Spaces of Holomorphic Immersions of Open Riemann Surfaces into the Complex Plane

Haripriya Sridharan

March 25, 2020

Thesis submitted for the degree of
Master of Philosophy
in
Pure Mathematics
at The University of Adelaide
Faculty of Engineering, Computer and Mathematical Sciences
School of Mathematical Sciences

SUB CRUCE LUMEN
THE UNIVERSITY of ADELAIDE

To my mother.

Your courage stuns me; your kindness is something I have never been able to comprehend. Words cannot describe how much you mean to me.

Contents

Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

I give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library Search and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship.

Signed: .... Date: 25 March 2020

Acknowledgements

It is impossible for me to find any way to adequately express my gratitude to my principal supervisor Finnur Lárusson. This thesis simply would not have existed without his expert guidance, endless patience, care, and constant encouragement. I knew precious little when I first met Finnur – he has so patiently guided me from there to where I am today, and helped me discover joy in mathematical research. It has truly been a privilege to learn from him, and for that I am grateful each and every day.

I also wish to extend my gratitude to my secondary supervisor Thomas Leistner for always being willing to have discussions, and for his diligently taught course on differential geometry. I thank my other lecturers Hang Wang, David Baraglia, Nicholas Buchdahl and Andy Hammerlindl – I have learnt so much from each.

Thank you to David Roberts for sharing his many valuable perspectives (and letting us befriend little Matilda), and to Daniel and Matt, for always being keen to have discussions related to my research. And of course, to all my other friends, for making it a delight to come in to work each day.

Finally, I thank my mother, father and lovely little sister. Nothing would have been possible without your immense love and support. Thank you for supporting me, caring so deeply for me, and believing in me even in those times when I did not. Mum – stay brave. This is, of course, for you.

Abstract

Let M be an open Riemann surface. A recent result due to Forstnerič and Lárusson [8] says that, for a closed conical subvariety $A \subset \mathbb{C}^n$ such that $A \setminus \{0\}$ is an Oka manifold, the weak homotopy type of the space of non-degenerate holomorphic A-immersions of M into $\mathbb{C}^n$ is the same as that of the space of holomorphic (or equivalently, continuous) maps from M into $A \setminus \{0\}$. In their paper, the authors sketch the proof of this theorem through claiming analogy with a related result, and invoking advanced results from complex and differential geometry, including seminal theorems from Oka theory.

The work contained in this thesis was motivated by the absence of a self-contained proof for the special case where $A = \mathbb{C}$ – as, perhaps, the first geometrically interesting case that one would consider. We remedy the absence by providing a fully detailed, self-contained proof of this case; namely, the parametric h-principle for holomorphic immersions of open Riemann surfaces into $\mathbb{C}$. We outline this more precisely as follows.

Take a holomorphic 1-form θ on M which vanishes nowhere. We denote by $\mathcal{I}(M, \mathbb{C})$ the space of holomorphic immersions of M into $\mathbb{C}$, and denote by $\mathcal{O}(M, \mathbb{C}^*)$ the space of nonvanishing holomorphic functions on M. We prove, in all detail, that the continuous map

$$\mathcal{I}(M, \mathbb{C}) \to \mathcal{O}(M, \mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a weak homotopy equivalence. This gives a full description of the weak homotopy type of $\mathcal{I}(M, \mathbb{C})$, as the target space $\mathcal{O}(M, \mathbb{C}^*)$ is known by algebraic topology (Remark 5.2.3).

Chapter 1

Introduction

1.1 Context and summary

The problem we are exploring fits into a long line of research that begins with a theorem of H. Whitney and W. C. Graustein of 1937.

This classical theorem states that immersions of the circle into the plane are characterised (up to 'regular homotopy', or homotopy through immersions) by the winding numbers of their tangential maps [25]. This can be stated more precisely as follows.

Denote by $\mathcal{I}(S^1, \mathbb{R}^2)$ the space, equipped with the C^1 topology, of C^1 immersions of the circle S^1 into $\mathbb{R}^2$. Denote by $C^0(S^1, \mathbb{R}^2_*)$ the space of continuous maps $S^1 \to \mathbb{R}^2_* := \mathbb{R}^2 \setminus \{(0,0)\}$ equipped with the compact-open topology. The Whitney-Graustein theorem says that the continuous map given by differentiation,

$$J : \mathcal{I}(S^1, \mathbb{R}^2) \to C^0(S^1, \mathbb{R}^2_*),$$

induces a bijection of path components. It is a basic result from algebraic topology that two loops in the punctured plane have equal winding number if and only if they are homotopic; that is, the set $\pi_0(C^0(S^1, \mathbb{R}^2_*))$ of path components of $C^0(S^1, \mathbb{R}^2_*)$ is in bijection with $\mathbb{Z}$ (we give details in Section 2.1). The Whitney-Graustein theorem thus forges a bijection between $\pi_0(\mathcal{I}(S^1, \mathbb{R}^2))$ and $\mathbb{Z}$.

In fact, as was proven much later in the 20th century, a significantly stronger statement can be made – the map J is a weak homotopy equivalence. As far as we know, the Whitney-Graustein theorem was the earliest manifestation of the homotopy principle (or *h-principle*), a notion first formalised by M. Gromov in [11, 12] (see also [6]). This important theory focuses on homotopical methods for solving partial differential relations $\mathcal{R}$, and is central to the problem we examine in this thesis.

To any differential relation $\mathcal{R}$, we may associate a unique algebraic relation by treating the derivatives as new independent variables. A solution to the algebraic relation is a *formal* solution of $\mathcal{R}$. The existence of a formal solution is a necessary condition for the solvability of $\mathcal{R}$. We try to deform a formal solution into a genuine solution of $\mathcal{R}$. The *basic h-principle* is said to hold for $\mathcal{R}$ if any formal solution can be deformed, through formal solutions, into a genuine solution.

We equip the set of formal solutions of $\mathcal{R}$ and the set of genuine solutions of $\mathcal{R}$ with the appropriate function space topology; denote these spaces by $\Gamma(\mathcal{R})$ and $S(\mathcal{R})$ respectively. The *parametric h-principle* is said to hold for $\mathcal{R}$ if the inclusion $S(\mathcal{R}) \hookrightarrow \Gamma(\mathcal{R})$ is a weak homotopy equivalence.

The stronger version of the Whitney-Graustein theorem (that the map J is a weak homotopy equivalence) may be viewed as a parametric h-principle for immersions of S^1 into $\mathbb{R}^2$. Indeed, for C^1 maps $f : S^1 \to \mathbb{R}^2$, consider the differential relation $\mathcal{R}$ defined by $f'(t) \neq (0,0)$ for any $t \in S^1$; then $\Gamma(\mathcal{R}) = C^0(S^1, \mathbb{R}^2_*)$, and the subspace $S(\mathcal{R})$ is homotopy equivalent to $\mathcal{I}(S^1, \mathbb{R}^2)$. The Whitney-Graustein theorem is a special case of Gromov's one-dimensional h-principle for *ample differential relations* (see [22]). We explore further details in Chapter 3.

While much of the foundational work in the theory of the h-principle is due to Gromov, the theory has since seen extensive developments, and is a significant part of modern differential geometry and topology. Given the concurrent developments in complex geometry, a natural theme to examine is how the h-principle manifests itself in a complex analytic setting.

In 1939, K. Oka proved (in modern terms) that a holomorphic line bundle on a Stein manifold (see Definition 2.2.8) is holomorphically trivial if it is topologically trivial. In fact, in a holomorphic line bundle which is topologically trivial, every continuous trivialisation can be continuously deformed into a holomorphic trivialisation. Oka's theorem was the first evidence of the h-principle in complex analysis.

Another form of the holomorphic h-principle – which may be viewed as a holomorphic analogue of the 1937 Whitney-Graustein theorem – appeared in a theorem of R. C. Gunning and R. Narasimhan in 1967 [13]. We provide an overview of the Gunning-Narasimhan theorem in Section 2.5; in Chapter 4 we further explore its role as a basic h-principle. The seminal theorem can be stated as follows. Let M be an open Riemann surface and θ a non-vanishing holomorphic 1-form on M. Given a holomorphic map $f : M \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, there exists a holomorphic immersion $F : M \to \mathbb{C}$ such that f can be deformed into dF/θ through nonvanishing holomorphic functions on M. (In Remark 4.2.9 we observe that this is equivalent to the more classical statement of the theorem, which is that, given a nonvanishing holomorphic 1-form θ on X, there exists a holomorphic function g on X such that the 1-form $e^g\theta$ is exact.)

As was proven much later by Alarcón and Forstnerič [1], the Gunning-Narasimhan theorem can be extended to more general target spaces. We let M and θ be as above. Let $A \subset \mathbb{C}^n$ be a closed conical subvariety (a subvariety A is conical if $cA = A$ for all $c \in \mathbb{C}^*$), and assume that $A \setminus \{0\}$ is an Oka manifold. We call a holomorphic immersion $F : M \to \mathbb{C}^n$ an A-*immersion* if its derivative with respect to any holomorphic coordinate lies in $A \setminus \{0\}$. Then, any holomorphic map $f : M \to A \setminus \{0\}$ can be deformed, through such maps, into dF/θ for some A-immersion $F : M \to \mathbb{C}^n$. This is the *basic Oka principle* for A-immersions.

The aim of this thesis is to explore a parametric h-principle for holomorphic immersions of open Riemann surfaces into the complex plane. This aim fits naturally into two contexts. We may view it as a holomorphic analogue of the stronger Whitney-Graustein theorem (that is, that J is a weak homotopy equivalence). It can equally be seen as a parametrisation of the Gunning-Narasimhan theorem, which is the basic h-principle for holomorphic immersions of open Riemann surfaces. Below we outline our goal more precisely.

As above, we let M be an open Riemann surface. We denote by $\mathcal{I}(M, \mathbb{C})$ the topological space of holomorphic immersions of M into $\mathbb{C}$, with the compact-open topology. Also equipped with the compact-open topology, we denote by $\mathcal{O}(M, \mathbb{C}^*)$ the space of holomorphic functions $M \to \mathbb{C}^*$.

Choose a nonvanishing holomorphic 1-form θ on M (cf. Theorem 2.2.14). We prove that the continuous map

$$\mathcal{I}(M, \mathbb{C}) \to \mathcal{O}(M, \mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a weak homotopy equivalence.

This theorem is a special case of a much stronger result due to Forstnerič and Lárusson, published in 2019 [8, Theorem 5.6], namely the *parametric Oka principle* for A-immersions (for a closed conical subvariety $A \subset \mathbb{C}^n$). In the paper, the authors sketch the proof for the general case by invoking advanced results; however there is no direct, self-contained proof in the literature as yet. As outlined above, we present in this thesis a fully detailed, self-contained proof for the geometrically interesting case when $A = \mathbb{C}$.

This parametric h-principle allows us to understand the rough shape (that is, the weak homotopy type) of the space $\mathcal{I}(M, \mathbb{C})$ through algebraic topology alone: the target space is easily seen to have the weak homotopy type of the space of nonvanishing continuous maps on M, $\mathcal{C}(M, \mathbb{C}^*)$ (Remark 5.2.3); in particular, the path components of $\mathcal{I}(M, \mathbb{C})$ are in bijection with $\mathbb{Z}^r$, where $r \in \{0, 1, 2, \ldots, \infty\}$ is the rank of $H_1(M, \mathbb{Z})$. (We make a remark on notation: in a smooth context, we denote the space of continuous maps from X to Y by $C^0(X, Y)$; in a holomorphic context, we write $\mathcal{C}(X, Y)$ for the same space. This is only convention; the difference carries no mathematical implication.)

Further, the nature of this special case, as we find, is such that the proof both admits elegant simplifications, and enlists advanced techniques from differential and complex geometry. The techniques we employ include convex integration theory and period dominating sprays; both of which, but particularly the latter, manifest themselves more transparently in our situation. We give these techniques a rigorous treatment in the thesis, and develop them through careful exposition for our special case.

1.2 Structure of the thesis

As per the above outline, the main contribution of this thesis is in providing an unabridged proof of a parametric h-principle for holomorphic immersions of open Riemann surfaces into $\mathbb{C}$. Specifically, where M is an open Riemann surface and θ is a nonvanishing holomorphic 1-form on M, we prove that the map induced by differentiation,

$$\mathcal{I}(M, \mathbb{C}) \to \mathcal{O}(M, \mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a weak homotopy equivalence. We proceed towards this goal step by step, developing necessary theory and illustrating the context on the way.

The basic ingredients come from real analysis, algebraic topology, differential topology, complex analysis and Riemann surface theory. While basic knowledge of these areas is assumed of the reader, we devote Chapter 2 to background, where we collate and present several specific results that are pertinent to our work.

The purpose of Chapter 3 is twofold: to provide context to our main problem, and develop advanced tools for solving it. In it, we present a proof of the Whitney-Graustein theorem by treating the result, at least partly, as a consequence of Gromov's one-dimensional h-principle for ample differential relations; we prove the basic version of the same h-principle through methods of convex integration theory. Convex integration theory was developed by Gromov in [11, 12] as a method for proving h-principles, and indeed it is an important element in our proof of the main theorem in Chapter 5. In Chapter 3, we develop a detailed exposition of the aspects of convex integration theory pertaining to the h-principles we engage with: the basic h-principle for ample differential relations (presented in the same chapter), and the h-principle for holomorphic immersions (the central problem of the thesis, in Chapter 5). Certainly the mathematics in Chapter 3 is not my invention, but due to Gromov (and, in the case of Theorem 3.5.2, Whitney and Graustein); my contribution there lies in the choice and presentation of the material, and the supporting details filled within many of the proofs. The contextual relevance of this chapter lies in the shared theme between the Whitney-Graustein theorem and the central problem of the thesis, as the theorem of Whitney and Graustein is the basic real analogue of the holomorphic h-principle we build up towards.

In Chapter 4, we make a step towards solving our primary problem, by laying out a complete proof of its basic form; that is, the basic h-principle for holomorphic immersions of open Riemann surfaces into $\mathbb{C}$. While this problem was essentially solved by Gunning and Narasimhan in their seminal 1967 paper [13] (see Remark 4.2.9), I found that their approach, perhaps due to its simplicity, did not lend itself in any clear way (if at all) to parametrisation. A major part of Chapter 4 is devoted to working out a different proof of this basic h-principle, based on very recent developments [1], using new methods which are better suited to parametric theorems. The chapter is divided into two sections. In the first section (4.1), we prove the basic h-principle using methods of Gunning and Narasimhan for the special case when $M = \mathbb{C}^*$; that is, we show that the differentiation map $\mathcal{I}(\mathbb{C}^*, \mathbb{C}) \to \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*), f \mapsto f'$, induces a surjection of path components. For the first section alone, we hone in on this special case for its relative transparency, allowing us to better illustrate the methods and glean further information. In the second section (4.2) we present a detailed proof of the basic h-principle for holomorphic immersions of open Riemann surfaces; that is,

$$\mathcal{I}(M,\mathbb{C}) \to \mathcal{O}(M,\mathbb{C}^*), \quad f \mapsto df/\theta,$$

induces a surjection at the π_0-level, using more advanced methods inspired by the work of Alarcón and Forstnerič in [1]. The format of this proof is adapted from the more general [1, Theorem 7.2]; however, in keeping with the goal of giving a self-contained proof, we cannot exclusively distill a proof of our special case from this original sketch, for we would then obtain a terse outline which invokes other advanced results from Oka theory (for example, the parametric Oka property) – in short, not self-contained. Thus, we use it as a guide and template, but fill in case-specific details and take different routes as necessary, and present a complete proof of the above basic h-principle with no black-box ingredients. We use this proof as the main springboard for the parametric h-principle of Chapter 5.

In Chapter 5 we arrive at the heart of the thesis. Here we prove the full parametric h-principle for immersions of open Riemann surfaces into $\mathbb{C}$ – more precisely, we provide a rigorous proof that the continuous map

$$\mathcal{I}(M,\mathbb{C}) \to \mathcal{O}(M,\mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a weak homotopy equivalence. This proof of the parametric h-principle in Chapter 5 is the most important contribution of the thesis. The theorem itself is a special case of the more general, very recent theorem of Forstnerič and Lárusson, [8, Theorem 5.6]; however, the authors provide only a sketch of the proof for the general result by drawing analogy with a related h-principle, and referring to various advanced results from complex and differential geometry (including other, equally advanced, parametric theorems from Oka theory). My work in Chapter 5 rectifies the absence of a full, self-standing proof for the case of holomorphic immersions into the complex plane $\mathbb{C}$ – which is perhaps the first geometrically interesting case one would consider. The proof of the basic h-principle in Section 4.2 is the guide and springboard for the proof of the parametric principle in Chapter 5.

In essence, we parametrise each step of the proof of the basic h-principle; in practice, this requires the use of advanced techniques, including convex integration theory and period dominating sprays. (The former is treated in Chapter 3; the latter is carefully developed for our special case in Chapters 4 and 5, although we do not refer to the term.

Period dominating sprays first appeared formally in the literature in 2014, in [1]; but Gunning and Narasimhan used an elementary form of this technique in [13].)

One further note is that the proof of the parametric h-principle in Chapter 5 does not use higher-dimensional complex analysis – for variety, we chose to use the Oka-Weil approximation theorem in the proof of the basic h-principle (Section 4.2), but it is not strictly needed, as we find in the proof of Theorem 5.2.1. Some of the ingredient lemmas of Chapter 5 are stated for Stein manifolds, to demonstrate their scope – however, the main theorem requires only classical results from Riemann surface theory.

I make a concluding remark on the proof. The structure of our proof shares many elements with the sketch of the general theorem in [8] – this is perhaps not surprising, as the proof in [1] of the basic h-principle guides the format of both. However, as outlined above, the two proofs differ in their details: our aim is to contribute to the literature a complete, self-contained proof of this particular case of geometric interest.

1.3 Directions for further research

The parametric h-principle we prove fully classifies the weak homotopy type of $\mathcal{I}(M, \mathbb{C}^*)$, by showing it to be weakly homotopy equivalent to a space we understand through algebraic topology (cf. Remark 5.2.3). Perhaps the next natural question to ask is when, if ever, is the weak homotopy equivalence promoted to a genuine homotopy equivalence? Forstnerič and Lárusson [8] give a sufficient condition – when M is of finite topological type, the map is a genuine homotopy equivalence. (Results of this type first appeared in [16]). This problem calls for further exploration; one could consider whether there are any other such cases, and what methods would be needed to establish it.

Another problem is whether we can find a reasonably concise, self-contained proof of the h-principle for holomorphic immersions of M into $\mathbb{C}^n$ when $n > 1$. The case we explore renders particular simplicity – some of our arguments are unique to the case $n = 1$; for example, whenever we use the universal covering map $\exp : \mathbb{C} \to \mathbb{C}^*$.

Finally, one could consider whether there is a different proof for the same h-principle. On this front, I did attempt to generalise Gunning and Narasimhan's proof; however, I could find no clear way to make that approach work, even for injectivity at the level of path components. There certainly could be another path to try, with scope for developing new methods.

Chapter 2

Background

In this chapter we present pertinent background material and outline some fundamental theorems that are used in proofs throughout the thesis.

2.1 Winding numbers

We recall the notion of a *winding number* of a continuous loop in the punctured plane. For simplicity of presentation, the results are stated for continuous loops in $\mathbb{C}^*$, but by making the natural identification $\mathbb{R}^2_* \simeq \mathbb{C}^*$, they also hold for loops in $\mathbb{R}^2_*$.

Definition 2.1.1. Assume that $c : [0, 1] \to \mathbb{C}^*$ is a continuous loop. Let $\theta : [0, 1] \to \mathbb{R}$ be an associated continuous function such that $\arg(c(t)) = \theta(t)$ for any $t \in [0, 1]$. The winding number of c is the integer

$$w(c) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

The following is an elementary result from algebraic topology.

Theorem 2.1.2. *Two continuous loops $c_1, c_2 : [0, 1] \to \mathbb{C}^*$ have equal winding number if and only if they are homotopic through continuous loops in $\mathbb{C}^*$.*

Proof. Let $\pi : \mathbb{R} \to S^1$ be the universal covering map defined by $t \mapsto e^{2\pi i t}$. We may assume, by composing with the continuous map $r : \mathbb{C}^* \to S^1, z \mapsto |z|$, that the images of c_1 and c_2 lie in $S^1 \subset \mathbb{C}^*$. (Note that composing a loop $[0, 1] \to \mathbb{C}^*$ with r changes neither the homotopy class nor the winding number of the loop.)

We first assume that $w(c_1) = w(c_2)$, and consider the loop $c : [0,1] \to S^1, t \mapsto c_1(t)/c_2(t)$. The loop c has winding number 0. If $\tilde{c} : [0,1] \to \mathbb{R}$ is a lifting of c with respect to $\pi : \mathbb{R} \to S^1$, then $\tilde{c}$ too is a loop. Since $\mathbb{R}$ is contractible, we can fix a homotopy $h : [0,1] \times [0,1] \to \mathbb{R}$ such that $h(0,t) = \tilde{c}(t)$, $h(1,t) = 0$ and $h(s,0) = h(s,1)$ for all $(s,t) \in [0,1] \times [0,1]$. (Note that the third condition is possible only because $\tilde{c}$ is a loop.) Then, the map $[0,1] \times [0,1] \to \mathbb{C}^*, (s,t) \mapsto c_2(t) \cdot \exp(2\pi i\, h(s,t))$, is a homotopy from c_1 to c_2 through loops in $\mathbb{C}^*$.

Conversely, assume that $k : [0,1] \times [0,1] \to S^1 \subset \mathbb{C}^*$ is a continuous map such that $k(0,t) = c_1(t)$, $k(1,t) = c_2(t)$ and $k(s,0) = k(s,1)$ for all $(s,t) \in [0,1] \times [0,1]$. Let $\tilde{k} : [0,1] \times [0,1] \to \mathbb{R}$ be a lifting of the homotopy k with respect to $\pi : \mathbb{R} \to S^1$. For any $s \in [0,1]$, the value $\tilde{k}(s,1) - \tilde{k}(s,0)$ is the winding number of the loop $k(s,\cdot)$. Since the image of the continuous map $[0,1] \to \mathbb{R}, s \mapsto \tilde{k}(s,1) - \tilde{k}(s,0)$, lies in $\mathbb{Z}$, it is a constant map. This implies that $w(c_1) = w(c_2)$. □

2.2 Open Riemann surfaces

2.2.1 Holomorphic approximation

In this subsection we state some classical holomorphic approximation theorems for open Riemann surfaces.

Definition 2.2.1. Let M be a complex manifold and $K \subset U$ a compact subset. Let

$$\hat{K} := \{x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \quad \text{for all } f \text{ holomorphic on } M\}.$$

We call $\hat{K}$ the *holomorphically convex hull* of K. The set K is *holomorphically convex* in M if $K = \hat{K}$.

Remark 2.2.2. For an open Riemann surface X, there is an equivalent definition of holomorphic convexity. If $K \subset X$ is a compact subset, then K is holomorphically convex if and only if none of the connected components of $X \setminus K$ are relatively compact in X.

Thus, for complex manifolds of one dimension, holomorphic convexity is a topological notion.

While the situation is not simple in higher dimensions, there are many known basic examples of holomorphically convex sets – one such example is a convex subset of $\mathbb{C}^n$. We use this fact in Theorem 4.2.8; the details are as follows.

Lemma 2.2.3. *If K is a convex subset of $\mathbb{C}^n$, then K is holomorphically convex.*

Proof. Let $w \in \mathbb{C}^n \setminus K$. We will find a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ such that $|f(w)| > \sup_{z \in K} |f(z)|$. Since K is a convex set, w is contained in an affine subspace $L \subset \mathbb{C}^n$ of real dimension $2n - 1$, such that $L \cap K = \varnothing$. As holomorphic functions are invariant under rotations and translations, we assume without loss of generality that $L := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \mathrm{Re}(z_1) = 0\}$ and $w = 0$. Then, L separates $\mathbb{C}^n$ into two regions, namely, $H_+ := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \mathrm{Re}(z_1) > 0\}$ and $H_- := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \mathrm{Re}(z_1) < 0\}$. Since $L \cap K = \varnothing$, K lies in either H_+ or H_-. Assume that $L \subset H_-$. The function $(z_1, \ldots z_n) \mapsto \mathrm{Re}(z_1)$ is continuous and negative on K, so it achieves a maximum $M < 0$ on K. Consider the holomorphic function $f : \mathbb{C}^n \to \mathbb{C}, z \mapsto e^{z_1}$. Then,

$$|f(z)| = \left|e^{\mathrm{Re}(z_1)}\right| \leq e^M < 1 = f(w).$$

This shows that K is holomorphically convex in $\mathbb{C}^n$. If $L \subset H_+$ we would consider $g : \mathbb{C}^n \to \mathbb{C}, z \mapsto e^{-z_1}$, to arrive at the same conclusion. □

Lemma 2.2.4. *Let M and N be complex manifolds. If $K \subset M$ and $L \subset N$ are compact, holomorphically convex subsets of M and N respectively, then $K \times L$ is a holomorphically convex subset of the product manifold $M \times N$.*

Proof. Consider any point $(x, y) \in (M \times N) \setminus (K \times L)$. It is either the case that $x \in M \setminus K$ or $y \in N \setminus L$. In the arguments below, we assume without loss of generality that $x \in M \setminus K$.

Since $K = \hat{K} \subset M$, there exists a holomorphic function $\alpha : M \to \mathbb{C}$ such that

$$|\alpha(x)| > \sup_{z \in K} |\alpha(z)|$$

Consider the holomorphic function $f : M \times N \to \mathbb{C}$ defined by $(z, w) \mapsto \alpha(z)$.

Notice that $\sup_{(z,w) \in K \times L} |f(z, w)| = \sup_{z \in K} |\alpha(z)|$. Thus,

$$|f(x, y)| = |\alpha(x)| > \sup_{(z,w) \in K \times L} |f(z, w)|.$$

This shows that $K \times L$ is holomorphically convex in $M \times N$. □

The classical Runge approximation theorem states that any holomorphic function on a simply connected domain $U \subset \mathbb{C}$ can be approximated, uniformly on U, by entire functions. The Runge approximation theorem was generalised to arbitrary open Riemann surfaces by H. Behnke and K. Stein [2]. We state this version below (Theorem 2.2.6); for a proof, see [7, §25].

Definition 2.2.5. Let X be an open Riemann surface. An open subset $Y \subset X$ is *Runge* in X if no connected component of $X \setminus Y$ is compact.

Theorem 2.2.6. *Let X be an open Riemann surface and $Y \subset X$ be Runge. Then any holomorphic function on Y can be approximated, uniformly on compact subsets of Y, by holomorphic functions on X.*

The following theorem is a significant strengthening of Theorem 2.2.6 (cf. Remark 2.2.2). For $X = \mathbb{C}$ this result is due to S. N. Mergelyan and the general version (as below) is due to E. Bishop [3].

Theorem 2.2.7. *Let X be an open Riemann surface and $K \subset X$ a holomorphically convex compact subset. Let $f : K \to \mathbb{C}$ be a continuous function that is holomorphic in the interior of K. Then f can be approximated, uniformly on K, by holomorphic functions on X.*

There is a further generalisation of Theorem 2.2.6 that is pertinent, particularly in Chapters 4 and 5. This is known as the Oka-Weil approximation theorem for Stein manifolds.

Definition 2.2.8. A complex manifold S is a *Stein manifold* if it satisfies the following:

(a) Given two points $x \neq x'$ in S, there exists a holomorphic function $f : S \to \mathbb{C}$ such that $f(x) \neq f(x')$, and

(b) for every compact subset $K \subset S$, the holomorphically convex hull $\hat{K}$ is also compact.

Remark 2.2.9. Open Riemann surfaces are precisely the Stein manifolds of one dimension (see [7, §26]).

Many characteristic properties of open Riemann surfaces also hold on Stein manifolds, including the solvability of the $\overline{\partial}$ problem.

Theorem 2.2.10. *If ω is a smooth $(0,1)$-form on a Stein manifold X with $\overline{\partial}\omega = 0$, then there exists a smooth function $f : X \to \mathbb{C}$ such that $\overline{\partial} f = \omega$.*

For a proof and further discussion, see [14, §5.2].

From this, we find the following. The proof is adapted directly from [15].

Theorem 2.2.11. *Let X be a Stein manifold and $f : X \to \mathbb{C}^*$ a continuous map. Then f can be deformed, through continuous maps $X \to \mathbb{C}^*$, into a holomorphic map.*

Proof. Fix an open cover $(U_\alpha)_{\alpha \in I}$ of coordinate balls on X, so that $f|_{U_\alpha} = e^{2\pi i \lambda_\alpha}$ for continuous maps $\lambda_\alpha : U_\alpha \to \mathbb{C}$. Define functions $n_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}$ by $n_{\alpha\beta} := \lambda_\alpha - \lambda_\beta$ for $\alpha, \beta \in I$. Since the maps $n_{\alpha\beta}$ are continuous and integer-valued, they are locally constant; moreover, for $\alpha, \beta, \gamma \in I$, they satisfy the *cocycle condition* $n_{\alpha\beta} + n_{\beta\gamma} = n_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. We only need to find holomorphic functions $\mu_\alpha : U_\alpha \to \mathbb{C}$ such that $n_{\alpha\beta} = \mu_\alpha - \mu_\beta$ on $U_\alpha \cap U_\beta$. Then, the well-defined continuous map $F : X \times [0,1] \to \mathbb{C}^*$ given by

$$F(x,t) = \exp(2\pi i((1-t)\lambda_\alpha(x) + t\mu_\alpha(x)))$$

for $(x,t) \in U_\alpha \times [0,1]$, would be a homotopy which takes f to a holomorphic function $g : X \to \mathbb{C}^*$. We construct $\mu_\alpha : U_\alpha \to \mathbb{C}$ as follows.

Choose a partition of unity (ϕ_α) subordinate to (U_α). We have well-defined smooth functions $\nu_\alpha := \sum_{\gamma \in I} n_{\alpha\gamma}\phi_\gamma$ which, by the cocycle condition, satisfy $\nu_\alpha - \nu_\beta = n_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.

Since $n_{\alpha\beta}$ is locally constant, we see that $\overline{\partial}\nu_\alpha - \overline{\partial}\nu_\beta = \overline{\partial} n_{\alpha\beta} = 0$ on $U_\alpha \cap U_\beta$. Thus we have a well-defined smooth (0,1)-form ω on X given by $\omega = \overline{\partial}\nu_\alpha$ on U_α. By Theorem 2.2.10, there is a smooth function $u : X \to \mathbb{C}$ such that $\overline{\partial} u = \omega$. Then, the functions $\mu_\alpha = \nu_\alpha - u$ are holomorphic on U_α, and satisfy $n_{\alpha\beta} = \mu_\alpha - \mu_\beta$ on $U_\alpha \cap U_\beta$. □

Note that, in calling a continuous function on a compact subset of a complex manifold *holomorphic*, we mean that it is defined and holomorphic on some open neighbourhood of the set.

