Peter Hochs and A. J. Roberts

Normal forms and invariant manifolds for nonlinear, non-autonomous PDEs, viewed as ODEs in infinite dimensions

Journal of Differential Equations, 2019; 267(12):7263-7312

Final publication at: http://dx.doi.org/10.1016/j.jde.2019.07.021

NORMAL FORMS AND INVARIANT MANIFOLDS FOR NONLINEAR, NON-AUTONOMOUS PDES, VIEWED AS ODES IN INFINITE DIMENSIONS

PETER HOCHS AND A.J. ROBERTS

ABSTRACT. We prove that a general class of nonlinear, non-autonomous ODEs in Fréchet spaces are close to ODEs in a specific normal form, where closeness means that solutions of the normal form ODE satisfy the original ODE up to a residual that vanishes up to any desired order. In this normal form, the centre, stable and unstable coordinates of the ODE are clearly separated, which allows us to define invariant manifolds of such equations in a robust way. The main motivation is the case where the Fréchet space in question is a suitable function space, and the maps involved in an ODE in this space are defined in terms of derivatives of the functions, so that the infinite-dimensional ODE is a finite-dimensional PDE. We show that our methods apply to a relevant class of nonlinear, non-autonomous PDEs in this way.

CONTENTS

1. Introduction 2
1.1. Main result 2
1.2. Compact polynomials and compactly differentiable maps 4
1.3. Motivation and outlook 4
1.4. Outline of this paper 5
1.5. Notation and conventions 6
2. Preliminaries and results 7
2.1. Nested sequences of Banach spaces 7
2.2. Setup and goal 8
2.3. Dynamics in a normal form 9
2.4. Invariant manifolds 11
2.5. Main result: an approximate normal form 12
2.6. A general class of PDEs in bounded domains 14
3. Derivatives and polynomials 16
3.1. First order derivatives 16
3.2. Higher order derivatives 16
3.3. Bounded polynomial maps 17
3.4. Standard monomials 18
4. Compact derivatives and polynomials 19
4.1. Compact multilinear maps 19
4.2. Compact polynomial maps 22

Date: July 17, 2019.
Key words and phrases. Nonlinear, non-autonomous PDE; invariant manifold; normal form; differentiable and polynomial maps between Fréchet spaces.

4.3. Compactly differentiable maps 23
4.4. A class of compactly differentiable maps 25
5. Polynomial and differentiable maps on graded Fréchet spaces 26
5.1. Properties of nested sequences of Banach spaces 26
5.2. Comparable sequences of Banach spaces 27
5.3. Sequences of Banach spaces comparable to sequences of Hilbert spaces 28
5.4. Example: Sobolev spaces and C^k-spaces 30
6. A coordinate transform 31
6.1. A residual 31
6.2. Construction of the coordinate transform 32
6.3. Proof of Proposition 6.4 33
7. Centre, stable and unstable coordinates 35
7.1. Centre, stable and unstable components of $\hat{F}^q$ 35
7.2. Update terms for ξ_p and F_p 36
7.3. Proof of Theorem 2.18 38
8. Dynamics of the normal form equation 38
9. Example: a non-autonomous version of Burgers' equation 41
9.1. Centre manifold via Theorem 2.18 41
9.2. Invariant manifolds via direct computations 43
Acknowledgement 45
References 45

1. Introduction

1.1. **Main result.** This paper is about a new approach to the study of invariant manifolds of nonlinear, non-autonomous PDEs. The main result in this paper, Theorem 1.1, applies more generally, to ODEs in possibly infinite-dimensional Fréchet spaces. If those spaces are suitable function spaces, and the maps involved in an ODE in such a space are differential operators, then we obtain a result for PDEs in finite-dimensional spaces as a corollary.

The idea behind our main result is that for a general class of ODEs in Fréchet spaces, we can construct a dynamical system in a useful normal form, as well as a time-dependent coordinate transformation, such that

- the invariant manifolds of the normal form system are equal to the invariant manifolds of its linearisation, and are hence clearly and robustly defined;
- a solution of the normal form system, after an application of the coordinate transform, satisfies the original ODE up to a residual term that vanishes to any desired order.

Furthermore, we allow flexibility in the definition of centre manifolds and subspaces, in the sense that the eigenvalues of the linearised equation with real parts up to a specified size may be considered central, rather than just the purely imaginary eigenvalues. Also, we allow solutions that are only defined on time intervals that are bounded above and/or below, rather than for all time.

The Fréchet spaces we work with are *graded Fréchet spaces*. Such a space is an intersection of a sequence of Banach spaces, each with a bounded inclusion map into the next. These spaces are both general enough to apply to various nonlinear PDEs, while being close enough to Banach spaces to allow us to define a meaningful notion of a residual vanishing up to a given order. Furthermore, in the proof of our main result, we iterate various maps between graded Fréchet spaces that occur. While those maps could be defined between fixed pairs of Banach spaces, in that setting we would not able to apply them iteratively.

The standard notions of differentiable maps, polynomial maps, and maps of a given order between Banach spaces, generalise directly to graded Fréchet spaces.

A more or less intuitive formulation of our main result on the existence of the type of normal form we aim for is the following Theorem 1.1. Theorem 2.18 is a precise formulation, and Corollary 2.23 is a special case that applies to nonlinear, non-autonomous PDEs.

Theorem 1.1 (Normal form theorem, intuitive formulation)**.** *Let* I *be an interval in* t*, and* V *a graded Fréchet space. Consider a nonlinear, non-autonomous* ODE

$$\dot{x}(t) = Ax(t) + f(t, x(t)) \tag{1.1}$$

for $x\colon I \to V$*, where* A *is a continuous linear operator on* V *(independent of* t*) whose eigenvectors are orthogonal with respect to a suitable inner product, and* $f\colon I \times V \to V$ *is infinitely differentiable, of order* 2 *in* $v \in V$*, and the derivative of* f *with respect to* $v \in V$ *is of order* 1*.*

Then for each $p \geq 2$*, there are both an* ODE

$$\dot{X}(t) = AX(t) + F_p(t, X(t)) \tag{1.2a}$$

for $X\colon I \to V$*, and a time-dependent, infinitely differentiable coordinate transformation*

$$\xi_p\colon I \times V \to V \tag{1.2b}$$

such that

- *if* $X(t)$ *satisfies (1.2a), then the map* $x\colon I \to V$ *given by* $x(t) = \xi_p(t, X(t))$ *satisfies*
$$\dot{x}(t) = Ax(t) + f(t, x(t)) + R_p(t, x(t)),$$
for an infinitely differentiable residual $R_p\colon I \times V \to V$ *of order* p *in* $v \in V$*;*
- *the nonlinearity* $F_p\colon I \times V \to V$ *is infinitely differentiable and of order* 2 *in* $v \in V$*;*
- *the component of a solution to (1.2a) in the stable subspace for* A *decays exponentially quickly to zero as* t *increases in* I*. Its component in the unstable subspace for* A *decays exponentially quickly to zero as* t *decreases in* I*. If the solution starts out in either the centre-stable or the centre-unstable subspace for* A*, then its component in the central subspace for* A *is bounded by a constant for all* $t \in I$*, or at worst by a specified, small exponential growth rate.*

The last point in Theorem 1.1 means that the centre, stable and unstable manifolds (in this case, linear subspaces) of (1.2a) are exactly the centre, stable and unstable spaces of A, respectively. (And similarly for the centre-stable and centre-unstable subspaces.) The centre, stable, unstable, centre-stable and centre-unstable subspaces for the dynamics in x described by (1.2a) and $x(t) = \xi_p(t, X(t))$ (which becomes an ODE in x if $\xi_p(t, -)$ is invertible for all t) are then obtained from these spaces via an application of the coordinate transform ξ_p. In this way, we show that any (non-autonomous) system of the form (1.1) is arbitrarily close to a system with robustly defined invariant manifolds.

1.2. **Compact polynomials and compactly differentiable maps.** Key ingredients of the proof of Theorem 1.1 are *compact polynomial maps* between Banach spaces and graded Fréchet spaces; and *compactly differentiable maps* between such spaces.

The coordinate transformation ξ_p and the nonlinearity F_p in Theorem 1.1 are polynomial maps, which we construct by adding (infinitely many) monomial terms with the right properties together. At the level of Banach spaces, a polynomial map can be naturally defined as a finite sum of restrictions to the diagonal of bounded multilinear maps. For example, Taylor polynomials of differentiable maps between normed vector spaces are polynomials of this type.

But not all such polynomial maps (for example, the identity map) can be approximated by sums of monomials in the natural norm extending the operator norm. This leads us to define *compact* polynomial maps, which can be approximated in this way in settings relevant to us. The notion of a compact polynomial map that we use seems natural, but we have not been able to find it elsewhere in the literature. Different notions of compact polynomial maps were developed and studied by Gonzalo, Jaramillo and Pełczyńsky [15, 24].

Our construction of the required coordinate transforms involves Taylor polynomials of differentiable maps between Banach spaces, and between graded Fréchet spaces. This construction is possible if those polynomials are compact in the sense just mentioned. That is the case for *compactly* differentiable maps, which we define for this purpose. We will see in Section 4.4 that, in applications to PDEs, the relevant differentiable maps are indeed compactly differentiable. This follows from various Sobolev embedding theorems.

1.3. **Motivation and outlook.** Various invariant manifolds are central to many areas of dynamical systems, including using centre manifolds to construct and justify reduced low-dimensional models of high-dimensional dynamics [28, e.g.]. Many dynamical systems involve PDEs in infinite dimensional state spaces of functions, and some applications require infinite dimensional centre manifolds [7, 27, e.g.]. In general we also want to cater for non-autonomous systems, with an aim to subsequently generalise to stochastic/rough dynamics [14, e.g.]. Further, encompassing unstable dynamics with both centre and stable is necessary for application to Saint Venant-like, cylindrical, problems [18, 19, 20, e.g.], and to deriving boundary conditions for approximate PDEs [26, e.g.]. Consequently, here we address the general challenge of constructing and justifying various infinite dimensional invariant manifolds for non-autonomous dynamical systems which have stable, unstable and centre modes. A crucial novel feature of the approach is that we further develop a backward theory recently initiated for finite dimensional systems [29]: analogous backward theory has been very useful in other domains [16, e.g.].

Applications of the extant forward theory in such a general setting is often confounded by impractical preconditions. Non-autonomous invariant manifold theories typically require bounded operators, and Lipschitz and/or uniformly bounded nonlinearities, [3, 4, 5, 10, 17, e.g.]. The extant boundedness requirement [17, Hypothesis 2.1(i) and 3.8(i), e.g.] arises from the general necessity of both forward and backward time convolutions with the semigroup (e.g., e^{At} for systems that linearise to $\dot{x} = Ax$), convolutions that must be continuous in extant forward theory, but cannot be continuous with unbounded operators. Despite many interesting specific scenarios having rigorous invariant manifolds established via strongly continuous semigroup operators and by mollifying nonlinearity [9, 31, e.g.], extant *non-autonomous* forward theory fails to rigorously apply in many practical cases.

The definition of the invariant manifolds made possible by Theorem 1.1 is a crucial reformation of the backward theory proposed. Classic definitions of un/stable and centre manifolds require the existence of limits as time goes to $\pm\infty$ [2, 6, 17, 25, e.g.]. This consequently requires solutions of the dynamical system to be well-behaved for all time, which requires constraints that in applications are often not found, or are hard to establish. For example, in stochastic systems very rare events will eventually happen over the infinite time requiring global Lipschitz and boundedness that are oppressive in applications. By modifying definitions we establish results for finite times, which are useful in many applications, and for a wider range of non-autonomous systems.

Figure 1 is a schematic depiction of what is achieved in the current paper, and includes some useful future developments. At a more technical level, we intend to generalise Theorem 1.1 to a larger class of ODEs and PDEs relevant to applications. For example, we plan to incorporate more general boundary conditions, and weaken the assumption that the linear operator A has orthogonal eigenvectors.

1.4. **Outline of this paper.** The main results of this paper, on normal forms and invariant manifolds of nonlinear, non-autonomous ODEs in Fréchet spaces, and of nonlinear, non-autonomous PDEs in finite-dimensional spaces, are stated in Section 2. We illustrate our results by applying them to an example PDE in Section 9.

In the rest of the paper, we prove our main results. We start by reviewing standard material on differentiable maps and polynomials on normed spaces in Section 3. In Sections 4 and 5, we develop technical tools we need for our proofs. Then in Sections 6 and 7, we use these tools to prove the main Theorems 2.18 and 2.22. We prove some properties of the normal form equation, which allow us to identify its invariant manifolds, in Section 8.

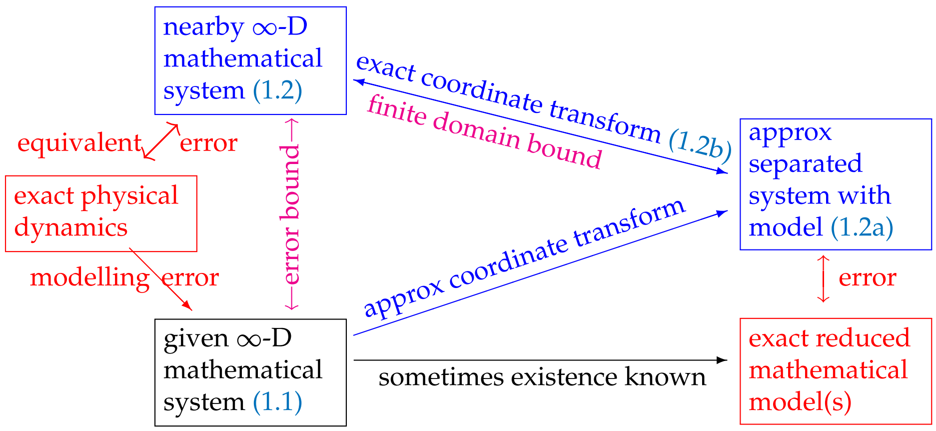

FIGURE 1. schematic diagram: blue, new theory and practice established by this article; magenta, for future research; black, mostly established extant theory and practice; red, practically unattainable (in general).

1.5. **Notation and conventions.** We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of nonnegative integers. We write $\mathbb{N}_0^\infty$ for the set of sequences $q = (q_j)_{j=1}^\infty$ in $\mathbb{N}_0$ with finitely many nonzero entries q_j, interpreted as multi-indices. For $q \in \mathbb{N}_0^\infty$, or in $\mathbb{N}_0^n$, we denote the (finite) sum of its elements by $|q|$.

We denote spaces of bounded linear operators by the letter $\mathcal{B}$, and spaces of compact linear operators by the letter $\mathcal{K}$.

When we mention a normed vector space V, the implicitly given norm is denoted by $\|\cdot\|_V$. Similarly, if V is an inner product space, then the inner product is denoted by $(-,-)_V$. Inner products on complex vector spaces are assumed to be linear in their second entries, and antilinear in their first entries. For maps $f, g \colon V \to \mathbb{R}$, when we write $f(v) = O(g(v))$, we implicitly mean that $f(v) = O(g(v))$ as $v \to 0$ in V.

If V is a normed vector space, and I is an open interval, and $f \colon I \to V$ and $f_j \colon I \to V$, for $j \in \mathbb{N}$, are maps, then we say that f_j converges to f if $f_j(t)$ converges to $f(t)$ in V uniformly in t in compact subsets of I. If f and f_j are smooth, then we say that f_j converges to f differentiably in t if $f_j^{(n)}$ converges to $f^{(n)}$ for every $n \in \mathbb{N}_0$, in this sense.

For maps $f, g \colon I \times V \to V$ and $h \colon V \to V$, the maps $f \circ g \colon I \times V \to V$ and $f \circ h \colon I \times V \to V$ are defined by

$$(f \circ g)(t, v) := f(t, g(t, v)), \quad (f \circ h)(t, v) := f(t, h(v)),$$

for $t \in I$ and $v \in V$.

If Ω is an open subset of $\mathbb{R}^d$ and $m \in \mathbb{N}$, then the Sobolev space of functions from Ω to $\mathbb{R}^m$ with weak derivatives up to order k in L^p is denoted by $W^{k,p}(\Omega; \mathbb{R}^m)$. The norm on this space is

$$\|u\|_{W^{k,p}} := \sum_{\alpha \in \mathbb{N}_0^n; |\alpha| \le k} \left\| \frac{\partial^\alpha u}{\partial x^\alpha} \right\|_{L^p}.$$

If $m = 1$, we write $W^{k,p}(\Omega) = W^{k,p}(\Omega; \mathbb{R})$.

2. PRELIMINARIES AND RESULTS

Our main result, Theorem 2.18, asserts that a broad class of nonlinear PDEs and ODEs in infinite-dimensional vector spaces may be effectively approximated by normal form systems via well-chosen, time-dependent, coordinate transformations. In this normal form, the centre, stable and unstable components of the PDE and ODE are clearly separated, which allows us to define centre manifolds for this class of equations in a robust way (Definition 2.11).

We first state our result on normal forms, and the definition of centre manifolds, for ODEs in a class of abstract vector spaces (Section 2.5). Our main reason for developing this theory is to apply it to the study of PDEs, for which the vector spaces used are spaces of functions, and the relevant maps between them are defined in terms of derivatives of functions. We discuss a relevant class of examples of such function spaces and maps in Section 2.6.

2.1. Nested sequences of Banach spaces.

The normal form we obtain in Theorem 2.18 is approximate in the sense that functions satisfying an equation transformed into that form satisfy the original equation up to a residual term. An important point in Theorem 2.18 is that this residual vanishes up to a specified order. To make precise what this vanishing up to a certain order means, we introduce in this subsection the type of topological vector spaces we consider. More details about these spaces and their properties are given in Section 5. A concrete class of examples of these spaces relevant to the study of PDEs is given in Section 2.6.

Definition 2.1. By a *nested sequence of Banach spaces*, we mean a sequence $\{V_k\}_{k=1}^{\infty}$ of Banach spaces such that

- for every k, $V_{k+1} \subset V_k$, where the inclusion map is bounded, and
- the intersection $V_\infty := \bigcap_{l=1}^{\infty} V_l$ is dense in V_k for every $k \in \mathbb{N}$.

We then consider V_∞ as a Fréchet space[1] with the seminorms (now actual norms) that are the restrictions of the norms on the spaces V_k. Such a space is a *graded Fréchet space*.

A *compactly* nested sequence of Banach spaces is such a sequence such that for every $l \in \mathbb{N}$, there is a $k \geq l$ such that the inclusion $V_k \subset V_l$ is compact.

Let $\{V_k\}_{k=1}^{\infty}$ be a nested sequence of Banach spaces.

Definition 2.2. The space $\mathcal{B}(V_\infty)$ of *bounded operators on* V_∞ consists of the linear maps $A\colon V_\infty \to V_\infty$ such that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that the linear map A extends continuously to a map in $\mathcal{B}(V_k, V_l)$.

[1] Much of what we write about Fréchet spaces of this form holds for more general projective limits of Banach spaces connected by bounded operators. But we do not need that degree of generality.

Remark 2.3. In Definition 2.2, if $k \leq l$, then the composition

$$V_l \hookrightarrow V_k \xrightarrow{A} V_l$$

is a bounded operator on V_l. So we may always take $k \geq l$ in this context, but this does not need to be assumed a priori. Similar remarks apply in analogous situations, such as Definitions 2.4 and 2.5 below.

Definition 2.4. A map $f\colon V_\infty \to V_\infty$ is *of order* n, written as $f = \mathcal{O}(n)$, if for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $\|f(v)\|_{V_l} = O(\|v\|^n_{V_k})$ as $v \to 0$ in V_k.

If I is an open interval, a map $f\colon I \times V_\infty \to V_\infty$ is *of order* n, written as $f = \mathcal{O}(n)$, if for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $\|f(t,v)\|_{V_l} = O(\|v\|^n_{V_k})$ as $v \to 0$ in V_k, uniformly in t in compact subsets of I.

