A NON-PERTURBATIVE SOLUTION FOR MØLLER SCATTERING

by

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A thesis submitted in accordance with the requirements
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in the faculty of Mathematical Sciences.

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SUMMARY.

The Dyson-Schwinger equations can be solved non-perturbatively for the electron propagator in the asymptotic region to give a finite value for the mass renormalisation. Similarly it is hoped that a non-perturbative solution of the vertex function and the photon propagator will produce a fully finite theory of Quantum Electrodynamics. In this report the solution found by Green, Cartier and Broyles[1] for the vertex function in the ladder approximation is used to determine the asymptotic behaviour of the Møller scattering amplitude.

To do this, I begin by presenting in some detail Green's non-perturbative treatment[2] of the electron propagator and the corresponding mass renormalisation. These methods form an important part of the second chapter, where I have applied them to deriving without perturbation theory the Dyson-Schwinger equations for the Compton and Møller scattering amplitudes; by making an approximation based on appropriate generalised Ward identities these become closed equations.

In chapter three Green's solution for the vertex function is presented along with the method by which it can be applied to Møller scattering; it is found that in the ladder approximation the equation derived for the Møller scattering amplitude reduces to the same equation as Green solved. Then the equations are iterated once to find the boundary conditions which apply to the general solution and hence

reflect the differences between the vertex function and Møller scattering.

These boundary conditions are found, in the interests of brevity, only for one particular polarisation state of the incident beam, but it is not anticipated that a much different result would be obtained for a general polarisation state.

Finally it is shown how this solution supports the hypothesis of asymptotic freedom whereas the perturbation solution certainly does not, and some remarks are made concerning the implications of Green's work on the vacuum polarisation integral.

STATEMENT.

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university, and to the best of my knowledge and belief, it contains no material previously published or written by another person, except where due reference has been made in the text.

Stephen Wilkinson September 10th, 1982.

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I am deeply indebted to all the members and students of the department of Mathematical Physics for their support during the year, and particularly so to my supervisor, Prof. H.S. Green, not only for his continued inspiration in academic matters, but also for his guidance and encouragement generally over the past three years.

I would like also to thank Mr. T. Saunders and Mr. D. Pearce for their help with the preparation of the thesis.

I take this opportunity to point out that during the rest of this report, work carried out and published by H.S. Green, J.F. Cartier and A.A. Broyles is referred to, for the sake only of convenience, as Green's work.

CHAPTER 0

INTRODUCTION.

Since it was first realised that without renormalisation the perturbation expansion gave infinite contributions to such quantities as the electron and photon self-energy in Quantum Electrodynamics, and that the series itself was divergent in many instances[3], attempts have been made to solve the various Dyson-Schwinger equations without the use of perturbation theory, anticipating that a better solution would produce finite (and just as accurate) answers before renormalisation.

The difficulty in doing this exactly arises in the nature of the Dyson-Schwinger equations : they form an infinite hierarchy in which each equation refers to an equation higher up than itself and thus finding the solution of a given equation involves terminating, by means of an approximation, its connection with the higher equations.

The earliest attempts in this direction[4] involved assuming some of the results of perturbation theory and thus could not strictly be called non-perturbative methods. Nonetheless they did produce a finite theory, and thus provided hope for more exact solutions.

In recent years these hopes have been fulfilled, at least partially, both along the same lines as above[5] and in more fundamental directions, *i.e.* without any use of the perturbation solution.

These later efforts can be divided conveniently into two groups : those that deal with the electron propagator and thus with mass renormalisation, and those which consider vacuum polarisation, the vertex function and the photon propagator.

Of the latter, a common conclusion was that reached originally by Gell-Mann and Low[6], *viz.* that QED may be finite provided the bare electron mass vanishes and the fine structure constant α is determined as a root of $f(x)$, where $f(x)x/2\pi$ is the coefficient of the sum of logarithmically divergent integrals in the vacuum polarisation, in terms of the coupling constant x.

Since then, refinements have from time to time been made on the original result. Thus Baker and Johnson[7] obtained a different function which they showed to have the same root as Gell-Mann and Low's condition; they later proved[8] that the root was unaltered by discarding contributions to the vacuum polarisation by closed electron loops. A similar result was also derived by Yock[9] and the eigenvalue function f was determined approximately by Blaha[10] and Adler[11]. The latest addition to this catalogue has been the solution proposed by Green, Cartier and Broyles[1] for the vertex function in the asymptotic region - they succeeded in solving an equation which reduces in an appropriate approximation to Yock's equation[12].

Perhaps more conclusive have been the solutions suggested for the electron propagator. Here the work of Johnson, Willey and Baker has been the most important - they found in 1964 by using a Bethe-Salpeter type equation for the vertex that in the asymptotic region the unrenormalised solution to the electron propagator had finite self-mass provided the bare mass was zero[13]; furthermore thay have shown[14] that their solution is consistent with assigning the physical mass m to the self-mass correction δm and that the usual divergent result obtained from perturbation theory is a direct consequence of the failure of the perturbation solution in the asymptotic region.

Other solutions, showing similar asymptotic properties, were obtained by Biswas and Vidhani[15] and Delbourgo[16], while more recently Green, Cartier and Broyles[2] published the first solution dealing also with values of the electron momentum comparable to the mass.

I have begun this thesis with a presentation of their solution for two reasons : firstly it demonstrates explicitly the success of non-perturbative methods in obtaining a finite result for the electron self-energy - or if renormalisation is used , in obtaining a renormalisation constant close to 1 - and secondly I have used the same methods later on to develop non-perturbative equations for Compton and Møller scattering.

CHAPTER 1

NON-PERTURBATIVE QUANTUM ELECTRODYNAMICS.

According to Dirac's equation, an electron of physical mass m and charge e_p is represented by a spinor field $\psi_r^p(x)$ where r takes the values 1 to 4. When the electron interacts with an electromagnetic field $A_\lambda^p(x)$, its own field satisfies the equations

$$(i\nabla - e_p\gamma^\lambda A_\lambda^p)\psi^p = m\psi^p$$

and

$$\bar{\psi}^p(-i\overleftarrow{\nabla} - e_p\gamma^\lambda A_\lambda^p) = m\bar{\psi}^p$$

where

$$\nabla = \gamma^\mu\partial_\mu,$$

$$\partial_\mu = \partial/\partial x^\mu$$

and

$$\bar{\psi}^p = \psi^{p\dagger}\gamma^0$$ is the Dirac conjugate of ψ^p.

Here, as throughout this thesis, products of field variables are to be interpreted in accordance with the time-ordering convention appropriate to their statistics.

The Dirac matrices γ^μ satisfy the usual anticommutation relations,

$$\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}$$

where $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$ are the non-vanishing components of the Lorentz metric, and the electromagnetic field is given by

$$\nabla^2 A^p_\mu - \theta\partial^\lambda\partial_\mu A^p_\lambda = e_p\bar{\psi}^p\gamma_\mu\psi^p$$

where θ is a gauge parameter undetermined by Maxwell's equations.

When self-interactions of the two fields are taken into account, it is well known[17] that the electron may equivalently be represented by the field

$$\psi = z^{-\frac{1}{2}}\psi^p$$

corresponding to bare mass m_b and charge e, provided the electromagnetic field be taken as

$$A_\lambda = y^{-\frac{1}{2}}A^p_\lambda \, .$$

These unrenormalised fields consequently satisfy the equations

$$(i\nabla - e\gamma^\lambda A_\lambda)\psi = m_b\psi$$

$$\bar{\psi}(-i\nabla - e\gamma^\lambda A_\lambda) = m_b\bar{\psi} \qquad (1.1)$$

and

$$\nabla^2 A_\lambda - \theta\partial_\lambda\partial^\mu A_\mu = ey^{-1}z\bar{\psi}\gamma_\lambda\psi \qquad (1.2)$$

while the requirement of Heisenberg's principle is that, when > is the vacuum state and $\bar{\psi}(x)$> the state representing the electron, the equal time anti-commutation relations

$$\{\psi(x),\psi(x')\} = \{\bar{\psi}(x),\bar{\psi}(x')\} = 0$$

$$\{\psi(x),\bar{\psi}(x')\} = \gamma^0\delta(\underline{x}-\underline{x}')/z \qquad (1.3)$$

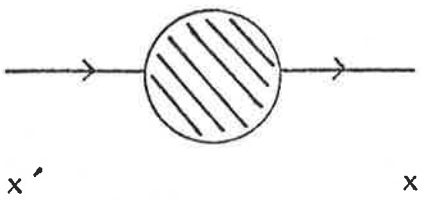
and the equal time commutation relations

$$[A_\lambda(x),A_\mu(x')] = [A_{\lambda,o}(x),A_{\mu,o}(x')] = 0$$

$$[A_\lambda(x),A_{\mu,o}(x')] = -ig_{\lambda\mu}\delta(\underline{x}-\underline{x}')/y \qquad (1.4)$$

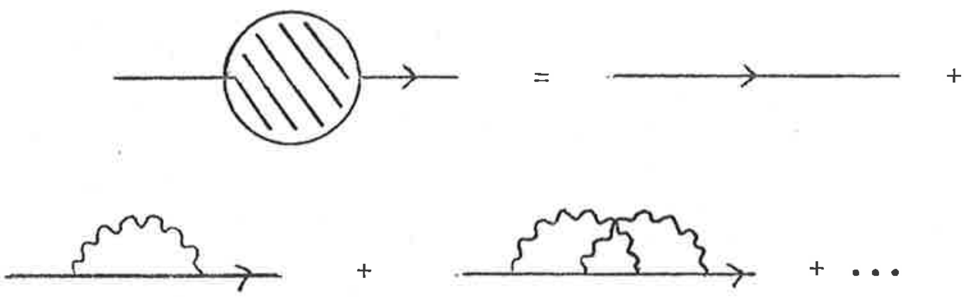
be satisfied (here of course $A_{\lambda,o} = \partial_o A_\lambda$ *etc.*).

Non-perturbative methods will take explicitly into account the self-interactions of the bare fields, and I will represent by the diagram



(1.5)

the sum of all Feynman diagrams with two external electron lines, thus



(1.6)

in the usual Feynman notation. In other words, the blob is a sort-of veil covering all those processes which occur between the external lines, in this case two electrons.

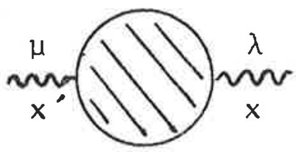
The contribution to the S-matrix element by (1.5) is therefore[18]

$$S(x-x') = \langle\psi(x)\bar{\psi}(x')\rangle \quad (1.7)$$

which of course is actually a Dirac matrix with components

$$S_{rs}(x-x') = \langle\psi_r(x)\bar{\psi}_s(x')\rangle \, .$$

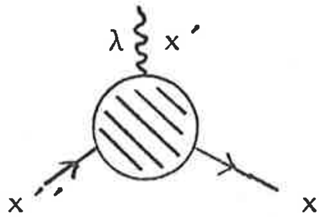
The other renormalised Feynman diagrams required in this section are

(1) the photon line 

corresponding to the propagator

$$D_{\lambda\mu}(x-x') = \langle A_\lambda(x)A_\mu(x')\rangle \quad (1.8)$$

and (2) the vertex function



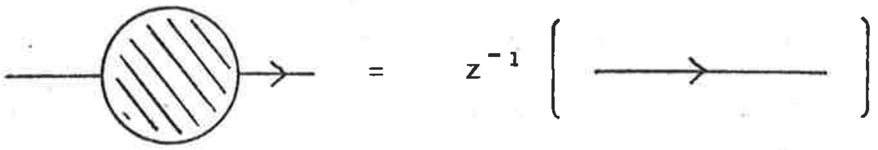
and its corresponding function

$$S_\lambda(x-x',x'-x'') = \langle\psi(x)A_\lambda(x')\bar{\psi}(x'')\rangle \quad (1.9)$$

which is also a Dirac matrix.

It is my aim in this chapter to present part of

Green's solution[2] to the equation satisfied by (1.7); thus the sum of the Feynman diagrams (1.6) will have been evaluated and (1.5) can be written



or

$$S(x-x') = z^{-1} S_F(x-x') \tag{1.10}$$

where the propagator on the left-hand side corresponds to the bare mass m_b and the Feynman propagator on the right has the physical electron mass m.

It will be shown that, under the conditions and approximations of Green's solution, z has a value very close to 1 and that m can be taken as the experimental value for the electron mass - thus a treatment of mass renormalisation will have been given which doesn't involve subtracting infinite quantities; rather it appears that the infinities are due to separating term by term the sum on the right-hand side of (1.6), which taken as a whole does converge.

1.1 THE ELECTRON PROPAGATOR EQUATION AND THE GENERALISED WARD IDENTITIES.

The field equations (1.1) can be applied to the electron propagator $S(x-x')$ to give its Dyson-Schwinger equation[18,19].

Since the operator $\psi(x)\bar{\psi}(x')$ is discontinuous, by virtue of the time-ordering convention, at $x_o = x'_o$, then its time derivative involves the derivative of a step function, *i.e.* it involves a delta function. Thus

$$\gamma^o\partial_o S(x-x') = \langle\gamma^o\partial_o\psi(x)\bar{\psi}(x')\rangle$$

$$+ \gamma^o\delta(x_o-x'_o)\langle \lim_{x^0\to+x^{0\prime}} \left[\psi(x)\bar{\psi}(x') - (-\bar{\psi}(x')\psi(x))\right]$$

$$= \langle\gamma^o\partial_o\psi(x)\bar{\psi}(x')\rangle + \delta(x-x')/z$$

and so

$$i\nabla S(x-x') = \langle\{m_b\psi(x) + e\gamma^\lambda A_\lambda(x)\psi(x)\}\bar{\psi}(x')\rangle$$

$$+ i\delta(x-x')/z$$

$$= m_b S(x-x') + e\gamma^\lambda S_\lambda(0,x-x') + i\delta(x-x')/z \ .$$

As mentioned in the introduction, the first requirement of the non-perturbative theory is that the bare mass vanishes - thus all the electron mass comes from its self-interactions. Under this condition, the Dyson-Schwinger equation for the electron propagator is

$$i\nabla S(x-x') = e\gamma^{\lambda}S_{\lambda}(0,x-x') + i\delta(x-x')/z$$

or by defining the Fourier transforms

$$iS(p) = \int S(x)e^{ip.x}d^4x$$

$$iS_{\lambda}(p,q) = \iint S_{\lambda}(x,-x')e^{i(p.x-q.x')}d^4xd^4x' \qquad (1.11)$$

etc.,

then the equation in momentum space is

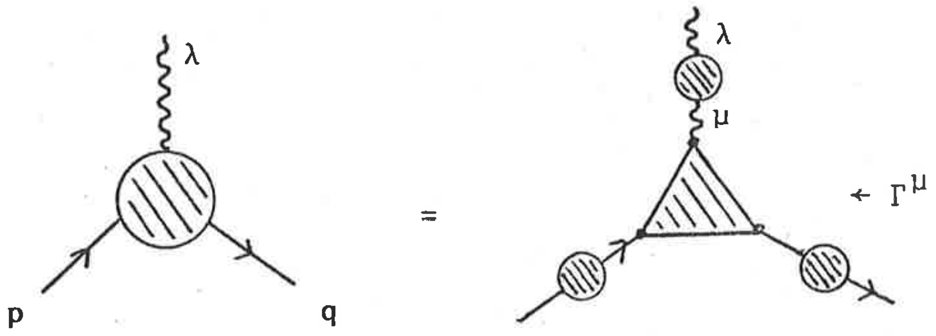
$$\not{p}S(p) = \frac{e\gamma^{\lambda}}{(2\pi)^4}\int S_{\lambda}(q,p)d^4q + z^{-1} \qquad (1.12)$$

where

$$\not{p} = p^{\mu}\gamma_{\mu}.$$

Of course (1.12) does no more than re-express S(p) in terms of a further unknown, the vertex function $S_{\lambda}(q,p)$. However the remaining field equations (1.2) are sufficient to link up the two functions S and S_{λ}, and by means of an approximation to yield an equation for S(p).

