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THE SAMPLING ERROR OF ESTIMATED DEVIATES, TOGETHER WITH OTHER ILLUSTRATIONS OF THE PROPERTIES AND APPLICATIONS OF THE INTEGRALS AND DERIVATIVES OF THE NORMAL ERROR FUNCTION

Author's Note (CMS 23.xxva) *

This section of the introduction to Volume I of the series of mathematical tables published by the British Association for the Advancement of Science is reprinted principally for the sake of the treatment in Application 1 of the *sampling error of estimated deviation*, which had become inaccessible.

The example is remarkable in that the quantities a and α are each defined as a function of the other, having, when the other is known, the property that, while being given functions of the statistics and the parameters, their distributions are independent of both parameters. The solution may be used directly to find the probability that the proportion of defective parts in a consignment of which a sample has been tested shall exceed any assigned value; it is therefore of practical importance in the critical drafting of specifications.

Properties of the Functions

Introductory

1. The Hermite polynomial, H_n, is defined by the equation

$$H_n = e^{\frac{1}{2}x^2}\left(-\frac{d}{dx}\right)^n (e^{-\frac{1}{2}x^2})$$

from this it follows that

$$H_{n+1} = -e^{\frac{1}{2}x^2}\frac{d}{dx}(H_n e^{-\frac{1}{2}x^2}) = xH_n - \frac{d}{dx}H_n \qquad (1)$$

from which the polynomials may easily be written down in succession, the coefficient of the highest power of x being unity.

Also

$$\frac{d}{dx}H_n = xH_n + e^{\frac{1}{2}x^2}\left(-\frac{d}{dx}\right)^n(-xe^{-\frac{1}{2}x^2}) = xH_n - xH_n + ne^{\frac{1}{2}x^2}\left(-\frac{d}{dx}\right)^{n-1}e^{-\frac{1}{2}x^2} = nH_{n-1} \qquad (2)$$

Hence is established the recurrence formula

$$H_{n+1} - xH_n + nH_{n-1} = 0 \qquad (3)$$

Mathematical Tables, 1: xxvi-xxxv, British Association for the Advancement of Science, (1931).

Writing this in the form

$$H_n - xH_{n-1} + (n-1)H_{n-2} = 0$$

and substituting

$$H_{n-2} = \frac{1}{n(n-1)} \frac{d^2H_n}{dx^2}$$

$$H_{n-1} = \frac{1}{n} \frac{dH_n}{dx}$$

we find the differential equation

$$\left(\frac{d^2}{dx^2} - x\frac{d}{dx} + n\right)H_n = 0 \tag{4}$$

an equation by means of which we can define the Hermite function for non-integral values of n.

Turning now to the closely related functions defined by

$$G_n = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} H_n$$

we have by definition the relation

$$G_{n+1} = -\frac{d}{dx}G_n \tag{5}$$

and evidently also the recurrence formula,

$$G_{n+2} - xG_{n+1} + (n+1)G_n = 0 \tag{6}$$

whence it follows that G_n satisfies the differential equation

$$\left(\frac{d^2}{dx^2} + x\frac{d}{dx} + (n+1)\right)G_n = 0 \tag{7}$$

The connection of this equation with equation (4) is most clearly shown by writing

$$I_n = G_{-(n+1)}$$

giving the equation

$$\left(\frac{d^2}{dx^2} + x\frac{d}{dx} - n\right)I_n = 0 \tag{8}$$

where (8) is the same equation in ix as (4) is in x.