The following is the Oka-Weil approximation theorem for Stein manifolds.

Theorem 2.2.12. *Let S be a Stein manifold, and let $K \subset S$ be a holomorphically convex, compact subset. If f is a holomorphic function on K, then f can be approximated, uniformly on K, by holomorphic functions on S.*

For a proof, see [14, Corollary 5.2.9].

2.2.2 Line bundles and 1-forms

Let X be a Riemann surface and $\{(U_i, z_i)\}_{i\in I}$ a covering by coordinate neighbourhoods. We begin by describing the construction of the so-called canonical line bundle of X.

On $U_i \cap U_j$ we have nonvanishing holomorphic functions $g_{ij} := dz_j/dz_i$. We construct, as laid out in [7], a holomorphic line bundle $p : E \to X$ with a holomorphic atlas $(h_i : E_{U_i} \to U_i \times \mathbb{C})_{i\in I}$ of E whose transition functions are g_{ij}.

Let $\tilde{E} := \bigcup_{i\in I} U_i \times \mathbb{C} \times \{i\}$ and equip $\tilde{E}$ with the topology induced from $X \times \mathbb{C} \times I$ (where I has the discrete topology). Define an equivalence relation on $\tilde{E}$ by setting $(x, t, i) \sim (x', t', j)$ when $x = x'$ and $t = g_{ij}(x)t'$.

We define $E = \tilde{E}/\sim$ and equip E with the quotient topology. We have a continuous map $p : E \to X$ induced by the projection $\tilde{E} \to X$. Let $\rho : \tilde{E} \to E$ be the canonical quotient map. Then we have

$$E_{U_i} = p^{-1}(U_i) = \rho(U_i \times \mathbb{C} \times \{i\})$$

and evidently, by restricting ρ to $U_i \times \mathbb{C} \times \{i\}$, we get a homeomorphism $U_i \times \mathbb{C} \times \{i\} \to E_{U_i}$.

The local trivialisations $\{h_i : E_{U_i} \to U_i \times \mathbb{C}\}_{i\in I}$ are given by the inverses of these homeomorphisms.

The line bundle so constructed is called the *holomorphic cotangent bundle* of X, and denoted $T^{*(1,0)}(X)$.

We can establish that the sheaf of holomorphic sections of $T^{*(1,0)}(X)$ is isomorphic to the sheaf of holomorphic 1-forms. (Note that there are different ways to define holomorphic 1-forms; we follow the definition laid out in [7, §9].) Let $U \subset X$ be open. First, consider a holomorphic 1-form ω defined on U and any coordinate neighbourhood (U_i, z_i) of X such that $U_i \cap U \neq \varnothing$. We have a holomorphic function $f_i : U_i \cap U \to \mathbb{C}$ such that $\omega = f_i\, dz_i$ on $U_i \cap U$. The collection $\{f_i : U_i \cap U \to \mathbb{C}\}_{i\in I}$ defines a holomorphic section of $T^{*(1,0)}(X)$ over U. Conversely, if we have a holomorphic section represented by a family $\{g_i : U_i \cap U \to \mathbb{C}\}_{i\in I}$ of holomorphic functions, we have a holomorphic 1-form on U given by $\omega = g_i\, dz_i$ on each $U_i \cap U$.

Now, note that the following fundamental theorem holds on every open Riemann surface.

Theorem 2.2.13. *Every holomorphic vector bundle on an open Riemann surface is trivial.*

For a proof, see [7, §30].

The above theorem allows us to draw the following significant result.

Theorem 2.2.14. *Every open Riemann surface X admits a nonvanishing holomorphic 1-form.*

Proof. By Theorem 2.2.13 every holomorphic vector bundle on X is trivial – in particular, the holomorphic cotangent bundle $p : E \to X$ is trivial. Fix a global linear trivialisation $h : E \to X \times \mathbb{C}$. We can define a nowhere-vanishing holomorphic section $F : X \to E$ by $h(F(x)) = (x, 1)$ for each $x \in X$. The global section F corresponds to a holomorphic 1-form that is defined on X and vanishes nowhere. □

Remark 2.2.15. By a mirror construction, we also get the *antiholomorphic cotangent bundle* $T^{*(0,1)}(X)$ of X, with transition functions $\overline{g_{ij}} = d\bar{z}_j/d\bar{z}_i$. Accordingly, antiholomorphic versions of Theorems 2.2.13 and 2.2.14 also hold.

2.2.3 Subharmonic exhaustion functions

In this section, let X be a Riemann surface and $Y \subset X$ an open subset.

Definition 2.2.16. A C^2 function $u : Y \to \mathbb{C}$ is *harmonic* if, with respect to any local coordinate $z = x + iy$, we have

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0.$$

If Y is simply connected and u is real valued, then u is the real part of a holomorphic function on Y.

The *Dirichlet problem* on X is the following. Let $f : \partial Y \to \mathbb{R}$ be a continuous function. We wish to find a continuous function $u : \overline{Y} \to \mathbb{R}$ which coincides with f on ∂Y and is harmonic in Y. Note that, if $\overline{Y}$ is compact and $Y \neq X$, then using the maximum principle we can show that, if a solution to the Dirichlet problem exists, it is unique.

Let $\mathrm{Reg}(Y)$ denote the set of all relatively compact, open subsets of Y on which a solution to the Dirichlet problem exists for any arbitrary continuous boundary condition. If $D \in \mathrm{Reg}(Y)$ and $u : Y \to \mathbb{R}$ is continuous, denote by $P_D(u) : \overline{D} \to \mathbb{R}$ the function which solves the Dirichlet problem on $\overline{D}$ (with boundary values $u|_{\partial D}$), and coincides with u on $Y \setminus D$. With this, we define the following.

Definition 2.2.17. A continuous function $u : Y \to \mathbb{R}$ is subharmonic (resp. strictly subharmonic) if $P_D(u) \geq u$ (resp. $P_D(u) > u$) on Y for all $D \in \operatorname{Reg}(Y)$.

For C^2 functions, we have a well-known equivalent definition of subharmonicity given by the following theorem.

Theorem 2.2.18. *A C^2 function $u : Y \to \mathbb{R}$ is subharmonic if and only if, with respect to any local coordinate $z = x + iy$,*

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u \geq 0. \tag{$*$}$$

Remark 2.2.19. For strict subharmonicity, the inequality in $(*)$ is replaced with strict equality.

We recall that, for a Hausdorff space M, a function $\omega : M \to \mathbb{R}$ is an *exhaustion function* if, for each $a \in \mathbb{R}$, the set $\{x \in M : \omega(x) < a\}$ is relatively compact in M.

The following theorem is fundamental.

Theorem 2.2.20. *Every open Riemann surface X admits a smooth, strictly subharmonic exhaustion function.*

We highlight the main ideas of the proof of Theorem 2.2.20 by providing a proof sketch below. For a more complete and detailed proof, see [20, §2.14].

Outline of proof. We sketch the construction of a smooth exhaustion function $\phi : X \to \mathbb{R}$ which satisfies, with respect to any local coordinate $z = x + iy$,

$$\Delta \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi > 0.$$

As the main step of the proof, we establish the following: for each point $p \in X$, there exists a smooth function $\alpha_p : X \to [0, \infty)$ such that $\Delta \alpha_p \geq 0$ on X, $\alpha_p(p) > 0$ and $\Delta \alpha_p(p) > 0$. Moreover, if $K \subset X$ is a smooth submanifold with boundary, we may choose $\operatorname{supp}(\alpha_p) \subset X \setminus K$.

Let $\chi : \mathbb{R} \to [0, \infty)$ be a smooth function such that $\chi = 0$ on $(-\infty, 0]$ and $\chi, \chi', \chi'' > 0$ on $(0, \infty)$. Given $r \in (0, 1)$, consider the smooth function $\gamma : \mathbb{P}^1 \to [0, \infty)$ defined by $\gamma(z) = \chi(e^{-|z|^2/r^2} - e^{-1/r^2})$ for $z \in \mathbb{C}$, and $\gamma(\infty) = 0$.

For any $a \in \mathbb{C}$ and $q \in (0, \infty)$, denote by $D(a, q) \subset \mathbb{C}$ the open disc of radius q that is centred at a. The function γ is such that $\gamma > 0$ on $D(0, 1)$ and $\gamma = 0$ on $\mathbb{P}^1 \setminus D(0, 1)$. Moreover, γ is strictly subharmonic on the open annulus $D(0, 1) - \overline{D(0, r)}$.

More generally, for any nonempty, relatively compact, open subset U of $D(0, 1)$, there exists a smooth function $\beta : \mathbb{P}^1 \to [0, \infty)$ such that $\beta > 0$ on $D(0, 1)$, $\beta = 0$ on $\mathbb{P}^1 \setminus D(0, 1)$ and β is strictly subharmonic on $D(0, 1) \setminus \overline{U}$. We can find this by fixing a point $c \in U$ and letting $\beta := \gamma \circ \Phi$ for sufficiently small $r \in (0, 1)$, where $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ is the automorphism of $\mathbb{P}^1$ defined by $z \mapsto (z - c)/(1 - \bar{c}z)$.

With these preparations in place, we can construct, for a given point $p \in X \setminus K$, a nonnegative smooth function α_p as above. Let $(V_m)_{m \in \mathbb{N}}$ be a locally finite collection of relatively compact, open subsets of $X \setminus K$ such that, for any $m \in \mathbb{N}$, $V_m \cap V_{m+1} \neq \varnothing$, and so that there exists a biholomorphism of a neighbourhood of V_m onto a neighbourhood of $\overline{D(0, 1)}$ which maps V_m onto $D(0, 1)$. Assume that $p \in V_1$. Choose a family of nonempty open sets $(W_m)_{m \in \mathbb{N} \cup \{0\}}$ with disjoint closures such that $p \in W_0 \subset\subset V_1$, and for each $m \in \mathbb{N}$, $W_m \subset\subset (V_m \cap V_{m+1})$. By the discussion above, we can find a family $(\beta_m)_{m \in \mathbb{N}}$ of smooth functions $\beta_m : X \to [0, \infty)$ such that $\beta_m = 0$ on $X \setminus V_m$, $\Delta\beta_m \geq 0$ on $X \setminus W_m$, and $\beta_m > 0$ and $\Delta\beta_m > 0$ on $\overline{W_{m-1}}$. Then, using induction, choose a sequence of positive constants $(K_m)_{m \in \mathbb{N}}$ such that, for each $m \in \mathbb{N}$, $\Delta\left(\sum_{j=1}^{m} K_j\beta_j\right) \geq 0$ on $X \setminus W_m$, and $\Delta\left(\sum_{j=1}^{m} K_j\beta_j\right) > 0$ on $\overline{W_0}$. Let $\alpha_p := \sum_{j=1}^{\infty} K_j\beta_j$; this function satisfies the desired properties.

Fix a strictly positive exhaustion function ρ on X. Let $(S_n)_{n \in \mathbb{N}}$ be a compact cover of X such that S_n lies in the interior of S_{n+1} for each $n \in \mathbb{N}$. For each $p \in X$ there exists $\nu \in \mathbb{N}$ such that $p \in S_\nu \setminus S_{\nu-1}$, and a smooth function $u_p : X \to [0, \infty)$ such that $\Delta u_p \geq 0$ on X, $\operatorname{supp}(u_p) \subset X \setminus S_{\nu-1}$, and $u_p > \rho$ and $\Delta u_p > \rho$ on a relatively compact neighbourhood $V_p \subset X \setminus S_{\nu-1}$ of p (we obtain this by setting $u_p := R\alpha_p$ for sufficiently large $R > 0$). In this way, choose a family of points $(p_j)_{j \in \mathbb{N}}$ in X with associated functions $(u_{p_j})_{j \in \mathbb{N}}$ and neighbourhoods $(V_{p_j})_{j \in \mathbb{N}}$ such that $(V_{p_j})_{j \in \mathbb{N}}$ is a locally finite cover of X.

The function $\phi := \sum_{j=1}^{\infty} u_{p_j}$ satisfies $\phi > \rho$ and $\Delta\phi > \rho$ on X. Thus, ϕ is a smooth, strictly subharmonic exhaustion function on X. □

2.3 Morse theory

Throughout this section, let X denote an open Riemann surface.

Note that most of the results in this section hold on arbitrary smooth manifolds, but for our purposes we consider noncompact Riemann surfaces.

Definition 2.3.1. If, at a critical point $p \in X$ of a smooth function $f : X \to \mathbb{R}$, the Hessian matrix $H_p f$ is nonsingular, then p is said to be a *nondegenerate* critical point. Otherwise, p is a *degenerate* critical point.

Definition 2.3.2. Let $f : X \to \mathbb{R}$ be a smooth function and $p \in X$ a nondegenerate critical point. The (Morse) index of p is the maximal dimension of a subspace of $T_p X$ on which $H_p f$ is negative definite.

We introduce the following notation. For a smooth function $g : X \to \mathbb{R}$ and a real number $a \in \mathbb{R}$ we let

$$X_a := g^{-1}((-\infty, a]) = \{x \in X : g(x) \leq a\}.$$

Lemma 2.3.3. (Morse lemma). *Let $f : X \to \mathbb{R}$ be a smooth real-valued function and $p \in X$ a nondegenerate critical point of f with index k_p. Then, there is a neighbourhood U of p and a chart $\phi : U \to \mathbb{C}$ with $\phi(p) = 0$, such that, for $z = x + iy \in \phi(U)$,*

$$f \circ \phi^{-1}(z) = \begin{cases} f(p) + x^2 + y^2 & \text{if } k_p = 0, \\ f(p) + x^2 - y^2 & \text{if } k_p = 1, \\ f(p) - x^2 - y^2 & \text{if } k_p = 2. \end{cases}$$

A proof can be found in [19, Lemma 2.2]. It follows that any nondegenerate critical point is isolated. From this, we note that there are at most countably many nondegenerate critical points on X, for any compact subset of X only has finitely many such points.

Definition 2.3.4. A smooth real-valued function $f : X \to \mathbb{R}$ is a *Morse function* if it has no degenerate critical points.

We focus on exhaustion functions which are Morse. As the following fundamental theorems illustrate, a Morse exhaustion function on an open Riemann surface X gives information on the homotopy type of X.

Theorem 2.3.5. *Let g be a smooth, real valued function on X, and let $a, b \in \mathbb{R}$ be such that $a < b$. If $g^{-1}([a,b])$ is a compact subset of X containing no critical points of g, then X_a is a strong deformation retract of X_b so that, in particular, the inclusion map $X_a \hookrightarrow X_b$ is a homotopy equivalence.*

Theorem 2.3.6. *Let $h : X \to \mathbb{R}$ be a smooth function, and let $p \in X$ be a nondegenerate critical point with index k. Set $c = f(p) \in \mathbb{R}$, and assume that $h([c-\epsilon, c+\epsilon])$ is compact for some $\epsilon > 0$. Then, for sufficiently small values of ϵ, the homotopy type of $X_{c+\epsilon}$ is the homotopy type of $X_{c-\epsilon}$ with a k-cell attached.*

For proofs of Theorems 2.3.5 and 2.3.6, see [19].

By definition, for a Morse exhaustion function on f on X, any critical point is nondegenerate, and $f^{-1}([a,b])$ is compact for $a < b$. Thus, when there is at most one critical point in $f^{-1}([a,b])$, the change in the homotopy type of the sublevel sets when going from X_a to X_b is fully described by the above theorems.

We now describe the Whitney C^k topology on the space of C^k functions $X \to \mathbb{R}$. Let

1. $\Phi = (U_i, \phi_i)_{i \in I}$ be a locally finite collection of charts on X,
2. $K = (K_i)_{i \in I}$ be a compact cover of X such that $K_i \subset U_i$ for each $i \in I$,
3. $\mathcal{V} = (V_i)_{i \in I}$ be an open cover of $\mathbb{R}$, and
4. $\epsilon = (\epsilon_i)_{i \in I}$ be a family of positive numbers.

If $f : X \to \mathbb{R}$ is a C^k function such that $f(K_i) \subset V_i$ for all $i \in I$, then the corresponding *strong basic neighbourhood* $\mathcal{N}^k(f; \Phi, \mathcal{V}, K, \epsilon)$ is the set of C^k functions $g : X \to \mathbb{R}$ such that, for each $i \in I$, $g(K_i) \subset V_i$, and

$$\left\| D^r(f \circ \phi_i^{-1})(x) - D^r(g \circ \phi_i^{-1})(x) \right\| < \epsilon_i$$

for all $x \in \phi_i(K_i)$ and each $r = 0, 1, \ldots, k$.

Sets constructed in this way form a basis for the so-called *Whitney C^k* topology (or the *strong C^k* topology) on the space of C^k functions $X \to \mathbb{R}$. We denote this topological space by $C^k_s(X, \mathbb{R})$.

The Whitney C^∞ topology on the space of smooth functions $X \to \mathbb{R}$ (which we denote by $C^\infty(X, \mathbb{R})$) is the union of the topologies induced by the inclusions $C^\infty(X, \mathbb{R}) \hookrightarrow C^k_s(X, \mathbb{R})$ for each $k \geq 0$.

The following theorem is a fundamental result from Morse theory.

Theorem 2.3.7. *The Morse functions on X form a dense subset of the space $C^k_s(X,\mathbb{R})$ of smooth, real-valued functions on X.*

For a proof, see [10, Chapter 2].

With this, we obtain the following well known and useful strengthening of Theorem 2.2.20.

Theorem 2.3.8. *Every open Riemann surface X admits a smooth, strictly subharmonic Morse exhaustion function.*

Proof. Let $f : X \to \mathbb{R}$ be a smooth, strictly subharmonic exhaustion function. Fix an atlas $\Phi = (U_i, \phi_i)_{i\in I}$ on X which is locally finite. Let $\Phi' = (U'_i, \phi_i|_{U'_i})_{i\in I}$ be a finer atlas on X such that $U'_i \subset\subset U_i$ for each $i \in I$.

Write $K_i := \overline{U'_i}$, so that $K = (K_i)_{i\in I}$ is a compact cover of X. By Theorem 2.2.18, on each chart (U_i, ϕ_i) of Φ,

$$\Delta f = \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\right) f \circ \phi_i^{-1} > 0,$$

where $\phi_i(z) = x_i + iy_i$. Thus, to each $i \in I$, we may associate a positive number $\epsilon_i < 1$ such that

$$\frac{\partial^2(f\circ\phi_i^{-1})}{\partial x_i^2} + \frac{\partial^2(f\circ\phi_i^{-1})}{\partial y_i^2} - 2\epsilon_i > 0 \quad \text{on } \phi_i(K_i).$$

Consider the neighbourhood $\mathcal{N}$ of f in $C^\infty(X,\mathbb{R})$ (with the Whitney C^∞ topology) comprising all smooth functions $g : X \to \mathbb{R}$ such that, for each $i \in I$,

$$\left\| D^r(f\circ\phi_i^{-1})(z) - D^r(g\circ\phi_i^{-1})(z)\right\| < \epsilon_i$$

for all $z \in \phi_i(K_i)$ and $r = 0, 1, 2$.

Using Theorem 2.3.7, choose a Morse function $g \in \mathcal{N}$ on X.

On $\phi_i(K_i)$ we have

$$\left\|\frac{\partial^2(f\circ\phi_i^{-1})}{\partial x_i^2} - \frac{\partial^2(g\circ\phi_i^{-1})}{\partial x_i^2}\right\| < \epsilon_i \text{ and } \left\|\frac{\partial^2(f\circ\phi_i^{-1})}{\partial y_i^2} - \frac{\partial^2(g\circ\phi_i^{-1})}{\partial y_i^2}\right\| < \epsilon_i,$$

and thus,

$$\frac{\partial^2(g\circ\phi_i^{-1})}{\partial x_i^2} + \frac{\partial^2(g\circ\phi_i^{-1})}{\partial y_i^2} > \frac{\partial^2(f\circ\phi_i^{-1})}{\partial x_i^2} + \frac{\partial^2(f\circ\phi_i^{-1})}{\partial y_i^2} - 2\epsilon_i > 0.$$

In other words, $g : X \to \mathbb{R}$ is a smooth, strictly subharmonic Morse function.

It remains to show that g is an exhaustion function on X. For any $a \in \mathbb{R}$, $\{x \in X : g(x) \leq a\} \subset \{x \in X : f(x) \leq a + 1\}$. Since f is an exhaustion function, the set $\{x \in X : f(x) \leq a + 1\}$ is relatively compact. This shows that g is also an exhaustion function on X. □

Remark 2.3.9. One main characteristic of a strictly subharmonic Morse exhaustion function $f : X \to \mathbb{R}$ is that every critical point p of f has an index of either 0 or 1 – indeed, $H_p f$ is positive definite on the subspace of T_pX spanned by $(1, -1)$ (where we have identified T_pX with $\mathbb{R}^2$). This feature is advantageous in certain arguments which rely on Morse theory; for example, the proofs of Theorems 4.2.8 and 5.2.1.

Remark 2.3.10. Note that we can choose a strictly subharmonic Morse exhaustion function $f : X \to \mathbb{R}$ so that, if p and q are distinct critical points of f in X, then $f(p) \neq f(q)$. We use this property in Theorems 4.2.8 and and 5.2.1; details are as follows.

Let $g : X \to \mathbb{R}$ be a positive, strictly subharmonic Morse exhaustion function. For each critical point p of g, take a coordinate disc U_p centred at p so that, if p and q are distinct critical points, $U_p \cap U_q = \varnothing$. Let $t_1, t_2, t_3, \dots$ be the critical values of g, ordered so that $t_n < t_{n+1}$ for each $n \geq 1$.

Consider an arbitrary $n \geq 1$. The value t_n is associated with a finite set of critical points $p_1^n, \dots, p_k^n$ of g. For ease of notation, we write $U_j^n := U_{p_j^n}$. Let $\phi_j^n : X \to [0, 1]$ be a smooth function which is compactly supported on U_j^n, and equals 1 on a smaller open disc $V_j^n \subset U_j^n$ containing p_j^n. Choose $0 < \epsilon_1^n < \dots < \epsilon_k^n < t_{n+1} - t_n$ small enough so that

(a) $\Delta(g + \epsilon_j^n\, \phi_j^n) > 0$ for all $j = 1, \dots k$, and

(b) if $x \in U_j^n$, then $d(g + \epsilon_j^n\, \phi_j^n)|_x$ vanishes if and only if $x = p_j^n$.

Note that dg is bounded away from $0 \in \mathbb{C}$ on $\overline{U_j^n} \setminus V_j^n$, allowing for condition (b). We can ask for condition (a) because Δg has a positive minimum on U_j^n.

For each $n \in \mathbb{N}$, let $\Psi_n := \sum\limits_i \epsilon_i^n\, \phi_i^n$. Then, set $f := g + \sum\limits_{n=1}^{\infty} \Psi_n$.

The map f is a well-defined, strictly subharmonic exhaustion function on X. It also has the desired property that if p and q are distinct critical points of f, then $f(p) \neq f(q)$.

2.4 Implicit function theorems

The inverse and implicit function theorems are fundamental (and equivalent) results from real analysis. The inverse function theorem gives a precise condition for a function to be invertible on a neighbourhood of a point in its domain, and the implicit function theorem allows us to represent relations by functions of several real (or complex) variables. There are multiple versions of the implicit function theorem – tighter assumptions will lead to stronger conclusions (for a summary, see [4]).

We first state, for the record, the classical versions of the inverse and implicit function theorems, and a corollary which allows us to define submanifolds based on constraints. We then present a variant of the classical implicit function theorem which is pertinent to the proof of the main result in Chapter 5.

Theorem 2.4.1. (Inverse function theorem.) *Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^n$ be a smooth map. If the differential df is an isomorphism at a point $x \in U$, then there exist open neighbourhoods $U' \subset U$ of x and $V' \subset \mathbb{R}^n$ of $f(x)$ such that $f(U') = V'$, and $f|_{U'} : U' \to V'$ is a diffeomorphism.*

Theorem 2.4.2. (Implicit function theorem.) *Let $U \subset \mathbb{R}^{n+m}$ be open, and let $f : U \to \mathbb{R}^m$ be a smooth map. Assume that at a point $(x, y) \in U$, $f(x, y) = 0$ and the differential $df|_{(x,y)}$ is surjective. Then there exist open neighbourhoods $V \subset \mathbb{R}^n$ of x and $W \subset \mathbb{R}^m$ of y, with a unique function $g : V \to W$ which is smooth, and satisfies $f(t, g(t)) = 0$ for all $t \in V$, so that, in particular, $g(x) = y$.*

Corollary 2.4.3. *Let $U \subset \mathbb{R}^{n+m}$ be open and $f : U \to \mathbb{R}^m$ a smooth function. If $y \in f(U)$ is a regular value, that is, $df|_x$ is surjective for all $x \in f^{-1}(y)$, then $f^{-1}(y)$ is a smooth submanifold of $\mathbb{R}^{n+m}$ of dimension n.*

Proofs of the above theorems may be found in any elementary real analysis textbook. For a proof of Corollary 2.4.3, see, for example, [24].

Theorem 2.4.2 is the classical form of the implicit function theorem. By relaxing the conditions of this theorem (in the precise sense below), we reach a broader conclusion, which can be more relevant in certain contexts; for example, in Theorem 5.2.1. The following theorem can be found in [18, pp. 230–231].

Theorem 2.4.4. (Variant of implicit function theorem.) *Let $X \subset \mathbb{R}^n$ be open, P a locally path-connected metric space, and $G : P \times X \to \mathbb{R}^n$ a continuous map. Assume that* *the derivative of G with respect to the second variable, D_2G, exists and is continuous on $P \times X$. Let $(a, b) \in P \times X$ be such that $G(a, b) = 0$ and $D_2G|_{(a,b)} : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then, there exist neighbourhoods $A \subset P$ and $B \subset X$ of a and b respectively, and a unique function $f : A \to B$ such that $G(p, f(p)) = 0$ for all $p \in P$; in particular, $f(a) = b$. Moreover, f is continuous on A.*

The proof uses a parametric version of the contraction mapping principle, which we give below, before proving Theorem 2.4.4.

Proposition 2.4.5. *Let (X, ρ) be a complete metric space, and $B \subset X$ any closed subset. Let (P, μ) be any metric space, and $K : P \times B \to X$ a continuous map which is a uniform contraction in its second variable. In other words, there exists $0 < C < 1$ such that for all $p \in P$ and $x, y \in X$,*

$$\rho(K(p, x), K(p, y))) \leq C\rho(x, y).$$

Then, for each $p \in P$, there exists a unique $s \in B$ such that $K(p, s) = s$, and the map $f : P \to B$, $p \mapsto s$, is continuous.

Proof. If we consider an arbitrary $p \in P$, we know by the classical contraction mapping principle on the map $K(p, \cdot)$, that there exists a unique point $s \in B$ such that $K(p, s) = s$. We only need to show that the map $f : P \to B$, $p \mapsto s$, is continuous.

We do this by showing continuity at each point of P. Accordingly, let $p \in P$ and $\epsilon > 0$ be arbitrary. Since K is continuous in its first variable, we may choose $\delta > 0$ so that if $\mu(p, q) < \delta$, then $\rho\left(K(p, f(p)), K(q, f(p))\right) < (1 - C)\epsilon$. We consider any $q \in P$ such that $\mu(p, q) < \delta$. Then,

$$\begin{aligned} \rho\left(f(p), f(q)\right) &= \rho\left(K(p, f(p)), K(q, f(q))\right) \\ &\leq \rho\left(K(p, f(p)), K(q, f(p))\right) + \rho\left(K(q, f(p)), K(q, f(q))\right) \\ &< (1 - C)\epsilon + C\rho(f(p), f(q)), \end{aligned}$$

which implies that $\rho(f(p), f(q)) < \epsilon$. □

Remark 2.4.6. If the map $K : P \times B \to X$ is *uniformly* continuous, then so too is f – we simply choose $\delta > 0$ independent of $p \in P$ and $f(p) \in B$. That is, δ is such that for *any* $q_1, q_2 \in P$ with $\mu(q_1, q_2) < \delta$, and any $x \in B$, we get $\rho(K(q_1, x), K(q_2, x)) < (1 - C)\epsilon$. The rest of the proof holds without change.

We can now prove the stated variant of the implicit function theorem. Note that while we have stated the theorem for $\mathbb{R}^n$, it is immediate that the statement holds for the affine space $\mathbb{C}^n$ also.

Proof of Theorem 2.4.4. Consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $v \mapsto D_2G|_{(a,b)}(v)$. We define

$$K : P \times X \to \mathbb{R}^n, \quad (p, x) \mapsto x - T^{-1}(G(p, x)).$$

Then, $K(a, b) = b$. Moreover, D_2K exists and is continuous on $P \times X$, and $D_2K|_{(a,b)} = 0$. Thus, we may choose a path-connected neighbourhood $A \times B \subset P \times X$ of (a, b) such that $\left\| D_2K|_{(p,y)} \right\| < \frac{1}{2}$ for each $(p, y) \in A \times B$ (where $\|\cdot\|$ is the Euclidean norm on matrices). We will assume that B is a closed subset of X, and thus complete. By the mean value theorem for functions of several variables, the map K satisfies

$$|K(p, x) - K(p, y)| \leq \tfrac{1}{2} |x - y|$$

for all $p \in A$ and $x, y \in B$. In other words, $K|_{A \times B}$ is a contraction in its second variable with constant $\frac{1}{2}$. Then, by Proposition 2.4.5, for each $p \in A$ there exists a unique $q \in B$ such that $K(p, q) = q$, and the map $f : A \to B$, $p \mapsto q$, is continuous. We note that $K(p, f(p)) = p$ holds if and only if $G(p, f(p)) = 0$, which concludes the proof. □

2.5 The Gunning-Narasimhan theorem

In their paper of 1967, Gunning and Narasimhan proved that every open Riemann surface admits a holomorphic immersion into the complex plane [13].

Specifically, the following theorem was shown to hold on any open Riemann surface.

Theorem 2.5.1. *Let X be an open Riemann surface and let θ be a nonvanishing holomorphic 1-form on X. There exists a holomorphic function $g : X \to \mathbb{C}$ such that the nonvanishing 1-form $e^g\theta$ is exact.*

This result is known as the Gunning-Narasimhan theorem. The proof uses basic homology theory and Riemann surface theory. In this section, we examine the outline of the proof (for details, refer to the original paper [13]).

