The Killing operator on locally homogeneous spaces

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Abstract

In this thesis we study a compatibility complex, derived form the Calabi complex, providing conditions for a symmetric 2-tensor on pseudo-Riemannian locally homogeneous space to be in the image of the Killing operator. In the first chapter, we describe general machinery to study the first cohomology group of the twisted de Rham complex of an arbitrary vector bundle with connection. This machinery will be applied in the last two chapters to the Killing bundle and the Killing connection, a vector bundle with connection that arises from a prolongation of the Killing equation.

In the second chapter we introduce the Killing bundle and the Killing connection, that provides an overdetermined system of linear partial differential equations for the Killing equation. We prove a theorem analogous to Hano's theorem on the splitting of the Lie algebra of Killing vector fields of a product Riemannian manifold [26], to arbitrary signature. Moreover, we study the structure of special subbundles of the Killing bundle and apply these results in Chapter 3 to provide a characterisation of pseudo-Riemannian locally homogeneous spaces in terms of the maximal parallel flat subbundle of the Killing bundle and to give new proof of the Ambrose-Singer theorem regarding homogeneous structures [3].

In the fourth chapter we construct a compatibility complex for the Killing operator, that arises from a modification of the Calabi complex, and establish its equivalence to the short twisted de Rham complex of the Killing connection. We make use of this equivalence to provide a characterisation of the image of the Killing operator on pseudo-Hermitian spaces of constant holomorphic sectional curvature by showing that the first twisted de Rham cohomology group are locally trivial. Even more, we provide several tools to study the first twisted de Rham cohomology group on product spaces. The last chapter is dedicated to Lorentzian locally symmetric spaces and locally homogeneous plane waves. We prove results on the Singer index of locally homogeneous plane waves and determine exactly which ones have Singer index equal to 0. We make use of this fact to show that the first twisted de Rham cohomology group of the Killing connections of locally homogeneous plane waves with Singer index 0 is locally trivial. Lastly, we provide a complete characterisation of the image of the Killing operator on Lorentzian locally symmetric spaces, showing in which cases first twisted de Rham cohomology group of the Killing connection is locally trivial and in which ones it is not.

Introduction

The concept of symmetry has been of great importance in mathematics and physics since their early beginnings. From a differential geometric point of view, they arise as a group of diffeomorphisms of a smooth manifold that leave invariant the geometric structure of interest. Particularly, in this thesis, the geometric structures in question will be pseudo-Riemannian metrics on smooth manifolds, and their "continuous symmetries" will be Lie groups of isometries. Their linear, or rather infinitesimal, counterpart are the *Killing vector fields*. They are given by Lie algebras comprised of vector fields whose local flow give rise to continuous symmetries of the metric.

Formally, a pseudo-Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a non-degenerate symmetric tensor on M, and its infinitesimal symmetries will be a vector fields in the kernel of the following first order linear differential operator:

$$\mathcal{K} : \Gamma(TM) \to \Gamma(\mathrm{Sym}^2 M), \quad \xi \mapsto L_\xi g, \tag{1}$$

where TM denotes the tangent bundle of M, $\mathrm{Sym}^2 M$ the bundle of symmetric 2-tensors on M and L_ξ the Lie derivative in the direction of the vector field ξ. Perhaps it is not evident at first sight that the operator defined above is indeed a linear differential operator on vector fields, hence we refer to Section 2.1 for a clarification.

The differential operator defined in equation (1) will be the object of study in this work, the so called *Killing operator*. Particularly, we will study differential conditions on pseudo-Riemannian manifolds with a "large amount" of Killing vector fields, for a symmetric 2-tensor field to be in the image of the Killing operator. The conceptual idea of "large amount" of infinitesimal symmetries on pseudo-Riemannian manifold is formalised by the notion of *locally homogeneity*. That is, a pseudo-Riemannian manifold is locally homogeneous if for each point in M, there exists an open neighbourhood U, with a local frame of the tangent bundle comprised of Killing vector fields in U.

On pseudo-Riemannian locally homogeneous spaces, the kernel of the Killing operator is well understood. However, we can wonder what could be said regarding its image. The image of the Killing operator has been studied by physicists in the context of *linearised gravity*. When analysing the linearised vacuum Einstein field equations for a metric tensor arising from a small perturbation of a "background metric", a tensor field that is in the image of the Killing operator can be regarded as an infinitesimal change in the background metric, due to an infinitesimal change of coordinates. Any two solutions of the linearised Einstein field equations arising from a perturbation of the background metric will be equivalent up to *gauge* if they differ by a tensor field that is in the image of the Killing operator. Thus tensor fields in the image of the Killing operator provide a set of local gauge transformations for the linearised Einstein equations (see [40, Section 5.7] or [43, Section 7.5] for more details).

In order to provide a complete characterisation of the image of the Killing operator, we need to find necessary and sufficient conditions for the inhomogeneous partial differential equation

$$\mathcal{K}(\xi) = h, \quad \text{for some} \quad h \in \Gamma(\mathrm{Sym}^2 M), \tag{2}$$

to have a solution. The standard approach to provide necessary conditions for the existence of solutions to equation (2) is finding a suitable *compatibility operator* $\mathcal{C}$, acting on symmetric 2-tensors, such that the sequence of differential operators

$$\Gamma(TM) \xrightarrow{\mathcal{K}} \Gamma(\mathrm{Sym}^2 M) \xrightarrow{\mathcal{C}} \Gamma(CM) \tag{3}$$

is in fact a differential complex, where CM is an appropriate vector bundle over M where $\mathcal{C}$ takes values. If such a complex exists, it will be referred to as a *compatibility complex* for the Killing operator. Therefore, any compatibility complex will provide us with necessary conditions for the inhomogeneous equation (2) to admit a solution and, moreover, these conditions will be sufficient if and only if the complex (3) is locally exact. In other words, the image of the Killing operator would be completely characterised by complex (3) when it is locally exact.

A first instance of a compatibility complex for the Killing operator is due to Barré de Saint Venant [4]. In 1864, studying conditions for a strain to arise from a linear displacement (within the context of linear deformations of solids), he introduced a second order linear differential operator providing necessary and sufficient conditions for a symmetric 2-tensor in Euclidean three-space to be in the image of the Killing operator. The result of Saint-Venant can then be stated as the complex

$$\Gamma(T\mathbb{R}^3) \xrightarrow{\mathcal{K}} \Gamma(\mathrm{Sym}^2\mathbb{R}^3) \xrightarrow{\mathcal{C}} \Gamma(\wedge^2\mathbb{R}^3 \otimes \wedge^2\mathbb{R}^3)$$

begin locally exact, where $\mathcal{C}$ denotes the differential operator given by the formula

$$(\mathcal{C}h)_{abcd} = \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac}.$$

Here, we have employed *Penrose abstract indices* notation. We refer to [40] or Chapter 4 for more details on this notation.

Later on, Eugenio Calabi introduced in his article [10], the complex of linear differential operators

$$\Gamma(TM) \xrightarrow{\mathcal{K}} \Gamma(\mathrm{Sym}^2 M) \xrightarrow{\mathcal{C}} \Gamma(RM) \longrightarrow \dots \tag{4}$$

where RM denotes the subbundle of $\wedge^2 M \otimes \wedge^2 M$, of tensors with the same symmetries of the Riemannian curvature tensor. The compatibility operator for the Killing operator in Calabi's complex, the *Calabi operator*, is the second order linear differential operator given by

$$(\mathcal{C}h)_{abcd} = \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} - R_{ab}{}^{e}{}_{[c}h_{d]e} - R_{cd}{}^{e}{}_{[a}h_{b]e},$$

where R denotes the curvature tensor of the Levi-Civita connection associated to g. We note that if the manifold in question is flat, the Calabi operator coincides with Saint-Venant's compatibility operator. In [10], Calabi proved that on Riemannian manifolds with constant sectional curvature, complex (4) is locally exact. In other words, on a Riemannian manifold with constant sectional curvature, the image of the Killing operator is completely characterised as the kernel of the Calabi operator.

Further results were obtained by Gasqui and Goldschmidt, on Riemannian locally symmetric spaces. In [23], they provided a locally exact compatibility complex in terms of a third order compatibility operator. These results have been recently improved in [12], by means of a locally exact compatibility complex defined by a second order linear operator that arises from the Calabi operator. This operator has also been studied in Lorentzian signature in [13], where analogous results to those of [12] have been obtain for Lorentzian locally symmetric spaces. In the context of linearised gravity, compatibility complexes for the Killing operator have been recently obtained for several Lorentzian manifolds of physical interest in [1, 31] and, in a more general setting, a compatibility operator derived from the Calabi operator is presented in [19].

Lastly, we would like to remark that the Killing operator viewed as a differential operator acting on differential 1-forms, instead of vector fields, has a hidden symmetry built into it. Under a suitable interpretation, the Killing operator is *projectively invariant* [18]. Noting that a Riemannian manifold is projectively flat if and only if it has constant sectional curvature, by Beltrami's Theorem [5], the Calabi complex can be seen as a Bernstein-Gelfand-Gelfand complex in flat projective differential geometry [16, 20].

In this thesis we will study a compatibility complex for the Killing operator on pseudo-Riemannian locally homogeneous spaces, that arises from a modification of the Calabi operator. Moreover, we will establish an equivalence between this compatibility complex and the twisted de Rham complex of a vector bundle with connection, which are derived from a prolongation of the Killing equation to an overdetermined system of partial differential equations. The first twisted de Rham cohomology group will be computed in several pseudo-Riemannian locally homogeneous spaces, providing necessary and sufficient conditions for a symmetric 2-tensor to be in the image of the Killing operator.

This thesis is organised as follows: In Chapter 1 we will describe the general machinery to be applied in Chapters 4 and 5 to study the image of the Killing operator. Specifically, we will provide conditions and tests to determine the image of a connection D, on an arbitrary smooth vector bundle E. Particularly, they will be described in terms of the first cohomology group of the twisted de Rham complex of (E, D).

In Chapter 2, we will introduce a vector bundle with connection which will provide us with an overdetermined system of partial differential equations for Killing vector fields. This bundle will be referred to as the Killing bundle, and its parallel sections will be in one to one correspondence with Killing vector fields. Moreover, we will establish its relation with the Killing operator and provide a description of special subbundles defined in terms of tensor fields on the base manifold, with particular emphasis on the kernel of its curvature. As an application, we present an extension of a theorem from Hano [26] on the decomposition of the Lie algebra of Killing vector fields on product spaces.

Chapter 3 will be dedicated to study pseudo-Riemannian locally homogeneous spaces and their relation with the Killing bundle and the Killing connection. Locally homogeneous spaces will be characterised in terms of the Killing bundle and a new proof of the famous theorem of Ambrose and Singer on homogeneous structures [3], by means of the results previously described in Chapter 2.

In Chapter 4 we will describe the Calabi complex for locally homogeneous spaces, and derive a compatibility complex for the Killing operator, by means of a modification of the Calabi operator. We will prove the equivalence between this complex and the twisted de Rham complex of the Killing bundle and study its local exactness on product spaces, given previous knowledge of the local exactness on each individual factor.

Lastly, Chapter 5 will be dedicated to Lorentzian symmetric spaces and locally homogeneous plane waves. Particularly, we will provide necessary and sufficient conditions for a symmetric 2-tensor on a class of locally homogeneous plane waves to be in the image of the Killing operator. Furthermore, we will make use of these results to characterise the image of the Killing operator on any Lorentzian locally symmetric space.

The author would like to remark that the majority of the results presented in this work have been obtained in collaboration with Micheal Eastwood, Thomas Leistner and Benjamin McMillan, in [12, 13]. A clarification of the author's contributions will be made clear at the beginning of each chapter before their presentation.

Chapter 1

The image of a connection on a vector bundle

In this chapter we will give a presentation of the necessary machinery applied in Chapters 4 and 5 to characterise the image of the Killing operator on certain pseudo-Riemannian locally homogeneous spaces. To this end, we will describe recent results obtained in [13], on the image of a connection on a vector bundle. This chapter is intended as a background chapter. However, Proposition 1.2.5 is a contribution from the author.

Letting E be a vector bundle, over a smooth manifold M, and D be a connection on E, we will aim to give a solution to the following problem: For a given differential 1-form ϕ with values on E, find necessary and sufficient conditions for the differential equation

$$D\eta = \phi$$

to have solutions. A system of overdetermined linear partial differential equations can be encoded in the equation for parallel sections of a vector bundle with connection (E, D) (see [7, 17]). The problem of finding necessary and sufficient conditions for a differentiaI 1-form with values in E to be in the image of D, will then be its inhomogeneous counterpart.

Particularly, in this thesis, the interest in this problem will become apparent in Section 2, when we introduce a vector bundle with connection, defining an overdetermined system of first order linear partial differential equations for Killing vector fields.

1.1 The twisted de Rham complex

Fixing notation, throughout this chapter, $E \to M$ will denote a smooth vector bundle E, over a smooth manifold M, and $D : \Gamma(E) \to \Gamma(\wedge^1 M \otimes E)$ will denote a connection on E. The curvature of D will be denoted by κ and it will be considered a section of $\wedge^2 M \otimes \mathrm{End}(E)$. Our convention for its definition will be

$$\kappa(X, Y)\eta = D_X D_Y \eta - D_Y D_X \eta - D_{[X,Y]}\eta,$$

with $X, Y \in \Gamma(TM)$ and $\eta \in \Gamma(E)$. We will usually consider the curvature also as the vector bundle homomorphism $\kappa : \wedge^k M \otimes E \to \wedge^2 M \otimes \wedge^k M \otimes E$ defined by

$$\kappa(\omega)(X, Y, X_1, \ldots, X_k) := \kappa(X, Y)\omega(X_1, \ldots, X_k).$$

The *exterior covariant derivative* will be the natural differential operator, induced by D, acting on E-valued differential forms

$$\mathrm{d}_D : \Gamma(\wedge^k M \otimes E) \to \Gamma(\wedge^{k+1} M \otimes E),$$

which generalises the usual exterior derivative on the differential k-forms, $\Gamma(\wedge^k M)$. It is invariantly defined by the formula

$$\begin{aligned}(\mathrm{d}_D\omega)(X_0, X_1, \ldots, X_k) \;=\; & \sum_{i=0}^{k}(-1)^i D_{X_i}\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) \\ & + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),\end{aligned}$$

where $X_0, \ldots, X_k$ are vector fields on M, ω is differential k-form with values in E, and the hat indicates omission. The exterior covariant derivative satisfies the Leibniz rule:

$$\mathrm{d}_D(\theta \wedge \omega) = \mathrm{d}\theta \wedge \omega + (-1)^j \theta \wedge \mathrm{d}_D\omega,$$

where $\theta \in \Gamma(\wedge^j M)$ and $\omega \in \Gamma(\wedge^k M \otimes E)$. Here $\mathrm{d} : \Gamma(\wedge^k M) \to \Gamma(\wedge^{k+1} M)$ denotes the exterior derivative on differential forms. With the aid of an auxiliary affine connection on the tangent bundle of M, we can extend D in a natural manner to $T^*M^{\otimes^k} \otimes E$, by

$$(D_X\omega)(X_1, \ldots, X_k) = D_X(\omega(X_1, \ldots, X_k)) - \omega(\nabla_X X_1, \ldots, X_k) - \cdots - \omega(X_1, \ldots, \nabla_X X_k).$$

In addition, if ∇ is torsion-free, the exterior covariant derivative can be expressed succinctly as

$$\mathrm{d}_D\omega = (k+1)!\,\Lambda^{(k+1)}(D\omega) \quad \text{with} \quad \omega \in \Gamma(\wedge^k M \otimes E), \tag{1.1}$$

where $\Lambda^{(k)} : T^*M^{\otimes^k} \to \wedge^k M$ denotes the skew-symmetrisation map. To be precise, if α is in $T^*M^{\otimes^k}$, $\Lambda^{(k)}\alpha$ will denote the projection of α into $\wedge^k M$. In the following, ∇ will denote a torsion-free affine connection, unless otherwise stated.

The compositions $\mathrm{d}_D^2 : \Gamma(\wedge^k M \otimes E) \to \Gamma(\wedge^{k+2} M \otimes E)$ are always vector bundle homomorphisms, which will be denoted by κ^k. The homomorphism κ^k will be referred to as the *k-curvature homomorphism* of D. To corroborate that the k-curvature homomorphisms of D are indeed vector bundle homomorphisms, the first thing to notice is that for any section of $\wedge^k M \otimes E$, we have that

$$(\Lambda^{(2)} \otimes \Lambda^{(k)})(D^2\omega) = \frac{1}{2}\kappa(\omega) \tag{1.2}$$

and
$$\Lambda^{(k+2)}(D^2\omega) = \Lambda^{(k+2)}((\Lambda^{(2)} \otimes \Lambda^{(k)})(D^2\omega)) = \Lambda^{(k+2)}(D\Lambda^{(k+1)}(D\omega)). \tag{1.3}$$
On the other hand, we can observe from equation (1.1) that
$$\kappa^k(\omega) = (k+2)!(k+1)!\Lambda^{(k+2)}(D\Lambda^{(k+1)}(D\omega)) = (k+2)!(k+1)!\Lambda^{(k+2)}(D^2\omega).$$
Combining the above equation with equations (1.2) and (1.3), we obtain the following formula for the k-curvature of D:
$$\kappa^k(\omega) = \frac{(k+2)!(k+1)!}{2}\,(\Lambda^{(k+2)} \circ \kappa)(\omega). \tag{1.4}$$
That $\kappa^k : \wedge^k M \otimes E \to \wedge^{k+2} M \otimes E$ is a vector bundle homomorphism, follows directly from equation (1.4), since κ^k is nothing but a linear combination of terms involving κ.

The exterior covariant derivative defines the *twisted de Rham sequence*:
$$0 \longrightarrow \Gamma(E) \xrightarrow{\;D\;} \Gamma(\wedge^1 M \otimes E) \xrightarrow{\;\mathrm{d}_D\;} \Gamma(\wedge^2 M \otimes E) \xrightarrow{\;\mathrm{d}_D\;} \dots \tag{1.5}$$
It is well know that the twisted de Rham sequence is a complex if and only if D is a flat connection [37]. To be precise, we refer to a connection being flat if its curvature tensor vanishes identically. It can be easily spotted from equation (1.4) that $\kappa^k = 0$ for all $k \geq 0$, when D is a flat connection, and also that D being a flat connection is a necessary condition for the twisted de Rham sequence to be a complex, since κ^0 coincides with the curvature of D. Generally, it is too strong of a condition for a connection to be flat. For this reason, under milder assumptions on D, we will construct a complex arising from the twisted de Rham sequence.

For each point p of M and each $k \geq 0$, the map $\kappa^k_p : \wedge^k_p M \otimes E_p \to \wedge^{k+2}_p M \otimes E_p$ will denote the restriction of κ^k to the fibre of $\wedge^k M \otimes E$ at p.

Definition 1.1.1. Let D be a connection on a smooth vector bundle E, over a smooth manifold M. We will say that D is *regular* if for each $k \geq 0$, the k-curvature of D has constant rank.

The notion of regularity of a connection is motivated by the need of building subbundles of $\wedge^k M \otimes E$, defined in terms of the k-curvature homomorphisms of D. The k-curvatures of a regular connection have, by definition, constant rank. For this reason, we can define the vector bundles $\ker \kappa^k$ and $\operatorname{Im} \kappa^k$ over M, with fibres $\ker \kappa^k_p$ and $\operatorname{Im} \kappa^k_p$, at p, respectively.

In what follows, we will always be considering regular connections on E. For a given section ω of $\operatorname{Im} \kappa^k$ and any section $\eta \in \Gamma(\wedge^k M \otimes E)$ such that $\kappa^k(\eta) = \omega$, we can observe that
$$\mathrm{d}_D\omega = \mathrm{d}_D\kappa^k(\eta) = \mathrm{d}_D^3\eta = \kappa^{k+1}(\mathrm{d}_D\eta). \tag{1.6}$$

This means that the image of d_D on the sections of $\operatorname{Im}\kappa^k$ is always contained in $\Gamma(\operatorname{Im}\kappa^{k+1})$. It follows from this argument that

$$0 \longrightarrow \Gamma(\operatorname{Im}\kappa^0) \xrightarrow{\mathrm{d}_D} \Gamma(\operatorname{Im}\kappa^1) \xrightarrow{\mathrm{d}_D} \Gamma(\operatorname{Im}\kappa^2) \xrightarrow{\mathrm{d}_D} \dots \tag{1.7}$$

is a well defined subsequence of the twisted de Rham sequence. The exterior covariant derivatives descend well to the sequence

$$0 \longrightarrow \Gamma(E) \xrightarrow{D} \Gamma(\wedge^1 M \otimes E) \xrightarrow{\mathrm{d}} \Gamma((\wedge^2 M \otimes E)/\operatorname{Im}\kappa^0) \xrightarrow{\mathrm{d}} \dots \tag{1.8}$$

obtained by quotienting the twisted de Rham sequence by the subsequence (1.7). Here we have denoted by d, the maps induced by the exterior covariant derivatives in the quotients.

For completeness, we will verify explicitly that the operators

$$\mathrm{d} : \Gamma((\wedge^k M \otimes E)/\operatorname{Im}\kappa^{k-2}) \to \Gamma((\wedge^{k+1} M \otimes E)/\operatorname{Im}\kappa^{k-1})$$

are indeed well defined. Choosing a section $[\omega]$ of $(\wedge^{k+2} M \otimes E)/\operatorname{Im}\kappa^k$ and a representative $\omega + \kappa^k(\eta)$ of $[\omega]$ such that $\omega \in \Gamma(\wedge^{k+2} M \otimes E)$ and $\eta \in \Gamma(\wedge^k M \otimes E)$, we know that $\mathrm{d}_D \circ \kappa^k = \kappa^{k+1} \circ \mathrm{d}_D$, from equation (1.6). Thus

$$\mathrm{d}_D(\omega + \kappa^k(\eta)) = \mathrm{d}_D\omega + \kappa^{k+1}(\mathrm{d}_D\eta),$$

which is a representative of $[\mathrm{d}_D\omega]$ in $\Gamma((\wedge^{k+3} M \otimes E)/\operatorname{Im}\kappa^{k+1})$. Therefore the exterior covariant derivative operators descend well to the quotients. Even more, sequence (1.8) is a complex by construction, which will be referred to as the *twisted de Rham complex*. The k-th cohomology group of the twisted de Rham complex will be denoted by $H^k(E, D)$, which is given by

$$H^k(E, D) := \frac{\ker(\mathrm{d} : \Gamma((\wedge^k M \otimes E)/\operatorname{Im}\kappa^{k-2}) \to \Gamma((\wedge^{k+1} M \otimes E)/\operatorname{Im}\kappa^{k-1}))}{\operatorname{Im}(\mathrm{d} : \Gamma((\wedge^{k-1} M \otimes E)/\operatorname{Im}\kappa^{k-3}) \to \Gamma((\wedge^k M \otimes E)/\operatorname{Im}\kappa^{k-2}))}$$

Remark 1.1.2. The zeroth twisted de Rham cohomology group can be defined also when D is not a regular connection and it is comprised by the parallel sections of E. From now on, we will always denote the vector space of parallel sections E by $H^0(E, D)$.

1.2 The first twisted de Rham cohomology group

In this section we will present conditions, in terms of the twisted de Rham cohomology, for an E-valued differential 1-form to be in the image of a regular connection. It is clear, from the definition of $H^k(E, D)$, that an E-valued differential k-form will be in the image of d_D if and only if it represents the zero cohomology class in $H^k(E, D)$. Particularly, a section ω, of $\wedge^1 M \otimes E$, will be in the image of D if and only if $[\omega] = [0]$ in $H^1(E, D)$.

This means that if the first cohomology group is trivial, any section of $\wedge^1 M \otimes E$ that is mapped to the image of the curvature will be in the image of D.

Whenever it be convenient, we will make a mild change of notation by denoting the differential operator

$$\mathrm{d} : \Gamma((\wedge^k M \otimes E)/\mathrm{Im}\ \kappa^{k-2}) \to \Gamma((\wedge^{k+1} M \otimes E)/\mathrm{Im}\ \kappa^{k-1}))$$

by d_k, so that $H^k(E, D) = \ker(\mathrm{d}_k)/\mathrm{Im}(\mathrm{d}_{k-1})$.

Definition 1.2.1. Let D be a regular connection on a vector bundle E. We will say that (E, D), or simply D, is *exact* if its first de Rham cohomology group is locally trivial.

In order to avoid carrying adjectives referring to the locality of the results, we would like to remark that in the reminder of this thesis, all results are local. This has been the major motivation for us to define an exact connection as having a locally trivial first twisted de Rham cohomology group rather than a globally trivial one.

The notion of exactness of a connection on a vector bundle can be defined in complete generality, without any assumption on the regularity of the connection. A connection D on a vector bundle E will be exact if for any section ω, of $\wedge^1 M \otimes E$, that is mapped by d_D to the image of the curvature in $\wedge^2 M \otimes E$, there exists a section η of E such that $D\eta = \omega$. The motivation behind this definition comes from considering the short complex

$$\Gamma(E) \xrightarrow{D} \Gamma(\wedge^1 M \otimes E) \xrightarrow{\mathrm{d}} \Gamma((\wedge^2 M \otimes E)/\mathrm{Im}(\kappa)). \tag{1.9}$$

To say that a connection is exact is equivalent to say that the complex (1.9) is exact. If D is an exact connection, the E-valued differential 1-forms that are in the image of D are exactly the ones that are mapped to the image of the curvature. To be precise $\omega \in \mathrm{Im}(D)$ if and only if $\mathrm{d}_D\omega = \kappa(\eta)$, for some $\eta \in \Gamma(E)$.

Definition 1.2.2. The complex defined in equation (1.9) will be referred to as the *short twisted de Rham complex* of (E, D).

It is straightforward that if $\mathrm{d}_D : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E)$ is injective, D is exact, since $\mathrm{d}_D\omega = \kappa(\eta)$ implies that $\mathrm{d}_D(\omega - D\eta) = 0$. In other words, $\omega = D\eta$, for some $\eta \in \Gamma(E)$. We have then proved the following proposition.

Proposition 1.2.3. *If* $\mathrm{d}_D : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E)$ *is injective, D is exact.*

It is worth noticing that κ will be injective when $\mathrm{d}_D : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E)$ is injective. From now on, E_0 will denote the kernel of the curvature $\kappa : E \to \wedge^2 M \otimes E$.

Proposition 1.2.4. *The first twisted de Rham cohomology group, $H^1(E, D)$, is isomorphic to* $\ker(\mathrm{d}_D)/\mathrm{Im}(D|_{E_0})$.

Proof. Let us consider the sequence

$$\mathrm{Im}(D|_{E_0}) \xrightarrow{\iota} \ker(\mathrm{d}_D) \xrightarrow{\pi} H^1(E,D) = \ker(\mathrm{d})/\mathrm{Im}(D) \tag{1.10}$$

where the ι and π denote the natural inclusion and projection, respectively. To be precise, here π is the composition $\ker(\mathrm{d}_D) \to \ker(\mathrm{d}) \to \ker(\mathrm{d})/\mathrm{Im}(D)$. To say that $H^1(E,D)$ is isomorphic to $\ker(\mathrm{d}_D)/\mathrm{Im}(D|_{E_0})$ is equivalent to say that π is surjective and that the sequence (1.10) is an exact complex, as ι is injective by definition.

Firstly, let us see that π is surjective. Letting us pick $[\omega] \in H^1(E,D)$ and any representative ω of $[\omega]$, from the definition of $H^1(E,D)$, there exists $\eta \in E$ such that $\mathrm{d}_D\omega = \kappa(\eta)$, or equivalently $\omega - D\eta \in \ker(\mathrm{d}_D)$. The image of $\omega - D\eta$ under π is then $[\omega]$, as needed. Clearly, this is independent of the choice of representative, since $\mathrm{d}_D(\omega + D\theta) = \kappa(\eta+\theta)$ again implies that $\omega - D\eta \in \ker(\mathrm{d}_D)$.

It is clear that sequence (1.10) is a complex, since $\mathrm{Im}(D|_{E_0}) \to \ker(\mathrm{d}_D) \to \ker(\mathrm{d})$ is a complex by construction. It is only left to show that $\ker(\pi) \subseteq \mathrm{Im}(\iota)$. By definition, any $\omega \in \ker(\pi)$ represents the class $[0]$ in $H^1(E,D)$, namely $\omega = D\eta$ for some $\eta \in \Gamma(E)$. Also, ω is contained in the kernel of d_D, which implies that

$$0 = \mathrm{d}_D\omega = \mathrm{d}_D D\eta = \kappa(\eta)$$

and thus $\eta \in \Gamma(E_0)$ and $\omega \in \mathrm{Im}(D|_{E_0})$. $\square$

The isomorphism between $H^1(E,D)$ and $\ker(\mathrm{d})/\mathrm{Im}(D)$ is in fact natural in the sense that it is the one which makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Im}(D|_{E_0}) \arrow[r, "\iota"] \arrow[d, "\iota"] & \ker(\mathrm{d}_D) \arrow[r, "\pi"] \arrow[d, "\iota"] & \ker(\mathrm{d}_D)/\mathrm{Im}(D|_{E_0}) \arrow[d] \\
\mathrm{Im}(D) \arrow[r, "\iota"] & \ker(\mathrm{d}) \arrow[r, "\pi"] & H^1(E,D).
\end{tikzcd}$$

An immediate consequence of Proposition 1.2.4 is that a connection D is exact if and only if the complex

$$\Gamma(E_0) \xrightarrow{D|_{E_0}} \Gamma(\wedge^1 M \otimes E) \xrightarrow{\mathrm{d}_D} \Gamma(\wedge^2 M \otimes E)$$

is exact. This provides us with many tools to test the exactness of a connection. For instance, if D has injective curvature, E_0 is trivial and thus $H^1(E,D)$ is exactly the kernel of d_D. On the other hand, if d_D is injective, it is immediate that $H^1(E,D)$ is trivial, providing an alternative point of view to Proposition 1.2.3.

We will say that a subbundle F of E is *parallel* if $\mathrm{Im}(D|_F) \subseteq \Gamma(\wedge^1 M \otimes F)$. In other words, the restriction of D to F is a connection on F. In this case, D descends well to a connection in the quotient bundle E/F.

Proposition 1.2.5. *Let E be a vector bundle with connection D and suppose that F is a parallel subbundle of E such that $D|_F$ is exact. If $\ker(\mathrm{d}_D) \subseteq \Gamma(\wedge^1 M \otimes F)$, then D is exact.*

Proof. Let $F_0 := F \cap E_0$ denote the kernel of κ, restricted to F. Under the assumption that $D|_F$ is exact, we know that $\ker(\mathrm{d}_D|_F) = \mathrm{Im}(D|_{F_0})$, by Proposition 1.2.4. Then, if $\ker(\mathrm{d}_D) \subseteq \Gamma(\wedge^1_M \otimes F)$, it follows that

$$\ker(\mathrm{d}_D) \subseteq \ker(\mathrm{d}_D|_F) = \mathrm{Im}(D|_{F_0}) \subseteq \mathrm{Im}(D|_{E_0}) \subseteq \ker(\mathrm{d}_D).$$

This means that the kernel of d_D is equal to the image D, restricted to E_0, and therefore by Proposition 1.2.4, $H^1(E, D) = \{0\}$. □

Proposition 1.2.6. *Given a vector bundle E with a connection D and a parallel subbundle F of E, if F and E/F are exact, and if the curvature on E/F is injective, then E is exact.*

Proof. In order to show that $H^1(E, D) \simeq \{0\}$, we will show that $\ker(\mathrm{d}_D)$ is contained in $\Gamma(\wedge^1 M \otimes F)$. Then, by Proposition 1.2.5, (E, D) will be exact as $(F, D|_F)$ is exact by assumption.

Choosing $\omega \in \ker(\mathrm{d}_D)$ and letting $[\omega]$ denote the class of ω in $\wedge^1 M \otimes E/F$, it is clear that $[\omega]$ is in the kernel of $\mathrm{d}_D : \wedge^1 M \otimes E/F \to \wedge^2 M \otimes E/F$. By assumption, $(E/F, D|_{E/F})$ is exact and $(E/F)_0 = \{0\}$. Then, by Proposition 1.2.4, we get

$$\{0\} = H^1(E/F, D|_{E/F}) = \ker(\mathrm{d}_D|_{E/F}),$$

which implies that ω represents the zero class in $\wedge^1 M \otimes E/F$. In other words, ω is a section of $\wedge^1 M \otimes F$ and therefore $\ker(\mathrm{d}_D) \subseteq \Gamma(\wedge^1 M \otimes F)$. □

1.2.1 Twisted de Rham cohomology of direct sums

The twisted de Rham cohomology groups of (E, D) can be studied in terms of parallel subbundles of E. Suppose that E splits as the direct sum of two parallel subbundles E_1 and E_2, and let D_i denote the restriction of D to E_i, for $i = 1, 2$. In other words (E, D) is isomorphic, as a vector bundle with connection, to the direct sum $(E_1 \oplus E_2, D_1 + D_2)$ of (E_1, D_1) and (E_2, D_2). Notice that d_D preserves the splitting of E, hence it splits as well as $\mathrm{d}_{D_1} + \mathrm{d}_{D_2}$. Consequently we obtain the splittings

$$\ker(\mathrm{d}_D) = \ker(\mathrm{d}_{D_1}) \oplus \ker(\mathrm{d}_{D_2}) \quad \text{and} \quad \mathrm{Im}(D|_{E_0}) = \mathrm{Im}(D_1|_{(E_1)_0}) \oplus \mathrm{Im}(D_2|_{(E_2)_0}).$$

Their quotient becomes

$$\ker(\mathrm{d}_D)/\mathrm{Im}(D|_{E_0}) \simeq \ker(\mathrm{d}_{D_1})/\mathrm{Im}(D_1|_{(E_1)_0}) \oplus \ker(\mathrm{d}_{D_2})/\mathrm{Im}(D_2|_{(E_2)_0}).$$

By Proposition 1.2.4, the left hand side is isomorphic to $H^1(E, D)$ and the right hand side to $H^1(E_1, D_1) \oplus H^1(E_2, D_2)$. In other words, the first twisted de Rham cohomology group splits accordingly to the splitting of (E, D). In fact, we will show that all cohomology groups split according to this direct sum decomposition of parallel subbundles.

