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A SYSTEM OF CONFOUNDING FOR FACTORS WITH MORE THAN TWO ALTERNATIVES, GIVING COMPLETELY ORTHOGONAL CUBES AND HIGHER POWERS *

* See Author's Note, Paper 189.

1. Introduction

In 1942 I called attention to the connexion between the theory of Abelian groups and the relations recognizable in the choice of interactions for confounding, or of treatment-combinations for use in the same block, when it is required to subdivide a complete replication into blocks without loss of information save on unimportant interactions.

The theory was developed in terms of factors having only two alternatives. This is, of course, the case of the greatest importance in practice, though factors with three alternatives also must frequently be used. The basic position developed so far is as follows:

(i) If there is no partial confounding, the interactions confounded, one less in number than the blocks, form with the identity a subgroup.

(ii) Any operator transforming one treatment to another treatment in the same block is orthogonal to the entire subgroup confounded, and these operators constitute the entire subgroup orthogonal to it. This is called the intrablock subgroup. It supplies the contents of the block containing the 'control', and, by multiplication, that containing any chosen treatment. The form of confounding adopted is often most concisely specified by means of the intrablock subgroup, or of its generators.

On the basis of these ideas it was possible to demonstrate the remarkable proposition that: Using blocks of 2^r plots, it is possible to test all combinations of so many as (2^r-1) *factors*, in such a way that all interactions confounded ($2^{2^r-1-r}-1$ in number) shall involve not less than three factors each.

(iii) Methods of confounding using fewer factors than the maximum possible may be found by eliminating unwanted factors. In the intrablock subgroup the symbol of any factor to be eliminated is simply deleted from any combination in which it occurs. In the subgroup confounded the whole combination containing any such symbol is deleted. Practical applications have been illustrated in the third edition of the *Design of Experiments*.

In my previous paper it was pointed out that, since, for example, blocks of eight would suffice for seven factors of two alternatives each, it follows that blocks of 27 must suffice for seven factors of three alternatives each. Indeed, several different arrangements suggested themselves. In the present note it will appear that blocks of 27 are large enough to accommodate a selection of treatment combinations out of 3^{13}, without confounding any interaction of less than three factors. I have not, however, listed the various arrangements using 5–12 factors obtainable by deletion.

2. Factors with p alternatives, where p is a prime

To arrange the combinations of $$(p^n-1)/(p-1)$$ factors, each of p alternatives, p being a prime number, in blocks of p^n units each, establish arbitrarily a 1 : 1 correspondence between the factors, or the letters representing them, each with one subgroup of order p of a group of order p^n, and with a particular element of that subgroup. E.g. taking $p = 3$, $n = 3$, we may use 13 factors $A, \ldots, M$ related as follows:

Table 1. *Association of thirteen factors with independent elements of an Abelian group*

A	α	C	$\alpha\beta$	E	γ	G	$\alpha\gamma^2$	I	$\beta\gamma^2$	K	$\alpha^2\beta\gamma$	M	$\alpha\beta\gamma^2$
B	β	D	$\alpha\beta^2$	F	$\alpha\gamma$	H	$\beta\gamma$	J	$\alpha\beta\gamma$	L	$\alpha\beta^2\gamma$		

Let S be any element of the group and X any element to be considered in relation to S. The sum of the products of the powers of $\alpha, \beta, \gamma, \ldots$ in the expressions in Greek letters of S and X, on dividing by p, leaves a remainder i equal to $0, 1, 2, \ldots$, or $(p-1)$. The combination of Latin letters involving each X to the corresponding power i shall be chosen as the intrablock interaction (or treatment in the control block) corresponding with S.

(i) The combination corresponding with the product of the Greek representations of S and S' is the interaction of the combinations corresponding with S and S'. For the index of any letter X, in relation to the product SS', will, if obtained as explained above, differ by zero or by a multiple of p from the sum of the indices in relation to S and S'; and this is sufficient to demonstrate the proposition.

(ii) The combinations chosen by this method form a subgroup of Latin letters, isomorphic with the entire group formed of Greek letters. This shall be used as the intrablock subgroup.

(iii) If u is the power of X in any interaction of Latin letters orthogonal to the combination corresponding with S, then $\Sigma(iu)$ is zero or a multiple of p.

