) \arrow[r] & \pi_0(p_*^{-1}(f),\tilde{f}) \arrow[r] & \pi_0(p_*^{-1}\mathscr{C}_f(X,Y),\tilde{f})
\end{tikzcd}$$

It is somewhat unconventional to include a base point for π_0 as done above. However, we are considering π_0 as a pointed set, so including the base point makes it explicit where our set is based, that is, $\pi_0(p_*^{-1}(f),\tilde{f})$ is based at the map $\tilde{f}$, $\pi_0(p_*^{-1}\mathscr{C}_f(X,Y),\tilde{f})$ is based at the path component $\mathscr{C}_{\tilde{f}}$ and $\pi_0(p_*^{-1}\mathscr{O}_f(X,Y),\tilde{f})$ is based at the path component $\mathscr{O}_{\tilde{f}}$.

Since $\pi_1(A)$ is abelian,

$$\pi_1(\mathscr{O}(X,A),\tilde{f}) \to \pi_1(\mathscr{C}(X,A),\tilde{f})$$

is an isomorphism and

$$\pi_0(p_*^{-1}\mathscr{O}_f(X,Y),\tilde{f}) \to \pi_0(p_*^{-1}\mathscr{C}_f(X,Y),\tilde{f})$$

is a bijection by Theorem 3.19. We also have that $\pi_0(p_*^{-1}(f),\tilde{f}) \to \pi_0(p_*^{-1}(f),\tilde{f})$ is the identity and hence a bijection. The initial instinct in this situation is to apply the five lemma to get $\pi_1(\mathscr{O}(X,Y),f) \to \pi_1(\mathscr{C}(X,Y),f)$ is an isomorphism, however the five lemma applies when all the arrows are between abelian groups, where in our case the last two arrows are simply between pointed sets. To show the same result applies we go through a careful consideration of the proof of the five lemma in this specific situation. Below we have labelled the arrows for reference.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_1(\mathscr{O}(X,A),\tilde{f}) \arrow[r, "g"] \arrow[d, "m"] & \pi_1(\mathscr{O}(X,Y),f) \arrow[r, "h"] \arrow[d, "n"] & \pi_0(p_*^{-1}(f),\tilde{f}) \arrow[r, "j"] \arrow[d, "q"] & \pi_0(p_*^{-1}\mathscr{O}_f(X,Y),\tilde{f}) \arrow[d, "r"] \\
0 \arrow[r] & \pi_1(\mathscr{C}(X,A),\tilde{f}) \arrow[r, "s"] & \pi_1(\mathscr{C}(X,Y),f) \arrow[r, "t"] & \pi_0(p_*^{-1}(f),\tilde{f}) \arrow[r, "u"] & \pi_0(p_*^{-1}\mathscr{C}_f(X,Y),\tilde{f})
\end{tikzcd}$$

The rows are exact, m is an isomorphism and r and q are bijections, and we wish to show that n is an isomorphism. We first show injectivity. Let $\alpha \in \pi_1(\mathscr{O}(X,Y),f)$ and suppose $n(\alpha)=0$. Then $\tilde{f} = t\circ n(\alpha) = q\circ h(\alpha)$, so $h(\alpha)\in \ker q$. Since $p_*^{-1}(f)$ is discrete and q is injective (in fact, q is the identity map), this tells us that $h(\alpha)=\tilde{f}$, so $\alpha\in\ker h$. Then there is $\beta\in\pi_1(\mathscr{O}(X,A),\tilde{f})$ such that $g(\beta)=\alpha$ by exactness. By commutativity we have $0=n(\alpha)=n\circ g(\beta)=s\circ m(\beta)$, so $m(\beta)\in\ker s$. Exactness tells us that s is injective, so $m(\beta)=0$, and since m is an isomorphism, $\beta=0$. Then $\alpha=g(\beta)=0$ and n is injective.

Recall that by Theorem 2.93, $\pi_1(\mathscr{C}(X,Y),f)=C_{f_*\pi_1(X)}$, the centraliser of $f_*\pi_1(X)$ in $\pi_1(Y)$. Thus, since $f_*\pi_1(X)$ is abelian, $\pi_1(\mathscr{C}(X,Y),f)$ is abelian either because $\pi_1(Y)$ is abelian or by Corollary 2.27. Since n is injective, this tells us that $\pi_1(\mathscr{O}(X,Y),f)$ is abelian also.

Now to show surjectivity. Take $\alpha'\in\pi_1(\mathscr{C}(X,Y),f)$. There is a map $f'\in p_*^{-1}(f)$ with $t(\alpha')=f'=q(f')$. By exactness we have $u\circ t(\alpha')=\mathscr{C}_{\tilde{f}}$, and so by commutativity $\mathscr{C}_{\tilde{f}}=u\circ t(\alpha')=u\circ q(f')=r\circ j(f')$ and so $j(f')\in\ker r$. Unravelling this, $j(f')\in\ker r$ tells us that $\mathscr{C}_{f'}=\mathscr{C}_{\tilde{f}}$, but injectivity of r tells us that if f' and f are homotopic through continuous maps then they are homotopic through holomorphic maps, that is, $\mathscr{O}_{f'}=\mathscr{O}_{\tilde{f}}$ and $f'\in\ker j$. Thus by exactness there is $\alpha\in\pi_1(\mathscr{O}(X,Y),f)$ with $h(\alpha)=f'$. So we have that $t\circ n(\alpha)=q\circ h(\alpha)=f'=t(\alpha')$. To proceed we have to understand the map $t:\pi_1(\mathscr{C}(X,Y),f)\to\pi_0(p_*^{-1}(f),\tilde{f})$.

