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ON THE MOTION OF AN ELECTRON IN SPATIALLY DEPENDENT ELECTROMAGNETOSTATIC FIELDS

by

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SUMMARY

This thesis is concerned with finding exact drift velocity expressions and trajectories for the motion of an electron in spatially dependent magnetic and electromagnetic field configurations.

Electron motion in static magnetic fields will be analysed first. The fields will be dependent on x and pointing in the Z direction. Exact drift velocities will be obtained for electron motion in an exponentially varying magnetic field. Such a field is monotonically increasing with x and pointing in the positive Z direction. Drift expressions will also be found for a magnetic field with a power law dependence, which is slightly more complicated than the first case. For x greater than zero the magnetic field may be either monotonically increasing or decreasing. This is governed by the power law dependence, α, being greater or less than zero. The drift velocity expressions obtained are compared with the Alfven drift velocity results in the limit of an adiabatically affected magnetic field. The exact drift expressions simplify to the perturbation results of Alfven.

The trajectories for the motion of an electron in the abovementioned magnetostatic fields will also be found, both for bound and unbound orbits.

The motion of an electron in a sinusoidal magnetic field varying with x will be analysed and it will be shown that the exact drift results degenerate to the Alfven drift velocity when the field is adiabatically affected. Trajectories for both bound and unbound orbits will again be considered.

All the drift velocity results obtained are in terms of well known functions of mathematical physics. The integral expressions obtained for the trajectories are found to be incomplete forms of the integrals obtained

in the drift velocity expressions.

In dealing with electromagnetic phenomena spatially dependent electromagnetic fields will be considered. It will be shown that if the electric scalar potential and the magnetic vector potentials have the same functional dependence, then for bound orbits the electron moves with a generalized electric drift velocity combined with an exact magnetic drift expression. Similar trajectory results exist for unbound orbits.

Tractable results will be shown to exist for field configurations in which the electric scalar potential varies as the square of the magnetic vector potential. Solutions to the electron motion will be shown to vary greatly with a parameter Ω which is dependent on the constant δ relating the scalar and vector potentials. It will then be shown how the integrals change when the speed of the electron approaches the speed of light. To illustrate these three results a magnetic field with a simple exponential dependence on x together with the corresponding electric fields will be used.

Further generalizations of the work by piecewise smoothing techniques will be indicated. It will also be shown how this work may be used in upper atmospheric physics, especially in the realm of electron motion in the magnetic tail of the earth. The applicability of the work to problems in laboratory physics will also be discussed, and it will be shown that for special cases the above results may be used to describe electron motion in the meridian plane of an axially symmetric field.

DECLARATION

This thesis contains no material which has been accepted for the award of any degree, and to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

Michael Headland

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CHAPTER 1

1.1 Introduction

Investigation of the non relativistic motion of individual charged particles in static inhomogeneous magnetic fields was initially stimulated by processes occuring in the magnetic field of the earth, particularly auroral displays in the polar regions.

Over a period of fifty years from about 1907, Carl Störmer published many papers in which he discussed the trajectories of protons in the terrestrial magnetic field. It was shown that although a completely analytical solution could not be found, the two constants of the motion, the energy and the canonical angular momentum of the particle, could be used to describe the path of the proton. Störmer showed that the particle was excluded from "forbidden regions" determined by the angular momentum of the proton, and that the path of the orbit could be analysed intuitively in terms of an effective potential. This potential was also related to the angular momentum of the particle. Trajectories of the motion of the particle, however, were obtained only by numerical integration of the equation of motion, as there was no third integral of motion. Analytical solutions were found for the special case of particles moving within the equatorial plane of the dipole field, and these results were expressed in terms of elliptic integrals. Störmer's work on this special example of axially symmetric field was summarized in his monograph, "The Polar Aurora" (1955) which has been reviewed by several authors. (Chamberlain; Ferraro and Plumpton).

As no exact analytical solution to this important problem was found, Alfven (1940) used perturbation techniques to describe charged particle motion in a magnetic field where the change in magnetic field strength with respect to the radius of gyration of the particle was much smaller than the actual strength of the magnetic field. Such a magnetic field was said to be 'adiabatically affected'.

Whilst the approximation was invalid for high energy cosmic rays, Alfven pointed out that it was a good assumption for many ionospheric situations in which the speed of the charged particle was relatively low. The perturbation of the homogeneous magnetic field generated an average drift of the zero order circular motion in a direction perpendicular to both the direction of the magnetic field and the gradient of the magnetic field strength. The magnetic dipole moment of the charged particle in its almost circular orbit was shown to be an adiabatic invariant within the limits of perturbation analysis. (Alfven 1955).

To describe static magnetic field situations in the ionosphere where an electric field is present, the most important result is the zero order electric drift velocity caused by spatially constant electric and magnetic fields at right angles. This simple drift expression also describes a particle drift caused by a forcefield, since the force field can be directly related to an equivalent electric field. A typical drift of this type in nature is that of a charged particle moving under the influence of the gravitational field of the earth in the magnetosphere. (Hess, Roederer).

The electric drift velocity is also used to describe drift processes due to charged particle motion along curved magnetic field lines, since as the particle moves along the curved field lines it experiences a centrifugal force. This force can be used to describe the curvature drift, in which the charged particle drifts in a direction perpendicular to both the direction of the field, and the radius of curvature of the magnetic field. Many authors have also used a similar mechanism to describe the gradient drift, the necessary force term being related to the magnetic dipole moment of the charged particle and the gradient of the magnetic field strength.

To obtain this magnetic force, Allis (1956), Lehnert (1964) and Schmidt (1966) assumed that for an adiabatically affected magnetic field the velocity of the charged particle could be represented as a superposition of the zero order circular velocity of a charged particle in a constant magnetic field, plus a first order guiding centre velocity due to the magnetic inhomogeneity. Having substituted this velocity into the equation of motion and used a first order Taylor's series expansion for the magnetic field, an instantaneous force term was obtained. When this force term was averaged over a full periodic orbit of the particle, it yielded the average magnetic force required to produce the gradient drift.

Another method of obtaining the magnetic drift velocity expressions is to assume that for an adiabatically affected field, the position of the charged particle may be represented by a guiding centre moving with an instantaneous drift velocity, about which the particle moves at a radial distance, in a circular orbit. By differentiating the first order expression for the guiding centre an instantaneous guiding centre is obtained. When averaged over a full cycle of the orbit of the charged particle, the instantaneous terms yield the required magnetic drift velocity expressions, with the cyclic component averaging to zero.

Rose and Clarke (1961) used this approach to analyse the gradient drift, whilst Seymour (1963) and Sivukhin (1965) obtained both the curvature and the gradient drift expressions using a curvilinear coordinate system in which the magnetic field was directed along the unit tangent vector of the Frenet-Serret trihedron.

Of particular interest in the present analysis is a magnetic field with straight and parallel magnetic field lines in the Z direction dependent on x. Alfven (1940) first studied the problem of charged particle motion in an adiabatically affected magnetic field using this particular type of field. He found the average motion of the charged particle in the X-Y plane by averaging the velocity components in the X and Y directions over a complete orbit with the aid of first order approximations for the magnetic field and the radius of gyration of the particle. The integrations were carried out with respect to the angle ψ at which the tangent to the trajectory intersected the X axis. Whilst the average drift along the X axis was zero, (being in the direction of the magnetic gradient) it was shown that there was a drift along the Y axis related to the magnetic dipole moment of the charged particle, and the gradient of the magnetic field.

Chandrasekhar (1960), Longmire (1963) and Kruskal (1963) discussed the problem by using the result that the average effect of the magnetic force in the direction of the gradient of the magnetic field along the X axis was zero. Thus with the aid of a first order approximation for the magnetic field they arrived at the same drift velocity expression first found by Alfven. Spitzer (1956) in the first edition of his book "Physics of Fully Ionized Gases" stated the Alfven drift velocity result and suggested that such orbits could be analysed only by perturbation techniques.

In answer to this Seymour (1959) chose to investigate the particular case of a magnetic field with a constant gradient $B_z = \lambda x$, and solved the equation of motion to obtain exact drift velocity expressions in terms of elliptic integrals. Spitzer (1962) in the second edition of the above-mentioned book stated, "The drift velocity expression can now be found in general only by means of an approximate theory". It will be shown in this thesis, that for this specific cartesian geometry, general exact solutions for the path of the particle can be obtained.

Schmidt and others have shown that three constants of motion exist for charged particle motion in electric and magnetic fields whose scalar and vector potentials respectively are dependent on x only. For the special case of an electrostatic field of this form solutions to the equation of motion are well documented. For example Landau and Lifschitz (1969) have solved this problem for specific functional dependences of electrostatic potentials, such as for example a power law dependence.

In the case of magnetostatic fields the Seymour (1959) problem was the first to be investigated. Hertweck (1959) in his analysis of charged particle motion in the magnetic field of an infinitely long current carrying wire has considered another special case of cartesian geometry. He found that for a charged particle confined to move in a fixed plane through the wire, with the magnetic field varying inversely as the distance from the wire, exact solutions were obtained in terms of Bessel functions. At present these are the only two magnetostatic problems solved for a field dependent on x. The only electro-magnetostatic problem of this nature which has been solved is the zero order electric drift result.

A few exact solutions for the motion of charged particles in axially symmetric fields have been studied because of the importance of this field type both in cosmological and laboratory plasmas (Delcroix). Three constants of motion in this particular case can be found when the magnetostatic and electrostatic potentials depend on the radius only. Hurley (1961) has shown that integral expressions describing the motion can be obtained for magnetic fields which are straight and parallel pointing in the Z direction and dependent on the radius. Solutions were obtained for a magnetic field varying inversely as the radius with the aid of an angular variable, θ, relating the radial and angular components of velocity as cosine and sine functions of θ respectively. The drift velocity obtained was a simple algebraic expression dependent on the initial conditions of the orbit of the particle.

Seymour (1971) indicated the physical significance of this angle as being the angle between the direction of the radius vector and the tangent to the trajectory at the tip of that vector. A generalized magnetic field was considered which varied as the power of the radius, $B_z = Ar^{-n}$, recovering Hurley's earlier result for n = 1, and also finding exact drift results for n = 3 in terms of complete elliptic integrals of the first and second kinds.

Morozov and Solovev (Leontovitch Vol. 2) found algebraic drift expressions for the motion of a charged particle within a uniformly charged cylinder located in a constant longitudinal magnetic field. Hertweck (1959), in his analysis of charged particle motion about an infinitely long wire found that in general the drift expression for motion along the wire was not expressible as well known functions of mathematical physics. The integrals were therefore analysed by numerical means. As already mentioned, however, for the special case of charged particle motion in a meridian plane through the wire, the problem degenerated into the special case, $B_z = \lambda/x$, in the cartesian geometry, the solution of which was analytical.

1.2 Discussion of Contents

From the above discussion it is apparent that very few exact analytical solutions for the motion of charged particles in magnetostatic fields exist. This is also true for charged particle motion in electromagnetostatic fields. Whilst general solutions exist for time dependent magnetic fields, (Seymour et al. (1965), Seymour (1966)), and also for time dependent electric fields in an homogeneous magnetic field, (Lehnert 1964, or Appendix 6) at present it appears that no generalized results are available for static fields.

In the present analysis it will be shown that general integral expressions for the time and displacement along the Y axis can be obtained for charged particle motions in electro-magnetostatic fields dependent on x. For magnetostatic fields which are straight, parallel and directed along the Z axis it will be shown that certain functional forms of the magnetic field lead to solutions of the motion of the charged particle in terms of well known functions of mathematical physics.

In Chapter 2 the three constants of motion found from the Lorentz force law will be utilized to obtain generalized integral expressions of the time and coordinate values for different combinations of electric and magnetic potentials. The integrals for electron motion in a magnetostatic field will be evaluated using the angle ψ at which the tangent to the trajectory intersects the X axis. Although this restricts the generality of the integrals for time and spatial coordinates it gives much physical insight into the analysis. For particular forms of magnetic field the use of the angle ψ makes the integrals more readily interpreted as well known functions of mathematical physics. The magnetostatic results will be used to describe electro-magnetostatic fields in which the electric and magnetic fields have the same functional dependence. This will lead to a generalized electric and magnetic drift expression for these spatially varying fields. Similarly, if the electric scalar potential is dependent on the square of the magnetic potential, the integrals for the time and displacement obtained are closely related to the integrals found in the magnetostatic case, but modified by a constant Ω. This constant contains the parameter which relates the electric potential and the square of the magnetic potential.

Exact drift velocity results for electrons will be found in Chapter

3. (Motion for protons will be in the opposite direction). The first magnetic field to be considered will be an exponentially varying field ($B_z = \lambda \ell^{\alpha x}$). This is always positive and monotonically increasing with x. Thus for orbits which are bound between a minimum value x_1 and a maximum value x_2 the electron will always turn in the same direction. The bound orbits will therefore be completely cyclic with ψ continuously increasing. Drift velocity solutions are expressed as a simple algebraic expression dependent on the initial conditions of the electron's orbit. A more complicated field to be considered is one with a power law dependence on x, ($B_z = \lambda x^{\alpha}$). For x always positive the magnetic field will be monotonically varying with x. Thus for bound orbits which do not cut the line $x = 0$, ψ will continuously increase and the orbits will be full cycles. If α is negative the electron moves in the positive Y direction. If α is positive and the orbit cuts the line $x = 0$, then for α an even integer the magnetic field is in the same direction on either side of the line $x = 0$, and the electron will again move through ever increasing values of the angle ψ. For α a positive odd integer the magnetic field on either side of the line $x = 0$ will be in opposite directions. Thus the electron will turn in opposite directions on either side of the zero line. Bound orbits will therefore not complete a revolution of ψ but the orbit angle will oscillate between the two angles $\Pi - \psi_o$ and ψ_o at which the electron leaves and re-enters the region of positive magnetic field. The path shape of the electron on either side of the line $x = 0$ is the same. For $\alpha \neq -1$ exact drift velocity expressions will be in terms of hypergeometric functions and gamma functions. For $\alpha = -1$ solutions to the motion of the electron will be in terms of confluent hypergeometric functions. This particular example will be compared with results obtained by Hertweck (1959).

If the electron's path is not bound by x and has no periodic nature, the concept of an exact drift velocity becomes quite meaningless. Thus to follow such an orbit it will be necessary to plot the path of the electron.

Indeed, if for some orbits more information is required about the orbit than a drift velocity expression then trajectories may be plotted. In Chapter 4 it will be pointed out how this may be achieved. Electron paths will be analysed for the two different forms of magnetic field in Chapter 3. For bound orbits the integral expressions for time and values of x and y will be "incomplete" forms of those obtained for the total increments as the particle motion is integrated to an arbitrary angle ψ between the limits of a cycle, and not the complete cycle.

For unbound orbits x becomes infinite as the electron moves along the Y axis and the electron asymptotes towards a limiting angular value ψ'. For the exponential magnetic field, logarithmic expressions for the path of the orbit are obtained which are dependent on ψ and ψ'. Similarly for a power law magnetic field in which $\alpha < -1$ the unbound orbits also give rise to points on the orbit related to ψ and the limiting angle ψ'.

In Chapter 5 electron motion in a sinusoidally varying magnetic field ($B_z = \lambda \sin \alpha x$) will be discussed. Due to the confining nature of the magnetic field within a full cycle of varying field strength $\left(-\frac{\pi}{\alpha} < x < \frac{\pi}{\alpha}\right)$ bound orbit drift expressions will be dependent on the maximum and minimum values of x. For orbits which are confined to move within a half cycle of magnetic variation, ψ will turn through 2π radians as the field is always in the same direction. If the electron crosses the line $x = 0$ it will go into a region of reversed magnetic field. The electron will therefore have the same path shape on either side of the line $B_z = 0$ as the magnetic field is antisymmetric about that line. The drift velocity expressions for these two types of motion will be in terms of elliptic integrals of the first and third kinds.

Orbits which penetrate the full cycle of varying magnetic field are no longer bound by x. They are however periodic in nature as the orbit shape will be identical in following periods of magnetic fluctuation. This type of motion may be represented by an exact drift velocity expression, not along the Y axis, but at an angle to it, due to the drift component in the direction of the magnetic gradient. As before the drift expressions are found in terms of elliptic integrals of the first and third kinds. The paths for the three types of motion are found in terms of incomplete elliptic integrals of the first and third kinds.

In conclusion electro-magnetostatic phenomena mentioned in Chapter 2 will be discussed in the light of the drift and trajectory analysis of Chapters 3 to 5. The motion of the electron will also be analysed for the case when the speed of the electron approaches the speed of light. For all these cases the one component exponentially varying magnetic field plus the corresponding electric fields will be used to illustrate the results.

CHAPTER 2

2.1 The Equation of Motion of an Electron in Static Electromagnetic Fields

For an electron of mass m, charge -e, positioned at $\underset{\sim}{r}(t)$ = $(x(t),y(t),z(t))$ moving with non-relativistic velocity $\underset{\sim}{v}$, the equation of motion is

$$m \frac{d\underset{\sim}{v}}{dt} = -e\{\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\} \qquad \text{(e.m.u.)}, \qquad \ldots(2.1)$$

where, since the fields are static,

$$\underset{\sim}{E} = -\nabla\phi \ , \qquad \ldots(2.2)$$

and

$$\underset{\sim}{B} = \text{curl } \underset{\sim}{A} \ , \qquad \ldots(2.3)$$

ϕ being the electric scalar potential, and $\underset{\sim}{A}$ the magnetic vector potential. Multiplying both sides of equation (2.1) by the velocity in the standard way, the conservation of energy

$$\varepsilon = \frac{1}{2} mv^2 - e\phi = \text{energy}$$

$$= \frac{1}{2} m \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\} - e\phi \ , \qquad \ldots(2.4)$$

is obtained.

2.2 Magnetic and Electric Potentials Dependent on x

If the electric scalar potential in equation (2.2) is a function of x only,

$$\phi = \phi(x) \ , \qquad \ldots(2.5)$$

then from equation (2.2)

$$\underset{\sim}{E} = (E(x),0,0) \ , \qquad \ldots(2.6)$$

where $$E(x) = -\frac{d\phi(x)}{dx} \quad . \qquad ...(2.6)$$

Similarly if the vector potential is chosen as a function of x only, and is of the form

$$\underset{\sim}{A} = (0, A_y(x), A_z(x)) , \qquad ...(2.7)$$

then from equation (2.3) the components of the magnetic field are

$$B_y(x) = -\frac{dA_z(x)}{dx} , \qquad ...(2.8)$$

$$B_z(x) = \frac{dA_y(x)}{dx} . \qquad ...(2.9)$$

The magnetic field is then of the form

$$\underset{\sim}{B} = (0, B_y(x), B_z(x)) , \qquad ...(2.10)$$

consistent with Maxwell's equation, div $\underset{\sim}{B} = 0$.

Using equations (2.6) and (2.10) the x,y and z components of the equation of motion are given by

$$m \frac{dv_x}{dt} = -e\{E(x) + B_z \frac{dy}{dt} - B_y \frac{dz}{dt}\} , \qquad ...(2.11)$$

$$m \frac{dv_y}{dt} = e\{B_z(x) \frac{dx}{dt}\}$$

$$= e \frac{dA_y(x)}{dt} , \qquad ...(2.12)$$

with the help of equation (2.9), and

$$m \frac{dv_z}{dt} = -eB_y(x) \frac{dx}{dt}$$

$$= e \frac{dA_z(x)}{dt} , \qquad ...(2.13)$$

with the aid of equation (2.8).

Equations (2.12) and (2.13) may be integrated immediately to give the Y and Z components of velocity as

$$v_y = \gamma_1 + \frac{e}{m} A_y(x) \ , \qquad \ldots(2.14)$$

$$v_z = \gamma_2 + \frac{e}{m} A_z(x) \ , \qquad \ldots(2.15)$$

where, for $x = x_o$ at time $t = t_o$

$$\gamma_1 = (v_y)_{t=t_o} - \frac{e}{m} A_y(x_o) \ , \qquad \ldots(2.16)$$

$$\gamma_2 = (v_z)_{t=t_o} - \frac{e}{m} A_z(x_o) \ . \qquad \ldots(2.17)$$

Substitution of equations (2.5), (2.14) and (2.15) into equation (2.4) yields for the energy expression

$$\varepsilon = \frac{m}{2} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\gamma_1 + \frac{e}{m} A_y(x)\right)^2 + \left(\gamma_2 + \frac{e}{m} A_z(x)\right)^2 \right\} - e\phi(x), \ldots(2.18)$$

which is dependent on x and its time derivative. From equation (2.18) the differential of time can be found in terms of x and the differential of x. Similarly by using equations (2.14) and (2.15) in conjunction with (2.18) the differentials of y and z can also be obtained. The y and z coordinates, and the time can be found by integrating these differential expressions.

2.3 Specialized Field Types

(i) Electrostatic Fields

If the magnetic field is zero, the components of the magnetic vector potential appearing in equations (2.8) and (2.9) are constant. Therefore the y and z components of the velocity are constants of the motion. Equation (2.18) yields

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(\varepsilon - e\phi(x))}} \ , \qquad \ldots(2.19)$$

where the y and z components of velocity are chosen to be zero.

Landau and Lifschitz and others have integrated equation (2.19) for different functional forms of electric scalar potentials. For example, it has been shown that for $e\phi(x) = A|x|^n$, integration of equation (2.19) leads to solutions for the period of oscillation of the electron in terms of beta functions.

(ii) Magnetostatic Fields

In the absence of an electric scalar potential, it follows from equation (2.4) that the speed of the electron

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 , \qquad \ldots(2.20)$$

is a constant of the motion.

(a) One Component Magnetic Field

For a magnetic field with only one component, $B_z(x)$, the magnetic field lines are straight and parallel and pointing in the z direction. In equation (2.17) the vector potential component $A_z(x)$ is constant. The z component of velocity in equation (2.15) is therefore constant and need not be considered explicitly. Equation (2.20) therefore becomes

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 . \qquad \ldots(2.21)$$

Substituting equation (2.14) into equation (2.21), or by manipulating equation (2.18), with $\phi(x) = v_z = 0$, the time differential becomes

$$dt = \frac{dx}{\sqrt{v^2 - (\gamma_1 + \frac{e}{m} A_y(x))^2}} . \qquad \ldots(2.22)$$

Similarly, using equations (2.14) and (2.22), the differential of y yields

$$dy = v_y dt ,$$

$$= \frac{\left(\gamma_1 + \frac{e}{m} A_y(x)\right) dx}{\sqrt{v^2 - \left(\gamma_1 + \frac{e}{m} A_y(x)\right)^2}} \quad . \qquad \ldots(2.23)$$

The x and y components of the velocity in equation (2.21) can be expressed in terms of the angle, ψ, at which the tangent to the trajectory cuts the x axis. Thus

$$\frac{dx}{dt} = v \cos\psi , \qquad \ldots(2.24)$$

$$\frac{dy}{dt} = v \sin\psi . \qquad \ldots(2.25)$$

While in general it will not be possible to substitute equations (2.24) and (2.25) into equations (2.22) and (2.23), for many important cases equations (2.24) and (2.25) can be used. This leads to expressions in terms of physically well-known quantities. From equations (2.24), (2.25), (2.9) and (2.22) the result

$$\frac{d\psi}{dt} = \omega = \frac{e}{m} B_z , \qquad \ldots(2.26)$$

is obtained, where $\omega = \frac{eB_z}{m}$ is the gyrofrequency of the electron. Equation (2.26) may be integrated to yield

$$t = \int \frac{d\psi}{\omega} \qquad \ldots(2.27)$$

as the time.

Using equations (2.25), (2.26) and (2.23) y becomes

$$y = \int \rho \sin\psi d\psi , \qquad \ldots(2.28)$$

where $\rho = v/\omega$ is the radius of gyration of the electron.

From equations (2.25), (2.14) and (2.16)

$$\sin\psi - \sin\psi_o = \frac{e}{mv}(A_y(x) - A_y(x_o)) \ . \qquad \text{...(2.29)}$$

Thus by using equations (2.29) and (2.9) the magnetic field can be obtained as a function of ψ for particular forms of magnetic potential. Alternatively, if a particular form of magnetic field is chosen, then by using equation (2.9) the magnetic vector potential may be found. In either case, from equation (2.29) x may be expressed as a function of ψ, and the orbit can be analysed in terms of this angle.

(b) Bound Orbits and Exact Drift Expressions

If the electron is bound between a minimum value x_1, and a maximum value x_2 of x, then from equation (2.27) the period for one orbit will be

$$T = \int_{\substack{\text{one}\\ \text{period}}} \frac{d\psi}{\omega} \ . \qquad \text{...(2.30)}$$

From equation (2.25) the drift along the y axis is given by

$$\Delta y = \int_{\substack{\text{one}\\ \text{period}}} \rho\sin\psi d\psi \ , \qquad \text{...(2.31)}$$

and from equation (2.29), $\Delta x = 0$, over one period. The motion of the electron along the y axis is therefore given by the drift expression

$$v_D = \frac{\Delta y}{T} \ . \qquad \text{...(2.32)}$$

If the motion of the electron is not bounded by x, and is not periodic, the trajectory of the electron cannot be described by the average drift expression of (2.32). The concept of a drift velocity for a non-periodic orbit is meaningless. In this case the orbit can only be analysed by plotting the path of the electron. This can be done by using equations (2.27), (2.28) and (2.29).

(c) Homogeneous Magnetic Field

For a constant magnetic field, $B_z = \lambda$, the orbit of the electron is circular and the gyrofrequency of the electron, $\omega_o = \frac{e\lambda}{m}$ is constant. Equation (2.30) gives

$$T_o = \frac{2\pi}{\omega_o} \quad , \qquad \ldots(2.33)$$

where T_o is the well-known period of an electron in an homogeneous magnetic field. The radius of gyration of the electron $\rho_o = v/\omega_o$ is also constant. From equations (2.27), (2.28) and (2.29) the trajectory of the electron becomes

$$\left.\begin{aligned} t(\psi) &= \frac{\psi}{\omega_o} + t_o \ , \\ y(\psi) &= \rho_o \cos\psi + (y_o - \rho_o) \ , \\ x(\psi) &= \rho_o \sin\psi + x_o \ , \end{aligned}\right) \qquad \ldots(2.34)$$

where $x = x_o$ at $\psi = 0$. $\underset{\sim}{r}_o = (x_o, y_o)$ is the point about which the electron circulates with angular frequency ω_o at a radius ρ_o. From equations (2.34) $\Delta y = \Delta x = 0$.

(d) Perturbation of the Zero Order Field

For a homogeneous magnetic field which is perturbed by a magnetic inhomogeneity such that the radius of gyration ρ of the orbit of the electron multiplied by the gradient of the magnetic field is much less than the magnetic field strength, or mathematically

$$\rho|\nabla B| \ll B_z \ , \qquad \ldots(2.35)$$

then the drift expression of equation (2.32) may be approximated by a first-order drift result.

To first order the period, T_o, of the orbit is given by equation (2.33). To find Δy, the magnetic field and the radius of gyration of the electron are approximated by first-order Taylor's series expansions about $x = x_o$.

$$\left.\begin{aligned} B_z &= B_o \left\{1 + \left(\frac{x-x_o}{B_o}\right) \frac{dB_z}{dx}\right\} , \\ \text{and} \qquad \rho(x) &= \rho_o \left\{1 - \frac{(x-x_o)}{B_o} \frac{dB_z}{dx}\right\} . \end{aligned}\right\} \qquad \ldots(2.36)$$

Using equation (2.36), and the zero order value of x from equation (2.34), equation (2.31) becomes

$$\Delta y = \int_o^{2\pi} \rho \sin\psi d\psi$$

$$= \frac{-\pi\rho_o}{B_o} \frac{dB_z}{dx} . \qquad \ldots(2.37)$$

The drift expression first obtained by Alfven (1940) is obtained by dividing equation (2.37) by equation (2.33). This leads to the well known result

$$v_D = \frac{\Delta y}{T_o} = - \frac{\mu}{B} \frac{|\nabla B|}{B} , \qquad \ldots(2.38)$$

or in vector form

$$\underset{\sim}{v}_D = - \frac{\mu}{e} \frac{\underset{\sim}{B} \times \nabla B}{B^2} , \qquad \ldots(2.38')$$

where $\mu = \frac{1}{2} mv^2/B$ is the magnetic dipole moment of the electron. It is found to be an adiabatic invariant of the motion of the electron.

(iii) Magnetic Field in the z Direction with an Electric Field Along the x Axis

(a) General Integral Expressions

The magnetic field in the y direction is zero. Thus from equations (2.8) and (2.15) the z component of the velocity is constant and again need not be considered explicitly. From equations (2.14) and (2.18) the time becomes

$$t = \int \frac{dx}{\sqrt{\frac{2}{m}\varepsilon - \frac{2e}{m}\phi(x) - \left(\gamma_1 + \frac{e}{m}A_y(x)\right)^2}} , \qquad \ldots(2.39a)$$

and y is given by

$$y = \int \frac{\left(\gamma_1 + \frac{e}{m}A_y(x)\right)dx}{\sqrt{\frac{2\varepsilon}{m} - \frac{2e}{m}\phi(x) - \left(\gamma_1 + \frac{e}{m}A_y(x)\right)^2}} . \qquad \ldots(2.39b)$$

This class of electro-magnetostatic fields includes the important example of constant electric and magnetic fields at right angles, in which the electron moves with the electric drift velocity. For the present analysis it will be more instructive to consider a larger group of fields in which there is a simple explicit relationship between the electric and magnetic potentials.

(b) Magnetic Vector Potential with the same Functional Form as the Electric Scalar Potential

If the electrostatic field is given by

$$\phi(x) = \phi_o\, f(x) , \qquad \ldots(2.40)$$

and the magnetostatic vector potential along the y axis is

$$A_y(x) = A_o\, f(x) , \qquad \ldots(2.41)$$

where ϕ_o and A_o are constants, and f(x) is an arbitrary function of x, then equations (2.40) and (2.41) yield

$$\phi(x) = \left(\frac{\phi_o}{A_o}\right) A_y(x) \ . \qquad \ldots(2.42)$$

Using equations (2.6), (2.9), (2.40) and (2.41) the electric and magnetic fields become

$$E(x) = \phi_o \frac{df(x)}{dx} \ , \qquad \ldots(2.43)$$

$$B_z(x) = A_o \frac{df(x)}{dx} \ , \qquad \ldots(2.44)$$

and

$$\frac{E(x)}{B_z(x)} = \frac{\phi_o}{A_o} = \text{constant} \ . \qquad \ldots(2.45)$$

Substituting equations (2.40) and (2.41) into equation (2.39a), the time becomes

$$t = \int \frac{dx}{\sqrt{\frac{2\varepsilon}{m} - \frac{2e}{m}\phi_o f(x) - \left(\gamma_1 + \frac{eA_o}{m} f(x)\right)^2}} \ . \qquad \ldots(2.46)$$

By rearranging f(x) in the denominator of equation (2.46), and using equation (2.45) the time yields

$$t = \int \frac{dx}{\sqrt{\frac{2\varepsilon}{m} - \gamma_1^2 + \gamma_1'^2 - \left(\gamma_1' + \frac{eA_o}{m} f(x)\right)^2}} \ , \qquad \ldots(2.47)$$

where

$$\gamma_1' = \gamma_1 + \frac{E}{B_z} \ . \qquad \ldots(2.48)$$

If a new coordinate system is defined such that

$$v_y' = v_y + \frac{E}{B_z} \ , \qquad \ldots(2.49)$$

$$v_x' = v_x \ , \qquad \ldots(2.50)$$

then in the dashed coordinate system the electron moves as if under the influence of a magnetic field only. From equation (2.47) the speed of the electron in the dashed coordinate system is

$$v'^2 = \frac{2\varepsilon}{m} - (\gamma_1^2 - \gamma_1'^2) \ . \qquad \text{...(2.51)}$$

Using equation (2.51), equation (2.47) may be rewritten

$$t = \int \frac{dx}{\sqrt{v'^2 - \left(\gamma_1' + \frac{e}{m} A_y(x)\right)^2}} \ . \qquad \text{...(2.52)}$$

The integrand has the familiar form of equation (2.22) except for the superscripts denoting the dashed coordinate system.

From equations (2.41), (2.42), (2.45), (2.48), (2.51), (2.52) and (2.39b)

$$y = -\frac{E}{B_z} t + \int \frac{\left(\gamma_1' + \frac{e}{m} A_y(x)\right) dx}{\sqrt{v'^2 - \left(\gamma_1' + \frac{e}{m} A_y(x)\right)^2}} \ . \qquad \text{...(2.53)}$$

The first term in equation (2.53) is a constant electric drift component. The second term is the magnetostatic component in the dashed coordinate system. Equations (2.49) and (2.50) can be rewritten

$$v_y' = v'\sin\psi \ , \qquad \text{...(2.54)}$$

$$v_x' = v'\cos\psi \ , \qquad \text{...(2.55)}$$

where

$$(v_x')^2 + (v_y')^2 = v'^2 \ . \qquad \text{...(2.56)}$$

From equations (2.54), (2.55) and (2.52) the time simplifies to

$$t = \int \frac{d\psi}{\omega} \ , \qquad \text{...(2.57)}$$

where $\omega = \frac{e}{m} B_z(\psi)$ is the gyrofrequency of the electron.

Using equations (2.54), (2.55), (2.57) and (2.53)

$$y = -\frac{E}{B_z}t + \int \rho'\sin\psi d\psi \ , \qquad \ldots(2.58)$$

where $\rho' = v'/\omega$ is the radius of gyration of the electron in the dashed coordinate system. Finally from equations (2.14), (2.49) and (2.54)

$$\sin\psi - \sin\psi_o = \frac{e}{mv'}(A_y(x) - A_y(x_o)) \ , \qquad \ldots(2.59)$$

where $\psi = \psi_o$ at $x = x_o$. Thus from equations (2.57), (2.58) and (2.59) use of the dashed coordinate system has transformed the static electromagnetic problem into a magnetostatic problem in the dashed coordinate system plus the electric drift component of equation (2.58), $-\frac{E}{B_z}t$.

(c) Bound Periodic Orbits

If the electron completes full cycles and is bound between a maximum value x_2, and a minimum value x_1, then over a complete orbit equation (2.57) yields

$$T = \int_{\substack{\text{one}\\ \text{period}}} \frac{d\psi}{\omega} \ , \qquad \ldots(2.60)$$

as the period of the orbit. In that time from equation (2.58) and (2.60) the increment of y becomes

$$\Delta y = -\frac{E}{B_z}T + \int_{\substack{\text{one}\\ \text{period}}} \rho'\sin\psi d\psi \ . \qquad \ldots(2.61)$$

Dividing equation (2.61) by (2.60) gives the drift velocity along the y axis

$$v_D = \frac{\Delta y}{T}$$

$$= -\frac{E}{B_z} + \frac{1}{T}\int_{\substack{\text{one}\\ \text{period}}} \rho'\sin\psi d\psi \ , \qquad \ldots(2.62a)$$

where $v_E = -E(x)/B_z(x)$ is a generalized electric drift velocity, and

$$v_D' = \frac{1}{T}\int_{\substack{\text{one}\\ \text{period}}} \rho' \sin\psi d\psi \ , \qquad \ldots(2.62b)$$

is the magnetostatic drift velocity of the electron in the dashed coordinate system.

(d) Constant Electric and Magnetic Fields

The most important example of such bound orbits is the case of an electric field $\underset{\sim}{E} = E_o \underset{\sim}{i}$ and magnetic field $\underset{\sim}{B}_z = \lambda \underset{\sim}{k}$ where E_o and λ are constants. Equation (2.57) becomes

$$t = \frac{\psi}{\omega_o} + t_o \ , \qquad \ldots(2.63)$$

where $\psi_o = 0$ at $t = t_o$, and $\omega_o = e\lambda/m$ is the zero order gyrofrequency. Using $\rho_o' = v/\omega_o$ and equation (2.63), equation (2.58) becomes

$$y = -\frac{E}{B_z}\left(\frac{\psi}{\omega_o} + t_o\right) + \rho_o' \cos\psi + (y_o - \rho_o') \ , \qquad \ldots(2.64)$$

with $y = y_o$ at $t = t_o$.

From equation (2.9) the magnetic vector potential due to a constant magnetic field is $A_y(x) = \lambda x$. Substituting this into equation (2.59) the value of x becomes

$$x = \rho_o' \sin\psi + x_o \ , \qquad \ldots(2.65)$$

where $x = x_o$ when $t = t_o$.

Integration of equation (2.63) over a full cycle of the orbit of the electron leads to the zero order result $T_o = 2\pi/\omega_o$. From equation (2.64) the electron moves a distance

$$\Delta y = -\frac{E}{B_z} T_o \ , \qquad \ldots(2.66)$$

along the y axis in time T_o.

Dividing equation (2.66) by the periodic time T_o gives the well known drift expression

$$v_D = \frac{\Delta y}{T_o} = -\frac{E}{B_z} , \qquad \ldots(2.67)$$

or vectorially

$$\underset{\sim}{v}_D = \frac{\underset{\sim}{E} \times \underset{\sim}{B}_z}{B_z^2} . \qquad \ldots(2.67')$$

This result is used extensively to describe many ionospheric and laboratory situations in which there are crossed electric and magnetic fields.

(e) Electric Scalar Potential Related to the Square of the Magnetic Vector Potential

Analytical solutions to the motion of an electron in static electro-magnetic fields exist if the relationship between the electric and magnetic potentials is

$$\phi(x) = \delta A_y^2(x) , \qquad \ldots(2.68)$$

where δ is a constant. From equations (2.18) and (2.68) the energy of the electron becomes

$$\frac{2\varepsilon}{m} = \left(\frac{dx}{dt}\right)^2 + \left(\gamma_1 + \frac{e}{m} A_y(x)\right)^2 - \frac{2e}{m}\phi(x)$$

$$= \left(\frac{dx}{dt}\right)^2 + \gamma_1^2 + \frac{2e}{m}\gamma_1 A_y(x) + \left(\frac{e}{m}\right)^2 \Omega A_y(x) , \qquad \ldots(2.69)$$

where

$$\Omega = 1 - \frac{2m\delta}{e} . \qquad \ldots(2.70)$$

The energy expression (2.69), on rearranging terms yields (for $\Omega \neq 0$)

$$\frac{2\varepsilon}{m} + \frac{2\gamma_1^2 m\delta}{e\Omega} = \left(\frac{dx}{dt}\right)^2 + \Omega\left(\frac{\gamma_1}{\Omega} + \frac{e}{m} A_y(x)\right)^2 . \qquad \ldots(2.71)$$

Defining

$$\xi^2 = \frac{2\varepsilon}{m} + \frac{2\gamma_1^2 m\delta}{e\Omega} \quad , \qquad \ldots(2.72)$$

equation (2.71) becomes

$$\xi^2 = \left(\frac{dx}{dt}\right)^2 + \Omega\left(\frac{\gamma_1}{\Omega} + \frac{e}{m}A_y(x)\right)^2 . \qquad \ldots(2.73)$$

Thus from equation (2.73) the time can be expressed as

$$t = \int \frac{dx}{\sqrt{\xi^2 - \Omega\left(\frac{\gamma_1}{\Omega} + \frac{e}{m}A_y(x)\right)^2}} \quad , \qquad \ldots(2.74)$$

and by using equations (2.14) and (2.73) y becomes

$$y = \int \frac{\left(\gamma_1 + \frac{e}{m}A_y(x)\right)dx}{\sqrt{\xi^2 - \Omega\left(\frac{\gamma_1}{\Omega} + \frac{e}{m}A_y(x)\right)^2}} \quad . \qquad \ldots(2.75)$$

Equations (2.74) and (2.75) are now in forms similar to the integrals of the magnetostatic equations (2.22) and (2.23). They have however been modified by a constant Ω. This constant contains the parameter δ, which relates the electric and magnetic potentials.

If the electric potential becomes small compared with the magnetic potential, δ approaches zero. Thus Ω approaches one, and ξ^2 simplifies to v^2. Equations (2.74) and (2.75) therefore reduces to integral forms of the magnetostatic equations (2.22) and (2.23). If the magnetic field becomes negligible in comparison with the electric field, δ approaches infinity. (δ being of the same order of magnitude as the energy of the electron). The parameter Ω therefore approaches $-2m\delta/e$ and equations (2.74), and (2.75) go over into the electrostatic result of equation (2.19).

For $0 < \Omega \leq \infty$, there are the useful substitutions

$$\xi\sin\theta = \Omega^{\frac{1}{2}}\left(\frac{\gamma_1}{\Omega} + \frac{e}{m} A_y(x)\right) , \qquad \ldots(2.76)$$

$$\xi\cos\theta = \left(\frac{dx}{dt}\right) . \qquad \ldots(2.77)$$

Differentiation of equation (2.76) and substitution of the result into equation (2.74) leads to, with the aid of equation (2.9)

$$t = \int \frac{d\theta}{\Omega^{\frac{1}{2}}\omega_\theta} , \qquad \ldots(2.78)$$

where $\omega_\theta = \frac{e}{m} B_z(\theta)$ is the gyrofrequency of the electron.

Similarly equations (2.75), (2.76), (2.9) and (2.78) give

$$y = \int \frac{\left(\left(\gamma_1 - \frac{\gamma_1}{\Omega}\right) + \left(\frac{\gamma_1}{\Omega} + \frac{e}{m} A_y(x)\right)\right) dx}{\sqrt{\xi^2 - \Omega\left(\frac{\gamma_1}{\Omega} + \frac{e}{m} A_y(x)\right)^2}}$$

$$= \frac{1}{\Omega^{\frac{1}{2}}} \left\{\left(\gamma_1 - \frac{\gamma_1}{\Omega}\right) (t - t_o) + \int \frac{\xi\sin\theta d\theta}{\Omega^{\frac{1}{2}} \omega_\theta} \right\} , \qquad \ldots(2.79)$$

and from equation (2.76) x is expressed as a function of θ.

(c) Constant Magnetic Field and an Electric Field with Constant Gradient

For a magnetic field $B_z = \lambda$, and an electric field $E = E_o x$ where λ and E_o are constants, equations (2.6), (2.9) and (2.68) give

$$\delta = \frac{E}{2\lambda^2} , \qquad \ldots(2.80)$$

and equation (2.78) becomes

$$t = \frac{\theta - \theta_o}{\Omega^{\frac{1}{2}} \omega_o} + t , \qquad \ldots(2.81)$$

where $\theta = \theta_o$ at $t = t_o$ and $\omega_o = e\lambda/m$ is the gyrofrequency of the electron. Equations (2.79) and (2.80) yield

$$y = \frac{1}{\Omega^{\frac{1}{2}}} \{ \left(\gamma_1 - \frac{\gamma_1}{\Omega}\right) (t-t_o) + \xi \frac{(\cos\theta - \cos\theta_o)}{\Omega^{\frac{1}{2}}\omega_o} \} + y_o \text{ , } \qquad \ldots(2.82)$$

where $y = y_o$ at $t = t_o$.

Also, from equations (2.76) and (2.9)

$$x = \frac{1}{\omega_o} \{ \frac{\xi\sin\theta}{\Omega^{\frac{1}{2}}} - \frac{\gamma_1}{\Omega} \} \text{ . } \qquad \ldots(2.83)$$

2.4 Discussion of Results

The analysis in this chapter clearly indicates that for specific forms of static magnetic fields solutions can be found for the motion of an electron in terms of the angular parameter ψ. This has then been extended to take into account static fields in which an electrostatic field is also present.

