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Dynamical model for correlated two-pion exchange in the pion-nucleon interaction

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A microscopic model for the \(NN \to \pi\pi\) process is presented in the meson exchange framework, which in the pseudophysical region agrees with available quasiempirical information. The scalar (\(\sigma\)) and vector (\(\rho\)) piece of correlated two-pion exchange in the pion-nucleon interaction is then derived via dispersion integrals over the unitarity cut. Inherent ambiguities in the method and implications for the description of pion-nucleon scattering data are discussed.

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I. INTRODUCTION

The interaction between a pion and a nucleon plays a prominent role in low and medium energy physics since it is an important ingredient in many other hadronic reactions, e.g., pion production in nucleon-nucleon collisions or scattering of a pion by a nucleus.

Recently we have presented a meson exchange model for \(\pi N\) scattering [1] which contains conventional direct and exchange pole diagrams [Figs. 1(a)–1(d)] plus \(\sigma\) and \(\rho\) exchange terms [Figs. 1(e), 1(f)], and is unitarized by means of the relativistic Schrödinger equation. The main difference from former models [2–6] is the evaluation of the scalar-isoscalar (\(\sigma\)) and vector-isovector (\(\rho\)) terms. While in Refs. [2–6] these contributions are treated as single exchanges with sharp masses, in Ref. [1] they were viewed as arising from a correlated pair of pions in the \(J=0\) (\(\sigma\)) and \(J=1\) (\(\rho\)) \(t\) channels (see Fig. 2). Their contribution was evaluated by using quasiempirical information about the \(t\)-channel \(NN \to \pi\pi\) amplitudes in the pseudophysical region, which has been obtained by Höhler et al. [7] from an analytical continuation of both \(\pi N\) and \(\pi\pi\) data, and performing a suitable dispersion integral over the unitarity cut.

In order to build in constraints from soft pion theorems, a subtracted dispersion relation was used in Ref. [1] for the scalar contribution. This leads to a specific feature apparently favored by the \(\pi N\) data: Namely, the resulting interaction is repulsive in \(S\) waves but attractive in \(P\) waves. The approach used in Ref. [1] led to a considerably stronger contribution from \(\rho\) exchange than used in former treatments. On the other hand, by defining effective coupling constants suitable for a sharp \(\rho\)-mass parametrization we found a rather small tensor-to-vector ratio of coupling strengths in the physical \(t\) region, in line with values used before in the \(\pi N\) system [2].

As shown in Ref. [1], a model based on the diagrams of Figs. 1 and 2 results in \(\pi N\) phase shifts in the elastic region that agree well with empirical information, as do the scattering lengths and the \(\pi N\) \(\Sigma\) term (\(\approx 65\) MeV).

Although the approach outlined above and described in detail in Ref. [1] for evaluating correlated \(2\pi\) exchange is certainly adequate for free \(\pi N\) scattering, problems arise when this \(\pi N\) interaction is used in other areas of physics. For example, modifications of the interaction in the nuclear medium, which come into play when a pion is scattered by a nucleus, cannot be taken into account. The study of such effects requires an explicit field-theoretic description.

The aim of the present work is to provide such an explicit model for the correlated \(2\pi\) and \(KK\) exchange process of Fig. 2. This requires as input realistic \(\pi\pi \to \pi\pi\) and \(\pi\pi \to KK\) \(T\) matrices, which we have generated from a potential model based similarly on meson exchange and involving coupling between \(\pi\pi\) and \(KK\) channels (see Fig. 3). The use of such a dynamical model for the \(\pi\pi\) interaction will facilitate future investigations of not only possible medium modifications of the pion and nucleon legs, but also of the interaction itself.

The paper is organized as follows: In the next section,
the microscopic model for the $NN \rightarrow 2\pi$ process is described and compared to the data in the pseudophysical region. Section III deals with the resulting pion-nucleon interaction terms arising from correlated $2\pi$ exchange and their implications for the description of empirical $\pi N$ data. Section IV contains a short summary and outlook.