(iv) If S_α is the element of which the Greek representation is α, then for any Latin letter X, the power i is the power to which α is raised in the representation of X. Hence the power of α in the Greek product of the interaction in which X is raised to the power of u is $\Sigma(iu)$, and the necessary and sufficient condition that the selection is orthogonal to S_α is that this Greek product does not involve α.

(v) Any interaction orthogonal to all the combinations corresponding with S_α, S_β, ... is such that the product of the Greek representations of its terms reduces to the identity. Such interactions are orthogonal to the entire intrablock subgroup and constitute the subgroup confounded.

(vi) The subgroup confounded contains interactions of not less than three factors, for no one element or product of two elements belonging to different cycles can reduce to the identity.

Applying this process to the problem of dividing 3^{13} different treatments in blocks of 27, so that the $3^{10}-1$ interactions confounded all involve at least three factors, we construct the following table, based on the arbitrary correspondences set out above:

Factor	Representative element of Greek group	Corresponding element of intrablock subgroup
A	α	$A^1B^0C^1D^1E^0F^1G^1H^0I^0J^1K^2L^1M^1 = ACDFGJK^2LM$
B	β	$A^0B^1C^1D^2E^0F^0G^0H^1I^1J^1K^1L^2M^1 = BCD^2HIJKL^2M$
E	γ	$A^0B^0C^0D^0E^1F^1G^2H^1I^2J^1K^1L^1M^2 = EFG^2HI^2JKLM^2$

From these three elements the intrablock subgroup may be generated without reference to the arbitrary correspondence established with the Greek group. The thirteen cycles are set out in full in §3 (Table 3).

Any interaction confounded, such as ABC^2, is orthogonal to these three, and therefore to the whole subgroup generated from them. There are $\frac{1}{6}(26.24)$ confounded interactions of three factors; that is, 104 interactions in 52 cycles.

Similarly, with blocks of 49, we may use 8 factors with 7 alternatives each, setting up the correspondence

$$A \ \alpha \qquad B \ \beta \qquad C \ \alpha\beta \qquad D \ \alpha\beta^2 \qquad E \ \alpha\beta^3 \qquad F \ \alpha\beta^4 \qquad G \ \alpha\beta^5 \qquad H \ \alpha\beta^6$$

Then the element of the intrablock subgroup corresponding with α will be

$$ACDEFGH,$$

and that corresponding with β will be

$$BCD^2E^3F^4G^5H^6$$

from which the intrablock subgroup, consisting of the identity and eight cycles of 6, may be generated. Interactions of three factors which are confounded, such as ABC^6, number $\frac{1}{6}(48.42) = 336$, in 56 cycles. In all, $7^6 - 1 = 117{,}648$ interactions are confounded in 19,608 cycles.

3. The orthogonal cube of 3

The system of confounding for 13 factors, each of three levels, in blocks of 27 is equivalent to the solution of the problem of an orthogonal cube of side 3 in 10 alphabets. If the first coordinate of the position of a point in this cube takes the values 0, 1 and 2, according to the power of A in any element of the intrablock subgroup, the second co-ordinate being likewise determined by the power of B, and the third by that of E, the power of any one of the remaining 10 letters specifies the values, 0, 1 or 2, of 10 distinct entries corresponding to each point of the cube. These are set out in three squares, representing successive layers, as follows:

Table 2

0 0 0 0 0 0 0 0 0 0	1 1 1 1 0 0 1 2 1 1	2 2 2 2 0 0 2 1 2 2
1 2 0 0 1 1 1 1 2 1	2 0 1 1 1 1 2 0 0 2	0 1 2 2 1 1 0 2 1 0
2 1 0 0 2 2 2 2 1 2	0 2 1 1 2 2 0 1 2 0	1 0 2 2 2 2 1 0 0 1

0 0 1 2 1 2 1 1 1 2	1 1 2 0 1 2 2 0 2 0	2 2 0 1 1 2 0 2 0 1
1 2 1 2 2 0 2 2 0 0	2 0 2 0 2 0 0 1 1 1	0 1 0 1 2 0 1 0 2 2
2 1 1 2 0 1 0 0 2 1	0 2 2 0 0 1 1 2 0 2	1 0 0 1 0 1 2 1 1 0