Whilst the purely electrostatic problem has been analysed in depth, (Marion (1970), Whittaker (1965)) and solved for specific forms of x dependence of scalar potential, the present research has indicated that equivalent analysis is possible for the complementary magnetostatic problems. The following chapters will reinforce this point of view by analysing specific forms of field dependence. Indeed, this thesis will indicate how analysis is possible with electromagnetostatic field configurations, and field configurations in which the particle is moving with relativistic velocities.

CHAPTER 3

3.1 Drift Velocity Expressions for Bound Orbits in a Static Magnetic Field

From Section (ii) parts (a) and (b) in Chapter 2, Part 2.3, the average motion of an electron in periodic time T can be found for bound orbits. The functions which constitute the drift velocity expressions are found to be related to the magnetic vector potential of the magnetic field. For an exponentially varying magnetic field, the drift velocity of the electron is related to the magnetic potential at $x = x_o$, whilst for a magnetic field with a power law dependence, $\alpha \neq -1$, the functions describing the drift are related to the vector potential at $x = x_2$ due to the choice of functional representation of the results in terms of hypergeometric functions. $\alpha = -1$ is a degenerate case and the solutions obtained depend on the parameter $\nu = \frac{mv}{e\lambda}$ which is independent of x.

3.2 Exponentially Varying Magnetic Field

For a magnetic field of the form

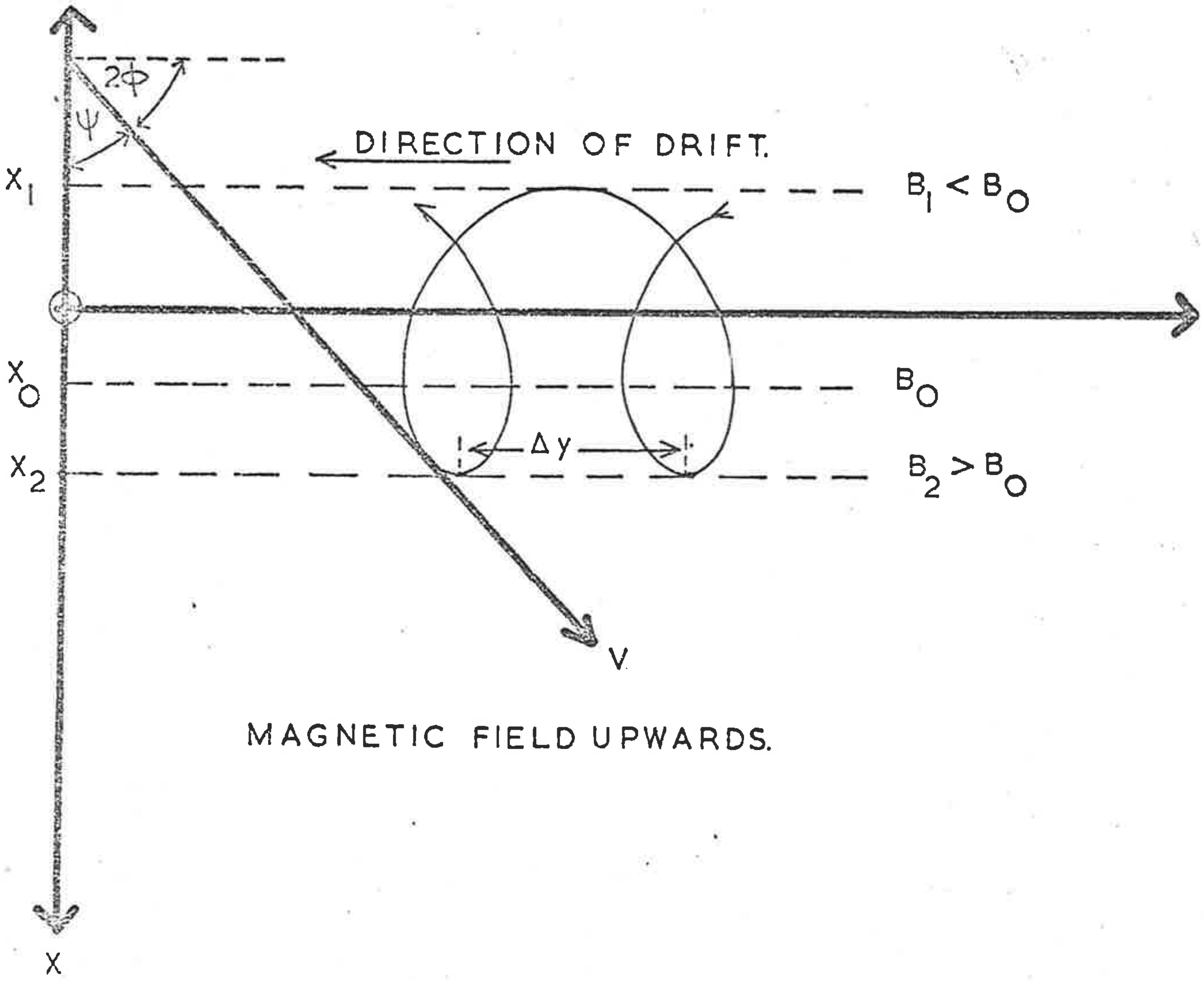
$$B_z = \lambda e^{\alpha x} , \qquad \ldots(3.1)$$

as shown in Figure 1(b), where λ and α are constants > 0, equation (2.29) becomes

$$\sin\psi - \sin\psi_o = \frac{e\lambda}{\alpha m v} (e^{\alpha x} - e^{\alpha x_o}) , \qquad \ldots(3.2)$$

with the aid of equation (2.9). For simplicity of mathematical formulation and without particular loss of generality, x_o is chosen to correspond to $\psi_o = 0$. Equation (3.2) then becomes

$$x = x_o + \frac{1}{\alpha} \ell n \left(1 + \frac{mv}{eA_o} \sin\psi\right) , \qquad \ldots(3.3)$$



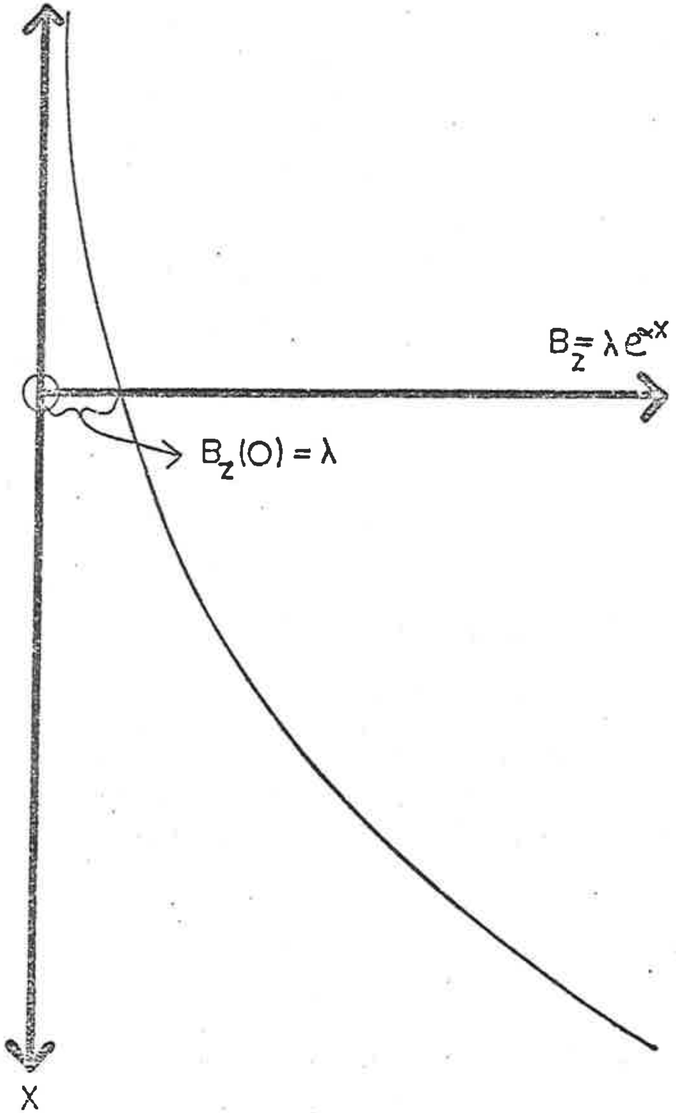


FIGURE I (a): BOUND ORBIT OF AN ELECTRON IN $B_z = \lambda e^{-\alpha x}$.

FIGURE I (b): SHAPE OF THE MAGNETIC FIELD where, $A_o = \lambda e^{\alpha x_o}/\alpha$ is the magnetic vector potential at $x = x_o$. For bound orbits $\left|\frac{mv}{eA_o}\right| < 1$. As shown in Figure 1(a), and from equation (3.3), the upper bound x_2 of the orbit at $\psi = \frac{\pi}{2}$ becomes

$$x_2 = x_o + \frac{1}{\alpha} \ln \left(1 + \frac{mv}{eA_o}\right) \quad . \qquad \ldots(3.4)$$

The lower bound, x_1, of the orbit at $\psi = \frac{3\pi}{2}$ is

$$x_1 = x_o + \frac{1}{\alpha} \ln \left\{1 - \frac{mv}{eA_o}\right\} \quad , \qquad \ldots(3.5)$$

and

$$- \infty < x_1 \leq x \leq x_2 < \infty \quad . \qquad \ldots(3.6)$$

(i) <u>The Exact Drift Velocity Expression</u>

Using equations (2.9), (2.26), (2.30), (3.1) and (3.3) the periodic time becomes

$$T = \frac{m}{e} \int_o^{2\pi} \frac{d\psi}{B_z}$$

$$= \frac{1}{\omega_o} \int_o^{2\pi} \frac{d\psi}{\left(1 + \frac{mv}{eA_o} \sin\psi\right)} \quad , \qquad \ldots(3.7)$$

where $\omega_o = \frac{eB_z(x_o)}{m}$ is the gyrofrequency of the electron at $x = x_o$.

Use of the new variable $\phi = \frac{\pi}{4} - \frac{\psi}{2}$ in equation (3.7) gives

$$T = \frac{4}{\omega_o} \int_o^{\frac{\pi}{2}} \frac{\sec^2\phi d\phi}{\left(1 + \frac{mv}{eA_o}\right) \left\{1 + \frac{\left(1 - \frac{mv}{eA_o}\right)}{\left(1 + \frac{mv}{eA_o}\right)} \tan^2\phi\right\}} \quad . \qquad \ldots(3.8)$$

If another variable is defined by $\tan\theta = \left\{\frac{\left(1 - \frac{mv}{eA_o}\right)}{\left(1 + \frac{mv}{eA_o}\right)}\right\}^{\frac{1}{2}} \tan\phi$, equation (3.8) gives

$$T = \frac{T_o}{\left(1 - \left(\frac{mv}{eA_o}\right)^2\right)^{\frac{1}{2}}} \qquad \ldots(3.9)$$

where $T_o = \frac{2}{\omega_o} = \frac{2\pi\rho_o}{v}$ is the periodic time of the orbit of an electron in a constant magnetic field, $B_o = \lambda e^{\alpha x_o}$.

Similarly, equation (2.31) yields

$$\Delta y = \frac{v}{\omega_o} \int_o^{2\pi} \frac{\sin\psi d\psi}{\left(1 + \frac{mv}{eA_o} \sin\psi\right)}$$

$$= \frac{vT_o}{\frac{mv}{eA_o}} \left[1 - \frac{1}{\left(1 - \left(\frac{mv}{eA_o}\right)^2\right)^{\frac{1}{2}}}\right] . \qquad \ldots(3.10)$$

Using equations(3.9), (3.10) and (2.32) the exact drift velocity becomes

$$v_D = \frac{\Delta y}{T}$$

$$= v\left(\frac{eA_o}{mv}\right)\left(\left(1 - \left(\frac{mv}{eA_o}\right)^2\right)^{\frac{1}{2}} - 1\right) . \qquad \ldots(3.11)$$

(ii) Alfven's Approximate Theory

For a magnetic field which is adiabatically affected, $\frac{mv}{eA_o} << 1$ and equation (3.11) reduces to

$$v_D \simeq v\left(\frac{eA_o}{mv}\right)\left(\left(1 - \frac{1}{2}\left(\frac{mv}{eA_o}\right)^2 + \ldots\right) - 1\right)$$

$$= - \frac{v}{2} \left(\frac{mv}{eA_o}\right) .$$

$$= - \frac{mv^2\alpha}{2eB_z(x_o)} , \qquad \ldots(3.12)$$

where from equation (2.9), $B_z(x_o) = \alpha A_o$.

Equation (3.12) agrees with the Alfven drift expression of equation (2.38) for an exponentially varying field.

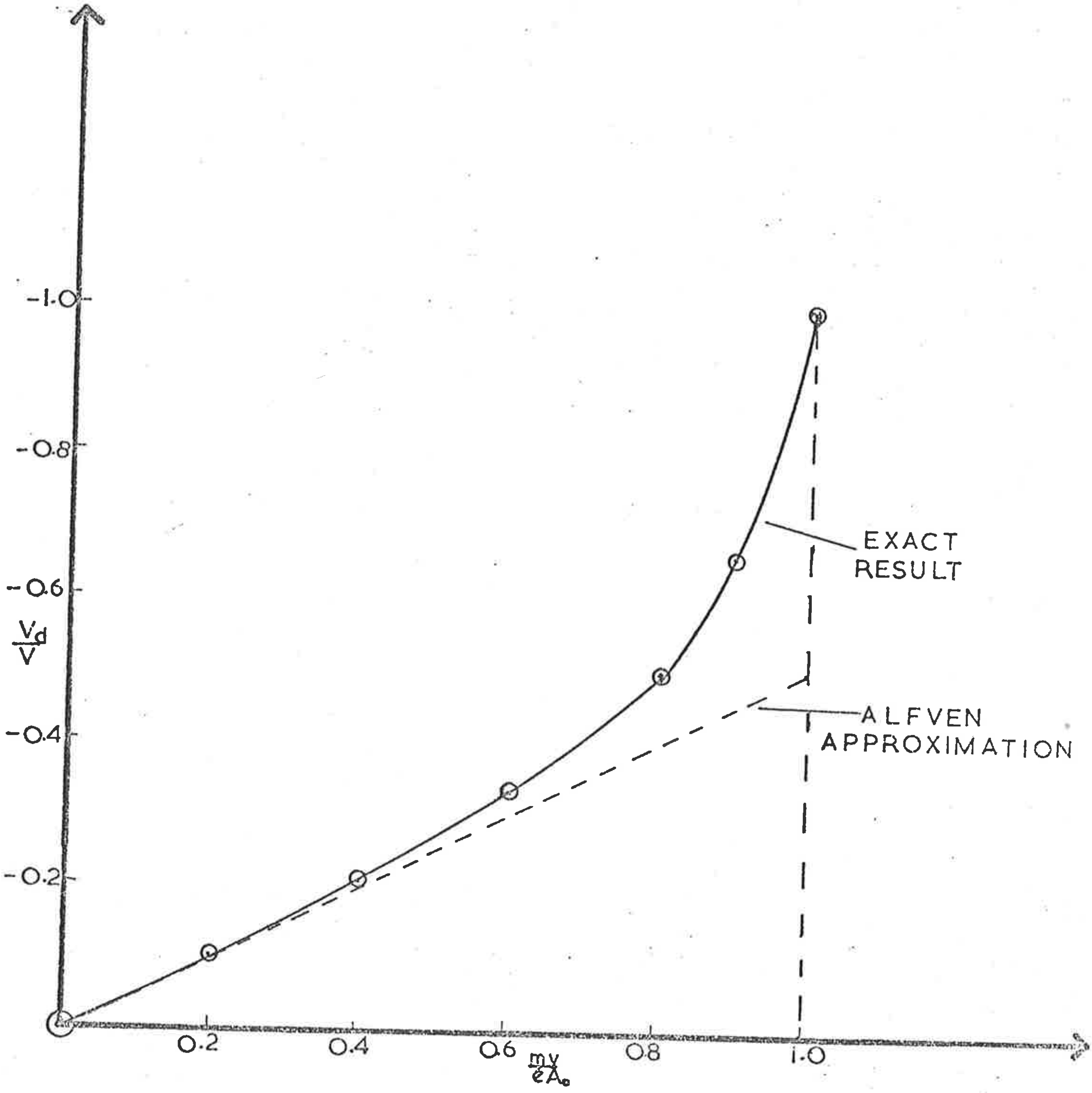


FIGURE 2: DRIFT VELOCITY OF AN ELECTRON IN AN EXPONENTIAL MAGNETIC FIELD

(iii) Discussion of Results

In Figure 2 the exact drift velocity expression of equation (3.11) is plotted against $\frac{mv}{eA_o} = \alpha\rho_o$ and compared with the perturbation result in equation (3.12). From Figure 2 the Alfven drift velocity is in good agreement with the exact result for $0 < \frac{mv}{eA_o} \lesssim 0.6$, the maximum error of the approximate result being less than ten percent.

3.3 Magnetic Field with Power Law Dependence

The magnetic field is of the form

$$B_z = \lambda x^{\alpha} \quad , \qquad \text{...(3.13)}$$

where λ and α are constants, and has the differing forms as sketched in Figure 8. Provided $\alpha \neq -1$ and $x_o > 0$ then equations (2.29) and (2.9) yield

$$x = x_o \left\{1 + \left(\frac{mv}{eA_o}\right) (\sin\psi - \sin\psi_o)\right\}^{\beta} \quad , \qquad \text{...(3.14)}$$

where

$$\beta = \frac{1}{\alpha + 1} \quad , \qquad \text{...(3.15)}$$

and

$$A_o = \lambda\beta x_o^{1/\beta} \qquad \text{...(3.16)}$$

is the magnetic vector potential at $x = x_o$ which, from equation (2.9) generates the magnetic field

$$B_o = \lambda x_o^{\alpha} \quad . \qquad \text{...(3.17)}$$

From equation (3.17), the radius of gyration of the electron at $x = x_o$ is

$$\rho_o = \frac{mv}{eB_o} \quad . \qquad \text{...(3.18)}$$

The electron motion sketched in Figure 3 is for $x_o > 0$. Again if x_o is chosen to correspond to $\psi_o = 0$ equation (3.14) yields

$$x = x_o \{1 + (\frac{mv}{eA_o}) \sin\psi\}^\beta . \qquad ...(3.19)$$

Recalling that equation (3.19) applies for all α except $\alpha = -1$, the electron orbits are then bounded between the limits

$$x_1 = x_o \left(1 - \frac{mv}{eA_o}\right)^\beta , \qquad ...(3.20)$$

for $\psi = \frac{3\pi}{2}$, and

$$x_2 = x_o \left(1 + \frac{mv}{eA_o}\right)^\beta , \qquad ...(3.21)$$

for $\psi = \frac{\pi}{2}$, so that

$$0 \leq x_1 < x < x_2 < \infty . \qquad ...(3.22)$$

For $\alpha > 0$ the magnetic field vanishes on the line $x = 0$, and the electron motion is sketched in Figure 3. For $\alpha < 0$ the field B_z becomes infinite on the line $x = 0$. The drift reverses direction and has the same general form as in Figure 4 for the case of $\alpha = -1$, to be considered in Case 2 of Section 2.

Another class of motions of practical interest (Seymour 1959) occurs when $x_o = 0$, and

$$\alpha > 0 , \qquad ...(3.23)$$

so that B_z is always zero on the line $x = 0$. Then with $\psi = \psi_o$ when $x_o = 0$ equation (2.29) with (3.13) gives

$$x = \{\frac{1}{\beta} \frac{mv}{e\lambda} (\sin\psi - \sin\psi_o)\}^\beta . \qquad ...(3.24)$$

Under these conditions the electron enters a region of reversed magnetic field when it crosses the neutral plane $x = 0$ if α is an odd integer, and symmetrical electron motions of the type shown in Figure 5 occur. From equation (3.24) x for any ψ is thus given by

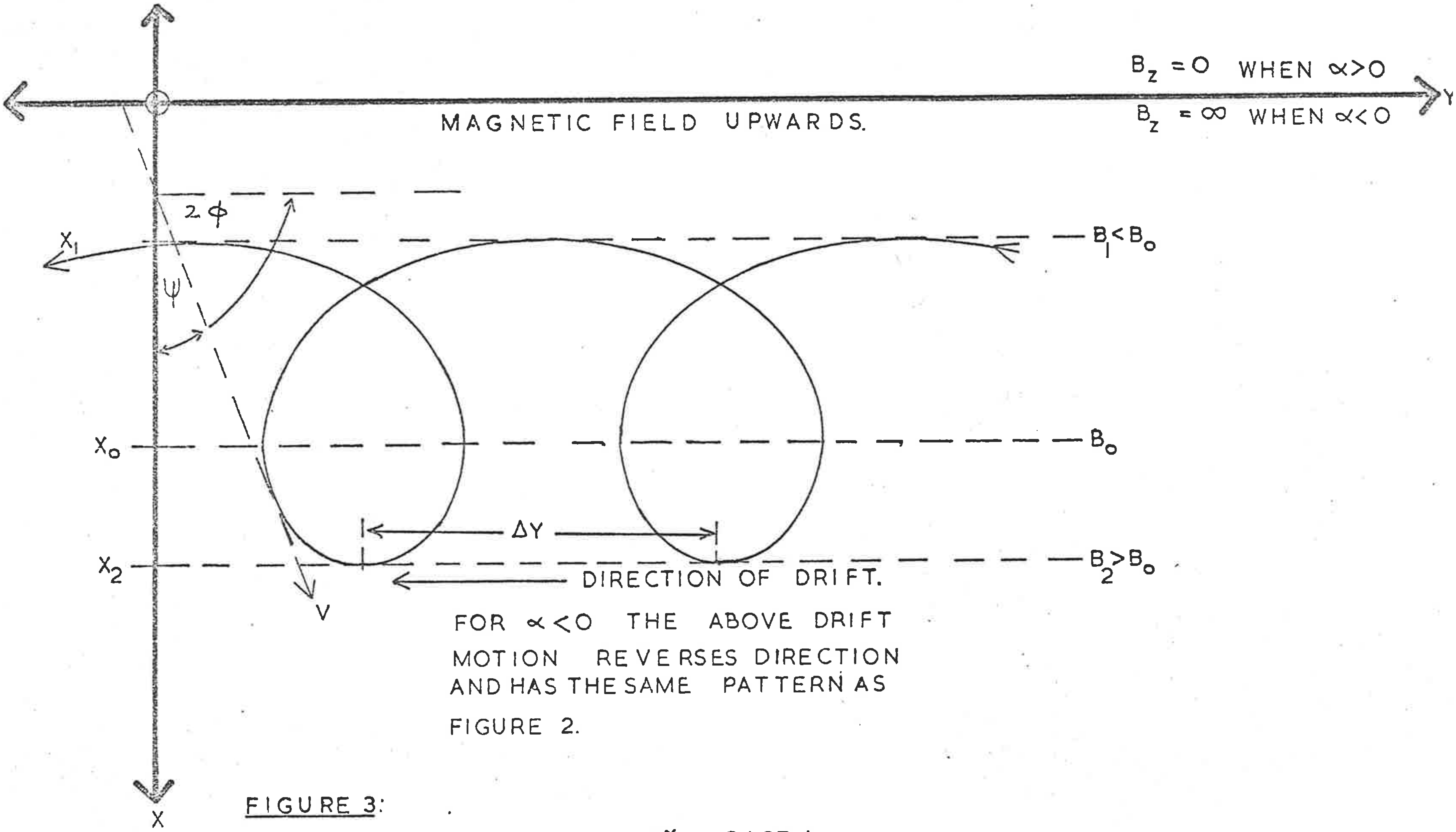


FIGURE 3:

ELECTRON DRIFT IN $B_z = \lambda X^{\alpha}$. CASE I: $\alpha \neq 1$, MOTION DOES NOT CROSS THE LINE $X=0$.

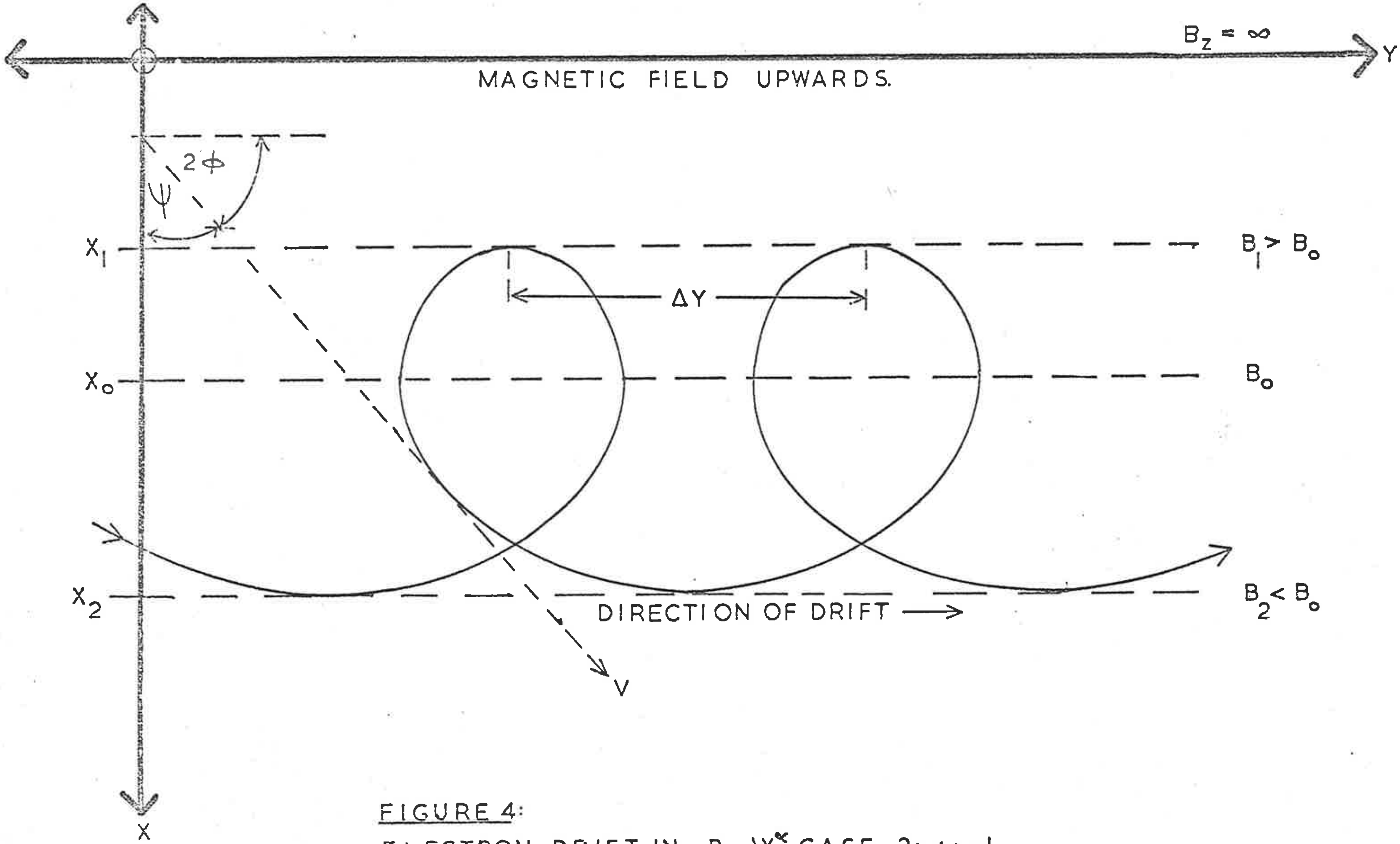


FIGURE 4:

ELECTRON DRIFT IN $B_z = \lambda X^{\alpha}$. CASE 2; $\alpha = -1$

MOTION DOES NOT CROSS LINE X=O.

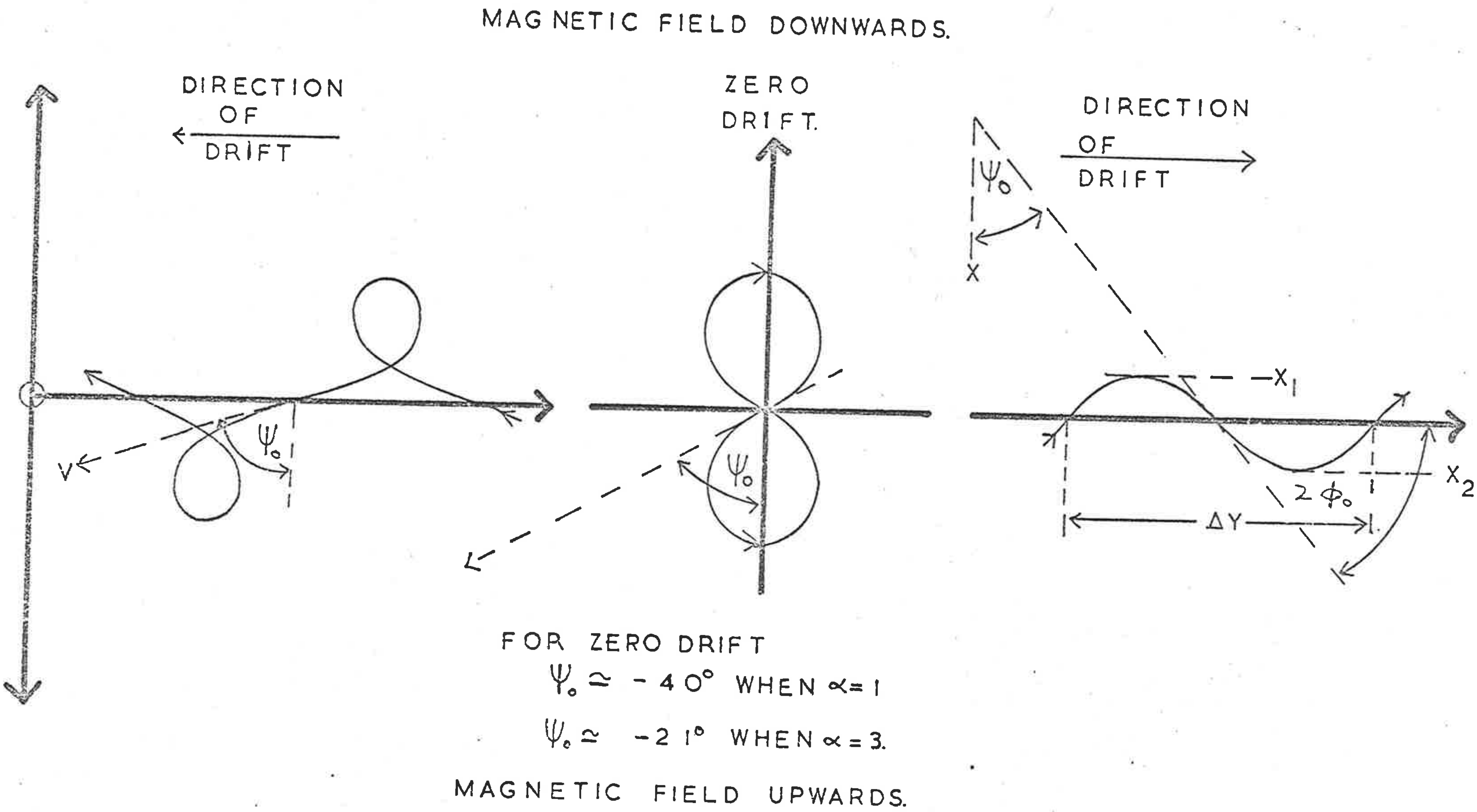


FIGURE 5:

ELECTRON DRIFT IN $B_z = \lambda X^\alpha$ CASE 3, $\alpha > 0$ AN ODD INTEGER.
MOTION CROSSES THE LINE X=O.

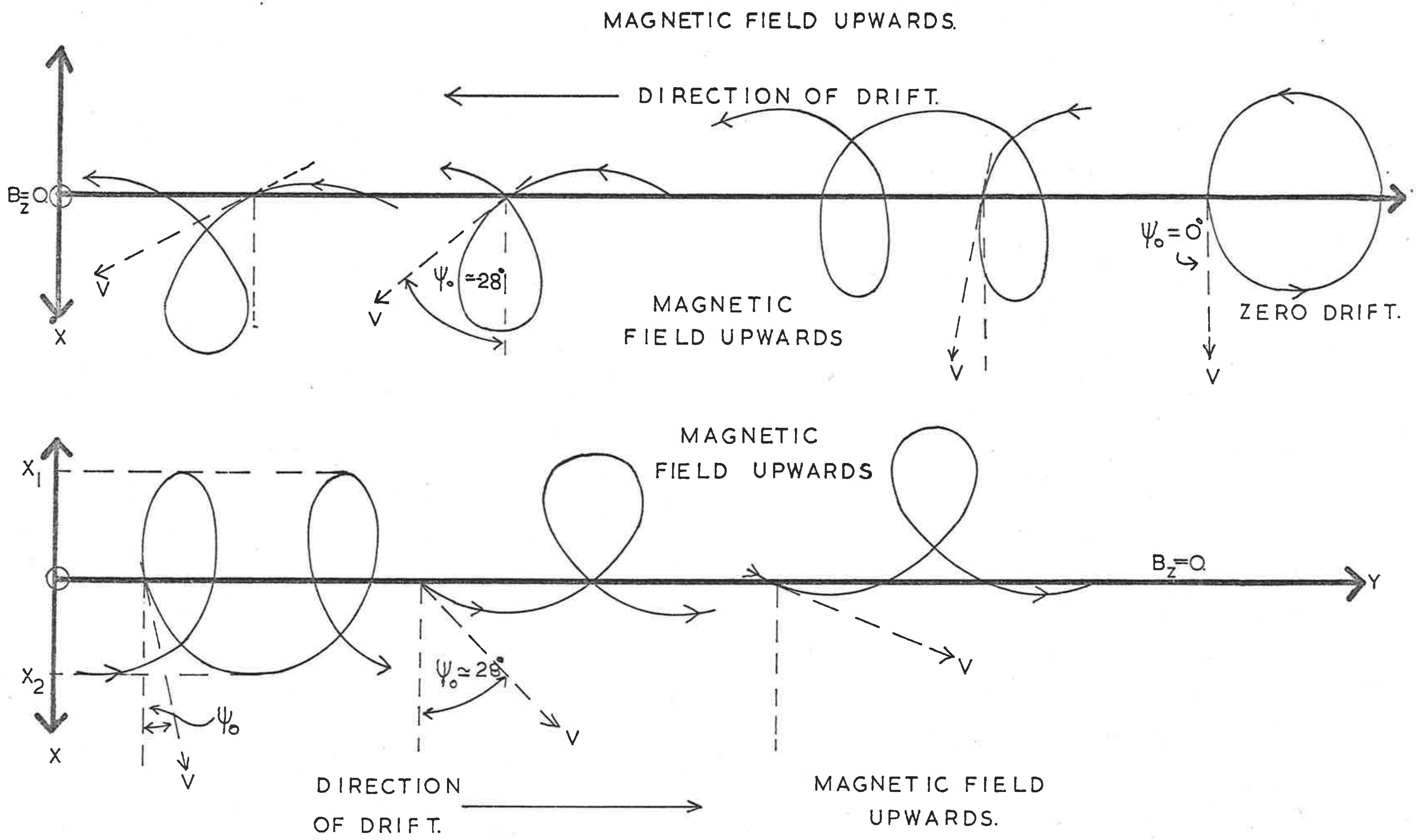


FIGURE 6:
ELECTRON DRIFT IN $B_z = \lambda X^{\alpha}$. CASE 4, $\alpha > 0$, AN EVEN INTEGER. MOTION CROSSES THE LINE $X = 0$.

$$x = \pm \{\frac{1}{\beta}\frac{mv}{e\lambda}(\sin\psi - \sin\psi_o)\}^{\beta} , \qquad ...(3.25)$$

with the limits

$$x_2 = -x_1 = \{\frac{1}{\beta}(\frac{mv}{e\lambda})(1 - \sin\psi_o)\}^{\beta} , \qquad ...(3.26)$$

for $\psi = \frac{\pi}{2}$, so that

$$0 > x_1 < x < x_2 < \infty . \qquad ...(3.27)$$

For α an even integer the magnetic field does not reverse direction when the electron crosses the neutral plane into the region $x < 0$. Particle motions of the type shown in Figure 6 now occur, and from equation (3.24) the corresponding limits are given by

$$x_1 = -\{\frac{1}{\beta}(\frac{mv}{e\lambda})(1 + \sin\psi_o)\}^{\beta} , \qquad ...(3.28)$$

for $\psi = \frac{3\pi}{2}$, and

$$x_2 = \{\frac{1}{\beta}(\frac{mv}{e\lambda})(1 - \sin\psi_o)\}^{\beta}, \qquad ...(3.29)$$

for $\psi = \frac{\pi}{2}$, so that

$$0 > x_1 < x < x_2 < \infty . \qquad ...(3.30)$$

<u>Exact Solutions in Terms of Hypergeometric Functions</u>

<u>Case 1</u>: $\alpha \neq -1$ Electron does not cross the line $x = 0$, on which $B_z = 0$ when $\alpha > 0$ and $B_z = \infty$ when $\alpha < 0$.

From equations (3.15), (3.16), (3.17), (3.18), (3.19) and (2.31) the exact drift in the y direction, from Figure 3 is given by

$$\Delta y = \rho_o \int_o^{2\pi} \frac{\sin\psi d\psi}{\left(1 + \frac{mv}{eA_o}\sin\psi\right)^{1-\beta}} . \qquad ...(3.31)$$

At $x = x_2$, equations (3.21), (3.13) and (3.16) respectively become

$B_2 = \lambda x_2^{\alpha}$, and $A_2 = \lambda\beta x_2^{1/\beta}$, with

$$\rho_2 = \frac{mv}{eB_2} = \frac{mv}{e\lambda x_2^{\frac{1-\beta}{\beta}}} \quad . \qquad \ldots(3.32)$$

Then use of the variable $\phi = \frac{\pi}{4} - \frac{\pi}{2}$ in equation (3.31), leads, with the help of (3.16), (3.17) and (3.21) to

$$\Delta y = 4\rho_2 \int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \sigma\sin^2\phi)^{1-\beta}} - 8\rho_2 \int_0^{\frac{\pi}{2}} \frac{\sin^2\phi d\phi}{(1 - \sigma\sin^2\phi)^{1-\beta}} , \qquad \ldots(3.33)$$

where

$$\sigma = \frac{2\rho_o}{\beta} \frac{x_o^{\frac{1-\beta}{\beta}}}{x_2^{\frac{1}{\beta}}} = \frac{2\rho_2}{x_2} = \frac{2mv}{eA_2} \quad . \qquad \ldots(3.34)$$

Combining (3.21) and (3.34) leads to the useful relationships

$$\frac{x_2}{x_o} = \frac{1}{(1 - \frac{\sigma}{2})^{\beta}} , \qquad \ldots(3.35)$$

and

$$\frac{x_2}{\rho_o} = \frac{2}{\sigma\beta} \left(1 - \frac{\sigma}{2}\right)^{1-\beta} . \qquad \ldots(3.36)$$

For $x_1 \geq 0$ (c.f. equation (3.22)), equations (3.20), (3.21) and (3.34) yield

$$x_2 > x_o \left(\frac{2\rho_o}{\beta x_o}\right)^{\beta} \geq x_2\sigma^{\beta} , \qquad \ldots(3.37)$$

so that the upper limit of $\frac{\rho_o}{\beta x_o}$ and of σ^{β} is unity. From the theory of hypergeometric functions one has the result, (see Appendix 1)

$$F(a,b;c;x) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\frac{\pi}{2}} \frac{(\sin\phi)^{2b-1}(\cos\phi)^{2c-2b-1}d\phi}{(1 - x\sin^2\phi)^{a}} , \qquad \ldots(3.38)$$

which for $b = \frac{1}{2}$, $c = 1$ reduces to

$$F(a,\tfrac{1}{2};1;x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - x\sin^2\phi)^a} . \qquad \ldots(3.39)$$

Noting that result (3.32) gives $\rho_2 = \frac{1}{2}\sigma\beta x_2$, expression (3.33) reduces to, by further manipulation

$$\Delta y = 2\beta x_2\left\{2\int_0^{\frac{\pi}{2}} (1 - \sigma\sin^2\phi)^\beta d\phi - (2-\sigma)\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-\sigma\sin^2\phi)^{1-\beta}}\right\} . \qquad \ldots(3.40)$$

Hence in terms of hypergeometric functions of the form (3.39), by putting $x = \sigma$, $a = -\beta$ for the first integral and $a = 1 - \beta$ for the second integral of equation (3.40), Δy can be written

$$\Delta y = \pi\beta x_2\left\{2(F(-\beta,\tfrac{1}{2};1;\sigma) - F(1-\beta,\tfrac{1}{2};1;\sigma)) + \sigma F(1-\beta,\tfrac{1}{2};1;\sigma)\right\} . \qquad \ldots(3.41)$$

From equation (2.30) and utilizing equations (3.15), (3.16), (3.17), (3.18) and (3.19) the periodic time is

$$T = \frac{\rho_o}{v}\int_0^{2\pi} \frac{d\psi}{\left(1 + \frac{mv}{eA_o}\sin\psi\right)^{1-\beta}} . \qquad \ldots(3.42)$$

And so with reference to Figure 3, the periodic time T corresponding to Δy is obtained as

$$T = \frac{2\sigma\beta x_2}{v}\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \sigma\sin^2\phi)^{1-\beta}} , \qquad \ldots(3.43)$$

with the help of equations (3.21) and (3.34). Putting $a = 1 - \beta$ and $x = \sigma$ in the result (3.39), equation (3.43) becomes

$$T = \frac{\pi\sigma\beta x_2}{v} F(1 - \beta, \tfrac{1}{2}; 1; \sigma) \ . \qquad \ldots(3.44)$$

Elimination of x_2 from equation (3.44) by means of equation (3.36) and use of the zero order orbit result

$$T_o = \frac{2\pi\rho_o}{v} \ , \qquad \ldots(3.45)$$

gives the interesting form

$$T = T_o\left(1 - \frac{\sigma}{2}\right)^{1-\beta} F(1 - \beta, \tfrac{1}{2}; 1; \sigma) \ . \qquad \ldots(3.46)$$

Equation (3.46) expresses the periodic time T for non-circular orbital motion in terms of the zero order time T_o corresponding to electron circular orbital motion in a constant magnetic field $B_o = \lambda x_o^{\alpha}$.

