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Effect of randomly distributed voids on effective linear and nonlinear elastic properties of isotropic materials

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Abstract

This study utilises a third-order expansion of the strain energy density function and finite strain elastic theory to derive an analytical solution for an isolated, spherical void subjected to axisymmetric loading conditions. The solution has been validated with previously published results for incompressible materials and hydrostatic loading. Using this new solution and a homogenisation methodology, the effective linear and nonlinear properties of a material containing a dilute distribution of voids are derived. The effective nonlinear elastic properties are shown to be typically much more sensitive to the concentration of voids than the linear elastic properties. The derived analytical expressions for effective material properties may be useful for the development and justification of new experimental methods for the evaluation of porosity and theoretical models describing the evolution of mechanical damage associated with void nucleation and growth (e.g. creep).

Keywords: finite deformation theory, nonlinear elasticity, micromechanics,

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void, effective properties

1. Introduction

It is well known from experimental studies that the properties of secondary phases may significantly influence the stiffness, strength, and ductility of inhomogeneous materials [1]. The exact details of the internal stress and strain fields associated with inhomogeneities are exceedingly complex, and therefore theoretical research has largely focussed on predicting the effective properties using various homogenisation methodologies and linear constitutive relationships [2, 3, 4]. However, very limited progress has been made towards solving the corresponding problem for nonlinear elastic materials, primarily due to the lack of closed-form nonlinear analytic solutions for non-trivial geometries [5].

The permanent interest in nonlinear phenomena in composites, and recent progress in advanced material manufacturing and nondestructive damage characterisation using nonlinear ultrasonic waves [6] motivate further investigations of the effective properties of inhomogeneous nonlinear elastic materials. In the spirit of the Hashin-Shtrikman bounds derived within linear elasticity, several attempts have been made to develop variational bounds for nonlinear elasticity, though they have been hindered by the absence of explicit extremum principles for nonlinear continuum mechanics. Ogden [7] developed upper and lower bounds for nonlinear elastic materials which can be described by a convex strain energy function. Ponte Castañeda [8] considered the more general class of polyconvex strain energy functions, though the obtained upper and lower bounds were generally far apart, and could in some cases only be developed for incompressible media. The variational bounds for materials with nonlinear constitutive relations expressed in terms of the linear strain tensor have also been investigated [9].

Many numerical techniques have also been developed in the past to determine effective material properties of heterogeneous materials. These include asymptotic, static, dynamic, stochastic, Voronoi cell FE and other homogenization methods, [10, 11, 12]. Such numerical models are extremely computationally intensive, and in the case of a dilute distribution, the numerical error becomes comparable with the effect of the secondary phases [13]. It has also been demonstrated that, at non-dilute concentrations, the averaged properties derived from numerical simulations are highly sensitive to the spatial distribution of the secondary phases [14]. However, recent studies based on elastomers [15, 16] have concluded that for cavities and composites where the inclusion phase is soft relative to the matrix, the influence of the spatial arrangement, and even the shape of the cavity, is not strong. The corresponding analysis for a compressible material has not, to the best knowledge of the authors, been conducted. An additional complication exists for weakly nonlinear materials, in which the loading which can be tolerated without causing permanent deformation is quite small, typically featuring strains less than 0.2%. Consequently, the departure from a linear elastic response associated with such low levels of strain is extremely small, and the nonlinear effects will be practically indistinguishable from numerical errors associated with a FE simulation. One proposed answer to these objections is the use of a multilevel FE method [17, 18], though such models require detailed analysis of the presence of modelling and discretisation errors, and the use of adaptive modelling to ensure accuracy [19, 20]. Additionally, it is expected that computational and numerical precision issues will be amplified in the case of weakly- nonlinear elastic materials, e.g. many common structural materials and composites, for which the magnitude of nonlinear effects is small compared to the dominant linear response.

Due to the unavailability of variational bounds, and the difficulties associated with applications of numerical modelling discussed above, analytical approaches appear to be more attractive. For incompressible materials, some exact solutions have been derived, such as the Neo-Hookean incompressible model for a 2D porous medium [21]. For weakly-nonlinear compressible materials, the most promising approach has focussed on a third-order expansion of the strain energy function can be utilised, which represents the lowest order of nonlinear response of elastic materials [6]. Hashin [22] investigated radially symmetric motions of incompressible media containing voids. The incompressible Mooney-Rivlin strain energy density function was employed, and results for expansion and compression were presented. Ogden [23] derived the effective second-order bulk modulus for a compressible material featuring a dilute distribution of spherical particles using referential volume averages and a second-order elastic solution. Using this method, it was possible to derive a simple solution for the effective second-order bulk modulus in closed form under the assumption that the overall material was isotropic. Chen & Jiang [24] used three independent boundary displacement conditions to calculate the third-order elastic constants (TOECs), though expressions for the effective properties were cumbersome, and instead of explicit expressions, graphs of the effective properties for a few pairs of materials were presented. Imam, Johnson, and Ferrari [25] found the effective constants for a particulate composite using the incompressible Mooney-Rivlin material. Displacement boundary conditions were applied to a medium containing incompressible spherical particles, and the resulting effective properties were calculated using a perturbation approach; a transformation [26] was used to reduce the perturbation problem to two linear elasticity problems for incompressible media. Recently, Giordano, Palla, and Colombo [27] generalised the Eshelby problem of linear elasticity to nonlinear inclusions under the assumption that the matrix phase is linear elastic. Using this approach, it was possible to derive closed-form expressions for the effective third-order elastic constants of a particulate composite in which the particle phase is nonlinear elastic, while the matrix is linear elastic. However, to the best knowledge of the authors there are no published results in the literature for the effective TOECs for a compressible material where the matrix, or both the matrix and the secondary phase are described by nonlinear elastic constitutive equations.

The current paper presents the analytical derivation of the effective linear and nonlinear properties of a material containing a dilute distribution of spherical voids. The voids are assumed to be randomly distributed, such that the overall material is isotropic. The paper is organised as follows. The next section provides the mathematical formulation and background of the second-order elasticity problem and homogenisation methodology. After that the perturbation solution for a spherical void subjected to axisymmetric loading is presented. This solution is validated against published results for hydrostatic loading. Using this new solution and a homogenisation methodology proposed by Hill [28], the effective linear and nonlinear properties of a material containing a dilute distribution of voids are finally derived. For the particular case of an incompressible material, the derived expressions for material properties reproduce previously published results [25]. The paper concludes with a numerical example and a discussion of the implications of the current work for different fields of mechanics of materials.

2. Formulation and background

2.1. Governing equations

The governing equations of nonlinear elasticity are briefly reviewed, and further details may be found in [29]. Let the material and spatial points of a body be given by X and x, respectively. The position vectors of X and x are denoted $\mathbf{Z}$ and $\mathbf{z}$, respectively. The motion is the mapping $\varphi : B_r \to B$ from a fixed reference configuration B_r to the configuration B. The local properties of the deformation are described by the deformation gradient, with component representation

$$F^a_{\ A} = \frac{\partial \varphi^a}{\partial X^A}. \tag{1}$$

The right Cauchy-Green deformation tensor is defined $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. The second Piola-Kirchhoff stress tensor $\mathbf{S}$ for an isotropic, hyperelastic material with strain energy density function W may be expressed in terms of $\mathbf{C}$, its principal invariants I_1, I_2, I_3, and the metric tensor of the material coordinates $\mathbf{G}$

$$\mathbf{S} = 2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{G} - 2\frac{\partial W}{\partial I_2}\mathbf{C} + 2I_3\frac{\partial W}{\partial I_3}\mathbf{C}^{-1} \tag{2}$$

where the invariants are

$$I_1 = \mathrm{tr}(\mathbf{C}) \quad , \quad I_2 = \tfrac{1}{2}\mathrm{tr}(\mathbf{C})^2 - \tfrac{1}{2}\mathrm{tr}(\mathbf{C}^2) \quad , \quad I_3 = \det\mathbf{C}. \tag{3}$$

In the absence of body forces, the equations of equilibrium in terms of the first Piola-Kirchhoff stress tensor are

$$\mathrm{Div}\,\mathbf{P} = \mathbf{0} \tag{4}$$

where the divergence is taken with respect to the coordinate system of the material points. The third order expansion of the strain energy density function [30] for a compressible, isotropic material is

$$\begin{aligned} W(I_1, I_2, I_3) = {} & \tfrac{1}{8}(K + \tfrac{4}{3}\mu)(I_1 - 3)^2 - \tfrac{1}{2}\mu(I_2 - 2I_1 + 3) + \tfrac{1}{24}(l + 2m)(I_1 - 3)^3 \\ & - \tfrac{1}{4}m(I_1 - 3)(I_2 - 2I_1 + 3) + \tfrac{1}{8}n(I_1 - I_2 + I_3 - 1). \end{aligned} \tag{5}$$

where the bulk modulus is $K = \lambda + \frac{2}{3}\mu$, the Lamé parameters are λ and μ, and l, m and n are the TOECs, expressed in Murnaghan's form [30].

