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SOME COMBINATORIAL THEOREMS AND ENUMERATIONS CONNECTED WITH THE NUMBERS OF DIAGONAL TYPES OF A LATIN SQUARE

Author's Note (CMS 41.394a) *

The series of theorems in this paper forms a little essay in the art of using the available mathematical apparatus (generating functions, bipartitional functions, etc.) to solve a number of problems of enumeration of some complexity. The author is convinced that more comprehensive methods remain to be developed, but that this will only be accomplished, and the accomplishment understood, by the aid of a study of particular problems, such as those here discussed.

The proofs in Sections 4 and 7 have been made more explicit than in the original, and some arithmetical slips in Table 3 have been corrected.

The problem in Section 6 suggests a very general class of enumeration, which so far as I know has not been discussed, namely: If a design consists of n objects at n loci, given a permutation group for loci, such that designs transformable into one another by any element of the group are regarded as equivalent, to enumerate the designs which can be made of objects corresponding with any given partition of n.

A great many practical problems are of this general type.

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SOME COMBINATORIAL THEOREMS AND ENUMERATIONS CONNECTED WITH THE NUMBERS OF DIAGONAL TYPES OF A LATIN SQUARE

By R. A. FISHER

1. If objects are of a different kinds, each available in unlimited numbers, the number of ways of choosing a set of n objects is

$$\frac{(n+a-1)!}{(a-1)!\,n!}.$$

For, if the selection consists of n_1 of the first kind, n_2 of the second, ..., n_a of the last, to every different selection there corresponds a linear arrangement of symbols, e.g. n noughts and $a-1$ crosses, arranged so that the first cross follows n_1 noughts, the second follows n_2 more, while the sequence ends with a series of n_a. Consequently, the number of ways of arranging such a sequence is the number of ways of making the selection required.

The number enumerated is clearly the coefficient of x^n in the expansion of the generating function

$$(1-x)^{-a}.$$

2. If

$$P = (p_1^{\pi_1} p_2^{\pi_2} \ldots), \quad \Sigma(\pi) = \rho,$$

is a partition of n, the number of ways of choosing n objects so that π_1 types are represented each p_1 times, π_2 types p_2 times, etc., is

$$\frac{a!}{\pi_1!\,\pi_2!\ldots(a-\rho)!},$$

for this is the number of ways of assigning the a kinds of objects to the groups represented a given number of times.

It may be noted that by multinomial expansion

$$(1+x+x^2+x^3+\ldots)^a = \Sigma \frac{a!}{\pi_1!\,\pi_2!\ldots(a-\rho)!} x^{\Sigma(p\pi)},$$

where the summation is taken over all integral values of $\pi_1, \pi_2, \ldots$. Hence

$$\sum_P \frac{a!}{\pi_1!\,\pi_2!\ldots(a-\rho)!}$$

is the coefficient of x^n in the expansion of

$$(1-x)^{-a};$$

this supplies an alternative demonstration of the expression

$$\frac{(n+a-1)!}{(a-1)!\,n!},$$

previously obtained.

3. If a_s is the number of types of weight s, then the number of ways of making a selection of total weight n is the coefficient of x^n in the expansion of

$$(1-x)^{-a_1}(1-x^2)^{-a_2}\ldots.$$

If we write

$$\phi(x) = a_1x + a_2x^2 + a_3x^3 + \ldots,$$

this generating function may be written alternatively in the form

$$\exp\{\phi(x) + \tfrac{1}{2}\phi(x^2) + \tfrac{1}{3}\phi(x^3) + \ldots\}.$$

Author's revised version of Section 4 (CMS 41.395a).

4. If we imagine a variate which takes the value x with frequency a_1, the value x^2 with frequency a_2, and so on, then

$$\phi(x), \quad \phi(x^2), \cdots$$

will be the sums of the first, second, and higher powers of the variate, i.e.,

$$s_1, \quad s_2, \quad s_3, \quad \text{etc.}$$

The product of the variate values of any selection will be x to the power of the weight of the selection. Hence the monomial symmetric function of the variates, $G(P)$, corresponding with any partition P is the generating function the coefficients of the powers of x in which give the frequencies with which different weights occur in selections of type P. But, the bipartitional function $Gs(P, Q)$ is defined to give the coefficients of the expansion of $G(P)$ in sums of powers. That is,