2. *The Orthogonal Relation of G and H.*—Since

$$G_n = -\frac{d}{dx}G_{n-1}$$

the integral

$$\int_{-\infty}^{\infty} H_m G_n dx = -\Big[H_m G_{n-1}\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} G_{n-1}\frac{d}{dx}H_m dx$$

of which the first part vanishes, for G_{n-1} is a polynomial multiplied by $e^{-\frac{1}{2}x^2}$. If m is less than n, the repetition of the process shows that

$$\int_{-\infty}^{\infty} H_m G_n dx = \int_{-\infty}^{\infty} G_{n-m-1}\left(\frac{d}{dx}\right)^{m+1} H_m dx$$

which is zero, for H_m is a polynomial of degree m. Since, moreover,

$$H_m G_n = H_n G_m$$

it follows that if m and n are any unequal integers, the integral is zero. If $m = n$, we have also

$$\int_{-\infty}^{\infty} H_n G_n dx = \int_{-\infty}^{\infty} G_0 n!\, dx = n! \tag{9}$$

3. *The Integrals of the Probability Integral.*—The functions I_n have hitherto been defined only when n is a negative integer; for such values equation (5) may be written

$$I_{n+1} = \int_x^{\infty} I_n dx \tag{10}$$

for I must tend to zero as x tends to infinity. We may therefore, commencing with $I_{-1}=G_0$, define

$$I_0=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}t^2}dt$$

and define I_n for positive integers by means of equation (10). We may then reduce I_n to a single integral by successive integrations by parts, for

$$I_n=\int_x^\infty I_{n-1}dt=\Big[(t-x)I_{n-1}\Big]_x^\infty+\int_x^\infty(t-x)I_{n-2}dt$$

and

$$\int_x^\infty(t-x)I_{n-2}dt=\left[\frac{(t-x)^2}{2!}I_{n-2}\right]_x^\infty+\int_x^\infty\frac{(t-x)^2}{2!}I_{n-3}dt$$

leading after n stages to

$$I_n=\frac{1}{\sqrt{2\pi}}\int_x^\infty\frac{(t-x)^n}{n!}e^{-\frac{1}{2}t^2}dt \tag{11}$$

and to the definite integral

$$I_n=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{t^n}{n!}e^{-\frac{1}{2}(t+x)^2}dt \tag{12}$$

To show that I_n satisfies the differential equation (8), observe that by differentiating equation (11) we have

$$\frac{d}{dx}I_n=\frac{1}{\sqrt{2\pi}}\int_x^\infty\left(-\frac{n}{t-x}\right)\frac{(t-x)^n}{n!}e^{-\frac{1}{2}t^2}dt=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{-nt^n}{t\,.\,n!}e^{-\frac{1}{2}(t+x)^2}dt$$

whence

$$\frac{d^2}{dx^2}I_n=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{n(t+x)t^n}{t\,.\,n!}e^{-\frac{1}{2}(t+x)^2}dt=\frac{1}{\sqrt{2\pi}}\int_x^\infty\frac{nt(t-x)^n}{(t-x)n!}e^{-\frac{1}{2}t^2}dt$$

therefore

$$\left(\frac{d^2}{dx^2}+x\frac{d}{dx}\right)I_n=\frac{1}{\sqrt{2\pi}}\int_x^\infty n\frac{(t-x)^n}{n!}e^{-\frac{1}{2}t^2}dt=nI_n$$

as is required. It will be noted that equations (11) and (12) define the function I_n for all values of n greater than -1 (using the ordinary generalisation of the factorial), and that the function so defined satisfies the differential equation (8) and the recurrence formula

$$(n+1)I_{n+1}+xI_n-I_{n-1}=0 \tag{13}$$

4. *The Value of* $I_n(0)$.—Substituting zero for x in equation (11), we have

$$I_n(0)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{t^n}{n!}e^{-\frac{1}{2}t^2}dt$$

or, if $z=\frac{1}{2}t^2$,

$$I_n(0)=\frac{2^{\frac{1}{2}(n-1)}}{\sqrt{2\pi}\,.\,n!}\int_0^\infty z^{\frac{1}{2}(n-1)}e^{-z}dz=\frac{\left(\frac{n-1}{2}\right)!\,2^{\frac{1}{2}(n-1)}}{\sqrt{2\pi}\,.\,n!}=\frac{1}{(\frac{1}{2}n)!\,2^{\frac{1}{2}(n+2)}}$$