Let X denote an arbitrary open Riemann surface.

Lemma 2.5.2. *Let D, D' be open subsets of X which are connected, relatively compact and Runge in X. Assume that the boundaries of D and D' are smooth, and $\overline{D} \subset D'$. Then, there exists a basis for the relative homology group $H_1(D', D)$ that is represented by simple, piecewise differentiable loops $\gamma_1, \ldots, \gamma_p$ in D'.*

Sketch of proof. A well-known topological classification (see Seifert and Threlfall [21]) is that every compact orientable surface with boundary is homeomorphic to a sphere with a finite number of handles attached and open discs removed. In particular, any connected component of $\overline{D'} \setminus D$ is homeomorphic to such a surface.

Let W_ν be a connected component of $\overline{D'} \setminus D$. On each handle of W_ν, choose a canonical pair of simple loops that are not nullhomotopic and not homotopic to each other (for a diagram, see [13]; loops for different handles are assumed to be disjoint). For each deleted disc Λ whose boundary is disjoint from $\partial\overline{D}$, choose a loop that forms the boundary of a slightly larger disc in D' which contains Λ. For the remaining discs, denoted by $\Lambda_1, \ldots, \Lambda_k$, choose simple arcs joining $\partial\Lambda_n$ to $\partial\Lambda_{n+1}$. These arcs are taken to be disjoint from each other and from all the other loops hitherto chosen; and, save for their endpoints, they lie in the interior of W_ν. We can extend these arcs through $\overline{D}$ to form simple loops.

There is a finite number of connected components of $\overline{D'} \setminus D$; on each component, we select loops as above. The images of these loops in $H_1(D', D)$ form a generating set for that relative homology group (one way to see this is by identifying $H_1(D', D)$ with the abelianisation of $\pi_1(D'/\overline{D})$). Extract a basis $\gamma_1, \ldots, \gamma_p$ from this set. □

Remark 2.5.3. We must note that the set $K := \overline{D} \cup \Gamma_1 \cup \cdots \cup \Gamma_p$, where Γ_k is the image of γ_k, is holomorphically convex in X – it can be shown that $D' - K$ has no relatively compact connected components. (If such a component U existed, basic homology arguments show that it would be a relatively compact connected component of $X \setminus \overline{D}$, which is a contradiction, for D is Runge in X.)

Lemma 2.5.4. *Let D, D' be connected, relatively compact, Runge subsets of X which have smooth boundaries, and which satisfy $\overline{D} \subset D'$. Let ω be a nonvanishing holomorphic 1-form on X such that $\int_\gamma \omega = 0$ for any piecewise differentiable loop γ in D. Then, given $\epsilon > 0$, there exists a holomorphic function f on X such that $\int_{\gamma'} \omega e^f = 0$ for any piecewise differentiable loop γ' in D', and $\|f\| < \epsilon$ on D.*

Sketch of proof. It is known that the topological groups D', D and D'/D each have the homotopy type of a bouquet of circles. This implies that the exact homology sequence of the pair (D', D) is of the form

$$0 \to H_1(D) \to H_1(D') \to H_1(D', D) \to 0.$$

As these homology groups are free abelian, the exact sequence splits. Let $\gamma_1, \ldots, \gamma_p$ be the loops obtained by applying Lemma 2.5.2 to the pair (D', D). Denote by $\gamma_{p+1}, \ldots, \gamma_q$ the loops associated with the pair $(D, \varnothing)$. The loops $\gamma_1, \ldots, \gamma_q$ represent generators for the homology group $H_1(D')$.

Let Λ_i be the image of $\gamma_i \subset D'$. Take continuous complex-valued functions $u_1, \ldots u_q$ on $\Lambda_1 \cup \ldots \cup \Lambda_q$, with disjoint supports, such that u_i is zero on $\overline{D}$ if $i = 1, \ldots p$, and u_i is zero on γ_j if $i \neq j$. The functions are specially required to satisfy (a) $\int_{\gamma_i} \omega e^{u_i} = 0$ and $\int_{\gamma_i} u_i \omega e^{u_i} \neq 0$ for $i = 1, \ldots p$, and (b) $\int_{\gamma_i} u_i \omega \neq 0$ for $i = p+1, \ldots q$ (elementary complex analysis arguments are used to show the existence of such u_i).

For $i = 1, \ldots, q$, we can construct holomorphic functions $f_i : X \to \mathbb{C}$ which approximate u_i arbitrarily closely on $\Lambda_1 \cup \ldots \cup \Lambda_q$ (we leave out the details; the idea is as follows). Both $K := \overline{D} \cup \Lambda_1 \cup \cdots \cup \Lambda_p$ and $L := \Lambda_{p+1} \cup \cdots \cup \Lambda_q$ are holomorphically convex in X (cf. Remark 2.5.3). For $i = 1, \ldots p$, we extend u_i to K by setting it to be zero on $\overline{D}$, and we apply Proposition 2.2.7 with respect to K. For $i = p+1, \ldots q$ we apply Proposition 2.2.7 twice: first with respect to L, then, with necessary modifications, with respect to K.

Let $a := (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^q$ (the first p coordinates are 1, the rest are 0). Assuming that each f_j is a sufficiently close approximation of u_j, we can find, for any $\delta > 0$, a point $x = (x_1, \ldots, x_q)$ with $|x - a| < \delta$ such that

$$\int_{\gamma_i} \omega e^{x_1 f_1 + \cdots + x_q f_q} = 0 \quad \text{for } i = 1, \ldots, q.$$

Set $f := x_1 f_1 + \cdots + x_q f_q$. By choosing sufficiently close approximations f_j and sufficiently small $\delta > 0$, we ensure that $\|f\| < \epsilon$ on D. □

Let $(D_n)_{n \in \mathbb{N}}$ denote an exhaustion of X by relatively compact, connected Runge subsets with smooth boundaries. Assume that the exhaustion satisfies $\overline{D_n} \subset D_{n+1}$ for each $n \in \mathbb{N}$ (such an exhaustion by Runge subsets always exists; see [7, §23]).

By Lemma 2.5.4, we inductively pick holomorphic functions $(g_k)_{k \in \mathbb{N}}$ on X such that, for each $k \in \mathbb{N}$, $\|g_k\| < 2^{-k}$ on D_k, and $\int_{\gamma'} e^{g_1 + \cdots + g_k} \theta = 0$ for each piecewise differentiable loop γ' in D_{k+1}. Setting $g := \sum_{k=1}^{\infty} g_k$, we have $\int_\gamma e^g\theta = 0$ for any piecewise differentiable loop γ in X; that is, $e^g\theta$ is exact.

Thus we close the sketch of the proof of Theorem 2.5.1. By Theorem 2.2.14, on any open Riemann surface X there is holomorphic 1-form θ which vanishes nowhere, so X admits a holomorphic immersion into the complex plane (simply any function $F : X \to \mathbb{C}$ such that $dF = e^g\theta$).

Remark 2.5.5. The proof of Theorem 2.5.1 is simple for the case $X = \mathbb{C}^*$. We prove this in Chapter 4 in greater detail (cf. Proposition 4.1.4).

Chapter 3

The Whitney-Graustein theorem

Whitney and Graustein's 1937 theorem classifies immersions of the circle in the plane up to regular homotopy by the winding numbers of their tangential maps. Today this result can be viewed as a relatively simple consequence of Gromov's one-dimensional h-principle for ample differential relations.

In this chapter, we present a proof of the basic h-principle for ample differential relations, through a systematic development of the methods of convex integration, and observe how it naturally implies the Whitney-Graustein theorem. For this, I have distilled and combined material mainly from [5, 9, 22], and supported it with further mathematical details at various parts for elucidation. The theory presented in this chapter supports and renders context to the main objective of the thesis, which is to find a parametric holomorphic analogue of the Whitney-Graustein theorem.

3.1 Introduction: the one-dimensional h-principle

We first state some definitions and acquaint ourselves with the one-dimensional h-principle for ample differential relations. The basic version of this h-principle will be used to prove the Whitney-Graustein theorem.

Definition 3.1.1. Let $A \subset \mathbb{R}^n$. For $a \in A$, we denote by $\mathrm{IntConv}(A, a)$ the interior of the convex hull of the connected component of A which contains a. We say A is *ample* if, for each $a \in A$, this convex hull is $\mathbb{R}^n$; equivalently, $\mathrm{IntConv}(A, a) = \mathbb{R}^n$.

Definition 3.1.2. Let P be a compact Hausdorff space and let $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a trivial vector bundle. Let $\mathcal{R}$ be a subset of E – in the case where P is a smooth manifold and E is a trivial smooth bundle, we refer to $\mathcal{R}$ as a *differential relation*. We say $\mathcal{R} \subset E$ is *ample* if, for all $p \in P$, $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$ is an ample subset of $\mathbb{R}^n$.

Definition 3.1.3. Let $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a trivial vector bundle as above, and consider $\mathcal{R} \subset E$. By a *section of* $\mathcal{R}$ we mean a continuous section f of E with $f(P) \subset \mathcal{R}$. We topologise the set of sections of $\mathcal{R}$ by equipping it with the compact-open topology, and denote this space by $\Gamma(\mathcal{R})$.

Remark 3.1.4. In a smooth setting (that is, when P is a smooth manifold and E is a trivial smooth vector bundle), we have the obvious analogous definition for the space of smooth sections of a differential relation $\mathcal{R} \subset E$; we denote this space by $\Gamma^\infty(\mathcal{R})$.

While some of the results we present are true in more general settings, we shall, with the goal of proving the Whitney-Graustein theorem in mind, mainly be interested in trivial bundles $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ over the circle $\mathbb{R}/\mathbb{Z}$, and the space of continuous sections $\Gamma(\mathcal{R})$ for a differential relation $\mathcal{R} \subset E$ that is open and ample.

Definition 3.1.5. Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a trivial vector bundle over the circle $\mathbb{R}/\mathbb{Z}$, and let $\mathcal{R} \subset E$ be a differential relation. A *solution of* $\mathcal{R}$ is a C^1 section $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ of E such that $f' \in \Gamma(\mathcal{R})$. The set of solutions of $\mathcal{R}$, which we denote by $\mathrm{Sol}(\mathcal{R})$, forms a topological space with the C^1 topology.

For reference, we describe the C^1 topology on the space $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Assume $f \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Take $\epsilon > 0$, a compact set $K \subset \mathbb{R}/\mathbb{Z}$ and an open set $V \subset \mathbb{R}^n$ such that $f(K) \subset V$. Consider the set $B^1(f; K, V, \epsilon)$ consisting of functions $g \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that

(i) $g(K) \subset V$, and

(ii) $\left\| f^{(k)} - g^{(k)} \right\|_K < \epsilon$ for $k = 0, 1$.

Sets constructed in this way form a neighbourhood subbasis for f in the so-called C^1 topology. We thus acquire an induced topology on $\mathrm{Sol}(\mathcal{R})$.

Here is a version of the one-dimensional h-principle, which is stronger than is necessary to prove the Whitney-Graustein theorem.

Theorem 3.1.6. *Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a trivial bundle and let $\mathcal{R} \subset E$ be an open and ample differential relation. Then, the natural map*

$$J : \mathrm{Sol}(\mathcal{R}) \to \Gamma(\mathcal{R}),$$

that is given by differentiation is a weak homotopy equivalence.

We pause to remark that if we were to take this powerful result for granted, we could prove the theorem of Whitney and Graustein with little effort. However, in the interests of proving the Whitney-Graustein theorem completely, presenting the basic ideas of convex integration theory and also minimising technical intricacies, we take a different approach. We give a full proof of the basic form of this h-principle (which is that J induces a surjection at the level of path components) using methods of convex integration theory. We then show that the induced map on path components is injective for the case where $\mathcal{R} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2_*$ which proves the Whitney-Graustein theorem.

I claim no originality for the mathematics that is presented in this chapter; my contribution lies in the choice and presentation of the material alone. The techniques of convex integration theory were developed by Gromov in [11, 12]. I gratefully acknowledge a significant reference for this material, Vincent Borrelli's lecture notes on one-dimensional convex integration [5] – Borrelli leaves out an intensive, technical approach in favour of intuitive explanations and helpful pictures; and although I have rather aimed for a rigorous, mathematically complete approach in this chapter, it would not have been possible without the understanding I acquired from this source. My exposition is, at times (particularly with parts of the proof of Lemma 3.2.4), also borrowed from the book of David Spring [22]. The proof of Whitney-Graustein, as presented in Section 3.5, I obtained from H. Geiges' book [9].

3.2 C-structures and the fundamental lemma

Consider an open subset $\mathcal{R} \subset \mathbb{R}^n$ with $\sigma \in \mathcal{R}$ and $z \in \mathrm{IntConv}(\mathcal{R}, \sigma)$. Let $g : [0, 1] \to \mathcal{R}$ be a continuous loop with basepoint σ and $z \in \mathrm{IntConv}(g[0, 1])$. We shall say that such a loop *strictly surrounds* z.

It is easy to construct such a loop g. Since $z \in \mathrm{IntConv}(\mathcal{R}, \sigma)$, there exists an n-simplex in $\mathbb{R}^n$ whose interior contains z, and whose vertices $y_0, \ldots, y_n$ lie in the connected component of $\mathcal{R}$ which contains σ. Any continuous loop λ in $\mathcal{R}$ with basepoint σ and which passes through $y_0, \ldots, y_n$ strictly surrounds the point z.

By setting $g := \lambda * \lambda^{-1}$, we have a *contractible* loop that strictly surrounds z. The contraction is explicitly described by the homotopy $g_u : [0,1] \to \mathcal{R}, u \in [0,1]$,

$$g_u(s) = \begin{cases} g(s) & s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}, 1], \\ g(\frac{u}{2}) & s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{cases}$$

(For reference, we describe the $*$ notation. If $u, v : [0,1] \to X$ are curves in a topological space X, the product curve $u * v : [0,1] \to X$ is defined by setting $u * v\,(t) = u(2t)$ when $t \in \left[0, \frac{1}{2}\right]$ and $u * v\,(t) = v(2t-1)$ when $t \in \left[\frac{1}{2}, 1\right]$.)

Definition 3.2.1. Let P be a compact Hausdorff space and $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ a trivial vector bundle, with $\mathcal{R} \subset E$ an open subset. Let $f \in \Gamma(E)$ and $\beta \in \Gamma(\mathcal{R})$ be such that $f(p) \in \operatorname{IntConv}(\mathcal{R}_p, \beta(p))$ for all $p \in P$. A *C-structure* over $B \subset P$ with respect to f, β is a pair of continuous maps (g, G) where:

(a) $g : [0,1] \times B \to \mathcal{R}$ is such that, for all $b \in B$, $g(\cdot, b)$ is a contractible loop in $\mathcal{R}_b$ that is based at $\beta(b)$ and that strictly surrounds $f(b)$, and

(b) $G : [0,1] \times [0,1] \times B \to \mathcal{R}$ describes the contraction of the loop $g(\cdot, b)$ to the constant loop of value $\beta(b)$; that is, for each $b \in B$, $G(\cdot, \cdot, b)$ lies in $\mathcal{R}_b$ and satisfies $G(\cdot, 1, \cdot) = g(\cdot, b)$, $G(\cdot, 0, \cdot) = \beta(b)$, and $G(0, \cdot, b) = G(1, \cdot, b) = \beta(b)$.

We topologise the set of C-structures by equipping it with the compact-open topology. It will be useful to make precise the following observation.

Proposition 3.2.2. *Let $E = P \times \mathbb{R}^n \to P$ be a trivial bundle as above, with $\mathcal{R} \subset E$. Assume we have maps $f \in \Gamma(E)$ and $\beta \in \Gamma(\mathcal{R})$ with respect to which we have two C-structures (g_0, G_0) and (g_1, G_1) over $K \subset P$ and $L \subset P$ respectively. Then, there is a homotopy $g_s : [0,1] \times (K \cap L) \to \mathcal{R}$, $s \in [0,1]$, such that g_s is a contractible loop in $\mathcal{R}_b$ with basepoint $\beta(b)$ and strictly surrounding f(b).*

Proof. If $K \cap L = \varnothing$, the claim is vacuously satisfied, so we assume that $K \cap L \neq \varnothing$. Using the contractibility of the loops g_0 and g_1, we define, for $(t, b) \in [0,1] \times (K \cap L)$,

$$g_s(t, b) = \begin{cases} g_0(t, b) * G_1(t, 2s, b) & s \in [0, \frac{1}{2}], \\ G_0(t, 2 - 2s, b) * g_1(t, b) & s \in [\frac{1}{2}, 1]. \end{cases}$$

This homotopy satisfies the desired properties. □

Remark 3.2.3. We see from the construction above that the loops g_s are continuously contractible with respect to s; so we in fact have a homotopy of C-structures $(g_s, G_s)_{s\in[0,1]}$ on $K \cap L$ which connects (g_0, G_0) to (g_1, G_1) (where it is understood that the maps g_0, g_1, G_0, G_1 are restricted to $K \cap L$).

Now we can prove the so-called *fundamental lemma*. This is a key stepping stone in the proof of the h-principle. Let us say we have, over a compact Hausdorff space P, a trivial bundle $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$. If $\mathcal{R} \subset E$ is an open subset, the fundamental lemma guarantees the existence of a C-structure (h, H) over P with respect to any $z \in \Gamma(E)$ and $\sigma \in \Gamma(\mathcal{R})$, provided only that $z(p) \in \mathrm{IntConv}(\mathcal{R}_p, \sigma(p))$ for each $p \in P$. Moreover, the fundamental lemma gives a representation of z as a Riemann integral with integrand h, which is of particular interest, given that h is a function with image in $\mathcal{R}$.

Lemma 3.2.4. (Fundamental lemma.) *Let P be a compact Hausdorff space, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ a trivial bundle and $\mathcal{R} \subset E$ an open subset such that, for all $p \in P$, $\mathcal{R}_p$ is a nonempty open subset of $\mathbb{R}^n$. Let $\sigma \in \Gamma(\mathcal{R})$ and $z \in \Gamma(E)$ be such that $z(p) \in$* $\mathrm{IntConv}(\mathcal{R}_p, \sigma(p))$ *for all $p \in P$. Then there exists a C-structure (h, H) over P with respect to z, σ which satisfies*

$$z = \int_0^1 h(s, \cdot)\, ds.$$

Proof. We begin by showing the existence of local maps $h_{\mathcal{U}} : [0,1] \times \mathcal{U} \to \mathcal{R}$ (over open sets $\mathcal{U} \subset P$) satisfying the property that for all $b \in \mathcal{U}$, $h_{\mathcal{U}}(\cdot, b)$ is a contractible loop in $\mathcal{R}_b$ with basepoint $\sigma(b)$ and strictly surrounding $z(b)$.

Consider an arbitrary $b \in P$, and take a contractible loop $g_b : [0,1] \to \mathcal{R}_b$ with basepoint $\sigma(b)$ and which strictly surrounds $z(b)$. (We discussed why such a loop exists at the outset of this section.) We denote by $G_b : [0,1] \times [0,1] \to \mathcal{R}_b$ the homotopy which takes the constant loop $\sigma(b)$ to g_b.

Let $L \subset \mathcal{R}_b$ denote the image of g_b (or G_b, for we may assume that they have the same image). Then $\{b\} \times L$ is a compact set in $\mathcal{R}$, and since $\mathcal{R}$ is open, there exists a neighbourhood $\mathcal{D}_b$ of b in P such that $\mathcal{D}_b \times L \subset \mathcal{R}$. For $y \in \mathcal{D}_b$ let g_y be the translation of g_b to $\{y\} \times L$. That is, if $g_b(t) = (b, x_t)$, we set $g_y(t) = (y, x_t)$. We may assume that $\mathcal{D}_b$ is small enough so that g_y strictly surrounds $z(y)$ (we can do this because $\mathrm{IntConv}(g_y([0,1]))$ is an open set and z is continuous). Of course, the loop g_y has basepoint $(y, \sigma(b))$, and not $(y, \sigma(y))$. To correct this, we conjugate with the parametrised line segment l_y which starts at $\sigma(b)$ and ends at $\sigma(y)$, so we have

$$h(t, y) = l_y * g_y * l_y^{-1}(t).$$

Note that, as necessary, we make $\mathcal{D}_b$ smaller to ensure that l_y is contained in both $\mathcal{R}_y$ and in $\mathcal{R}_b$. By construction, we can see that the loops $h(\cdot, y)$ may be continuously deformed in $\mathcal{R}_y$ to the constant loop with value $\sigma(y)$; in other words, given any $b \in P$, there exists a neighbourhood $\mathcal{U}$ of b on which a C-structure $(h_{\mathcal{U}}, H_{\mathcal{U}})$ exists.

Since P is compact, we only need a finite number of such open sets (together with C-structures) to cover P. We then use the contractibility of the loops to piece together the local maps into a well-defined global structure – we do this by induction, the details are as follows.

Let $\mathcal{U} = (U_i)_{i=0,\ldots,m}$ be an open cover of P on which C-structures with respect to z, σ exist. Let $\mathcal{W} = (W_i)_{i=0,\ldots,m}$ be a finer open cover such that $\overline{W_i} \subset U_i$. We will assume that sets U_i and W_i are both connected. Consider any two open sets from the cover $\mathcal{U}$ – without loss of generality, take U_0 and U_1, and assume that $U_0 \cap U_1 \neq \varnothing$ (if the intersection is empty, the result is evident). Denote by (g_0, G_0) the C-structure on U_0 and by (g_1, G_1) the C-structure on U_1. By Proposition 3.2.2, we have a homotopy of C-structures $(g_s, G_s)_{s\in[0,1]}$on $U_0 \cap U_1$.

Fix open subsets $B_0, B_1 \subset P$ such that $\overline{W_0} \subset B_0 \subset \overline{B_0} \subset U_0$ and $\overline{W_1} \subset B_1 \subset \overline{B_1} \subset U_1$. Also choose a larger open set $B_0' \subset U_0$ which is relatively compact in U_0, with $\overline{B_0} \subset B_0'$. Let $\xi : B_0 \cup B_1 \to [0, 1]$ be a continuous map such that $\xi = 1$ on $B_1 \setminus B_0'$ and $\xi = 0$ on B_0.

Define $H : [0, 1] \times [0, 1] \times (B_0 \cup B_1) \to \mathcal{R}$ by $H(s, t, b) = G_{\xi(b)}(s, t, b)$. The map H is continuous, for ξ is continuous.

If we define $h : [0, 1] \times (B_0 \cup B_1) \to \mathcal{R}$ by $h(s, t, b) = H(s, 1, b)$, then (h, H) is a C-structure on $B_0 \cup B_1$ with respect to z, σ.

Assume now that for any set consisting of k open sets on P (together with C-structures on each such open set) we can suitably modify the homotopies, as above, to obtain a C-structure on the union of these sets – this is our inductive hypothesis. In particular, we assume that a C-structure has been constructed on an open neighbourhood U_0' of $\overline{W_0} \cup \cdots \cup \overline{W_{k-1}}$. We relabel U_k as U_1', and we repeat the arguments above with U_0' and U_1'. Hence, by induction, we may find a global C-structure (h, H) with respect to z, σ on P.

It remains to reparametrise h to ensure that

$$z = \int_0^1 h(s, \cdot)\, ds.$$

For clarity, let us relabel the C-structure above as (g, G) until we reparametrise as required.

Fix an arbitrary $b \in P$. We have, by assumption, that $g(\cdot, b)$ is a loop in $\mathcal{R}_b$ that strictly surrounds $z(b)$. In particular, this means that we can find a neighbourhood U of b together with a partition $0 < s_1 < s_2 < \cdots < s_{n+1} < 1$ of $[0, 1]$ such that:

(i) for any $y \in U$, $g(s_1, y), g(s_2, y), \ldots, g(s_{n+1}, y)$ form the vertices of an n-simplex $\Delta(y)$ in $\mathbb{R}^n$,

(ii) $z(y)$ is an interior point of the n-simplex $\Delta(y)$. [1]

Note that we can certainly find a partition and neighbourhood U such that (i) is satisfied, for $\mathrm{IntConv}(g[0, 1], b)$ is a nonempty *open* set in $\mathbb{R}^n$. Likewise, we can ask for (ii) because z and g are continuous maps and $z(b)$ lies in the *interior* of $\Delta(b)$.

Since g and z are both continuous, the barycentric coordinates of $z(y)$ in the n-simplex $\Delta(y)$ can be expressed as continuous positive functions on U.

For positive numbers η_1, η_2 which are suitably small (as we specify later), choose, for each integer $1 \leq i \leq n + 1$, a function $f_i : [0, 1] \to (0, \infty)$ satisfying

(i) $f_i < \eta_1$ on $[0, 1] \setminus [s_k - \eta_2, s_k + \eta_2]$,

(ii) $\int_0^1 f_i(s)\, ds = 1$.

For $y \in U$ define

$$b_i(y) := \int_0^1 g(s, y) f_i(s)\, ds.$$

By a suitable choice of f_i, $b_i(y)$ can be made arbitrarily close to $g(s_i, y)$. We may shrink U as necessary and choose η_1 and η_2 small enough so that $b_1(y), \ldots, b_{n+1}(y)$ for $y \in U$ define the vertices of an n-simplex $\Delta'(y)$ in $\mathbb{R}^n$ whose interior contains $z(y)$ (once again, this is possible because of the continuity of z and each b_i, and the fact that $z(b)$ is chosen to lie in the *interior* of $\Delta'(b)$).

[1]Parts of the following exposition are borrowed from [22], pp. 29-31.

By the continuity of the functions $b_1, \ldots, b_{n+1}$ and z, the barycentric coordinates of z (with respect to $\Delta'(y)$) are continuous, strictly positive functions on U. From these continuous coordinates, we get family of functions $p_i : P \to [0,1]$ for $1 \leq i \leq n+1$ satisfying the following:

(a) each p_i is compactly supported in U,

(b) there is a neighbourhood $W(b)$ of the fixed b such that $\overline{W(b)} \subset U$ and on which the family (p_i) are the barycentric coordinates of z with respect to Δ'; that is, if $y \in W(b)$ then

$$z(y) = \sum_{i=1}^{n+1} p_i(y)\, b_i(y) \text{ with } \sum_{i=1}^{n+1} p_i(y) = 1. \tag{$*$}$$

The collection $(W(b))_{b \in P}$ is an open cover of P. The space P is compact, so we pick a finite subcover $(W_j)_{j \in J}$. Since we have the above construction for each W_j, for clarity we rewrite $(*)$ as

$$z(y) = \sum_{i=1}^{n+1} p_i^j(y)\, b_i^j(y) \text{ with } \sum_{i=1}^{n+1} p_i^j(y) = 1.$$

We also assign the notation f_i^j to the functions we constructed earlier, to indicate the open set to which they are associated.

Let $(q_j : P \to [0,1])$ be a partition of unity subordinate to $(W_j)_{j \in J}$. Define, for any $y \in P$, a map $\alpha_y : [0,1] \to (0, \infty)$ by

$$\alpha_y(s) = \sum_{j \in J} q_j(y) \sum_{i=1}^{n+1} p_i^j(y)\, f_i^j(s).$$

We see easily that the map α_y is positive for each $y \in P$ – on each W_j which contains y, we have $\sum_{i=1}^{n+1} p_i^j(y)\, f_i^j(s) > 0$.

We also see that $\int_0^1 \alpha_y(s)\, ds = 1$. Moreover, for each $y \in P$,

$$\begin{aligned}
\int_0^1 g(s,y)\, \alpha_y(s)\, ds &= \sum_{j \in J} q_j(y) \sum_{i=1}^{n+1} p_i^j(y) \int_0^1 g(s,y)\, f_i^j(s)\, ds \\
&= \sum_{j \in J} q_j(y) z(y) \\
&= z(y).
\end{aligned}$$

Let $\lambda : [0,1] \times P \to [0,1] \times P$ be the continuous function defined by

$$\lambda(t,b) = (\lambda_1(t,b), \lambda_2(t,b)) = \left(\int_0^t \alpha_b(r)\, dr\,,\, b\right).$$

Since $\lambda_1(0,b) = 0, \lambda_1(1,b) = 1$ and $\dfrac{\partial \lambda_1}{\partial t} > 0$, we know, for fixed $b \in P$, that $\lambda_1(\cdot, b)$ has a continuous inverse $\mu_b : [0,1] \to [0,1]$. The map $\lambda^{-1} : [0,1] \times P \to [0,1] \times P$ given by $\lambda^{-1}(t,b) = (\mu_b(t),\, b)$ is the inverse of λ. Since $[0,1] \times P$ is compact, λ is in fact a *closed*, bijective, continuous map. Thus λ^{-1} is continuous.

Define

$$h : [0,1] \times P \to \mathcal{R}, \quad (s,b) \mapsto g(\lambda^{-1}(s,b)).$$

Then, by a change of coordinates $t = \lambda_1(s,y)$, we get

$$\begin{aligned} z(y) &= \int_0^1 g(s,y)\, \alpha_y(s)\, ds \\ &= \int_0^1 h(t,y)\, dt, \quad y \in P. \end{aligned}$$

Finally note that h is continuous and satisfies $h(0,\cdot) = h(1,\cdot) = \sigma$.

By setting

$$H : [0,1] \times [0,1] \times P \to \mathcal{R}, \quad (s,t,b) \mapsto G(\mu_b(s), t, b),$$

we have a C-structure (h,H) on P with the desired integral property. □

If, in the setting above, P is a smooth manifold, and the maps σ and z happen to be C^∞, then we can ask for the map $h : P \times [0,1] \to \mathcal{R}$ to also be C^∞. This is not pertinent to the Whitney-Graustein theorem, but for completeness we provide the details below.

Lemma 3.2.5. *Let P be a compact smooth manifold, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ a trivial smooth vector bundle and $\mathcal{R} \subset E$ an open differential relation such that, for all $p \in P$, $\mathcal{R}_p$ is a nonempty subset of $\mathbb{R}^n$. Let $\sigma \in \Gamma^\infty(\mathcal{R})$ and $z \in \Gamma^\infty(E)$ be such that for all $p \in P$, $z(p) \in \mathrm{IntConv}(\mathcal{R}_p, \sigma(p))$. Then there exists a C^∞ map $k : P \times [0,1] \to \mathcal{R}$ such that:*

(a) $k(\cdot, 0) = k(\cdot, 1) = \sigma$,

(b) $k(p,\cdot)$ *is a contractible loop in* $\mathcal{R}_p$ *which strictly surrounds* $z(p)$, *and*

(c) $z(p) = \int_0^1 k(p,s)\,ds$.