$$G(P) = \sum_Q Gs(P, Q) s_{q_1}^{\chi_1} s_{q_2}^{\chi_2} \cdots,$$

the summation extending over all partitions Q. Hence

$$\sum_Q Gs(P, Q) \phi^{\chi_1}(x^{q_1}) \phi^{\chi_2}(x^{q_2}) \cdots$$

is the generating function for weight of selections of type P, where Q is any partition of the number of variates chosen, defined by

$$Q = (q_1^{\chi_1} q_2^{\chi_2} \cdots).$$

Combinatorily, as Sukhatme (1938) has shown,

$$(-)^{\rho-\sigma} \pi_1! \pi_2! \cdots Gs(P, Q)$$

is the number of ways of making directed circuits q out of undivided parts p.

Since, summing for all partitions (P) of the same partible number

$$\sum_P Gs(P, Q) = \frac{1}{q_1^{\chi_1} q_2^{\chi_2} \cdots \chi_1! \chi_2! \cdots},$$

it follows that the generating function for a given number of objects selected is

$$\sum_Q \frac{\phi^{\chi_1}(x^{q_1}) \phi^{\chi_2}(x^{q_2}) \cdots}{q_1^{\chi_1} q_2^{\chi_2} \cdots \chi_1! \chi_2! \cdots}$$

where Q is any partition of the number of objects.

Thus the generating function for a choice of one object is $\phi(x)$. For two objects we have

$$\tfrac{1}{2}\phi^2(x) + \tfrac{1}{2}\phi(x^2);$$

for three objects

$$\tfrac{1}{6}\phi^3(x) + \tfrac{1}{2}\phi(x^2)\phi(x) + \tfrac{1}{3}\phi(x^3),$$

and so on.

* 4. If we imagine a variate which takes the value x with frequency a_1, the value x^2 with frequency a_2, and so on, then

$$\phi(x), \quad \phi(x^2), \quad \ldots$$

will be the sums of the first, second, and higher powers of all the variates. Consequently, the generating function for the number of ways of choosing objects of partitional type P, and given total weight, must be

$$\Sigma Gs(P,Q)\,\phi^{\chi_1}(x^{q_1})\,\phi^{\chi_2}(x^{q_2})\ldots,$$

where Q stands for the partition of which the partible number is the weight,

$$(q_1^{\chi_1} q_2^{\chi_2}\ldots), \quad \Sigma(\chi) = \sigma,$$

and Gs is the bipartitional function giving the coefficient of the expansion of the monomial symmetric function P in terms of sums of powers (Macmahon, 1915). Combinatorily, as Sukhatme (1938) has shown,

$$(-)^{\rho-\sigma}\pi_1!\,\pi_2!\ldots Gs(P,Q)$$

is the number of ways of making directed circuits q out of undivided parts p.

Since
$$\sum_P Gs(P,Q) = \frac{1}{q_1^{\chi_1} q_2^{\chi_2}\ldots x_1!\,x_2!},$$

it follows that, if we sum for all partitions P of a given partible number, the generating function is

$$\sum_Q \frac{\phi^{\chi_1}(x^{q_1})\,\phi^{\chi_2}(x^{q_2})\ldots}{q_1^{\chi_1} q_2^{\chi_2}\ldots x_1!\,x_2!},$$

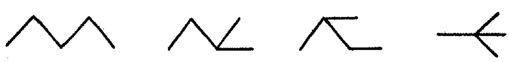
where the summation is taken over all partitions Q of the same partible number.

Thus the generating function for a choice of one object is $\phi(x)$. For two objects we have

$$\tfrac{1}{2}\phi^2(x) + \tfrac{1}{2}\phi(x^2);$$

for three objects
$$\tfrac{1}{6}\phi^3(x) + \tfrac{1}{2}\phi(x^2)\,\phi(x) + \tfrac{1}{3}\phi(x^3),$$
and so on.