0 0 2 1 2 1 2 2 2 1	1 1 0 2 2 1 0 1 0 2	2 2 1 0 2 1 1 0 1 0
1 2 2 1 0 2 0 0 1 2	2 0 0 2 0 2 1 2 2 0	0 1 1 0 0 2 2 1 0 1
2 1 2 1 1 0 1 1 0 0	0 2 0 2 1 0 2 0 1 1	1 0 1 0 1 0 0 2 2 2

Corresponding with rows and columns of a Latin or Graeco-Latin square, the cube consists of plane strata of 9 points each, which may be referred to as rows, columns and layers respectively, for the sake of expressing its properties in language analogous to that used for the squares. In every one of the 10 sites, each number occurs thrice in every row, column or layer, and thrice with each possible number at each of the remaining 9 sites.

In other words we may say that each of 13 principles of classification is made to divide 27 objects into 3 sets of 9, in each of which all the classes of every other principle of classification are equally represented.

It should be noted that sets of three objects alike in any two respects are alike also in two other characters, but differ in the remaining nine.

Since the orthogonal cube of 3 affords 13 ways of dividing 27 objects into 3 groups of 9 each in such a way that every two objects fall 4 times into the same class, it provides at once a solution of the problem in incomplete blocks of arranging 27 varieties in blocks of 9, using 13 replications and 39 blocks.

Table 3. *Intrablock subgroup for blocks of* 27; *the identity with* 13 *cycles of* 2

$ACDFGJK^2LM$
BCD^2HIJKL^2M
$ABC^2FGHIJ^2M^2$
$AB^2D^2FGH^2I^2KL^2$
$EFG^2HI^2JKLM^2$
$ACDEF^2HI^2J^2L^2$
$ACDE^2G^2H^2IKM^2$
$BCD^2EFG^2H^2J^2K^2$
$BCD^2E^2F^2GI^2LM^2$
$ABC^2EF^2H^2KLM$
$A^2BDEGH^2JL^2M^2$
$AB^2D^2EF^2IJK^2M^2$
$ABC^2E^2G^2I^2JK^2L^2$

Table 4. 40 *treatments in* 40 *blocks of* 13

C	b	e	h	i	o	o′	r	r′	u	u′	v	v′
b	e	h	i	n	p	q	s	t	w	x′	y	z
b	e	h	i	n′	p′	q′	s′	t′	w′	x	y′	z′
C	a	e	f	g	n	n′	r	r′	s	s′	t	t′
a	e	f	g	o	p	q′	u	v	w	x	y′	z
a	e	f	g	o′	p′	q	u′	v′	w′	x′	y	z′
C	a	b	c	d	n	n′	o	o′	p	p′	q	q′
a	b	c	d	r	s	t′	u	v′	w	x	y	z′
a	b	c	d	r′	s′	t	u′	v	w′	x′	y′	z
C	a	h	j	k	n	n′	u	u′	w	w′	x	x′
a	h	j	k	o	p	q′	r′	s′	t	v′	y	z′
a	h	j	k	o′	p′	q	r	s	t′	v	y′	z
C	a	i	l	m	n	n′	v	v′	y	y′	z	z′
a	i	l	m	o	p	q′	r	s	t′	u′	w′	x′
a	i	l	m	o′	p′	q	r′	s′	t	u	w	x
C	b	f	j	l	o	o′	s	s′	w	w′	y	y′
b	f	j	l	n	p	q	r′	t′	u′	v	x	z′
b	f	j	l	n′	p′	q′	r	t	u	v′	x′	z
C	b	g	k	m	o	o′	t	t′	x	x′	z	z′
b	g	k	m	n	p	q	r	s′	u	v′	w′	y′
b	g	k	m	n′	p′	q′	r′	s	u′	v	w	y

