189

THE THEORY OF CONFOUNDING IN FACTORIAL EXPERIMENTS IN RELATION TO THE THEORY OF GROUPS

Author's Note (CMS 39.340a)

The first three sections of Paper 189 occupying less than three pages give the general theory which is extended to more complex cases in Paper 202. The remainder of Paper 189 is mainly occupied by a catalogue completing the solution of a group of problems which in the literature had received only rather unsystematic and adventitious treatment.

Group properties are so abstract that the language in which the ideas of the theory are expressed requires the utmost care if the ideas are to be conveyed to the reader's mind. Every effort has been made in these pages to make clear the necessary connection of each idea with those preceding.

Practical workers with experiments involving confounding should gain familiarity through the practical approach developed in *The Design of Experiments*, which in the third edition illustrates the full use of factors at two levels.

THE THEORY OF CONFOUNDING IN FACTORIAL EXPERIMENTS IN RELATION TO THE THEORY OF GROUPS

BY R. A. FISHER

1. INTRODUCTION

When the treatments in an experiment introduce all combinations of n factors, each at two levels, so that they are 2^n in number, it has long been known that it is possible in many cases to divide each replication into two, four or more blocks in such a way that the contrasts between blocks constitute only such interactions between the primary factors as are of minor importance to the experimenter. The great advantage of limiting the size of the block lies in the fact that in this way the contents of each block may be made much more homogeneous than if it had been of larger size.

In a very large class of experiments of this type we are concerned to evaluate the effects of each individual factor with high precision, and to discover whether any pair of the factors tested show an appreciable interaction. If in these respects the precision of the experiment can be increased, it is usually advantageous to do so at the expense of foregoing information as to the reality of one or more of the interactions involving three or more factors.

Various systems of confounding, using factors up to six in number, have been discussed by Barnard (1936) and Yates (1937). In the present paper I propose to develop the connexion of the subject with that of Abelian groups, to prove a general proposition connecting the minimal size of block required with the number of factors involved, and to supply a catalogue of systems of confounding available up to fifteen factors.

2. THE GROUP PROPERTIES OF TREATMENTS AND COMPARISONS

A group may be formed, of which the elements are all the selections that can be made of none or more out of n letters. The order of the group is 2^n . The product of any two elements is formed by combining the letters they contain, deleting any they may have in common. The group is, therefore, Abelian.

As is well known, the treatments of a factorial experiment correspond one to one with the elements of this group. Any treatment combination chosen as control is designated by the symbol (1), while the symbols for all other treatment combinations are designated by the letters representing the factors in which they differ from the control. Thus (abc) stands for the treatment differing from the control in factors A , B , C , and agreeing with it in all other factors.

Every element of the group, other than the identity, divides the whole into two halves, having respectively even and odd numbers of letters in common with that element. The identity always belongs to the even half. The elements other than the identity thus each correspond with a particular comparison in which half of the treatments tested is contrasted with the other half. The comparisons of particular interest to the experimenter comprise those due to single factors, n in number, represented by single letters, A , B , C , D , ... and to the interactions of these, two at a time, AB , AC , BC , etc. We shall be concerned that these shall not be confounded.

Now, if a treatment has an even number of letters in common with a contrast S_1 and also an even number of letters in common with S_2 , it must equally have an even number of letters in common with the product S_1S_2 . Consequently the contrasts with which it has an even number of letters in common constitute, with the identity, an entire subgroup. The halvings corresponding to any two elements are orthogonal, i.e. if the letter A occurs in S_1 , but not in S_2 , then if g

is the aggregate of elements even for both, Ag is an equal aggregate even only for the second. Consequently, the number of elements even for all members of a subgroup of order 2^p is 2^{n-p} , and these must form a subgroup.

In this connexion it may be noted that the number of subgroups of order 2^a is equal to the number of order 2^{n-a} , and is

$$\prod_{s=1}^{n-a} \frac{2^{n-s+1} - 1}{2^s - 1}.$$

If there are 2^c blocks it is obvious that $2^c - 1$ interactions will be confounded, and that these with the identity will form a subgroup of order 2^c .

Let C be any interaction confounded, and let B be the operator needed to change any treatment into any other treatment occupying the same block, then since C is a contrast between whole blocks, any two members of the same block must have either both an odd or both an even number of letters in common with C . Hence B has an even number of letters in common with C .

There are only 2^{n-c} operators in the subgroup having an even number of letters in common with all confounded interactions. Hence any treatment when operated on by this subgroup of operators yields the set of treatments in the same block. This subgroup of operators is therefore called the intrablock subgroup.

When the number of factors is large, the number of plots in each block may be much less than the number of blocks. Hence it is easier to specify the intrablock subgroup than the subgroup of confounded interactions, which may be derived from it. If in any solution we desire to omit any factor, say X , without diminishing the size of block, we may always do this by omitting the symbol x from the specification of any element of the intrablock subgroup in which it may occur. At the same time all elements containing X of the subgroup confounded are deleted, so dividing its order by two, and halving the number of blocks required to carry out the experiment. These operations do not introduce any new interaction among those confounded; consequently, if, in the original solution, no interaction involving less than three factors was confounded, this will still remain true after any chosen factor has been omitted.

3. THE RELATION BETWEEN THE NUMBER OF FACTORS AND THE NECESSARY SIZE OF THE BLOCK

It is a remarkable fact that so long as the number of units in a block exceeds the number of factors used, arrangements can be found such that no interaction confounded involves less than three factors. Thus blocks of 8 may be used with so many as 7 factors, or 128 treatment combinations, so that the number of blocks in the replication would be 16, and blocks of 16 may be used with so many as 15 factors, or 32,768 treatment combinations, there being then 2048 blocks in a replication. The general proof of this proposition may be developed as follows:

To arrange the combinations of $2^n - 1$ factors in blocks of 2^n units each; first establish a 1 : 1 correspondence between the factors, or the letters by which they are represented, and the elements, other than the identity, of a group of order 2^n . For example, using Greek letters for this latter group we might have for 15 factors:

A	α	F	$\beta\gamma$	L	$\alpha\beta\delta$
B	β	G	$\alpha\beta\gamma$	M	$\gamma\delta$
C	$\alpha\beta$	H	δ	N	$\alpha\gamma\delta$
D	γ	J	$\alpha\delta$	O	$\beta\gamma\delta$
E	$\alpha\gamma$	K	$\beta\delta$	P	$\alpha\beta\gamma\delta$

Corresponding with each Latin letter, or with each element other than the identity of the

Greek group, there is the subgroup of order 2 to which that element belongs, and the subgroup of order 2^{n-1} , or in this case 2^3 , all of the elements of which have an even number of Greek letters in common with it.