We may describe t as follows. Given a representative loop $\gamma:[0,1]\to\mathscr{C}(X,Y)$ with $\gamma(0)=\gamma(1)=f$ we lift by the fibration $p_*:p_*^{-1}\mathscr{C}_f(X,Y)\to\mathscr{C}_f(X,Y)$ to get a continuous map $\tilde{\gamma}:[0,1]\to p_*^{-1}\mathscr{C}_f(X,Y)$ such that $\tilde{\gamma}(0)=\tilde{f}$. Then we define $t[\gamma]=\tilde{\gamma}(1)$. By this construction, since $t(\alpha')=t\circ n(\alpha)=f'$, the group element $\alpha'-n(\alpha)$ is such that $t(\alpha'-n(\alpha))=\tilde{f}$, so $\alpha'-n(\alpha)\in\ker t$. By exactness there is $\beta'\in\pi_1(\mathscr{C}(X,A),\tilde{f})$ with $s(\beta')=\alpha'-n(\alpha)$. Since m is an isomorphism there is $\beta\in\pi_1(\mathscr{O}(X,A),\tilde{f})$ with $m(\beta)=\beta'$, so $n\circ g(\beta)=s\circ m(\beta)=\alpha'-n(\alpha)$. Then $n(g(\beta)+\alpha)=\alpha'$ and n is surjective. This shows n is an isomorphism. □

Lemma 3.35. *Let X and Y be Riemann surfaces, $Y\neq\mathbb{P}$. Let $f_1,f_2:X\to Y$ be holomorphic maps that are homotopic through continuous maps and suppose $f_{1*}\pi_1(X)$ is abelian. Then f_1 and f_2 are homotopic through holomorphic maps.*

Proof. Let $p:A\to Y$ be the covering space associated with $(f_1)_*\pi_1(X)<\pi_1(Y)$. Then the fundamental group of A is abelian since $(f_1)_*\pi_1(X)$ is abelian. Since $(f_1)_*\pi_1(X)=p_*\pi_1(A)$ there is a holomorphic lifting $\tilde{f}_1:X\to A$ of f_1 by p. Since p is a fibration, given a homotopy $H:X\times[0,1]\to Y$ from f_1 to f_2, there is a lifting $\tilde{H}:X\times[0,1]\to A$ of H by p with $\tilde{H}(\cdot,0)=\tilde{f}_1$. By construction, $\tilde{H}(\cdot,1)=\tilde{f}_2:X\to A$ is a holomorphic lifting of f_2 by p. Since $\tilde{f}_1$ and $\tilde{f}_2$ are homotopic through continuous maps and A is abelian, there is a homotopy $\tilde{G}:X\times[0,1]\to A$ from $\tilde{f}_1$ to $\tilde{f}_2$ through holomorphic maps by Theorem 3.19. Then $G=p\circ\tilde{G}:X\times[0,1]\to Y$ is a homotopy from f_1 to f_2 through holomorphic maps, as required. □

Corollary 3.36. *Let X be a Riemann surface with $\pi_1(X) = \mathbb{Z}$ and $Y \neq \mathbb{P}$ be a Riemann surface. Then (X, Y) satisfies hPOP.*

Proof. By Lemma 3.35, $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X, Y)$ induces a bijection of path components. By Theorem 3.33, $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}_{\mathscr{O}}(X, Y)$ induces an isomorphism of all homotopy groups. Thus (X, Y) satisfies hPOP. □

3.5 Summary

We collect our results in a table with references to summarise the work done. Where appropriate, we also calculate the homotopy groups $\pi_n\mathscr{O}(X, Y)$ for Riemann surfaces X and Y. We can categorise our Riemann surfaces into those pairs which satisfy BOP and those which do not. Within those that satisfy BOP, we further separate into the subcategories of the topological pairs, the Gromov pairs and the non-Gromov pairs. The following table covers all possible pairs of Riemann surfaces.

Pairs that satisfy BOP	**POP**
Topological pairs	
• $X = \mathbb{D}$ or $\mathbb{C}$ or $Y = \mathbb{D}$ or $\mathbb{C}$	✓ by Thms. 3.6 and 3.7 and Cor. 3.4
• $X = \mathbb{P}$, $Y \neq \mathbb{P}$	✓ by Cor. 3.5 and Thm. 3.9
Gromov pairs	
• X open, $Y = \mathbb{C}^*$ or a torus	✓ by Thm. 3.11 and Cor. 3.21
• X open, $Y = \mathbb{P}$	π_0-mono[a] by Cor. 3.13, π_1-epi[b] for $X = \mathbb{C}^*$ by Cor. 3.17
Non-Gromov pairs	
• $X = \bar{X} \backslash \bigcup D_i$, $\bar{X}$ compact, (D_i) a non-empty finite collection of mutually disjoint closed disks, $Y = \mathbb{D}^*$	✓ by Cor. 3.22
Pairs for which BOP fails	h**POP**
• X compact, $Y = \mathbb{P}$	hPOP fails for $X = \mathbb{P}$ by Sec. 3.4.1
• X compact, X and Y not simply connected	✓ by Thm. 3.30 and Cor. 3.32
• X open, $\pi_1(X) \neq 0$, Y an annulus or $\pi_1(Y)$ not abelian	✓ for $\pi_1(X)$ or $\pi_1(Y)$ abelian by Cor. 3.36 and Thm. 3.19
• $X = X' \backslash \{p\}$ for any X', $Y = \mathbb{D}^*$	✓ by Thm. 3.19
• $\pi_1(X)$ is infinitely generated, $Y = \mathbb{D}^*$	✓ by Thm. 3.19

[a] By π_0-mono we mean a π_0-monomorphism, that is, an injection on path components.
[b] By π_1-epi we mean a π_1-epimorphism, that is, a surjection on fundamental groups.