5. *Branches*. Let a_s stand for the number of ways of arranging s lines in a connected figure, so that after the first the others may follow in sequence, or branch off in any number from the end of any previous line. Thus, with four lines we should have the arrangements



Then, in making up branches of $(s+1)$ lines we may follow the first by a selection of any number of branches containing a total number s; consequently, the sequence of numbers a_s may be found by equating the coefficients of x^n in the identity

$$\sum_{s=1}^{\infty} a_s x^s = x \prod_{s=1}^{\infty} (1-x^s)^{-a_s},$$

or, symbolically,
$$\log\frac{\phi(x)}{x} = \phi(x) + \tfrac{1}{2}\phi(x^2) + \tfrac{1}{3}\phi(x^3) + \ldots.$$

In arithmetical calculation, when each coefficient up to a_s has been obtained, the generating function, so far as it has been calculated, is multiplied by

$$(1-x^s)^{-a_s},$$

to obtain a new series, including the next coefficient, a_{s+1}, the earlier coefficients being unaltered. Table 1 shows the calculation up to branches of 16 lines, set out in a form from which it may easily be extended to higher values.

* See previous page for revised version of Section 4.

A similar problem arises in the chemistry of carbon compounds, in which, however, the modification is introduced that the number of branches arising at any point is limited to three. The recurrence relationship by which the number of branches may be enumerated, subject to this limitation, is evidently

$$\frac{1}{x}\phi(x) = 1+\phi(x)+\tfrac{1}{2}\phi^2(x)+\tfrac{1}{2}\phi(x^2)+\tfrac{1}{6}\phi^3(x)+\tfrac{1}{2}\phi(x)\phi(x^2)+\tfrac{1}{3}\phi(x^3).$$

Table 2 shows the calculation of the first 16 terms of this series.

Table 1. *Calculation of the number of different branches to be made of n elements*

											Final values $\phi(x)/x$
x^0	1	.	.	.	.	.	.	.	.	.	1
x^1	1	.	.	.	.	.	.	.	.	.	1
x^2	1	2	.	.	.	.	.	.	.	.	2
x^3	1	2	4	.	.	.	.	.	.	.	4
x^4	1	3	5	9	.	.	.	.	.	.	9
x^5	1	3	7	11	20	.	.	.	.	.	20
x^6	1	4	11	19	28	48	.	.	.	.	48
x^7	1	4	13	29	47	67	115	.	.	.	115
x^8	1	5	17	47	83	123	171	286	.	.	286
x^9	1	5	23	61	142	222	318	433	719	.	719
x^{10}	1	6	27	91	235	415	607	837	1123	1842	1842
x^{11}	1	6	33	125	341	741	1173	1633	2205	2924	4766
x^{12}	1	7	42	180	531	1301	2261	3296	4440	5878	12486
x^{13}	1	7	48	230	833	1983	4287	6587	9161	12037	32973
x^{14}	1	8	57	315	1269	3349	7741	13261	18981	25452	87811
x^{15}	1	8	69	411	1890	5570	12650	25875	39603	53983	235381

Table 2. *Number of branches such that no more than four elements meet at any point. Enumeration of alkyl radicals*

Power of x	1	ϕ	$\frac{1}{2}\phi_1^2$	$\frac{1}{2}\phi_2$	$\frac{1}{6}\phi_1^3$	$\frac{1}{2}\phi_1\phi_2$	$\frac{1}{3}\phi_3$	ϕ/x
0	1	.	.	.	.	.	.	1
1	.	1	.	.	.	.	.	1
2	.	1	$\frac{1}{2}$	$\frac{1}{2}$	.	.	.	2
3	.	2	1	.	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	4
4	.	4	$2\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	.	8
5	.	8	6	.	$1\frac{1}{2}$	$1\frac{1}{2}$	.	17
6	.	17	14	1	$4\frac{1}{6}$	$2\frac{1}{2}$	$\frac{1}{3}$	39
7	.	39	33	.	11	6	.	89
8	.	89	80	2	$28\frac{1}{2}$	$11\frac{1}{2}$	.	211
9	.	211	194	.	$73\frac{5}{6}$	$27\frac{1}{2}$	$\frac{2}{3}$	507
10	.	507	478	4	190	59	.	1238
11	.	1238	1188	.	490	141	.	3057
12	.	3057	$2979\frac{1}{2}$	$8\frac{1}{2}$	$1265\frac{2}{3}$	327	$1\frac{1}{3}$	7639
13	.	7639	7528	.	3278	796	.	19241
14	.	19241	$19161\frac{1}{2}$	$19\frac{1}{2}$	$8513\frac{1}{2}$	$1929\frac{1}{2}$	.	48865
15	.	48865	49060	.	$22182\frac{1}{3}$	4796	$2\frac{1}{3}$	124906

6. *The number of ways of arranging the units of a partition, P, in a ring.*

The number of ways of arranging the units of a partition, P, of the partible number t in open sequence is

$$\frac{t!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\dots}.$$

From this it follows that, provided the numbers p have no common factor, the number of ways of arranging them in a closed sequence is

$$\frac{1}{t}\frac{t!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\dots},$$

since each closed sequence corresponds with t open sequences, in this case all different, formed by breaking the ring at t different points.