Definition 2.5. An n times *differentiable map from* V_∞ *to itself* is a map $f\colon V_\infty \to V_\infty$ such that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that f extends to an n times differentiable map from V_k to V_l. If a map is n times differentiable for every $n \in \mathbb{N}$, then it is *infinitely differentiable*.

Basic material on differentiable maps between normed vector spaces is reviewed in Section 3.

Definition 2.6. Two nested sequences $\{V_k\}_{k=1}^\infty$ and $\{W_k\}_{k=1}^\infty$ of Banach spaces are *comparable* if for every $k \in \mathbb{N}$, there are $l_1, l_2, l_3, l_4 \in \mathbb{N}$ such that we have bounded inclusions $V_{l_1} \subset W_k \subset V_{l_2}$ and $W_{l_3} \subset V_k \subset W_{l_4}$.

In the setting of this definition, $V_\infty = W_\infty$.

2.2. **Setup and goal.** Let $\{V_k\}_{k=1}^\infty$ be a compactly nested sequence of Banach spaces, such that V_1 is a Hilbert space. Let $A \in \mathcal{B}(V_\infty)$. Let $I \subset \mathbb{R}$ be an open interval in t containing $t = 0$, and let $f\colon I \times V_\infty \to V_\infty$ be infinitely differentiable with respect to V_∞ and I. Suppose that $f = \mathcal{O}(2)$, and that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $f\colon V_k \to V_l$ is differentiable, and

$$\|f'_{V_\infty}(t,v)\|_{\mathcal{B}(V_k,V_l)} = O(\|v\|_{V_k}), \tag{2.1}$$

uniformly in t in compact subsets of I.

Suppose that $\{e_j\}_{j=1}^\infty \subset V_\infty$ is a set of eigenvectors of A which is a Hilbert basis of V_1 (an orthonormal set that spans a dense subspace of V_1). We assume that the sequence $\{V_k\}_{k=1}^\infty$ is comparable to a nested sequence of separable Hilbert spaces in which the vectors e_j are orthogonal. However, we will see in Remark 6.1 that we may equivalently make the seemingly stronger but more concrete assumption that the spaces V_k themselves are separable Hilbert spaces. The assumption that $\{V_k\}_{k=1}^\infty$ is comparable to a nested sequence of separable Hilbert spaces is easier to check in practice than the condition that every space V_k can be chosen to be a separable Hilbert space itself. For example, Section 2.6 discusses a class of relevant cases where the spaces V_k are not Hilbert spaces for $k \geq 2$. In this sense, the notion of comparable sequences of Banach spaces is a tool that makes it easier to check the conditions of Theorem 2.18.

We study smooth maps $x\colon I \to V_\infty$ satisfying the non-autonomous dynamical system differential equation

$$\dot{x}(t) = Ax(t) + f(t, x(t)) \quad \text{for all } t \in I\,. \tag{2.2}$$

Since the nonlinearity f satisfies (2.1), $x = 0$ is an equilibrium of the system (2.2). We provide a novel backward approach to establish invariant manifolds in a finite domain about the equilibrium $x = 0$. For these invariant manifolds to be useful in applications, the time interval I will be long enough for transient dynamics to decay to insignificance in the context of the application. The proofs of our main results simplify considerably if the time interval I is short, or bounded. But we emphasise that we only aim this theory to support the many applications where the time interval I is long enough, or unbounded, so that the theorems are non-trivially useful in the application.

Remark 2.7. Orthogonality of the eigenvectors e_j of A is used in some places in this paper; see for example Lemma 4.5 and the proofs of Lemmas 6.7 and 7.7. We believe that the results in this paper hold under a weaker condition, however, and intend to address this in future work.

2.3. **Dynamics in a normal form.** We define invariant manifolds, or sets, for dynamical systems in a particular normal form, and show that this definition captures the essence of such manifolds. In Section 2.5, we show that a very general class of ODEs of the form (2.2) can be brought into this normal form, modulo residuals that vanish to a desired order.

2.3.1. *Spectral gap in an exponential trichotomy.* Let $\alpha, \beta, \gamma, \tilde{\mu}$ be such that $0 \le \alpha < \tilde{\mu} < \min(\beta, \gamma)$, and no eigenvalues of A have real parts in the intervals $(-\beta, -\alpha)$ and (α, γ) (depending upon the circumstances, β or γ could be ∞, and/or α may be zero). For every $j \in \mathbb{N}$, let α_j be the eigenvalue of A corresponding to e_j. With respect to the parameters α, β and γ, we define the sets of indices of *central*, *stable* and *unstable* eigenvalues and eigenvectors, respectively, as

$$\begin{aligned} J_c &:= \{j \in \mathbb{N} : |\Re(\alpha_j)| \le \alpha\}; \\ J_s &:= \{j \in \mathbb{N} : \Re(\alpha_j) \le -\beta\}; \\ J_u &:= \{j \in \mathbb{N} : \Re(\alpha_j) \ge \gamma\}. \end{aligned}$$

For $a = c, s, u$, let V_a be the closure in V_1 of the span of the eigenvectors e_j, for $j \in J_a$. For any map g into V_1 and $a \in \{c, s, u\}$, we write g_a for its components in V_a. The sets V_c, V_s, V_u are respectively called the *centre/stable/unstable subspaces*. Further, we define the *centre-stable subspace* $V_{cs} := V_c \oplus V_s$, and the *centre-unstable subspace* $V_{cu} := V_c \oplus V_u$.

For $v \in V_\infty$ and a multi-index $q \in \mathbb{N}_0^\infty$, we set

$$v^q := \prod_{j=1}^{\infty} (e_j, v)_{V_1}^{q_j}.$$

This number is a monomial expression of degree $|q|$ in the coefficients $\{(e_j, v)_{V_1}\}_{j=1}^\infty$ of the vector v with respect to the basis $\{e_j\}$. For $q \in \mathbb{N}_0^\infty$, write $q = q^c + q^s + q^u$, for $q^c, q^s, q^u \in \mathbb{N}_0^\infty$ such that $q_j^c = 0$ if $j \notin J_c$, $q_j^s = 0$ if $j \notin J_s$ and $q_j^u = 0$ if $j \notin J_u$.

2.3.2. *Normal form dynamics.* The nonlinear term in the normal form of the equation (2.2) that we will construct *separates invariant subspaces* in the following sense. This is the key property of that normal form, and implies that the invariant subsets of the normal form equation equal those of the linearised equation.

Definition 2.8. Let $F\colon I \times V_\infty \to V_\infty$ be a smooth map. Write $F = F_c + F_s + F_u$, where $F_a \in V_a$ for $a \in \{c, s, u\}$. We say that F *separates invariant subspaces* if the components F_c, F_s and F_u of F are of the forms

$$F_c(t, v) = \sum_{\substack{q \in \mathbb{N}_0^\infty: \\ |q| \leq p \text{ and} \\ q^s = q^u = 0 \text{ or} \\ q^s \neq 0 \neq q^u}} v^q F^q(t), \tag{2.3a}$$

$$F_s(t, v) = \sum_{\substack{q \in \mathbb{N}_0^\infty: \\ |q| \leq p \text{ and} \\ q^s \neq 0}} v^q F^q(t), \tag{2.3b}$$

$$F_u(t, v) = \sum_{\substack{q \in \mathbb{N}_0^\infty: \\ |q| \leq p \text{ and} \\ q^u \neq 0}} v^q F^q(t), \tag{2.3c}$$

for all $t \in I$ and $v \in V_\infty$, for smooth maps $F^q\colon I \to V_\infty$, where the series converge in $\mathrm{Pol}(V_\infty)$, differentiably in t.

Consider a polynomial map $F\colon I \times V_\infty \to V_\infty$ that separates invariant subspaces, and the ODE

$$\dot{X}(t) = AX(t) + F(t, X(t)), \tag{2.4}$$

in smooth maps $X\colon I \to V_\infty$. Because F separates invariant subspaces, this ODE has very explicit invariant manifolds, by Lemma 2.9 and Proposition 2.10 below.

Lemma 2.9. *Suppose that* $X\colon I \to V_\infty$ *satisfies* (2.4)*, where* F *separates invariant subspaces. Let* $a \in \{c, s, u\}$*. If there exists a* $t \in I$ *such that* $X(t) \in V_a$*, then* $X(t) \in V_a$ *for all* $t \in I$.

Proposition 2.10. *There is a neighbourhood* $D_{\tilde{\mu}}$ *of* $I \times \{0\}$ *in* $I \times V_\infty$*, with the following property. Let* $X\colon I \to V_\infty$ *be a solution of* (2.4)*, for some open interval* I *containing* 0*, and where* F *sepearates invariant subspaces. Write* $X = X_c + X_s + X_u$*, with* $X_a \in V_a$ *for* $a = c, s, u$*.*

- *If* $(t, X(t)) \in D_{\tilde{\mu}}$ *for all* $t \in I$ *with* $t \geq 0$*, then for all such* t*,* $\|X_s(t)\|_{V_1} \leq \|X_s(0)\|_{V_1} e^{-(\beta - \tilde{\mu})t}$.
- *If* $(t, X(t)) \in D_{\tilde{\mu}}$ *for all* $t \in I$ *with* $t \leq 0$*, then for all such* t*,* $\|X_u(t)\|_{V_1} \leq \|X_u(0)\|_{V_1} e^{(\gamma - \tilde{\mu})t}$.
- *Suppose that* $X_s(0) = 0$ *or* $X_u(0) = 0$*. If* $(t, X(t)) \in D_{\tilde{\mu}}$ *for all* $t \in I$*, then for all* $t \in I$*,* $\|X_c(t)\|_{V_1} \leq \|X_c(0)\|_{V_1} e^{(\alpha + \tilde{\mu})|t|}$.

Since $\beta - \tilde{\mu}$ and $\gamma - \tilde{\mu}$ are positive, this proposition in particular states that stable solutions decrease to zero exponentially quickly as t increases in I, while unstable solutions decrease to zero exponentially quickly as

t decreases in I. The numbers α and $\tilde{\mu}$ represent bounds on what one takes to be relatively small real parts of eigenvalues of A (classically, these numbers are zero), so that the third point in Proposition 2.10 intuitively states that central solutions, at worst, only grow relatively slowly as $|t|$ increases.

Lemma 2.9 and Proposition 2.10 are proved in Section 8. The specific form of the set $D_{\tilde{\mu}}$ is also specified there, see (8.7).

2.4. **Invariant manifolds.** Lemma 2.9 and Proposition 2.10 show that, for every $a = c, s, u$, the set

$$D_{\tilde{\mu}} \cap (I \times V_a)$$

is a centre, stable or unstable submanifold of $I \times V_\infty$ for (2.4), respectively. Furthermore, for $a = cs$ and $a = cu$, we obtain centre-stable and centre-unstable manifolds, respectively. (Here we use the cases of the third point in Proposition 2.10 where $X_u(0) = 0$ and $X_s(0) = 0$, respectively.) This motivates Definition 2.11 of invariant subspaces of dynamical systems of a certain form. To state it precisely, we incorporate existence of solutions of (2.4).

For $v \in V_\infty$, we write a_v for the infimum of the set of all $a > 0$ such that there is a solution $X : (-a, 0] \to V_\infty$ of (2.4), with $X(0) = v$. Similarly, b_v is the supremum of the set of all $b > 0$ such that there is a solution $X : [0, b) \to V_\infty$ of (2.4), with $X(0) = v$. If such a and b exist, we set $I_v := (-a_v, b_v)$. (In particular, $I_v = \mathbb{R}$ if such a solution exists for all $a, b > 0$.) If such an a exists but no b, we set $I_v := (-a_v, 0)$, and if such a b exists but no a, we set $I_v := (0, b_v)$. If there are no such $a, b > 0$, we set $I_v := \emptyset$.

Definition 2.11. Let $\xi \colon I \times V_\infty \to V_\infty$ be a smooth map, and let $F \colon I \times V_\infty \to V_\infty$ be a polynomial map that separates invariant subspaces. Consider the dynamical system for smooth maps $x \colon I \to V_\infty$ determined by

$$x(t) = \xi(t, X(t)), \tag{2.5}$$

for $t \in I$, for a smooth map $X \colon I \to V_\infty$ satisfying (2.4). Let $D_{\tilde{\mu}}$ be as in Proposition 2.10. For every $a = c, s, u, cs, cu$, set

$$E_a := \left\{(t, \xi(t, v)) : t \in I_v, v \in V_a, (t, v) \in D_{\tilde{\mu}}\right\} \subset \mathbb{R} \times V_\infty.$$

The set E_c is a *centre subset* of the dynamical system in x; the set E_s is a *stable subset* of the system; and the set E_u is an *unstable subset* of the system. The set E_{cs} is a *centre-stable subset*, and $E_{cu,p}$ is a *centre-unstable subset* of the system. Such spaces are *invariant* or *integral subsets* of the dynamical system in x.

Remark 2.12. If the map $\xi_t := \xi(t, -)$ in Definition 2.11 is invertible for all $t \in I$ (on a suitable domain), then the dynamical system in x in that definition is equivalent to the ODE

$$\frac{d}{dt}(\xi_t^{-1} \circ x)(t) = A(\xi_t^{-1} \circ x)(t) + F(t, (\xi_t^{-1} \circ x)(t)).$$

Remark 2.13. In general, existence and uniqueness of solutions of (2.4) is not guaranteed, hence the careful definition of I_ν. Existence and uniqueness of solutions is an assumption in previous definitions [17, Theorem 2.9, e.g.]; see Hypothesis 2.7 in that reference. There are existence and uniqueness results if f satisfies a local Lipschitz condition, but that is not the case in many applications to PDEs. Under additional assumptions, Vanderbauwhede & Iooss [32, proof of Theorem 3] showed such a local Lipschitz condition holds.

Remark 2.14. Invariant subsets or submanifolds are not unique in general; here this non-uniqueness is due to various possibilities for I_ν, $D_{\tilde{\mu}}$ and ξ_p, and is reflected in the use of the indefinite article in Definition 2.11.

Example 2.15. For one example of the non-uniqueness engendered via ξ, consider the classic example system of $\dot{x} = -x^2$ and $\dot{y} = -y$ in the role of (2.2) (and let the step function $H(x) := 1$ when $x > 0$, and $H(x) := 0$ when $x \leq 0$). This ODE system may be given, for every C, as the coordinate transformation, (2.5), $x = X$ and $y = Y + CH(X)e^{-1/X}$ together with the system, (2.4), $\dot{X} = -X^2$ and $\dot{Y} = -Y$ (by design, here symbolically identical to the original xy-system). Lemma 2.9 identifies $Y = 0$ as the centre subspace of this XY-system. Definition 2.11 then gives the classic non-uniqueness that, for every C, $y = CH(x)e^{-1/x}$ are centre manifolds for the xy-system.

Remark 2.16. In the setting of Definition 2.11, if ξ is a local diffeomorphism in the Fréchet manifold sense, then the subsets E_j in Definition 2.11 are Fréchet manifolds. This would justify the more specific terminology *invariant submanifolds* rather than just invariant subsets.

2.5. **Main result: an approximate normal form.** Our main result, Theorem 2.18, states that for an ODE of the form (2.2), there is a dynamical system in the normal form used to define invariant manifolds in Definition 2.11, such that solutions of the normal form system satisfy (2.2) up to a residual term that vanishes to any desired order. In this sense, (2.2) is arbitrarily close to a dynamical system with clearly and robustly defined invariant manifolds.

Definition 2.17. A function $f\colon I \to \mathbb{R}$ *grows at most polynomially* if there are $C, r > 0$ such that for all $t \in I$, $|f(t)| \leq C(1 + |t|^r)$.

An infinitely differentiable map $\varphi\colon I \times V_\infty \to V_\infty$ *has polynomial growth* if for every $v \in V_\infty$, every $k \in \mathbb{N}$, and every $l \in \mathbb{N}_0$, the function

$$\|\varphi(-, v)^{(l)}\|_{V_k}\colon I \to [0, \infty)$$

grows at most polynomially.

We use the term $\tilde{\mu}$-*regular integral* for an integral of the form

$$\int_a^\infty e^{-\mu t} f(t)\, dt,$$

where $\Re(\mu) > \tilde{\mu}$ and f grows at most polynomially. The larger $\tilde{\mu}$, the better the convergence properties of $\tilde{\mu}$-regular integrals.

Theorem 2.18. *Let* $p \in \mathbb{N}$ *be such that* $p \geq 2$, $\beta - (p+1)\alpha > \tilde{\mu}$ *and* $\gamma - (p+1)\alpha > \tilde{\mu}$. *Suppose that* f *has polynomial growth. Then there are three infinitely differentiable maps* $F_p, \xi_p, R_p \colon I \times V_\infty \to V_\infty$, *such that*

- $F_p = \mathcal{O}(2)$ *and* F_p *separates invariant manifolds;*
- $R_p = \mathcal{O}(p)$,

and if a smooth map $x \colon I \to V_\infty$ *is given by*

$$x(t) = \xi_p(t, X(t)) \tag{2.6}$$

for all $t \in I$, *for a smooth map* $X \colon I \to V_\infty$ *satisfying*

$$\dot{X}(t) = AX(t) + F_p(t, X(t)) \tag{2.7}$$

for all $t \in I$, *then for all* $t \in I$,

$$\dot{x}(t) = Ax(t) + f(t, x(t)) + R_p(t, X(t)). \tag{2.8}$$

Finally, there is a construction of the map ξ_p *in which all integrals over* I *that occur are* $\tilde{\mu}$*-regular.*

We prove this theorem in Sections 3 to 7; see in particular Section 7.3.

In Theorem 7.9, the map F_p is the *nonlinearity* of the normal form ODE (2.7); the map ξ_p is a *(time-dependent) coordinate transformation*, and the map R_p is the *residual* modulo which a transformed solution (2.6) of (2.7) satsfies the original equation (2.2). The maps F_p and ξ_p can be chosen to be polynomials of a certain type. The conclusion that the integrals occurring are $\tilde{\mu}$-regular is more than just convenient: this is clear in the classical case where $\alpha = 0$ (see Remark 2.19).

Remark 2.19. In cases where the centre eigenvalue bound α equals zero, we can always choose $\tilde{\mu}$ so small that the conditions on $\alpha, \beta, \gamma, \tilde{\mu}$ and p in Theorem 2.18 are satisfied. In these cases, the residual R_p can be made to vanish to arbitrarily large order p. Furthermore, the integrals that occur in the construction of ξ_p (see Definition 7.2) are $\tilde{\mu}$-regular for some $\tilde{\mu} > 0$ precisely if they converge. Hence $\tilde{\mu}$-regularity for some $\tilde{\mu} > 0$ is a necessary condition for the construction to make sense.

Choosing the centre eigenvalue bound α positive, which imposes a positive lower bound on $\tilde{\mu}$, restricts the vanishing order p of R_p, but also makes the construction of the coordinate transform ξ_p more robust, in the sense that the integrals over I in its construction are $\tilde{\mu}$-regular. Many researchers choose to phrase problems as singular perturbations [8, 23, 33, e.g.]. In such cases the bounds on the hyperbolic rates $\beta, \gamma \propto \frac{1}{\varepsilon} \to \infty$ as the perturbation parameter $\varepsilon \to 0$. Consequently, choosing $\tilde{\mu}, p \propto 1/\sqrt{\varepsilon}$ (say) then the residual R_p again can be made to vanish to arbitrarily large order for small enough ε.

However, in applications we generally require an invariant manifold in some chosen domain of interest that resolve phenomena on chosen time scales of interest. Such subjective choices, informed by the governing equations, generally dictate the chosen bound α separated by a big enough gap from the bounds β, γ so that the centre manifold evolution, constructed to a valid order p, provides a useful model over the chosen domain for the desired phenomena.

Remark 2.20. The derivative at $0 \in V_\infty$ of the coordinate transformation ξ_p is the identity map, and hence invertible. If a suitable generalisation of the inverse function theorem applies to ξ_p, such as a version of the Nash–Moser theorem, then it follows that ξ_p is a local diffeomorphism at zero. Then it would be justified to call the invariant subsets of Definition 2.11 invariant submanifolds in this setting (at least in a neighbourhood of zero), see Remark 2.16.