For negative integers also, if $I_n=G_{-(n+1)}$, noting that $G_m(0)=\frac{1}{\sqrt{2\pi}}H_m(0)$, and that $H_m(0)$ is zero when m is odd, but when m is even reduces to

$$(-1)^{\frac{1}{2}m}(m-1)(m-3)\,.\,.\,.\,5.3.1$$

or, if $m=-(n+1)$, to

$$(n+2)(n+4)\,.\,.\,.\,(-3)(-1)=\frac{2^{-\frac{1}{2}(n+1)}(\frac{1}{2})!}{(\frac{1}{2}n)!}$$

we should still have

$$I_n(0)=\frac{1}{(\frac{1}{2}n)!\,2^{\frac{1}{2}(n+2)}} \tag{14}$$

so that a function I_n defined for negative values of n so as to have this value when x is zero, and to satisfy the differential equation (8), will reduce at integral values to the function $G_{-(n+1)}$.

5. *The Solutions of the Differential Equations.*—The differential equation (8) yields an indicial equation $t(t-1)=0$, the two roots corresponding to odd and even functions which satisfy the equation; if

$$y_1 = x + A_3x^3 + A_5x^5 + \cdots$$

is the odd solution, then substituting in the equation we find

$$2r(2r+1)A_{2r+1} + (2r-1-n)A_{2r-1} = 0$$

or

$$y_1 = x + \frac{n-1}{3!}x^3 + \frac{(n-1)(n-3)}{5!}x^5 + \cdots$$

a series which is absolutely convergent for all values of n and x, and reduces to a polynomial if n is an odd positive integer. Similarly the even solution is seen to be

$$y_2 = 1 + \frac{n}{2!}x^2 + \frac{n(n-2)}{4!}x^4 + \cdots$$

which also is absolutely convergent, and reduces to a polynomial for even positive integral values of n. Since the function I_n has at $x=0$ a known value and a known differential coefficient, we may express it in terms of y_1 and y_2 in the form

$$I_n = \frac{2^{-\frac{1}{2}(n+2)}}{\left(\frac{n}{2}\right)!}y_2 - \frac{2^{-\frac{1}{2}(n+1)}}{\left(\frac{n-1}{2}\right)!}y_1 \tag{15}$$

which shall define I_n for all values of n. We may note that

$$I_n(x) + I_n(-x) = \frac{2^{-\frac{1}{2}n}}{(\frac{1}{2}n)!}y_2$$

and this will be a polynomial in x when n is an even positive integer, while

$$I_n(-x) - I_n(x) = \frac{2^{-\frac{1}{2}(n-1)}}{\left(\frac{n-1}{2}\right)!}y_1$$

and this will be a polynomial in x when n is an odd positive integer.

Reversing the order of the terms in these polynomials we have, when n is even

$$I_n(x) + I_n(-x) = \frac{1}{n!}\left[x^n + \frac{n(n-1)}{2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4}x^{n-4} + \cdots\right]$$

$$= \frac{x^n}{n!} + \frac{1}{2}\frac{x^{n-2}}{(n-2)!} + \frac{1}{2 \cdot 4}\frac{x^{n-4}}{(n-4)!} + \cdots \tag{16}$$

which is also the form for $I_n(-x) - I_n(x)$, when n is odd.

The polynomials represented by (16) correspond closely to the Hermite polynomials, and may be written $H_n^* \div n!$. Then it is easy to verify that

$$H_n^* = e^{-\frac{1}{2}x^2}\frac{d^n}{dx^n}(e^{\frac{1}{2}x^2})$$

and that

$$H_{n+1}^* = xH_n^* + \frac{d}{dx}H_n^*$$

and so to derive the relations

$$\frac{d}{dx}H_n^* = nH_{n-1}^*$$

$$H_{n+1}^* - xH_n^* - nH_{n-1}^* = 0$$

$$\left(\frac{d^2}{dx^2} + x\frac{d}{dx} - n\right)H_n^* = 0$$

It is of more interest that, defining the Hermite function H_n as

$$H_n = \sqrt{2\pi}e^{\frac{1}{4}x^2}I_{-(n+1)}$$

substituting in equation (15) and identifying the odd and even parts with the corresponding solutions of the differential equation for H_n, we have