This provides a classification into ten cases, with only the last two having overlap. Despite this overlap, it appears neither group subsumes the other. For instance, there are definitely punctured surfaces with $\pi_1(X)$ finitely generated, and it is plausible that there are surfaces with $\pi_1(X)$ infinitely generated that are not punctured surfaces, however to prove this rigorously would be a detour too far afield into surface topology.

Let X and Y be Riemann surfaces, $Y \neq \mathbb{P}$, and let $f : X \to Y$ be holomorphic. By Theorem 2.93, whenever hPOP or POP holds for the pair (X, Y) then $\mathscr{O}_f(X, Y)$ is an Eilenberg-MacLane space of type $(\pi_1(\mathscr{O}(X, Y), f), 1)$. Combining the results of Theorem 2.93 and Lemma 2.29, we can calculate that

$$\pi_1(\mathscr{O}(X,Y), f) = \begin{cases} 0 & \text{if } f_*\pi_1(X) \text{ is not abelian,} \\ \mathbb{Z} & \text{if } f_*\pi_1(X) = \mathbb{Z} \text{ and } Y \text{ is not a torus,} \\ \pi_1(Y) & \text{if } f_*\pi_1(X) = 0. \end{cases}$$

In the case Y is a torus, we have $\pi_1(\mathscr{O}(X, Y), f) = \mathbb{Z} \oplus \mathbb{Z}$. This gives us the following table (where T denotes a torus):

Pairs (X, Y)	$\pi_1(\mathscr{O}(X, Y), f)$
• $(X, \mathbb{C})$ or $(X, \mathbb{D})$	0
• $(\mathbb{C}, Y)$ or $(\mathbb{D}, Y)$	$\pi_1(Y)$
• $(\mathbb{P}, Y)$	$\pi_1(Y)$
• (X, Y) with $\pi_1(X)$ or $\pi_1(Y)$ abelian and both non-trivial	$\mathbb{Z}$ if $Y \neq T$, $\mathbb{Z} \oplus \mathbb{Z}$ if $Y = T$
• (X, Y), X compact	$\begin{cases} 0 & \text{if } f \text{ non-constant} \\ \pi_1(Y) & \text{if } f \text{ constant} \end{cases}$

Remark 3.37. It is of interest to note that the homotopy groups of $\mathscr{C}(X, \mathbb{P})$ are quite complicated for any topological space X. For $x_0 \in X$, we get a sequence

$$\mathbb{P} \hookrightarrow \mathscr{C}(X, \mathbb{P}) \xrightarrow{\mathrm{ev}_{x_0}} \mathbb{P}$$

Applying the homotopy group functor to this, we see that the identity $\pi_n(\mathbb{P}) \to \pi_n(\mathbb{P})$ factors through $\pi_n\mathscr{C}(X, \mathbb{P})$, that is, we get

$$\pi_n(\mathbb{P}) \longrightarrow \pi_n\mathscr{C}(X, \mathbb{P}) \longrightarrow \pi_n(\mathbb{P})$$

This shows that $\pi_n\mathscr{C}(X, \mathbb{P}) \neq 0$ for $n \geq 2$. Moreover, this shows the homotopy groups of $\mathscr{C}(X, \mathbb{P})$ are at least as complicated as those of $\mathbb{P}$, which are not all known. The complicated nature of these homotopy groups are part of the difficulty in constructing an accessible proof for POP when X is an open Riemann surface.

Appendix A

The parametric Dolbeault lemma

Dolbeault's lemma (Lemma 2.12) is a fundamental result in complex analysis which, in one dimension, states that for a given smooth $(0,1)$-form $\omega \in \mathscr{E}^{(0,1)}(X)$ on an open Riemann surface X there is a smooth function $u \in \mathscr{E}(X)$ with $\bar{\partial} u = \omega$.

In [2, p. 104], Forster gives the construction of the unique solution (up to addition of a holomorphic function) to the $\bar{\partial}$-equation on the complex plane that goes to 0 at infinity when the right hand side has compact support. Given $g : \mathbb{C} \to \mathbb{C}$ smooth with compact support, the unique solution is given by the integral formula

$$\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{z - \zeta} dz \wedge d\bar{z} \,.$$

To see this solution is unique, suppose there is another solution that goes to 0 at infinity. Then the difference of these solutions would be holomorphic on $\mathbb{C}$ and go to 0 at infinity, and is thus the constant 0 by Liouville's theorem. This solution is then extended by an exhaustion process to arbitrary smooth functions $\mathbb{C} \to \mathbb{C}$.

We wish to follow this construction and generalise Dolbeault's lemma to a continuous family of smooth functions. To understand what we mean by this, consider the following definition (where $\mathscr{E}(X) \subset \mathscr{C}(X)$[1] is endowed with the subspace topology and $\mathscr{C}(X)$ carries the compact-open topology).

Definition A.1. Let X be a Riemann surface and P a Hausdorff topological space. Then a *continuous family of smooth functions* on X parametrised by P is a continuous map $g : P \to \mathscr{E}(X)$, that is, a continuous map $X \times P \to \mathbb{C}$ such that $g_t = g(t) : X \to \mathbb{C}$ is smooth for all $t \in P$.

Lemma A.2. *Let P be a locally compact Hausdorff space and $g : P \to \mathscr{E}(\mathbb{C})$ be a continuous family of smooth functions with compact support, that is, for each $t \in P$, g_t*

[1] Here, $\mathscr{C}(X)$ denotes the set of continuous functions $X \to \mathbb{C}$.

has compact support. Suppose that for any compact set $K \subset P$, the set

$$\bigcup_{t \in K} \operatorname{supp} g_t$$

is relatively compact in $\mathbb{C}$. Then there is a continuous family of smooth functions $f : P \to \mathscr{E}(\mathbb{C})$ such that $\bar{\partial} f_t = g_t$ for each $t \in P$.