Proof. From Lemma 3.2.4, we already have a map $h : P \times \mathbb{R}/\mathbb{Z} \to \mathcal{R}$ satisfying properties (a), (b) and (c), but we are not guaranteed that it is C^∞. We take the steps below to smooth the map h from the lemma.

If $\epsilon > 0$ is given, there exists, by the Whitney approximation theorem (see, for example, [17]), a smooth map $h_\epsilon : P \times \mathbb{R}/\mathbb{Z} \to \mathcal{R}$ with $h_\epsilon(\cdot, 0) = \sigma$, such that for all $p \in P$, $h_\epsilon(p,\cdot) \subset \mathcal{R}_p$, $\|h - h_\epsilon\| < \epsilon$, and $h_\epsilon(p,\cdot)$ is contractible (that is, nullhomotopic in $\mathcal{R}_p$) and strictly surrounds $z(p)$.

Set

$$z_\epsilon(p) := \int_0^1 h_\epsilon(p,t)\,dt,$$

and define

$$k_\epsilon : P \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n, \quad (p,t) \mapsto h_\epsilon(p,t) + (z(p) - z_\epsilon(p))f(t),$$

for a smooth function $f : [0,1] \to [0,\infty)$ such that $f(0) = f(1) = 0$ and $\int_0^1 f(t)\,dt = 1$.

Then, we have

$$\int_0^1 k_\epsilon(p,t)\,dt = z(p).$$

Consider an arbitrary $p \in P$. Since $\mathcal{R}_p$ is open, we can choose ϵ small enough so that $k_\epsilon(p,\cdot)$ is contractible, strictly surrounds $z(p)$, and its image lies inside $\mathcal{R}_p$. Since k_ϵ is a continuous map, there exists a neighbourhood V of p such that $k_\epsilon(y,\cdot)$ lies inside $\mathcal{R}_p \cap \mathcal{R}_y$ for all $y \in V$. This is possible because $\mathcal{R}$ is open and the image of the loop $(p, k_\epsilon(p,\cdot))$ is compact – expressly, we find an open set $W \times A \subset \mathcal{R}$ containing the image of $(p, k_\epsilon(p,\cdot))$; thus, for all y in some neighbourhood $W' \subset W$ of p, the image of $k_\epsilon(y,\cdot)$ lies entirely within A. This holds for all sufficiently small $\epsilon > 0$.

The manifold P can be covered with *finitely* many open sets constructed in this fashion. Thus we can choose ϵ that is independent of the point $p \in P$. Simply set $k(p,t) := (p, k_\epsilon(p,t))$ and we have a smooth map with the desired properties. □

3.3 C^0-closeness and convex integration

Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a trivial bundle and let $\mathcal{R} \subset E$ be an open differential relation. Say we are given $\sigma \in \Gamma(\mathcal{R})$ (resp. $\Gamma^\infty(\mathcal{R})$) and a C^1 (resp. C^∞) map $f_0 : \mathbb{R}/\mathbb{Z} \to$

$\mathbb{R}^n$ such that $f_0'(s) \in \mathrm{IntConv}(\mathcal{R}_s, \sigma(s))$ for all $s \in \mathbb{R}/\mathbb{Z}$. Then Lemma 3.2.4 and Lemma 3.2.5 guarantee the existence of a C^0 (resp. C^∞) map $h : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathcal{R}$ such that, for all $s \in \mathbb{R}/\mathbb{Z}$, $h(\cdot, s)$ is a contractible loop in $\mathcal{R}_s$ based at $\sigma(s)$ and

$$f_0' = \int_0^1 h(t, \cdot)\, dt.$$

Definition 3.3.1. The map

$$F_N : [0,1] \to \mathbb{R}^n, \quad t \mapsto f_0(0) + \int_0^t h(Ns, s)\, ds,$$

for $N \in \mathbb{N}$ is said to be obtained from f_0 by *convex integration.*

We have $F_N'(t) = h(Nt, t) \in \mathcal{R}_t$. The convex integration process lends F_N another special property: by increasing N, the map F_N can be made arbitrarily close to the initial map f_0 in the C^0 sense. Both of these properties are necessary for proof of the h-principle, and we make the statement of the latter precise in the Propositions 3.3.2 and 3.3.3 below.

Proposition 3.3.2. *Let $f_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a C^1 map as above. If $\epsilon > 0$, then for all sufficiently large $N \in \mathbb{N}$, the map $F_N : [0,1] \to \mathbb{R}^n$ obtained from f_0 by convex integration as above, satisfies $\|F_N - f_0\| < \epsilon$.*

Proof. Let (h, H) be the C-structure associated with f_0 as above; in particular so that $f_0'(t) = \displaystyle\int_0^1 h(s,t)\, ds$ and $F_N(t) = f_0(0) + \displaystyle\int_0^t h(Ns, s)\, ds$.

We define step functions $l : [0,1] \to \mathbb{R}^n$ and $k : [0,1] \to \mathbb{R}^n$ as follows. If $t \in [0,1]$, let

$$l(t) = f_0(0) + \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} f_0'\left((i-1)/N\right) \quad \text{and} \quad k(t) = f_0(0) + \int_0^{\frac{\lfloor Nt \rfloor}{N}} h(Ns, s)\, ds.$$

(Here, $\lfloor Nt \rfloor$ denotes the largest integer n such that $n \leq Nt$.) We first show that $\|l - f_0\|$ and $\|k - F_N\|$ are arbitrarily small for a sufficiently large integer N.

Since h is continuous on the compact set $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, it is bounded; let $M > 0$ be such that $\|h\| < M$. Then, for any $t \in [0,1]$,

$$|k(t) - F_N(t)| = \left| \int_{\frac{\lfloor Nt \rfloor}{N}}^t h(Ns, s)\, ds \right| \leq M \left| \frac{\lfloor Nt \rfloor}{N} - t \right| \leq M \cdot \frac{1}{N};$$

and as M is independent of t, this bound is uniform.

Now consider the auxiliary step function $l_0 : [0,1] \to \mathbb{R}^n$ given by $t \mapsto f_0\left(\frac{\lfloor Nt \rfloor}{N}\right)$. Since f_0 is uniformly continuous on the compact set $\mathbb{R}/\mathbb{Z}$, the value $\left| f_0(t) - f_0\left(\frac{\lfloor Nt \rfloor}{N}\right)\right|$ can be chosen arbitrarily and uniformly small over all $t \in [0,1]$, provided N is sufficiently large. We rewrite $l_0(t)$ as the telescoping sum $f_0(0) + \sum_{i=1}^{\lfloor Nt \rfloor} (f_0(i/N) - f_0((i-1)/N))$. Since f_0' is uniformly continuous, we can find, by the mean value theorem, an arbitrarily small bound for

$$|l_0(t) - l(t)| = \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} \left(\frac{f_0(i/N) - f_0((i-1)/N)}{N} - f_0'((i-1)/N) \right)$$

which holds uniformly for all $t \in [0,1]$, and sufficiently large $N > 0$. Expressly, given $\epsilon' > 0$, let N be large enough so that, for all $t \in [0,1]$, if $|s-t| \leq 1/N$, then $|f_0'(t) - f_0'(s)| < \epsilon'$. By the mean value theorem, $|l_0(t) - l(t)| < \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} \epsilon' \leq \epsilon'$. Hence $\|l - f_0\| \to 0$ as $N \to \infty$.

To prove the proposition, it remains only to show that the value $\|l - k\|$ is arbitrarily small, when N is chosen sufficiently large. Through a change of variables, we have

$$\begin{aligned} k(t) &= f_0(0) + \sum_{i=1}^{\lfloor Nt \rfloor} \int_{\frac{i-1}{N}}^{\frac{i}{N}} h(Ns, s)\, ds \\ &= f_0(0) + \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} \int_0^1 h\left(u, \frac{i-1+u}{N}\right) du. \end{aligned}$$

Rewrite $l(t)$ as $f_0(0) + \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} \int_0^1 h\left(u, \frac{i-1}{N}\right) du$. Let $\epsilon' > 0$. If $t \in \mathbb{R}/\mathbb{Z}$ and $\delta > 0$, denote by $\overline{B}_\delta(t)$ the set $\{t' \in \mathbb{R}/\mathbb{Z} : |t' - t| \leq \delta\}$. Since h is a continuous function on a compact domain, it is uniformly continuous; so, for all $(u,t) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and every sufficiently large $N > 0$, $|h(u,s) - h(u,t)| < \epsilon'$ if $s \in \overline{B}_{1/N}(t)$. Then,

$$\begin{aligned} |k(t) - l(t)| &\leq \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} \int_0^1 \left| h\left(u, \frac{i-1+u}{N}\right) - h\left(u, \frac{i-1}{N}\right) \right| du \\ &< \frac{1}{N} \sum_{i=1}^{\lfloor Nt \rfloor} \epsilon' = \frac{\lfloor Nt \rfloor \epsilon'}{N} \leq \epsilon'. \end{aligned}$$

Since N is independent of t, as $N \to \infty$, the map k converges uniformly to l.

Thus, F_N converges uniformly to f_0 on $[0,1]$. □

In the case where $h : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathcal{R}$ is continuously differentiable, we can find a more precise bound as follows.

Proposition 3.3.3. *If the maps $f_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$, $h : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathcal{R}$ and $F_N : [0,1] \to \mathbb{R}^n$ as above are C^1, we have*

$$\|F_N - f_0\|_{C^0} \leq \frac{1}{N}\left(2\|h\|_{C^0} + \left\|\frac{\partial h}{\partial t}\right\|_{C^0}\right),$$

where $\dfrac{\partial h}{\partial t}$ denotes the partial derivative of h with respect to its second variable.

Proof. Assume $t \in [0,1]$ and let $n := \lfloor Nt \rfloor$.

Define $I_0 := \left[0, \frac{1}{N}\right]$, $I_1 := \left[\frac{1}{N}, \frac{2}{N}\right]$, $\ldots$, $I_{n-1} := \left[\frac{n-1}{N}, \frac{n}{N}\right]$ and $I_n := \left[\frac{n}{N}, t\right]$.

We also associate, to each integer $0 \leq j \leq n$, the integrals

$$S_j := \int_{I_j} h(Nv, v)\, dv \text{ and } s_j := \int_{I_j}\int_0^1 h(u,x)\, du\, dx.$$

Through the change of variables $u = Nv - j$ for $0 \leq j \leq n-1$ we get

$$S_j = \frac{1}{N}\int_0^1 h\left(u, \frac{u+j}{N}\right) du = \int_{I_j}\int_0^1 h\left(u, \frac{u+j}{N}\right) du\, dx.$$

The mean value theorem yields the inequality

$$\left|h\left(u, \frac{u+j}{N}\right) - h(u,x)\right| \leq \left\|\frac{\partial h}{\partial t}\right\|_{C^0} \cdot \left|\frac{u+j}{N} - x\right|.$$

Hence, for $0 \leq j \leq n-1$,

$$\begin{aligned}
|S_j - s_j| &= \left|\int_{I_j}\int_0^1 h\left(u, \frac{u+j}{N}\right) - h(u,x)\, du\, dx\right| \\
&\leq \int_{I_j}\int_0^1 \left|h\left(u, \frac{u+j}{N}\right) - h(u,x)\right| du\, dx \\
&\leq \int_{I_j}\int_0^1 \left\|\frac{\partial h}{\partial t}\right\|_{C^0} \cdot \left|\frac{u+j}{N} - x\right| du\, dx \\
&\leq \left\|\frac{\partial h}{\partial t}\right\|_{C^0} \frac{1}{N}\int_{I_j}\int_0^1 1\, du\, dx \\
&= \left\|\frac{\partial h}{\partial t}\right\|_{C^0} \frac{1}{N^2}.
\end{aligned}$$

We also note that

$$
\begin{aligned}
|S_n - s_n| &= \left| \int_{n/N}^{t} h(Nv, v)\, dv - \int_{n/N}^{t} \int_0^1 h(u, v)\, du\, dv \right| \\
&\leq \left| \int_{n/N}^{t} h(Nv, v)\, dv \right| + \left| \int_{n/N}^{t} \int_0^1 h(u, v)\, du\, dv \right| \\
&\leq \|h\|_{C^0} \cdot \left| \frac{n}{N} - t \right| + \|h\|_{C^0} \cdot \left| \frac{n}{N} - t \right| \\
&\leq 2\, \|h\|_{C^0} \cdot \left| \frac{1}{N} \right|.
\end{aligned}
$$

And so we get

$$
\|F_N - f_0\|_{C^0} \leq \sum_{j=0}^{n} |S_j - s_j| \leq \frac{1}{N} \left(2\, \|h\|_{C^0} + \left\| \frac{\partial h}{\partial t} \right\|_{C^0} \right)
$$

as required. □

3.4 A basic h-principle

The following theorem is a version of Gromov's basic h-principle for ample differential relations. A considerably more complicated parametric version of its proof allows us to obtain the full one-dimensional h-principle that we stated in the introduction. We shall, however, only need the basic principle to prove the Whitney-Graustein theorem, and we can now present the proof in detail.

Theorem 3.4.1. *Let* $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ *be a trivial bundle and let* $\mathcal{R} \subset E$ *be a differential relation that is open and ample. Assume further that for each* $t \in \mathbb{R}/\mathbb{Z}$, $\mathcal{R}_t$ *is an nonempty open subset of* $\mathbb{R}^n$. *If* $\sigma \in \Gamma(\mathcal{R})$, *then there exists a* C^1 *map* $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ *such that:*

(i) $f \in \mathrm{Sol}(\mathcal{R})$,

(ii) f' *is homotopic to* σ *in* $\Gamma(\mathcal{R})$.

Proof. Take any C^1 map $f_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$. By the definition of an ample differential relation, if $t \in \mathbb{R}/\mathbb{Z}$, then $f_0'(t) \in \mathbb{R}^n = \mathrm{IntConv}(\mathcal{R}_t, \sigma(t))$.

We consider the map

$$F : [0,1] \to \mathbb{R}^n, \quad t \mapsto f_0(0) + \int_0^t h(Ns, s)\, ds,$$

obtained by convex integration. The map $h : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathcal{R}$ is the one that is given by the Fundamental Lemma – in particular, such that $h(\cdot, t)$ is a contractible loop in $\mathcal{R}_t$ based at $\sigma(t)$ and

$$f_0' = \int_0^1 h(u, \cdot)\, du.$$

We modify F as follows to make it a loop:

$$f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n, \quad t \mapsto f_0(0) + \int_0^t h(Ns, s)\, ds - t \int_0^1 h(Ns, s)\, ds.$$

Note that if N is sufficiently large, then $f'(t) \in \mathcal{R}_t$. This is because $\mathcal{R}_t$ is open and Proposition 3.3.2 tells us that the value

$$\left| \int_0^1 h(Ns, s)\, ds \right| = |F(1) - f_0(1)|$$

can be made arbitrarily small. In other words, for sufficiently large N, we have that $f \in \mathrm{Sol}(\mathcal{R})$, satisfying requirement (i).

Define, for any $u \in [0,1]$, the map

$$f_u : [0,1] \to \mathbb{R}^n, \quad t \mapsto f_0(0) + \int_0^t h_u(Ns, s)\, ds - u \cdot t \int_0^1 h(Ns, s)\, ds,$$

where h_u is the homotopy that describes the contraction of h (as in part (b) of Definition 3.2.1), with $h_1(s,t) = h(s,t)$ and $h_0(s,t) = \sigma(t)$ for all $s, t \in \mathbb{R}/\mathbb{Z}$. While f_u is not necessarily a loop, we can see that

$$\begin{aligned} f_u'(0) &= \sigma(0) - u \int_0^1 h(Ns, s)\, ds \\ &= \sigma(1) - u \int_0^1 h(Ns, s)\, ds \\ &= f_u'(1). \end{aligned}$$

So $f_u' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$, $u \in [0,1]$, is a homotopy which takes f' to σ. Once again, by Proposition 3.3.2, the term

$$\left| u \int_0^1 h(Ns, s)\, ds \right| = |u|\, |F(1) - f_0(1)|$$

can be made arbitrarily small. Thus $f'_u(t)$ can be made arbitrarily close to $h_u(Nt, t) \in \mathcal{R}_t$ by choosing a sufficiently large N.

In other words, by choosing an appropriate N, we ensure that $f'_u \in \Gamma(\mathcal{R})$. This way we also satisfy requirement (ii). □

Remark 3.4.2. Say $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ is a trivial bundle and let $\mathcal{R} \subset E$ be a differential relation. There is an alternative notion of sections and of solutions of $\mathcal{R}$ which fits more naturally into advanced frameworks. According to this notion, a section of $\mathcal{R}$ is a pair of continuous sections (f_0, f_1) of E where $f_1(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$. A solution of $\mathcal{R}$ is a section of $\mathcal{R}$ such that f_0 is C^1 and $f'_0 = f_1$. Theorem 3.4.1 then tells us that every section of $\mathcal{R}$ can be continuously deformed through sections of $\mathcal{R}$ into a solution of $\mathcal{R}$.

The basic h-principle (Theorem 3.4.1) is often stated succinctly as in the following corollary. Since any two elements σ_1, σ_2 of $\Gamma(\mathcal{R})$ belong to the same path component if and only if $\sigma_1, \sigma_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ are homotopic through sections of $\mathcal{R}$, this corollary is an immediate consequence of Theorem 3.4.1. We provide the elementary details of the proof for completeness.

Corollary 3.4.3. *Let E and $\mathcal{R}$ be as in Theorem 3.4.1. The natural map induced by differentiation*

$$G : \pi_0(\mathrm{Sol}(\mathcal{R})) \to \pi_0(\Gamma(\mathcal{R}))$$

is well defined and surjective.

Proof. Recall that the sets $M_{A,U} = \{f \in \Gamma(\mathcal{R}) \,|f(A) \subset U\}$ (where A is compact in $\mathbb{R}/\mathbb{Z}$ and U is open in $\mathbb{R}^n$) form a subbasis for the compact-open topology on $\Gamma(\mathcal{R})$.

We first note that G is well defined. Say $f_0, f_1 \in \mathrm{Sol}(\mathcal{R})$ belong to the same path component. Let $p : [0, 1] \to \mathrm{Sol}(\mathcal{R})$ be a continuous path such that $p(0) = f_0$ and $p(1) = f_1$.

Define

$$D : \mathrm{Sol}(\mathcal{R}) \to \Gamma(\mathcal{R}), \quad f \mapsto f'.$$

We easily show that D is continuous. Take any $f \in \mathrm{Sol}(\mathcal{R})$ and assume $f' \in \Gamma(\mathcal{R})$ is contained in some $M_{A,U}$ (where A is compact in $\mathbb{R}/\mathbb{Z}$ and U is open in $\mathbb{R}^n$). Choose a compact set B in $\mathbb{R}/\mathbb{Z}$ and an open set $V \subset \mathbb{R}^n$ such that $f(B) \subset V$.

Let $N_{B,V} := \{g \in \mathrm{Sol}(\mathcal{R}) \,|\, g(B) \subset V,\, g'(A) \subset U\}$. Notice that set $N_{B,V}$ is open in $\mathrm{Sol}(\mathcal{R})$.

As $D(N_{B,V}) \subset M_{A,U}$, we see that D is continuous. So $D \circ p$ is a continuous path that connects f'_0 to f'_1. In other words, G is well defined.

We now show that G is surjective. Assume $\sigma \in \Gamma(\mathcal{R})$ and let $[\sigma]$ denote its equivalence class in $\pi_0(\Gamma(\mathcal{R}))$. By Theorem 3.4.1 we have a map $f \in \mathrm{Sol}(\mathcal{R})$ together with a homotopy $h : [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ that takes f' to σ through sections of $\mathcal{R}$.

To establish that G is surjective we only need to see that the path $p : [0,1] \to \Gamma(\mathcal{R})$ given by $p(t) := h(t, \cdot)$ is continuous in $\Gamma(\mathcal{R})$. Let $t \in [0,1]$, and choose an open set $M_{A,U} \subset \Gamma(\mathcal{R})$ such that $p(t) \in M_{A,U}$. In other words, $h(t,a) \in U$ for all $a \in A$, so $\{t\} \times A \subset h^{-1}(U)$.

Since $h^{-1}(U)$ is open in the product topology on $[0,1] \times \mathbb{R}/\mathbb{Z}$ and A is compact in $\mathbb{R}/\mathbb{Z}$, we can find an open set U' in [0,1] such that $(\{t\} \times A) \subset (U' \times A) \subset h^{-1}(U)$.

This tells us directly that $p(U') \subset M_{A,U}$.

Thus p is continuous, which implies that $f' \in [\sigma]$. So G is surjective. □

Remark 3.4.4. As we stated, the map G from Corollary 3.4.3 is bijective, but the proof of injectivity in this generality is far from simple. In the special case $\mathcal{R} := \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2_*$, however, the proof of injectivity is considerably simplified. We present the details in the following section.

3.5 The Whitney-Graustein theorem

Definition 3.5.1. Let $\mathcal{I}(S^1, \mathbb{R}^2)$ denote the space of C^1 immersions $S^1 \to \mathbb{R}^2$ with the C^1 topology. We say that two immersions f_0, f_1 are *regularly homotopic* if they belong to the same path component of $\mathcal{I}(S^1, \mathbb{R}^2)$.

In other words, f_0, f_1 are regularly homotopic if there exists $H : S^1 \times [0,1] \to \mathbb{R}^2$ with $H(\cdot, 0) = f_0$ and $H(\cdot, 1) = f_1$, where for all $t \in [0,1]$, $H(\cdot, t)$ is an immersion.

Theorem 3.5.2. (Whitney-Graustein [25]) *Associating to an immersion $S^1 \to \mathbb{R}^2$ the winding number of its tangential map $S^1 \to \mathbb{R}^2_*$ induces a bijection between the path components of the space $I(S^1, \mathbb{R}^2)$ and the integers. In short,*

$$\pi_0(\mathcal{I}(S^1, \mathbb{R}^2)) \simeq \mathbb{Z}.$$

Proof. Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z}$ be the trivial bundle and define $\mathcal{R} := \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2_*$. Note that $\mathcal{R}$ is open and ample with

$$\mathrm{Sol}(\mathcal{R}) = \mathcal{I}(S^1, \mathbb{R}^2) \text{ and } \Gamma(\mathcal{R}) = C^0(S^1, \mathbb{R}^2_*).$$

It is well known that two continuous maps $\mathbb{R}/\mathbb{Z} \to \mathbb{R}^2_*$ are homotopic if and only their winding numbers are equal; so $\pi_0(C^0(S^1, \mathbb{R}^2_*)) \simeq \mathbb{Z}$.

Corollary 3.4.3 tells us that the map

$$J : \mathrm{Sol}(\mathcal{R}) \to \Gamma(\mathcal{R}), \quad \gamma \mapsto \gamma',$$

induces a surjection of path components. We show below that J in fact induces a bijection of path components.

Let $g_0, g_1 \in \mathcal{I}(S^1, \mathbb{R}^2)$, and assume that there exists a homotopy $H : S^1 \times [0, 1] \to \mathbb{R}^2_*$ with $H(\cdot, 0) = g_0'$ and $H(\cdot, 1) = g_1'$. This assumption means that g_0' and g_1' have the same winding number n.

Let us assume without loss of generality that g_0, g_1 are parametrised by arc length. Then the images of g_0' and g_1' lie in S^1, and we may assume that H lies entirely in $S^1 \subset \mathbb{R}^2_*$.

Consider the C^1 map $F : S^1 \times [0, 1] \to \mathbb{R}^2$ given by

$$F(s, t) = \int_0^s H(u, t)\, du - s \int_0^1 H(u, t)\, du.$$

Notice that (i) the map F is continuous, (ii) $F(s, 0) = g_0(s)$, $F(s, 1) = g_1(s)$ and, (iii) for all $t \in [0, 1]$, $F(\cdot, t)$ is a loop.

So, F is a homotopy of C^1 loops between g_0 and g_1. In fact, we now argue that it is a *regular* homotopy. We have

$$\frac{d}{ds} F(s, t) = H(s, t) - \int_0^1 H(u, t)\, du.$$

If $n \neq 0$, $H(\cdot, t)$ is necessarily nonconstant for any $t \in [0, 1]$. Thus,

$$\left| \int_0^1 H(u, t)\, du \right| < 1.$$

(One could use the equality case of the Cauchy-Schwarz inequality on the L^2 inner product space to verify that if $f : S^1 \to S^1$ is such that $\left| \int_{[0,1]} f(t)\, dt \right| = 1$, then f is constant.) So

$\frac{d}{ds}F(s,t) \neq 0$ for all $s, t \in [0,1]$; that is, when $n \neq 0$, F is a regular homotopy between g_0 and g_1.

Now consider the case $n = 0$.

Let $h_0, h_1 : S^1 \to \mathbb{R}$ be lifts of g_0', g_1' respectively with respect to the universal covering

$$\Phi : \mathbb{R} \to S^1, \quad t \mapsto (\cos t, \sin t).$$

(These lifts are loops precisely because the winding numbers of g_0' and g_1' are zero.)

Modify, if necessary, the initial maps g_0 and g_1 through regular homotopies so that their basepoints match and so that $g_0' = g_1'$ on $[0, \epsilon]$ for some small ϵ. Moreover, our modifications should ensure that g_0 and g_1 are nonconstant on the interval $[0, \epsilon]$. Thus, by choosing the liftings h_0, h_1 to have the same initial point, we get $h_0 = h_1$ on $[0, \epsilon]$.

If we set, for $t \in [0,1]$ and $x \in S^1$, $h_t(x) := (1-t)h_0(x) + th_1(x)$, then the map

$$H : S^1 \times [0,1] \to S^1, \quad (x,t) \mapsto (\cos(h_t(x)), \sin(h_t(x))),$$

is a homotopy between g_0' and g_1' which is nonconstant for each $t \in [0,1]$. By arguing as we did in the previous case, we obtain a map F which is a regular homotopy between g_0 and g_1.

So,

$$\pi_0(\mathcal{I}(S^1, \mathbb{R}^2)) \to \pi_0(C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2_*)$$

is a bijection, which proves the theorem. □

Chapter 4

The basic h-principle for immersions

We let M be an open Riemann surface and θ a nonvanishing holomorphic 1-form on M. The focus of this chapter is on proving that the continuous map $J : \mathcal{I}(M, \mathbb{C}) \to \mathcal{O}(M, \mathbb{C}^*), f \mapsto df/\theta$, induces a surjection at the level of path components.

We present two distinct proofs for this, in Sections 4.1 and 4.2 respectively, but the first proof (using methods of Gunning-Narasimhan) will be examined specially in the setting of $M = \mathbb{C}^*$. While the first proof can be generalised to all open Riemann surfaces (a brief sketch of this is given in Section 2.5), we focus on $\mathbb{C}^*$, for its simplicity renders transparency to the steps, and allows us to glean more information. The methods used in this first proof, however, do not lend themselves in any apparent way to more general parametric theorems. The second proof, presented in Section 4.2, is inspired by methods introduced by Alarcón and Forstnerič in [1] in the proof of the more general basic Oka principle. We work out a detailed, self-contained proof for our special setting. Although ostensibly more advanced, it is more conducive to parametrisation than the first proof. In Chapter 5, we build upon the core ideas of this proof to prove the parametric h-principle, which says that the map J is a weak homotopy equivalence.

4.1 Immersions of the punctured complex plane

In this section we find a complete description of the weak homotopy type of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$, before showing, analogous to Whitney-Graustein and using techniques from [13], that the continuous differentiation map $J : \mathcal{I}(\mathbb{C}^*, \mathbb{C}) \to \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ induces a surjection of path components.

The larger objective of this thesis, as outlined in Chapter 1, is to prove the parametric h-principle for holomorphic immersions of open Riemann surfaces into $\mathbb{C}$; of that, this is the simplest nontrivial case. We devote attention to this special case because of its relative transparency – with ease we find that we can completely describe the weak homotopy type of the target space $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$, and give a clear, self-contained proof of the desired h-principle.

4.1.1 Path components of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ and winding numbers

To any holomorphic map $f : \mathbb{C}^* \to \mathbb{C}^*$ we can associate a winding number $w(f)$ by defining it as the winding number of the loop $\gamma : [0, 2\pi] \to \mathbb{C}^*, t \mapsto f(e^{it})$. Equivalently,

$$w(f) = \frac{1}{2\pi} \int_\gamma \frac{dz}{z}.$$

In this subsection, we show that that two elements $f_1, f_2 \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ belong to the same path component if and only if $w(f_1) = w(f_2)$; in other words, that the map $\kappa : \pi_0(\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)) \to \mathbb{Z}, [f] \mapsto w(f)$, is a well-defined bijection.

Assume first that $f_1, f_2 \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ belong to the same path component. This means that there exists a homotopy $H : [0, 1] \times \mathbb{C}^* \to \mathbb{C}^*$ such that $H(0, \cdot) = f_1, H(1, \cdot) = f_2$ and $H(t, \cdot) \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ for all $t \in [0, 1]$. We know (cf. Section 2.1) that $\pi_1(\mathbb{C}^*)$ is isomorphic to $\mathbb{Z}$, with correspondence given by winding numbers. So, by simply considering the map $G : [0, 2\pi] \times I \to \mathbb{C}^*, G(s, t) = H(e^{it}, t)$, we see that $w(f_1) = w(f_2)$. That is, κ is well defined.

Surjectivity of κ is easy to see: if $n \in \mathbb{Z}$, then the n-th power map $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n$, has winding number n.

We now address injectivity of κ. Consider two maps $f_1, f_2 \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ of equal winding number, and let $\exp : \mathbb{C} \to \mathbb{C}^*$ denote the exponential map. From covering space theory we know that, given any holomorphic map $F : \mathbb{C}^* \to \mathbb{C}^*$, there exists a lifting of F with respect to exp if and only if $F_* \pi_1(\mathbb{C}^*) = 0$. Evidently, this condition is satisfied precisely when F has winding number 0.

Since f_1 and f_2 have the same winding number, the map $h : \mathbb{C}^* \to \mathbb{C}^*$ given by $h := f_1/f_2$ has winding number 0. Let $\tilde{h} : \mathbb{C}^* \to \mathbb{C}$ be a lifting of h with respect to the universal covering map $\exp : \mathbb{C} \to \mathbb{C}^*$. Since h is holomorphic, we know that $\tilde{h}$ is also holomorphic. Moreover, since $\mathbb{C}$ is contractible, h is nullhomotopic through maps in $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$. Let $K : [0,1] \times \mathbb{C}^* \to \mathbb{C}^*$ (with $K(0,\cdot) = h, K(1,\cdot) = 1$) denote such a homotopy; the map $\hat{K} : [0,1] \times \mathbb{C}^* \to \mathbb{C}^*, (t,z) \mapsto f_2(z)\, H(z,t)$, is then a homotopy between f_1 and f_2 with $\hat{K}(t,\cdot) \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ for all $t \in [0,1]$. Thus κ is injective also.