2.2. Axisymmetric deformations

The material configuration is described by the spherical coordinates (R, Θ, Φ) and the spatial configuration is described by the spherical coordinates (r, θ, ϕ). The basis vectors, metric tensor, and shifting tensors associated with these coordinate systems are presented in Appendix A. Axisymmetric deformations are described by the mapping

$$r = r(R, \Theta), \quad \theta = \theta(R, \Theta), \quad \phi = \Phi. \tag{6}$$

Consider a spherical cavity of radius a embedded in an infinite medium with a strain energy function given by (5). The cavity, located at the origin of the coordinates and with a radius of a, must be free of traction at its surface, which requires

$$\mathbf{P}\mathbf{d}_R = \mathbf{0}, \quad R = a. \tag{7}$$

If the medium is subjected to a homogeneous deformation which is axisymmetric with respect to the polar axis $\Theta = 0$, the boundary condition at infinity is

$$\mathbf{z} - \mathbf{Z} = U(\alpha X \mathbf{d}_X + \alpha Y \mathbf{d}_Y + Z \mathbf{d}_Z), \quad R \to \infty \tag{8}$$

where α is a parameter, and $\mathbf{d}_R$ is the radial unit vector in the material configuration. The boundary conditions reduce to a radially symmetric motion when $\alpha = 1$, a simple extension without lateral contraction when $\alpha = 0$, and a pure extension consistent with isochoric deformation when $\alpha = -\frac{1}{2}$.

2.3. Homogenisation methodology

Effective constitutive relations for nonlinear materials are formulated by relating macroscale variables to volume averages of microscale fields over a representative volume. Hill [28] discussed the development of effective constitutive relations for nonlinear materials at finite strain, finding that the appropriate field variables are the first Piola-Kirchhoff stress and the deformation gradient, as these quantities possess the property that their representative volume averages over a body are dependent only on data prescribed on the surface of the body.

The referential volume average of the deformation gradient in a body B_r with surface ∂B_r is denoted $\bar{\mathbf{F}}$, and the referential volume average of the first Piola-Kirchhoff stress tensor is denoted $\overline{\mathbf{P}(\mathbf{F})}$, where

$$\bar{\mathbf{F}} = \frac{1}{V} \int_{B_r} \mathbf{F} dV = \frac{1}{V} \int_{\partial B_r} \mathbf{z} \otimes \mathbf{n} dS \tag{9}$$

$$\overline{\mathbf{P}} = \frac{1}{V} \int_{B_r} \mathbf{P} dV = \frac{1}{V} \int_{\partial B_r} (\mathbf{P}\mathbf{n}) \otimes \mathbf{Z} dS \tag{10}$$

where **n** is the outward unit normal to the surface in the material coordinate system. In the above equations, the divergence theorem has been used to express the deformation gradient and the first Piola-Kirchhoff stress in terms of surface data [31, 32]. Ogden [23] discussed the application of Hill's results to a hyperelastic material, showing that the referential volume average of the strain energy function $\overline{W(\mathbf{F})}$ serves as a potential for the effective material. Nemat-Nasser [33] discussed the choice of kinematical and dynamical variables for homogenisation, with a particular focus on finite deformation plasticity and phase transformation problems.

In this paper, the solutions for the effective elastic properties are obtained from the elastic field of an isolated spherical void in a compressible nonlinear elastic material of infinite extent, following the methodology of a previous solution for incompressible materials [25]. Using this solution, it is possible to develop a solution for the representative spherical shell in Figure 1 and the effective homogeneous material in Figure 2. Following [25], the effective elastic properties are developed by imposing identical boundary displacement conditions, and equating the referential volume average of the strain energies in each model. As discussed by Chen & Jiang [24], the necessary geometric and boundary conditions for this model to provide exact results are not generally satisfied; the results for non-dilute concentrations must be interpreted as an approximation.

3. Perturbation solution for single spherical void

As discussed in section 1, exact solutions for nonlinear elasticity are, in general, only available for simple geometries and loading conditions. How-

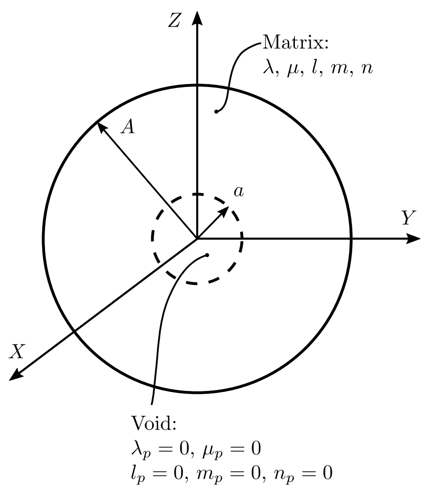


Figure 1: Isolated void model.

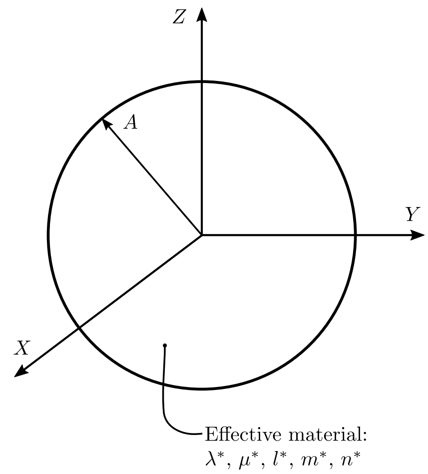


Figure 2: Effective homogeneous material.

ever, approximate solutions may be derived using perturbation methods (for example, [34, 35, 36]). Applying this procedure to the problem described in section 2.2, the boundary condition (8) may be expanded as a power series provided that the displacement gradient is small [24]. Using the shifting tensor (Appendix A, equation (A.4)) and expanding the displacement as a series in physical components,

$$r g_r^A \mathbf{d}_A - R\mathbf{d}_R = U\mathbf{u}_1 + U^2\mathbf{u}_2 \tag{11}$$

where the summation convention is applied to the index A, and the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ represent the first-order displacement vector and the second-order displacement vector, respectively. The condition $|\mathbf{u}_2| \ll |\mathbf{u}_1|$ is assumed to ensure the validity of the perturbation expansion. For axisymmetric deformations, these vectors have the form

$$\mathbf{u}_1 = u_1(R,\Theta)\mathbf{d}_R + \frac{v_1(R,\Theta)}{R}\mathbf{d}_\Theta \quad , \quad \mathbf{u}_2 = u_2(R,\Theta)\mathbf{d}_R + \frac{v_2(R,\Theta)}{R}\mathbf{d}_\Theta \tag{12}$$

where the functions u_1, v_1, u_2, v_2 are the physical components of the respective displacement vectors, and U is assumed to be a small parameter. The expansions for the functions $r(R,\Theta)$ and $\theta(R,\Theta)$ in terms of u_1, v_1, u_2, v_2 are,

$$r(R,\Theta) = R + Uu_1(R,\Theta) + U^2\left(u_2(R,\Theta) + \frac{v_1(R,\Theta)^2}{2R}\right) \tag{13}$$

$$\theta(R,\Theta) = \Theta + U\frac{v_1(R,\Theta)}{R} + U^2\left(\frac{u_1(R,\Theta)}{R} - \frac{u_1(R,\Theta)v_1(R,\Theta)}{R^2}\right). \tag{14}$$