The k-curvature homomorphisms, κ^k, will split as

$$\kappa^k = \mathrm{d}_{D_1}^2 + \mathrm{d}_{D_2}^2 + \mathrm{d}_{D_1} \circ \mathrm{d}_{D_2} + \mathrm{d}_{D_2} \circ \mathrm{d}_{D_1} = \kappa_1^k + \kappa_2^k + \mathrm{d}_{D_1} \circ \mathrm{d}_{D_2} + \mathrm{d}_{D_2} \circ \mathrm{d}_{D_1}, \tag{1.11}$$

for all $k \geq 0$, where $\kappa_i^k := \mathrm{d}_{D_i}^2 : \wedge^k M \otimes E \to \wedge^{k+2} M \otimes E$. By taking a section $\omega = \omega_1 + \omega_2$ of $\wedge^k M \otimes E$, with $\omega_i \in \Gamma(\wedge^k M \otimes E_i)$, we observe that

$$(\mathrm{d}_{D_1} \circ \mathrm{d}_{D_2} + \mathrm{d}_{D_2} \circ \mathrm{d}_{D_1})(\omega) = \mathrm{d}_{D_1}\mathrm{d}_{D_2}\omega_2 + \mathrm{d}_{D_2}\mathrm{d}_{D_1}\omega_1 = 0,$$

since $\mathrm{d}_{D_i}\omega_i$ is a section of $\wedge^{k+1} M \otimes E_i$. Therefore, the splitting

$$\kappa^k = \kappa_1^k + \kappa_2^k$$

of the k-curvature operators, follows from equation (1.11).

Proposition 1.2.7. *Let E_1 and E_2 be parallel subbundles of E, such that $E = E_1 \oplus E_2$. Then, there is an isomorphism*

$$H^k(E, D) \simeq H^k(E_1, D_1) \oplus H^k(E_2, D_2),$$

where $D_i = D|_{E_i}$ for $i = 1, 2$.

Proof. Recall that under the assumption that (E, D) splits as $(E_1 \oplus E_2, D_1 + D_2)$, where E_1 and E_2 are parallel subbundles of E, the induced operators by D, d_D and κ^k, split as well. Since $\mathrm{Im}(\kappa^k) = \mathrm{Im}(\kappa_1^k) \oplus \mathrm{Im}(\kappa_2^k)$, the splitting $\mathrm{d} = \mathrm{d}|_{E_1} + \mathrm{d}|_{E_2}$ of the operatros d are well defined. Therefore, it follows from the definition of $H^k(E, D)$ that

$$H^k(E, D) = \frac{\ker \mathrm{d}_k}{\mathrm{Im}\, \mathrm{d}_{k-1}} = \frac{\ker \mathrm{d}_k|_{E_1}}{\mathrm{Im}\, \mathrm{d}_{k-1}|_{E_1}} \oplus \frac{\ker \mathrm{d}_k|_{E_2}}{\mathrm{Im}\, \mathrm{d}_{k-1}|_{E_2}} = H^k(E_1, D_1) \oplus H^k(E_2, D_2),$$

as claimed. □

An immediate corollary from the above proposition:

Corollary 1.2.8. *Let E_1 and E_2 be parallel subbundles of E, such that (E, D) decomposes as*

$$(E, D) = (E_1 \oplus E_2, D_1 + D_2).$$

Then (E, D) is exact if and only if both (E_1, D_1) and (E_2, D_2) are exact.

Proposition 1.2.9. *Let D be an exact connection on the vector bundle $E \to M$ and denote by π, the natural projection from the product manifold $M \times \bar{M}$ to M. Then $\pi^* D$ is an exact connection on the vector bundle $\pi^* E$ over $M \times \bar{M}$.*

Proof. This is a particular instance of [13, Proposition 2.7]. □

1.2.2 κ^1-injectivity

In this subsection we will provide sufficient algebraic conditions for a connection to be exact. Applications of these conditions will appear in Chapter 4, to prove that the Killing connection is exact in certain pseudo-Riemannian symmetric spaces.

Definition 1.2.10. We will say that a connection D, on a vector bundle E, is κ^1*-injective* if its 1-curvature homomorphism is injective.

κ^1-injectivity will provide us with a simple algebraic test for a connection to be exact. We remark that not all connections are κ^1-injective. It can be observed from equation (1.4) that, for a given $\omega \in \wedge^1 M \otimes E$, κ^1 takes the form

$$\kappa^1(\omega)(X, Y, Z) = 2(\kappa(X, Y)\omega(Z) + \kappa(Z, X)\omega(Y) + \kappa(Y, Z)\omega(X)).$$

Example 1.2.11. On any pseudo-Riemannian manifold (M, g), we can equip the tangent bundle of M with the Levi-Civita connection associated to g. Then, letting ω be the element of $\wedge^1_p M \otimes T_p M$ identified with the identity endomorphism of $T_p M$, we get

$$\kappa^1(\omega)(X, Y, Z) = 2(R(X, Y)Z + R(Z, X)Y + R(Y, Z)X) = 0,$$

where the last equality is nothing but the first Bianchi identity and thus $\kappa^1(\omega) = 0$.

Lemma 1.2.12. *Let D be a κ^1-injective connection on a vector bundle E. Then D is exact and has injective curvature.*

Proof. By definition, the 1-curvature operator, κ^1 is the vector bundle homomorphism $\mathrm{d}_D^2 : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^3 M \otimes E)$, so it is immediate that

$$\ker(\mathrm{d}_D : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E) \subseteq \ker(\kappa^1).$$

Moreover, it implies that if κ^1 is injective, so is $\mathrm{d}_D : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E)$. Under the assumption of D being κ^1-injective, it follows from Proposition 1.2.3 that D is exact.

It is only left to prove that D has injective curvature. Recall that $\kappa^{k+1} \circ \mathrm{d}_D = \mathrm{d}_D \circ \kappa^k$ for all $k \geq 0$ and, particularly, for $k = 0$ we have that

$$\mathrm{d}_D \kappa(\eta) = \kappa^1(D\eta), \tag{1.12}$$

for some section η of E. Supposing that η is a section of E_0, equation (1.12) implies that $D\eta = 0$, since κ^1 is injective by hypothesis. By choosing any non-constant function f on M, we have that

$$\kappa(f\eta) = 0 \quad \text{and} \quad D(f\eta) = \mathrm{d}f \otimes \eta.$$

Then, from equation (1.12), we can see that

$$0 = \mathrm{d}_D \kappa(f\eta) = \kappa^1(\mathrm{d}f \otimes \eta).$$

In other words, $\mathrm{d}f \otimes \eta \in \ker(\kappa^1)$, which is a contradiction. $\square$

1.3 The curvature filtration

In this subsection we will build a filtration of subbundles of E, starting with the kernel of the curvature, in order to study the exactness of connections with non-injective curvature. Assuming D is a regular connection, the kernel of the curvature is a vector subbundle of E. Then, $E_1 := \{\phi \in E \ : \ \kappa(\phi) \in \wedge^2 M \otimes E_0\}$ is also a subbundle of E, since it is the preimage of a vector bundle by κ. Inductively, we define for each $\ell \geq 0$ the subbundle

$$E_{\ell+1} := \{\phi \in E \ : \ \kappa(\phi) \in \wedge^2 M \otimes E_\ell\},$$

of E, that contains E_ℓ by definition. The family of vector subbundles $\{E_\ell\}_{\ell=0}^{\infty}$, defines a filtration

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$$

of subbundles of E, that will be referred to as the *curvature filtration* of (E, D). For later convenience we will set $E_{-1} = \{0\}$.

We will be interested in vector bundles with connections that have a parallel curvature filtration. In general, the curvature filtration is not parallel. For instance, in the following chapters, we will see that for the Killing bundle it is rather unusual for the kernel of the curvature to be parallel.

When the curvature filtration of (E, D) is parallel, namely E_ℓ is parallel for all $\ell \geq 0$, the restriction of D to $E_{\ell+1}$ and $E_{\ell+1}/E_\ell$ is again a connection, and

$$\begin{tikzcd} E_{\ell+1} \arrow[r, "\kappa"] \arrow[d] & \wedge^2 M \otimes E_{\ell+1} \arrow[d] \\ E_{\ell+1}/E_\ell \arrow[r, "\kappa"] & \wedge^2 M \otimes E_{\ell+1}/E_\ell \end{tikzcd}$$

is a well defined commutative diagram. By definition, $\kappa(\phi)$ is in $\wedge^2 M \otimes E_\ell$ for all ϕ in $E_{\ell+1}$, so the restriction of D to $E_{\ell+1}/E_\ell$ is in fact a flat connection. On the other hand $\kappa : E_{\ell+1}/E_\ell \to \wedge^2 M \otimes E_\ell/E_{\ell-1}$ is well defined and injective, for all $\ell \geq 1$. To see this, let $\phi \in E_{\ell+1}$ be a representative of a class $[\phi]$ in $E_{\ell+1}/E_\ell$. Then, by definition, $\kappa(\phi)$ is in $\wedge^2 M \otimes E_\ell$ and $[\kappa(\phi)] = [0]$ if and only if $\kappa(\phi)$ is in $\wedge^2 \otimes E_{\ell-1}$, i.e. $\phi \in E_\ell$.

Lemma 1.3.1. *If the curvature filtration of (E, D) is parallel, then for each $\ell \geq 0$, the map*

$$\mathrm{d} : \ker(\mathrm{d}_D : \wedge^1 \otimes E_{\ell+1}/E_\ell \to \wedge^2 \otimes E_{\ell+1}/E_\ell) \to \wedge^2 M \otimes E_\ell/E_{\ell-1},$$

is injective.

Proof. As it has been already noted, the connection induced on $E_\ell/E_{\ell-1}$ is flat and therefore exact, so for $\phi \in \ker(\mathrm{d}_D : \wedge^1 \otimes E_{\ell+1}/E_\ell \to \wedge^2 \otimes E_{\ell+1}/E_\ell)$, there exists a section ψ of $E_\ell/E_{\ell-1}$ for which $D\psi = \phi$. But if $\mathrm{d}_D\phi = 0$, we have that

$$\kappa(\psi) = \mathrm{d}_D D\psi = \mathrm{d}_D\phi = 0 \in \Gamma(\wedge^2 M \otimes E_{\ell-1}/E_{\ell-2}),$$

which means that the curvature maps ψ so $\Gamma(\wedge^2 M \otimes E_{\ell-2})$, hence ψ is a section of $E_{\ell-1}$. Therefore $\phi = D\psi = 0 \in \Gamma(\wedge^1 M \otimes E_\ell / E_{\ell-1})$. □

The curvature filtration of (E, D) will stabilise after a finite number of steps, since E has finite rank. We will denote by L, the first integer such that $E_L = E_{L+1}$.

Proposition 1.3.2. *If the curvature filtration of (E, D) is parallel and $E_L = E$, D is exact.*

Proof. Any class of $H^1(E, D)$ can be represented by a section ϕ, of $\wedge^1 M \otimes E$ such that $\mathrm{d}_D \phi = 0$. By assumption $E_L = E$, so $\phi \in \wedge^1 M \otimes E_L$. Applying Lemma 1.3.1 inductively, we conclude that ϕ is a section of $\wedge^1 M \otimes E_0$. Since D is flat on E_0 and $\mathrm{d}_D \phi = 0$, there exists a section η of E_0 such that $D\eta = \phi$ and thus D is exact. □

The above proposition will become the key to deal with the exactness of the Killing connection on locally homogeneous plane waves spacetimes, in Chapter 5.

Corollary 1.3.3. *Suppose that the curvature filtration of (E, D) is parallel and that it stabilises in E_L. Then $(E_L, D|_{E_L})$ is exact.*

Proof. It is clear from its definition that the curvature filtration of $(E_L, D|_{E_L})$ will be parallel and will stabilise in the L-th step to $(E_L)_L = E_L$. The exactness of $(E_L, D|_{E_L})$ then follows from Proposition 1.3.2. □

The below theorem is an immediate consequence from Proposition 1.2.6 and Corollary 1.3.3.

Theorem 1.3.4. *If the curvature filtration of (E, D) is parallel and the induced connection on E/E_L is exact, then D is exact on E.*

Corollary 1.3.5. *If the kernel of the curvature of D is parallel and E admits a parallel complement C, to E_0, then E_0 is equal to E_1. In this case (E, D) is exact if and only if the induced connection on C is exact.*

Proof. If E_0 admits a parallel complement C in E, clearly the curvature will map $E_1 \oplus \cap C$ into a subset of C, as a consequence of C being parallel, which implies that $E_1 = E_0$. That (E, D) is exact if and only if $(C, D|_C)$ follows from identifying C with E_0. □

Lastly, we provide characterisation for connections with parallel curvature filtration.

Lemma 1.3.6. *For a vector bundle with connection, (E, D), the following are equivalent:*

(1) The curvature filtration of (E, D) is parallel.

(2) For each $\ell \geq 0$, $\phi \in \Gamma(E_\ell)$ implies that $(D\kappa)(\phi) \in \Gamma(\wedge^1 M \otimes \wedge^2 M \otimes E_{\ell-1})$.

Proof. Let us assume that the curvature filtration of (E, D) is parallel. Firstly, for any section ϕ of E_0, clearly $(D_Z\kappa)(X,Y)\phi = -\kappa(X,Y)D_Z\phi = 0$. For any fixed $\ell \geq 1$, choosing a section ϕ of E_ℓ, we have that

$$(D_Z\kappa)(X,Y)\phi = D_Z\kappa(X,Y)\phi - \kappa(\nabla_Z X, Y)\phi - \kappa(X, \nabla_Z Y)\phi - \kappa(X,Y)D_Z\phi,$$

for some auxiliary affine connection ∇. That $(D\kappa)(\phi) \in \Gamma(\wedge^1 M \otimes \wedge^2 M \otimes E_{\ell-1})$, can be observed from the above equation, since the curvature filtration is parallel and $\kappa(X,Y)(E_\ell)$ is contained in $E_{\ell-1}$, by definition.

Conversely, if $\phi \in \Gamma(E_\ell)$ implies that $(D\kappa)(\phi) \in \Gamma(\wedge^1 M \otimes \wedge^2 M \otimes E_{\ell-1})$, for $\ell = 0$, we have

$$0 = (D_Z\kappa)(X,Y)\phi = -\kappa(X,Y)D_Z\phi,$$

which implies that $D\Gamma(E_0) \subseteq \Gamma(\wedge^1 M \otimes E_0)$. Inductively, let us assume the statement holds up to ℓ. If $\phi \in \Gamma(E_{\ell+1})$, we have that

$$(D_Z\kappa)(X,Y)\phi = D_Z\kappa(X,Y)\phi - \kappa(\nabla_Z X, Y)\phi - \kappa(X, \nabla_Z Y)\phi - \kappa(X,Y)D_Z\phi \in \Gamma(E_\ell)$$

implies that $\kappa(X,Y)D_Z\phi \in \Gamma(E_{\ell-1})$, since $\kappa(\phi) \in \Gamma(\wedge^2 M \otimes E_\ell)$ and E_ℓ was parallel, by assumption. $\square$

$==$

Chapter 2

The Killing bundle

In this chapter we will introduce the Killing bundle and the Killing connection of a pseudo-Riemannian manifold. Specifically, in Section 2.1 we will derive a prolongation of the Killing equation giving rise to the Killing bundle and the Killing connection. Section 2.2 will be dedicated to the study of certain algebra structures on the space of sections of the Killing bundle and of special subbundles of it, defined in terms of tensor fields on the base manifold. Lastly, in Section 2.3, the curvature of the Killing connection will be studied, in terms of the results described in Section 2.2.

2.1 The prolongation of the Killing equation

Let (M, g) be a pseudo-Riemannian manifold. The *Killing operator* on vector fields is the first order linear differential operator

$$\mathcal{K} : \Gamma(TM) \to \Gamma(\mathrm{Sym}^2 M), \quad \xi \mapsto L_\xi g, \tag{2.1}$$

where L_ξ denotes the Lie derivative in the direction of the vector field ξ. The *Killing equation* is the first order partial differential equation

$$\mathcal{K}(\xi) = 0$$

for the vector fields in the kernel of the Killing operator. Solutions to the Killing equation are called *Killing vector fields*. These are the infinitesimal isometries of the pseudo-Riemannian manifold (M, g), or more precisely, the vector fields whose local flow is given by one-parameter subgroups of the isometry group of (M, g). The set of Killing vector fields of (M, g), with the usual vector field bracket, forms a Lie algebra which will be denoted by $\mathfrak{kill}(M, g)$.

The choice of an affine connection, ∇, on TM, permits us to define the family of endomorphisms of TM

$$A_\xi^\nabla := -\nabla\xi - \tau_\xi^\nabla, \quad \xi \in \Gamma(TM),$$

that will be referred to as *Nomizu operators*. Here $\tau^\nabla \in \Gamma(\wedge^2 M \otimes TM)$ denotes the torsion tensor of ∇ and we will use the following convention for its definition:

$$\tau^\nabla_X Y = \nabla_X Y - \nabla_Y X - [X,Y].$$

Conveniently, the Lie derivative in the direction of ξ can be expressed in terms of ∇ and A^∇_ξ as

$$L_\xi = \nabla_\xi + A^\nabla_\xi. \tag{2.2}$$

If ∇ is a metric connection, it follows from equation (2.2) that the Killing operator can be expressed as

$$\mathcal{K}(\xi) = A^\nabla_\xi \cdot g,$$

where $\cdot$ denotes the natural action of the endomorphisms of TM on covariant tensors. This action is given explicitly by

$$(A \cdot \alpha)(X_1, \dots, X_r) = -\alpha(AX_1, \dots, X_r) - \dots - \alpha(X_1, \dots, AX_r),$$

where $\alpha \in T^*M^{\otimes^r}$. Then, the Killing equation takes the form

$$(A^\nabla_\xi \cdot g)(X,Y) = -g(A^\nabla_\xi X, Y) - g(X, A^\nabla_\xi Y) = 0, \tag{2.3}$$

for all $X, Y \in TM$. From this point of view, we can see that the Killing operator is a linear differential operator acting on vector fields. Indeed, for any $\xi, \eta \in \Gamma(TM)$ and $a \in \mathbb{R}$, we have

$$A^\nabla_{a\xi+\eta} = -\nabla(a\xi + \eta) - \tau^\nabla_{a\xi+\eta} = -a\nabla\xi - \nabla\eta - a\tau^\nabla_\xi - \tau^\nabla_\eta = aA^\nabla_\xi + A^\nabla_\eta,$$

and thus

$$\mathcal{K}(a\xi + \eta) = aA^\nabla_\xi \cdot g + A^\nabla_\eta \cdot g = a\mathcal{K}(\xi) + \mathcal{K}(\eta).$$

From now on, we will fix ∇ to denote the Levi-Civita connection associated to g, unless otherwise stated. Also, to simplify notation, we will omit all upper scripts on the Nomizu operators that indicate their relation to ∇. The endomorphisms of the tangent bundle of M, which annihilate the metric g when acted upon, form the vector bundle $\mathfrak{so}(TM, g)$ of skew-symmetric endomorphisms of TM. Then a vector field ξ is a Killing vector field if and only if the endomorphism A_ξ is in $\mathfrak{so}(TM, g)$.

For a given endomorphism A of TM, we will denote by $\hat{A}$ the projection of A to $\mathfrak{so}(TM, g)$, given explicitly by

$$\hat{A} = \frac{1}{2}(A - A^*),$$

where A^* denotes the g-adjoint endomorphism of A. Analogously, we will denote the projection of A to the symmetric endomorphisms of TM, which we shall denote by $\mathrm{Sym}^2(TM, g)$, by

$$\dot{A} := \frac{1}{2}(A + A^*).$$

To avoid confusion, we remark that we denote the bundle of symmetric 2-tensors on M by $\mathrm{Sym}^2 M$, and $\mathrm{Sym}^2(M,g)$ denotes the bundle of g-symmetric endomorphisms of TM.

Remark 2.1.1. In this notation, we could define the Killing operator to be the differential operator $\mathrm{k} : \Gamma(TM) \to \Gamma(\mathrm{Sym}(TM,g))$, defined by

$$\mathrm{k}(\xi) = -\dot{A}_\xi, \tag{2.4}$$

which will take values on the symmetric endomorphisms of TM, instead of symmetric 2-tensors. The minus sign in the right hand side of the above equation has been taken for convenience in the following chapters.

The action of any given skew-symmetric endomorphism A, by definition, annihilates the metric tensor. Then, it follows that its covariant derivatives will also be skew-symmetric, since

$$0 = \nabla_X(A \cdot g) = (\nabla_X A) \cdot g + A \cdot \nabla_X g = (\nabla_X A) \cdot g. \tag{2.5}$$

In general, for any vector fields $X, Y, \xi \in \Gamma(TM)$, it follows from a straightforward calculation that

$$(\nabla_X A_\xi)(Y) - (\nabla_Y A_\xi)(X) = -R(X,Y)\xi. \tag{2.6}$$

For future convenience, we will define the differential operator

$$\mathrm{d}^\nabla : \Gamma(\mathrm{End}(TM)) \to \Gamma(\wedge^1 M \otimes \mathrm{End}(TM)),$$

acting on the endomorphisms of TM, to be

$$(\mathrm{d}^\nabla A)(X)Y = (\nabla_X A)(Y) - (\nabla_Y A)(X).$$

Therefore, equation (2.6) takes the rather simple form

$$(\mathrm{d}^\nabla A_\xi)(X)Y = -R(X,Y)\xi, \tag{2.7}$$

in terms of d^∇.

Lemma 2.1.2. *Let (M,g) be a pseudo-Riemannian manifold and let ξ be a vector field on M. Then*

$$\nabla_X \hat{A}_\xi = -R(X,\xi) - (\mathrm{d}^\nabla \dot{A}_\xi)(X) + (\mathrm{d}^\nabla \dot{A}_\xi)(X)^*. \tag{2.8}$$

for all $X \in TM$.

Proof. For any vector field X on M, the endomorphism $\nabla_X \hat{A}_\xi$ is skew-symmetric, since $\hat{A}_\xi$ is skew-symmetric by definition. Making use of the symmetries of $\nabla_X \hat{A}_\xi$ and contracting it with the metric tensor, we have

$$\begin{aligned} g((\nabla_X \hat{A}_\xi)(Y), Z) &= g((\nabla_X \hat{A}_\xi)(Y), Z) - g((\nabla_Y \hat{A}_\xi)(X), Z) \\ &\quad - g((\nabla_Y \hat{A}_\xi)(Z), X) + g((\nabla_Z \hat{A}_\xi)(Y), X) \\ &\quad + g((\nabla_Z \hat{A}_\xi)(X), Y) - g((\nabla_X \hat{A}_\xi)(Z), Y) \\ &\quad - g((\nabla_X \hat{A}_\xi)(Y), Z), \end{aligned}$$

which, after a rearrangement of terms, takes the form

$$2g((\nabla_X \hat{A}_\xi)(Y), Z) = g((\mathrm{d}^\nabla \hat{A}_\xi)(X)Y, Z) - g((\mathrm{d}^\nabla \hat{A}_\xi)(Y)Z, X) + g((\mathrm{d}^\nabla \hat{A}_\xi)(Z)X, Y).$$

Letting us replace $\hat{A}_\xi$ by $A_\xi - \dot{A}_\xi$ in the right hand side of the above equation, we get

$$\begin{array}{rcl} 2g((\nabla_X \hat{A}_\xi)(Y), Z) & = & g((\mathrm{d}^\nabla A_\xi)(X)Y, Z) - g((\mathrm{d}^\nabla \dot{A}_\xi)(X)Y, Z) \\ & & -g((\mathrm{d}^\nabla A_\xi)(Y)Z, X) + g((\mathrm{d}^\nabla \dot{A}_\xi)(Y)Z, X) \\ & & +g((\mathrm{d}^\nabla A_\xi)(Z)X, Y) - g((\mathrm{d}^\nabla \dot{A}_\xi)(Z)X, Y). \end{array} \tag{2.9}$$

A direct computation reveals that for any symmetric endomorphism H, it holds that

$$g((\mathrm{d}^\nabla H)(Y)Z, X) = -g((\mathrm{d}^\nabla H)(X)Y, Z) - g((\mathrm{d}^\nabla H)(Z)X, Y) \tag{2.10}$$

and also, since $(\mathrm{d}^\nabla H)(Z)X$ is skew-symmetric in X and Z,

$$g((\mathrm{d}^\nabla H)(Y)Z, X) = -g((\mathrm{d}^\nabla H)(X)Y, Z) + g((\mathrm{d}^\nabla H)(X)^*Y, Z). \tag{2.11}$$

Then, we notice from equations (2.7) and (2.11) that equation (2.9) becomes

$$\begin{array}{rcl} 2g((\nabla_X \hat{A}_\xi)(Y), Z) & = & -g(R(X,Y)\xi, Z) + g(R(Y,Z)\xi, X) - g(R(Z,X)\xi, Y) \\ & & -2g((\mathrm{d}^\nabla \dot{A}_\xi)(X)Y, Z) + 2g((\mathrm{d}^\nabla \dot{A}_\xi)(X)^*Y, Z). \end{array}$$

Finally, making use of the Bianchi identity, we obtain

$$g((\nabla_X \hat{A}_\xi)(Y), Z) = -g(R(X,\xi)Y, Z) - g((\mathrm{d}^\nabla \dot{A}_\xi)(X)Y, Z) + 2g((\mathrm{d}^\nabla \dot{A}_\xi)(X)^*Y, Z),$$

or

$$\nabla_X \hat{A}_\xi = -R(X,\xi) - (\mathrm{d}^\nabla \dot{A}_\xi)(X) + (\mathrm{d}^\nabla \dot{A}_\xi)(X)^*,$$

as required. □

If we consider a Killing vector field ξ, Lemma 2.1.2 together with equation (2.3) provide us with the overdetermined system of partial differential equations

$$\left\{ \begin{array}{rclr} \nabla_X \xi & = & -AX, & \xi \in \Gamma(TM) \\ \nabla_X A & = & -R(X,\xi), & A \in \Gamma(\mathfrak{so}(TM,g)) \end{array} \right. \tag{2.12}$$

for a vector field ξ and a skew-symmetric endomorphism A, of TM, whose solutions define Killing vector fields. This is, if the first line of equation (2.12) is satisfied, the second line is automatically satisfied by equation (2.8). This result was first obtained by Bertram Kostant in [34] and later by Robert Geroch in [24].

The system of partial differential equations (2.12) allows to define a connection on a vector bundle over M, which was firstly introduced in [24] as *Killing transport*, whose parallel sections define Killing vector fields.

Definition 2.1.3. Let (M, g) be a pseudo-Riemannian manifold. We will say that the vector bundle $E := TM \oplus \mathfrak{so}(TM, g)$, over M, is the *Killing bundle* of (M, g). The connection $D : \Gamma(E) \to \Gamma(\wedge^1 M \otimes E)$ on E, defined by

$$D_X \begin{bmatrix} \xi \\ A \end{bmatrix} = \begin{bmatrix} \nabla_X \xi + AX \\ \nabla_X A + R(X, \xi) \end{bmatrix}, \quad X \in \Gamma(TM). \tag{2.13}$$

will be referred to as the *Killing connection*.

For convenience we will denote sections of E, of the form $(\xi, \hat{A}_\xi)$, by ϕ_ξ. Particularly, when ξ is a Killing vector field, $\hat{A}_\xi = A_\xi$ and ϕ_ξ is a parallel section of E. It follows by equation (2.12), these are precisely all parallel sections of E, namely

$$H^0(E, D) = \{\phi_\xi \in \Gamma(E) \ : \ \xi \in \mathfrak{kill}(M, g)\}.$$

The curvature of the Killing connection $\kappa \in \Gamma(\wedge^2 M \otimes \mathrm{End}(E))$ will be referred to as the *Killing curvature*. It has the form

$$\kappa(X, Y) \begin{bmatrix} \xi \\ A \end{bmatrix} = - \begin{bmatrix} 0 \\ (\nabla_\xi R)(X, Y) + (A \cdot R)(X, Y) \end{bmatrix}. \tag{2.14}$$

In what follows K will always denote the maximal parallel flat subbundle of E. Equipped with $D|_K$, $(K, D|_K)$ becomes a flat vector bundle over M, such that $H^0(E, D)$ is contained in $\Gamma(K)$. Consequently, Killing vector fields are uniquely determined by the value of ϕ_ξ at a point in the sense that, for a given Killing vector field ξ, by knowing the values of ξ and $\nabla\xi$ at a point p, one can always recover ξ by parallel transporting $(\phi_\xi)_p$.

The space of parallel sections of E can be equiped with a bracket operation defined as

$$\{\phi_\xi, \phi_\eta\} := \phi_{[\xi,\eta]} \tag{2.15}$$

for two given sections ϕ_ξ and ϕ_η in $H^0(E, D)$. It is immediate that $H^0(E, D)$ is closed under the above bracket, since the Lie bracket of ξ and η is again a Killing vector field. Moreover, it satisfies the Jacobi identity

$$\{\phi_\xi, \{\phi_\eta, \phi_\zeta\}\} = \phi_{[\xi,[\eta,\zeta]]} = \phi_{[[\xi,\eta],\zeta]+[\eta,[\xi,\zeta]]} = \{\{\phi_\xi, \phi_\eta\}, \phi_\zeta\} + \{\phi_\eta, \{\phi_\xi, \phi_\zeta\}\}$$

and thus $H^0(E, D)$, equiped with the bracket defined in equation (2.15), is a Lie algebra over the real numbers. We have proved that:

Proposition 2.1.4. *The map* $\iota : (\mathfrak{kill}(M, g), [\cdot, \cdot]) \to (H^0(E, D), \{\cdot, \cdot\})$, $\xi \mapsto \phi_\xi$ *is a Lie algebra isomorphism.*

Remark 2.1.5. The above proposition shows that the dimension of the Lie algebra of Killing vector fields is bounded by above by the rank of E.

It is a well know result by Hano [26] (see also [33, Theorem 3.5]), that if (M,g) is a complete and simply connected Riemannian manifold with de Rham decomposition [14] given by

$$(M,g) = (M_0,g_0) \times \cdots \times (M_k,g_k),$$

where (M_0,g_0) is an Euclidean factor, the connected component of its isometry group decomposes accordingly:

$$\mathrm{Iso}_0(M,g) \simeq \mathrm{Iso}_0(M_0,g_0) \times \cdots \times \mathrm{Iso}_0(M_k,g_k),$$

where $\mathrm{Iso}_0(M_i,g_i)$ denotes the connected component of the isometry group of (M_i,g_i), with $i = 0,\ldots,k$. In terms of Killing vector fields, it means that they split as well in terms of the de Rham decomposition of (M,g), i.e.

$$\mathfrak{kill}(M,g) = \mathfrak{kill}(M_0,g_0) \times \cdots \times \mathfrak{kill}(M_k,g_k).$$

Even though the de Rham decomposition was generalised by Wu to pseudo-Riemannian manifolds [44], where the factors are flat or indecomposable, i.e. when the action of the holonomy group with basepoint p, on T_pM, leaves no non-trivial non-degenerate subspaces, Hano's theorem does not directly generalise. In fact, in Lorentzian signature, a product of Euclidean space with an indecomposable Cahen-Wallach space (see Remark 5.1.5) has more Killing vector fields than just the Killing vector fields of the factors, see [35, Remark 3.6]. Below we provide decomposition theorem, similar to the one from Hano, in arbitrary signature.

Theorem 2.1.6. *Let (M,g) be a pseudo-Riemannian manifold with de Rham-Wu decomposition*

$$(M,g) = (M_1,g_1) \times \cdots \times (M_k,g_k).$$

If (E_i,D_i) denotes the Killing bundle and connection of (M_i,g_i) and

$$\{X \in T_pM \ :\ R(X,\cdot) = 0\} = \{0\} \quad \textit{for all } p \in M,$$

then

$$H^0(E,D) \simeq H^0(E_1,D_1) \oplus \cdots \oplus H^0(E_k,D_k).$$

Proof. For the proof of Theorem 2.1.6 we refer to Theorem 4.3.9 in Section 4.3. □

The Killing bundle can be endowed with a fibrewise bilinear map $[\![\cdot,\cdot]\!]_p : E_p \times E_p \to E_p$, defined by

$$\left[\!\!\left[\begin{bmatrix} X \\ A \end{bmatrix}, \begin{bmatrix} Y \\ B \end{bmatrix} \right]\!\!\right]_p = \begin{bmatrix} AY - BX \\ [A,B] - R(X,Y) \end{bmatrix}, \tag{2.16}$$

where $[\cdot,\cdot]$ denotes the commutator of endomorphisms, which turns its fibres into algebras over the real numbers. We will refer to the bracket, defined in equation (2.16) as the *Killing bracket*. In general, the Killing bracket is not a Lie bracket but we will prove in the following sections that on the fibres of certain subbundles of E, the Killing bracket becomes a Lie bracket.

Remark 2.1.7. The Killing bracket is $C^\infty(M)$-linear and thus it can be extended to sections of E.