The remainder of the Greek group is a set of 2^{n-1} (eight) elements, not including the identity, and the corresponding set of Latin letters will be chosen to specify a treatment in the same block as the control treatment.

Every Latin letter of the set of Latin letters which corresponds with the interaction of any two elements of the Greek group, contains an odd number of Greek letters in common with this interaction, and therefore an odd number of Greek letters in common with one but not both of the elements of which it is the interaction. The set is therefore constituted of the Latin letters which appear in one but not in both of the sets corresponding with these elements. It is therefore their interaction. Hence the $2^n - 1$ sets with the identity form a group. This is chosen as the intrablock subgroup of the group formed by all combinations of the Latin letters.

If S and T are any two Latin letters, and ξ any Greek letter which occurs in the representation of one of them, but not of both, then the set of Latin letters corresponding with ξ contains all of which the representations contain ξ , and therefore includes one but not both of the letters S and T . Hence the interaction ST is not confounded.

Evidently, also, the main action T is not confounded, for T will occur in the set corresponding with any single Greek letter which the Greek representation of T may contain.

Hence with only 2^n treatments in each block, we may test $2^n - 1$ factors and all their interactions without confounding any interaction of less than three factors.

The same argument provides the means of identifying the interactions which are confounded. For, if $STU\dots$ is any interaction, and if the product of the Greek representations corresponding with the letters it contains involves the letter ξ , then ξ must be involved in an odd number of these letters; consequently the interaction has an odd number of letters in common with the set corresponding with ξ , and is therefore not confounded. The interactions confounded are those, and those only, for which the product of the Greek representations of the Latin elements involved reduces to the identity.

4. EXTENSION TO CASES IN WHICH EACH FACTOR OCCURS AT MORE THAN TWO LEVELS

If each factor is tested at p levels, where p is a prime number, the same for all factors, the correspondence with the theory of Abelian groups remains and many of the same conclusions follow.

The chief difference arises from the fact that each factor is now associated, not with a single comparison, but with a cycle of $p - 1$ independent comparisons, represented by the powers, excluding zero, of a single symbol. The interactions of any two factors will then constitute $p - 1$ such cycles, representable by the products of these powers. Whereas in the case of two levels the interaction of such comparisons as AB and BC will never involve the factor B , with more than two levels, if we consider the interactions of AB and BC , it appears that only one of the interaction cycles will be free from B and involve only the two factors A and C , while the remainder will involve all three factors. It should be noted that the number of factors involved is never less than the number in the case of experimentation at two levels, though it may be more.

Consequently, if we experiment with the p^n treatment combinations of n factors, arranging the treatments in blocks of p^a units in such a way that $p^{n-a} - 1$ treatment comparisons are confounded, we shall have a subgroup confounded of order p^{n-a} , and an intrablock subgroup of order p^a .

It appears on consideration that if $A^iB^jC^k \dots$ is any one of the comparisons confounded, and $A^x B^y C^z$ is a member of the intrablock subgroup, or a treatment in the same block as the control, then

$$ix + jy + kz + \dots = 0 \pmod{p}.$$

Given the subgroup confounded, the intrablock subgroup may then be constructed as in the case $p = 2$. Further, any of the solutions available when $p = 2$ will be available in general, with the assurance that no interaction will involve fewer factors than the corresponding interaction when $p = 2$.

Consequently, no interaction of less than three factors need be confounded if 2^a exceeds n .

There will, however, be some variety in the numbers of interactions of different orders confounded, according to which set is chosen. Thus, with 7 factors arranged in blocks of p^3 units, we might choose for confounding a subgroup of order p^4 derivable from the generators

$$ABC \quad ADE \quad AFG \quad BD^iF^j.$$

The cases in which the 0, 1 or 2 of the relations

$$1 + i = p, \quad 1 + j = p, \quad i + j = p, \quad 1 + i + j = p$$

is or are satisfied, lead in general to three different solutions, though, when $p = 3$, only two of these can occur, since i and j can only take the values 1 or 2.

The distribution of the forty cycles of two interactions each, confounded in these two cases, according to the numbers of factors involved are as follows:

No. of factors	$i = j = 1$	Three other cases	Total no. of cycles
1	—	—	7
2	—	—	42
3	5	6	140
4	15	11	280
5	9	15	336
6	8	4	224
7	3	4	64
Total	40	40	1093

The solutions possible seem, however, to be numerous.

5. SEVEN FACTORS

The solution for 7 factors at two levels, tested in blocks of 8, follows directly from the general theorem proved in § 3. The intrablock group contains, in addition to the control, the 7 treatments

$$(abdg) \quad (abef) \quad (acdf) \quad (aceg) \quad (bcde) \quad (bcfg) \quad (defg).$$

It may be remarked that these form a solution of the problem of arranging 7 treatments in balanced incomplete blocks of 4 units each. However the matter be viewed, there are 30 such selections possible, and of these 8 have no interaction in common with any chosen one. For that set out above these are:

$$\begin{array}{llllll} (abcd) & (abeg) & (acfg) & (adef) & (bcef) & (bdfg) & (cdeg) \\ (abcd) & (abfg) & (acef) & (adeg) & (bceg) & (bdef) & (cdfg) \\ (abce) & (abdf) & (acfg) & (adeg) & (bcdg) & (befg) & (cdef) \\ (abce) & (abfg) & (acd g) & (adef) & (bcd f) & (bdeg) & (cefg) \\ (abcf) & (abeg) & (acde) & (adfg) & (bcdg) & (bdef) & (cefg) \\ (abcf) & (abde) & (acd g) & (aefg) & (bceg) & (bdfg) & (cdef) \\ (abcg) & (abdf) & (acde) & (aefg) & (bcef) & (bdeg) & (cdfg) \\ (abcg) & (abde) & (acef) & (adfg) & (bcd f) & (befg) & (cdeg) \end{array}$$

If it were possible to construct a balanced set, using only 5 replications, each of the 35 selections of 4 letters would have to occur in only 1 replication. Since, however, every pair of 8 sets listed above has one element in common, a balanced set in 5 replications is not possible.