Using (12), the left Cauchy-Green deformation tensor and the Cauchy stress tensor are expanded up to second order in U to provide [37]

$$\mathbf{B} = \mathbf{G} + 2\boldsymbol{\varepsilon}(\mathbf{u}_1 + \mathbf{u}_2) + \nabla\mathbf{u}_1\left(\nabla\mathbf{u}_1\right)^T \tag{15}$$

$$\mathbf{T} = \boldsymbol{\sigma}(\mathbf{u}_1 + \mathbf{u}_2) - \vartheta\boldsymbol{\sigma}(\mathbf{u}_1) + 2\boldsymbol{\varepsilon}(\mathbf{u}_1)\boldsymbol{\sigma}(\mathbf{u}_1) + \mathbf{T}'(\mathbf{u}_1) \tag{16}$$

$$\begin{aligned}\mathbf{T}'(\mathbf{u}_1) = &\left(\lambda\boldsymbol{\omega}\cdot\boldsymbol{\omega} + \left(l - m + \tfrac{1}{2}n\right)\vartheta^2 + \tfrac{1}{2}\left(\lambda + 2m - n\right)\right)\mathbf{G} \\ &+ (2m-n)\vartheta\boldsymbol{\varepsilon}(\mathbf{u}_1) + n\boldsymbol{\varepsilon}(\mathbf{u}_1)^2 + \mu\nabla\mathbf{u}_1(\nabla\mathbf{u}_1)^T\end{aligned} \tag{17}$$

where $\mathbf{G}$ is the metric tensor of the material coordinate system, and ϑ and $\boldsymbol{\omega}$ are the linear dilatation and the linear rotation vector corresponding to $\mathbf{u}_1$. The functions $\boldsymbol{\varepsilon}(\mathbf{u})$ and $\boldsymbol{\sigma}(\mathbf{u})$ are the strain and stress tensors for linear isotropic elasticity,

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \tfrac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda(\operatorname{Div}\mathbf{u})\mathbf{G} + \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right).$$

The terms in the expansion of the boundary condition (7) at $R = a$ are therefore

$$\boldsymbol{\sigma}(\mathbf{u}_1)\mathbf{d}_R = \mathbf{0} \tag{18a}$$

$$\boldsymbol{\sigma}(\mathbf{u}_2)\mathbf{d}_R = \left[\vartheta\boldsymbol{\sigma}(\mathbf{u}_1) - 2\boldsymbol{\varepsilon}(\mathbf{u}_1)\boldsymbol{\sigma}(\mathbf{u}_1) - \mathbf{T}'(\mathbf{u}_1)\right]\mathbf{d}_R. \tag{18b}$$

where the fact that the first Piola-Kirchhoff traction is zero has been used to write the boundary condition in terms of the Cauchy traction. Expansion of the boundary condition (8) leads to the following boundary conditions for the linear and second-order problems:

$$\mathbf{u}_1 = \tfrac{1}{2}[1 + \alpha + (1 - \alpha)\cos 2\Theta]\mathbf{d}_R - \tfrac{1}{2}(1 - \alpha)\sin 2\Theta\mathbf{d}_\Theta \tag{19a}$$

$$\mathbf{u}_2 = \mathbf{0}. \tag{19b}$$

Using the perturbation expansion, the nonlinear elastic problem has been reduced to two sequential linear elasticity problems, which may be solved using standard techniques.

3.1. Linear solution

The linear solution to the axisymmetric deformation of an infinite medium containing an isolated spherical void is a standard problem in elasticity, which may be solved using potential functions. The displacement solution is constructed using the axisymmetric potentials for isotropic materials discussed by Green and Zerna [38],

$$\mathbf{u}_1 = \frac{1}{2\mu}\nabla(\chi + Z\eta) - \frac{2(1-\nu)}{\mu}\eta\mathbf{d}_Z \tag{20}$$

where χ and η are potential functions which can be represented in terms of the axisymmetric spherical harmonic functions. The forms of the potentials

χ and η which are consistent the linear boundary conditions (18a) and (19a) are

$$\chi = F_0 a^3 R^{-1} + (A_2 R^2 + F_2 a^5 R^{-3}) P_2(\cos\Theta) \tag{21a}$$

$$\eta = (B_1 R + G_1 a^3 R^{-2}) P_1(\cos\Theta) \tag{21b}$$

where $P_n(\cos\Theta)$ is the zonal spherical harmonic of degree n, and F_0, A_2, F_2, B_1 and G_1 are arbitrary constants to be determined from the boundary conditions (18a) and (19a). The constants are found to satisfy

$$\begin{aligned} A_2 = -2U\alpha\mu \quad , \quad B_1 = -\tfrac{1}{3}U(1+2\alpha)(3K+\mu) \\ F_0 = \frac{5(3\lambda+5\mu)}{9\lambda+14\mu}A_2 + \frac{3(3\lambda^2-8\mu^2)}{2(\lambda+\mu)(9\lambda+14\mu)}B_1 \\ F_2 = \frac{6(\lambda+\mu)}{9\lambda+14\mu}A_2 - \frac{4\mu}{9\lambda+14\mu}B_1 \\ 5F_2 + 2G_1 = 0. \end{aligned} \tag{22}$$

3.2. Second-order solution

The second-order solution requires solving a linear elasticity problem for u_2, v_2 subjected to a body force and surface tractions dependent on the linear solution $\mathbf{u}_1$. Additionally, the linear solution may be shown to no longer satisfy the equilibrium condition when considering second order terms. As such, it is necessary to impose an additional displacement to achieve equilibrium. Using (16), the second-order equation of equilibrium may be shown to be

$$\mathrm{Div}\,[\boldsymbol{\sigma}(\mathbf{u}_2) + \boldsymbol{\sigma}(\mathbf{u}_1)\nabla\mathbf{u}_1 + \mathbf{T}'(\mathbf{u}_1)] = \mathbf{0}. \tag{23}$$

The particular solution necessary to maintain equilibrium at second order is formed using a potential ζ, and a Galerkin vector $\mathbf{w}$, with the particular solution for the displacement having the form

$$\mathbf{u}_2^{(P)} = \frac{\lambda + 3\mu + 2m}{2\mu(\lambda + 2\mu)}\nabla\zeta + \frac{\lambda + 2\mu}{2\mu}\left(\frac{1}{\lambda + \mu}\nabla^2\mathbf{w} - \frac{1}{\lambda + 2\mu}\nabla\mathrm{div}\,\mathbf{w}\right). \quad (24)$$