From equations (3.41) and (3.44) the exact drift velocity for electron motion in a magnetic field $B_z = \lambda x^{(1-\beta)/\beta}$ is given by

$$v_D = \frac{\Delta y}{T} = -v\left\{\frac{2}{\sigma}\left(1 - \frac{F(-\beta, \frac{1}{2}; 1; \sigma)}{F(1-\beta, \frac{1}{2}; 1; \sigma)}\right) - 1\right\} \ . \qquad \ldots(3.47)$$

When $\alpha = 1$ equation (3.13) reduces to a magnetic field of constant gradient λ, and the results (3.41), (3.44) and (3.47) reduce to those obtained by Seymour (1959, Section 3, Case 1). Since $\alpha = 1$, $\beta = \frac{1}{2}$, then from equation (3.39), with $x = \sigma = k_1^2$,

$$\left.\begin{aligned} \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_1^2) &= \int_o^{\frac{\pi}{2}} \frac{d\phi}{(1 - k_1^2 \sin^2\phi)^{\frac{1}{2}}} = K \ , \\ \text{and} \qquad \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k_1^2) &= \int_o^{\frac{\pi}{2}} (1 - k_1^2 \sin^2\phi)^{\frac{1}{2}} \, d\phi = E \ , \end{aligned}\right\} \qquad \ldots(3.48)$$

are complete elliptic integrals of the first and second kinds respectively, both of modulus k_1 where k_1 is bounded by the inequality

$$0 \leq k_1 \leq 1 \ . \qquad \ldots(3.49)$$

It has been seen that for $\alpha > 0$ the electron drift has the pattern and direction shown in Figure 3, and that for $\alpha < 0$ the drift has the pattern and direction shown in Figure 4. When $\alpha = 0$ equation (3.13) gives the magnetic field as $B_z = \lambda =$ constant. Then with $\beta = 1$, $F(0,\frac{1}{2};1;\sigma) = 1$; $F(-1,\frac{1}{2};1;\sigma) = 1 - \sigma/2$, equations (3.19), (3.20), (3.41), (3.46) and (3.47) respectively reduce to, $x_1 = x_o - \rho_o$, $x_2 = x_o + \rho_o$, $\Delta y = 0$, $T = T_o$, and $v_D = 0$, as is correct for circular electron orbital motion in a constant magnetic field.

<u>Case 2:</u> $\alpha = -1$ Electron does not cross the line $x = 0$ on which $B_z = \infty$.

When $\alpha = -1$ equation (3.13) becomes $B_z = \lambda/x$, and β becomes infinite. For electron motion in this magnetic field the results (3.41), (3.44) and (3.47) for Δy, T and v_D are invalid. The infinite value of β here suggests solutions for these quantities in terms of confluent hyper-geometric functions. Again choosing x_o to correspond to $\psi_o = 0$, use of equations (3.13), (3.18) and (2.9) in (2.29) yields in this case

$$x = x_o\, e^{\rho_o/x_o \sin\psi} \quad . \qquad \text{...(3.50)}$$

From this result the electron motion of Figure 4 in $B_z = \lambda/x$ is bounded by

$$x_1 = x_o e^{-\rho_o/x_o} \qquad \text{for } \psi = \frac{3\pi}{2} \;, \qquad \text{...(3.51)}$$

and

$$x_2 = x_o e^{+\rho_o/x_o} \qquad \text{for } \psi = \frac{\pi}{2} \;. \qquad \text{...(3.52)}$$

From equations (2.31) and (3.50),

$$y = \rho_o \int e^{\rho_o/x_o \sin\psi} \sin\psi d\psi \;, \qquad \text{...(3.53)}$$

and similarly equation (2.30) yields

$$t = \frac{\rho_o}{v} \int e^{\rho_o/x_o \sin\psi} d\psi \;. \qquad \text{...(3.54)}$$

Thus, with reference to Figure 4, equation (3.53) gives

$$\Delta y = 2\rho_o \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\rho_o/x_o \sin\psi} \sin\psi d\psi \quad . \qquad \text{...(3.55)}$$

From equations (3.13) and (3.32) it is found in this case that $\frac{mv}{e\lambda} = \frac{\rho_o}{x_o} = \frac{\rho_2}{x_2}$. Putting $\nu = \frac{2\rho_o}{x_o} = \frac{2\rho_2}{x_2}$ and $\tau = \sin\psi$, Δy becomes

$$\Delta y = 2\rho_o \int_{-1}^{1} \frac{e^{\frac{\nu\tau}{2}} \tau d\tau}{(1-\tau^2)^{\frac{1}{2}}} \quad . \qquad \text{...(3.56)}$$

From confluent hypergeometric function theory we have the result (Appendix 2)

$$M(\tfrac{1}{2},1;-\nu) = \frac{e^{-\frac{\nu}{2}}}{\pi} \int_{-1}^{1} \frac{e^{\frac{\nu\tau}{2}} d\tau}{(1-\tau^2)^{\frac{1}{2}}} \quad . \qquad \text{...(3.57)}$$

By means of the standard result (e.g. Hochstadt, 1971)

$$\frac{d}{d\nu} M(a,b;-\nu) = -\frac{a}{b} M(a+1,b+1;-\nu) \qquad \text{...(3.58)}$$

and the previous result (3.57), equation (3.56) becomes

$$\Delta y = 2\pi\rho_o e^{\frac{\nu}{2}} \{M(\tfrac{1}{2},1;-\nu) - M(\tfrac{3}{2},2;-\nu)\} \quad . \qquad \text{...(3.59)}$$

Similarly equations (3.45), (3.54) and (3.57) yield for the periodic time

$$T = T_o \, e^{\frac{\nu}{2}} M(\tfrac{1}{2},1,-\nu) \quad , \qquad \text{...(3.60)}$$

and so the drift of an electron in a magnetic field $B_z = \lambda/x$ becomes from equations (3.45), (3.59) and (3.60)

$$v_D = \frac{\Delta y}{T} = v \, \{1 - \frac{M(\frac{3}{2},2;-\nu)}{M(\frac{1}{2},1,-\nu)}\} \quad . \qquad \text{...(3.61)}$$

Case 3: $\alpha > 0$ an odd integer. Electron crosses the line $x = 0$ on which $B_z = 0$ and enters a region of reversed magnetic field.

When $\alpha > 0$ is an odd integer, equation (3.13) shows that the magnetic field reverses when $x < 0$, and the symmetrical motions of Figure 5 occur. Here equations (3.24) and (2.31) give for $x > 0$

$$y = \left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \int \frac{\sin\psi d\psi}{(\sin\psi - \sin\psi_o)^{1-\beta}} , \qquad \ldots(3.62)$$

and similarly equation (2.30) yields

$$t = \frac{1}{v} \left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \int \frac{d\psi}{(\sin\psi - \sin\psi_o)^{1-\beta}} . \qquad \ldots(3.63)$$

Using $\psi = \frac{\pi}{2} - 2\phi$, then $\psi_o = \frac{\pi}{2} - 2\phi_o$, where from Figure 5, $-\frac{\pi}{2} \leq \psi_o \leq \frac{\pi}{2}$, $0 \leq \phi_o \leq \frac{\pi}{2}$. In terms of ϕ_o the limits given by equation (3.26) reduce to

$$x_2 = -x_1 = \left(\frac{2mv}{\beta e\lambda}\right)^{\beta} \sin^{2\beta}\phi_o . \qquad \ldots(3.64)$$

Utilizing the symmetry of the drift patterns of Figure 5, equation (3.62) gives

$$\Delta y = 4\left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \int_{\psi_o}^{\frac{\pi}{2}} \frac{\sin\psi d\psi}{(\sin\psi - \sin\psi_o)^{1-\beta}}$$

$$= 2^{3+\beta}\left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \left\{\int_o^{\phi_o} \frac{\cos^2\phi d\phi}{(\sin^2\phi - \sin^2\phi_o)^{1-\beta}} - \frac{1}{2}\int_o^{\phi_o} \frac{d\phi}{(\sin^2\phi_o - \sin^2\phi)^{1-\beta}}\right\} . \qquad \ldots(3.65)$$

If another variable of integration is defined by $\sin\phi = \sin\phi_o \sin\theta$,

$$\Delta y = 2^{3+\beta}\left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta}\gamma^{2\beta-1} \left\{\int_o^{\frac{\pi}{2}} \frac{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}}{\cos^{1-2\beta}\theta} d\theta - \frac{1}{2}\int_o^{\frac{\pi}{2}} \frac{d\theta}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}\cos^{1-2\beta}\theta}\right\}, \qquad \ldots(3.66)$$

where

$$\gamma = \sin\phi_o = \{\frac{\beta}{2}\frac{x_2}{\rho_2}\} = (\frac{eA_2}{2mv})^{\frac{1}{2}} , \qquad \ldots(3.67)$$

using equations (3.32), (3.64) and (2.9).

The result (3.38) enables (3.65) to be written as

$$\Delta y = 2\Sigma\gamma^{2\beta-1}\{2F_1 - F_2\} , \qquad \ldots(3.68)$$

where

$$\sum = \frac{\pi^{\frac{1}{2}}2^{\beta}\left(\frac{mv}{e\lambda}\right)^{\beta}\beta^{1-\beta}\Gamma(\beta)}{\Gamma(\beta + \frac{1}{2})} , \qquad \ldots(3.69)$$

and

$$F_1 = F(-\frac{1}{2},\frac{1}{2};\beta + \frac{1}{2};\gamma^2) , \qquad \ldots(3.70)$$

$$F_2 = F(\frac{1}{2},\frac{1}{2};\beta + \frac{1}{2};\gamma^2) . \qquad \ldots(3.71)$$

Similarly equation (3.63) yields for the periodic time

$$T = \frac{2}{v}\sum \gamma^{2\beta-1} F_2 . \qquad \ldots(3.72)$$

From the results (3.68) and (3.72) the drift velocity for $\alpha > 0$ an odd integer may be written

$$v_D = \frac{\Delta y}{T} = v \left(2\frac{F_1}{F_1} - 1\right) . \qquad \ldots(3.73)$$

Recalling the forms (3.48), when $\alpha = 1$, $\beta = \frac{1}{2}$ and the results (3.68), (3.72) and (3.73) reduce to those obtained by Seymour (1959, Section III, Case 2).

<u>Case 4:</u> $\alpha > 0$ an even integer. Electron crosses the line $x = 0$ on which $B_z = 0$ and enters a region of non-reversed fields.

When $\alpha > 0$ is an even integer equation (3.13) shows that the magnetic field does not reverse direction when $x < 0$. Here the magnetic field itself has symmetry about the neutral plane defined by $x = 0$, and typically the electron drift patterns are as shown in Figure 6.

From equations (3.24) and (2.31) the form (3.62) is again obtained for y, while use of equation (2.30) gives for t the form (3.63).

In terms of ϕ_o the limits given by equations (3.28) and (3.29) are

$$x_1 = -\left(\frac{2mv}{\beta e\lambda}\right)^{\beta} \cos^{2\beta}\phi_o \,, \qquad \ldots(3.74)$$

and

$$x_2 = \left(\frac{2mv}{\beta e\lambda}\right)^{\beta} \sin^{2\beta}\phi_o \,. \qquad \ldots(3.75)$$

From the drift pattern of Figure 6, Δy can be expressed as

$$\Delta y = 2\left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \left\{\int_{\psi_o}^{\frac{\pi}{2}} \frac{\sin\psi d\psi}{(\sin\psi-\sin\psi_o)^{1-\beta}} + \int_{\frac{3\pi}{2}}^{\pi-\psi_o} \frac{\sin\psi d\psi}{(\sin\psi-\sin\psi_o)^{1-\beta}}\right\} . \qquad \ldots(3.76)$$

Again using $\psi = \frac{\pi}{2} - 2\phi$, and $\psi_o = \frac{\pi}{2} - 2\phi_o$, equation (3.76) becomes

$$\Delta y = 2^{1+\beta}\left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \left\{\int_{o}^{\phi_o} \frac{(2\cos^2\phi-1)d\phi}{(\sin^2\phi_o-\sin^2\phi)^{1-\beta}} + \int_{-\phi_o}^{-\frac{\pi}{2}} \frac{(1-2\sin^2\phi)d\phi}{(\cos^2\phi-\cos^2\phi_o)^{1-\beta}}\right\} . \qquad \ldots(3.77)$$

Changing the variable ϕ to θ in the first integral of equation (3.77) by means of the relationship $\sin\phi = \sin\phi_o\sin\theta$, and changing ϕ to μ in the second integral by use of $\cos\phi = \cos\phi_o\sin\mu$, equation (3.77) can be written

$$\Delta y = 2^{2+\beta} \left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \left\{\gamma^{2\beta-1} \int_o^{\frac{\pi}{2}} \frac{\cos^{2\beta-1}\theta d\theta}{(1-\gamma^2\sin^2\theta)^{-\frac{1}{2}}} - \frac{\gamma^{2\beta-1}}{2} \int_o^{\frac{\pi}{2}} \frac{\cos^{2\beta-1}\theta d\theta}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}}\right.$$

$$\left. + \frac{\xi^{2\beta-1}}{2} \int_o^{\frac{\pi}{2}} \frac{\cos^{2\beta-1}\mu d\mu}{(1-\xi^2\sin^2\mu)^{\frac{1}{2}}} - \xi^{2\beta-1} \int_o^{\frac{\pi}{2}} \frac{\cos^{2\beta-1}\mu d\mu}{(1-\xi^2\sin^2\mu)^{-\frac{1}{2}}}\right\} , \qquad \ldots(3.78)$$

where, from equations (3.32), (3.75) and (2.9)

$$\gamma = \sin\phi_o = \left(\frac{\beta}{2}\frac{x_2}{\rho_2}\right)^{\frac{1}{2}} = \left(\frac{eA_2}{2mv}\right)^{\frac{1}{2}} \text{ , and } \xi = \cos\phi_o = (1 - \gamma^2)^{\frac{1}{2}} .$$

In terms of the results (3.38), equation (3.78) finally becomes

$$\Delta y = \sum\{\gamma^{2\beta-1}(2F_1 - F_2) - \xi^{2\beta-1}(2F_3 - F_4)\} , \qquad \text{...(3.79)}$$

where $\sum$ is given by equation (3.69), F_1 and F_2 are given by equations (3.70) and (3.71) respectively (but with β determined by $\alpha > 0$ an even integer), and

$$F_3 = F(-\tfrac{1}{2},\tfrac{1}{2};\beta + \tfrac{1}{2},\xi^2) \quad , \qquad \text{...(3.80)}$$

$$F_4 = F(\tfrac{1}{2},\tfrac{1}{2};\beta + \tfrac{1}{2};\xi^2) \quad . \qquad \text{...(3.81)}$$

Similarly, reference to Figure 6 and equation (3.63) enables the periodic time T in this case to be expressed as

$$T = \frac{1}{v}\sum(\gamma^{2\beta-1} F_2 + \xi^{2\beta-1} F_4) . \qquad \text{...(3.82)}$$

From results (3.79) and (3.82) the drift velocity for $\alpha > 0$ an even integer becomes

$$v_D = \frac{\Delta y}{T} = \left\{\frac{\gamma^{2\beta-1}(2F_1-F_2) - \xi^{2\beta-1}(2F_3-F_4)}{\gamma^{2\beta-1}F_2 + \xi^{2\beta-1} F_4}\right\} . \qquad \text{...(3.83)}$$

Since α cannot be assigned the value unity in this case, there is no solution here which can be expressed in terms of the complete elliptic integrals given by the forms (3.48).

3.3 Alfven's Approximate Drift Velocity

In Cases 1 and 2 of the previous section the electron does not cross the neutral plane, and Alfven's drift velocity is readily obtained from the exact results (3.47) and (3.61) as follows.

Considering first the ratio of hypergeometric functions appearing in the result (3.47), use of series A1 of Appendix 1 and the binomial expansion $(1 + X)^{-1} = 1 - X + X^2 - X^3 + \ldots$ leads , for $\sigma << 1$, to the approximate result

$$\frac{F(-\beta,\frac{1}{2};1;\sigma)}{F(1-\beta,\frac{1}{2};1;\sigma)} \doteqdot 1 - \frac{1}{2}\sigma - \frac{1}{8}(1-\beta)\sigma^2 , \qquad \ldots(3.84)$$

where it can be seen from equation (3.34) that the smallness of σ relative to unity implies orbital motion far from the neutral plane. Insertion of the result (3.84) into the equation (3.47) yields the first order β dependent Alfven drift velocity expression

$$\frac{v_D}{v} = -\frac{1}{4}(1-\beta)\sigma . \qquad \ldots(3.85)$$

When $\alpha = 1$, $\beta = \frac{1}{2}$ equation (3.34) above gives $\sigma = \frac{4\rho_2}{x_2}$ and equation (3.84) reduces to

$$\frac{v_D}{v} = -\frac{1}{8}\sigma , \qquad \ldots(3.86)$$

as obtained by Seymour (1959, Section III, Case 1).

Consider now the ratio of confluent hypergeometric functions appearing in the result (3.61) for $\alpha = -1$, the use of series (A6) of Appendix 2 similarly leads for $\nu << 1$, to the approximate result

$$\frac{M(\frac{3}{2};2;-\nu)}{M(\frac{1}{2};1;-\nu)} \doteqdot 1 - \frac{\nu}{4} . \qquad \ldots(3.87)$$

Substitution of this result into equation (3.61) gives the Alfven drift velocity expression

$$\frac{v_D}{v} = \frac{\nu}{4} \qquad \text{for } \alpha = -1 . \qquad \ldots(3.88)$$

From equation (3.34)

$$\frac{\rho_o}{\rho_2} = \left(\frac{x_2}{x_o}\right)^{\frac{1-\beta}{\beta}} \quad . \qquad \ldots(3.89)$$

Elimination of σ from equation (3.85) by means of equation (3.34) leads to

$$\frac{v_D}{v} = -\frac{1}{2}\,\frac{1-\beta}{\beta}\,\frac{\rho_2}{x_2} \quad . \qquad \ldots(3.90)$$

When α approaches -1, β approaches infinity, and the limiting forms of equations (3.89) and (3.90) lead to the conclusion that

$$\frac{v_D}{v} = \frac{\nu}{4} \quad , \qquad \ldots(3.91)$$

consistent with result (3.88) in the small-perturbation limit.

4. Discussion of Results

In Cases 1 and 3 of Section 2, which yielded Δy, T and v_D/v in terms of hypergeometric functions, it was noted that when $\alpha = 1$ these results assumed forms containing the well-tabulated complete elliptic integrals E and K (see e.g. Byrd and Friedman, 1971).

For $\alpha = -2$ in Case 1 equation (3.47) simplifies by means of linear transformations of hypergeometric functions (Abramowitz and Stegun, 1965) to

$$\frac{v_D}{v} = \frac{\sigma}{\sigma - 2} \qquad \ldots(3.92)$$

where, from equation (3.34), $\sigma = -\dfrac{2\rho_2}{x_2}$,

with

$$-\infty \leq \sigma \leq 0 \ . \qquad \ldots(3.93)$$

Similarly, when $\alpha = -3$ in Case 1 equation (3.47) becomes

$$\frac{v_D}{v} = -\left\{\frac{2}{\sigma}\left(1 - \frac{K}{E}\right) - 1\right\} \quad , \qquad \ldots(3.94)$$

where the complete elliptic integrals K and E have modulus $k = \left(\frac{\sigma}{\sigma - 1}\right)^{\frac{1}{2}}$; with $\sigma = -\frac{4\rho_2}{x_2}$ and

$$0 \leq k \leq 1 \quad . \qquad \ldots(3.95)$$

The results (3.92) and (3.94) were used respectively to obtain the curves for $\alpha = -2$ and $\alpha = -3$ in Figure 7. However, in general, when particular values of α are chosen in Cases 1, 3 and 4 the hypergeometric functions so determined are not related to well tabulated functions of mathematical physics, and since F(a,b;c;x) is not extensively tabulated for the great range and variety of combinations of a,b,c and x that may be encountered in practical situations, the most effective way of utilizing the principal results for v/v_D given by equations (3.47), (3.73) and (3.83) of Cases 1, 3 and 4 of Section 2 is to first calculate the specific F(a,b;c;x) that are required.

Although the confluent hypergeometric function in Case 2 is better tabulated (see e.g. Slater, 1960; and Abramowitz and Stegun, 1965, pp503-535) the above conclusion appeared to apply to the numerical evaluation of equation (3.61), expressing v_D/v in terms of M(a,b;x). Accordingly, P.W. Seymour developed programs for use with the Monroe 1656 desk top computer/calculator (having 256 program step capacity) to obtain F(a,b;c;x) and M(a,b;x), and hence normalized drift velocity characteristics for selected values of α in Cases 1,2,3 and 4 as shown in Figure 7. While calculations leading to Figure 7 were tedious, it is a simple matter to extend the Alfven drift velocity region to smaller values of v_D/v for selected values of α by employing the small perturbation results (3.85) and (3.88).

The motion of a charged particle in the magnetic field of a straight current carrying conductor of infinite length has been investigated by

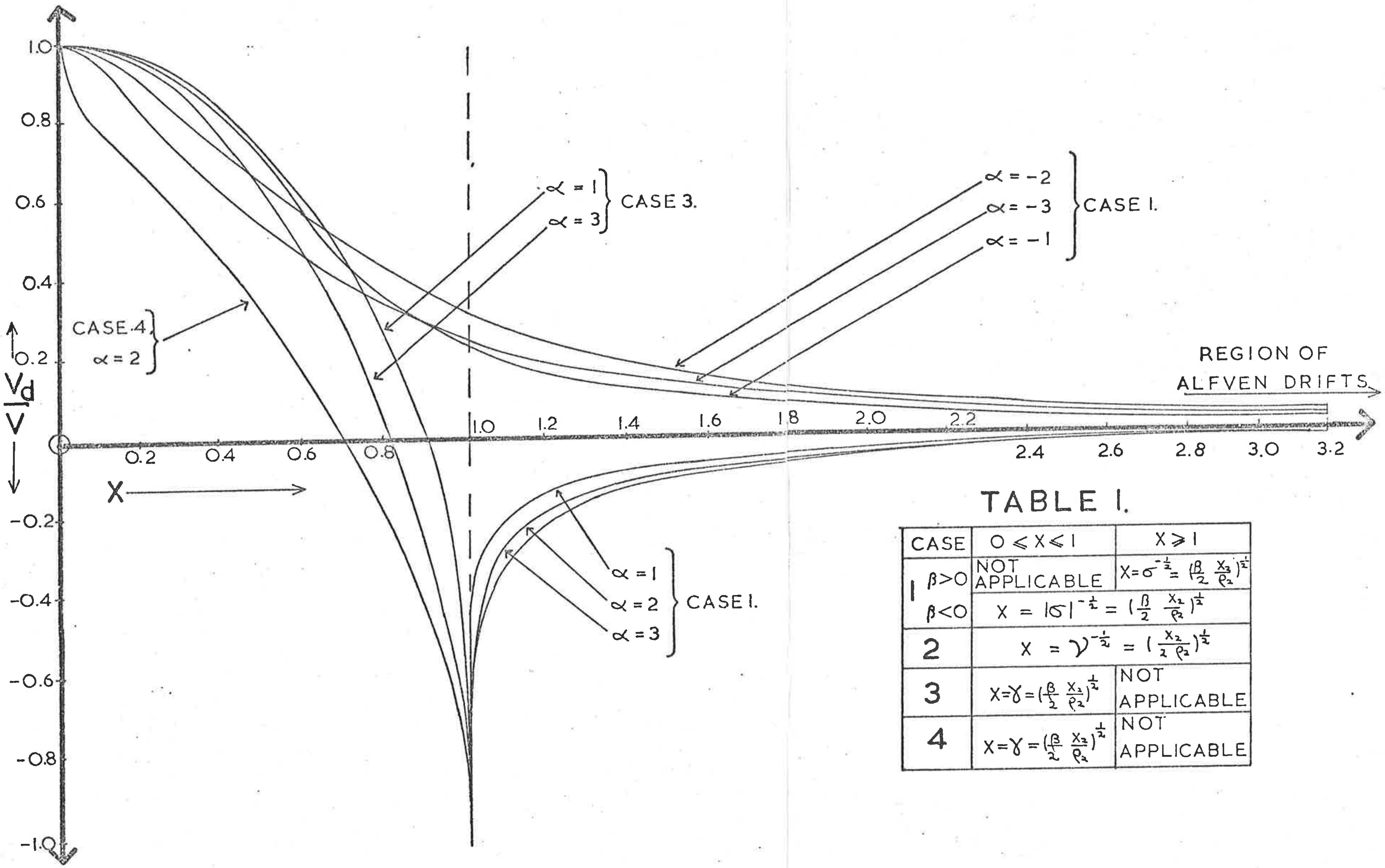


TABLE I.

CASE	$0 \leqslant X \leqslant 1$	$X \geqslant 1$
1 $\beta>0$	NOT APPLICABLE	$X=\sigma^{-\frac{1}{2}} = (\frac{\beta}{2}\frac{X_2}{\rho_2})^{\frac{1}{2}}$
1 $\beta<0$	$X = \lvert\sigma\rvert^{-\frac{1}{2}} = (\frac{\beta}{2}\frac{X_2}{\rho_2})^{\frac{1}{2}}$	
2	$X = \nu^{-\frac{1}{2}} = (\frac{X_2}{2\rho_2})^{\frac{1}{2}}$	
3	$X=\gamma=(\frac{\beta}{2}\frac{X_2}{\rho_2})^{\frac{1}{2}}$	NOT APPLICABLE
4	$X=\gamma=(\frac{\beta}{2}\frac{X_2}{\rho_2})^{\frac{1}{2}}$	NOT APPLICABLE

FIGURE 7: ELECTRON DRIFT VELOCITY CURVES V_d/V PLOTTED AGAINST THE PARAMETER X AS DEFINED IN TABLE I FOR CASES 1,2,3 AND 4.

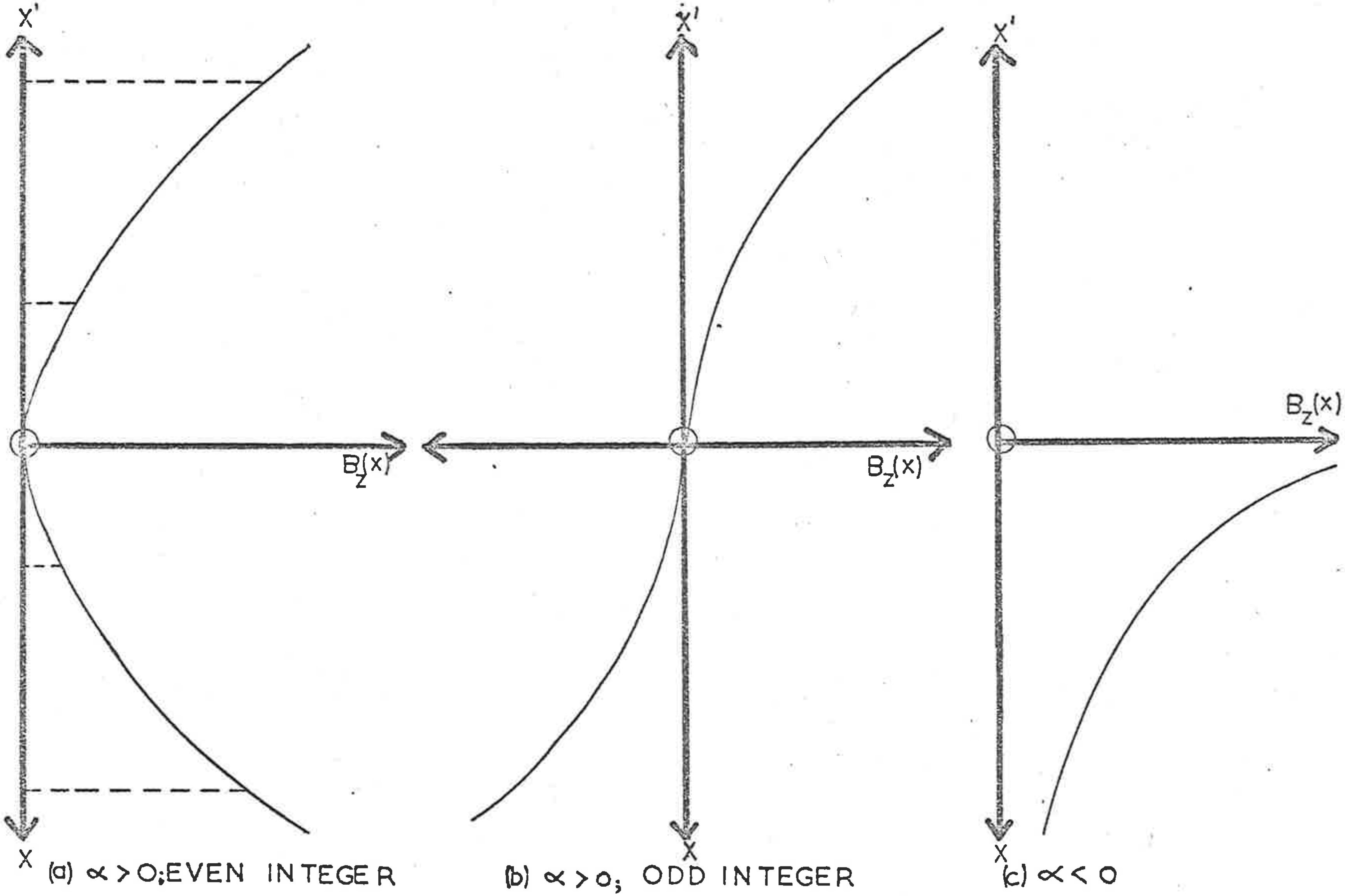


FIGURE 8: VARIATION OF MAGNETIC FIELD $B_z = \lambda x^{\alpha}$ FOR DIFFERENT α

Hertweck (1959). In terms of a parameter λ (the reduced angular momentum of the particle) Hertweck considers four special cases of particle motion in the magnetic field external to the conductor. The only case which is analytically tractable is that of $\lambda = 0$, in which the particle motion is confined to a meridian plane. Detailed examination of Hertweck's case of $\lambda = 0$ shows that it corresponds in effect to the present Case 2 of $\alpha = -1$. Expressing the ratio v_D/v for an electron in terms of Hertweck's analysis parameters,

$$\frac{v_D}{v} = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}\frac{\Delta\zeta}{\Delta\tau} = -\frac{iJ_1(i(2\varepsilon)^{\frac{1}{2}})}{J_0(i(2\varepsilon)^{\frac{1}{2}})}, \qquad \ldots(3.96)$$

where ε is a dimensionless parameter formed from the ratio of two energies. Using standard Bessel function theory (e.g. Bell, 1968), equation (3.96) can be expressed in terms of modified Bessel functions as

$$\frac{v_D}{v} = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}\frac{\Delta\zeta}{\Delta\tau} = \frac{I_1[(2\varepsilon)^{\frac{1}{2}}]}{I_0[(2\varepsilon)^{\frac{1}{2}}]} . \qquad \ldots(3.97)$$

It is readily shown that $(2\varepsilon)^{\frac{1}{2}} = \nu/2$, and so

$$\frac{v_D}{v} = \frac{I_1\left(\frac{\nu}{2}\right)}{I_0\left(\frac{\nu}{2}\right)} , \qquad \ldots(3.98)$$

an alternative form of our result (3.61). In direct confirmation of this result one can show that equation (3.59) may be further expressed as

$$\Delta y = 2\pi\rho_0 I_1\left(\frac{\nu}{2}\right) , \qquad \ldots(3.99)$$

and equation (3.60) as

$$T = \frac{2\pi\rho_0}{v} I_0\left(\frac{\nu}{2}\right) , \qquad \ldots(3.100)$$

after a number of transformations, whereupon, with $v_D = \Delta y/T$ the result (3.98) is immediately obtained. Calculations of v_D/v from equation

(3.98) using suitable mathematical tables of modified Bessel functions (1937, 1952) leads to results agreeing generally to four decimal places with those calculated from equation (3.61) by means of the Monroe 1656 desk top computer/calculator approach described above. The advantage of this latter approach is of course, that for all characteristics appearing in Figure 7 plotted points can be obtained as required for any selected values of the variable X, whereas for the particular characteristics which may be plotted using appropriate mathematical tables one does not have this freedom of choice, which can prove inconvenient.

Referring to Table 1 of Figure 7, for the region $0 \leq X \leq 1$ the angle ψ_o corresponding to zero drift velocity is found from the numerical work to be approximately -40^o for $\alpha = 1$, precisely zero for $\alpha = 2$ and approximately -21^o for $\alpha = 3$. Referring to Figure 6, for $\alpha = 2$ the electron drift in the negative direction of the y-axis the left-hand part of the drift cycle has zero displacement when $\psi_o \approx -28^o$. For electron drift in the positive direction of the y-axis the right-hand part of the cycle has zero displacement when $\psi_o \approx 28^o$.

This work on a magnetic field with power law dependence has been accepted for publication in the Australian Journal of Physics (1975).

CHAPTER 4

4.1 Trajectory of an Electron in a Spatially Dependent Magnetic Field

The orbits of the electron will be analysed in terms of ϕ which is connected to ψ by the relation $\psi = \frac{\pi}{2} + 2\phi$. Only half of the orbit will be analysed for bound orbits as the rest of the orbit can be obtained from the first half cycle. Similarly only half of the trajectory will be considered when analysing unbound orbits. The position, P, of the electron at time, $t = t(\phi)$ is given by

$$P(\phi) = (x(\phi),y(\phi)) \,, \qquad \ldots(4.1a)$$

and for bound orbits $t = y = 0$ for $\psi = \frac{\pi}{2}$ and $x = x_2$. Thus $\phi = 0$, and

$$P(0) = (x_2,0) \,. \qquad \ldots(4.1b)$$

Equation (4.1b) will also give the initial conditions for an unbound path in an exponential magnetic field in which the trajectory of the electron asymptotes to an angle ψ', (and ϕ'), as x approaches $-\infty$.

For a magnetic field with a power law dependence in which $x > 0$ and $\alpha < -1$, unbound orbits will asymptote to an angle ψ' as x approaches $+\infty$. For this type of trajectory the initial conditions will be taken at $\psi = \frac{3\pi}{2}$ where x has its minimum value x_1 and $t = y = 0$. Using the substitution $\psi = \frac{3\pi}{2} + 2\theta$ the position of the electron at time $t = t(\theta)$ is

$$P(\theta) = (x(\theta),y(\theta)), \qquad \ldots(4.2a)$$

with the initial condition at $t(0) = 0$ given by

$$P(0) = (x_1,0) \,. \qquad \ldots(4.2b)$$

As ψ approaches ψ', θ approaches θ' so that $\psi' = \frac{3\pi}{2} + 2\theta'$.

The appropriate equations required to analyse the trajectories are found in Section 2.3, part (II), (a) of Chapter 2.

4.2 Exponentially Varying Magnetic Field

The magnetic field is again of the form

$$B_z = \lambda e^{\alpha x} \quad , \qquad \ldots(3.1)$$

where λ and $\alpha > 0$, as shown in Chapter 3, Figure 1(b).

(i) Bound Orbits

A typical orbit is shown in Figure 1(a). From equations (2.27), (3.1), (3.2), (3.3), (3.4) and (3.5) the time is given by

$$t(\psi) = \int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{\left(1 + \frac{mv}{eA_o} \sin\psi\right)} \quad , \qquad \ldots(4.3)$$

and $\psi = \frac{\pi}{2}$ when $t = 0$. Using the substitution $\psi = \frac{\pi}{2} + 2\phi$ followed by $\tan\theta = \left\{\frac{1 - \frac{mv}{eA_o}}{1 + \frac{mv}{eA_o}}\right\}^{\frac{1}{2}} \tan\phi$, the time becomes

$$t(\phi) = \frac{2}{\omega_o\left(1 - \left(\frac{mv}{eA_o}\right)^2\right)^{\frac{1}{2}}} \tan^{-1}\left\{\left(\frac{1 - \frac{mv}{eA_o}}{1 + \frac{mv}{eA_o}}\right)^{\frac{1}{2}} \tan\phi\right\} \quad . \qquad \ldots(4.4)$$

Similarly from equations (2.28), (3.1), (3.2), (3.3), (3.4), (3.5) and (4.3) y is given by

$$y(\psi) = \frac{v}{\omega_o} \int_{\frac{\pi}{2}}^{\psi} \frac{\sin\psi d\psi}{\left(1 + \frac{mv}{eA_o} \sin\psi\right)} \quad ,$$

or

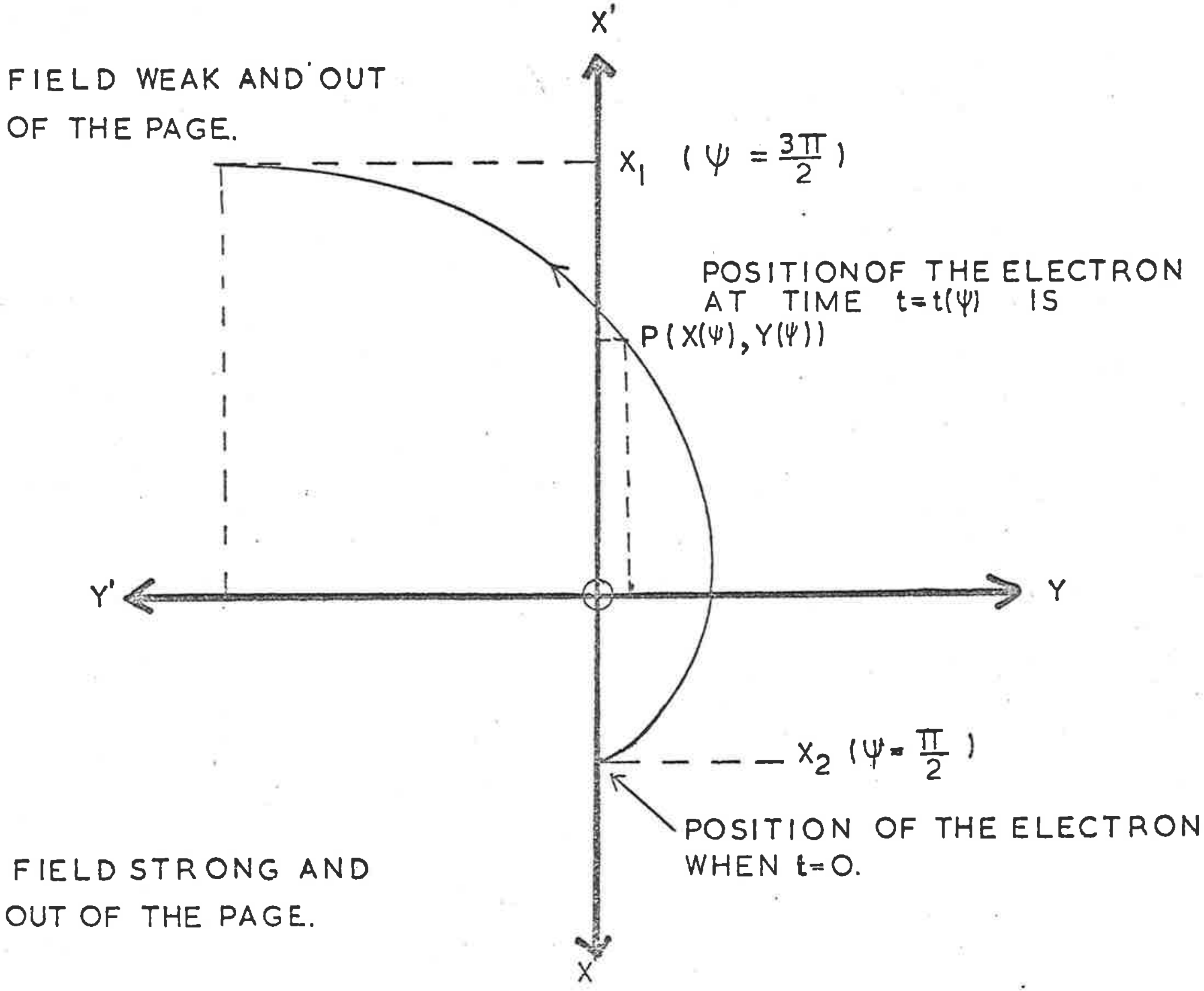


FIGURE 1(a): BOUND ORBIT OF AN ELECTRON IN THE FIELD $B_z = \lambda e^{\alpha x}$

$$y(\phi) = \frac{2}{\alpha}\left\{\phi - \frac{\tan^{-1}}{\left(1 - \frac{mv}{eA_o}\right)^{2}\Big)^{\frac{1}{2}}} \left\{\left(\frac{1 - \frac{mv}{eA_o}}{1 + \frac{mv}{eA_o}}\right)^{\frac{1}{2}} \tan\phi\right\}\right\}. \qquad \ldots(4.5)$$

From equation (3.3) x becomes

$$x(\phi) = x_2 + \frac{1}{\alpha}\,\ell n\,(\cos^2\phi(1 + \tan^2\theta)) \quad , \qquad \ldots(4.6)$$

with the aid of equation (3.4). The position $P = (x(\phi), y(\phi))$ of the electron at time $t(\phi)$ is given by equations (4.5) and (4.6). Orbits are bound when

$$\frac{mv}{eA_o} < 1 \;. \qquad \ldots(4.7)$$

(b) Unbound Orbits

If equation (4.7) is invalid the electron path is unbounded and ψ approaches ψ' as x approaches $-\infty$. From equation (3.2) and with reference to Figure 1(b)

$$\sin\psi - \sin\psi' = \frac{e\lambda}{mv}\,\frac{\ell^{\alpha x}}{\alpha} \quad , \qquad \ldots(4.8)$$

with ψ_o chosen to be the angle ψ' at $x' = -\infty$.