So, we have that $\pi_0(\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)) \simeq \mathbb{Z}$.

4.1.2 The weak homotopy type of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$

Denote by $\mathcal{O}_n(\mathbb{C}^*, \mathbb{C}^*)$ the path component of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ associated with winding number $n \in \mathbb{Z}$. We have seen that $\pi_0(\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)) \simeq \mathbb{Z}$. In this section, we show that $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ has the weak homotopy type of a countable disjoint union of circles.

To do so, we need the following lemma.

Lemma 4.1.1. *The map* $\exp_* : \mathcal{O}(\mathbb{C}^*, \mathbb{C}) \to \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$*, induced by the exponential map, is a covering map.*

Proof. It is simple to see that $\exp_*$ is continuous. Let $g \in \mathcal{O}(\mathbb{C}^*, \mathbb{C})$ and $q = \exp_*(g) \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$. Take any compact set $K \subset \mathbb{C}^*$ and open set $V \subset \mathbb{C}^*$ such that $q \in N_{K,V} := \{p \in \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*) : p(K) \subset V\}$. Write $U := \exp^{-1}(V)$ and $M_{K,U} := \{f \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}) : f(K) \subset U\} \subset \mathcal{O}(\mathbb{C}^*, \mathbb{C})$. We see that $g(K) \subset \exp^{-1}(V)$, that is, $g \in M_{K,U}$. Evidently, $\exp_*(M_{K,U}) \subset N_{K,V}$; this shows that $\exp_*$ is continuous at each point.

Earlier we argued that a map $p \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ has a lifting with respect to the covering map $\exp : \mathbb{C} \to \mathbb{C}^*$ if and only if the winding number of p is zero. Thus, $\exp_* : \mathcal{O}(\mathbb{C}, \mathbb{C}^*) \to \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$ is a continuous surjection.

Assume $p \in \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$ is arbitrary. Define $N := \{q \in \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*) | q(1) \subset W\}$, where $W \subset \mathbb{C}^*$ is a neighbourhood of $p(1)$ that is evenly covered by $\exp : \mathbb{C} \to \mathbb{C}^*$. In other words, we have open sets $(U_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$ such that $\exp^{-1}(W) = \coprod_{n\in\mathbb{Z}} U_n$ and that, for each $n \in \mathbb{Z}$, $\exp : U_n \to W$ is a homeomorphism.

Define $M_n := \{g \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}) | g(1) \subset U_n\}$. If $n \neq k$, then $U_n \cap U_k = \varnothing$, and thus $M_n \cap M_k = \varnothing$. When $n \in \mathbb{Z}$ is given, there exists a unique lifting f of p with respect to $\exp : \mathbb{C} \to \mathbb{C}^*$ which satisfies $f(1) \in U_n$. Moreover, if $g \in M_n$ then $\exp(g(1)) \in V$.

So, $\exp_*^{-1}(N) = \coprod_{n\in\mathbb{Z}} M_n$, and $\exp_* |_{M_n} : M_n \to N$ is a continuous bijection. It is easy to see that it is also open. If $U'_n \subset U_n$ is open and $M'_n := \{g \in \mathcal{O}(\mathbb{C}^*, \mathbb{C}) | g(1) \in U'_n\}$, then $\exp_*(M'_n) = \{q \in \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*) |\, q(1) \in \exp(U'_n)\}$, which is open in $\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$.

So, $\exp_* : \mathcal{O}(\mathbb{C}^*, \mathbb{C}) \to \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$ is a covering map. □

With this, we can establish the weak homotopy type of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$.

Theorem 4.1.2. *The space $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ is weakly homotopy equivalent to a countable disjoint union of circles.*

Proof. Consider the continuous map $K : \mathbb{C}^* \to \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$, where $K(z_0)$ for $z_0 \in \mathbb{C}^*$ is the constant map $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z_0$. We show that K is a weak homotopy equivalence.

Since $\mathcal{O}(\mathbb{C}^*, \mathbb{C})$ is contractible and $\pi_n(\mathcal{O}(\mathbb{C}^*, \mathbb{C})) \to \pi_n(\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*))$ is an isomorphism for $n \geq 2$ by Lemma 4.1.1, we have that $\pi_n(\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)) = 0$ for all $n \geq 2$. Thus we only need to show that the induced map on fundamental groups, $K_* : \pi_1(\mathbb{C}^*) \to \pi_1(\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*))$, is an isomorphism.

We address surjectivity first. Take an element $[\omega] \in \pi_1(\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*))$ represented by a loop $\omega : [0,1] \to \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$, $t \mapsto (\omega_t : \mathbb{C}^* \to \mathbb{C}^*)$. Let $\tilde{\omega} : [0,1] \to \mathcal{O}(\mathbb{C}^*, \mathbb{C})$, $t \mapsto (\tilde{\omega}_t : \mathbb{C}^* \to \mathbb{C})$ be a lifting of the loop with respect to the covering map $\exp_*$. Since $\mathbb{C}$ is contractible, the lifting is homotopic to the loop $\mu : [0,1] \to \mathcal{O}(\mathbb{C}^*, \mathbb{C}), t \mapsto \mu_t$, where $\mu_t : \mathbb{C}^* \to \mathbb{C}, z \mapsto \tilde{\omega}_t(1)$.

Consequently, $[\omega] = [\exp_* \circ \mu] \in \pi_1(\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*))$. Then, the element of $\pi_1(\mathbb{C}^*)$ represented by the loop $[0,1] \to \mathbb{C}^*, t \mapsto \omega_t(1)$ maps to $[\omega]$ under K_*. That is, K_* is surjective.

To see injectivity, consider $[f], [g] \in \pi_1(\mathbb{C}^*)$ represented by loops $f, g : [0,1] \to \mathbb{C}^*$. Assume that there exists a homotopy $H : [0,1] \times [0,1] \to \mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$ such that $H(0, \cdot) = f, H(1, \cdot) = g$. Lift H to $\tilde{H} : [0,1] \times [0,1] \to \mathcal{O}(\mathbb{C}^*, \mathbb{C})$ via the covering map $\exp_*$, and continuously deform $\tilde{H}$ to the continuous map $\tilde{G} : [0,1] \times [0,1] \to \mathcal{O}(\mathbb{C}^*, \mathbb{C})$, where $\tilde{G}(s,t) : \mathbb{C}^* \to \mathbb{C}, z \mapsto \tilde{H}(s,t)(1)$.

Then $G := \exp_* \circ \tilde{G}$ is a homotopy such that $G(0, \cdot) = K \circ f$ and $G(1, \cdot) = K \circ g$. Moreover, since $G(s, \cdot)$ is a loop of constant maps for any $s \in [0,1]$, there exists a continuous map $h : [0,1] \times [0,1] \to \mathbb{C}^*$ with $K \circ h = G$. We have $h(0, \cdot) = f$ and $h(1, \cdot) = g$, so $[f] = [g]$ in $\pi_1(\mathbb{C}^*)$. In other words, K_* is injective.

We have established so far that $\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$ is weakly homotopy equivalent to the punctured plane (and hence the circle). Any two path components of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ are homeomorphic – multiplication by f, where f has winding number n, gives an explicit homeomorphism from $\mathcal{O}_0(\mathbb{C}^*, \mathbb{C}^*)$ to $\mathcal{O}_n(\mathbb{C}^*, \mathbb{C}^*)$. Thus $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ has the weak homotopy type of a countable disjoint union of circles. □

4.1.3 Surjectivity of $\pi_0(\mathcal{I}(\mathbb{C}^*, \mathbb{C})) \to \pi_0(\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*))$

Now equipped with better knowledge about the target space of J, we turn our attention to the induced map $J_* : \pi_0(\mathcal{I}(\mathbb{C}^*, \mathbb{C})) \to \pi_0(\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*))$. In this section, we establish that J_* is surjective.

Take a path component $\mathcal{O}_n(\mathbb{C}^*, \mathbb{C}^*)$ of $\mathcal{O}(\mathbb{C}^*, \mathbb{C}^*)$ associated with $n \in \mathbb{Z} \setminus \{-1\}$. Note that the n-th power map $z \mapsto z^n$ is contained in $\mathcal{O}_n(\mathbb{C}^*, \mathbb{C}^*)$. Clearly, the path component of $\mathcal{I}(\mathbb{C}^*, \mathbb{C})$ that contains the immersion $z \mapsto \dfrac{z^{n+1}}{n+1}$ is mapped to $\mathcal{O}_n(\mathbb{C}^*, \mathbb{C}^*)$ under J_*.

The case of $\mathcal{O}_{-1}(\mathbb{C}^*, \mathbb{C}^*)$ is less obvious. We proceed by reducing the proof of Lemma 4 of [13] to our special case (the details are in Proposition 4.1.4). We also make use of the following observation (the proof is elementary; see [13], Lemma 3):

Lemma 4.1.3. *Given $[a, b] \subset \mathbb{R}$, $c \in \mathbb{C}$ and a continuous complex-valued function f on $[a, b]$, there exists a continuous complex-valued function g with compact support in (a, b) such that*

$$\int_a^b e^{f(x)+g(x)}\, dx = c \quad \textit{and} \quad \int_a^b g(x)\, e^{f(x)+g(x)}\, dx \neq 0.$$

The following proposition is key. For a more general version, see [13].

Proposition 4.1.4. *Let $h : \mathbb{C}^* \to \mathbb{C}^*$ be a holomorphic function. Then there exists a holomorphic map $g : \mathbb{C}^* \to \mathbb{C}$ such that he^g has a primitive on $\mathbb{C}^*$.*

Proof. Let γ be a simple closed curve in $\mathbb{C}^*$ that winds around $0 \in \mathbb{C}$.

By Lemma 4.1.3, we have a continuous function u on the image of $\gamma \subset \mathbb{C}^*$, such that

$$\int_\gamma h(z)\, e^{u(z)}\, dz = 0 \quad \text{and} \quad \int_\gamma u(z)\, h(z)\, e^{u(z)}\, dz \neq 0.$$

Take the entire function ϕ defined by

$$\phi(s) = \int_\gamma h(z)\, e^{su(z)}\, dz, \quad s \in \mathbb{C}.$$

We denote by K the image of γ, and notice that neither of the connected components of $\mathbb{C}^* \setminus K$ are relatively compact in $\mathbb{C}^*$. Then we have, by the Mergelyan approximation theorem, a sequence of holomorphic functions $(w_\nu)_{\nu \in \mathbb{N}}$ on $\mathbb{C}^*$ that converges uniformly to u on $K \subset \mathbb{C}^*$.

Consider an associated sequence of entire functions,

$$\phi_\nu(s) = \int_\gamma h(z)\, e^{s w_\nu(z)}\, dz.$$

Note that the functions ϕ_ν converge to ϕ uniformly on compact subsets of $\mathbb{C}$. Note also that $\phi(1) = 0$ and $\dfrac{\partial \phi}{\partial s}(1) \neq 0$. Thus, by a well-known theorem of Hurwitz, given $\delta > 0$ we can find $\nu_0 \in \mathbb{N}$ and a point $s_0 \in \mathbb{C}^*$ with $|s_0 - 1| < \delta$ such that, setting $g := s_0\, w_{\nu_0}$, we have

$$\int_\gamma h(z)\, e^{g(z)}\, dz = 0. \qquad \square$$

Equivalently, he^g has a primitive on $\mathbb{C}^*$.

Let $h : \mathbb{C}^* \to \mathbb{C}^*$ denote the holomorphic map $z \mapsto 1/z$. Proposition 4.1.4 guarantees the existence of holomorphic functions $f : \mathbb{C}^* \to \mathbb{C}$ and $F : \mathbb{C}^* \to \mathbb{C}$ such that $F' = h\, e^f$. The map F can be explicitly defined by

$$F(z) = \int_1^z h(z)\, e^{f(z)}\, dz,$$

where the integral is taken over any path in $\mathbb{C}^*$ from 1 to $z \in \mathbb{C}^*$. Since e^f has winding number 0 (simply consider the homotopy $e^{tf}, t \in [0,1]$, to see this) the function $g\, e^f$ belongs to $\mathcal{O}_{-1}(\mathbb{C}^*, \mathbb{C}^*)$.

Thus $J_*([F]) \in \mathcal{O}_{-1}(\mathbb{C}^*, \mathbb{C}^*)$, and J_* is surjective.

4.2 Immersions of open Riemann surfaces

Let M be an open Riemann surface, and θ a nonvanishing holomorphic 1-form on M. We prove that the continuous map

$$\mathcal{I}(M, \mathbb{C}) \to \mathcal{O}(M, \mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a surjection of path components. This is the main result of this section, and appears as Theorem 4.2.8.

This is in fact a restatement of the seminal result due to Gunning and Narasimhan [13] (Remark 4.2.9 explains this in detail). In this section we give a different proof, by treating the result as a special case of the basic Oka principle for holomorphic immersions [1, Theorem 2.6].

The format of the proof of Theorem 4.2.8 is adapted from [1]. As a special situation, however, our proof differs from the general sketch in [1], in that some steps can be achieved, as we find, through more direct, self-contained arguments, allowing us to avoid black-box ingredients. If we were to extract the proof for our special case from the sketch in [1] without any modification, we would have a concise outline which refers to some advanced results from Oka theory, including the parametric Oka property – in other words, not a self-contained proof. Instead, we use the proof of [1, Theorem 7.2] as a template and work out a complete proof of our special case. Further explanations and supportive details pertaining to our case have been added throughout, both within the main proof, and by providing preparatory Lemmas 4.2.2, 4.2.4, 4.2.5 and 4.2.7. (Lemma 4.2.3, the period correction lemma, was adapted from [1]; and I have made both simplifications and developments as necessary for this special setting.) Some results in the background chapter are also intended to support this theorem.

We find that this proof of Theorem 4.2.8 lends itself more naturally to parametrisation than the one presented by Gunning and Narasimhan.

Throughout this section and in Chapter 5, we denote by I the unit interval $[0, 1]$.

Definition 4.2.1. Let L be a paracompact space, M an open Riemann surface and θ a nonvanishing holomorphic 1-form on M. We take a compact submanifold with boundary $D \subset M$, and consider simple, piecewise-smooth loops $\gamma_1, \dots, \gamma_l : I \to D$ which form a basis of $H_1(D, \mathbb{Z})$. For a continuous map $f : L \times D \to \mathbb{C}$ with $f(t, \cdot)$ holomorphic for each $t \in L$, we define the *period map* $\mathcal{P} : L \to \mathbb{C}^l$ of f as follows.

Let $\mathcal{P}_i : L \to \mathbb{C}, t \mapsto \int_{C_i} f(t, \cdot)\, \theta$. The period map of f with respect to $\gamma_1, \dots, \gamma_l$ is given by

$$\mathcal{P} : L \to \mathbb{C}^l, \quad t \mapsto (\mathcal{P}_1(t), \dots, \mathcal{P}_l(t)).$$

Lemma 4.2.2. *Let X be an open Riemann surface. Consider a compact subset $D \subset X$ such that the inclusion $\iota : D \to X$ induces an injection of the homology groups $H_1(\cdot, \mathbb{Z})$, and the boundary ∂D can be expressed as a finite union of piecewise-smooth arcs in X. Then D is holomorphically convex in X.*

Proof. We show that $X \backslash D$ contains no relatively compact connected components. Assume for a contradiction that $U \subset X$ is a relatively compact connected component of $X \setminus D$.

Then, $\partial\overline{U}$ is represented by a cycle $\gamma \in Z_1(D)$ whose image in $H_1(X,\mathbb{Z})$ is trivial.

Let θ be a nonvanishing holomorphic 1-form on X. Fix $a \in U$, and let f be a meromorphic function on X which has a simple pole at a and is holomorphic on $X \setminus \{a\}$. Let $V \subset U$ be an open coordinate disc centred at a, and let γ_0 be a simple loop in V whose interior domain contains a. By the residue theorem, $\int_{\gamma_0} f\theta \neq 0$. We orient γ_0 so that the cycle $\gamma + \gamma_0 \in Z_1(D)$ forms a piecewise-smooth boundary of a compact subset $U_0 \subset U \setminus \{a\}$. By Stokes' theorem,

$$\iint_{U_0} d(f\theta) = \int_\gamma f\theta + \int_{\gamma_0} f\theta.$$

(While Stokes' theorem is usually stated for subsets with smooth boundary, we may simply consider, instead of γ, a smooth loop $\gamma_1 : I \to X$ which is homotopic to γ through an arbitrarily small deformation in X, and note that $\int_{\gamma_1} f\theta = \int_\gamma f\theta$.) Since $d(f\theta) = 0$ on $X \setminus \{a\}$, $\int_\gamma f\theta = -\int_{\gamma_0} f\theta \neq 0$, which implies that γ represents a nontrivial element of $H_1(D,\mathbb{Z})$. This is a contradiction, for the induced map $\iota_* : H_1(D,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ is an injection, and γ is nullhomologous in X. □

The following period correction lemma is key to the proof of Theorem 4.2.8.

Lemma 4.2.3. *Let X be an open Riemann surface, $M \subset X$ be a compact submanifold with boundary and $F : M \to \mathbb{C}$ a holomorphic immersion. Fix a nowhere-vanishing holomorphic 1-form θ on X and let $f : M \to \mathbb{C}^*$ be given by $dF = f\theta$. Also choose simple, piecewise-smooth loops $\gamma_1, \dots, \gamma_l : I \to M$ which form a basis of $H_1(M,\mathbb{Z})$. Then, there exists a holomorphic map $\Phi_f : \mathbb{C}^l \times M \to \mathbb{C}^*$, such that $\Phi_f(0,\cdot) = f$ and the period map $\mathcal{P} : \mathbb{C}^l \to \mathbb{C}^l$ of Φ_f with respect to $\gamma_1, \dots, \gamma_l$ has full rank at $0 \in \mathbb{C}^l$.*

Proof. Denote by $C_i \subset M$ the image of γ_i and $C = \bigcup_{j=1}^{l} C_j$. We assume that the subsets C_i meet only at one common point $p \in M$, such that $\gamma_i(0) = \gamma_i(1) = p$.

To each $i = 1, \dots, l$, we associate a continuous function $h_i : C \to \mathbb{C}$ that is identically zero on $C \setminus C_i$ and whose values on C_i we will specify shortly. In particular, the functions h_i vanish at $p \in M$.

We write $\zeta = (\zeta_1, \dots, \zeta_l) \in \mathbb{C}^l$, and define a map

$$\Psi : \mathbb{C}^l \times C \to \mathbb{C}^*, \quad (\zeta, x) \mapsto f(x)\, e^{\zeta_1 h_1(x) + \cdots + \zeta_l h_l(x)}.$$

Notice that $\Psi(0, x) = f(x)$, and if we fix $x \in C$, then $\Psi(\cdot, x) : \mathbb{C}^l \to \mathbb{C}^*$ is a holomorphic function.

We have, for any $x \in C$,

$$\left.\frac{\partial \Psi(\zeta, x)}{\partial \zeta_i}\right|_{\zeta=0} = h_i(x) f(x) \in \mathbb{C}.$$

Note that, when $x \in C \setminus C_i$, we have $\left.\frac{\partial \Psi(\zeta, x)}{\partial \zeta_i}\right|_{\zeta=0} = 0$.

Define $\tilde{\mathcal{P}}_i : \mathbb{C}^l \to \mathbb{C}, \zeta \mapsto \int_{C_i} \Psi(\zeta, \cdot)\,\theta$, and consider the period map of Ψ,

$$\tilde{\mathcal{P}} : U \to \mathbb{C}^l, \quad \zeta \mapsto (\tilde{\mathcal{P}}_1(\zeta), \ldots, \tilde{\mathcal{P}}_l(\zeta)).$$

Clearly, when $j \neq i$, $\left.\frac{\partial \tilde{\mathcal{P}}_i(\zeta)}{\partial \zeta_j}\right|_{\zeta=0}$ vanishes.

We first show that the value

$$\begin{aligned} \left.\frac{\partial \tilde{\mathcal{P}}_i(\zeta)}{\partial \zeta_i}\right|_{\zeta=0} &= \int_{C_i} h_i f\,\theta \\ &= \int_0^1 h_i(\gamma_i(t)) f(\gamma_i(t)) \theta(\gamma_i(t), \dot{\gamma}_i(t))\,dt \end{aligned}$$

is nonzero for all $i = 1, \ldots, l$.

We can do so by choosing the functions h_i appropriately. Let $\eta_i : I \to [0, \infty)$ be a function supported on a neighbourhood $V_i \subset (0,1)$ of $\frac{1}{2}$ such that $\int_0^1 \eta_i(t)\,dt = 1$. We let $h_i \circ \gamma_i = \eta_i$ on C_i and let $h_i \circ \gamma_i = 0$ on $C \setminus C_i$. By making V_i small enough, the value $\int_{C_i} h_i f\,\theta$ can be made arbitrarily close to $f(\gamma_i(\frac{1}{2}))\theta(\gamma_i(\frac{1}{2}), \dot{\gamma}_i(\frac{1}{2})) \neq 0$.

By Lemma 4.2.2, we may use the Mergelyan-Bishop theorem to approximate the functions $h_i : C \to \mathbb{C}$ by holomorphic functions $g_i : M \to \mathbb{C}$ uniformly on C.

We define

$$\Phi_f : \mathbb{C}^l \times M \to \mathbb{C}^*, \quad (\zeta, x) \mapsto f(x)\, e^{\zeta_1 g_1(x) + \cdots + \zeta_l g_l(x)}.$$

If the approximations g_i are sufficiently close to h_i on C, then the corresponding period map

$$\mathcal{P} : \mathbb{C}^l \to \mathbb{C}^l, \quad \zeta \mapsto \left(\int_{C_1} \Phi_f(\zeta, \cdot)\,\theta, \ldots, \int_{C_l} \Phi_f(\zeta, \cdot)\,\theta \right),$$

has full rank at $\zeta = 0$.

Moreover, notice that $\Phi_f(0, x) = f(x)$, and if we fix $x \in M$, then $\Phi_f(\cdot, x) : \mathbb{C}^l \to \mathbb{C}^*$ is a holomorphic function. This concludes our proof. □

Lemma 4.2.4. *Given $\epsilon > 0$, a continuous map $h : I \to \mathbb{C}^*$ and a complex number $z \in \mathbb{C}$, there exists a homotopy $h_s : I \to \mathbb{C}^*, s \in I$, with fixed endpoints, such that $h_0 = h$ and*

$$\int_0^1 h_1(t)\, dt = z.$$

Proof. Let $\eta > 0$ be a positive number, which we will make as small as is necessary. Consider the curves $g_1 : I \to \mathbb{C}^*, g_1(t) = h\left(\frac{t}{2}\right)$, and $g_2 : I \to \mathbb{C}^*, g_2(t) = h\left(\frac{1+t}{2}\right)$. Then $h = g_1 * g_2$.

First consider the case where $z \neq 0$. Let $c : I \to \mathbb{C}^*$ be any arc such that $c(0) = h\left(\frac{1}{2}\right)$ and $c(1) = z$.

Let

$$\tilde{h}_1(s) = \begin{cases} g_1\left(\frac{s}{\eta}\right) & \text{if } s \in [0, \eta], \\ c\left(\frac{s-\eta}{\eta}\right) & \text{if } s \in [\eta, 2\eta], \\ z & \text{if } s \in [2\eta, 1-2\eta], \\ c\left(\frac{1-\eta-s}{\eta}\right) & \text{if } s \in [1-2\eta, 1-\eta], \\ g_2\left(\frac{s-1+\eta}{\eta}\right) & \text{if } s \in [1-\eta, 1]. \end{cases}$$

We see that $\tilde{h}_1$ is homotopic to h, and the value $\left|\int_0^1 h_1(t)\, dt - z\right|$ can be made arbitrarily small. Fix $[a, b] \subset (2\eta, 1-2\eta)$ and let $\alpha : I \to [0, \infty)$ be a continuous function that is supported on $[a, b]$, with $\int_0^1 \alpha(t)\, dt = 1$. (Note that we will keep the curve α fixed throughout, even if we make η smaller.) Let $\epsilon_1, \epsilon_2 \in \mathbb{R}$ be such that

$$\int_0^1 \tilde{h}_1(t)\, dt + \epsilon_1 + i\epsilon_2 = z.$$

Note that, by an appropriate choice of η, we can make $|\epsilon_1|$ and $|\epsilon_2|$ arbitrarily small. Take $\delta > 0$ with $|z| > \delta$. By making η smaller if necessary, we ensure that $|\epsilon_1 M| < \delta/2$ and $|\epsilon_2 M| < \delta/2$, where $M := \max_{t \in I} |\alpha(t)|$. Let $h_1 : I \to \mathbb{C}^*$ be given by

$$h_1(t) = \tilde{h}_1(t) + (\epsilon_1 + i\epsilon_2)\, \alpha(t).$$

Then, h_1 is homotopic to h and satisfies $\int_0^1 h_1(t)\, dt = z$.

In the case where $z = 0$, the idea remains the same. We first construct a function $\tilde{h}_1$, which spends equal amounts of time at -1 and 1 (rather than at z), is homotopic to h, and is such that $\left|\int_0^1 \tilde{h}_1(t)\, dt\right|$ is arbitrarily small. Similar to the above, we modify $\tilde{h}_1$ on a subset $[a, b]$ on which $\tilde{h}_1$ is constant, and obtain a curve h_1 such that $\int_0^1 h_1(t)\,dt = 0$. The details are very much analogous, and so we will not explicitly write them here. □

The following is a simple parametric version of the Oka-Weil approximation theorem.

Lemma 4.2.5. *Let X be a Stein manifold and $K \subset X$ a compact, holomorphically convex subset. For a neighbourhood U of K, let $h : I \times U \to \mathbb{C}$ be a continuous function such that for each $t \in I$, $h(t, \cdot)$ is holomorphic. Then, given $\epsilon > 0$, there exists a continuous map $w : I \times X \to \mathbb{C}$ with $w(t, \cdot)$ holomorphic on X for each $t \in I$, such that $|w - h| < \epsilon$ on $I \times K$.*

Proof. Consider an arbitrary $t \in I$. By Theorem 2.2.12, given $\epsilon > 0$, we can find some holomorphic $f : X \to \mathbb{C}$ such that $|f - h(t, \cdot)| < \epsilon$ on K. Since this is an open condition, there exists some neighbourhood W of t in I such that, for $s \in W$, we have $|f - h(s, \cdot)| < \epsilon$ on K.

Thus we can cover I with open sets $W_1, \dots, W_m$, where each open set W_k is associated with a function f_k as above. Take a partition of unity $(p_k)_{k=1,\dots,m}$ subordinate to this open cover. Let

$$w : I \times X \to \mathbb{C}, \quad (t, x) \mapsto \sum_{k=1}^{m} p_k(t) f_k(x).$$

We have, for $(t, x) \in I \times K$,

$$\begin{aligned} |w(t, x)) - h(t, x)| &= \left| \sum_{k=1}^{m} p_k(t) \left(f_k(x) - h(t, x)\right) \right| \\ &\leq \sum_{k=1}^{m} p_k(t) \left|f_k(z) - h(t, x)\right| < \epsilon. \end{aligned}$$

Thus the map w satisfies the above requirements. □

Remark 4.2.6. Note that, by analogous arguments, we have a parametric version of the Mergelyan-Bishop theorem: that is, if X is an open Riemann surface, $K \subset X$ is holomorphically convex and $h : I \times K \to \mathbb{C}$ is a continuous function such that $h(t, \cdot)$ is holomorphic on the interior of K for each $t \in I$, then, as above, we have a map $w : I \times X \to \mathbb{C}$ which uniformly approximates h on $I \times K$.

We provide one further ingredient for Theorem 4.2.8; the following lemma can be viewed as a parametrisation of the special case of the Oka property with approximation that is of interest to us. By using covering space theory in this special situation, we are able to prove the following result in a straightforward manner, avoiding the advanced arguments used to prove the general Oka property with approximation.

Lemma 4.2.7. *Let X be a Stein manifold and $D \subset X$ a compact, holomorphically convex set. Let $h : I \times X \to \mathbb{C}^*$ be a continuous map and $W \subset X$ a neighbourhood of D such that, for each $t \in I$, $h(t, \cdot)$ is holomorphic on W. Then, given $\epsilon > 0$, there exists a continuous map $f : I \times X \to \mathbb{C}^*$ such that, for all $t \in I$, $f(t, \cdot)$ is holomorphic on X and $|h - f| < \epsilon$ on $I \times D$.*

In the case where D is a strong deformation retract of X, we have the following additional property. There exists a positive $\epsilon' \leq \epsilon$ such that, if $g_0, g_1 : X \to \mathbb{C}^*$ are holomorphic maps with $\|g_0 - h(0, \cdot)\|_D < \epsilon'$ and $\|g_1 - h(1, \cdot)\|_D < \epsilon'$, then f can be chosen to satisfy $f(0, \cdot) = g_0$ and $f(1, \cdot) = g_1$.

Proof. By Theorem 2.2.11, there is a homotopy which takes $h(0, \cdot)$ to a holomorphic map $\gamma : X \to \mathbb{C}^*$ through continuous maps $X \to \mathbb{C}^*$. Define a continuous map $k : I \times X \to \mathbb{C}^*$ by $k(t, x) = h(t, x)/\gamma(x)$. Then k is nullhomotopic, so there exists a continuous lifting $\tilde{k} : I \times X \to \mathbb{C}$ of k with respect to the universal covering map $\exp : \mathbb{C} \to \mathbb{C}^*$. Note that $\tilde{k}(t, \cdot)$ is holomorphic on W for any $t \in I$. By Lemma 4.2.5, there is a continuous map $w : I \times X \to \mathbb{C}$ such that, for each $(t, x) \in I \times D$,

$$|k(t, x) - \exp(w(t, x))| < \epsilon / \|\gamma\|_D . \qquad (*)$$

By setting $f : I \times X \to \mathbb{C}^*$, $(t, x) \mapsto \gamma(x) \cdot \exp(w(t, x))$, we find that $|h - f| < \epsilon$ on $I \times D$.

Now assume that X strongly deformation retracts onto D. Choose $0 < \epsilon' \leq \epsilon$ such that, if $g_1, g_0 : X \to \mathbb{C}^*$ are holomorphic maps satisfying $\|g_0 - h(0, \cdot)\|_D < \epsilon'$ and $\|g_1 - h(1, \cdot)\|_D < \epsilon'$, then $g_0|_D$ (resp. $g_1|_D$) is homotopic through continuous maps to $h(0, \cdot)|_D$ (resp. $h(1, \cdot)|_D$).