II. MICROSCOPIC MODEL FOR THE $N\bar{N} \rightarrow \pi\pi$ PROCESS

We will generate the amplitude for the process of Fig. 2 by solving the scattering equation

$$T_{NN \rightarrow \pi\pi} = V_{NN \rightarrow \pi\pi} + \sum_{pp=\pi\pi,\bar{K}K} T_{pp \rightarrow \pi\pi} g_{pp} V_{NN \rightarrow pp},$$

(2.1)

where $V_{NN \rightarrow pp}$ is the transition interaction and $T_{pp \rightarrow \pi\pi}$ the transition amplitudes from $\pi\pi$ and $\bar{K}K$ to $\pi\pi$; both will be specified below (we use $p$ to denote a generic pseudoscalar meson, $\pi, K$, or $\bar{K}$). Equation (2.1) could be considered to be a four-dimensional Bethe-Salpeter-type equation. However, we use the Blankenbecler-Sugar (BbS) technique \cite{8} to reduce the dimensionality of the integral to 3, which simplifies the calculation while maintaining unitarity. More explicitly, we have, in the c.m. system and in the helicity representation,

$$\langle q00|V_{NN \rightarrow \pi\pi}(t)|\bar{p}\lambda_N\lambda_{\bar{N}}\rangle = \langle q00|V_{NN \rightarrow \pi\pi}(t)|\bar{p}\lambda_N\lambda_{\bar{N}}\rangle + \sum_{pp} \int d^3k \frac{\langle q00|T_{pp \rightarrow \pi\pi}(t)|\bar{k}00\rangle \langle k00|V_{NN \rightarrow pp}(t)|\bar{p}\lambda_N\lambda_{\bar{N}}\rangle}{(2\pi)^3 \omega_p(k)[t - 4\omega_p^2(k)]},$$

(2.2)

with

$$\omega_p(k) = \sqrt{k^2 + m_p^2},$$

(2.3)

where $m_p = m_\pi, m_K$ for $p = \pi, K$ respectively. Thus, $k$ is the magnitude of the three-momentum part $\bar{k}$ of the relative four-momentum of the intermediate two-meson state. The four-momenta of the two intermediate mesons $k_1$ and $k_2$ are related to $\bar{k}$ by

$$k_1 = \left(\sqrt{t/2}, \bar{k}\right),$$

$$k_2 = \left(\sqrt{t/2}, -\bar{k}\right).$$

(2.4)

The helicity of the nucleon (antinucleon) is denoted by $\lambda_N$ ($\lambda_{\bar{N}}$). We perform a partial wave decomposition by writing

$$\langle q00|V_{NN \rightarrow \pi\pi}(t)|\bar{p}\lambda_N\lambda_{\bar{N}}\rangle = \frac{1}{4\pi} \sum_J [2J + 1] d_{J0}^J(\cos \theta) \langle 00|V_{NN \rightarrow pp}^{J}(q,p;t)|\lambda_N\lambda_{\bar{N}}\rangle,$$

(2.5)

with a similar expression for $T_{NN \rightarrow \pi\pi}$. Here, $d_{J0}^J$ are the conventional reduced rotation matrices, $\theta$ is the angle between $\bar{p}$ and $\bar{q}$, and $\lambda = \lambda_N - \lambda_{\bar{N}}$. Using these expressions, Eq. (2.2) becomes

$$\langle 00|T_{NN \rightarrow \pi\pi}^{J}(q,p;t)|\lambda_N\lambda_{\bar{N}}\rangle = \langle 00|V_{NN \rightarrow \pi\pi}^{J}(q,p;t)|\lambda_N\lambda_{\bar{N}}\rangle + \sum_{pp} \int_0^\infty dk 2 \frac{\langle k00|T_{pp \rightarrow \pi\pi}(q,k;t)|00\rangle \langle 00|V_{NN \rightarrow pp}^{J}(k,p;t)|\lambda_N\lambda_{\bar{N}}\rangle}{(2\pi)^3 \omega_p(k)[t - 4\omega_p^2(k)]},$$

(2.6)

The $N\bar{N} \rightarrow 2\pi$ on-shell amplitudes are related to the Frazer-Fulco helicity amplitudes $f_{J}^{\pm}$ \cite{9} via

\[\text{FIG. 2. Correlated } \pi\pi (K\bar{K}) \text{ exchange contributions.}\]

\[\text{FIG. 3. The contributions to the potential of the coupled } \pi\pi-K\bar{K} \text{ model.}\]
\[ f^{J^z}(t) = \frac{p_{on}^{m_N}}{4(2\pi)^2} \langle 00 | T^{J^z}_{N\pi\pi}(q_{on}; t) | \frac{1}{2} \frac{1}{2} \rangle, \]
\[ f^{J^z}(t) = -\frac{p_{on}^{m_N}}{2(2\pi)^2 \sqrt{t}} \langle 00 | T^{J^z}_{N\pi\pi}(q_{on}; t) | \frac{1}{2} \frac{1}{2} \rangle, \]  

(7.7)

with
\[ q_{on} = \frac{t}{4} - m_N^2, \]
\[ p_{on} = \frac{t}{4} - m_N^2. \]  