$$e^{\frac{1}{2}x^2}\left(1 - \frac{n+1}{2!}x^2 + \frac{(n+1)(n+3)}{4!}x^4 - \ . \ . \ .\right) = 1 - \frac{n}{2!}x^2 + \frac{n(n-2)}{4!}x^4 - \ . \ . \ . = y_2^*$$

and

$$e^{\frac{1}{2}x^2}\left(x - \frac{n+2}{3!}x^2 + \frac{(n+2)(n+4)}{5!}x^4 - \ . \ . \ .\right) = x - \frac{n-1}{3!}x^3 + \frac{(n-1)(n-3)}{5!}x^5 - \ . \ . \ . = y_1^*$$

and

$$H_n = \frac{2^{\frac{1}{2}n}\sqrt{\pi}}{\left(-\frac{n+1}{2}\right)!}y_2^* - \frac{2^{\frac{1}{2}(n+1)}\sqrt{\pi}}{\left(-\frac{n+2}{2}\right)!}y_1^*$$

which reduces to the polynomial form for positive integers.

Applications

1. *The Sampling Error of Estimated Deviates.*—If the population sampled is specified by

$$df = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x-m)^2}dx$$

then from a sample we can calculate the estimates

$$\bar{x} = \frac{1}{n}S(x) \qquad s^2 = \frac{1}{n}S(x-\bar{x})^2$$

and the simultaneous distribution of $\bar{x}$ and s is known to be

$$df = \frac{1}{\left(\frac{n-3}{2}\right)!}\left\{\frac{ns^2}{2\sigma^2}\right\}^{\frac{1}{2}(n-3)} e^{-\frac{ns^2}{2\sigma^2}}\frac{nsds}{\sigma^2} \cdot \sqrt{\frac{n}{2\pi}}e^{-\frac{n}{2\sigma^2}(\bar{x}-m)^2}\frac{d\bar{x}}{\sigma} \tag{17}$$

If now τ is the true deviation of any point in terms of the true standard deviation σ, and t is the apparent deviation in terms of s,

$$\bar{x} + st = m + \sigma\tau$$

and for each value of τ, t will have a determinate distribution.

Changing the variates from $\bar{x}$ and s to t and s, we have

$$d\bar{x} = -sdt$$

and

$$\bar{x} - m = \sigma\tau - st$$

so that the index of the exponential term is

$$-\frac{n}{2}\left\{\frac{s^2}{\sigma^2} + \left(\tau - t\frac{s}{\sigma}\right)^2\right\} = -\frac{n}{2}\left\{(1+t^2)\left(\frac{s}{\sigma} - \frac{\tau t}{1+t^2}\right)^2 + \frac{\tau^2}{1+t^2}\right\}$$

and, writing u for s/σ, we obtain

$$df = \frac{{}^{\frac{1}{2}n}}{\left(\frac{n-3}{2}\right)!\, 2^{\frac{1}{2}(n-2)}\sqrt{\pi}} e^{-\frac{n}{2}\frac{\tau^2}{1+t^2}}dt \,.\, u^{n-1}e^{-\frac{n(1+t^2)}{2}\left(u-\frac{\tau t}{1+t^2}\right)^2}du \tag{18}$$

Now u varies in value from 0 to ∞, and the integral between these limits of the frequency factor involving u may be written

$$(n-1)!\{n(1+t^2)\}^{-\frac{1}{2}n} \,.\, I_{n-1}\left(-\frac{\tau t\sqrt{n}}{\sqrt{1+t^2}}\right) . \sqrt{2\pi}$$

giving the distribution of t in the form

$$\frac{(n-1)!}{2^{\frac{1}{2}(n-3)}\left(\frac{n-3}{2}\right)!} \,.\, (1+t^2)^{-\frac{1}{2}n} \,.\, e^{-\frac{n}{2}\frac{\tau^2}{1+t^2}} \,.\, I_{n-1}\left(-\frac{\tau t\sqrt{n}}{\sqrt{1+t^2}}\right) . dt \tag{19}$$

Since

$$I_{n-1}(0) = \frac{2^{-\frac{1}{2}(n+1)}}{\left(\frac{n-1}{2}\right)!}$$

when $\tau = 0$ the distribution reduces to

$$\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!\sqrt{\pi}}(1+t^2)^{-\frac{1}{2}n}dt$$

which is "Student's" distribution.