Remark 2.21. In the proof of Theorem 2.18, explicit constructions of the maps F_p and ξ_p are given. In practice, however, it can be easier to determine these maps in more direct ways. This is illustrated in an example in Section 9. Theorem 2.18 implies that one can always find these maps. We prove this by giving a construction that always leads to an answer, even though more direct constructions may exist in specific situations.

Similarly, the domain D_μ in Proposition 2.10, defined in (8.7), is guaranteed to have the properties in Proposition 2.10. In practice, these properties often hold on much larger domains.

2.6. **A general class of PDEs in bounded domains.** Because Theorem 2.18 applies to abstract Banach spaces V_k, it gives one the flexibility to choose these spaces such that, for specific PDE applications,

(1) the residual R_p is of order p with respect to norms relevant to the problem, and
(2) the spaces V_k incorporate the relevant boundary conditions.

This subsection explores a class of nonlinear PDEs to which Theorem 2.18, and hence Definition 2.11, apply.

Let $d \in \mathbb{N}$ be the dimension of the domain of the PDEs to be considered. Let Ω be a bounded, open subset of $\mathbb{R}^d$, or of a d-dimensional manifold, with C^1 boundary. Let $m \in \mathbb{N}$, and let $1 \leq p < \infty$. For $k \in \mathbb{N}$, let V_k be the Sobolev space $W^{k-1,k+1}(\Omega, \mathbb{R}^m)$.

Let $A\colon C^\infty_c(\Omega;\mathbb{R}^m) \to C^\infty_c(\Omega;\mathbb{R}^m)$ be a linear partial differential operator. (Here the subscript c denotes compactly supported functions.) Let $s \in \mathbb{N}$, with $s \geq 2$, be the 'polynomial' order of the nonlinearities in the PDEs. Let

$$D_1, \dots, D_s\colon C^\infty_c(\Omega;\mathbb{R}^m) \to C^\infty_c(\Omega;\mathbb{R})$$

be linear partial differential operators. For index-vector $q \in \mathbb{N}_0^s$ and $u \in C^\infty_c(\Omega, \mathbb{R}^m)$, we set

$$(Du)^q := (D_1u)^{q_1} \cdots (D_su)^{q_s}. \tag{2.9}$$

Let α, β, γ and $\tilde{\mu}$ be as in Section 2.5. Fix smooth functions[2] $a^j_q\colon \mathbb{R} \to \mathbb{C}$, for $q \in \mathbb{N}_0^s$, with $|q| \leq s$, such that these functions and all their derivatives grow at most polynomially. Define $f\colon \mathbb{R} \times C^\infty_c(\Omega;\mathbb{R}^m) \to C^\infty_c(\Omega;\mathbb{R}^m)$ by $f(t,u) := (f_1(t,u), \dots, f_m(t,u))$, where for each j,

$$f_j(t,u) = \sum_{q \in \mathbb{N}_0^s, |q| \leq s} a^j_q(t)(Du)^q$$

for $t \in \mathbb{R}$ and $u \in C^\infty_c(\Omega;\mathbb{R}^m)$. Suppose that $f = \mathcal{O}(2)$.

[2] The real line may be replaced by a smaller open interval.

We write

$$W^{\infty}(\Omega;\mathbb{R}^m) := \bigcap_{k=1}^{\infty} W^{k-1,k+1}(\Omega;\mathbb{R}^m).$$

Then $C_c^{\infty}(\Omega;\mathbb{R}^m) \subset W^{\infty}(\Omega;\mathbb{R}^m) \subset C^{\infty}(\Omega;\mathbb{R}^m)$. The maps A and f extend continuously to $W^{\infty}(\Omega;\mathbb{R}^m)$. Suppose that that the eigenfunctions $\{e_j\}_{j=1}^{\infty}$ of this extension of A form a Hilbert basis of $L^2(\Omega,\mathbb{R}^m)$.

Theorem 2.22. *The spaces V_k and the maps A and f satisfy the hypotheses of Theorem 2.18.*

We prove Theorem 2.22 in Section 4.4. Together with Theorem 2.18, it has the following immediate consequence.

Corollary 2.23. *Let $p \in \mathbb{N}$ be such that $p \geq 2$. Suppose that α, β, γ and $\tilde{\mu}$ satisfy $\beta - (p+1)\alpha > \tilde{\mu}$ and $\gamma - (p+1)\alpha > \tilde{\mu}$ (as in Theorem 2.18). Then there are infinitely differentiable maps*

$$F_p, \xi_p, R_p \colon \mathbb{R} \times W^{\infty}(\Omega;\mathbb{R}^m) \to W^{\infty}(\Omega;\mathbb{R}^m),$$

where F_p is a polynomial map that separates invariant subspaces, such that if X and x are as in (2.7) and (2.6), then

$$\dot{x}(t) = Ax(t) + f(t, x(t)) + R_p(t, X(t))$$

for all $t \in I$. Further, for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that for all $u \in W^{\infty}(\Omega;\mathbb{R}^m)$,

$$\|R_p(t,v)\|_{W^{l-1,l+1}} = O(\|v\|^p_{W^{k-1,k+1}})$$

as $v \to 0$ in $W^{k-1,k+1}(\Omega,\mathbb{R}^m)$. There is a construction of the map ξ_p in which all integrals over I that occur are $\tilde{\mu}$-regular.

This corollary shows that any PDE of the form (2.2), with A and f as in this subsection, is equivalent up to a residual of order p to a PDE with clear invariant manifolds, as in Definition 2.11.

Example 2.24. Suppose that $\Omega = S^1$, the circle. This amounts to imposing periodic boundary conditions. Take $m = 1$, and let $A \colon C^{\infty}(S^1) \to C^{\infty}(S^1)$ by any linear partial differential operator with constant coefficients. Its eigenfunctions, $e_j(\theta) = e^{ij\theta}$ for $j \in \mathbb{Z}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, are orthogonal in the Sobolev spaces $W^{k,2}(S^1)$. For a map f as in Theorem 2.22, that is, a polynomial expression in derivatives of functions, whose polynomial coefficients increase at most polynomially, Theorem 2.22 implies that the conditions of Theorem 2.18 are satisfied in this case, so Corollary 2.23 applies. This generalises directly to cases where Ω is a higher-dimensional torus; that is, to problems in $\mathbb{R}^d$ with periodic boundary conditions. Here we used the case where the domain Ω is a manifold, rather than an open subset of $\mathbb{R}^d$.

Most of the rest of this paper is devoted to proofs of Theorems 2.18 and 2.22, and developing the tools used in these proofs. In Section 8, we prove Lemma 2.9 and Proposition 2.10. In Section 9 we illustrate Corollary 2.23 by working out an example.

3. Derivatives and polynomials

In this section we review standard material on derivatives of maps between normed vector spaces. We also briefly discuss polynomial maps between normed vector spaces. Throughout this section, $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are normed vector spaces, possibly infinite-dimensional. Let $U \subset V$ be an open subset, and let $f\colon U \to W$ be a map. We fix an element $u \in U$.

3.1. **First order derivatives.** This subsection and the next contain some standard definitions and facts about derivatives of maps between normed vector spaces. Details and proofs can be found in various textbooks [34, e.g.].

For a map $\varphi\colon V \supset \mathrm{dom}(\varphi) \to W$, we use the notation $\varphi(h) = o(h)$ for the statement

$$\lim_{h \to 0} \frac{\|\varphi(h)\|_W}{\|h\|_V} = 0,$$

were h runs over $\mathrm{dom}(\varphi) \setminus \{0\}$.

Definition 3.1. The map $f : U \to W$ is *differentiable* at u, if there is an operator $f'(u) \in \mathcal{B}(V, W)$ such that

$$f(u+h) = f(u) + f'(u)h + o(h).$$

Then $f'(u)$ is the *derivative* of f at u. If f is differentiable at every point in U, then we say that f is *differentiable*. In that case, the *derivative* of f is the map

$$f'\colon U \to \mathcal{B}(V, W) \tag{3.1}$$

mapping $u \in U$ to $f'(u)$.

The derivative of a map at a point is unique, if it exists.

Lemma 3.2 (Chain rule). *Let $(X, \|\cdot\|_X)$ be a third normed vector space. Let $A \subset W$ be an open subset containing $f(U)$. If $g\colon A \to X$ is differentiable at $f(u)$ and f is differentiable at u, then $g \circ f$ is differentiable at u, and*

$$(g \circ f)'(u) = g'(f(u)) \circ f'(u).$$

Definition 3.3. The map f is a *near-identity* at u if the map (3.1) is continuous in a neighbourhood of u, and

$$f'(u)h = h + o(h).$$

3.2. **Higher order derivatives.** Fix a positive integer $n \in \mathbb{N}$. We write $\mathcal{B}^n(V, W)$ for the space of multilinear maps $\lambda\colon V^n \to W$ for which the norm

$$\|\lambda\| := \sup_{\substack{v_1, \ldots, v_n \in V \\ \|v_1\|_V = \cdots = \|v_n\|_V = 1}} \|\lambda(v_1, \ldots, v_n)\|_W \tag{3.2}$$

is finite. There is a natural isometric isomorphism

$$\mathcal{B}(V, \mathcal{B}(V, \ldots, \mathcal{B}(V, W) \cdots)) \xrightarrow{\cong} \mathcal{B}^n(V, W) \tag{3.3}$$

mapping an operator T in the left-hand side to the operator $\lambda \in \mathcal{B}^n(V, W)$ given by

$$\lambda(v_1, \ldots, v_n) = T(v_1)(v_2) \cdots (v_n),$$

for $v_1, \dots, v_n \in V$.

Suppose $f : U \to W$ is differentiable. The map f is *twice differentiable* at u if the map (3.1) is differentiable at u. Then we write

$$f^{(2)}(u) := (f')'(u) \in \mathcal{B}(V, \mathcal{B}(V, W)) \cong \mathcal{B}^2(V, W).$$

Inductively, for $n \geq 2$, f is defined to be n times differentiable at u if it is $n-1$ times differentiable, and the map

$$f^{(n-1)} \colon U \to \mathcal{B}^{n-1}(V, W)$$

is differentiable at u. We then set

$$f^{(n)}(u) := (f^{(n-1)})'(u) \quad \in \mathcal{B}^n(V, W).$$

In this case, we write

$$f^{(n)}(u)h^n := f^{(n)}(u)(h, h, \dots, h). \tag{3.4}$$

As before, we say that f is n *times differentiable* if it is n times differentiable at every point in U. And *infinitely differentiable* means n times differentiable for every $n \in \mathbb{N}$.

Theorem 3.4 (Taylor's theorem)**.** *Suppose f is $n+1$ times differentiable. Suppose that $\|f^{(n+1)}(\xi)\| \leq M$ for all ξ in a closed ball around u contained in U. Then for every h in this ball,*

$$\Big\| f(u+h) - \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(u)h^j \Big\|_W \leq \frac{M}{(n+1)!} \|h\|_V^{n+1}.$$

3.3. **Bounded polynomial maps.** The space of polynomial functions on a finite-dimensional vector space can be identified with the symmetric algebra of the dual space. The correspondence is given by restricting symmetric multilinear functions to the diagonal. This identification motivates Definition 3.6 of bounded polynomial maps between possibly infinite-dimensional normed vector spaces. This definition involves bounded multilinear operators, because we need to estimate norms in various places.

An operator in $\mathcal{B}^n(V, W)$ is said to be *symmetric* if it is invariant under permutations of its arguments. Let $\mathcal{SB}^n(V, W)$ be the subspace of symmetric operators in $\mathcal{B}^n(V, W)$. An example of such a symmetric operator is the nth derivative of a map.

Lemma 3.5. *If f is n times differentiable at u, then $f^{(n)}(u)$ is symmetric.*

We denote the permutation group of $\{1, \dots, n\}$ by Σ_n.

Let $S \colon \mathcal{B}^n(V, W) \to \mathcal{SB}^n(V, W)$ be the symmetrisation operator: for every $\lambda \in \mathcal{B}^n(V, W)$ and $v_1, \dots, v_n \in V$,

$$(S\lambda)(v_1, \dots, v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \lambda(v_{\sigma(1)}, \dots, v_{\sigma(n)}).$$

The operator S is continuous, and that $\mathcal{SB}^n(V, W)$ is the zero level set of S minus the identity, and hence closed in $\mathcal{B}^n(V, W)$. So $\mathcal{SB}^n(V, W)$ is a Banach space if V and W are.

An element $\lambda \in \mathcal{B}^n(V, W)$ defines a map $p_\lambda \colon V \to W$ by

$$p_\lambda(v) = \lambda(v, \dots, v). \tag{3.5}$$

We have $p_{S\lambda} = p_\lambda$, and the map $\lambda \mapsto p_\lambda$ is injective on $\mathcal{SB}^n(V, W)$.

Definition 3.6. A *bounded homogeneous polynomial map* of degree n from V to W is a map of the form p_λ as in (3.5). We write $\mathrm{Pol}^n(V, W)$ for the space of such maps. It inherits a norm from the space $\mathcal{SB}^n(V, W)$ via the linear isomorphism $\lambda \mapsto p_\lambda$. If $\lambda \neq 0$, then the *degree* of p_λ is n.

A *bounded polynomial map* from V to W is a finite sum of bounded homogeneous polynomial maps. The *degree* of a bounded polynomial map is the degree of its highest-degree homogeneous term.

We write $\mathrm{Pol}(V, W)$ for the space of all bounded polynomial maps from V to W. This is the algebraic direct sum of the spaces $\mathrm{Pol}^n(V, W)$.

Because $\mathcal{SB}^n(V, W)$ is a Banach space if V and W are, so is $\mathrm{Pol}^n(V, W)$. We could define $\mathrm{Pol}^0(V, W)$ as the space of constant maps into W, but we only consider homogeneous polynomials of order at least one.

If f is n times differentiable at u, then we have the map

$$h \mapsto f^{(n)}(u)h^n \in \mathrm{Pol}^n(V, W).$$

Lemma 3.7–3.10 below are basic facts showing that bounded polynomials and their orders and compositions behave as one would expect. Their proofs are short and straightforward.

Lemma 3.7. *Every bounded polynomial map is infinitely differentiable.*

Lemma 3.8. *If $p \in \mathrm{Pol}^n(V, W)$, then there is a constant $C > 0$ such that for all $v \in V$,*

$$\|p(v)\|_W \leq C\|v\|_V^n.$$

Lemma 3.9. *If p is a polynomial map from V to W of order lower than n, and*

$$\|p(v)\|_W = O(\|v\|_V^n),$$

as $v \to 0$ in V, then $p = 0$.

Lemma 3.10. *If $p_1 \in \mathrm{Pol}^m(U, V)$ and $p_2 \in \mathrm{Pol}^n(V, W)$, then $p_2 \circ p_1 \in \mathrm{Pol}^{mn}(U, W)$.*

3.4. **Standard monomials.** Let $V^* := \mathcal{B}(V, \mathbb{C})$ be the continuous dual of V. We denote the pairing between V^* and V by $\langle -, - \rangle$. For every $j \in \mathbb{N}$, let $e^j \in V^*$ be given. What follows is most natural if V is a Hilbert space and e^j is given by taking inner products with an element e_j of a Hilbert basis, but it applies more generally.

Consider a multi-index $q \in \mathbb{N}_0^\infty$. If $|q| = n$, and m is the largest number for which $q_m \neq 0$, then we define the element

$$e^q := \underbrace{e^1 \otimes \cdots \otimes e^1}_{q_1 \text{ factors}} \otimes \cdots \otimes \underbrace{e^m \otimes \cdots \otimes e^m}_{q_m \text{ factors}} \in \mathcal{B}^n(V, \mathbb{C}). \tag{3.6}$$

In other words, for all $v_1, \ldots, v_n \in V$,

$$e^q(v_1, \ldots, v_n) =$$
$$\langle e^1, v_1 \rangle \cdots \langle e^1, v_{q_1} \rangle \langle e^2, v_{q_1+1} \rangle \cdots \langle e^2, v_{q_1+q_2} \rangle \cdots \langle e^m, v_{q_1+\cdots+q_{m-1}+1} \rangle \cdots \langle e^m, v_n \rangle.$$

We write $p^q := p_{e^q}$ for the corresponding homogeneous polynomial. One could call this the standard q-monomial with respect to the set $\{e^j\}_{j=1}^{\infty}$. (If $V = \mathbb{C}^k$ and the elements e^j are the standard coordinates, then the monomial functions in the usual sense are precisely the scalar multiples of the maps p^q.)

For $v \in V$, we write

$$v^q := p^q(v) = \prod_{j=1}^{\infty} \langle e^j, v \rangle^{q_j}. \tag{3.7}$$

This product is finite (since q has finitely many nonzero terms) and depends on the set $\{e^j\}$. The following lemma follows from the definition of the derivative.

Lemma 3.11. *The derivative of* p^q *in* (3.7) *is given by*

$$(p^q)'(u)(h) = \sum_{j=1}^{\infty} q_j \langle e^j, u \rangle^{q_j - 1} \langle e_j, h \rangle \Big(\prod_{k \neq j} \langle e^k, u \rangle^{q_k} \Big),$$

for all $u, h \in V$.

4. Compact derivatives and polynomials

It is a nontrivial question in what sense differentiable maps between normed vector spaces can be approximated by polynomial maps [1, 11, 12, 13, 21, 22, e.g.]. In this section we discuss an approach to this problem that is suitable for our purposes. This discussion includes the further problem of approximating a polynomial by sums of the standard monomials of Section 3.4. The polynomials for which this is possible are the *compact polynomials* introduced in Section 4.2.

Section 4.3 introduces *compactly differentiable* maps. We combine these with Taylor's theorem to express the lowest order parts of such maps in terms of standard monomials. We discuss a class of examples of compactly differentiable maps relevant to the study of PDEs.

4.1. Compact multilinear maps.

Let V and W be Banach spaces. Further, let $\mathcal{K}^n(V, W) \subset \mathcal{B}^n(V, W)$ be the image of the space

$$\mathcal{K}(V, \mathcal{K}(V, \dots, \mathcal{K}(V, W) \cdots))$$

under the isomorphism (3.3). Using induction on n, one can show that $\mathcal{K}^n(V, W)$ is closed in $\mathcal{B}^n(V, W)$, and hence a Banach space.

Let $\{e^j\}_{j=1}^{\infty} \subset V^*$ and $\{f_k\}_{k=1}^{\infty} \subset W$ be countable subsets whose spans are dense. (So V^* and W are separable.) For any $\alpha \in \mathbb{N}^n$, consider the multilinear map

$$e^{\alpha} := e^{\alpha_1} \otimes \cdots \otimes e^{\alpha_n} : V \times \cdots \times V \to \mathbb{C}. \tag{4.1}$$

A Banach space has the approximation property if every compact operator on the space is a norm-limit of finite-rank operators. This is always true for Hilbert spaces, but we need to consider more general Banach spaces for applications. The following result is standard in the case where V and W are Hilbert spaces.

Proposition 4.1. *If V^* has the approximation property, then the space* $\mathrm{span}\{e^j \otimes f_k : j, k \in \mathbb{N}\}$ *is dense in* $\mathcal{K}(V, W)$.

Proof. Since V^* has the approximation property, the space of finite-rank operators (linear operators with finite-dimensional images) is dense in $\mathcal{K}(V, W)$. See for example Proposition 4.12(b) in the book by Ryan [30]. The space $\mathrm{span}\{e^j \otimes f_k : j, k \in \mathbb{N}\}$ is dense in the space of finite-rank operators, so the claim follows. □

Lemma 4.2. *If V^* has the approximation property, then for every $n \in \mathbb{N}$, the span of* $\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, k \in \mathbb{N}\}$ *is dense in* $\mathcal{K}^n(V, W)$.