Using the substitution $\psi = \frac{\pi}{2} + 2\phi$, equation (4.8) yields

$$x(\phi) = \frac{1}{\alpha}\,\ell n\,\left\{\frac{2\alpha mv}{e\lambda}\,(\sin^2\phi' - \sin^2\phi)\right\} \quad , \qquad \ldots(4.9)$$

as the x coordinate. This has a maximum value x_2 such that

$$x_2 = \frac{1}{\alpha}\,\ell n\,\left(\frac{2\alpha mv}{e\lambda}\,\sin^2\phi'\right) \quad , \qquad \ldots(4.10)$$

when $\phi = 0$. From equations (3.1), (4.8), (4.9) and (2.27) and using the substitution $\psi = \frac{\pi}{2} + 2\phi$, the time becomes

$$t(\psi) = \frac{1}{\alpha v}\int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{(\sin\psi - \sin\psi')}$$

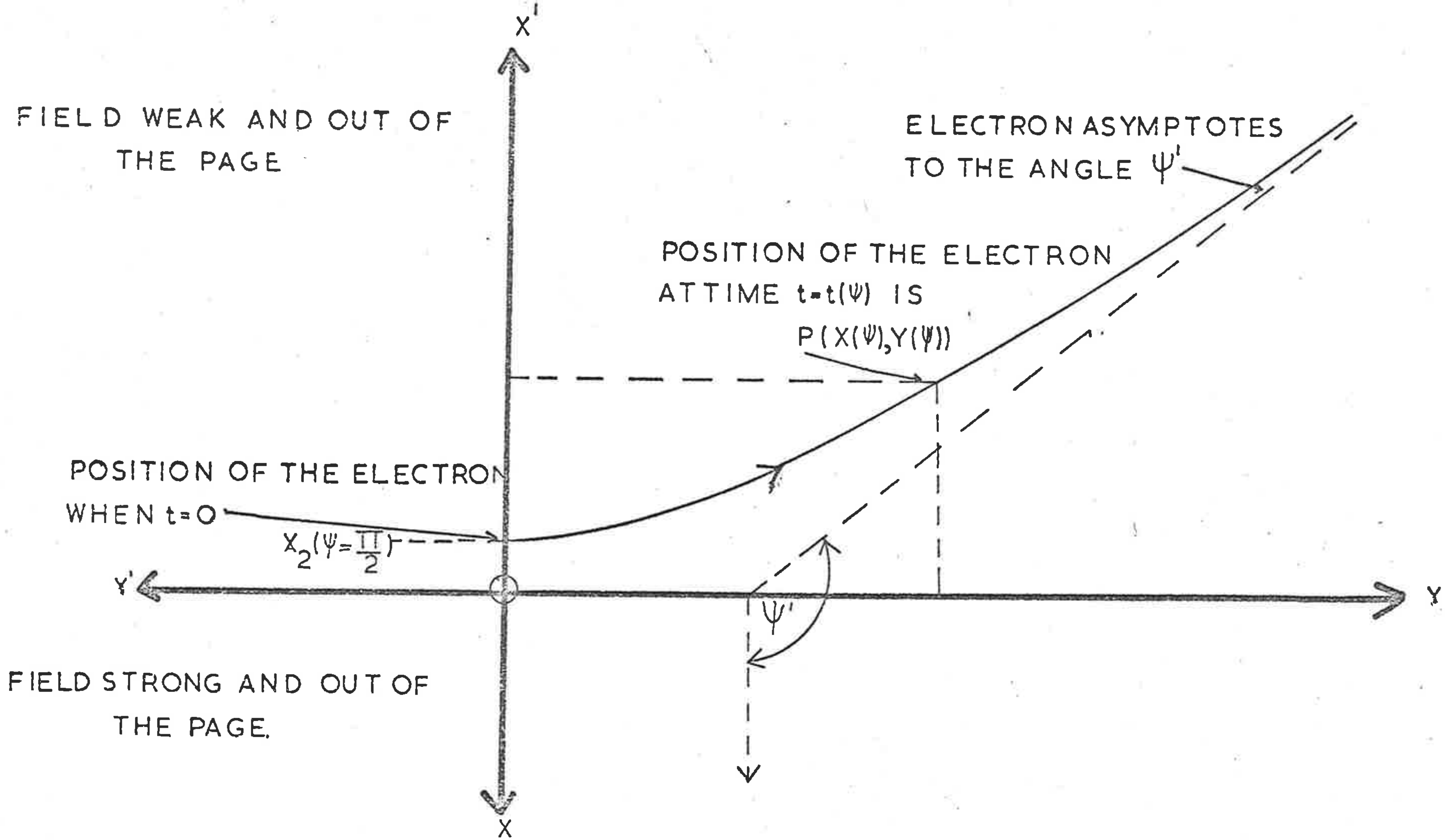


FIGURE 1(b): UNBOUND ORBIT OF AN ELECTRON IN THE FIELD $B_z = \lambda e^{\alpha x}$

$$= \frac{1}{\alpha v \sin^2\phi'} \int_o^{\phi} \frac{\sec^2\phi d\phi}{(1 - (\cot^2\phi')\tan^2\phi)} , \qquad \ldots(4.11)$$

and with the aid of the substitution r = tanφ equation (4.11) gives

$$t(r) = \frac{1}{\alpha v \sin^2\phi'} \int_o^{y} \frac{dy}{(1 - (\cot^2\phi')r^2)}$$

$$= \frac{1}{2\alpha v \sin^2\phi'} \left\{ \int_o^{y} \frac{dy}{(1 - (\cot\phi')r)} + \int_o^{y} \frac{dy}{(1 + (\cot\phi')r)} \right\} .$$

...(4.12)

Finally, if $X = 1 - (\cot\phi')r$ is substituted into the first integral $X' = 1 + (\cot\phi')r$ is substituted into the second integral of equation (4.12), the time reduces to

$$t(\phi) = \frac{1}{\alpha v \sin 2\phi'} \left\{ \ell n \left(\frac{1 + \cot\phi' \tan\phi}{1 - \cot\phi' \tan\phi} \right) \right\} . \qquad \ldots(4.13)$$

Similarly from equations (3.1), (4.8), (4.9), (2.28) and (4.13) the y coordinate becomes

$$y(\psi) = \frac{1}{\alpha} \int_{\frac{\pi}{2}}^{\psi} \frac{\sin\psi d\psi}{(\sin\psi - \sin\psi')} ,$$

and thus in terms of ϕ, y is given by

$$y(\phi) = \frac{2\phi}{\alpha} + \frac{\cot 2\phi'}{\alpha} \ell n \left\{ \frac{1 + \cot\phi' \tan\phi}{1 - \cot\phi' \tan\phi} \right\} . \qquad \ldots(4.14)$$

(c) Discussion of Solutions

Figure 1(f) shows a typical bound orbit for which $mv/eA_o = 0.5$. Figures 1(c), (d) and (e) show different unbound orbits for $\alpha mv/e\lambda = 1$ and ϕ' ranging through 30^o, 45^o and 60^o respectively. From equation (4.14), as $\phi \to \phi'$, $1 - \cot\phi'\tan\phi \to 0$. Therefore y becomes infinite if $\phi' \neq 45^o$. When $\phi' = 45^o$ the orbit is such that the maximum value of y is $y_{max} = \frac{\pi}{2\alpha}$ when $\phi' = \frac{\pi}{4}$ radians. This is analogous to a reflection at right angles

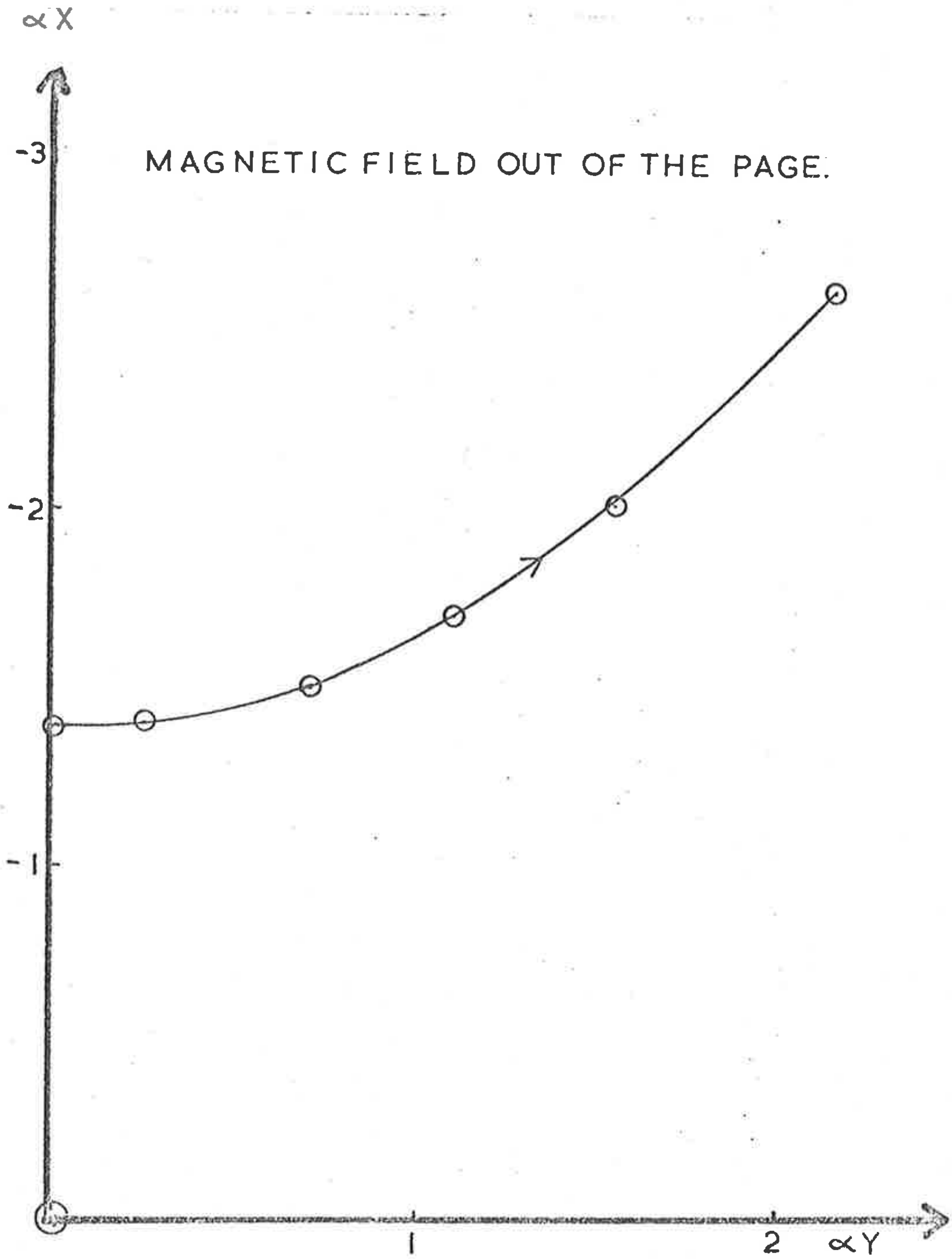


FIGURE 1(c): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda e^{\alpha x}$.
UNBOUND ORBIT IN WHICH $\phi' = 30°$

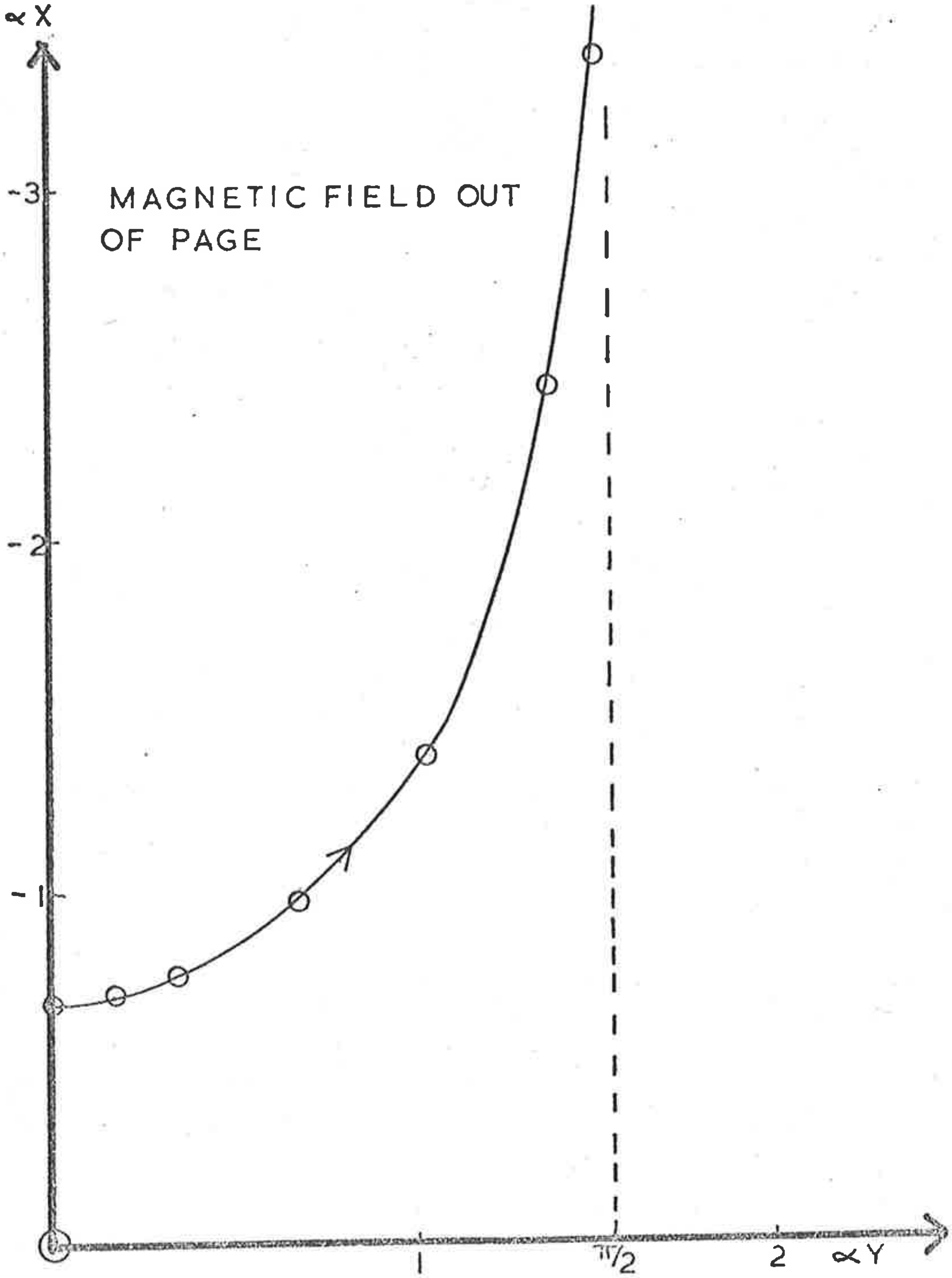


FIGURE 1(d) ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda e^{\alpha x}$
UNBOUND ORBIT IN WHICH $\phi' = 45°$

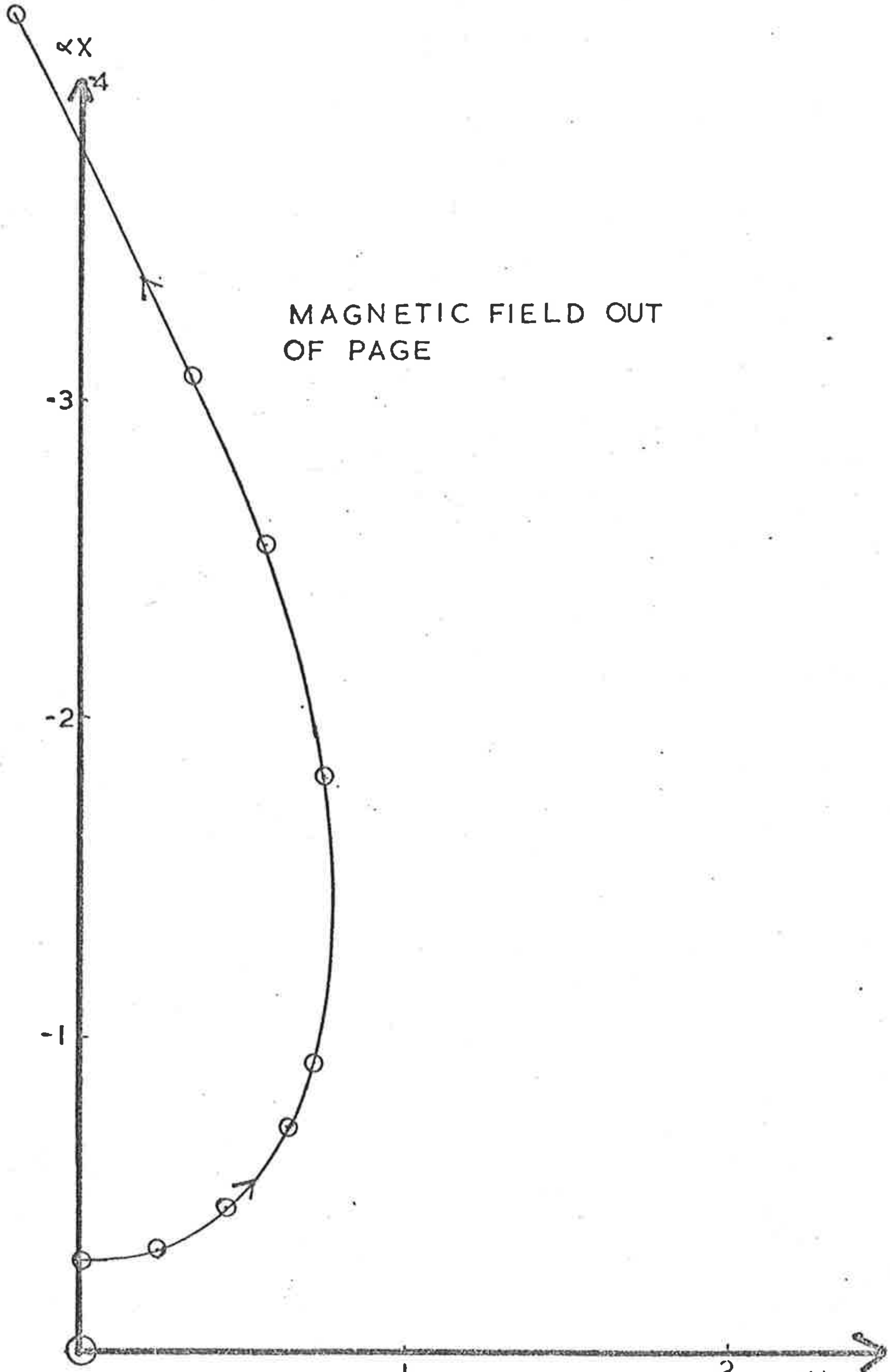


FIGURE I(c) ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda e^{\alpha x}$ UNBOUND ORBIT IN WHICH $\phi' = 60°$

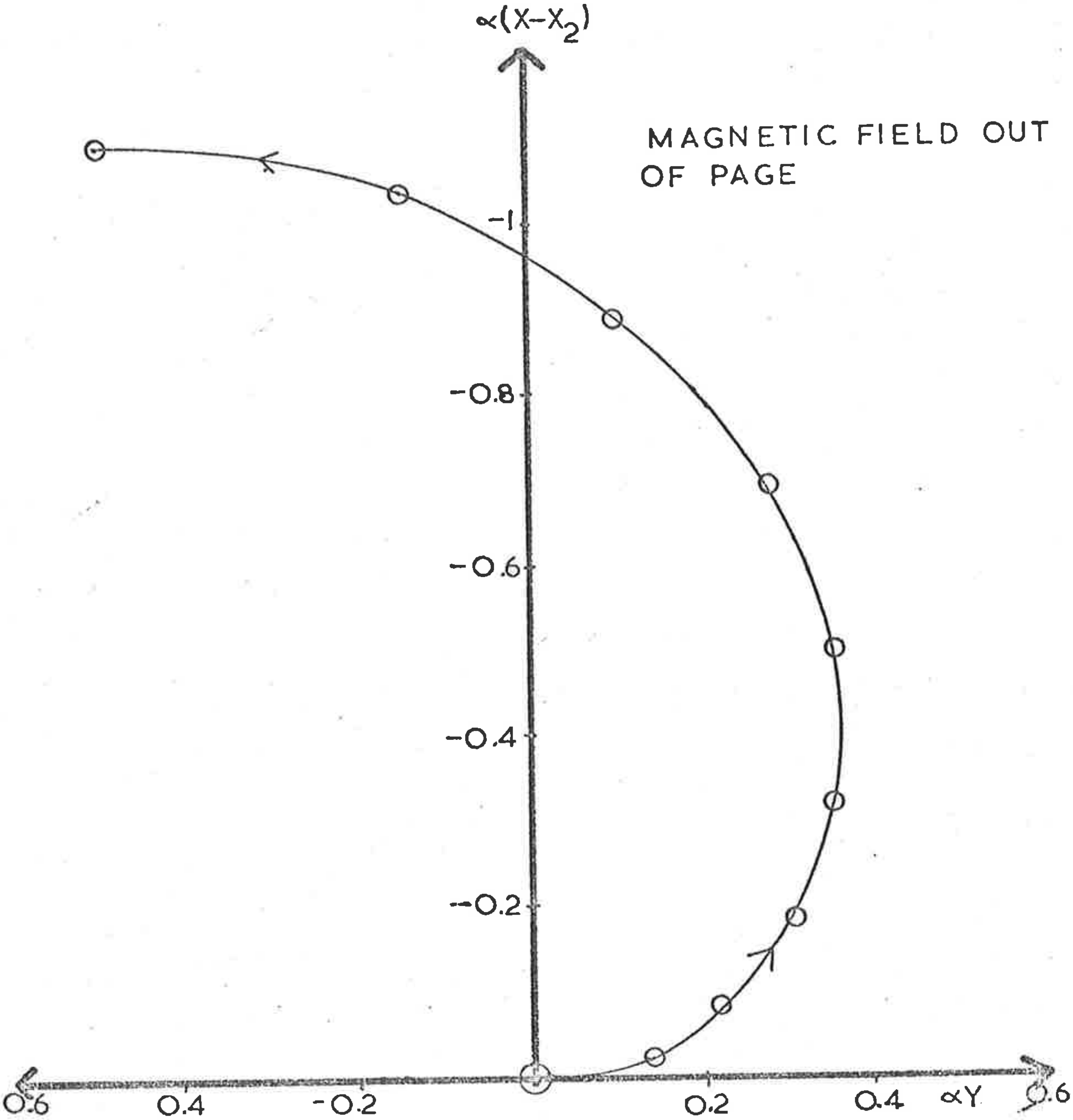


FIGURE 1(f) ELECTRON TRAJECTORY IN THE FIELD B_z
BOUND ORBIT IN WHICH $\frac{mv}{eA_0} = 0.5$

to the line $y = 0$. If $0 < \phi' < \frac{\pi}{4}$, then the electron moves always in the positive y direction, whilst if $\frac{\pi}{4} < \phi' < \frac{\pi}{2}$ the electron orbit loops and eventually moves in the negative y direction.

4.3 Magnetic Field with a Power Law Dependence $B_z = \lambda x^{\alpha}$

Case 1: Bound Orbits not Crossing tne Line $x = 0, \alpha \neq -1$.

Figures 2(a) and (b) show the appropriate half cycles for both $\alpha > 0$ and $\alpha < 0$ respectively. From equations (3.19), (3.21), (3.34) and the substitution $\psi = \frac{\pi}{2} + 2\phi$, x becomes

$$x(\psi) = x_o\left(1 + \frac{mv\sin\psi}{eA_o}\right)^{\beta}$$

$$= x_2(1 - \sigma\sin^2\phi)^{\beta} . \qquad ...(4.17)$$

Using equations (3.15), (3.16), (3.17), (3.19) and (2.27) the time becomes

$$t(\psi) = \frac{1}{\omega_o}\int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{\left(1 + \frac{mv}{eA_o}\sin\psi\right)^{\frac{1}{\beta}}} ,$$

or

$$t(\phi) = \frac{\sigma\beta x_2}{v}\int_o^{\phi} \frac{d\phi}{(1 - \sigma\sin^2\phi)^{1-\beta}} , \qquad ...(4.18)$$

and similarly equations (3.15), (3.16), (3.17), (3.19) and (2.28) give

$$y(\psi) = \frac{v}{\omega_o}\int_{\frac{\pi}{2}}^{\psi} \frac{\sin\psi d\psi}{\left(1 + \frac{mv}{eA_o}\sin\psi\right)^{\frac{1}{\beta}}} ,$$

thus

$$y(\phi) = \beta x_2\left\{2\int_o^{\phi}(1 - \sigma\sin^2\phi)^{\beta}d\phi - (2-\sigma)\int_o^{\phi}\frac{d\phi}{(1 - \sigma\sin^2\phi)^{1-\beta}}\right\} , \qquad ...(4.19)$$

with the same notation as in Chapter 3.

If ϕ were chosen to be $\frac{\pi}{2}$ equations (4.18) and (4.19) would give precisely half the results of equations (3.43) and (3.33). This is

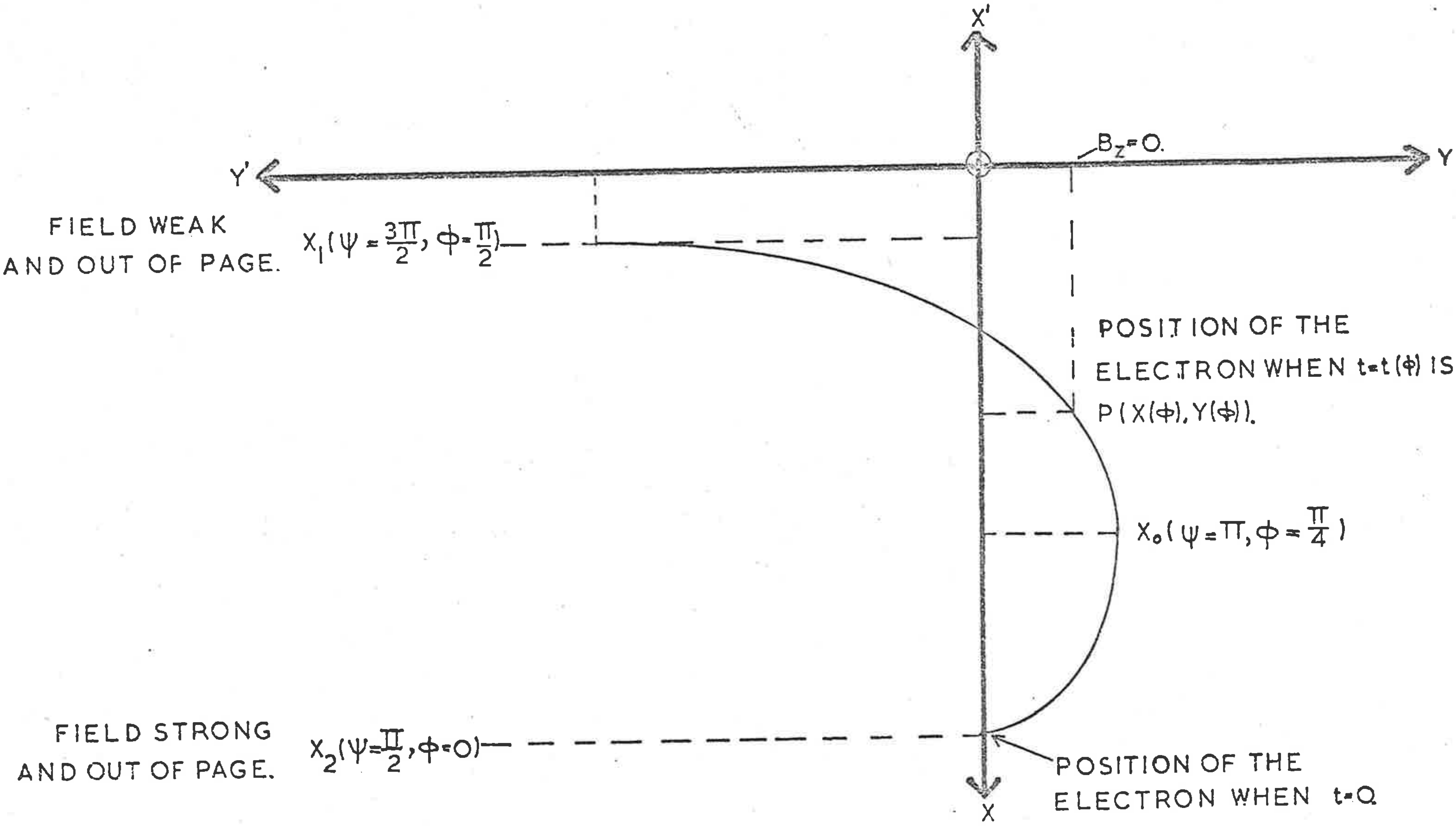


FIGURE 2(a) BOUND ORBIT OF AN ELECTRON IN THE FIELD $B_z = \lambda X^{\alpha}$; $\alpha > 0$.
TRAJECTORY DOES NOT CROSS THE LINE $X = 0$

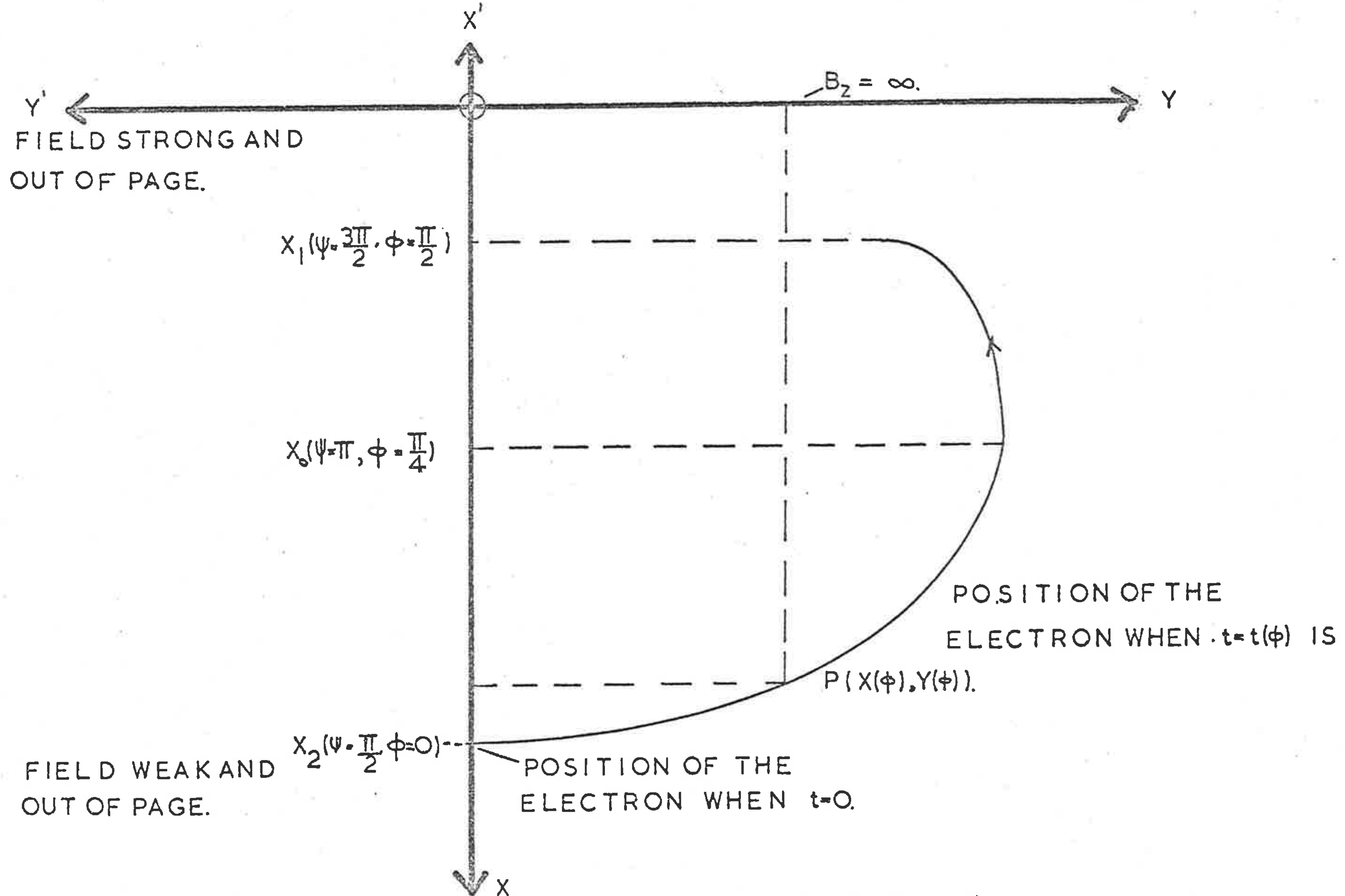


FIGURE 2(b): BOUND ORBIT OF AN ELECTRON IN THE FIELD $B_z = \lambda X^{\alpha}$; $\alpha < 0$. TRAJECTORY CANNOT CROSS THE LINE $X = 0$.

because the trajectory results have been analysed over a half cycle only.

Defining

$$\mathcal{F}(a,b;c;x;\phi) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^{\phi} \frac{(\sin\phi)^{2b-1}(\cos\phi)^{2c-2b-1}d\phi}{(1 - x\sin^2\phi)^a} , \qquad \ldots(4.20)$$

as a curly-F function, (see Appendix 3) then equations (4.18) and (4.19) become

$$t(\phi) = \frac{\pi\sigma\beta x_2}{2v} \mathcal{F}\ (1-\beta,\tfrac{1}{2};1;\sigma;\phi) \quad , \qquad \ldots(4.21)$$

and

$$y(\phi) = \frac{\pi\beta x_2}{2} \{2(\mathcal{F}(-\beta,\tfrac{1}{2};1;\sigma;\phi) - \mathcal{F}(1-\beta,\tfrac{1}{2};1;\sigma;\phi))$$

$$+ \sigma\mathcal{F}(1-\beta,\tfrac{1}{2};1;\sigma;\phi)\} \quad . \qquad \ldots(4.22)$$

In particular if $\alpha = 1$ and $\beta = \frac{1}{2}$

$$\frac{\pi}{2} \mathcal{F}(\tfrac{1}{2},\tfrac{1}{2};1;k_1^2;\phi) = \int_o^{\phi} \frac{d\phi}{(1 - k_1^2\sin^2\phi)^{\frac{1}{2}}} = F(k_1/\phi) ,$$

and $\qquad \ldots(4.23)$

$$\frac{\pi}{2} \mathcal{F}(-\tfrac{1}{2},\tfrac{1}{2};1;k_1^2;\phi) = \int_o^{\phi} (1 - k_1^2\sin^2\phi)^{\frac{1}{2}}d\phi = E(k_1/\phi) ,$$

where $F(k_1/\phi)$ and $E(k_1/\phi)$ are incomplete elliptic integrals of the first and second kinds respectively. Equations (4.21) and (4.22) therefore reduce to

$$t(\phi) = \frac{1}{\omega_2} F(k_1/\phi) \quad , \qquad \ldots(4.24)$$

$$y(\phi) = x_2(2(E(k_1/\phi) - F(k_1/\phi)) + k_1^2F(k_1/\phi)) , \qquad \ldots(4.25)$$

and equation (4.17) becomes

$$x(\phi) = x_2(1 - k_1^2\sin^2\phi)^{\frac{1}{2}} , \qquad \ldots(4.26)$$

with

$$k_1^2 = \frac{2mv}{eA_2} = \frac{mv}{ex_2^2} . \qquad \ldots(4.27)$$

Case 2: $\alpha = -1$. Electron does not cross the line $x = 0$ on which $B_z = \infty$

The orbit of the electron is shown in Figure 2(b). From equation (3.50) the x value becomes

$$x = x_o \, \ell^{\frac{mv}{e\lambda}\sin\psi} , \qquad \ldots(3.50)$$

which on using the substitution $\psi = \frac{\pi}{2} + \phi$ becomes

$$x(\phi) = x_o \, \ell^{\frac{mv}{e\lambda}\cos\phi} . \qquad \ldots(4.28)$$

From equation (3.54) the time yields

$$t(\psi) = \frac{1}{\omega_o} \int_{\frac{\pi}{2}}^{\psi} \ell^{\frac{mv}{e\lambda}\sin\psi} d\psi ,$$

$$= \frac{1}{\omega_o} \int_{o}^{\phi} \ell^{\frac{mv}{e\lambda}\cos\phi} d\phi , \qquad \ldots(4.29)$$

and similarly the value of y from equation (3.53) is given by

$$y(\phi) = \rho_o \int_{o}^{\phi} \ell^{\frac{mv}{e\lambda}\cos\phi} \cos\phi d\phi . \qquad \ldots(4.30)$$

In a similar manner as with the bound orbits $\alpha \neq -1$ a curly-M function is defined such that

$$\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)} \mathcal{M}(a,b;z;\theta) = 2^{1-b} \, \ell^{\frac{z}{2}} \int_{o}^{\theta} \ell^{-\frac{z}{2}\cos\phi} (\sin\theta)^{b-1} \left(\cot\left(\tfrac{\theta}{2}\right)\right)^{b-2a} d\theta , \qquad \ldots(4.31)$$

as shown in Appendix 4.

Thus equations (4.29) and (4.30) can be written

$$t(\phi) = T_o \, \frac{\ell^{\nu/2}}{2} \, \mathcal{M}(\tfrac{1}{2},1;-\nu;\phi) , \qquad \ldots(4.32)$$

and

$$y(\phi) = \pi\rho_o \ell^{\nu/2} \left(M(\tfrac{1}{2},1;-\nu;\phi) - M(\tfrac{3}{2},2;-\nu;\phi)\right) , \qquad \ldots(4.33)$$

using the nomenclature of Chapter 3 together with equation (4.31). If $\phi = \pi$, then equations (4.32) and (4.33) are precisely half the increments in equations (3.59) and (3.60) of Chapter 3.

Case 3: $\alpha > 0$. Electron crosses the line $x = 0$ on which $B_z = 0$.

The section of the trajectory to be considered as shown in Figure 3(a) is the region of ψ varying from $\frac{\pi}{2}$ to $\pi - \psi_o$. The remainder of a full cycle can be found in a similar manner for α both even and odd. From equation (3.63) the time becomes

$$t(\psi) = \frac{1}{v}\left(\frac{mv}{e\lambda}\right)^{\beta} \beta^{1-\beta} \int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{(\sin\psi - \sin\psi_o)^{1-\beta}} , \qquad \ldots(4.34)$$

and using $\psi = \frac{\pi}{2} + 2\phi$, with $\psi_o = \frac{\pi}{2} + 2\phi_o$ equation (4.34) yields

$$t(\phi) = \frac{2^{\beta-1}}{v}\left(\frac{mv}{e\lambda}\right)\beta^{1-\beta} \int_o^{\phi} \frac{d\phi}{(\sin^2\phi_o - \sin^2\phi)^{1-\beta}} . \qquad \ldots(4.35)$$

Substitution of $\sin\phi_F = \dfrac{\sin\phi}{\sin\phi_o}$ in equation (4.35) gives

$$t(\phi_F) = 2^{\beta}\left(\frac{mv}{e\lambda}\right)^{\beta} \gamma^{2\beta-1} \int_o^{\phi_F} \frac{(\cos\phi_F)^{2\beta-1} d\phi_F}{(1 - \sin^2\phi_o \sin^2\phi_F)^{\frac{1}{2}}} . \qquad \ldots(4.36)$$

Thus from equations (3.69), (4.20) and (4.36) the time becomes

$$t(\phi_F) = \frac{\Sigma\gamma^{2\beta-1}}{2v} \mathcal{F}_2(\gamma^2;\phi_F) , \qquad \ldots(4.37)$$

with

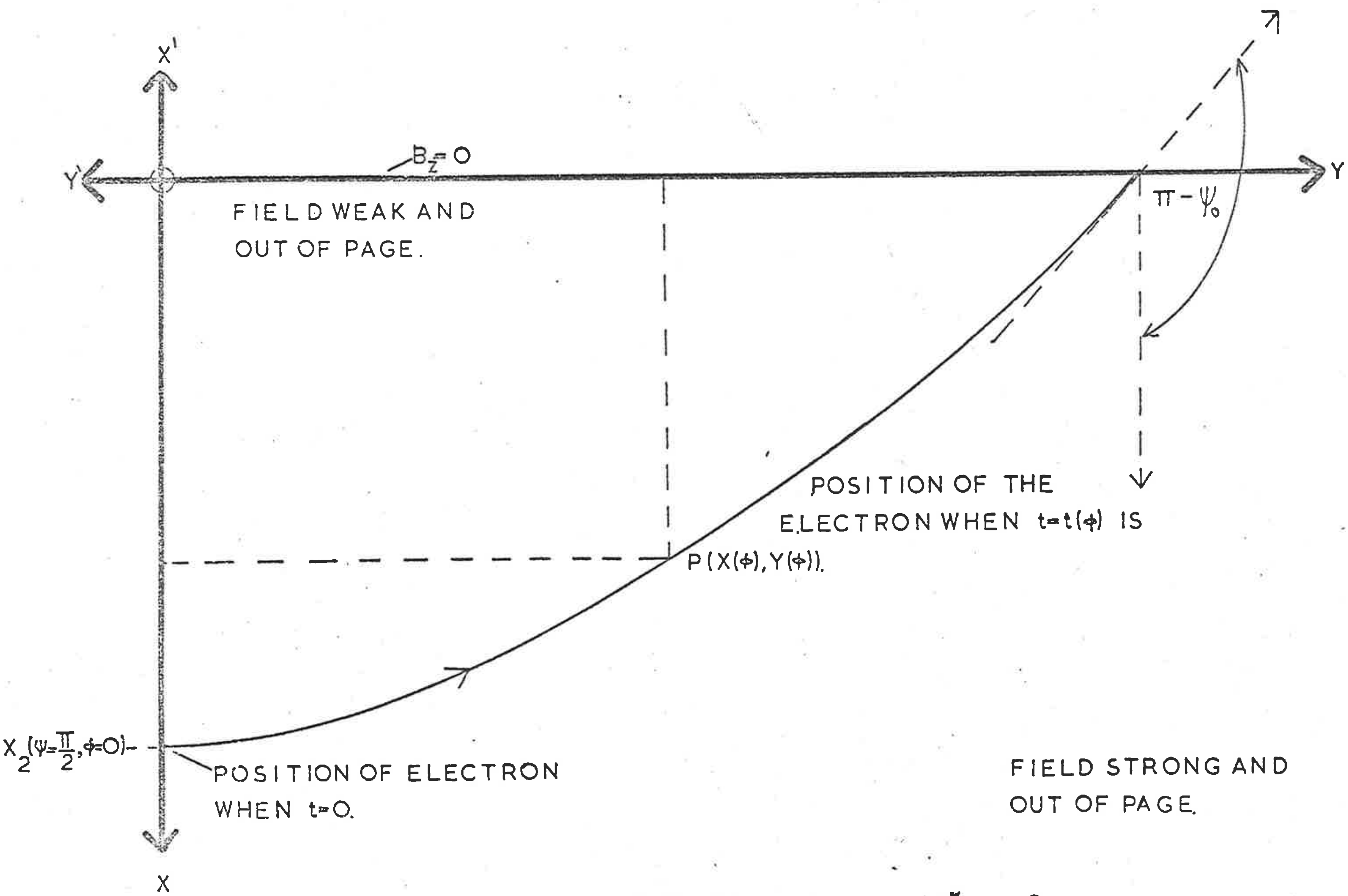


FIGURE 3(a): BOUND ORBIT OF AN ELECTRON IN THE FIELD $B_z = \lambda X^{\alpha}$, $\alpha > 0$. TRAJECTORY CROSSES THE LINE $X = 0$.

$$\left.\begin{aligned} \mathcal{F}_2(\gamma^2;\phi_F) &= \mathcal{F}(\tfrac{1}{2},\tfrac{1}{2};\beta+\tfrac{1}{2};\gamma^2;\phi_F) , \\ \text{and} \qquad & \\ \mathcal{F}_1(\gamma^2;\phi_F) &= \mathcal{F}(-\tfrac{1}{2},\tfrac{1}{2};\beta+\tfrac{1}{2};\gamma^2;\phi_F) . \end{aligned}\right\} \qquad \ldots(4.38)$$

Similarly from equation (3.62)

$$y(\phi_F) = \sum \gamma^{2\beta-1}(\mathcal{F}_1(\gamma^2;\phi_F) - \tfrac{1}{2}\mathcal{F}_2(\gamma^2;\phi_F)) . \qquad \ldots(4.39)$$

Using equation (3.24) and the substitution $\psi = \frac{\pi}{2} + 2\phi$, x becomes

$$x = \left(\frac{1}{\beta}\left(\frac{mv}{e\lambda}\right)(\sin\psi - \sin\psi_o)\right)^{\beta} ,$$

or

$$x(\phi_F) = \left\{\left(\frac{2mv}{\beta e\lambda}\right)\right\}^{\beta}\gamma^{2\beta}\cos^{2\beta}\phi_F . \qquad \ldots(4.40)$$

Thus equations (4.37), (4.39) and (4.40) constitute any point $P(\phi_F) = (x(\phi_F),y(\phi_F))$ of the orbit of the electron at a time $t = t(\phi_F)$.