To see this, it is necessary to express $S_{\lambda}(q,p)$ in terms of its 'irreducible' part $\Gamma^{\lambda}(q,p)$: diagrammatically it is clear that



where the term

contains no single (irreducible) photon or electron lines. Thus the corresponding functions are given by

$$S_\lambda(q,p) \quad = \quad ieD_{\lambda\mu}(p-q)S(q)\Gamma^\mu(q,p)S(p) \ . \qquad (1.13)$$

From the definition (1.8) it is easily shown that

$$\partial_\nu D_{\lambda\mu}(x-x') \quad = \quad \langle \partial_\nu A_\lambda(x)A_\mu(x')\rangle$$

and so

$$y\left[\nabla^2\delta^\nu{}_\lambda \ - \ \theta\partial_\lambda\partial^\nu\right]D_{\nu\mu}(x-x')$$

$$= \quad ze\langle\bar\psi(x)\gamma_\lambda\psi(x)A_\mu(x')\rangle \quad + \quad ig_{\lambda\mu}\delta(x-x')$$

or

$$y(1-\theta)\nabla^2\partial^\lambda D_{\lambda\mu}(x-x') \quad = \quad i\partial_\mu\delta(x-x') \qquad (1.14)$$

using the conservation of charge.

Moreover it follows from (1.9) that

$$y\left[\nabla^2\delta^\lambda{}_\mu \ - \ \theta\partial^\lambda\partial_\mu\right]S_\lambda(x'-x,x-x'')$$

$$= \quad ze\langle\psi(x')\bar\psi(x)\gamma_\lambda\psi(x)\bar\psi(x'')\rangle$$

and so

$$y(1-\theta)\nabla^2\partial^\lambda S_\lambda(x'-x,x-x'')$$
$$= e\delta(x-x'')S(x'-x) - e\delta(x'-x)S(x-x'') \,. \qquad (1.15)$$

In terms of the Fourier transforms (1.11), equation (1.14) becomes

$$y(1-\theta)(p-q)^2(p^\lambda-q^\lambda)D_{\lambda\mu}(p-q) \;=\; q_\mu - p_\mu \qquad (1.16)$$

while (1.15) is

$$y(1-\theta)(p-q)^2(p^\lambda-q^\lambda)S_\lambda(q,p) \;=\; ie\{S(p) - S(q)\}.$$

By substituting (1.13) and making use of (1.16), there results

$$(q_\mu-p_\mu)\Gamma^\mu(q,p) \;=\; S^{-1}(q) - S^{-1}(p) \,. \qquad (1.17)$$

This is the first generalised Ward identity[4], derived here without recourse to perturbation theory. This clearly establishes a link between S_λ and S in (1.12); it will be shown below how an approximate equation for S can thereby be formed.

1.2 AN APPROXIMATE SOLUTION FOR S(P).

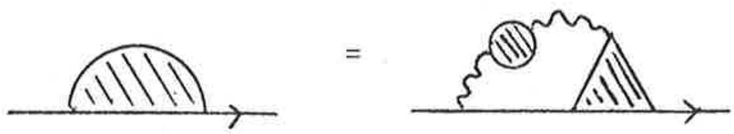
By substituting (1.13) into the propagator equation (1.12), it results that

$$\not{p}S(p) = \{1 + \Sigma(p)S(p)\}/z \tag{1.18}$$

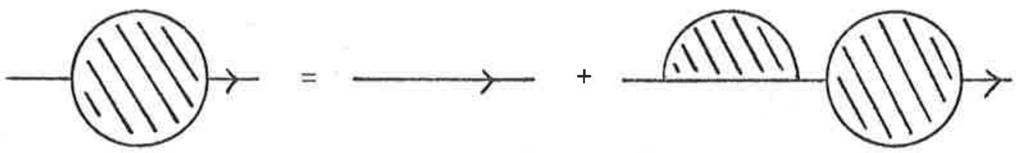
where

$$\Sigma(p) = i\frac{e^2 z\gamma^\lambda}{(2\pi)^4}\int D_{\lambda\mu}(p-q)S(q)\Gamma^\mu(q,p)d^4q \tag{1.19}$$

represents the self-energy contributions to S(p); $\Sigma(p)$ corresponds to the renormalised diagram



so that (1.18) may be written



bearing in mind that the single line represents the Feynman propagator $1/\not{p}$.

From (1.18) it follows that

$$S^{-1}(p) = z\not{p} - \Sigma(p) . \tag{1.20}$$

Two approximations are used to solve this equation. Firstly, the photon propagator is taken to have its lowest order form :

$$D_{\lambda\mu}(k) \quad = \quad -\{g_{\lambda\mu} - (1-\theta)k_\lambda k_\mu/k^2\}/k^2 \tag{1.21}$$

and secondly the vertex function $\Gamma^\mu(q,p)$ is replaced in the integrand of (1.19) by its value at the pole of $D_{\lambda\mu}(p-q)$, *i.e.*

$$\Gamma^\mu(q,p) \rightarrow \Gamma^\mu(q,q) \ . \tag{1.22}$$

Both of these approximations have the advantage that they become exact as the electron momentum p^2 becomes large, and consequently the solution obtained should be exact in the asymptotic limit; however a numerical solution of (1.20) suggests[2] that the approximation is good over a large part of the domain.

It can be shown[2,21] that the solution to (1.20) exists only if the gauge parameter $\theta = 0$, *i.e.* in the Landau-Maxwell gauge.

Thus with these approximations

$$S^{-1}(p) \quad = \quad z\not{p} + \frac{ie^2 z}{(2\pi)^4}\int\left[\gamma^\lambda - \frac{(\not{p}-\not{q})(p^\lambda - q^\lambda)}{(p-q)^2 + i\varepsilon}\right]\frac{F_\lambda(q)}{(p-q)^2 + i\varepsilon}d^4q$$

where from (1.17)

$$F^\lambda(q) \quad = \quad S(q)\Gamma^\lambda(q,q)$$

$$= \quad S(q)\frac{\partial}{\partial q_\lambda}S^{-1}(q) \ . \tag{1.23}$$

– finally an equation has been obtained explicitly for S(p).

This can be written

$$S^{-1}(p) \quad = \quad z\not{p} + \frac{e^2 z}{4\pi^2}\left\{\gamma_\lambda I^\lambda(p) - J(p)\right\}$$

where

$$I^\lambda(p) \quad = \quad \frac{i}{4\pi^2}\int\frac{F^\lambda(q)d^4q}{(p-q)^2+i\varepsilon}$$

and

$$J(p) \quad = \quad \frac{i}{4\pi^2}\int\frac{(\not{p}-\not{q})(p^\lambda-q^\lambda)}{\{(p-q)^2+i\varepsilon\}^2}F_\lambda(q)d^4q$$

$$= \quad -\tfrac{1}{2}\nabla\frac{i}{4\pi^2}\int\frac{(p^\lambda-q^\lambda)F_\lambda(q)}{(p-q)^2+i\varepsilon}d^4q \quad + \quad \tfrac{1}{2}\gamma_\lambda I^\lambda(p)$$

where here $\nabla = \gamma^\mu\partial/\partial p^\mu$.

Since $\quad \nabla^2\frac{p^\lambda-q^\lambda}{(p-q)^2} \quad = \quad 2\partial^\lambda\left[\frac{1}{(p-q)^2}\right]$

then

$$\nabla^2 J(p) \quad = \quad -\nabla\partial_\lambda I^\lambda(p) + \tfrac{1}{2}\gamma_\lambda\nabla^2 I^\lambda(p) \tag{1.24}$$

and

$$\nabla^2 S^{-1}(p) \quad = \quad \varepsilon z\{\tfrac{1}{2}\gamma_\lambda\nabla^2 I^\lambda(p) + \nabla\partial_\lambda I^\lambda(p)\}$$

$$= \quad \varepsilon z\{\tfrac{1}{2}\gamma_\lambda F^\lambda(p) + G(p)\} \tag{1.25}$$

where

$$\nabla G(p) \quad = \quad \nabla^2\partial_\lambda I^\lambda(p) \quad = \quad \partial_\lambda F^\lambda(p)$$

and

$$\varepsilon \quad = \quad e^2/4\pi \ .$$

This equation can be solved by using the Lorentz invariance of S(p) to write

$$S^{-1}(p) \quad = \quad \sigma(p^2)\not{p} - \rho(p^2)$$

whence from (1.23)

$$F^{\lambda}(p) = 2A\not{p}p^{\lambda} + 2Bp^{\lambda} + C\gamma^{\lambda}$$

for

$$A = (\rho\sigma' - \sigma\rho')/D$$

$$B = (p^2\sigma\sigma' - \rho\rho')/D$$

$$C = (\sigma^2\not{\phi} + \rho\sigma)/D$$

and

$$D = \sigma^2 p^2 - \rho^2 .$$

Then by comparing coefficients of the various Dirac matrices on both sides of (1.25), the following equations are obtained :

$$x\rho'' + 2\rho' + (x\tau)' = \frac{\lambda a}{a^2 - x}$$

$$x\sigma'' + 3\sigma' = \frac{\lambda\sigma'}{\sigma} + a\tau'$$

and

$$\tau' = \frac{\lambda a'}{a^2 - x} \qquad (1.26)$$

where

$$x = p^2$$

$$\lambda = 3\varepsilon z/4$$

$$a = \rho/\sigma$$

and $'$ means d/dx.

In fact the solution to these equations is not especially important for the work which follows; however two features of the solution are particularly relevant.

Firstly, the solutions for σ and ρ are both finite, hence the integral in (1.20) converges. Thus because of the $1/p^2$ dependence in $\Sigma(p)$, for large values of p^2 it is clear that $\Sigma(p) \to 0$; this gives the so-called asymptotic freedom for an electron - this will be discussed further in chapter four.

Secondly, it was mentioned in the introduction to this chapter that the object of a renormalised theory is to describe equivalently the processes of self-interaction by including a self-mass and self-charge term in the theory and then ignoring the self-interactions.

Thus when m is the physical mass of the electron, the boundary conditions for equations (1.26) are that

$$S^{-1}(p) = \not{p} - m$$

when

$$p^2 = m^2$$

and

$$\Sigma(p) \to 0 \quad \text{as} \quad p^2 \to -\infty$$

- that is to say

$$\rho(m^2) = m\sigma(m^2)$$

$$\sigma(m^2) = 1$$

and $\sigma_\infty = z$

where $$\sigma_\infty = \lim_{x \to -\infty} \sigma(x) \qquad (1.27)$$

Unfortunately choosing the bare mass zero means there is no unit of p^μ, hence these equations do not determine m.

However when equations (1.26) are approximately solved by expansion in powers of λ (which turns out to be very much less than 1) then to second order the function σ satisfies

$$x^2\sigma = \lambda^2\int_0^x xf(x)dx + x^2\sigma_\infty \qquad (1.28)$$

where

$$f(x) = -\int_{-\infty}^{-x} \frac{dx}{x^2(1-x)}\left[\log(1-x)+x\right] .$$

Thus $\sigma - \sigma_\infty = \lambda^2\int_0^1 xf(x)dx$

or $1 - z = \frac{1}{2}\lambda^2(\pi^2/3 - 1)$ using (1.27)

\- in other words to the order of accuracy obtained z can be taken as 1.

This is an important result; not only does it show the essential finiteness of QED, at least as far as the mass renormalisation is concerned, but it provides a good explanation of the success of the renormalisation procedure when it is applied to the divergent integrals appearing in perturbation theory.

Moreover, from (1.28) σ never differs from the constant σ_∞ by terms of order greater than λ^2 and hence to a good approximation the electron propagator can be given by

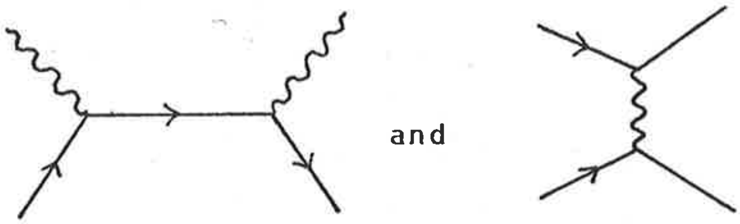
$$S^{-1}(p) = \not{p} - a(p^2) . \qquad (1.29)$$

This result will be useful later in determining the Møller scattering amplitude.

CHAPTER 2

TWO SCATTERING EQUATIONS.

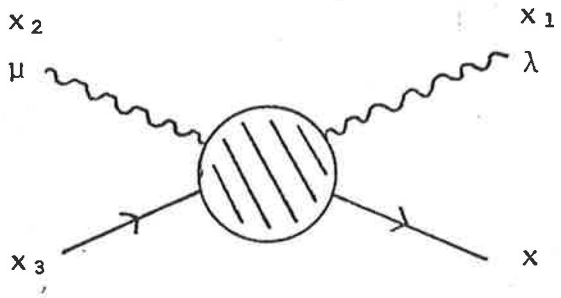
In what remains of this report, I hope to apply the methods of chapter one to two problems, *viz.* Compton and Møller scattering. Although the graphs for these two processes are similar, *e.g.*



are comparable first order diagrams for each process respectively, there are essential differences between the two which require separate treatment.

2.1 THE COMPTON SCATTERING EQUATION.

Compton scattering involves an in-coming and an out-going electron and photon, and thus the appropriate amplitude, which might be represented by the following renormalised diagram,



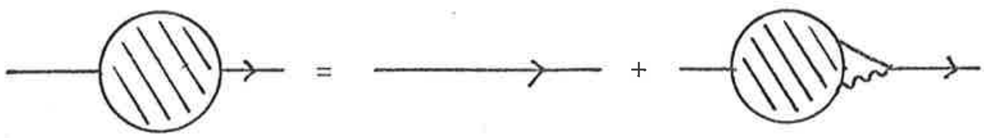
(2.1)

is

$$S_{\lambda\mu}(x-x_1,x_1-x_2,x_2-x_3) = <\psi(x)A_\lambda(x_1)A_\mu(x_2)\bar{\psi}(x_3)>$$

- as for the electron propagator and the vertex this is also a Dirac matrix.