(2.8)

**A. \(N\bar{N} \rightarrow \pi\pi, K\bar{K} \) transition potentials**

The ingredients of the dynamical model for the transition interactions \(V_{N\pi\pi} \) and \(V_{N\pi\pi\pi} \) employed in this paper are displayed graphically in Fig. 4. The potential \(V_{N\pi\pi\pi} \) consists of \(\Lambda \) and \(\Sigma \) exchange terms plus \(\rho\)-meson pole diagrams. Their evaluation is based on the following spin-momentum-dependent parts of the interaction Lagrangians:

\[ \mathcal{L}_{BB\rho} = \frac{f_{BB\rho}}{m_p} \bar{\psi}_B \gamma^5 \gamma^\mu \psi_B \partial_\mu \phi_B, \]  

(2.9a)

\[ \mathcal{L}_{N\pi\pi} = g_{N\pi\pi} \bar{\psi}_N \gamma^\mu \psi_B \partial_\mu \phi_B, \]
\[ + \frac{f_{N\pi\pi}}{4m_N} \bar{\psi}_N \sigma^{\mu\nu} \psi_B (\partial_\mu \phi_{B\rho} - \partial_\nu \phi_{B\mu}), \]  

(2.9b)

\[ \mathcal{L}_{N\Delta\pi} = \frac{f_{N\Delta\pi}}{m_\pi} \bar{\psi}_\Delta (g_{1\Delta\gamma} + z \Delta \gamma_{\gamma\nu}) \psi_N \partial_\nu \phi_\pi \]  

(2.9c)

\[ \mathcal{L}_{\rho\rho} = g_{\rho\rho} \phi_B \rho \phi_B. \]  

(2.9d)

Here, \(\psi_B\) are the field operators for spin-1/2 particles \((N, \Lambda, \Sigma)\), \(\psi_\Delta\) is the spin-3/2 \(\Delta\)-isobar operator, and \(\phi_B\) are the corresponding operators for pseudoscalar \((\pi, \bar{K})\) mesons, while \(\phi_{B\rho}\) denotes the \(\rho\) meson. Also, \(\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]\). The \(N\Delta\) coupling [Eq. (2.9c)] includes off-shell contributions, whose strength is characterized by the parameter \(z\). For the propagators, we have

\[ S_{B}(p) = \frac{p + m_B}{p^2 - m_B^2}, \]  

(2.10a)

\[ S_{\Delta}^{\mu\nu}(p) = \frac{p + m_{\Delta}}{p^2 - m_{\Delta}^2} \left[ -g^{\mu\nu} + \frac{1}{3} \gamma^\mu \gamma^\nu + \frac{2}{3m_{\Delta}^2} p^\mu p^\nu \right. \]
\[ - \left. \frac{1}{3m_{\Delta}} (p^\mu \gamma^\nu - p^\nu \gamma^\mu) \right], \]  

(2.10b)

\[ S_{\rho}^{\mu\nu}(p) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_{\rho}^2}. \]  

(2.10c)

In this work, we omit the nonpole contributions to the spin-3/2 propagator [Eq. (2.10b)] since it is known [10] that their effect can be taken into account by the second term of the interaction Lagrangian [Eq. (2.9c)].

As usual, the resulting vertex functions are modified by phenomenological form factors \(F\) to account for the extended vertex structure. For the baryon exchange diagrams in Fig. 4 we choose

\[ F_{BB\rho}(q^2) = \frac{\left( n_{BB\rho} \Lambda_{BB\rho}^2 - m_B^2 \right)^2}{n_{BB\rho} \Lambda_{BB\rho}^2 - q^2}, \]  

(2.11)

where \(m_B\) is the mass (four-momentum) of the exchanged baryon (in the BbS framework adopted here, \(q^2 = -q^2\)). The cutoff masses \(\Lambda_{BB\rho}\) and powers \(n_{BB\rho}\) will be adjusted later. For the \(\rho\)-pole diagrams we introduce form factors at the meson-meson-meson vertices as follows:

\[ F_{\rho\rho}(q^2) = \frac{\left( n_{\rho\rho} \Lambda_{\rho\rho}^2 + m_\rho^2 \right)^2}{n_{\rho\rho} \Lambda_{\rho\rho}^2 + q^2}, \]  

(2.12)

with

\[ \Lambda_{\rho\rho} = \left( \frac{\Lambda_{\rho\rho}^2 + m_\rho^2}{4} \right)^{1/2}. \]  

(2.13)

In order to judge the behavior of these form factors it is \(\Lambda_{\rho\rho}\) which should be compared with \(\Lambda_{BB\rho}\) of Eq. (2.11) or the conventional monopole cutoff parameters.