If $\alpha = 1$, $\beta = \frac{1}{2}$ then equation (4.37) becomes

$$t(\phi_F) = \frac{1}{2v}\left(\frac{mv}{e\lambda}\right)^{\frac{1}{2}}\int_o^{\phi_F}\frac{d\phi}{(1-\gamma^2\sin^2\phi)^{\frac{1}{2}}}$$

$$= \frac{1}{2v}\left(\frac{mv}{e\lambda}\right)^{\frac{1}{2}} F(\gamma/\phi_F) , \qquad \ldots(4.41)$$

and from equation (4.39)

$$y(\phi_F) = \left(\frac{mv}{e\lambda}\right)^{\frac{1}{2}}(E(\gamma/\phi_F) - \tfrac{1}{2}F(\gamma/\phi_F)) . \qquad \ldots(4.42)$$

Also from equations (4.40) and (3.67)

$$x(\phi_F) = \left(\frac{4mv}{e\lambda}\right)^{\frac{1}{2}}\gamma\cos\phi_F , \qquad \ldots(4.43)$$

and thus

$$x(\phi_F) = x_2\cos\phi_F . \qquad \ldots(4.44)$$

The expressions $F(\gamma/\phi_F)$ and $E(\gamma/\phi_F)$ are elliptic integrals of the first and second kinds as defined in equation (4.23). When $\phi = \phi_o$, $\phi_F = \frac{\pi}{2}$ and equations (4.41) and (4.42) reduce to precisely one quarter the results obtained by Seymour (1959) (Section III, Case 2), as the present analysis ranges over one quarter of a cycle of the orbit of the electron.

Case 4: Unbound orbits $\alpha < -1$. As x approaches infinity the electron orbit asymptotes to a straight line cutting the x axis at an angle ψ'.

A typical unbound orbit is shown in Figure 3(b). From equations (2.29), (2.9) and (3.13) the x coordinate in terms of ψ and ψ' is

$$\sin\psi - \sin\psi' = \beta\left(\frac{e}{mv}\right) x^{1/\beta} , \qquad \ldots(4.45)$$

since $\alpha < -1$ and x is infinite at $\psi = \psi'$. From equation (4.45)

$$x = \left\{\frac{1}{\beta}\left(\frac{mv}{e\lambda}\right)(\sin\psi - \sin\psi')\right\}^{\beta} , \qquad \ldots(4.46)$$

and using the substitutions $\psi = \frac{3\pi}{2} + 2\phi$ and $\psi' = \frac{3\pi}{2} + 2\phi'$ equation (4.46) yields

$$x(\phi) = \left(\frac{1}{\beta}\left(\frac{2mv}{e\lambda}\right)(\sin^2\phi - \sin^2\phi')\right)^{\beta} . \qquad \ldots(4.47)$$

The minimum value of x is at $\psi = \frac{3\pi}{2}$ when $x = x_1$ and

$$x_1 = \left(-\frac{1}{\beta}\frac{2mv}{e\lambda}\sin^2\phi'\right)^{\beta} , \qquad \ldots(4.48)$$

where $\beta < 0$.

From equations (4.46), (3.13), (2.26), (2.27) and the substitution $\psi = \frac{3\pi}{2} + 2\phi$ and $\psi_o = \frac{3\pi}{2} + 2\phi_o$, the time becomes

$$t(\psi) = \frac{1}{v}\left(\frac{mv}{e\lambda}\right)^{\beta}(-\beta)^{1-\beta}\int_{\frac{3\pi}{2}}^{\psi}\frac{d\psi}{(\sin\psi'-\sin\psi)^{1-\beta}}$$

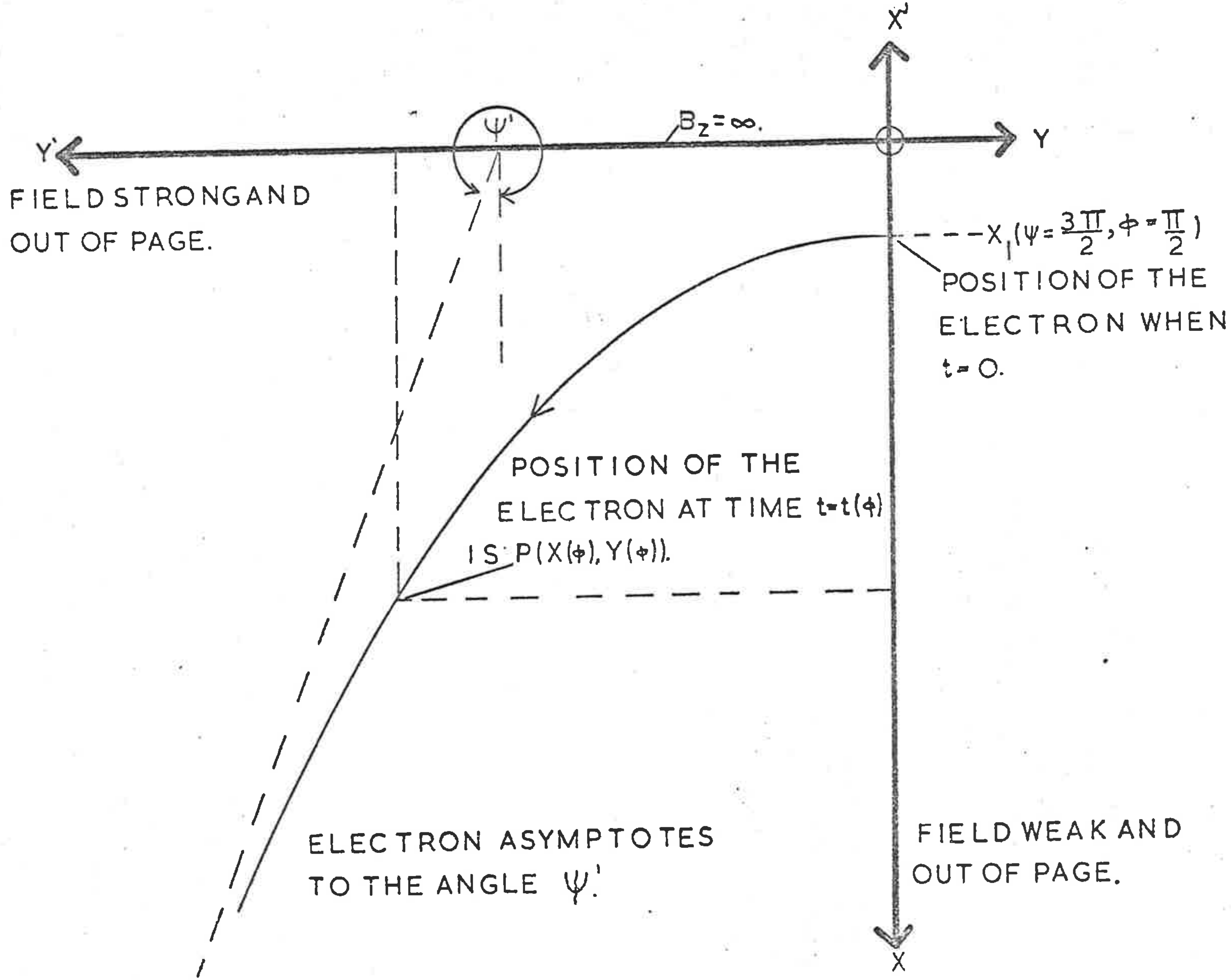


FIGURE 3(b): UNBOUND ORBIT OF AN ELECTRON IN THE FIELD $B_z = \lambda X^{\alpha}$; $\alpha < -1$. TRAJECTORY CANNOT CROSS THE LINE X=O.

$$= \frac{2^{\beta}}{v} \left(\frac{mv}{e\lambda}\right)^{\beta} (-\beta)^{1-\beta} \int_{o}^{\phi} \frac{d\phi}{(\sin^2\phi' - \sin^2\phi)^{1-\beta}} . \qquad \ldots(4.49)$$

Substitution of $\sin\phi_T = \frac{\sin\phi}{\sin\phi'}$ in equation (4.49) gives

$$t(\phi_T) = \frac{2^{\beta}}{v} \left(\frac{mv}{e\lambda}\right)^{\beta} (-\beta)^{1-\beta} (\sin\phi')^{2\beta-1} \int_{o}^{\phi_T} \frac{(\cos\phi_T)^{2\beta-1} d\phi_T}{(1-\sin^2\phi'\sin^2\phi_T)^{\frac{1}{2}}} . \qquad \ldots(4.50)$$

In the same manner equations (4.46), (3.13), (2.28) and (2.26) yield

$$y(\psi) = \left(\frac{mv}{e\lambda}\right)^{\beta} (-\beta)^{1-\beta} \int_{\frac{3\pi}{2}}^{\psi} \frac{\sin\psi d\psi}{(\sin\psi' - \sin\psi)^{1-\beta}} ,$$

or

$$y(\phi_T) = -2^{\beta+1} \left(\frac{mv}{e\lambda}\right)^{\beta} (-\beta)^{1-\beta} (\sin\phi')^{2\beta-1} \left\{ \int_{o}^{\phi_T} \frac{(1-\sin^2\phi'\sin^2\phi_T)^{\frac{1}{2}} d\phi_T}{\cos^{1-2\beta}\phi_T} - \frac{1}{2} \int_{o}^{\phi_T} \frac{\cos^{2\beta-1}\phi_T d\phi_T}{(1-\sin^2\phi'\sin^2\phi_T)^{\frac{1}{2}}} \right\} . \qquad \ldots(4.51)$$

From equation (4.47) x can be expressed as

$$x(\phi_T) = \left(\frac{2mv}{-\beta e\lambda}\right)^{\beta} (\sin\phi')^{2\beta} (\cos\phi_T)^{2\beta} . \qquad \ldots(4.52)$$

Utilizing equations (4.37) and (4.38) with γ replaced by $\Omega = \sin\phi'$ and ϕ_F replaced by ϕ_T, equation (4.50) becomes

$$t(\phi_T) = \frac{\Sigma}{2v} \Omega^{2\beta-1} \mathcal{F}_2(\Omega^2;\phi_T) , \qquad \ldots(4.53)$$

and equation (4.51) yields

$$y(\phi_T) = -\sum \Omega^{2\beta-1} (\mathcal{F}_1(\Omega^2;\phi_T) - \frac{1}{2} \mathcal{F}_2(\Omega^2;\phi_T)) , \qquad \ldots(4.54)$$

where from equations (4.48) and (2.9)

$$\Omega = \sin\phi' = \left(\frac{-eA_1}{2mv}\right)^{\frac{1}{2}} , \qquad \ldots(4.55)$$

and A_1 is the magnetic vector potential of the magnetic field at $x = x_1$.

Discussion of Results

The bound orbits of cases 1 and 3 yielded the position of the electron, $P(\phi) = (x(\phi),y(\phi))$ at a time $t(\phi)$ in terms of the newly defined curly-F function. It was shown that when $\alpha = 1$, the results assumed forms containing incomplete elliptic integrals of the first and second kinds, which are well tabulated (see e.g. Byrd and Friedman,1971). Differing trajectories for $\alpha = 1$ are shown in Figures 4 and 5. When $\alpha = -2$, $\beta = -1$ and the orbits are bound, equation (4.18) gives

$$t(\phi) = -\frac{\sigma x_2}{2(1-\sigma)^{3/2}} \left\{(2-\sigma)\tan^{-1}((1-\sigma)^{\frac{1}{2}}\tan\phi) - \frac{\sigma}{2}\sin[2(\tan^{-1}((1-\sigma)^{\frac{1}{2}}\tan\phi))]\right\} , \qquad \ldots(4.56)$$

with the help of Appendix 5, equation (A.31).

Similarly equation (4.19) yields

$$y(\phi) = -\frac{x_2}{2(1-\sigma)^{3/2}} \left\{-\sigma^2\tan^{-1}((1-\sigma)^{\frac{1}{2}}\tan\phi) + \frac{(2-\sigma)}{2}\sigma\sin[2(\tan^{-1}((1-\sigma)^{\frac{1}{2}}\tan\phi))]\right\} , \qquad \ldots(4.57)$$

and from equation (4.17) the value of x is

$$x(\phi) = \frac{x_2}{(1-\sigma\sin^2\phi)} . \qquad \ldots(4.58)$$

If the orbits for $\alpha = -2$, $\beta = -1$ are unbounded, then from equation (4.49) and Appendix 5 equation (A.38), the time yields

$$t(\phi\ ) = \left(\frac{e\lambda}{2mv}\right) \int_o^{\phi} \frac{d\phi}{(\sin^2\phi'-\sin^2\phi)^2}$$

$$= \frac{e\lambda}{4mv\sin^4\phi'} \{(1-\tan^2\phi')\ln \{\frac{1+\cot\phi'\tan\phi}{1-\cot\phi'\tan\phi}\}$$

$$+ (1 + \tan^2\phi') \frac{\tan\phi}{(1-\cot^2\phi'\tan^2\phi)}\} , \qquad \ldots(4.59)$$

and using equation (A.34) and (A.38) of Appendix 5, equation (4.51) becomes

$$y(\phi) = \frac{e\lambda}{2m\sin^4\phi'} \{(1 - 2\tan^2 2\phi') \ln \left(\frac{1+\cot\phi'\tan\phi}{1-\cot\phi'\tan\phi}\right)$$

$$+ \frac{\cos 2\phi'}{2\cos^2\phi'} \left(\frac{\tan\phi}{1-\cot^2\phi'\tan^2\phi}\right)\} . \qquad \ldots(4.60)$$

From equation (4.47)

$$x(\phi) = \frac{2mv}{e\lambda} (\sin^2\phi' - \sin^2\phi)^{-1} . \qquad \ldots(4.61)$$

Solutions for both bound and unbound trajectories are in terms of trigonometric and logarithmic functions of the boundary conditions of the orbit, and also the angular parameter ϕ.

When $\alpha = -3$, $\beta = -\frac{1}{2}$, and equations (4.18) and (4.19) give the bound orbit results

$$\left.\begin{aligned} t(\phi) &= -\frac{\sigma x_2}{2v(1-\sigma)^{\frac{1}{2}}} E(k/\theta) , \\ \text{and} \quad y(\phi) &= -\frac{x_2}{2(1-\sigma)^{\frac{1}{2}}} (2F(k/\theta)-(2-\sigma)E(k/\theta)) , \end{aligned}\right\} \qquad \ldots(4.62)$$

where $F(k/\theta)$ and $E(k/\theta)$ are incomplete elliptic integrals of the first and second kind respectively, with $k = (\frac{\sigma}{\sigma-1})^{\frac{1}{2}}$ and $\theta = \tan^{-1} \{(1-\sigma)^{\frac{1}{2}}\tan\phi\}$. From equation (4.17) the value of x becomes

$$x(\phi) = \frac{x_2}{(1-\sigma\sin^2\phi)^{\frac{1}{2}}} . \qquad \ldots(4.63)$$

In general however it will not be possible to obtain results in terms of well known functions of mathematical physics, and it will be necessary to compute curly-$\mathcal{F}$ and curly-$\mathcal{M}$ functions. For $\alpha \neq -1$ and

$b > c > 0$ the curly-F functions can be found using the series expansion

$$\mathcal{F}(a,b,c;x;\theta) = \sum \frac{(a)_r (b)_r}{(c)_r r!} \nu_{r,\theta} x^r \quad , \qquad \ldots(4.64)$$

which is closely related to the series expansion for the Gaussian hypergeometric function, but every term in the series is modified by the normalized incomplete beta function

$$\nu_{r,\theta} = \frac{B_\theta(b+r,c-b)}{B(b+r,c-b)} \quad , \qquad \ldots(4.65)$$

as shown in equation (A14). Equation (4.64) simplifies to incomplete elliptic integrals when $a = \pm \frac{1}{2}$, $b = \frac{1}{2}$ and $c = 1$. (Hancock,(1917), pp26).

If $b \nmid c \nmid 0$ which is the case for unbound orbits, ($\alpha < -1$) the series expansion (4.64) cannot be applied. In this case it will be necessary to integrate equation (4.20) by standard quadrature methods.

The curly-M function can be written as in Appendix 4

$$M(a,b;x;\theta) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \mu\xi(n+a,b-a) \frac{x^n}{n!} \quad , \qquad \ldots(4.66)$$

where again

$$\mu_\xi(n+a,b-a) = \frac{B_\xi(n+a,b-a)}{B(n+a,b-a)} \quad , \qquad \ldots(4.67)$$

is a modified beta function. Thus when $\alpha = -1$ electron trajectories of equations (4.32) and (4.33) can be found using equations(4.66) and (4.67). The incomplete beta function is well tabulated (Pearson, K; (1934)) or it can be readily computed.

Figure 4 shows trajectories for $\alpha = 1$, in which the electron does not cut the line $x = 0$. The values of k_1 are $\frac{1}{2}$ and $\frac{1}{\sqrt{2}}$. Figure 5 shows orbits which cut the line $x = 0$. ϕ_o varies from 20^o to 70^o.

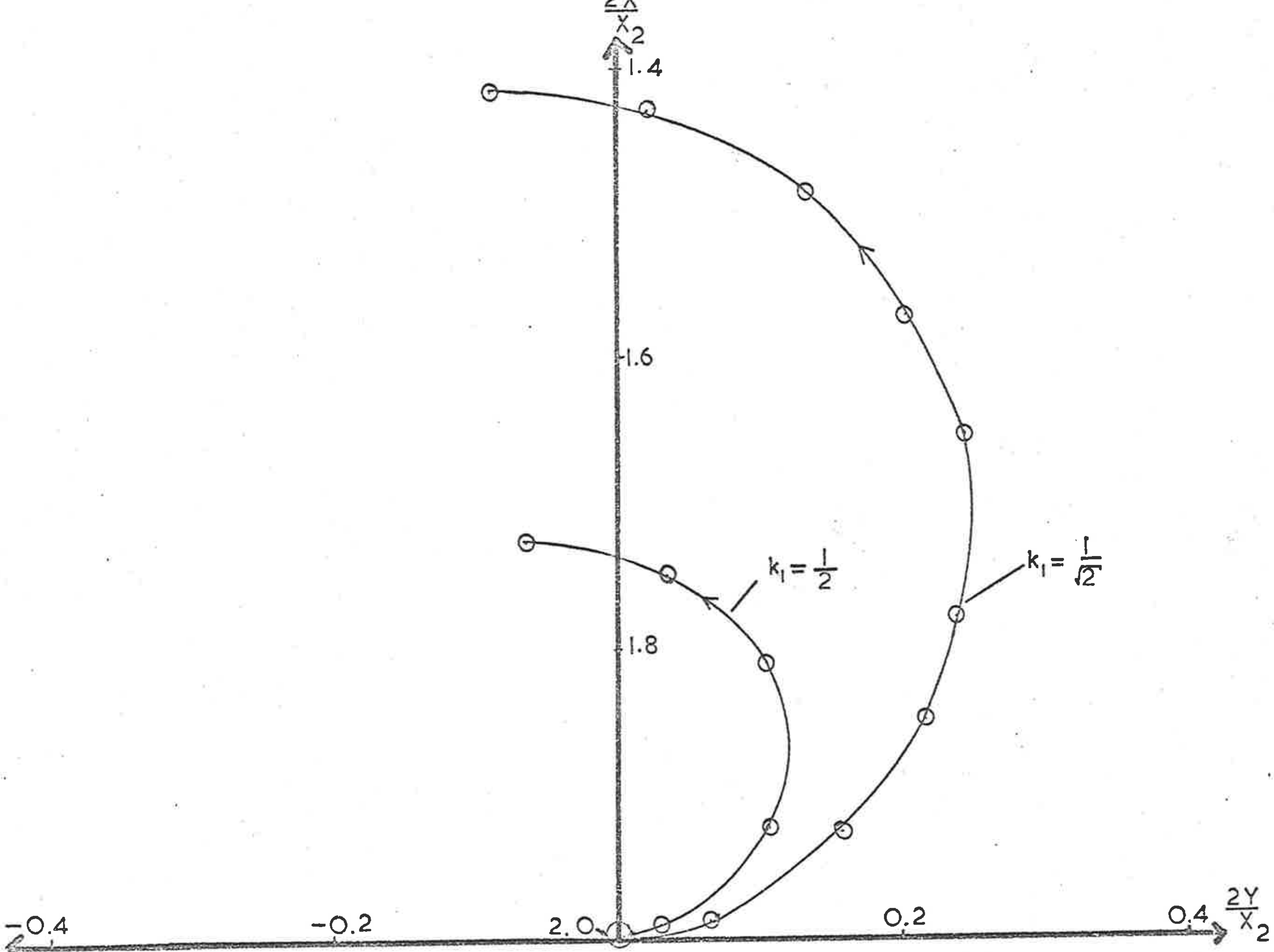


FIGURE 4 : ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT NOT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$.

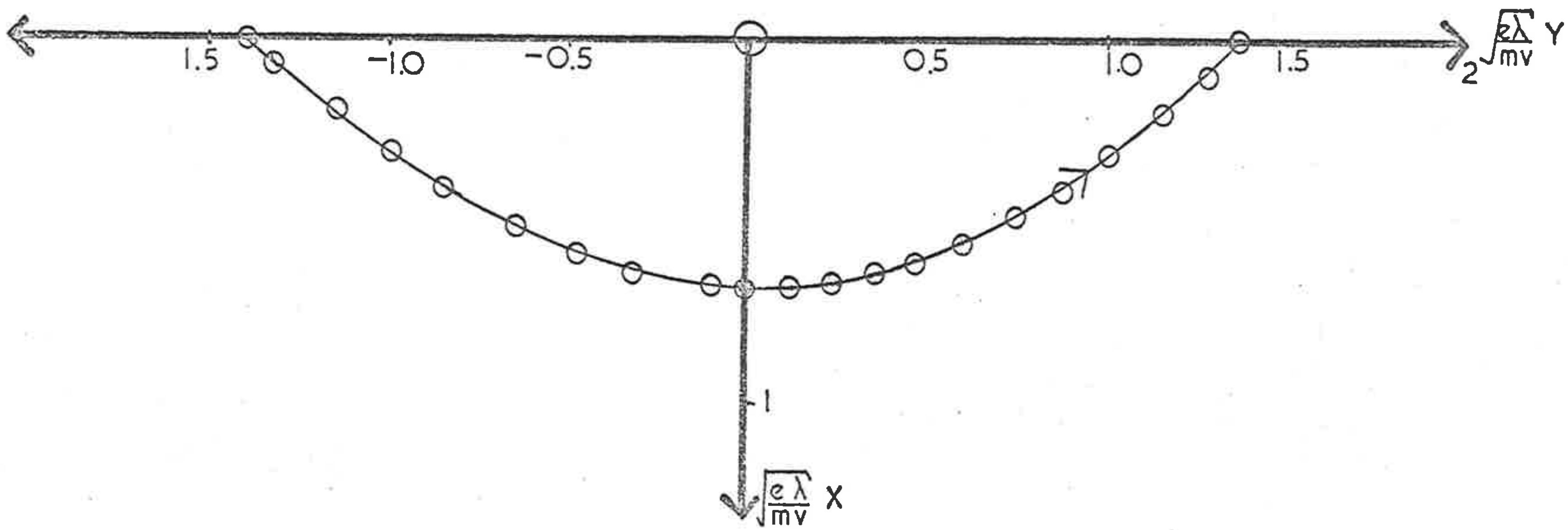


FIGURE 5(a): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$. $\phi_0 = -20°$

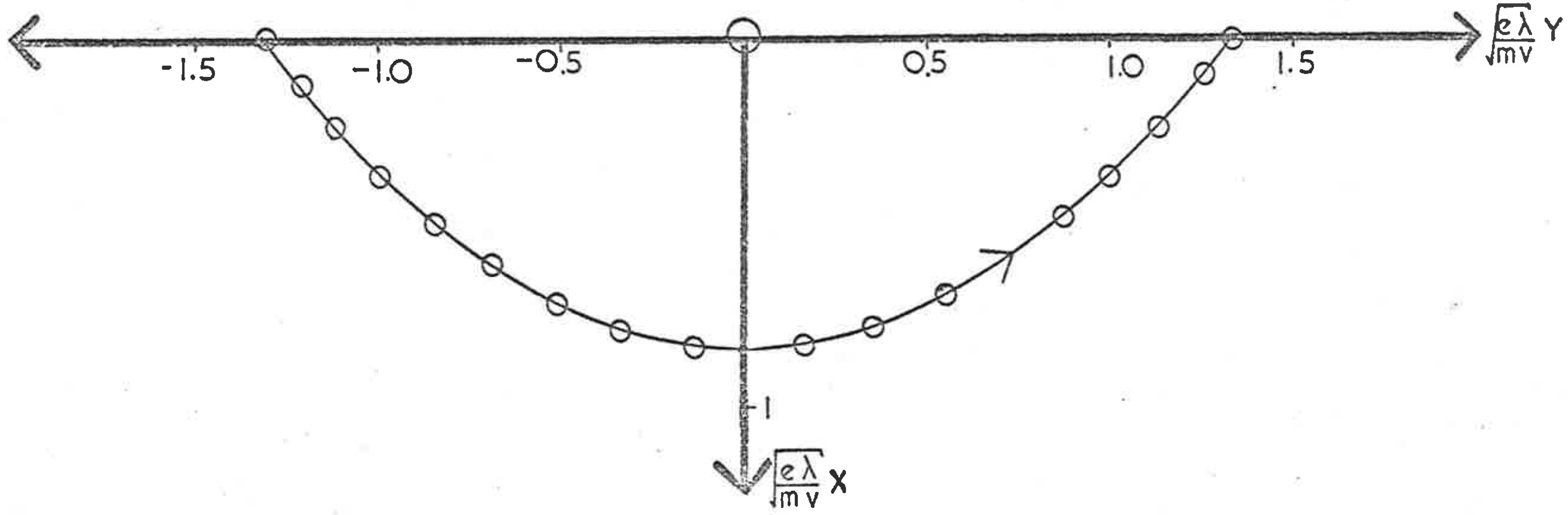


FIGURE 5(b): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$. $\phi_o = -25°$

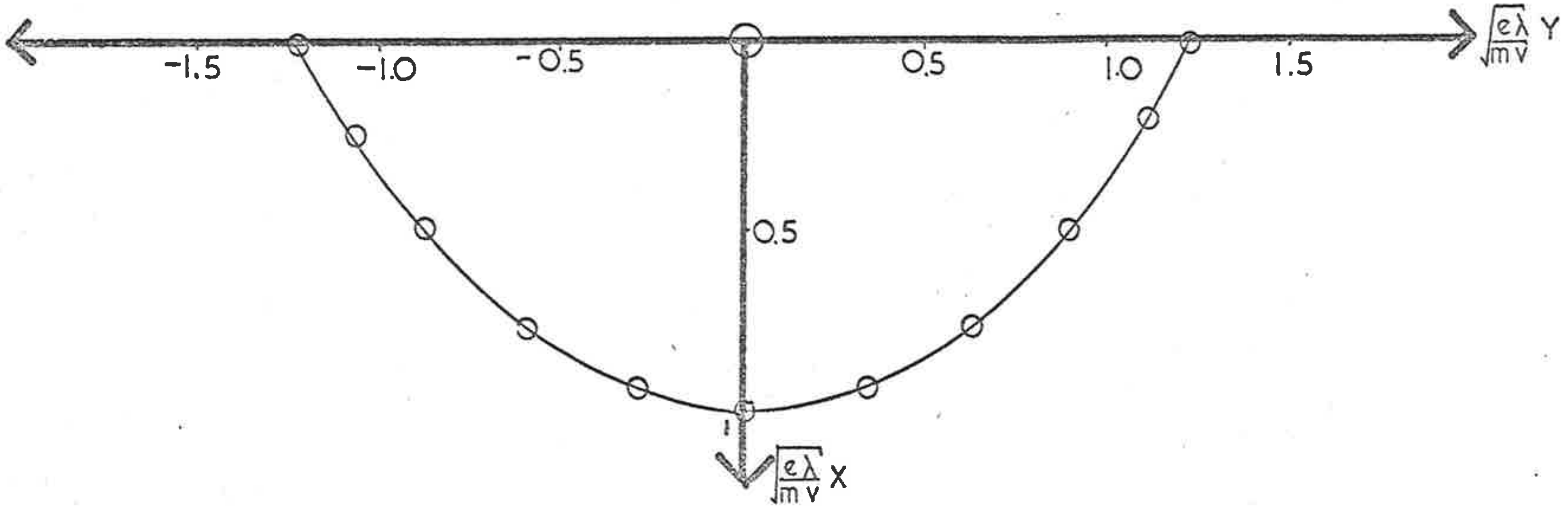


FIGURE 5(c): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$ BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$. $\phi_0 = -30^\circ$

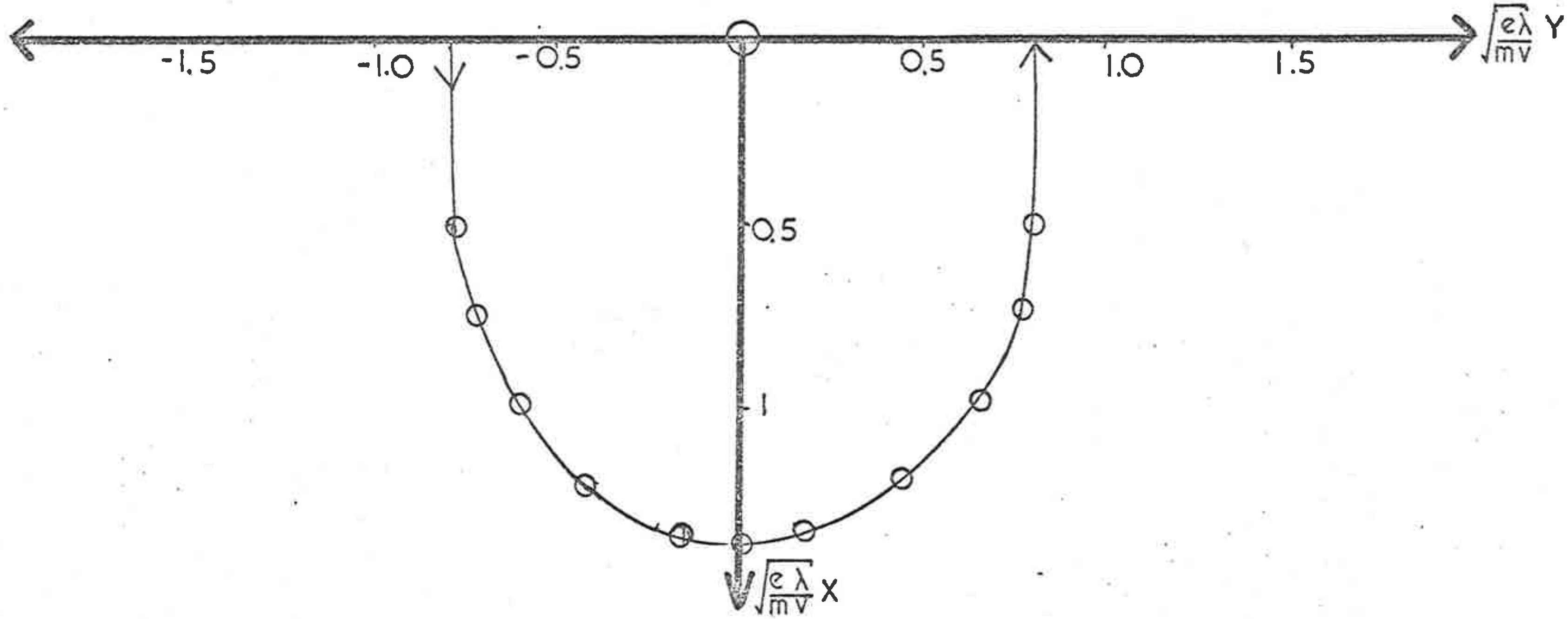


FIGURE 5(d): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$. $\phi_0 = -45°$

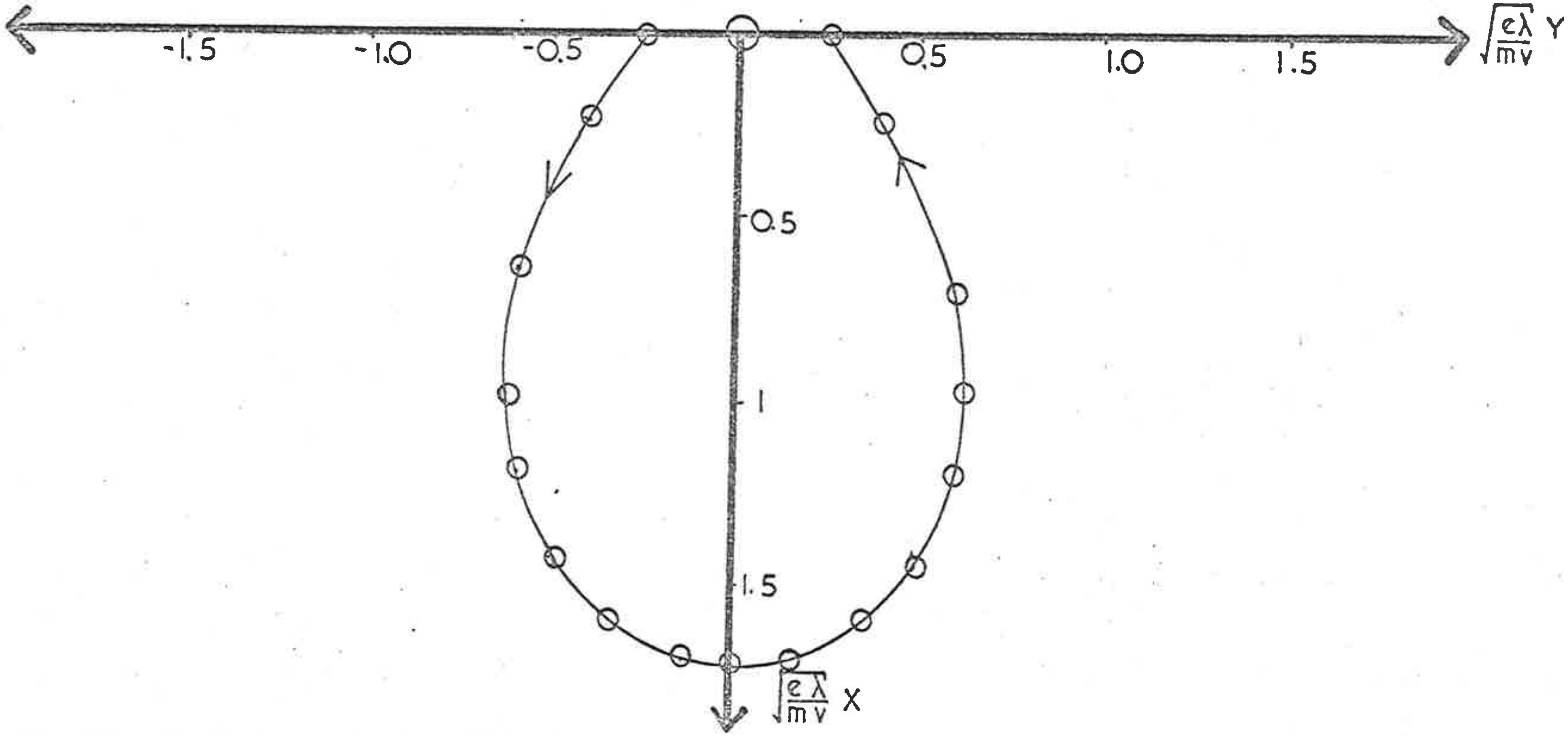


FIGURE 5(e): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$ $\phi_0 = -60^\circ$

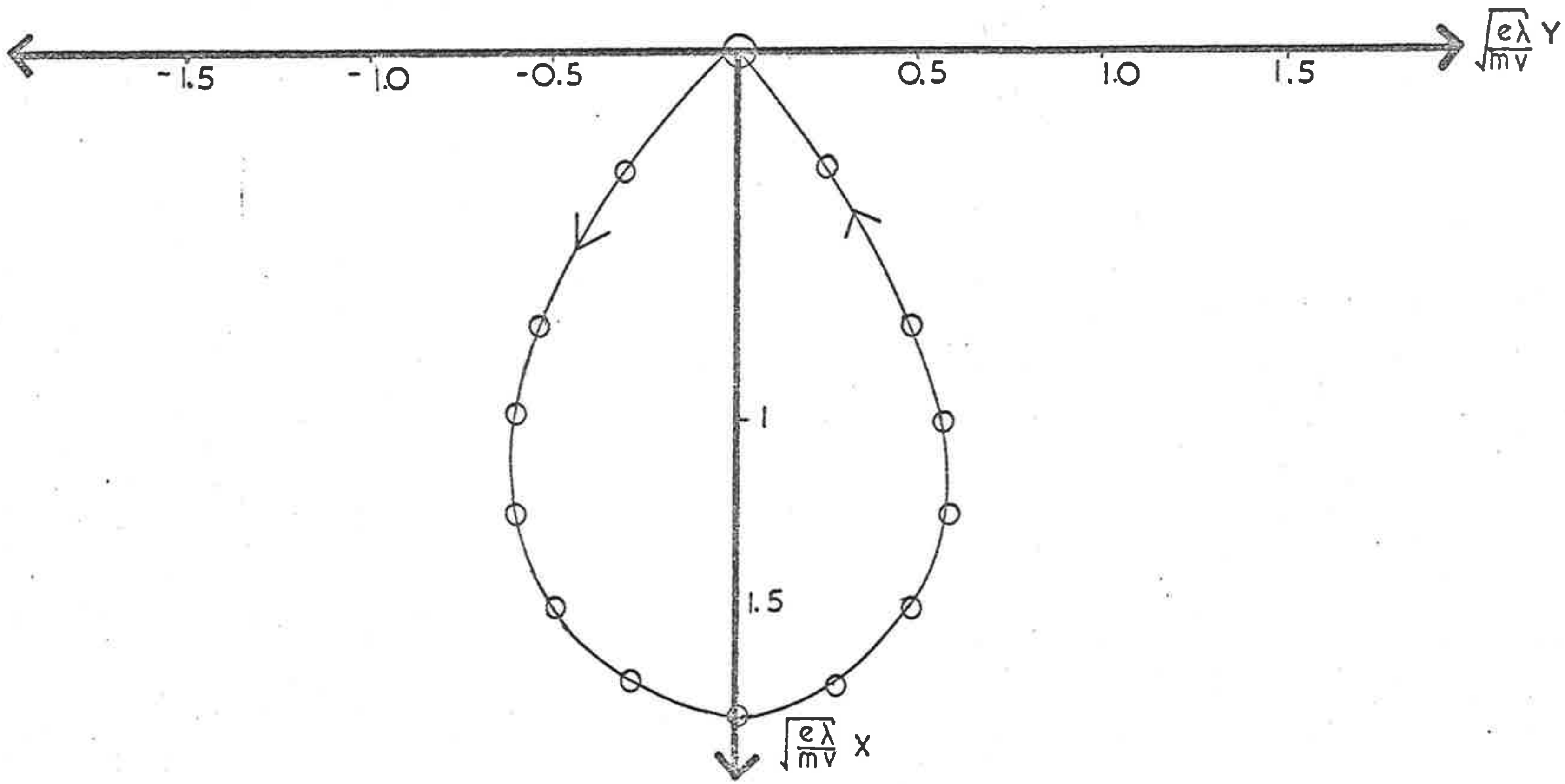


FIGURE 5(f): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$. $\phi_0 = -65^\circ$.

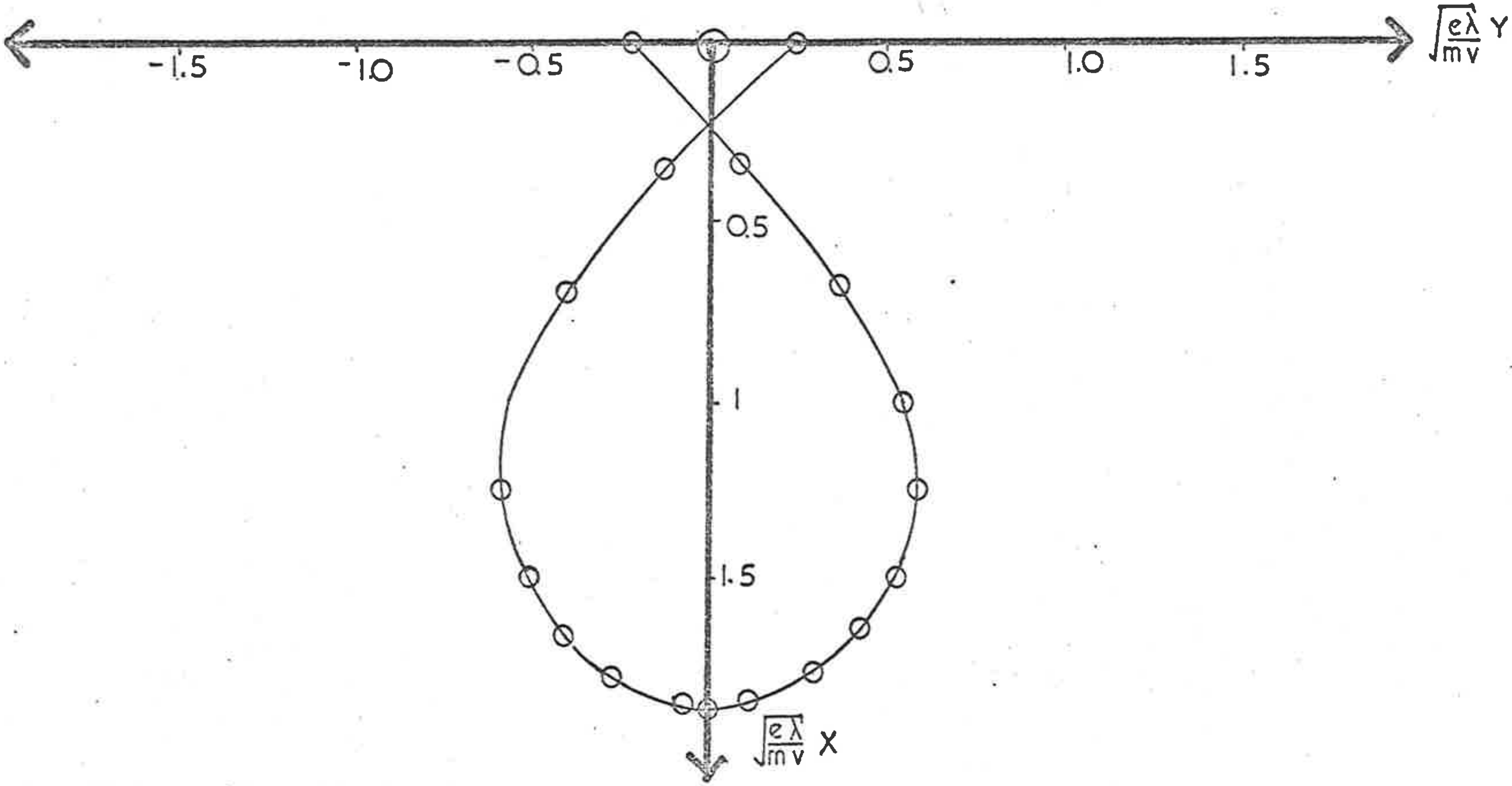


FIGURE 5(g): ELECTRON TRAJECTORY IN THE FIELD $B_z = \lambda X$. BOUND ORBIT CROSSING THE LINE $B_z = 0$ WHEN $X = 0$. $\phi_0 = -70°$.