Proof. Define the family f_t by

$$f_t(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g_t(z)}{z - \zeta} dz \wedge d\bar{z} \,.$$

We wish to show the following three properties:

1. $f_t \in \mathscr{E}(\mathbb{C})$ for each $t \in P$,
2. $\bar{\partial} f_t = g_t$ for each $t \in P$,
3. f_t is continuous in t.

By [2, p. 104] we know that conditions (1) and (2) are satisfied for each $t \in P$, so consider the problem of continuity.

Fix $t_0 \in P$ and let $K \subset \mathbb{C}$ be compact. It suffices to show that for all $\epsilon > 0$, $\|f_t - f_{t_0}\|_K < \epsilon$ whenever t is sufficiently close to t_0, where $\|\cdot\|_K$ is the supremum norm on K. We see that

$$\begin{aligned}
\|f_t - f_{t_0}\|_K &= \sup_{\zeta \in K} |f_t(\zeta) - f_{t_0}(\zeta)| \\
&= \sup_{\zeta \in K} \left| \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g_t(z) - g_{t_0}(z)}{z - \zeta} dz \wedge d\bar{z} \right| \\
&= \sup_{\zeta \in K} \left| -\frac{1}{\pi} \iint (g_t(\zeta + re^{i\theta}) - g_{t_0}(\zeta + re^{i\theta})) e^{-i\theta} dr d\theta \right| ,
\end{aligned}$$

by making the substitution $z = \zeta + re^{i\theta}$. By local compactness of P there is a compact neighbourhood V of t_0. By the assumption $\bigcup_{t \in V} \operatorname{supp} g_t \Subset \mathbb{C}$, there is R sufficiently large such that $g_t(\zeta + re^{i\theta}) = g_{t_0}(\zeta + re^{i\theta})$ outside the rectangle $[0, R] \times [0, 2\pi]$, for all $t \in V$. Using this gives

$$\begin{aligned}
\|f_t - f_{t_0}\|_K &\leq \sup_{\zeta \in K} \frac{1}{\pi} \iint \left| g_t(\zeta + re^{i\theta}) - g_{t_0}(\zeta + re^{i\theta}) \right| dr d\theta \\
&\leq \frac{1}{\pi} \iint \sup_{\zeta \in K} \left| g_t(\zeta + re^{i\theta}) - g_{t_0}(\zeta + re^{i\theta}) \right| dr d\theta \\
&\leq 2R \sup_{\zeta \in K} \left| g_t(\zeta + re^{i\theta}) - g_{t_0}(\zeta + re^{i\theta}) \right| \\
&= 2R \, \|g_t - g_{t_0}\|_K \,.
\end{aligned}$$

By continuity of the family g_t there is a neighbourhood $U \subset V$ of t_0 such that $\|g_t - g_{t_0}\|_K < \frac{\epsilon}{2R}$ whenever $t \in U$. Thus we have $\|f_t - f_{t_0}\|_K < \epsilon$ whenever $t \in U$. This shows the family f_t is continuous at t_0, and as $t_0 \in P$ was arbitrary this shows that f_t is a continuous family. □

Lemma A.3. *Let X be an open Riemann surface and $K \subset X$ a compact subset. Then there is an open cover $(U_j)_{j=1}^N$ of K, a finer open cover $(V_j)_{j=1}^N$ and holomorphic functions $\psi_j : X \to \mathbb{C}$ such that*

(1) $\psi_j|_{U_j} \to \mathbb{D}$ *is a biholomorphism,*

(2) $\psi_j|_{V_j} \to \frac{1}{2}\mathbb{D}$ *is a biholomorphism,*

(3) $\psi_j \left(\bigcup_{k=1}^N U_k \backslash U_j \right) \cap \mathbb{D} = \emptyset.$

This lemma was brought to our attention by the work of John in [15]. The following proof is the same given by John and is included for completeness.

Proof. For every $z \in X$ there is a function $\psi'_z : X \to \mathbb{C}$ with a simple zero at z and no other zeros by Weierstrass' theorem (Theorem 2.9). Also, since z is a simple zero of ψ'_z, the inverse function theorem gives us a relatively compact open neighbourhood U'_z of z such that $\psi'_z|_{U'_z}$ is a biholomorphism onto its image. Since $\psi'_z(U'_z) \subset \mathbb{C}$ is open and contains the origin we may take ψ'_z and U'_z such that $\psi'_z(U'_z) = \mathbb{D}$. Then $G = \bigcup_{z \in K} U'_z$ is relatively compact in X.

Since z is the only zero of ψ'_z, and since G is relatively compact, $\psi'_z(G \backslash U'_z)$ is bounded away from zero, that is, there is $\epsilon_z \in (0, 1]$ such that

$$|\psi'_z(x)| > \epsilon_z \text{ for all } x \in G \backslash U'_z \,.$$

Define $\psi_z : X \to \mathbb{C}$ by

$$\psi_z = \frac{1}{\epsilon_z} \psi'_z \,.$$

Then $|\psi_z(x)| > 1$ for all $x \in G \backslash U'_z$ and $\mathbb{D} \subset \psi_z(U'_z)$. Let $U_z = \psi_z^{-1}(\mathbb{D})$ and $V_z = \psi_z^{-1}(\frac{1}{2}\mathbb{D})$. Since K is compact, there are finitely many $z_1, \ldots, z_N \in K$ with $K \subset V_{z_1} \cup \cdots \cup V_{z_N}$. Then letting $U_j = U_{z_j}$, $V_j = V_{z_j}$ and $\psi_j = \psi_{z_j}$ gives the collections $(U_j)_{j=1}^N$, $(V_j)_{j=1}^N$ and $(\psi_j)_{j=1}^N$ which satisfy the conditions of the lemma. □

Definition A.4. Let X be a Riemann surface and P a Hausdorff topological space. A *continuous family of smooth* $(0,1)$*-forms* on X is a map $\omega : P \to \mathscr{E}^{(0,1)}(X)$ such that for any chart (U, φ) on X, the map $(\varphi^{-1})^*\omega : P \to \mathscr{E}(\varphi(U))$ is given by $(\varphi^{-1})^*\omega_t = \omega'_t d\bar{z}$ for a continuous family of smooth functions $\omega'_t : \varphi(U) \to \mathbb{C}$ as defined in Definition A.1.