Proof. We prove this by induction on n. If $n = 1$, then the claim is precisely Proposition 4.1. Now suppose that the claim holds for a given n. By definition,

$$\mathcal{K}^{n+1}(V, W) = \mathcal{K}(V, \mathcal{K}^n(V, W)).$$

By the induction hypothesis, the set $\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, k \in \mathbb{N}\}$ has dense span in $\mathcal{K}^n(V, W)$. Therefore, Proposition 4.1, with W replaced by $\mathcal{K}^n(V, W)$, implies that the set

$$\{e^j \otimes e^\alpha \otimes f_k : j, k \in \mathbb{N}, \alpha \in \mathbb{N}^n\}$$

has dense span in $\mathcal{K}^{n+1}(V, W)$. This is precisely the claim for $n + 1$. □

A Schauder basis of a Banach space V is a subset $\{e_j\}_{j=1}^\infty \subset V$ such that for each $v \in V$, there are unique complex numbers $\{v^j\}_{j\in\mathbb{N}}$ such that

$$\left\| v - \sum_{j=1}^{n} v^j e_j \right\|_V \to 0 \quad \text{as } n \to \infty.$$

A space with a Schauder basis has the approximation property.

Lemma 4.3. *If $\{e^j\}_{j=1}^\infty$ is a Schauder basis of V^* and $\{f_k\}_{k=1}^\infty$ is a Schauder basis of W, then for every $n \in \mathbb{N}$, the set $\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, k \in \mathbb{N}\}$ is a Schauder basis of $\mathcal{K}^n(V, W)$.*

Proof. Lemma 4.2 implies that $\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, k \in \mathbb{N}\}$ has dense span. So it remains to show that if $a_\alpha^k \in \mathbb{C}$ are such that

$$\sum_{\alpha\in\mathbb{N}^n} \sum_{k=1}^{\infty} a_\alpha^k e^\alpha \otimes f_k = 0,$$

then $a_\alpha^k = 0$ for all α and k. Since $\{f_k\}_{k=1}^\infty$ is a Schauder basis of W, this reduces to the case where $W = \mathbb{C}$. In that case, one can prove the claim by induction on n, using the fact that $\{e^j\}_{j=1}^\infty$ is a Schauder basis of V^*. □

Remark 4.4. In the induction step in the proof of the special case of Lemma 4.2 where W is a Hilbert space, we still need the general version of Proposition 4.1, where W is a Banach space. This is because $\mathcal{K}^n(V, W)$ is only a Banach space, even if W is a Hilbert space.

The subspace $\mathcal{SK}^n(V,W)$ of symmetric operators in $\mathcal{K}^n(V,W)$ is closed in $\mathcal{B}^n(V,W)$, since it is the intersection of the closed subspaces $\mathcal{SB}^n(V,W)$ and $\mathcal{K}^n(V,W)$. Hence $\mathcal{SK}^n(V,W)$ is a Banach space with respect to the norm (3.2).

A Schauder basis $\{e_j\}_{j=1}^\infty$ of a Banach space V is unconditional if there is a constant $C > 0$ such that for all $a^j, \varepsilon_j \in \mathbb{C}$ with $|\varepsilon_j| = 1$, and all $n \in \mathbb{N}$,

$$\Big\|\sum_{j=1}^n \varepsilon_j a^j e_j\Big\|_V \le C\Big\|\sum_{j=1}^n a^j e_j\Big\|_V.$$

In that case, convergence of $\sum_{j=1}^n a^j e_j$ implies convergence of $\sum_{j\in A} a^j e_j$, for every $A \subset \mathbb{N}$.

Lemma 4.5. *Suppose that V and W are Hilbert spaces, and that $\{e_j\}_{j=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ are orthogonal sets in V and W respectively, with dense spans. Let $e^j \in V^*$ be defined by taking inner products with e_j. Then $\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, k \in \mathbb{N}\}$ is an unconditional Schauder basis of $\mathcal{K}^n(V,W)$.*

Proof. The set $\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, k \in \mathbb{N}\}$ is a Schauder basis of $\mathcal{K}^n(V,W)$ by Lemma 4.3. It remains to show that it is unconditional. By rescaling the vectors e_j and f_k, we reduce the proof to the case where $\{e_j\}_{j=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ are Hilbert bases. In that case, for all finite subsets $A \subset \mathbb{N}_0^n \times \mathbb{N}$ and all $a_\alpha^k \in \mathbb{C}$,

$$\Big\|\sum_{(\alpha,k)\in A} a_\alpha^k e^\alpha \otimes f_k\Big\|^2_{\mathcal{B}^n(V,W)} = \sup_{\alpha\in\mathbb{N}_0^m} \sum_{k\in\mathbb{N};\,(\alpha,k)\in A} |a_\alpha^k|^2.$$

□

Lemma 4.6. *Let U, V and W be normed vector spaces, and $n \in \mathbb{N}$. Let $\lambda \in \mathcal{B}^n(V,W)$, and $a_1,\dots,a_n \in \mathcal{K}(U,V)$. Define $\nu\colon U \times \cdots \times U \to W$ by*

$$\nu(u_1,\dots,u_n) = \lambda(a_1u_1,\dots,a_nu_n),$$

for all $u_1,\dots,u_n \in U$. Then $\nu \in \mathcal{K}^n(U,W)$.

Proof. We use induction on n. For $n = 1$, $\nu \in \mathcal{K}^1(U,W)$ because the composition of a compact operator and a bounded operator is compact. Suppose that the claim holds for a given n. Let $\lambda \in \mathcal{B}^{n+1}(V,W)$, and $a_1,\dots,a_{n+1} \in \mathcal{K}(U,V)$. For a fixed $u \in U$, define $\nu_u \in \mathcal{B}^n(U,W)$ by

$$\nu_u(u_1,\dots,u_n) = \lambda(a_1u_1,\dots,a_nu_n,a_{n+1}u),$$

for $u_1,\dots,u_n \in U$. For a fixed $v \in V$, define $\lambda_v \in \mathcal{B}^n(V,W)$ by

$$\lambda_v(v_1,\dots,v_n) = \lambda(v_1,\dots,v_n,v),$$

for $v_1,\dots,v_n \in V$. Then for all $u_1,\dots,u_n \in U$,

$$\nu_u(u_1,\dots,u_n) = \lambda_{a_{n+1}u}(a_1u_1,\dots,a_nu_n).$$

So by the induction hypothesis, $\nu_u \in \mathcal{K}^n(U,W)$. In this way, we obtain the map

$$\tilde\nu\colon U \to \mathcal{K}^n(U,W),$$

mapping $u \in U$ to ν_u. It remains to show that $\tilde\nu$ is a compact operator.

Define $\tilde{\lambda} \in \mathcal{B}(V, \mathcal{K}^n(U, W))$ by

$$\tilde{\lambda}(v) \colon (u_1, \ldots, u_n) \mapsto \lambda(a_1 u_1, \ldots, a_n u_n, v),$$

for $v \in V$ and $u_1, \ldots, u_n \in U$. (This map takes values in $\mathcal{K}^n(U, W)$ by the induction hypothesis.) Since a_{n+1} is compact and $\tilde{\lambda}$ is bounded, we find that $\tilde{v} = \tilde{\lambda} \circ a_{n+1}$ is a compact operator. □

4.2. **Compact polynomial maps.** In our construction of the coordinate transformation ξ_p and the nonlinearity F_p in Theorem 2.18, we use the fact that certain polynomial maps can be approximated in norm by standard monomials of the form (3.7). Contrary to the finite-dimensional case, this is not possible for every bounded polynomial map. This leads us to the notion of *compact polynomial maps* in Definition 4.7. They can be approximated in norm by standard monomials, as we show in Proposition 4.8.

Definition 4.7. A *compact homogeneous polynomial map* of degree n from V to W is a map of the form p_λ as in (3.5), for $\lambda \in \mathcal{K}^n(V, W)$. We write $\mathcal{K}\,\mathrm{Pol}^n(V, W)$ for the space of such maps. This space inherits a norm from the space $\mathrm{Pol}^n(V, W)$ it is contained in.

If $\lambda \neq 0$, then the *degree* of p_λ is n. A *compact polynomial map* from V to W is a finite sum of compact homogeneous polynomial maps. The *degree* of a compact polynomial map is the degree of its highest-degree homogeneous term. We write $\mathcal{K}\,\mathrm{Pol}(V, W)$ for the space of all compact polynomial maps between these spaces.

The isometric isomorphism $S\mathcal{B}^n(V, W) \cong \mathrm{Pol}^n(V, W)$ restricts to an isometric isomorphism $S\mathcal{K}^n(V, W) \cong \mathcal{K}\,\mathrm{Pol}^n(V, W)$. So $\mathcal{K}\,\mathrm{Pol}^n(V, W)$ is a closed subspace of the Banach space $\mathrm{Pol}^n(V, W)$, and hence is a Banach space itself.

For every $w \in W$, an operator of the form $e^q \otimes w$, with e^q as in (3.6), is an element of $\mathcal{K}^n(V, W)$. Indeed, $e^q \otimes w$ is an iteration of rank-one operators, So $p^q \otimes w \in \mathcal{K}\,\mathrm{Pol}^n(V, W)$.

The following proposition is the reason why we are interested in compact polynomial maps.

Proposition 4.8. *Suppose that V and W are Banach spaces, that V^* has a Schauder basis $\{e^j\}_{j=1}^\infty$, and that W has a Schauder basis $\{f_k\}_{k=1}^\infty$. Then the elements*

$$p^q \otimes f_k \quad \in \mathcal{K}\,\mathrm{Pol}^n(V, W), \tag{4.2}$$

where the multi-index q ranges over the elements of $\mathbb{N}_0^\infty$ with $|q| = n$, and k ranges over the positive integers, form a Schauder basis of $\mathcal{K}\,\mathrm{Pol}^n(V, W)$.

Proof. Consider the space

$$X := \overline{\mathrm{span}\{e^\alpha \otimes f_k : \alpha \in \mathbb{N}^n, \alpha_1 \leq \cdots \leq \alpha_n, k \in \mathbb{N}\}}.$$

Lemma 4.3 implies that the set of $e^\alpha \otimes f_k$ where $\alpha_1 \leq \cdots \leq \alpha_n$ is a Schauder basis of X. And $S \colon X \to S\mathcal{K}^n(V, W)$ is a bounded linear isomorphism with bounded inverse. Since such isomorphisms map Schauder bases to Schauder bases, we find that the set $Se^\alpha \otimes f_k$, for non-decreasing α as above, is a Schauder basis of $S\mathcal{K}^n(V, W)$.

For $\alpha \in \mathbb{N}$ with non-decreasing entries, define $q(\alpha) \in \mathbb{N}_0^\infty$ by

$$q(\alpha)_j = \#\{m \in \mathbb{N} : \alpha_m = j\}.$$

Then $e^\alpha = e^{q(\alpha)}$. (Note that e^α, for $\alpha \in \mathbb{N}^n$, and e^q, for $q \in \mathbb{N}_0^\infty$, are defined differently; compare (3.6) and (4.1).) Every sequence in $\mathbb{N}_0^\infty$ occurs in exactly one way as $q(\alpha)$, for alpha as above, so $Se^q \otimes f_k$, where $q \in \mathbb{N}_0^\infty$ and $k \in \mathbb{N}$, is a Schauder basis of $S\mathcal{K}^n(V, W)$. Since $p^q = p_{Se^q}$, the claim follows. □

A reformulation of Proposition 4.8 is that for every compact polynomial map $p \in \mathcal{K}\,\mathrm{Pol}^n(V, W)$, there are unique complex numbers a_q^k such that

$$p = \sum_{q,k} a_q^k p^q \otimes f_k,$$

where the sum converges in the norm on $\mathrm{Pol}^n(V, W)$. Conversely, all polynomial maps p of this form are compact.

Lemma 4.9. *In the setting of Lemma 4.5, the set $\{p^q \otimes f_k : |q| = n, k \in \mathbb{N}\}$ is an unconditional Schauder basis of $\mathcal{K}\,\mathrm{Pol}^n(V, W)$.*

Proof. The proof is analogous to the proof of Proposition 4.8, where we now use Lemma 4.5 instead of Lemma 4.3, and we use the fact that bounded linear isomorphisms with bounded inverses map unconditional Schauder bases to unconditional Schauder bases. □

Lemma 4.10. *If $p_1 \in \mathcal{K}\,\mathrm{Pol}^m(U, V)$ and $p_2 \in \mathcal{K}\,\mathrm{Pol}^n(V, W)$, then $p_2 \circ p_1 \in \mathcal{K}\,\mathrm{Pol}^{mn}(U, W)$.*

Proof. The proof is similar to the proof of Lemma 3.10, with bounded multilinear maps replaced by compact ones. □

Remark 4.11. Other notions of compact polynomial maps were studied by Gonzalo, Jaramillo and Pełczyńsky in [15, 24].

4.3. **Compactly differentiable maps.** Let V and W be normed vector spaces, let $U \subset V$ be an open subset containing a vector u, and let $f: U \to W$ be n times differentiable at u.

We will work with a special kind of differentiable maps between normed spaces.

Definition 4.12. The map f is n times *compactly differentiable* at u if

$$f^{(n)}(u) \in \mathcal{K}^n(V, W).$$

The main property of compactly differentiable maps that we use in the proof of Theorem 2.18 is that they satisfy versions of Taylor's theorem in which the Taylor polynomials that occur are compact, and can hence be approximated by standard monomials. See Corollaries 4.13 and 5.11.

If f is n times compactly differentiable at u, then by Lemma 3.5,

$$f^{(n)}(u) \in S\mathcal{K}^n(V, W).$$

Then the map $h \mapsto f^{(n)}(u)h^n$ of (3.4) is the compact polynomial map associated to $f^{(n)}(u)$. Together with Theorem 3.4 and Proposition 4.8, this leads to the following conclusion.

Corollary 4.13. *Suppose that* V *and* W *are Banach spaces, that* V^* *has a Schauder basis* $\{e^j\}_{j=1}^\infty$*, and that* W *has a Schauder basis* $\{f_k\}_{k=1}^\infty$*. Suppose* f *is* $n+1$ *times differentiable, and* k *times compactly differentiable for every* $k \leq n$*. Then there are unique complex numbers* a_q^k *such that*

$$f(u+h) = \sum_{q \in \mathbb{N}_0^\infty;\, |q| \leq n} \sum_{k=1}^{\infty} a_q^k h^q f_k + O(\|h\|_W^{n+1}), \tag{4.3}$$

where the part of the sum where $|q| = m$ *converges as a function of* h *in the norm on* $\mathrm{Pol}^m(V, W)$*, for* $m = 0, \ldots, n$*.*

(Note that, in (4.3), on the left-hand side f is a map from U to W, whereas on the right-hand side, f_k is an element of W.)

Lemma 4.14. *A compact polynomial map is infinitely compactly differentiable.*

Proof. We show that the derivative of every homogeneous compact polynomial $p_\lambda \in \mathcal{K}\,\mathrm{Pol}^n(V, W)$, for $\lambda \in S\mathcal{K}^n(V, W)$, is a compact polynomial in $\mathcal{K}\,\mathrm{Pol}^{n-1}(V, \mathcal{K}(V, W))$. This implies the claim by induction on n. As in the proof of Lemma 3.7, $p_\lambda'(u)h = n\lambda(h, u, \ldots, u)$ for all $u, h \in V$. In other words, $p_\lambda'(u) = p_\nu$, with $\nu = n\lambda$, where we view λ as an element of $\mathcal{K}^{n-1}(V, \mathcal{K}(V, W))$. This shows that $p_\lambda'(u) \in \mathcal{K}\,\mathrm{Pol}^{n-1}(V, \mathcal{K}(V, W))$. □

Lemma 4.15. *Let* U*,* V *and* W *be normed vector spaces and let* $f\colon U \to V$ *and* $g\colon V \to W$ *be differentiable maps. If either* f *or* g *is compactly differentiable, then so is* $g \circ f$*.*

Proof. Lemma 3.2 implies that for all $u \in U$,

$$(g \circ f)'(u) = g'(f(u)) \circ f'(u).$$

If f is compactly differentiable, then $f'(u) \in \mathcal{K}(U, V)$. If g is compactly differentiable, then $g'(f(u)) \in \mathcal{K}(V, W)$. In either case, we find that $(g \circ f)'(u) \in \mathcal{K}(U, V)$. □

Lemma 4.16. *Let* U*,* V *and* W *be normed vector spaces, let* $f\colon V \to W$ *be* n *times differentiable, and suppose that* U *is a subspace of* V*, with compact inclusion map* $j\colon U \hookrightarrow V$*. Then* $f \circ j$ *is* n *times compactly differentiable as a map from* U *to* W*.*

Proof. Let $u, h_1, \ldots, h_n \in U$. Then

$$(f \circ j)^{(n)}(u)(h_1, \ldots, h_n) = f^{(n)}(j(u))(j(h_1), \ldots, j(h_n)).$$

So the claim follows from Lemma 4.6. □

Remark 4.17. It is possible for a map to be n times compactly differentiable, even though not $n-1$ times. For example, the first derivative of the identity operator on an infinite-dimensional Banach space V is the identity map itself, and not a compact operator. But its higher-order derivatives are zero, and hence compact.

4.4. **A class of compactly differentiable maps.** We end this section by discussing a class of compactly differentiable maps (specifically, compact polynomials) that are relevant to the study of nonlinear PDEs. These maps are polynomial expressions in derivatives of functions; see Proposition 4.19 below.

Let $\Omega \subset \mathbb{R}^m$ be a bounded open subset with C^1 boundary. For $k \in \mathbb{N}_0$ and $p > 0$, consider the Sobolev space $W^{k,p}(\Omega)$.

Lemma 4.18. *Let $n \in \mathbb{N}$ and $p > 1$. Pointwise multiplication of n functions defines a map $\mu \in \mathcal{B}^n(W^{k,np}(\Omega), W^{k,p}(\Omega))$.*

Proof. Let $n \in \mathbb{N}$. By Hölder's inequality, for all $u_1, \dots, u_n \in L^{np}(\Omega)$,

$$\|u_1 \cdots u_n\|_{L^p(\Omega)} \leq \|u_1\|_{L^{np}(\Omega)} \cdots \|u_n\|_{L^{np}(\Omega)}. \tag{4.4}$$

For $\alpha \in \mathbb{N}_0^m$, there are combinatorial constants c^α_β, for $\beta = (\beta^{(1)}, \dots, \beta^{(n)})$, with $\beta^{(1)}, \dots, \beta^{(n)} \in \mathbb{N}_0^m$ such that $|\beta^{(1)}| + \cdots + |\beta^{(n)}| \leq |\alpha|$, such that for all $u_1, \dots, u_n \in C^\infty_c(\Omega)$, we have the generalised Leibniz rule

$$\frac{\partial^\alpha (u_1 \cdots u_n)}{\partial x^\alpha} = \sum_{|\beta^{(1)}|+\cdots+|\beta^{(n)}| \leq |\alpha|} c^\alpha_\beta \frac{\partial^{\beta^{(1)}} u_1}{\partial x^{\beta^{(1)}}} \cdots \frac{\partial^{\beta^{(n)}} u_n}{\partial x^{\beta^{(n)}}}.$$

Together with (4.4), this implies that

$$\|u_1 \cdots u_n\|_{W^{k,p}} \leq \Big(\sum_{|\alpha| \leq k} \sum_{|\beta^{(1)}|+\cdots+|\beta^{(n)}| \leq |\alpha|} c^\alpha_\beta \Big) \|u_1\|_{W^{k,np}} \cdots \|u_n\|_{W^{k,np}}.$$

□

Proposition 4.19. *Let $k, l, m_1, m_2, n \in \mathbb{N}$ and $p > 1$, with $k \leq l$. Let*

$$D_1, \dots, D_n \colon C^\infty_c(\Omega; \mathbb{R}^{m_1}) \to C^\infty_c(\Omega; \mathbb{R}^{m_2})$$

be linear partial differential operators of orders smaller than k. Fix complex numbers a^j_q, for $q \in \mathbb{N}_0^n$ with $|q| \leq n$, and $j \in \{1, \dots, m_2\}$. Define $f \colon C^\infty_c(\Omega; \mathbb{R}^{m_1}) \to C^\infty_c(\Omega; \mathbb{R}^{m_2})$ by $f(u) = (f_1(u), \dots, f_{m_2}(u))$, where for each j,

$$f_j(u) = \sum_{q \in \mathbb{N}_0^n, |q| \leq n} a^j_q (Du)^q \tag{4.5}$$

for $u \in W^{l,np}(\Omega; \mathbb{R}^{m_1})$, and with $(Du)^q$ as in (2.9), defines a compact polynomial map

$$f \in \mathcal{K}\,\mathrm{Pol}^n(W^{l,np}(\Omega; \mathbb{R}^{m_1}), W^{l-k,p}(\Omega; \mathbb{R}^{m_2})).$$

Proof. We first consider the case where $m_2 = 1$, and $a_q = 1$ if $q = (1, \dots, 1)$, and zero otherwise. By Lemma 4.18, pointwise multiplication defines a map $\mu \in \mathcal{B}^n(W^{l-k,np}(\Omega), W^{l-k,p}(\Omega))$.