The evaluation of the relevant transition potentials based on Eqs. (2.9)–(2.12) is involved but straightforward. The resulting expressions have to be multiplied by appropriate isospin factors derived from SU(3). More details can be found in Ref. [11]. Some slight modifications occur since now we use the BbS framework.

**B. \(\pi\pi \rightarrow \pi\pi, K\bar{K} \) amplitude**

The starting point for the evaluation of \(T_{\pi\pi \rightarrow \pi\pi} \) and \(T_{\pi\bar{K} \rightarrow \pi\bar{K}} \) is the driving terms shown in Fig. 3. Such a model, involving the coupled channels \(\pi\pi\) and \(K\bar{K}\), was
constructed by our group some time ago [12] based on
time-ordered perturbation theory. Here we use a model
with essentially the same physical input, which alterna-
tively uses the BbS technique. This procedure proved to
be advantageous when studying the scalar form factor of
the pion, kaon, and nucleon [13] since it has the correct
analytic behavior in the unphysical region (below the \( \pi \pi \)
threshold). The interaction Lagrangians used are (again
without isospin)

\[
\mathcal{L}_{\text{np}} = \frac{g_{\text{np}}}{2m_p} \phi \phi \partial_\mu \phi \partial^\mu \phi, \tag{2.14a}
\]

\[
\mathcal{L}_{\text{pp}} = g_{\text{pp}} \phi \phi \partial_\mu \phi \partial^\mu \phi, \tag{2.14b}
\]

\[
\mathcal{L}_{f_{2\text{pp}}} = \frac{g_{f_{2\text{pp}}}}{2m_p} \phi^\ast \phi \partial_\mu \phi \partial^\mu \phi, \tag{2.14c}
\]

where \( \nu \) denotes the vector mesons \( \omega, \rho, \phi, \) and \( K^* \) while \( f_2 \) is the tensor meson. As before, form factors are at-
tached to each vertex. For \( t- \) \([s-]\) channel exchanges, form
factors of the form given in Eq. (2.11) [Eq. (2.12)] are
used. For the \( s \)-channel pole diagrams in our interaction
model, bare masses have to be used. These pole con-
tributions then get renormalized to reproduce the physi-

cal resonance parameters by the iteration in the scatter-
ing equation. Values for bare masses, coupling constants
(with some constraints from SU(3) symmetry), and cut-
off masses have been adjusted to reproduce the empirical
\( \pi \pi \) phase shifts and inelasticities. These parameters are
given in Tables I–IV. The description of the data is as
successful as in Ref. [12]. Figure 5 shows the phases for
the \( J=0 \) partial waves of relevance in this paper, as well
as the \( S \)-wave inelasticity around 1 GeV. (In \( P \) waves, the
inelasticity is rather small in this energy region.)

C. Model in the pseudophysical region

In order to evaluate the \( NN \rightarrow \pi \pi \) amplitudes it re-
main to specify the parameters in the \( NN \rightarrow \pi \pi, K \bar{K} \)
transition potentials. Masses and most coupling con-
stants are not treated as fit parameters but are taken
from other sources, using SU(3) symmetry arguments
wherever possible. The \( \rho NN \) coupling \( g_{\rho NN}^{(0)} \) is taken
to be equal to the \( \rho \pi \pi \) coupling. The parameter \( x_\Delta \)
[Eq. (2.9c)], the bare tensor-vector coupling constant ra-
tio \( \kappa_\rho^{(0)} \equiv f_{\rho NN}^{(0)} g_{\rho NN}^{(0)} \), and the cutoff masses \( \Lambda_{NN\pi}, \Lambda_{N\Delta\pi} \) have been adjusted to the quasiempirical results
obtained by Höhler et al. [7] from analytic continuation