Lemma A.5. *Let X be an open Riemann surface, $Y \Subset X$ an open subset and let P be a locally compact Hausdorff space. Then for any continuous family of smooth $(0,1)$-forms $\omega : P \to \mathscr{E}^{(0,1)}(X)$ there is a continuous family of smooth functions $u : P \to \mathscr{E}(Y)$ with $\bar{\partial} u_t = \omega_t$ for each $t \in P$.*

Proof. Take open covers $(U_j)_{j=1}^N$, $(V_j)_{j=1}^N$ of $\bar{Y}$ and holomorphic functions $\psi_j : X \to \mathbb{C}$ as in Lemma A.3. Let $G = \bigcup_{j=1}^N U_j$ and $G_0 = \bigcup_{j=1}^N V_j$. Choose smooth functions $\varphi_j : X \to [0,1]$ such that $\operatorname{supp} \varphi_j \subset U_j$ and $\sum \varphi_j = 1$ on G_0.

Define

$$\omega_{t,j} = \begin{cases} \varphi_j \omega_t & \text{on } U_j\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\omega_{t,j}$ is a continuous family of smooth $(0,1)$-forms that can be written locally as

$$\omega_{t,j} = \psi_j^*(w_{t,j} d\bar{z})$$

for a continuous family of smooth functions $w_{t,j} : \mathbb{D} \to \mathbb{C}$. Since $\operatorname{supp} w_{t,j} \subset \mathbb{D}$ for each $t \in P$, we may consider $w_{t,j} : \mathbb{D} \to \mathbb{C}$ as a continuous family of compactly supported smooth functions $w_{t,j} : \mathbb{C} \to \mathbb{C}$. By Lemma A.2 there is a continuous family of smooth functions $g_{t,j} : \mathbb{C} \to \mathbb{C}$ with

$$\frac{\partial g_{t,j}}{\partial \bar{z}} = w_{t,j}$$

for each $t \in P$. Define $\eta_{t,j} : G \to \mathbb{C}$ by $\eta_{t,j} = (\psi_j|_G)^* g_{t,j}$. Then $\eta_{t,j}$ is continuous in t since $g_{t,j}$ is, and $\eta_{t,j}$ is smooth for each $t \in P$ since ψ_j is holomorphic. Noting $g_{t,j}$ is holomorphic outside $\operatorname{supp} w_{t,j}$ and thus outside $\mathbb{D}$, and that $\psi_j(G \backslash U_j) \cap \mathbb{D} = \varnothing$ by Lemma A.3, we have $\eta_{t,j}$ is holomorphic on $G \backslash U_j$. This gives then that $\bar{\partial} \eta_{t,j} = \omega_{t,j}$ on G.

Finally setting $u_t = \sum_{j=1}^N \eta_{t,j}$ we have a continuous family of smooth functions such that

$$\bar{\partial}\left(u_t|_{G_0}\right) = \omega_t|_{G_0}\,,$$

and in particular, $\bar{\partial} u_t = \omega_t$ on Y. □

We now approach the main result of this section, the Dolbeault lemma with parameters. The following theorem gives a generalisation of the classic Dolbeault lemma on an open Riemann surface with quite a general parameter space, with the only requirements being the Hausdorff property, local compactness and paracompactness. This means we can solve the Dolbeault equation with a continuous family of smooth functions parametrised by $\mathbb{R}$, the spheres S^n for all $n \in \mathbb{N}$, and in fact for any manifold or possibly much more general space. This theorem was proved with hopes to find an explicit proof of the parametric Oka principle in the case of an open Riemann surface mapping into $\mathbb{C}^*$, however this was fruitless.

Theorem A.6 (The parametric Dolbeault lemma)**.** *Let X be an open Riemann surface and let P be a locally compact, paracompact, Hausdorff space. Then for any continuous family of smooth $(0,1)$-forms $\omega : P \to \mathscr{E}^{(0,1)}(X)$ there is a continuous family of smooth functions $u : P \to \mathscr{E}(X)$ with $\bar{\partial} u_t = \omega_t$ for each $t \in P$.*

Proof. Let $Y_1 \Subset Y_2 \Subset \cdots \Subset X$ be an exhaustion of X by relatively compact Runge domains. We know by Lemma A.5 that on each domain Y_n we can find a continuous family of smooth functions $g^{(n)} : P \to \mathscr{E}(Y_n)$ such that $\bar{\partial} g_t^{(n)} = \omega_t|_{Y_n}$ and thus we have a sequence of solutions $(g_t^{(n)})_{n\in\mathbb{N}}$. We wish to adjust this sequence so that it converges to a solution on X. The aim is to construct a sequence $(f_t^{(n)})_{n\in\mathbb{N}}$ such that:

(1) $f_t^{(n)}$ is a continuous family of smooth functions,

(2) $\bar{\partial} f_t^{(n)} = \omega_t|_{Y_n}$ for every $t \in P$,

(3) $\left\| f_t^{(n+1)} - f_t^{(n)} \right\|_{Y_{n-1}} < 2^{-n}$ for every $t \in P$.

