Singularity Structure of Scalar Field Cosmologies

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Abstract

The classical dynamical structure of cosmological models in which the matter content of the universe consists of a scalar field with arbitrary non-negative potential is analyzed in detail. Emphasis is placed on the global features of solutions including the existence of an initial space-time singularity and particle horizons.

A natural class of potentials is defined which includes as an elementary subset all non-negative polynomial potentials. For the special case where the space-time metric is of Bianchi type I it is shown that almost all solutions of these models possess initial space-time singularities and particle horizons. Close to the singularity the dynamics is essentially independent of the form of the potential and the space of solutions may be identified with that of the exactly integrable model for which potential is identically zero. Furthermore, it is shown that there exist at least two solutions for which no particle horizons exist. These special solutions are isotropic and are exponential attractors in solution space. They are associated with the existence of inflationary behavior in the system, which they completely characterize.

These results, which are consistent with earlier work done for particular potentials, demonstrate the robustness of scalar field cosmological models and help to clarify the significance of the large number of horizon free and singularity free exact solutions.

Implications to more general space-times are considered making use of a form of long wavelength approximation to the Einstein Field Equations and an attempt is made to construct a general asymptotic solution. This is shown to be valid for general spatially homogeneous space-times subject to a generic condition on the spatial curvature.

In addition, an interesting class of exponential potentials which do not fall into the above mentioned class are investigated and their behavior compared with that above. These solutions possess an oscillatory behavior near an initial space-time singularity which seems to be associated with the inability of the scalar field to negotiate the walls of extremely steep potentials.

Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan or photocopying.

SIGNED: DATE: 5/3/96

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Contents

Chapter 1

Scalar Field Cosmologies

1.1 Introduction

This thesis concerns itself with the properties of scalar field cosmological models. By a scalar field cosmological model we shall mean, more precisely, any theory determined by variation of the action

$$S[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}(R + \mathcal{L}_\phi) \tag{1.1}$$

where R is the Ricci scalar, g is the determinate of the metric $g_{\mu\nu}$ and $\mathcal{L}_\phi$ is the covariant form of the classical scalar field Lagrangian;

$$\mathcal{L}_\phi = \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi). \tag{1.2}$$

V is an arbitrary non-negative potential. (Note that we shall always work with units $c = 8\pi G = 1$ and adopt the metric signature $(- + + +)$.)

Any cosmological solution $(g_{\mu\nu}, \phi)$ on a space-time manifold M for which the functional derivative of the action (1.1) vanishes will be called a scalar field cosmology.

The choice of potential V represents the only freedom in the specification of the theory so V may be understood as characterizing the model. Frequently studied examples include the massless scalar field ($V(\phi) = 0$), the massive scalar field ($V(\phi) = \frac{1}{2}m^2\phi^2$) and the Higgs Field ($V(\phi) = A(\lambda\phi^2 - \Lambda)^2$). Theories such as these arise naturally in particle physics and have attracted interest in cosmology, primarily, as a mechanism for the inflationary universe scenario. (The role of scalar fields in inflation will be discussed in Section 1.5.)

Cosmologies generated by (1.1) are often referred to as minimally coupled scalar field cosmologies indicating that (1.1) is, essentially, the simplest action that could be chosen to model the interaction of a scalar field with gravity.

There do, however, exist physically interesting theories which include a non-minimal coupling between the scalar field and gravity. One could therefore consider the more general action

$$S[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}(f(\phi)R + \mathcal{L}_\phi) \tag{1.3}$$

where f is a smooth function of ϕ. However, it can be shown that any theory for which $f(\phi) \neq 0$ for all ϕ may be obtained by a conformal transformation $g \mapsto \lambda(\phi)^2 g$ of the metric and a smooth transformation $\phi \mapsto \tilde{\phi}(\phi)$ of the scalar field from some unique minimally coupled model V. Examples of such transformations appear in the literature for a number of theories including conventional Brans-Dicke type theories [2, 1, 3], Brans-Dicke theory with torsion [4] and scalar field theories with non-negative non-minimal coupling term $|\xi|\phi^2 R$ [5]. Since the field equations obtained from (1.1) are generally much simpler than those of (1.3), the analysis of these solutions is a natural starting point when considering more general scalar-tensor theories. In particular, the causal structure of solutions of (1.1) will correspond to those of (1.3) so the results obtained below concerning the existence of particle horizons will be applicable to a far greater class of models than those determined by (1.1).