By Rellich's lemma, boundedness of Ω implies that the maps $D_1, \dots, D_n$ extend to compact operators

$$D_1, \dots, D_n \colon W^{l,p}(\Omega) \to W^{l-k,p}(\Omega).$$

The map

$$\nu \colon W^{l,np}(\Omega) \times \cdots \times W^{l,np}(\Omega) \to W^{l-k,p}(\Omega)$$

defined by

$$\nu(u_1, \dots, u_n) = \mu(D_1 u_1, \dots, D_n u_n)$$

for $u_1, \dots, u_n \in W^{l,np}(\Omega)$, is in $\mathcal{K}^n(W^{l,np}(\Omega), W^{l-k,p}(\Omega))$ by Lemma 4.6. Hence p_ν is an element of $\mathcal{K}\,\mathrm{Pol}^n(W^{l,np}(\Omega), W^{l-k,p}(\Omega))$.

Every component of a general map of the form (4.5) is a finite sum of maps of the form p_ν as above, applied to the components of f. Hence it is in $\mathcal{K}\,\mathrm{Pol}^n(W^{l,np}(\Omega; \mathbb{R}^{m_1}), W^{l-k,p}(\Omega; \mathbb{R}^{m_2}))$. □

Example 4.20. Let $k \geq 1$, $l \geq k$ and $p > 1$ be integers. Let $\Omega \subset \mathbb{R}$ be a bounded open interval. Consider the map f from $W^{l,2p}(\Omega)$ to $W^{l-k,p}(\Omega)$, mapping $u \in W^{l,2p}(\Omega)$ to $u'u$. Taking $m_1 = m_2 = 1$, $n = 2$, $D_1 u = u'$ and $D_2 u = u$, for $u \in W^{l,2p}(\Omega)$, in Proposition 4.19, we find that f is a compact polynomial in $\mathcal{K}\,\mathrm{Pol}^2(W^{l,2p}(\Omega), W^{l-k,p}(\Omega))$ for every $k > 1$. Hence, by Lemma 4.14, f is in particular infinitely compactly differentiable. For $k = 1$, the map f is only a bounded polynomial in $\mathrm{Pol}^2(W^{l,2p}(\Omega), W^{l-1,p}(\Omega))$.

Remark 4.21. Proposition 4.19 extends directly to relatively compact open subsets Ω of manifolds. The latter extension is relevant, for example, if one uses periodic boundary conditions, so that one works with with functions on a torus.

5. Polynomial and differentiable maps on graded Fréchet spaces

Apart from polynomial and differentiable maps between normed vector spaces, we also use such maps between graded Fréchet spaces, defined in terms of nested sequences of Banach spaces, as in Definition 2.1. In this section, we discuss some further properties of such spaces, and in particular what it means for two sequences of Banach spaces defining such a space to be comparable.

5.1. **Properties of nested sequences of Banach spaces.** The definitions of bounded and compact polynomials and (compactly) differentiable maps between Banach spaces generalise directly to graded Fréchet spaces; see Definitions 5.2 and 5.3.

Let $\{V_k\}_{k=1}^\infty$ be a nested sequence of Banach spaces, as in Definition 2.1. Their intersection V_∞ is a graded Fréchet space.

Definition 5.1. The space $\mathcal{B}^n(V_\infty)$ of *bounded, n-multilinear operators on* V_∞ consists of the multilinear maps $\lambda\colon V_\infty \times \cdots \times V_\infty \to V_\infty$ (with n factors V_∞) such that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that λ extends continuously to a map in $\mathcal{B}^n(V_k, V_l)$.

Note that $\mathcal{B}(V_\infty) = \mathcal{B}^1(V_\infty)$.

Definition 5.2. We write $\mathrm{Pol}^n(V_\infty)$ for the space of all maps $p\colon V_\infty \to V_\infty$ such that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that p extends continuously to a polynomial in $\mathrm{Pol}^n(V_k, V_l)$. The space $\mathcal{K}\,\mathrm{Pol}^n(V_\infty)$ is defined analogously.

A feature of the spaces $\mathrm{Pol}^n(V_\infty)$ and $\mathcal{K}\,\mathrm{Pol}^n(V_\infty)$ that is useful to us, is that they admit natural compositions. If $p_1 \in \mathrm{Pol}^m(V_\infty)$ and $p_2 \in$

$\mathrm{Pol}^n(V_\infty)$, then Lemma 3.10 implies that $p_2 \circ p_1$ lies in $\mathrm{Pol}^{mn}(V_\infty)$. Similarly, Lemma 4.10 implies that $p_2 \circ p_1 \in \mathcal{K}\,\mathrm{Pol}^{mn}(V_\infty)$ if $p_1 \in \mathcal{K}\,\mathrm{Pol}^m(V_\infty)$ and $p_2 \in \mathcal{K}\,\mathrm{Pol}^n(V_\infty)$.

Definition 5.3. An n times *(compactly) differentiable map from* V_∞ *to itself* is a map $f\colon V_\infty \to V_\infty$ such that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that f extends to an n times (compactly) differentiable map from V_k to V_l.

Lemmas 3.2 and 4.15 imply that the spaces of differentiable and compactly differentiable maps from V_∞ to itself are closed under composition. Lemma 4.16 implies that if $\{V_k\}_{k=1}^\infty$ is a compactly nested sequence of Banach spaces, then all n times differentiable maps from V_∞ to itself are n times compactly differentiable.

It follows directly from Definition 2.4 that if $f, g\colon V_\infty \to V_\infty$ are of orders m and n, respectively, then $g \circ f$ is of order $m+n$. We also have the following lemma.

Lemma 5.4. *Consider two maps $f, g\colon V_\infty \to V_\infty$, where f is differentiable and g is of order m. Suppose that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $f\colon V_k \to V_l$ is differentiable, and $\|f'(v)\|_{\mathcal{B}(V_k,V_l)} = O(\|v\|_{V_k}^n)$ as $v \to 0$ in V_k. Then the map $f' \circ g\colon V_\infty \to V_\infty$, mapping $v \in V_\infty$ to $f'(v)(g(v))$ is of order $n+m$.*

Proof. Let $l \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $f\colon V_k \to V_l$ is differentiable, and its derivative satisfies the estimate in the lemma. Let $k' \in \mathbb{N}$ be such that $\|g(v)\|_{V_k} = O(\|v\|_{V_{k'}}^m)$ as $v \to 0$ in $V_{k'}$. Then $\|(f' \circ g)(v)\|_{V_l} = O(\|v\|_{V_{k'}}^{m+n})$ as $v \to 0$ in $V_{k'}$. □

An example of a situation where the condition on f in Lemma 5.4 is satisfied is the following.

Lemma 5.5. *Let $p \in \mathrm{Pol}^n(V_\infty)$, for $n \geq 2$. Then for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $\|p'(v)\|_{\mathcal{B}(V_k,V_l)} = O(\|v\|_{V_k}^{n-1})$ as $v \to 0$ in V_k.*

Proof. Let $p \in \mathrm{Pol}^n(V_\infty)$, and let $k, l \in \mathbb{N}$ be such that $p \in \mathrm{Pol}^n(V_k, V_l)$. As in the proof of Lemma 3.7, $p' \in \mathrm{Pol}^{n-1}(V_k, \mathcal{B}(V_k, V_l))$. So the claim follows from Lemma 3.8. □

Remark 5.6. Everything in this subsection generalises directly to polynomial and (compactly) differentiable maps between two different Fréchet spaces that are given as intersections of nested sequences of Banach spaces. We will not need this generalisation, however.

5.2. **Comparable sequences of Banach spaces.** In the rest of this section, we discuss some relevant properties and examples of comparable sequences of Banach spaces (Definition 2.6). Particularly relevant to Theorem 2.18 are sequences of Banach spaces comparable to sequences of separable Hilbert spaces, which we discuss in Section 5.3. We will see relevant examples in Section 5.4.

Suppose that $\{V_k\}_{k=1}^\infty$ and $\{W_k\}_{k=1}^\infty$ are comparable nested sequences of Banach spaces. Then $V_\infty = W_\infty$ as sets.

Lemma 5.7. *The two spaces V_∞ and W_∞ are equal as Fréchet spaces.*

Proof. Let $(v_j)_{j=1}^\infty$ be a sequence in V_∞ such that for every $k \in \mathbb{N}$, $\lim_{j\to\infty} \|v_j\|_{V_k} = 0$. Let $k \in \mathbb{N}$, and choose $l \in \mathbb{N}$ such that we have a bounded inclusion $V_l \subset W_k$. Then there is a constant $C > 0$ such that for every j, $\|v_j\|_{W_k} \leq C\|v_j\|_{V_l}$, which goes to zero as $j \to \infty$. □

The following lemma follows directly from the definitions, and the fact that the classes of maps in question are closed under composition with bounded linear maps.

Lemma 5.8. *If $f: V_\infty \to V_\infty$ is a (compact) polynomial map or a (compactly) differentiable map, then it also defines a map of the same type on W_∞.*

This lemma in particular states that $\mathrm{Pol}^n(V_\infty) = \mathrm{Pol}^n(W_\infty)$ as vector spaces. We will use the fact that this equality includes natural topologies on these spaces (Corollary 5.10 below) to prove Corollary 5.13.

Lemma 5.9. *For all $l \in \mathbb{N}$, there is an $l' \in \mathbb{N}$ such that for every $k' \in \mathbb{N}$ with $k' \geq l'$, there is a $k \in \mathbb{N}$ such that we have a bounded inclusion map $\mathcal{B}^n(V_{k'}, V_{l'}) \subset \mathcal{B}^n(W_k, W_l)$*

Proof. Let $l \in \mathbb{N}$. Choose $l' \in \mathbb{N}$ and $C_1 > 0$ such that for every $v \in V_\infty$, $\|v\|_{W_l} \leq C_1\|v\|_{V_{l'}}$. Let $k' \geq l'$. Choose $k \in \mathbb{N}$ and $C_2 > 0$ such that for every $v \in V_\infty$, $\|v\|_{V_{k'}} \leq C_2\|v\|_{W_k}$. Then for all $\lambda \in \mathcal{B}^n(V_{k'}, V_{l'})$,

$$\begin{aligned}\sup_{\|w_1\|_{W_k},\ldots,\|w_n\|_{W_k}\leq 1} &\|\lambda(w_1,\ldots,w_n)\|_{W_l} \\ &\leq C_1C_2^n \sup_{\|v_1\|_{V_{k'}},\ldots,\|v_n\|_{V_{k'}}\leq 1} \|\lambda(v_1,\ldots,v_n)\|_{V_{l'}} \\ &\leq C_2C_2^n\|\lambda\|_{\mathcal{B}^n(V_{k'},V_{l'})}.\end{aligned}$$

□

For a sequence $(p_j)_{j=1}^\infty$ in $\mathrm{Pol}^n(V_\infty)$, we define $p_j \to 0$ in $\mathrm{Pol}^n(V_\infty)$ to mean that for every $l \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $p_j \to 0$ in $\mathrm{Pol}^n(V_k, V_l)$. (This includes the requirement that $p_j \in \mathrm{Pol}^n(V_k, V_l)$ for every j.)

Corollary 5.10. *We have $\mathrm{Pol}^n(V_\infty) = \mathrm{Pol}^n(W_\infty)$, including topologies.*

Proof. Let $(p_j)_{j=1}^\infty$ be a sequence in $\mathrm{Pol}^n(V_\infty)$ converging to zero. Let $l \in \mathbb{N}$. Choose l' as in Lemma 5.9. Choose $k' \in \mathbb{N}$ such that $p_j \to 0$ in $\mathrm{Pol}^n(V_{k'}, V_{l'})$. Choose $k \in \mathbb{N}$ as in Lemma 5.9.

For each j, write $p_j = p_{\lambda_j}$, for $\lambda_j \in S\mathcal{B}^n(V_{k'}, V_{l'})$. By Lemma 5.9, there is a $C > 0$ such that for every j,

$$\|\lambda_j\|_{\mathcal{B}(W_k,W_l)} \leq C\|\lambda_j\|_{\mathcal{B}(V_{k'},V_{l'})},$$

which goes to zero as $j \to \infty$. Hence $p_j \to 0$ in $\mathrm{Pol}^n(W_\infty)$. □

5.3. **Sequences of Banach spaces comparable to sequences of Hilbert spaces.** As before, we suppose that $\{V_k\}_{k=1}^\infty$ and $\{W_k\}_{k=1}^\infty$ are comparable nested sequences of Banach spaces. Now we make the additional assumption that the spaces W_k are separable Hilbert spaces. Suppose that $\{e_j\}_{j=1}^\infty$ is a subset of W_∞ that is orthogonal in all spaces W_k, with dense span. Then taking inner products with e_j defines bounded functionals, all denoted by e^j, on all spaces W_k, and hence in V_k for k large enough.

One of the main reasons why we introduced compactly differentiable maps on sequences of Banach spaces is the following version of Taylor's theorem. This is a key technical tool in our proof of Theorem 2.18: we use this to find the maps a^q in (6.2).

Corollary 5.11. *Suppose f is an $n+1$ times differentiable map from V_∞ to itself, and that f is m times compactly differentiable for every $m \leq n$. Then there are unique complex numbers a^k_q such that*

$$f(u+h) = \sum_{q \in \mathbb{N}_0^\infty;\, |q| \leq n} \sum_{j=1}^{\infty} a^j_q h^q e_j + \rho(h), \tag{5.1}$$

where $\rho\colon V_\infty \to V_\infty$ is of order $n+1$, and the part of the sum where $|q| = m$ converges as a function of h in $\mathrm{Pol}^m(V_\infty)$, for $m = 0, \dots, n$.

Proof. Let $l \in \mathbb{N}$. Choose $k, k', l' \in \mathbb{N}$ be such that we have bounded inclusions $V_{l'} \subset W_l$ and $W_k \subset V_{k'}$, and $f\colon V_{k'} \to V_{l'}$ is $n+1$ times differentiable, and m times compactly differentiable for every $m \leq n$. Then the same is true for $f\colon W_k \to W_l$. So Corollary 4.13 implies that (5.1) holds, for unique a^j_q, where the sum converges in $\mathrm{Pol}(W_\infty)$, and hence in $\mathrm{Pol}(V_\infty)$ by Corollary 5.10, and ρ is of order $n+1$ as a map from W_∞ to itself, and hence as a map from V_∞ to itself. □

Remark 5.12. A useful feature of Corollary 5.11 is that it is not assumed that the spaces V_k have the approximation property. The point is that the separable Hilbert spaces W_k do have this property.

The following corollary is an important way in which we use comparable sequences of Banach spaces. It is used in the proof of Lemma 7.7.

Corollary 5.13. *Let $a^j_q \in \mathbb{C}$ be given such that*

$$\sum_{q \in \mathbb{N}_0^\infty;\, |q| = n} \sum_{j=1}^{\infty} a^j_q p^q \otimes e_j \tag{5.2}$$

converges in $\mathrm{Pol}^n(V_\infty)$. Then for all subsets $A \subset \{q \in \mathbb{N}_0^\infty : |q| = n\} \times \mathbb{N}$, the series

$$\sum_{(q,j) \in A} a^j_q p^q \otimes e_j \tag{5.3}$$

converges in $\mathrm{Pol}^n(V_\infty)$.

Proof. By Corollary 5.10, the series (5.2) converges in $\mathrm{Pol}^n(W_\infty)$, and it is enough to show that (5.3) converges in $\mathrm{Pol}^n(W_\infty)$. And that follows from Lemma 4.9. □

Remark 5.14. In Corollary 5.13, the two series converge to elements of $\mathcal{K}\,\mathrm{Pol}^n(V_\infty)$.

5.4. **Example: Sobolev spaces and C^k-spaces.** Let Ω be a bounded open subset of $\mathbb{R}^d$ or of an d-dimensional Riemannian manifold, and suppose that the boundary of Ω is C^1. Set

$$V_k := W^{k-1,2}(\Omega) \quad \text{and} \quad W_k := W^{k-1,k+1}(\Omega).$$

Lemma 5.15. *The above sequences $\{V_k\}_{k=1}^{\infty}$ and $\{W_k\}_{k=1}^{\infty}$ of Banach spaces are comparable.*

Proof. Set $r := \lceil d/2 \rceil$. By a Sobolev embedding theorem, we have bounded inclusions

$$W^{l+\frac{d}{p}(1-p/2),p}(\Omega) \subset W^{l,2}(\Omega) \quad \text{and} \quad W^{l+r,2}(\Omega) \subset W^{l+r-\frac{d}{2}(1-2/q),q}(\Omega),$$

for all $1 \leq p < 2 < q < \infty$ and every $l \in \mathbb{N}_0$. Now for all such p, q and l,

$$l + \frac{d}{p}(1-p/2) < l + r \quad \text{and} \quad l + r - \frac{d}{2}(1-2/q) > l \geq 1.$$

So we have bounded inclusions

$$W^{l+r,p}(\Omega) \subset W^{l,2}(\Omega) \quad \text{and} \quad W^{l+r,2}(\Omega) \subset W^{l,q}(\Omega).$$

Furthermore, since Ω has finite volume, for every l and all $p' \geq p''$ we have bounded inclusions $W^{l,p'}(\Omega) \subset W^{l,p''}(\Omega)$.

The above arguments imply that we have bounded inclusions

$$W_{k+r} \subset V_k \subset W_1 \quad \text{and} \quad V_{k+r} \subset W_k \subset V_k.$$

□

Remark 5.16. In this example, the spaces V_k are separable Hilbert spaces.

For another example of comparable sequences of Banach spaces, fix $p > n$. For $k \in \mathbb{N}$, set

$$V_k := W^{k,p}(\Omega) \quad \text{and} \quad W_k := C^k(\overline{\Omega}).$$

The space $C^k(\overline{\Omega})$ is complete in the norm given by the maximum of the sup-norms of the partial derivatives of functions up to order k.

Lemma 5.17. *The sequences $\{V_k\}_{k=1}^{\infty}$ and $\{W_k\}_{k=1}^{\infty}$ of Banach spaces are comparable.*

Proof. We have a bounded inclusion $W_k \subset V_k$ for every k. So it remains to show that for every $k \in \mathbb{N}$, there are $l_1, l_2 \in \mathbb{N}$ such that we have bounded inclusions

$$\begin{aligned} V_{l_1} &\subset W_k; \\ V_k &\subset W_{l_2}. \end{aligned} \tag{5.4}$$

By a Sobolev embedding theorem, we have a bounded inclusion

$$W^{k,p}(\Omega) \subset C^l(\overline{\Omega})$$

for all $k, l \in \mathbb{N}$ such that

$$l + \frac{n}{p} < k \leq l + 1 + \frac{n}{p}.$$

For $k \in \mathbb{N}$, set $l_1 := k + 1 + \lceil n/p \rceil$ and $l_2 := \max\{k - 1 - \lfloor n/p \rfloor, 1\}$. Then this Sobolev embedding theorem yields the desired inclusions (5.4). □

Remark 5.18. If $n = 1$, then we may take $p = 2$, so that the spaces V_k are Hilbert spaces.