Let $f_t^{(1)} = g_t^{(1)}$ and suppose that $f_t^{(2)}, \dots, f_t^{(n)}$ have been constructed. Since $\bar{\partial} g_t^{(n+1)} = \bar{\partial} f_t^{(n)}$ on Y_n we have that $g_t^{(n+1)} - f_t^{(n)}$ is holomorphic on Y_n. By the Runge approximation theorem [2, p. 200, Theorem 25.5], for each $t \in P$, we have a function $h_t \in \mathscr{O}(Y_{n+1})$ with

$$\left\| g_t^{(n+1)} - f_t^{(n)} - h_t \right\|_{Y_{n-1}} < 2^{-n-1}\,.$$

Fix $t_0 \in P$. Then

$$\begin{aligned}
\left\| g_t^{(n+1)} - f_t^{(n)} - h_{t_0} \right\|_{Y_{n-1}} &= \| g_t^{(n+1)} - g_{t_0}^{(n+1)} - \left(f_t^{(n)} - f_{t_0}^{(n)} \right) \\
&\qquad + g_{t_0}^{(n+1)} - f_{t_0}^{(n)} - h_{t_0} \|_{Y_{n-1}} \\
&\le \left\| g_t^{(n+1)} - g_{t_0}^{(n+1)} \right\|_{Y_{n-1}} + \left\| f_t^{(n)} - f_{t_0}^{(n)} \right\|_{Y_{n-1}} \\
&\qquad + \left\| g_{t_0}^{(n+1)} - f_{t_0}^{(n)} - h_{t_0} \right\|_{Y_{n-1}} \\
&< 2^{-n-2} + 2^{-n-2} + 2^{-n-1} = 2^{-n}
\end{aligned}$$

for all t in a sufficiently small neighbourhood $V_{t_0} \subset P$ of t_0 by continuity of $g_t^{(n+1)}$ and $f_t^{(n)}$. Doing this for each $t \in P$ gives us an open cover $(V_t)_{t\in P}$ of P. Therefore, given any $t \in P$ there is $k \in P$ such that

$$\left\| g_t^{(n+1)} - f_t^{(n)} - h_k \right\|_{Y_{n-1}} < 2^{-n}\,.$$

Since P is paracompact there is a continuous partition of unity $(\chi_k)_{k\in P}$ subordinate to $(V_k)_{k\in P}$. Then $\tilde{h}_t = \sum_{k\in P} \chi_k(t) h_k$ is well defined as this sum is locally finite in t, and is thus

a continuous family of holomorphic functions on Y_{n+1}. Define $f_t^{(n+1)} = g_t^{(n+1)} - \tilde{h}_t$. Then $f_t^{(n+1)}$ is a continuous family of smooth functions and $\bar{\partial} f_t^{(n+1)} = \bar{\partial} g_t^{(n+1)} = \omega_t$ on Y_{n+1}. We also have that

$$\begin{aligned}
\left\| f_t^{(n+1)} - f_t^{(n)} \right\|_{Y_{n-1}} &= \left\| g_t^{(n+1)} - f_t^{(n)} - \tilde{h}_t \right\|_{Y_{n-1}} \\
&= \left\| \sum_{k \in P} \chi_k(t)(g_t^{(n+1)} - f_t^{(n)}) - \sum_{k \in P} \chi_k(t) \tilde{h}_k \right\|_{Y_{n-1}} \\
&\leq \sum_{k \in P} \chi_k(t) \left\| g_t^{(n+1)} - f_t^{(n)} - \tilde{h}_k \right\|_{Y_{n-1}} \\
&< \sum_{k \in P} \chi_k(t) 2^{-n} = 2^{-n}
\end{aligned}$$

as required.

Since every $x \in X$ is contained in almost all Y_n, for each $t \in P$ the pointwise limit $u_t(x) = \lim_{n \to \infty} f_t^{(n)}(x)$ exists, and as in [2, p. 106, p. 201] we see that $u_t : X \to \mathbb{C}$ is a smooth function. Further we have that $\bar{\partial} u_t = \omega_t$ on X, so we wish to show u_t is continuous in t. Let $\epsilon > 0$, $t_0 \in P$ and $K \subset X$ compact. Then there is $N \in \mathbb{N}$ such that $K \subset Y_{N-1}$ and $2^{-N+2} < \epsilon$. On Y_N we may write $u_t = f_t^{(N)} + \sum_{k=N}^{\infty} \left(f_t^{(k+1)} - f_t^{(k)} \right)$. From this we see that

$$\begin{aligned}
\|u_t - u_{t_0}\|_K &\leq \|u_t - u_{t_0}\|_{Y_{N-1}} \\
&= \left\| f_t^{(N)} + \sum_{k=N}^{\infty} \left(f_t^{(k+1)} - f_t^{(k)} \right) - f_{t_0}^{(N)} - \sum_{k=N}^{\infty} \left(f_{t_0}^{(k+1)} - f_{t_0}^{(k)} \right) \right\|_{Y_{N-1}} \\
&\leq \left\| f_t^{(N)} - f_{t_0}^{(N)} \right\|_{Y_{N-1}} + \sum_{k=N}^{\infty} \left\| f_t^{(k+1)} - f_t^{(k)} \right\|_{Y_{N-1}} + \sum_{k=N}^{\infty} \left\| f_{t_0}^{(k+1)} - f_{t_0}^{(k)} \right\|_{Y_{N-1}} \\
&\leq \left\| f_t^{(N)} - f_{t_0}^{(N)} \right\|_{Y_{N-1}} + \sum_{k=N}^{\infty} \left\| f_t^{(k+1)} - f_t^{(k)} \right\|_{Y_{k-1}} + \sum_{k=N}^{\infty} \left\| f_{t_0}^{(k+1)} - f_{t_0}^{(k)} \right\|_{Y_{k-1}} \\
&< \left\| f_t^{(N)} - f_{t_0}^{(N)} \right\|_{Y_{N-1}} + 2 \sum_{k=N}^{\infty} 2^{-k} \\
&= \left\| f_t^{(N)} - f_{t_0}^{(N)} \right\|_{Y_{N-1}} + 2 \left(\frac{1}{2^{N-1}} \right) \\
&< \epsilon - \frac{1}{2^{N-2}} + \frac{1}{2^{N-2}} = \epsilon \,,
\end{aligned}$$