2.2 Algebras of sections of the Killing bundle

The sections of the Killing bundle define derivations on the sections of the algebra of tensors over M, in a natural way. Since M is equipped with a metric tensor, without loss of generality, we will always assume that the tensors which we are working with are covariant, unless otherwise stated. For a section $\phi = (\xi, A)$ of E and a tensor field $\alpha \in \Gamma(T^*M^{\otimes^r})$, we define

$$\phi \cdot \alpha := \nabla_\xi \alpha + A \cdot \alpha.$$

These are indeed derivations of the algebra of sections of the tensor algebra since they preserve the tensor type and satisfy the Leibniz rule: For any α in $\Gamma(T^*M^{\otimes^r})$, β in $\Gamma(T^*M^{\otimes^s})$ and f in $C^\infty(M)$ we have that

$$\phi \cdot (\alpha \otimes \beta) = \nabla_\xi(\alpha \otimes \beta) + A \cdot (\alpha \otimes \beta) = (\nabla_\xi \alpha) \otimes \beta + \alpha \otimes (\nabla_\xi \beta) + (A \cdot \alpha) \otimes \beta + \alpha \otimes (A \cdot \beta).$$

Therefore we get

$$\phi \cdot (\alpha \otimes \beta) = (\phi \cdot \alpha) \otimes \beta + \alpha \otimes (\phi \cdot \beta)$$

and also

$$\phi \cdot f\alpha = \nabla_\xi f\alpha + A \cdot f\alpha = \xi(f)\alpha + f\nabla_\xi \alpha + fA \cdot \alpha = \xi(f)\alpha + f\phi \cdot \alpha,$$

where we have defined $A \cdot f = 0$. In general, for $(\alpha_1, \ldots, \alpha_k) \in \Gamma(T^*M^{\otimes^{r_1}} \oplus \cdots \oplus T^*M^{\otimes^{r_k}})$, it is given by

$$\phi \cdot (\alpha_1, \ldots, \alpha_k) = (\phi \cdot \alpha_1, \ldots, \phi \cdot \alpha_k).$$

Particularly, since E is contained in the tensor algebra, for any element (X, A) of E and any section (Y, B) of E, we have

$$\begin{bmatrix} X \\ A \end{bmatrix} \cdot \begin{bmatrix} Y \\ B \end{bmatrix} = \begin{bmatrix} \nabla_X Y + AY \\ \nabla_X B + [A, B] \end{bmatrix}. \tag{2.17}$$

It is worth noticing that the operation defined in equation (2.17) makes the pair $(\Gamma(E), \cdot)$ an algebra over the real numbers. The product operation on $\Gamma(E)$, defined in equation (2.17) will be referred to as the *Killing product*. Also, for any pair of subbundles E_1 and E_2 of E, we will use the notation

$$\Gamma(E_1) \cdot \Gamma(E_2) := \{\phi_1 \cdot \phi_2 \in \Gamma(E) \ : \ \phi_1 \in \Gamma(E_1),\ \phi_2 \in \Gamma(E_2)\}.$$

In the rest of this section we will focus our attention on the derivations of the sections of the tensor algebra which annihilate an arbitrary (but fixed) tensor field and all of its iterated covariant derivatives at a point, however later on we will pay special attention to the Riemannian curvature tensor. Also, since TM is equiped with a metric tensor, without loss of generality we will always consider covariant tensors, unless otherwise stated.

Let us fix an arbitrary tensor field α of type $(0,r)$ and let $\nabla^\ell\alpha$ denote the ℓ-th iterated covariant derivative of α. To be precise, for a tensor field α of rank $(0,r)$, its ℓ-th iterated covariant derivative is the tensor field $\nabla^\ell\alpha$ of rank $(0,r+\ell)$. For α, a point p of M and a non-negarive integer number ℓ, we define the vector subspace

$$K_p^{\alpha,\ell} := \{\phi \in E_p \ : \ \phi\cdot\nabla^i\alpha,\ 0\le i\le \ell\},$$

of E_p. For convenience, when $\ell=0$ we will simply write K_p^α. Notice that the vector spaces $K_p^{\alpha,\ell}$ correspond to the intersections

$$K_p^{\alpha,\ell} = \bigcap_{i=0}^{\ell} K_p^{\nabla^i\alpha},$$

thus these subspaces of E_p define the non-increasing sequence

$$E_p \supseteq K_p^\alpha \supseteq K_p^{\alpha,1} \supseteq K_p^{\alpha,2} \supseteq \dots \tag{2.18}$$

of vector subspaces of E_p which will stabilise to a subspace of E_p, since E_p is finite dimensional. We will denote the aforementioned space by $K_p^{\alpha,\infty}$.

For a given $X\in TM$ we will denote by ι_X, the contraction map

$$\iota_X \ : \ T^*M^{\otimes^{r+1}} \to T^*M^{\otimes^r}, \quad (\iota_X\alpha)(X_1,\dots,X_r) = \alpha(X,X_1,\dots,X_r),$$

for some $\alpha\in T^*M^{\otimes^r}$. Also, we will use the formulas

$$\iota_Y(\nabla_X\nabla\alpha) - \iota_X(\nabla_Y\nabla\alpha) = R(X,Y)\cdot\alpha \quad \text{and} \quad \iota_X(\phi\cdot\nabla\alpha) = \phi\cdot\nabla_X\alpha - \nabla_{\phi\cdot X}\alpha. \tag{2.19}$$

Here $\phi\in E$ and $R(X,Y)\cdot\alpha$ denotes the usual endomorphism action of $R(X,Y)$ on α.

Lemma 2.2.1 (Leibniz rule). *Let α be a tensor field on M and let ϕ be a section of E. Then*

$$\nabla_X(\phi\cdot\alpha) = (D_X\phi)\cdot\alpha + \iota_X(\phi\cdot\nabla\alpha), \tag{2.20}$$

for all $X\in\Gamma(TM)$.

Proof. Let us choose a section $\phi=(\xi,A)$, of the Killing bundle E. Then, it follows from the definition of D that

$$(D_X\phi)\cdot\alpha = \nabla_{\nabla_X\xi+AX}\alpha + (\nabla_X A + R(X,\xi))\cdot\alpha. \tag{2.21}$$

Noticing that

$$\begin{aligned}\nabla_X A\cdot\alpha &= \nabla_X(A\cdot\alpha) - A\cdot\nabla_X\alpha \\ &= \nabla_X(A\cdot\alpha) - \iota_X(A\cdot\nabla\alpha) - \nabla_{AX}\alpha,\end{aligned} \tag{2.22}$$

and

$$\begin{aligned} R(X,\xi)\cdot\alpha &= \nabla_X\nabla_\xi\alpha - \nabla_\xi\nabla_X\alpha - \nabla_{\nabla_X\xi}\alpha + \nabla_{\nabla_\xi X}\alpha \\ &= \nabla_X\nabla_\xi\alpha - \iota_X(\nabla_\xi\nabla\alpha) - \nabla_{\nabla_X\xi}\alpha \end{aligned} \tag{2.23}$$

we can replace equations (2.22) and (2.23) in equation (2.21) to obtain, after a rearrangement,

$$\nabla_X(\phi\cdot\alpha) = (D_X\phi)\cdot\alpha + \iota_X(\phi\cdot\nabla\alpha),$$

as claimed. □

Even though the dimensions of $K_p^{\alpha,\ell}$ can vary from point to point, our interest will be placed in manifolds and tensors where the dimension of $K_p^{\alpha,\ell}$ is constant on M.

Definition 2.2.2. Let α be a tensor field on M. We will say that α is *Killing-regular* on (M,g) if, for each $\ell \geq 0$, the map $p \mapsto \dim K_p^{\alpha,\ell}$ is constant.

If α is a Killing-regular tensor field, for each $\ell \geq 0$, we can build a vector bundle over M with fibres $K_p^{\alpha,\ell}$. We will denote such bundle by $K^{\alpha,\ell}$.

Lemma 2.2.3. *Let α be a Killing-regular tensor field on (M,g). Then $K^{\alpha,\infty}$ is the unique maximal parallel subbundle of E, which is contained in K^α.*

Proof. Firstly, let us see that $K^{\alpha,\infty}$ is a parallel subbundle of E. Let ϕ be a section of $K^{\alpha,\infty}$, then equation (2.20) implies that

$$0 = -\iota_X(\phi\cdot\nabla^{\ell+1}\alpha) = (D_X\phi)\cdot\nabla^\ell\alpha$$

for all $\ell \geq 0$. Therefore $D\Gamma(K^{\alpha,\infty}) \subseteq \Gamma(\wedge^1 M\otimes K^{\alpha,\infty})$, i.e. $K^{\alpha,\infty}$ is a parallel subbundle of E, contained in K^α. It is only left to prove that $K^{\alpha,\infty}$ is the maximal subbundle with this property. Let K' be another parallel subbundle of E contained in K^α, and let $\phi \in \Gamma(K')$. Then

$$\begin{aligned} 0 &= (D_{X_\ell}\dots D_{X_1}\phi)\cdot\alpha \\ &= -\iota_{X_\ell}((D_{X_{\ell-1}}\dots D_{X_1}\phi)\cdot\nabla\alpha) \\ &\vdots \\ &= (-1)^\ell\iota_{X_\ell}\dots\iota_{X_1}(\phi\cdot\nabla^\ell\alpha), \end{aligned}$$

which means that ϕ is a section of $K^{\nabla^\ell\alpha}$ for all $\ell \geq 0$. Consequently $K' \subseteq K^{\alpha,\infty}$, completing the proof. □

If α is a Killing-regular tensor field, analogously to sequence (2.18), we define the non-increasing sequence of vector bundles

$$E \supseteq K^{\alpha,0} \supseteq K^{\alpha,1} \supseteq K^{\alpha,2} \supseteq \dots \tag{2.24}$$

which will converge to $K^{\alpha,\infty}$. Since E has finite rank, sequence (2.24) stabilises after a finite number of steps and we will denote by s_α, the first integer such that $K^{\alpha,s_\alpha} = K^{\alpha,s_\alpha+1}$.

Proposition 2.2.4. *Let α be a Killing-regular tensor field on (M,g). Then K^{α,s_α} coincides with $K^{\alpha,\infty}$.*

Proof. Any section ϕ of K^{α,s_α} will satisfy, by definition,

$$\phi \cdot \nabla^\ell \alpha = 0 \quad \text{for all} \quad \ell \leq s_{\alpha+1}.$$

It follows, from Lemma 2.2.1, that

$$0 = \nabla_X(\phi \cdot \nabla^\ell \alpha) - \iota_X(\phi \cdot \nabla^{\ell+1}\alpha) = (D_X\phi)\cdot \nabla^\ell \alpha,$$

for all $\ell \leq s_\alpha$, which implies that $D\phi \in \Gamma(\wedge^1 M \otimes K^{\alpha,s_\alpha})$ and hence K^{α,s_α} is parallel. Since K^{α,s_α} is parallel and it is contained in K^α, by Lemma 2.2.3, it must be contained in $K^{\alpha,\infty}$. By definition $K^{\alpha,\infty} \subseteq K^{\alpha,s_\alpha}$ and therefore $K^{\alpha,s_\alpha} = K^{\alpha,\infty}$. $\square$

The space of sections of the bundles $K^{\alpha,\infty}$, equiped with the Killing bracket, are in fact algebras over $C^\infty(M)$. However, in general, the Killing bracket does not satisfy the Jacobi identity.

Proposition 2.2.5. *Let α be a Killing-regular tensor field on (M,g). Then $\Gamma(K^{\alpha,\infty})$ is closed under the Killing bracket, inherited from $\Gamma(E)$. In other words, $(\Gamma(K^{\alpha,\infty}), [\![\cdot,\cdot]\!])$ is a subalgebra of $(\Gamma(E), [\![\cdot,\cdot]\!])$.*

Proof. Let (X,A) and (Y,B) be sections of $K^{\alpha,\infty}$. Then, it follows from the definition of the Killing bracket that

$$[\![(X,A),(Y,B)]\!] \cdot \nabla^\ell \alpha = \nabla_{AY}\nabla^\ell\alpha - \nabla_{BX}\nabla^\ell\alpha + [A,B]\cdot\nabla^\ell\alpha - R(X,Y)\cdot\nabla^\ell\alpha. \tag{2.25}$$

Since (X,A) and (Y,B) are sections of $K^{\alpha,\infty}$, we know that

$$\iota_Y(\nabla_X\nabla^{\ell+1}\alpha) = -\iota_Y(A\cdot\nabla^{\ell+1}\alpha) \quad \text{and} \quad \iota_X(\nabla_Y\nabla^{\ell+1}\alpha) = -\iota_X(B\cdot\nabla^{\ell+1}\alpha).$$

Recall, from equation (2.19), that $\iota_Y(A\cdot\nabla^{\ell+1}\alpha) = A\cdot\nabla_Y\nabla^\ell\alpha - \nabla_{AY}\nabla^\ell\alpha$. Therefore

$$\begin{aligned} \iota_Y(\nabla_X\cdot\nabla^{\ell+1}\alpha) &= -A\cdot\nabla_Y\nabla^\ell\alpha + \nabla_{AY}\nabla^\ell\alpha \\ &= A\cdot(B\cdot\nabla^\ell\alpha) + \nabla_{AY}\nabla^\ell\alpha \\ &= [A,B]\cdot\nabla^\ell\alpha + B\cdot(A\cdot\nabla^\ell\alpha) + \nabla_{AY}\nabla^\ell\alpha \end{aligned} \tag{2.26}$$

and analogously

$$\iota_X(\nabla_Y\cdot\nabla^{\ell+1}\alpha) = B\cdot(A\cdot\nabla^\ell\alpha) + \nabla_{BX}\nabla^\ell\alpha. \tag{2.27}$$

Substracting equation (2.27) from equation (2.26), we obtain

$$\iota_Y(\nabla_X\nabla^{\ell+1}\alpha) - \iota_X(\nabla_Y\nabla^{\ell+1}\alpha) = \nabla_{AY}\nabla^\ell\alpha - \nabla_{BX}\nabla^\ell\alpha + [A,B]\cdot\nabla^\ell\alpha.$$

Recall that

$$\iota_Y(\nabla_X\nabla^{\ell+1}\alpha) - \iota_X(\nabla_Y\nabla^{\ell+1}\alpha) = R(X,Y)\cdot\nabla^\ell\alpha,$$

from equation (2.19). It follows that $[\![(X,A),(Y,B)]\!]\cdot\nabla^\ell\alpha = 0$ and, as ℓ is arbitrary, therefore $[\![(X,A),(Y,B)]\!] \in \Gamma(K^{\alpha,\infty})$. $\square$

The Killing bundle is equipped with natural projections onto TM and $\mathfrak{so}(TM,g)$. Let us denote them by $\pi_{TM} : E \to TM$ and $\pi_{\mathfrak{so}} : E \to \mathfrak{so}(TM,g)$, respectively.

Lemma 2.2.6. *Let ϕ and ψ be sections of E. Then*

$$\phi \cdot \psi = D_{\pi_{TM}(\phi)}\psi + [\![\phi,\psi]\!]. \tag{2.28}$$

Proof. The result follows directly from the definitions. □

Proposition 2.2.7. *Let α be a Killing-regular tensor on (M,g). Then $(\Gamma(K^{\alpha,\infty}),\cdot)$ is a subalgebra of $(\Gamma(E),\cdot)$.*

Proof. Let ϕ and ψ be sections of $K^{\alpha,\infty}$. By Lemma 2.2.6, we know that the Killing product relates to the Killing connection and Killing bracket by $\phi\cdot\psi = D_{\pi_{TM}(\phi)}\psi + [\![\phi,\psi]\!]$, hence it will be enough to show that $D_{\pi_{TM}(\phi)}\psi$ and $[\![\phi,\psi]\!]$ are sections of $K^{\alpha,\infty}$. We have showed in Lemma 2.2.3 that $K^{\alpha,\infty}$ is a parallel subbundle of E, hence $D_{\pi_{TM}(\phi)}\psi$ is a section of $K^{\alpha,\infty}$. Also, $\Gamma(K^{\alpha,\infty})$ is closed under the Killing bracket, by Proposition 2.2.5. Therefore, $\phi\cdot\psi = D_{\pi_{TM}(\phi)}\psi + [\![\phi,\psi]\!]$ is a section of $K^{\alpha,\infty}$, as claimed. □

In what follows we will introduce a metric tensor on the Killing bundle, in terms of the pseudo-Riemannian metric on TM, and its associated Killing form, B^g on $\mathfrak{so}(TM,g)$.

Definition 2.2.8. On a pseudo-Riemannian manifold (M,g), we will say that the tensor field $g_E : E\times E \to \mathbb{R}$, defined by

$$g_E((X,A),(Y,B)) := g(X,Y) - B^g(A,B), \quad (X,A),(Y,B)\in E, \tag{2.29}$$

is the *Killing metric* of E.

Remark 2.2.9. The choice of the minus sign in the summand corresponding to the Killing form was made for g_E to be positive definite when (M,g) is a Riemannian manifold. Moreover, when (M,g) is Riemannian, the restriction of g_E to any subbundle of E remains positive definite.

The Killing metric enjoys desired compatibility conditions with $(\Gamma(E),\cdot)$. It is clear that $(X,A)\cdot g = \nabla_X g + A\cdot g = 0$ for all $(X,A)\in\Gamma(E)$. The Killing form on $\mathfrak{so}(TM,g)$ is $\mathrm{ad}_{\mathfrak{so}(TM,g)}$-invariant and, by definition, the metric in $\mathfrak{so}(TM,g)$ induced by g. This means that $(X,A)\cdot B^g = \nabla_X B^g + A\cdot B^g = 0$ for all $(X,A)\in\Gamma(E)$, just as for g. The proposition below follows.

Proposition 2.2.10. *The Killing metric is annihilated by the action of $(\Gamma(E),\cdot)$. To be precise*

$$\phi\cdot g_E = 0,$$

for all $\phi\in\Gamma(E)$.

In terms of the natural projections, and for a given Killing-regular tensor field α, we can define the vector subbundle

$$H^{\alpha,\ell} := \ker(\pi_{TM} : K^{\alpha,\ell} \to TM)$$

of $K^{\alpha,\ell}$. It is explicitly by

$$H^{\alpha,\ell} = \{(0, A) \in E \ : \ A \cdot \nabla^i \alpha = 0, \text{ for all } 0 \le i \le \ell\}.$$

The projection of $H^{\alpha,\ell}$ into $\mathfrak{so}(TM, g)$ will be denoted by $\mathfrak{h}^{\alpha,\ell}$, i.e.

$$\mathfrak{h}^{\alpha,\ell} = \{A \in \mathfrak{so}(TM, g) \ : \ A \cdot \nabla^i \alpha = 0, \text{ for all } 0 \le i \le \ell\}.$$

When the restriction of the Killing form on $\mathfrak{so}(TM, g)$ to $\mathfrak{h}^{\alpha,\ell}$ is non-degenerate, the vector bundle $K^{\alpha,\ell}$ admits an orthogonal direct sum decomposition

$$K^{\alpha,\ell} = H^{\alpha,\ell} \oplus C^{\alpha,\ell}, \tag{2.30}$$

where $C^{\alpha,\ell}$ will denote the orthogonal complement of $H^{\alpha,\ell}$ in $K^{\alpha,\ell}$, with respect to $g_E|_{K^{\alpha,\ell}}$.

We will restrict ourselves to Killing-regular tensor fields such that the Killing form restricted to $\mathfrak{h}^{\alpha,\infty}$ is non-degenerate.

Proposition 2.2.11. *Let α be a Killing-regular tensor field on (M, g) such that the Killing form on $\mathfrak{h}^{\alpha,\infty}$ is non-degenerate and let*

$$K^{\alpha,\infty} = H^{\alpha,\infty} \oplus C^{\alpha,\infty}$$

be an orthogonal direct sum decomposition. Then, the orthogonal direct sum decomposition is preserved by $\Gamma(K^{\alpha,\infty})$, i.e.

$$\Gamma(K^{\alpha,\infty}) \cdot \Gamma(H^{\alpha,\infty}) \subseteq \Gamma(H^{\alpha,\infty}) \quad \textit{and} \quad \Gamma(K^{\alpha,\infty}) \cdot \Gamma(C^{\alpha,\infty}) \subseteq \Gamma(C^{\alpha,\infty}).$$

Proof. By Proposition 2.2.7, $(\Gamma(K^{\alpha,\infty}), \cdot)$ is a subalgebra of $(\Gamma(E), \cdot)$. To show that $\Gamma(K^{\alpha,\infty})$ preserves the splitting of $K^{\alpha,\infty}$, the first notice that if ϕ is a section of $K^{\alpha,\infty}$ and $(0, A)$ one of $H^{\alpha,\infty}$, we have that

$$\phi \cdot \begin{bmatrix} 0 \\ A \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \cdot A \end{bmatrix}$$

is a section of $K^{\alpha,\infty}$. Since its TM component is equal to 0, it is in fact a section of $H^{\alpha,\infty}$.

It is only left to show that $\Gamma(K^{\alpha,\infty}) \cdot \Gamma(C^{\alpha,\infty}) \subseteq \Gamma(C^{\alpha,\infty})$. By Proposition 2.2.10, $\phi \cdot g_E = 0$ for all $\phi \in \Gamma(E)$. Then, choosing $\psi \in \Gamma(H^{\alpha,\infty})$ and $\eta \in \Gamma(C^{\alpha,\infty})$, we get

$$0 = (\phi \cdot g_E)(\psi, \eta) = \phi \cdot (g_E(\psi, \eta)) - g_E(\phi \cdot \psi, \eta) - g_E(\psi, \phi \cdot \eta) = -g_E(\psi, \phi \cdot \eta).$$

Since $\psi \in \Gamma(H^{\alpha,\infty})$, this shows that $\phi \cdot \eta \in \Gamma(C^{\alpha,\infty})$ for all $\phi \in \Gamma(K^{\alpha,\infty})$. □

Interestingly enough (for the following sections), the above proposition shows that

$$\Gamma(H^{\alpha,\infty}) \cdot \Gamma(C^{\alpha,\infty}) \subseteq \Gamma(C^{\alpha,\infty}).$$

The following corollary shows us that an analogous statement as the one of the above equation holds for $(\Gamma(K^{\alpha,\infty}), [\![\cdot,\cdot]\!])$.

Corollary 2.2.12. *Let α be a Killing-regular tensor field on (M, g) such that the Killing form on $\mathfrak{h}^{\alpha,\infty}$ is non-degenerate and let*

$$K^{\alpha,\infty} = H^{\alpha,\infty} \oplus C^{\alpha,\infty}$$

be an orthogonal direct sum decomposition. Then,

$$[\![H^{\alpha,\infty}, C^{\alpha,\infty}]\!] \subseteq C^{\alpha,\infty}.$$

Proof. Let us chosing $\phi \in \Gamma(H^{\alpha,\infty})$ and $\psi \in \Gamma(C^{\alpha,\infty})$. By Lemma 2.2.6, we can see that $[\![\phi, \psi]\!] = \phi \cdot \psi - D_{\pi_{TM}(\phi)}\psi$. However, $\pi_{TM}(\phi) = 0$ from the definition of $H^{\alpha,\infty}$ and, consequently, $[\![\phi, \psi]\!] = \phi \cdot \psi$ which is contained in $\Gamma(C^{\alpha,\infty})$, by Proposition 2.2.11. □

In the reminder of this section, we will present a procedure to build metric connections from sections of 1-forms taking values on subbundles of the Killing bundle. In general, we will consider the 1-forms with values in E as

$$\wedge^1 M \otimes E \;=\; \begin{array}{c} \wedge^1 M \otimes TM \\ \oplus \\ \wedge^1 M \otimes \mathfrak{so}(TM, g) \end{array} \;=\; \begin{array}{c} \mathrm{End}(TM) \\ \oplus \\ \wedge^1 M \otimes \mathfrak{so}(TM, g) \end{array} \tag{2.31}$$

Abusing notation, the projections to each summand will be

$$\pi_{TM} : \wedge^1 M \otimes E \to \wedge^1 M \otimes TM \quad \text{and} \quad \pi_{\mathfrak{so}} : \wedge^1 M \otimes E \to \wedge^1 M \otimes \mathfrak{so}(TM, g)$$

and also their restrictions to subbundles of $\wedge^1 M \otimes E$. By considering $\wedge^1 M \otimes TM$ as the bundle of endomorphisms of TM, we can see from equation (2.31) that it is always possible to build sections of $\wedge^1 M \otimes E$ such that its $\wedge^1 M \otimes TM$ component is equal to Id_{TM}, the identity endomorphism of the tangent bundle viewed as a section of $\wedge^1 M \otimes TM$. In other words, this is equivalent to say that there exists a section

$$\sigma \in \Gamma(\wedge^1 M \otimes E) \quad \text{such that} \quad \pi_{TM}(\sigma) = \mathrm{Id}_{TM}. \tag{2.32}$$

Given any $\sigma \in \Gamma(\wedge^1 M \otimes E)$ satisfying equation (2.32), we will let S denote its projection into $\wedge^1 \otimes \mathfrak{so}(TM, g)$, i.e. $S = \pi_{\mathfrak{so}}(\sigma)$. When contracted with a vector X, we will write S_X instead of $S(X)$. Notice that

$$\sigma(X) \cdot \alpha = \nabla_X \alpha + S_X \cdot \alpha, \quad \text{for any} \quad X \in TM \quad \text{and} \quad \alpha \in \Gamma(T^*M^{\otimes^r}).$$

Since S is a section of $\wedge^1 M \otimes \mathfrak{so}(TM, g)$, it is evident that

$$\sigma(X) \cdot g = \nabla_X g + S_X \cdot g = 0, \quad \text{for all} \quad X \in TM,$$

which means that in fact σ defines a metric connection $\tilde{\nabla} := \nabla + S$.

Remark 2.2.13. The above construction holds if we replace the Killing bundle for any of its subbundles that projects surjectively onto TM.

We will pay special attention to subbundles of E associated to families of Killing-regular tensor. Analogously to how we have previously defined the bundles K^α and H^α, if $\alpha_1, \ldots, \alpha_k$ are Killing-regular tensors, we can define the subbundles

$$K^{(\alpha_1,\ldots\alpha_k)} := \bigcap_{i=1}^{k} K^{\alpha_i} \quad \text{and} \quad H^{(\alpha_1,\ldots\alpha_k)} := \ker(\pi_{TM} : K^{(\alpha_1,\ldots\alpha_k)} \to TM),$$

of the Killing bundle. In the case when the Killing form on $\mathfrak{h}^{(\alpha_1,\ldots,\alpha_k)}$, the image of $\pi_{\mathfrak{so}}$ on $H^{(\alpha_1,\ldots\alpha_k)}$, is non-degenerate we will denote the orthogonal complement of $H^{(\alpha_1,\ldots\alpha_k)}$ in $K^{(\alpha_1,\ldots\alpha_k)}$ by $C^{(\alpha_1,\ldots\alpha_k)}$.

Lemma 2.2.14. *Let $\alpha_1, \ldots, \alpha_k$ be a Killing-regular tensor fields such that the natural projection $\pi_{TM} : K^{(\alpha_1,\ldots,\alpha_k)} \to TM$ is surjective and the Killing form on $\mathfrak{h}^{(\alpha_1,\ldots,\alpha_k)}$ is non-degenerate. Then, there exists a unique section σ, of $\wedge^1 M \otimes C^{(\alpha_1,\ldots,\alpha_k)}$, such that $\pi_{TM}(\sigma) = \mathrm{Id}_{TM}$.*

Proof. Since $H^{(\alpha,\ldots,\alpha_k)}$ is by definition the kernel of $\pi_{TM} : K^{(\alpha,\ldots,\alpha_k)} \to TM$ and $K^{(\alpha,\ldots,\alpha_k)}$ projects surjectively onto TM, so does $C^{(\alpha,\ldots,\alpha_k)}$. By the above construction, the vector bundle $\wedge^1 M \otimes C^{(\alpha_1,\ldots,\alpha_k)}$ admits a section satisfying equation (2.32)(see Remark 2.2.13).

Let σ be a section of $\wedge^1 M \otimes C^{(\alpha_1,\ldots,\alpha_k)}$ such that $\pi_{TM}(\sigma) = \mathrm{Id}_{TM}$. To prove that σ is unique, suppose there exists a section $\sigma' \in \Gamma(\wedge^1 M \otimes C^{(\alpha_1,\ldots,\alpha_k)})$ such that $\pi_{TM}(\sigma') = \mathrm{Id}_{TM}$. Then, if we let S and S' be the $\wedge^1 M \otimes \mathfrak{so}(TM, g)$ components of σ and σ', respectively, we observe that

$$0 = \sigma(X) \cdot \alpha_i - \sigma'(X) \cdot \alpha_i = (S_X - S'_X) \cdot \alpha_i, \quad \text{for all} \quad i = 1, \ldots, k.$$

The above equation implies that $S_X - S'_X$ is in $\mathfrak{h}^{(\alpha_1,\ldots,\alpha_k)}$ for all $X \in TM$, which contradicts our assumption of σ and σ' being sections of $\wedge^1 M \otimes C^{(\alpha_1,\ldots,\alpha_k)}$. Consequently, $S = S'$ and hence $\sigma = \sigma'$. $\square$

At first sight, the above lemma may appear disconnected to the narrative of this section. However, we have previously showed in Proposition 2.2.4 that the subbundle $K^{\alpha,\infty}$ is in fact equal to K^{α,s_α} for some integer $s_\alpha < \infty$, which is nothing but the bundle $K^{(\alpha,\nabla\alpha,\ldots,\nabla^{s_\alpha}\alpha)}$. This means that in the instances where $K^{\alpha,\infty}$ satisfies all the hypothesis imposed on $K^{(\alpha_1,\ldots,\alpha_k)}$ in Lemma 2.2.14, the existence of a unique section of $\wedge^1 M \otimes C^{\alpha,\infty}$ which solves equation (2.32) is guaranteed.

Lemma 2.2.15. *Let α be a Killing-regular tensor such that $\pi_{TM} : K^{\alpha,\infty} \to TM$ is surjective and the Killing form on $\mathfrak{h}^{\alpha,\infty}$ is non-degenerate. Then, if σ is a section of $\wedge^1 M \otimes C^{\alpha,\infty}$ of the form $\sigma = (\mathrm{Id}_{TM}, S)$, we have*

$$\nabla_X S + S_X \cdot S = 0.$$

Proof. We have shown in Proposition 2.2.11 that $\Gamma(K^{\alpha,\infty}) \cdot \Gamma(C^{\alpha,\infty}) \subseteq \Gamma(C^{\alpha,\infty})$ and, particularly, $\Gamma(C^{\alpha,\infty}) \cdot \Gamma(C^{\alpha,\infty}) \subseteq \Gamma(C^{\alpha,\infty})$. By Lemma 2.2.14 there exists a unique section of $\wedge^1 M \otimes C^{\alpha,\infty}$ which is of the forms (Id_{TM}, S). Let us denote this section by σ. It is straightforward to see that

$$\sigma(X)\cdot\sigma(Y) = \begin{bmatrix} \nabla_X Y + S_X Y \\ \nabla_X S_Y + [S_X, S_Y] \end{bmatrix} = \begin{bmatrix} \nabla_X Y + S_X Y \\ S_{\nabla_X Y + S_X Y} \end{bmatrix} + \begin{bmatrix} 0 \\ (\nabla_X S)_Y + (S_X \cdot S)_Y \end{bmatrix} \in \Gamma(C^{\alpha,\infty}).$$

Lastly, that $\nabla_X S + S_X \cdot S = 0$ follows from the uniqueness of σ. □

Lastly, for a given Killing-regular tensor field, we will say that $\sigma \in \Gamma(\wedge^1 M \otimes E)$ is an *α-Killing section* if it solves the system or partial differential equations

$$\pi_{TM}(\sigma) = \mathrm{Id}_{TM}, \quad \sigma(X)\cdot\alpha = 0 \quad \text{and} \quad \sigma(X)\cdot\sigma = 0 \quad \text{for all } X \in TM \tag{2.33}$$

Lemma 2.2.16. *Let α be a Killing-regular tensor field on (M,g) and let $\sigma \in \Gamma(\wedge^1 M \otimes E)$ be an α-Killing section. Then σ is a section of $\wedge^1 M \otimes K^{\alpha,\infty}$.*

Proof. From the first two equations in (2.33) we deduce that $\sigma = (\mathrm{Id}_{TM}, S)$ and a section of $\wedge^1 M \otimes K^\alpha$, for some $S \in \Gamma(\wedge^1 M \otimes \mathfrak{so}(TM, g)$. Expanding the third equation, we can see that

$$\sigma(X)\cdot\sigma = \begin{bmatrix} \nabla_X \mathrm{Id}_{TM} + [S_X, \mathrm{Id}_{TM}] \\ \nabla_X S + S_X \cdot S \end{bmatrix} = \begin{bmatrix} 0 \\ \nabla_X S + S_X \cdot S \end{bmatrix}$$

implies that $\sigma(X)\cdot S = 0$ for all $X \in TM$. In other words, σ is a section of $\wedge^1 M \otimes K^{(\alpha,S)}$. It is only left to show that $K^{(\alpha,S)}$ is in fact equal to $K^{\alpha,\infty}$. Recall that for any section ϕ of E we have

$$\iota_X(\phi\cdot\nabla\alpha) = \phi\cdot\nabla_X\alpha - \nabla_{\phi\cdot X}\alpha \quad \text{and} \quad \nabla_X\alpha = -S_X\cdot\alpha.$$

Then, a direct calculation reveals

$$\iota_Y(\sigma(X)\cdot\nabla\alpha) = \sigma(X)\cdot\nabla_Y\alpha - \nabla_{\sigma(X)\cdot Y}\alpha = -\sigma(X)\cdot(S_Y\cdot\alpha) + S_{\sigma(X)\cdot Y}\alpha = -(\sigma(X)\cdot S)_Y\cdot\alpha$$

which is equal to 0, since $\sigma \in \Gamma(\wedge^1 M \otimes K^{(\alpha,S)})$. Inductively, we can see that

$$\iota_Y(\sigma(X)\cdot\nabla^{\ell+1}\alpha) = -(\sigma(X)\cdot S)_Y\cdot\nabla^\ell\alpha$$

for all $\ell \geq 0$. This means that σ is a section of $\wedge^1 M \otimes K^{\alpha,\infty}$. □

2.3 The Killing curvature

The kernel of the Killing curvature at a point p is, by definition, the vector subspace of E_p comprised by elements (X, A) such that

$$\nabla_X R + A\cdot R = 0,$$

namely, K_p^R. In [38], Katsumi Nomizu introduced the notion of *Killing generators* of a pseudo-Riemannian manifold. These are elements $(X, A) \in E_p$, solutions to the following equation

$$\nabla_X \nabla^\ell R + A \cdot \nabla^\ell R = 0 \tag{2.34}$$

for all $\ell \geq 0$, i.e. the space of Killing generators at p is precisely $K_p^{R,\infty}$. Moreover, in [39], it was proven that the space of Killing generators at a point p, equiped with the Killing bracket, becomes a real Lie algebra.