Proposition 4.19 and Lemma 5.15 together imply Theorem 2.22. The extension of Proposition 4.19 to coeficients a_q depending on a real (time) parameter t in a smooth way, and the extension of Lemma 5.15 to vector-valued functions, are straightforward.

6. A COORDINATE TRANSFORM

6.1. **A residual.** Recall the setting of Section 2.2. In this section and the next, based upon the details of some given dynamical system (2.2) we construct both a coordinate transformation (2.6) and a corresponding 'normal form' system (2.7), such that solutions X to (2.7), transformed by (2.6), satisfy the given dynamical system (2.2) up to residuals of a specified order p. See Theorem 2.18. We do this inductively, by showing how to construct such a transformed system to satisfy (2.2) with residual of order $p + 1$ from a version with residual of order p.

In Section 7, we construct a more specific choice of the general coordinate transform constructed in this section, in order to establish exact invariant manifolds, and study their properties, for constructed systems arbitrarily close to the given system (2.2).

Remark 6.1. In Section 2.2, we assumed that the sequence $\{V_k\}_{k=1}^{\infty}$ is comparable to a nested sequence $\{W_k\}_{k=1}^{\infty}$ of separable Hilbert spaces in which the vectors e_j are orthogonal. Lemma 5.8 implies that we may equivalently assume that $\{V_k\}_{k=1}^{\infty}$ itself is a nested sequence of separable Hilbert spaces, because all maps from V_∞ to itself we use transfer to maps from W_∞ to itself of the same type (e.g. compact polynomial and compactly differentiable maps). However, the formulation where $\{V_k\}_{k=1}^{\infty}$ is only comparable to a nested sequence of separable Hilbert spaces makes it clearer that we have the flexibility to consider maps between Banach spaces. This is natural for example in the context of Proposition 4.19.

Let $p \in \mathbb{N}$, with $p \geq 2$. Let $\xi_p, F_p \colon I \times V_\infty \to V_\infty$ be such that $\xi_p - \mathrm{id}$ and F_p are compact polynomial maps of order at most $p - 1$ in the V_∞ component, and infinitely differentiable in I. Suppose, furthermore, that ξ_p is a near-identity at zero, and that $F_p = \mathcal{O}(2)$.

Recall that our goal is to relate the dynamics of maps x satisfying (2.2) to the dynamics of maps $X \colon I \to V_\infty$ satisfying (2.7) when x and X are related by the coordinate transform ξ_p as in (2.6).

For maps $f, g \colon I \times V_\infty \to V_\infty$, with f differentiable, we write $f'_{V_\infty} \circ g$ for the map from $I \times V_\infty$ to V_∞ given by

$$(f'_{V_\infty} \circ g)(t, v) = f'_{V_\infty}(t, v)(g(t, v)),$$

for all $t \in I$ and $v \in V_\infty$. (Note that this is different from $(f \circ g)'_{V_\infty}(t, v) = f'_{V_\infty}(t, g(t, v))(g(t, v))$.) If g is a map from V_∞ to V_∞ to itself, then the composition $f'_{V_\infty} \circ g$ is defined analogously. Also recall the notation for compositions of maps to and from $I \times V_\infty$ and V_∞ under *Notation and conventions* in Section 1.5.

Define the maps $\Phi_p, R_p \colon I \times V_\infty \to V_\infty$ by

$$\Phi_p := (\xi_p)'_I + (\xi_p)'_{V_\infty} \circ (A + F_p) \quad \text{and} \quad R_p := -A \circ \xi_p - f \circ \xi_p + \Phi_p.$$

The map R_p is the *residual* of the transformed ODE, in the following sense.

Lemma 6.2. *For all smooth maps* $X \colon I \to V_\infty$ *satisfying* (2.7)*, and with* $x \colon I \to V_\infty$ *determined from* X *by* (2.6)*,*

$$\dot{x}(t) = Ax(t) + f(t, x(t)) + R_p(t, X(t)). \tag{6.1}$$

Proof. For X and x as in the lemma, the chain rule (Lemma 3.2) and (2.7) imply that for all $t \in I$,

$$\dot{x}(t) = \Phi_p(t, X(t)) = Ax(t) + f(t, x(t)) + R_p(t, X(t)).$$

□

Lemma 6.3. *The maps* Φ_p *and* R_p *are infinitely compactly differentiable.*

Proof. Because the Banach spaces V_k are compactly nested, it is enough to show that Φ_p and R_p are infinitely differentiable. And that is true by the chain rule, because f is infinitely differentiable, and so are F_p and ξ_p, by Lemma 3.7. □

To recursively construct (2.7) and (2.6), suppose that $R_p = \mathcal{O}(p)$. We proceed to show that (2.7) and (2.6) may be refined to make the new residual of $\mathcal{O}(p+1)$. By Lemma 6.3, Corollary 5.11 (where $u = 0$ and $h = v$), and Lemmas 3.8 and 3.9, there are unique, infinitely differentiable maps

$$a^q \colon I \to V_\infty$$

for all multi-indices $q \in \mathbb{N}_0^\infty$ with $|q| = p$, and a map

$$\rho_p \colon I \times V_\infty \to V_\infty$$

such that for all $t \in I$, and $v \in V_\infty$,

$$R_p(t, v) = -\sum_{q \in \mathbb{N}_0^\infty;\, |q| = p} v^q a^q(t) + \rho_p(t, v), \tag{6.2}$$

where the sum converges in $\mathrm{Pol}^p(V_\infty)$, differentiably in t, and $\rho_p = \mathcal{O}(p+1)$. The monomial v^q is defined as in (3.7), with respect to the set of functionals $e^j = (e_j, -)_{V_1}$, for $j \in \mathbb{N}$.

In Section 6.2, we construct maps ξ_{p+1}, F_{p+1} such that the order p term $R_p(t, X(t))$ in (6.1) may be replaced by an order $p+1$ term $R_{p+1}(t, X(t))$, if (2.7) and (2.6) hold with p replaced by $p+1$.

6.2. **Construction of the coordinate transform.** For each $j \in \mathbb{N}$, let α_j be the eigenvalue of A corresponding to e_j. For all $q \in \mathbb{N}_0^\infty$ with $|q| = p$, define $\mu^q \in \mathbb{C}$ by

$$\mu^q = \sum_{j=1}^{\infty} q_j \alpha_j \quad \in \mathbb{C}. \tag{6.3}$$

(This sum has at most p nonzero terms.) For such $q \in \mathbb{N}_0^\infty$, let a^q be as in (6.2). Let $\hat{\xi}^q, \hat{F}^q \colon I \to V_\infty$ be smooth maps such that

$$\hat{F}^q + (\hat{\xi}^q)' + \mu^q \hat{\xi}^q - A\hat{\xi}^q = a^q. \tag{6.4}$$

Suppose that the sums

$$\hat{F}(t,\nu) = \sum_{q\in\mathbb{N}_0^\infty;|q|=p} \nu^q \hat{F}^q(t) \quad \text{and} \quad \hat{\xi}(t,\nu) = \sum_{q\in\mathbb{N}_0^\infty;|q|=p} \nu^q \hat{\xi}_q(t)$$

converge in $\mathrm{Pol}^p(V_\infty)$, differentiably in t.

Define a new coordinate transform map $\xi_{p+1}\colon I \times V_\infty \to V_\infty$ and corresponding map $F_{p+1}\colon I \times V_\infty \to V_\infty$ that replaces F_p in (2.7), by

$$\xi_{p+1} = \xi_p + \hat{\xi} \quad \text{and} \quad F_{p+1} = F_p + \hat{F}. \tag{6.5}$$

The following result is the main step in the construction of the coordinate transform we are looking for.

Proposition 6.4. *If X and x are as in (2.7) and (2.6), with p replaced by $p+1$, then (6.1) holds, with the residual R_p replaced by a residual R_{p+1} satisfying*

$$R_{p+1} = \mathcal{O}(p+1).$$

The maps $\xi_{p+1} - \mathrm{id}$ and F_{p+1} are compact polynomials in $\mathcal{K}\,\mathrm{Pol}(V_\infty)$ of order at most p, and ξ_{p+1} is a near-identity.

Remark 6.5. The maps $\hat{\xi}^q$ and $\hat{F}^q$ can be found explicitly if we decompose (6.4) with respect to the basis $\{e_j\}_{j=1}^\infty$. This will be done in Section 7. One solution to (6.4) is $\hat{F}^q = a^q$ and $\hat{\xi}^q = 0$. However, for our purposes, we need the function F^q to be of a specific form. The main purpose of this work is to find $\hat{F}^q$ such that the e_j-component of F^q is zero for certain combinations of q and j, in such a way that an exact separation of stable, centre and unstable modes is maintained. See Proposition 7.1.

6.3. **Proof of Proposition 6.4.** Define the map $\hat{\Phi}\colon I \times V_\infty \to V_\infty$ by

$$\hat{\Phi} = \hat{\xi}'_I + \hat{\xi}'_{V_\infty} \circ (A + F_p) + (\xi_p)'_{V_\infty} \circ \hat{F} + \hat{\xi}'_{V_\infty} \circ \hat{F}. \tag{6.6}$$

Lemma 6.6. *We have*

$$\hat{\Phi} - (\hat{\xi}'_I + \hat{\xi}'_{V_\infty} \circ A + \hat{F}) = \mathcal{O}(p+1). \tag{6.7}$$

Proof. The left-hand side of (6.7) equals

$$\hat{\xi}'_{V_\infty} \circ F_p + \hat{\xi}'_{V_\infty} \circ \hat{F} + ((\xi_p)'_{V_\infty} - \mathrm{id}) \circ \hat{F}.$$

By Lemma 5.5, the derivative $\hat{\xi}'_{V_\infty}$ satisfies the condition of Lemma 5.4, with $n = p-1$. Since $F_p = \mathcal{O}(2)$, Lemma 5.4 implies that $\hat{\xi}'_{V_\infty} \circ F_p = \mathcal{O}(p+1)$. Similarly, $\hat{\xi}'_{V_\infty} \circ \hat{F} = \mathcal{O}(2p-1)$. Now ξ_p is a polynomial map, and a near-identity. By Lemma 5.5, this implies that $(\xi_p)'_{V_\infty} - \mathrm{id}$ satisfies the condition of Lemma 5.4, with $n = 1$. So Lemma 5.4 implies that $((\xi_p)'_{V_\infty} - \mathrm{id}) \circ \hat{F} = \mathcal{O}(p+1)$. □

Lemma 6.7. *For $q \in \mathbb{N}_0^\infty$ such that $|q| = p$, let a^q be as in (6.2). Then for all $t \in I$ and $\nu \in V_\infty$,*

$$\hat{\xi}'_I(t,\nu) + \hat{\xi}'_{V_\infty}(t,\nu)Au + \hat{F}(t,\nu) = A\hat{\xi}(t,\nu) + \sum_{q\in\mathbb{N}_0^\infty;|q|=p} \nu^q a^q(t). \tag{6.8}$$

Proof. First of all, $\hat{\xi}_I'(t,v) = \sum_{q\in\mathbb{N}_0^\infty;|q|=p} v^q(\hat{\xi}^q)'(t)$. By Lemma 3.11,

$$\hat{\xi}_{V_\infty}'(t,v)Av = \sum_{q\in\mathbb{N}_0^\infty;|q|=p} \Big(\sum_{j=1}^\infty q_j(e_j,v)_{V_1}^{q_j-1}(e_j,Av)_{V_1}\prod_{j'\neq j}(e_{j'},v)_{V_1}^{q_{j'}}\Big)\hat{\xi}^q(t).$$

Now, because the vectors $\{e_j\}_{j=1}^\infty$ are orthogonal with respect to $(-,-)_{V_1}$, we have $(e_j,Av)_V = \alpha_j(e_j,v)_{V_1}$ for every j. So

$$\hat{\xi}_{V_\infty}'(t,v)Av = \sum_{q\in\mathbb{N}_0^\infty;|q|=p}\Big(\sum_{j=1}^\infty q_j\alpha_j\Big)v^q\hat{\xi}^q(t).$$

We find that the left-hand side of (6.8) equals

$$\sum_{q\in\mathbb{N}_0^\infty;|q|=p} v^q\Big((\hat{\xi}^q)'(t)+\hat{F}^q(t)+\mu^q\hat{\xi}^q(t)\Big),$$

with μ^q as in (6.3). So the claim follows from (6.4). □

Define the maps $\Phi_{p+1}, R_{p+1}\colon I\times V_\infty \to V_\infty$ by

$$\Phi_{p+1} := \Phi_p + \hat{\Phi} \quad \text{and} \quad R_{p+1} := -A\circ\xi_{p+1} - f\circ\xi_{p+1} + \Phi_{p+1}.$$

Lemma 6.8. *The residual* R_{p+1} *satisfies* $R_{p+1} = \mathcal{O}(p+1)$.

Proof. By Lemmas 6.6 and 6.7,

$$\begin{aligned} R_{p+1} &= -A\circ\xi_p - f\circ\xi_{p+1} + \Phi_p - \tilde{R}_p + \mathcal{O}(p+1) \\ &= R_p - f\circ\xi_{p+1} + f\circ\xi_p - \tilde{R}_p + \mathcal{O}(p+1), \end{aligned} \tag{6.9}$$

where, for $t\in I$ and $v\in V_\infty$,

$$\tilde{R}_p(t,v) := -\sum_{q\in\mathbb{N}_0^\infty;|q|=p} v^q a^q(t).$$

By (6.2), the last expression in (6.9) equals

$$f\circ\xi_{p+1} - f\circ\xi_p + \mathcal{O}(p+1).$$

Let $l\in\mathbb{N}$ be given, and choose $k,k'\in\mathbb{N}$ such that $f\colon I\times V_{k'}\to V_l$ is differentiable, and $\xi_p,\xi_{p+1}\in \mathrm{Pol}(V_k,V_{k'})$. Using Theorem 3.4, we write

$$\big\|f(t,\xi_{p+1}(t,v)) - f(t,\xi_p(t,v)) - f'(t,\xi_p(t,v))\hat{\xi}(t,v)\big\|_{V_l} = O(\|\hat{\xi}(t,v)\|^2),$$

for $t\in I$ and $v\in V_{k'}$. Lemma 3.8 implies that $\|\hat{\xi}(t,v)\|_{V_{k'}} = O(\|v\|_{V_k})$ uniformly in t in compact sets. So

$$\big\|f(t,\xi_{p+1}(t,v)) - f(t,\xi_p(t,v))\big\|_{V_l} = \big\|f'(t,\xi_p(t,v))\hat{\xi}(t,v)\big\|_{V_l} + O(\|v\|_{V_k}^{2p}),$$

uniformly in t in compact sets. And then, as in the proof of Lemma 5.4, the assumption (2.1) and Lemma 5.4 imply that $\|f'(t,\xi_p(t,v))\hat{\xi}(t,v)\|_{V_l} = O(\|v\|_{k'}^{p+1})$, uniformly in t in compact sets. □

Proof of Proposition 6.4. The correction terms $\hat{\xi}$ and $\hat{F}$ lie in $\mathcal{K}\,\mathrm{Pol}^p(V_\infty)$. Hence $\xi_{p+1} - \mathrm{id}$ and F_{p+1} are compact polynomials, because $\xi_p - \mathrm{id}$ and F_p are. By Lemma 3.8, this also implies that ξ_{p+1} is a near-identity because ξ_p is. The desired property of R_{p+1} is Lemma 6.8. □

7. CENTRE, STABLE AND UNSTABLE COORDINATES

There is considerable flexibility in choosing the maps $\hat{\xi}^q$ and $\hat{F}^q$ in Section 6.2. In this section, we discuss how to make specific choices, in terms of the eigenvalues of A, so that the normal form (2.7) is useful for detecting invariant manifolds.

7.1. Centre, stable and unstable components of $\hat{F}^q$.

We use the notation from Section 2.5. In particular, let α, β, γ and $\tilde{\mu}$ be spectral gap parameters defined there. Recall the definition of polynomial growth in Definition 2.17.

Proposition 7.1. *Suppose that $\beta-(p+1)\alpha > \tilde{\mu}$ and $\gamma-(p+1)\alpha > \tilde{\mu}$. Suppose that R_p has polynomial growth. The maps $\hat{\xi}^q$ and $\hat{F}^q$ in Section 6.2 can be chosen such that*

- *if either $q^s = 0$ and $q^u \neq 0$, or $q^u = 0$ and $q^s \neq 0$, then $\hat{F}^q_c = 0$;*
- *if $q^s = 0$, then $\hat{F}^q_s = 0$; and*
- *if $q^u = 0$, then $\hat{F}^q_u = 0$.*

Proposition 7.1 is proved in Section 7.2, after some preparation done in this subsection.

In the construction of the coordinate transform ξ_p in Theorem 2.18, we use the following convolution operation to construct explicit solutions to the ODE (6.4).

Definition 7.2. Let $\mu \in \mathbb{C}$, such that $|\Re(\mu)| > \tilde{\mu}$. Set $a := \inf I$ and $b := \sup I$. Let u be a continuous function on $\mathbb{R}$ such that $u(t) = O(e^{\tilde{\mu}|t|})$ as $t \to \infty$ if $b = \infty$ and as $t \to -\infty$ if $a = -\infty$. Then we define the function $e^{\mu(\cdot)} \star u$ on I by

$$(e^{\mu(\cdot)} \star u)(t) := \begin{cases} \int_a^t e^{\mu(t-\tau)}u(\tau)\,d\tau & \text{if } \Re(\mu) < -\tilde{\mu}; \\ \int_t^b e^{\mu(t-\tau)}u(\tau)\,d\tau & \text{if } \Re(\mu) > \tilde{\mu}. \end{cases}$$

The integrals occurring in this definition are $\tilde{\mu}$-regular, in the sense defined in Section 2.5.

Lemma 7.3. *In the setting of Definition 7.2,*

$$(e^{\mu(\cdot)} \star u)' = \mu(e^{\mu(\cdot)} \star u) - \operatorname{sgn}(\Re(\mu))u.$$

Proof. This is a straightforward computation. □

Lemma 7.4. *Let $u\colon I \to \mathbb{C}$ be a smooth function, and suppose that u and all its derivatives grow at most polynomially. Then, for every μ as in Definition 7.2, $e^{\mu(\cdot)} \star u$ and all its derivatives grow at most polynomially.*

Proof. First of all, because $(e^{\mu(\cdot)} \star u)^{(k)} = e^{\mu(\cdot)} \star (u^{(k)})$, it is enough to consider the function u itself rather than all its derivatives.

Let $C > 0$ and $n \in \mathbb{N}_0$ be such that for all $t \in I$, $|u(t)| \leq C(1+|t|^n)$. We prove by induction on n that there is a constant $C' > 0$ such that for all $t \in I$, $|(e^{\mu(\cdot)} \star u)(t)| \leq C'(1+|t|^n)$. We consider the case where $\Re(\mu) < -\tilde{\mu}$; the case where $\Re(\mu) > \tilde{\mu}$ is similar.

If $n = 0$, then for all $t \in I$,

$$\left|(e^{\mu(\cdot)} \star u)(t)\right| \leq 2C \int_{-\infty}^t e^{\mu(t-\tau)}\,d\tau = \frac{-2C}{\mu}.$$

Suppose that the claim holds for n, and suppose that $|u(t)| \leq C(1+|t|^{n+1})$ for a constant C. Using integration by parts, we find that

$$\begin{aligned} \left|(e^{\mu(\cdot)} \star u)(t)\right| &\leq C \int_{-\infty}^{t} e^{\mu(t-\tau)}(1+|\tau|^{n+1})\, d\tau \\ &= \frac{-C}{\mu}\Big(1+|t|^{n+1} - (n+1)\int_{-\infty}^{t} e^{\mu(t-\tau)} \operatorname{sgn}(\tau)|\tau|^{n}\, d\tau\Big), \end{aligned}$$

which implies the claim by the induction hypothesis. □

Let $q \in \mathbb{N}_0^\infty$, with $|q| = p$. Consider the differentiable maps

$$a_j^q \colon I \to \mathbb{C} \quad \text{such that} \quad a^q(t) = \sum_{j=1}^{\infty} a_j^q(t) e_j,$$

where the sum converges in V_1, uniformly and differentiably in t in compact sets in I.