In considering the approximation to normality of the general distribution (19), we shall require an approximate value of the integral function I_{n-1}, with negative argument proportional to $\sqrt{n}$, for the case when n is large. For this purpose we shall use the formula, wherein $x = -2\sqrt{n}\sinh\phi$,

$$I_{n-1}(x) = \frac{n^{\frac{1}{2}n}}{n!}\frac{\sqrt{2\pi n}}{\sqrt{2\cosh\phi}}e^{-\frac{1}{2}\phi}e^{n(\phi-\frac{1}{2}e^{-2\phi})} \tag{20}$$

or, neglecting terms which do not contain ϕ,

$$\log I_{n-1} = n(\phi - \tfrac{1}{2}e^{-2\phi}) + C$$

Differentiating with respect to x, since

$$\frac{d}{dx} = -\frac{1}{2\sqrt{n}\cosh\phi}\frac{d}{d\phi}$$

$$\frac{d}{dx}(\log I_{n-1}) = -\sqrt{n}\,e^{-\phi}$$

and since

$$\frac{dx}{dt} = -\frac{\tau\sqrt{n}}{(1+t^2)^{3/2}}$$

the approximate equation for the mode is

$$-\frac{nt}{1+t^2} + \frac{n\tau^2 t}{(1+t^2)^2} + \frac{n\tau}{(1+t^2)^{3/2}}e^{-\phi} = 0 \tag{21}$$

which is satisfied by

$$t = \tau$$

for at this point

$$2\sinh\phi = \frac{\tau^2}{\sqrt{1+\tau^2}}$$

whence

$$2\cosh\phi = \frac{2+\tau^2}{\sqrt{1+\tau^2}}$$

and

$$e^{-\phi} = \frac{1}{\sqrt{1+\tau^2}}$$

Using this relation,

$$\frac{d}{dx}(\log I) = -\sqrt{n}\,.\,e^{-\phi} \quad = \frac{-\sqrt{n}}{\sqrt{1+\tau^2}}$$

$$\frac{d^2}{dx^2}(\log I) = \frac{-e^{-\phi}}{2\cosh\phi} \quad = \frac{-1}{2+\tau^2}$$

$$\frac{d^3}{dx^3}(\log I) = \frac{-1}{4\sqrt{n}\cosh^3\phi} = \frac{-2(1+\tau^2)^{3/2}}{\sqrt{n}(2+\tau^2)^3}$$

$$\frac{d^4}{dx^4}(\log I) = \frac{-3\sinh\phi}{8n\cosh^5\phi} \quad = \frac{-6\tau^2(1+\tau^2)^2}{n(2+\tau^2)^5} \tag{22}$$

and

$$\frac{\partial x}{\partial t} = \frac{-\sqrt{n}\tau}{(1+t^2)^{3/2}} \qquad \frac{\partial^2 x}{\partial t^2} = \frac{3\sqrt{n}\tau t}{(1+t^2)^{5/2}}$$

$$\frac{\partial^3 x}{\partial t^3} = \frac{3\sqrt{n}\tau(1-4t^2)}{(1+t^2)^{7/2}} \qquad \frac{\partial^4 x}{\partial t^4} = \frac{15\sqrt{n}\tau t(4t^2-3)}{(1+t^2)^{9/2}} \tag{23}$$

consequently, at the mode,

$$\frac{\partial^2}{\partial t^2}(\log I) = -\frac{3n\tau^2}{(1+\tau^2)^3} - \frac{n\tau^2}{(1+\tau^2)^3(2+\tau^2)}$$