Balanced sets do, however, exist, both in 10 and in 15 replications. One type of solution for 10 replications is obtained by observing that the 28 treatments common to the 28 pairs of the 8 sets listed are all different, so that we have only to repeat the first solution twice, and each of the other solutions having no element in common with it once each, to obtain a set of 10 in which each of the 35 eligible treatments occurs twice in the control block.

This equalization also extends to the interactions confounded, for 7 of these are represented by the same symbols as the 7 treatments of the control block, while the remaining 8 are respectively the interaction of all 7 treatments and the 7 interactions of 3 factors complementary to those already used.

Obviously, there are 30 balanced sets of the type set out above. A second system, of which there are 252, is illustrated below, where we specify the 3-factor interactions confounded:

<i>ABE</i>	<i>ACD</i>	<i>AFG</i>	<i>BCG</i>	<i>BDF</i>	<i>CEF</i>	<i>DEG</i>
<i>ABE</i>	<i>ACD</i>	<i>AFG</i>	<i>BCF</i>	<i>BDG</i>	<i>CEG</i>	<i>DEF</i>
<i>ABC</i>	<i>ADF</i>	<i>AEG</i>	<i>BDE</i>	<i>BFG</i>	<i>CDG</i>	<i>CEF</i>
<i>ABC</i>	<i>ADG</i>	<i>AEF</i>	<i>BDE</i>	<i>BFG</i>	<i>CDF</i>	<i>CEG</i>
<i>ABG</i>	<i>ACE</i>	<i>ADF</i>	<i>BCD</i>	<i>BEF</i>	<i>CFG</i>	<i>DEG</i>
<i>ABF</i>	<i>ACE</i>	<i>ADG</i>	<i>BCD</i>	<i>BEG</i>	<i>CFG</i>	<i>DEF</i>
<i>ABD</i>	<i>ACF</i>	<i>AEG</i>	<i>BCG</i>	<i>BEF</i>	<i>CDE</i>	<i>DFG</i>
<i>ABD</i>	<i>ACG</i>	<i>AEF</i>	<i>BCF</i>	<i>BEG</i>	<i>CDE</i>	<i>DFG</i>
<i>ABG</i>	<i>ACF</i>	<i>ADE</i>	<i>BCE</i>	<i>BDF</i>	<i>CDG</i>	<i>EFG</i>
<i>ABF</i>	<i>ACG</i>	<i>ADE</i>	<i>BCE</i>	<i>BDG</i>	<i>CDF</i>	<i>EFG</i>

Of balanced sets of 15 replications, three types may be mentioned, one of which is unique in that it is unaltered by any even permutation, so that there are only two solutions in the set, of which the second is, of course, constituted by the 15 solutions not used in the first:

<i>ABC</i>	<i>ADE</i>	<i>AFG</i>	<i>BDF</i>	<i>BEG</i>	<i>CDG</i>	<i>CEF</i>
<i>ABC</i>	<i>ADF</i>	<i>AEG</i>	<i>BDG</i>	<i>BEF</i>	<i>CDE</i>	<i>CFG</i>
<i>ABC</i>	<i>ADG</i>	<i>AEF</i>	<i>BDE</i>	<i>BFG</i>	<i>CDF</i>	<i>CEG</i>
<i>ABF</i>	<i>ADE</i>	<i>ACG</i>	<i>BCE</i>	<i>BDG</i>	<i>CDF</i>	<i>EFG</i>
<i>ABG</i>	<i>ADE</i>	<i>ACF</i>	<i>BCD</i>	<i>BEF</i>	<i>CEG</i>	<i>DFG</i>
<i>ABD</i>	<i>ACE</i>	<i>AFG</i>	<i>BCG</i>	<i>BEF</i>	<i>CDF</i>	<i>DEG</i>
<i>ABE</i>	<i>ACD</i>	<i>AFG</i>	<i>BCF</i>	<i>BDG</i>	<i>CEG</i>	<i>DEF</i>
<i>ABE</i>	<i>ACF</i>	<i>ADG</i>	<i>BDF</i>	<i>BCG</i>	<i>CDE</i>	<i>EFG</i>
<i>ABG</i>	<i>ACD</i>	<i>AEF</i>	<i>BDF</i>	<i>BCE</i>	<i>CFG</i>	<i>DEG</i>
<i>ABD</i>	<i>ACG</i>	<i>AEF</i>	<i>BCF</i>	<i>BEG</i>	<i>CDE</i>	<i>DFG</i>
<i>ABF</i>	<i>ACE</i>	<i>ADG</i>	<i>BCD</i>	<i>BEG</i>	<i>CFG</i>	<i>DEF</i>
<i>ABG</i>	<i>ACE</i>	<i>ADF</i>	<i>BCF</i>	<i>BDE</i>	<i>CDG</i>	<i>EFG</i>
<i>ABD</i>	<i>ACF</i>	<i>AEG</i>	<i>BCF</i>	<i>BFG</i>	<i>CDG</i>	<i>DEF</i>
<i>ABF</i>	<i>ACD</i>	<i>AEG</i>	<i>BCG</i>	<i>BDE</i>	<i>CEF</i>	<i>DFG</i>
<i>ABE</i>	<i>ACG</i>	<i>ADF</i>	<i>BCD</i>	<i>BFG</i>	<i>CEF</i>	<i>DEG</i>