<table>
<thead>
<tr>
<th>Particle</th>
<th>Mass (MeV)</th>
<th>Particle</th>
<th>Mass (MeV)</th>
<th>Particle</th>
<th>Mass (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>139.57</td>
<td>( \rho^{(0)} )</td>
<td>1151.3</td>
<td>( f_2^{(0)} )</td>
<td>1710.0</td>
</tr>
<tr>
<td>( K )</td>
<td>495.82</td>
<td>( \omega )</td>
<td>782.6</td>
<td>( N )</td>
<td>938.926</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>1400.0</td>
<td>( K^* )</td>
<td>895.0</td>
<td>( \Delta )</td>
<td>1232.0</td>
</tr>
<tr>
<td>( \epsilon^{(0)} )</td>
<td>1505.0</td>
<td>( \phi )</td>
<td>1020.0</td>
<td>( \Delta )</td>
<td>1115.6</td>
</tr>
<tr>
<td>( \rho )</td>
<td>770.0</td>
<td>( f_2 )</td>
<td>1270.0</td>
<td>( \Sigma )</td>
<td>1193.0</td>
</tr>
</tbody>
</table>

of \( \pi N \) and \( \pi \pi \) data. The values used for the baryon
exchange contributions are given in Table V. The value
used for \( \kappa_\rho^{(0)} \) is 4.136. Note that the functional form of
the form factors has been chosen such that the dependence
on the power \( n \) is quite weak [the factor \( n \) multiplying
\( \Lambda^2 \) in Eqs. (2.11) and (2.12) ensures that an expansion
of \( F(q^2)/F(0) \) in powers of \( q^2 \) is independent of \( n \) up to
order \( q^2 \)]. We take \( n_{NN\pi} (n_{N\Delta\pi}) \) to be 1 (2). Since
the influence of the \( K \bar{K} \) intermediate state is small anyhow,
\( \Lambda_{NNK} \) and \( \Lambda_{N\Delta K} \) are arbitrarily put to 2.5 GeV. This
rather large value implies that the \( K \bar{K} \) contribution as
evaluated here is probably an upper limit. For consis-
tency, the parameters at the \( \rho \pi \pi \) and \( \rho K \bar{K} \) vertex are
taken to be the same as in the \( \pi \pi \rightarrow \pi \pi, K \bar{K} \) model
described in the last section.

We mention that the baryon-baryon-meson form factor
parameters should not be expected to agree with values
employed in the Bonn potential [14] and its extension to
the hyperon-nucleon case [15]. The reason is that for the
\( t \)-channel baryon exchange process considered here, one
is in a quite different kinematic regime. The fact that we
cannot establish a definite relation for the cutoff parame-
ters in different kinematic domains is the price we have to
pay for our simplified treatment of the vertex structure,
which makes the form factor depend on the momentum
of only one particle. This is a general problem, which, in
our opinion, is difficult to avoid, since a reliable QCD
calculation of the full momentum dependence of the vertex
does not exist.

There is one amplitude, \( f_0 \), for the scalar (\( \sigma \)) channel
whereas there are two, \( f_1 \) and \( f_2 \), for the vector (\( \rho \))
channel. In Fig. 6 we show the results in the pseudophy-

cisical region (\( t \geq 4m_N^2 \)) obtained from our dynamical
model, for both the real and imaginary parts.

Given that we have only four free parameters (\( \kappa_\rho^{(0)}, \)
\( x_\Delta, \Lambda_{NN\pi}, \) and \( \Lambda_{N\Delta\pi} \)), there is remarkable
agreement with the quasiempirical result [7] in all amplitudes. Some
disagreement occurs in the scalar amplitude, especially

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coupling constant ( \alpha_\beta \gamma )</th>
<th>Form factor power ( n_{\alpha_\beta \gamma} )</th>
<th>Cutoff ( \Lambda_{\alpha_\beta \gamma} (\text{MeV}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi \pi \rho ), t-channel ( \rho ) exchange</td>
<td>2.1</td>
<td>2</td>
<td>1650</td>
</tr>
<tr>
<td>( \pi \pi \rho ), s-channel ( \rho ) exchange</td>
<td>2.1</td>
<td>2</td>
<td>3300</td>
</tr>
<tr>
<td>( \pi \pi \epsilon ), s-channel ( \epsilon ) exchange</td>
<td>0.004</td>
<td>2</td>
<td>2000</td>
</tr>
<tr>
<td>( \pi \pi f_2 ), s-channel ( f_2 ) exchange</td>
<td>0.040</td>
<td>2</td>
<td>2000</td>
</tr>
</tbody>
</table>
at higher \( t \). Fortunately, as we will demonstrate below, these do not severely affect our final result, the correlated \( \pi \pi \) (and \( K \bar{K} \)) exchange potential in \( \pi N \) scattering. Furthermore one should keep in mind that the quasiphotonic result is subject to considerable uncertainty at large values of \( t \).

### III. \( \pi N \) Interaction Arising from Correlated \( 2\pi \) Exchange

In order to derive the effective \( \sigma \) and \( \rho \) exchange potentials we use the same procedure as in Ref. [1]; namely, we first perform dispersion integrals over the unitarity cut using as input the \( N\bar{N} \to \pi \pi \) amplitudes derived in the foregoing section. Corresponding \( \pi N \) potentials are then obtained in a straightforward way. We refer the reader to Ref. [1] for details.