For completeness, we will show that the Killing bracket, defined in equation 2.16 is indeed a Lie bracket on $K_p^{R,\infty}$. That $K_p^{R,\infty}$ is closed under the Killing bracket follows from Proposition 2.2.5, with the Riemannian curvature tensor taking the place of α. To show that the bracket defined in equation (2.16) satisfies the Jacobi identity, we compute

$$\left[\!\!\left[\begin{bmatrix} X \\ A \end{bmatrix}, \left[\!\!\left[\begin{bmatrix} Y \\ B \end{bmatrix}, \begin{bmatrix} Z \\ C \end{bmatrix} \right]\!\!\right] \right]\!\!\right] = \begin{bmatrix} ABZ - ACY - [B,C]X + R(Y,Z)X \\ [A,[B,C]] - [A, R(Y,Z)] + R(BZ,X) + R(X,CY) \end{bmatrix}.$$

A direct computation reveals that the TM component of the cyclic sum of the above equation vanishes identically. Letting $\mathfrak{C}$ denote the cyclic sum, its $\mathfrak{so}(TM, g)$ component becomes

$$(\pi_{\mathfrak{so}} \circ \mathfrak{C}) \left[\!\!\left[\begin{bmatrix} X \\ A \end{bmatrix}, \left[\!\!\left[\begin{bmatrix} Y \\ B \end{bmatrix}, \begin{bmatrix} Z \\ C \end{bmatrix} \right]\!\!\right] \right]\!\!\right] = -(A \cdot R)(Y,Z) - (B \cdot R)(Z,X) - (C \cdot R)(X,Y). \tag{2.35}$$

Since (X, A), (Y, B) and (Z, C) are Killing generators, it follows that

$$A \cdot R = -\nabla_X R, \quad B \cdot R = -\nabla_Y R \quad \text{and} \quad C \cdot R = -\nabla_Z R$$

and therefore

$$(\pi_{\mathfrak{so}} \circ \mathfrak{C}) \left[\!\!\left[\begin{bmatrix} X \\ A \end{bmatrix}, \left[\!\!\left[\begin{bmatrix} Y \\ B \end{bmatrix}, \begin{bmatrix} Z \\ C \end{bmatrix} \right]\!\!\right] \right]\!\!\right] = (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y).$$

which, by the second Bianchi identity, vanishes identically. It is worth noticing that equation (2.35) is the only obstruction for the Killing bracket on $K_p^{\alpha,\infty}$, to be a Lie bracket. The proposition below follows directly from this argument.

Proposition 2.3.1. *Let $\alpha \in \Gamma(T^*M^{\otimes^r})$. Then $(K_p^{\alpha,\infty} \cap K_p^{R,\infty}, [\![\cdot,\cdot]\!])$ is a Lie subalgebra of $(K_p^{R,\infty},, [\![\cdot,\cdot]\!])$.*

Interested in pseudo-Riemannian manifolds whose curvature tensors are Killing-regular, we obtained the following relation between the maximal parallel flat subbundle K, of the Killing bundle, and the bundles of derivations of R.

Proposition 2.3.2. *Let (M, g) be a pseudo-Riemannian manifold and let K be the maximal parallel flat subbundle of E. Then, for each $p \in M$, $K_p \subseteq K_p^{R,\infty}$ and the equality holds if and only if R is Killing-regular.*

Proof. Let ϕ be any section of K. For each $p \in M$, by definition K_p is contained in K_p^R, the kernel of the Killing curvature at p. Since K is parallel we obtain

$$\begin{aligned} 0 &= (D_{X_\ell} \dots D_{X_1}\phi) \cdot R \\ &= -\iota_{X_\ell}((D_{X_{\ell-1}} \dots D_{X_1}\phi) \cdot \nabla R) \\ &\vdots \\ &= (-1)^\ell \iota_{X_\ell} \dots \iota_{X_1}(\phi \cdot \nabla^\ell R) \end{aligned}$$

which follows from Lemma 2.19. This is independent of the point and of ℓ and therefore $K_p \subseteq K_p^{R,\infty}$.

If in addition we assume that R is a Killing-regular tensor field, the kernel of the Killing curvature is the vector bundle K^R. By Lemma 2.2.3, the bundle $K^{R,\infty}$ is the maximal parallel subbundle of E which contained in K^R. This is precisely the definition of K. □

Corollary 2.3.3. *If (M, g) is a pseudo-Riemannian manifold whose curvature tensor is Killing-regular, $K = K^{R,\infty}$.*

The following proposition is a well known result, which we have taken from the book of Kobayashi and Nomizu [33, Proposition 2.6, Chapter VI]. We should say that there is a sign mistake in the statement of the proposition.

Proposition 2.3.4. *Let ξ and η be Killing vector fields. Then*

$$A_{[\xi,\eta]} = [A_\xi, A_\eta] - R(\xi, \eta). \tag{2.36}$$

Proof. In general, the bracket of vector fields can be expressed in terms of the Nomizu operators as

$$[\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi = A_\xi \eta - A_\eta \xi. \tag{2.37}$$

Then, it is straightforward to see that

$$A_{[\xi,\eta]}X = -\nabla_X(A_\xi \eta - A_\eta \xi) = -(\nabla_X A_\xi)(\eta) - A_\xi \nabla_X \eta + (\nabla_X A_\eta)(\xi) + A_\eta \nabla_X \xi. \tag{2.38}$$

Notice that in the right hand side of equation (2.38) we have that

$$-A_\xi \nabla_X \eta + A_\eta \nabla_X \xi = A_\xi A_\eta X - A_\eta A_\xi X = [A_\xi, A_\eta]X.$$

Since ξ and η are Killing vector fields, we have that

$$(\nabla_X A_\xi)(\eta) = -R(X, \xi)\eta \quad \text{and} \quad (\nabla_X A_\eta)(\xi) = -R(X, \eta)\xi$$

by equation (2.12). Then, the remaining terms of the right hand side of equation (2.38) become

$$-(\nabla_X A_\xi)(\eta) + (\nabla_X A_\eta)(\xi) = R(X, \xi)\eta - R(X, \eta)\xi = -R(\xi, \eta)X,$$

where the last equality follows from the Bianchi identity. It follows that equation (2.38) becomes

$$A_{[\xi,\eta]}X = [A_\xi, A_\eta]X - R(\xi,\eta)X$$

as claimed. $\square$

Corollary 2.3.5. *The natural inclusion map* $(H^0(E,D),\{\cdot,\cdot\}) \to (\Gamma(K).[\![\cdot,\cdot]\!])$ *is a Lie algebra monomorphism.*

Proof. Let ξ and η be Killing vector fields. From the definition of the bracket in $H^0(E,D)$, we have that

$$\left\{\begin{bmatrix}\xi\\ A_\xi\end{bmatrix},\begin{bmatrix}\eta\\ A_\eta\end{bmatrix}\right\} = \begin{bmatrix}[\xi,\eta]\\ A_{[\xi,\eta]}\end{bmatrix}.$$

From equations (2.36) and (2.37) in Proposition (2.3.4), it follows that

$$\begin{bmatrix}[\xi,\eta]\\ A_{[\xi,\eta]}\end{bmatrix} = \begin{bmatrix}A_\xi\eta - A_\eta\xi\\ [A_\xi, A_\eta] - R(\xi,\eta)\end{bmatrix} = \left[\!\!\left[\begin{bmatrix}\xi\\ A_\xi\end{bmatrix},\begin{bmatrix}\eta\\ A_\eta\end{bmatrix}\right]\!\!\right],$$

where the last equality follows from restricting the Killing bracket to sections of K. $\square$

To conclude this chapter, let $\mathfrak{k}_p^{R,\infty}$ denote the projection of $K_p^{R,\infty}$ to $\mathfrak{so}(T_pM,g_p)$ and let $\mathfrak{hol}_p(M,g)$ be the holonomy Lie algebra of (M,g) with base point p.

Proposition 2.3.6. *The projection of* $K_p^{R,\infty}$ *to* $\mathfrak{so}(T_pM,g_p)$ *is contained in*

$$\mathfrak{n}_p := \{A \in \mathfrak{so}(T_pM,g_p)\ :\ [A,H] \in \mathfrak{hol}_p(M,g),\ \forall H \in \mathfrak{hol}_p(M,g)\}, \tag{2.39}$$

the normaliser of the holonomy algebra of (M,g).

Proof. Choosing (ξ,A) in $K_p^{R,\infty}$, it follows by equation (2.34) that

$$A\cdot\nabla^\ell R = -\nabla_\xi\nabla^\ell R \in (\wedge_p^1 M)^{\otimes^{\ell+1}} \otimes \wedge_p^2 M \otimes \mathfrak{hol}_p(M,g).$$

A closer look at the above equation reveals that

$$\begin{aligned}[A,(\nabla^\ell R)(X_1,\ldots,X_\ell;X,Y)] \;=\; & -(\nabla^{\ell+1}R)(\xi,X_1,\ldots,X_\ell;X,Y)\\ & +(\nabla^\ell R)(AX_1,\ldots,X_\ell;X,Y)\\ & \vdots\\ & +(\nabla^\ell R)(X_1,\ldots,X_\ell;X,AY)\end{aligned}$$

The right hand side of the above equation is contained in the holonomy algebra of (M,g), since it is contained in the span of the curvature tensor and its derivatives. It follows that

$$[A,(\nabla^\ell R)(X_1,\ldots,X_\ell;X,Y)] \in \mathfrak{hol}_p(M,g),$$

for all $X_1,\ldots,X_\ell,X,Y \in T_pM$, which implies that A is in the normaliser of holonomy algebra, $\mathfrak{n}_p$. $\square$

Chapter 3

Pseudo-Riemannian locally homogeneous spaces

In this chapter we provide a characterisation of pseudo-Riemannian locally homogeneous spaces by means of the Killing bundle and the Killing connection, and give a new proof of the Ambrose-Singer on homogeneous structures [22], under slightly different assumptions. In the last Section, we discuss the Singer index of pseudo-Riemannian locally homogeneous spaces and relate it to properties of subbundles of the Killing bundle.

3.1 The Killing bundle of pseudo-Riemannian locally homogeneous spaces

A pseudo-Riemannian manifold (M, g) is called *locally homogeneous* if for any given pair of points p and q of M, there exist open neighbourhoods U and V of p and q, respectively, and a local isometry $f : U \to V$ such that $f(p) = q$. Equivalently, in terms of local Killing vector fields, (M, g) is a pseudo-Riemannian locally homogeneous space if and only if for each point $p \in M$, there exists an open neighbourhood of p such that its Killing vector fields provide a frame of the tangent bundle of M. Particularly, (M, g) will be called (globally) *homogeneus* if there exist a subgroup of the isometry group $\mathrm{Iso}(M, g)$, acting transitively by isometries on M. In other words, for any two given points p and q of M, there exists $g \in G$ such that $g \cdot p = q$. A pseudo-Riemannian homogeneous space might admit many subgroups of its isometry group acting transitively, for which the same homogeneous space could be represented by different quotient spaces. If $M = G/H$, and we let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively, we will say that the pair $(\mathfrak{g}, \mathfrak{h})$ is *reductive* if $\mathfrak{g}$ admits a direct sum decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{such that} \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

We have shown in Proposition 2.1.4 that the Lie algebra $\mathfrak{kill}(M, g)$, of Killing vector fields of (M, g), is isomorphic to the Lie algebra of the parallel sections of its Killing bundle, equiped with the bracket defined in equation (2.16). Therefore we can provide a characterisation of the local homogeneity of (M, g) in terms of the local parallel sections of its Killing bundle. In general, for any open subset U of M, the map

$$\mathfrak{kill}(U, g|_U) \to \Gamma(U, K), \quad \xi \mapsto \begin{bmatrix} \xi \\ A_\xi \end{bmatrix}.$$

is an injection of the local Killing vector fields on U into the local sections of the maximal parallel flat subbundle K, of E. Letting $\mathfrak{kill}(U, g|_U)_p$ denote the vector subspace of T_pM, obtained by evaluating the local Killing fields of U at p, it follows that when (M, g) is locally homogeneous, the composition

$$\mathfrak{kill}(U, g|_U)_p \to K_p \to T_pM$$

is surjective for all p in M. In other words, the natural projection from K to TM is surjective. On the other hand, K being a flat and parallel subbundle of the Killing bundle implies that, locally, there exist a frame of K composed of local parallel sections of E. If in addition $K \to TM$ is a surjection, for each point of M, there exist a local frame of TM comprised of Killing vector fields. Therefore, we have proven the following proposition.

Proposition 3.1.1. *A pseudo-Riemannian manifold (M, g) is locally homogeneous if and only if the natural projection $K \to TM$ is a surjection.*

The previous proposition was known to Nomizu in [38] for the Riemannian setting. However, he enunciated his result in a slightly different language than ours. In our terminology, he showed that a Riemannian manifold (M, g) is locally homogeneous if and only if

(1) R is Killing-regular.

(2) The projection $K_p^{R,\infty} \to T_pM$ is surjective for all p in M.

These two conditions are in fact equivalent to $K \to TM$ being a surjection. Indeed, condition (1) implies that we can build the vector bundle $K^{R,\infty}$ whose fibres are given by Killing generators which, by condition (2), will project subjectively onto TM. That $K \to TM$ is a surjection follows from Corollary 2.3.3, which provides us with the equality between $K^{R,\infty}$ and K.

In the previous chapter we provided a description of subbundles of the Killing bundle, associated to Killing-regular tensors. Now, we will focus on tensor fields on pseudo-Riemannian locally homogeneous spaces, which are invariant by local isometries. To be precise, we will say that a tensor field $\alpha \in \Gamma(T^*M^{\otimes^r})$ is invariant by local isometries if for any local isometry of (M, g), $f : U \to V$, then $f^*\alpha|_U = \alpha|_U$.

If α is a tensor field invariant by local isometries of a pseudo-Riemannian locally homogeneous space, we have a natural isomorphism between $K_p^{\alpha,\ell}$ and $K_q^{\alpha,\ell}$, for any pair of points p, q in M and consequently, for each $\ell \geq 0$, the map $p \mapsto \dim K_p^{\alpha,\ell}$ is a constant function on M. In other words, invariant tensor fields on locally homogeneous spaces are Killing-regular, which will allow us to define the vector bundles $K^{\alpha,\ell}$ over M, with fibres $K_p^{\alpha,\ell}$ at $p \in M$. Particularly, we will apply the results obtained in Chapter 2 to build metric connections that parallelise invariant tensor fields.

Recall that the kernel of the Killing curvature at a point p of M is given by K_p^R. The curvature tensor of the Levi-Civita connection is of course invariant by local isometries and, for this reason, local homogeneity guarantees us that the kernel of the Killing curvature is in fact a vector subbundle of E. Moreover, for each $\ell \geq 0$, $K_p^{R,\ell}$ is defined solely in terms of R and its iterated covariant derivatives and, for this reason, $K^{R,\ell}$ will be subbundle of E.

Proposition 3.1.2. *Let (M, g) be a pseudo-Riemannian locally homogeneous space. Then the maximal parallel flat subbundle K of E is the vector bundle over M whose fibres consist of Killing generators.*

Proof. Since (M, g) is locally homogeneous, its curvature tensor is invariant by local isometries and ,therefore, it is Killing-regular. It follows by Corollary 2.3.3, that the maximal parallel flat subbundle of E is equal to $K^{R,\infty}$ which, by definition, has fibres comprised by the Killing generators. □

Establishing conventions for the reminder of this chapter, we will simplify our notation when considering locally homogeneous spaces. We write K instead of $K^{R,\infty}$ and analogously H and $\mathfrak{h}$ will take the place of $H^{R,\infty}$ and $\mathfrak{h}^{R,\infty}$, respectively. Lastly, if the Killing form on $\mathfrak{h}$ is non-degenerate, C will denote the orthogonal complement of H in K.

3.2 Homogeneous structures

It is a well known result, due to E. Cartan, that a connected, simply connected and complete Riemannian manifold is a symmetric space if and only if its curvature tensor is covariantly constant. In [3], W. Ambrose and I. M. Singer provided a characterisation of a Riemannian homogeneous space (M, g), extending the aforementioned result from E. Cartan, to homogeneous spaces. To be precise, they proved that a connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if there exists a tensor field $S \in \Gamma(\wedge^1 M \otimes \mathrm{End}(TM))$, that is a solution to the system of partial differential equations

$$S_X \cdot g = 0, \quad \nabla_X R + S_X \cdot R = 0 \quad \text{and} \quad \nabla_X S + S_X \cdot S = 0, \tag{3.1}$$

for all $X \in \Gamma(TM)$. The above system of partial differential equations will be referred to as the *Ambrose-Singer equations.* These conditions for the tensor field S are in fact equivalent to the existence of a metric connection $\nabla^S := \nabla + S$ with covariantly constant curvature and torsion tensors, with respect to ∇^S. Namely,

$$\nabla^S g = 0, \quad \nabla^S T^S = 0 \quad \text{and} \quad \nabla^S R^S = 0, \tag{3.2}$$

where T^S and R^S are the torsion and curvature tensors of ∇^S, respectively. Any tensor field satisfying equations (3.1) will be referred to as a *homogeneous structure* on (M, g) and its associated connection or, equivalently, a connection satisfying equation (3.2) will be called an *Ambrose-Singer connection*. The torsion and curvature of ∇^S relate to R and S by the formulas

$$T^S_X Y = S_X Y - S_Y X \quad \text{and} \quad R^S(X,Y) = R(X,Y) - [S_X, S_Y] + S_{T^S_X Y} \tag{3.3}$$

and, contracting S with the metric tensor, we can recover the homogeneous structure S from T^S by

$$2g(S_X Y, Z) = g(T^S_X Y, Z) - g(T^S_Y Z, X) + g(T^S_Z X, Y). \tag{3.4}$$

In the book [42], of Tricerri and Vanhecke, a new proof of the results of Ambrose and Singer was given. In the pseudo-Riemannian setting, Gadea and Oubiña showed in [22] that a homogeneous structure on a pseudo-Riemannian manifold exists if and only if it is a reductive homogeneous space. The most general statement, due to Kiričenko [32], shows that the Ambrose-Singer connection also preserves any tensor field that is invariant by local isometries. We also refer to the book of Calvaruso and Castrillón López [11] for a proof of the most general, Ambrose-Singer-Kiričenko theorem.

In what follows, we will apply the results obtained in Chapter 2 to provide a description of the Ambrose-Singer equations and the Ambrose-Singer-Kiričenko theorem in the language of the Killing bundle. Recall that a section of $\wedge^1 M \otimes E$, solving the system of differential equations

$$\pi_{TM}(\sigma) = \mathrm{Id}_{TM}, \quad \sigma(X) \cdot R = 0 \quad \text{and} \quad \sigma(X) \cdot \sigma = 0, \tag{3.5}$$

has been called an R-Killing section. The existence of such a section is in fact equivalent to the existence of a homogeneous structure on (M, g), and equation (3.5) is in a sense, an embedding of the Ambrose-Singer equations into the Killing bundle setting. Indeed, the first of the above equations implies that $\sigma = (\mathrm{Id}_{TM}, S)$, where S is some section of $\wedge^1 M \otimes \mathfrak{so}(TM, g)$. Therefore

$$\sigma(X) \cdot R = \nabla_X R + S_X \cdot R$$

and

$$\sigma(X) \cdot \sigma = \begin{bmatrix} \nabla_X \mathrm{Id}_{TM} + [S_X, \mathrm{Id}_{TM}] \\ \nabla_X S + S_X \cdot S \end{bmatrix} = \begin{bmatrix} 0 \\ \nabla_X S + S_X \cdot S \end{bmatrix} \tag{3.6}$$

show that equations (3.1) and (3.5) are equivalent. For future convenience, we note that equation (3.6) is equivalent to

$$\sigma(X) \cdot \sigma = \begin{bmatrix} 0 \\ \sigma(X) \cdot S \end{bmatrix}. \tag{3.7}$$

Proposition 3.2.1. *Let (M, g) be a pseudo-Riemannian manifold let σ be an R-Killing section. Then (M, g) is locally homogeneous.*

Proof. In order to show that (M, g) is locally homogeneous, we will prove that σ is in fact a section of $\wedge^1 M \otimes K$. Then, as $\pi_{TM}(\sigma) = \mathrm{Id}_{TM}$, we would get that the projection $K \to TM$ is surjective and therefore, by Proposition 3.1.1, (M, g) will be locally homogeneous.

Let us denote the $\wedge^1 M \otimes \mathfrak{so}(TM, g)$ component of σ by S. From $\sigma(X) \cdot R = 0$, we observe that

$$\iota_Y(\sigma(X)\cdot\nabla R) = \sigma(X)\cdot\nabla_Y R - \nabla_{\sigma(X)\cdot Y} R = -(\sigma(X)\cdot S_Y - S_{\sigma(X)\cdot Y})\cdot R = -(\sigma(X)\cdot S)_Y \cdot R.$$

From Lemma 2.2.1 we obtain

$$(D_Y\sigma(X)) \cdot R = -\iota_Y(\sigma(X) \cdot \nabla R) = (\sigma(X) \cdot S)_Y \cdot R = 0,$$

where the last equality follows from equation (3.7). Inductively, we can see that

$$\sigma(X) \cdot \nabla^\ell R = 0 \quad \text{for all} \quad \ell \geq 0.$$

In other words, σ is a section of $\wedge^1 M \otimes K^{R,\infty}$. From Proposition 3.1.2, we then conclude that σ is in fact a section of $\wedge^1 M \otimes K$. As claimed before, $K \to TM$ is surjective and by Proposition 3.1.1, (M, g) is locally homogeneous. □

For the converse statement, we will provide a proof in the more restrictive case when the Killing form on $\mathfrak{h}$ is non-degenerate, instead of assuming reductivity.

Proposition 3.2.2. *Let (M, g) be a pseudo-Riemannian locally homogeneous space such that the Killing form on $\mathfrak{h}$ is non-degenerate. Then, there exists a unique section σ of $\wedge^1 M \otimes C$, solution to equation (3.5). Moreover*

$$\sigma(X) \cdot \alpha = 0,$$

for any tensor field which is invariant by local isometries of (M, g).

Proof. Under the assumption of (M, g) being locally homogeneous, by Proposition 3.1.2 we get the equality $K = K^{R,\infty}$. Proposition 2.2.4 tells us that $K = K^{R,s_R}$ for some integer $s_R < \infty$, hence $K = K^{R,s_R} = K^{(R,\nabla R,\dots,\nabla^{s_R} R)}$. Setting $K = H \oplus C$ and applying Lemma 2.2.14 we know that the exists a unique section σ, of $\wedge^1 M \otimes C$, such that $\pi_{TM}(\sigma)$ is equal to Id_{TM}. Lastly, that

$$\sigma(X) \cdot \sigma = 0 \quad \text{and} \quad \sigma(X) \cdot R = 0$$

follows by Lemma 2.2.15 and by construction, respectively.

It is only left to prove that if α is a tensor field which is invariant by local isometries of (M, g), then $\sigma(X) \cdot \alpha = 0$ for all $X \in TM$. Recall that the Lie derivative can always be expressed as $L_X = \nabla_X + A_X$ for any vector field X. Particularly, if ξ is a local Killing vector field, we have that

$$\phi_\xi \cdot \alpha = \nabla_\xi \alpha + A_\xi \cdot \alpha = L_\xi \alpha = 0, \quad \text{where} \quad \phi_\xi = (\xi, A_\xi) \in \Gamma(K).$$

It follows that $\phi \cdot \alpha = 0$ for all sections of $\Gamma(K)$ and, particularly $\sigma(X)$. □

3.3 The Singer homogeneous index

Let (M, g) be a pseudo-Riemannian manifold and let p be a point in M. We will define a non-increasing sequence of subalgebras of $\mathfrak{so}(T_pM, g_p)$, analogously to the sequence introduced in equation (2.18), depending on the curvature tensor of (M, g). Recall, from Chapter 2, that the Lie algebra of skew-symmetric endomorphisms which annihilate the curvature tensor and its iterated covariant derivatives at a point p and up to order ℓ, when acted upon, has been defined as

$$\mathfrak{h}_p^{R,\ell} = \{A \in \mathfrak{so}(T_pM, g_p) \ : \ A \cdot \nabla^i R = 0, \ \forall 0 \leq i \leq \ell\}.$$

It is clear from their definition that $\mathfrak{h}_p^{R,\ell+1}$ is contained in $\mathfrak{h}_p^{R,\ell}$ and, consequently, they define the non-increasing sequence of subalgebras

$$\mathfrak{so}(T_pM, g_p) \supseteq \mathfrak{h}_p^{R} \supseteq \mathfrak{h}_p^{R,1} \supseteq \mathfrak{h}_p^{R,2} \supseteq \mathfrak{h}_p^{R,3} \supseteq \dots \tag{3.8}$$

It is clear that the above sequence will stabilise after a finite amount of steps. We will denote by $k_g(p)$, the first integer such that $\mathfrak{h}_p^{R,k_g(p)} = \mathfrak{h}_p^{R,k_g(p)+\ell}$, for all $\ell \geq 0$, and we will refer to $k_g(p)$ as the *Singer index* of (M, g) at p. We note that in the literature it is usually referred to as the Singer invariant.

In a general setting, the Lie algebras $\mathfrak{h}_p^{R,\ell}$ depend on p and so does $k_g(p)$. Nevertheless, it will be of our interest to focus on pseudo-Riemannian manifolds such that for each pair of points, p and q in M, there exists an isomorphism between $\mathfrak{h}_p^{R,\ell}$ and $\mathfrak{h}_q^{R,\ell}$. Particularly, when (M, g) is locally homogeneous, the curvature tensor and its covariant derivatives are invariant by local isometries. Consequently, any local isometry f that maps a point p to a point q, will induce an isomorphism between $\mathfrak{h}_p^{R,\ell}$ and $\mathfrak{h}_q^{R,\ell}$. It follows from this argument that the Singer index map, $p \mapsto k_g(p)$, will be constant on locally homogeneous spaces. In general, when the Singer index is constant, we will ignore the point and we will simply denote it by k_g.

The curvature tensor of a locally homogeneous space is always Killing-regular and therefore, for each $\ell \geq 0$, we can define the vector bundle $\mathfrak{h}^{R,\ell}$ with fibre $\mathfrak{h}_p^{R,\ell}$ at p. Considering $H^{R,\ell}$ instead, we have the sequence

$$E \supseteq H^R \supseteq H^{R,1} \supseteq H^{R,2} \supseteq H^{R,3} \supseteq \dots \tag{3.9}$$

of subbundles of E, that will clearly stabilise in the k_g step.

Proposition 3.3.1. *Let (M, g) be a pseudo-Riemannian locally homogeneous space. Then $k_g \leq s_R$ and the equality holds when (M, g) is reductive.*

Proof. We have shown in Proposition 2.2.4 that the sequence defined in equation (2.24) stabilises at the s_R-th step. That is, $K^{R,s_R} = K^{R,s_R+k}$ for all $k \geq 0$. For each $\ell \geq 0$, $H^{R,\ell}$

is contained in $K^{R,\ell}$ and particularly for $\ell = s_R$. Consequently, $H^{R,s_R} = H^{R,s_R+k}$ for all $k \geq 0$ which implies that $k_g \leq s_R$.

It is only left to prove that, in the case when (M,g) is reductive, $k_g = s_R$. When (M,g) is reductive we have

$$K^R = H^R \oplus C^{R,\infty}, \tag{3.10}$$

as a consequence of the existence of a homogeneous structure. It follows that from the decomposition of K^R in equation (3.10) that the sequence for K^R will stabilise in the same step as the sequence for H^R, hence $k_g = s_R$. □

Examples in the literature for locally homogeneous spaces of arbitrarily high Singer index can be found in [36]. Locally homogeneous spaces with $k_g = 0$ will be of our particular interest in the rest of this thesis. Non-trivial examples of such spaces are scarce in the literature, a few of them can be found in [28].

Example 3.3.2. Locally symmetric spaces are pseudo-Riemannian manifolds whose curvature tensor is parallel with respect to the Levi-Civita connection. Particularly they are locally homogeneous. Since $\nabla R = 0$, $A \cdot \nabla R = 0$ for all $A \in \mathfrak{so}(TM,g)$, which implies that $\mathfrak{h}^R = \mathfrak{h}^{R,1}$. Consequently, locally symmetric spaces have Singer index equal to 0.

The following corollary is an immediate consequence of Proposition 3.3.1.

Corollary 3.3.3. *Let (M,g) be a reductive pseudo-Riemannian locally homogeneous space. Then K^R is parallel if and only if $k_g = 0$.*

Proof. By Proposition 3.3.1, $k_g = s_R$. Also, by the Ambrose-Singer theorem, if (M,g) is a reductive locally homogeneous space it admits a homogeneous structure S. Then, we observe that $C_p^R = C_p^{R,\infty} = \{(X, S_X) \ : \ X \in T_pM\}$, which implies that s_R will depend only on the sequence

$$H^R \supseteq H^{R,1} \supseteq H^{R,2} \supseteq \dots$$

and that $K^R = H^R \oplus C^{R,\infty}$. By Proposition 2.2.4, the kernel of the Killing curvature is parallel if and only if it is equal to $K^{R,\infty}$, and this equality will happen if and only if $H^R = H^{R,\infty}$, i.e. $k_g = 0$. □

Corollary 3.3.3 provides us with a criterion to find locally homogeneous spaces such that the kernel of their Killing curvatures are parallel. In Chapter 5 we will present examples of a class of locally homogeneous Lorentzian manifolds which have Singer index equal to 0.

Chapter 4

The Calabi complex

In this chapter we will give an introduction and present results obtained on the main topic of this thesis, concerning the problem of finding necessary and sufficient conditions for a symmetric 2-tensor to be in the image of the Killing operator. The results presented in Sections 4.1 and 4.2 have already appeared in [12, 13] for pseudo-Riemannian locally symmetric spaces. The author's contribution has been the adaptation of the aforementioned results found in [12, 13] to reductive pseudo-Riemannian locally homogeneous spaces. Most of Section 4.3.2 has already appeared in [13], and the majority has been of the author's contribution.

Oftentimes, throughout this chapter, we will employ Penrose's abstract index notation. At first glance, this change of notation may appear inelegant for the exposition, however its usefulness will become apparent in the proofs of the main results obtained in this chapter and Chapter 5. We will proceed to give a brief description of Penrose's notation.

Upper indices will denote covariant tensors and lower indices will denote contravariant ones. For instance, a vector field will be denoted by ξ^a and, with the aid of a metric tensor, $\xi_a = g_{ab}\xi^b$ will denote the 1-form that is dual to ξ^a, with respect to g_{ab}. We will always raise or lower indices with the metric tensor without expressly saying it. Enclosing indices with round and square brackets will indicate to take the symmetric and skew-symmetric part of a tensor, respectively. For instance, if $\xi_{ab} \in \wedge^1 M \otimes \wedge^1 M$, $\xi_{(ab)}$ and $\xi_{[ab]}$ will be the tensor fields given by the formulas

$$\xi_{(ab)} = \frac{1}{2}(\xi_{ab} + \xi_{ba}) \quad \text{and} \quad \xi_{[ab]} = \frac{1}{2}(\xi_{ab} - \xi_{ba}).$$

The curvature tensor $R_{abcd} \in \Gamma(RM)$, of an affine torsion-free connection ∇_a, will be defined by the formulas

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\xi^c = R_{ab}{}^c{}_d \xi^d \quad \text{and} \quad (\nabla_a \nabla_b - \nabla_b \nabla_a)\xi_c = -R_{ab}{}^d{}_c \xi_d$$

for $\xi^a \in TM$ and $\xi_a \in \wedge^1 M$. For more details about Penrose abstract indices notation we refer to the book of Roger Penrose and Wolfgang Rindler [40].

The role played by Killing vector fields in the former chapters will be replaced by Killing 1-forms. For this reason, the Killing operator will be consider as the first order linear differential operator

$$\mathcal{K} : \Gamma(\wedge^1 M) \to \Gamma(\mathrm{Sym}^2 M) \qquad \sigma_b \mapsto \nabla_{(a}\sigma_{b)}, \tag{4.1}$$

acting on differential 1-forms, rather than on vector fields. We note that if we let $X^a = g^{ab}\sigma_b$, the Killing operator acting on σ_a is nothing but

$$(\mathcal{K}\sigma)_{ab} = \nabla_{(a}\sigma_{b)} = \frac{1}{2}(L_X g)_{ab}.$$

4.1 The twisted de Rham complex and a Calabi complex

In this section, we will derive a complex of linear differential operators providing necessary conditions for a symmetric 2-tensor on pseudo-Riemannian locally homogeneous space to be in the image of the Killing operator. Furthermore, we will establish a relation between the Calabi complex and the short twisted de Rham complex of the Killing bundle with its Killing connection (see Definition 1.2.2). Before proceeding, we remark that the results presented below have already appeared in [12, 13].

Recall that we have defined the Killing operator, as a differential operator acting on vector fields and taking values on the symmetric endomorphisms of the pseudo-Riemannian manifold (M, g), to be the first order linear differential operator

$$\mathrm{k} : \Gamma(TM) \to \Gamma(\mathrm{Sym}(TM, g)),$$

defined by

$$\mathrm{k}(\xi) = -\dot{A}_\xi, \quad \text{with} \quad A_\xi = -\nabla\xi \quad \text{and} \quad \dot{A}_\xi = \frac{1}{2}(A_\xi + A_\xi^*). \tag{4.2}$$

The kernel of the Killing operator on locally homogeneous spaces is well understood, but what about its image? For a given symmetric endomorphism Q of the tangent bundle of M, we can ask ourselves the following question:

Is there a vector field on M such that $\mathrm{k}(\xi) = Q$?

This problem can be formulated as an existence problem for a solution to the linear first order the partial differential equation:

$$\mathrm{k}(\xi) = Q, \tag{4.3}$$

for a given symmetric endomorphism Q, of TM. We will refer to equation (4.3) as the *inhomogeneous Killing equation.* The first thing to notice from equation (4.2) is that the inhomogeneous Killing equation will have a solution if and only if there exists a vector field $\xi \in \Gamma(TM)$ such that

$$QX = -\dot{A}_\xi X = \nabla_X \xi + \hat{A}_\xi X, \quad \text{for all} \quad X \in \Gamma(TM).$$

The right hand side of the above equation suggests an embedding of the inhomogeneous Killing equation into the Killing bundle. Indeed, the following theorem provides us with an overdetermined system of partial differential equations, built into the Killing bundle, that is equivalent to the inhomogeneous Killing equation.