Even models for which $f(\phi)$ vanishes on a countable set of points such as the conformally coupled scalar field [7] and its generalization discussed by Madsen [6] ($f(\phi) = 1 - \xi\phi^2$ with ξ a constant) can be shown to be locally conformally equivalent to (1.1) for some V (see [8] for the special case $\xi = 1/6$). Thus the local behavior of solutions for these theories is completely determined by the solutions generated by (1.1). For the theory discussed by Madsen, at least, it is possible to obtain global solutions by stitching together solutions on the domains $f > 0$ and $f < 0$ obtained via a conformal transformation [9].

Effective scalar field theories generated by an action of the form (1.1) also arise naturally in the context of a number of alternative theories of gravity including higher order gravity theories [10, 11] and dimensionally reduced

Kaluza Klein theories [12, 13]. For example, it was shown by Barrow and Cotsakis [11], generalising an earlier result by Whitt [10], that any higher order gravity theory with action of the form

$$S[g_{\mu\nu}] = \int d^4x\sqrt{-g}f(R) \tag{1.4}$$

where $f(R)$ is an analytic function, may be derived from an effecive theory with action (1.1) via a conformal transformation of the metric related implicitely to the effective scalar field.

Higher order modifications to general relativity, of which (1.4) and (1.3) provide examples, are likely to become important at high energies. Scalar field cosmologies may therefore provide a fundamental insight into the behavior of gravity and its interactions with matter at very high energies. The fact that they provide a mechanism for generating inflation in the early universe provides evidence of this.

In accordance with this view, scalar field cosmologies should be a natural choice of matter field for studying the global structures of space-time such as singularities and particle horizons. It is therefore of interest that certain conditions on the stress energy tensor which are essential for the proof of the famous singularity theorems of Hawking and Ellis [15] are not obeyed by scalar fields.

The exploration of the global properties of scalar field cosmologies, particularly the past asymptotic behavior, is the primary concern of this thesis.

The asymptotic behavior of scalar field cosmologies has received relatively little attention in the literature. This may be partly due to the view amongst cosmologists that close to an initial singularity general relativity is no longer valid and a quantum description of gravity is appropriate. However, in the absence of a satisfactory quantum theory the classical description which has been so useful in understanding inflation seems a good starting point. Classical behavior invariably provides important insights into a quantum system and the quantum theories which have generally been most successful are those for which the classical dynamics are well understood.

Furthermore, the relevant time scales for inflationary cosmology are generally several orders of magnitude greater than the Plank time, which is the natural time scale in quantum gravity. From the point of view of inflation the (full blown) quantum era can probably be treated as being, for all intensive purposes, at the initial singularity. The singularity structure of classical scalar field cosmologies may therefore be helpful in understanding certain aspects of pre-inflationary physics, particularly the outstanding problem of initial conditions [22]. In fact, we shall see examples where an initial singularity does not even exist so the quantum era may never be reached.

In any case, the discussion below shall be entirely classical. It is hoped that it provides some insight into general relativistic dynamics and the evolution of our universe.

As a final comment before proceeding I note that a natural (and perhaps more realistic) generalization of (1.1) would be to consider a system of N coupled scalar fields or a complex scalar field. This has not been done because it would make some technical details of the analysis rather more complicated without, in my opinion, significantly altering the essential features of the theory, at least with regard to asymptotic behavior. For the same reason I have not included a perfect fluid term in the Lagrangian which might be included in a more realistic theory to model the interaction of the scalar field with more conventional matter.

1.2 The Field Equations

Let us now proceed with the task of developing the theory of scalar field cosmologies generated by (1.1). Variation of the action with respect to $g_{\mu\nu}$ yields the Einstein Field Equations (EFE);

$$G_{\mu\nu} = T_{\mu\nu} \tag{1.5}$$

where $G_{\mu\nu}$ is the Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \tag{1.6}$$

and $T_{\mu\nu}$ is the stress-energy tensor for a scalar field

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}_\phi . \tag{1.7}$$

If we take the trace of (1.5) we find;

$$T \equiv T^\mu_\mu = -R. \tag{1.8}$$

The EFE may thus be expressed in alternative form

$$R_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}. \tag{1.9}$$

Variation of the action with respect to ϕ yields the additional field equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = V'(\phi) \tag{1.10}$$

where ∇_μ is the covariant derivative operator (Levi Civita connection). The prime, here and throughout this thesis, denotes differentiation with respect to ϕ. Note that equation (1.10) can alternatively be derived from the contracted Bianchi Identities which imply that $\nabla_\nu T^{\mu\nu} = 0$.