If, however, the numbers p have any common factor f, the number of ways of forming an open sequence consisting of a repetition of f equivalent parts is

$$N_f = \frac{\frac{t}{f}!}{\left(\frac{p_1}{f}!\right)^{\pi_1}\left(\frac{p_2}{f}!\right)^{\pi_2}\dots},$$

and if p/f have further prime common factors, $F_1, F_2, \dots$, the number of these which will consist of a succession of fF_1 equivalent parts is N_{fF_1}, and of these again the number consisting of fF_1F_2 equivalent parts is $N_{fF_1F_2}$.

Consequently, in the enumeration of the open sequences, consisting of successions of only f equivalent parts,

$$N_{fF_1\dots F_s}$$

will appear with coefficient $(-)^s$; and in the enumeration of closed sequences with only f repetitions, it will appear with coefficient

$$(-)^s\frac{f}{t}.$$

Hence, in the enumeration of all closed sequences, N_f will be involved with coefficient

$$\frac{f}{t}\left(1-\frac{1}{f_1}\right)\left(1-\frac{1}{f_2}\right)\dots,$$

where $f_1, f_2, \dots$ are the prime factors of f. The total number of closed sequences is, therefore,

$$n(P) = \sum_f \frac{\nu(f)}{t}N_f,$$

where the summation is for all common factors of the parts, including unity, and

$$\nu(f) = f\left(1-\frac{1}{f_1}\right)\left(1-\frac{1}{f_2}\right)\dots.$$

For example, with the partition $(24, 12^2)$, the values of N_f for all factors of 12 are:

Table 3. *Form for the enumeration of rings composed of elements of a given partition*

f	1	2	3	4	6	12
N_f	12,457507,283610,994800	2498,640144	900900	18480	420	12
$\nu(f)$	1	1	2	2	2	4
$N_f\,\nu(f)/48$	259531,401741,895725	.52,055003	$37537\frac{1}{2}$	770	$17\frac{1}{2}$	1
With f-fold symmetry	259531,401689,821962	104,109219	56280	1539	51	3

The two last lines both give the total 259531,401793,989054 as the value of $n(P)$, when $P = (24, 12^2)$; the fourth line gives the terms of the formula, which need not individually be integral; the fifth line shows the arrangements divided according to the highest symmetry each shows, i.e.

$$\tfrac{1}{48}(N_1-N_2-N_3+N_6),\quad \tfrac{1}{24}(N_2-N_4-N_6+N_{12}),\quad \tfrac{1}{16}(N_3-N_6),$$

and so on.

7. *Rings of branches*. The number of distinct directed rings of which the elements are branches, chosen so that the total weight is given, may be found by multiplying the number of arrangements for a given partition P, by the generating function for the number of ways of choosing elements of that partition. This gives the generating function,

$$\sum_{f=1}^{\infty} \sum_{P,Q} \frac{\nu(f)}{t} N_f \, Gs(P,Q) \phi_{q_1}^{\chi_1} \phi_{q_2}^{\chi_2} \ldots;$$

but

$$\sum_P N_f(P) Gs(P,Q)$$

is the coefficient of

$$s_{q1}^{\chi_1} s_{q2}^{\chi_2} \ldots$$

in the expansion of

$$(\alpha^f + \beta^f + \gamma^f + \cdots)^{t/f},$$

and this is clearly zero, save when $q = f$, $\chi = t/f$, when it is unity.