Theorem 4.1.1. *Let (M, g) be a pseudo-Riemannian manifold. Then, $\xi \in \Gamma(TM)$ is a solution to the inhomogeneous Killing equation*

$$\mathrm{k}(\xi) = Q \quad \textit{for a given} \quad Q \in \Gamma(\mathrm{Sym}(TM, g))$$

if and only if

$$D_X \begin{bmatrix} \xi \\ A \end{bmatrix} = \begin{bmatrix} QX \\ (\mathrm{d}^\nabla Q)(X) - (\mathrm{d}^\nabla Q)(X)^* \end{bmatrix} \quad \textit{for some} \quad A \in \Gamma(\mathfrak{so}(TM, g)).$$

Proof. Suppose that $\xi \in \Gamma(TM)$ is a solution to the inhomogeneous Killing equation with inhomogeneous term $Q \in \Gamma(\mathrm{Sym}(TM, g))$. Inspecting equation (4.2), we can observe that $QX = -\dot{A}_\xi X$ and also $QX = \nabla_X \xi + \hat{A}_\xi X$. By Lemma 2.1.2 we obtain

$$\nabla_X \hat{A}_\xi + R(X, \xi) = -(\mathrm{d}^\nabla \dot{A}_\xi)(X) + (\mathrm{d}^\nabla \dot{A}_\xi)(X)^* = (\mathrm{d}^\nabla Q)(X) - (\mathrm{d}^\nabla Q)(X)^*.$$

Consequently, it follows immediately that

$$\begin{bmatrix} QX \\ (\mathrm{d}^\nabla Q)(X) - (\mathrm{d}^\nabla Q)(X)^* \end{bmatrix} = \begin{bmatrix} \nabla_X \xi + \hat{A}_\xi X \\ \nabla_X \hat{A}_\xi + R(X, \xi) \end{bmatrix} = D_X \begin{bmatrix} \xi \\ \hat{A}_\xi \end{bmatrix}.$$

To prove that the converse statement holds, it will be required only to look at the TM component of

$$D_X \begin{bmatrix} \xi \\ A \end{bmatrix} = \begin{bmatrix} \nabla_X \xi + AX \\ \nabla_X A + R(X, \xi) \end{bmatrix} = \begin{bmatrix} QX \\ (\mathrm{d}^\nabla Q)(X) - (\mathrm{d}^\nabla Q)(X)^* \end{bmatrix}.$$

Since Q is symmetric by assumption, the same must be true for the endomorphism $\nabla\xi + A$. This means that

$$Q + \dot{A}_\xi = -\hat{A}_\xi + A,$$

since $\nabla\xi = -\hat{A}_\xi - \dot{A}_\xi$. The left hand side of the above equation is a section of $\mathrm{Sym}(TM, g)$ and the one on the right hand side is on $\Gamma(\mathfrak{so}(TM, g))$, hence $A = \hat{A}_\xi$ and $Q = -\dot{A}_\xi = \mathrm{k}(\xi)$, as required. The equation on the $\mathfrak{so}(TM, g)$ component is automatically satisfied by Lemma 2.1.2. $\square$

The above theorem provides us with an overdetermined system of linear partial differential equations, embedded into the Killing bundle, which is equivalent to the inhomogeneous Killing equation, however before investigating the consequences of Theorem 4.1.1, we will change the notation for the reminder of this section. In the following we will employ Penrose's abstract index notation and establish a few conventions for the objects that we have been working with.

The Killing bundle will be considered as the vector bundle with connection

$$E = \wedge^1 M \oplus \wedge^2 M \qquad D_a \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a \sigma_d \end{bmatrix}, \tag{4.4}$$

instead of the one defined previously. It is worth noticing that the Killing connection defined as above differs from the one defined throughout this work by a sign. To be precise, in abstract indices notation, the Killing connection defined in Chapter 2 would be

$$D_a \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma_b + \mu_{ab} \\ \nabla_a \mu_{bc} + R_{bc}{}^d{}_a \sigma_d \end{bmatrix}.$$

Nevertheless, our interest has been placed in the kernel and image of this connection, for which this sign discrepancy becomes irrelevant.

The role played by Killing vector fields in the former chapters will be replaced by Killing 1-forms. For this reason, the Killing operator will be consider as the linear first order differential operator

$$\mathcal{K} : \Gamma(\wedge^1 M) \to \Gamma(\mathrm{Sym}^2 M) \qquad \sigma_b \mapsto \nabla_{(a}\sigma_{b)}, \tag{4.5}$$

acting on differential 1-forms, rather than on vector fields, just as it has been defined in the introduction of this chapter. It can be observed from its definition in equation (4.5) that

$$\nabla_{(a}\sigma_{b)} = 0 \quad \text{if and only if} \quad \nabla_a \sigma_b = \nabla_{[a}\sigma_{b]}. \tag{4.6}$$

Therefore, the isomorphism provided in Proposition 2.1.4 can be translated into this new construction by

$$\ker(\mathcal{K}) \to H^0(E, D) \qquad \sigma_a \mapsto \begin{bmatrix} \sigma_a \\ \nabla_a \sigma_b \end{bmatrix}.$$

In other words, the parallel sections of the new Killing bundle are exactly those of the form

$$\begin{bmatrix} \sigma_a \\ \nabla_a \sigma_b \end{bmatrix} \quad \text{such that} \quad \nabla_a \sigma_b = \nabla_{[a}\sigma_{b]}.$$

The curvature of the Killing connection will take the form

$$(D_a D_b - D_b D_a) \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sigma^e(\nabla_e R_{abcd}) + 2R_{ab}{}^e{}_{[c}\mu_{d]e} + 2R_{cd}{}^e{}_{[a}\mu_{b]e} \end{bmatrix}, \tag{4.7}$$

and the exterior covariant derivative, acting on $\Gamma(\wedge^1 M \otimes E)$, will be taken to be the differential operator $D^\wedge : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E)$ given by

$$D^\wedge_a \begin{bmatrix} \eta_{bc} \\ \psi_{bcd} \end{bmatrix} := D_{[a} \begin{bmatrix} \eta_{b]c} \\ \psi_{b]cd} \end{bmatrix} = \begin{bmatrix} \nabla_{[a}\eta_{b]c} + \psi_{[ab]c} \\ \nabla_{[a}\psi_{b]cd} - R_{cd}{}^{e}{}_{[a}\eta_{b]e} \end{bmatrix}. \tag{4.8}$$

With these conventions $2D^\wedge$ is equal to the exterior covariant derivative operator

$$\mathrm{d}_D : \Gamma(\wedge^1 M \otimes E) \to \Gamma(\wedge^2 M \otimes E),$$

defined in Chapter 1, and the curvature of the Killing connection can be expressed, in terms of D and $D^\wedge$, as the composition $2D^\wedge \circ D = \kappa$.

We remark that the terms involving μ in the formula for the curvature of the Killing connection are precisely μ, thought as a skew-symmetric endomorphism of TM, acting on the Riemannian curvature tensor in the usual sense. Moreover, it defines the homomorphism of vector bundles $\mathcal{R} : \wedge^2 M \to RM$, given by

$$(\mathcal{R}\mu)_{abcd} := 2R_{ab}{}^{e}{}_{[c}\mu_{d]e} + 2R_{cd}{}^{e}{}_{[a}\mu_{b]e}, \tag{4.9}$$

where RM denotes the vector subbundle of $\wedge^2 M \otimes \wedge^2 M$ of tensors with the symmetries of the Riemannian curvature tensor, i.e.

$$RM := \{T_{abcd} \in \wedge^2 M \otimes \wedge^2 M \ : \ T_{abcd} = T_{cdab} \ \text{ and } \ T_{[abc]d} = 0\}.$$

Lastly, we remark that differential forms taking values on the Killing bundle will be denoted with Greek upper scripts, namely ϕ^α will be a section of E and $\phi_a{}^\alpha$ will denote a section of $\wedge^1 M \otimes E$.

Given that our new conventions are settled, we will return to the consequences of Theorem 4.1.1. In abstract indices notation it can be restated as follows: Let h_{ab} be a given symmetric 2-tensor field on a pseudo-Riemannian manifold (M, g). Then

$$h_{ab} = \nabla_{(a}\sigma_{b)}, \quad \text{for some} \quad \sigma_b \in \Gamma(\wedge^1 M),$$

if and only if

$$D_a \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} h_{ab} \\ 2\nabla_{[b}h_{c]a} \end{bmatrix}, \quad \text{for some} \quad \sigma_b \in \Gamma(\wedge^1 M) \text{ and } \mu_{bc} \in \Gamma(\wedge^2 M).$$

We are now in condition to consider the existence problem for solutions to the inhomogeneous Killing equation in terms of the Killing bundle. This problem can be stated as follows: For a given section of $\wedge^1 M \otimes E$ of the form

$$\omega_a{}^\alpha = \begin{bmatrix} h_{ab} \\ 2\nabla_{[b}h_{c]a} \end{bmatrix} \quad \text{for some} \quad h_{bc} \in \Gamma(\mathrm{Sym}^2 M). \tag{4.10}$$

Does there exists a section ω^α of the Killing bundle such that $D_a\omega^\alpha = \omega_a{}^\alpha$?

Conditions for such a problem to have a solution have been presented in Chapter 1, for arbitrary vector bundles with connections. We will proceed to investigate these conditions specifically for the Killing bundle. For $\omega_a{}^\alpha \in \Gamma(\wedge^1 M \otimes E)$ of the form given in equation (4.10), we can see that a *necessary* condition for the inhomogeneous problem to have a solution is that

$$D_a^\wedge \begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ -(\nabla_e R_{abcd})\sigma^e + 2R_{ab}{}^e{}_{[c}\mu_{d]e} + 2R_{cd}{}^e{}_{[a}\mu_{b]e} \end{bmatrix} \quad \text{for some} \quad \begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} \in \Gamma(E).$$

A straightforward (but rather long) calculation shows that

$$D_a^\wedge \begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{C}h)_{abcd} \end{bmatrix}, \tag{4.11}$$

where $\mathcal{C} : \Gamma(\mathrm{Sym}^2 M) \to \Gamma(RM)$ denotes the Calabi operator. Recall that the Calabi operator is the second order linear differential operator acting on symmetric 2-tensors which is defined by the formula

$$(\mathcal{C}h)_{abcd} = \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} - R_{ab}{}^e{}_{[c}h_{d]e} - R_{cd}{}^e{}_{[a}h_{b]e}.$$

If we let $\mu_{ab} = \nabla_{[a}\sigma_{b]}$ for some differential 1-form σ_a, a straightforward but rather lengthy calculation shows that the composition

$$\Gamma(\wedge^1 M) \xrightarrow{\mathcal{K}} \Gamma(\mathrm{Sym}^2 M) \xrightarrow{\mathcal{C}} \Gamma(RM) \tag{4.12}$$

takes the simple form

$$(\mathcal{C}\circ\mathcal{K})(\sigma)_{abcd} = -(\nabla_e R_{abcd})\sigma^e + 2R_{ab}{}^e{}_{[c}\mu_{d]e} + 2R_{cd}{}^e{}_{[a}\mu_{b]e} = -(\nabla_e R_{abcd})\sigma^e + (\mathcal{R}\mu)_{abcd}.$$

In the special cases when (M, g) is a reductive locally homogenous space, we can say more about sequence (4.12). By the Ambrose-Singer theorem [11, Theorem 2.2.1], there exists a tensor field $S_{abc} \in \Gamma(\wedge^1 M \otimes \wedge^2 M)$ (a homogeneous structure) solution to the Ambrose-Singer equations, so let us assume that (M, g) is a reductive locally homogeneous space with homogeneous structure S_{abc}. In abstract indices, the Ambrose-Singer equations (see equation (3.1)) take the form

$$\sigma^e\nabla_e R_{abcd} = \sigma^f S_{fa}{}^e R_{ebcd} + \sigma^f S_{fb}{}^e R_{aecd} + \sigma^f S_{fc}{}^e R_{abed} + \sigma^f S_{fd}{}^e R_{abce} \tag{4.13}$$

$$\sigma^e\nabla_e S_{abc} = \sigma^f S_{fa}{}^e S_{ebc} + \sigma^f S_{fb}{}^e S_{aec} + \sigma^f S_{fc}{}^e S_{abe} \tag{4.14}$$

for all $\sigma^a \in TM$. After a rearrangement of terms on the right hand side of equation (4.13), it takes the simpler form

$$\sigma^e\nabla_e R_{abcd} = -2R_{ab}{}^f{}_{[c}\sigma^e S_{|e|d]f} - 2R_{cd}{}^f{}_{[a}\sigma^e S_{|e|b]f},$$

which is nothing but $\mathcal{R}$ applied to the 2-form $-\sigma^e S_{eab}$. Here the vertical bars indicate that the indices enclosed by them are not being affected by the skew-symmetrisation. For instance $2S_{[a|b|c]} = S_{abc} - S_{cba}$. Defining the linear differential operator $\mathcal{S} : \wedge^1 M \to \wedge^2 M$ to be the map

$$\sigma_b \mapsto \sigma^e S_{eab} + \nabla_{[a}\sigma_{b]}, \tag{4.15}$$

we have proved the following proposition:

Proposition 4.1.2. *Let (M, g) be a reductive pseudo-Riemannian locally homogeneous space with homogeneous structure S_{abc}. Then the following diagram commutes*

$$\begin{array}{ccc} \Gamma(\wedge^1 M) & \xrightarrow{\mathcal{K}} & \Gamma(\mathrm{Sym}^2 M) \\ \downarrow{\scriptstyle \mathcal{S}} & & \downarrow{\scriptstyle \mathcal{C}} \\ \Gamma(\wedge^2 M) & \xrightarrow{\mathcal{R}} & \Gamma(RM). \end{array} \tag{4.16}$$

The vector bundle homomorphism $\mathcal{R}$, is defined solely in terms of the Riemannian curvature tensor which means that, in the cases when (M, g) is locally homogeneous, it will have constant rank. When this is the case, the kernel and image of $\mathcal{R}$ will be in fact well defined vector subbundles of $\wedge^2 M$ and RM, respectively. Defining the quotient bundle $CM := RM/\mathrm{Im}(\mathcal{R})$, the composition map

$$\Gamma(\mathrm{Sym}^2 M) \xrightarrow{\mathcal{C}} \Gamma(RM) \longrightarrow \Gamma(CM)$$

which will be denoted by $\mathcal{L} : \Gamma(\mathrm{Sym}^2 M) \to \Gamma(CM)$, provides us with the following complex of linear differential operators

$$\Gamma(\wedge^1 M) \xrightarrow{\mathcal{K}} \Gamma(\mathrm{Sym}^2 M) \xrightarrow{\mathcal{L}} \Gamma(CM). \tag{4.17}$$

The complex (4.17) will be called the *Calabi complex*.

Before proceeding to the main theorem of this section, we remark that if we write $\mathrm{d}\sigma_{ab} := \nabla_{[a}\sigma_{b]}$, then $\sigma^c S_{cab} = (\mathcal{S}\sigma)_{ab} - \mathrm{d}\sigma_{ab}$ and by the Ambrose-Singer equation (4.13), the curvature of the Killing connection takes the form

$$\begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} \quad \mapsto \quad - \begin{bmatrix} 0 \\ \mathcal{R}(\mathcal{S}\sigma - \mathrm{d}\sigma + \mu)_{abcd} \end{bmatrix}. \tag{4.18}$$

Theorem 4.1.3. *Let (M, g) be a reductive pseudo-Riemannian locally homogeneous space. Then, the Calabi complex*

$$\Gamma(\wedge^1 M) \xrightarrow{\mathcal{K}} \Gamma(\mathrm{Sym}^2 M) \xrightarrow{\mathcal{L}} \Gamma(CM) \tag{4.19}$$

is locally exact if and only if the short twisted de Rham complex

$$\Gamma(E) \xrightarrow{D} \Gamma(\wedge^1 M \otimes E) \xrightarrow{\mathrm{d}} \Gamma((\wedge^2 M \otimes E)/\kappa(E)) \tag{4.20}$$

is locally exact.

Proof. If the short twisted de Rham complex of the Killing bundle is locally exact, the Calabi complex will be exact by construction. Suppose that h_{ab} is in the kernel of $\mathcal{L}$, then $(\mathcal{C}h)_{abcd} = 2R_{ab}{}^{e}{}_{[c}\omega_{d]e} + 2R_{cd}{}^{e}{}_{[a}\omega_{b]e} = (\mathcal{R}\omega)_{abcd}$ for some 2-form ω_{ab}. Embedding h_{ab} into the Killing bundle, we can see that

$$D_a^{\wedge}\begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{C}h)_{abcd} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{R}\omega)_{abcd} \end{bmatrix} \in \kappa(E).$$

Since the short twisted de Rham complex is exact by assumption, there exist a 1-form σ_a and a 2-form μ_{ab} such that

$$D_a\begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} \nabla_a\sigma_b - \mu_{ab} \\ \nabla_a\mu_{bc} - R_{bc}{}^{d}{}_{a}\sigma_d \end{bmatrix} = \begin{bmatrix} h_{ab} \\ 2\nabla_{[b}h_{c]a} \end{bmatrix},$$

which implies that $\nabla_{(a}\sigma_{b)} = h_{ab}$, since μ_{ab} is a 2-form. Thus the Calabi complex is exact.

To show the converse, let us choose $\eta_{ab} \in \Gamma(\wedge^1 M \otimes \wedge^1 M)$ and $\phi_{abc} \in \Gamma(\wedge^1 M \otimes \wedge^2 M)$ such that

$$D_a^{\wedge}\begin{bmatrix} \eta_{bc} \\ \phi_{bcd} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{R}\mu)_{abcd} \end{bmatrix}, \quad \text{for some} \quad \mu_{ab} \in \wedge^2 M.$$

If we denote by ω_{ab} and h_{ab} the skew-symmetric and symmetric part of η_{ab}, respectively, by adding a convenient section of $\wedge^2 M \otimes E$ to the above equation, we get

$$D_a^{\wedge}\left(\begin{bmatrix} \omega_{bc} + h_{bc} \\ \phi_{bcd} \end{bmatrix} + D_b\begin{bmatrix} 0 \\ \omega_{cd} \end{bmatrix}\right) = D_a^{\wedge}\begin{bmatrix} h_{bc} \\ \phi_{bcd} + \nabla_b\omega_{cd} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{R}\mu + \mathcal{R}\omega)_{abcd} \end{bmatrix}.$$

A closer inspection reveals that

$$D_a^{\wedge}\begin{bmatrix} h_{bc} \\ \phi_{bcd} + \nabla_b\omega_{cd} \end{bmatrix} = \begin{bmatrix} \nabla_{[a}h_{b]c} + \phi_{[ab]c} + \nabla_{[a}\omega_{b]c} \\ \nabla_{[a}\phi_{b]cd} + \nabla_{[a}\nabla_{b]}\omega_{cd} - R_{cd}{}^{e}{}_{[a}h_{d]e} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{R}\mu + \mathcal{R}\omega)_{abcd} \end{bmatrix},$$

which implies that

$$\phi_{[ab]c} + \nabla_{[a}\omega_{b]c} = -\nabla_{[a}h_{b]c} \in \Gamma(\wedge^2 M \otimes \wedge^1 M). \tag{4.21}$$

The map $\wedge^1 M \otimes \wedge^2 M \to \wedge^2 M \otimes \wedge^1 M$ defined as $T_{abc} \mapsto T_{[ab]c}$ is an isomorphism with inverse given by $\tilde{T}_{abc} \mapsto \tilde{T}_{abc} + 2\tilde{T}_{c(ab)}$. Applying this isomorphism to equation (4.21), we find that

$$\phi_{bcd} + \nabla_b\omega_{cd} = -\nabla_{[a}h_{b]c} - \nabla_{[c}h_{a]b} - \nabla_{[c}h_{b]a} = 2\nabla_{[c}h_{b]a},$$

where the last equality follows from the identity $\nabla_{[a}h_{b]c} + \nabla_{[b}h_{c]a} + \nabla_{[c}h_{a]b} = 0$, which is nothing but a consequence of the symmetries of h_{ab}. Lastly, we can see from

$$D_a^{\wedge}\begin{bmatrix} h_{bc} \\ \phi_{bcd} + \nabla_b\omega_{cd} \end{bmatrix} = D_a^{\wedge}\begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ (\mathcal{C}h)_{abcd} \end{bmatrix},$$

and from the exactness of the Calabi complex and Theorem 4.1.1 that there exist a 1-form σ_b and a 2-form μ_{bc} such that

$$D_a \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = \begin{bmatrix} h_{bc} \\ 2\nabla_{[c} h_{d]b} \end{bmatrix} = \begin{bmatrix} \omega_{bc} + h_{bc} \\ \phi_{bcd} \end{bmatrix} + D_b \begin{bmatrix} 0 \\ \omega_{cd} \end{bmatrix},$$

which after a rearrangement yields

$$\begin{bmatrix} \eta_{bc} \\ \phi_{bcd} \end{bmatrix} = D_a \begin{bmatrix} \sigma_b \\ \mu_{bc} - \omega_{bc} \end{bmatrix}.$$

Consequently, the short twisted de Rham complex is exact. □

4.2 The curvature filtration of the Killing connection

In this section we will apply the results described in Section 1.3, on the curvature filtration of a vector bundle with connection, to the Killing bundle and the Killing connection on a special class of pseudo-Riemannian locally homogeneous spaces. The results presented in this section are a slightly more general to the ones that have appeared previously in [13], for the Killing connection on pseudo-Riemannian locally symmetric spaces. Here we will present analogous results which extend the locally symmetric case to locally homogeneous, under extra assumptions on their holonomy algebras. We will assume that (M, g) is a pseudo-Riemannian locally homogeneous space whose holonomy algebra with base point p is contained in $\mathfrak{h}_p^R = \{A \in \mathfrak{so}(T_pM, g_p) \ : \ A \cdot R = 0\}$ for all p in M.

This condition on the holonomy algebra may appear to be quite a strong of condition, however this is always the case for any pseudo-Riemannian locally symmetric space. Recall that a pseudo-Riemannian locally symmetric space is characterised by possesing a curvature tensor which is invariant by parallel transport, i.e. $\nabla R = 0$. The famous holonomy theorem from Ambrose and Singer [2] tells us that the holonomy algebra of (M, g) is generated by its curvature endomorphisms, hence $\nabla R = 0$ implies that $H \cdot R = 0$ for all $H \in \mathfrak{hol}_p(M, g)$.

Instances of non-symmetric locally homogeneous spaces whose holonomy algebras are contained in $\mathfrak{h}_p^R$ are locally homogeneous plane waves, which shall be properly presented in Chapter 5. It will be observed in equation (5.7) that the holonomy algebra of an $(n+2)$-dimensional locally homogeneous plane wave, which is defined by a rank $k \le n$ symmetric endomorphism of the tangent bundle, is

$$\mathfrak{hol}_p(M, g) \simeq \left\{ \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1, n+1) \ : \ x \in \mathbb{R}^k \right\},$$

and in Proposition 5.2.3 we showed that

$$\mathfrak{h}_p^R \simeq \mathfrak{z}_{\mathfrak{so}(\mathbb{E}_p)}(Q_p) \ltimes (\mathbb{E} \wedge X_-)_p \simeq \left\{ \begin{pmatrix} 0 & x^t & 0 \\ 0 & A & -x \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1, n+1) \ : \ A \in \mathfrak{z}_{\mathfrak{so}(n)}(Q_0) x \in \mathbb{R}^n \right\},$$

where Q_0 is a constant symmetric $n \times n$ matrix. Thus the inclusion of $\mathfrak{hol}_p(M, g)$ in $\mathfrak{h}_p^R$ becomes evident. We refer to Sections 5.1 and 5.2 for more details.

Remark 4.2.1. When $\mathfrak{hol}_p(M, g)$ is contained in $\mathfrak{h}_p^R$, the vector space

$$\mathfrak{r}_p := \operatorname{span}\{R(X, Y) \ : \ X, Y \in T_pM\},$$

is in fact an ideal in $\mathfrak{h}_p^R$.

We will proceed to study the curvature filtration of the Killing bundle of pseudo-Riemannian locally homogeneous spaces. Recall that the curvature filtration $\{E_\ell\}_{\ell=0}^\infty$ is defined by setting $E_0 := \ker \kappa$, which for the Killing bundle is K^R, inductively by

$$E_\ell := \{\phi \in E \ : \ \kappa(\phi) \in \wedge^2 M \otimes E_{\ell-1}\}, \quad \text{for each } \ell \geq 1.$$

The curvature of the Killing connection is given by

$$\kappa(\phi) = -\begin{bmatrix} 0 \\ \phi \cdot R \end{bmatrix}, \quad \text{for some } \phi \in E.$$

As usual, we will let $\pi_{TM} : E \to TM$ and $\pi_{\mathfrak{so}} : E \to \mathfrak{so}(TM, g)$ denote the natural projections from E to TM and $\mathfrak{so}(TM, g)$, respectively. The projection onto the tangent bundle allows us to define

$$\mathfrak{h}_\ell := \ker(\pi_{TM}|_{E_\ell} : E_\ell \to TM), \quad \text{for each } \ell \geq 0.$$

Particularly, $\mathfrak{h}_0 = \mathfrak{h}^R$ and for aesthetic reasons we shall simply use $\mathfrak{h}_0$ rather than $\mathfrak{h}^R$. Inspecting the definitions of E_ℓ and $\mathfrak{h}_\ell$, we notice that for an arbitrary element ϕ of E,

$$\phi \in E_\ell \quad \text{if and only if} \quad \phi \cdot R \in \wedge^2 M \otimes \mathfrak{h}_{\ell-1}.$$

More precisely,

$$E_\ell = \{\phi \in E \ : \ \phi \cdot R \in \wedge^2 M \otimes \mathfrak{h}_{\ell-1}\}.$$

Notice that the family of vector bundle $\{\mathfrak{h}_\ell\}_{\ell=0}^\infty$, defines a the filtration

$$\mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \mathfrak{h}_2 \subseteq \dots$$

of subbundles of $\mathfrak{so}(TM, g)$. Also, in general, a direct calculation reveals that

$$(\phi \cdot R)(X, Y) = [A, R(X, Y)] \qquad \text{mod } \mathfrak{hol}(M, g), \tag{4.22}$$

for any element ϕ of E such that $\pi_{\mathfrak{so}}(\phi) = A$. If in addition we assume the contention $\mathfrak{hol}(M,g) \subseteq \mathfrak{h}_0$, we get that

$$(\phi \cdot R)(X,Y) \in \mathfrak{h}_\ell \quad \text{if and only if} \quad [A, R(X,Y)] \in \mathfrak{h}_\ell.$$

For the reminder of this section, we will only consider locally homogeneous spaces with the properties described in the first paragraph of this section, namely spaces whose holonomy algebras are contained in $\mathfrak{h}_0$.

By letting $\mathfrak{r}$ denote the subbundle of $\mathfrak{so}(TM,g)$ whose fibres at p are given by the $\mathfrak{r}_p$ as in Remark 4.2.1, we have showed that

$$\mathfrak{h}_\ell = \{A \in \mathfrak{so}(TM,g) \ : \ [A,B] \in \mathfrak{h}_{\ell-1}, \ \forall\, B \in \mathfrak{r}\}, \quad \ell \geq 1.$$

Lastly, notice that the Killing curvature will map any element ϕ in E such that $\pi_{\mathfrak{so}}(\phi) = 0$, to $\wedge^2 M \otimes E_1$. Since E_1 is contained in E_ℓ for all $\ell \geq 1$, we have proved the following proposition.

Proposition 4.2.2. *Let (M,g) be a pseudo-Riemannian locally homogeneous space such that $\mathfrak{hol}(M,g)$ is contained in $\mathfrak{h}_0$. Then*

$$\mathfrak{h}_\ell = \{A \in \mathfrak{so}(TM,g) \ : \ [A,B] \in \mathfrak{h}_{\ell-1}, \ \forall\, B \in \mathfrak{r}\}, \quad \textit{for all } \ell \geq 1. \tag{4.23}$$

Moreover

$$E_\ell = TM \oplus \mathfrak{h}_\ell, \quad \textit{for all } \ell \geq 1. \tag{4.24}$$

We recall, before the proposition below, that k_g denotes the Singer homogeneous index of a pseudo-Riemannian locally homogeneous space (M,g).

Proposition 4.2.3. *Let (M,g) be a reductive pseudo-Riemannian locally homogeneous space such that $\mathfrak{hol}(M,g)$ is contained in $\mathfrak{h}_0$. Then the curvature filtration of (E,D) is parallel if and only if $k_g = 0$.*

Proof. That $k_g = 0$ when the curvature filtration of (E,D) is parallel follows directly from Corollary 3.3.3, since $E_0 = K^R$. To show the converse, we will prove that $\phi \in \Gamma(E_\ell)$ implies that $(D\kappa)(\phi) \in \Gamma(\wedge^1 M \otimes \wedge^2 M \otimes E_{\ell-1})$ for all $\ell \geq 0$, where we have set $E_{-1} := \{0\}$. Then, that the curvature filtration of (E,D) is parallel, would follow from Lemma 1.3.6.

First of all, we will calculate $D\kappa$. Choosing $\phi \in \Gamma(E)$ and $X,Y,Z \in \Gamma(TM)$, by definition we have

$$(D_Z\kappa)(X,Y)\phi = D_Z\kappa(X,Y)\phi - \kappa(\nabla_Z X, Y)\phi - \kappa(X, \nabla_Z Y)\phi - \kappa(X,Y)D_Z\phi.$$

Expanding the above equation, it is not hard to see that

$$(D_Z\kappa)(X,Y)\phi = \begin{bmatrix} -(\phi \cdot R)(X,Y)Z \\ -(\nabla_Z(\phi \cdot R))(X,Y) + ((D_Z\phi) \cdot R)(X,Y) \end{bmatrix}.$$

Applying Lemma 2.2.1 to the $\mathfrak{so}(TM,g)$ component of $(D_Z\kappa)(X,Y)\phi$, we obtain

$$(D_Z\kappa)(X,Y)\phi = -\begin{bmatrix}(\phi\cdot R)(X,Y)Z\\(\phi\cdot\nabla R))(Z;X,Y)\end{bmatrix}.$$

When $k_g=0$, it is immediate that $\phi\in\Gamma(E_0)$ implies that

$$(D\kappa)(\phi)\in\Gamma(\wedge^1M\otimes\wedge^2M\otimes E_{-1})=\{0\}.$$

For $\ell\geq 1$, we can see from equation (4.22) that $\phi\cdot\nabla R\in\Gamma(\wedge^1M\otimes\wedge^2M\otimes\mathfrak{h}_{\ell-1})$ when $\phi\in\Gamma(E_\ell)$. To conclude, by Proposition 4.2.2 we have that $E_\ell=TM\oplus\mathfrak{h}_\ell$ for $\ell\geq 1$ and therefore $(D\kappa)(\phi)\in\Gamma(\wedge^1M\otimes\wedge^2M\otimes E_{\ell-1})$ when $\phi\in\Gamma(E_\ell)$, as desired. □

When the restriction of the Killing form of $\mathfrak{so}(TM,g)$ to $\mathfrak{h}_0$ is non-degenerate, it provides us with the orthogonal decompositions

$$\mathfrak{so}(TM,g)=\mathfrak{h}_0\oplus\mathfrak{h}_0^\perp \quad\text{and}\quad E=E_0\oplus\mathfrak{h}_0^\perp$$

of $\mathfrak{so}(TM,g)$ and E, respectively. Such decompositions will be of great help for in the reminder of this chapter, when dealing with Riemannian locally homogeneous spaces.

Proposition 4.2.4. *Let (M,g) be a pseudo-Riemannian locally symmetric space such that the Killing form on $\mathfrak{h}_0$ is non-degenerate. Then $E_0=E_1$ and, in addition, D is exact if and only if the connection that is induced on $\mathfrak{h}_0^\perp$ from the Levi-Civita connection of g is exact.*

Proof. Since (M,g) is locally symmetric, $\nabla R=0$, which implies that $E_0=TM\oplus\mathfrak{h}_0$. Choosing $\phi\in E_1$ such that A is its component in $\mathfrak{h}_0^\perp$, we get that

$$\kappa(\phi)=-\begin{bmatrix}0\\A\cdot R\end{bmatrix}\in\wedge^2M\otimes E_0$$

and thus $A\cdot R\in\wedge^2M\otimes\mathfrak{h}_0$. This implies that $[A,R(X,Y)]\in\mathfrak{h}_0$ for all $X,Y\in TM$. On the other hand, $R(X,Y)\in\mathfrak{h}_0$ and $[\mathfrak{h}_0,\mathfrak{h}_0^\perp]\subseteq\mathfrak{h}_0^\perp$, since $\mathfrak{so}(TM,g)=\mathfrak{h}_0\oplus\mathfrak{h}_0^\perp$ is an orthogonal decomposition. Hence $[A,R(X,Y)=0$. By the pairwise symmetry of R, we also get that

$$R(AX,Y)+R(X,AY)=0\quad\text{for all}\quad X,Y\in TM.$$

Consequently $A\in\mathfrak{h}_0\cap\mathfrak{h}_0^\perp=\{0\}$.

Lastly, from Theorem 1.3.4, (E,D) will be exact if the connection induced in $E/E_0=\mathfrak{h}_0^\perp$ is exact. Also, we have that

$$D_X\begin{bmatrix}0\\A\end{bmatrix}=\begin{bmatrix}AX\\\nabla_XA\end{bmatrix}=\begin{bmatrix}0\\\nabla_XA\end{bmatrix}\qquad\text{mod } E_0.$$

Since the Killing form on $\mathfrak{so}(TM,g)$ is induced by g, $\mathfrak{h}_0^\perp$ is parallel with respect to the Levi-Civita connection, so the connection on $\mathfrak{h}_0^\perp$, induced by D, is indeed the Levi-Civita connection. □

4.3 The exactness of the Killing connection

4.3.1 Spaces of constant holomorphic sectional curvature

In this subsection we will study the exactness of the Killing connection on pseudo-Hermitian manifolds with constant holomorphic sectional curvature. A pseudo-Hermitian manifold is a triplet (M, g, J), where (M, g) is a pseudo-Riemannian manifold and J is complex structure, namely an endomorphism of TM such that $J^2 = -\mathrm{Id}_{TM}$, which is compatible with the metric in the sense that $g(JX, JY) = g(X, Y)$, for all $X, Y \in TM$. We will say that (M, g, J) has constant holomorphic sectional curvature if the sectional curvatures of (M, g, J) restricted to complex planes are constant. Here, by complex planes we mean a subspace of T_pM that is spanned by pairs of vectors of the form X, JX.