$$\frac{\partial^3}{\partial t^3}(\log I) = -\frac{3n\tau(1-4\tau^2)}{(1+\tau^2)^4} + \frac{9n\tau^3}{(1+\tau^2)^4(2+\tau^2)} + \frac{2n\tau^3}{(1+\tau^2)^3(2+\tau^2)^3}$$

$$\frac{\partial^4}{\partial t^4}(\log I) = \frac{-15n\tau^2(4\tau^2-3)}{(1+\tau^2)^5} - \frac{n\tau^2(75\tau^2-12)}{(1+\tau^2)^5(2+\tau^2)} - \frac{36n\tau^4}{(1+\tau^2)^4(2+\tau^2)^3} - \frac{6n\tau^6}{(1+\tau^2)^4(2+\tau^2)^5} \tag{24}$$

The second derivative of the ordinate at the mode is therefore

$$-\frac{n(1-t^2)}{(1+t^2)^2} + \frac{n\tau^2(1-3t^2)}{(1+t^2)^3} - \frac{3n\tau^2}{(1+\tau^2)^3} - \frac{n\tau^2}{(1+\tau^2)^3(2+\tau^2)} = -\frac{2n}{2+\tau^2}$$

for large samples we infer that the variance of t is given by

$$nV(t) = 1 + \tfrac{1}{2}\tau^2 \tag{25}$$

The third derivative is

$$\frac{2nt(3-t^2)}{(1+t^2)^3} - \frac{12n\tau^2 t(1-t^2)}{(1+t^2)^4} - \frac{3n\tau(1-4\tau^2)}{(1+\tau^2)^4} + \frac{9n\tau^3}{(1+\tau^2)^4(2+\tau^2)} + \frac{2n\tau^3}{(1+\tau^2)^3(2+\tau^2)^3} = \frac{2n\tau(12+5\tau^2)}{(2+\tau^2)^3}$$

* Equating this to $\gamma_1\sigma_t^3$, we find

$$\gamma_1 = \frac{\tau(12+5\tau^2)}{\sqrt{2n(2+\tau^2)^3}} \tag{26}$$

Finally, the fourth derivative is

$$\frac{6n(1-6t^2+t^4)}{(1+t^2)^4} - \frac{12n\tau^2(1-10t^2+5t^4)}{(1+t^2)^5} + \frac{15n\tau^2(3-4\tau^2)}{(1+\tau^2)^5} + \frac{3n\tau^2(4-25\tau^2)}{(1+\tau^2)^5(2+\tau^2)}$$

$$- \frac{36n\tau^4}{(1+\tau^2)^4(2+\tau^2)^3} - \frac{6n\tau^6}{(1+\tau^2)^4(2+\tau^2)^5}$$

$$= \frac{6n}{(2+\tau^2)^5}(32-32\tau^2-40\tau^4-9\tau^6)$$

† and on equating this to $\gamma_2\sigma_t^4$, we have

$$\gamma_2 = \frac{3}{2n}\,\frac{32-32\tau^2-40\tau^4-9\tau^6}{(2+\tau^2)^3} \tag{27}$$

* For $\gamma_1\sigma_t^3$, read γ_1/σ_t^3.

† For $\gamma_2\sigma_t^4$, read γ_2/σ_t^4.

The value of γ_1, measuring asymmetry of the third moment, ranges from zero when $\tau=0$, to $5/\sqrt{2n}$ when τ is infinite, thus showing a very moderate asymmetry. The value of γ_2, which measures the departure of the fourth moment from its normal value, varies from $6/n$ when $\tau=0$, to $-27/2n$ when τ is infinite. The nature of the approximations found above, appropriate for large samples, is illustrated by the fact that when $\tau=0$, the exact value of γ_2 for "Student's" distribution is $6/(n-5)$.

2. *Truncated Normal Distribution.*—If, in a normal distribution with mean m and standard deviation σ, all record is omitted of individuals below a given value, which we may write $m+\sigma\xi$, the frequency of the truncated distribution in the range dx will be

$$df=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}dx \div \int_\xi^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt=\frac{1}{\sigma\sqrt{2\pi}I_0(\xi)}e^{-\frac{(x-m)^2}{2\sigma^2}}dx \tag{28}$$

provided that x exceeds $m+\sigma\xi$.