Let g_0 and g_1 be as above. Since the space $(X \times \{0, 1\}) \cup (D \times I)$ is a strong deformation retract of $X \times I$ [23, Theorem 6], the homotopy between $g_0|_D$ and $h(0, \cdot)|_D$ extends to a homotopy between g_0 and $h(0, \cdot)$ on X, through maps $X \to \mathbb{C}^*$. Similarly we have a homotopy between g_1 and $h(1, \cdot)$ on X. Thus, g_0/γ and g_1/γ can be lifted with respect to $\exp : \mathbb{C} \to \mathbb{C}^*$ to holomorphic maps $v_0 : X \to \mathbb{C}$ and $v_1 : X \to \mathbb{C}$ respectively. Assuming that the liftings v_0 and v_1 are chosen appropriately close to $\tilde{k}(0,\cdot)$ and $\tilde{k}(1,\cdot)$ respectively on D, we approximate, as above, $\tilde{k}$ by a continuous map $w : I \times X \to \mathbb{C}$ so that $(*)$ holds, but specially choosing $w(0,\cdot) = v_0$ and $w(1,\cdot) = v_1$. The map $f : I \times X \to \mathbb{C}^*$, as defined above, satisfies $|h - f| < \epsilon$ on $I \times D$, $f(0,\cdot) = g_0$ and $f(1,\cdot) = g_1$. □

We now arrive at the main theorem. With further arguments, it is shown to be a restatement of the Gunning-Narasimhan theorem (cf. Remark 4.2.9). However, as outlined in the outset of this section, the proof presented below is different. The format of this proof is inspired by that of the basic Oka principle for holomorphic immersions [1], of which the following theorem is a special case. To elucidate certain points and simplify others (with regard to the nature of our special case), and give a self-contained proof, I have carefully added details to – and at times, necessarily deviated from – the proof outline of the more general principle. This I have done both within the proof below, and by separately introducing the previous ingredient Lemmas 4.2.2, 4.2.4, 4.2.5 and 4.2.7. We note in particular that in our situation allows us to address the critical case of the following proof with more elementary arguments.

Theorem 4.2.8. *Let M be an open Riemann surface. Fix a nowhere-vanishing holomorphic 1-form θ on M. Given any holomorphic map $f : M \to \mathbb{C}^*$, there exists a holomorphic immersion $\tilde{F} : M \to \mathbb{C}$ such that the map $\tilde{f} := d\tilde{F}/\theta : M \to \mathbb{C}^*$ is homotopic to f through holomorphic maps $M \to \mathbb{C}^*$.*

Proof. Choose a coordinate disc $T \subset M$ in a coordinate neighbourhood in M, and let V_0 be an open neighbourhood of T on which $f\theta$ is exact. Fix a smooth, strictly subharmonic Morse exhaustion function $\tau : M \to \mathbb{R}$ such that

(a) 0 is a regular value of τ, and

(b) $\tau < 0$ on T and $D_0 := \{x \in M | \tau(x) \leq 0\} \subset V_0$.

If p and q are distinct critical points of τ in $M \setminus D_0$, we assume that $\tau(p) \neq \tau(q)$. (Given τ satisfying (a) and (b), we may modify it to ensure that this assumption holds, as per Remark 2.3.10.) Denote by $p_1, p_2, \ldots$ the critical points of τ in $M \setminus D_0$, ordered so that $0 < \tau(p_1) < \tau(p_2) < \cdots$. (Note that it is possible that τ only has finitely many critical points, or none.)

We choose a strictly increasing and divergent sequence $(a_n)_{n\in\mathbb{N}}$ of regular values of τ, such that for any integer n associated with critical points p_n and p_{n+1}, we have $\tau(p_n) < a_{n+1} < \tau(p_{n+1})$.

Denote by D_n the compact submanifold with boundary $\{x \in M \mid \tau(x) \leq a_n\}$. Our goal is to construct a sequence of holomorphic functions $f_n : V_n \to \mathbb{C}^*, n \geq 0$, on neighbourhoods V_n of D_n, such that

(i) $f_n\theta$ is exact on V_n,

(ii) there is a homotopy $h_n : I \times V_n \to \mathbb{C}^*$, where each $h_n(t, \cdot)$ is holomorphic, with $h_n(0, \cdot) = f_n$ and $h_n(1, \cdot) = f|_{V_n}$, and

(iii) $\|h_n - h_{n+1}\|_{D_n} < \epsilon_n \cdot 2^{-(n+2)}$,

where $\epsilon_n > 0$ is defined as follows. Let

$$\delta_n = \min\left\{1, \inf_{(t,x)\in I\times D_n} |h_n(t,x)|\right\}$$

for each $n \geq 0$. Then set $\epsilon_n = \min\{\delta_0, \delta_1, \ldots, \delta_n\}$.

Since $f\theta$ is exact on V_0, we may set $f_0 := f|_{V_0}$ and define the homotopy $h_0 : I\times V_0 \to \mathbb{C}^*$ by $h_0(t, x) = f(x)$. We now proceed by induction; assuming that we have constructed $f_0, \ldots, f_n$ that satisfy conditions (i), (ii) and (iii) above, we will construct f_{n+1} on a neighbourhood V_{n+1} of D_{n+1}. We must consider two separate cases. Case 1, the noncritical case, is for when there is no change in the homotopy type of the sublevel sets when going from D_n to D_{n+1}. Case 2, the critical case, is for when the homotopy type does change.

Case 1 (the noncritical case): Assume that $D_{n+1} \setminus D_n$ has no critical points of τ, and that we have $f_0, \ldots, f_n$ satisfying the above requirements. We may also assume that D_{n+1} is connected (if not, we apply the following arguments to each path component).

It is a basic result of Morse theory that D_n is a strong deformation retract of D_{n+1} (Theorem 2.3.5; for a proof see [19]). Fix a basis $C_1, \ldots, C_l$ of $H_1(D_n, \mathbb{Z})$ as in Lemma 4.2.3, and pick an open ball W around the origin in $\mathbb{C}^l$. By the same lemma, there exists a holomorphic map $\Phi_{f_n} : W \times D_n \to \mathbb{C}^*$ such that $\Phi_{f_n}(0, \cdot) = f_n$, and whose period map $\mathcal{P}^\Phi : W \to \mathbb{C}^l$ with respect to $C_1, \ldots, C_l$ has full rank at $0 \in \mathbb{C}^l$.

Pick an open neighbourhood $V_{n+1} \subset M$ of D_{n+1} which strongly deformation retracts onto D_{n+1}. Let B_0 and B be closed balls around $0 \in \mathbb{C}^l$ such that $B_0 \subset B \subset W$. By

Lemma 4.2.2, D_n is holomorphically convex in the Riemann surface V_{n+1}, since D_{n+1} is a strong deformation retract of V_{n+1} and the inclusion $D_n \hookrightarrow D_{n+1}$ is a homotopy equivalence. The closed ball B is convex, and thus also holomorphically convex in W (cf. Lemma 2.2.3). So, by Lemma 2.2.4, the compact set $B \times D_n$ is holomorphically convex in the manifold $W \times V_{n+1}$. Note moreover that since $B \times D_n$ is a strong deformation retract of $W \times V_{n+1}$, Φ_{f_n} can be extended continuously to a nonzero map on $W \times V_{n+1}$. Therefore, by Lemma 4.2.7, we can get a holomorphic map $\Psi : W \times V_{n+1} \to \mathbb{C}^*$ which uniformly approximates Φ_{f_n} arbitrarily closely on $B_0 \times D_n$.

Since we are working in the noncritical setting, $C_1, \ldots, C_l$ form a basis for $H_1(V_{n+1}, \mathbb{Z})$, and we have a period map $\mathcal{P}^\Psi : W \to \mathbb{C}^l$ of Ψ with respect to $C_1, \ldots, C_l$. When $\mathcal{P}^\Phi : W \to \mathbb{C}^l$ (the period map of Φ_{f_n}) is restricted to a small enough closed ball B_0 around the origin, it maps bijectively onto a closed neighbourhood of the origin in $\mathbb{C}^l$ (as it has maximal rank at $0 \in \mathbb{C}^l$). We may thus choose our approximation $\Psi : W \times V_{n+1} \to \mathbb{C}^*$ close enough so that the image of $\mathcal{P}^\Psi|_{B_0}$ also contains the origin. That is, there exists $\zeta_0 \in B_0$ such that the holomorphic 1-form $\Psi(\zeta_0, \cdot)\theta$ on V_{n+1} is exact.

Thus, we set $f_{n+1} := \Psi(\zeta_0, \cdot)$; by appropriately choosing B_0 and Ψ, we ensure that f_{n+1} approximates f_n arbitrarily closely on D_n.

It remains to show that f_{n+1} is homotopic to $f|_{D_{n+1}}$. By assumption, f_n is connected to $f|_{D_n}$ via a homotopy $h_n : I \times D_n \to \mathbb{C}^*$. Since D_n is holomorphically convex in V_{n+1}, we can, by Lemma 4.2.7, approximate $h_n : I \times D_n \to \mathbb{C}^*$ by a homotopy $h_{n+1} : I \times V_{n+1} \to \mathbb{C}^*$ of holomorphic maps $V_{n+1} \to \mathbb{C}^*$, such that $h_{n+1}(0, \cdot) = f_{n+1}$, $h_{n+1}(1, \cdot) = f|_{V_{n+1}}$ and $\|h_{n+1} - h_n\| < \epsilon_n \cdot 2^{-(n+2)}$ on D_n.

Case 2 (the critical case): We start with compact subsets $D_n \subset D_{n+1}$ with exactly one critical point $p_{n+1} \in D_{n+1} \setminus D_n$. Again, we assume that we have $f_1, \ldots, f_n$ satisfying (i), (ii) and (iii). Since τ is strictly subharmonic, the Morse index of p_{n+1} is either 0 or 1 (cf. Remark 2.3.9).

If the Morse index of p_{n+1} is 0, then a new connected component of the sublevel set $\{x \in M | \tau(x) \leq t\}$ appears when t passes p_{n+1}. We extend f_n as follows. Let K be a connected component of D_{n+1}. If $K \cap D_n \neq \varnothing$, the map f_n is extended onto K as in Case 1. If $K \cap D_n = \varnothing$, we may choose, as an extension of f_n on K, any holomorphic function which is homotopic to f – which could simply be f.

Now say that the Morse index of p_{n+1} is 1. Then the homotopy type of D_{n+1} is described by attaching a smooth, embedded arc C to the set D_n. More specifically, assuming that $a_{n+1} - a_n$ is sufficiently small, we have that $D_n \cup C$ is a strong deformation retract of D_{n+1} (see [19, Part I, §3]).

Denote by $q_1, q_2 \in D_n$ the endpoints of C. We may assume that they are distinct, and $C \cap D_n = \{q_1, q_2\}$. By assumption, we have a homotopy $h_n : I \times D_n \to \mathbb{C}^*$ such that $h(0,\cdot) = f_n|_{D_n}$ and $h(1,\cdot) = f|_{D_n}$. Denote by L the subset $(I \times \{q_1, q_2\}) \cup (\{1\} \times C)$ of $I \times C$, and let $r : I \times C \to L$ be a continuous surjection from $I \times C$ onto L (cf. [23, Theorem 6]) so that $r(t,a) = (t,a)$ for all $(t,a) \in L$. Consider $p \circ r : I \times C \to \mathbb{C}^*$, where $p : L \to \mathbb{C}^*$ is given by

$$p(t,x) = \begin{cases} h_n(t,x) & \text{if } (t,x) \in I \times \{q_1, q_2\}, \\ f(x) & \text{if } (t,x) \in \{1\} \times C. \end{cases}$$

At this stage, we consider two separate cases. The first is where q_1 and q_2 lie in different connected components of D_n, in which case we define a continuous auxiliary map $\beta_n : D_n \cup C \to \mathbb{C}^*$ by letting $\beta_n = f_n|_{D_n}$ on D_n, and $\beta_n = p \circ r\,(0,\cdot)$ on C.

On the other hand, assume that q_1 and q_2 belong to the same connected component of D_n. We may assume that D_{n+1} (and thus $D_n \cup C$) is connected, for every connected component of D_{n+1} which does not contain C may be treated as in the noncritical case. This also implies that D_n is connected, since we can find a path in D_n taking q_1 to q_2. Let $x_0 \in D_n$, and define

$$F_n : D_n \to \mathbb{C}, \quad x \mapsto \int_{x_0}^{x} f_n \theta,$$

where the integral is taken over any path in D_n from x_0 to x. We let $p \circ r$ be as defined above. Then, by Theorem 4.2.4, we get a continuous map $k : I \times C \to \mathbb{C}^*$ with $k(\cdot, q_1) = f_n(q_1)$, $k(\cdot, q_2) = f_n(q_2)$ and $k(1,\cdot) = p \circ r(0,\cdot)$, such that $\int_C k(0,\cdot)\theta = F_n(q_2) - F_n(q_1)$. (In this integral we assume that C is traversed from q_1 to q_2.) The auxiliary map $\beta_n : D_n \cup C \to \mathbb{C}^*$ in this case is defined by setting $\beta_n = f_n|_{D_n}$ on D_n, and $\beta_n = k(0,\cdot)$ on C.

In both scenarios, we have a continuous map $\beta_n : D_n \cup C \to \mathbb{C}^*$ such that $\int_\gamma \beta_n \theta = 0$ over any closed loop γ in $D_n \cup C$; moreover, β_n is homotopic to $f|_{D_n \cup C}$, and the homotopy can be chosen arbitrarily close to h_n on D_n.

Although $D_n \cup C$ is not a Riemann surface with boundary, the arguments in the proof of Lemma 4.2.3 still hold with respect to the map $\beta_n : D_n \cup C \to \mathbb{C}^*$. This allows us to approximate β_n by a nonvanishing holomorphic map α_n on an open neighbourhood of W of $D_n \cup C$, whose associated 1-form $\alpha_n \theta$ is exact. The details are as follows.

Choose piecewise-smooth loops $\gamma_1, \ldots, \gamma_l$ in $D_n \cup C$ which form a basis for $H_1(D_n \cup C, \mathbb{Z})$. Following the proof of Lemma 4.2.3, we obtain functions g_i that are holomorphic on an open neighbourhood of $D_n \cup C$, and a deformation map $\Phi_{\beta_n} : \mathbb{C}^l \times (D_n \cup C) \to \mathbb{C}^*$ with $\Phi(0, \cdot) = \beta_n$, so that its period map with respect to $\gamma_1, \ldots, \gamma_l$ has full rank at $0 \in \mathbb{C}^l$. By Lemma 4.2.2 and Remark 4.2.6, we can approximate β_n, uniformly on $D_n \cup C$, by a holomorphic map $\tilde{\beta}_n : \tilde{V} \to \mathbb{C}^*$ on an open neighbourhood $\tilde{V}$ of $D_n \cup C$. Let W be an open neighbourhood of $D_n \cup C$ such that $W \subset\subset \tilde{V}$, which strongly deformation retracts onto $D_n \cup C$. By Lemma 4.2.3 again, we can construct a deformation map $\Phi_{\tilde{\beta}_n} : \mathbb{C}^l \times W \to \mathbb{C}^*$, so that $\Phi_{\tilde{\beta}_n}(0, \cdot) = \tilde{\beta}_n$, using the same holomorphic functions g_i used to make Φ_{β_n}. Assuming that $\tilde{\beta}_n$ approximates β_n sufficiently closely on $D_n \cup C$, the period map $\tilde{\mathcal{P}} : \mathbb{C}^l \to \mathbb{C}^l$ of $\Phi_{\tilde{\beta}_n}$ will vanish at a point $\zeta \in \mathbb{C}^l$ arbitrarily close to the origin in $\mathbb{C}^l$. Thus we can approximate β_n arbitrarily closely on $D_n \cup C$ by a holomorphic map $\alpha_n : W \to \mathbb{C}$, such that the periods of $\alpha_n \theta$ vanish.

Finally, we repeat the arguments from Case 1 (the noncritical step) to construct f_{n+1} on a neighbourhood of D_{n+1} that approximates α_n on W. We assume that the approximations are close enough so that the maps α_n, β_n and f_{n+1} can be deformed to one another on $D_n \cup C$ through nonvanishing continuous maps. Then, as in Case 1, we have a homotopy $h_{n+1} : I \times D_{n+1} \to \mathbb{C}^*$ such that $h_{n+1}(0, \cdot) = f_{n+1}$, $h_{n+1}(1, \cdot) = f|_{D_{n+1}}$ and $\|h_{n+1}(t, \cdot) - h_n(t, \cdot)\|_{D_n} < \epsilon_n \cdot 2^{-(n+2)}$.

This completes the induction. The desired $\tilde{f} : M \to \mathbb{C}$ is given by the limit of f_n as $n \to \infty$, and a homotopy h which takes $\tilde{f}$ to f is given by the limit of $h_n(t, \cdot)$ as $n \to \infty$. By construction, h takes values in $\mathbb{C}^*$ – since each $(t, x) \in I \times M$ lies in some $I \times D_m$, we have $|h(t, x) - h_m(t, x)| \leq \delta_m/2$. Moreover, since $f_n \theta$ is exact for each $n \geq 0$, $\tilde{f}\theta$ is also exact. Fixing $p \in M$, we set, for any $x \in M$, $\tilde{F}(x) := \int_p^x \tilde{f}\theta$. □

Remark 4.2.9. The seminal result due to Gunning and Narasimhan of 1967 states that every open Riemann surface M admits a holomorphic immersion into the complex plane. Specifically, it gives that, if θ is a nonvanishing holomorphic 1-form on M, then there exists a holomorphic function $g : M \to \mathbb{C}$ such that $e^g \theta$ is exact. We find that this statement is equivalent to that of Theorem 4.2.8; the details are as follows.

Let θ be a nonvanishing holomorphic 1-form on M and $f : M \to \mathbb{C}^*$ a holomorphic map with no zeros. Then $f\theta$ is a nonvanishing holomorphic 1-form on M, and by the above statement of the Gunning-Narasimhan theorem, there exists a holomorphic function $g : M \to \mathbb{C}$ and an immersion $\tilde{G} : M \to \mathbb{C}$ such that $d\tilde{G} = e^g f\theta$. The homotopy $e^{tg} f$,

$t \in I$, joins $d\tilde{G}/\theta$ to f through holomorphic maps $M \to \mathbb{C}^*$.

Conversely, assume that Theorem 4.2.8 holds. Then there is a holomorphic immersion $\tilde{F} : M \to \mathbb{C}$ such that $\tilde{f} := d\tilde{F}/\theta$ is homotopic (through nonvanishing holomorphic maps on M) to the constant map of value 1. This implies that $\tilde{f}_*(\pi_1(M))$ is trivial, where $\tilde{f}_* : \pi_1(M) \to \pi_1(\mathbb{C}^*)$ is the induced map on fundamental groups. So $\tilde{f}$ can be lifted with respect to the exponential map, $\exp : \mathbb{C} \to \mathbb{C}^*$. That is, $d\tilde{F}/\theta$ can be written as e^g for some holomorphic function $g : M \to \mathbb{C}$.

Chapter 5

The parametric h-principle for immersions

Let M be an open Riemann surface, and θ a nonvanishing holomorphic 1-form on M. The purpose of this chapter is prove that the continuous map

$$\mathcal{I}(M, \mathbb{C}) \to \mathcal{O}(M, \mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a weak homotopy equivalence: this is the *parametric* h-principle for holomorphic immersions of open Riemann surfaces into $\mathbb{C}$.

In essence, we achieve this by parametrising the arguments in the previous section, where we showed that the above map induces a surjection of path components. While the fundamental ideas are inherited from the the basic principle, the parametric h-principle is a more intricate mathematical problem, involving further analytic and topological arguments, and methods of convex integration theory.

5.1 Preparatory lemmas

Let L be a paracompact space, $D \subset M$ be a compact submanifold with boundary, and consider simple, piecewise-smooth loops $\gamma_1, \ldots, \gamma_l : I \to D$ which form a basis of $H_1(D, \mathbb{Z})$. For a continuous map $f : L \times D \to \mathbb{C}$ with $f(t, \cdot)$ holomorphic for each $t \in L$, recall that the *period map* $\mathcal{P} : L \to \mathbb{C}^l$ of f with respect to $\gamma_1, \ldots, \gamma_l$ is given by $t \mapsto (\mathcal{P}_1(t), \ldots, \mathcal{P}_l(t))$, where $\mathcal{P}_i : L \to \mathbb{C}$, $t \mapsto \int_{C_i} f(t, \cdot)\, \theta$.

Lemma 5.1.1. *Let M be an open Riemann surface and $D \subset M$ a compact submanifold with boundary. Let K be a compact space. We fix a nowhere-vanishing holomorphic 1-form θ on M, and consider a continuous map $f : K \times D \to \mathbb{C}^*$ such that, for each $t \in I$, $f(t, \cdot)\theta$ is an exact holomorphic 1-form. Choose simple, piecewise-smooth loops $\gamma_1, \ldots, \gamma_l : I \to D$ which form a basis of $H_1(D, \mathbb{Z})$. Then, there exists a continuous map $\Phi : K \times \mathbb{C}^l \times D \to \mathbb{C}^*$, such that for any $t \in K$,*

1. *$\Phi(t, \cdot, \cdot)$ is holomorphic,*
2. *$\Phi(t, 0, \cdot) = f(t, \cdot)$,*
3. *if $\mathcal{P} : K \times \mathbb{C}^l \to \mathbb{C}^l$ is the period map of Φ with respect to $\gamma_1, \ldots \gamma_l$, then its derivative with respect to the second variable, $D_2\mathcal{P}$, is continuous on $K \times \mathbb{C}^l$, and*
4. *$D_2\mathcal{P}$ is invertible at each $(t, 0) \in K \times \{0\}$.*

Proof. Denote by $C_i \subset D$ the image of γ_i, and write $C = \bigcup_{j=1}^{l} C_j$. We may assume that the subsets C_i meet only at one common point $p \in D$, and $\gamma_i(0) = \gamma_i(1) = p$.

As in Lemma 4.2.3, to each $i = 1, \ldots, l$ we associate a continuous function $h_i : C \to \mathbb{C}$ that is identically zero on $C \setminus C_i$ and whose values on C_i we will specify shortly. In particular, the functions h_i vanish at p.

Writing $\zeta = (\zeta_1, \ldots, \zeta_l) \in \mathbb{C}^l$, we define the continuous map

$$\Psi : K \times \mathbb{C}^l \times C \to \mathbb{C}^*, \quad (t, \zeta, x) \mapsto f(t, x)\, e^{\zeta_1 h_1(x) + \cdots + \zeta_l h_l(x)}.$$

We then have, for any $(t, x) \in K \times C$,

$$\left.\frac{\partial \Psi(t, \zeta, x)}{\partial \zeta_i}\right|_{\zeta=0} = h_i(x) f(t, x) \in \mathbb{C}.$$

Note that when $x \in C \setminus C_i$, we have $\left.\frac{\partial \Psi(t, \zeta, x)}{\partial \zeta_i}\right|_{\zeta=0} = 0$.

Write $\tilde{\mathcal{P}}_i : K \times \mathbb{C}^l \to \mathbb{C}, (t, \zeta) \mapsto \int_{C_i} \Psi(t, \zeta, \cdot)\, \theta$, and consider the map

$$\tilde{\mathcal{P}} : K \times \mathbb{C}^l \to \mathbb{C}^l, \quad (t, \zeta) \mapsto (\tilde{\mathcal{P}}_1(t, \zeta), \ldots, \tilde{\mathcal{P}}_l(t, \zeta)).$$

Clearly, when $j \neq i$, $\left.\dfrac{\partial \tilde{\mathcal{P}}_i(t,\zeta)}{\partial \zeta_j}\right|_{\zeta=0}$ vanishes for all $t \in K$. We ensure that the value

$$\left.\frac{\partial \tilde{\mathcal{P}}_i(t,\zeta)}{\partial \zeta_i}\right|_{\zeta=0} = \int_{C_i} h_i f(t,\cdot)\,\theta \\ = \int_0^1 h_i(\gamma_i(s)) f(t,\gamma_i(s)) \theta(\gamma_i(s),\dot{\gamma}_i(s))\,ds$$

is nonzero for all $i = 1,\ldots,l$. We can do so by choosing the functions h_i appropriately. Let $\eta_i : I \to [0,\infty)$ be a continuous function, compactly supported on an arc $V_i \subset (0,1)$ containing $\frac{1}{2}$, such that $\int_0^1 \eta_i(t)\,dt = 1$. Define h_i by $h_i \circ \gamma_i = \eta_i$ on C_i and $h_i \circ \gamma_i = 0$ on $C \setminus C_i$. Since $f(t,\gamma_i(\frac{1}{2}))\theta(\gamma_i(\frac{1}{2}),\dot{\gamma}_i(\frac{1}{2})) \neq 0$, and the parameter space K is compact, we may choose V_i small enough so that the integral $\int_{C_i} h_i f(t,\cdot)\,\theta$ is nonvanishing for all $t \in K$.

By Lemma 4.2.2, C is holomorphically convex in a suitable open neighbourhood of D; thus we may approximate, using the Mergelyan-Bishop theorem, the functions $h_i : C \to \mathbb{C}$ by holomorphic functions $\sigma_i : D \to \mathbb{C}$ uniformly on C.

Define

$$\Phi : K \times \mathbb{C}^l \times D \to \mathbb{C}^*, \quad (t,\zeta,x) \mapsto f(t,x)\, e^{\zeta_1\sigma_1(x)+\cdots+\zeta_l\sigma_l(x)}.$$

Notice that for each $(t,x) \in K \times D$, $\Phi(t,\cdot,x) : \mathbb{C}^l \to \mathbb{C}^*$ is a holomorphic function.

Take a compact ball $\overline{V} \subset \mathbb{C}^l$ with nonempty interior, centred at $0 \in \mathbb{C}^l$. The map Φ is uniformly close to Ψ on $K \times \overline{V} \times C$ in the compact-open topology; so, if the approximations σ_i are sufficiently close to h_i on C, then the corresponding period map,

$$\mathcal{P} : K \times \mathbb{C}^l \to \mathbb{C}^l, \quad (t,\zeta) \mapsto \left(\int_{C_1} \Phi(t,\zeta,\cdot)\,\theta, \ldots, \int_{C_l} \Phi(t,\zeta,\cdot)\,\theta \right),$$

has full rank at (t,ζ) for all $(t,\zeta) \in K \times \overline{V}$, provided that $\overline{V}$ is chosen sufficiently small. This is by the compactness of K and $\overline{V}$, and the nature of the compact-open topology on the space of holomorphic functions (for uniform convergence of holomorphic functions on compact subsets – in this case, $\overline{V}$ – is equivalent to uniform convergence of all the derivatives on compact subsets). This concludes our proof. □

We now present a series of further lemmas that are used in the proof of the main theorem. Below we have a parametric version of the Oka-Weil approximation theorem.

Lemma 5.1.2. *Let P be a compact Hausdorff space. Consider a Stein manifold X and a compact, holomorphically convex subset $D \subset X$. For a neighbourhood U of D, let $h : P \times U \to \mathbb{C}$ be a continuous function such that, for each $p \in P$, $h(t, \cdot)$ is holomorphic. Then, given $\epsilon > 0$, there exists a continuous map $w : P \times X \to \mathbb{C}$ with $w(t, \cdot)$ holomorphic on X for each $t \in P$, satisfying $|w - h| < \epsilon$ on $P \times D$.*

Moreover, assume A is a closed subset of P, contained in an open neighbourhood $W_0 \subset P$; and say we are given a continuous function $v : W_0 \times X \to \mathbb{C}$ with $v(t, \cdot)$ holomorphic for all $t \in W_0$, and $|v - h| < \epsilon$ on $W_0 \times D$. Then we may choose $w = v$ on $A \times X$.

Proof. Consider an arbitrary $t \in P$. By the Oka-Weil approximation theorem (Theorem 2.2.12), given $\epsilon > 0$, we can find some holomorphic function $g : X \to \mathbb{C}$ such that $|g - h(t, \cdot)| < \epsilon$ on D.

We first consider the case where no map v, as above, is specified (that is, $A = \varnothing$). Take a neighbourhood $W \subset P$ of t such that, for all $s \in W$, $|g - h(s, \cdot)| < \epsilon$ on D. By doing this at each point $t \in P$, we obtain an open cover of P, from which we take a finite subcover $W_1, \dots, W_m$. Each open set W_k is associated with a holomorphic function g_k as above; that is, for all $s \in W_k$, $|g_k - h(s, \cdot)| < \epsilon$ on D. Let $(p_k)_{k=1,\dots m}$ be a partition of unity subordinate to this open cover. If we set

$$w : P \times X \to \mathbb{C}, \quad (t, x) \mapsto \sum_{k=1}^{m} p_k(t) g_k(x),$$

we have, for $(t, x) \in P \times D$,

$$\begin{aligned} |w(t, x) - h(t, x)| &= \left| \sum_{k=1}^{m} p_k(t) \left(g_k(x) - h(t, x) \right) \right| \\ &\leq \sum_{k=1}^{m} p_k(t) \left| g_k(x) - h(t, x) \right| < \epsilon. \end{aligned}$$

Now say $A \neq \varnothing$, and we are given $v : W_0 \times X \to \mathbb{C}$ such that $|v - h| < \epsilon$ on $W_0 \times D$. If $t \in P \setminus W_0$, we find, as above, a neighbourhood $W \subset P$ of t and holomorphic function $f : X \to \mathbb{C}$ such that $\|f - h(s, \cdot)\|_D < \epsilon$ for all $s \in W$. We further assume that $W \cap A = \varnothing$. In this way, we find an open cover of $P \setminus W_0$ and take a finite subcover $W_1, \dots, W_m$. Each W_k (for $k = 1, \dots, m$) is associated with a holomorphic function f_k as above.

Note that the collection $W_0, W_1, \dots, W_m$ covers P. Let $(q_k)_{k=0,\dots m}$ be a partition of unity subordinate to this open cover, and set

$$w : P \times X \to \mathbb{C}, \quad (t, x) \mapsto q_0(t)v(t, x) + \sum_{k=1}^{m} q_k(t) f_k(x).$$

Then w satisfies the required approximation condition as above, and we have $w = v$ on $A \times X$. □

Remark 5.1.3. In the special case where X is an open Riemann surface, it suffices to assume that $h(t, \cdot)$ is holomorphic on the interior of D, for each $t \in P$. The ensuing result is a parametric form of the classical Mergelyan-Bishop theorem, and the proof is analogous to the above: where we have employed the Oka-Weil approximation theorem, we use the Mergelyan-Bishop theorem (Theorem 2.2.7) instead.