It was seen in chapter one that applying the field equations to the amplitude corresponds to 'drawing back' partially the veil covering the interaction area; thus equation (1.12) can be represented as



showing that the first part of the veil consists of the simple vertex.

A similar procedure is needed here for the Compton scattering amplitude :

From the field equations (1.1), the Dyson-Schwinger equation is

$$i\nabla S_{\lambda\mu}(x-x_1,x_1-x_2,x_2-x_3) =$$

$$e\gamma^{\nu}\langle\psi(x)A_{\nu}(x)A_{\lambda}(x_1)A_{\mu}(x_2)\bar{\psi}(x_3)\rangle + i\delta(x-x_3)D_{\lambda\mu}(x_1-x_2)/z \; . \qquad (2.2)$$

The Fourier transform of $S_{\lambda\mu}$ can be defined by

$$S_{\lambda\mu}(x-x_1,x_1-x_2,x_2-x_3) =$$

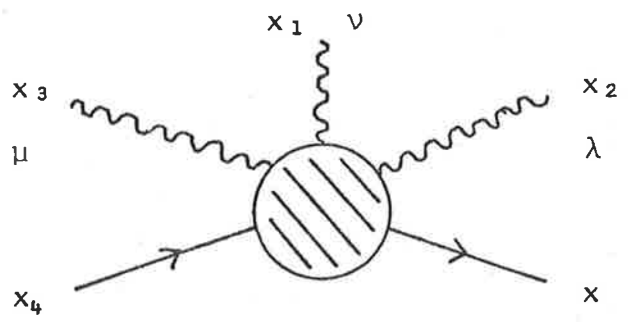
$$\frac{i}{(2\pi)^{12}}\iiint S_{\lambda\mu}(p,q,r)e^{-ip.(x-x_1)-iq.(x_1-x_2)} \times e^{-ir.(x_2-x_3)}d^4p\,d^4q\,d^4r \qquad (2.3)$$

and a similar definition applies for

$$S_{\nu\lambda\mu}(x-x_1,x_1-x_2,x_2-x_3,x_3-x_4)$$

$$= \langle\psi(x)A_{\nu}(x_1)A_{\lambda}(x_2)A_{\mu}(x_3)\bar{\psi}(x_4)\rangle$$

corresponding to a five-point process with diagram



Thus the Fourier transform of (2.2) is

$$\not{p}_f S_{\lambda\mu}(p_f,q,p_i) = \frac{e\gamma^\nu}{(2\pi)^4}\int S_{\nu\lambda\mu}(r,p_f,q,p_i)d^4r$$

$$+ (2\pi)^4 iD_{\lambda\mu}(k_i)\delta(k_i-k_f)/z \qquad (2.4)$$

where, according to the definitions (2.1) and (2.3), $S_{\lambda\mu}(p_f,q,p_i)$ is the amplitude for

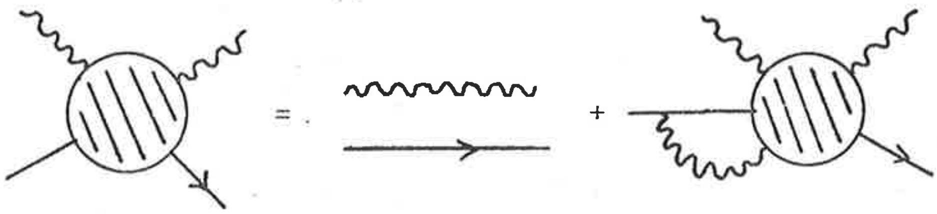
(1) an electron with initial momentum p_i and final momentum p_f interacting with

(2) a photon of initial momentum k_i and final momentum k_f,

and

$k_i = q - p_i$
$k_f = q - p_f$.

Thus, diagrammatically,



Equation (2.4) is made into an approximate equation for $S_{\lambda\mu}$ by forming another generalised Ward identity from $S_{\nu\lambda\mu}$.

From (1.2) it follows that

$$y(1-\theta)\nabla^2\partial^\nu S_{\nu\lambda\mu}(x_1-x,x-x_2,x_2-x_3,x_3-x_4)$$
$$= e\{\delta(x-x_4)S_{\lambda\mu}(x_1-x_2,x_2-x_3,x_3-x)$$
$$- \delta(x_1-x)S_{\lambda\mu}(x-x_2,x_2-x_3,x_3-x_4)\}$$

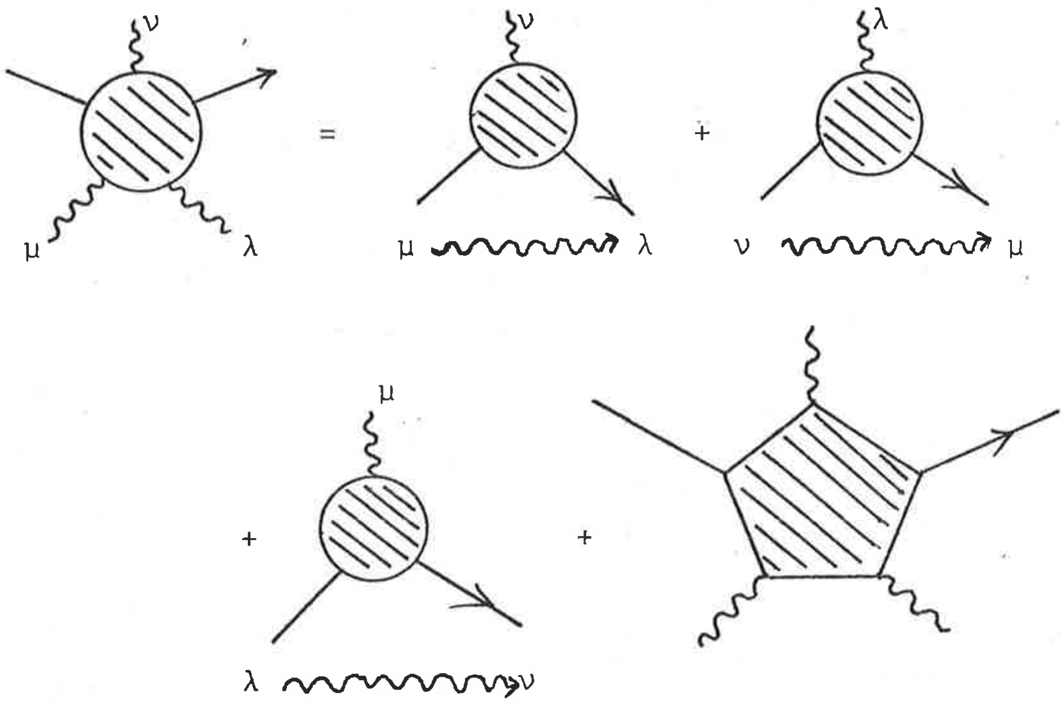
or in momentum space

$$iy(1-\theta)(p_f-r)^2(p_f^{\nu}-r^{\nu})S_{\nu\lambda\mu}(r,p_f,q,p_i)$$

$$= e\{S_{\lambda\mu}(r,r-k_f,r-p_f+p_i) - S_{\lambda\mu}(p_f,q,p_i)\}$$

$$+ (2\pi)^4\{(p_{f\lambda}-r_{\lambda})\delta(r-q)S_{\mu}(q,r) + (P_{f\mu}-r_{\mu})\delta(r+k_f-p_i) \times S_{\lambda}(p_i-k_f,p_i)\} \; . \qquad (2.5)$$

To obtain an expression analogous to (1.17), $S_{\nu\lambda\mu}$ must be decomposed into a sum of its irreducible parts :



where here, as below, the renormalised propagators are represented by the bare lines

 and  .

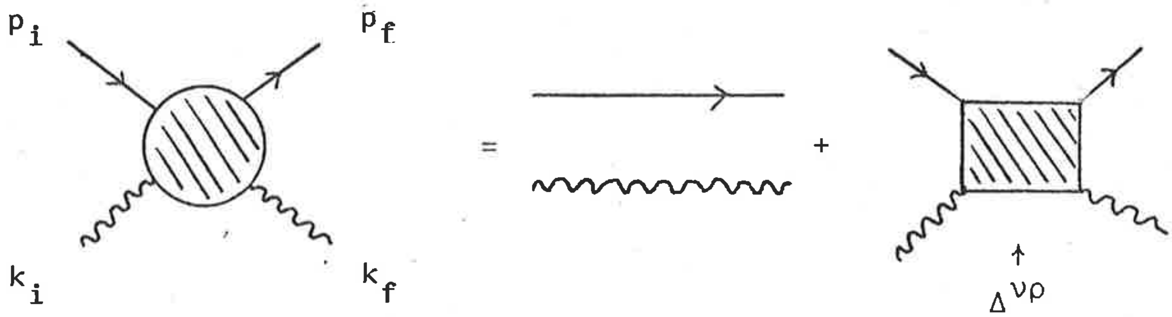
This means that the irreducible part $\Delta^{\rho\sigma\tau}$ of $S_{\nu\lambda\mu}$ can be defined by

$$S_{\nu\lambda\mu}(r,p_f,q,p_i) = (2\pi)^4 i\Big[D_{\lambda\nu}(r-p_f)S_\mu(r,p_i)\delta(r-q)$$

$$+ D_{\nu\mu}(p_f-r)S_\lambda(r,p_i)\delta(r+k_f-p_i)$$
$$+ D_{\mu\lambda}(p_i-q)S_\nu(r,p_i)\delta(p_i-p_f)\Big]$$

$$-ie^3 D_{\nu\rho}(p_f-r)D_{\lambda\sigma}(k_f)S(r)\Delta^{\rho\sigma\tau}(r,p_f,q,p_i)S(p_i)D_{\tau\mu}(p_i). \tag{2.6}$$

Similarly the irreducible part of $S_{\lambda\mu}$ is



i.e.

$$S_{\lambda\mu}(p_f,q,p_i) = (2\pi)^4 i D_{\lambda\mu}(k_f)S(p_f)\delta(p_f-p_i)$$

$$- e^2 D_{\lambda\nu}(k_f)S(p_f)\Delta^{\nu\rho}(p_f,q,p_i)S(p_i)D_{\rho\mu}(k_i). \tag{2.7}$$

When these substitutions have been made, and use made again of (1.16), equation (2.5) reduces to

$$(r_\nu - p_{f\nu})S(r)\Delta^{\nu\lambda\mu}(r,p_f,q,p_i)S(p_i)$$

$$= S(p_f)\Delta^{\lambda\mu}(p_f,q,p_i)S(p_i)$$
$$- S(r)\Delta^{\lambda\mu}(r,r+k_f,r-p_f+p_i)S(r-p_f+p_i). \tag{2.8}$$

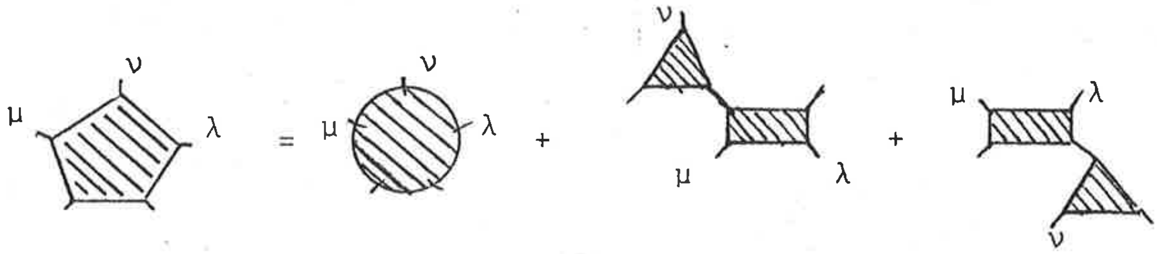
This is the generalised Ward identity to be used in forming from (2.4) the equation for $S_{\lambda\mu}$. To see how this is done, a further reduced amplitude $\Gamma^{\nu\lambda\mu}$ is defined by

$$\Delta^{\nu\lambda\mu}(r,p_f,q,p_i) \;=\; \Gamma^{\nu\lambda\mu}(r,p_f,q,p_i)$$

$$+\; \Gamma^{\nu}(r,p_f)S(p_f)\Delta^{\lambda\mu}(p_f,q,p_i)$$

$$+\; \Delta^{\lambda\mu}(r,r+k_f,r-p_f+p_i)S(r-p_f+p_i)\Gamma^{\nu}(r-p_f+p_i,p_i)$$

i.e.



$\Delta\uparrow$, $\Gamma\uparrow$

whence from (1.16) and (2.8)

$$(r_\nu - p_{f\nu})\Gamma^{\nu\lambda\mu}(r,p_f,q,p_i) \;=\; \Delta^{\lambda\mu}(p_f,q,p_i) - \Delta^{\lambda\mu}(r,r+k_f,r-p_f+p_i). \tag{2.9}$$

Now by substituting (2.6) and (2.7) into (2.4), there results

$$\not{p}_f S(p_f)\Delta^{\lambda\mu}(p_f,q,p_i) \;=\; \gamma^{\lambda}S(q)\Gamma^{\mu}(q,p_i)$$

$$+\; \gamma^{\mu}S(p_f-k_i)\Gamma^{\lambda}(p_f-k_i,p_i)$$

$$+\; \frac{ie^2\gamma^\rho}{(2\pi)^4}\int D_{\rho\nu}(p_f-r)S(r)\Delta^{\nu\lambda\mu}(r,p_f,q,p_i)d^4r$$

and so from (1.12) and (1.13),

$$z^{-1}\Delta^{\lambda\mu}(p_f,q,p_i)$$

$$= \gamma^{\lambda}S(q)\Gamma^{\mu}(q,p_i) + \gamma^{\mu}S(p_f-k_i)\Gamma^{\lambda}(p_f-k_i,p_i)$$

$$+ \frac{ie^2\gamma^{\rho}}{(2\pi)^4}\int D_{\rho\nu}(p_f-r)S(r)F^{\nu\lambda\mu}(r,p_f,q,p_i)d^4r \qquad (2.10)$$

where

$$F^{\nu\lambda\mu}(r,p_f,q,p_i) = S(r)\Big\{\Gamma^{\nu\lambda\mu}(r,p_f,q,p_i)$$

$$+ \Delta^{\lambda\mu}(r,r+k_f,r-p_f+p_i)S(r-p_f+p_i)\Gamma^{\nu}(r-p_f+p_i,p_i)\Big\}.$$

The approximation to be used is then the same as that made (1.22) for the electron propagator, *i.e.* in $F^{\nu\lambda\mu}$, p_f is taken to have its value at the pole of $D_{\lambda\mu}(p_f-r)$, so by (2.9) and (1.17)

$$F^{\nu\lambda\mu}(r,r,q,p_i) = -S(r)\frac{\partial}{\partial r_{\nu}}\Big\{\Delta^{\lambda\mu}(r,q,p_i)S(p_i)\Big\}S^{-1}(p_i)$$

provided $q-r$ and p_i-r are kept constant during the differentiation. Thus under this approximation (2.10) is an integro-differential equation for $\Delta^{\lambda\mu}$.