C	c	e	j	m	p	p′	r	r′	w	w′	z	z′
c	e	j	m	n	o′	q′	s	t	u′	v′	x	y′
c	e	j	m	n′	o	q	s′	t′	u	v	x′	y
C	c	f	i	k	p	p′	s	s′	v	v′	x	x′
c	f	i	k	n′	o	q	r	t	u′	w	y′	z′
c	f	i	k	n	o′	q′	r′	t′	u	w′	y	z
C	c	g	h	l	p	p′	t	t′	u	u′	y	y′
c	g	h	l	n	o′	q′	r	s′	v	w	x′	z′
c	g	h	l	n′	o	q	r′	s	v′	w′	x	z
a	b	c	d	e	f	g	h	i	j	k	l	m
C	d	e	k	l	q	q′	r	r′	x	x′	y	y′
d	e	k	l	n	o	p′	s	t	u	v	w′	z′
d	e	k	l	n′	o′	p	s′	t′	u′	v′	w	z
C	d	f	h	m	q	q′	s	s′	u	u′	z	z′
d	f	h	m	n	o	p′	r′	t′	v′	w	x′	y′
d	f	h	m	n′	o′	p	r	t	v	w′	x	y
C	d	g	i	j	q	q′	t	t′	v	v′	w	w′
d	g	i	j	n	o	p′	r	s′	u′	x	y	z
d	g	i	j	n′	o′	p	r′	s	u	x′	y′	z′

The sets of 3 also, into which the 27 objects are divided by a double classification, supply a solution of the problem of testing 27 varieties in 13 replications in 117 blocks of 3 each, since sets of 4 principles of classification, any pair of which make the same subdivision into 9 threes, can be chosen in 13 ways. These sets of 4 correspond 1 to 1 with the 13 primary principles of classification, the relation being that between any element of the Abelian group to which the factors correspond and the cycles of the subgroup orthogonal to that element; thus, to the single principle of classification A there corresponds the set of four cycles, $BEHI$, with B there corresponds $AEFG$, and so on.

In addition to these two solutions, there are two more randomized block solutions to be derived from the orthogonal cube. If we designate the elements of the intrablock subgroup set out above

Table 5. 40 *treatments in* 130 *blocks of* 4

C a n n′	C b o o′	C c p p′	C d q q′	C e r r′	C f s s′	C g t t′
a o p q′	b n p q	c n o′ q′	d n o p′	e n s t	f o w y′	g o x z
a o′ p′ q	b n′ p′ q′	c n′ o q	d n′ o′ p	e n′ s′ t′	f o′ w′ y	g o′ x′ z′
a r s t′	b r u v′	c r w z′	d r x y	e o u v	f n r′ t′	g n r s′
a r′ s′ t	b r′ u′ v	c r′ w′ z	d r′ x′ y′	e o′ u′ v′	f n′ r t	g n′ r′ s
a u w x	b s w y	c s v′ x	d s u z′	e p w z	f p v x	g p u y′
a u′ w′ x′	b s′ w′ y′	c s′ v x′	d s′ u′ z	e p′ w′ z′	f p′ v′ x′	g p′ u′ y
a v′ y z′	b t x′ z	c t u′ y′	d t v w′	e g x′ y	f q′ u z	g q′ v w
a v y′ z	b t′ x z′	c t′ u y	d t′ v′ w	e g′ x y′	f q u′ z′	g q v′ w′
b e h i	a e f g	d e k l	c e j m	a b c d	b g k m	b f j l

a i l m	a h j k	d g i j	c f i k	c g h l	d f h m
C h u u′	C i v v′	C j w w′	C k x x′	C l y y′	C m z z′
h n w x′	i n y z	j o′ s y′	k o t z′	l o s w′	m o′ t x
h n′ w′ x	i n′ y′ z′	j o s′ y	k o′ t′ z	l o′ s′ w	m o t′ x′
h o r′ v′	i o r u′	j p r′ z′	k q r y′	l q r′ x	m p r w′
h o′ r v	i o′ r′ u	j p′ r z	k q′ r′ y	l q′ r x′	m p′ r′ w
h p t y	i p s x′	j n u′ x	k n u w′	l n v z′	m n v′ y′
h p′ t′ y′	i p′ s′ x	j n′ u x′	k n′ u′ w	l n′ v′ z	m n′ v y
h q s z	i q t w	j q′ t v′	k p′ s v	l p′ t u	m q′ s u′
h q′ s′ z′	i q′ t′ w′	j q t′ v	k p s′ v′	l p t′ u′	m q s′ u