For $q \in \mathbb{N}_0^\infty$ with $|q| = p$, let $J^q \subset \mathbb{N}$ be the set of $j \in \mathbb{N}$ such that either

- $j \in J_c$ and either $q^s = 0$ and $q^u \neq 0$, or $q^u = 0$ and $q^s \neq 0$;
- $j \in J_s$ and $q^s = 0$; or
- $j \in J_u$ and $q^u = 0$.

For $q \in \mathbb{N}_0^\infty$ with $|q| = p$, and $j \in \mathbb{N}$, write $\mu_j^q := \mu^q - \alpha_j$, with μ^q as in (6.3).

Lemma 7.5. *If* $\beta - (p+1)\alpha > \tilde{\mu}$ *and* $\gamma - (p+1)\alpha > \tilde{\mu}$, *then for every* $j \in J^q$, $|\Re(\mu_j^q)| > \tilde{\mu}$.

Proof. If $q^s = 0$ and $q^u \neq 0$, and $j \in J_c$, then

$$\Re(\mu_j^q) = \sum_{k \in J_c} q_k \Re(\alpha_k) + \sum_{k \in J_u} q_k \Re(\alpha_k) - \Re(\alpha_j) \geq \gamma - (p+1)\alpha > \tilde{\mu}.$$

If $q^u = 0$ and $q^s \neq 0$, and $j \in J_c$, then

$$\Re(\mu_j^q) = \sum_{k \in J_c} q_k \Re(\alpha_k) + \sum_{k \in J_s} q_k \Re(\alpha_k) - \Re(\alpha_j) \leq -\beta + (p+1)\alpha < -\tilde{\mu}.$$

If $q^s = 0$, and $j \in J_s$, then

$$\Re(\mu_j^q) = \sum_{k \in J_c} q_k \Re(\alpha_k) + \sum_{k \in J_u} q_k \Re(\alpha_k) - \Re(\alpha_j) \geq \beta - p\alpha > \tilde{\mu}.$$

And if $q^u = 0$, and $j \in J_u$, then

$$\Re(\mu_j^q) = \sum_{k \in J_c} q_k \Re(\alpha_k) + \sum_{k \in J_s} q_k \Re(\alpha_k) - \Re(\alpha_j) \leq -\gamma + p\alpha < \tilde{\mu}.$$

□

7.2. **Update terms for ξ_p and** F_p**.** Suppose that R_p has polynomial growth. Then the functions a_j^q and their derivatives grow at most polynomially, uniformly in q and j.

For every $q \in \mathbb{N}_0^\infty$ with $|q| = p$ and $j \in \mathbb{N}$, consider the ODE for $\hat{\xi}_j^q$ and $\hat{F}_j^q$

$$\hat{F}_j^q + (\hat{\xi}_j^q)' + \mu_j^q \hat{\xi}_j^q = a_j^q. \tag{7.1}$$

Define the maps $\hat{\xi}_j^q, \hat{F}_j^q : I \to \mathbb{C}$ as follows. If $j \in J^q$, then

$$\hat{\xi}_j^q = \operatorname{sgn}(\Re(\mu_j^q))e^{-\mu_j^q(\cdot)} \star a_j^q \quad \text{and} \quad \hat{F}_j^q = 0. \tag{7.2}$$

This definition makes sense because of Lemma 7.5 and the growth behaviour of the functions a_j^q. If $j \notin J^q$, then we set

$$\hat{\xi}_j^q = 0 \quad \text{and} \quad \hat{F}_j^q = a_j^q. \tag{7.3}$$

Lemma 7.6. *With the above definitions, the* ODE (7.1) *is satisfied for all* q *and* j.

Proof. If $j \notin J^q$, the statement is immediate from the definitions. If $j \in J^q$, it follows from Lemma 7.3. □

Lemma 7.7. *Suppose that* $\beta - (p+1)\alpha > \tilde{\mu}$ *and* $\gamma - (p+1)\alpha > \tilde{\mu}$. *Then the sums*

$$\hat{F}(t, \nu) = \sum_{q \in \mathbb{N}_0^\infty; |q|=p} \sum_{j \in \mathbb{N}} \nu^q \hat{F}_j^q(t) e_j \quad \textit{and} \quad \hat{\xi}(t, \nu) = \sum_{q \in \mathbb{N}_0^\infty; |q|=p} \sum_{j \in \mathbb{N}} \nu^q \hat{\xi}_j^q(t) e_j \tag{7.4}$$

converge in $\mathrm{Pol}^p(V_\infty)$, *differentiably in* t.

Proof. The first sum in (7.4) equals

$$\sum_{q \in \mathbb{N}_0^\infty; |q|=p} \sum_{j \in \mathbb{N} \setminus J^q} \nu^q a_j^q(t) e_j. \tag{7.5}$$

Since $\{V_k\}_{k=1}^\infty$ is comparable to a nested sequence of separable Hilbert spaces in which the set $\{e_j\}_{j=1}^\infty$ is orthogonal, Corollary 5.13 implies that this series converges in $\mathrm{Pol}^p(V_\infty)$.

Write $J_\pm^q := \{j \in J^q : \pm\Re(\mu_j^q) > \tilde{\mu}\}$. Lemma 7.5 states that $J^q = J_+^q \cup J_-^q$. So the second sum in (7.4) equals

$$\sum_{q \in \mathbb{N}_0^\infty; |q|=p} \nu^q \sum_{j \in J_+^q} \int_{-\infty}^t e^{-\mu_j^q(t-\tau)} a_j^q(\tau)\, d\tau\, e_j \\ + \sum_{q \in \mathbb{N}_0^\infty; |q|=p} \nu^q \sum_{j \in J_-^q} \int_t^\infty e^{-\mu_j^q(t-\tau)} a_j^q(\tau)\, d\tau\, e_j. \tag{7.6}$$

Tonelli's theorem implies that convergence of the first of these sums is equivalent to convergence of

$$\int_{-\infty}^t e^{-\mu_j^q(t-\tau)} \Big(\sum_{q \in \mathbb{N}_0^\infty; |q|=p} \nu^q \sum_{j \in J_+^q} a_j^q(\tau) e_j \Big)\, d\tau \tag{7.7}$$

The sum inside the brackets converges in $\mathrm{Pol}^p(V_\infty)$, uniformly in τ, by convergence of (6.2) and Corollary 5.13. Since the functions a_j^q grow at most polynomially, uniformly in q and j, the value of that sum grows at most polynomially as well, when viewed as a convergent series in $\mathrm{Pol}^p(V_k, V_l)$. So the integral over τ converges in $\mathrm{Pol}^p(V_\infty)$, by completeness of the spaces $\mathrm{Pol}^p(V_k, V_l)$. By continuity of (7.7) in t, the convergence is uniform in t on compact subsets of I. The derivatives of (7.7) with respect to t are linear combinations of (7.7) and

$$\sum_{q\in\mathbb{N}_0^\infty;|q|=p} \nu^q \sum_{j\in J_+^q} a_j^q(t)e_j$$

and therefore converge as well.

By an analogous argument, the second sum in (7.6) converges as well, differentiably in t. □

Proposition 7.1 follows from Lemmas 7.6 and 7.7.

7.3. Proof of Theorem 2.18.

Lemma 7.8. *If f, ξ_p, F_p, Φ_p and R_p have polynomial growth, then so do ξ_{p+1}, F_{p+1}, Φ_{p+1} and R_{p+1}.*

Proof. If R_p has polynomial growth, then the functions a_j^q and all their derivatives grow at most polynomially, uniformly in q and j. Hence, by (7.2) and (7.3), the map $\hat{F}$ has polynomial growth. By Lemma 7.4, the same is true for $\hat{\xi}$. So F_{p+1} and ξ_{p+1} have polynomial growth.

Polynomial growth is preserved under composition and derivatives in the I and V_∞ directions. Hence the map $\hat{\Phi}$ as in (6.6) has polynomial growth, and therefore so do Φ_{p+1} and R_{p+1}. □

Combining Lemma 7.8 with Propositions 6.4 and 7.1, we prove the following slightly more explicit version of Theorem 2.18.

Theorem 7.9. *Let $p \in \mathbb{N}$ be such that $p \geq 2$, $\beta-(p+1)\alpha > \tilde{\mu}$ and $\gamma-(p+1)\alpha > \tilde{\mu}$. Suppose that f has polynomial growth. Then there are infinitely differentiable maps*

$$F_p, \xi_p, R_p \colon I \times V_\infty \to V_\infty,$$

where $R_p = \mathcal{O}(p)$, F_p is a polynomial map that separates invariant subspaces, ξ_p is a near-identity and $\xi_p - \mathrm{id}$ and F_p are compact polynomials of orders at most $p-1$, such that if X and x are as in (2.7) and (2.6), then (6.1) holds. Finally, there is a construction of the map ξ_p in which all integrals over I that occur are $\tilde{\mu}$-regular.

Proof. We use induction on p to prove that the claim holds for every p, including the auxiliary statement that ξ_p, F_p, Φ_p and R_p have polynomial growth.

If $p = 2$, then we may take $F_p(t,\nu) = 0$ and $\xi_p(t,\nu) = \nu$ for all $t \in I$ and $\nu \in V_0$. Then $R_2 = f$, so ξ_p, F_p, Φ_p and R_p have polynomial growth because f does.

The induction step follows from Lemma 7.8 and Propositions 6.4 and 7.1. □

8. Dynamics of the normal form equation

It remains to prove Lemma 2.9 and Proposition 2.10, which we use to justify Definition 2.11 based on Theorem 2.18. Throughout this section, we suppose that $F \colon I \times V_\infty \to V_\infty$ is a smooth map that separates invariant subspaces.

Proof of Lemma 2.9. First, suppose that $a = c$. For all $v \in V_c$ and all $q \in \mathbb{N}_0^\infty$ with $|q| \leq p$ and $q^s \neq 0$ or $q^u \neq 0$, we have $v^q = 0$. So the properties (2.3) of the map F imply that $F(I \times V_c) \subset V_c$. This, in turn, implies that for all maps $X\colon I \to V_\infty$ satisfying (2.7), if $X(t) \in V_c$ for a given t then $\dot{X}(t) \in V_c$. So $X(t) \in V_c$ for all $t \in I$.

Next, suppose that $a = s$. If $v \in V_s$ and $q \in \mathbb{N}_0^\infty$, then $v^q = 0$ if $q^u \neq 0$. So $F_u(t, v) = 0$ for all $t \in I$. And the components of $F_c(t, v)$ with $q^u \neq 0$ are zero for the same reason, while its components with $q^s = 0$ are zero since $v \in V_s$. Hence $F_c(t, v) = 0$. We conclude that $F(I \times V_s) \subset V_s$. As in the case $a = c$, this implies the claim for $a = s$.

The argument for $a = u$ is entirely analogous to the case $a = s$. □

To prove Proposition 2.10, we start with a general comparison estimate for solutions of ODEs in Hilbert spaces.

Lemma 8.1. *Let V be a Hilbert space, $W \subset V$ a subspace, I an open interval containing 0, and g a map from $I \times V$ into the space of linear operators from W to V. Let $X\colon I \to W$ be a differentiable map (as a map into V), such that for all $t \in I$,*

$$\dot{X}(t) = g(t, X(t))X(t).$$

If $\zeta \in \mathbb{R}$ is such that $g(t, w) + \zeta$ is negative semidefinite for all $t \in I$ and $w \in W$, then for all $t \in I$ with $t \geq 0$,

$$\|X(t)\|_V \leq \|X(0)\|_V e^{-\zeta t}$$

Proof. First, suppose that $\zeta = 0$. Then for all $t \in I$,

$$\frac{d}{dt}\|X(t)\|_V^2 = 2\mathfrak{R}(\dot{X}(t), X(t))_V = 2\mathfrak{R}(g(t, X(t))X(t), X(t))_V \leq 0.$$

So $\|X\|_V^2$ is a nonnegative, non-increasing function on I, and the claim for $\zeta = 0$ follows.

Next, let $\zeta \in \mathbb{R}$ be arbitrary. Then

$$\frac{d}{dt}(X(t)e^{\zeta t}) = \big(g(t, X(t)) + \zeta\big)X(t)e^{\zeta t}.$$

Applying the claim for $\zeta = 0$, with $X(t)$ replaced by $X(t)e^{\zeta t}$ and $g(t, w)$ by $g(t, w) + \zeta$, now yields the claim for ζ. □

For any homogeneous polynomial map $p = p_\lambda$ between normed vector spaces V and W, where $\lambda \in \mathcal{SB}^n(V, W)$, define the map $\tilde{p}\colon V \to \mathcal{B}(V, W)$ by

$$\tilde{p}(v_1)v_2 := \lambda(v_1, \ldots, v_1, v_2). \tag{8.1}$$

Here $n - 1$ copies of v_1 are inserted into λ on the right hand side.

For all $t \in I$, the map $F(t, -)$ lies in $\mathrm{Pol}(V_k, V_1)$ for some k. The operator A lies in $\mathcal{B}(V_l, V_1)$ for some l. By replacing the smaller of k or l by the larger of these two numbers, we henceforth assume $k = l$. Applying the construction (8.1) to each homogeneous term of $F(t, -)$ and adding the resulting maps, we obtain a map $\tilde{F}\colon V_k \to \mathcal{B}(V_k, V_1)$, such that for all $v \in V_k$,

$$F(t, v) = \tilde{F}(t, v)v.$$

For $a \in \{c, s, u\}$, we write $\tilde{F}_a$ for $\tilde{F}$ composed with orthogonal projection onto V_a.

For $j \in \mathbb{N}$, let $q^{(j)} \in \mathbb{N}_0^\infty$ be defined by $q^{(j)}_m = 1$ if $m = j$, and $q^{(j)}_m = 0$ otherwise.

Lemma 8.2. *Let $v \in V_k$. Write $v = v_c + v_s + v_u$, where $v_a \in V_a$ for $a \in \{c, s, u\}$. Then for all $t \in I$, the components of $F(t, v)$ in V_s, V_u and V_c satisfy*

$$(F(t,v))_s = \tilde{F}(t,v)v_s; \tag{8.2}$$

$$(F(t,v))_u = \tilde{F}(t,v)v_u; \tag{8.3}$$

$$(F(t,v))_c = \tilde{F}(t,v_c)v_c \quad \text{if } v_s = 0 \text{ or } v_u = 0. \tag{8.4}$$

Proof. Let $t \in I$ and $v \in V_k$. To prove (8.2), we use the fact that by (2.3b),

$$F_s(t,v) = \sum_{j \in J_s} v_j \sum_{q \in \mathbb{N}_0^\infty : |q| \leq p-1} v^q F^{q+q^{(j)}}(t).$$

So

$$\tilde{F}_s(t,v) = \sum_{j \in J_s} \sum_{q \in \mathbb{N}_0^\infty : |q| \leq p-1} v^q F^{q+q^{(j)}}(t) \otimes e^j,$$

where $\{e^j\}_{j \in \mathbb{N}}$ is the basis of V_1^* dual to $\{e_j\}_{j \in \mathbb{N}}$. Hence

$$\tilde{F}_s(t,v)v = \tilde{F}(t,v)v_s,$$

which implies (8.2). The equality (8.3) can be proved analogously.

To prove (8.4), we note that by (2.3a),

$$F_c(t,v) = F_{c,1}(t,v) + F_{c,2}(t,v),$$

where

$$F_{c,1}(t,v) = \sum_{q \in \mathbb{N}_0^\infty : |q| \leq p, q^s = q^u = 0} v^q F^q(t); \tag{8.5}$$

$$F_{c,2}(t,v) = \sum_{q \in \mathbb{N}_0^\infty : |q| \leq p, q^s \neq 0 \neq q^u} v^q F^q(t). \tag{8.6}$$

The right hand side of (8.5) only depends on v_c, and the right hand side of (8.6) is zero if $v_s = 0$ or $v_u = 0$. So, under that condition, $F_c(t,v) = F_c(t,v_c) = \tilde{F}(t,v_c)v_c$. □

Proof of Proposition 2.10. Set

$$D_{\tilde{\mu}} := \left\{(t,v) \in I \times V_\infty : \|\tilde{F}(t,v)\|_{\mathcal{B}(V_k,V_1)} < \tilde{\mu}\right\}. \tag{8.7}$$

Because F is a sum of polynomials of degrees at least two, we have $\tilde{F}(t,0) = 0$ for all t. So $D_{\tilde{\mu}}$ contains $I \times \{0\}$. It is open by continuity of $\tilde{F}$.

Let $X \colon I \to V_\infty$ be a solution of the constructed system (2.7). As in the proof of Lemma 2.9,

$$\dot{X}_s(t) = AX(t) + F_s(t, X(t)) = \left(A + \tilde{F}(t, X(t)\right)X_s(t), \tag{8.8}$$

where we used the first equality in Lemma 8.2 and the fact that A preserves V_s. For all $(t,v) \in D_{\tilde{\mu}}$, the operator

$$A + \tilde{F}_s(t,v) + \beta - \tilde{\mu} \colon V_k \cap V_s \to V_1 \cap V_s$$

is negative semidefinite. Hence the claim about X_s follows from the second part of Lemma 8.1. The claim about X_u can be proved similarly, via a version of Lemma 8.1 for positive-definite operators.

Next, suppose that $X_s(0) = 0$ or $X_u(0) = 0$. By Lemma 2.9, either $X_s(t) = 0$ for all $t \in I$ or $X_u(t) = 0$ for all $t \in I$. Similarly to (8.8), the third equality in Lemma 8.2 implies that

$$\dot{X}_c(t) = \big(A + \tilde{F}(t, X_c(t)\big)X_c(t),$$

for all $t \in I$. And for all $(t, \nu) \in D_{\tilde{\mu}}$, the operator

$$A + \tilde{F}_c(t, \nu) - \alpha - \tilde{\mu} \colon V_k \cap V_c \to V_1 \cap V_c$$

is negative semidefinite. So by Lemma 8.1,

$$\|X_c(t)\|_{V_1} \leq e^{(\alpha+\tilde{\mu})t}\|X_c(0)\|_{V_1}$$

for all $t \geq 0$ in I. It similarly follows that for all $t \leq 0$ in I,

$$\|X_c(t)\|_{V_1} \leq e^{-(\alpha+\tilde{\mu})t}\|X_c(0)\|_{V_1}.$$

□

9. Example: a non-autonomous version of Burgers' equation

Let $r \in \mathbb{R}$, and consider the non-autononomous, nonlinear PDE

$$\partial_t u(t, \theta) = \partial_\theta^2 u(t, \theta) + ru(t, \theta) - \frac{t}{2}(\partial_\theta u(t, \theta))^2, \tag{9.1}$$

with 2π-periodic boundary conditions in θ. Then Theorem 2.22 applies, where Ω is the circle.

Using Theorem 2.18, we compute the centre manifold of the normal form system approximating (9.1) up to residuals of order three, in Section 9.1. Via a direct approach, we compute all invariant manifolds for residuals of orders three and four, in Section 9.2. We find that the order three centre manifolds computed in the two ways agree. These computations illustrate Remark 2.21, that the construction from Theorem 2.18 is guaranteed to give a result, while a direct computation may be more efficient in concrete situations.