A second set, this time of 70 solutions, may be formed by replacing the first three solutions used above by

<i>ABC</i>	<i>ADE</i>	<i>AFG</i>	<i>BDG</i>	<i>BEF</i>	<i>CDF</i>	<i>CEG</i>
<i>ABC</i>	<i>ADG</i>	<i>AEF</i>	<i>BDF</i>	<i>BEG</i>	<i>CDE</i>	<i>CFG</i>
<i>ABC</i>	<i>ADF</i>	<i>AEG</i>	<i>BDE</i>	<i>BFG</i>	<i>CDG</i>	<i>CEF</i>

while a third system of balanced replications is provided by

<i>ABC</i>	<i>ADE</i>	<i>AFG</i>	<i>BDF</i>	<i>BEG</i>	<i>CDG</i>	<i>CEF</i>
<i>ABC</i>	<i>ADF</i>	<i>AEG</i>	<i>BDG</i>	<i>BEF</i>	<i>CDE</i>	<i>CFG</i>
<i>ABC</i>	<i>ADG</i>	<i>AEF</i>	<i>BDE</i>	<i>BFG</i>	<i>CDF</i>	<i>CEG</i>
<i>ABF</i>	<i>ADE</i>	<i>ACG</i>	<i>BCE</i>	<i>BDG</i>	<i>CDF</i>	<i>EFG</i>
<i>ABG</i>	<i>ADE</i>	<i>ACF</i>	<i>BCD</i>	<i>BEF</i>	<i>CEG</i>	<i>DFG</i>
<i>ABF</i>	<i>ACE</i>	<i>ADG</i>	<i>BCG</i>	<i>BDE</i>	<i>CDF</i>	<i>EFG</i>
<i>ABD</i>	<i>ACF</i>	<i>AEG</i>	<i>BCG</i>	<i>BEF</i>	<i>CDE</i>	<i>DFG</i>
<i>ABE</i>	<i>ACF</i>	<i>ADG</i>	<i>BCD</i>	<i>BFG</i>	<i>CEG</i>	<i>DEF</i>
<i>ABG</i>	<i>ACD</i>	<i>AEF</i>	<i>BCD</i>	<i>BCE</i>	<i>CFG</i>	<i>DEG</i>
<i>ABD</i>	<i>ACG</i>	<i>AEF</i>	<i>BCF</i>	<i>BEG</i>	<i>CDE</i>	<i>DFG</i>
<i>ABF</i>	<i>ACD</i>	<i>AEG</i>	<i>BCE</i>	<i>BDG</i>	<i>CFG</i>	<i>DEF</i>
<i>ABG</i>	<i>ACE</i>	<i>ADF</i>	<i>BCF</i>	<i>BDE</i>	<i>CDG</i>	<i>EFG</i>
<i>ABE</i>	<i>ACD</i>	<i>AFG</i>	<i>BCG</i>	<i>BDF</i>	<i>CEF</i>	<i>DEG</i>
<i>ABD</i>	<i>ACE</i>	<i>AFG</i>	<i>BCF</i>	<i>BEG</i>	<i>CDG</i>	<i>DEF</i>
<i>ABE</i>	<i>ACG</i>	<i>ADF</i>	<i>BCD</i>	<i>BFG</i>	<i>CEF</i>	<i>DEG</i>

It may be that there are other balanced arrangements beyond these three listed.

6. METHODS OF CONFOUNDING EIGHT TO FIFTEEN FACTORS
IN BLOCKS OF SIXTEEN PLOTS

Eight factors

There are five arrangements with 8 factors, in addition to the one mentioned under (j) below.

(a) The first example possesses some remarkable features. We derive all blocks from that containing the control, i.e.

(<i>I</i>)	(<i>aceg</i>)	(<i>bceh</i>)	(<i>cdef</i>)
(<i>abcd</i>)	(<i>acfh</i>)	(<i>befg</i>)	(<i>cdgh</i>)
(<i>abef</i>)	(<i>adfg</i>)	(<i>bdeg</i>)	(<i>efgh</i>)
(<i>abgh</i>)	(<i>adeh</i>)	(<i>bdfh</i>)	(<i>abcdefgh</i>)

The corresponding interactions constitute a subgroup of order 16, the intrablock subgroup. This is a group of operators which transform any treatment into one occurring in the same block. These operators are therefore

<i>I</i>	<i>ABCD</i>	<i>ABEF</i>	<i>ABGH</i>
<i>ACEG</i>	<i>ACFH</i>	<i>ADEH</i>	<i>ADFG</i>
<i>BCEH</i>	<i>BCFG</i>	<i>BDEG</i>	<i>BDFH</i>
<i>CDEF</i>	<i>CDGH</i>	<i>EFGH</i>	<i>ABCDEF GH</i>

The interactions which are confounded (including the identity) always constitute a subgroup, each of the elements of which has an even number (including zero) of symbols in common with every element of the intrablock subgroup. Since in this case every element contains a number divisible by 4, and consequently has an even number of symbols in common with every other,

while the subgroup is of order 2^4 out of 2^8 in the entire group, it follows that the confounded subgroup is identical with the intrablock subgroup.

The remaining blocks are derived from the first by applying operators not belonging to the intrablock subgroup. In particular, the contents of the block containing any treatment may be obtained by using the corresponding operator.

The arrangement chosen is unaltered by a group of permutations such as

$$\begin{aligned}(ab)(cd)(ef)(gh), \\ (ac)(bd)(eg)(fh), \\ (ae)(bf)(cg)(dh), \\ (bc)(fg), (bce)(dgh), \text{ etc.}\end{aligned}$$

which generate the permutation group representing the symmetry of a perfect cube. This group is of order 48. Consequently we may make $8!/48 = 840$ selections not mutually related by this group. These may be generated by choosing one of the 35 ways of dividing eight objects into 2 sets of 4, representing the alternate apices of the cube, and then choosing one of the 24 one to one correspondences between the two sets of objects.