2.2 THE MØLLER SCATTERING EQUATION.

The second equation with which I am concerned is that for the amplitude

$$S(x-x_1,x_1-x_2,x_2-x_3) \quad = \quad \langle\psi(x)\psi(x_1)\bar{\psi}(x_2)\bar{\psi}(x_3)\rangle$$

corresponding to Møller scattering of two electrons.

The essential differences between the Møller scattering equation and the one derived in §2.1 are due to two causes :

(1) Whereas $S_{\lambda\mu}$ is a Dirac matrix, the Møller scattering amplitude has four spinor affixes ; its components are

$$S_{rstu}(x-x_1,x_1-x_2,x_2-x_3) \\ = \langle\psi_r(x)\psi_s(x_1)\bar{\psi}_t(x_2)\bar{\psi}_u(x_3)\rangle \ .$$

While statements of the form $\not{p}_f S_{\lambda\mu}(p_f,q,p_i)$ are unambiguous, it is necessary to record, when the 4-spinor S is multiplied by a Dirac matrix, on which of the spinor affixes that matrix acts. In what follows, I will denote by

$$T \quad = \quad \gamma_a^\lambda S(x-x_1,x_1-x_2,x_2-x_3)$$

the object with components

$$T_{rstu} \quad = \quad \sum_{a=1}^{4} \gamma_{ra}^\lambda S_{astu}(x-x_1,x_1-x_2,x_2-x_3)$$

and by

$$U = \gamma_b^\lambda S(x-x_1, x_1-x_2, x_2-x_3)$$

the object with components

$$U_{rstu} = \sum_{a=1}^{4} \gamma_{sa}^\lambda S_{ratu}(x-x_1, x_1-x_2, x_2-x_3),$$

and so on.

(2) The amplitude must be anti-symmetric under exchanges of both pairs of electrons, thus

$$S_{rstu}(x-x_1, x_1-x_2, x_2-x_3) = -S_{rsut}(x-x_1, x_1-x_3, x_3-x_2)$$

and so on. To take advantage of this, two equations will be developed : one for the full amplitude S and another for a sort-of reduced amplitude from which the original may be derived by anti-symmetrising. It is worth noting that it is sufficient to anti-symmetrise with respect to only one pair of electrons - either the in-coming or the out-going - since then the amplitude is automatically anti-symmetric under exchanges within the other pair.

Proceeding similarly to the work in §2.1, the Dyson-Schwinger equation for Møller scattering is

$$i\nabla^a S(x-x_1, x_1-x_2, x_2-x_3) = e\gamma_a^\lambda \langle \psi(x) A_\lambda(x) \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(x_3) \rangle$$

$$+ iz^{-1}\{\delta(x-x_3) S^b(x_1-x_2) - \delta(x-x_2) S^b(x_1-x_3)\}$$

which is to say

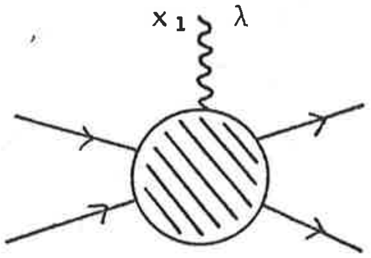
$$\not{p}_1^a S(p_1,q,p_1') = \frac{e\gamma^\lambda_a}{(2\pi)^4}\int S_\lambda(r,p_1,q,p_1')\,d^4r$$

$$+ (2\pi)^4 i z^{-1} S^b(p_2)\{\delta(p_2-p_2') - \delta(p_2-p_1')\} \qquad (2.11)$$

where $S_\lambda(r,p_1,q,p_1')$ is the Fourier transform of the five-point function

$$S_\lambda(x-x_1,x_1-x_2,x_2-x_3,x_3-x_4)$$
$$= \langle\psi(x)A_\lambda(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\rangle$$

corresponding to the diagram



and $S(p_1,q,p_1')$ is the Fourier transform of $S(x-x_1,x_1-x_2,x_2-x_3)$ and consequently corresponds to the scattering of electrons with momenta p_1', p_2' into two electrons with momenta p_1 and p_2, provided $p_2 = q-p_1$ and and $p_2' = q-p_1'$.

The appropriate generalised Ward identity is found to be

$$y(1-\theta)\nabla^2\partial^\lambda S_\lambda(x_1-x,x-x_2,x_2-x_3,x_3-x_4)$$

$$= e\Big\{-\delta(x-x_1)S(x-x_2,x_2-x_3,x_3-x_4)$$
$$-\delta(x-x_2)S(x_1-x,x-x_3,x_3-x_4)$$
$$+\delta(x-x_3)S(x_1-x_2,x_2-x,x-x_4)$$
$$+\delta(x-x_4)S(x_1-x_2,x_2-x_3,x_3-x)\Big\}$$

so that in the momentum representation,

$$iy(1-\theta)(p_1-r)^2(p_1^\lambda-r^\lambda)S_\lambda(r,p_1,q,p_1')$$

$$= e\Big[-S(p_1,q,p_1') - S(r,q,p_1')$$

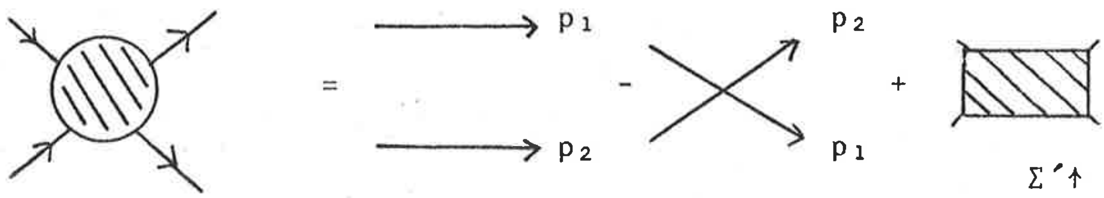
$$+ S(r,r+p_2,p_1') + S(r,r+p_2,r-p_1+p_1')\Big]. \qquad (2.12)$$

It will be useful to express this in terms of the irreducible amplitudes Σ' and Δ'^μ defined respectively by

$$S(p_1,q,p_1') = (2\pi)^4 i S^a(p_1)S^b(p_2)$$

$$\times\{\delta(p_1-p_1') - \delta(p_1-p_2')\} - e^2\Sigma'(p_1,q,p_1') \qquad (2.13)$$

i.e.



and

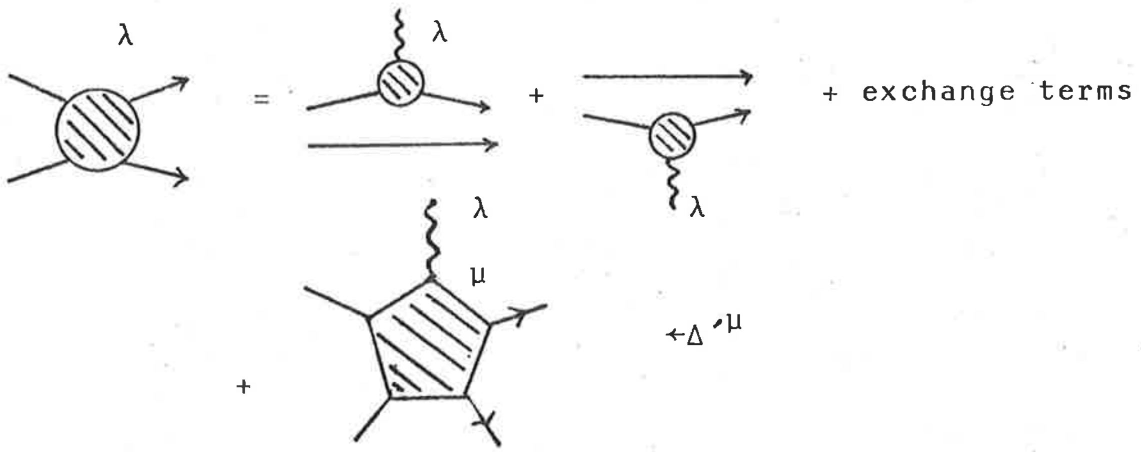
$$S_\lambda(r,p_1,q,p_1') = (2\pi)^4 i\Big[S_\lambda^a(r,p_1)S^b(p_2)$$

$$\times\{\delta(p_1-p_1') - \delta(p_1-p_2')\} + S^a(r)S_\lambda^b(p_2,q-r)$$

$$\times\{\delta(r-p_1') - \delta(r-p_2')\}\Big] - ie^3D_{\lambda\mu}(p_1-r)\Delta'^\mu(r,p_1,q,p_1')$$

(2.14)

or in diagrammatic terms,



By substituting these into (2.12) there results

$$(p_{1_\mu}-r_\mu)\Delta'^{\mu}(r,p_1,q,p_1') \;=\; -\Sigma'(p_1,q,p_1') - \Sigma'(r,q,p_1')$$

$$+ \Sigma'(r,r+p_2,p_1') + \Sigma'(r,r+p_2,r-p_1+p_1') \; . \qquad (2.15)$$

It was mentioned above that a completely anti-symmetric amplitude is obtained by anti-symmetrising with respect to only one pair of electrons; thus there exists an amplitude $\Sigma(p_1,q,p_1')$ anti-symmetric under exchanges of p_1' and p_2' such that

$$\Sigma'(p_1,q,p_1') \;=\; S^a(p_1)S^b(p_2)\Sigma(p_1,q,p_1') \qquad (2.16)$$

and an analogous amplitude Δ^μ satisfying

$$\Delta'^{\mu}(r,p_1,q,p_1') \;=\; S^a(r)S^b(p_2)\Delta^{\mu}(r,p_1,q,p_1'). \qquad (2.17)$$

By substituting (2.13), (2.14), (2.16) and (2.17) into (2.11) the following equation for $\Sigma(p_1,q,p_1')$ can be derived :

$$z^{-1}\Sigma(p_1,q,p_1') = \gamma_a^\lambda D_{\lambda\mu}(p_1-p_1')\Gamma_b^\mu(p_2,p_2')S^a(p_1')S^b(p_2')$$

$$- \gamma_a^\lambda D_{\lambda\mu}(p_1-p_2')\Gamma_b^\mu(p_2,p_1')S^a(p_2')S^b(p_1')$$

$$+ \frac{ie^2\gamma_a^\lambda}{(2\pi)^4}\int D_{\lambda\mu}(p_1-r)F^\mu(r,p_1,q,p_1')d^4r \qquad (2.18)$$

where

$$F^\mu(r,p_1,q,p_1') = S^a(r)\Big[\Delta^\mu(r,p_1,q,p_1')$$

$$- \Gamma_a^\mu(r,p_1)S^a(p_1)\Sigma(p_1,q,p_1')\Big] .$$

As before, the approximation in the integrand $p_1 \to r$ makes this a closed equation for Σ : from the generalised Ward identity (2.15) it follows that

$$(p_{1\mu}-r_\mu)\Big[\Delta^\mu(r,p_1,q,p_1') - \Gamma_a^\mu(r,p_1)S^a(p_1)\Sigma(p_1,q,p_1')$$

$$- \Gamma_b^\mu(p_2,q-r)S_b(q-r)\Sigma(r,q,p_1')\Big]$$

$$= - \Sigma(p_1,q,p_1') - \Sigma(r,q,p_1') + \Sigma(r,r+p_2,p_1')$$

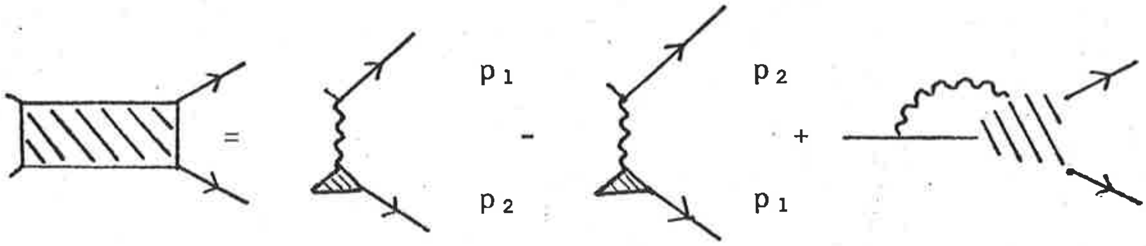
$$+ \Sigma(r,r+p_2,r-p_1+p_1')$$

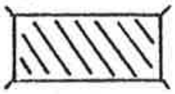
and hence that

$$F^\mu(r,r,q,p_1') = - S^a(r)\left[\frac{\partial}{\partial r_\mu} + 2\frac{\partial}{\partial q_\mu} + \frac{\partial}{\partial p_{1\mu}'}\right]\Sigma(r,q,p_1')$$

$$- S^a(r)S_b^{-1}(q-r)\left[\frac{\partial}{\partial(q-r)_\mu}S^b(q-r)\right]\Sigma(r,q,p_1') .$$

Before proceeding to the next section I will make one further simplification. Equation (2.18) retains the exchange terms due to Fermi statistics - in fact the corresponding diagram equation is



where the blob  is a function of  . However this equation can be generated by anti-symmetrising a similar equation which doesn't assume any exchange symmetry; the two out-going electron lines  and  in this equation are therefore common factors to each term and may be removed. In algebraic terms, there exists an amplitude without exchange symmetry $\Sigma_A(p_1,q,p_1')$ from which $\Sigma(p_1,q,p_1')$ can be obtained by anti-symmetrising, *i.e.*

$$\Sigma(p_1,q,p_1') = \Sigma_A(p_1,q,p_1')S^a(p_1')S^b(p_2')$$

$$- \Sigma_A(p_1,q,p_2')S^a(p_2')S^b(p_1') .$$

Defining the comparable quantity $\Delta_A^\mu(r,p_1,q,p_1')$ then Σ_A satisfies

$$z^{-1}\Sigma_A(p_1,q,p_1') = \gamma_a^\lambda D_{\lambda\mu}(p_1-p_1')\Gamma_b^\mu(p_2,p_2')$$

$$+ \frac{ie^2\gamma_a^\lambda}{(2\pi)^4}\int D_{\lambda\mu}(p_1-r)F_A^\mu(r,p_1,q,p_1')d^4r \qquad (2.19)$$

where

$$F^{\mu}_{A}(r,r,q,p_1') = - S^{a}(r)\Big[\{\frac{\partial}{\partial r_{\mu}} + 2\frac{\partial}{\partial q_{\mu}} + \frac{\partial}{\partial p'_{1\mu}}\}\Sigma_{A}(r,q,p_1')$$

$$+ S_{b}^{-1}(q-r)\frac{\partial}{\partial(q-r)_{\mu}}\Sigma_{A}(r,q,p_1')$$

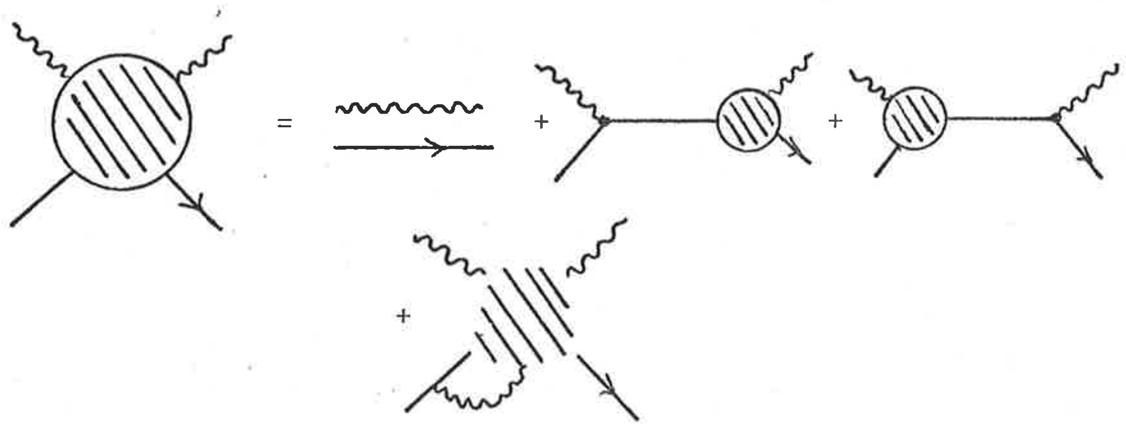
$$- \Sigma_{A}(r,q,p_1')\{\frac{\partial}{\partial p'_{2\mu}}S^{b}(p_2)\}S_{b}^{-1}(p_2)$$

$$- \Sigma_{A}(r,q,p_1')\{\frac{\partial}{\partial p'_{1\mu}}S^{a}(p_1)\}S_{a}^{-1}(p_1)\Big]$$