CHAPTER 5

5.1 Motion of the Electron in a Periodically Varying Magnetic Field

If the magnetic field is

$$B_z = \lambda \sin \alpha x \quad , \qquad \ldots(5.1)$$

and λ and α are constants greater than zero, then the magnetic field varies periodically and has points of zero magnetic field at $x = 0$, $\pm \frac{\pi}{\alpha}, \frac{2\pi}{\alpha}, \ldots, \pm \frac{N\pi}{\alpha}$.

Using equations (2.29), (2.9) and (5.1) the relationship between ψ and x becomes

$$\sin\psi - \sin\psi_o = -\frac{e\lambda}{\alpha m v}(\cos\alpha x - \cos\alpha x_o) \quad , \qquad \ldots(5.2)$$

with $x = x_o$ when $\psi = \psi_o$. From equation (5.2)

$$x = \frac{1}{\alpha}\cos^{-1}\{\cos\alpha x_o - \frac{\alpha m v}{e\lambda}(\sin\psi - \sin\psi_o)\} \quad , \qquad \ldots(5.3)$$

and for orbits not crossing the line $B_z = 0$, $\psi_o = 0$ at $x = x_o$. Equation (5.3) then becomes

$$x = \frac{1}{\alpha}\cos^{-1}\left(\cos\alpha x_o\left(1 + \frac{mv}{eA_o}\sin\psi\right)\right) \quad , \qquad \ldots(5.4)$$

where $A_o = -\dfrac{\lambda\cos\alpha x_o}{\alpha} < 0$, is the magnetic vector potential of the magnetic field at x_o. When $\psi = \frac{3\pi}{2}$, equation (5.4) yields the minimum value

$$x_1 = \frac{1}{\alpha}\cos^{-1}\left(\cos\alpha x_o\left(1 - \frac{mv}{eA_o}\right)\right) \quad , \qquad \ldots(5.5)$$

of the orbit of the electron, and when $\psi = \frac{\pi}{2}$, equation (5.4) yields the maximum value

$$x_2 = \frac{1}{\alpha}\cos^{-1}\left\{\cos\alpha x_o\left(1 + \frac{mv}{eA_o}\right)\right\} \quad , \qquad \ldots(5.6)$$

of the orbit. x is bounded by the inequality

$$0 < x_1 \le x \le x_2 < \pi/\alpha \quad . \qquad \ldots(5.7)$$

From equations (5.5) and (5.6) the electron is bounded within the half cycle when

$$|\cos\alpha x_o| < 1 - \frac{\alpha m v}{e\lambda} \quad . \qquad \ldots(5.8)$$

For trajectories which cross the line $B_z = 0$ at $x = 0$ and enter a region of reversed magnetic field, $x_o = 0$ when $\psi = \psi_o$, and equation (5.3) yields

$$x = \pm \frac{1}{\alpha} \cos^{-1} \left(1 - \left(\frac{\alpha m v}{e\lambda}\right) (\sin\psi - \sin\psi_o)\right) \quad . \qquad \ldots(5.9)$$

Equation (5.9) has a maximum value x_2 when $\psi = \pi/2$ given by

$$x_2 = -x_1 = \frac{1}{\alpha} \cos^{-1} \left(1 - \left(\frac{\alpha m v}{e\lambda}\right) (1 - \sin\psi_o)\right) \quad , \qquad \ldots(5.10)$$

and is bounded such that

$$-\frac{\pi}{\alpha} < x_1 \le x \le x_2 < \frac{\pi}{\alpha} \quad . \qquad \ldots(5.11)$$

The inequality (5.11) is valid if from equation (5.9)

$$\sin\psi_o > 1 - \frac{2e\lambda}{\alpha m v} \quad . \qquad \ldots(5.12)$$

If equation (5.12) is not valid the electron will cut the lines of zero magnetic field at $x = \pm \pi/\alpha$ with a new crossing angle ψ_o'. From equation (5.9)

$$\sin\psi_o' = \sin\psi_o + 2e\lambda/(\alpha m v) \quad . \qquad \ldots(5.13)$$

5.2 Exact Drift Velocities for Bound Orbits

Case 1: Electron Does Not Cross the Line $B_z = 0$ at $x = 0, \pi/\alpha$.

For orbits shown in Figure (1), equations (2.30), (5.1) and (5.4) yield

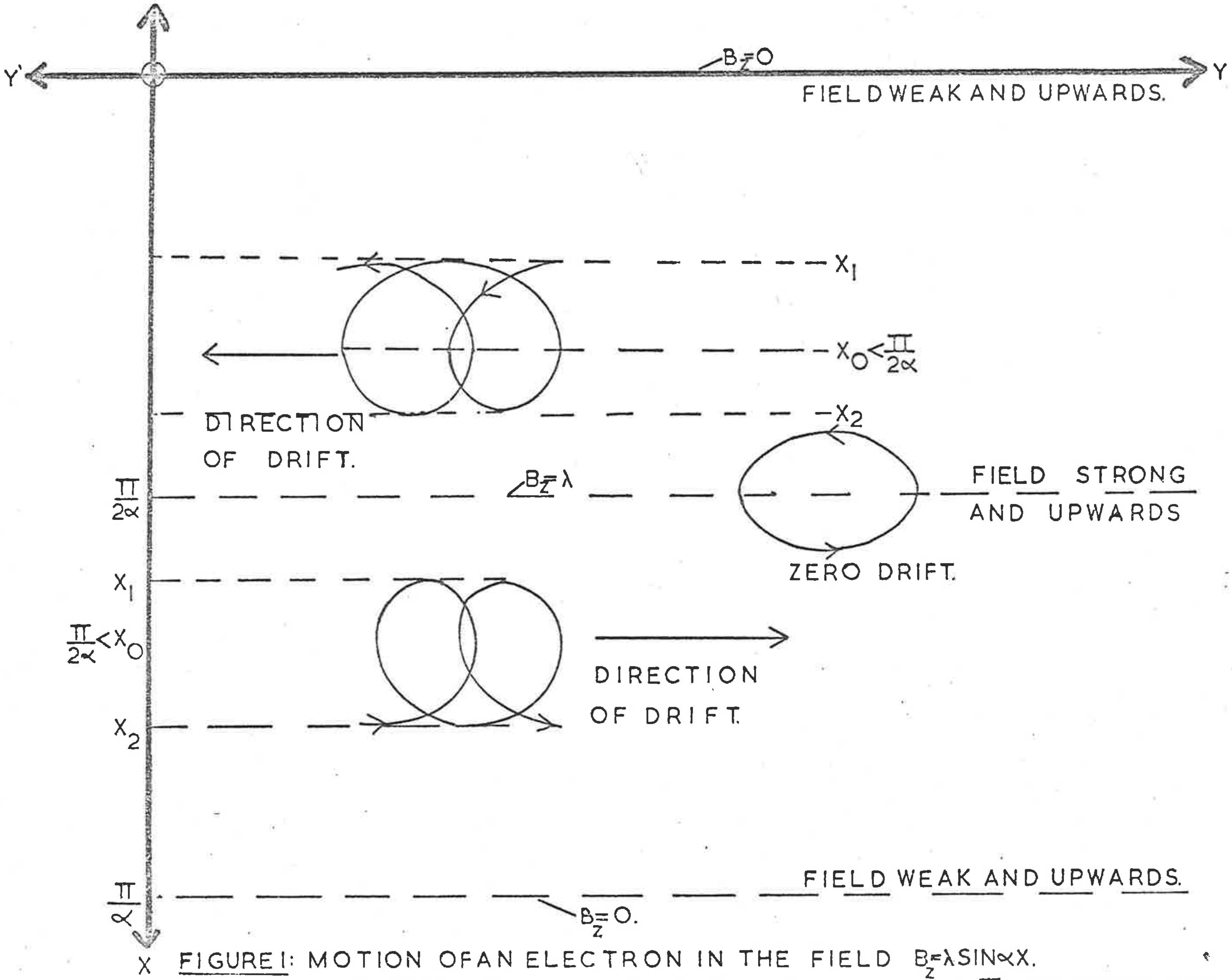


FIGURE I: MOTION OF AN ELECTRON IN THE FIELD $B_z = \lambda \sin \alpha X$.
ELECTRON DOES NOT CROSS THE LINES $X = 0, \frac{\pi}{\alpha}$ ON WHICH $B_z = 0$.

$$T = \frac{m}{e\lambda}\int_{o}^{2\pi} \frac{d\psi}{\sin\alpha x}$$

$$= \frac{m}{e\lambda}\int_{o}^{2\pi} \frac{d\psi}{(1-\cos^2\alpha x)^{\frac{1}{2}}}$$

$$= \frac{m}{e\lambda}\int_{o}^{2\pi} \frac{d\psi}{(1+\cos\alpha x_o - \frac{\alpha mv}{e\lambda}\sin\psi)^{\frac{1}{2}}(1-\cos\alpha x_o + \frac{\alpha mv}{e\lambda}\sin\psi)^{\frac{1}{2}}} \quad , \qquad \ldots(5.14)$$

as the periodic time.

Using the substitution $\psi = \frac{\pi}{2} - 2\phi$ equations (5.14), (5.6) and (5.1) become

$$T = \frac{4mv}{eB_2}\int_{o}^{\frac{\pi}{2}} \frac{d\phi}{(1+s^2\sin^2\phi)^{\frac{1}{2}}(1-t^2\sin^2\phi)^{\frac{1}{2}}} \quad , \qquad \ldots(5.15)$$

where $B_2 = \lambda\sin\alpha x_2$ is the magnetic field strength at x_2

$$s^2 = \frac{2\alpha mv/(e\lambda)}{(1 + \cos\alpha x_2)} \quad , \qquad \ldots(5.16)$$

and

$$t^2 = \frac{2\alpha mv/(e\lambda)}{(1 - \cos\alpha x_2)} \quad . \qquad \ldots(5.17)$$

Equation (5.15) may be written

$$T = \frac{4mv}{eB_2}\int_{o}^{\frac{\pi}{2}} \frac{\sec^2\phi\, d\phi}{(1+(1+s^2)\tan^2\phi)^{\frac{1}{2}}(1+(1-t^2)\tan^2\phi)^{\frac{1}{2}}} \qquad \ldots(5.18)$$

Substituting $\tan\xi = (1+s^2)^{\frac{1}{2}}\tan\phi$ into equation (5.18) gives

$$T = \frac{4m}{eB_2(1+s^2)}\int_{o}^{\frac{\pi}{2}} \frac{d\xi}{\left(1 - \left(\frac{s^2+t^2}{1+s^2}\right)\sin^2\xi\right)^{\frac{1}{2}}} \quad ,$$

$$= 2\left(\frac{m}{\alpha ve\lambda}\right)^{\frac{1}{2}} qK(q) \quad , \qquad \ldots(5.19)$$

where $K(q)$ is a complete elliptic integral of the first kind, modulus q. From equations (5.5), (5.6), (5.16) and (5.17)

$$q^2 = \frac{s^2 + t^2}{1 + s^2} = \frac{4\alpha mv/(e\lambda)}{(1-\cos\alpha x_2)(1+\cos\alpha x_1)} < 1 . \qquad ...(5.20)$$

Similarly from equations (2.31), (5.1), (5.4), (5.16), (5.17) and the substitution $\psi = \frac{\pi}{2} - 2\phi$, Δy yields

$$\Delta y = \frac{mv}{e\lambda}\int_o^{2\pi} \frac{\sin\psi d\psi}{(1+\cos\alpha x_o - \frac{\alpha mv}{e\lambda}\sin\psi)^{\frac{1}{2}}(1-\cos\alpha x_o + \frac{\alpha mv}{e\lambda}\sin\psi)^{\frac{1}{2}}}$$

$$= 2\left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} q\left\{\left(1 + \frac{2}{s^2}\right)\int_o^{\frac{\pi}{2}} \frac{d\xi}{(1-q^2\sin^2\xi)^{\frac{1}{2}}} - \frac{2}{s^2}\int_o^{\frac{\pi}{2}} \frac{d\xi}{(1-n\sin^2\xi)(1-q^2\sin^2\xi)^{\frac{1}{2}}}\right\} , \qquad ...(5.21)$$

where

$$n = \frac{s^2}{s^2 + 1} = \frac{2\ mv/(e\lambda)}{1 + \cos\alpha x_1} , \qquad ...(5.22)$$

and

$$q^2 = \frac{2}{(1 - \cos\alpha x_2)}\ n . \qquad ...(5.23)$$

From the theory of complete elliptic integrals,the elliptic integral of the third kind is defined as

$$\Pi\left(\frac{\pi}{2},n;q\right) = \int_o^{\frac{\pi}{2}} \frac{d\phi}{(1-n\sin^2\phi)(1-q^2\sin^2\phi)^{\frac{1}{2}}} . \qquad ...(5.24)$$

Thus in terms of elliptic integrals of the first and third kinds equation (5.21) becomes

$$\Delta y = 2\left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} q\left(\left(1 + \frac{2}{s^2}\right) K(q) - \frac{2}{s^2}\,\Pi\left(\frac{\pi}{2},n;q\right)\right) . \qquad ...(5.25)$$

From equations (2.32), (5.19) and (5.25) the exact drift velocity is given by

$$v_D = -v\left\{\frac{2}{s^2}\left(\frac{\Pi\left(\frac{\pi}{2},n;q\right)}{K(q)} - 1\right) - 1\right\} . \qquad ...(5.26)$$

<u>Case 2</u>: Electron Moves into a Region of Reversed Magnetic Field in Crossing $B_z = 0$ on $x = 0$.

For an orbit as sketched in Figure 2, equations (5.1), (5.9) and

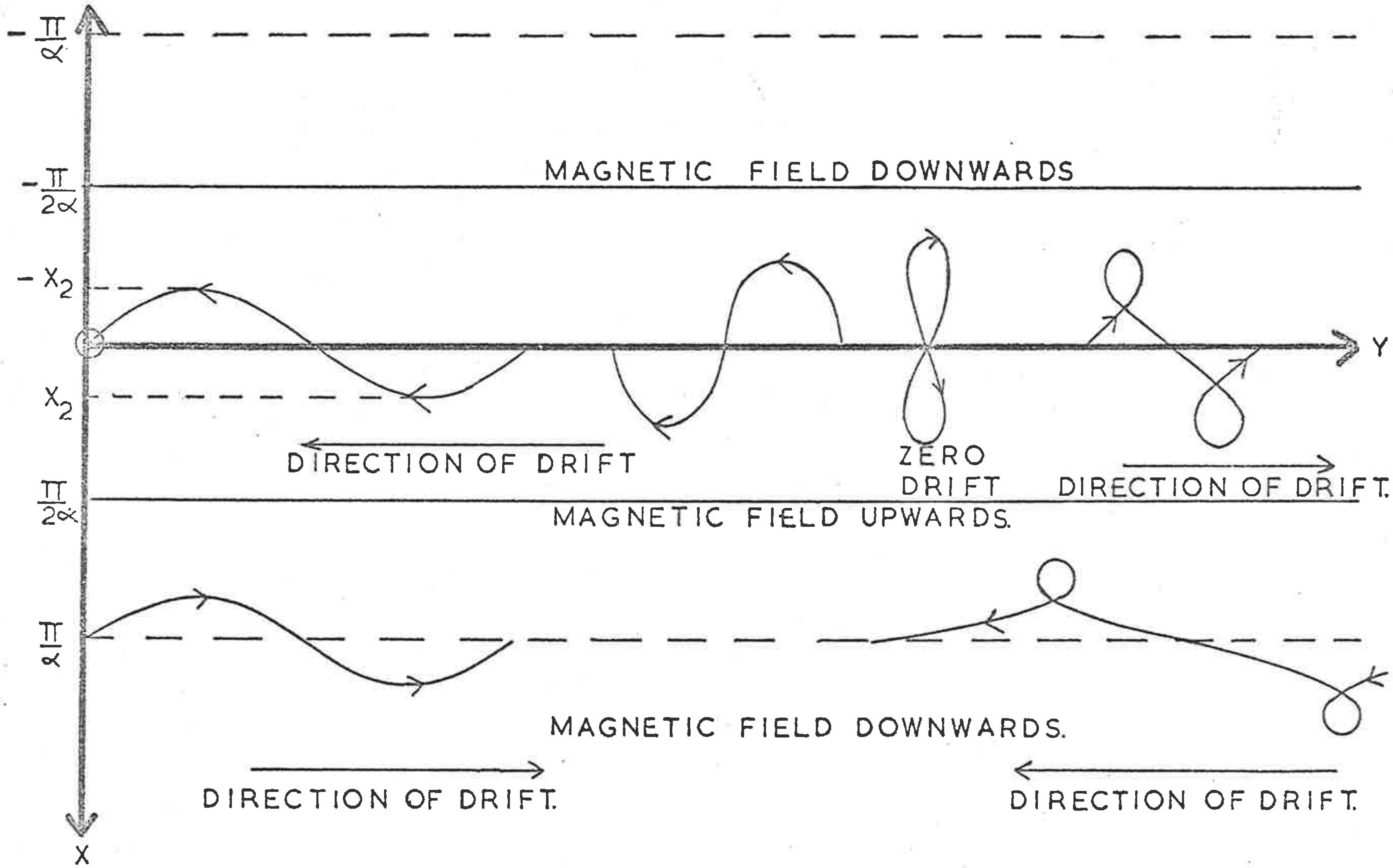


FIGURE 2: MOTION OF AN ELECTRON IN THE FIELD $B_z = \lambda \sin \alpha X$. ELECTRON CROSSES A LINE $B_z = 0$ AT $X = 0$ OR $X = \frac{\pi}{\alpha}$.

(2.30) yield

$$T = \frac{2}{\alpha v}\int_{\psi_o}^{\pi-\psi_o} \frac{d\psi}{(\sin-\sin\psi_o)^{\frac{1}{2}}(2e\lambda/(\alpha mv)+\sin\psi-\sin\psi_o)^{\frac{1}{2}}} \quad . \qquad \ldots(5.27)$$

Using the substitution $\psi = \frac{\pi}{2} - 2\phi$ followed by $\sin\phi = \sin\phi_o\sin\theta$, equation (5.27) yields

$$T = \frac{4}{\alpha\delta v\sin\phi_o}\int_o^{\frac{\pi}{2}} \frac{d\theta}{(1+\cos^2\phi_o\tan^2\theta)^{\frac{1}{2}}\left(1+\ (\frac{1+\delta^2}{\delta^2})\tan^2\theta\right)^{\frac{1}{2}}} \quad , \qquad \ldots(5.28)$$

with

$$\delta = \left\{\frac{e\lambda}{\alpha mv\sin^2\phi_o} - 1\right\} \quad . \qquad \ldots(5.29)$$

Finally with the aid of the substitution $\tan\mu = ((1 + \delta^2)/\delta^2)^{\frac{1}{2}}\tan\theta$, equation (5.29) yields

$$T = 4\ \left(\frac{m}{\alpha ve\lambda}\right)^{\frac{1}{2}} \int_o^{\frac{\pi}{2}} \frac{d\mu}{(1\ -\ k^2\sin^2\mu)^{\frac{1}{2}}}$$

$$= 4\ \left(\frac{m}{\alpha ve\lambda}\right)^{\frac{1}{2}} K(k) \quad , \qquad \ldots(5.30)$$

where K(k) is a complete elliptic integral of the first kind, modulus k, and

$$k = \sin\phi_o\left(1 + \left(\frac{\alpha mv}{e\lambda}\right)\ \cos^2\phi_o\right)^{\frac{1}{2}} \quad . \qquad \ldots(5.31)$$

From equation (5.12) and using $\psi_o = \frac{\pi}{2} - 2\phi_o$

$$\cos^2\phi_o < 1 - \left(\frac{e\lambda}{\alpha mv}\right) \quad , \qquad \ldots(5.32)$$

and therefore equation (5.31) yields

$$0 < k < 1 \quad . \qquad \ldots(5.33)$$

From equations (5.1), (5.9) and (2.31) Δy is given by

$$\Delta y = \frac{2}{\alpha}\int_{\psi_o}^{\pi-\psi_o} \frac{\sin\psi d\psi}{(\sin\psi-\sin\psi_o)^{\frac{1}{2}}(2e\lambda/(\alpha mv)+\sin\psi-\sin\psi_o)^{\frac{1}{2}}}$$

$$= 4\left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}}\left\{(1+2\delta^2\sin^2\phi_o)\int_0^{\frac{\pi}{2}} \frac{d\mu}{(1-k^2\sin^2\mu)^{\frac{1}{2}}}\right.$$

$$\left. - 2\delta^2\sin^2\phi_o\int_0^{\frac{\pi}{2}} \frac{d\mu}{(1-p\sin^2\mu)(1-k^2\sin^2\mu)^{\frac{1}{2}}}\right\} , \quad \ldots(5.34)$$

and

$$p = \frac{1}{1+\delta^2} = \left(\frac{\alpha mv}{e\lambda}\right)\sin^2\phi_o < 1 . \quad \ldots(5.35)$$

Thus equation (5.34) becomes

$$\Delta y = 4\left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}}\left\{K(k) + 2\delta^2\sin^2\phi_o\left(K(k) - \Pi\left(\frac{\pi}{2},p;k\right)\right)\right\} , \quad \ldots(5.36)$$

where $\Pi\left(\frac{\pi}{2},p;k\right)$ is an elliptic integral of the third kind as defined in equation (5.24).

From equations (5.30), (5.36) and (2.32) the drift velocity becomes

$$v_D = \frac{\Delta y}{T} = -v\left\{2\delta^2\sin^2\phi_o\left(\frac{\Pi\left(\frac{\pi}{2},p;k\right)}{K(k)} - 1\right) - 1\right\} . \quad \ldots(5.37)$$

(a) The Alfven Drift Approximation

For the orbits as described in Case 1 the electron does not cross the line $B_z = 0$, and in the limit of

$$\frac{\alpha mv}{e\lambda} \ll |\cos\alpha x_o| \ll 1 , \quad \ldots(5.38)$$

the orbits are adiabatically affected.

From equations (5.20) and (5.22)

$$n \simeq \frac{2\, mv/(e\lambda)}{1+\cos\alpha x_o} , \quad \ldots(5.39)$$

and

$$q^2 \simeq \frac{2}{1 - \cos\alpha x_o} n \quad , \qquad \text{...(5.40)}$$

with

$$s^2 = \frac{n}{1 - n} \quad , \qquad \text{...(5.41)}$$

and

$$n < q << 1 \quad . \qquad \text{...(5.42)}$$

Using the complete elliptic integral expansions

$$K(q^2) = \frac{\pi}{2} \{1 + q^2/4 + \left(\frac{3}{8}\right)^2 q^4 + \ldots\} \, , \qquad \text{...(5.43)}$$

and

$$\Pi\left(\frac{\pi}{2}, n, q^2\right) = \frac{\pi}{2} \{1 + \frac{1}{2}\left(n + \frac{q^2}{2}\right) + \frac{3}{8}\left(\frac{nq^2}{2} + n^2 + \frac{3}{8} q^4\right) + \ldots\} \, , \qquad \text{...(5.44)}$$

equation (5.25) yields

$$\Delta y = \Pi\rho_o\{(1-n)\left(1 + \frac{3}{8} q^2 + \frac{3}{4} n + \ldots\right) - \left(1 + \frac{q^2}{4} + \ldots\right)\}$$

$$= - \Pi\rho_o \frac{q^2}{8} \cos\alpha x_o \, , \qquad \text{...(5.45)}$$

with the aid of equations (5.39) and (5.40).

Similarly equation (5.19) yields the result $T_o = 2\Pi\rho_o/v$ and therefore equation (5.26) simplifies to

$$v_D = -v\{\frac{q^2}{8} \cos\alpha x_o\}$$

$$= - \frac{mv^2}{2\lambda} \frac{\cos\alpha x_o}{\sin^2\alpha x_o} \, , \qquad \text{...(5.46)}$$

in agreement with the Alfven drift velocity expression of equation (2.38) when applied to a sinusoidal magnetic field.

(b) Discussion of Results

For orbits confined to move within the bounds $0 < x < \frac{\pi}{\alpha}$ as in Case 1, when $0 < x_o < \frac{\pi}{2\alpha}$ as shown in Figure 1 , the electron drifts in the negative direction along the Y axis. If $x_o = \frac{\pi}{2\alpha}$, then for all values of

$\alpha mv/(e\lambda)$ the electron has a zero drift velocity, due to the symmetry of the field strength about that line. If the electron is confined such that $\frac{\pi}{2\alpha} < x_o < \frac{\pi}{\alpha}$, the gradient of the magnetic field becomes negative and the electron drifts in the positive direction along the Y axis. Figure 3 shows typical drift velocity curves for $x_o = \frac{\pi}{6\alpha}$ and $x_o = \frac{\pi}{3\alpha}$. For these values $\cos\alpha x_o$ is 0.5 and -0.5 respectively and $\alpha mv/(e\lambda) < 0.5$ in agreement with equation (5.35). The respective curves are symmetric about the axis on which $v_D = 0$.

Orbits which cut a line on $B_z = 0$ are shown in Figure 2. Figure 4 shows drift velocity curves for varying ϕ_o. $\alpha mv/(e\lambda)$ has been chosen as 0.4, 1 and 2. The elliptic integrals were obtained from well known tables (Belyakov, V.B. et al. (1965)). Alternatively the curves can be computed.

5.3 Trajectories of an Electron in a Sinusoidal Magnetic field

Case 1: Bound Orbit in Which the Electron Does not Cut the Line $B_z = 0$ on $x = 0, \frac{\pi}{\alpha}$.

From Figure 1 and equations (2.27), (5.1) and (5.4) the time becomes

$$t(\psi) = \frac{m}{e\lambda}\int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{\sin\alpha x}$$

$$= \frac{m}{e\lambda}\int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{\left(1+\cos\alpha x_o - \frac{\alpha mv}{e\lambda}\sin\psi\right)^{\frac{1}{2}}\left(1-\cos\alpha x_o + \frac{\alpha mv}{e\lambda}\sin\psi\right)^{\frac{1}{2}}}, \quad \ldots(5.48)$$

and using the substitution $\psi = \frac{\pi}{2} + 2\phi$ followed by $\tan\xi = (1 + s^2)^{\frac{1}{2}}\tan\phi$ equation (5.48) becomes

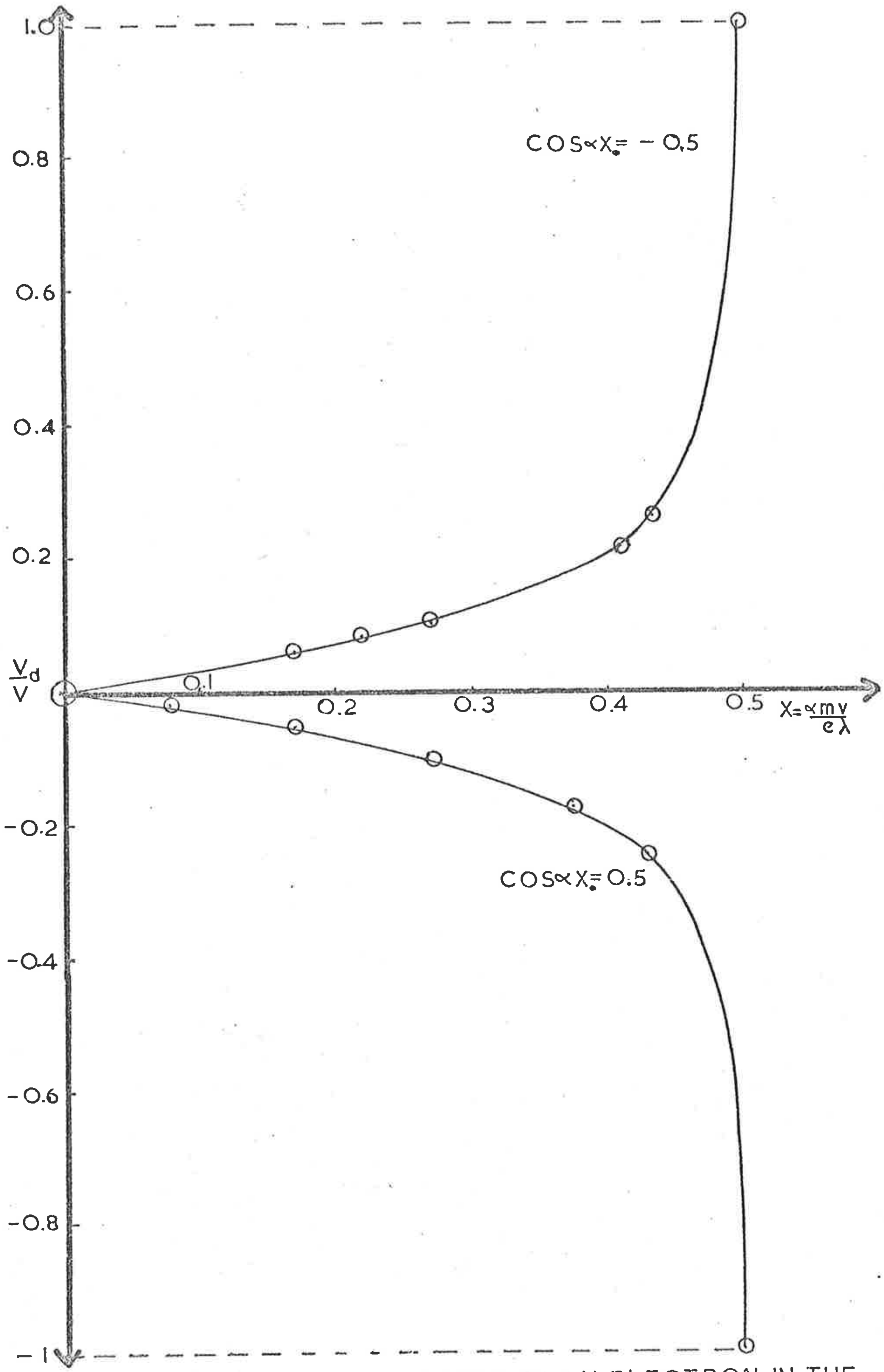


FIGURE 3 DRIFT VELOCITY OF AN ELECTRON IN THE FIELD $B_z = \lambda \sin\alpha X$. ELECTRON DOES NOT CUT THE LINE $X=0$ ON WHICH $B_z = 0$.

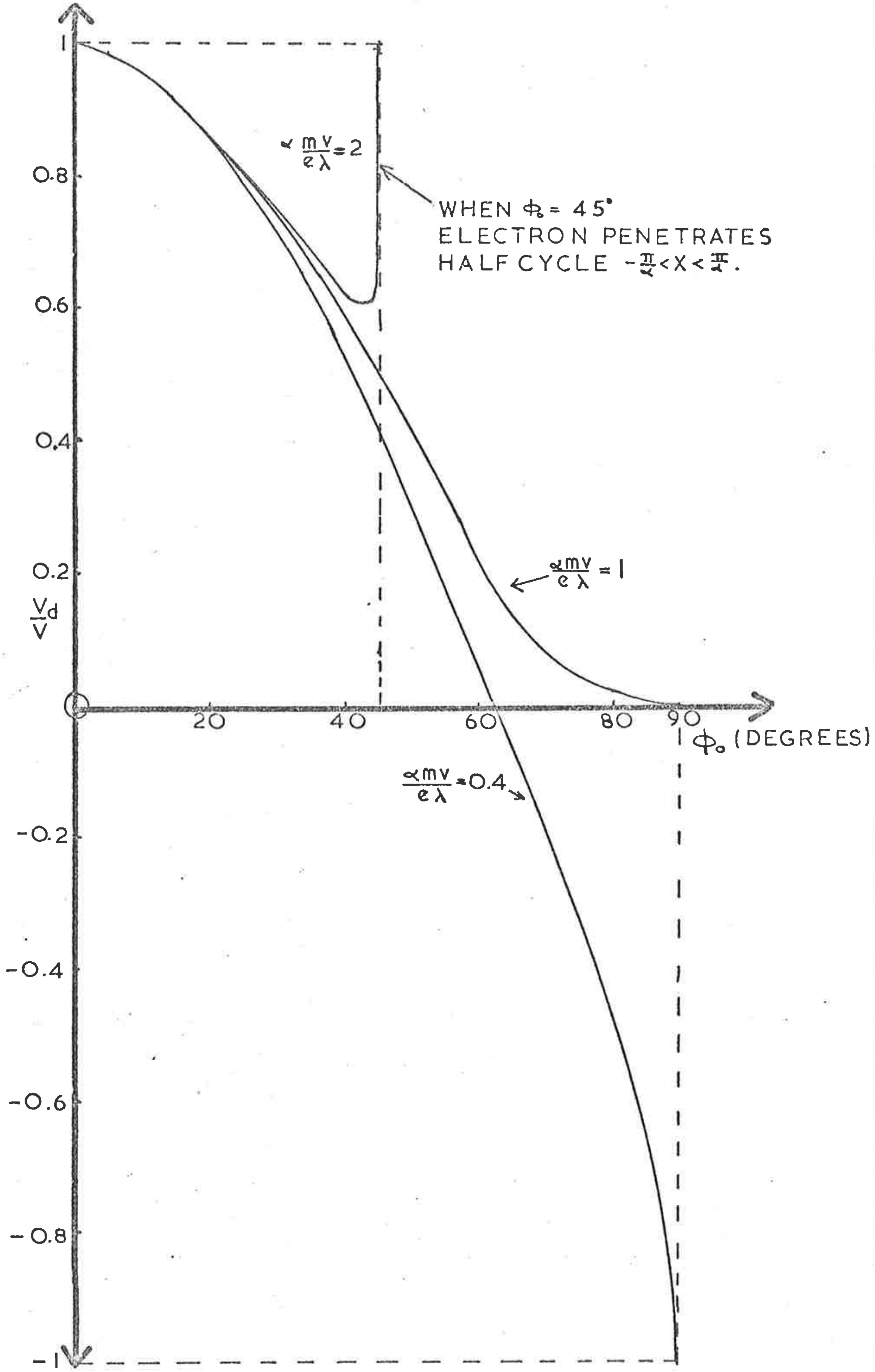


FIGURE 4: DRIFT VELOCITY OF AN ELECTRON IN THE FIELD $B_z = \lambda \sin \alpha X$. ELECTRON CUTS LINE $X = 0$.

$$t(\xi) = \left(\frac{m}{\alpha e v \lambda}\right)^{\frac{1}{2}} qF(q/\xi) \qquad \ldots(5.49)$$

where

$$F(q/\xi) = \int_o^{\xi} \frac{d\xi}{(1 - q^2\sin^2\xi)^{\frac{1}{2}}} , \qquad \ldots(5.50)$$

is an incomplete elliptic integral of the first kind and

$$\xi = \tan^{-1}\{(1 + s^2)^{\frac{1}{2}}\tan\phi\} \quad . \qquad \ldots(5.51)$$

Similarly from equations (2.28), (5.1) and (5.4)

$$y(\xi) = \left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} \frac{q}{s^2}\{(2 + s^2)F(q/\xi) - 2\Pi(n,q/\xi)\} , \qquad \ldots(5.52)$$

and

$$\Pi(n,q/\xi) = \int_o^{\xi} \frac{d\xi}{(1-n\sin^2\xi)(1-q^2\sin^2\xi)^{\frac{1}{2}}} , \qquad \ldots(5.53)$$

is an incomplete elliptic integral of the third kind.

From equation (5.4) and the relation $\psi = \frac{\pi}{2} + 2\phi$ the value of x becomes

$$x(\phi) = \frac{1}{\alpha} \cos^{-1} \left\{\cos\alpha x_o + \left(\frac{\alpha mv}{e\lambda}\right) (1-2\sin^2\phi)\right\}$$

$$= \frac{1}{\alpha} \cos^{-1} \left\{\cos\alpha x_2 - \frac{2\alpha mv}{e\lambda} \sin^2\phi\right\} , \qquad \ldots(5.54)$$

with the help of equation (5.6). Equations (5.52) and (5.54) give the point $P(\phi) = (x(\phi),y(\phi))$ of the particle at a time $t(\phi)$ represented by equation (5.49).

Case 2: Bound Orbit in which the Electron Cuts the Line $x = 0$ on Which $B_z = 0$.

With reference to Figure 2, using equations (2.27), (5.1), and (5.9), and following the same procedure as for Case 2 of Section 5.2, the time becomes

$$t(\psi) = \frac{1}{\alpha v}\int_{\frac{\pi}{2}}^{\psi} \frac{d\psi}{(\sin\psi-\sin\psi_o)^{\frac{1}{2}}(2e\lambda/(\alpha mv)+\sin\psi-\sin\psi_o)^{\frac{1}{2}}} ,$$

or

$$t(\theta) = \left(\frac{m}{\alpha ve\lambda}\right)^{\frac{1}{2}} \int_o^{\theta} \frac{d\theta}{(1 - k^2\sin^2\theta)^{\frac{1}{2}}} . \qquad \ldots(5.55)$$

Thus from equation (5.55) and (5.50)

$$t(\theta) = \left(\frac{m}{\alpha ve\lambda}\right)^{\frac{1}{2}} F(k/\theta) , \qquad \ldots(5.56)$$

and

$$\theta = \tan^{-1}\left\{\left(1 - \frac{\alpha mv}{e\lambda}\sin^2\phi_o\right) \frac{\sin\phi}{(\sin^2\phi_o-\sin^2\phi)^{\frac{1}{2}}}\right\} . \qquad \ldots(5.57)$$

Similarly from equations (2.28), (5.9), (5.50) and (5.53)

$$y(\theta) = \left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} (F(k/\theta) + 2\delta^2\sin^2\phi_o(F(k/\theta) - \Pi(p,k/\theta))), \qquad \ldots(5.58)$$

using the nomenclature of Section 5.2, Case 2. From equation (5.9) and using the substitution $\psi = \frac{\pi}{2} + 2\phi$ with $\psi_o = \frac{\pi}{2} + 2\phi_o$, x becomes

$$x(\phi) = \frac{1}{\alpha}\cos^{-1}\left\{1 - \frac{2\alpha mv}{e\lambda}(\sin^2\phi_o - \sin^2\phi)\right\} , \qquad \ldots(5.59)$$

within the limits of equation (5.12). In terms of ϕ_o, equation (5.12) becomes

$$\sin^2\phi_o < \frac{e\lambda}{\alpha mv} , \qquad \ldots(5.60)$$

in agreement with equation (5.33).

Case 3: Unbound Orbit in which the Electron Passes through the Full Cycle of Magnetic Variation.

If equation (5.60) is invalid the electron will follow a path as sketched in Figure 5. From equation (5.13) and $\psi = \frac{\pi}{2} + 2\phi$ we find

$$\sin^2\phi_o' = \sin^2\phi_o - e\lambda/(\alpha mv) , \qquad \ldots(5.61)$$

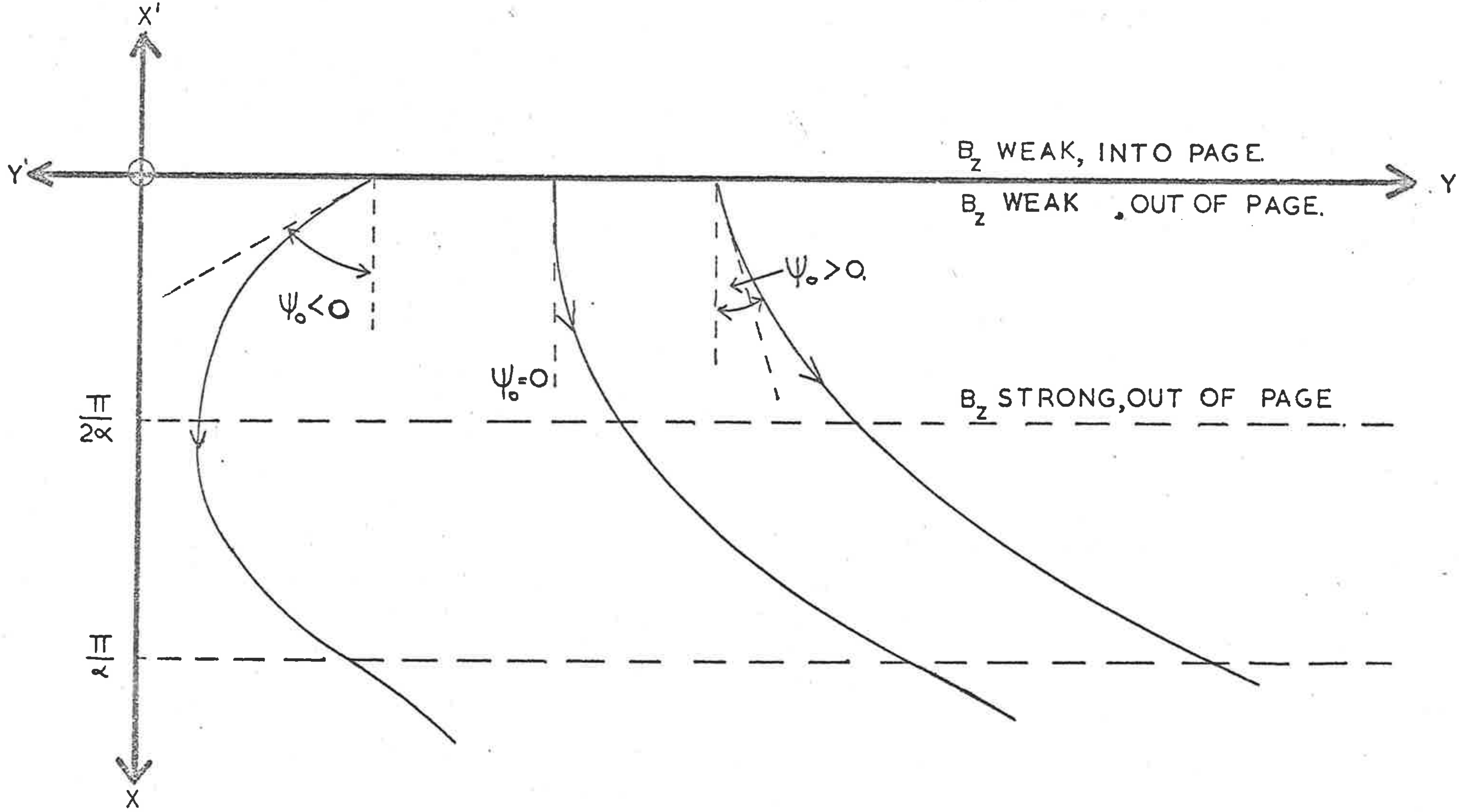


FIGURE 5: MOTION OF AN ELECTRON IN THE FIELD $B_z = \lambda \sin \alpha x$. ELECTRON PENETRATES THE HALF CYCLE X $[0, \pi/\alpha]$.

and from equations (2.27), (5.1), (5.13) and (5.61) the time becomes

$$t(\psi) = \frac{1}{\alpha v}\int_{\psi_o}^{\psi} \frac{d\psi}{(\sin\psi-\sin\psi_o)^{\frac{1}{2}}(\sin\psi_o'-\sin\psi)^{\frac{1}{2}}}$$

$$= \frac{1}{\alpha v}\int_{\phi}^{\phi_o} \frac{d\phi}{(\sin^2\phi_o-\sin^2\phi)^{\frac{1}{2}}(\sin^2\phi-\sin^2\phi_o')^{\frac{1}{2}}} \quad . \qquad \ldots(5.62)$$

With the aid of the substitution $\sin\phi = \sin\phi_o\sin\theta$ followed by $\cos\xi = (\alpha mv/(e\lambda))^{\frac{1}{2}}\sin\phi_o'\cot\theta$, equation (5.62) yields

$$t(\xi) = \left(\frac{m}{\alpha e\lambda v}\right)^{\frac{1}{2}} k'\int_{\xi}^{\frac{\pi}{2}} \frac{d\xi}{(1 - k'^2\sin^2\xi)^{\frac{1}{2}}} \quad , \qquad \ldots(5.63)$$

with

$$k'^2 = \frac{1}{(1 + \alpha mv/(e\lambda)\sin^2\phi_o'\cos^2\phi_o)} \quad , \qquad \ldots(5.64)$$

and

$$0 < k'^2 < 1. \qquad \ldots(5.65)$$

Equation (5.63) may be rewritten

$$t(\xi) = \left(\frac{m}{\alpha e\lambda v}\right)^{\frac{1}{2}} k'\{K(k') - F(k'/\xi)\} \quad , \qquad \ldots(5.66)$$

where $K(k')$ and $F(k'/\xi)$ are respectively complete and incomplete elliptic integrals of the first kind. The parameter ξ is given by

$$\xi = \cos^{-1}\{\left(\frac{\alpha mv}{e\lambda}\right)^{\frac{1}{2}} \sin\phi_o'\cot\theta\}$$

$$= \cos^{-1}\{\left(\frac{\alpha mv}{e\lambda}\right)^{\frac{1}{2}} \frac{\sin\phi_o'}{\sin\phi} (\sin^2\phi_o - \sin^2\phi)\}^{\frac{1}{2}} \quad , \qquad \ldots(5.67)$$

thus as $\phi \to \phi_o'$ and with the aid of equation (5.61), $\cos\xi \to 1$ and therefore $\xi \to 0$.