Table 6. *Incomplete blocks of* 11–13 *replications, existing or possibly existing*

r	v	b	k	λ	
11	23	23	11	5	
11	45	99	5	1	
11	45	55	9	2	
11	56	56	11	1	
11	100	110	10	1	o.s. } ?
11	111	111	11	1	o.s.
12	19	57	4	2	
12	22	33	8	4	
12	25	100	3	1	
12	33	44	9	3	
12	37	111	4	1	
12	61	122	6	1	
12	67	67	12	2	
12	121	132	11	1	o.s.
12	133	133	12	1	o.s.
13	27	117	3	1	o.c.
13	27	39	9	4	o.c.
13	27	27	13	6	
13	40	130	4	1	o.c.
13	40	52	10	3	
13	40	40	13	4	o.c.
13	53	53	13	3	
13	66	143	6	1	
13	66	78	11	2	
13	79	79	13	2	
13	144	156	12	1	o.s. } ?
13	157	157	13	1	o.s.

(Table 3) by the letters n to z, and their squares by the letters n' to z', the identity being represented by C, while the 13 first letters of the alphabet stand for the 13 principles of classification in accordance with which these 27 have been subdivided, we may associate each group of 9 with the four letters of the subgroup orthogonal to the principle of classification from which it was derived, thus making 39 blocks of 13 letters each, which with a fortieth block consisting of the first 13 letters of the alphabet themselves, involves each of the letters 13 times and every pair 4 times in association, thus providing 40 treatments in 40 blocks of 13 with 13 replications.

Equally, each set of 3 obtained by a double subdivision may be associated with the symbol for the principle of classification orthogonal to the two already used, to make 117 blocks of 4 with which are associated 13 more blocks of 4, representing the cycles of the subgroups orthogonal to each classification used. In this case, 40 treatments are subdivided in 130 blocks of 4 in 13 replications such that every 2 treatments occur once in the same block. Table 6 shows these 4 solutions derivable from the orthogonal cube of 3 in a list of the possible incomplete block solutions with 11–13 replications. The two pairs of solutions related to orthogonal squares of sides 10 to 12 have been included as fulfilling the arithmetical condition, although presumably no combinatorial solutions exist for them.

4. Factors having a number of levels which is a power of a prime

In § 2 it was shown that, using factors having any prime number p of different levels, blocks of p^s units will suffice for use with

$$(p^s-1)/(p-1)$$

different factors, without any interaction of less than three factors being confounded.

This proposition may be extended with full generality to factors having a number of variants, p^r, which is a power of a prime. Using the fact that a field of p^r symbols can be constructed unambiguously subject to arithmetical operations, we shall show that blocks of p^{rs} plots suffice for use with

$$(p^{rs}-1)/(p^r-1)$$

factors, each at p^r levels.

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be s field variables each taking p^r values, then out of p^{rs} combinations one and only one has all values zero. Let

$$\sum_{j=1}^{s} a_j \lambda_j$$

be any linear function of the variables λ, such that not all the coefficients a are zero. Then the number of sets of coefficients which may possibly be chosen is $p^{rs}-1$.

Since, however,
$$u\sum_{j=1}^{s} a_j\lambda_j = \sum_{j=1}^{s} \alpha_j\lambda_j,$$

where $\alpha_j = ua_j$, it appears that each possible linear function is related by simple multiplication with p^r-1 others, of which one is zero; or, in other words, belongs to a set of p^r-1 non-zero functions.

There are, therefore, $(p^{rs}-1)/(p^r-1)$ different sets, and these are associated as indices with an equal number of letters, or, otherwise stated, each is used to specify the level of application of an equal number of experimental factors, in the $p^{rs}-1$ different treatment combinations occurring in the same block with the control.

To show that such an intrablock subgroup will confound no interaction of less than three factors; consider that if

$$A^{i_\alpha} B^{i_\beta} \dots$$

specify any interaction, this interaction will be confounded if, and only if

$$S\{i_\alpha \Sigma a_j \lambda_j\} = 0,$$

where S stands for summation over all letters, for all combinations of λ. This will be true when, and only when,

$$S(i_\alpha a_j) = 0, \quad \text{for all } j \text{ from } 1 \text{ to } s.$$

If i could be zero for all letters save two, take them to be A and B; then for these two letters

$$b_j = u a_j, \quad \text{for all } j,$$

or, b and a would belong to the same set, contrary to the construction. Hence no interaction of less than three factors has been confounded.