Any solution contains fourteen interactions of 4 factors, consisting of seven pairs. Any one of these pairs may be taken to represent sets of alternate apices. Consequently, the solution may be obtained from each of 7 of the 35 subdivisions. In fact, in addition to those representing the symmetry of the cube the permutation of 7 letters (*bfcchgd*) also leaves the solution unchanged. Further, within each of these seven subdivisions 4 of the 24 correspondences give the same solution. For example, if in the solution above we regard (*adfg*) and (*bceh*) as representing alternate apices, the permutation group of order four containing (*af*)(*dg*), (*ag*)(*df*), and (*ad*)(*fg*) leaves the example unaltered.

There are thus only 30 different solutions, corresponding with the 30 solutions for 7 factors in blocks of 8, the confounded subgroup of which can be derived from that of the present example merely by deleting the letter *H*.

In this case only interactions of 4 factors are confounded. The 70 interactions of 4 factors may all be confounded equally, i.e. in one-fifth of the replications, by using 10 or 15 replications as in the analogous solution with 7 factors.

(b) The intrablock group is

$$\begin{array}{llll} I & ABCD & BCFG & ABDFH \\ BGH & ABEF & BDEG & ACDGH \\ CFH & ACEG & CDEF & AEFHG \\ DEH & ADFG & ABCEH & BCDEFGH \end{array}$$

The group of confounded interactions is

$$\begin{array}{llll} I & BCEH & CDEF & ABEFH \\ ABG & BCFG & CDGH & ACEGH \\ ACF & BDEG & EFGH & ADFGH \\ ADE & BDFH & ABCDH & ABCDEFG \end{array}$$

These subgroups are now distinct, though for 8 factors in blocks of 16 plots they are necessarily of the same order. They have a common subgroup of order 4 containing the interactions *BCFG*, *BDEG*, *CDEF*.

The solution is unaltered by the permutations (*bc*)(*fg*), (*bcd*)(*egf*), (*bg*)(*cf*), a group of order 24. Hence there are 1680 solutions in the set.

(c) The intrablock block is

<i>I</i>	<i>DEF</i>	<i>BCDH</i>	<i>ACDFG</i>
<i>AEH</i>	<i>ABCF</i>	<i>BEFG</i>	<i>BCEFH</i>
<i>BDG</i>	<i>ACEG</i>	<i>ABCDE</i>	<i>ABDEGH</i>
<i>CGH</i>	<i>ADFH</i>	<i>ABFGH</i>	<i>CDEFGH</i>

The group of confounded interactions

<i>I</i>	<i>BDF</i>	<i>CEFH</i>	<i>BCDEH</i>
<i>ACH</i>	<i>ABDE</i>	<i>DEGH</i>	<i>BEFGH</i>
<i>AEF</i>	<i>ABGH</i>	<i>ACDEG</i>	<i>ABCDFH</i>
<i>BCG</i>	<i>CDFG</i>	<i>ADFGH</i>	<i>ABCEFG</i>

The solution is unaltered by the group of order 8 generated by $(ab)(de)(gh)$, $(cf)(dg)(eh)$, $(ac)(bf)(eg)$; the number of solutions is 5040.

(d) The intrablock group is

<i>I</i>	<i>ABEG</i>	<i>BCFG</i>	<i>ACDGH</i>
<i>EH</i>	<i>ABGH</i>	<i>ABDEF</i>	<i>BCDEH</i>
<i>BCD</i>	<i>ACEF</i>	<i>ABDFH</i>	<i>DEFGH</i>
<i>DFG</i>	<i>ACFH</i>	<i>ACDEG</i>	<i>BCEFGH</i>

Unlike the group of confounded interactions this may contain interactions of two factors. The group of confounded interactions is

<i>I</i>	<i>BDG</i>	<i>BCEH</i>	<i>CDEGH</i>
<i>ABC</i>	<i>CDF</i>	<i>BCFG</i>	<i>ABDEGH</i>
<i>AEH</i>	<i>ABDF</i>	<i>EFGH</i>	<i>ACDEFH</i>
<i>AFG</i>	<i>ACDG</i>	<i>BDEFH</i>	<i>ABCEFGH</i>

The one interaction *BCFG* is common to the two subgroups, the solution is invariant for a permutation group of order 8 generated by (eh) , $(bc)(fg)$, $(bf)(cg)$; there are therefore 5040 solutions of this type.

(e) The intrablock group is

<i>I</i>	<i>ACFH</i>	<i>BCEF</i>	<i>CDEH</i>
<i>FG</i>	<i>ACGH</i>	<i>BCEG</i>	<i>ABCD</i>
<i>ABCD</i>	<i>ADEF</i>	<i>BDFH</i>	<i>ABEFGH</i>
<i>ABEH</i>	<i>ADEG</i>	<i>BDGH</i>	<i>CDEFGH</i>

The group of confounded interactions is

<i>I</i>	<i>BDE</i>	<i>CDEH</i>	<i>ADEFG</i>
<i>ACE</i>	<i>ABCD</i>	<i>CDFG</i>	<i>BCEFG</i>
<i>ADH</i>	<i>ABEH</i>	<i>EFGH</i>	<i>BDFGH</i>
<i>BCH</i>	<i>ABFG</i>	<i>ACFGH</i>	<i>ABCEFGH</i>

There is a common subgroup of order 4.

The group of permutations for which the solution is invariant is generated by

$$(fg), (ab)(cd), (ab)(eh), (ac)(bd), (ace)(bdh).$$

This is a group of order 48, so that there are 840 solutions of this type.

Nine factors

With 9 factors also five types of solution are possible.