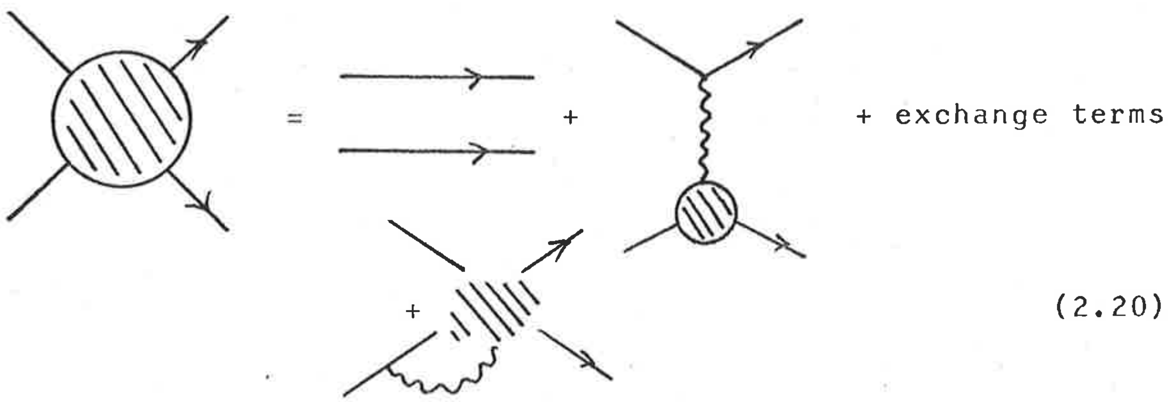
by virtue again of (2.15).

2.3 SUMMARY OF RESULTS.

In this chapter non-perturbative methods have been used to derive equations representing Compton and Møller scattering, and thus by extension the Bethe-Salpeter equation[22] has been derived; in fact the results obtained have done no more than confirm algebraically what would, on the basis of Feynman diagrams, be expected in any case. Thus, the Compton scattering equation (2.10) would be drawn, restoring the external lines, as



while the Møller scattering equation is



(2.20)

where in each case the blob is a function of other renormalised Feynman diagrams. If each of these diagrams is expanded in powers of the fine structure constant, then it is clear that all the topologically distinct Feynman graphs corresponding to the appropriate process are produced.

In the latter equation, the 'a' stream of particles, consisting of the initial and final electrons with momentum p_1' and p_1 respectively has clearly been treated differently to the 'b' stream, with the result that in the second order term on the right of (2.20) the 'a' electrons pass through a simple vertex while the 'b' electrons go across a renormalised vertex. There are two ways in which this situation can be rectified :

(1) The simple vertex in (2.20) can be made into a full vertex by adding an extra term into the 'blob'. This disguises the asymmetry between a and b but doesn't remove it.

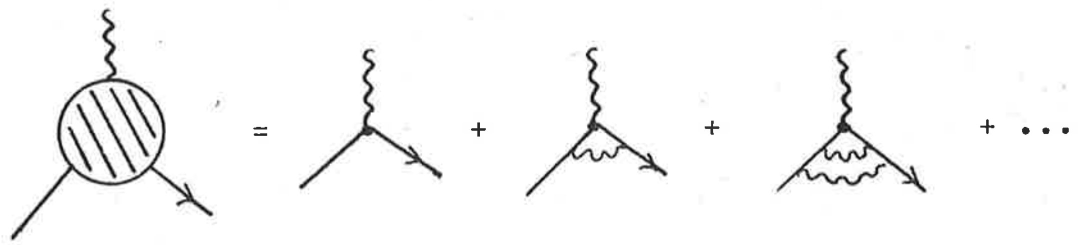
(2) A symmetric treatment of a and b requires premultiplying (2.11) by ϕ_2^b; this corresponds in diagrammatic terms to 'drawing back the veil' on both the out-going electron lines and hence produces many more terms underneath the veil than are necessary (all the required information is contained in (2.19)).

Neither of these remedies is particularly elegant; however in the next chapter I intend to solve, in the asymptotic region, an approximate version of (2.19) and by means of this approximation the asymmetry will be removed.

CHAPTER 3

ASYMPTOTIC SOLUTION TO THE MØLLER SCATTERING EQUATION.

It has recently been shown[1] by Green *et al.* that a solution to the non-perturbative equations for the vertex function $\Gamma^{\lambda}(p_1,p_2)$ can be obtained in the ladder approximation, *i.e.* when the vertex function is taken as the infinite sum



Green's solution proceeds in three steps, *viz.*

(1) Convert the non-perturbative integro-differential equation into a second order partial differential equation.

(2) Make the appropriate approximations and thereby obtain a set of component equations not involving Dirac matrices.

(3) Reduce the vectors and tensors in step (2) to scalars and solve these equations; the boundary conditions are obtained by comparison with the perturbation solution in the region where the two solutions overlap.

I intend in solving the Møller scattering equation to follow an identical plan, though there will necessarily be some differences within each step.

The result of the vertex solution most relevant to this chapter is that in the asymptotic limit p_1^2, $p_2^2 >> m^2$,

$$\Gamma^{\lambda}(p_1,p_2) \sim \gamma^{\lambda} \tag{3.1}$$

- an example of the asymptotic freedom mentioned in the introduction. This result is also consistent with the solution for the electron propagator discussed in §1.2; from (1.29)

$$S^{-1}(p) \simeq \not{p} - a(p^2)$$

where the function $a(p^2)$ rapidly approaches zero as $-p^2$ becomes large. Thus by the Ward identity

$$\Gamma^{\lambda}(p,p) \to \gamma^{\lambda}$$

in the asymptotic limit, as required.

3.1 THE LADDER APPROXIMATION.

By following the same procedure as that used in §1.2 it is possible to change equation (2.19) for Σ_A into a second order partial differential equation.

Thus by setting

$$
\begin{aligned}
p_1 &= p + k \\
p_2 &= p - k \\
p_1' &= p + \ell \\
p_2' &= p - \ell
\end{aligned}
\qquad (3.2)
$$

(2.19) can be written

$$
\Sigma_A(p,k,\ell) = \gamma_a^{\lambda} D_{\lambda\mu}(k-\ell)\Gamma_b^{\mu}(p-k,p-\ell) + \frac{ie^2}{(2\pi)^4}\gamma_a^{\lambda}\int D_{\lambda\mu}(p+k-r)F_A^{\mu}(r,p,k,\ell)d^4r \qquad (3.3)
$$

where $\Sigma_A(p,k,\ell)$ and $F_A^{\mu}(r,p,k,\ell)$ mean the same as $\Sigma_A(p_1,q,p_1')$ and $F_A^{\mu}(r,p_1,q,p_1')$ respectively provided equations (3.2) are satisfied.

If the photon propagator is taken to have its lowest order form (1.21) then (3.3) may be written

$$
\Sigma_A(p,k,\ell) = \gamma_a^{\lambda} D_{\lambda\mu}(k-\ell)\Gamma_b^{\mu}(p-k,p-\ell) - \varepsilon\left\{\gamma_{\mu}^{a} I^{\mu}(p,k,\ell) - J(p,k,\ell)\right\}
$$

for integral functions I^{μ} and J.

Furthermore, in the asymptotic region Γ^{μ}_{b} can be taken as γ^{μ}_{b} from (3.1), and thus when $\nabla^{a} = \gamma^{a}_{\lambda}\partial/\partial p_{\lambda}$ it follows from (1.24) that

$$\nabla^{2}\Sigma_{A}(p,k,\ell) = -\varepsilon\left[\tfrac{1}{2}\gamma^{a}_{\mu}F^{\mu}_{A}(p,k,\ell) + G(p,k,\ell)\right] \qquad (3.4)$$

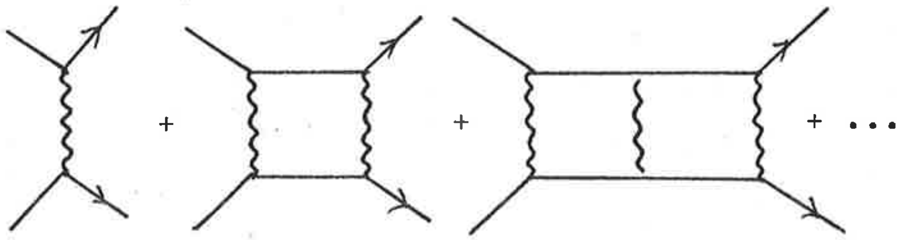
where

$$F^{\mu}_{A}(p,k,\ell) = F^{\mu}_{A}(p,p,k,\ell)$$

and

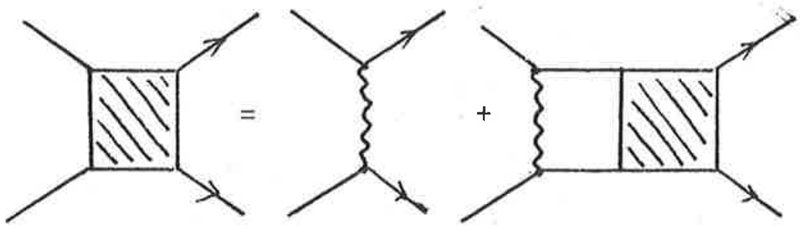
$$\nabla^{a}G(p,k,\ell) = \partial_{\mu}F^{\mu}_{A}(p,k,\ell) \; .$$

It is hoped to solve this equation in the ladder approximation, *i.e.* to compute the sum of diagrams



(3.5)

Since the diagram equation for this sum is



then the appropriate substitution for F^{μ}_{A} is

$$F^{\mu}_{A}(p,k,\ell) = \gamma^{\mu}_{b}S^{a}(p+k)S^{b}(p-k)\Sigma_{A}(p,k,\ell) \qquad (3.6)$$

Further justification of this approximation can be found in ref. 1.

Thus (3.4) together with (3.6) are the asymptotic equations for Møller scattering in the ladder approximation. It will be necessary to make one further change before the component equations can be found : by choosing the pure imaginary representation of the Dirac matrices, which has the property that γ_0 is symmetric whereas γ_1, γ_2 and γ_3 are anti-symmetric, the matrix $\gamma' = \gamma^0\gamma^5$ satisfies

$$\gamma_\mu\gamma' = \gamma'\gamma_\mu^t \qquad (\gamma_\mu^t \text{ is the transpose of } \gamma_\mu) .$$

Now define a new amplitude

$$\Sigma'_A = \gamma'_b\Sigma_A \tag{3.7}$$

(note this is quite distinct from the Σ' in §2.2).
From (3.4),

$$\nabla^2\Sigma'_A = -\varepsilon\{\tfrac{1}{2}\gamma^a_\mu F'^\mu_A + G'\}$$

where

$$F'^\mu_A = \gamma'_b\gamma^\mu_b S^a(p+k)S^b(p-k)\Sigma_A$$

$$= \gamma^{\mu t}_b S^a(p+k)S^t_b(p-k)\Sigma'_A$$

so that

$$\nabla^2\Sigma'_A = -\varepsilon\{\tfrac{1}{2}\gamma^a_\mu\gamma^{\mu t}_b F'_A + G'\} \tag{3.8}$$

with

$$F'_A = S^a(p+k)S^{bt}(p-k)\Sigma'_A \tag{3.9}$$

and

$$\nabla^a G' = \gamma_b^{\mu t}\partial_\mu F'_A \quad .$$

Now an arbitrary Dirac matrix Γ can be decomposed into scalar, pseudo-scalar, vector, pseudo-vector and tensor components by defining the anti-symmetric tensors

$$\gamma_{\lambda\mu} = \tfrac{1}{2}[\gamma_\lambda,\gamma_\mu]$$

$$\gamma_{\lambda\mu\nu} = \tfrac{1}{2}\{\gamma_{\lambda\mu},\gamma_\nu\} = i\varepsilon_{\lambda\mu\nu\rho}\gamma^5\gamma^\rho$$

and

$$\gamma_{\lambda\mu\nu\rho} = \tfrac{1}{2}[\gamma_{\lambda\mu\nu},\gamma_\rho] = -i\varepsilon_{\lambda\mu\nu\rho}\gamma^5 \quad .$$

Then the components

$$g = \tfrac{1}{4}\mathrm{tr}\Gamma$$

$$g_\lambda = \tfrac{1}{4}\mathrm{tr}\Gamma\gamma_\lambda$$

$$g_{\lambda\mu} = \tfrac{1}{4}\mathrm{tr}\Gamma\gamma_{\lambda\mu}$$

$$g_{\lambda\mu\nu} = \tfrac{1}{4}\mathrm{tr}\Gamma\gamma_{\lambda\mu\nu}$$

and

$$g_{\lambda\mu\nu\rho} = \tfrac{1}{4}\mathrm{tr}\Gamma\gamma_{\lambda\mu\nu\rho} \tag{3.10}$$

have the fortunate property that

$$\Gamma = g + g_\lambda\gamma^\lambda + g_{\lambda\mu}\gamma^{\lambda\mu} + g_{\lambda\mu\nu}\gamma^{\lambda\mu\nu} + g_{\lambda\mu\nu\rho}\gamma^{\lambda\mu\nu\rho} \quad .$$

This method can be applied to (3.8), which of course is a rank 4 spinor equation with components

$$\nabla^2(\Sigma'_A)_{rstu} = -\varepsilon\{\tfrac{1}{2}(\gamma_\mu)_{rv}(\gamma^\mu)_{ws}(F'_A)_{vwtu} + (G')_{rstu}\} .$$

If a rank 4 spinor such as $(F'_A)_{vwtu}$ is considered as a Dirac matrix of elements, labelled by v and w, which are themselves Dirac matrices labelled by t and u, then this equation can be written in the so-called co-spinor representation :

$$\nabla^2\Sigma'_A = -\varepsilon\{\tfrac{1}{2}\gamma_\mu F'_A\gamma^\mu + G'\} \qquad (3.11)$$

\- hence the reason for introducing the γ' matrix in (3.7) : otherwise the second γ^μ in (3.11) would have been $\gamma^{\mu t}$.