Instances of peudo-Hermitian manifolds with constant holomorphic sectional curvature are, in the Riemannian setting, $\mathbb{C}P^n$, $\mathbb{C}H^n$ and $\mathbb{C}^n$ for positive, negative and zero constant curvatures, respectively. In general, if we let (M, g, J) be a pseudo-Hermitian manifold of constant holomorphic sectional curvature k and $\omega(X, Y) := g(JX, Y)$ be its fundamental 2-form. Its curvature tensor is of the form

$$R(X, Y) = -\frac{k}{4}(X \wedge Y + JX \wedge JY + 2\omega(X, Y)J), \tag{4.25}$$

where $X \wedge Y = g(X, \cdot)Y - g(Y, \cdot)X$. We refer to [41] for more details on the curvature tensors of almost-Hermitian manifolds. Notice that the terms $X \wedge Y$ and $JX \wedge JY$ correspond to curvature tensors of pseudo-Riemannian manifolds with constant sectional curvature. Therefore, if we let A be any skew-symmetric endomorphism of TM, a straightforward computation reveals that its action on the curvature tensor takes the rather simple form:

$$(A \cdot R)(X, Y) = -\frac{k}{2}\omega(X, Y)[A, J] - \frac{k}{2}g([A, J]X, Y)J. \tag{4.26}$$

All spaces of constant holomorphic sectional curvature are Kähler and, particularly, locally symmetric. This can be easily seen from the formula for the curvature tensor given in equation (4.25), since R is constructed entirely in terms of the metric tensor and the complex structure, which are parallel tensor fields.

We have shown in the previous sections that locally symmetric spaces have Singer index equal to 0 (see Example 3.3.2), hence the kernel of the Killing curvature on locally symmetric spaces is equal to the maximal parallel flat subbundle of E and it is given by $K = TM \oplus \mathfrak{h}^R$. Particularly, in the case of spaces of constant holomorphic sectional curvature, $\mathfrak{h}^R_p$ coincides with the holonomy algebra of (M, g) at p, which is equal to

$$\mathfrak{u}(T_pM, g_p, J_p) := \{A \in \mathfrak{so}(T_pM, g_p) \ : \ [A, J_p] = 0\}.$$

Therefore, the kernel of the Killing curvature is given by

$$K = TM \oplus \mathfrak{u}(TM, g, J), \tag{4.27}$$

where $\mathfrak{u}(TM, g, J)$ denotes the vector bundle over M, with fibre $\mathfrak{u}(T_pM, g_p, J_p)$ at p. To simplify notation, from now on, we will use $\mathfrak{h}^R$ instead of $\mathfrak{u}(TM, g, J)$.

The goal of this subsection is to prove the following theorem.

Theorem 4.3.1. *The Killing connection of a pseudo-Hermitian manifold of constant holomorphic sectional curvature is exact.*

The approach to this problem we will be the following: Since K is a parallel flat subbundle of E, $(K, D|_K)$ is trivially exact. By Proposition 1.2.6, if $(E/K, D|_{E/K})$ is exact and the curvature homomorphism,

$$\kappa : E/K \to \wedge^2 M \otimes E/K$$

induced on the quotient bundle is injective, (E, D) will be exact.

First of all, we will show that $\kappa : E/K \to \wedge^2 M \otimes E/K$ is injective. Notice that, from equation (4.27), the quotient E/K is equal to $\mathfrak{so}(TM, g)/\mathfrak{h}^R$. Choosing a representative A of a class $[A] \in E/K$, we can see that the Killing curvature on E/K can be expressed as

$$\kappa(X, Y)(A) = -(A \cdot R)(X, Y) = \frac{k}{2}\omega(X, Y)[A, J] \qquad \mod \mathfrak{h}^R, \tag{4.28}$$

where the last equality follows from the fact that $\mathfrak{hol}(M, g) = \mathfrak{h}^R$ and from equation (4.26). From the above equation follows that $[A]$ will be in the kernel of $\kappa : E/K \to \wedge^2 M \otimes E/K$ if and only if $[A, J]$ is in $\mathfrak{h}^R$. Particularly, κ will be injective if $\mathfrak{h}^R$ is self-normalising in $\mathfrak{so}(TM, g)$.

Proposition 4.3.2. $\mathfrak{u}(TM, g, J)$ *is self-normalising in* $\mathfrak{so}(TM, g)$.

Proof. Suppose that $A \in \mathfrak{so}(TM, g)$ is in the normaliser of $\mathfrak{u}(TM, g, J)$ in $\mathfrak{so}(TM, g)$. By definition, $[A, U]$ will be in $\mathfrak{u}(TM, g, J)$ for all $U \in \mathfrak{u}(TM, g, J)$. Making use of the Jacobi identity, we can see that

$$0 = [A, [U, J]] = [[A, U], J] + [U, [A, J]] = [U, [A, J]], \quad \text{for all } U \in \mathfrak{u}(TM, g, J).$$

Because $[A, J]$ is in $\mathfrak{u}(TM, g, J)$, the above equation implies that $[A, J]$ is in the centre of $\mathfrak{u}(TM, g, J)$, which is nothing but $\mathbb{R}J$. In other words, $[A, J]$ is a multiple of J. If we let $[A, J] = cJ$, for some $c \in \mathbb{R}$, multiplying by J and tracing we obtain

$$\mathrm{tr}([A, J]J) = c\,\mathrm{tr}(J^2) = -c\,\mathrm{tr}(\mathrm{Id}) = -2nc.$$

A closer look at the left hand side of the above equation reveals

$$\mathrm{tr}([A, J]J) = \mathrm{tr}(AJ^2) - \mathrm{tr}(JAJ) = -\mathrm{tr}(A) + \mathrm{tr}(JAJ^{-1}) = 0,$$

which implies that $c = 0$. It follows that A commutes with J and therefore A is in $\mathfrak{u}(TM, g, J)$. $\square$

We have showed that the curvature on E/K is injective, as a consequence of the above proposition. It is only left to show that $(E/K, D)$ is exact. In order to do this, we will investigate $\kappa^1 : \wedge^1 M \otimes E/K \to \wedge^3 M \otimes E/K$. We conclude with the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. Recall that $\kappa^1 : \wedge^1 M \otimes E/K \to \wedge^3 M \otimes E/K$ is defined by

$$\kappa^1(\phi)(X, Y, Z) = 2\kappa(X, Y)\phi(Z) + 2\kappa(Z, X)\phi(Y) + 2\kappa(Y, Z)\phi(X),$$

for an E/K-valued 1-form ϕ. Explicitly, by equation (4.26), $\kappa^1(\phi)$ takes the form

$$\kappa^1(\phi)(X, Y, Z) = k\omega(X, Y)[\phi(Z), J] + k\omega(Z, X)[\phi(Y), J] + k\omega(Y, Z)[\phi(X), J] \mod \mathfrak{h}^R.$$

Letting us choose a g-orthonormal frame $\{X_i, JX_i\}_{i=1}^n$, of TM, and fixing $\ell \in \{1, \ldots, n\}$, for any $Z \in \operatorname{span}\{X_\ell, JX_\ell\}^\perp$ we get

$$\kappa^1(\phi)(X_\ell, JX_\ell, Z) = k\omega(X_\ell, JX_\ell)[\phi(Z), J] = k[\phi(Z), J] \mod \mathfrak{h}^R.$$

We can observe from the above equation that $[\phi]$ will be in the kernel of

$$\kappa^1 : \wedge^1 M \otimes E/K \to \wedge^3 M \otimes E/K$$

if and only if $[\phi(Z), J] \in \mathfrak{h}^R$. However, by Proposition 4.3.2, $\mathfrak{h}^R$ is self-normalising in $\mathfrak{so}(TM, g)$, which implies that $[\phi] = [0]$ and therefore $\kappa^1 : \wedge^1 M \otimes E/K \to \wedge^3 M \otimes E/K$ is injective. By Lemma 1.2.12, the exactness of $(E/K, D)$ follows from the injectivity of $\kappa^1 : \wedge^1 M \otimes E/K \to \wedge^3 M \otimes E/K$. By Proposition 1.2.6, (E, D) is exact, since $(E/K, D|_{E/K})$ is exact and the curvature homomorphism,

$$\kappa : E/K \to \wedge^2 M \otimes E/K$$

induced on the quotient bundle is injective. □

4.3.2 Product spaces

This subsection will be dedicated to study the exactness of the Killing connection for products of pseudo-Riemannian spaces, provided previous knowledge of each individual de Rham-Wu factor [44]. Throughout this subsection, we will return to Penrose abstract index notation (see Section 4.1 or [40] for more details). To work with product spaces, we will borrow standard notation from complex geometry as follows: Letting (M, g) be a pseudo-Riemannian manifold which is a product of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2), that is, $M = M_1 \times M_2$ and $g = g_1 + g_2$, we will write

$$\wedge^{1,0} M := \pi_1^*(\wedge^1 M_1) \quad \text{and} \quad \wedge^{0,1} M := \pi_2^*(\wedge^1 M_2)$$

for the pullback bundles of 1-forms over M_1 and M_2, respectively. Here $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$ denotes the natural projection. Analogously, for the bundles of 2-forms we will write

$$\wedge^{2,0}M := \pi_1^*(\wedge^2 M_1) \quad \text{and} \quad \wedge^{0,2} M := \pi_2^*(\wedge^2 M_2)$$

and its complement in $\wedge^2 M$ will be denoted by $\wedge^{1,1}M$. That is, the bundle of 2-forms on M decomposes as

$$\wedge^2 M = \wedge^{2,0}M \oplus \wedge^{1,1}M \oplus \wedge^{0,2}M.$$

The curvature tensor will be regarded as a section of $(\wedge^{2,0}M \otimes \wedge^{2,0}M) \oplus (\wedge^{0,2}M \otimes \wedge^{0,2}M)$. In regards with the Killing bundle, it will have the decomposition

$$E = E^{1,0} \oplus E^{0,1} \oplus \wedge^{1,1}M, \tag{4.29}$$

inherited from the bundles of 1-forms and 2-forms. Here $E^{1,0} = \wedge^{1,0}M \oplus \wedge^{2,0}M$ and $E^{0,1} = \wedge^{0,1}M \oplus \wedge^{0,2}M$ correspond to the pullbacks of the Killing bundles of M_1 and M_2 by the natural projections. We will denote indices from $\wedge^{1,0}M$ with a, b,..., indices from $\wedge^{0,1}M$ with $\bar{a}$, $\bar{b}$, ..., and A, B,... for both groups of indices.

Lemma 4.3.3. *The vector bundles $E^{1,0}$ and $E^{0,1}$ are parallel subbundles of E.*

Proof. It follows from their definitions. □

Recall that in Proposition 1.2.5 we have showed that if F is a parallel and exact subbundle of E and the kernel of $D^\wedge : \Gamma(\wedge^1 \otimes E) \to \Gamma(\wedge^2 \otimes E)$ is contained in $\Gamma(\wedge^1 \otimes F)$, then (E, D) is exact. The aim for the rest of this subsection will be to find conditions for the kernel of $D^\wedge : \Gamma(\wedge^1 \otimes E) \to \Gamma(\wedge^2 \otimes E)$ to be contained in $\Gamma(\wedge^1 \otimes K)$, where K denotes the maximal parallel flat subbundle of E. Since K is flat, it is trivially exact, hence the exactness of the Killing connection on E would follow from Proposition 1.2.5. For simplicity we will denote the kernel of $D^\wedge : \Gamma(\wedge^1 \otimes E) \to \Gamma(\wedge^2 \otimes E)$ by $\ker(D^\wedge)$.

First of all, let us choose sections η_{BC} of $\wedge^1 M \otimes \wedge^1 M$, ψ_{BCD} of $\wedge^1 M \otimes (\wedge^{2,0}M \oplus \wedge^{0,2}M)$ and ϕ_{BCD} of $\wedge^1 M \otimes \wedge^{1,1}M$, and let us considered them as the sections

$$\begin{bmatrix} \eta_{BC} \\ \psi_{BCD} \end{bmatrix} \in \Gamma(\wedge^1 \otimes (E^{1,0} \oplus E^{0,1})) \quad \text{and} \quad \begin{bmatrix} 0 \\ \phi_{BCD} \end{bmatrix} \in \Gamma(\wedge^1 M \otimes \wedge^{1,1}) \subseteq \Gamma(\wedge^1 M \otimes E),$$

of the Killing bundle. We will find constraints on η_{BC}, ψ_{BCD} and ϕ_{BCD} for

$$\Omega_B{}^\alpha := \begin{bmatrix} \eta_{BC} \\ \psi_{BCD} + \phi_{BCD} \end{bmatrix} \in \Gamma(\wedge^1 M \otimes E),$$

to be in the kernel of $D^\wedge$. The map $D^\wedge : \Gamma(\wedge^1 \otimes E) \to \Gamma(\wedge^2 \otimes E)$, on $\Omega_B{}^\alpha$ is given by

$$D^\wedge_A \Omega_B{}^\alpha = \begin{bmatrix} \nabla_{[A}\eta_{B]C} + \psi_{[AB]C} + \phi_{[AB]C} \\ \nabla_{[A}\psi_{B]CD} - R_{CD}{}^E{}_{[A}\eta_{B]E} \end{bmatrix} + \begin{bmatrix} 0 \\ \nabla_{[A}\phi_{B]CD} \end{bmatrix},$$

where the first summand of the right hand side is a section of $\wedge^2 M \otimes (E^{2,0} \oplus E^{0,2})$ and the second one is of $\wedge^2 M \otimes \wedge^{1,1} M \subset \wedge^2 M \otimes E$. Then $\Omega_B{}^\alpha$ will be in the kernel of $D^\wedge$ if and only if η_{BC}, ψ_{BCD} and ϕ_{BCD} are solutions to the system of partial differential equations:

$$\nabla_{[A}\eta_{B]C} + \psi_{[AB]C} + \phi_{[AB]C} = 0, \quad \nabla_{[A}\psi_{B]CD} - R_{CD}{}^{E}{}_{[A}\eta_{B]E} = 0, \quad \nabla_{[A}\phi_{B]CD} = 0. \quad (4.30)$$

The following lemmas will provide us with helpful constraints on sections $\wedge^1 M \otimes E$, to be in the kernel of $D^\wedge$ on product manifolds.

Lemma 4.3.4. *On a product of pseudo-Riemannian manifolds with curvature tensor R_{ABCD}, let $\phi_{Ab\bar{c}} \in \Gamma(\wedge^1 M \otimes \wedge^{1,1} M)$ be a solution of*

$$\nabla_{[A}\phi_{B]CD} = 0. \quad (4.31)$$

Then

$$R_{ab}{}^{e}{}_{c}\phi_{\bar{a}e\bar{b}} = 0, \quad \textit{and} \quad R_{\bar{a}\bar{b}}{}^{\bar{e}}{}_{\bar{c}}\phi_{ab\bar{e}} = 0. \quad (4.32)$$

Proof. Re-writing equation (4.31) using barred and unbarred indices, we obtain

$$\nabla_{[a}\phi_{b]c\bar{a}} = 0, \quad \nabla_{[\bar{a}}\phi_{\bar{b}]a\bar{c}} = 0 \quad \text{and} \quad \nabla_a\phi_{\bar{a}b\bar{b}} - \nabla_{\bar{a}}\phi_{ab\bar{b}} = 0.$$

Differentiating the first equation with respect to the barred indices, we have

$$0 = \nabla_{\bar{a}}(\nabla_a\phi_{bc\bar{b}} - \nabla_b\phi_{ac\bar{b}}) = \nabla_a\nabla_{\bar{a}}\phi_{bc\bar{b}} - \nabla_b\nabla_{\bar{a}}\phi_{ac\bar{b}}.$$

The third equation implies that

$$\nabla_a\nabla_{\bar{a}}\phi_{bc\bar{b}} - \nabla_b\nabla_{\bar{a}}\phi_{ac\bar{b}} = \nabla_a\nabla_b\phi_{\bar{a}c\bar{b}} - \nabla_b\nabla_a\phi_{\bar{a}c\bar{b}} = -R_{ab}{}^{e}{}_{c}\phi_{\bar{a}e\bar{b}},$$

and therefore $R_{ab}{}^{e}{}_{c}\phi_{\bar{a}e\bar{b}} = 0$. Analogously, differentiating the second equation with respect to the unbarred indices, we obtain $R_{\bar{a}\bar{b}}{}^{\bar{e}}{}_{\bar{c}}\phi_{ab\bar{e}} = 0$, as claimed. □

Remark 4.3.5. We can observe from equations (4.32) that a section ϕ of $\wedge^1 M \otimes \wedge^{1,1} M$ will be a solution of equation (4.31) if and only if $\phi_c = X^{\bar{a}}Y^{\bar{b}}\phi_{\bar{a}c\bar{b}}$ and $\phi_{\bar{c}} = X^a Y^b \phi_{ab\bar{c}}$ are contained in $\{X_D \in \wedge^1 M \ : \ R_{AB}{}^{D}{}_{C}X_D = 0\}$ for all $X_A, Y_B \in \wedge^1 M$. In other words, if the curvature endomorphisms $R_{ab}{}^{d}{}_{c}$ are injective, the map $\phi_{BCD} \mapsto \nabla_{[A}\phi_{B]CD}$ will be injective, which will allow us to reduce the problem of the exactness of the Killing connection to the equations on $\wedge^M \otimes (E^{1,0} \oplus E^{0,1})$.

This leads us to the *nullity* of a pseudo-Riemannian manifold. If we let R be the curvature tensor of a pseudo-Riemannian manifold, the nullity of (M, g) at a point p will be the vector subspace of T_pM defined as $\nu_p(M, g) := \{X \in T_pM \ : \ \iota_X R = 0\}$. In the cases when $p \mapsto \dim \nu_p(M, g)$ is a constant map, the nullities define a vector subbundle of TM, which shall be referred to as the *nullity bundle* of (M, g) and it will be denoted by $\nu(M, g)$. To simplify notation we will write ν instead of $\nu(M, g)$, unless otherwise necessary.

Particular instances of pseudo-Riemannian manifolds admitting a nullity bundle are locally homogeneous ones. Indeed, for any pair of points $p, q \in M$ there exists a local isometry $f : U \to V$ mapping p to q such that $f^*\nu_q = \nu_p$, which is a simple consequence of the curvature tensor being invariant by local isometries.

On a product pseudo-Riemannian manifold $(M, g) = (M_1, g_1) \times (M_2, g_2)$, we will write $\nu^{1,0} := \pi_1^*\nu(M_1, g_1)$ and $\nu^{0,1} := \pi_2^*\nu(M_2, g_2)$, in order to fit the nullity bundles in our conventions for vector bundles over product manifolds.

In terms of the nullity bundles, Lemma 4.3.4 could be stated as follows.

Lemma 4.3.6. *On a product $(M_1, g_1) \times (M_2, g_2)$ of pseudo-Riemannian manifolds with nullity bundles $\nu^{1,0}$ and $\nu^{0,1}$, respectively, let $\phi \in \Gamma(\wedge^1 M \otimes \wedge^{1,1} M)$ be a solution of*

$$\nabla_{[A}\phi_{B]CD} = 0.$$

Then ϕ is a section of $(\wedge^{1,0}M \otimes (\wedge^{1,0}M \wedge \nu^{0,1})) \oplus (\wedge^{0,1}M \otimes (\wedge^{0,1}M \wedge \nu^{1,0}))$.

Lemma 4.3.6 suggests that the study of the exactness of the Killing connection on product spaces is simplified if the manifolds in question have zero nullity, namely $\nu = \{0\}$. We will proceed to do that but before, we note that (M, g) will have zero nullity if and only if each local de Rham-Wu decomposition of (M, g) has factors with zero nullity bundles. The below lemma follows from this observation and Lemma 4.3.6.

Lemma 4.3.7. *Let (M, g) be a pseudo-Riemannian manifold with $\nu = \{0\}$. If (M, g) splits as a product, i.e. $(M, g) = (M_1, g_1) \times (M_2, g_2)$, then*

$$\ker(D^\wedge) \subseteq \Gamma(\wedge^1 M \otimes (E^{1,0} \oplus E^{0,1})).$$

The following theorem will provide us with a characterisation of the exactness of the Killing connection on spaces with zero nullity.

Theorem 4.3.8. *Let (M, g) be a pseudo-Riemannian manifold with zero nullity bundle. Suppose that (M, g) is the product of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2), then the first twisted de Rham cohomology group of (E, D) splits as*

$$H^1(E, D) = H^1(E^{1,0}, D|_{E^{1,0}}) \oplus H^1(E^{0,1}, D|_{E^{0,1}}).$$

Particularly, the Killing connection of (M, g) is exact if and only if the Killing connections of (M_1, g_1) and (M_2, g_2) are exact.

Proof. The idea behind the proof is to make use of Proposition 1.2.4, which states that the first twisted de Rham cohomology group of (E, D) is isomorphic to $\ker(D^\wedge)/\mathrm{Im}(D|_{E_0})$.

First of all, we will show that $E_0 = E_0^{1,0} \oplus E_0^{0,1}$, which will yield the decomposition $\mathrm{Im}(D|_{E_0}) = \mathrm{Im}(D|_{E_0^{1,0}}) \oplus \mathrm{Im}(D|_{E_0^{1,0}})$, since $E^{1,0}$ and $E^{0,1}$ are parallel subbundles of E, by Lemma 4.3.3. Suppose that there exists a section $\phi_{a\bar{b}}$ of $\wedge^{1,1}M$ such $(0, \phi_{a\bar{b}})$ is in the kernel of the Killing curvature. Then, we would get that

$$R_{AB}{}^{E}{}_{[C}\phi_{D]E} + R_{CD}{}^{E}{}_{[A}\phi_{B]E} = 0. \tag{4.33}$$

By taking $ABCD = abc\bar{d}$ and $ABCD = \bar{a}\bar{b}\bar{c}d$, the above equation becomes

$$R_{ab}{}^{e}{}_{c}\phi_{\bar{d}e} = 0 \quad \text{and} \quad R_{\bar{a}\bar{b}}{}^{\bar{e}}{}_{\bar{c}}\phi_{d\bar{e}} = 0.$$

By assumption, the nullity bundles of the curvature tensors of (M_1, g_1) and (M_2, g_2) were trivial, thus $\phi_{a\bar{b}} = 0$. We have proved that $E_0 = E_0^{1,0} \oplus E_0^{0,1}$.

By Proposition 4.3.7, we know that $\ker(D^\wedge) \subseteq \Gamma(\wedge^1 M \otimes (E^{1,0} \oplus E^{0,1}))$ and, by Lemma 4.3.3, $E^{1,0}$ and $E^{0,1}$ are parallel subbundles of E. Therefore we get that $\ker(D^\wedge)$ splits as

$$\ker(D^\wedge) = \ker(D|^\wedge_{E^{1,0}}) \oplus \ker(D|^\wedge_{E^{0,1}}).$$

The isomorphism provided in Proposition 1.2.4 reveals that the first twisted de Rham cohomology group of (E, D) will be given by

$$H^1(E,D) \simeq \ker(D^\wedge)/\mathrm{Im}(D|_{E_0}) = \ker(D|^\wedge_{E^{1,0}})/\mathrm{Im}(D|_{E_0^{1,0}}) \oplus \ker(D|^\wedge_{E^{0,1}})/\mathrm{Im}(D|_{E_0^{1,0}}),$$

which again, by Proposition 1.2.4, yields the desired isomorphism

$$H^1(E,D) \simeq H^1(E^{1,0}, D|_{E^{1,0}}) \oplus H^1(E^{0,1}, D|_{E^{0,1}}).$$

That the Killing connection of (M, g) is exact if and only if the Killing connections of (M_1, g_1) and (M_2, g_2) are exact follows directly from Proposition 1.2.9. □

Before proceeding to the consequences of Theorem 4.3.8, we would like to remark that hidden in its proof is the key to prove the following:

Theorem 4.3.9. *Let (M, g) be a pseudo-Riemannian manifold with zero nullity bundle. Suppose that (M, g) is the product of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2), then the zeroth twisted de Rham cohomology group of (E, D) splits as*

$$H^0(E,D) = H^0(E^{1,0}, D|_{E^{1,0}}) \oplus H^0(E^{0,1}, D|_{E^{0,1}}).$$

Proof. Let ϕ^α be a section in $H^0(E, D)$ of the form

$$\phi^\alpha = \begin{bmatrix} \sigma_B \\ \mu_{BC} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_{BC} \end{bmatrix}, \quad \text{with } \mu_{BC} \in \Gamma(\wedge^{2,0}M \oplus \wedge^{0,2}M) \text{ and } \omega_{BC} \in \Gamma(\wedge^{1,1}M).$$

Since ϕ^α is in $H^0(E,D)$, we know that

$$0 = -\sigma^E\nabla_E R_{ABCD} + R_{AB}{}^E{}_{[C}\mu_{D]E} + R_{CD}{}^E{}_{[A}\mu_{B]E} + R_{AB}{}^E{}_{[C}\omega_{D]E} + R_{CD}{}^E{}_{[A}\omega_{B]E}.$$

Noticing that the above equation decouples as

$$0 = -\sigma^E\nabla_E R_{ABCD} + R_{AB}{}^E{}_{[C}\mu_{D]E} + R_{CD}{}^E{}_{[A}\mu_{B]E} \in \Gamma(\odot^2 \wedge^{2,0} M \oplus \odot^2 \wedge^{0,2} M)$$

and

$$0 = R_{AB}{}^E{}_{[C}\omega_{D]E} + R_{CD}{}^E{}_{[A}\omega_{B]E} \in \Gamma((\wedge^{2,0} \oplus \wedge^{0,2}) \odot \wedge^{1,1}),$$

where $\odot$ denotes the symmetric product, we are in the same situation as in equation (4.33). Since (M,g) has zero nullity, from the above equation we get that

$$R_{ab}{}^e{}_c\omega_{\bar{d}e} = 0 \quad \text{and} \quad R_{\bar{a}\bar{b}}{}^{\bar{e}}{}_{\bar{c}}\omega_{d\bar{e}} = 0$$

imply that $\omega_{AB} = 0$. In other words $H^0(E,D) = H^0(E^{1,0} \oplus E^{0,1}, D|_{E^{1,0}\oplus E^{0,1}})$. Lastly, that $H^0(E,D) = H^0(E^{1,0}, D|_{E^{1,0}}) \oplus H^0(E^{0,1}, D|_{E^{0,1}})$ follows from the fact that $E^{1,0}$ and $E^{0,1}$ are parallel, by Lemma 4.3.3. □

Returning to the consequences of Theorem 4.3.8, the simplest instances of pseudo-Riemannian manifolds with zero nullity bundles are spaces with constant sectional curvature. They have curvature tensors of the form

$$R_{abcd} = k(g_{ac}g_{bc} - g_{ad}g_{bc}), \quad k \in \mathbb{R}.$$

If they are non-flat, their nullity spaces are always equal to $\{0\}$. In the complex setting, analogously to spaces of constant sectional curvature, are pseudo-Hermitian manifolds with constant holomorphic sectional curvature. Letting ω_{ab} denote their fundamental 2-firm, their curvature tensors are of the form

$$R_{abcd} = k(g_{ac}g_{bc} - g_{ad}g_{bc} + \omega_{ac}\omega_{bc} - \omega_{ad}\omega_{bc} + 2\omega_{ab}\omega_{cd}), \quad k \in \mathbb{R},$$

and also they will have zero nullity bundles, if they are non-flat.

Corollary 4.3.10. *The Killing connection of the product of pseudo-Riemannian manifolds with non-zero constant sectional curvature and pseudo-Hermitian manifolds with non-flat constant holomorphic sectional curvature is exact.*

Proof. The Killing connection of a space of constant sectional curvature is flat and thus trivially exact. On the other hand, we have proved in Theorem 4.3.1 that the Killing connection of spaces of constant holomorphic sectional curvature are exact. The exactness of the Killing connection follows straightforwardly from Theorem 4.3.8. □

In the Riemannian case, it was proved in [15, Proposition C] that if $(G/H, g)$ is a simply connected homogeneous Riemannian manifold without an Euclidean de Rham factor [14], in the following cases the nullity bundle of $(G/H, g)$ is trivial:

(1) The Lie algebra of G is reductive, i.e., the direct sum of a semisimple and an abelian ideal.

(2) The Lie algebra of G is 2-step nilpotent.

These spaces provide us with large families of examples where the above proposition reduces the problem of studying the exactness of the Killing connection is completely reduced to their irreducible de Rham factors. Particularly, it is well known that all non-flat irreducible Riemannian symmetric spaces have a simple Lie group of isometries and, in fact, they are all classified by the real simple Lie algebras. We refer to the book of Helgason [27] for more details about the classification of Riemannian symmetric spaces. The exactness of the Killing connection in such spaces was addressed in [12], where it was proved that the Killing connection of a Riemannian locally symmetric space is exact, unless they have at least one Hermitian and one flat factor in its local de Rham decomposition. Clearly, non-flat irreducible Riemannian symmetric spaces are a particular cases of those, so Theorem 4.3.8 provides us with an alternative proof of the exactness of the Killing connection for their products, given a priori knowledge of the exactness of each irreducible de Rham factor.

Corollary 4.3.11. *The Killing connection of a Riemannian symmetric spaces without Euclidean de Rham factors is exact.*

Before finishing this section, we will make a last comment on the implications of Lemma 4.3.4. Suppose that $\phi_{BCD} \in \Gamma(\wedge^1 M \otimes \wedge^{1,1} M)$ is a solution to equation (4.31). By tracing equations (4.32) in Lemma 4.3.4 we observe that

$$R_{ac}\phi_{\bar{a}\ \bar{b}}^{\ c} = 0 \quad \text{and} \quad R_{\bar{a}\bar{c}}\phi_{ab}^{\ \ \bar{c}} = 0$$

imply that $\phi_{BCD} \mapsto \nabla_{[A}\phi_{B]CD}$ will be injective when the Ricci tensor is non-degenerate.

Proposition 4.3.12. *Suppose that $(M, g) = (M_1, g_1) \times (M_2, g_2)$ has a non-degenerate Ricci tensor. Then* $\ker(D^\wedge) \subseteq \Gamma(\wedge^1 M \otimes (E^{1,0} \oplus E^{0,1}))$.

Perhaps the simplest instance of pseudo-Riemannian manifolds with non-degenerate Ricci tensor are Einstein manifolds with non-zero scalar curvature.

To conclude, we prove the following lemma providing conditions for exactness of the Killing connection of a product space, given previous knowledge on the de Rham-Wu factors.

Lemma 4.3.13. *Let $(M,g) = (M_1,g_1)\times(M_2,g_2)$ be a product of two pseudo-Riemannian manifolds such that the Killing connection of each factor is exact and $E_0 = K$. Then*

$$\ker(D^\wedge)\cap\Gamma(\wedge^1 M\otimes(E^{1,0}\oplus E^{0,1}))\subseteq\Gamma(\wedge^1 M\otimes K).$$

Proof. Let us pick an element (η_{AB},ψ_{ABC}) of $\ker(D^\wedge)\cap\Gamma(\wedge^1 M\otimes(E^{1,0}\oplus E^{0,1}))$. From equation (4.30) with all indices unbarred and all indices barred, i.e. $ABC = abc$ and $ABC = \bar a\bar b\bar c$, we know that

$$\begin{bmatrix}\eta_{ab}\\ \psi_{abc}\end{bmatrix} = \begin{bmatrix}\nabla_a\sigma_b-\mu_{ab}\\ \nabla_a\mu_{bc}-R_{bc}{}^{e}{}_{a}\sigma_e\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}\eta_{\bar a\bar b}\\ \psi_{\bar a\bar b\bar c}\end{bmatrix} = \begin{bmatrix}\nabla_{\bar a}\sigma_{\bar b}-\mu_{\bar a\bar b}\\ \nabla_{\bar a}\mu_{\bar b\bar c}-R_{\bar b\bar c}{}^{\bar e}{}_{\bar a}\sigma_{\bar e}\end{bmatrix}, \tag{4.34}$$

by the exactness of each factor. Here, the sections (η_{ab},ψ_{abc}) and $(\eta_{\bar a\bar b},\psi_{\bar a\bar b\bar c})$, are still depending on the unbarred and barred coordinates respectively. By assumption, E_0 is equal to K, and by the isomorphism from Proposition 1.2.4, we get that (σ_a,μ_{ab}) and $(\sigma_{\bar a},\mu_{\bar a\bar b})$ are sections of K. Furthermore, since K is parallel, (η_{ab},ψ_{abc}) and $(\eta_{\bar a\bar b},\psi_{\bar a\bar b\bar c})$ are sections of $\wedge^1\otimes K$. The first equation in (4.30) with $ABC = \bar a bc$ tells us that

$$\psi_{\bar a bc} = \nabla_b\eta_{\bar a c}-\nabla_{\bar a}\eta_{bc} = \nabla_b\eta_{\bar a c}-\nabla_{\bar a}(\nabla_b\sigma_c-\mu_{bc}).$$

Since $\psi_{\bar a bc} = \psi_{\bar a[bc]}$ and ∇_b and $\nabla_{\bar a}$ commute, we can rewrite the above equation as

$$\psi_{\bar a bc} = \nabla_b\theta_{\bar a c}+\nabla_{\bar a}\mu_{bc} = \nabla_{[b}\theta_{|\bar a|c]}+\nabla_{\bar a}\mu_{bc} \tag{4.35}$$

where $\theta_{\bar a a} := \eta_{\bar a a}-\nabla_{\bar a}\sigma_a$. From equation (4.35) we can see that

$$\nabla_{(b}\theta_{|\bar a|c)} = 0$$

This means that $\theta_{\bar a a}$ is a Killing 1-form with respect to the unbarred index. To be precise, picking a vector field $X^{\bar a}$ that is the pull-back of a vector field on M_2 and such that $\nabla_a X^{\bar a} = 0$, the 1-form $\theta_a := X^{\bar a}\theta_{\bar a a}$ satisfies

$$0 = X^{\bar a}\nabla_{(b}\theta_{|\bar a|c)} = \nabla_{(b}\theta_{c)}.$$

For this reason we can conclude that

$$\begin{bmatrix}\eta_{\bar a b}\\ \psi_{\bar a bc}\end{bmatrix} = \begin{bmatrix}\theta_{\bar a b}+\nabla_{\bar a}\sigma_b\\ \nabla_b\theta_{\bar a c}+\nabla_{\bar a}\mu_{bc}\end{bmatrix} = \begin{bmatrix}\theta_{\bar a b}\\ \nabla_b\theta_{\bar a c}\end{bmatrix}+D_{\bar a}\begin{bmatrix}\sigma_b\\ \mu_{bc}\end{bmatrix}\in\Gamma(\wedge^1 M\otimes K).$$

In an analogous way, interchanging the barred and unbarred indices, we can see that $(\eta_{a\bar b},\psi_{a\bar b\bar c})$ is contained in $\Gamma(\wedge^1 M\otimes K)$ and therefore $(\eta_{AB},\psi_{ABC})\in\Gamma(\wedge^1 M\otimes K)$, as required. $\square$

Chapter 5

Lorentzian symmetric spaces and plane waves spacetimes

In this chapter we will study the exactness of the Killing connection on locally homogeneous Lorentzian manifolds. Specifically, in Section 5.1, we described known results about locally homogeneous plane wave spacetimes. Section 5.2 will be dedicated to the Singer index of locally homogeneous plane waves, where we characterise all locally homogeneous plane waves with Singer index equal to 0. Lastly, in Section 5.3 we study exactness of the Killing connection of locally homogeneous plane wave with Singer index 0 is exact and Lorentzian locally symmetric spaces. In addition, the author would like to remark that the results obtained in Section 5.3, regarding Lorentzian locally symmetric spaces, have appeared previously in [13], and are one of his contributions to the article.