The potential ζ and the Galerkin vector $\mathbf{w} = w_R\mathbf{d}_R + R^{-1}w_\Theta\mathbf{d}_\Theta$ may be found using trial solutions of the form

$$\begin{aligned}\zeta = {} & C_1a^6R^{-4} + C_2a^3R^{-1}P_2(\cos\Theta) + C_3a^5R^{-3}P_4(\cos\Theta) \\ & + C_4a^{10}R^{-8}(4\cos^4\Theta + \sin^4\Theta)\end{aligned} \quad (25a)$$

$$\begin{aligned}w_R = {} & \left(a^3D_1 + \frac{a^6D_2}{R^3} + \frac{a^8D_3}{R^5}\right) + \left(a^3D_4 + \frac{a^6D_5}{R^3} + \frac{a^8D_6}{R^5}\right)P_2(\cos\Theta) \\ & + \left(a^3D_7 + \frac{a^6D_8}{R^3} + \frac{a^8D_9}{R^5}\right)P_4(\cos\Theta)\end{aligned} \quad (25b)$$

$$\begin{aligned}w_\Theta = {} & \left(a^3E_1 + \frac{a^6E_2}{R^3} + \frac{a^8E_3}{R^5}\right)\frac{d}{d\Theta}[P_2(\cos\Theta)] \\ & + \left(a^3E_4 + \frac{a^6E_5}{R^3} + \frac{a^8E_6}{R^5}\right)\frac{d}{d\Theta}[P_4(\cos\Theta)]\end{aligned} \quad (25c)$$

where the C, D and E coefficients are to be determined by substitution into the second-order equation of equilibrium (23). Complete expressions are presented for each constant in Appendix B. As with the linear solution, the homogeneous solution to the second order problem may be derived using potential functions,

$$\mathbf{u}_2^{(h)} = \frac{1}{2\mu}\nabla(\chi_2 + Z\eta_2) - \frac{2(1-\nu)}{\mu}\eta_2\mathbf{d}_Z \quad (26)$$

where the potentials χ_2 and η_2 are constructed using axisymmetric spherical harmonic functions

$$\chi_2 = a^3 F_1 R^{-1} + a^5 F_3 R^{-3} P_2(\cos\Theta) + a^7 F_5 R^{-5} P_4(\cos\Theta) \tag{27a}$$

$$\eta_2 = a^3 G_2 R^{-2} P_1(\cos\Theta) + a^5 G_4 R^{-4} P_3(\cos\Theta) \tag{27b}$$

and the values of the constants F_1, F_3, F_5, G_2, and G_4 are determined through the second-order boundary conditions. The particular solution (24) may be shown to vanish at infinity, and therefore only singular harmonics need to be included in the expressions for χ_2 and η_2. The complete second-order displacement is

$$\mathbf{u}_2 = \mathbf{u}_2^{(h)} + \mathbf{u}_2^{(P)}, \tag{28}$$

the equilibrium equation at second order reduces to

$$\mathrm{Div}\, \boldsymbol{\sigma}(\mathbf{u}_2^{(h)}) = \mathbf{0}, \tag{29}$$

and the second order boundary conditions become

$$\boldsymbol{\sigma}(\mathbf{u}_2)\mathbf{d}_R = [\vartheta\boldsymbol{\sigma}(\mathbf{u}_1) - 2\boldsymbol{\varepsilon}(\mathbf{u}_1)\boldsymbol{\sigma}(\mathbf{u}_1) - \mathbf{T}'(\mathbf{u}_1)]\, \mathbf{d}_R \tag{30}$$

$$\mathbf{u}_2^{(h)} = \mathbf{0}. \tag{31}$$

Though the calculations to determine the constants in equation (27) are extremely cumbersome, the problem is a standard linear elasticity problem which, having found the particular solution, may be solved using standard methods. Due to the length of the expressions involved, the explicit solutions for each coefficient are omitted.

The present solution, of the spherical void problem, may be generalised to the more general problem of a spherical inhomogeneity embedded in a medium of infinite extent. The particular solution presented in equations (25) may also be used to construct the solution for a spherical inhomogeneity embedded in an infinite medium subjected to homogeneous axisymmetric displacement conditions at infinity: the second-order solution in the matrix phase may be found using the results as they are presented in Appendix B, after making the appropriate adjustments to the values of the constants F_0, A_2, F_2, B_1 and G_1 for the inhomogeneity problem. The second-order deformation of the particulate phase does not require a particular solution, as the contribution of the linear solution to the second-order equilibrium equation vanishes. The only second-order solution that is necessary for the particulate phase is the homogeneous solution, which is required to satisfy additional continuity conditions at the matrix-particle interface.

3.3. Accuracy of the perturbation solution

The perturbation approach used to derive the solution presented in Section 3.2 relies on an expansion of the displacement as a power series about the unstressed configuration of the body. Similarly, the strain energy function which has been employed, equation (5), consists of a power series expansion which is terminated at third-order in the principal extensions [34]. The validity of employing these expansions rests upon the condition $|\mathbf{u}_2| \ll |\mathbf{u}_1|$, i.e. that the displacement associated with the second-order correction be small in comparison with the displacement associated with the linear elasticity solution [36]. One consequence of this condition is that the results of the present study are not suitable for characterising the effective behaviour of materials which can sustain large deformations, as the perturbation solution is not appropriate to capture the response of such a material.

The materials to which the present study is directed are weakly-nonlinear materials subjected to loading which does not cause yielding. Such conditions are appropriate for many common structural materials, including steel and aluminium alloys, and composite materials [6]. Under these conditions, the deformations are sufficiently small (e.g. below 0.2%) that a perturbation approach may be expected to yield accurate results. Some possible practical applications of the current work is in the acoustoelasticity and the propagation of nonlinear ultrasonic waves in structures, which have been of great interest in the area of nondestructive testing and evaluation in recent years, and has been the subject of many recent experimental studies [6, 39].

3.4. Comparison with previously published results

The above solution presented for axisymmetric deformations may be reduced to a purely radial expansion by setting the parameter $\alpha = 1$. This case is of particular interest, as the second-order solution for this problem has been derived in previous studies [23]. For an infinite medium containing a spherical void subjected to radial expansion, the coefficients in the linear elasticity solution (22) are

$$A_2 = -2U\mu\,, \quad B_1 = -3U(\lambda+\mu)\,, \quad F_0 = \tfrac{1}{2}U(3\lambda+2\mu)$$
$$F_2 = G_1 = 0.$$

The potential function and biharmonic vector involved in the particular solution for the displacement at second order (25) are

$$\zeta = \frac{1}{32}\frac{a^6}{R^4}(3\lambda+2\mu)^2U^2\,, \quad \mathbf{w} = \mathbf{0}$$

such that

$$u_R^{(P)} = \frac{a^6(3\lambda + 2\mu)^2(\lambda + 3\mu + 2m)}{16\mu^2(\lambda + 2\mu)R^5}$$

which may be shown to have the same form as the second-order solution presented by Ogden [23].

4. Effective homogeneous material

The solution for an isolated spherical void in an infinite matrix is adapted to the methodology for calculating effective elastic constants described in section 2.3 by considering the spherical shell $a \leq R \leq A$, where $a^3/A^3 = c$, to replicate the volume fraction of voids in the composite medium. The displacement on the surface $R = A$ is imposed as a boundary condition on the effective homogeneous medium. Due to (9), this boundary condition establishes the condition

$$\overline{\mathbf{F}^*} = \left(1 - \frac{V_v}{V}\right)\overline{\mathbf{F}} \tag{32}$$

where V is the volume of the effective homogeneous medium, and V_v is the volume of the void in the RVE.

Due to the arrangement of the voids, the effective body may be assumed to possess isotropic symmetry, such that the strain energy density function of the effective medium is assumed to take the form

$$\begin{aligned} W^*(I_1, I_2, I_3) = {} & \tfrac{1}{8}(K^* + \tfrac{4}{3}\mu^*)(I_1 - 3)^2 - \tfrac{1}{2}\mu^*(I_2 - 2I_1 + 3) \\ & + \tfrac{1}{24}(l^* + 2m^*)(I_1 - 3)^3 - \tfrac{1}{4}m^*(I_1 - 3)(I_2 - 2I_1 + 3) \\ & + \tfrac{1}{8}n^*(I_1 - I_2 + I_3 - 1). \end{aligned} \tag{33}$$

where K^*, μ^*, l^*, m^*, and n^* are the elastic constants of the effective material. The displacement solution for the effective medium is constructed using spherical harmonic functions

$$\mathbf{u}_1^{(H)} = \frac{1}{2\mu^*}\nabla(\chi^{(H)} + Z\eta^{(H)}) - \frac{2(1-\nu^*)}{\mu^*}\eta^{(H)}\mathbf{d}_Z \tag{34a}$$

$$\mathbf{u}_2^{(H)} = \frac{1}{2\mu^*}\nabla(\chi_2^{(H)} + Z\eta_2^{(H)}) - \frac{2(1-\nu^*)}{\mu^*}\eta_2^{(H)}\mathbf{d}_Z \tag{34b}$$