Now let the components of Σ'_A be the matrices whose elements are

$$\sigma_{tu} = \tfrac{1}{4}(\Sigma'_A)_{rrtu}$$

$$(\sigma_\lambda)_{tu} = \tfrac{1}{4}(\Sigma'_A\gamma_\lambda)_{rrtu}$$

$$(\sigma_{\lambda\mu})_{tu} = \tfrac{1}{4}(\Sigma'_A\gamma_{\lambda\mu})_{rrtu}$$

$$(\sigma_{\lambda\mu\nu})_{tu} = \tfrac{1}{4}(\Sigma'_A\gamma_{\lambda\mu\nu})_{rrtu}$$

and those of F'_A be

$$\eta_{tu} = \tfrac{1}{4}(F'_A)_{rrtu}$$

and similarly for η_λ, $\eta_{\lambda\mu}$ and $\eta_{\lambda\mu\nu}$ (here I have anticipated that the pseudo-scalar components of Σ'_A vanish).

Then by using the relation

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$$

from which

$$[\gamma_{\mu\nu},\gamma_\rho] = 2(\gamma_\mu g_{\nu\rho} - \gamma_\nu g_{\mu\rho})$$

and

$$\{\gamma_{\lambda\mu\nu},\gamma_\rho\} = 2(\gamma_{\lambda\mu} g_{\nu\rho} + \gamma_{\mu\nu} g_{\lambda\rho} + \gamma_{\nu\lambda} g_{\mu\rho})$$

it results from (3.11) that

$$\nabla^2\sigma = -3\epsilon\eta ,$$

$$\nabla^2\sigma_\lambda = 2\epsilon(\eta_\lambda - \partial_\lambda\theta)$$

where

$$\nabla^2\theta = \partial^\lambda\eta_\lambda ,$$

$$\nabla^2\sigma_{\lambda\mu} + 2\partial_\lambda\partial^\nu\sigma_{\mu\nu} - 2\partial_\mu\partial^\nu\sigma_{\lambda\nu} = -\epsilon\eta_{\lambda\mu}$$

where

$$\nabla^2\partial^\lambda\sigma_{\lambda\mu} = \epsilon\partial^\lambda\eta_{\lambda\mu} ,$$

and finally

$$\nabla^2\partial^\nu\sigma_{\lambda\mu\nu} = -2\epsilon\partial^\nu\eta_{\lambda\mu\nu} . \tag{3.12}$$

On the other hand, Σ'_A and F'_A are related by (3.9) and so by substituting the approximate form (1.29) of the electron propagator, it is easily shown that

$$\eta = (p_+\sigma + p_+^\lambda\sigma_\lambda - p_1^\lambda p_2^\mu\sigma_{\lambda\mu})/D$$

$$\eta_\lambda = (p_{+\lambda}\sigma - p_-\sigma_\lambda + p_{+\lambda}^\mu\sigma_\mu + p_-^\mu\sigma_{\lambda\mu} + p_1^\mu p_2^\nu\sigma_{\lambda\mu\nu})/D$$

$$\eta_{\lambda\mu} = (-p_{-\lambda\mu}\sigma - p_{-\lambda}\sigma_\mu + p_{-\mu}\sigma_\lambda + p_+\sigma_{\lambda\mu} + p_{+\lambda}^\nu\sigma_{\mu\nu}$$
$$- p_{+\mu}^\nu\sigma_{\lambda\nu} + p_+^\nu\sigma_{\lambda\mu\nu})/D$$

and

$$\eta_{\lambda\mu\nu} = (p_{-\lambda\mu}\sigma_\nu + p_{-\mu\nu}\sigma_\lambda + p_{-\nu\lambda}\sigma_\mu + p_{+\lambda}\sigma_{\mu\nu}$$
$$+ p_{+\mu}\sigma_{\nu\lambda} + p_{+\nu}\sigma_{\lambda\mu} - p_-\sigma_{\lambda\mu\nu} + p_{+\lambda}^\rho\sigma_{\mu\nu\rho}$$
$$+ p_{+\mu}^\rho\sigma_{\nu\lambda\rho} + p_{+\nu}^\rho\sigma_{\lambda\mu\rho})/D \qquad (3.13)$$

where, if $a_1 = a(p_1^2)$ and $a_2 = a(p_2^2)$, then

$$D = (p_1^2 - a_1^2)(p_2^2 - a_2^2)$$

$$p_\pm = p_1 . p_2 \pm a_1 a_2$$

$$p_{\pm\mu} = a_2 p_{1\mu} \pm a_1 p_{2\mu}$$

and

$$p_{\pm\mu\nu} = p_{1\mu}p_{2\nu} \pm p_{2\mu}p_{1\nu} .$$

Thus equations (3.12) are the required equations to be solved, together with the substitutions (3.13) on the right-hand side; as promised, they contain explicitly no Dirac matrices even though each equation is indeed a matrix equation.

Disregarding this fact, the equations are identical to the equations for the vertex solved by Green, Cartier and Broyles. It is their solution which I intend to present in the following two sections.

3.2 GENERAL SOLUTION OF THE ASYMPTOTIC EQUATIONS - FIRST SET.

In the asymptotic region the function $a(p^2)$ in (1.29) goes rapidly to zero and so in (3.13) the substitutions

$$D \to p_1 p_2$$

$$p_\pm \to p_1 . p_2$$

$$p_{\pm\mu} \to 0$$

$$p_{\pm\mu\nu} \to p_{\pm\mu\nu} \tag{3.14}$$

can be made.

Thus equations (3.12) become

$$\nabla^2\sigma = -3\varepsilon(p_1 . p_2\sigma - p_1^\lambda p_2^\mu \sigma_{\lambda\mu})/p_1^2 p_2^2 \tag{3.15}$$

$$\nabla^2\sigma_\lambda = 2\varepsilon(-p_1 . p_2\sigma_\lambda + p_{+\lambda}^\nu \sigma_\nu + p_1^\mu p_2^\nu \sigma_{\lambda\mu\nu})/p_1^2 p_2^2 - 2\varepsilon\partial_\lambda\theta \tag{3.16}$$

where

$$\nabla^2\theta = \partial^\lambda\left[(-p_1 . p_2\sigma_\lambda + p_{+\lambda}^\nu \sigma_\nu + p_1^\mu p_2^\nu \sigma_{\lambda\mu\nu})/p_1^2 p_2^2\right]$$

$$\nabla^2\sigma_{\lambda\mu} + 2\partial_\lambda\partial^\nu\sigma_{\mu\nu} - 2\partial_\mu\partial^\nu\sigma_{\lambda\nu}$$

$$= -\varepsilon(-p_{-\lambda\mu}\sigma + p_1 . p_2\sigma_{\lambda\mu} + p_{+\lambda}^\nu \sigma_{\mu\nu} - p_{+\mu}^\nu \sigma_{\lambda\nu})/p_1^2 p_2^2 \tag{3.17}$$

and

$$\nabla^2 \partial^\nu \sigma_{\lambda\mu\nu} = -2\varepsilon\partial^\nu(p_{-\lambda\mu}\sigma_\nu + p_{-\mu\nu}\sigma_\lambda + p_{-\nu\lambda}\sigma_\mu - p_1 \cdot p_2 \sigma_{\lambda\mu\nu}$$

$$+ p^\rho_{+\lambda}\sigma_{\mu\nu\rho} + p^\rho_{+\mu}\sigma_{\nu\lambda\rho} + p^\rho_{+\nu}\sigma_{\lambda\mu\rho})/p_1^2 p_2^2 \qquad (3.18)$$

yielding two independent sets of equations (3.15), (3.17) and (3.16), (3.18). The second set will be dealt with later; I will give here the solution to the first.

The first step, as discussed in the introduction to this chapter, is to reduce the tensor $\sigma_{\lambda\mu}$ to its scalar components. Since it is an anti-symmetric tensor function of p_1 and p_2 (or equivalently p_1' and p_2') then the only independent scalars to be formed from it are

$$s_1 = p_1^\lambda \partial^\mu \sigma_{\lambda\mu}$$

$$s_2 = p_2^\lambda \partial^\mu \sigma_{\lambda\mu}$$

and

$$s_3 = p_1^\lambda p_2^\mu \sigma_{\lambda\mu} .$$

Clearly from (3.15)

$$\nabla^2 \sigma = 3\varepsilon(s_3 - p_1 \cdot p_2 \sigma)/p_1^2 p_2^2 . \qquad (3.19)$$

Now under the conditions (3.14)

$$p_1^\lambda n_{\lambda\mu} = \left[(p_1 \cdot p_2 p_{1\mu} - p_1^2 p_{2\mu})\sigma - p_{1\mu} s_3\right]/p_1^2 p_2^2$$

$$- p_2^\lambda \sigma_{\lambda\mu}/p_2^2$$

$$= -p_{1\mu}\nabla^2\sigma/3\varepsilon - (p_{2\mu}\sigma + p_2^\lambda\sigma_{\lambda\mu})/p_2^2 \quad \text{from (3.19)}$$

and similarly

$$p_2^\lambda\eta_{\lambda\mu} = p_{2\mu}\nabla^2\sigma/3\varepsilon + (p_{1\mu}\sigma - p_1^\lambda\sigma_{\lambda\mu})/p_1^2 \ .$$

Thus, since (3.17) implies

$$\nabla^2\partial^\mu\sigma_{\lambda\mu} = \varepsilon\partial^\mu\eta_{\lambda\mu}$$

it follows that

$$\nabla^2 s_1 = -(p_1.\partial + 4)\nabla^2\sigma/3 - \varepsilon(p_2.\partial + 4)\sigma/p_2^2 - \varepsilon s_2/p_2^2$$

$$= -\nabla^2(p_1.\partial + 2)\sigma/3 - \frac{\varepsilon}{p_2^2}(p_2.\partial + 2)\sigma - \varepsilon s_2/p_2^2$$

and

$$\nabla^2 s_2 = \nabla^2(p_2.\partial + 2)\sigma/3 + \frac{\varepsilon}{p_1^2}(p_1.\partial + 2)\sigma - \varepsilon s_1/p_1^2$$

where

$$p_1.\partial = p_1^\mu\partial/\partial p^\mu \ \textit{etc.}$$

In other words, s_1 and s_2 satisfy the coupled equations

$$\nabla^2(s_1+p_1.\partial\sigma/3+2\sigma/3) = -\varepsilon(s_2+p_2.\partial\sigma+2\sigma)/p_2^2 \qquad (3.20)$$

$$\nabla^2(s_2-p_2.\partial\sigma/3-2\sigma/3) = -\varepsilon(s_1-p_1.\partial\sigma-2\sigma)/p_1^2 \ . \qquad (3.21)$$

Furthermore,

$$p_1^\lambda p_2^\mu \eta_{\lambda\mu}, \quad = \quad -\sigma + p_1 . p_2 (p_1 . p_2 \sigma - s_3)/p_1^2 p_2^2$$

$$= \quad -(1 + \frac{p_1 . p_2}{3\varepsilon} \nabla^2)\sigma$$

and so

$$\nabla^2 s_3 \quad = \quad -\varepsilon p_1^\lambda p_2^\mu \eta_{\lambda\mu} + 2(p_2 . \partial s_1 - p_1 . \partial s_2 + 2s_1 - 2s_2)$$

or

$$\nabla^2 s_3 - p_1 . p_2 \nabla^2 \sigma/3 \quad = \quad \varepsilon\sigma + 2(p_2 . \partial s_1 + 2s_1 - p_1 . \partial s_2 - 2s_2) \tag{3.22}$$

- equations (3.19) - (3.22) are the equations for the four unknowns σ, s_1, s_2 and s_3.

At this stage it is convenient to change the independent variables to

$$x = (p_1^2 p_2^2)^{\frac{1}{2}}$$

and

$$z = (p_1^2/p_2^2)^{\frac{1}{2}} \quad . \tag{3.23}$$

If f is a function of p_1^2 and p_2^2, then

$$\partial_\lambda f \quad = \quad 2p_{1\lambda} f_1 + 2p_{2\lambda} f_2$$

where

$$f_1 \quad = \quad \partial f/\partial p_1^2 \quad \textit{etc.}$$

and so

$$\nabla^2 f \quad = \quad 4p_1^2 f_{11} + 8p_1 . p_2 f_{12} + 4p_2^2 f_{22} + 8(f_1 + f_2) \quad .$$

In the asymptotic region where k^2 can be neglected in comparison with p^2,

$$2p_1 \cdot p_2 \quad = \quad p_1^2 + p_2^2 \tag{3.24}$$

so that

$$\nabla^2 f \quad = \quad 4p_1^2(f_{11}+f_{12}) + 4p_2^2(f_{22}+f_{12}) + 8(f_1+f_2) \ . \tag{3.25}$$

Define

$$D_x \quad \equiv \quad x\partial/\partial x$$

and

$$D \quad = \quad (1-y^2)\partial/\partial y + yD_x$$

where

$$y \quad = \quad \tfrac{1}{2}(z + z^{-1}) \quad . \tag{3.26}$$

Clearly

$$D_x f \quad = \quad p_1^2 f_1 + p_2^2 f_2$$

and

$$Df \quad = \quad (1-y^2)\frac{dz}{dy}\frac{\partial f}{\partial z} + yD_x f$$

$$= \quad -(y^2-1)^{\frac{1}{2}}(xzf_1 - xz^{-1}f_2) + y(xzf_1 + xz^{-1}f_2)$$

$$= \quad x(f_1+f_2) \ .$$

Thus by (3.25)

$$\nabla^2 f \quad = \quad 4(D_x+2)(f_1+f_2)$$

i.e.

$$x\nabla^2 f \quad = \quad 4(D_x+1)Df \quad . \tag{3.27}$$

Similarly in the asymptotic region

$$p_1.\partial f \quad = \quad (zD+D_x)f$$

and

$$p_2.\partial f \quad = \quad (z^{-1}D+D_x)f \quad . \tag{3.28}$$

It is clear from the form of the equations that the following are appropriate substitutions :

$$\sigma \quad = \quad a\left(\frac{x}{k_p^2}\right)^{\alpha}/k_p^2$$

$$s_1 \quad = \quad (b+zc)\left(\frac{x}{k_p^2}\right)^{\alpha}/k_p^2$$

$$s_2 \quad = \quad -(b+z^{-1}c)\left(\frac{x}{k_p^2}\right)^{\alpha}/k_p^2$$

$$s_3 \quad = \quad d\left(\frac{x}{k_p^2}\right)^{\alpha+1} \tag{3.29}$$

where a, b, c and d are functions only of y, and k_p is the momentum of the first order photon in (3.5), *i.e.* $k_p = k-\ell$.