5.1 Locally homogeneous plane waves

Throughout this section, (M, g) will denote a Lorentzian manifold. For a given a vector field $X \in \Gamma(TM)$ and a given point p of M, we will let

$$X^{\perp} := \{Y \in TM \ : \ g(X, Y) = 0\}$$

denote the vector subbundle of TM with fibre $\mathbb{R}X_p^{\perp}$ at a point p of M. A Lorentzian manifold (M, g) is called a *pp-wave spacetime* (where pp stands for plane fronted with parallel rays) if it admits a vector field, which shall be denoted by X_-, that is parallel, null and such that its curvature tensor is non-null and it vanishes identically on $X_-^{\perp} \wedge X_-^{\perp}$, namely

$$\nabla X_- = 0, \quad g(X_-, X_-) = 0 \quad \text{and} \quad R(X, Y) = 0 \quad \text{for all} \quad X, Y \in \Gamma(X_-^{\perp}).$$

Originally, four dimensional pp-waves spacetimes were discovered by Brinkmann [8] as a class of Einstein manifolds that can be mapped conformally to each other, however, their name was introduced in [30], by Jordan, Ehlers and Kundt, in the English republication of their article [29].

Our convention for bivectors, considered as skew-symmetric endomorphisms of the tangent bundle of M, will be

$$X \wedge Y := (\iota_X g) \otimes Y - (\iota_X g) \otimes X.$$

In general, any two vector subbundles of the tangent bundle, V_1 and V_2, will define the subbundle

$$V_1 \wedge V_2 := \operatorname{span}\{X \wedge Y \in \mathfrak{so}(TM, g) \ : \ X \in V_1 \text{ and } Y \in V_2\}$$

of the skew-symmetric endomorphisms of TM.

An $(n+2)$-dimensional pp-wave spacetime always admits a special set of local coordinates $(x_-, x_1, \ldots, x_n, x_+)$, such that its metric tensor is of the following form:

$$g = 2\mathrm{d}x_-\mathrm{d}x_+ + 2H(x_1, \ldots, x_n, x_+)\mathrm{d}x_+^2 + \sum_{i=1}^{n} \mathrm{d}x_i^2. \tag{5.1}$$

Here H denotes a smooth function on M that does not depend on the coordinate x_-. In the literature, these coordinate charts are referred to as *Brinkmann coordinates*, see for example [21]. In Brinkmann coordinates, the Levi-Civita connection, associated to g, is given by

$$\nabla\partial_- = 0, \quad \nabla\partial_i = (\partial_i H)\mathrm{d}x_+ \otimes \partial_- \quad \text{and} \quad \nabla\partial_+ = \mathrm{d}H \otimes \partial_- - \mathrm{d}x_+ \otimes \operatorname{grad}(H), \tag{5.2}$$

where grad is taken with respect to the Euclidean metric in $\operatorname{span}\{\partial_1, \ldots, \partial_n\}$. It is straightforward to verify from equations (5.1) and (5.2), for the metric tensor and its Levi-Civita connection, that the coordinate vector field ∂_- is null and parallel. Particularly, from equation (5.2), we can observe that the vector subbundle $X_-^\perp := \partial_-^\perp$ is parallel with respect to the Levi-Civita connection.

The Riemannian curvature tensor and of a pp-wave spacetime and its first covariant derivative, in Brinkmann coordinates, are given by the formulas

$$R = \sum_{i,j=1}^{n} 4H_{ij}(\mathrm{d}x_i \wedge \mathrm{d}x_+) \cdot (\mathrm{d}x_j \wedge \mathrm{d}x_+) \quad \text{and} \quad \nabla R = \sum_{i,j=1}^{n} 4\mathrm{d}H_{ij} \otimes (\mathrm{d}x_i \wedge \mathrm{d}x_+) \cdot (\mathrm{d}x_j \wedge \mathrm{d}x_+),$$

where H_{ij} denotes $\partial_i\partial_j H$.

A special class of pp-waves spacetimes are the so called *plane waves*. These are pp-waves spacetimes for which, in addition, the curvature tensor is parallel in the directions of the subbundle $X_-^\perp$ of TM. To be precise, for the curvature tensor of a plane wave, it must hold that

$$\nabla_X R = 0, \quad \text{for all} \quad X \in \Gamma(X_-^\perp).$$

This implies that it is a necessary and sufficient condition, for a pp-wave spacetime to be a plane wave, that the function H satisfies

$$\partial_i\partial_j\partial_k H = 0, \quad \text{for all} \quad i,j,k = 1,\dots,n,$$

forcing H to be a quadratic function on the variables $x_1,\dots,x_n$. Therefore, the function H will be of the form

$$H = \frac{1}{2}\sum_{i,j=1}^{n} Q_{ij}(x^+)x_i x_j,$$

where Q_{ij} corresponds to the ij-entry of an $n\times n$ symmetric matrix Q, that is a function of only the coordinate x^+. Letting $\mathbb{E}$ denote the vector subbundle of TM, spanned by $\{\partial_1,\dots,\partial_n\}$, we will often consider Q as a symmetric endomorphism of $\mathbb{E}$, with respect to the restriction of g to $\mathbb{E}$. Abusing notation, we will also consider Q as the symmetric 2-tensor on M defined

$$Q = \sum_{i\le j=1}^{n} Q_{ij}(x_+)\mathrm{d}x_i\mathrm{d}x_j.$$

From now on, we will fix once and for all, the frame $\{X_-, X_1,\dots,X_n,X_+\}$ of TM which is given by

$$X_- = \partial_-, \quad X_i = \partial_i, \quad X_+ = -H\partial_- + \partial_+. \tag{5.3}$$

In this frame, the non-vanishing components of the curvature tensor and its iterated covariant derivatives are

$$(\nabla^\ell R)(X_+,\dots,X_+;X,X_+,Y,X_+) = g(Q^{(\ell)}X,Y), \quad \text{with} \quad X,Y\in\Gamma(\mathbb{E}) \tag{5.4}$$

where $Q^{(\ell)}$ denotes the symmetric endomorphism of $\mathbb{E}$, defined by

$$Q = \sum_{i\le j=1}^{n} (\partial_+^\ell Q_{ij})(x_+)\mathrm{d}x_i\mathrm{d}x_j.$$

The endomorphisms $Q^{(\ell)}$, in this frame, will be

$$Q = \sum_{i,j=1}^{n} Q_{ij}(\iota_{X_i}g)\otimes X_j.$$

Regarding the skew-symmetric endomorphisms of TM, their matrices will take the form

$$\begin{pmatrix} a & x^t & 0 \\ y & A & -x \\ 0 & -y^t & -a \end{pmatrix} \quad \text{with} \quad a\in\mathbb{R},\ x,y\in\mathbb{R}^n \text{ and } A\in\mathfrak{so}(n), \tag{5.5}$$

when expressed in the frame (5.3).

In order to simplify our notation, we will usually write $\nabla^\ell_+ R$ for the iterated covariant derivatives of R in the direction of X_+, i.e.

$$(\nabla^\ell_+ R)(U,V,X,Y) := (\nabla^\ell R)(X^+,\dots,X^+;U,V,X,Y).$$

Notice that all of the covariant derivatives of R are of the same algebraic type, in the sense that $\nabla^\ell_+ R$ is completely determined by a symmetric endomorphism which, in this case, is $Q^{(\ell)}$. The curvature endomorphisms can be expressed in terms of bivectors as

$$(\nabla^\ell_+ R)(X,X_+) = (Q^{(\ell)}X)\wedge X_-.$$

Remark 5.1.1. When (M,g) is indecomposable, Q is injective. It follows straightforwardly that the holonomy algebra of an indecomposable plane wave is isomorphic to $\mathbb{R}^n$:

$$\mathfrak{hol}_p(M,g) \simeq (\mathbb{E}\wedge X_-)_p \simeq \left\{ \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1,n+1) \ : \ x\in\mathbb{R}^n \right\}. \tag{5.6}$$

In the case that (M,g) is a decomposable plane wave, i.e. the product of an indecomposable plane wave and a flat factor $(\mathbb{R}^k, g_{\mathbb{R}^k})$, in terms of Q, it will mean that $\dim\ker(Q)=k$. In this case, we will let $\mathbb{H}_p := \ker(Q_p)^\perp \subseteq \mathfrak{so}(T_pM,g_p)$ and $\mathbb{H}$ will denote the vector bundle over M with fibre $\mathbb{H}_p$ at p. Therefore, we have that the holonomy algebra is

$$\mathfrak{hol}_p(M,g) \simeq (\mathbb{H}\wedge X_-)_p \simeq \left\{ \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1,n+1) \ : \ x\in\mathbb{R}^{n-k} \right\}. \tag{5.7}$$

For a given point p in M, we will let $\mathfrak{n}_p$ denote the normaliser in $\mathfrak{so}(T_pM,g_p)$ of the holonomy algebra of (M,g), with basepoint p. That is

$$\mathfrak{n}_p := \{A\in\mathfrak{so}(T_pM,g_p) \ : \ [A,H]\in\mathfrak{hol}_p(M,g), \ \forall\, H\in\mathfrak{hol}_p(M,g)\}.$$

Proposition 5.1.2. *Let (M,g) be an indecomposable plane wave spacetime. Then, $\mathfrak{n}_p$ is isomorphic to the parabolic subalgebra*

$$\mathfrak{p} := \left\{ \begin{pmatrix} a & x^t & 0 \\ 0 & A & -x \\ 0 & 0 & -a \end{pmatrix} \in \mathfrak{so}(1,n+1) \ : \ a\in\mathbb{R},\ x\in\mathbb{R}^n,\ A\in\mathfrak{so}(n) \right\},$$

of $\mathfrak{so}(1,n+1)$.

Proof. From equations (5.5) and (5.6), and a straightforward calculation we can observe that

$$\left[\begin{pmatrix} a & x^t & 0 \\ y & A & -x \\ 0 & -y^t & -a \end{pmatrix}, \begin{pmatrix} 0 & u^t & 0 \\ 0 & 0 & -u \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} -y^tu & u^t(a\mathrm{Id}-A) & 0 \\ 0 & yu^t-uy^t & -(a\mathrm{Id}+A)u \\ 0 & 0 & y^tu \end{pmatrix}$$

will be in the holonomy algebra of (M,g) if and only if $y^tu=0$. Since a, x and A can be chosen freely, it follows that $\mathfrak{n}_p$ will be isomorphic to $\mathfrak{p}$. $\square$

We will denote by $\mathfrak{p}_p$, the Lie algebra spanned by the skew-symmetric endomorphisms of T_pM, spanned by

$$\{(X_- \wedge X_+)_p, (X_i \wedge X_j)_p, (X_i \wedge X_-)_p\}_{i,j=1}^n,$$

which is isomorphic to $\mathfrak{p}$, the parabolic subalgebra of $\mathfrak{so}(1, n+1)$.

Corollary 5.1.3. *Let (M, g) be a locally homogeneous plane wave, and let $\mathfrak{k}_p^{R,\infty}$ be the projection of $K_p^{R,\infty}$ to $\mathfrak{so}(T_pM, g_p)$. Then $\mathfrak{k}_p^{R,\infty}$ is contained in $\mathfrak{p}_p$.*

Proof. By assumption, (M, g) is a locally homogeneous plane wave and therefore it follows from Proposition 5.1.2 that $\mathfrak{n}_p \simeq \mathfrak{p}$. On the other hand, by Proposition 2.3.6, the projection of $K_p^{R,\infty}$ to $\mathfrak{so}(T_pM, g_p)$ is always contained in the normaliser of the holonomy algebra of (M, g). It follows that $\mathfrak{k}_p^{R,\infty}$ is contained in $\mathfrak{p}_p$, as claimed. □

In the rest of this section we will place our attention on the plane waves which are locally homogeneous. In [6], Blau and O'Laughlin provided a classification of homogeneous plane waves spacetimes and, later on, Globke and Leistner showed that an indecomposable locally homogeneous pp-wave such that the rank of the curvature endomorphism is greater than 1, is a plane wave [25]. Therefore, with the exception of the cases when the curvature endomorphism has rank equal to 1, indecomposable locally homogeneous pp-waves are completely classified in terms of the symmetric matrices Q. In adequate coordinates, an $(n+2)$-dimensional indecomposable locally homogeneous pp-wave is defined by a matrix that takes the form

$$Q(x_+) = e^{x_+ P} Q_0 e^{-x_+ P} \tag{5.8}$$

or

$$Q(x_+) = \frac{1}{(x_+ + x_0)^2} \; e^{\log(x_+ + x_0)P} Q_0 e^{-\log(x_+ + x_0)P}, \tag{5.9}$$

where $x_0 \in \mathbb{R}$, $Q_0 \in \mathrm{Sym}(n)$ with trivial kernel and $P \in \mathfrak{so}(n)$. For Q defined in equation (5.9), we will always take $\{x_+ \in \mathbb{R} \ : \ x_+ + x_0 > 0\}$ as its domain. We will say that a locally homogeneous pp-wave (M, g) is *regular*, if Q is of the form (5.8), and we will say that (M, g) is singular if Q is of the form (5.9). We will usually write Q, in generality, as

$$Q = (\phi')^2 e^{\phi P} Q_0 e^{-\phi P}, \tag{5.10}$$

where ϕ is a smooth function, depending only on x_+. In this notation, a regular locally homogeneous plane wave will correspond to the endomorphism Q, with $\phi(x_+) = x_+$, and a singular one will correspond to the one with $\phi(x_+) = \log(x_+ + x_0)$. For convenience, and without loss of generality, we will always choose singular locally homogeneous plane waves with $x_0 = 1$. In this case, it is worth noticing that $Q(0) = Q_0$.

Remark 5.1.4. Singular locally homogeneous singular plane waves with the same matrices P and Q_0 are isometric.

Remark 5.1.5. A special class of indecomposable locally homogeneous plane waves are the so called *Cahen-Wallach spaces*, introduced in [9]. They are indecomposable, in the sense of the de Rham-Wu decomposition theorem [44], and they are locally symmetric spaces, i.e. $\nabla R = 0$. They are the locally homogeneous plane waves which are defined by a pair (Q_0, P) such that $[P, Q_0] = 0$. Results regarding the image of the Killing operator on Lorentzian locally symmetric spaces were obtained in [13], and in Section 5.5 we extend this results to locally homogeneous plane waves with Singer index equal to 0. In fact, $P = 0$ can be chosen.

We will let $\mathfrak{z}_{\mathfrak{so}(n)}(A) := \{B \in \mathfrak{so}(n) \;:\; [A, B] = 0\}$ denote the centraliser of a matrix, $A \in \mathfrak{gl}(n, \mathbb{R})$, in $\mathfrak{so}(n)$. For the centraliser of a family of matrices, $\{A_1, \ldots, A_\ell\}$, we will write

$$\mathfrak{z}_{\mathfrak{so}(n)}(A_1, \ldots, A_\ell) := \cap_{i=1}^{\ell} \mathfrak{z}_{\mathfrak{so}(n)}(A_i).$$

Recall that if A and B are matrices in $\mathfrak{gl}(n, \mathbb{R})$ which commute, then $e^{A+B} = e^A e^B$. If (M, g) is a locally homogeneous plane wave, defined by the pair (Q_0, P), we can always add elements from the centraliser of Q_0 and P, to P, leaving Q invariant. This follows by inspecting

$$Q = (\phi')^2 e^{\phi(P+Z)} Q_0 e^{-\phi(P+Z)} = (\phi')^2 e^{\phi P} Q_0 e^{-\phi P}.$$

Consequently, any other locally homogeneous plane wave defined in terms of the pair $(Q_0, P + Z)$, with $Z \in \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P)$, will be isometric to (M, g).

5.2 The Singer index of locally homogeneous plane waves

In this subsection we will study the Singer index of locally homogeneous plane waves. Particularly, we will determine exactly those with Singer homogeneous index equal to 0, in terms of the pairs of matrices (Q_0, P) defining them.

In order to compute the Singer homogeneous index of locally homogeneous plane waves, first we will describe the Lie algebras of automorphisms of algebraic curvature tensors of the same type of locally homogeneous plane waves. We will let V^{n+2} denote a real vector space of dimension $n + 2$, equiped with a non-degenerate symmetric bilinear form of signature $(1, n+1)$, which shall be denoted by $\langle \cdot, \cdot \rangle$. The Lie algebra of endomorphisms of V^{n+2}, i.e. $\mathfrak{so}(V^{n+2}) := \{A \in \mathrm{End}(V^{n+2}) \;:\; \langle Ax, y\rangle + \langle x, Ay\rangle, \text{ for all } x, y \in V^{n+2}\}$, is naturally isomorphic to $\mathfrak{so}(1, n + 1)$. We will fix an orthogonal basis $\{e_-, e_1, \ldots, e_n, e_+\}$ of V^{n+2} such that

$$\langle e_-, e_-\rangle = \langle e_-, e_i\rangle = \langle e_i, e_+\rangle = 0, \quad \langle e_-, e_+\rangle = 1 \quad \text{and} \quad \langle e_i, e_j\rangle = \delta_{ij},$$

and will denote by V^n, the vector subspace of V^{n+2} spanned by $\{e_1, \ldots, e_n\}$. In this basis, a generic element of $\mathfrak{so}(V^{n+2})$ will be of the form

$$\begin{pmatrix} a & x^t & 0 \\ y & A & -x \\ 0 & -y^t & -a \end{pmatrix}, \quad a \in \mathbb{R},\ u, v \in \mathbb{R}^n,\ A \in \mathfrak{so}(n). \tag{5.11}$$

We will consider the algebraic curvature tensors on V^{n+2} of the same algebraic type of the curvature tensor of an indecomposable locally homogeneous plane wave at a point. The vector space of tensors with the symmetries of Riemannian curvature tensors on a vector space V will be denoted by RV. Explicitly,

$$RV := \{T \in \mathrm{Sym}^2 \wedge^2 V^* \ :\ T(w,x,y,z) + T(y,w,x,z) + T(x,y,w,z) = 0,\ \forall\, w,x,y,z \in V\}.$$

Let Q be a symmetric endomorphism of V^n with trivial kernel and let $\mathcal{R}^Q \in RV^{n+2}$ be the algebraic curvature tensor such that its only non-zero components, up to symmetries, are given by

$$\mathcal{R}^Q(x, e_+, y, e_+) = (Qx, y), \quad \text{with} \quad x, y \in V^n. \tag{5.12}$$

Here $(\cdot,\cdot)$ denotes the restriction of $\langle\cdot,\cdot\rangle$ to V^n. We will denote the Lie algebra of skew-symmetric automorphisms of $\mathcal{R}^Q$ by

$$\mathfrak{aut}(\mathcal{R}^Q) = \{A \in \mathfrak{so}(V^{n+2}) \ :\ A \cdot \mathcal{R}^Q = 0\}.$$

Proposition 5.2.1. *Let $\mathcal{R}^Q \in RV^{n+2}$ be an algebraic curvature tensor defined by a symmetric endomorphism Q of V^n, as in equation (5.12). Then*

$$\mathfrak{aut}(\mathcal{R}^Q) \simeq \mathfrak{z}_{\mathfrak{so}(V^n)}(Q) \ltimes V^n,$$

where

$$\mathfrak{z}_{\mathfrak{so}(V^n)}(Q) \ltimes V^n = \left\{ \begin{pmatrix} 0 & u^t & 0 \\ 0 & A & -u \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(V^{n+2}) \ :\ A \in \mathfrak{z}_{\mathfrak{so}(V^n)}(Q),\ u \in V^n \right\}.$$

Proof. Let $\tilde{A}$ be an element of $\mathfrak{aut}(\mathcal{R}^Q)$ which, in the basis $\{e_-, e_1, \ldots, e_n, e_+\}$, takes the form

$$\begin{pmatrix} a & u^t & 0 \\ v & A & -u \\ 0 & -v^t & -a \end{pmatrix}, \quad a \in \mathbb{R},\ u, v \in \mathbb{R}^n,\ A \in \mathfrak{so}(n),$$

and let x and y be elements of V^n. By inspection, we can see that

$$\tilde{A}e_- = ae_- + v, \quad \tilde{A}x = (u,x)e_- + Ax - (v,x)e_+, \quad \tilde{A}e_+ = -u - ae_+.$$

By definition, the action of $\tilde{A}$ annihilates $\mathcal{R}^Q$. Expanding $(\tilde{A}\cdot\mathcal{R}^Q)(x, e_+, y, e_+) = 0$ reveals that

$$0 = \mathcal{R}^Q(Ax, e_+, y, e_+) - a\mathcal{R}^Q(x, e_+, y, e_+) + \mathcal{R}^Q(x, e_+, Ay, e_+) - a\mathcal{R}^Q(x, e_+, y, e_+),$$

which, by equation (5.12) is nothing but

$$0 = (QAx, y) + (Qx, Ay) - 2a(Qx, y) = (([Q, A] - 2aQ)x, y), \quad \text{for all } x, y \in V^n.$$

The inner product $(\cdot, \cdot)$ is of definite signature and therefore it must hold that

$$[A, Q] + 2aQ = 0.$$

Multiplying the above equation by Q and taking its trace we get

$$0 = \mathrm{tr}([A, Q]Q) = -2a\ \mathrm{tr}(Q^2) \neq 0,$$

which implies that $a = 0$ and $A \in \mathfrak{z}_{\mathfrak{so}(n)}(Q)$. Lastly, we can observe that

$$0 = (\tilde{A} \cdot \mathcal{R}^Q)(e_-, e_+, x, e_+) = -\mathcal{R}^Q(v, e_+, x, e_+) = -(Qv, x)$$

for all $x \in V^n$. Since Q is injective, v must be equal to 0. The remaining cases do not reveal any extra constraint for $\tilde{A}$ and thus

$$\tilde{A} = \begin{pmatrix} 0 & u^t & 0 \\ 0 & A & -u \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad u \in V^n \quad \text{and} \quad A \in \mathfrak{z}_{\mathfrak{so}(V^n)}(Q),$$

as claimed. $\square$

If (M, g) is a locally homogeneous plane wave defined by the pair (Q_0, P), by choosing a point p in M, we can observe that R_p and $\mathcal{R}^{Q_p}$ are of the same algebraic type in the sense that they are equal as elements of RT_pM. Consequently, by Proposition 5.2.1, we have obtained the isomorphism

$$\mathfrak{h}_p^R = \mathfrak{aut}(\mathcal{R}^{Q_p}) \simeq \mathfrak{z}_{\mathfrak{so}(V^n)}(Q_p) \ltimes V^n.$$

More generally, the iterated covariant derivatives $(\nabla_+^\ell R)_p$, of the curvature tensor, are of the same algebraic type as $\mathcal{R}^{Q_p^{(\ell)}}$, which yields the isomorphism

$$\mathfrak{h}_p^{\nabla_+^\ell R} \simeq \mathfrak{z}_{\mathfrak{so}(V^n)}(Q_p^{(\ell)}) \ltimes V^n,$$

again by Proposition 5.2.1. Identifying $\mathfrak{z}_{\mathfrak{so}(V^n)}(Q_p^{(\ell)}) \ltimes V^n$ with $\mathfrak{z}_{\mathfrak{so}(\mathbb{E}_p)}(Q_p^{(\ell)}) \ltimes (\mathbb{E} \wedge X_-)_p$, we have showed that

$$\mathfrak{h}^{\nabla_+^\ell R} = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q^{(\ell)}) \ltimes \mathbb{E} \wedge X_-. \tag{5.13}$$

Lemma 5.2.2. *Let $Q : I \subseteq \mathbb{R} \to \mathrm{Sym}(n)$ be a smooth function of the form*

$$Q(t) = \phi'(t)^2 e^{\phi(t)P} Q_0 e^{-\phi(t)P},$$

where $Q_0 \in \mathrm{Sym}(n)$, $P \in \mathfrak{so}(n)$ and $\phi \in C^\infty(I)$ such that $\phi'(t) \neq 0$ for all $t \in I$. Then, for each $\ell \geq 0$,

$$\mathfrak{z}_{\mathfrak{so}(n)}(Q(t), Q'(t), \ldots, Q^{(\ell)}(t)) = \mathfrak{z}_{\mathfrak{so}(n)}(Q(t), \mathrm{ad}(P)Q(t), \ldots, \mathrm{ad}(P)^\ell Q(t)). \tag{5.14}$$

Proof. Letting $q : I \subseteq \mathbb{R} \to \mathrm{Sym}^2(n)$ be the map $q = e^{\phi P} Q_0 e^{-\phi P}$ and recalling that

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\phi P} = \phi' P e^{\phi P} = \phi' e^{\phi P} P,$$

it is straightforward from the Leibniz rule that

$$q' = \phi' P e^{\phi P} Q_0 e^{-\phi P} - \phi' e^{\phi P} Q_0 e^{-\phi P} P = \phi' [P, q].$$

Since $Q = (\phi')^2 q$, we have that Q' is given by

$$Q' = 2\phi'\phi'' q + (\phi')^3 \mathrm{ad}(P) q = 2\phi''(\phi')^{-1} Q + \phi' \mathrm{ad}(P) Q. \tag{5.15}$$

Inductively, it is clear that $Q^{(\ell)}$ will be a linear combination of $Q, \mathrm{ad}(P)Q, \ldots, \mathrm{ad}(P)^\ell Q$, since Q' is a linear combination of Q and $\mathrm{ad}(P)Q$, namely

$$Q^{(\ell)}(t) = f_{\ell,0} Q(t) + f_{\ell,1}(t)\mathrm{ad}(P)Q(t) + \cdots + f_{\ell,\ell}(t)\mathrm{ad}(P)^\ell Q(t), \tag{5.16}$$

where $f_0, \ldots, f_\ell$ are smooth functions on I. Now, suppose that equation (5.14) holds for all ℓ up to k. Then

$$\mathfrak{z}_{\mathfrak{so}(n)}(Q, Q', \ldots, Q^{(k+1)}) = \mathfrak{z}_{\mathfrak{so}(n)}(Q, \mathrm{ad}(P)Q, \ldots, \mathrm{ad}(P)^k Q) \cap \mathfrak{z}_{\mathfrak{so}(n)}(Q^{(k+1)}).$$

Since $Q^{(k+1)}$ is of the form found in equation (5.16), the right hand side of the above equation is exactly equal to $\mathfrak{z}_{\mathfrak{so}(n)}(Q, \mathrm{ad}(P)Q, \ldots, \mathrm{ad}(P)^k Q, \mathrm{ad}(P)^{k+1} Q)$. □

Proposition 5.2.3. *Let (M, g) be a locally homogeneous plane wave. Then*

$$\mathfrak{h}^{R,\ell} = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q, \mathrm{ad}(P)Q, \ldots, \mathrm{ad}(P)^\ell Q) \ltimes \mathbb{E} \wedge X_-.$$

Proof. Let us choose an element A of $\mathfrak{h}^{R,\ell}$. By equation (5.13), for any vector X in TM, it is clear that $AX \in X_-^\perp$, since $\mathfrak{h}^{R,\ell} \subseteq \mathfrak{h}^R$. Then

$$\iota_X(A \cdot \nabla^{\ell+1} R) = A \cdot \nabla_X \nabla^\ell R - \nabla_{AX} \nabla^\ell R = A \cdot \nabla_X \nabla^\ell R,$$

follows from the fact that, for each $\ell \geq 0$, $\nabla^\ell R$ is parallel in the directions of $X_-^\perp$. The above equation implies that A is in $\mathfrak{h}^{R,\ell+1}$ if and only if A is in $\mathfrak{h}^{\nabla_{X^+}\nabla^\ell R}$. Particularly, when $\ell = 0$, this yields the isomorphism

$$\mathfrak{h}^{R,1} \simeq \mathfrak{h}^R \cap \mathfrak{h}^{\nabla_+ R}.$$

Inductively, we observe that for each $\ell \geq 0$

$$\mathfrak{h}^{R,\ell} = \bigcap_{i=0}^{\ell} \mathfrak{h}^{\nabla_+^i R}, \tag{5.17}$$

which, by equation (5.13), yields

$$\mathfrak{h}^{R,\ell} = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q, \dots, Q^{(\ell)}) \ltimes \mathbb{E} \wedge X_-.$$

Applying Lemma 5.2.2 to each fibre of $\mathfrak{h}^{R,\ell}$, we get the desired isomorphism:

$$\mathfrak{h}^{R,\ell} \simeq \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q, \operatorname{ad}(P)Q, \dots, \operatorname{ad}(P)^\ell Q) \ltimes \mathbb{E} \wedge X_-.$$

□

The above proposition allows us to construct examples of locally homogeneous plane waves with Singer homogeneous index equal to 0. The simplest one, perhaps, is a locally homogeneous plane wave defined by a Q_0 with all of its eigenvalues possessing multiplicity 1. In this case, the centraliser of Q_0 in $\mathfrak{so}(n)$ is trivial and, therefore, $\mathfrak{h}^R = \mathfrak{h}^{R,\infty} = \mathbb{E} \wedge X_-$. Here is a non-trivial example:

Example 5.2.4. The locally homogeneous plane wave with $Q_0 \in \operatorname{Sym}^2(n)$ and $P \in \mathfrak{so}(n)$ given by

$$Q_0 = \begin{pmatrix} \mathrm{Id}_{n-2} & 0 \\ 0 & q \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$$

with $q \in \operatorname{Sym}^2(2)$ and $p \in \mathfrak{so}(2)$, such that q has different non-zero eigenvalues and $p \neq 0$. The commutator of P and Q_0 is

$$[P, Q_0] = \begin{pmatrix} 0 & 0 \\ 0 & [p, q] \end{pmatrix}.$$

Clearly, P and Q_0 do not commute, since the commutator of p and q does not vanish, and their centralisers in $\mathfrak{so}(n)$ coincide and are given by

$$\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) = \mathfrak{z}_{\mathfrak{so}(n)}(P) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \;:\; A \in \mathfrak{so}(n-2) \right\}$$

Consequently, $\mathfrak{h}^R = \mathfrak{h}^{R,\infty}$ or, in other words, the locally homogeneous plane wave defined by the pair (Q_0, P) has a Singer homogeneous index equal to 0.

We will show in Theorem 5.2.7 that all locally homogeneous plane wave with $k_g = 0$ are, in a sense, of the same form as in Example 5.2.4.

Lemma 5.2.5. *Let $Q \in \mathfrak{gl}(n, \mathbb{R})$ and $P \in \mathfrak{z}_{\mathfrak{so}(n)}(Q)^\perp$. Then $\mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q]) = \mathfrak{z}_{\mathfrak{so}(n)}(Q, P)$.*

Proof. The inclusion of $\mathfrak{z}_{\mathfrak{so}(n)}(Q, P)$ in $\mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q])$ follows directly from the Jacobi identity. To verify that $\mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q]) \subseteq \mathfrak{z}_{\mathfrak{so}(n)}(Q, P)$, let us choose an element A, of $\mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q])$. Since $[A, Q] = [A, [P, Q]] = 0$, by the Jacobi identity we can observe that

$$0 = [A, [P, Q]] = [[A, P], Q] + [P, [A, Q]] = [[A, P], Q],$$

which implies that $[A, P] \in \mathfrak{z}_{\mathfrak{so}(n)}(Q)$. As $A \in \mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q])$ and $P \in \mathfrak{z}_{\mathfrak{so}(n)}(Q)^\perp$, the commutator $[A, P]$ is in $\mathfrak{z}_{\mathfrak{so}(n)}(Q)^\perp$, since

$$\mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q]) \subseteq \mathfrak{z}_{\mathfrak{so}(n)}(Q) \quad \text{and} \quad [\mathfrak{z}_{\mathfrak{so}(n)}(Q), \mathfrak{z}_{\mathfrak{so}(n)}(Q)^\perp] \subseteq \mathfrak{z}_{\mathfrak{so}(n)}(Q)^\perp.$$

It follows from this argument that A commutes with P and thus $\mathfrak{z}_{\mathfrak{so}(n)}(Q, [P, Q])$ is equal to $\mathfrak{z}_{\mathfrak{so}(n)}(Q, P)$, as claimed. $\square$

The above lemma will provide us with a useful tool to characterise the locally homogeneous plane waves with Singer homogeneous index equal to 0. For the reminder of this section, we will let P_0 denote the $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0)^\perp$ component of P.

Proposition 5.2.6. *Let (M, g) be a locally homogeneous plane wave defined by the pair (Q_0, P). Then, $k_g = 0$ if and only if $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$.*

Proof. On a locally homogeneous space, to posses Singer index equal to 0 is equivalent to have $\mathfrak{h}_p^R = \mathfrak{h}_p^{R,1}$ for an arbitrary point p. To prove this proposition, we will show that $\mathfrak{h}_p^R = \mathfrak{h}_p^{R,1}$ if and only if $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$.