The moments of the distribution about its terminus are easily expressed in I functions, for

$$\mu'_r=\frac{1}{\sigma\sqrt{2\pi}I_0(\xi)}\int_{m+\sigma\xi}^\infty x^r e^{-\frac{(x-m)^2}{2\sigma^2}}dx=\sigma^r r!\frac{I_r(\xi)}{I_0(\xi)} \tag{29}$$

It is noteworthy that in the estimation of the unknown parameters ξ and σ, the method of moments gives in this case the same solution as the method of maximum likelihood; and is therefore, in this case, efficient.

For, apart from a constant, taking the origin at the terminus,

$$L=-n\log\sigma-\frac{S(x+\sigma\xi)^2}{2\sigma^2}-n\log I_0$$

so that the equations of maximum likelihood are

$$-\frac{n}{\sigma}+\frac{1}{\sigma^2}S\left\{x\left(\xi+\frac{x}{\sigma}\right)\right\}=0 \tag{30}$$

$$-S\left(\xi+\frac{x}{\sigma}\right)+n\frac{I_{-1}}{I_0}=0 \tag{30 bis}$$

but

$$I_{-1}=\xi I_0+I_1$$

so that equation (30 bis) may be written

$$\bar{x}=\sigma\frac{I_1}{I_0}$$

which is the equation obtained by equating the first moment of the sample to that of the population. Substituting, now, in (30) the value found for x, we have

$$n\xi\frac{I_1}{I_0}+\frac{1}{\sigma^2}S(x^2)=n$$

but

$$2I_2=I_0-\xi I_1$$

hence

$$S(x^2)=2n\sigma^2\frac{I_2}{I_0} \tag{31}$$

which is the equation obtained by equating the second moment of the sample to that of the population.

Eliminating σ between the two equations, the maximum likelihood solution for ξ must satisfy the equation

$$\frac{nS(x^2)}{S^2(x)}=\frac{2I_0I_2}{I_1^2} \tag{32}$$

and the practical solution consists in entering the observed value of the quantity on the left in a table showing, for values of ξ, the quantity on the right.

The precision is determined by the second differential coefficients

$$\frac{\partial^2 L}{\partial\sigma^2} = -\frac{n}{\sigma^2}\left\{2\xi\frac{\bar{x}}{\sigma} + 3\frac{\mu_2'}{\sigma^2} - 1\right\} = -\frac{n}{\sigma^2}\left\{2\xi\frac{I_1}{I_0} + 6\frac{I_2}{I_0} - 1\right\}$$

$$\frac{\partial^2 L}{\partial\sigma\partial\xi} = \frac{n\bar{x}}{\sigma^2} = \frac{n}{\sigma}\frac{I_1}{I_0}$$

$$\frac{\partial^2 L}{\partial\xi^2} = -n\left\{1 + \frac{I_{-2}I_0 - I_{-1}^2}{I_0^2}\right\} \tag{33}$$

Using

$$I_{-2} = \xi I_{-1}$$

and

$$I_{-1} - \xi I_0 = I_1$$

it is clear that

$$I_{-2}I_0 - I_{-1}^2 = -I_{-1}I_1$$

and

$$I_0^2 - I_{-1}I_1 = I_0^2 - I_1(I_1 + \xi I_0) = 2I_0I_2 - I_1^2$$

If, therefore,

$$\Delta = \begin{vmatrix} \frac{n}{\sigma^2}\left(2\frac{I_2}{I_0} + 1\right) & -\frac{n}{\sigma}\frac{I_1}{I_0} \\ -\frac{n}{\sigma}\frac{I_1}{I_0} & n\left(2\frac{I_2}{I_0} - \frac{I_1^2}{I_0^2}\right) \end{vmatrix} = \begin{vmatrix} \frac{n}{\sigma^4}\left(\mu_2' + \sigma^2\right) & -\frac{n}{\sigma^2}\mu_1'^2 \\ -\frac{n}{\sigma^2}\mu_1' & \frac{n}{\sigma^2}\left(\mu_2' - \mu_1'^2\right) \end{vmatrix}$$