where the potential functions are

$$\chi^{(H)} = A_2^{(H)}R^2P_2(\cos\Theta) + A_4^{(H)}A^{-2}R^4P_4(\cos\Theta) \tag{35a}$$

$$\eta^{(H)} = B_1^{(H)}R^1P_1(\cos\Theta) + B_3^{(H)}A^{-2}R^3P_3(\cos\Theta) \tag{35b}$$

$$\chi_2^{(H)} = A_3^{(H)}R^2P_2(\cos\Theta) + A_5^{(H)}A^{-2}R^4P_4(\cos\Theta) + A_7^{(H)}A^{-4}R^6P_6(\cos\Theta) \tag{35c}$$

$$\eta_2^{(H)} = B_2^{(H)}R^1P_1(\cos\Theta) + B_4^{(H)}A^{-2}R^3P_3(\cos\Theta) + B_6^{(H)}A^{-4}R^5P_5(\cos\Theta) \tag{35d}$$

and no particular solution at second order is necessary, due to the form of $\mathbf{u}_1^{(H)}$. The coefficients in (35) are identified using the boundary conditions, and the results for the linear solution are

$$A_2^{(H)} = \frac{\mu^*}{\mu}\left(A_2 + cF_0 - 3\frac{3K^*-2\mu^*}{6K^*+17\mu^*}cG_1 - \frac{21}{2}\frac{3K^*+\mu^*}{6K^*+17\mu^*}c^{5/3}F_2\right) \tag{36a}$$

$$A_4^{(H)} = \frac{\mu^*(3K^*-2\mu^*)}{\mu(6K^*+17\mu^*)}\left(2cG_1 - 5c^{5/3}F_2\right) \tag{36b}$$

$$B_1^{(H)} = \frac{3K^*+\mu^*}{\mu}\left(\frac{\mu}{3(\lambda+\mu)}B_1 + \frac{1}{2}cF_0 + \frac{3\lambda+5\mu}{6(\lambda+\mu)}cG_1\right) \tag{36c}$$

$$B_3^{(H)} = \frac{7\mu^*(3K^*+\mu^*)}{2\mu(6K^*+17\mu^*)}\left(2cG_1 + 5c^{5/3}F_2\right). \tag{36d}$$

The corresponding coefficients for the second-order solution are omitted, as they are the solution to a standard linear elasticity problem.

4.1. Volume averaged stress fields

The referential volume average of the first Piola-Kirchhoff stress in the effective medium is

$$\overline{\mathbf{P}^*} = \frac{1}{V}\int_{B^*} \mathbf{P}^* dV \tag{37}$$

and the total strain energy stored in the part of the matrix $a \leq R \leq A$ is

$$\overline{\mathbf{P}} = \frac{1}{V - V_v}\int_{B} \mathbf{P} dV. \tag{38}$$

In (37) and (38), the respective tensors are written in terms of a fixed Cartesian basis. After evaluation of the above integrals, the effective properties are derived using

$$\overline{\mathbf{P}^*} = \left(1 - \frac{V_v}{V}\right)\overline{\mathbf{P}}. \tag{39}$$

The averaged strain energies consist of terms proportional to U, corresponding to the linear solution of the problem, and terms proportional to U^2,
205 corresponding to the second order part of the solution. Using the solution presented in section 3, expressions for the three effective elastic properties l^*, m^*, n^* in terms of the volume fraction of voids c may be derived.

4.2. Effective elastic constants

Solving equation (39) considering first order terms in U allows for the calculation of the effective bulk modulus and the effective shear modulus. By considering radially symmetric deformation, $\alpha = 1$ in (8), the referential volume averages of the first Piola-Kirchhoff stress in the matrix and in the effective homogeneous medium are

$$\overline{\mathbf{P}} = 3KU\left[\mathbf{e}_z \otimes \mathbf{d}_Z + \mathbf{e}_x \otimes \mathbf{d}_X + \mathbf{e}_y \otimes \mathbf{d}_Y\right]$$
$$\overline{\mathbf{P}^*} = 3K^*U\left(1 + \frac{3cK}{4\mu}\right)\left[\mathbf{e}_z \otimes \mathbf{d}_Z + \mathbf{e}_x \otimes \mathbf{d}_X + \mathbf{e}_y \otimes \mathbf{d}_Y\right]$$

where the volumetric concentration of voids is $c = a^3/A^3$. Hence the effective bulk modulus K^* is

$$K^* = \frac{4(1-c)K\mu}{4\mu + 3Kc}. \tag{40a}$$

A similar calculation for $\alpha = -\frac{1}{2}$ provides

$$\begin{aligned}
\overline{\mathbf{P}} &= \mu U \left[2\mathbf{e}_z \otimes \mathbf{d}_Z - (\mathbf{e}_x \otimes \mathbf{d}_X + \mathbf{e}_y \otimes \mathbf{d}_Y)\right] \\
\overline{\mathbf{P}^*} &= \mu^* U \left(1 + \frac{6c(K+2\mu)}{9K+8\mu}\right) \left[2\mathbf{e}_z \otimes \mathbf{d}_Z - (\mathbf{e}_x \otimes \mathbf{d}_X + \mathbf{e}_y \otimes \mathbf{d}_Y)\right]
\end{aligned}$$

and hence the effective shear modulus,

$$\mu^* = \frac{\mu(1-c)(9K+8\mu)}{9K+8\mu+6c(K+2\mu)}. \tag{40b}$$

Equations (40a) and (40b) are identical to previously published dilute concentration results for the effective shear modulus of a solid containing a randomly-arranged distribution of voids [40], and therefore demonstrate that the homogenisation method considered here is consistent with results derived within linear elasticity.

The effective Murnaghan constants are identified using the terms associated with the second-order solution in (39). For radially symmetric deformations, $\alpha = 1$, an expression for the second-order bulk modulus $l + \frac{1}{9}n$ may be written in the simple form

$$\begin{aligned}
l^* + \tfrac{1}{9}n^* = \frac{27c(1-c)^2K^3\mu}{(3cK+4\mu)^3} - \frac{4(1-c)\mu(9cK^2-16\mu^2)}{(3cK+4\mu)^3}\left(l + \tfrac{1}{9}n\right) + \frac{72c(1-c)K^2\mu}{(3cK+4\mu)^3}m \\
- \frac{c(1-c)K^2(3(c+1)K+8\mu)}{(3cK+4\mu)^3}n + \frac{36c(1-c)K^2\mu}{(3cK+4\mu)^3}l.
\end{aligned} \tag{41}$$

The solution (41) is equivalent to a result found by Ogden [23], Chen & Jiang [24], though those authors considered an inhomogeneity rather than a void. The remaining effective constants may be found by solving a system of equations based on considering two other values of α. The closed form solutions are too cumbersome to present in full, however expansions correct to first order in the concentration c for three linearly independent third-order elastic constants are

$$l^* + \tfrac{1}{9}n^* = l + \tfrac{1}{9}n - c\left[\left(1 + \frac{9K}{4\mu}\right)\left(l + \tfrac{1}{9}n\right) - \frac{9K^2}{8\mu^2}\left(m - \tfrac{1}{6}n\right) + \frac{3K^3}{64\mu^3}n - \frac{27K^3}{64\mu^2}\right] \tag{42a}$$

$$\begin{aligned} m^* - \tfrac{1}{6}n^* = m - \tfrac{1}{6}n - c\Bigg[& \frac{243K^3 + 792K^2\mu + 1572K\mu^2 + 640\mu^3}{4\mu(9K+8\mu)^2}\left(m - \tfrac{1}{6}n\right) \\ & - \frac{120\mu^2}{(9K+8\mu)^2}\left(l + \tfrac{1}{9}n\right) - \frac{3K(33K^2 + 132K\mu + 92\mu^2)}{8\mu(9K+8\mu)^2}n \\ & - \frac{K(63K^2 + 222K\mu + 247\mu^2)}{2(9K+8\mu)^2}\Bigg] \end{aligned} \tag{42b}$$

$$\begin{aligned} n^* = n - c\Bigg[& \frac{45(300K^3 + 1149K^2\mu + 1376K\mu^2 + 536\mu^3)}{7(9K+8\mu)^3}n \\ & + \frac{7200\mu^3}{7(9K+8\mu)^3}\left(l + \tfrac{1}{9}n\right) - \frac{60\mu^2(-9K^2 + 84K\mu + 134\mu^2)}{7(9K+8\mu)^3} \\ & - \frac{540\mu(33K^2 + 132K\mu + 92\mu^2)}{7(9K+8\mu)^3}\left(m - \tfrac{1}{6}n\right)\Bigg]. \end{aligned} \tag{42c}$$

