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Note on the Numerical Evaluation of a Bessel Function Derivative

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As early as 1870, J. W. L. Glaisher had distinctly formulated a scheme for the systematic tabulation of a group of fundamental functions, in addition to the algebraic, logarithmic, and exponential functions. His purpose was to choose such fundamental functions as commended themselves by the elegance of their properties, and their naturally central position as the products of the simplification of integral expressions. His table of the error function appeared in 1871 [*Phil. Mag.* (4), 42, 436], following on the publication of the sine integral, the cosine integral, and the exponential integral in 1870 [*Phil. Trans.* 160, 367]. How well chosen was this group of functions will be familiar to all who have had occasion to require the numerical values of integral expressions.

The present note calls attention to the existence of a previously unsuspected connexion between the derivative of the Bessel function

$$\frac{\partial}{\partial \nu} J_\nu(x)$$

and the sine and cosine integrals, which enables the values of the former, when ν is the half of an odd integer, to be calculated directly from tables of the latter. In any expression of the form

$$f_1(\nu)+f_2(\nu)\,J_\nu$$

if, for any value of ν, f_1, f_2 tend to infinity, while the whole expression remains finite, we are liable to require the numerical value of the derivative. The case, of this sort, which led to the following results, occurred in the investigation of the efficiency in small samples of certain statistics ["Theory of Statistical Estimation," *Proc. Camb. Phil. Soc.*, 22 (1925), 700–725].

The identity may perhaps best be derived from Hankel's formula

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}} \cos zt\, dt,$$

by direct differentiation and the substitution $\nu = \frac{1}{2}$. Then

$$\frac{\partial}{\partial \nu} J_\nu(z) = J_\nu(z) \left\{ \log \frac{z}{2} - \frac{\Gamma'(\nu+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \right.$$

$$\left. + \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \log(1-t^2) \cos zt\, dt, \right.$$

which, when $\nu = \frac{1}{2}$, reduces to

$$\sqrt{\left(\frac{2}{\pi z}\right)} \left\{ \sin z \left(\log \frac{z}{2} + \gamma \right) + \tfrac{1}{2} z \int_{-1}^{1} \log(1-t^2) \cos zt\, dt \right\}.$$

The latter term may be reduced successively to

$$\int_{-1}^{1} \log(1+t) \cos zt\, z\, dt = \int_{-z}^{z} \log \frac{z+t}{z} \cos t\, dt$$

$$= \int_{0}^{2z} \log \frac{u}{z} \cos(u-z)\, du.$$

Now

$$\int_{\epsilon}^{2z} \log \frac{u}{z} \cos(u-z) du = \left[\log \frac{u}{z} \sin(u-z) \right]_{\epsilon}^{2z} - \int_{\epsilon}^{2z} \sin(u-z) \frac{du}{u},$$

and when ϵ tends to zero, may be reduced to

$$\sin z \left(\log 2 + \log \frac{\epsilon}{z} \right) - \cos z\, Si(2z) + \sin z \{ Ci(2z) - Ci(\epsilon) \},$$

where $$Si(x) = \int_0^x \sin x \frac{dx}{x}, \quad Ci(x) = -\int_x^\infty \cos x \frac{dx}{x};$$

and since, when $\epsilon \to 0$ through real values,

$$Ci(\epsilon) - \log \epsilon \to \gamma,$$

we have

$$\int_0^{2z} \log \frac{u}{z} \cos(u-z)\, du = \sin z \left(\log \frac{2}{z} - \gamma \right) - \cos z\, Si(2z) + \sin z\, Ci(2z),$$

and consequently

$$\left. \begin{aligned} \left[\frac{\partial}{\partial \nu} J_\nu(z) \right]_{\nu=\frac{1}{2}} &= \sqrt{\left(\frac{2}{\pi z}\right)} \{ \sin z\, Ci(2z) - \cos z\, Si(2z) \} \\ &= J_{\frac{1}{2}}(z)\, Ci(2z) - J_{-\frac{1}{2}}(z)\, Si(2z) \end{aligned} \right\}. \qquad \text{(I)}$$

The corresponding formula for $\nu = -\frac{1}{2}$ may be obtained from

$$\frac{\partial}{\partial z} J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z),$$

whence
$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \nu} J_\nu(z)\right] = \frac{\partial}{\partial \nu} J_{\nu-1}(z) - \frac{1}{z} J_\nu(z) - \frac{\nu}{z}\frac{\partial}{\partial \nu} J_\nu(z).$$

Putting $\nu = \frac{1}{2}$, we have

$$\left[\frac{\partial}{\partial \nu} J_\nu(z)\right]_{\nu=-\frac{1}{2}} = \frac{\partial}{\partial z}\left\{\sqrt{\left(\frac{2}{\pi z}\right)}\left(\sin z\, Ci(2z) - \cos z\, Si(2z)\right)\right\}$$

$$+\frac{1}{z}\sqrt{\left(\frac{2}{\pi z}\right)}\sin z + \frac{1}{2z}\sqrt{\left(\frac{2}{\pi z}\right)}\left(\sin z\, Ci(2z) - \cos z\, Si(2z)\right)$$

$$= \frac{1}{z}\sqrt{\left(\frac{2}{\pi z}\right)}\sin z$$

$$+\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\cos z\, Ci(2z) + \sin z\, Si(2z) + \frac{\sin z \cos 2z - \cos z \sin 2z}{z}\right\};$$

therefore

$$\left.\begin{aligned}\left[\frac{\partial}{\partial \nu} J_\nu(z)\right]_{\nu=-\frac{1}{2}} &= \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\cos z\, Ci(2z) + \sin z\, Si(2z)\right\}\\ &= J_{-\frac{1}{2}}(z)\, Ci(2z) + J_{\frac{1}{2}}(z)\, Si(2z)\end{aligned}\right\} \qquad \text{(II)}$$

Using the general recurrence formula

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{z}{2\nu} J_\nu(z),$$

the values of $\partial J_\nu(z)/\partial\nu$ may be obtained for all values of ν equal to the half of an odd integer.

Glaisher's tables of the sine integral give 18 decimal places for values of x proceeding by intervals of ·01 from 0 to 1, and 11 places proceeding by intervals of ·1 from 1 to 5; both parts of the table may be interpolated to the full number of figures given, using second, fourth, and sixth differences. Unfortunately, the central difference formulae were not known to Glaisher, and he gives the first three differences only. The most serious gap in his tables is from 5 to 15, where he gives 11 places for integer values only, and direct interpolation is useless. The table of the cosine integral is not so satisfactory, since in the range from 0 to 1 accurate interpolation is no longer possible, and it would have been wiser to tabulate $Ci(x) - \log x$. Beyond 15 the asymptotic formulae are good for seven places for both functions.