1.3 Hydrodynamics of the Scalar Field

1.3.1 The Stress Energy Tensor

Consider a unit timelike vector field U^μ representing a family of observers in M (or a *frame of reference*). We can decompose the gradient vector $\phi^\mu = g^{\mu\nu}\partial_\nu\phi$ into components tangential and orthogonal to U^μ so that we have

$$\phi^\mu = \alpha U^\mu + N^\mu \tag{1.11}$$

where

$$\alpha = -\phi^\mu U_\mu$$

and

$$N_\mu U^\mu = 0. \tag{1.12}$$

The stress-energy tensor may now be written

$$\begin{aligned} T_{\mu\nu} &= \alpha^2 U_\mu U_\nu + 2\alpha N_{(\mu}U_{\nu)} + N_\mu N_\nu - \mathcal{L}_\phi g_{\mu\nu} \\ &= (\alpha^2 + \mathcal{L}_\phi)U_\mu U_\nu + 2\alpha N_{(\mu}U_{\nu)} - \mathcal{L}_\phi h_{\mu\nu} + N_\mu N_\nu \end{aligned} \tag{1.13}$$

where the curved parentheses around the tensor indices denote symmetrisation (square brackets will indicate anti-symmetrisation) and

$$h_{\mu\nu} = U_\mu U_\nu + g_{\mu\nu} \tag{1.14}$$

is the usual projection tensor onto the local orthogonal subspace of U^μ and satisfies

$$h_{\mu\nu}U^\nu = h_{\mu\nu}U^\mu = 0. \tag{1.15}$$

The energy density as measured by an observer with 4-velocity U^μ is just

$$\begin{aligned}\rho &= T_{\mu\nu}U^\mu U^\nu \\ &= \alpha^2 + \mathcal{L}_\phi. \end{aligned} \quad (1.16)$$

At this point we shall impose an additional constraint on the stress energy tensor. We shall require, on heuristic grounds, that the energy density as measured by any observer is always non-negative. We therefore impose the condition:

$$T_{\mu\nu}U^\mu U^\nu \geq 0 \ \text{ for all timelike vectors } \ U^\mu. \quad (1.17)$$

The above inequality is a form of the Weak Energy Condition [15]. It must be stressed that not all solutions of the EFE satisfy the Weak Energy Condition and so its imposition requires some justification. As stated above, the primary motivation is a heuristic aversion to negative energy solutions but we shall see below that the class of solutions selected by the Weak Energy Condition is a very natural one and has interesting and appealing cosmological features.

Let us consider a space-time point p where the energy density is zero. Then (1.16) and (1.2) imply that

$$\tfrac{1}{2}\phi^\mu\phi_\mu + V(\phi) = -\alpha^2$$

at p. Thus $\phi^\mu\phi_\mu \leq 0$ with equality occurring only if $V(\phi) = \alpha^2 = 0$ (recalling that $V \geq 0$). However, if $\alpha^2 = 0$, ϕ^μ must either be space-like, thereby violating our previous assertion, or vanish identically. It follows that ϕ^μ is timelike (or identically zero) at all points p for which the energy density vanishes in some frame U^μ.

If, alternatively, the energy density is greater than zero in all frames then a theorem due to Synge [14] asserts that there exists a unit timelike eigenvector U^μ_E of T^μ_ν at p. That is;

$$T^\mu_\nu U^\nu_E = \lambda U^\mu_E.$$

Contracting (1.13) with U^μ_E and raising the index we see that N_ν must vanish in this frame. This implies that at p

$$\phi^\mu = \alpha U^\mu_E \quad (1.18)$$

where the condition that U_E^μ must be of unit length gives $\alpha = \pm\sqrt{-\phi_\rho \phi^\rho}$.

The Weak Energy Condition thus implies that ϕ^μ is everywhere timelike (Note that the converse is also true. If ϕ^μ is everywhere timelike then the Weak Energy Condition follows from (1.16)). Since ϕ is smooth on M it follows that the contours of ϕ will provide a space-like foliation of M if there exist no open regions with ϕ_μ identically zero (ie ϕ constant). Such regions may only occur for values of ϕ for which $V'(\phi) = 0$.