5. Solution using 64 plots to a block, and 21 factors at 4 levels each

The abstract process of § 4 may be illustrated by the problem of the cube of 4 in 18 alphabets; or, in other words, the subdivision of the 63 comparisons among 64 objects into 21 orthogonal sets of 3, each being comparisons among 4 lots of 16 into which the whole may be divided.

The rules for addition and multiplication of the four field symbols, which will be written 0, 1, p, q, are shown below.

Addition table

	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

Multiplication table

	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	q	1
q	0	q	1	p

The 64 combinations of three field values consist of one in which they are all zero, and 21 sets of 3 such that one member of each set is a simple multiple of the others. These sets are:

21 *sets of coefficients*

A	1	0	0	H	1	0	p	O	1	1	q
B	0	1	0	I	1	0	q	P	1	p	1
C	0	0	1	J	1	1	0	Q	1	p	p
D	0	1	1	K	1	p	0	R	1	p	q
E	0	1	p	L	1	q	0	S	1	q	1
F	0	1	q	M	1	1	1	T	1	q	p
G	1	0	1	N	1	1	p	U	1	q	q

We now take any one of these and find the sum of the products of the coefficients with each of the 21 in turn, to find the index of the corresponding letter in the corresponding treatment in the block containing the 'control'. Instead of a^1, a^p, a^q, I shall now write a, a', a''. We find thus the

21 treatment combinations, each representing a set of 3 which, with the control, occupy a single block:

$(aghijklmnopqrstu)$
$(bdefjk'l''mnop'q'r's''t''u'')$
$(cde'f''gh'i''mn'o''pq'r''st'u'')$
$(bce''f'gh'i''jk'l''n''o'p''rs't)$
$(bc'd''e'g'h''ijk''l''m''n'qr''su')$
$(bc''d'f''g''hi'jkl''m'o''pq''t'u)$
$(acde'f''h''i'jkln''o'q''r't''u')$
$(ac'd'e''fg''h'jklm''n'p''q's''t')$
$(ac''d''ef'g'i''jklm'o''p'r''s'u'')$
$(abdefghik''l'p''q''r''s't'u')$
$(ab'd'e'f'ghij''k'm''n''o''p'q'r')$
$(ab''d''e''f''ghij'l''m'n'o's''t''u'')$
$(abce''f'h''i'k''l'mn'o''p'qs''u)$
$(abc'd''e'g''h'k''l'm'n''opr'tu'')$
$(abc''d'f''g'i''k''l'm''no'q'rst'')$
$(ab'cd''fh''i'j''k'm'np''rst'u'')$
$(ab'c'ef''g''h'j''k'mo'qr''s't''u)$
$(ab'c''de''g'i''j''k'n'opq''s''tu')$
$(ab''cd'eh''i'j'l''m''opq'r''s't)$
$(ab''c'df'g''h'j'l''no''p'q''rsu')$
$(ab''c''e'fg'i''j'l''mn''p''qr't'u)$

These may alternatively be generated from the first three; thus the simple interaction of the first two gives the tenth as written above; the other interactions of these two give the eleventh and twelfth.

The completely orthogonal cube of 4 may then be constructed directly by taking the four phases of any factor, such as A, to specify position in one direction, B for a second and C for the third, and indicating the phase of the 18 following letters by level designations such as 0, 1, 2, 3 in the 18 cells assigned to each of the 64 points of the cube. Such a cube could, of course, be used to generate four distinct incomplete block solutions, these all having 21 replications.

Summary

The system of confounding a number of factors each of only two alternatives, developed in the previous paper, is here extended (i) to factors having any prime number of alternatives, and (ii) to the case in which the number of alternatives is any power of a prime.

In the first case the factors may be chosen to correspond with subgroups of order p of an Abelian group of order p^n, equal to the number of plots in each block. In the second, each factor corresponds with a combination of s values, not all zero, each taking the p^r values of the field, this being the number of levels for each factor. Any selection which is a simple multiple of a second belongs to the same factor; thus $(p^{rs}-1)/(p^r-1)$ different factors may be used, without confounding any interaction of less than three factors.

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