By applying equations (3.27) and (3.28), noting that D_x can be replaced by the eigenvalue α when applied to the first three equations of (3.29) and by $\alpha+1$ when applied to the fourth, there results from (3.19)

$$4(\alpha+1)Da = 3\varepsilon(d-ya) \tag{3.30}$$

whereas (3.20) and (3.21) yield

$$4(\alpha+1)\{Db+zDc+c(1-yz)+(zD+1-yz+\alpha+2)Da/3\}$$

$$= \varepsilon\{zb+c-(D+\alpha z+2z)a\} \tag{3.31}$$

and

$$4(\alpha+1)\{-Db-z^{-1}Dc-c(1-z^{-1}y)-(z^{-1}D+1-yz^{-1}+\alpha+2)Da/3\}$$

$$= \varepsilon\{-z^{-1}b-c+(D+\alpha z^{-1}+2z^{-1})a\} \tag{3.32}$$

respectively. These two equations can be combined to give

$$4(\alpha+1)(D-y)(c+Da/3) = \varepsilon\{b-(\alpha+2)a\} \tag{3.33}$$

and

$$4(\alpha+1)\{Db+c+(\alpha+3)Da/3)\} = \varepsilon(c-Da) . \tag{3.34}$$

Finally, (3.22) implies that

$$4(\alpha+2)(D+y)d - y\varepsilon(d-ya)$$

$$= \varepsilon a + 4(\alpha+2)(b+yc) + 4(yDb+Dc) . \tag{3.35}$$

With this equation, (3.33) and (3.34) can be used to eliminate Dc and Db respectively; then from (3.30) it follows that

$$(D+y)d = b+yc \tag{3.36}$$

provided the condition holds that

$$4(\alpha+1)(\alpha+2) + \varepsilon \neq 0 .$$

Then (3.35) reduces to

$$4(D+y)(c+yb-d) = -\{4(\alpha+1)yD/3 + \varepsilon\}a . \tag{3.37}$$

It is now a question of eliminating all but one of the variables. This can be done by forming the difference of (3.31) and (3.32) and then solving by means of (3.30) and (3.36) for yb+c-d to find

$$\varepsilon(yb+c-d) = \frac{16}{\varepsilon}(\alpha+1)^2 D(D+y)Da + 4(\alpha+1)D(4yD+\alpha+4)a/3$$
$$+ \varepsilon(D+\alpha y+y)a . \tag{3.38}$$

Applying (3.37) then leaves a fourth order equation for a, which fortunately factorises to give

$$\{4(D+y)D + \varepsilon\}a' = 0 \tag{3.39}$$

where

$$a' = \{4(\alpha+1)^2(D+y)D + (\alpha+1)\varepsilon(4yD+3) + 3\varepsilon^2/4\}a . \tag{3.40}$$

The solution of this equation can be found in terms of the associated Legendre functions. To see this, note that these functions satisfy

$$\{(1-y^2)d^2/dy^2 - 2yd/dy + \nu(\nu+1) - \mu^2/(1-y^2)\}Q^{\mu}_{\nu}(y) = 0$$

and hence

$$(D-\alpha y-\nu y)(D-\alpha y+\nu y)Q^{\mu}_{\nu}$$

$$= \{D^2 - 2\alpha yD - (\alpha-\nu) + y^2(\alpha-\nu)(\alpha+\nu+1)\}Q^{\mu}_{\nu}$$

$$= (\mu^2-\nu^2)Q^{\mu}_{\nu} . \qquad (3.41)$$

Now suppose

$$a'(y) = (y^2-1)^q Q_1(y) .$$

Then (3.39) implies that

$$\{D^2 + y(1-4q)D + 4q^2y^2 - 2q + \varepsilon/4\}Q_1 = 0$$

which, on comparison with (3.41), has the solution

$$Q_1 = \text{const.}Q^{\mu}_{-\frac{1}{2}} \qquad (3.42)$$

where $\mu^2 = \frac{1}{4}(1-\varepsilon)$, provided

$$q = \tfrac{1}{2}(\alpha-\tfrac{1}{2}) .$$

Secondly, the homogeneous solution to (3.40) is found by setting

$$a(y) = (y^2-1)^p Q_2(y) ;$$

then comparison with (3.41) gives the two independent solutions

$$a_1(y) \quad = \quad (y^2-1)^{\frac{1}{2}(\alpha-\frac{1}{2}+\delta)} Q^{\frac{1}{2}(1-\delta)}_{-\delta-\frac{1}{2}}(y)$$

and

$$a_2(y) \quad = \quad (y^2-1)^{\frac{1}{2}(\alpha-\frac{1}{2}+\delta)} Q^{\frac{1}{2}(1-\delta)}_{\delta-\frac{1}{2}}(y) \qquad (3.43)$$

where

$$\delta \quad = \quad \varepsilon/2(\alpha+1) \ .$$

The remaining functions can readily be determined from (3.30), (3.36) and (3.38); this is left until §3.5 when the boundary conditions can be used to fix the various constants involved.

3.3 GENERAL SOLUTION OF THE ASYMPTOTIC EQUATIONS - SECOND SET.

The second pair of equations (3.16) and (3.18) have a similar solution, found in this case by considering the independent scalars

$$t_1 = p_1^\lambda \sigma_\lambda$$

$$t_2 = p_2^\lambda \sigma_\lambda$$

and

$$u = -\tfrac{1}{2} p_1^\lambda p_2^\mu \partial^\nu \sigma_{\lambda\mu\nu} \;.$$

It follows from (3.16) that

$$\nabla^2 \partial^\lambda \sigma_\lambda = 0$$

and since the only non-trivial, scalar solutions of $\nabla^2 f = 0$ involve delta-function singularities at $p^2 = 0$, this is taken to mean

$$\partial^\lambda \sigma_\lambda = 0 \;.$$

Thus there results from (3.16)

$$\nabla^2 t_1 = 2\varepsilon(t_2/p_2^2 - p_1.\partial\theta)$$

$$\nabla^2 t_2 = 2\varepsilon(t_1/p_1^2 - p_2.\partial\theta)$$

$$\nabla^2 \theta = (p_1.\partial t_2 + p_2.\partial t_1 + t_1 + t_2 - 2u)/p_1^2 p_2^2 \qquad (3.44a)$$

and from (3.18) it follows that

$$(\nabla^2 + 2\varepsilon p_1 . p_2/p_1^2 p_2^2)u = \frac{\varepsilon}{p_1^2 p_2^2}\Big[-p_1^2(p_2.\partial+1)t_2$$

$$-p_2^2(p_1.\partial+1)t_1 + p_1.p_2(p_1.\partial+1)t_2 + p_1.p_2(p_2.\partial+1)t_1\Big] . \qquad (3.44b)$$

In terms of the variables x, y and z defined by (3.23) and (3.26), the following forms can be assumed :

$$t_1 = m(e+zf)\left(\frac{x}{k_p^2}\right)^\beta$$

$$t_2 = m(e+z^{-1}f)\left(\frac{x}{k_p^2}\right)^\beta$$

$$\theta = mg\left(\frac{x}{k_p^2}\right)^\beta / x$$

and

$$u = mh\left(\frac{x}{k_p^2}\right)^\beta$$

where e, f, g and h are functions only of y and m is the electron mass included to keep the dimensions correct.

By a similar procedure to the above, it results from (3.44) that the new variables satisfy the equations

$$4(\beta+1)(D-y)f = 2\varepsilon\{e - (D-y)g\} \qquad (3.45)$$

$$De + (\beta+2)f = j$$

where

$$j = (\beta+1)f + \delta\{f - (\beta-1)g\} \qquad (3.46)$$

$$4\beta(D-y)g = 2\{yj + (D-y)f + (\beta+1)e - h\} \tag{3.47}$$

and

$$2(\beta+1)Dh + \varepsilon yh = \varepsilon(y^2-1)j \ . \tag{3.48}$$

The important difference between this set of equations and the comparable set (3.33), (3.34), (3.36) and (3.37) is that in the so-called 'outer' asymptotic region where $(p.k)^2$ is large compared with p^2k^2, the functions h and j satisfy the coupled equations (3.48) and

$$\{2(\beta+1)(D-y) + \varepsilon y\}j = \varepsilon h$$

which have, by the method discussed in §3.2, the two independent solutions

$$h_1(y) = (y^2-1)^{\frac{1}{2}(\beta-\frac{1}{2}+\delta)} Q^{\frac{1}{2}}{}_{\delta-\frac{1}{2}}(y)$$

and

$$h_2(y) = (y^2-1)^{\frac{1}{2}(\beta-\frac{1}{2}+\delta)} Q^{\frac{1}{2}}{}_{-\delta-\frac{1}{2}}(y) \tag{3.49}$$

with similar results for j (in this context $\delta = \varepsilon/2(\beta+1)$).

It is sufficient at this stage to note that the solution of the remaining two equations (3.45) and (3.46) can be found from (3.49) by standard techniques; it is clear in any case that the result will again be a combination of associated Legendre functions.

Further discussion of these two solutions will be made in the final chapter, where it will be shown that the details of the solution have a bearing on the convergence of the vacuum polarisation within this model.

3.4 THE PERTURBATION SOLUTION.

When equation (3.4) for the Møller scattering amplitude was formed, the first order term corresponding to

(3.50)

was lost during the differentiation.

Its effect on the solution must therefore be recorded by the boundary conditions applied to (3.4); the method used here of determining these is to iterate equations (3.15) - (3.18) using the first order substitution of (3.50), *i.e.*

$$\Sigma'_A = \gamma_a^\lambda D_{\lambda\mu}(k_p)\gamma_b^{\mu t}\gamma' \tag{3.51}$$

and then to match up this solution in the asymptotic region with the one obtained in §§3.2 - 3.3.

The Landau term $k_\lambda k_\mu / k^2$ in equation (1.21) for the photon propagator introduces a numerical factor (3/4) into (3.51); for convenience the factor is taken as 1 so that the components of Σ'_A are

$$\sigma = -\gamma'/k_p^2$$

$$\sigma_\lambda = -\gamma'\gamma_\lambda/k_p^2 \tag{3.52a}$$

$$\sigma_{\lambda\mu} = -\gamma'\gamma_{\lambda\mu}/k_p^2$$

and

$$\sigma_{\lambda\mu\nu} = -\gamma'\gamma_{\lambda\mu\nu}/k_p^2 \;. \qquad (3.52b)$$

To remove these remaining Dirac matrices, it is necessary to define components in a similar manner to (3.10) of σ, σ_λ *etc.* Thus I set, including the γ' for convenience

$$\sigma_{;} = \tfrac{1}{4}\mathrm{tr}\gamma'\sigma$$

$$\sigma_{;\lambda} = \tfrac{1}{4}\mathrm{tr}\gamma'\sigma\gamma_\lambda$$

$$\sigma_{\lambda;} = \tfrac{1}{4}\mathrm{tr}\gamma'\sigma_\lambda$$

$$\sigma_{\lambda;\mu} = \tfrac{1}{4}\mathrm{tr}\gamma'\sigma_\lambda\gamma_\mu$$

and so on.

Making use of the first order substitutions (3.52) it then results that the following 16 equations are satisfied by the components of Σ_A' :

(a) $\nabla^2\sigma_{;} = 3\varepsilon p_1.p_2/k_p^2 D$

(b) $\nabla^2\sigma_{\lambda;} = 0$

(c) $\nabla^2\sigma_{\lambda\mu;} - 2\partial_\mu\partial^\nu\sigma_{\lambda\nu;} + 2\partial_\lambda\partial^\nu\sigma_{\mu\nu;} = -\varepsilon p_{-\lambda\mu}/k_p^2 D$

(d) $\nabla^2 \partial^\nu \sigma_{\lambda\mu\nu;} = 0$

(e) $\nabla^2 \sigma_{;\lambda} = 0$

(f) $\nabla^2 \sigma_{\lambda;\mu} = -\varepsilon\left[\frac{2}{D}(-p_1 . p_2 g_{\lambda\mu} + p_{+\lambda\mu}) - 2\partial_\lambda \theta_\mu\right]/k_p^2$

where

$$\nabla^2 \theta_\mu = 4p_\mu/D$$

(g) $\nabla^2 \sigma_{\lambda\mu;\nu} = 0$

(h) $\nabla^2 \partial^\nu \sigma_{\lambda\mu\nu;\rho} = 0$

(i) $\nabla^2 \sigma_{;\lambda\mu} = -3\varepsilon p_{-\mu\lambda}/k_p^2 D$

(j) $\nabla^2 \sigma_{\lambda;\mu\nu} = 0$

(k) $\nabla^2 \sigma_{\lambda\mu;\nu\rho} - 2\partial_\mu \partial^\alpha \sigma_{\lambda\alpha;\nu\rho} + 2\partial_\lambda \partial^\alpha \sigma_{\mu\alpha;\nu\rho}$

$$= \varepsilon\left[p_1 . p_2 (g_{\lambda\rho} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\rho}) + p_{+\lambda}^\alpha (g_{\mu\rho} g_{\alpha\nu} - g_{\mu\nu} g_{\alpha\rho}) - p_{+\mu}^\alpha (g_{\lambda\rho} g_{\alpha\nu} - g_{\lambda\nu} g_{\alpha\rho})\right]/k_p^2 D$$

(ℓ) $\nabla^2 \partial^\nu \sigma_{\lambda\mu\nu;\rho\alpha} = 0$

(m) $\nabla^2 \sigma_{;\lambda\mu\nu} = 0$

(n) $\nabla^2 \sigma_{\lambda;\mu\nu\rho} = -2\varepsilon p_1^\alpha p_2^\beta g_{\lambda\alpha\beta;\mu\nu\rho}/k_p^2 D$

where $g_{\lambda\alpha\beta;\mu\nu\rho}$ is the tensor obtained by antisymmetrising

$$g_{\lambda\mu}g_{\alpha\rho}g_{\beta\nu}$$

with respect to $(\lambda\alpha\beta)$ and $(\mu\nu\rho)$.

(o) $$\nabla^2\sigma_{\lambda\mu;\nu\rho\alpha} = 0$$

(p) $$\nabla^2\partial^\nu\sigma_{\lambda\mu\nu;\rho\alpha\beta} = -4\varepsilon\Big[p_1^\nu p_2^\gamma(p_{1\lambda}g_{\mu\nu\gamma;\rho\alpha\beta} - p_{1\mu}g_{\lambda\nu\gamma;\rho\alpha\beta})/p_1^2$$

$$+ p_1^\nu p_2^\gamma(p_{2\lambda}g_{\mu\nu\gamma;\rho\alpha\beta} - p_{2\mu}g_{\lambda\nu\gamma;\rho\alpha\beta})/p_2^2\Big]/k_p^2 D$$

(3.53a-p)

I intend here to solve only for the functions $\sigma_;$, $\sigma_{\lambda;}$, $\sigma_{\lambda\mu;}$ and $\sigma_{\lambda\mu\nu;}$ which represent a particular polarisation state of the incident beam of electrons (the spinor affixes corresponding to these particles are t and u, and these are the affixes which label the components of the matrices σ, σ_λ *etc.*). These are given by equations (3.53a-d).