An example is provided for the massive scalar field ($V = \frac{1}{2}m^2\phi^2$) by setting $\phi = 0$ everywhere on M. The evolution equation (1.10) is satisfied and the gravitational field obeys the vacuum Einstein Equations $R_{\mu\nu} = 0$.

A solution such as this for which ϕ_μ vanishes identically on M will be termed a *trivial* scalar field cosmology. Such solutions seem neither interesting nor particularly physical and in general we expect the set of scalar field cosmologies for which ϕ is constant on *any* open set to constitute a vanishingly small subset of all possible solutions. We will therefore assume henceforth that ϕ_μ vanishes on at most a set of measure zero in M. The contours of ϕ are thus space-like hypersurfaces which foliate space-time and the timelike vector field U_E^μ may be uniquely defined everywhere on M as the future directed unit normal to the hypersurfaces of constant ϕ. It may be evaluated from (1.18) at all points where ϕ_μ is not identically zero.

Returning to the stress-energy tensor we see that U_E^μ defines the invariant decomposition

$$\begin{aligned} T_{\mu\nu} &= (-\phi_\rho\phi^\rho + \mathcal{L}_\phi)U_{E\mu}U_{E\nu} - \mathcal{L}_\phi h_{E\mu\nu} \\ &= (-\tfrac{1}{2}\phi_\rho\phi^\rho + V(\phi))U_{E\mu}U_{E\nu} + (-\tfrac{1}{2}\phi_\rho\phi^\rho - V(\phi))h_{E\mu\nu} \end{aligned} \quad (1.19)$$

where it is understood that $h_{E\mu\nu}$ is the projection operator onto the hypersurface orthogonal to U_E^μ. (1.19) should be recognized as the stress-energy tensor of a perfect fluid flowing along the integral curves of U_E^μ, with energy density ρ and pressure p (as measured by an observer co-moving with the fluid) given by

$$\begin{aligned} \rho &= -\tfrac{1}{2}\phi_\rho\phi^\rho + V(\phi) \\ p &= -\tfrac{1}{2}\phi_\rho\phi^\rho - V(\phi). \end{aligned} \quad (1.20)$$

In general, it is not possible to write down an equation of state $p = f(\rho)$ for the fluid but the special case $V = 0$ provides an important exception to this rule. In this case the scalar field behaves like a perfect fluid with equation of state $\rho = p$. The scalar field with V identically zero will be referred to below as the massless scalar field.

Notice that when $V(\phi)$ becomes large compared to $\phi^\rho\phi_\rho$, the pressure becomes large and negative. Such pressures are usually ruled out for classical fluids due to the fact that they have never been observed in nature. This is expressed precisely by the following condition on the stress energy tensor, the so called Strong Energy Condition [15];

$$(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu})U^\mu U^\nu \geq 0 \text{ for all timelike vectors } U^\mu \tag{1.21}$$

which for a perfect fluid reduces to

$$\rho + 3p \geq 0. \tag{1.22}$$

Thus, the strong energy condition is equivalent to the statement that no negative pressure may have magnitude greater than 1/3 of the energy density. This particular value is chosen for convenience rather than any fundamental reason since the part in brackets on the left hand side of (1.21) may be equated with the Ricci Tensor via the EFE. The Strong Energy Condition thus becomes a condition on the gravitational field which may be loosely interpreted via the Raychaudhuri equation [16] as saying that the gravitational force is attractive. Substituting (1.20) into(1.22) we see that for a scalar field the Strong Energy Condition becomes

$$-\phi^\rho\phi_\rho - V(\phi) \geq 0. \tag{1.23}$$

The Strong Energy Condition seems less fundamental than the Weak Energy Condition and there is no *a priori* reason why we would expect the above inequality to hold in general for scalar fields. As we shall see, violation of the Strong Energy Condition at some time during their evolution appears to be a generic property of scalar field cosmologies and is, in fact, precisely what makes them interesting as models of the early universe.