Remark 5.1.4. If we additionally assume that the parameter space P has a smooth structure (say P is a compact smooth manifold with boundary), and the maps h and v are smooth, then we can find a *smooth* approximation $w : P \times X \to \mathbb{C}$. We only need to select a smooth partition of unity in the proof, and this conclusion follows.

From Lemma 5.1.2, we obtain the following stronger result for families of nonvanishing holomorphic maps.

Lemma 5.1.5. *Let P be a simply connected, compact Hausdorff space. Let X be a Stein manifold and $D \subset X$ a compact, holomorphically convex subset. Consider a continuous map $h : P \times X \to \mathbb{C}^*$, and a neighbourhood $W \subset X$ of D such that, for each $t \in P$, $h(t, \cdot)$ is holomorphic on W. Then, given $\epsilon > 0$, there exists a continuous map $f : P \times X \to \mathbb{C}^*$ such that, for all $t \in P$, $f(t, \cdot)$ is holomorphic on X and $|h - f| < \epsilon$ on $P \times D$.*

In the case where D is a strong deformation retract of X, we have the following additional property. Let $A \subset P$ be a closed subset and let $A_0 \subset P$ be an open neighbourhood of A. There exists a positive $\epsilon' \leq \epsilon$ such that, if we are given a continuous $g : A_0 \times X \to \mathbb{C}^*$, holomorphic for each fixed $t \in A_0$ and with $|g - h| < \epsilon'$ on $A_0 \times D$, then we may choose $f = g$ on $A \times X$.

Proof. Take any $t_0 \in P$. By Theorem 2.2.11, there is a homotopy which takes $h(t_0, \cdot)$ to a holomorphic map $\gamma : X \to \mathbb{C}^*$ through continuous maps $X \to \mathbb{C}^*$. Define a continuous map $k : P \times X \to \mathbb{C}^*$ by $k(t, x) = h(t, x)/\gamma(x)$. As P is simply connected, there exists a continuous lifting $\tilde{k} : P \times X \to \mathbb{C}$ of k with respect to the universal covering map exp :

$\mathbb{C} \to \mathbb{C}^*$. (We may see this by considering a loop $p = (p_1, p_2)$ in $P \times X$; assuming that p_1 has basepoint t_0, we can deform $k \circ p$ to the nullhomotopic loop $s \mapsto h(t_0, p_2(s))/\gamma(p_2(s))$.) Note that $\tilde{k}(t, \cdot)$ is holomorphic on W for any $t \in P$. Since $P \times D$ is compact and the exponential map is continuous, there exists $\delta > 0$ such that, if $z : P \times D \to \mathbb{C}$ satisfies $\|\tilde{k} - z\|_{P \times D} < \delta$, then $\|k - e^z\|_{P \times D} < \epsilon / \|\gamma\|_D$.

By Lemma 5.1.2, there is a continuous map $w : P \times X \to \mathbb{C}$ such that $w(t, \cdot)$ is holomorphic for each $t \in P$, and $|\tilde{k} - w| < \delta$ on $P \times D$, which implies that for each $(t, x) \in P \times D$,

$$|k(t, x) - \exp(w(t, x))| < \epsilon / \|\gamma\|_D .$$

By setting $f : P \times X \to \mathbb{C}^*$, $(t, x) \mapsto \gamma(x) \cdot \exp(w(t, x))$, we get that $|h - f| < \epsilon$ on $P \times D$.

Now assume that X strongly deformation retracts onto D. Choose $0 < \epsilon' \leq \epsilon$ such that, if $g : A_0 \times X \to \mathbb{C}^*$ is a family of holomorphic maps satisfying $\|g - h\| < \epsilon'$ on $A_0 \times D$, then $g|_{A_0 \times D}$ is homotopic to $h|_{A_0 \times D}$ through continuous maps $A_0 \times D \to \mathbb{C}^*$. (Since h is bounded away from 0 on the compact set $\overline{A_0} \times D$, any function that is sufficiently close to $h|_{A_0 \times D}$ also vanishes nowhere, and can be deformed by convex linear combinations into h on $A_0 \times D$.)

Assume that g as above is given. Since the space $(\{0, 1\} \times A_0 \times X) \cup (I \times A_0 \times D)$ is a strong deformation retract of $I \times A_0 \times X$ (see [23, Theorem 6]), the homotopy between $g|_{A_0 \times D}$ and $h|_{A_0 \times D}$ extends to a homotopy between g and $h|_{A_0 \times X}$, through continuous maps $A_0 \times X \to \mathbb{C}^*$.

Thus, g/γ can be lifted with respect to $\exp : \mathbb{C} \to \mathbb{C}^*$ to a map $v : A_0 \times X \to \mathbb{C}$, which, for suitably small ϵ', can be chosen so that $|\tilde{k} - v| < \delta$ on $A_0 \times D$. We approximate, as above, the lifting $\tilde{k}$ of k by a continuous $w : P \times X \to \mathbb{C}$, but specially choosing $w = v$ on $A \times X$ (cf. Lemma 5.1.2). Then the map $f : P \times X \to \mathbb{C}^*$, as defined above, satisfies $|h - f| < \epsilon$ on $P \times D$ and $f = g$ on $A \times X$. □

Remark 5.1.6. In the above lemma, as is the case with Lemma 5.1.2, if X is an open Riemann surface, then we only need to assume that $h(t, \cdot)$ is holomorphic on the interior of D for each $t \in P$ (and not necessarily on some neighbourhood W). The proof is only a slight variation of the above – we find the auxiliary map $w : P \times X \to \mathbb{C}$ which approximates $\tilde{k}$ by employing the parametric version of the Mergelyan-Bishop theorem, in place of Lemma 5.1.2: see Remark 5.1.3.

We now present two lemmas related to the question of constructing (or deforming) families of paths in order to satisfy given integral representation conditions. Ideas from convex integration theory will be necessary – these techniques are studied in depth in Chapter 3.

Lemma 5.1.7. *Let P be a compact metric space. We consider a closed subset $R \subset P$ which is a strong deformation retract of P, and a continuous family of paths $w : R \times I \to \mathbb{C}^*$. Let $\alpha : P \to \mathbb{C}$ be a continuous function such that $\alpha(p) = \int_0^1 w(p,t)\,dt$ for all $p \in R$, and let $h_0, h_1 : P \to \mathbb{C}^*$ be nonvanishing continuous maps with $h_0(p) = w(p,0)$ and $h_1(p) = w(p,1)$ for all $p \in R$.*

Then, given $\epsilon > 0$, there exists a continuous map $\sigma : P \times I \to \mathbb{C}^*$ such that

1. *$\sigma = w$ on $R \times I$,*
2. *$\left|\int_0^1 \sigma(\cdot,t)\,dt - \alpha\right| < \epsilon$, and*
3. *$\sigma(\cdot,0) = h_0$ and $\sigma(\cdot,1) = h_1$.*

Proof. We begin by observing that, as P strongly deformation retracts onto R, the set $K := (R \times I) \cup (P \times \{0,1\})$ is a strong deformation retract of $P \times I$ [23, Theorem 6]. So, we can find a continuous function $\tilde{w} : P \times I \to \mathbb{C}^*$ such that $\tilde{w} = w$ on $R \times I$, $\tilde{w} = h_0$ on $P \times \{0\}$, and $\tilde{w} = h_1$ on $P \times \{1\}$.

Choose neighbourhoods W, U of R in P such that $\overline{W} \subset U$, and for all $p \in U$, $\left|\int_0^1 \tilde{w}(p,t)\,dt - \alpha(p)\right| < \epsilon/2$. Let $\chi : P \to [0,1]$ be a continuous function which is supported on U, and equals 1 on W.

Consider the continuous family of paths $\tau : P \times I \to \mathbb{C}$ given by

$$\tau(p,t) = \chi(p)\,\tilde{w}(p,t) + (1 - \chi(p))\alpha(p).$$

For any $p \in P$, notice that the definite integral

$$\int_0^1 \tau(p,t)\,dt = \chi(p)\int_0^1 \tilde{w}(p,t)\,dt + (1 - \chi(p))\alpha(p)$$

satisfies

$$\left|\int_0^1 \tau(p,t)\,dt - \alpha(p)\right| < \epsilon/2.$$

We first modify τ into a map $\tilde{\sigma} : P \times I \to \mathbb{C}$ so that condition 3 holds. Since P is a metric space, it is perfectly normal, and we may choose a continuous function $\eta : P \to [0,\infty)$

which vanishes precisely on R. Assume that η is bounded above by $q < 1/4$, and write $\eta_p := \eta(p)$.

We define $\tilde{\sigma}$ as follows. On $[0, \eta_p]$ (resp. $[1-\eta_p, 1]$), let $\tilde{\sigma}(p, \cdot)$ traverse the straight line segment from $h_0(p)$ to $\tau(p, 0)$ (resp. from $\tau(p, 1)$ to $h_1(p)$). Note that for all $p \in R$, $h_0(p) = \tau(p, 0)$ and $h_1(p) = \tau(p, 1)$. Let $\tilde{\sigma}(p, t) = \tau(p, (t - \eta_p)/(1 - 2\eta_p))$ for all $(p, t) \in P \times [\eta_p, 1 - \eta_p]$.

With a suitable bound q on the function η, we have the approximation

$$\left| \int_0^1 \tilde{\sigma}(p, t)\, dt - \alpha(p) \right| < \epsilon/2.$$

The goal is to deform $\tilde{\sigma}$ into the required nonvanishing map $\sigma : P \times I \to \mathbb{C}^*$, ensuring that $\left| \int_0^1 \tilde{\sigma}(p, t)\, dt - \int_0^1 \sigma(p, t)\, dt \right| < \epsilon/2$ and keeping the deformation fixed on K. For this we need to employ ideas from convex integration theory.

Let $B := P \times I$. We know by Lemma 3.2.4 that there exists a C-structure (h, H) in $\mathbb{C}^*$, over B, with respect to $\tilde{\sigma}, \tilde{w}$, such that

$$\tilde{\sigma}(p, t) = \int_0^1 h(s, (p, t))\, ds.$$

For any $p \in P$, consider the map

$$h_p : \mathbb{R}/\mathbb{Z} \times I \to \mathbb{C}^*, \quad (s, t) \mapsto h(s, (p, t)).$$

With this, we define the function

$$f_p : I \to \mathbb{C}, \quad t \mapsto \int_0^t h_p(Ns, s)\, ds.$$

Provided that N is sufficiently large (cf. Proposition 3.3.2), for all $(p, t) \in B$,

(a) $\gamma_p(t) := \dfrac{d}{dt} f_p(t)$ is nonzero, and

(b) $\left| \int_0^r \gamma_p(t)\, dt - \int_0^r \tilde{\sigma}(p, t)\, dt \right| < \epsilon' < \epsilon/4$,

for a positive value $\epsilon' > 0$ which we specify shortly. Note that although the underlying manifold in Proposition 3.3.2 is taken to be the circle $\mathbb{R}/\mathbb{Z}$ (in order to be consistent with the goal of Chapter 3), this assumption is stronger than is necessary for the proof, which holds without change for the parameter space I, as we have here. Noting that P is also compact, we may force condition (b) to hold.

For $\delta > 0$, let $N(\delta)$ denote the δ-neighbourhood in $\mathbb{C}$ of the set $\{\tilde{w}(p,t) \in \mathbb{C}^* : (p,t) \in K\}$. We choose δ so small that $N(\delta) \subset \mathbb{C}^*$.

Assume that the C-structure (h, H) has been chosen to satisfy

$$H([0,1]^2 \times \overline{V}) \subset N(\delta/4) \qquad (\dagger)$$

on a sufficiently small neighbourhood $V \subset P \times I$ of K. (Note: from the proof of Lemma 3.2.4, it is clear how to make (†) hold. On K, $\tilde{\sigma} = \tilde{w}$, so, in the notation of the same lemma, for each $b \in K$ we choose the local C-structures (h_b, H_b) over $\mathcal{D}_b$ so that the image of H_b lies in $N(\delta/4)$, before applying the inductive proof procedure to obtain a global C-structure.) We also assume that V is so small that

$$|\tilde{\sigma}(p,t) - \tilde{w}(p,t)| < \delta/4 \qquad (*)$$

for all $(p,t) \in V$.

Pick a C^1 function $\lambda : P \times I \to [0,1]$ that is compactly supported on V, and equals 1 on a neighbourhood $V' \subset V$ of K.

Define, for $p \in P$,

$$\begin{aligned} &\xi_p : I \to \mathbb{C}, \\ &\quad t \mapsto \int_0^t \gamma_p(r)\,dr + \lambda(p,t)\left(\int_0^t \tilde{\sigma}(p,t)dr - \int_0^t \gamma_p(r)\,dr\right). \end{aligned}$$

As $\left|\int_0^t \gamma_p(r)\,dr - \int_0^t \tilde{\sigma}_p(r)\,dr\right| < \epsilon/4$, $\dfrac{\partial}{\partial t}\xi_p$ satisfies conditions 1, 2 and 3.

It remains to ensure that the image of $\dfrac{\partial}{\partial t}\xi_p$ is in $\mathbb{C}^*$. For $(p,t) \in P \times I$, $\dfrac{\partial}{\partial t}\xi_p(t)$ is given by

$$\gamma_p(t) + \lambda(p,t)(\tilde{\sigma}(p,t) - \gamma_p(t)) + \frac{\partial}{\partial t}\lambda(p,t)\left(\int_0^t \tilde{\sigma}(p,t)dr - \int_0^t \gamma_p(r)\,dr\right).$$

With an appropriate upper bound ϵ' on $\left|\int_0^t \gamma_p(r)\,dr - \int_0^t \tilde{\sigma}_p(r)\,dr\right|$, we claim that

$$\left|\frac{\partial}{\partial t}\xi_p(t) - \gamma_p(t)\right| < \delta/2$$

for all $(p,t) \in V$.

By (†) and (∗), the inequality

$$|\gamma_p(t) - \tilde{\sigma}(p,t)| < |\gamma_p(t) - \tilde{w}(p,t)| + |\tilde{w}(p,t) - \tilde{\sigma}(p,t)| < \delta/2$$

holds for all $(p,t) \in V$. So, by choosing ϵ' suitably small, the above claim holds.

So $\left|\frac{\partial}{\partial t}\xi_p(t) - \tilde{w}(p,t)\right| < \delta$ when $(p,t) \in V$, and $\frac{\partial}{\partial t}\xi_p = \gamma_p$ on $(P \times I) \setminus V$. Thus, $\frac{\partial}{\partial t}\xi_p$ lies in $\mathbb{C}^*$, and we set $\sigma(p,t) := \frac{\partial}{\partial t}\xi_p(t)$. □

We can now correct the error in Lemma 5.1.7, and make families of paths precisely satisfy a given integral formula.

Lemma 5.1.8. *Let P be a compact metric space, and $R \subset P$ a closed subset which is a strong deformation retract of P. We consider a continuous function $z : P \to \mathbb{C}, z \mapsto z_p$. Let $K := (P \times \{0,1\}) \cup (R \times I)$, and assume we have a continuous nonvanishing map $f : K \to \mathbb{C}^*$ such that $\int_0^1 f(p,t)\,dt = z_p$ whenever $p \in R$. Then, there exists a continuous map $h : P \times I \to \mathbb{C}^*$ such that $h = f$ on K and, for all $p \in P$,*

$$\int_0^1 h(p,t)\,dr = z_p.$$

Proof. We first extend f continuously to $\tilde{f} : \tilde{K} \to \mathbb{C}^*$ where

$$\tilde{K} = (P \times ([0, \tfrac{1}{3}] \cup \{1\})) \cup (R \times I).$$

This is evidently possible since $\tilde{K}$ strongly deformation retracts onto K.

Consider the continuous function

$$w : P \to \mathbb{C}, \quad p \mapsto w_p = z_p - \int_0^{\frac{1}{3}} \tilde{f}(p,t)\,dt.$$

Let $\epsilon > 0$ be given. By Lemma 5.1.7, there exists a continuous map $g : P \times \left[\frac{1}{3}, 1\right] \to \mathbb{C}^*$ such that $g = \tilde{f}$ on $\left(P \times \left\{\frac{1}{3}, 1\right\}\right) \cup \left(R \times \left[\frac{1}{3}, 1\right]\right)$, and

$$\left|\int_{\frac{1}{3}}^1 g(p,t)\,dt - w_p\right| < \epsilon$$

for all $p \in P$. Define an auxiliary function $\tilde{h} : P \times I \to \mathbb{C}^*$ by setting $\tilde{h} = \tilde{f}$ on $\tilde{K}$ and $\tilde{h} = g$ on $P \times \left[\frac{1}{3}, 1\right]$. Let $\epsilon_1, \epsilon_2 : P \to \mathbb{R}$ be continuous functions defined by

$$\int_0^1 \tilde{h}(p,t)\,dt + \epsilon_1(p) + i\epsilon_2(p) = z_p.$$

Note that the values $\|\epsilon_1\|_P$ and $\|\epsilon_2\|_P$ can be made arbitrarily small by choosing ϵ sufficiently small, and moreover, for $p \in R$, $\epsilon_1(p) = \epsilon_2(p) = 0$.

Choose numbers $0 < a < b < 1/3$, and let $\alpha : \left[0, \frac{1}{3}\right] \to [0, \infty)$ be a smooth curve which is compactly supported on $[a, b]$ and satisfies $\int_0^1 \alpha(t)\, dt = 1$. Take $\delta > 0$ such that $|\tilde{f}| > \delta$ on $\tilde{K}$. Let $M := \max_{t \in [0,1]} |\alpha(t)|$.

Assume that we have constructed the map g so that $M \cdot \|\epsilon_1\|_P < \delta/2$ and $M \cdot \|\epsilon_2\|_P < \delta/2$. Then, for $(p, t) \in P \times I$, define $h : P \times I \to \mathbb{C}^*$ by

$$h(p,t) = \tilde{h}(p,t) + (\epsilon_1(p) + i\epsilon_2(p))\, \alpha(t).$$

We note that $h = f$ on K, and if $(p, t) \in P \times I$, then

$$\int_0^1 h(p,t)\, dt = \int_0^1 \tilde{h}(p,t)\, dt + \epsilon_1(p) + i\epsilon_2(p) = z_p. \qquad \square$$

The following lemma is proven using basic ideas from covering space theory, adapted from [7, Theorem 4.22]. We use this result in the proof of Lemma 5.1.10.

Lemma 5.1.9. *Let X and Y be Hausdorff spaces, and $p : X \to Y$ a surjective local homeomorphism. Assume that $p : X \to Y$ is closed and $p^{-1}(y)$ is finite for any $y \in Y$. Then $p : X \to Y$ is a covering map.*

Proof. Consider an arbitrary $y \in Y$, and let $\zeta_1, \ldots, \zeta_n$ be the distinct points in the fibre $p^{-1}(y)$. As $p : X \to Y$ is a local homeomorphism, we may choose pairwise disjoint open neighbourhoods $W_1, \ldots, W_n$ in X of $\zeta_1, \ldots, \zeta_n$ respectively, so that $p|_{W_i} : W_i \to U_i$ is a homeomorphism (where each U_i is a neighbourhood of y). Since the neighbourhood $V := W_1 \cup \cdots \cup W_n$ of $p^{-1}(y)$ is open, $p(X \setminus V)$ is closed, and thus $U := U_1 \cap \cdots \cap U_n \cap (Y \setminus p(X \setminus V))$ is an open neighbourhood of y with $p^{-1}(U) \subset V$. If we set $V_j := W_j \cap p^{-1}(U)$, then $p^{-1}(U) = V_1 \cup \cdots \cup V_n$, and each $p|_{V_i} : V_i \to U$ is a homeomorphism. $\square$

Below we have the last of the supporting lemmas – the final ingredient which we present as a standalone result. The proof uses a variant of the classical implicit function theorem (Theorem 2.4.4).

Lemma 5.1.10. *Let A be an open, relatively compact subset of $\mathbb{C}^n$, $\Omega \subset \mathbb{C}^n$ a neighbourhood of $\overline{A}$, and K a locally path-connected metric space. We consider a continuous map $F : K \times \Omega \to \mathbb{C}^n$ which is holomorphic in the second variable. Assume that D_2F (the derivative with respect to the second variable) is continuous on $K \times \Omega$, and for all $b \in K$,*

1. *there exists* $\zeta \in A$ *with* $F(b,\zeta) = 0$,

2. $F(b,\cdot)$ *vanishes nowhere on the boundary* ∂A, *and*

3. *if* $t \in \overline{A}$, *then* $D_2 F|_{(b,t)}$ *is invertible.*

Then, if we write $P := \left(F|_{K\times A}\right)^{-1}(0)$, *the map* $\mathrm{pr} : P \to K$ *given by projection is a finite-sheeted covering map.*

Proof. By condition 1, the map $\mathrm{pr} : P \to K$ is surjective. Moreover, condition 3, together with the fact that $\overline{A}$ is compact, imply that for any $b \in K$, the set of points in A on which $F(b,\cdot)$ vanishes is finite. (This is a simple application of the inverse function theorem to the holomorphic map $F(b,\cdot) : \Omega \to \mathbb{C}^n$.)

We first show that $\mathrm{pr} : P \to K$ is a local homeomorphism. Consider an arbitrary $(b,\zeta) \in K \times A$ such that $F(b,\zeta) = 0$. By Theorem 2.4.4, there exist open neighbourhoods $U \subset K$ and $W' \subset A$ of b and ζ respectively, with a unique function $\varphi : U \to W'$ which is continuous, and satisfies $F(c,\varphi(c)) = 0$ for all $c \in U$; in particular, $\varphi(b) = \zeta$.

We let $W := (U \times W') \cap \mathrm{pr}^{-1}(U)$. Then, we can see that $\mathrm{pr}|_W : W \to U$ is a homeomorphism, for the uniqueness of the function $\varphi : U \to W'$ means that, if $(c,\xi) \in W$, then $\xi = \varphi(c)$; the inverse of $\mathrm{pr}|_W : W \to U$ is given by (id,φ). In other words, $\mathrm{pr} : P \to K$ is a local homeomorphism.

Finally we observe that $\mathrm{pr} : P \to K$ is a closed map: this follows from the fact that the projection $K \times \overline{A} \to K$ is a closed map (since $\overline{A}$ is compact), and $F^{-1}(0)$ is a closed set. Lemma 5.1.9 then tells us that $\mathrm{pr} : P \to K$ is a covering map. □

5.2 Main theorem

We now arrive at the focal result of this chapter. The following theorem directly gives us that the differentiation map $\mathcal{I}(M,\mathbb{C}) \to \mathcal{O}(M,\mathbb{C}^*), f \mapsto df/\theta$, is a weak homotopy equivalence: see Corollary 5.2.2.

Theorem 5.2.1. *Let* M *be an open Riemann surface. Fix a nowhere vanishing holomorphic 1-form* θ *on* M. *We denote by* B *the closed unit ball in* $\mathbb{R}^n, n \geq 1$, *and let* b_0 *be a point in the sphere* ∂B. *We let* $f_0 : M \to \mathbb{C}^*$ *be a nonvanishing holomorphic function such that* $f_0\theta$ *is exact.*

Assume we have a continuous map $f : B \times M \to \mathbb{C}^*$ with $f(b_0, \cdot) = f_0$, such that $f(b, \cdot)$ is holomorphic for each $b \in B$, and $f(b, \cdot)\theta$ is exact whenever $b \in \partial B$. Then, there exists a continuous map $h : B \times I \times M \to \mathbb{C}^*$ such that, for all $(b, t) \in B \times I$,

1. *$h(b, 0, \cdot) = f(b, \cdot)$,*
2. *$h(b_0, t, \cdot) = f_0$,*
3. *$h(b, t, \cdot)$ is holomorphic,*

and, if $(b, t) \in (\partial B \times I) \cup (B \times \{1\})$, then $h(b, t, \cdot)\theta$ is exact.

Proof. As with Theorem 4.2.8, we start by choosing a relatively compact, open coordinate disc $V_0 \subset M$ and a smaller closed disc $T \subset V_0$. Fix a smooth, strictly subharmonic Morse exhaustion function $\tau : M \to \mathbb{R}$ such that

(a) 0 is a regular value of τ, and

(b) $\tau < 0$ on T and $D_0 := \{x \in M : \tau(x) \leq 0\} \subset V_0$.

Let $p_1, p_2, \ldots$ be the critical points of τ in $M \setminus D_0$, ordered so that $0 < \tau(p_1) < \tau(p_2) < \cdots$, and let $(a_n)_{n \in \mathbb{N}}$ be a strictly increasing, divergent sequence of regular values of τ, such that $\tau(p_n) < a_{n+1} < \tau(p_{n+1})$ for any integer n associated with critical points $p_n < p_{n+1}$.

We denote by D_n the sublevel set $\{x \in M \,|\, \tau(x) \leq a_n\}$. Our goal is to inductively construct a sequence of continuous maps $h_n : B \times I \times V_n \to \mathbb{C}^*, n \geq 0$, on neighbourhoods V_n of D_n, such that

(i) $h_n(b, t, \cdot)$ is holomorphic for all $(b, t) \in B \times I$,

(ii) $h_n(b, 0, \cdot) = f(b, \cdot)|_{V_n}$ and $h_n(b_0, t, \cdot) = f_0|_{V_n}$ for all $(b, t) \in B \times I$,

(iii) if $(b, t) \in (\partial B \times I) \cup (B \times \{1\})$, then $h_n(b, t, \cdot)\theta$ is exact, and

(iv) $\|h_n - h_{n+1}\|_{B \times I \times D_n} < \epsilon_n \cdot 2^{-(n+2)}$,

where $\epsilon_n > 0$ is defined as follows. Let

$$\delta_n = \min\big\{1, \inf_{(b,t,x) \in B \times I \times D_n} |h_n(s, t, x)|\,\big\}$$

for each $n \geq 0$. Then set $\epsilon_n = \min\{\delta_0, \delta_1, \ldots, \delta_n\}$.

Since V_0 is simply connected, every holomorphic 1-form on V_0 is exact; so we may define $h_0 : B \times I \times V_0 \to \mathbb{C}^*, (b,t,x) \mapsto f(b,x)$.

For the inductive step, assume we have constructed $h_0, \ldots, h_n$ that satisfy conditions (i), (ii), (iii) and (iv) above, and consider two separate cases for the construction of h_{n+1}. Case 1, the noncritical case, is where $D_{n+1} \setminus D_n$ has no critical points of τ, so there is no change in the homotopy type of the sublevel sets in going from D_n to D_{n+1}. Case 2, the critical case, is where there is a critical point $p_{n+1} \in D_{n+1} \setminus D_n$; that is, when the homotopy type does change.

Case 1 (the noncritical case): Assume that $D_{n+1} \setminus D_n$ has no critical points of τ, and that we have $h_0, \ldots, h_n$ as above. We will assume that D_{n+1} is connected – if not, we may repeat the following arguments on each path component.

Fix a basis $C_1, \ldots, C_l$ of $H_1(D_n, \mathbb{Z})$ as in Lemma 5.1.1. From this lemma, we get a continuous deformation map $\Phi : B \times I \times \mathbb{C}^l \times D_n \to \mathbb{C}^*$ such that, for all $(b,t,x) \in B \times I \times D_n$, $\Phi(b,t,0,x) = h_n(b,t,x)$; and, letting $\mathcal{P}^\Phi : B \times I \times \mathbb{C}^l \to \mathbb{C}^l$ denote the period map with respect to $C_1, \ldots, C_l$, $D_3\mathcal{P}^\Phi$ is continuous on $B \times I \times \mathbb{C}^l$, and $D_3\mathcal{P}^\Phi$ is invertible at $(b,t,0)$ for each $(b,t) \in B \times I$. (Here, $D_3\mathcal{P}^\Phi$ denotes the derivative of $\mathcal{P}^\Phi$ with respect to the variable in $\mathbb{C}^l$.) We note, from the construction given in the proof, that Φ is of the form

$$\Phi : B \times I \times \mathbb{C}^l \times D_n \to \mathbb{C}^*, \quad (b,t,\zeta,x) \mapsto h_n(b,t,x)\, e^{\zeta_1\sigma_1(x)+\cdots+\zeta_l\sigma_l(x)},$$

where $\sigma_1, \ldots, \sigma_l$ are holomorphic functions on a neighbourhood of D_n. In the present case, we may choose an open neighbourhood $V_{n+1} \subset M$ of D_{n+1} which strongly deformation retracts onto D_{n+1}, and assume $\sigma_1, \ldots, \sigma_l$ to be defined on V_{n+1}.

We define the sets

$$K_1 := (B \times \{0\}) \cup (\{b_0\} \times I)$$

and

$$K_2 := (\partial B \times I) \cup (B \times \{1\}).$$

By Lemma 4.2.2, D_n is holomorphically convex in the Riemann surface V_{n+1}. Since $(K_1 \times V_{n+1}) \cup (B \times I \times D_n)$ is a strong deformation retract of $B \times I \times V_{n+1}$ (cf. [23, Theorem 6]), h_n can be extended continuously to a map $\tilde{h}_n : B \times I \times V_{n+1} \to \mathbb{C}^*$ with $\tilde{h}_n(b,t,x) = f(b,x)$ for all $(b,t,x) \in K_1 \times V_{n+1}$. Then, by Lemma 5.1.5, we have a continuous map $g_{n+1} : B \times I \times V_{n+1} \to \mathbb{C}^*$ which approximates h_n arbitrarily closely on $B \times I \times D_n$, such that $g_{n+1} = \tilde{h}_n$ on $K_1 \times V_{n+1}$, and $g_{n+1}(b, t, \cdot)$ is holomorphic for each $(b, t) \in B \times I$.

Define

$$\Psi : B \times I \times \mathbb{C}^l \times V_{n+1} \to \mathbb{C}^*, \quad (b, t, \zeta, x) \mapsto g_{n+1}(b, t, x)\, e^{\zeta_1\sigma_1(x)+\cdots+\zeta_l\sigma_l(x)}.$$

Choose a compact ball $A \subset \mathbb{C}^l$ centred at $0 \in \mathbb{C}^l$. The map Ψ then approximates Φ arbitarily closely on $B \times I \times A \times D_n$. We denote by $\mathcal{P}^\Psi : B \times I \times \mathbb{C}^l \to \mathbb{C}^l$ the period map of Ψ with respect to $C_1, \ldots, C_l$. Taking a suitably close approximation $\Psi : B \times I \times V_n$ as above, we let $A_0 \subset A$ be an open ball centred at $0 \in \mathbb{C}^l$ such that,

(1) at each $(b, t, \zeta) \in B \times I \times \overline{A_0}$, $D_3\mathcal{P}^\Psi$ is invertible,

(2) if $(b, t) \in K_2$, there exists $\zeta' \in A_0$ such that $\mathcal{P}^\Psi(b, t, \zeta') = 0$, and

(3) $\mathcal{P}^\Phi(b, t, \cdot)$ is nonvanishing on $\overline{A_0} \setminus \{0\}$, and $\mathcal{P}^\Psi(b, t, \cdot)$ is nonvanishing on ∂A_0, for all $(b, t) \in K_2$.