4.3. Incompressible material

The procedure used in section 3 applies for compressible media. Imam, Johnson, and Ferrari [25] addressed the incompressible case, using a transformation [26] to absorb the conservative part of the body force into the

Lagrange multiplier associated with the incompressibility constraint. An alternate approach is to apply a limiting procedure to each of the elastic constants [41]. The strain energy function for the incompressible Mooney-Rivlin material is

$$W_{MR} = \tfrac{1}{2}\left(\tfrac{1}{2}\mu + \xi\right)(I_1 - 3) + \tfrac{1}{2}\left(\tfrac{1}{2}\mu - \xi\right)(I_2 - 3). \tag{43}$$

where μ and ξ are the two incompressible elastic constants. In the incompressible limit, the two linear elastic and the three third-order elastic constants of the compressible case collapse into two constants, μ and ξ. The appropriate limits necessary to specialise the strain energy function (5) to (43) have been discussed in [41], finding that in addition to the classical constraint $\nu \to \frac{1}{2}$,

$$(1-2\nu)B \to -\mu\,, \quad (1-2\nu)^3 C \to 0\,, \quad \tfrac{1}{4}A + \tfrac{3}{2}\mu \to \xi \tag{44}$$

where ν is the classical Poisson ratio and the TOECs A, B, and C are expressed in the form of Landau & Lifshitz [41], related to the Murnaghan constants by

$$A = n\,, \quad B = m - \tfrac{1}{2}n\,, \quad C = l - m + \tfrac{1}{2}n. \tag{45}$$

Expressing the TOECs in the form (45) and applying the limits to the matrix material in equations (40b), (42a), (42b) and (42c), the effective properties of the incompressible medium containing voids are

$$\mu^* = \frac{3\mu(1-c)}{3+2c} \tag{46a}$$

$$\nu^* = \frac{6+c}{11+12c} \tag{46b}$$

$$A^* = A - \tfrac{20}{189}\left(25A + 33\mu\right)c + \dots \tag{46c}$$

$$(1-2\nu^*)B^* = -\mu + \tfrac{1}{72}(174\mu - 7A)c + \dots \tag{46d}$$

$$(1-2\nu^*)^3 C^* = -\tfrac{3}{64}(15\mu + A)c \tag{46e}$$

The five elastic constants of the Murnaghan constitutive relation reduce to two, and the properties of the effective medium are consistent with an incompressible material when $c = 0$. After making the substitution given in Equation (44), the expansion for the effective third-order elastic constant ξ^* is identical to the result presented for incompressible materials in [25] (after setting the elastic constants of the particulate phase to zero).

4.4. Incompressible Neo-Hookean material

Another common incompressible material model is the Neo-Hookean material, with the strain energy density function

$$W_{NH} = \tfrac{1}{2}\mu(I_1 - 3)$$

and which can be obtained as a special case of the Mooney-Rivlin material (43) by applying $\xi = \frac{1}{2}\mu$. After applying this condition to the matrix material, the effective third-order elastic constants become

$$\begin{aligned} A^* &= -4\mu - \tfrac{1340}{189}\mu c + \dots \\ (1-2\nu^*)B^* &= -\mu + \tfrac{101}{36}\mu c + \dots \\ (1-2\nu^*)^3 C^* &= -\tfrac{33}{64}\mu c. \end{aligned}$$

and the effective strain energy function may be written in terms of the macroscopic deformation gradient $\overline{\mathbf{F}}$. Considering purely radial deformations, i.e. $\overline{\mathbf{F}} = \overline{J}^{1/3}\mathbf{I}$, where $\mathbf{I}$ is the unit tensor, the effective strain energy function has the form

$$\overline{W}(\overline{J}^{1/3}\mathbf{I}) = \tfrac{9}{8}(\overline{J}^{2/3} - 1)^2 K^* + \tfrac{9}{8}(\overline{J}^{2/3} - 1)^3 (l^* + \tfrac{1}{9}n^*).$$

Then, equation (41) may be used in combination with the appropriate incompressible limits to show that

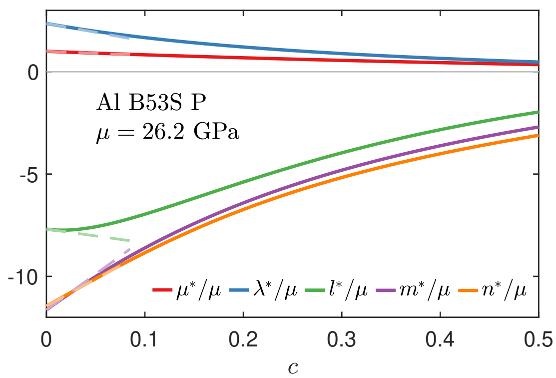
$$\overline{W}(\overline{J}^{1/3}\mathbf{I}) = \mu\left[\frac{3(1-c)}{2c}(\overline{J}^{2/3} - 1)^2 - \frac{(1-c)(11+5c)}{8c^2}(\overline{J}^{2/3} - 1)^3\right] \quad (48)$$

which, considering terms up to third order in $\overline{J} - 1$, is identical to the result presented in [16] for the effective strain energy of a Neo-Hookean material containing cavities.

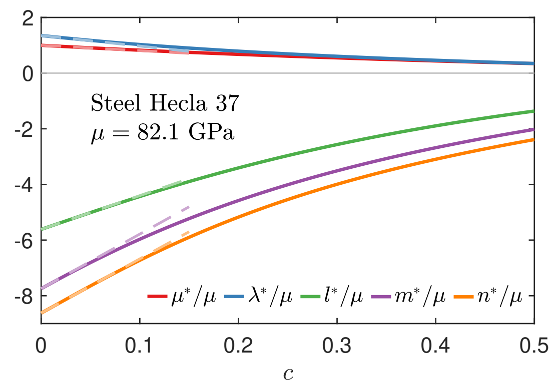
5. Discussion

5.1. Effect of voids on linear and nonlinear properties: numerical example

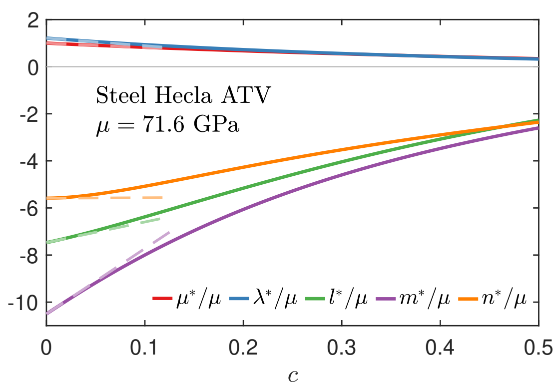
It is difficult to draw general conclusions from equations (42a), (42b) and (42c) as they feature five independent elastic constants, except to note that the linear elastic constants must satisfy the conditions $K > 0$, $\mu >$



(a)



(b)



(c)

Figure 3: The effective elastic constants of the a porous medium with a matrix phase consisting of (a) aluminium alloy B53S P, (b) steel alloy Hecla 37, (c) steel alloy Hecla ATV using the elastic constants reported by [42]. Dashed lines are used to indicate the linear approximations to each property, given for the TOECs by (42a), (42b), and (42c). Each effective property reduces to zero at $c = 1$, though they are only displayed for volume concentrations less than 50%, due to the dilute distribution assumption involved in the derivation.