1.3.2 Kinematics of the Fluid

We conclude this section with a brief discussion of the kinematic properties of the fluid 4-velocity U^μ_E. These may be obtained in a standard way from the purely spatial relativistic strain tensor;

$$S_{E\mu\nu} = U_{E\rho;\sigma} h^{\rho}_{E\mu} h^{\sigma}_{E\nu} \tag{1.24}$$

(where the semi-colon is just the usual shorthand notation for covariant differentiation) and the acceleration vector;

$$\dot{U}^{\nu}_{E} = U^{\nu}_{E;\rho} u^{\rho}. \tag{1.25}$$

Using the fact that $U^{\mu}_{E} = \pm(-\phi_{\rho}\phi^{\rho})^{-\frac{1}{2}}\phi^{\mu}$ we may express $S_{E\mu\nu}$ in terms of the scalar field and its derivatives [6]. We find

$$S_{E\mu\nu} = \alpha^{-1}\phi_{\mu;\nu} - \alpha^{-3}(\phi_{\nu}\phi_{\rho;\nu} + \phi_{\mu}\phi_{\rho;\mu})\phi^{\rho} + \alpha^{-5}\phi_{\rho;\sigma}\phi^{\rho}\phi^{\sigma}\phi_{\mu}\phi_{\nu}. \tag{1.26}$$

Where $\alpha = \pm(-\phi_{\rho}\phi^{\rho})^{\frac{1}{2}}$. From inspection of (1.26), remembering that ϕ_{μ} is exact, we see that $S_{E\mu\nu}$ is symmetric. Thus the fluid is irrotational, ie

$$\omega_{E\mu\nu} \equiv S_{E[\mu\nu]} = 0. \tag{1.27}$$

This result is not unexpected since U^{μ}_{E} is hypersurface orthogonal. A full list of the kinematic quantities associated with $S_{E\mu\nu}$ may be found in [6].

Evaluating the acceleration vector explicitly we find

$$\dot{U}^{\mu}_{E} = \alpha^{-2}\phi^{\lambda}\phi^{\mu}_{;\lambda} - \alpha^{-4}\phi^{\rho}\phi^{\lambda}\phi_{\lambda;\rho}\phi^{\mu}. \tag{1.28}$$

Clearly this will be non-zero in general implying that the worldlines of the fluid are typically not geodesics and may in fact be quite complicated curves.

1.4 The Synchronous Coordinate System.

We now turn to the task of reformulating the EFE as a set of dynamical evolution equations. This involves choosing a global 3+1 splitting of space-time, ie; a family of space-time hypersurfaces on M and family of timelike curves along which physical fields on the hypersurfaces evolve. One obvious way to proceed would be to take the integral curves of U^{μ}_{E} as the coordinate curves for a global cosmological time coordinate t so that $\phi = \phi(t)$ and each hypersurface of constant ϕ represents the instantaneous 3-space. Such a procedure can always be carried out and has a clear physical interpretation since an observer co-moving with the fluid will be at rest, although, since the time-coordinate curves are not geodesics the coordinate t will not be the proper time in the rest frame of the observer. An advantage of working with such coordinates is that all but the zeroth component of ϕ^μ vanishes, however the gravitational and scalar field degrees of freedom become tangled up in the lapse function $N^2(t,x) = -g_{00}$ in a rather complicated way. Also, we shall frequently be interested in studying the behavior of families of timelike geodesics and such curves may look quite complicated in coordinates defined within this framework.

1.4.1 Construction of the Synchronous System

It turns out to be more convenient to work with a time coordinate whose coordinate curves are timelike geodesics. Such a 3+1 splitting of space-time may be constructed as follows. Let Σ_0 be a space-like hypersurface in M and let U^μ be its unit normal. Then at each point $p \in \Sigma_0$, U^μ is tangent to a unique rotation free timelike geodesic γ_p. Σ_0 therefore generates a unique rotation free geodesic congruence Γ_t where the affine parameter t is chosen so that $\gamma_p(0) = p$ for all $p \in \Sigma_0$ and the tangent vector to the congruence at each point, U^μ, satisfies

$$U^\rho U_\rho = -1. \tag{1.29}$$

We now make the following assumption: There exists such a hypersurface Σ_0 for which Γ_t is smooth and onto. ie U^μ is a smooth vector field on M.

The congruence Γ_t may be represented by the smooth map $\Gamma : R \times \Sigma_0 \to M$, $\Gamma(t, p_0) = \gamma_{p_0}(t)$ †.

The map Γ defines a smooth family of space-like hypersurfaces Σ_t on M according to $\Sigma_t = \Gamma(t, \Sigma_0)$. We may therefore view t as a function on M with contours Σ_t. Clearly $t = 0$ on Σ_0 and we shall frequently refer to Σ_0 as the initial hypersurface.