in a sufficiently small neighbourhood of t_0 by (1) and (3). Thus u_t is continuous in t and we have a continuous family of smooth functions $u : P \to \mathscr{E}(X)$ with $\bar{\partial} u_t = \omega_t$ on X for every $t \in P$. $\square$

Appendix B

Abelian Fuchsian groups

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the upper-half plane. It is a well-known fact that $\mathrm{Aut}\,\mathbb{H}$ may be identified as $\mathrm{PSL}_2\mathbb{R}$ by assigning the Möbius transformation $z \mapsto \dfrac{az+b}{cz+d}$, $ad - bc = 1$, to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The classification of discrete abelian subgroups of $\mathrm{PSL}_2\mathbb{R}$, also known as abelian *Fuchsian groups*, is well-known and we include it here for completeness following the exposition of [27]. Our interest in these subgroups is to classify the Riemann surfaces with abelian fundamental group whose universal covering is $\mathbb{H}$ (or equivalently, the unit disc).

We begin by exploring the elements of $\mathrm{PSL}_2\mathbb{R} = \mathrm{SL}_2\mathbb{R}\,/\{\pm I\}$. For an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2\mathbb{R}$ we define an action on $\mathbb{H}$ by

$$Az = \frac{az+b}{cz+d}$$

for all $z \in \mathbb{H}$. One way to classify elements of $\mathrm{PSL}_2\mathbb{R}$ is to consider their fixed points. We see that $Az = z$ if $cz^2 + (d-a)z - b = 0$, so A has at most 2 fixed points if A is not the identity. If $c \neq 0$, then

$$z = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c}\,.$$

Rearranging the discriminant and using the fact $ad - bc = 1$ we find that

$$z = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}\,,$$

thus the multiplicity of z is determined by the absolute trace $|\,\mathrm{Tr}(A)| = |a+d|$. If $|a+d| < 2$ then A has precisely one fixed point in $\mathbb{H}$. If $|a+d| = 2$ then A has precisely one fixed point in $\mathbb{R}$. Finally if $|a+d| > 2$, A has two fixed points in $\mathbb{R}$.

Now consider the case $c = 0$. Then $d = a^{-1}$ and $Az = a^2 z + ab$. Thus $Az = z$ gives the equation $(a^2 - 1)z + ab = 0$. If $a = 1$ then the only solution to this is ∞. Note in this situation $|\operatorname{Tr}(A)| = |a + a^{-1}| = 2$. If $a \neq 1$ then the solutions to this are $z = \frac{ab}{1-a^2}$ and ∞, so A has two fixed points in $\mathbb{R} \cup \{\infty\}$. Again we notice that $|\operatorname{Tr}(A)| = |a + a^{-1}| > 2$. This leads to the following classification.

Definition B.1. Let $A \in \mathrm{PSL}_2\mathbb{R} \setminus \{I\}$.

1. A is *elliptic* if $|\operatorname{Tr}(A)| < 2$, that is, A has precisely one fixed point in $\mathbb{H}$.
2. A is *parabolic* if $|\operatorname{Tr}(A)| = 2$, that is, A has precisely one fixed point in $\mathbb{R} \cup \{\infty\}$.
3. A is *hyperbolic* if $|\operatorname{Tr}(A)| > 2$, that is, A has precisely two fixed points in $\mathbb{R} \cup \{\infty\}$.

Lemma B.2. *Let $A \in \mathrm{PSL}_2\mathbb{R} \setminus \{I\}$. Then A is conjugate in $\mathrm{PSL}_2\mathbb{R}$ to one of the following three standard forms:*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

depending on whether A is elliptic, parabolic or hyperbolic respectively, where $x, \lambda > 0$, $\lambda \neq 1$.

Proof. First suppose A is elliptic with fixed point $z_0 \in \mathbb{H}$. Let $C \in \mathrm{PSL}_2\mathbb{R}$ be such that $Cz_0 = i$. Then i is a fixed point of CAC^{-1}. If $CAC^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we have

$$\frac{ai + b}{ci + d} = i\,,$$

so $(a - d)i + (b + c) = 0$. Thus $a = d$ and $b = -c$, so $ad - bc = a^2 + b^2 = 1$. Picking $\theta \in [0, 2\pi)$ such that $a = \cos\theta$ and $b = \sin\theta$ we have

$$CAC^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now suppose A is parabolic with fixed point $x_0 \in \mathbb{R} \cup \{\infty\}$. Choose $C \in \mathrm{PSL}_2\mathbb{R}$ with $Cx_0 = \infty$. Then ∞ is fixed point of CAC^{-1}. If $CAC^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we have

$$\frac{a\infty + b}{c\infty + d} = \infty\,,$$

which gives $c = 0$. Thus

$$CAC^{-1} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

and we can easily choose another conjugate that normalises a to 1.