Hence,

$$\sum_{f=1}^{\infty} \sum_{P,Q} \frac{\nu(f)}{t} N_f(P) Gs(P,Q) \phi_{q1}^{\chi_1} \phi_{q1}^{\chi_1} \cdots = \sum_{f=1}^{\infty} \frac{\nu(f)}{f} \sum_{t/f=1}^{\infty} \frac{f}{t} \phi^{t/f}(x^f)$$

$$= \sum_{f=1}^{\infty} \frac{\nu(f)}{f} \{-\log(1-\phi(x^f))\}$$

Hence, if

$$\psi(x) = \phi(x) + \tfrac{1}{2}\phi^2(x) + \tfrac{1}{3}\phi^3(x) + \ldots,$$

or, symbolically,

$$-\log(1-\phi),$$

the generating function required is

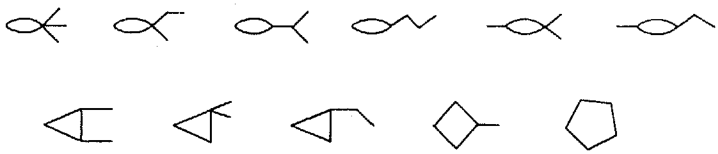
$$\sum_{f=1}^{\infty} \frac{\nu(f)}{f} \psi(x^f),$$

of which the coefficient of x^u will enumerate the number of rings of branches of total weight u. It should be noted that this enumeration includes 'rings' of a single element, i.e. of simple branches. The numbers of proper rings of two or more branches may be obtained by subtraction. The coefficients of $\psi(x)$ and the enumeration of the numbers of 'rings and branches' are shown in Table 4.

Table 4. *Enumeration of rings consisting of branches*

Power of x	ψ_1	$\frac{1}{2}\psi_2$	$\frac{2}{3}\psi_3$							Rings and branches	Proper rings
1	1	.	.	.	.	.	.	.	.	1	0
2	$1\frac{1}{2}$	$\frac{1}{2}$	.	.	.	.	.	.	.	2	1
3	$3\frac{1}{3}$	.	$\frac{2}{3}$	.	.	.	.	.	.	4	2
4	$7\frac{3}{4}$	$\frac{3}{4}$	.	$\frac{1}{2}$	.	.	.	.	.	9	5
5	$19\frac{1}{5}$	.	.	.	$\frac{4}{5}$	.	.	.	.	20	11
6	48	$1\frac{2}{3}$	1	.	.	$\frac{1}{3}$	.	.	.	51	31
7	$124\frac{1}{7}$	.	.	.	.	.	$\frac{6}{7}$	.	.	125	77
8	$323\frac{7}{8}$	$3\frac{7}{8}$	.	$\frac{3}{4}$	.	.	.	$\frac{1}{2}$	.	329	214
9	$859\frac{1}{9}$	.	$2\frac{2}{9}$	.	.	.	.	.	$\frac{2}{3}$	862	576
10	$2299\frac{4}{5}$	$9\frac{3}{5}$	.	.	$1\frac{1}{5}$	.	.	.	$\frac{2}{5}$	2311	1592
11	$6216\frac{1}{11}$	.	.	.	.	.	.	.	$\frac{10}{11}$	6217	4375
12	$16917\frac{1}{3}$	24	$5\frac{1}{6}$	$1\frac{2}{3}$	.	$\frac{1}{2}$	.	.	$\frac{1}{3}$	16949	12183
13	$46349\frac{1}{13}$	.	.	.	.	.	.	.	$\frac{12}{13}$	46350	33864
14	$127650\frac{3}{14}$	$62\frac{1}{4}$	.	.	.	.	$1\frac{2}{7}$	.	$\frac{3}{7}$	127714	94741
15	353256	.	$12\frac{4}{5}$	.	$2\frac{2}{3}$	.	.	.	$\frac{8}{15}$	353272	265461
16	$981585\frac{15}{16}$	$161\frac{15}{16}$	.	$3\frac{7}{8}$	.	.	.	$\frac{3}{4}$	$\frac{1}{2}$	981753	746372

It will be seen, for example, that of weight 5 there are nine possible branches, and twenty possible rings and branches. There must therefore be eleven possible proper rings. These are:



The configurations may also be denoted by giving letters to the termini of the constituent lines in such order as to show which follow which in order round the ring, or from the branches to their base. Thus the eleven configurations figured above may be denoted as follows:

$$..C\left\{\begin{array}{l}D.\\E.\\F.\\B..\end{array}\right. \qquad ..C\left\{\begin{array}{l}DE.\\F.\\B..\end{array}\right. \qquad ..C\left\{\begin{array}{l}D\left\{\begin{array}{l}E.\\F.\end{array}\right.\\B..\end{array}\right. \qquad ..C\left\{\begin{array}{l}DEF.\\B..\end{array}\right. \qquad ..B\left\{\begin{array}{l}C\left\{\begin{array}{l}B..\\E.\\F.\end{array}\right.\\D.\end{array}\right. \qquad ..B\left\{\begin{array}{l}C\left\{\begin{array}{l}B..\\EF.\end{array}\right.\\D.\end{array}\right.$$

$$..C\left\{\begin{array}{l}D.\\E.\left\{\begin{array}{l}F.\\B..\end{array}\right.\end{array}\right. \qquad ..CD\left\{\begin{array}{l}E.\\F.\\B..\end{array}\right. \qquad ..CD\left\{\begin{array}{l}EF.\\B..\end{array}\right. \qquad ..CDE\left\{\begin{array}{l}F.\\B..\end{array}\right. \qquad ..BCDEF..$$

Here any termination is shown by a single stop, recurrence by two dots. For clarity B has been repeated in two cases, but with experience in the notation this is unnecessary. The same configuration may evidently have more than one equivalent formula, but the formula determines the configuration uniquely.

8. *Diagonal types of a Latin square.* In a Latin square written in the standard form, with corner at A, the letters forming the diagonal possess a definite set of relationships, or configuration, which is unchanged by any permutation of the rows, columns and letters which leaves the corner element unchanged, and which does not permute the categories. Such a transformation is called an intramutation.

Taking B to represent any letter used in the square, other than A, we may note that the row and column containing B in the first column and row intersect on the diagonal at an element which has some letter other than B. Consequently, the type of any diagonal of a Latin square may be denoted by a sequence of letters in which B follows whatever letter is thus found on the diagonal. Any A on the diagonal will form the commencement of a terminating branch, but the configuration may consist wholly or partly of rings of branches. The total weight of the configuration found in any diagonal of an $n \times n$ square is evidently $n-1$, and the number of different diagonal types on which a Latin square can possibly be built is the number of ways of selecting elements of total weight $n-1$ from those enumerated in Table 4.

Thus, if b_s stands for the number of different rings or branches of weight s, the coefficient of x^{n-1} in the expansion of

$$(1-x)^{-b_1}(1-x^2)^{-b_2}(1-x^3)^{-b_3}\ldots$$

will be the number of possible diagonal types of an $n \times n$ Latin square.

The enumeration of the number of diagonal types available for squares of different sizes thus proceeds as in Table 1, using the series obtained in Table 4. The generating function for this series is shown in Table 5.

Table 5. *Calculation of the numbers of diagonal types*

Power of x									Diagonal types	Side of square
0	1	.	.	.	.	.	.	.	1	1
1	1	.	.	.	.	.	.	.	1	2
2	1	3	.	.	.	.	.	.	3	3
3	1	3	7	.	.	.	.	.	7	4
4	1	6	10	19	.	.	.	.	19	5
5	1	6	18	27	47	.	.	.	47	6
6	1	10	32	59	79	130	.	.	130	7
7	1	10	44	107	167	218	343	.	343	8
8	1	15	69	204	344	497	622	951	951	9
9	1	15	105	312	692	1049	1424	1753	2615	10
10	1	21	141	564	1314	2283	3158	4145	7318	11
11	1	21	201	912	2302	4699	7074	9377	20491	12
12	1	28	283	1519	4289	9644	15519	21770	57903	13
13	1	28	367	2287	7837	17680	33930	49393	163898	14
14	1	36	495	3699	13929	35451	70576	113346	466199	15
15	1	36	659	5603	24093	68667	138667	251514	1,328993	16
16	1	45	833	8630	40800	133008	287758	546681	3,799624	17

Summary

The note contains a sequence of theorems in combinatorial analysis connected with the numbers of selections which can be made from objects of given numbers of types.

Enumerations are given of

(*a*) the number of different branches to be formed of n elements;

(*b*) the related problem of the enumeration of alkyl radicals;

(*c*) the number of rings consisting of branches; and

(*d*) the numbers of diagonal types of Latin squares up to seventeen elements in the side.

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