The solutions of the two non-trivial equations in this set may be found by noting that if $x_a^2 = (x+a)^2$ and $x = p^2$, then

$$\nabla^2(x_a^{-1}F(x_a)) = 4F''(x_a)$$

$$= G(x_a) \quad \text{say}$$

so that

$$x_a^{-1}F(x_a) = \tfrac{1}{4}x_a^{-1}\int_0^{x_a}dx_a\int_0^{x_a}dx_a G(x_a) \qquad (3.54)$$

where 0 and x_a are the required bounds of integration if $x_a^- F(x_a)$ is to be non-singular at the origin.

In the region where perturbation theory is applicable, the approximation

$$D \to p_1^2 p_2^2$$

is no longer valid; instead the renormalised theory must be used so that

$$D = (p_1^2 - m^2)(p_2^2 - m^2) .$$

Then the following three results are useful :

Firstly, if

$$\nabla^2 \phi_1(x) = 1/(x - m^2)$$

then from (3.54)

$$\phi_1(x) = \tfrac{1}{4}(1 - m^2/x)\log(1 - x/m^2) . \qquad (3.55)$$

Secondly,

$$(p_1^2 - m^2)^{-1}(p_2^2 - m^2)^{-1} = \tfrac{1}{2}\int_{-1}^{1} d\lambda/(x_\lambda - \mu_\lambda^2)^2 \qquad (3.56)$$

where

$$x_\lambda = (p + \lambda k)^2$$

and

$$\mu_\lambda^2 = m^2 - (1 - \lambda^2)k^2$$

Thirdly, if

$$\nabla^2\phi_2(x) \;=\; \int_{-1}^{1} d\lambda/(x_\lambda-\mu_\lambda^2)$$

then by (3.54)

$$\phi_2(x) \;=\; \tfrac{1}{4}\int_{-1}^{1} d\lambda(1-\mu_\lambda^2/x_\lambda)\log(1-x_\lambda/\mu_\lambda^2) \;. \qquad (3.57)$$

Hence from (3.53a)

$$\nabla^2\sigma_; \;=\; 3\varepsilon\left[(p_1^2-m^2)^{-1} + (p_2^2-m^2)^{-1} - 4k^2/D\right]/2k_p^2$$

- while the last term in this may be integrated with the help of (3.56), its contribution in the asymptotic region is negligible so that in that region

$$\sigma_; \;=\; -(1 - \tfrac{3\varepsilon}{8}\log(p_1^2p_2^2/m^4) + \ldots)/k_p^2 \qquad (3.58)$$

using (3.55) and the first order term given by (3.52).

On the other hand, the first order substitution for $\sigma_{\lambda\mu;}$ vanishes, so that to second order

$$\nabla^2\sigma_{\lambda\mu;} \;=\; -\varepsilon p_{-\lambda\mu}/k_p^2D$$

$$\;=\; 2\varepsilon(\tilde{p}_\lambda k_\mu - \tilde{p}_\mu k_\lambda)/k_p^2D$$

where

$$\tilde{p}_\lambda \;=\; p_\lambda - p.kk_\lambda/k^2 \qquad (3.59)$$

is the transverse component of p_λ.

Bearing in mind (3.56),

$$\tilde{p}_\lambda/D = -\tfrac{1}{4}\tilde{\partial}_\lambda \int_{-1}^{1} d\lambda/(x_\lambda-\mu_\lambda^2)$$

so that by (3.57)

$$\sigma_{\lambda\mu;} = \frac{\varepsilon}{8k_p^2}p_{-\lambda\mu}\int_{-1}^{1} d\lambda\left[\mu_\lambda^2\log(1-x_\lambda/\mu_\lambda^2)/x_\lambda + 1\right]/x_\lambda$$

$$\simeq \frac{\varepsilon}{4k_p^2}p_{-\lambda\mu}\log(p_1^2/p_2^2)/(p_1^2-p_2^2) \qquad (3.60)$$

when $p_1^2, p_2^2 >> m^2$.

Finally, the first order substitutions imply from (3.53b) and (3.53d) that

$$\sigma_{\lambda;} = \sigma_{\lambda\mu\nu;} = 0 .$$

These are the results I hope to use in determining exactly the asymptotic behaviour of the four functions $\sigma_{;}$, $\sigma_{\lambda;}$, $\sigma_{\lambda\mu;}$ and $\sigma_{\lambda\mu\nu;}$.

3.5 JOINING THE TWO SOLUTIONS.

The perturbation solution just obtained is clearly a divergent sum for large values of p_1^2 and p_2^2. It remains to be seen at what point the power series expansion becomes invalid and a more accurate solution, such as the non-perturbative one already described, takes over.

In this section I will join up these two solutions in the 'inner' asymptotic region, where p_1^2 and p_2^2 are large but close to each other : thus x is large but y and z are both close to 1.

To make the join, note that in §§3.2 and 3.3, the general solution was obtained for the component matrices σ, σ_λ *etc.* Since the equations they satisfy do not involve any Dirac matrices, the components $\sigma_;$, $\sigma_{;\lambda}$, $\sigma_{\lambda;}$ *etc.* have the same general solution as the corresponding matrix.

According then to (3.29),

$$\sigma_; \quad = \quad a(y)\left(\frac{x}{k_p^2}\right)^\alpha / k_p^2$$

which can be written as a power series in α,

$$\sigma_; \quad = \quad a(y)(1 + \alpha\log(x/k_p^2) + \ldots)/k_p^2 \; .$$

Clearly this joins very well with (3.58), since in the inner region $k_p^2 \simeq m^2$, provided

$$\alpha = -3\varepsilon/4 + o(\varepsilon^2)$$

$$\simeq -3\delta/2$$

and

$$a(y) = -1 + o(\varepsilon^2) \ . \qquad (3.61)$$

Fortunately these conditions are consistent : by letting the constant in (3.42) vanish, the solution (3.43) for a(y) can be expressed in terms of the hypergeometric function[23] $F(a,b;c;z^{-2})$ to show that near $y = 1$,

$$a_1(y) \sim K(y^2-1)^{-\delta/2} z^{-1-\delta/2} \qquad (3.62)$$

where K is a constant. Because of the analytic properties of F, the second solution doesn't have this desirable behaviour.

Another independent solution may however be found by considering the symmetry of the perturbation solution under exchanges of p_1^2 and p_2^2: both the non-zero components $\sigma_{;}$ and $\sigma_{\lambda\mu;}$ are unchanged by such transformations and it is clear that the final solution must have the same property - however $a_1(y)$ is not suitable in this respect since the Legendre functions have a branch point at $y = 1$.

Thus an appropriate function $a_1'(y)$ is defined by expressing $a_1(y)$ in terms of $F(a,b;c;z^{-2})$ and then replacing z by z^{-1}; then a(y) must be taken to be the sum of a_1 and a_1'. It is clear that a_1' must also be a solution of (3.39) because of the symmetry in that equation between z and z^{-1}.

It remains to check that with this choice of solution, the remaining components $\sigma_{\lambda\mu}$; join in the inner region.

From (3.30), the dominant term in the function d corresponding to a is

$$d(y) = \frac{2K}{3}(y^2-1)^{\frac{1}{2}-\delta/2}.\tfrac{1}{4}\Big\{z^{3\delta/2}F(-3\delta/2,-\delta/2;1-\delta;z^{-2})$$

$$- z^{-3\delta/2}F(-3\delta/2,-\delta/2;1-\delta;z^2)\Big\}$$

and again near y = 1 this can be approximated by

$$d(y) \sim \frac{K}{2}\sigma(y^2-1)^{\frac{1}{2}-\delta/2}\log z .$$

From (3.61) and (3.62) it follows that K = -1, and this solution is clearly consistent with the perturbation expansion

$$s_3 = -\sigma x(y^2-1)^{\frac{1}{2}}\log z/2k_p^2$$

obtained from (3.60).

The remaining components b+yc and c+yb can be found from (3.36) and (3.38), and it is easily checked that they too have the appropriate behaviour near y = 1 to be consistent with the perturbation solution.

CHAPTER 4

SOME CONCLUSIONS.

4.1 THE NON-PERTURBATIVE THEORY.

The asymptotic solution found in chapter three is of particular interest for two reasons :

Firstly, it indicates the region where the perturbation theory is valid and works well; for the polarisation state I have considered this region is enormous, since a match has been obtained right into the inner asymptotic region where both p_1^2 and p_2^2 are much larger than m^2 but p.k is small.

Secondly, although Dyson has shown[3] not only that the perturbation series has divergent coefficients but that the series itself is divergent for large values of the incident momenta, the solution given here does not have this property :

As z becomes large, both a_1 and a_1' become small, since for large z

$$a_1 \sim (z-z^{-1})^{-\delta/2} z^{-1-\delta/2}$$

and

$$a_1' \sim (z-z^{-1})^{-1-\delta/2} z^{-1-\delta/2}$$

and so for large x and z,

$$\sigma_{;} \to 0$$

- similarly the remaining scalars become small, so

$$\sigma_{\lambda\mu;} \to 0$$

and this lends to the theory (according at least to the solutions considered) the property of asymptotic freedom, *i.e.* for large incident momenta the particles behave as though the coupling constant were zero. It must be remembered here that because of the definition (2.13) of the reduced amplitude, a factor ε^2 is included in the functions used - the first order term (3.50) contains two vertices.

Moreover, the other components of this polarisation state, *viz.* $\sigma_{\lambda;}$ and $\sigma_{\lambda\mu\nu;}$ have not been found exactly according to the general solution (3.49) since for these functions the second order perturbation solution was zero and therefore they remain small, provided they are convergent, compared with $\sigma_{;}$ and $\sigma_{\lambda\mu;}$. Thus this doesn't affect the issue of asymptotic freedom since all the functions of (3.49) and the remaining scalars have the desired asymptotic behaviour.

It was mentioned in chapter one that because the bare mass is set to zero there is no mass scale incorporated in the theory. Nonetheless it is known[24] that theories which exhibit asymptotic freedom define a mass renormalisation group in the manner of Weinberg[25].

Thus in spite of the non-determinism of equation (1.27) for the electron mass there may be room for its determination by other methods[26].

This is particularly interesting in the light of Coleman's and Gross' work[27] on asymptotically free theories : they find that the only asymptotically free theories are those with non-Abelian gauge groups - the present work is an apparent contradiction of this. In fact of course they assumed the asymptotic behaviour of Quantum Electrodynamics as found by perturbation theory, and this is clearly an incorrect result.

4.2 VACUUM POLARISATION AND CHARGE RENORMALISATION.

The aspect of the theory of quantum electrodynamics not yet covered by this work is that of the other renormalisation constant, y. In fact Källén has shown[28] that at least one of the constants y and z must be infinite; however his proof excludes explicitly the use of the Landau-Maxwell gauge taken here. The most important result in this respect of Green's work on the vertex function is his solution for the second set of asymptotic equations, *i.e.* the solution presented in §3.3 :

The remaining functions f and g of that section which for the vertex function are the scalars associated with

$$C^{\lambda}_{\mu} = \tfrac{1}{4}\mathrm{tr}\Gamma^{\lambda}\gamma_{\mu}$$

can be shown to have the solution

$$K\delta f = 4(y^2-1)^{\frac{1}{2}(\beta+\delta-2)}\left[b_1 z^{\delta} + b_2 z^{-\delta} + \int_{-1}^{1} d\lambda F(\lambda) z_{\lambda}^{-\delta}\right]$$

and

$$(\beta-1)g = 4(y^2-1)^{\frac{1}{2}(\beta+\delta-2)} \int_{-1}^{1} d\lambda F(\lambda) z_{\lambda}^{-\delta-1}$$

where

$$K = -1 - 2\beta/\delta$$

$$z_{\lambda} = y + \lambda(y^2-1)^{\frac{1}{2}}$$

β-1 is the power of x in the solution, and $F(\lambda)$ can be expressed in terms of Legendre functions.

Clearly the value

$$\delta \simeq \varepsilon/2$$

obtained in §3.5 results in a solution f which is inversely proportional to the fine structure constant - it is difficult to see how this might join with the perturbation solution; however Green found that while b_1 and b_2 were constants of integration in the outer asymptotic region (where the solution is valid), they could be taken to have the form

$$-(p.k)^2\tilde{p}^2k^2$$

where $\tilde{p}^\lambda$ is the transverse component of p^λ defined in (3.59). This function is constant where p.k is large but reduces to

$$-x(y^2-1)/4yk^2$$

in the inner asymptotic region.

Thus a consistent match with perturbation theory for the vertex function was obtained by setting

$$\beta \simeq -1 - \tfrac{1}{2}\varepsilon$$

so that

$$\delta \simeq -1 + \tfrac{1}{2}\varepsilon.$$

With this matching condition, the vacuum polarisation integral, which according to Valatin[29] and Schwinger[30] is

$$\Pi^{\mu}{}_{\lambda}(k) = \frac{ie^2}{(2\pi)^4}\int \mathrm{tr}\Big[(\gamma_\lambda - \partial_\lambda \nabla^{-2}\nabla)S(p+k)\Gamma^{\mu}(p+k,p-k)$$

$$\times S(p-k)\Big]d^4k \qquad (4.1)$$

becomes convergent :

Defining $\eta^{\mu}{}_{\lambda}$ in terms of $C^{\mu}{}_{\lambda}$ *etc.* by analogy with (3.13), (4.1) becomes

$$\Pi^{\mu}{}_{\lambda}(k) = \frac{4ie^2}{(2\pi)^4}\int(\eta^{\mu}{}_{\lambda} - \partial_\lambda\theta^{\mu})d^4p$$

$$= \frac{4ie^2}{(2\pi)^4}\int \nabla^2\eta^{\mu}{}_{\lambda}d^4p/2\varepsilon$$

and since the power of x in $\eta^{\mu}{}_{\lambda}$ is $-2-\frac{1}{2}\varepsilon$, this is convergent (more detail concerning this can be found in Green's 1982 paper).

Thus it appears that by taking a more accurate form for the asymptotic behaviour of the vertex function (and hence of the Møller scattering amplitude) a convergent result is found for the vacuum polarisation integral and the contribution of this term to the charge renormalisation is therefore finite; it must of course be remembered that this solution follows from adopting the ladder approximation and it is not known how this solution takes into account the eigenvalue condition on the fine structure constant mentioned in the introduction. It is possible that when more terms are included in the sum (3.5) the asymptotic behaviour of these functions may be significantly altered.

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