#### A. Potential in the scalar channel

Here, a subtracted dispersion relation is used to impose the chiral symmetry constraint at the Cheng-Dashen point, with \( f_+^0(2m_\pi^2) \) put to zero, i.e.,

\[
\frac{f_+^0(t)}{t - 4m_N^2} = \frac{t - 2m_\pi^2}{\pi} \int_{4m_N^2}^{t_c} \frac{\text{Im} f_+^0(t')}{(t' - t)(t' - 4m_N^2)(t' - 2m_\pi^2)} dt',
\]

(3.1)

with \( t_c = 50m_\pi^2 \). Due to the slightly different \( \text{Im} f_+^0 \) predicted by the dynamical model compared to the pseudoequilibrium data of Ref. [7] (see Fig. 6) the resulting potential is now a bit stronger compared to that obtained in Ref. [1]. This is demonstrated in Fig. 7, for the on-shell case and some selected partial waves.

#### B. Potential in the vector channel

As in Ref. [1] we first start from

\[
f_\pm^1(t) = \frac{1}{\pi} \int_{4m_N^2}^{t_c} \frac{\text{Im} f_\pm^1(t')}{t' - t} dt'.
\]

(3.2)

As expected from the excellent agreement of our model amplitudes \( f_\pm^1 \) with the quasiphotonic ones of Ref. [7] (cf. again Fig. 6), the present results for the \( \pi N \) potential in the \( \rho \) channel are practically the same as those obtained in Ref. [1].

However, it was already pointed out in Ref. [1] that there is a considerable ambiguity in this result. Alternatively, we could have used a method proposed by Frazer and Fulco [9] and applied by Höhler and Pietarinen [16]. Here, one first constructs combinations \( \Gamma_{1,2}(t) \) corresponding to vector (\( \Gamma_1 \)) and tensor (\( \Gamma_2 \)) coupling amplitudes,

\[
\Gamma_1(t) = -\frac{m_N}{p_{\text{on}}^2} \left( f_+^1(t) - \frac{t}{4\sqrt{2m_N}} f_+^1(t) \right),
\]

(3.3a)

\[
\Gamma_2(t) = -\frac{m_N}{p_{\text{on}}^2} \left( f_+^1(t) - \frac{t m_N}{\sqrt{2}} f_+^1(t) \right),
\]

(3.3b)

and then performs the dispersion integrals over the unitarity cut,

\[
\bar{\Gamma}_{1,2}(t) = \frac{1}{\pi} \int_{4m_N^2}^{t_c} \frac{\text{Im} \Gamma_{1,2}(t')}{t' - t} dt'.
\]

(3.4)

Differences in the resulting potentials originate from the

### TABLE III. Parameters used in the \( \pi \pi \to K \bar{K} \) potential.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coupling constant ( g_{\alpha\beta\gamma} )</th>
<th>Form factor power ( n_{\alpha\beta\gamma} )</th>
<th>Cutoff ( \Lambda_{\alpha\beta\gamma} ) (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K K \rho, t\text{-channel } \rho ) exchange</td>
<td>0.525</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \omega, t\text{-channel } \omega ) exchange</td>
<td>-0.525</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \phi, t\text{-channel } \phi ) exchange</td>
<td>-1.050</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \rho, s\text{-channel } \rho ) exchange</td>
<td>0.525</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \epsilon, s\text{-channel } \epsilon ) exchange</td>
<td>0.001</td>
<td>2</td>
<td>2000</td>
</tr>
<tr>
<td>( K K f_2, s\text{-channel } f_2 ) exchange</td>
<td>0.010</td>
<td>2</td>
<td>2000</td>
</tr>
</tbody>
</table>