(f) Intrablock group

<i>I</i>	<i>CDGH</i>	<i>ABCFG</i>	<i>ACEGJ</i>
<i>BCHJ</i>	<i>CEFH</i>	<i>ABDFH</i>	<i>ADEHJ</i>
<i>BDGJ</i>	<i>DEFG</i>	<i>ABEGH</i>	<i>AFGHJ</i>
<i>BEFJ</i>	<i>ABCDE</i>	<i>ACDFJ</i>	<i>BCDEFGHJ</i>

Group of confounded interactions

<i>I</i>	<i>BDFH</i>	<i>DEFG</i>	<i>ACFGJ</i>
<i>ABJ</i>	<i>BDGJ</i>	<i>DEHJ</i>	<i>ADFHJ</i>
<i>ACH</i>	<i>BEFJ</i>	<i>FGHJ</i>	<i>AEGHJ</i>
<i>ADG</i>	<i>BEGH</i>	<i>ABCDG</i>	<i>ABCDGHJ</i>
<i>AEF</i>	<i>CDFJ</i>	<i>ABCEF</i>	<i>ABCEF HJ</i>
<i>BCDE</i>	<i>CDGH</i>	<i>ABDFH</i>	<i>ABDEFGJ</i>
<i>BCFG</i>	<i>CEFH</i>	<i>ABFGH</i>	<i>ACDEFGH</i>
<i>BCHJ</i>	<i>CEGJ</i>	<i>ACDEJ</i>	<i>BCDEFGHJ</i>

The common subgroup is of order 8.

If we omit the factor *A* we delete all interactions containing *A* from those confounded, and delete the letter *A* from all containing it in the intrablock group. The solution for 8 factors so obtained is of type (a). There are therefore only 270 solutions of this type. Omitting any of the other letters leads to a solution of type (b).

(g) Intrablock group

<i>I</i>	<i>AEFG</i>	<i>CDEJ</i>	<i>ABDEGH</i>
<i>ABFJ</i>	<i>BCDG</i>	<i>CFGH</i>	<i>ACDFGJ</i>
<i>ACEH</i>	<i>BDFH</i>	<i>ABCDEF</i>	<i>BCEF HJ</i>
<i>ADHJ</i>	<i>BEGJ</i>	<i>ABCGHJ</i>	<i>DEFGHJ</i>

Group of confounded interactions

<i>I</i>	<i>ABGH</i>	<i>ABCDG</i>	<i>CFGHJ</i>
<i>AEJ</i>	<i>ACDF</i>	<i>ABCEH</i>	<i>ABDFHJ</i>
<i>AFH</i>	<i>ACGJ</i>	<i>ABCFJ</i>	<i>ABEFGJ</i>
<i>BDJ</i>	<i>BCEF</i>	<i>ADEFG</i>	<i>ACDEHJ</i>
<i>BFG</i>	<i>BCHJ</i>	<i>ADGHJ</i>	<i>ACEFGH</i>
<i>CDH</i>	<i>DEGH</i>	<i>BDEFH</i>	<i>BCDEGJ</i>
<i>CEG</i>	<i>DFGJ</i>	<i>BEGHJ</i>	<i>BCDFGH</i>
<i>ABDE</i>	<i>EFHJ</i>	<i>CDEFJ</i>	<i>ABCDEFGHJ</i>

The common subgroup is of order 8. Deleting any letter reduces the solution to (c); there are therefore 5040 solutions. One-fourteenth of the interactions of 3, 4, 5 and 6 factors is confounded in addition to the interaction of all 9 factors.

(h) The intrablock group is

<i>I</i>	<i>ACFG</i>	<i>ACDGJ</i>	<i>ABDEFG</i>
<i>DFJ</i>	<i>BCDE</i>	<i>ACEFH</i>	<i>ACDEHJ</i>
<i>EGH</i>	<i>ABDFH</i>	<i>BCDGH</i>	<i>BCFGHJ</i>
<i>ABHJ</i>	<i>ABEGJ</i>	<i>BCEFJ</i>	<i>DEFGHJ</i>

The group of confounded interactions may be generated from ABC , AFJ , AGH , BDJ , and BEH . It contains 7 interactions of three factors, 9 of four, 6 of five, 6 of six and 3 of seven. The letters A , B and C occur in none of the triple interactions of the intrablock group; deleting any of these we have solution (c). Deleting the other letters gives (d) in each case.

(i) The intrablock group is

I	$BCFG$	$ABEFJ$	$BDEHJ$
EGH	$BDGJ$	$ACEGJ$	$ABCDGH$
$ACHJ$	$CDFJ$	$ADEFG$	$ABFGHJ$
$ADFH$	$ABCDE$	$BCEFFH$	$CDEFGHJ$

Deleting E we have solution (e); G and H give (d); A and B give (b), and C , D , F and J give (c).

(j) The intrablock group is

I	$DEGJ$	$ACDGH$	$BCGHJ$
FH	$ACEFJ$	$BCDEF$	$ABDFHJ$
$ABDJ$	$ACEHJ$	$BCDEH$	$ABEFGH$
$ABEG$	$ACDFG$	$BCFGJ$	$DEFGHJ$

Deleting C gives solution (e); the six letters A , B , D , E , G and J give (d).

There are 8 interactions of three, 10 of four, 4 of five, six and seven, and 1 of eight factors.

The group of confounded interactions is generated by ABC , CDJ , CEG , CFH , and ADE . Since FH is one of the intrablock interactions, all confounded interactions contain either both or neither of these letters, hence if either of these letters is deleted, the other also disappears from the group of confounded interactions. There are thus 15 confounded interactions involving only the other seven factors, the solution being derivable from that of seven factors in blocks of eight merely by splitting each plot for the eighth factor. This really constitutes a sixth method of arranging eight factors in blocks of sixteen plots.

Ten factors in blocks of sixteen plots

There are four solutions for ten factors.

(k) The intrablock group is

I	$ADFGJ$	$BCFHJ$	$ABCDGH$
$CDJK$	$ACEHK$	$BDEGK$	$ABEFJK$
$EFGH$	$ACFGK$	$BDFHK$	$ABGHJK$
$ADEHJ$	$BCEGJ$	$ABCDEF$	$CDEFGHJK$

The 63 confounded interactions may be generated from ACJ , ADK , AEG , AFH , BCK and BEH . There are 8 of three factors, 18 of four, 26 of five, 8 of six and seven, and 5 of eight. Deleting A or B gives solution (f); the other eight factors lead to solution (i).