Thus from equation (5.66) the time required for the electron to cross the full cycle is given by

$$T = \left(\frac{m}{\alpha e\lambda v}\right)^{\frac{1}{2}} k'K(k') \quad . \qquad \ldots(5.68)$$

Similarly using equations (2.28), (5.1) and (5.9)

$$y = \frac{1}{\alpha}\int_{\psi_o}^{\psi} \frac{\sin\psi d\psi}{(\sin\psi-\sin\psi_o)^{\frac{1}{2}}(\sin\psi_o'-\sin\psi)^{\frac{1}{2}}}$$

$$= \left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} k' \left\{\int_{\xi}^{\frac{\pi}{2}} \frac{d\xi}{(1 - k'^2\sin^2\xi)^{\frac{1}{2}}}\right.$$

$$\left. - 2k_o^2\left(1 - \frac{1}{k'^2}\right)\int_{\xi}^{\frac{\pi}{2}} \frac{d\xi}{(1-k_o^2\sin^2\xi)(1-k'^2\sin^2\xi)^{\frac{1}{2}}}\right\} , \qquad \text{...(5.69)}$$

where

$$k_o^2 = \frac{1}{(1+\alpha mv/(e\lambda)\sin^2\phi_o')} = \frac{e\lambda}{\alpha mv\sin^2\phi_o} , \qquad \text{...(5.70)}$$

and since equation (5.60) is invalid

$$0 < k_o < 1 . \qquad \text{...(5.71)}$$

From equation (5.69)

$$y(\xi) = \left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} k' \left\{(K(k') - F(k'/\xi))\right.$$

$$\left. - 2k_o^2\left(1 - \frac{1}{k'^2}\right)(\Pi(k_o,k') - \Pi(k_o,k'/\xi))\right\} . \qquad \text{...(5.72)}$$

$K(k')$ and $\Pi(k_o,k')$ are complete elliptic integrals of the first and third kinds respectively, while $F(k'/\xi)$ and $\Pi(k_o,k'/\xi)$ are incomplete elliptic integrals of the first and third kinds as shown in equations (5.50) and (5.53).

If the electron crosses the half cycle $x\varepsilon[0,\pi/\alpha]$ then the increment of y from equation (5.72) becomes

$$\Delta y = \left(\frac{mv}{\alpha e\lambda}\right)^{\frac{1}{2}} k' \left\{K(k') - 2k_o^2\left(1 - \frac{1}{k'^2}\right)\Pi(k_o,k')\right\} . \qquad \text{...(5.73)}$$

The value of x is given by equation (5.54) and when $\phi = \phi_o'$, $x = \pi/\alpha$.

Thus from equations (5.68) and (5.73) and with $\Delta x = \pi/\alpha$ there is an exact drift result.

$$v_D = \frac{\Delta x}{T}\underset{\sim}{i} + \frac{\Delta y}{T}\underset{\sim}{j} \quad ,$$

$$= \frac{\pi}{\alpha T}\underset{\sim}{i} + v\left\{1 - 2k_o^2\left(1 - \frac{1}{k'^2}\right)\frac{\pi(k_o,k')}{K(k')}\right\}\underset{\sim}{j} \quad . \qquad \ldots(5.74)$$

Unlike the drift results evaluated earlier this velocity is not in a direction perpendicular to the gradient of the magnetic field, but moves at an angle Ξ to the x axis. This angle can be found directly from equations (5.74) and (5.68), or from equation (5.73), and the result $\Delta x = \pi/\alpha$, and is given by

$$\tan\Xi = \frac{\Delta x}{\Delta y}$$

$$= \frac{\Pi}{(\alpha mv/(e\lambda))^{\frac{1}{2}}k'(K(k') - 2k_o^2(1 - \frac{1}{k'^2})\Pi(k_o,k'))} \quad , \qquad \ldots(5.75)$$

or

$$\Xi = \tan^{-1}\left\{\frac{\Pi}{\left(\frac{\alpha mv}{e\lambda}\right)^{\frac{1}{2}}k'(K(k') - 2k_o^2(1 - \frac{1}{k'^2})\Pi(k_o,k'))}\right\} \quad .$$

$$\ldots(5.76)$$

5.4 Discussion of Results

The incomplete elliptic integrals of the first and third kinds are well tabulated, (Belyakov, V.B. et al. (1965)), or alternatively can be computed. All the trajectories for the motion of an electron in a sinusoidal magnetic field can be readily plotted. Figure 6 shows the differing orbits considered. They are compared to show the change in the orbit with varying initial conditions.

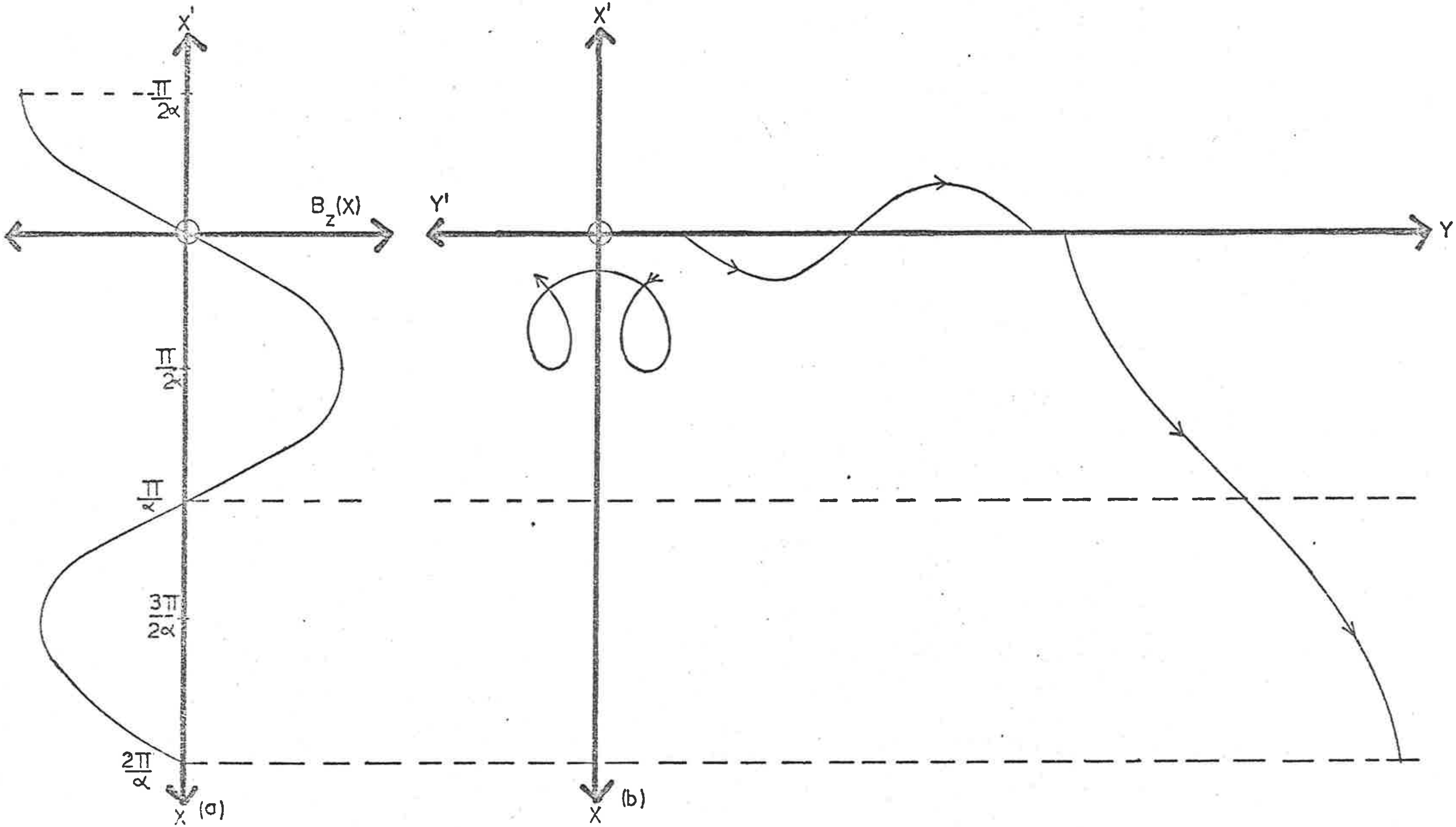


FIGURE 6: (a) VARIATION OF THE FIELD $B_z = \lambda \sin \alpha X$ WITH X.
(b) THE DIFFERING TRAJECTORIES IN A SINUSOIDAL FIELD.

CHAPTER 6

6.1 Motion of an Electron in Static Electromagnetic Fields

Electric scalar potential and magnetic vector potential of the same functional form.

Equation (2.1) has the components

$$\frac{dv_x}{dt} = -\frac{e}{m}\left(E(x) + v_y B_z(x)\right) \,, \qquad \ldots(6.1)$$

and

$$\frac{dv_y}{dt} = +\frac{e}{m} v_x B_z(x) \,, \qquad \ldots(6.2)$$

and since the respective electric and magnetic potentials have the same functional form, from equation (2.45)

$$E(x) = \frac{\phi_o}{A_o} B_z(x) \,, \qquad \ldots(6.3)$$

where ϕ_o/A_o is constant. For

$$v_y = v_y' - E/B_z \,, \qquad \ldots(6.4)$$

equations (6.1) and (6.2) become

$$\frac{dv_x}{dt} = -\frac{e}{m} v_y' B_z(x) \,, \qquad \ldots(6.5)$$

and

$$\frac{dv_y'}{dt} = \frac{e}{m} v_x B_z(x) \,, \qquad \ldots(6.6)$$

which is the equation of motion of an electron in a static magnetic field dependent on x. From equation (2.51) the speed of the electron in the dashed coordinates is

$$v'^2 = \frac{2\varepsilon}{m} - (\gamma_1^2 - \gamma_1'^2) \,. \qquad \ldots(6.7)$$

Equations (6.5) and (6.6) in terms of the angle ψ as shown in

Chapter 2 yield

$$v_x = v'_x = v'\cos\psi \ , \qquad \ldots(2.54)$$

and

$$v'_y = v'\sin\psi \ . \qquad \ldots(2.55)$$

where

$$(v'_x)^2 + (v'_y)^2 = v'^2 \ . \qquad \ldots(2.56)$$

In all the analysis of electron motion in Chapters 3 - 5, the speed, charge and mass of the electron and the initial or boundary conditions of the orbit dictate the shape of the trajectory, its boundedness or unboundedness. Thus introduction of an electric field and changing v to v' in the magnetostatic equations of motion will modify the orbit and may cause a large variation of orbit shape. A simple example will illustrate this.

(1) Motion of an Electron in Exponentially Varying Electric and Magnetic Fields

(a) Bound Orbits:

If

$$E(x) = (\phi_o/A_o)\,B_z(x) = E_o \ell^{\alpha x} \ , \qquad \ldots(6.8)$$

then with the results of Chapter 2, Section III part (b), (c) the orbit of the electron is readily found. Using equations (2.57), (2.58), (2.59) together with equations (4.4), (4.5), (4.6) and the speed in equation (6.7) the time becomes

$$t(\phi) = \frac{2}{\omega_o\left(1 - \left(\frac{mv'}{eA_o}\right)^2\right)^{\frac{1}{2}}} \tan^{-1}\left\{\left[\frac{1 - \left(\frac{mv'}{eA_o}\right)}{1 + \left(\frac{mv'}{eA_o}\right)}\right]^{\frac{1}{2}} \tan\phi\right\} \ , \qquad \ldots(6.9)$$

y gives

$$y(\phi) = -\frac{E}{B_z}\,t(\phi) + \frac{2v'}{\omega_o\left(\frac{mv'}{eA_o}\right)}\,\frac{1}{\left(1-\left(\frac{mv'}{eA_o}\right)^2\right)^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{1-\frac{mv'}{eA_o}}{1+\frac{mv'}{eA_o}}\right)^{\frac{1}{2}}\tan\phi\right\} \quad \ldots(6.10)$$

and from equation (4.6) x is given by

$$x(\phi) = x_2 + \frac{1}{\alpha}\,\ell n\left\{\cos^2\phi\left[1+\frac{\left(1-\frac{mv'}{eA_o}\right)}{\left(1+\frac{mv'}{eA_o}\right)}\tan^2\phi\right]\right\}, \quad \ldots(6.11)$$

where

$$x_2 = x_o + \frac{1}{\alpha}\,\ell n\left(1+\frac{mv'}{eA_o}\right). \quad \ldots(6.12)$$

When $\phi = \frac{\pi}{2}$ equations (6.9) and (6.10) become

$$T/2 = \frac{\Pi}{\omega_o\left(1-\left(\frac{mv'}{eA_o}\right)^2\right)^{\frac{1}{2}}}, \quad \ldots(6.13)$$

and

$$\Delta y/2 = -\frac{E}{B_z}\frac{T}{2} + \frac{v'\Pi}{\omega_o\left(\frac{mv'}{eA_o}\right)}\left[1-\frac{1}{\left(1-\left(\frac{mv'}{eA_o}\right)^2\right)^{\frac{1}{2}}}\right], \quad \ldots(6.14)$$

which give rise to the drift expression

$$v_D = -\frac{E}{B_z} + \frac{v'}{\left(\frac{mv'}{eA_o}\right)}\left(\left(1-\left(\frac{mv'}{eA_o}\right)^2\right)^{\frac{1}{2}}-1\right), \quad \ldots(6.15)$$

The introduction of the electric field has therefore given rise to the generalized electric drift velocity $v_E = E/B_z$ and the magnetostatic drift dependent on v'. Since v' from equation (6.7), (2.51) and (2.48) is dependent on the ratio $\frac{E}{B_z}$, suitable choices of the electric field will cause v' to become smaller. Thus for specific orbits an unbound magnetostatic orbit may become bound when an electrostatic field is introduced or vice versa. For bound orbits

$$\left(\frac{mv'}{eA_o}\right) < 1, \qquad \ldots(6.16)$$

or in terms of E/B_z, equations (6.7), (6.16) and (2.48) give

$$\left(\frac{m}{eA_o}\right)\left(\frac{2\varepsilon}{m} - \left(\gamma_1^2 - \left(\gamma_1 - \frac{E}{B_z}\right)^2\right)\right)^{\frac{1}{2}} < 1 . \qquad \ldots(6.17)$$

If $E/B_z = 0$, then $\frac{2\varepsilon}{m} = v^2$ is constant. If $E/B_z > 0$ then v' is smaller than $2\varepsilon/m$ and the orbits are likely to become "more bounded". For $E/B_z < 0$, v' is greater than $2\varepsilon/m$ and orbits will tend to become unbounded.

(2) Unbound Orbits

Using equations (4.13), (4.14), (2.57) and (2.58) the time and the y value become

$$t(\phi) = \frac{1}{\alpha v'\sin 2\phi'}\left\{\ell n\left\{\frac{1 + \cot\phi'\tan\phi}{1 - \cot\phi'\tan\phi}\right\}\right\} , \qquad \ldots(6.17)$$

and

$$y(\phi) = -\frac{E}{B_z}t(\phi) + \frac{2\phi}{\alpha} + \frac{\cot 2\phi'}{\alpha}\left\{\ell n\left(\frac{1 + \cot\phi'\tan\phi}{1 - \cot\phi'\tan\phi}\right)\right\} . \qquad \ldots(6.18)$$

From equation (4.9) and with v' as the speed of the electron x becomes

$$x(\phi) = \frac{1}{\alpha}\ \ell n\left\{\frac{2\alpha mv'}{e\lambda}\cos^2\phi\sin^2\phi'\{1-\cot^2\phi'\tan^2\phi\}\right\} . \qquad \ldots(6.19)$$

For $\phi \to \phi'$ the orbit of the electron is almost a straight line. The limiting angle ψ'' is therefore found by dividing equation (6.19) by (6.18) for $\phi \sim \phi'$. Therefore

$$\tan\psi'' = \lim_{\phi\to\phi'} \frac{x(\phi)}{y(\phi)}$$

$$= \lim_{\phi\to\phi'}\left\{\frac{\ell n\left\{\left(\frac{2\alpha mv'}{e\lambda}\right)\cos^2\phi'\sin^2\phi(1-\cot^2\phi'\tan^2\phi)\right\}}{2\phi + (-E/(B_z v'\sin 2\phi') + \cot 2\phi')\left\{\ell n\left(\frac{1+\cot\phi'\tan\phi}{1-\cot\phi'\tan\phi}\right)\right\}}\right\}$$

$$= \frac{-1}{\left(\frac{-E}{B_z v'\sin 2\phi'} + \cot 2\phi'\right)} . \qquad \ldots(6.20)$$

Substituting $\psi'' = \pi/2 + 2\phi''$, equation (6.20) becomes

$$\tan 2\phi'' = \frac{1}{\cot 2\phi' - E/(B_z v' \sin 2\phi')} \quad . \qquad \ldots(6.21)$$

For a zero electric field equation (6.21) gives $\phi'' = \phi'$. In general however there is a marked modification to the limiting angle because of the presence of the electric field.

6.2 Electric Scalar Potential Related to the Square of the Magnetic Vector Potential

If the electric scalar and magnetic vector potentials are related as

$$\phi(x) = \delta(A_y(x))^2 \quad , \qquad \ldots(2.68)$$

where δ is constant, then the trajectories of an electron in such fields can be found by using the results of Chapter 2, Section 2.3, part (d). An important parameter is

$$\Omega = 1 - \frac{2m}{e}\delta \ , \qquad \ldots(2.70)$$

which can take any value. The variation of the trajectories with Ω will be analysed for an exponential field.

(i) Motion of an Electron in an Exponential Electric Field $E(x) = \delta\lambda^2/(2\alpha)\ell^{2\alpha x}$ and a Magnetic Field $B_z = \lambda e^{\alpha x}$.

For a magnetic field

$$B_z = \lambda e^{\alpha x} \quad , \qquad \ldots(3.1)$$

the scalar electric potential from equations (2.68), (3.1) and (2.9) becomes

$$\phi(x) = \frac{\delta\lambda^2}{\alpha^2} e^{2\alpha x} \quad . \qquad \ldots(6.22)$$

Case 1: The Electromagnetic Parameter, $\Omega > 0$.

(a) Bound Orbits for the Electron

From equation (2.16) $\gamma_1 = v_{yo} - e/m\ A_y(x)$ and choosing $v_{yo} = 0$, then using equations (3.1) and (2.9)

$$\gamma_1 = -\frac{e}{m} A_y(x_o)$$

$$= -\frac{e\lambda}{\alpha m} e^{\alpha x_o} = -\gamma_2 < 0 , \qquad \text{...(6.23)}$$

since λ and $\alpha > 0$. Thus $\gamma_2 > 0$. Equation (2.76), (3.1) and (2.9) yield

$$\xi \sin\theta = \Omega^{\frac{1}{2}} \{-\frac{\gamma_2}{\Omega} + \frac{e\lambda}{m\alpha} e^{\alpha x}\} , \qquad \text{...(6.24)}$$

and using equation (6.24) x becomes

$$x = \frac{1}{\alpha} \ln \{\frac{\alpha m}{e\lambda\Omega^{\frac{1}{2}}} \{\frac{\gamma_2}{\Omega^{\frac{1}{2}}} + \xi\sin\theta\}\} . \qquad \text{...(6.25)}$$

At $\theta = \pi/2$, x has its maximum value x_2, where

$$x_2 = \frac{1}{\alpha} \ln \{\frac{\alpha m}{e\lambda\Omega^{\frac{1}{2}}} \{\frac{\gamma_2}{\Omega^{\frac{1}{2}}} + \xi\}\} , \qquad \text{...(6.26)}$$

and its minimum value x_1, at $\theta = 3\pi/2$, given by

$$x_1 = \frac{1}{\alpha} \ln \{\frac{\alpha m}{e\lambda\Omega^{\frac{1}{2}}} \{\frac{\gamma_2}{\Omega^{\frac{1}{2}}} - \xi\}\} . \qquad \text{...(6.27)}$$

Using equations (2.78), (3.1), (3.25) and (2.9), the time becomes

$$t(\theta) = \frac{1}{\alpha} \int_{\frac{\pi}{2}}^{\theta} \frac{d\theta}{(\gamma_2/\Omega^{\frac{1}{2}} + \xi\sin\theta)} , \qquad \text{...(6.28)}$$

and using the substitution $\theta = \pi/2 + 2\phi$ followed by the substitution

$$\tan\mu = \left(\frac{\frac{\gamma_2}{\Omega^{\frac{1}{2}}} - \xi}{\frac{\gamma_2}{\Omega^{\frac{1}{2}}} + \xi}\right)^{\frac{1}{2}} \tan\phi , \qquad \text{...(6.29)}$$

equation (6.28) yields

$$t(\mu) = \frac{2}{\alpha\left(\frac{\gamma_2^2}{\Omega} - \xi^2\right)^{\frac{1}{2}}} \mu \quad . \qquad \ldots(6.30)$$

Similarly from equations (2.79), (3.1), (6.56) and (2.9)

$$y(\mu) = -\gamma_2 t(\mu) + \frac{2\phi}{\alpha\Omega^{\frac{1}{2}}} \quad , \qquad \ldots(6.31)$$

and also from equation (6.25)

$$x(\phi) = \frac{\ell n}{\alpha} \left\{\frac{\alpha m}{e\lambda\Omega^{\frac{1}{2}}} (\gamma_2/\Omega^{\frac{1}{2}} + \xi)(1 + \tan^2\mu)\cos^2\phi\right\} . \qquad \ldots(6.32)$$

In a half cycle $\mu = \pi/2$ and equation (6.30) becomes

$$\frac{T}{2} = \frac{\Pi}{\alpha\left(\frac{\gamma_2^2}{\Omega} - \xi\right)^{\frac{1}{2}}} \quad , \qquad \ldots(6.33)$$

with equation (6.31) yielding

$$\frac{\Delta y}{2} = -\gamma_2 \frac{T}{2} + \frac{\Pi}{\alpha\Omega^{\frac{1}{2}}} \quad . \qquad \ldots(6.34)$$

Dividing equation (6.34) by equation (6.33) gives

$$v_D = \frac{\Delta y}{T} = -\gamma_2 + \frac{\left(\frac{\gamma_2^2}{\Omega} - \xi\right)^{\frac{1}{2}}}{\Omega^{\frac{1}{2}}} \quad , \qquad \ldots(6.35)$$

as the drift velocity. If $\Omega = 1$ and the electric field is zero, equation (6.35) gives the drift result of equation (3.11) for an electron in a static exponential magnetic field only.

(b) Unbound Orbits

For unbound orbits $\gamma_2 = -\gamma_1$ but $v_{yo} \neq 0$ in general. $\gamma_2/\Omega^{\frac{1}{2}} < \xi$ and thus $\xi - \gamma_2/\Omega^{\frac{1}{2}} > 0$. Using the substitution $\psi = \frac{\pi}{2} + 2\phi$ the time in equation (6.28) becomes

$$t(\phi) = \frac{2}{\alpha(\xi+\gamma_2/\Omega^{\frac{1}{2}})}\int_o^\phi \frac{\sec^2\phi d\phi}{\left[1 - \left(\frac{\xi-\gamma_1/\Omega^{\frac{1}{2}}}{\xi+\gamma_1/\Omega^{\frac{1}{2}}}\right)\tan^2\phi\right]}$$

$$= \frac{1}{\alpha(\xi^2-\gamma_2^2/\Omega)^{\frac{1}{2}}} \ell n \left\{\frac{1 + \left(\frac{\xi - \frac{\gamma_2}{\Omega^{\frac{1}{2}}}}{\xi + \frac{\gamma_2}{\Omega^{\frac{1}{2}}}}\right)^{\frac{1}{2}} \tan\phi}{1 - \left(\frac{\xi - \frac{\gamma_2}{\Omega^{\frac{1}{2}}}}{\xi + \frac{\gamma_2}{\Omega^{\frac{1}{2}}}}\right)^{\frac{1}{2}} \tan\phi}\right\} , \qquad ...(6.36)$$

and similarly equations (2.79), (6.26), (3.1), (2.9) and (6.36) give

$$y(\phi) = -\gamma_2 t(\phi) + \frac{2\phi}{\alpha\Omega^{\frac{1}{2}}} , \qquad ...(6.37)$$

with the x coordinate given by

$$x(\phi) = \frac{1}{\alpha} \ell n \left\{\frac{\alpha m\left(\xi + \frac{\gamma_2}{\Omega^{\frac{1}{2}}}\right)}{e\lambda\Omega^{\frac{1}{2}}} \cos^2\phi \left[1 - \left(\frac{\xi - \frac{\gamma_2}{\Omega^{\frac{1}{2}}}}{\xi + \frac{\gamma_2}{\Omega^{\frac{1}{2}}}}\right) \tan^2\phi\right]\right\} . \qquad ...(6.38)$$

If the electric field is "switched off", then $\Omega = 1$, $\xi = v$, $\gamma_2 = -v\sin\psi' = v(1-2\cos^2\phi')$ and therefore

$$\left.\begin{aligned} \xi + \frac{\gamma_2}{\Omega^{\frac{1}{2}}} &= 2v\cos^2\phi' , \\ \xi - \frac{\gamma_2}{\Omega^{\frac{1}{2}}} &= 2v\sin^2\phi' . \end{aligned}\right\} \qquad ...(6.39)$$

Equations (6.39) then give

$$\left.\begin{aligned} \left(\xi^2 - \frac{\gamma_2^2}{\Omega}\right)^{\frac{1}{2}} &= v\sin 2\phi' \\ \text{and} \qquad \frac{(\xi - \gamma_2/\Omega^{\frac{1}{2}})}{(\xi + \gamma_2/\Omega^{\frac{1}{2}})} &= \cot^2\phi' \end{aligned}\right\} \qquad ...(6.40)$$

Hence from equations (6.39) and (6.40), equations (6.36), (6.37) and (6.38) respectively simplify to the magnetostatic equations of (4.13), (4.14) and (4.9) in Chapter 4.

From equations (6.36), (6.37), (6.38) and (6.23) the limiting angle ψ_E' to which the electron asymptotes is given by

$$\tan\psi_E' = \lim_{\phi\to\phi'} \frac{x(\phi)}{y(\phi)}$$

$$= + \frac{(\xi - \gamma_2^2/\Omega)^{\frac{1}{2}}}{\gamma_1}$$

$$= \frac{(2\varepsilon/m - \gamma_1^2)^{\frac{1}{2}}}{\gamma_1} \quad . \qquad \ldots(6.41)$$

γ_1 is the speed of the electron in the y direction at $x = \infty$ and $2\varepsilon/m$ is the total speed of the electron at $x = \infty$.

Case 2: The Electromagnetic Parameter, $\Omega = 0$ and $2m\delta/e = 1$.

From equation (2.69) and with $\Omega = 0$, the energy becomes

$$\frac{2\varepsilon}{m} = \left(\frac{dx}{dt}\right)^2 + \gamma_1^2 + \frac{2e}{m}\gamma_1 A_y(x) \quad . \qquad \ldots(6.42)$$

Defining

$$\mu = \frac{2\varepsilon}{m} - \gamma_1^2 \, , \qquad \ldots(6.43)$$

equation (6.42) yields

$$\mu = \left(\frac{dx}{dt}\right)^2 + \frac{2e}{m}\gamma_1 A_y(x) \quad . \qquad \ldots(6.44)$$

For an exponential magnetic field equations (6.44), (3.1) and (2.9) yield

$$t = \int_{x_o}^{x} \frac{dx}{\mu\left\{1 - \left(\frac{2e\gamma_1\lambda e^{\alpha x}}{m\mu\alpha}\right)\right\}^{\frac{1}{2}}} \, , \qquad \ldots(6.45)$$

as the time. Substitution of $X = 2e\gamma_1\lambda e^{\alpha x}/(m\mu\alpha)$ in equation (6.45) gives

$$t(X) = \int_{X_o}^{X} \frac{dX}{\mu\alpha X\sqrt{(1-X)}} = \frac{1}{\mu\alpha}\,\ell n \left|\frac{(1-X)^{\frac{1}{2}}-1}{(1-X)^{\frac{1}{2}}+1}\right| \Bigg|_{X_o}^{X} \quad . \qquad \ldots(6.46)$$

Similarly equations(6.44) and (2.14) yield

$$y(X) = \left(\gamma_1 + \frac{\mu}{2\gamma_1}\right) t(X) + \mu/(2\gamma_1\alpha)\ [X - \log X] \Big|_{X_o}^{X} , \qquad \ldots(6.47)$$

with the help of equations (6.44) and (6.46), and standard integral expressions (see equation (3.3.30), Abramowitz and Stegun).

Case 3: The Electromagnetic Parameter, $\Omega < 0$ and $2m\delta/e > 1$.

Since $\Omega < 0$ a new parameter

$$\Omega_1 = -\Omega > 0 , \qquad \ldots(6.47)$$

is defined. From equation (2.74) the time becomes

$$t = \int \frac{dx}{\left(\xi^2 + \Omega_1\left(-\frac{\gamma_1}{\Omega_1} + \frac{e}{m} A_y(x)\right)^2\right)^{\frac{1}{2}}} , \qquad \ldots(6.48)$$

and the quantity

$$\xi \sinh\theta = \Omega_1^{\frac{1}{2}} \left(-\gamma_1/\Omega_1 + \frac{e}{m} A_y(x)\right) , \qquad \ldots(6.49)$$

gives on differentiation

$$\Omega_1^{\frac{1}{2}} \frac{e}{m} B_z(x)dx = \xi \cosh\theta d\theta . \qquad \ldots(6.50)$$

Equations (6.50) and (6.48) give

$$t(\theta) = \int \frac{d\theta}{\Omega_1^{\frac{1}{2}}\omega_\theta} , \qquad \ldots(6.51)$$

as the time, whilst equation (2.75) becomes

$$y(\theta) = \gamma_1\left\{1 + \frac{1}{\Omega_1^{\frac{1}{2}}}\right\} t(\theta) + \frac{1}{\Omega_1} \int \frac{\xi \sinh\theta d\theta}{\omega_\theta} . \qquad \ldots(6.52)$$

For an exponential magnetic field equations (6.49), (3.1) and (2.9) yield

$$\xi \sinh\theta = \Omega_1^{\frac{1}{2}}(-\gamma_1/\Omega_1 + e\lambda/(m\alpha)\ell^{\alpha x}) , \qquad \ldots(6.53)$$

from which the value of x becomes

$$x(\theta) = \frac{1}{\alpha} \ell n \left\{ \alpha m/(e\lambda\Omega_1^{\frac{1}{2}}) \left(\frac{\gamma_1}{\Omega_1^{\frac{1}{2}}} + \xi \sinh\theta \right) \right\} , \qquad \ldots(6.54)$$

and from equation (6.51), (6.53), (3.1) and (2.9) the time is given by

$$t(\theta) = \int_{\theta_o}^{\theta} \frac{d\theta}{\alpha(\gamma_1/\Omega_1^{\frac{1}{2}} + \xi \sinh\theta)} . \qquad \ldots(6.55)$$

Finally, with the aid of the substitution

$$\tanh\mu = \frac{1}{\left(1 + \frac{\Omega_1 \xi^2}{\gamma_1^2}\right)^{\frac{1}{2}}} \left\{ \tanh 2\theta - \Omega_1^{\frac{1}{2}} \xi/\gamma_1 \right\} , \qquad \ldots(6.56)$$

the time becomes

$$t(\mu) = \frac{2}{\alpha(\gamma_1^2/\Omega_1 + \xi^2)^{\frac{1}{2}}} \mu . \qquad \ldots(6.57)$$

Similarly from equations (6.52), (6.53) and (6.57) the value of y is

$$y(\mu) = \gamma_1 t(\mu) + \theta/(\alpha\Omega^{\frac{1}{2}}) . \qquad \ldots(6.58)$$

Variation of Ω has led to a complete change of functional dependence of the results from trigonometric to hyperbolic tangent functions. This indicates that the field may be predominantly electrostatic (Ω large), magnetostatic ($\Omega \sim 1$), or singular , having no dominant field type ($\Omega = 0$).

6.3 Electromagnetostatics and Special Relativity

If the speed, v, of the electron approaches the speed of light, then the present non-relativistic approach is not applicable. In special relativity the static fields are represented by a covariant vector potential

$$A_\nu = (0, A_2(x_1), A_3(x_1), A_4(x_1)) , \qquad \ldots(6.59)$$

where $x = x_1$, $y = x_2$, $z = x_3$ and $ict = x_4$. A_ν consists of

$$\underset{\sim}{A} = (A_1, A_2, A_3) = (0, A_2(x_1), A_3(x_1)) \ , \qquad \ldots(6.60)$$

and

$$A_4 = i\phi(x_1) \ , \qquad \ldots(6.61)$$

consistent with Maxwell's equations.

The relativistic components of velocity are

$$u_\nu = \frac{dx_\nu}{d\tau} \ , \qquad \ldots(6.62)$$

and τ is the proper time given by

$$(d\tau)^2 = -dx_\nu dx^\nu \ , \qquad \ldots(6.63)$$

and is related to the time t as

$$\frac{d\tau}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \gamma d\tau = dt \ , \qquad \ldots(6.64)$$

with

$$\gamma = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \ . \qquad \ldots(6.65)$$

For an electron with rest mass m and charge -e the relativistic Lagrangian is given by

$$L' = \frac{1}{2} m u_\nu u^\nu - e u_\lambda A^\lambda \ , \qquad \ldots(6.66)$$

and the relativistic equations of motion are

$$\frac{d}{d\tau}\left\{\frac{\partial L'}{\partial u_\nu}\right\} - \frac{\partial L'}{\partial x_\nu} = 0 \ . \qquad \ldots(6.67)$$

The three constants of motion for the relativistic potentials in equations (6.60) and (6.61) are given by

$$m u_2 - e A_2(x_1) = m\gamma v_2 - e A_2(x_1) = p_2 \ , \qquad \ldots(6.68)$$

$$m u_3 - e A_3(x_1) = m\gamma v_3 - e A_3(x_1) = p_3 \ , \qquad \ldots(6.69)$$

and

$$m\gamma c^2 - e\phi(x_1) = \varepsilon \, . \qquad \ldots(6.70)$$

ε is the energy of the electron while p_2 and p_3 are the relativistic canonical momenta in the x_2 and x_3 directions.

From equation (6.70) γ becomes

$$\gamma = \frac{1}{mc^2} (\varepsilon + e\phi(x_1)) \quad , \qquad \ldots(6.71)$$

and from equations (6.68) and (6.69) the x_2 and x_3 components of the velocity may be written

$$v_2 = \frac{dx_2}{dt} = \frac{1}{m\gamma} (p_2 + eA_2(x_1)) \, , \qquad \ldots(6.72)$$

and

$$v_3 = \frac{dx_3}{dt} = \frac{1}{m\gamma} (p_3 + eA_3(x_1)) \, . \qquad \ldots(6.73)$$

Equation (6.71) gives

$$v^2 = \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 = \frac{-c^2}{\gamma^2} (1 - \gamma^2) , \qquad \ldots(6.74)$$

as the speed of the electron, and therefore

$$\left(\frac{dx_1}{dt}\right)^2 = \frac{-c^2}{\gamma^2} (1-\gamma^2) - \frac{1}{(m\gamma)^2} [(p_2+eA_2(x_1))^2+(p_2+eA_2(x_1))^2]$$

$$= \frac{c^2}{\gamma^2} \left\{-1 - \frac{1}{(mc)^2} \left[\frac{-(\varepsilon+e\phi(x_1))^2}{c^2} + (p_2+eA_2(x_1))^2+(p_3+eA_3(x_1))^2\right]\right\} .$$

$$\ldots(6.75)$$

From equation (6.75) the differential of the time is given by

$$dt = \frac{\gamma dx_1}{c\left\{-1 - \frac{1}{(mc)^2}\left[\frac{-(\varepsilon+e\phi(x_1))^2}{c^2} + (p_2+eA_2(x_1))^2+(p_3+eA_3(x_1))^2\right]\right\}^{\frac{1}{2}}}$$

$$= \frac{(\varepsilon+\phi(x_1))dx_1}{mc^3\left\{-1 - \frac{1}{(mc)^2}\left[\frac{-(\varepsilon+e\phi(x_1))^2}{c^2} + (p_2+eA_2(x_1))^2+(p_3+eA_3(x_1))^2\right]\right\}^{\frac{1}{2}}}$$

$$\ldots(6.76)$$

Using equations (6.72) and (6.76) the differential of x_2 becomes

$$dx_2 = v_2 dt$$

$$= \frac{(p_2 + eA_2(x_1))dx_1}{mc\left\{1 - \frac{1}{(mc)^2}\left[\frac{-(\varepsilon+e\phi(x_1))^2}{c^2} + (p_2+eA_2(x_1))^2 + (p_3+eA_3(x_1))^2\right]\right\}^{\frac{1}{2}}},$$

...(6.77)

and from equations (6.73) and (6.76)

$$dx_3 = v_3 dt$$

$$= \frac{(p_3+eA_3(x_1))dx_1}{mc\left\{1 - \frac{1}{(mc)^2}\left[\frac{-(\varepsilon+e\phi(x_1))^2}{c^2} + (p_2+eA_2(x_1))^2 + (p_3+eA_3(x_1))^2\right]\right\}^{\frac{1}{2}}}.$$

...(6.78)

Equations (6.76), (6.77) and (6.79) may now be integrated to obtain the trajectory of the electron for the covariant vector potential of equation (6.59).

If $v \ll c$ then the energy expression (6.70) degenerates to the non-relativistic energy equation (2.4). Similarly the canonical momenta in equations (6.68) and (6.69) simplify to the non-relativistic results of equations (2.14) and (2.15).

(a) One Component Magnetic Field

If

$$A_\nu = (0, A_2(x_1), 0, 0) \quad , \qquad ...(6.79)$$

then the magnetic field is

$$B3(x_1) = \frac{dA_2(x_1)}{dx_1} \quad . \qquad ...(6.80)$$

From equations (6.69) and (6.70)

$$\left.\begin{aligned} m\gamma v_3 &= p_3 \quad , \\ m\gamma c^2 &= \varepsilon, \text{ or } \quad \gamma = \varepsilon/mc^2 \quad , \\ \text{and} \quad \eta^2 &= \gamma^2 - 1 > 0. \end{aligned}\right\} \qquad ...(6.81)$$

Equation (6.76) yields on integration

$$t = \frac{\gamma}{c}\int \frac{dx_1}{\left(\eta^2 - \frac{1}{(mc)^2}(p_2+eA_2(x_1))^2\right)^{\frac{1}{2}}} \quad , \qquad \ldots(6.82)$$

and equation (6.77) similarly yields

$$y = \frac{1}{mc}\int \frac{(p_2+eA_2(x_1))dx_1}{\left(\eta^2 - \frac{(p_2+eA_2(x_1))^2}{(mc)^2}\right)^{\frac{1}{2}}} \quad . \qquad \ldots(6.83)$$

Since from equation (6.81) γ is constant, equation (6.72) is a simple relationship. If the parameter κ is defined such that

$$\eta \sin\kappa = (p_2 + eA_2(x_1))/(mc) \quad , \qquad \ldots(6.84a)$$

then

$$\eta \cos\kappa d\kappa = \frac{eB_3(x_1)}{(mc)} dx_1 \quad , \qquad \ldots(6.84b)$$

and therefore equations (6.82) and (6.83) simplify to

$$t = \int \frac{d\kappa}{\omega_\kappa} \quad , \qquad \ldots(6.85a)$$

and

$$y = \gamma^{-1}\int \frac{(\eta c)\sin\kappa d\kappa}{\omega_\kappa} \quad , \qquad \ldots(6.85b)$$

where

$$\omega_\kappa = \frac{e}{\gamma m} B_3(\kappa) \quad , \qquad \ldots(6.86)$$

is the relativistic gyrofrequency in terms of κ.

(b) Exponentially Varying Magnetic Field

For a magnetic field of the form

$$B_z = \lambda e^{\alpha x_1} \quad , \qquad \ldots(6.87)$$

where λ and $\alpha > 0$, equation (6.84a)gives

$$\eta \sin\kappa = (p_2 + \frac{e\lambda}{\alpha} e^{\alpha x_1})/(mc), \qquad \ldots(6.88)$$

using equation (2.9). From equation (6.88) x becomes

$$x = \frac{1}{\alpha}\ \ell n\ \{\frac{\alpha}{e\lambda}\ (-p_2 + \eta(mc)\sin\kappa)\} \qquad \ldots(6.89)$$

Using equations (6.87), (6.88) and (6.89), equations (6.85a) and (6.85b) become

$$t = \frac{\gamma m}{\alpha}\int \frac{d\kappa}{(-p_2 + \eta(mc)\sin\kappa)}, \qquad \ldots(6.90)$$

and

$$y = \frac{\eta c}{\alpha}\int \frac{m \sin\kappa d\kappa}{(-p_2 + \eta(mc)\sin\kappa)} \qquad \ldots(6.91)$$

For bound orbits $-p_2 > \eta(mc) > 0$ in which case, with the aid of the substitution $\kappa = \frac{\pi}{2} + 2\phi$, equations (6.90) and (6.91) yield

$$t(\phi) = \frac{2\gamma m}{\alpha(-p_2)\left(1 - \left(\frac{mc\eta}{p_2}\right)^2\right)^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{1 - \left(\frac{mc\eta}{-p_2}\right)}{1 + \left(\frac{mc\eta}{-p_2}\right)}\right)^{\frac{1}{2}}\tan\phi\right\}, \qquad \ldots(6.92)$$

and

$$y(\phi) = \frac{+2\phi}{\alpha} + \frac{p_2}{m\gamma}\ t(\phi) \qquad \ldots(6.93)$$

From equation (6.89) the maximum value x_2 of x occurs when $\kappa = \frac{\pi}{2}$; thus

$$x_2 = \frac{1}{\alpha}\ \ell n\ \{\frac{\alpha}{e\lambda}\ (-p_2 + \eta mc)\}, \qquad \ldots(6.94)$$

and the minimum value x_1 occurs when $\kappa = 3\pi/2$ and is given by

$$x_1 = \frac{1}{\alpha}\ \ell n\ \{\frac{\alpha}{e\lambda}\ (-p_2 - \eta mc)\} \qquad \ldots(6.95)$$

For unbound orbits, $\eta(mc) > -p_2$ and equation (6.90) yields

$$t(\phi) = \frac{\gamma m}{\alpha(-p_2)\left(\left(\frac{mc\eta}{p_2}\right)^2 - 1\right)^{\frac{1}{2}}}\ \ell n\ \{\frac{1 + X}{1 - X}\}, \qquad \ldots(6.96)$$

with y given by equation (6.93), but for the expression of time which is from equation (6.96), and $X = \left\{\left(\frac{mc\eta}{-p_2} - 1\right) / \left(\frac{mc\eta}{-p_2} + 1\right)\right\}^{\frac{1}{2}} \tan\phi$.