### TABLE IV. Parameters used in the \( K \bar{K} \to K \bar{K} \) potential.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coupling constant ( g_{\alpha\beta\gamma} )</th>
<th>Form factor power ( n_{\alpha\beta\gamma} )</th>
<th>Cutoff ( \Lambda_{\alpha\beta\gamma} ) (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K K \rho, t\text{-channel } \rho ) exchange</td>
<td>0.525</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \omega, t\text{-channel } \omega ) exchange</td>
<td>-0.525</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \phi, t\text{-channel } \phi ) exchange</td>
<td>-1.050</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \rho, s\text{-channel } \rho ) exchange</td>
<td>0.525</td>
<td>2</td>
<td>3100</td>
</tr>
<tr>
<td>( K K \epsilon, s\text{-channel } \epsilon ) exchange</td>
<td>0.001</td>
<td>2</td>
<td>2000</td>
</tr>
<tr>
<td>( K K f_2, s\text{-channel } f_2 ) exchange</td>
<td>0.010</td>
<td>2</td>
<td>2000</td>
</tr>
</tbody>
</table>
TABLE V. Parameters used in the $N\bar{N} \rightarrow \pi\pi, KK$ transition potentials: $t$-channel baryon exchanges.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coupling constant</th>
<th>Form factor power</th>
<th>Cutoff $\Lambda_{\alpha\beta\gamma}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NN\pi$</td>
<td>0.0790</td>
<td>1</td>
<td>1780</td>
</tr>
<tr>
<td>$N\Delta\pi$</td>
<td>0.36</td>
<td>2</td>
<td>1705</td>
</tr>
<tr>
<td>$N\Delta K$</td>
<td>0.0718</td>
<td>1</td>
<td>2500</td>
</tr>
<tr>
<td>$NSS$</td>
<td>0.00247</td>
<td>1</td>
<td>2500</td>
</tr>
</tbody>
</table>

*$\pi_{\Delta} = -0.847$. 

FIG. 5. $\pi\pi$ phase shifts obtained for $J = 0$ and $J = 1$ from our coupled channel $\pi\piKK$ model and the $S$-wave inelasticity. For references to the data, see Ref. [12].

FIG. 6. $N\bar{N} \rightarrow \pi\pi$ helicity amplitudes in the pseudophysical region. The solid lines denote the imaginary parts of the model amplitudes and the dashed lines the real parts. Squares and triangles denote the quasiempirical amplitudes taken from Ref. [7].
The intermediate state is the source of a branch cut in the complex \( t \) plane extending from \(-\infty\) to \(-70.5m_P^2\). In fact, there is an infinite number of such left hand cuts generated by all processes contributing to correlated two-pion exchange and it is by far impossible to include these pieces. Anyhow, \( \rho \) exchange is defined by the integral over the unitarity cut only. Therefore it is unavoidable that the results induced by Eqs. (3.2) and (3.4), respectively, will differ. (Cutting off the integration over the unitarity cut at \( t_c \) turns out to play a minor role only.)

These differences can be nicely demonstrated by parametrizing the resulting potentials in terms of effective \( t \)-dependent \( \rho \)-coupling strengths \( g_{1,2}(t) \) defined by

\[
g_{1,2}(t) = 12\pi(m_\rho^2 - t)\Gamma_{1,2}(t) ,
\]

![FIG. 7. On-shell potentials in various \( \pi N \) partial waves arising from correlated 2\( \pi \) exchange in the scalar channel. The solid lines are the result if the input from the dynamical model is used; the dashed lines are based on the pseudoempirical input given in [17].](image)

![FIG. 8. A diagram contributing to correlated two-pion exchange and its cuts.](image)

![FIG. 9. Effective coupling strengths for \( \rho \) exchange: (a) vector coupling strength \( g_1/4\pi \), (b) tensor coupling strength \( g_2/4\pi \), (c) \( \kappa = g_2/g_1 \). The solid lines denote the results if the dispersion integrals are performed for the \( f \) amplitudes [Eq. (3.2)]; the dashed lines show the results if the form factors \( \Gamma_{1,2} \) are dispersed [Eq. (3.4)].](image)
where $\Gamma_{1,2}$ is either obtained by inserting $f^1_{\pm}$ calculated using Eq. (3.2) into Eqs. (3.3) or alternatively by dispersing $\Gamma_{1,2}$ [cf. Eq. (3.4)]. (For the motivation of the definition of $g_{1,2}$, see Ref. [1].) In Fig. 9 we have plotted the effective vector coupling strength $g_1(t)/4\pi$, the effective tensor coupling strength $g_2(t)/4\pi$, and their ratio $\kappa = g_2/g_1$, choosing $m_p = 770$ MeV. Since the $t$ dependence in $\Gamma_{1,2}$ of Eq. (3.3) is rather weak, the resulting $g_2$ does not differ much. But the factor of $t$ in $\Gamma_1$ leads to a much smaller $g_1$ if $\Gamma_{1,2}$ are dispersed.