Finally suppose A is hyperbolic. Choose an element $C \in \mathrm{PSL}_2\mathbb{R}$ taking the fixed points to 0 and ∞. Then by observing $CAC^{-1}\infty = \infty$ and $CAC^{-1}0 = 0$ we obtain $c = b = 0$ and so

$$CAC^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in (0, \infty)\backslash\{1\}$. □

Lemma B.3. *Let $A, B \in \mathrm{Aut}\,\mathbb{H}\backslash\{I\}$ commute. Regarding A and B as elements of $\mathrm{Aut}\,\mathbb{P}$, if A has one fixed point in $\mathbb{P}$, then B has one fixed point in $\mathbb{P}$.*

Proof. Let $z_0 \in \mathbb{P}$ be the fixed point of A. Since A and B commute we have $ABz_0 = BAz_0 = Bz_0$, so Bz_0 is a fixed point of A. Thus $Bz_0 = z_0$ since A has one fixed point, so z_0 is a fixed point of B.

Now suppose $w \in \mathbb{P}$ is a fixed point of B. Then $BAw = ABw = Aw$ and Aw is also a fixed point of B. Then either $Aw = w$ or $Aw = z_0$ since B has at most two fixed points. If $Aw = w$ then $w = z_0$ since A has only one fixed point. If $Aw = z_0 = Az_0$ then $w = z_0$ since A is an automorphism and thus injective. Thus if w is a fixed point of B then $w = z_0$, that is, B has one fixed point. □

Corollary B.4. *Let $A, B \in \mathrm{PSL}_2\mathbb{R}\backslash\{I\}$ commute. Then A and B have the same number of fixed points.*

Lemma B.5. *Let $A, B \in \mathrm{PSL}_2\mathbb{R}\backslash\{I\}$. Then A and B commute if and only if they have the same fixed points in $\mathbb{P}$.*

Proof. Suppose that A and B commute. Then by Corollary B.4, A and B have the same number of fixed points. If A has one fixed point, z_0 say, then z_0 is also a fixed point of B since Bz_0 is a fixed point of A. So suppose A and B have two fixed points. Pick an element $C \in \mathrm{PSL}_2\mathbb{R}$ that takes the fixed points of A to 0 and ∞. Then

$$CAC^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in (0, \infty)\backslash\{1\}$, and

$$CBC^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$. Since $AB = BA$, we have $CAC^{-1}CBC^{-1} = CBC^{-1}CAC^{-1}$, so

$$\begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1}c & \lambda^{-1}d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda^{-1}b \\ \lambda c & \lambda d^{-1} \end{pmatrix}.$$

Thus $\lambda^{-1}b = \lambda b$ and $\lambda^{-1}c = \lambda c$. Since $A \neq I$, $\lambda \neq 1$ and so $b = c = 0$. Hence CBC^{-1} fixes 0 and ∞ also and it follows that A and B fix the same points.

Now suppose A and B have the same fixed point set. Then we can conjugate by an element $C \in \mathrm{PSL}_2\mathbb{R}$ such that CAC^{-1} and CBC^{-1} are both of one of the three standard forms:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Clearly these standard forms commute with elements of the same type, and so it follows that A and B commute also. □

Theorem B.6. *Let $G < \mathrm{PSL}_2\mathbb{R}$ be a discrete abelian subgroup. Then G is cyclic.*

Proof. Since G is abelian, every element of G has the same fixed point set. Thus we can pick an element $C \in \mathrm{PSL}_2\mathbb{R}$ such that the conjugate subgroup CGC^{-1} has one of the following three forms:

$$\left\{ \begin{pmatrix} \cos\theta_g & -\sin\theta_g \\ \sin\theta_g & \cos\theta_g \end{pmatrix} : g \in G \right\}, \left\{ \begin{pmatrix} 1 & x_g \\ 0 & 1 \end{pmatrix} : g \in G \right\}, \left\{ \begin{pmatrix} \lambda_g & 0 \\ 0 & \lambda_g^{-1} \end{pmatrix} : g \in G \right\}.$$

Now, since conjugation in $\mathrm{PSL}_2\mathbb{R}$ preserves the topological group structure, CGC^{-1} is still a discrete abelian subgroup, so let us consider each of these groups in turn.

First suppose G is an elliptic subgroup, so

$$CGC^{-1} = \left\{ \begin{pmatrix} \cos\theta_g & -\sin\theta_g \\ \sin\theta_g & \cos\theta_g \end{pmatrix} : g \in G \right\}.$$

We can define an injective group homomorphism $CGC^{-1} \to S^1$ by $g \mapsto \cos\theta_g + i\sin\theta_g$ to get a discrete subgroup of $S^1 \subset \mathbb{C}$. Since the discrete subgroups of S^1 are all finite abelian and thus cyclic, it follows that CGC^{-1}, and hence G, is cyclic also.

Now suppose G is a parabolic subgroup, so

$$CGC^{-1} = \left\{ \begin{pmatrix} 1 & x_g \\ 0 & 1 \end{pmatrix} : g \in G \right\}.$$

Again we define an injective group homomorphism $CGC^{-1} \to (\mathbb{R}, +)$, $g \mapsto x_g$, to get a discrete abelian subgroup of $\mathbb{R}$. Since the discrete abelian subgroups of $\mathbb{R}$ are cyclic, G must be cyclic.

Finally, if G is a hyperbolic subgroup then

$$CGC^{-1} = \left\{ \begin{pmatrix} \lambda_g & 0 \\ 0 & \lambda_g^{-1} \end{pmatrix} : g \in G \right\}.$$

We then define an injective group homomorphism $CGC^{-1} \to ((0,\infty), \cdot)$, $g \mapsto \lambda_g^2$, to get a discrete abelian subgroup of $(0,\infty)$. Since the discrete abelian subgroups of $(0,\infty)$ are cyclic, G must be cyclic. This completes the proof. □

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