0. However, for common structural materials, the Poisson's ratio may be reasonably approximated as $\nu = \frac{1}{3}$. Using this condition in the equations for the effective properties, the first order expansions are

$$\mu^* = \mu - \frac{15}{8}c\mu + \dots \tag{49a}$$

$$K^* = K - 3cK + \dots \tag{49b}$$

$$l^* + \tfrac{1}{9}n^* = (1 - 7c)\left(l + \tfrac{1}{9}n\right) + 8c\left(m - \tfrac{1}{6}n\right) - \frac{8}{9}cn + 8c\mu + \dots \tag{49c}$$

$$m^* - \tfrac{1}{6}n^* = \left(1 - \frac{471c}{128}\right)\left(m - \tfrac{1}{6}n\right) + \frac{509}{768}cn + \frac{15}{128}c\left(l + \tfrac{1}{9}n\right) + \frac{429}{256}c\mu + \dots \tag{49d}$$

$$n^* = \left(1 - \frac{101615c}{28672}\right)n + \frac{22905}{14336}c\left(m - \tfrac{1}{6}n\right) - \frac{225}{7168}c\left(l + \tfrac{1}{9}n\right) + \frac{315}{4096}c\mu + \dots \tag{49e}$$

It may be noted that certain coefficients in equations (49d), and (49e) are close to zero; for practical purposes these dependences may be neglected. Previous studies have found that the values of the Murnaghan constants for many structural materials e.g. aluminium and steel alloys are often negative, and typically have a numerical value that is one order of magnitude larger than the linear elastic constants. The equations (42a), (42b), and (42c) suggest that the influence of the void volume fraction on the TOECs will be much stronger than on the linear elastic constants. This conclusion may be illustrated with reference to specific materials: Smith, Stern, and Stephens [42] measured the TOECs of a range of steel and aluminium alloys using the acoustoelastic effect. For three steel and aluminium alloys, the effective elastic properties and the corresponding first order expansions in c are plotted in Figure 3, demonstrating that the effective TOECs may be significantly more sensitive to the presence of voids. For example for AL B53S P, TOEC l initially decreases with the increase of void concentration meanwhile two other elastic constants, m and n, are increasing with void concentration. For some materials, the slope of the growth of these constants can be of one order or even two orders of magnitude larger than for their elastic counterparts, see Figure 3. It is interesting to note that the linear elastic constants, λ and μ, always decrease monotonically with the void concentration. However, the TOEC dependencies can be monotonic or non-monotonic, have tendencies to increase or decrease and can have extrema. These dependences and non-monotonic features can be utilised in the development of new methods for the evaluation of porosity as well as assisting in the explanation of experimental tendencies on non-linear response for porous materials, which can be based on the measurement of TOECs. It is important to note that all dependencies shown in Figure 3 have been derived under the assumption of a dilute distribution of voids, and therefore the values of elastic constants at high void concentration should be treated with caution.

6. Conclusion

The ultimate objective of this research is to develop nonlinear micromechanical theory for the evaluation of non-linear elastic properties of inhomogeneous materials, similar to classical micromechanical theories, which have been developed for linear elastic constants. These theories have found many applications in Physics, Applied Mechanics, material modelling and engineering. It is expected that the nonlinear micromechanical theory to be developed may have a similar impact on many research areas due to growing interest to nonlinear properties. As mentioned in the introduction, the micromechanics of nonlinear materials is an area where analytical approaches have significant advantages over numerical methods, which justifies the present derivations supported by a careful validation.

In this work, the effective third-order elastic constants of isotropic materials containing spherical voids have been derived using referential volume average theorems. The results were derived using finite strain elastic theory and are valid for dilute distributions of voids. Due to the complexity of the explicit expressions for the effective properties, first-order expansions with respect to the volumetric concentration have been presented. The effective elastic constants of the incompressible Mooney-Rivlin material were derived by applying limits to each elastic constant; these expressions were found to be identical to previously published results for incompressible materials [25]. The authors were unable to find any numerical results of the problem under consideration, which is not surprising due to many obstacles associated with numerical modelling of nonlinear response in the case of dilute distribution of weakly-nonlinear materials.

An analysis of the derived expressions indicates that for common structural materials the effective third-order elastic constants may be much more sensitive to the concentration of voids than the effective linear elastic constants. This suggests that evaluation of the porosity, void nucleation and monitoring of void growth in such materials, e.g. due to low-cycle fatigue loading [43, 44] or ductile damage [45] or creep, would be more effective based on the measurements of the change of the TOECs than the linear elastic constants. The measurement of TOECs is typically conducted with linear and nonlinear ultrasonic techniques utilising acoustoelastic effects and nonlinear effects, e.g. higher harmonic generation, respectively [46, 47, 48]. Indeed, many recent experimental studies utilising ultrasonic methods have indicated that the TOECs are more sensitive to the accumulation of mechanical damage (creep, fatigue, radiation damage) than linear properties [49], supporting the conclusions of the present study. However, many kinds of mechanical damage are also associated with other damage mechanisms, e.g. accumulation of localised plastic deformations in the case of fatigue damage. These mechanisms are disregarded in the current work, which is only focused on the effect of voids. Therefore, though the current results may not be directly applicable to damage mechanics, they can contribute to understanding the mechanisms involved.

The obtained analytical results can also be utilised to verify advanced numerical techniques, which could be developed to evaluate the effective nonlinear characteristics of materials and composites. As discussed in the introduction, the reproduction of the current analytical results can represent a serious challenge for numerical approaches. Therefore, exact analytical results, such as the one presented in this paper, can provide the necessary benchmark against which the numerical techniques can be calibrated or compared.

Further research to meet the ultimate objective formulated in the beginning of this Section may include investigation of explicit first-order expressions for the effective properties of composite material (for example, particle-reinforced composites or fibre-reinforced composites) using the same approach as well as experimental studies directed to validate the developed approach and analytical solutions. These experimental studies could be based on ultrasonics methods for measurement of TOECs as it was discussed previously. For the metal fatigue field, it would be of interest to derive the second-order elasticity solution for a point dilatation and the corresponding expressions for the effective nonlinear properties for dilute distribution of dilatation in the nonlinear elastic material, the latter probably represents the most elementary micromechanics model of accumulated fatigue damage in the high-cycle fatigue regime.

Funding

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Appendix A.

The basis vectors for the spherical coordinates of the material and spatial coordinate systems are

$$\mathbf{d}_1 = \mathbf{d}_R = \sin\Theta\cos\Phi\mathbf{i} + \sin\Theta\sin\Phi\mathbf{j} + \cos\Theta\mathbf{k} \tag{A.1a}$$

$$\mathbf{d}_2 = \mathbf{d}_\Theta = R(\cos\Theta\cos\Phi\mathbf{i} + \cos\Theta\sin\Phi\mathbf{j} - \sin\Theta\mathbf{k}) \tag{A.1b}$$

$$\mathbf{d}_3 = \mathbf{d}_\Phi = R(-\sin\Theta\sin\Phi\mathbf{i} + \sin\Theta\cos\Phi\mathbf{j}) \tag{A.1c}$$

$$\mathbf{e}_1 = \mathbf{e}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k} \tag{A.1d}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = r(\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}) \tag{A.1e}$$

$$\mathbf{e}_3 = \mathbf{e}_\phi = r(-\sin\theta\sin\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j}) \tag{A.1f}$$

and the metric tensors are

$$\mathbf{G} = G_{AB}\mathbf{d}^A \otimes \mathbf{d}^B = \mathbf{d}^R \otimes \mathbf{d}^R + R^2 \mathbf{d}^\Theta \otimes \mathbf{d}^\Theta + R^2 \sin\Theta \mathbf{d}^\Phi \otimes \mathbf{d}^\Phi \tag{A.2a}$$

$$\mathbf{g} = g_{ab}\mathbf{e}^a \otimes \mathbf{e}^b = \mathbf{e}^r \otimes \mathbf{e}^r + r^2 \mathbf{e}^\theta \otimes \mathbf{e}^\theta + r^2 \sin\theta \mathbf{e}^\phi \otimes \mathbf{e}^\phi. \tag{A.2b}$$