$$= \frac{n^2}{\sigma^6}\{\mu_2'^2 - \mu_2'\mu_1'^2 + \sigma^2\mu_2' - 2\sigma^2\mu_1'^2\}$$

then

$$V(\sigma) = \frac{\Delta_{22}}{\Delta} = \frac{\sigma^4\mu_2}{n\{\mu_2'\mu_2 + \sigma^2(2\mu_2 - \mu_2')\}} \tag{34}$$

$$V(\xi) = \frac{\Delta_{11}}{\Delta} = \frac{\sigma^2(\mu_2' + \sigma^2)}{n\{\mu_2'\mu_2 + \sigma^2(2\mu_2 - \mu_2')\}} \tag{34 bis}$$

and the correlation between the sampling errors of σ and ξ will be

$$+\frac{\sigma\mu_1'}{\sqrt{\mu_2(\mu_2' + \sigma^2)}} \tag{35}$$

3. *Modified Poisson Series.*—In the simple Poisson series the frequency with which the variate takes the integral value x is

$$\frac{e^{-m}m^x}{x!}$$

If, now, m is a variate with distribution specified by

$$df = \frac{1}{2^{\frac{1}{2}(q-1)}\left(\frac{q-1}{2}\right)!}\left(\frac{m}{\sigma}\right)^q e^{-\frac{m^2}{2\sigma^2}}\frac{dm}{\sigma} \tag{36}$$

as, for example, is the standard deviation as estimated from a sample, the frequency of the value x will be

$$\frac{e^{\frac{1}{2}\sigma^2}\sigma^x}{x!\,2^{\frac{1}{2}(q-1)}\left(\frac{q-1}{2}\right)!}\int_0^\infty \left(\frac{m}{\sigma}\right)^{x+q} e^{-\frac{1}{2}\left(\frac{m}{\sigma}+\sigma\right)^2}\frac{dm}{\sigma} \tag{37}$$

and this is equal to

$$\frac{e^{\frac{1}{2}\sigma^2}\sqrt{2\pi}}{2^{\frac{1}{2}(q-1)}\left(\frac{q-1}{2}\right)!}\cdot\frac{(q+x)!}{x!}\sigma^x I_{q+x}(\sigma) \tag{38}$$

putting $x = 0, 1, 2, \ldots$ we have the modified series, in which evidently

$$\sum_{x=0}^{\infty}\left\{\frac{(q+x)!}{x!}\sigma^x I_{q+x}(\sigma)\right\} = \frac{1}{\sqrt{2\pi}}\left(\frac{q-1}{2}\right)!\ 2^{\frac{1}{2}(q-1)}e^{-\frac{1}{2}\sigma^2} \tag{39}$$

A more general distribution of m which develops a modified series in I functions is

$$df = \frac{1}{\sigma^{q+1}I_q(a)q!\sqrt{2\pi}}m^q e^{-\frac{1}{2}\left(\frac{m}{\sigma}+a\right)^2}dm$$

a being a real parameter which may be either positive or negative. Then the frequency of the value x is

$$\frac{e^{a\sigma+\frac{1}{2}\sigma^2}}{\sigma^{q+1}I_q(a)q!\ x!\sqrt{2\pi}}\int_0^{\infty} m^{q+x}e^{-\frac{1}{2}\left(\frac{m}{\sigma}+a+\sigma\right)^2}dm \tag{40}$$

which is

$$\frac{e^{a\sigma+\frac{1}{2}\sigma^2}}{I_q(a)q!}\cdot\frac{(q+x)!}{x!}\sigma^x I_{q+x}(\sigma+a) \tag{41}$$

giving the interesting identity

$$\sum_{x=1}^{\infty}\frac{(q+x)!}{x!}\sigma^x I_{q+x}(\sigma+a) = q!\ I_q(a)e^{-a\sigma-\frac{1}{2}\sigma^2} \tag{42}$$