We have showed in Proposition 5.2.3, that

$$\mathfrak{h}^R = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q) \ltimes \mathbb{E} \wedge X_- \quad \text{and} \quad \mathfrak{h}^{R,1} = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q, [P, Q]) \ltimes \mathbb{E} \wedge X_-.$$

The choice of a point such that $x_+ = 0$, yields

$$\mathfrak{h}_p^R \simeq \mathfrak{z}_{\mathfrak{so}(n)}(Q_0) \ltimes \mathbb{R}^n \quad \text{and} \quad \mathfrak{h}_p^{R,1} \simeq \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, [P, Q_0]) \ltimes \mathbb{R}^n = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, [P_0, Q_0]) \ltimes \mathbb{R}^n.$$

Lastly, by Lemma 5.2.5 it follows that $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0, [P_0, Q_0]) = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$ which implies that $\mathfrak{h}_p^R = \mathfrak{h}_p^{R,1}$ if and only if $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$. $\square$

To conclude this chapter, for a given Q_0, we will characterise explicitly the matrices P for which the pair (Q_0, P) defines a locally homogeneous plane wave with Singer index equal to 0. In order to do this, we will make use of Proposition 5.2.6, by finding all possibles $P_0 \in \mathfrak{z}_{\mathfrak{so}(n)}(Q_0)^\perp$ such that $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$. To be clear, we will find conditions on P_0 such that $[A, P_0] = 0$ for all $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0)$. However, before stating the theorem, we will set up the notation.

We will let $\lambda_1, \ldots, \lambda_k$ denote all the different eigenvalues of Q_0, i.e. not counting multiplicities, and W_i will denote the eigenspace associated to the eigenvalue λ_i. If we let $\mu(\lambda_i)$ denote the multiplicity of λ_i, without loss of generality, we will assume that the eigenvalues are ordered in a way such that $\mu(\lambda_1) \geq \mu(\lambda_2) \geq \cdots \geq \mu(\lambda_k)$. Also we will assume that $\mu(\lambda_1) \geq \cdots \geq \mu(\lambda_\ell) > 1$ and $\mu(\lambda_{\ell+1}) = \cdots = \mu(\lambda_k) = 1$ for some ℓ and we will set

$$W_{\mu\neq 1} := \bigoplus_{i=1}^{\ell} W_i \qquad \text{and} \qquad W_{\mu=1} := \bigoplus_{i=\ell+1}^{k} W_i.$$

Theorem 5.2.7. *Let (M, g) be a locally homogeneous plane wave defined by the pair (Q_0, P). Then, $k_g = 0$ if and only if the projection of P into $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0)^\perp$, i.e. P_0, lies in $\mathfrak{so}(W_{\mu=1}) \subseteq \mathfrak{so}(n)$.*

Proof. Let $\pi_{ij} : \mathfrak{so}(n) \to \mathfrak{so}(W_i \oplus W_j)$ be the natural projection, for $i, j = 1, \ldots k$. The centraliser of Q_0 in $\mathfrak{so}(n)$ is then given by $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) \simeq \mathfrak{so}(W_1) \oplus \cdots \oplus \mathfrak{so}(W_\ell)$ and the projection of P_0 to $\mathfrak{so}(W_i \oplus W_j)$ will be of the form

$$\pi_{ij}(P_0) = \begin{pmatrix} 0 & p_{ij} \\ -p_{ij}^t & 0 \end{pmatrix} \quad \text{for some } p_{ij} \in W_i \otimes W_j.$$

Choosing an arbitrary element of $\mathfrak{so}(W_i) \subseteq \mathfrak{z}_{\mathfrak{so}(n)}(Q_0)$, we have that

$$\left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p_{ij} \\ -p_{ij}^t & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & Ap_{ij} \\ -p_{ij}^t A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } \ A \in \mathfrak{so}(W_i)$$

will hold if and only if $p_{ij} = 0$. It follows that in order to get the equality between $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0)$ and $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$, we need $p_{ij} = 0$ and thus $\pi_{ij}(P_0) = 0$. Repeating this process for all $i, j = 1, \ldots, \ell$, we conclude that $\pi_{\mu\neq 1}(P_0) = 0$, where $\pi_{\mu\neq 1} : \mathfrak{so}(n) \to \mathfrak{so}(W_{\mu\neq 1})$ is the natural projection.

Lastly, writing P_0 in blocks as

$$P_0 = \begin{pmatrix} 0 & p \\ -p^t & p_1 \end{pmatrix}, \quad \text{with} \quad p \in W_{\mu\neq 1} \otimes W_{\mu=1} \quad \text{and} \quad p_1 \in \mathfrak{so}(W_{\mu=1}),$$

reveals that

$$\left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p \\ -p^t & p_1 \end{pmatrix}\right] = \begin{pmatrix} 0 & Ap \\ -p^t A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } \ A \in \mathfrak{so}(W_1) \oplus \cdots \oplus \mathfrak{so}(W_\ell)$$

will hold if and only if $p = 0$. Consequently, $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, P_0)$ if and only if $P_0 \in \mathfrak{so}(W_{\mu=1})$. Lastly, the claim follows from Proposition 5.2.6. $\square$

5.3 Exactness on Lorentzian symmetric spaces and plane waves spacetimes

In this section we will study the exactness of the Killing connection on Lorentzian locally symmetric spaces and locally homogeneous plane waves with Singer index equal to 0. Particularly, the main result obtained on locally homogeneous plane wave is the following theorem:

Theorem 5.3.1. *The Killing connection of a locally homogeneous plane wave spacetime with Singer index equal to 0 is exact.*

Particular instances of locally homogeneous plane waves with Singer index 0 are the locally symmetric ones, i.e. Cahen-Wallach spaces (see Remark 5.1.5). We therefore get the following corollary.

Corollary 5.3.2. *The Killing connection of a Cahen-Wallach space is exact.*

Lorentzian symmetric spaces were classified in 1970 by Cahen and Wallach [9]. In their article they proved that if (M, g) is a simply connected Lorentzian symmetric space, then (M, g) is the product of a Riemannian symmetric space with one of the following:

(1) $\mathbb{R}$ with metric $-\mathrm{d}t^2$.

(2) The universal covering space of an n-dimensional de Sitter or anti de Sitter space with $n \geq 2$.

(3) An n-dimensional Cahen-Wallach space, with $n \geq 3$.

Particularly, any Lorentzian locally symmetric spaces is universally covered by a simply connected Lorentzian symmetric space. Regarding the exactness of the Killing connection on Lorentzian locally symmetric spaces, the main result obtained in this section is the following theorem:

Theorem 5.3.3. *Let (M, g) be a Lorentzian locally symmetric space. Then the Killing connection is exact unless the de Rham-Wu decomposition of (M, g) contains a Hermitian factor and a factor that is flat or a Cahen-Wallach space, in which case the Killing connection is not exact.*

We will first proceed to prove Theorem 5.3.1. In order to achieve this goal, we will compute explicitly the curvature filtrations of locally homogeneous plane waves spacetimes and apply the machinery previously described in Sections 1.3 and 4.2. The proof of Theorem 5.3.1 will be organised as follows: Locally homogeneous plane waves have their holonomy algebras contained in $\mathfrak{h}_0$ (see Remark 5.1.1) and it has been proved in [25] that they are reductive. If in addition we assume that $k_g = 0$, by Proposition 4.2.3 the curvature filtration of (E, D) will be parallel. Lastly, we will compute their curvature filtrations and show that they stabilise at the second step and that $E_2 = E$. Then, that (E, D) is exact would follow from Proposition 1.3.2. Before proceeding to calculate the curvature filtration of locally homogeneous plane waves, we will set up the notation for this subsection and recall previous results about locally homogeneous plane waves.

We will let (M, g) be an $(n+2)$-dimensional locally homogeneous plane wave, defined by the pair (Q_0, P), and we will let P_0 denote the projection of P to $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0)^{\perp}$. We will often refer to Q_0 and P as endomorphisms of the vector bundle $\mathbb{E}$, or its matrices in an appropriate orthogonal frame. If the rank of Q is equal to k, we will let $\mathbb{H}$ to be the subbundle of $\mathbb{E}$ of rank k with fibre $\ker(Q_p)^{\perp} \subseteq \mathbb{E}_p$ at p. The bundle with fibre $\mathfrak{hol}_p(M, g)$ at p will be $\mathfrak{hol}(M, g) = \mathbb{H} \wedge X_-$ and $\mathfrak{h}_0 = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q) \ltimes (\mathbb{E} \wedge X_-)$. We will make use of the identifications

$$\mathfrak{hol}_p(M,g) \simeq \left\{ \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1, n+1) \ : \ x \in \mathbb{R}^k \right\} =: \mathfrak{hol}$$

and

$$(\mathfrak{h}_0)_p \simeq \left\{ \begin{pmatrix} 0 & x^t & 0 \\ 0 & A & -x \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1, n+1) \ : \ A \in \mathfrak{z}_{\mathfrak{so}(n)}(Q_0) \ x \in \mathbb{R}^n \right\} = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0) \ltimes \mathbb{R}^n$$

We will let $\mathfrak{p} := (X_- \wedge X_+ \oplus \mathfrak{so}(\mathbb{E})) \ltimes \mathbb{E} \wedge X_-$ be the subbundle of $\mathfrak{so}(TM, g)$ whose fibres will be identified by the isomorphism

$$\mathfrak{p}_p \simeq \left\{ \begin{pmatrix} a & x^t & 0 \\ 0 & A & -x \\ 0 & 0 & -a \end{pmatrix} \in \mathfrak{so}(1, n+1) \ : \ a \in \mathbb{R}, \ A \in \mathfrak{so}(n), \ x \in \mathbb{R}^n \right\} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.$$

Lastly, we also remark that $\mathfrak{hol}_p(M, g)$ is generated by the curvature endomorphisms and therefore $\mathfrak{hol}(M, g)$ coincides with $\mathfrak{r} = \{R(X, Y) \ : \ X, Y \in TM\}$.

We are now in condition to proceed to compute the curvature filtration of (E, D). Firstly, we will calculate the kernel of the Killing curvature and, for the remaining steps of the curvature filtration, we have proved in Proposition 4.2.2 that for each $\ell \geq 1$ the curvature filtration is given by $E_\ell = TM \oplus \mathfrak{h}_\ell$, with

$$\mathfrak{h}_\ell = \{A \in \mathfrak{so}(TM, g) \ : \ [A, B] \in \mathfrak{h}_{\ell-1}, \ \forall \ B \in \mathfrak{r}\}.$$

Thus, it will only be required to calculate $\mathfrak{h}_\ell$.

Proposition 5.3.4. *Let (M,g) be a locally homogeneous plane wave, defined by the pair (Q_0,P). Then, the kernel of the Killing curvature is $K^R = H^R \oplus C^R$, where*

$$H^R = \begin{bmatrix} 0 \\ \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q) \ltimes \mathbb{E}\wedge X_- \end{bmatrix} \quad \text{and} \quad C^R = \begin{bmatrix} X_-^{\perp} \\ 0 \end{bmatrix} \oplus \mathbb{R}\begin{bmatrix} X_+ \\ A_+ \end{bmatrix},$$

with A_+ given by

$$A_+ = -\sum_{i<j}(P_0)_{ij}X_i\wedge X_j$$

in the case when (M,g) is a regular plane wave and

$$A_+ = -\frac{1}{x_+ + 1}\left(X_-\wedge X_+ + \sum_{i<j}(P_0)_{ij}X_i\wedge X_j\right),$$

when (M,g) is singular plane wave.

Proof. Recall that the bundle H^R is the subbundle of E, defined by elements of the form $(0,A)$, such that $A\cdot R = 0$. We have showed, in Proposition 5.2.3, that $\mathfrak{h}^R$ is equal to $\mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q)\ltimes \mathbb{E}\wedge X_-$. Therefore, the bundle H^R is given exactly by

$$H^R = \begin{bmatrix} 0 \\ \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q) \ltimes \mathbb{E}\wedge X_- \end{bmatrix}.$$

The subbundle C^R of K^R, complementary to H^R is given by elements (X,A) of E, solutions to the equation

$$\nabla_X R + A\cdot R = 0.$$

Notice that if the TM component of $(X,A)\in C^R$ is in $X_-^{\perp}$, the endomorphism A can be taken to be equal to 0, since R is parallel in the directions of $X_-^{\perp}$. It is immediate that $X_-^{\perp}$ injects into C^R as

$$X_-^{\perp} \hookrightarrow \begin{bmatrix} X_-^{\perp} \\ 0 \end{bmatrix} \subseteq C^R.$$

It is only left to find an element, A_+ in $\mathfrak{so}(TM,g)$, such that

$$\nabla_{X_+}R + A_+\cdot R = 0. \tag{5.18}$$

At each $p\in M$, Proposition 2.3.6 implies that A_+ must be in the normaliser of the holonomy algebra of (M,g), which is isomorphic to $\mathfrak{p}_p$, by Corollary 5.1.3. Also, A_+ lies in a complement of $\mathfrak{h}^R$ in $\mathfrak{so}(TM,g)$ hence, without loss of generality, we can assume that A_+ is of the form

$$A_+ = -a\,X_-\wedge X_+ + \sum_{i<j}A_{ij}\,X_i\wedge X_j = \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix},$$

such that $a \in \mathbb{R}$ and $A \in \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q)^{\perp}$. We note that if (X_+, B_+) is another element in C^R, $A_+ - B_+$ would lie in $\mathfrak{h}^R$, which implies that A_+ will be unique modulo $\mathfrak{h}^R$.

A closer look at equation (5.18) reveals that

$$\begin{aligned} 0 &= (\nabla_{X_+} R)(X, X_+, Y, X_+) + (A_+ \cdot R)(X, X_+, Y, X_+) \\ &= g(Q'X, Y) + g([A, Q]X, Y) + 2a\, g(QX, Y). \end{aligned}$$

The vector fields $X, Y \in \Gamma(\mathbb{E})$ were arbitrary, hence

$$Q' + [A, Q] + 2aQ = 0. \tag{5.19}$$

must hold. Recall that $Q = (\phi')^2 e^{\phi P} Q_0 e^{-\phi P}$. Then, a direct calculation yields

$$Q' = 2\phi''(\phi')^{-1} Q + \phi'[P, Q],$$

for what equation (5.19) becomes

$$[\phi' P + A, Q] + 2(\phi''(\phi')^{-1} + a)Q = 0. \tag{5.20}$$

The endomorphism A_+ is unique modulo $\mathfrak{h}^R$, so we will take

$$a = -\phi''(\phi')^{-1} \quad \text{and} \quad A = -\phi' P_0,$$

since $A \in \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q)^{\perp}$, by assumption. In the case that (M, g) is regular plane wave, $\phi(x_+) = x_+$ and thus $a = 0$. It follows that A_+ is given by

$$A_+ = -\sum_{i<j} (P_0)_{ij}\, X_i \wedge X_j.$$

Lastly, in the case when (M, g) a singular plane wave, $\phi(x_+) = \log(x_+ + 1)$ and therefore $a = (x_+ + 1)^{-1}$ and

$$A_+ = -\frac{1}{x_+ + 1} \left(X_- \wedge X_+ + \sum_{i<j} (P_0)_{ij} X_i \wedge X_j \right).$$

We can conclude that

$$C^R = \begin{bmatrix} X_-^{\perp} \\ 0 \end{bmatrix} \oplus \mathbb{R} \begin{bmatrix} X_+ \\ A_+ \end{bmatrix}.$$

□

Now we will proceed to calculate $\mathfrak{h}_\ell$ for $\ell \geq 1$. We will show that $\mathfrak{h}_1 = \mathfrak{p}$ and $\mathfrak{h}_2 = \mathfrak{so}(TM, g)$, hence $\mathfrak{h}_\ell = \mathfrak{so}(TM, g)$ for all $\ell \geq 2$. Following the notation used in Section 4.2, and using the identifications established in the beginning of this subsection, we have showed that $(\mathfrak{h}_0)_p$ is isomorphic to $\mathfrak{z}_{\mathfrak{so}(n)}(Q_0) \ltimes \mathbb{R}^n$. For locally homogeneous plane waves we can observe that, for $\ell = 1$, we have

$$(\mathfrak{h}_1)_p \simeq \{A \in \mathfrak{so}(1, n+1) \ : \ [A, B] \in \mathfrak{z}_{\mathfrak{so}(n)}(Q_0) \ltimes \mathbb{R}^n \ \forall \ B \in \mathfrak{hol}\}.$$

Choosing arbitrary elements $A \in \mathfrak{so}(1, n+1)$ and $B \in \mathfrak{hol}$, of the form

$$A = \begin{pmatrix} a & x^t & 0 \\ y & Z & -x \\ 0 & -y^t & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & u^t & 0 \\ 0 & 0 & -u \\ 0 & 0 & 0 \end{pmatrix},$$

we get that their commutator is given by

$$[A, B] = \begin{pmatrix} -y^t u & (au + Zu)^t & 0 \\ 0 & yu^t - uy^t & -(au + Zu) \\ 0 & 0 & y^t u \end{pmatrix}. \tag{5.21}$$

The first thing to notice is that there are no constraints in a, x or Z for $[A, B]$ to be in $\mathfrak{hol}$, whereas for y we have that it must solve the following equations

$$y^t u = 0 \quad \text{and} \quad [yu^t - uy^t, Q_0] = 0 \quad \text{for all } u \in \operatorname{Im}(Q_0) = \ker(Q_0)^{\perp}.$$

The first equation implies that $y \in \ker(Q_0)$, since $u \in \ker(Q_0)^{\perp}$. This implies that $yu^t - uy^t$ takes an off diagonal form, but the centraliser of Q_0 is block diagonal, hence y needs to be equal to 0. Consequently, we have showed that $(\mathfrak{h}_1)_p$ is isomorphic to $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$, which translated into the vector bundles becomes

$$\mathfrak{h}_1 = \mathfrak{p} = (X_- \wedge X_+ \oplus \mathfrak{so}(\mathbb{E})) \ltimes \mathbb{E} \wedge X_-.$$

Analogously as for the case of $\mathfrak{h}_1$, we can observe from equation (5.21) that there are not constraints $A \in \mathfrak{so}(1, n+1)$ for it to satisfy $[A, B] \in (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ for all $B \in \mathfrak{hol}$, hence $(\mathfrak{h}_2)_p \simeq \mathfrak{so}(1, n+1)$. We have showed that $\mathfrak{h}_2 = \mathfrak{so}(TM, g)$.

Proof of Theorem 5.3.1. It was noted in Remark 5.1.1 that the holonomy algebras of locally homogeneous plane waves are always contained in $\mathfrak{h}_0$, hence $E_\ell = TM \oplus \mathfrak{h}_\ell$ for all $\ell \geq 1$, by Proposition 4.2.2. We have proved in Proposition 5.3.4 that

$$\mathfrak{h}_0 = \mathfrak{z}_{\mathfrak{so}(\mathbb{E})}(Q) \ltimes \mathbb{E} \wedge X_-,$$

and also we have computed

$$\mathfrak{h}_1 = (X_- \wedge X_+ \oplus \mathfrak{so}(\mathbb{E})) \ltimes \mathbb{E} \wedge X_- \quad \text{and} \quad \mathfrak{h}_2 = \mathfrak{so}(TM, g),$$

which implies that the curvature filtration of (E, D) stabilises at the second step and that $E = E_2$. By assumption, the Singer index of (M, g) is equal 0, hence that the curvature filtration is parallel follows from Proposition 4.2.3. Lastly, it follows from Proposition 1.3.2 that (E, D) is exact, since the curvature filtration is parallel and $E = E_2$. $\square$

Before moving into the proof of Theorem 5.3.3, we need to remark that Cahen-Wallach spaces and of course flat manifolds, have a non-trivial nullity bundle. To deal with products of such spaces we will need to prove a rather technical proposition and, for convenience, we will enunciate a crucial lemma from [12]:

Lemma 5.3.5. *[12, Lemma 3] Supouse (M, g) is a Riemannian locally symmetric space with neither Hermitian not flat factors. If μ_{bc} is a 2-form on M so that*

$$\nabla_a \mu_{bc} - R_{bc}{}^d{}_a \sigma_d, \quad \textit{for some uniquely determined 1-form } \sigma_c,$$

then $\nabla_b \sigma_c = \mu_{bc}$.

Proposition 5.3.6. *Let (M_1, g_1) be a locally symmetric space of dimension > 1 that is either one of the following:*

1. *Pseudo-Riemannian of non-zero constant sectional curvature.*

2. *Non-Hermitian indecomposable Riemannian.*

If $(M_2, g_{2)}$ is another pseudo-Riemannian locally symmetric space whose Killing connection is exact, then the Killing connection of $(M, g) = (M_1, g_1) \times (M_2, g_2)$ is exact.

Proof. The proof is structure as follows: First, we will show that the kernel of $D^\wedge$ is contained in $\Gamma(\wedge^1 M \otimes (E^{1,0} \oplus E^{0,1}))$. Then, by Lemma 4.3.13, the kernel of $D^\wedge$ would be contained in $\Gamma(\wedge^1 M \otimes K)$. Since K is a parallel flat sub-bundle of E, $D|_K$ is trivially exact and therefore , by Proposition 1.2.5, D would be exact.

Let $\Omega_B{}^\alpha \in \Gamma(\wedge^1 M \otimes E)$ be in the kernel of $D^\wedge$, with

$$\Omega_B{}^\alpha = \begin{bmatrix} \eta_{BC} \\ \psi_{BCD} \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_{BCD} \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} \eta_{BC} \\ \psi_{BCD} \end{bmatrix} \in \Gamma(\wedge^1 M \otimes (E^{1,0} \oplus E^{0,1}))$$

and $\phi_{BCD} \in \Gamma(\wedge^1 M \otimes \wedge^{1,1} M)$. By hypothesis, M_1 is pseudo-Riemannian of non-zero constant sectional curvature or a non-Hermitian indecomposable Riemannian symmetric space, for which $R_{ab}{}^e{}_c$ has trivial kernel as an endomorphism acting on 1-forms. Lemma 4.3.4 guarantees us that $R_{ab}{}^e{}_c \phi_{\bar{a}e\bar{b}} = 0$, hence $\phi_{\bar{a}a\bar{b}} = 0$.

Now we will show that $\phi_{ab\bar{a}} = \phi_{(ab)\bar{a}}$. Since $\nabla_{[b}\phi_{c]d\bar{e}} = 0$, we have

$$0 = \nabla_{[a} \nabla_b \phi_{c]d\bar{e}} = -R^e{}_{d[ab}\phi_{c]e\bar{e}}. \tag{5.22}$$

Fixing $X^{\bar{e}}$, set $X^{\bar{e}}\phi_{ab\bar{e}} = h_{ab} + \omega_{ab}$, with $h_{ab} = X^{\bar{e}}\phi_{(ab)\bar{e}}$ and $\omega_{ab} = X^{\bar{e}}\phi_{[ab]\bar{e}}$. Contracting equation (5.22) with $X^{\bar{e}}$ we obtain

$$0 = R^e{}_{d[ab}h_{c]e} + R^e{}_{d[ab}\omega_{c]e}. \tag{5.23}$$

In the case of (1) that (M_1, g_1) is pseudo-Riemannian of non-zero constant sectional curvature, we have that R_{abcd} is a non-zero constant multiple of $g_{ac}g_{bd} - g_{ad}g_{bc}$, so that equation (5.23) gives

$$0 = g_{e[a}h_c{}^e g_{b]d} - g_{e[b}h_c{}^e g_{a]d} + g_{e[a}\omega_c{}^e g_{b]d} - g_{e[b}\omega_c{}^e g_{a]d} = -2g_{e[a}\omega_b{}^e g_{c]d},$$

since h_{ab} is symmetric. But this is nothing else than

$$0 = \omega_{[ab}g_{c]d},$$

which implies that $\omega_{ab} = 0$.

In case (2), when (M_1, g_1) is non-Hermitian indecomposable Riemannian, tracing equation (5.23) over bd we have

$$0 = 2R^e{}_{[a}h_{c]e} + 2R^e{}_{[a}\omega_{c]e} + R_{ac}{}^{ef}\omega_{ef}. \tag{5.24}$$

As (M_1, g_1) is an indecomposable Riemannian symmetric space, it is Einstein and without loss of generality, we can assume $R_{ab} = \pm g_{ab}$. Then equation (5.24) becomes the eigenvalue equation

$$R_{ac}{}^{ef}\omega_{ef} = \pm 2\omega_{ac} \tag{5.25}$$

for a 2-form on M_1. It was proven in [12, Theorem 2] that if equation (5.25) holds on M_1, then ω_{ab} has to be parallel, and hence (M_1, g_1) is a Hermitian locally symmetric space with ω_{ab} a constant multiple of its Kähler form. Therefore, by our assumption on (M_1, g_1), we obtain as well that $\omega_{ab} = 0$.

Hence, in both cases we have $X^{\bar{c}}\phi_{[ab]\bar{c}} = \omega_{ab} = 0$. This holds for every vector field $X^{\bar{a}}$ and therefore it must hold that $\phi_{[ab]\bar{a}} = 0$.

In a similar way as in Lemma 4.3.13, we know that

$$\eta_{ab} = \nabla_a\sigma_b - \mu_{ab} \quad \text{and} \quad \psi_{abc} = \nabla_a\mu_{bc} - R_{bc}{}^e{}_a\sigma_e \tag{5.26}$$

by the exactness of the Killing connection of (M_1, g_1) for some $\sigma_a \in \wedge^1$ and $\mu_{ab} \in \wedge^2$. From the first equation in (4.30) with $ABC = \bar{a}bc$ and equation (5.26) we have

$$\nabla_{\bar{a}}(\nabla_b\sigma_c - \mu_{bc}) - \nabla_b\eta_{\bar{a}c} + \psi_{\bar{a}bc} + \phi_{bc\bar{a}} = 0.$$

Defining $\theta_{\bar{a}b} := \eta_{\bar{a}b} - \nabla_{\bar{a}}\sigma_b$, it takes the form

$$-\nabla_b\theta_{\bar{a}c} - \nabla_{\bar{a}}\mu_{bc} + \psi_{\bar{a}bc} + \phi_{bc\bar{a}} = 0. \tag{5.27}$$

Symmetrising and skew-symmetrising Equation (5.27) in bc, we obtain

$$\psi_{\bar{a}bc} = \nabla_{[b}\theta_{|\bar{a}|c]} + \nabla_{\bar{a}}\mu_{bc} \quad \text{and} \quad \phi_{bc\bar{a}} = \nabla_{(b}\theta_{|\bar{a}|c)}, \tag{5.28}$$

as $\psi_{\bar{a}bc} = \psi_{\bar{a}[bc]}$ and $\phi_{ab\bar{a}} = \phi_{(ab)\bar{a}}$. Then

$$\begin{aligned} \nabla_{\bar{a}}\psi_{bcd} - \nabla_b\psi_{\bar{a}cd} &= \nabla_{\bar{a}}(\nabla_b\mu_{cd} - R_{cd}{}^e{}_b\sigma_e) - \nabla_b(\nabla_{[c}\theta_{|\bar{a}|d]} + \nabla_{\bar{a}}\mu_{cd}) \\ &= (\nabla_{\bar{a}}\nabla_b - \nabla_b\nabla_{\bar{a}})\mu_{cd} - R_{cd}{}^e{}_b\nabla_{\bar{a}}\sigma_e - \nabla_b\nabla_{[c}\theta_{|\bar{a}|d]} \\ &= -R_{cd}{}^e{}_b\nabla_{\bar{a}}\sigma_e - \nabla_b\nabla_{[c}\theta_{|\bar{a}|d]} \end{aligned} \tag{5.29}$$

The second equation in (4.30) with $ABCD = \bar{a}bcd$ becomes

$$\nabla_{\bar{a}}\psi_{bcd} - \nabla_b\psi_{\bar{a}cd} = -R_{cd}{}^e{}_b\eta_{\bar{a}e}. \tag{5.30}$$

Combining above equations we obtain

$$R_{cd}{}^e{}_b(\eta_{\bar{a}e} - \nabla_{\bar{a}}\sigma_e) = \nabla_b\nabla_{[c}\theta_{|\bar{a}|d]},$$

which is nothing but

$$R_{cd}{}^e{}_b\theta_{\bar{a}e} = \nabla_b\nabla_{[c}\theta_{|\bar{a}|d]}. \tag{5.31}$$

Since (M_1, g_1) is a space of non-zero constant sectional curvature or a non-Hermitian indecomposable Riemannian symmetric space, by Lemma 5.3.5, the above equation implies that

$$\nabla_b\theta_{\bar{a}c} = \nabla_{[b}\theta_{|\bar{a}|c]} \tag{5.32}$$

This means that $\phi_{bc\bar{a}} = \nabla_{(b}\theta_{|\bar{a}|c)} = 0$ and therefore $\phi_{ABC} = 0$.

We have shown that the kernel of $D^\wedge$ is contained in $\Gamma(\wedge^1 M \otimes (E^{1,0} \oplus E^{0,1}))$. Then, by Lemma 4.3.13, $\ker(D^\wedge)$ is contained in $\Gamma(\wedge^1 M \otimes K)$. Moreover, by Proposition 1.2.5, D is exact, as $D|_K$ is exact.

□

The following example illustrates the need of excluding Hermitian symmetric spaces in the previous theorem. A different version of it appeared in [12, Proposition 4].

Example 5.3.7. Let (M_1, g_1) be an indecomposable Riemannian Hermitian symmetric space and let (M_2, g_2) be a symmetric space with non-injective curvature $R_{\bar{a}\bar{b}}{}^{\bar{d}}{}_{\bar{c}}$. In other words, M_2 has a parallel differential 1-form $\xi_{\bar{a}}$. Then the Killing connection of $(M, g) = (M_1 \times M_2, g_1 + g_2)$ is not exact.

To see this, and here we follow [12, Proposition 4], let $\omega_{AB} = \omega_{ab}$ be the Kähler form of (M_1, g_1) with Kähler potential $\phi_B = \phi_b$, i.e. $\nabla_{[A}\phi_{B]} = \omega_{AB}$, and $\xi_A = \xi_{\bar{a}}$ be the parallel vector field on M_2. Then we set $h_{AB} = \phi_{(A}\xi_{B)}$,

$$\psi_{BCD} = 2\nabla_{[C}h_{D]B} = \omega_{CD}\xi_B + \tfrac{1}{2}\left(\nabla_C\phi_B\xi_D - \nabla_D\phi_B\xi_C\right) \quad \text{and} \quad \eta_B{}^\alpha = \begin{bmatrix} h_{BC} \\ \psi_{BCD} \end{bmatrix}.$$

Then $\psi_{[BCD]} = 0$, and for the exterior covariant derivative of $\eta_B{}^\alpha$ it is

$$D^\wedge_A\eta_B{}^\alpha = \begin{bmatrix} 0 \\ \nabla_{[A}\psi_{B]CD} - R_{CD}{}^E{}_{[A}h_{B]E} \end{bmatrix}.$$

Since ω_{AB} is parallel, it is $\nabla_A\nabla_B\phi_C = \nabla_A\nabla_C\phi_B$, so that, together with $\nabla_A\xi_B = 0$,

$$\nabla_{[A}\psi_{B]CD} = \tfrac{1}{2}\left(\nabla_{[A}\nabla_{B]}\phi_C\xi_D - \nabla_{[A}\nabla_{B]}\phi_D\xi_C\right) = -\tfrac{1}{2}R_{AB}{}^{E}{}_{[C}\xi_{D]}\phi_E.$$

On the other hand,

$$R_{CD}{}^{E}{}_{[A}h_{B]E} = \tfrac{1}{2}R_{CD}{}^{E}{}_{[A}\xi_{B]}\phi_E,$$

so that, for $\mu_{AB} = \phi_{[A}\xi_{B]}$,

$$D^{\wedge}_{A}\eta_B{}^{\alpha} = \begin{bmatrix} 0 \\ R_{AB}{}^{E}{}_{[C}\mu_{D]E} + R_{CD}{}^{E}{}_{[A}\mu_{B]E} \end{bmatrix}$$

is in the range of the curvature of the Killing connection. However, if $\eta_B{}^{\alpha}$ was in the range of the Killing connection, then there would be a one-form $\sigma_C = \sigma_c + \sigma_{\bar{c}}$ such that $h_{BC} = \nabla_B\sigma_C - \mu_{BC}$, and this, by the definition of h_{BC} and μ_{BC}, implies that $\nabla_B\sigma_C = \phi_B\xi_C = \phi_b\xi_{\bar{c}}$. Hence

$$0 = \nabla_b\sigma_c = \nabla_{\bar{b}}\sigma_c = \nabla_{\bar{b}}\sigma_{\bar{c}} \quad \text{and} \quad \nabla_b\sigma_{\bar{c}} = \phi_b\xi_{\bar{c}}.$$

Therefore, σ_c is a lift of a parallel vector field on M_1, and, with M_1 being indecomposable and Riemannian, must be zero. The last equation implies that

$$0 = \nabla_{[a}\nabla_{b]}\sigma_{\bar{c}} = \omega_{ab}\xi_{\bar{c}},$$

which is a contradiction, as the Kähler form and the parallel vector field are both not zero.

We conclude with the proof of main theorem of this section.

Proof of Theorem 5.3.3. Let $(M_1, g_1) \times \cdots \times (M_k, g_k) \times (L, g_L)$ be the local de Rham-Wu decomposition of (M, g) into irreducible Riemannian factors (M_i, g_i) and a Lorentzian factor (L, g_L), such that (L, g_L) does not contain a non-flat Riemannian factor, that is, (L, g_L) is either one of the following:

1. Indecomposable Lorentzian, i.e. with non-zero constant sectional curvature or a Cahen-Wallach space.
2. A product of an indecomposable Lorentzian symmetric space with a Euclidean factor.
3. Minkowski space.

In all three cases, Corollaries 4.3.10 and 5.3.2 imply that the Killing connection of (L, g_L) is exact. Moreover, by [12] the same holds for (M_k, g_k), so that we can apply Proposition 5.3.6 to obtain that the Killing connection is exact for $(M_k, g_k) \times (L, g_L)$, provided that (M_k, g_k) is an irreducible Riemannian symmetric space that is non-Hermitian if L admits a parallel vector field. Inductively, it follows that the Killing connection of (M, g) is exact unless it contains a Hermitian factor in its local de Rham decomposition and L admits a parallel vector field.

□

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