(l) The intrablock group is

I	$ABFGK$	$BDFGJ$	$ACDFHK$
$ADJK$	$ACEGJ$	$CDEGK$	$BCEFJK$
$BEHK$	$AEFGH$	$ABCDEF$	$ABCGHJK$
$CFHJ$	$BCDGH$	$ABDEHJ$	$DEFGHJK$

The confounded interactions are those of the group generated from ADG , AFJ , AEK , BEG , BFH and CEH . There are 9 interactions of three factors, 16 of four, 15 of five, 12 of six, 7 of seven, 3 of eight, and 1 of nine factors. Deleting A , B , C , D , E or F gives solution (i), deleting H , J or K gives solution (h), deleting G gives solution (g).

(m) The intrablock group is

<i>I</i>	<i>ABEGJ</i>	<i>BCFGK</i>	<i>ACEFJK</i>
<i>FHJ</i>	<i>ACDGH</i>	<i>ABDFHK</i>	<i>BCDEHJ</i>
<i>DEGK</i>	<i>ACEHK</i>	<i>ABEFGH</i>	<i>BCGHJK</i>
<i>ABDJK</i>	<i>BCDEF</i>	<i>ACDFGJ</i>	<i>DEFGHJK</i>

The 63 interactions confounded are generated from *ABC*, *ADE*, *AGK*, *AHJ*, *BDG* and *BFJ*. There are 10 of three factors, 16 of four, 12 of five and six, 10 of seven, and 3 of eight. Deleting *F*, *H*, or *J* leads to solution (*j*); *D*, *E*, *G* and *K* give (*h*), *A*, *B*, and *C*, give (*i*).

(n) The intrablock group is

<i>I</i>	<i>DEFH</i>	<i>ABFGHJ</i>	<i>BCDEFG</i>
<i>ABEK</i>	<i>DGJK</i>	<i>ACDEHJ</i>	<i>BCDHJK</i>
<i>ACFJ</i>	<i>ABDEGJ</i>	<i>ACDFGK</i>	<i>BCEFJK</i>
<i>BCGH</i>	<i>ABDFHK</i>	<i>ACEGHK</i>	<i>EFGHJK</i>

The interactions confounded may be generated from the six triple interactions *ABC*, *AEF*, *AJK*, *BEH*, *BGK* and *CFH*. There are 10 of three factors, 15 of four, 12 of five, 15 of six, 10 of seven, and 1 of all ten. Deleting any one letter leads to solution (*h*).

Eleven factors in blocks of sixteen plots

There are three solutions for eleven factors.

(o) The intrablock group is

<i>I</i>	<i>BEGHK</i>	<i>ABEFJK</i>	<i>BCFGKL</i>
<i>ADEKL</i>	<i>CDGJK</i>	<i>ACDFHK</i>	<i>ABCDEFG</i>
<i>AFGHJ</i>	<i>CEFHL</i>	<i>ACEGJL</i>	<i>ABCHJKL</i>
<i>BDFJL</i>	<i>ABDGHL</i>	<i>BCDEHJ</i>	<i>DEFGHJKL</i>

There are 127 interactions confounded; these may be generated from the seven triple interactions *ADJ*, *AEH*, *AFL*, *AGK*, *BDK*, *BEL* and *CDL*. There are 12 interactions of three factors, 26 of four, 28 of five, 24 of six, 20 of seven, 13 of eight and 4 of nine factors. Deleting *A*, *B* and *C* produces solution (*k*), *D*, *E*, *F*, *G*, *H*, *J*, *K* and *L* give solution (*l*).

(p) The intrablock group is

<i>I</i>	<i>ABEGJL</i>	<i>ACEGHK</i>	<i>BCEFJK</i>
<i>DGJK</i>	<i>ABFGHJ</i>	<i>ACFGKL</i>	<i>BCHJKL</i>
<i>EFHL</i>	<i>ABDEKL</i>	<i>BCDEFG</i>	<i>ACDEHJ</i>
<i>ABDFHK</i>	<i>ACDFJL</i>	<i>BCDGHL</i>	<i>DEFGHJKL</i>

The group of confounded interactions may be generated from *ABC*, *ADG*, *AEF*, *AHL*, *AJK*, *BDJ* and *BFL*. There are 13 interactions of three factors, 26 of four, 24 of five and six, 26 of seven, 13 of eight and 1 of all eleven factors. Deleting *A*, *B* or *C* gives solution (*k*), the eight other letters lead to solution (*m*).

(q) The intrablock group is

<i>I</i>	<i>BCGHL</i>	<i>ACDEHJ</i>	<i>ABDEGJL</i>
<i>DGJK</i>	<i>DEFHL</i>	<i>ACEGHK</i>	<i>ACDFGKL</i>
<i>ABEKL</i>	<i>ABDFHK</i>	<i>BCDEFG</i>	<i>BCDHJKL</i>
<i>ACFJL</i>	<i>ABFGHJ</i>	<i>BCEFJK</i>	<i>EFGHJKL</i>

The confounded interactions may be generated from *ABC*, *AEF*, *AHL*, *AJK*, *BEH*, *BGK*, and *CFH*. There are 13 interactions of three, 25 of four and five, 27 of six, 23 of seven, 10 of eight, 3 of nine and 1 of ten factors. Deleting *A*, *B*, *C*, *E*, *F* or *H* leads to solution (*l*), *D*, *G*, *J* or *K* to (*m*), while deleting *L* leads to (*n*).

Twelve factors in blocks of sixteen plots

With twelve factors there are only two solutions.