C. Implications for $\pi N$ scattering

Our model for correlated $2\pi$ exchange is supplemented by direct and exchange pole diagrams involving the nucleon and $\Delta$ isobar, and is then unitarized by means of a relativistic Schrödinger equation. We refer to Ref. [1] for details. It has been shown in that paper that, based on the quasiempirical input for the $N N \rightarrow \pi\pi$ process, a good description of all $\pi N$ partial waves is obtained by adjusting open form factor parameters. In that paper, $\rho$

![Fig. 10. $\pi N$ scattering phase shifts in $S$ and $P$ waves, as functions of the pion laboratory momentum. The solid lines originate from the model using the first ansatz for $\rho$ exchange [Eq. (3.2)]; the dashed lines denote the results if the second ansatz is used [Eq. (3.4)]. Empirical information is taken from Ref. [18].](image-url)
TABLE VI. The scattering lengths and volumes in units $m^{-3(L+1)}$

<table>
<thead>
<tr>
<th>Model</th>
<th>Koch and Pietarinan [18]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{11}$</td>
<td>0.165</td>
</tr>
<tr>
<td>$S_{21}$</td>
<td>−0.092</td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>−0.080</td>
</tr>
<tr>
<td>$P_{31}$</td>
<td>−0.042</td>
</tr>
<tr>
<td>$P_{33}$</td>
<td>0.210</td>
</tr>
</tbody>
</table>

exchange as defined by Eqs. (3.2) has been used.

We first want to discuss what happens when we now replace the quasipirical input for correlated $2\pi$ exchange by our dynamical model. The slight increase in the $\sigma$-channel potential (Fig. 7) leads to comparably weakly modified phase shifts. This effect can be compensated by a small readjustment of the cutoff parameter (introduced in addition for the $\sigma$ potential; see Ref. [1]), from 1200 MeV to 1120 MeV. There is almost no change in the $\rho$ channel provided the same ansatz is used as in Ref. [1]. Therefore a quantitative description of $S$ and $P$ waves is obtained with precisely the same values for parameters in pole and exchange diagrams as in Ref. [1] (solid lines in Fig. 10). Corresponding scattering lengths and volumes are given in Table VI.

However, a dramatic change occurs if the $\rho$ exchange potential is evaluated using Eq. (3.4). There is a strong reduction in the $S_{11}$ phase shift predictions, with smaller modifications in other partial waves (dashed lines of Fig. 10). The latter can be eliminated by suitably readjusting parameters in the pole and exchange diagrams, but the discrepancy in $S_{11}$ essentially remains.

In view of this situation, one may ask if the $\pi N$ data can discriminate between the different formulations for $\rho$ exchange. Within the strict confines of our model, it could be argued that it does. On the other hand, the discrepancy could be an indicator of the absence of an important ingredient still missing in the $S_{11}$ interaction. Indeed, there is empirically well-established resonant structure in that partial wave at higher energies, which cannot be reproduced by either model. One source for the required additional attraction in $S_{11}$ is the strong coupling of this partial wave to the reaction channel $\eta N$. A second source of attraction is provided by $N^*\Sigma$ (1535, 1650) pole diagrams in the $\pi N$ interaction. If direct coupling of the form

$$L_{N^*N^π} = g_{N^*N^π} Ψ_N^*Ψ_N^π + H.c.$$  \hspace{1cm} (3.6)

is assumed at the $N^*N\pi$ vertex, this process gives rise to attraction in the $S_{11}$ partial wave of $\pi N$ scattering starting from the $\pi N$ threshold.

To demonstrate the power of these additional degrees of freedom, in Fig. 11 the result of a simple calculation starting from the second model for $\rho$ exchange is plotted where an additional $N^*$ pole diagram has been included. The parameters used here are $m_{N^*}^0 = 1550$ MeV, $(g_{N^*N})^2/4\pi = 0.1$, and $\Lambda_{N^*} = 2000$ MeV with the form factor parametrization of Ref. [1]. Obviously such a model can describe low-energy $\pi N$ scattering. Therefore, to discard the second model of $\rho$ exchange on the basis of the current discrepancies is certainly not justified.

IV. SUMMARY

We have presented a dynamical model for the $NN \rightarrow \pi\pi$ process in the meson exchange framework, which in the pseudophysical region agrees with available quasipirical information. The scalar ($\sigma$) and vector ($\rho$) piece of correlated two-pion exchange in the pion-nucleon interaction is then derived via a dispersion integral over the unitarity cut. Concerning $\rho$ exchange, there is a sizable ambiguity in the prediction for its effective strength, which is due to different formulations of the coupling to the nucleon. While the restricted low-energy model we have used favors one formulation, mechanisms such as coupling to the $\eta N$ channel and inclusion of higher $N^*$ resonances, not treated in our model but necessary to explain the data at higher energies, could significantly alter this result, and suggest a direction of future investigation.