The contravariant basis vectors follow from the relations $\mathbf{d}^A = G^{AB}\mathbf{d}_B$, $\mathbf{e}^k = g^{km}\mathbf{e}_m$. The shifting tensors have the components

$$g^k_A = \mathbf{e}^k \cdot \mathbf{d}_A \quad , \quad g^A_k = \mathbf{e}_k \cdot \mathbf{d}^A. \tag{A.3}$$

Using the basis vectors (A.1), the non-zero components of the shifting tensors are

$$g^r_R = \cos(\theta - \Theta)\,, \quad g^R_r = \cos(\theta - \Theta) \tag{A.4a}$$

$$g^r_\Theta = R\sin(\theta - \Theta)\,, \quad g^R_\theta = -r\sin(\theta - \Theta) \tag{A.4b}$$

$$g^\theta_R = -\frac{\sin(\theta - \Theta)}{r}\,, \quad g^\Theta_r = \frac{\sin(\theta - \Theta)}{R} \tag{A.4c}$$

$$g^\theta_\Theta = \frac{R}{r}\cos(\theta - \Theta)\,, \quad g^\Theta_\theta = \frac{r}{R}\cos(\theta - \Theta) \tag{A.4d}$$

$$g^\phi_\Phi = \frac{R\sin\Theta}{r\sin\theta}\,, \quad g^R_r = \frac{r\sin\theta}{R\sin\Theta}. \tag{A.4e}$$

where the axisymmetry condition $\phi = \Phi$ has been applied.

Appendix B.

The constants in the particular solution (24) are calculated using the equilibrium condition (4). In the following equations, the relation

$$5F_2 + 2G_1 = 0$$

has been applied, which is valid for the spherical cavity considered in the current paper as well as the more general case of a spherical inhomogeneity embedded in an infinite elastic medium. By comparing coefficients, the constants for the conservative part of the body force, (25a), are calculated as

$$C_1 = -\frac{1}{8}F_0^2 \quad , \quad C_2 = \left(\frac{1}{2}A_2 - \frac{\mu}{3(\lambda+\mu)}B_1\right)F_0$$

$$C_3 = \frac{18F_2}{7F_0}C_2 = \frac{18}{7}\left(\frac{1}{2}A_2 - \frac{\mu}{3(\lambda+\mu)}B_1\right)F_2 \quad , \quad C_4 = \frac{9F_2^2}{4F_0^2}C_1 = -\frac{9}{32}F_2^2.$$

The constants involved in the components of the Galerkin vector of the body force, (25b) and (25c), may be shown to satisfy the following equations:

$$D_1 = \left(\frac{n}{84\mu^2} - \frac{m(63\lambda+65\mu)}{420\mu^2(\lambda+2\mu)} - \frac{63\lambda^2+214\lambda\mu+115\mu^2}{840\mu^2(\lambda+2\mu)}\right)A_2G_1$$
$$+\left(\frac{2l}{21(\lambda+\mu)(\lambda+2\mu)} + \frac{m(11\lambda+15\mu)}{210\mu(\lambda+\mu)(\lambda+2\mu)} + \frac{\lambda^2+38\lambda\mu+25\mu^2}{420\mu(\lambda+\mu)(\lambda+2\mu)}\right)B_1G_1$$

$$D_2 = -\frac{150F_0(\lambda+\mu)(3\lambda+5\mu)+5G_1(219\lambda^2+742\lambda\mu+665\mu^2)}{24G_1(\lambda+\mu)(4\lambda+5\mu)}D_3$$
$$+\frac{7}{3}D_8 - \frac{35(3\lambda+4\mu)}{54(\lambda+\mu)}D_9 + \frac{(\lambda+4\mu)G_1^2}{12\mu^2}$$

$$D_3 = \frac{(4\lambda+5\mu)(\lambda+3\mu+2m)}{30\mu^2(\lambda+2\mu)}F_2G_1$$

$$D_4 = \left(-\frac{n}{21\mu^2} + \frac{m(33\lambda+67\mu)}{84\mu^2(\lambda+2\mu)} + \frac{33\lambda^2+134\lambda\mu+137\mu^2}{168\mu^2(\lambda+2\mu)}\right)A_2G_1$$
$$+\left(\frac{(5\lambda+7\mu)(\lambda-5\mu)}{84\mu(\lambda+\mu)(\lambda+2\mu)} - \frac{m(3\lambda+17\mu)}{42\mu(\lambda+\mu)(\lambda+2\mu)} - \frac{8l}{21(\lambda+\mu)(\lambda+2\mu)}\right)B_1G_1$$

$$D_5 = \frac{30F_0(\lambda+\mu)(3\lambda-5\mu)+5G_1(105\lambda^2+344\lambda\mu+310\mu^2)}{12G_1(\lambda+\mu)(4\lambda+5\mu)}D_3$$
$$-\frac{10}{3}D_8 + \frac{35}{54}\left(\frac{\mu}{\lambda+\mu} - \frac{3F_0}{G_1}\right)D_9 - \frac{(\lambda+4\mu)G_1^2}{12\mu^2}$$

$$D_6 = \frac{15F_0(\lambda+\mu)+G_1(23\lambda+35\mu)}{G_1(4\lambda+5\mu)}D_3 - \frac{2}{9}D_9$$

$$D_7 = \frac{(\lambda-5\mu)(\lambda+3\mu+2m)}{105\mu(\lambda+\mu)(\lambda+2\mu)}B_1G_1 - \frac{(\lambda-5\mu)(\lambda+3\mu+2m)}{70\mu^2(\lambda+2\mu)}A_2G_1$$

$$D_8 = \left(\frac{m(2\lambda-\mu)(3\lambda+7\mu)}{105\mu^2(\lambda+\mu)(\lambda+2\mu)} - \frac{n(8\lambda+11\mu)}{140\mu^2(\lambda+\mu)} + \frac{2l}{35(\lambda+\mu)(\lambda+2\mu)} - \frac{-3\lambda^3+49\lambda^2\mu+199\lambda\mu^2+195\mu^3}{210\mu^2(\lambda+\mu)(\lambda+2\mu)}\right)G_1^2$$

$$D_9 = \left(\frac{9n}{56\mu^2} - \frac{3m(12\lambda+25\mu)}{35\mu^2(\lambda+2\mu)} - \frac{36\lambda^2+138\lambda\mu+135\mu^2}{70\mu^2(\lambda+2\mu)}\right)F_2G_1$$

$$E_1 = -\frac{1}{4}D_1 + \frac{1}{16}D_7$$

$$E_2 = -\frac{70F_0(\lambda+\mu)(33\lambda+50\mu)+5G_1(1263\lambda^2+4402\lambda\mu+3980\mu^2)}{168G_1(\lambda+\mu)(4\lambda+5\mu)}D_3 + \frac{5}{3}D_8 + \frac{5}{108}\left(\frac{7F_0}{G_1}+\frac{11\lambda}{\lambda+\mu}-33\right)D_9 + \frac{(7\lambda+32\mu)G_1^2}{168\mu^2}$$

$$E_3 = -\frac{15(\lambda+\mu)F_0}{4(4\lambda+5\mu)G_1}D_3 + \frac{1}{28}\left(\frac{19\lambda}{4\lambda+5\mu}-39\right)D_3 + \frac{1}{9}D_9$$

$$E_4 = \frac{2\lambda-\mu}{2(\lambda-5\mu)}D_7$$

$$E_5 = \frac{9}{56}\left(38-\frac{51\lambda}{\lambda+\mu}+\frac{100\lambda}{4\lambda+5\mu}\right)D_3 - \frac{1}{2}D_8 + \left(\frac{3}{4}-\frac{\lambda}{4(\lambda+\mu)}\right)D_9 - \frac{9G_1^2}{280\mu}$$

$$E_6 = \frac{9}{70}D_3\left(5-\frac{8\lambda}{4\lambda+5\mu}\right) - \frac{3D_9}{20}$$

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