(*r*) The intrablock group is

<i>I</i>	<i>ACFHJL</i>	<i>BCFGKL</i>	<i>CDGHLM</i>
<i>ABEFLM</i>	<i>ADFGJM</i>	<i>BDEGJL</i>	<i>ABCDEFGH</i>
<i>ABGHJK</i>	<i>ADEHKL</i>	<i>BDFHKM</i>	<i>ABCDJKLM</i>
<i>ACEGKM</i>	<i>BCEHJM</i>	<i>CDEFJK</i>	<i>EFGHJKLM</i>

The 255 interactions confounded may be generated from the eight *DEM*, *AEJ*, *AFK*, *AGL*, *AHM*, *BEK*, *BGM* and *CEL*. There are 16 interactions of three factors, 39 of four, 48 of five, six of seven, 39 of eight, 16 of nine and 1 of all twelve factors. Deletion of any letter leads to solution (*o*).

(*s*) The intrablock group is

<i>I</i>	<i>ABGHJK</i>	<i>BCFGKL</i>	<i>ACDGHLM</i>
<i>DEHKL</i>	<i>ACEGKM</i>	<i>ABDEGJL</i>	<i>BCDEFGH</i>
<i>DFGJM</i>	<i>ACFHJL</i>	<i>ABDFHKM</i>	<i>BCDJKLM</i>
<i>ABEFLM</i>	<i>BCEHJM</i>	<i>ACDEFJK</i>	<i>EFGHJKLM</i>

The group of interactions confounded is that generated by *ABC*, *AFG*, *AEH*, *AJM*, *AKL*, *BEK*, *BFJ* and *CEL*. There are 17 interactions of 3 factors, 38 of four, 44 of five, 52 of six, 54 of seven, 33 of eight, 12 of nine, 4 of ten and 1 of eleven factors. Deleting *A*, *B* or *C* gives solution (*o*), deleting *D* gives (*p*), while the other eight letters all lead to solution (*q*).

Thirteen factors

With 13 factors we have one solution.

(*t*) The intrablock group is

<i>I</i>	<i>HJKLMN</i>	<i>ABEFJKN</i>	<i>ACEGJLN</i>
<i>BCDEHJ</i>	<i>ABDGHLN</i>	<i>ACDFJLM</i>	<i>BCDEKLMN</i>
<i>BCFGKL</i>	<i>ABDGJKM</i>	<i>ACDFHKN</i>	<i>BCFGHJMN</i>
<i>DEFGMN</i>	<i>ABEFHLM</i>	<i>ACEGHJM</i>	<i>DEFGHJKL</i>

The 511 interactions confounded by this arrangement may be generated from nine chosen from them, such as *ABC*, *ADE*, *AFG*, *AHJ*, *AKL*, *AMN*, *BDF*, *BHK* and *CDG*. There are 22 interactions of three factors, 55 of four, one-thirteenth of the total in each case, 72 of five, 96 of six, 116 of seven, 87 of eight, 40 of nine, 16 of ten, 6 of eleven, and 1 of twelve factors. Deleting the factor *A* we have solution (*r*); deleting any of the other twelve letters leads to solution (*s*).

Fourteen factors

With 14 factors we have the one solution.

(u) With the intrablock group

<i>I</i>	<i>ACEGJLN</i>	<i>HJKLMNO</i>	<i>ACEGHKMO</i>
<i>ABDGJKM</i>	<i>BCDEHJO</i>	<i>ABDGHLNO</i>	<i>BCDEKLMN</i>
<i>ABEFHLM</i>	<i>BCFGKLO</i>	<i>ABEFJKNO</i>	<i>BCFGHJMN</i>
<i>ACDFHKN</i>	<i>DEFGMNO</i>	<i>ACDFJLMO</i>	<i>DEFGHJKL</i>

The 1023 interactions confounded may be generated from *ABC*, *ADE*, *AFG*, *AHJ*, *AKL*, *AMN*, *BDF*, *BHK*, *BMO* and *CDG*. They comprise 28 interactions of three factors, 77 of four, or one-thirteenth of each of these classes, 112 of five, 168 of six, 232 of seven, 203 of eight, 112 of nine, 56 of ten, 28 of eleven and 7 of twelve factors. Deletion of any letter leads necessarily to solution (t).

Fifteen factors

With 15 factors we have the complete solution from which all the others have been derived by deletion; the intrablock group is

(v)	<i>I</i>	<i>ABEFJKNO</i>	<i>ACEGJLNP</i>	<i>BCFGKLOP</i>
	<i>ABDGHLNO</i>	<i>ACDFHKNP</i>	<i>BCDEHJOP</i>	<i>DEFGHJKL</i>
	<i>ABDGJKMN</i>	<i>ACDFJLMO</i>	<i>BCDEKLMN</i>	<i>DEFGMNOP</i>
	<i>ABEFHLP</i>	<i>ACEGHKMO</i>	<i>BCFGHJMN</i>	<i>HJKLMNOP</i>

The 2047 interactions confounded may be generated from *ABC*, *ADE*, *AFG*, *AHJ*, *AKL*, *AMN*, *AOP*, *BDF*, *BHK*, *BMO* and *CDG*. There are 35 triple interactions among those confounded, and it illustrates the manifold connexions among these combinatorial problems, that these constitute the most symmetrical of the solutions of the problem of choosing 35 blocks of 3 from among 15 varieties such that each pair of varieties occurs equally frequently in the same block (Fisher, 1940).

There are also 35 interactions of twelve factors comprising the complementary solution; of interactions of four and of eleven factors 105 are confounded; of five and ten 168, of six and nine 280, and of seven and eight factors 435 each.

REFERENCES

- M. M. BARNARD (1936). An enumeration of the confounded arrangements in the $2 \times 2 \times 2 \dots$ factorial designs. *J. Roy. Statist. Soc. Suppl.* 3, 195-202.
 R. A. FISHER (1940). An examination of the different possible solutions of a problem in incomplete blocks. *Ann. Eugen., Lond.*, 10, 52-75.
 F. YATES (1937). The design and analysis of factorial experiments. *Tech. Commun. Bur. Soil Sci., Harpenden*, no. 35.