Similar results can be obtained from equations (6.76), (6.77) and

(6.78) if a one component magnetic field and an electrostatic field with the same functional dependence is considered.

(b) Discussion of Results

The relativistic nature of the work makes it possible to investigate the motion of fast moving electrons in magnetostatic and electromagnetostatic fields with a cartesian geometry. Such work may be useful in analysing cosmic rays.

CHAPTER 7

7.1 Conclusions

Clearly exact solutions exist for the motion of an electron in chosen static spatially varying magnetic fields. From the three forms of magnetic field considered many different trajectories have been encountered.

For an exponential field, $B_z = \lambda\ell^{\alpha x}$, only two types of orbits were found. They were either bound cyclic orbits drifting in the negative Y direction, or unbound orbits in which the electron is "reflected" about the x axis and moves to infinity by asymptoting to an angle ψ' with the x axis. It was found that one curve was required to represent all the drift velocities of the electron for bound orbits. All the trajectories were found in terms of simple trigonometric and logarithmic functions. The simple results reflect the simple nature of the magnetic field which is always in the same direction and monotonically increasing with x.

A magnetic field with the power law dependence $B_z = \lambda x^\alpha$ is a much more complicated field and varies greatly if α is positive or negative. If α is greater than zero then at $x = 0$, $B_z = 0$ and for α an odd integer, an electron crossing the line $x = 0$ enters a region of reversed magnetic field. This creates a symmetrical orbit in which the electron oscillates about the line $x = 0$. For α an odd integer the electron always experiences a magnetic field in the same direction, and therefore the orbits are cyclic. Bound orbits not crossing the line $x = 0$ for $\alpha > 0$ and bound orbits for $\alpha < 0$ drift in opposite directions. For positive α the electrons drift in the negative y direction while for negative α the electrons drift in the positive y direction. Also when $\alpha < -1$ there are unbound orbits in which the electron moves to infinity, asymptoting to an angle ψ' with the x axis. The magnetic field monotonically varies with x on either side of the line

$x = 0$. If α is positive the magnetic field is monotonically increasing, and if α is negative it is monotonically decreasing.

A sinusoidal field dependence $B_z = \lambda \sin\alpha x$ presents the greatest variation type, since when $0 < x < \frac{\pi}{2\alpha}$ the magnetic field is positive and monotonically increasing, while if $\frac{\pi}{2\alpha} < x < \frac{\pi}{\alpha}$ the magnetic field is positive and monotonically decreasing. Thus for orbits confined within the half cycle, $0 < x < \frac{\pi}{\alpha}$, and if $0 < x_o < \frac{\pi}{2\alpha}$ the electron drifts in the negative y direction, but if $\frac{\pi}{(2\alpha)} < x_o < \frac{\pi}{\alpha}$, then the electron drifts in the positive y direction. If $x_o = \frac{\pi}{2\alpha}$ the electron does not drift, but moves in an elliptic type orbit about that line. Orbits which cut the line $x = 0$ will enter a region of reversed magnetic field. Trajectories will therefore be symmetric about $x = 0$, with the direction of the drift governed by the initial conditions of the orbit of the electron. If the electron has enough energy to penetrate the half cycle, then a completely different type of trajectory is discovered. This orbit is unbounded by x but is periodic in nature. Thus each different form of magnetic field considered yields quite different orbit types.

In dealing with spatially dependent electric and magnetic fields it was found that for fields with the same functional dependence there was a generalized electric drift velocity coupled with the magnetostatic drift result. To illustrate this an exponential field dependence was used. Similarly the exponential magnetic field was used with the motion of the electron in an electro-magnetostatic field where the electric scalar potential was related by a constant to the square of the magnetic vector potential. The variation of the orbit with the constant $\Omega = 1 - 2m\delta/e$ was of prime importance in this example. It was shown that if $\Omega \approx 1$ the orbits are closely magnetostatic in nature, while if Ω approaches positive infinity, (and is of the same order as the energy), the electron behaves as if acted upon by an attractive potential. If Ω approaches negative infinity it moves as if under the influence of a repulsive potential. The value $\Omega = 0$ is a singular case in which the electric field and magnetic field interact in such a way that the electric potential and the magnetic potential squared terms cancel. Integral expressions for the motion of the electron have also been analysed for the case when the speed of the electron approaches the speed of light. Results have been analysed for the case of a one component magnetic vector potential $A_2(x_1)$.

7.2 Extension of Results by Piecewise Fitting

If in a problem the type of field being considered is not closely one which can be considered exactly, but can be obtained by piecewise fitting different forms of magnetic field together, then the trajectories can also be piecewise fitted, provided that the initial conditions of the motion of the electron are known. The restrictions of the magnetic field at $x = x_o$ where the fields are being piecewise fitted are

$$B_{z-}(x_o) = B_{z+}(x_o) \;, \qquad \ldots(7.1)$$

and

$$\frac{dB_{z-}(x_o)}{dx} = \frac{dB_{z+}(x_o)}{dx} \;, \qquad \ldots(7.2)$$

where - signifies the limit as $x \to x_o$ for $x < x_o$ and + signifies the limit $x \to x_o$ for $x > x_o$.

Two simple examples are magnetic fields of the form

$$B_z(x) = \lambda|x|^{\alpha} \quad \alpha > 1 \;, \qquad \ldots(7.3)$$

and

$$B_z(x) = \lambda x|x|^{\alpha} \quad \alpha > 1 \;, \qquad \ldots(7.4)$$

where α is not necessarily an integer. Thus for any $\alpha > 1$ the electron crossing the line $x = 0$ can either move into a region of reversing or non-reversing fields. Given initial conditions in the region $x > 0$, then the initial conditions for the region $x < 0$ are the final conditions for the region $x > 0$. These can be readily found. The forms of magnetic field need not be so closely related, since for example the first field may be exponential, $B_z = \lambda \ell^{\alpha x}$, and the second field may have a power law dependence, $B_z = \lambda' x^{\alpha'}$. Thus from equations (7.1) and (7.2)

$$\lambda e^{\alpha x_o} = \lambda' x_o^{\alpha'} \quad , \qquad \ldots(7.5)$$

and

$$\lambda \alpha e^{\alpha x_o} = \lambda' \alpha' x_o^{\alpha' - 1} \quad . \qquad \ldots(7.6)$$

From equation (7.5) and (7.6)

$$\alpha = \alpha'/x_o \quad , \qquad \ldots(7.7)$$

is the necessary condition for piecewise smooth fitting of curves. To fit trajectories it would be necessary to use the trajectory results of Chapter 4. (A process akin to the trapezoid rule in numerical analysis could be useful in analysing magnetic fields).

7.3 Applicability of Results in Upper Atmospheric Research

Cowley (1970) has already used Seymour's (1959) paper to discuss processes in the upper atmospheric regions, in particular the magnetic tail of the earth. Stevenson and Comstock, (1968) have also used the magnetic field with constant gradient to describe particle penetration through the magnetospheric boundary.

Cole (1974) has utilized the adiabatic drift approximation to describe methods in which particles may penetrate into the magnetosphere. It is considered that the present analysis gives another approach to this important problem. Indeed, much of the motion of the electron at the interface of the earth's magnetic field and the solar wind is non adiabatic, and therefore it may be more profitable to use the exact approach discussed in this thesis. Seymour (1975) is at present using the drift expressions of the power law magnetic field to investigate non-adiabatic motion of an electron in the magnetic tail of the magnetosphere.

7.4 Application in Laboratory Plasmas: The Axially Symmetric Field (R,θ,z)

For specific combinations of axially symmetric electro-magnetic fields and electron orbit types the axially symmetric solutions simplify to the case of electron motion in fields with cartesian symmetry. Hertweck (1959) found one such example of this when discussing the motion of an electron in the magnetic field of an infinitely long current carrying wire. For the special case of electron motion in a meridian plane of the wire solutions to the electron motion were in terms of Bessel functions. As shown in Chapter 3 this is the particular case in cartesian geometry of motion of an electron in a magnetic field in the z direction varying as the inverse of x.

The vector potential

$$\underset{\sim}{A} = (0, A_\theta(R), A_z(R)) \ , \qquad \ldots(7.8)$$

generates the magnetic field

$$\underset{\sim}{B} = (0, B_\theta(R), B_z(R)) \ , \qquad \ldots(7.9)$$

and from equation (2.3)

$$\left.\begin{aligned} B_\theta(R) &= -\frac{\partial A_z(R)}{\partial R} \ , \\ \text{and} \quad B_z(R) &= -\frac{1}{R}\frac{\partial (RA_\theta)}{\partial R} \ . \end{aligned}\right\} \qquad \ldots(7.10)$$

The electric potential

$$\phi = \phi(R) \ , \qquad \ldots(7.11)$$

generates the electric field

$$(E(R),0,0) \quad , \qquad ...(7.12)$$

where

$$E(R) = \frac{d\phi(R)}{dR} \quad . \qquad ...(7.13)$$

The Lagrangian for an electron in these axially symmetric fields is

$$L = \frac{1}{2}\, m(\dot{R}^2+(R\dot{\theta})^2+\dot{z}^2) + e\phi(R) - e(R\dot{\theta}A_\theta(R)+\dot{z}A_z(R)), \qquad ...(7.14)$$

and from the Lagrangian equations of motion

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_j}\right\} - \frac{\partial L}{\partial q_j} = 0 \; , \qquad ...(7.15)$$

where q_j are generalized coordinates, the three constants of motion are

$$\frac{1}{2}\, m(\dot{R}^2+(R\dot{\theta})^2+\dot{z}^2) - e\phi(R) = \varepsilon \; , \qquad ...(7.16)$$

as the energy,

$$R(R\dot{\theta} - \frac{e}{m}\, A_\theta(R)) = \gamma_\theta \; , \qquad ...(7.17)$$

as the canonical angular momentum, and

$$\dot{z} - \frac{e}{m}\, A_z(R) = \gamma_o \; , \qquad ...(7.18)$$

as the z component of the canonical momentum. Using equations (7.16), (7.17) and (7.18) the time differential becomes

$$dt = \frac{dR}{\sqrt{\dot{R}^2}}$$

$$= \frac{dR}{\left(\frac{2\varepsilon}{m} + \frac{2e}{m}\,\phi(x) - \frac{1}{R^2}\left(\gamma_\theta + \frac{eR}{m}\, A_\theta(x)\right)^2 - \left(\gamma_o + \frac{e}{m}\, A_z(R)\right)^2\right)^{\frac{1}{2}}} \qquad ...(7.19)$$

From equation (7.17) and (7.19)

$$d\theta = \frac{1}{R}\, v_\theta dt$$

$$= \frac{1}{R^2} \frac{(\gamma + (e/m)RA_\theta(R))dR}{\left\{\frac{2\varepsilon}{m} + \frac{2e}{m}\phi(R) - \frac{1}{R^2}\left(\gamma_\theta + \frac{eR}{m}A_\theta(R)\right)^2 - \left(\gamma_o + \frac{e}{m}A_z(R)\right)^2\right\}^{\frac{1}{2}}}, \quad \ldots(7.20)$$

and from equations (7.18) and (7.19)

$$dz = v_z dt$$

$$= \frac{(\gamma_o + \frac{e}{m}A_z(R))dR}{\left\{\frac{2\varepsilon}{m} + \frac{2e}{m}\phi(R) - \frac{1}{R^2}\left(\gamma_\theta + \frac{eR}{m}A_\theta(R)\right)^2 - \left(\gamma_o + \frac{e}{m}A_z(R)\right)^2\right\}^{\frac{1}{2}}} \quad \ldots(7.21)$$

If however $A_\theta(R) = 0 = B_z(R)$ and $\gamma_\theta = 0$ so that the electron is confined to move in a meridian plane, θ = constant in equation (7.20), then on integration equation (7.19) becomes

$$t = \int \frac{dR}{\left\{\frac{2\varepsilon}{m} + \frac{2e}{m}\phi(R) - \left(\gamma_o + \frac{e}{m}A_z(R)\right)^2\right\}^{\frac{1}{2}}}, \quad \ldots(7.22)$$

and equation (7.21) similarly yields

$$z = \int \frac{(\gamma_o + (e/m)A_z(R))dR}{\left\{\frac{2\varepsilon}{m} + \frac{2e}{m}\phi(R) - \left(\gamma_o + (e/m)A_z(R)\right)^2\right\}^{\frac{1}{2}}}. \quad \ldots(7.23)$$

Equations (7.22) and (7.23) are the same as equations (2.39a) and (2.39b) which describe the motion of an electron in a magnetic field in the Z direction dependent on x with an electric field along the x axis also dependent on x. The axial parameter γ_o becomes the cartesian parameter γ_1. Axially symmetric fields are also prevalent in work on thermonuclear containment (Simon 1958, Bishop 1955).

7.5 Other Field Configurations

Very few fields have been considered. Lehnert (1964) has investigated charged particle motion in a hyperbolic magnetic field, $B_z = B_o(y,x,0)$ where $B_o = (2m/q)c_o$, and has shown that exact solutions exist for electron motion in the plane $y = 0$. These solutions are identical in form to the results obtained by Seymour (1959).

There are however no exact analytical solutions at present for magnetic fields which can be described most simply by a curvilinear coordinate representation. Landau and Lifschitz in their book called "Mechanics" have shown that by using the Hamilton-Jacobi theory several interesting solutions can be obtained for the motion of a particle in scalar potentials which could most easily be described by using the parabolic and elliptic curvilinear coordinate systems. For example, a solution to the motion of a charged particle in a coulomb field of two fixed point charges at a distance 2σ apart was analysed.

At present little is known about the solubility of problems involving magnetostatic vector potentials easily representable in curvilinear coordinates. If such solutions existed it might shed some light on the solubility of such important problems as the motion of the electron in the magnetic field of the earth. The toroidal or elliptic coordinate systems might lead to useful results.

APPENDIX 1

DERIVATION OF RESULT (3.38)

The hypergeometric differential equation has for solution the series expansion

$$F(a,b;c;x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r x^r}{(c)_r r!} , \qquad \ldots(A.1)$$

where the Pochhammer symbol is

$$(a)_r = a(a+1)(a+2)\ldots.(a+r-1) . \qquad \ldots(A.2)$$

In terms of gamma functions

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)} \ ; \ r \text{ is a positive integer}, \qquad \ldots(A.3)$$

and hence using the beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} , \qquad \ldots(A.4)$$

where $x > 0$, $y > 0$, the solution (A.1) may be written as

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}dt \sum_{r=0}^{\infty} \frac{(a)_r (tx)^r}{r!} ,$$

and since by expansion

$$\sum \frac{(a)_r (tx)^r}{r!} = (1 - tx)^{-a} ,$$

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a} dt , \qquad \ldots(A.5)$$

where $c > b > 0$. The trigonometrical substitution $t = \sin^2\phi$ reduces the form (A.5) to the result (3.38) of Chapter 3 as quoted by Erdélyi (1953).

APPENDIX 2

DERIVATION OF EQUATION (3.57)

The confluent hypergeometric differential equation has a series solution of the form

$$M(a,b;x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \frac{x^r}{r!} \quad . \qquad \ldots (A.6)$$

Following the procedure of Appendix 1, (A.6) may be written

$$M(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_o^1 e^{xt} t^{a-1} (1-t)^{b-a-1} dt \;, \qquad \ldots (A.7)$$

since

$$\sum_{r=0}^{\infty} \frac{(xt)^r}{r!} = e^{xt} \; .$$

Changing the variable by means of the relationship $2t = 1 - \tau$, the solution (A.7) becomes

$$M(a,b;x) = \frac{2^{1-b}\Gamma(b)e^{\frac{x}{2}}}{\Gamma(a)\Gamma(b-a)} \int_{-1}^1 e^{-\frac{x\tau}{2}} (1-\tau)^{a-1}(1+\tau)^{b-a-1} d\tau, \quad ..(A.8)$$

and in particular, when $a = \frac{1}{2}$, $b = 1$, and $\nu = -x$ one obtains the result (3.57).

APPENDIX 3

DISCUSSION OF THE CURLY F FUNCTION

The curly-F function is defined by the integral

$$\mathcal{F}(a,b;c;x;k) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^k \frac{t^{b-1}(1-t)^{c-b-1}dt}{(1-xt)^a} , \qquad \text{...(A.9)}$$

where $0 < k < 1$. On substitution of $t = \sin^2\phi$ equation (A.9) becomes

$$\mathcal{F}(a,b;c;x;\theta) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^\theta \frac{\sin^{2b-1}\phi\cos^{2c-2b-1}\phi d\phi}{(1-x\sin^2\phi)^a} , \qquad \text{...(A.10)}$$

with $\theta = \sin^{-1}\sqrt{k}$.

Since $(1 - x\sin^2\phi)^{-a} = \sum_{r=0}^{\infty} \frac{(a)_r}{r!} (x\sin^2\phi)^r$, equation (A.10) becomes

$$\mathcal{F}(a,b;c;x;\theta) = \frac{\Gamma(a)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \{\frac{(a)_r x^r}{r!} 2\int_o^\theta \sin^{2b+2r-1}\theta\cos^{2c-2b-1}\theta d\theta\} .$$

...(A.11)

Hence from the definition of the incomplete beta function

$$B_\theta(x,y) = 2\int_o^\theta \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta , \qquad \text{...(A.12)}$$

which is valid for $x > 0$, $y > 0$ equation (A.11) yields

$$\mathcal{F}(a,b;c;x;\theta) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \frac{(a)_r x^r}{r!} B_\theta(b+r,c-b) . \qquad \text{...(A.13)}$$

Using the normalized incomplete beta function

$$\nu_{r,\theta} = \frac{B_\theta(b+r,c-b)}{B(b+r,c-b)} , \qquad \text{...(A.14)}$$

equation (A.13) yields

$$\mathcal{F}(a,b;c;x;\theta) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r r!} \nu_r x^r , \qquad \text{...(A.15)}$$

as the power series representation of the curly-F function, where $b > c > 0$. If $k = 1$ in equation (A.9) then the integral becomes the Gaussian hypergeometric form discussed in Appendix 1, and $\nu_{r,\frac{\pi}{2}} = 1$ in equation (A.15).

(b) Negative Argument $x < 0$

For an integral of the form

$$\mathcal{F}(a,\tfrac{1}{2},1;x;\phi) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^\phi \frac{d\phi}{(1-x\sin^2\phi)^a}$$

$$= \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^\phi \frac{\sec^{2a}\phi d\phi}{(1+(1-x)\tan^2\phi)^a} , \qquad \text{...(A.16)}$$

where $x < 0$, substitution of $y = (1-x)^{\frac{1}{2}}\tan\phi$ in equation (A.16) gives

$$\mathcal{F}(a,\tfrac{1}{2},1;x;\phi) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^y \frac{\left(1 + \frac{y^2}{1-x}\right)^{a-1} dy}{(1-x)^{\frac{1}{2}}(1+y^2)^a} . \qquad \text{...(A.17)}$$

Finally, use of $y = \tan\theta$ in equation (A.17) yields

$$\mathcal{F}(a,\tfrac{1}{2},1;x;\theta) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_o^\theta \left(1 - \frac{x}{x-1} \sin^2\theta\right)^{a-1} d\theta . \qquad \text{...(A.16)}$$

APPENDIX 4

THE CURLY - M FUNCTION

The curly-M function is defined by the integral

$$\mathcal{M}(a,b;x;\theta) = \frac{2^{1-b}\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \ell^{\frac{x}{2}} \int_0^\theta \ell^{-\frac{x}{2}\cos\theta} (\sin\theta)^{b-1} \cot^{b-2a}\left(\frac{\theta}{2}\right) d\theta$$

$$= \frac{2^{1-b}\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \ell^{\frac{x}{2}} \int_0^\theta \ell^{-\frac{x}{2}\cos\theta} \sin^{2a-1}\left(\frac{\theta}{2}\right) \cos^{2b-2a-1}\left(\frac{\theta}{2}\right) d\theta.$$

...(A.19)

Using the expansion $\ell^z = \sum_{n=0}^{\infty} z^n/n!$ equation (A.19) becomes

$$\mathcal{M}(a,b;x;\theta) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{x^n}{n!} 2\int_0^\xi \sin^{2(n+a)-1}\xi \cos^{2b-2a-1}\xi d\xi ,$$

...(A.20)

where $\xi = \theta/2$. Again using the normalized beta function for $b > a > 0$

$$\mu_\xi(n+a,b-a) = \frac{B_\xi(n+a,b-a)}{B(n+a,b-a)} , \qquad \text{...(A.21)}$$

equation (A.20) becomes

$$\mathcal{M}(a,b;x;\theta) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \mu_\xi(n+a,b-1) \frac{x^n}{n!} . \qquad \text{...(A.22)}$$

When $\theta = \pi$, $\xi = \frac{\pi}{2}$ and equation (A.19) simplifies to the confluent hypergeometric function, and (A.22) becomes the series expansion of the confluent hypergeometric function, with $\mu_\xi = 1$, as shown in Appendix 2.

APPENDIX 5

SPECIAL INTEGRALS

The integral

$$I_1 = \int_o^\phi \frac{d\phi}{(1 - \sigma\sin^2\phi)}$$

$$= \int_o^\phi \frac{\sec^2\phi d\phi}{(1+(1-\sigma)\tan^2\phi)} \quad . \qquad \text{...(A.23)}$$

Using the substitution $y = (1-\sigma)^{\frac{1}{2}}\tan\phi$ equation (A.23) becomes

$$I = \frac{1}{(1-\sigma)^{\frac{1}{2}}} \int_o^y \frac{dy}{(1+y^2)} \quad , \qquad \text{...(A.24)}$$

and using $y = \tan\theta$ equation (A.24) yields

$$I_1 = \frac{\theta}{(1-\sigma)^{\frac{1}{2}}} \quad . \qquad \text{...(A.25)}$$

The integral

$$I_2 = \int_o^\phi \frac{d\phi}{(1-\sigma\sin^2\phi)^2}$$

$$= \int_o^\phi \frac{\sec^4\phi d\phi}{(1+(1-\sigma)\tan^2\phi)^2} \quad . \qquad \text{...(A.26)}$$

Using the substitution $y = \tan\phi$ equation (A.26) becomes

$$I_2 = \int_o^y \frac{(1+y^2)dy}{(1+(1-\sigma)y^2)^2}$$

$$= \int_o^y \frac{dy}{(1+(1-\sigma)y^2)} + \sigma \int_o^y \frac{y^2dy}{(1+(1-\sigma)y^2)^2}$$

$$= \int_o^y \frac{dy}{(1+(1-\sigma)y^2)} + \frac{\sigma}{2} \int_o^y \frac{ydy^2}{(1+(1-\sigma)y^2)^2} \quad . \qquad \text{...(A.27)}$$

Using integration by parts equation (A.27) becomes

$$I_2 = \int_o^y \frac{dy}{(1+(1-\sigma)y^2)} + \frac{\sigma}{2}\left\{\frac{-y/(1-\sigma)}{(1+(1-\sigma)y^2)}\Big|_o^y + \int_o^y \frac{dy/(1-\sigma)}{1+(1-\sigma)y^2}\right\}$$

$$= -\frac{\sigma y}{2(1-\sigma)(1+(1-\sigma)y^2)}\Big|_o^y + \frac{(2-\sigma)}{2(1-\sigma)}\int_o^y \frac{dy}{(1+(1-\sigma)y^2)} \quad . \qquad \ldots(A.28)$$

From the substitution $(1-\sigma)^{\frac{1}{2}}y = \tan\theta$ equation (A.29) yields

$$I_2 = -\frac{\sigma}{4(1-\sigma)^{\frac{3}{2}}}\sin 2\theta + \frac{(2-\sigma)}{2(1-\sigma)^{\frac{3}{2}}}\theta \quad . \qquad \ldots(A.29)$$

From equation (4.19) with $\beta = -1$

$$y(\phi) = -x_2\{2I_1 - (2-\sigma)I_2\}$$

$$= \frac{-x_2}{(1-\sigma)^{\frac{3}{2}}}\left(2(1-\sigma)\theta - (2-\sigma)\left\{\frac{(2-\sigma)}{2}\theta - \frac{\sigma\sin 2\theta}{4}\right\}\right)$$

$$= \frac{-x_2}{2(1-\sigma)^{\frac{3}{2}}}\left(-\sigma^2\theta + \frac{(2-\sigma)\sigma}{2}\sin 2\theta\right) , \qquad \ldots(A.30)$$

and from equation (4.18)

$$t(\phi) = -\sigma x_2 \quad I_2$$

$$= \frac{-\sigma x_2}{2(1-\sigma)^{\frac{3}{2}}}\left((2-\sigma)\theta - \frac{\sigma}{2}\sin 2\theta\right) \quad . \qquad \ldots(A.31)$$

(2) For an integral of the form

$$I_3 = \int_o^\phi \frac{d\phi}{(\sin^2\phi' - \sin^2\phi)}$$

$$= \int_o^\phi \frac{d\phi}{\sin^2\phi'\cos^2\phi - \cos^2\phi'\sin^2\phi}$$

$$= \int_o^{\phi} \frac{\sec^2\phi d\phi}{\sin^2\phi'(1 - \cot^2\phi'\tan^2\phi)} \quad . \qquad \ldots(A.32)$$

substituting $y = \tan\phi$ equation (A.32) yields

$$I_3 = \frac{1}{\sin^2\phi'} \int_o^{y} \frac{dy}{(1 - \cot^2\phi' y^2)}$$

$$= \frac{1}{2\sin^2\phi'} \int_o^{y} \left(+ \frac{dy}{1-\cot\phi' y} + \frac{dy}{1+\cot\phi' y}\right) . \qquad \ldots(A.33)$$

Substituting $X = 1-\cot\phi' y$ in the first integral and $X = 1+\cot\phi' y$ in the second integral equation (A.33) becomes

$$I_3 = \frac{1}{\sin^2\phi'} \quad \ell n \left\{\frac{1+\cot\phi' y}{1-\cot\phi' y}\right\} . \qquad \ldots(A.34)$$

The integral

$$I_4 = \int_o^{\phi} \frac{d\phi}{(\sin^2\phi'-\sin^2\phi)^2}$$

$$= \int_o^{\phi} \frac{\sec^4\phi d\phi}{\sin^4\phi'(1-\cot^2\phi'\tan^2\phi)^2} \quad . \qquad \ldots(A.35)$$

Using the substitution $y = \tan\phi$ equation (A.35) becomes

$$I_4 = \int_o^{y} \frac{(1+y^2)dy}{\sin \ \phi'(1-\cot^2\phi' y^2)^2}$$

$$= \frac{1}{\sin^4\phi'} \left\{\int_o^{y} \frac{dy}{(1-\cot^2\phi' y^2)} + (1+\cot^2\phi')\int_o^{y} \frac{y^2 dy}{(1-\cot^2\phi' y^2)^2}\right\}$$

$$= \frac{1}{\sin^4\phi'} \left\{\int_o^{y} \frac{dy}{(1-\cot^2\phi' y^2)} + \left(\frac{1+\cot^2\phi'}{2}\right)\int_o^{y} \frac{y dy^2}{(1-\cot^2\phi' y^2)^2}\right\} . \qquad \ldots(A.36)$$

Using integration by parts equation (A.36) becomes

$$= \frac{1}{\sin^4\phi'} \left\{\int_o^{y} \frac{dy}{(1-\cot^2\phi' y^2)} + \left(\frac{1+\cot^2\phi'}{2\cot^2\phi'}\right)\left\{\frac{y}{1-\cot^2\phi' y^2}\right|_o^y - \int_o^{y} \frac{dy}{(1-\cot^2\phi' y^2)} \right\}\right\}$$

$$= \frac{1}{\sin^4\phi'} \left\{ \left(\frac{1-\tan^2\phi'}{2}\right) \int_o^y \frac{dy}{(1-\cot^2\phi y^2)} + \left(\frac{1+\tan^2\phi'}{2}\right) \frac{y}{(1-\cot^2\phi' y^2)} \Bigg|_o^y \right\} .$$

...(A.37)

Using result (A.34) equation (A.37) becomes

$$I_4 = \frac{1}{\sin^4\phi'} \left\{ \left(\frac{1-\tan^2\phi'}{2}\right) \ell n \left\{\frac{1+\cot\phi'\tan\phi}{1-\cot\phi'\tan\phi}\right\} + \left(\frac{1+\tan^2\phi'}{2}\right) \frac{\tan\phi}{(1-\cot^2\phi'\tan^2\phi)} \right\} . \quad ...(A.38)$$

APPENDIX 6

MOTION OF THE ELECTRON IN ELECTROMAGNETIC FIELDS

6.1 Electric and Polarization Drift Velocities

When dealing with electromagnetic phenomena much use is made of the zero order electric drift velocity, the equation of motion in an electric field, $\underset{\sim}{E}$, and a constant magnetic field $\underset{\sim}{B}_o$ is

$$\frac{m d\underset{\sim}{v}}{dt} = -e\{\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}_o\} \quad , \quad (\text{e.m.u.}) \; . \qquad \ldots(A.39)$$

The velocity of the electron is taken to be a superposition of the zero order circular orbit in a constant magnetic field, $\underset{\sim}{v}_o$, and a constant electric drift velocity, $\underset{\sim}{v}_E$, such that

$$\underset{\sim}{v} = \underset{\sim}{v}_o + \underset{\sim}{v}_E \quad , \qquad \ldots(A.40)$$

and

$$\underset{\sim}{v}_E = \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} \quad . \qquad \ldots(A.41)$$

Substituting equation (A.40) into equation (A.39) yields

$$\frac{m d\underset{\sim}{v}_o}{dt} = -e\underset{\sim}{v} \; \times \underset{\sim}{B}_o - \frac{m d\underset{\sim}{v}_E}{dt} \; . \qquad \ldots(A.42)$$

For a constant electric field, $\underset{\sim}{E}_o$, $\frac{m d\underset{\sim}{v}_E}{dt} = 0$ and equation (A.42) simplifies to the zero order result of an electron in a constant magnetic field, equation (A.41) giving the zero order electric drift velocity. If however the electric field is not constant, the second component on the right hand side of equation (A.42) may be interpreted as an equivalent force

$$\underset{\sim}{F} = -\frac{m d\underset{\sim}{v}_E}{dt} \quad , \qquad \ldots(A.43)$$

which causes an equivalent electric field

$$\underset{\sim}{E}_1 = -\underset{\sim}{F}_1/e = \frac{m}{e}\frac{d\underset{\sim}{v}_E}{dt} \quad . \qquad \ldots(A.44)$$

Defining

$$v_o = \underset{\sim}{v}_1 + \underset{\sim}{v}_P \quad , \qquad \ldots(A.45)$$

where

$$\underset{\sim}{v}_P = \left(\frac{m}{e}\frac{d\underset{\sim}{v}_E}{dt}\right) \times \underset{\sim}{B}_o$$

$$= -\frac{m}{eB_o^2}\frac{d\underset{\sim}{E}}{dt} \quad , \qquad \ldots(A.46)$$

then substitution of equation (A.45) into equation (A.42) yields

$$\frac{md\underset{\sim}{v}_1}{dt} = -\frac{md\underset{\sim}{v}_P}{dt} - e\{\underset{\sim}{v}_1 \times \underset{\sim}{B}\} \quad , \qquad \ldots(A.47)$$

and for an electric field which varies slowly, $\frac{md\underset{\sim}{v}_P}{dt} \sim 0$ and equation (A.47) simplifies to the zero order result

$$\frac{md\underset{\sim}{v}_1}{dt} = -e\,\underset{\sim}{v}_1 \times \underset{\sim}{B} \quad . \qquad \ldots(A.48)$$

Equations (A.41) and (A.46) are the well known electric and polarization drift expressions, as analysed by Longmire (1963), and others.

6.2 Generalization of the Electric and Polarization Drifts

The two drift expressions (A.41) and (A.46) are the first and second order results, and higher order results may be found by repeating the above process ad infinitum. If the electric field does not vary slowly then $\frac{md\underset{\sim}{v}_P}{dt}$ is no longer negligible and as before may be interpreted as a force term

$$\underset{\sim}{F}_2 = -\frac{md\underset{\sim}{v}_P}{dt} = \frac{m^2}{eB_o^2}\frac{d^2\underset{\sim}{E}}{dt^2} \quad , \qquad \ldots(A.49)$$

which causes an equivalent electric field

$$\underset{\sim}{E}_2 = -\underset{\sim}{F}_2/e = -\frac{m^2}{e^2B_o^2}\frac{d^2\underset{\sim}{E}}{dt^2} \quad , \qquad \ldots(A.50)$$

where induction effects associated with the time variation of $\underset{\sim}{E}$ are neglected. Defining, $\underset{\sim}{v}_1 = \underset{\sim}{v}_2 + \underset{\sim}{v}_{E_2}$, ...(A.51)

where

$$\underset{\sim}{v}_{E_2} = \frac{\underset{\sim}{E}_2 \times \underset{\sim}{B}_o}{B_o^2}$$

$$= - \frac{m^2}{e^2 B_o^4} \frac{d^2 \underset{\sim}{E}}{dt^2} \times \underset{\sim}{B}_o \quad , \qquad \text{...(A.52)}$$

then equation (A.47) yields

$$\frac{md\underset{\sim}{v}_2}{dt} = - \frac{md\underset{\sim}{v}_{E_2}}{dt} - e(\underset{\sim}{v}_2 \times \underset{\sim}{B}_o) \quad . \qquad \text{...(A.53)}$$

If $\frac{md\underset{\sim}{v}_{E_2}}{dt} \sim 0$ equation (A.53) simplifies to the equation of motion of an electron in a constant magnetic field. If this is not the case then a new force term

$$\underset{\sim}{F}_3 = - \frac{md\underset{\sim}{v}_{E_2}}{dt} \quad , \qquad \text{...(A.54)}$$

is defined which causes an electric field

$$\underset{\sim}{E}_3 = \frac{m}{e} \frac{d\underset{\sim}{v}_{E_2}}{dt} \quad , \qquad \text{...(A.55)}$$

and as before a velocity

$$\underset{\sim}{v}_2 = \underset{\sim}{v}_3 + \underset{\sim}{v}_{E_3} \quad , \qquad \text{...(A.56)}$$

is used where

$$\underset{\sim}{v}_{E_3} = \frac{\underset{\sim}{E}_3 \times \underset{\sim}{B}_o}{B_o^2} = + \frac{m^3}{e^3 B_o^6} \frac{d^3 \underset{\sim}{E}}{dt^3} \quad . \qquad \text{...(A.57)}$$

The velocity of the electron becomes

$$\underset{\sim}{v} = \underset{\sim}{v}_3 + \underset{\sim}{v}_{E_3} + \underset{\sim}{v}_{E_2} + \underset{\sim}{v}_P + \underset{\sim}{v}_{E_o} \quad , \qquad \text{...(A.58)}$$

and if $\frac{md\underset{\sim}{v}_{E_3}}{dt} \sim 0$ equation (A.53) becomes

$$\frac{m d\underset{\sim}{v}_3}{dt} = -e\underset{\sim}{v}_3 \times \underset{\sim}{B}_o , \qquad \text{...(A.59)}$$

which is again the zero order equation. This process may be continued to higher orders until the desired accuracy is reached, provided Maxwell's equations are not seriously violated by the requirement B_o = constant.

If the N^{th} order drift expression is taken to be

$$\underset{\sim}{v}_N = \underset{\sim}{v}_{N-1} + \underset{\sim}{v}_{E_N} , \qquad \text{...(A.60)}$$

then the force term becomes

$$\underset{\sim}{F}_N = - \frac{m d\underset{\sim}{v}_{E_{N-1}}}{dt} , \qquad \text{...(A.61}$$

which causes an equivalent electric field

$$\underset{\sim}{E}_N = \frac{m}{e} \frac{d\underset{\sim}{v}_{E_{N-1}}}{dt} . \qquad \text{...(A.62)}$$

and an N^{th} order drift

$$\underset{\sim}{v}_{E_N} = \frac{m}{eB_o^2} \left(\frac{d\underset{\sim}{v}_{E_{N-1}}}{dt}\right) \times \underset{\sim}{B}_o . \qquad \text{...(A.63)}$$

Thus to the N^{th} order the velocity of the electron becomes

$$\underset{\sim}{v} = \underset{\sim}{v}_N + \sum_{r=0}^{N} \underset{\sim}{v}_{E_N} , \qquad \text{...(A.64)}$$

and if $\frac{m d\underset{\sim}{v}_{E_N}}{dt} \sim 0$, $\underset{\sim}{v}_N$ will be closely the radial velocity of an electron in a constant magnetic field. The N^{th} order drift expression from equation (A.64) is

$$\underset{\sim}{v}_{D_N} = \sum_{r=0}^{N} \underset{\sim}{v}_{E_N} . \qquad \text{...(A.65)}$$

If N = 2n where n is a positive integer, the drift velocity expression (A.65) becomes

$$\underset{\sim}{v}_{D_{2n}} = \sum_{n=0}^{n} \frac{(-1)^n}{B_o^2 \omega_c^{2n}} \left\{ \frac{d^{2n}\underset{\sim}{E}}{dt^{2n}} \times \underset{\sim}{B}_o - \frac{m}{e} \frac{d^{2n+1}\underset{\sim}{E}}{dt^{2n+1}} \right\} , \qquad \text{...(A.66)}$$

where $\omega_c = \frac{eB_o}{m}$ is the zero order gyrofrequency of the electron.

If the electric field has a characteristic frequency $\nu > 0$ such that $\underset{\sim}{E} \simeq \underset{\sim}{E}_o \, \ell^{i\nu t}$ then as $n \to \infty$ equation (A.66) yields

$$\underset{\sim}{v}_{D2n} \simeq \sum_{r=0}^{n} \left(\frac{\nu}{\omega_c}\right)^{2n} \left\{\frac{\underset{\sim}{E}_o \times \underset{\sim}{B}_o}{B_o^2} - \frac{im\nu \underset{\sim}{E}_o}{eB_o^2}\right\} \quad , \qquad \ldots(A.67)$$

which is convergent if $\nu < \omega_c$. Thus as $n \to \infty$

$$\underset{\sim}{v} = \underset{\sim}{v}_\infty + \underset{\sim}{v}_{D\infty} \quad , \qquad \ldots(A.68)$$

where from equation (A.66)

$$\underset{\sim}{v}_{D\infty} = \sum_{r=0}^{\infty} \frac{(-1)^n}{B_o^2 \omega_c^{2n}} \left\{\frac{d^{2n}\underset{\sim}{E}}{dt^{2n}} \times \underset{\sim}{B}_o - \frac{m}{e} \frac{d^{2n+1}\underset{\sim}{E}}{dt^{2n+1}}\right\} \quad . \qquad \ldots(A.69)$$

6.3 Different Functional Forms of Electric Field.

(a) Exponentially Varying Field $\underset{\sim}{E} = \underset{\sim}{E}_o \ell^{\alpha t}$.

Equation (A.68) becomes

$$\underset{\sim}{v}_{D\infty} = \left\{\sum_{r=0}^{\infty} (-1)^n \left(\frac{\alpha}{\omega_c}\right)^{2n}\right\} \cdot \left[\frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} - \frac{m\alpha}{eB_o^2} \underset{\sim}{E}\right] \quad , \qquad \ldots(A.70)$$

and using the expansion $\frac{1}{1 + X^2} = \sum_{n=0}^{\infty} (-1)^n X^{2n}$, equation (A.70) yields

$$\underset{\sim}{v}_{D\infty} = \frac{1}{\left(1 + \left(\frac{\alpha}{\omega_c}\right)^{2n}\right)} \left\{ \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} - \frac{m}{eB_o^2} \underset{\sim}{E} \right\} \quad , \qquad \ldots(A.71)$$

which is valid for $|\alpha| < \omega_c$.

(b) Oscillating Electric Field $\underset{\sim}{E} = \underset{\sim}{E}_o \ell^{i\nu t}$.

Equation (A.68) gives

$$\underset{\sim}{v}_{D\infty} = \left\{\sum_{r=0}^{\infty} \left(\frac{\nu}{\omega_c}\right)^{2n}\right\} \cdot \left\{ \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} - \frac{im\nu}{eB_o^2} \underset{\sim}{E} \right\} , \qquad \ldots(A.72)$$

and from the expansion $\frac{1}{1-X^2} = \sum_{n=0}^{\infty} X^{2n}$, equation (A.72) may be written

$$\underset{\sim}{v}_{D\infty} = \frac{1}{1 - \left(\frac{\nu}{\omega_c}\right)^2} \left\{ \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} - \frac{im\nu}{eB_o^2} \underset{\sim}{E} \right\} , \qquad \ldots (A.73)$$

where $\nu < \omega_c$. This result is in agreement with the expression found by Schmidt (1966; pp44).

(c) Power Law Dependence.

If the electric field is of the form

$$\underset{\sim}{E} = E\underset{\sim}{e} = (E_o + c_1 t + c_2 t^2 + c_3 t^3)\underset{\sim}{e} \qquad \ldots (A.74)$$

where E_o, c_1, c_2 and c_3 are constants and $\underset{\sim}{e}$ is the unit vector in the direction of the electric field, then the drift expression becomes

$$\underset{\sim}{v}_D = \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} - \frac{m\underset{\sim}{e}}{eB_o^2}(c_1 + 2c_2 t + 3c_3 t^2) - \frac{2c_2 + 6c_3 t}{\omega_c^2 B_o^2}\underset{\sim}{e} \times \underset{\sim}{B}_o + \frac{m6c_3}{eB_o^2\omega_c^2}\underset{\sim}{e} \qquad \ldots (A.75)$$

This result may be rearranged to give

$$\underset{\sim}{v}_D = \left(\frac{E}{B_o^2} - \frac{2c_2 + 6c_3 t}{\omega_c^2}\right)\underset{\sim}{e} \times \underset{\sim}{B}_o - \frac{m\underset{\sim}{e}}{eB_o^2}\left(c_1 - \frac{6c_3}{\omega_c^2} + 2c_2 t + 3c_3 t^2\right) . \qquad \ldots (A.76)$$

Similar results can be obtained for higher-order polynomials if required. If the electric field becomes adiabatically affected then equation (A.66) degenerates to the electric and polarization drifts.

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