Condition (2) is made possible by the compactness of the parameter space $B \times I$, and the fact that the holomorphic function $\mathcal{P}^\Phi(b, t, \cdot)$, for each $(b, t) \in K_2$, is a local bijection around $0 \in \mathbb{C}^l$ which maps the origin to itself. We also briefly observe why condition (1) is possible, knowing that $\mathcal{P}^\Psi|_{B\times I\times A}$ approximates $\mathcal{P}^\Phi|_{B\times I\times A}$ only in the compact-open topology. Note first that the statement holds near a *fixed* $(b, t) \in B \times I$, by the nature of the compact-open topology on the space of holomorphic functions: if Ψ is a sufficiently close approximation, $D_3\mathcal{P}^\Psi$ is invertible at each point in $V_{b,t} \times C_{b,t}$, for a suitable neighbourhood $V_{b,t} \subset B \times I$ of (b, t), and a ball $C_{b,t}$ centred at $0 \in \mathbb{C}^l$. Then, by the compactness of the parameter space $B \times I$, we find A_0 and an approximation Ψ satisfying (1) for all $(b, t) \in B \times I$.

Finally, for condition (3), again consider a fixed $(b, t) \in K_2$, and choose an open ball $A_{b,t}$ around the origin in $\mathbb{C}^l$ on whose closure $\mathcal{P}^\Phi(b, t, \cdot)$ is bijective. Since $\mathcal{P}^\Phi(b, t, 0) = 0$ and $D_3\mathcal{P}^\Phi|_{(b,t,0)}$ is invertible, we know by Theorem 2.4.4 that there exists a connected neighbourhood $M_{b,t}$ of (b, t) in $B \times I$ together with a map $r : M_{b,t} \to A_{b,t}$, such that $r(b, t) = 0$ and $\mathcal{P}^\Phi(c, s, r(c, s)) = 0$ for all $(c, s) \in M_{b,t}$. Moreover, for suitably small $A_{b,t}$, the map r is both continuous and unique. In our case, since $\mathcal{P}^\Phi(c, s, 0) = 0$ for all $(c, s) \in K_2$, the map r vanishes throughout $K_2 \cap M_{b,t}$. In other words, by the uniqueness of r, $\mathcal{P}^\Phi(c,s,\cdot)|_{\overline{A_{b,t}}}$ vanishes *only* at the origin for all $(c,s) \in M_{b,t} \cap K_2$; and since K_2 is compact, we can find an A_0 satisfying this requirement for all $(b,t) \in K_2$. If Ψ is a sufficiently close approximation of Φ, then for all $(b,t) \in K_2$, $\mathcal{P}^\Psi(b,t,\cdot)$ does not vanish on the boundary ∂A_0. We will assume Ψ is chosen so.

We now show that there exists a continuous map $\rho : B \times I \to A_0$ with $\rho(b,t) = 0$ whenever $(b,t) \in K_1$, and $\mathcal{P}^\Psi(b,t,\rho(b,t)) = 0$ for all $(b,t) \in K_2$. We first note that $\mathcal{P}^\Psi$ satisfies the assumptions of Theorem 2.4.4 (taking the parameter space to be $B \times I$) at any point $(b,t,\zeta) \in B \times I \times A_0$ for which $\mathcal{P}^\Psi(b,t,\zeta) = 0$. In particular, this observation is relevant on the subset K_2, where condition (2) applies. It is easy to see K_2 is simply connected – one way is by showing that $\pi_1(K_2,(b_0,1))$ is trivial (the basepoint is not of importance, for K_2 is path connected). Explicitly, let $l : I \to K_2$ be a loop with $l(0) = l(1) = (b_0,1)$, and write $l = (l_1,l_2)$. The map $I \times I \to K_2, (s,t) \mapsto (l_1(t),(1-s)l_2(t)+s)$, continuously deforms l to a loop in the contractible space $B \times \{1\}$ while preserving the basepoint $(b_0,1)$ – thus K_2 is simply connected. If we write $P := (K_2 \times A_0) \cap \left(\mathcal{P}^\Psi\right)^{-1}(0)$, we know by Lemma 5.1.10 that the projection map $\mathrm{pr} : P \to K_2$ is a covering map. Then, since K_2 is simply connected, covering space theory gives that P is a disjoint union of homeomorphic copies of K_2; more precisely, if we take any connected component R of P, $\mathrm{pr}|_R : R \to K_2$ is a homeomorphism. Denote by $(\mathrm{id},\varphi) : K_2 \to R$ the inverse.

We may extend $\varphi : K_2 \to A_0$ continuously to a map $\tilde{\varphi} : \partial(B \times I) \to A_0$ by setting $\varphi = 0$ on $B \times \{0\}$ and $\tilde{\varphi} = \varphi$ on K_2 (note that $\tilde{\varphi}$ is well defined, for φ vanishes throughout $\partial B \times \{0\}$). Let N be a neighbourhood of $\partial(B \times I)$ in $B \times I$; we then further extend the map $\tilde{\varphi}$ continuously to the desired map $\rho : B \times I \to A_0$, so that ρ vanishes on $(B \times I) \setminus N$. For suitably chosen N (and of course, A_0 and Ψ), if we set $h_{n+1}(b,t,\cdot) := \Psi(b,t,\rho(b,t),\cdot)$ we have a homotopy $h_{n+1} : B \times I \times V_{n+1} \to \mathbb{C}^*$ which satisfies the desired properties.

Case 2 (the critical case): We start with compact subsets $D_n \subset D_{n+1}$ with exactly one critical point $p_{n+1} \in D_{n+1} \setminus D_n$. Again, we assume that we have $h_0,\dots,h_n$ satisfying (i), (ii), (iii) and (iv). Since τ is strictly subharmonic, the Morse index of p_{n+1} is either 0 or 1 (cf. Remark 2.3.9).

First consider the case where the Morse index of p_{n+1} is 0. As t passes p_{n+1}, the sublevel sets $\{x \in M | \tau(x) \leq t\}$ acquire a new connected component. We can extend h_n as follows. Take a connected component K of D_{n+1}. If $K \cap D_n \neq \varnothing$, the map g_n is extended onto K as in Case 1. If $K \cap D_n = \varnothing$, we simply choose $B \times I \times K, (b,t,x) \mapsto f(b,x)$, as the extension of h_n to K.

Now assume that the Morse index of p_{n+1} is 1. Then the homotopy type of D_{n+1} is described by attaching a smooth embedded arc C to the set D_n. More specifically, assuming, as we may, that $a_{n+1} - a_n$ is sufficiently small, $D_n \cup C$ is a strong deformation retract of D_{n+1} (see [19, Part I, §3]), and C meets D_n precisely at the endpoints of C.

We denote by $q_1, q_2 \in D_n$ the endpoints of C, which we assume are distinct. We may further assume that D_{n+1} is connected (for any connected component of D_{n+1} which does not contain C can be treated as in the noncritical case). Consider two different situations: (A) where q_1 and q_2 belong to the same connected component of D_n, and (B) where q_1 and q_2 lie in different connected components of D_n.

Say we are in situation A: here, since by assumption $D_n \cup C$ is connected, so too is D_n. Therefore, we may pick $x_0 \in D_n$ and obtain a well-defined map

$$H_n : K_2 \times D_n \to \mathbb{C}, \quad ((b,t),x) \mapsto \int_{x_0}^{x} h_n(b,t,\cdot)\theta.$$

Parametrise C by a curve $\gamma : I \to C$ with $\gamma(0) = q_1$ and $\gamma(1) = q_2$. We treat the two situations similarly, to construct a continuous auxiliary map $\beta_n : B \times I \times (D_n \cup C) \to \mathbb{C}^*$ which equals h_n on $B \times I \times D_n$, and satisfies $\beta_n(b,t,\cdot) = f(b,\cdot)$ for all $(b,t) \in K_1$. The difference lies in that, with situation A, we further require β_n to satisfy $\int_\gamma \beta_n(b,t,\cdot)\theta = H_n((b,t),q_2) - H_n((b,t),q_1)$ for all $(b,t) \in K_2$.

By Lemma 5.1.8, we can find a continuous map $\kappa : B \times I \times C \to \mathbb{C}^*$ such that:

1. for all $(b,t,x) \in K_1 \times C$, $\kappa(b,t,x) = f(b,x)$,
2. for all $(b,t) \in B \times I$, $\kappa(b,t,q_1) = h_n(b,t,q_1)$ and $\kappa(b,t,q_2) = h_n(b,t,q_2)$, and
3. if in situation A, then $\int_\gamma \kappa(b,t,\cdot)\theta = H_n((b,t),q_2) - H_n((b,t),q_1)$ for all $(b,t) \in K_2$.

Expressly, in the notation of Lemma 5.1.8, we let $P := B \times I$, $R := K_1$, and identify C with the unit interval. We then choose $z : P \to \mathbb{C}$ to be a continuous function which equals $\int_\gamma f(b,\cdot)\theta$ for all $(b,t) \in K_1$ (this suffices for situation B); with the added requirement in situation A that $z(b,t) = H_n((b,t),q_2) - H_n((b,t),q_1)$ whenever $(b,t) \in K_2$.

We then have a well-defined map $\beta_n : B \times I \times (D_n \cup C) \to \mathbb{C}^*$ given by $\beta_n = h_n$ on $B \times I \times D_n$, and $\beta_n = \kappa$ on $B \times I \times C$. Moreover, for each $(b,t) \in K_2$, the 1-form $\beta_n(b,t,\cdot)\theta$ is exact.

Although $D_n \cup C$ is not a Riemann surface with boundary, we can adapt the arguments in the proof of Lemma 5.1.1 to the map $\beta_n : B \times I \times (D_n \cup C) \to \mathbb{C}^*$. This, as we show, allows us to approximate β_n by a map $h_{n+1} : B \times I \times V_{n+1} \to \mathbb{C}^*$ on an open neighbourhood of V_{n+1} of $D_n \cup C$, such that $h_{n+1}(b,t,\cdot) = f|_{V_{n+1}}$ if $(b,t) \in K_1$, and $h_{n+1}(b,t,\cdot)\theta$ is an exact holomorphic 1-form on V_{n+1} for all $(b,t) \in K_2$. The details are as follows.

We fix piecewise-smooth loops $\gamma_1, \ldots, \gamma_l$ in $D_n \cup C$ which form a basis for $H_1(D_n \cup C, \mathbb{Z})$. By following the proof of Lemma 5.1.1, we get functions σ_i which are holomorphic on an open neighbourhood of D_{n+1}, from which we construct a deformation map $\Phi_{\beta_n} : B \times I \times \mathbb{C}^l \times (D_n \cup C) \to \mathbb{C}^*$ with $\Phi_{\beta_n}(\cdot,\cdot,0,\cdot) = \beta_n$, such that, if $\mathcal{P} : B \times I \times \mathbb{C}^l \to \mathbb{C}^l$ is its period map with respect to $\gamma_1, \ldots, \gamma_l$, $D_3\mathcal{P}$ is invertible at $(b,t,0)$ for each $(b,t) \in B \times I$. By Lemma 4.2.2 and Remark 5.1.6, we can approximate β_n on $B \times I \times (D_n \cup C)$ by a continuous nonvanishing map $\tilde{\beta}_n : B \times I \times \tilde{V} \to \mathbb{C}^*$, for a suitable open neighbourhood $\tilde{V}$ of D_{n+1}. The map $\tilde{\beta}_n$ is such that $\tilde{\beta}_n(b,t,\cdot)$ is holomorphic for each $(b,t) \in B \times I$, and we further choose it to satisfy $\tilde{\beta}_n(b,t,\cdot) = f(b,\cdot)|_{\tilde{V}}$ if $(b,t) \in K_1$.

Let $V_{n+1} \subset\subset \tilde{V}$ be an open neighbourhood of D_{n+1} which strongly deformation retracts onto D_{n+1}; by Lemma 5.1.1, we find a deformation map $\Phi_{\tilde{\beta}_n} : B \times I \times \mathbb{C}^l \times V_{n+1} \to \mathbb{C}^*$ using the same holomorphic functions σ_i used to construct Φ_{β_n} (we may assume that the maps σ_i are defined on $\tilde{V}$). For a compact ball $E \subset \mathbb{C}^l$ centred at $0 \in \mathbb{C}^l$, the period maps $B \times I \times E \to \mathbb{C}^l$ of $\Phi_{\tilde{\beta}_n}$ and Φ_{β_n} (with respect to $\gamma_1, \ldots, \gamma_l$) are arbitrarily close in the compact-open topology. Denoting by $\tilde{\mathcal{P}}$ the period map of $\Phi_{\tilde{\beta}_n}$, and shrinking E as necessary, we have, if $\tilde{\beta}_n$ is a sufficiently close approximation of β_n, that $D_3\tilde{\mathcal{P}}$ is invertible at each $(b,t,\zeta) \in B \times I \times E$. (We justified this claim in Case 1: it holds true by the compactness of $B \times I$, and the nature of the compact-open topology on the space of holomorphic functions, where closeness implies C^∞-closeness on compact subsets.) Now, the important step can be taken: if $\tilde{\beta}_n$ and β_n are sufficiently close on $B \times I \times (D_n \cup C)$, we can find a continuous map $\xi : B \times I \to E$ which vanishes on K_1 and yields exact holomorphic 1-forms $\Phi_{\tilde{\beta}_n}(b,t,\xi(b,t),\cdot)\theta$ on U for all $(b,t) \in K_2$. We achieve this by employing the variant of the implicit function theorem, Theorem 2.4.4 – the same arguments are laid out in detail in Case 1, where we constructed a similar map $\rho : B \times I \to A_0$.

Set $h_{n+1}(b,t,x) := \Phi_{\tilde{\beta}_n}(b,t,\xi(b,t),x)$. The map $h_{n+1} : B \times I \times V_{n+1} \to \mathbb{C}^*$ can be chosen arbitrarily close to h_n on $B \times I \times D_n$, and satisfies properties (i), (ii), (iii) and (iv).

Hence we have completed the induction. The desired homotopy h is given by the limit of h_n as $n \to \infty$. By construction, h takes values in $\mathbb{C}^*$ – since each $(b,t,x) \in B \times I \times M$ lies in some $B \times I \times D_m$, we have $|h(b,t,x) - h_m(b,t,x)| \leq \delta_m/2$. Moreover, if $(b,t) \in K_2$, since $h_n(b,t,\cdot)\theta$ is exact for all $n \geq 0$, $h(b,t,\cdot)\theta$ is exact. This concludes our proof. □

Corollary 5.2.2. *Let M be an open Riemann surface and θ a nonvanishing holomorphic 1-form on M. The continuous map*

$$\mathcal{I}(M,\mathbb{C}) \to \mathcal{O}(M,\mathbb{C}^*), \quad f \mapsto df/\theta,$$

is a weak homotopy equivalence.

Proof. We first prove the 1-parametric case. By Theorem 4.2.8, we have surjectivity at the level of path components. Theorem 5.2.1 now gives us injectivity. Indeed, consider holomorphic immersions $f_0, f_1 : M \to \mathbb{C}$ such that df_0/θ and df_1/θ belong to the same path component of $\mathcal{O}(M,\mathbb{C}^*)$. We let $h : I \to \mathcal{O}(M,\mathbb{C}^*)$ be a continuous map such that $h(s,\cdot)$ is holomorphic for all $s \in I$, $h(0) = df_0/\theta$ and $h(0) = df_1/\theta$. Then, by Theorem 5.2.1, there exists a continuous map $H : I \times I \to \mathcal{O}(M,\mathbb{C}^*)$ such that $H(s,0) = h(s)$, $H(0,t) = df_0/\theta$ for all $(s,t) \in I \times I$, and $H(s,t)\theta$ is exact whenever $(s,t) \in (\{0,1\} \times I) \cup (I \times \{1\})$. For $0 \leq \eta \leq 1$, we write $L := (I \times \{\eta\}) \cup (\{0,1\} \times [0,\eta])$, and define a homeomorphism $f_\eta : I \to L$ by

$$f_\eta(x) = \begin{cases} (0,3s) & s \in \left[0, \frac{\eta}{3}\right], \\ \left(\frac{3s-\eta}{3-2\eta}, \eta\right) & s \in \left[\frac{\eta}{3}, 1 - \frac{\eta}{3}\right], \\ (1, 3-3s) & s \in \left[1 - \frac{\eta}{3}, 1\right]. \end{cases}$$

The homotopy $G : I \times I \to \mathcal{O}(M,\mathbb{C}^*), (s,\eta) \mapsto H \circ f_\eta(s)$, takes h to the path $g := H \circ f_1$. Note that g is such that $g(0) = df_0/\theta$, $g(1) = df_1/\theta$, and, for all $s \in I$, $g(s)\theta$ is exact.

Fix $x_0 \in M$ and assume without loss of generality that, for all $x \in M$, $\int_{x_0}^x df_0 = f_0(x)$ and $\int_{x_0}^x df_1 = f_1(x)$. We then have a continuous path $I \to \mathcal{I}(M,\mathbb{C})$, $t \mapsto F_t$, where $F_t(x) = \int_{x_0}^x g(t)\,\theta$. So, $[f_0] = [f_1]$ in $\pi_0(\mathcal{I}(M,\mathbb{C}))$.

By applying these arguments to the special case where $f_0 = f_1$, we find that the map $\mathcal{I}(M,\mathbb{C}) \to \mathcal{O}(M,\mathbb{C}^*)$ induces a surjection at the level of fundamental groups. We thus have our *1-parametric* h-principle for holomorphic immersions: that is, the induced map $\pi_0(\mathcal{I}(M,\mathbb{C})) \to \pi_0(\mathcal{O}(M,\mathbb{C}^*))$ is a bijection, and the induced map $\pi_1(\mathcal{I}(M,\mathbb{C})) \to \pi_1(\mathcal{O}(M,\mathbb{C}^*))$ is a group epimorphism.

Now, let $n \geq 1$. We denote by B^n the closed unit ball in $\mathbb{R}^n$, and let $S^n = \partial B^{n+1}$ be the unit sphere. Consider a map $\varphi : S^n \to \mathcal{I}(M, \mathbb{C})$ which represents an equivalence class in the kernel of the homomorphism $\pi_n(\mathcal{I}(M, \mathbb{C})) \to \pi_n(\mathcal{O}(M, \mathbb{C}^*))$. In other words, $f : S^n \to \mathcal{O}(M, \mathbb{C}^*), b \mapsto d\varphi(b)/\theta$, is nullhomotopic in $\mathcal{O}(M, \mathbb{C}^*)$, which means that there exists a continuous map $\tilde{f} : B^{n+1} \to \mathcal{O}(M, \mathbb{C}^*)$ such that $\tilde{f}|_{S^n} = f$.

If $b_0 \in S^n$ is a given basepoint, by Theorem 5.2.1 there exists a continuous map $k : B^{n+1} \times I \to \mathcal{O}(M, \mathbb{C}^*)$ such that $k(\cdot, 0) = \tilde{f}$, $k(b_0, \cdot) = f(b_0)$, and $k(b,t)\theta$ is an exact 1-form for all $(b,t) \in (\partial B^{n+1} \times I) \cup (B^{n+1} \times \{1\})$. Fix $x_0 \in M$, and note that $\tilde{\varphi} : S^n \to \mathcal{I}(M, \mathbb{C}), b \mapsto \int_{x_0}^x k(b,1)\theta$, represents the identity in $\pi_n(\mathcal{I}(M, \mathbb{C}))$. We may assume without loss of generality that $\int_{x_0}^x f(b)\theta = \varphi(b)$ for all $b \in S^n$. The homotopy $S^n \times I \to \mathcal{I}(M, \mathbb{C}), (b,t) \mapsto \int_{x_0}^x k(b,t)\theta$, then takes φ to $\tilde{\varphi}$. This tells us that the kernel of $\pi_n(\mathcal{I}(M, \mathbb{C})) \to \pi_n(\mathcal{O}(M, \mathbb{C}^*))$ is trivial; that is, the map is a monomorphism.

It remains to show that $\pi_n(\mathcal{I}(M, \mathbb{C})) \to \pi_n(\mathcal{O}(M, \mathbb{C}^*))$ is an epimorphism. Here we assume that $n \geq 2$, having already established the 1-parametric principle. We consider any continuous map $\beta : S^n \to \mathcal{O}(M, \mathbb{C}^*)$ such that $\beta(b_0)\theta$ is exact for a fixed basepoint $b_0 \in S^n$. Without loss of generality, we let $b_0 \in S^n$ be the north pole $(1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $q : B^n \to S^n$ be given by $(x_1, \ldots, x_n) \mapsto (2p^2 - 1, cx_1, \ldots, cx_n)$, where $p = \sqrt{x_1^2 + \cdots + x_n^2}$ and $c = 2\sqrt{1 - p^2}$; this induces the well-known homeomorphism $\tilde{q} : B^n/\partial B^n \to S^n$. Define a map $\tilde{\beta} : B^n \to \mathcal{O}(M, \mathbb{C}^*), x \mapsto \beta \circ q(x)$. By Theorem 5.2.1, there exists a continuous map $l : B^n \times I \to \mathcal{O}(M, \mathbb{C}^*)$ such that $l(\cdot, 0) = \tilde{\beta}$, $l(c_0, \cdot) = \tilde{\beta}(c_0)$ for some fixed $c_0 \in \partial B^n$, and $l(b,t)\theta$ is an exact 1-form for all $(b,t) \in (\partial B^n \times I) \cup (B^n \times \{1\})$.

For $0 \leq \eta \leq 1$, write $M_\eta := (\partial B^n \times [0, \eta]) \cup (B^n \times \{\eta\})$, and observe that $M_\eta/(\partial B^n \times \{0\})$ is homeomorphic to S^n as follows. Each point $x \in B^n$ is uniquely identified by its radius $p(x) \in [0,1]$ and $n-1$ angular coordinates $\varphi_1(x), \ldots, \varphi_{n-1}(x)$, where $\varphi_1(x), \ldots, \varphi_{n-2}(x) \in [0, \pi]$ and $\varphi_{n-1}(x) \in [0, 2\pi)$. We define a continuous function $\psi_\eta : M_\eta \to B^n$ by mapping $(b,t) \in M_\eta$ (where $b \in B^n$ and $t \in I$) to the point $x \in B^n$ with radius $p(x) := \dfrac{p(b) + \eta - t}{1 + \eta}$, and angular coordinates $\varphi_1(x) := \varphi_1(b), \ldots, \varphi_{n-1}(x) := \varphi_{n-1}(b)$. Observe that this map is a homeomorphism with the property that if $c \in \partial B^n$, then $\psi(c, 0) = c$. This tells us that we have a well-defined induced map $\tilde{\psi}_\eta : M_\eta/(\partial B^n \times \{0\}) \to B^n/\partial B^n$ which is also a homeomorphism, and expressly, $\tilde{q} \circ \tilde{\psi}_\eta$ is a homeomorphism from $M_\eta/(\partial B^n \times \{0\})$ to S^n.

Consider the map

$$\Phi : B^n \times I \to \mathcal{O}(M, \mathbb{C}^*), \quad (b, \eta) \mapsto l \circ (\psi_\eta)^{-1}(b).$$

By construction, Φ induces a well-defined map $\Phi_* : S^n \times I \to \mathcal{O}(M, \mathbb{C}^*)$ such that, for all $(b, t) \in S^n \times I$,

(a) $\Phi_*(b, 0) = \beta(b)$,

(b) $\Phi_*(b_0, t) = \beta(b_0)$, and

(c) $\Phi_*(b, 1)\theta$ is exact.

We fix $x_0 \in M$; the element of $\pi_n(\mathcal{I}(M, \mathbb{C}))$ represented by

$$S^n \times I \to \mathcal{I}(M, \mathbb{C}), \quad \int_{x_0}^{x} \Phi_*(b, 1)\theta,$$

maps to the equivalence class of β in $\pi_n(\mathcal{O}(M, \mathbb{C}^*))$. Thus, $\pi_n(\mathcal{I}(M, \mathbb{C})) \to \pi_n(\mathcal{O}(M, \mathbb{C}^*))$ is an epimorphism. □

Remark 5.2.3. It is not difficult to show that $\mathcal{O}(M, \mathbb{C}^*)$ has the same weak homotopy type as $\mathcal{C}(M, \mathbb{C}^*)$, the space of nonvanishing continuous maps on M, which is a space we understand through algebraic topology.

Consider the inclusion $\mathcal{O}(M, \mathbb{C}^*) \hookrightarrow \mathcal{C}(M, \mathbb{C}^*)$. Denote by B the closed unit ball in $\mathbb{R}^n, n \geq 1$, and let $f : B \times M \to \mathbb{C}^*$ be a continuous map with $f(b, \cdot)$ holomorphic for all $b \in \partial B$. We fix a basepoint $b_0 \in \partial B$. Then, the map $k : B \times M \to \mathbb{C}^*, (b, x) \mapsto f(b, x)/f(b_0, x)$, is nullhomotopic; we can thus lift k with respect to the universal covering map $\exp : M \to \mathbb{C}^*$ to a continuous map $v : B \times M \to \mathbb{C}$, so that $v(b_0, \cdot)$ is a map of constant value 0. Let $\tilde{h} : B \times I \times M \to \mathbb{C}$ be given by $(b, t, x) \mapsto (1 - t)\, v(b, x)$. The map $h : B \times I \times \mathbb{C}^* \to \mathbb{C}^*, (b, t, x) \mapsto f(b_0, x) \exp(\tilde{h}(b, t, x))$, is such that

1. $h(b, t, \cdot) = f(b, \cdot)$ for all $(b, t) \in (B \times \{0\}) \cup (\{b_0\} \times I)$, and
2. $h(b, t, \cdot)\theta$ is holomorphic for all $(b, t) \in (\partial B \times I) \cup (B \times \{1\})$.

Then, arguing as in Corollary 5.2.2, we have that $\mathcal{O}(M, \mathbb{C}^*) \hookrightarrow \mathcal{C}(M, \mathbb{C}^*)$ induces an isomorphism at the level of π_n for all $n \geq 1$, and an injection at the level of π_0. We get surjectivity at the level of π_0 from Theorem 2.2.11.

This way, we have a complete description of the weak homotopy type of $\mathcal{I}(M, \mathbb{C})$. In particular, knowing that M has the homotopy type of a bouquet of circles, standard algebraic topology arguments tell us that $\pi_0(\mathcal{I}(M, \mathbb{C}^*))$ is in bijection with $\mathbb{Z}^r$, the free abelian group on r elements, where $r \in \{0, 1, 2, \ldots, \infty\}$ is the number of circles in the bouquet. In other words, r is the rank of the free abelian group $H_1(M, \mathbb{Z})$.

Bibliography

[1] A. Alarcón and F. Forstnerič. Null curves and directed immersions of open Riemann surfaces. *Inventiones mathematicae*, 196(3):733–771, 2014.

[2] H. Behnke and K. Stein. Entwicklung analytischer Funktionen auf Riemannschen Flächen. *Mathematische Annalen*, 120(1):430–461, 1947.

[3] E. Bishop. Subalgebras of functions on a Riemann surface. *Pacific J. Math.*, 8(1):29–50, 1958.

[4] K. C. Border. *Notes on the Implicit Function Theorem.* Lecture notes, unpublished, 2019 (accessed 20 January 2020). http://www.its.caltech.edu-/∼kcborder/Notes/IFT.pdf.

[5] V. Borelli. *One-dimensional convex integration.* Lecture notes, unpublished, 2012 (accessed 20 January 2020). http://math.univ-lyon1.fr/∼borrelli/Diablerets/1-diablerets.pdf.

[6] Y. Eliashberg and M. Gromov. Removal of singularities of smooth mappings. *Mathematics of the USSR-Izvestiya*, 5(3):615–639, 1971.

[7] O. Forster. *Lectures on Riemann Surfaces*, volume 81. Springer-Verlag, New York, 1981.

[8] F. Forstnerič and F. Lárusson. The parametric h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$. *Communications in Analysis and Geometry*, 27(1):1–45, 2019.

[9] H. Geiges. *H-principles and Flexibility in Geometry.* Memoirs of the American Mathematical Society, no. 779. American Mathematical Society, Providence, R.I., 2003.

[10] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities*. Graduate Texts in Mathematics, 14. Springer-Verlag, New York, 1973.

[11] M. Gromov. A topological technique for the construction of solutions of differential equations and inequalities. *Proceedings of the International Congress of Mathematicians*, 2:221–225, 1970.

[12] M. Gromov. *Partial Differential Relations*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge. Springer-Verlag, 1986.

[13] R. Gunning and R. Narasimhan. Immersion of open Riemann surfaces. *Mathematische Annalen*, 174(2):103–108, 1967.

[14] L. Hörmander. *An Introduction to Complex Analysis in Several Variables*. North-Holland mathematical library, 7. North-Holland, second edition, 1979.

[15] F. Lárusson. *Eight lectures on Oka manifolds*, 2014 (accessed 20 October 2019). http://www.maths.adelaide.edu.au/finnur.larusson/misc/Beijing.pdf.

[16] F. Lárusson. Absolute neighbourhood retracts and spaces of holomorphic maps from Stein manifolds to Oka manifolds. *Proceedings of the American Mathematical Society*, 143(3):1159–1167, 2015.

[17] J. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics; 218. Springer-Verlag, second edition, 2012.

[18] L. H. Loomis and S. Sternberg. *Advanced Calculus*. Addison-Wesley, Reading, Massachusetts, 1968.

[19] J. Milnor. *Morse theory*. Annals of Mathematics Studies, 51. Princeton University Press, 1969.

[20] T. Napier and M. Ramachandran. *An Introduction to Riemann Surfaces*. Cornerstones. Birkhäuser, Boston, 2012.

[21] H. Seifert and W. Threlfall. *Lehrbuch der Topologie*. Chelsea, New York, 1947.

[22] D. Spring. *Convex Integration Theory: Solutions to the h-principle in geometry and topology*, volume 92 of *Modern Birkhäuser Classics*. Springer Basel, 1998.

[23] A. Strøm. Note on cofibrations II. *Mathematica Scandinavia*, 22:130–142, 1968.

[24] L. W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York, New York, 2008.

[25] H. Whitney. On regular closed curves in the plane. *Compositio Mathematica*, 4:276–284, 1937.