9.1. Centre manifold via Theorem 2.18.

In this setting,

$$Au = u'' + ru \quad \text{and} \quad f(t, u) = -\frac{t}{2}(u')^2,$$

where a prime denotes the derivative in the θ-direction. The eigenfunctions of A are e_j, for $j \in \mathbb{Z}$, given by $e_j(\theta) := e^{ij\theta}$. The eigenvalue corresponding to e_j is $\alpha_j = r - j^2$ (which has multiplicity two when $j \neq 0$). Choose α, β, γ and $\tilde{\mu}$ such that $0 \leq \alpha < \tilde{\mu} < \beta = \gamma < 1$, and $\alpha < \frac{1}{2}$. Suppose that r lies within α of an integer of the form n^2, for a nonzero $n \in \mathbb{Z}$. Then the eigenvalue α_n is central up to precision α.

We determine a corresponding centre manifold for a system that approximates (9.1) up to a third-order residual. This involves the coordinate transform ξ_3. To compute this centre manifold, we only need to apply ξ_3 to elements of $V_c = \text{span}\{e_n, e_{-n}\}$. In other words, we only need to compute $\xi_3(t, X_n e_n + X_{-n}e_{-n})$, for $t \in \mathbb{R}$ and $X_n, X_{-n} \in \mathbb{C}$. (We do not determine the domain $D_{\tilde{\mu}}$ here.)

For $p = 2$, the map ξ_2 is the identity map. So

$$\xi_3(t, X_n e_n + X_{-n}e_{-n}) = X_n e_n + X_{-n}e_{-n} + \hat{\xi}(X_n e_n + X_{-n}e_{-n}),$$

where

$$\hat{\xi}(X_n e_n + X_{-n} e_{-n}) = \sum_{q \in \mathbb{Z}^\infty; |q|=2} \sum_{j \in \mathbb{Z}} \hat{\xi}_j^q(t) e_j (X_n e_n + X_{-n} e_{-n})^q.$$

For $j \in \mathbb{Z}$, let $q^{(j)} \in \mathbb{Z}^\infty$ be defined by $q^{(j)}_m = 1$ if $m = j$, and $q^{(j)}_m = 0$ otherwise. Then, for $q \in \mathbb{Z}^\infty$ with $|q| = 2$,

$$(X_n e_n + X_{-n} e_{-n})^q = \begin{cases} X_n^2 & \text{if } q = 2q^{(n)}; \\ X_{-n}^2 & \text{if } q = 2q^{(-n)}; \\ X_n X_{-n} & \text{if } q = q^{(n)} + q^{(-n)} \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\hat{\xi}(X_n e_n + X_{-n} e_{-n}) = \sum_{j \in \mathbb{Z}} \Big(X_n^2 \hat{\xi}_j^{2q^{(n)}}(t) + X_{-n}^2 \hat{\xi}_j^{2q^{(-n)}}(t) + X_n X_{-n} \hat{\xi}_j^{q^{(n)}+q^{(-n)}}(t) \Big) e_j.$$

The map $\hat{\xi}_j^{2q^{(n)}}$ is expressed in terms of the map $a_j^{2q^{(n)}}$ in

$$R_2(t,u) = -\sum_{j \in \mathbb{Z}} \sum_{q \in \mathbb{Z}^\infty; |q|=2} a_j^q(t) e_j u^q.$$

(The order three term in (6.2) now equals zero.) See (7.2) and (7.3). If $u = \sum_{l \in \mathbb{Z}} x_l e_l$, then

$$R_2(t,u) = -\frac{t}{2}(u')^2 = -\frac{t}{2} \sum_{j \in \mathbb{Z}} \Big(\sum_{k \in \mathbb{Z}} k(j-k) x_k x_{j-k} \Big) e_j.$$

The equality $x_k x_{j-k} = u^{2q^{(n)}} = x_n^2$ holds precisely if $k = n$ and $j = 2n$. Hence

$$a_{2n}^{2q^{(n)}}(t) = \frac{t}{2} n^2$$

and $a_j^{2q^{(n)}} = 0$ if $j \neq 2n$. An analogous argument shows that

$$a_{-2n}^{2q^{(-n)}}(t) = \frac{t}{2} n^2$$

and $a_j^{2q^{(-n)}} = 0$ if $j \neq -2n$. The equality $x_k x_{j-k} = u^{q^{(n)}+q^{(-n)}} = x_n x_{-n}$ holds precisely if $j = 0$ and either $k = n$ or $k = -n$. Hence

$$a_0^{q^{(n)}+q^{(-n)}}(t) = -tn^2$$

and $a_j^{q^{(n)}+q^{(-n)}} = 0$ if $j \neq 0$.

The relevant numbers μ_j^q as in Section 7.1 equal

$$\begin{aligned} \mu_{2n}^{2q^{(n)}} &= 2\alpha_n - \alpha_{2n} = r + 2n^2; \\ \mu_{-2n}^{2q^{(-n)}} &= 2\alpha_{-n} - \alpha_{-2n} = r + 2n^2; \\ \mu_0^{q^{(n)}+q^{(-n)}} &= \alpha_n + \alpha_{-n} - \alpha_0 = r - 2n^2 \end{aligned}$$

(note that $\alpha_j = \alpha_{-j}$ for every j). Because $n^2 \geq 1$ and $\alpha < \frac{1}{2}$, the real parts of $\mu_{2n}^{2q^{(n)}}$ and $\mu_{2n}^{2q^{(-n)}}$ are greater than α, whereas the real part of $\mu_0^{q^{(n)}+q^{(-n)}}$ is smaller than $-\alpha$.

And with J^q as in Section 7.1, we have $2n \in J^{2q^{(n)}}$. Indeed,

$$J_s = \{j \in \mathbb{Z} : |j| \geq n+1\},$$

so $2n \in J_s$ and $(2q^{(n)})^s = 0$. Similarly, $2n \in J^{2q^{(-n)}}$. And $0 \in J_u$ and $(q^{(n)} + q^{(-n)})^u = 0$, so $0 \in J^{q^{(n)}+q^{(-n)}}$.

Hence, by (7.2),

$$\begin{aligned}\hat{\xi}_{2n}^{2q^{(n)}}(t) &= \int_{-\infty}^{t} e^{-(r+2n^2)(t-\tau)} \frac{\tau}{2} n^2 \, d\tau \\ &= \frac{n^2}{2(r+2n^2)} \Big(t - \frac{1}{r+2n^2} \Big).\end{aligned}$$

The integral converges since $\Re(r + 2n^2) > 0$, and is $\tilde{\mu}$-regular. Similarly,

$$\hat{\xi}_{-2n}^{2q^{(-n)}}(t) = \frac{n^2}{2(r+2n^2)} \Big(t - \frac{1}{r+2n^2} \Big).$$

And because $\Re(\mu_0^{q^{(n)}+q^{(-n)}}) < -\alpha$,

$$\begin{aligned}\hat{\xi}_{0}^{q^{(n)}+q^{(-n)}}(t) &= -\int_{t}^{\infty} e^{-(r-2n^2)(t-\tau)} (-\tau n^2) \, d\tau \\ &= \frac{-n^2}{r-2n^2} \Big(t - \frac{1}{r-2n^2} \Big).\end{aligned}$$

We conclude that for all $t \in \mathbb{R}$ and $X_n, X_{-n} \in \mathbb{C}$,

$$\begin{aligned}&\xi_3(t, X_n e_n + X_{-n} e_{-n}) = \\ &\quad X_n e_n + X_{-n} e_{-n} + \frac{n^2}{2(r+2n^2)} \Big(t - \frac{1}{r+2n^2} \Big) (X_n^2 e_{2n} + X_{-n}^2 e_{-2n}) \\ &\qquad - \frac{n^2}{r-2n^2} \Big(t - \frac{1}{r-2n^2} \Big) X_n X_{-n}.\end{aligned} \tag{9.2}$$

(The last term is a scalar multiple of the constant function e_0.) If $r = n^2$, this simplifies to

$$\begin{aligned}&\xi_3(t, X_n e_n + X_{-n} e_{-n}) = \\ &\quad X_n e_n + X_{-n} e_{-n} + \frac{1}{6} \Big(t - \frac{1}{3n^2} \Big) (X_n^2 e_{2n} + X_{-n}^2 e_{-2n}) + \Big(t + \frac{1}{n^2} \Big) X_n X_{-n}.\end{aligned}$$

9.2. **Invariant manifolds via direct computations.** For order of residual $p = 2$, the map ξ_2 is the identity map, $x_j = X_j$.

Proceeding to order of residual $p = 3$ we construct quadratic corrections to the identity ξ_2 to form ξ_3. In the eigenvector basis the field $u(t, \theta) = \sum_j x_j(t) e^{ij\theta}$ (all sums in this section are over $\mathbb{Z}$), and the PDE (9.1) becomes

$$\dot{x}_j = \alpha_j x_j + \frac{t}{2} \sum_k b_{jk} x_{j-k} x_k \quad \text{where } b_{jk} := k(j-k). \tag{9.3}$$

Writing

$$x_j(t) = \xi_3(t, X(t))_j = X_j + \sum_{k,l \in \mathbb{Z}} g_j^{kl}(t) X_k X_l,$$

and solving[3] for g_j^{kl} such that x_j satisfies (9.3) up to terms of order three if $\dot X_k = \alpha_k X_k$, we find that $x(t) = \xi_3(t, X(t))$ is given by

$$x_j = X_j + \frac{1}{2}\sum_{k:|d_{jk}^{-1}|>\tilde\mu} b_{jk}[d_{jk}t - d_{jk}^2]X_{j-k}X_k\,, \tag{9.4}$$

$$\text{where } d_{jk} := 1/[-\alpha_j + \alpha_k + \alpha_{j-k}] = 1/[r + 2jk - k^2].$$

For $r \approx n^2$ and odd n, the denominators in d_{jk} are not small. Then this map, combined with the linear $\dot X_j = \alpha_j X_j$, matches the PDE (9.1) to third-order errors.

However, for $r \approx n^2$ and even $n > 0$, some denominators are small, becoming zero when $r = n^2$. Then the divisor being zero becomes $k(k-j) = n^2/2$ and hence has zeros for every pair of integer factors of $n^2/2$ (including negative pairs). Consequently these terms are excluded from the sum (9.4), and instead lead to nonlinearly modifying the evolution for some j via

$$\dot X_j = \alpha_j X_j + \frac{t}{2}\sum_{k:|d_{jk}^{-1}|<\tilde\mu} b_{jk}X_{j-k}X_k\,.$$

Often the centre manifold is of most interest, so in ξ_3 setting all $X_j = 0$ except $X_{\pm n}$, gives the quadratic approximate centre manifold to be $x_j = X_j$ for all j except

$$x_0 = X_0 - n^2\left[\frac{1}{r-2n^2}t - \frac{1}{(r-2n^2)^2}\right]X_nX_{-n}\,,$$
$$x_{\pm 2n} = X_{\pm 2n} + \frac{1}{2}n^2\left[\frac{1}{r+2n^2}t - \frac{1}{(r+2n^2)^2}\right]X_{\pm n}^2\,.$$

This is the same result as (9.2).

Proceeding to order of residual $p = 4$ we may construct cubic corrections to ξ_3 to form ξ_4. For simplicity, restrict attention to the cases of n odd. It is straightforward but tedious to construct that for ξ_4

$$x_j = \xi_{3,j} + \sum_{k,l:|d_{jkl}^{-1}|>\tilde\mu} b_{jl}b_{lk}c_{jkl}(t)X_{j-l}X_{l-k}X_k\,, \tag{9.5}$$

$$\text{where } c_{jkl} := \tfrac{1}{2}d_{lk}d_{jkl}t^2 - (d_{lk}d_{jkl}^2 + \tfrac{1}{2}d_{lk}^2d_{jkl})t + (d_{lk}d_{jkl}^3 + \tfrac{1}{2}d_{lk}^2d_{jkl}^2),$$
$$d_{jkl} := 1/[-\alpha_j + \alpha_k + \alpha_{l-k} + \alpha_{j-l}] = 1/[2r + 2jl + 2kl - 2k^2 - 2l^2].$$

The terms excluded from (k, l) in the sum (9.5) must cause cubic terms in the evolution. For example, when $r = n^2 = 1$ then[4]

$$\dot X_0 = X_0, \tag{9.6a}$$
$$\dot X_{\pm 1} = (\tfrac{1}{9}t - \tfrac{1}{3}t^2)X_{-1}X_{\pm 1}^2\,, \tag{9.6b}$$
$$\dot X_{\pm 2} = -3X_{\pm 2} + (\tfrac{104}{225}t - \tfrac{8}{15}t^2)X_{\mp 1}X_{\pm 1}X_{\pm 2}\,, \tag{9.6c}$$
$$\dot X_{\pm 3} = -8X_{\pm 2} + (\tfrac{594}{1225}t - \tfrac{18}{35}t^2)X_{\mp 1}X_{\pm 1}X_{\pm 3}\,, \tag{9.6d}$$

[3]The computer algebra code used for the computations in this section is available on http://www.maths.adelaide.edu.au/anthony.roberts/pBurgers.txt.

[4]The apparent pattern in these ODEs becomes more complicated—at $\dot X_{\pm 6}$ for example.

$$\dot{X}_{\pm 4} = -15X_{\pm 3} + (\tfrac{1952}{3969}t - \tfrac{32}{63}t^2)X_{\mp 1}X_{\pm 1}X_{\pm 3}\,, \tag{9.6e}$$

$$\vdots$$

By construction, in this case of $r = 1$, the coordinate transform (9.5) together with the ODEs (9.6) creates a dynamical system in $u(t, \theta) = \sum_j x_j e^{ij\theta}$ which is the same as the PDE (9.1) to a residual of order four. In the combined system (9.5) and (9.6) for $r = 1$, by definition Definition 2.11 three invariant manifolds are: the 1D unstable manifold parametrised by X_0 with all other $X_j = 0$; the 2D centre manifold parametrised by $X_{\pm 1}$ with all other $X_j = 0$; and the stable manifold with $X_0 = X_{\pm 1} = 0$.

Acknowledgement. Part of this research was supported by the Australian Research Council grant DP150102385.

References

[1] R. M. Aron and J. B. Prolla. Polynomial approximation of differentiable functions on Banach spaces. *J. Reine Angew. Math.*, 313:195–216, 1980.
[2] Bernd Aulbach, Martin Rasmussen, and Stefan Siegmund. Invariant manifolds as pullback attractors of nonautonomous differential equations. *Discrete and Continuous Dynamical Systems*, 15(2):579–596, 2006.
[3] Bernd Aulbach and Thomas Wanner. Integral manifolds for Caratheodory type differential equations in Banach spaces. In B. Aulbach and F. Colonius, editors, *Six Lectures on Dynamical Systems*, pages 45–119. World Scientific, Singapore, 1996.
[4] Bernd Aulbach and Thomas Wanner. Invariant foliations for Caratheodory type differential equations in Banach spaces. In V. Lakshmikantham and A. A. Martynyuk, editors, *Advances of Stability Theory at the End of XX Century*. Gordon & Breach Publishers, 1999. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.45.5229&rep=rep1&type=pdf.
[5] Bernd Aulbach and Thomas Wanner. The Hartman–Grobman theorem for Caratheodory-type differential equations in Banach spaces. *Nonlinear Analysis*, 40:91–104, 2000.
[6] Luis Barreira and Claudia Valls. *Stability of Nonautonomous Differential Equations*, volume 1926 of *Lecture Notes in Mathematics*. Springer, 2007.
[7] J. E. Bunder and A. J. Roberts. Nonlinear emergent macroscale PDEs, with error bound, for nonlinear microscale systems. Technical report, [https://arxiv.org/abs/1806.10297], June 2018.
[8] V. Bykov and V. Gol'dshtein. Fast and slow invariant manifolds in chemical kinetics. *Computers & Mathematics with Applications*, 2013.
[9] J. Carr. *Applications of centre manifold theory*, volume 35 of *Applied Math. Sci.* Springer–Verlag, 1981.
[10] C. Chicone and Y. Latushkin. Center manifolds for infinite dimensional nonautonomous differential equations. *J. Differential Equations*, 141:356–399, 1997.
[11] Stefania D'Alessandro. Polynomial algebras and smooth functions in Ba- nach spaces. Ph.D. thesis, Universitá degli studi di Milano, 2013.
[12] Stefania D'Alessandro and Petr Hájek. Polynomial algebras and smooth functions in Banach spaces. *J. Funct. Anal.*, 266(3):1627–1646, 2014.
[13] Stefania D'Alessandro, Petr Hájek, and Michal Johanis. Corrigendum to the paper "Polynomial algebras on classical Banach spaces" [MR1656881]. *Israel J. Math.*, 207(2):1003–1012, 2015.
[14] Peter Friz and Martin Hairer. *A Course on Rough Paths. With an introduction to regularity structures*. Springer, 2014.
[15] Raquel Gonzalo and Jesús Angel Jaramillo. Compact polynomials between Banach spaces. *Proc. Roy. Irish Acad. Sect. A*, 95(2):213–226, 1995.

[16] Joseph F. Grcar. John von Neumann's analysis of Gaussian elimination and the origins of modern numerical analysis. *SIAM Review*, 53(4):607–682, 2011.
[17] Mariana Haragus and Gerard Iooss. *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*. Springer, 2011.
[18] M. Hărăguş. Reduction of pdes on unbounded domains application: unsteady water waves problem. *J. Nonlinear Sci.*, 8:353–374, 1998.
[19] A. Mielke. *On Saint-venant's Problem And Saint-venant's Principle In Nonlinear Elasticity*. Trends In Appl Of Maths To Mech, 1988.
[20] A. Mielke. Reduction of PDEs on domains with several unbounded directions: A first step towards modulation equations. *A. angew Math Phys*, 43(3):449–470, 1992.
[21] A. S. Nemirovskiĭ. The polynomial approximation of functions on Hilbert space. *Funkcional. Anal. i Priložen.*, 7(4):88–89, 1973.
[22] A. S. Nemirovskiĭ and S. M. Semenov. The polynomial approximation of functions on Hilbert space. *Mat. Sb. (N.S.)*, 92(134):257–281, 344, 1973.
[23] G. A. Pavliotis and A. M. Stuart. *Multiscale methods: averaging and homogenization*, volume 53 of *Texts in Applied Mathematics*. Springer, 2008.
[24] A. Pełczyński. On weakly compact polynomial operators on B-spaces with Dunford-Pettis property. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 11:371–378, 1963.
[25] Christian Potzsche and Martin Rasmussen. Taylor approximation of integral manifolds. *Journal of Dynamics and Differential Equations*, 18:427–460, 2006.
[26] A. J. Roberts. Boundary conditions for approximate differential equations. *J. Austral. Math. Soc. B*, 34:54–80, 1992.
[27] A. J. Roberts. Macroscale, slowly varying, models emerge from the microscale dynamics in long thin domains. *IMA Journal of Applied Mathematics*, 80(5):1492–1518, 2015.
[28] A. J. Roberts. *Model emergent dynamics in complex systems*. SIAM, Philadelphia, jan 2015.
[29] A. J. Roberts. Backwards theory supports modelling via invariant manifolds for non-autonomous dynamical systems. Technical report, [http://arxiv.org/abs/1804.06998], April 2018.
[30] Raymond A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.
[31] A. Vanderbauwhede. Centre manifolds, normal forms, and elementary bifurcations. *Dynamics Reported*, 2:89–169, 1989.
[32] A. Vanderbauwhede and G. Iooss. Center manifold theory in infinite dimensions. In *Dynamics reported. Expositions in dynamical systems. New series. Volume 1*, pages 125–163. Berlin etc.: Springer-Verlag, 1992.
[33] Ferdinand Verhulst. *Methods and applications of singular perturbations: boundary layers and multiple timescales*, volume 50 of *Texts in Applied Maths*. Springer, 2005.
[34] Vladimir A. Zorich. *Mathematical analysis. II*. Universitext. Springer, Heidelberg, second edition, 2016. Translated from the fourth and the sixth corrected (2012) Russian editions by Roger Cooke and Octavio Paniagua T.

School of Mathematical Sciences, University of Adelaide
E-mail address: peter.hochs@adelaide.edu.au

School of Mathematical Sciences, University of Adelaide
E-mail address: anthony.roberts@adelaide.edu.au
URL: http://orcid.org/0000-0001-8930-1552