Any family of geodesics $G \in \Gamma_t$ may be represented as the restriction of Γ to some subset $R \times \Sigma_0(G)$ of $R \times \Sigma_0$ where $\Sigma_0(G) \subset \Sigma_0$. We can endow Γ_t with a topology by defining its open sets to be all collections of trajectories G for which $\Sigma_0(G)$ is an open subset of Σ_0. In particular, since Σ_0 is a

†In general the map γ_{p_0} may not be defined for all $t \in R$ so the domain of Γ will actually be some set $D \subset R \times \Sigma_0$. When we write the image of some set $P_0 \subset \Sigma_0$ under Γ as $\Gamma(R, P_0)$ it can be read as $\Gamma((R \times P_0) \cap D)$

3-D Reimmanian manifold, for any $\gamma \in \Gamma_t$ there exists a neighbourhood G such that $\Sigma_0(G)$ is homeomorphic to some subset $D \subset R^3$. Define $M(G) = \Gamma(R, \Sigma_0(G))$ then there is a natural coordinate system on $M(G)$ for which t is the time coordinate and its coordinate curves are the family of geodesics Γ_t. The homeomorphism $\Sigma_0(G) \to D$ defines a coordinate system (x^i) on $\Sigma_o(G)$. Let $\mathbf{X}_i$ be the tangent vector fields to the respective x^i then $\mathbf{X}_i$ span the tangent space of $\Sigma_0(G)$ and

$$[\mathbf{X}_i, \mathbf{X}_j] = 0$$

for all $i, j = 1, 2, 3$. The bold face notation for vectors has been introduced here for convenience and we shall frequently interchange between bold face and index notation below.

We may now obtain a coordinate system on $M(G)$ by Lie transporting $\mathbf{X}_i$ along the congruence Γ_t. That is, we define the vector fields $\mathbf{X}_i$ on P to be the solutions of the differential equations;

$$\pounds_{\mathbf{U}} \mathbf{X}_i = [\mathbf{U}, \mathbf{X}_i] = 0. \tag{1.30}$$

(This is equivalent to extending the x^i-coordinate curves onto $M(G)$ by defining $x^i(\Gamma(t, p_0)) = x^i(p_0)$ for $p_0 \in \Sigma_0(G)$). It follows immediately that

$$\pounds_{\mathbf{U}}[\mathbf{X}_i, \mathbf{X}_j] = [\mathbf{U}, [\mathbf{X}_i, \mathbf{X}_j]] = 0$$

by the Jacobi Identity and (1.30). Thus the X_i commute everywhere on $M(G)$ and we may define the coordinate system (t, x^i) on $M(G)$ with

$$\mathbf{U} = \frac{\partial}{\partial t} \qquad \mathbf{X}_i = \frac{\partial}{\partial x^i}. \tag{1.31}$$

Clearly the space-like tangent vectors $\mathbf{X}_i$ are everywhere tangent to the hypersurfaces Σ_t. Furthermore, $\mathbf{U}$ is normal to Σ_t. In order to see this first recall that, by construction, $\mathbf{U}$ is normal to the initial hypersurface Σ_0. Contracting $\mathbf{U}$ with $\mathbf{X}_i$ and differentiating with respect to time we find;

$$\frac{D}{dt} U_\rho X_i^\rho = U_{\rho;\sigma} U^\sigma X_i^\rho + U_\rho X^\rho_{i;\sigma} U^\sigma. \tag{1.32}$$

The first term on the right hand side vanishes immediately by the geodesic equation. In order to see that the second term also vanishes observe that by (1.30)

$$\pounds_{\mathbf{U}} X_i^\mu = U^\rho X^\mu_{i;\rho} - X_i^\rho U^\mu_{;\rho} = 0. \tag{1.33}$$

Thus, (1.32) becomes

$$\begin{aligned} \frac{D}{dt} U_\rho X_i^\rho &= U_\rho U^\rho_{;\sigma} X_i^\sigma \\ &= \tfrac{1}{2}(U_\rho U^\rho)_{;\sigma} X_i^\sigma \\ &= 0. \end{aligned} \tag{1.34}$$

Since $\mathbf{X}_i$ span the tangent space on $\Sigma_0(G)$ and $\mathbf{U}$ is normal to Σ_0, it follows that $\mathbf{U}$ is normal to the surfaces Σ_t everywhere. Note that this is true globally since it holds for coordinates generated by all coordinate patches on Σ_0.

In such a coordinate system the metric tensor takes on a particularly simple form since,

$$\begin{aligned} g_{00} &= g_{\mu\nu} U^\mu U^\nu = -1 \\ g_{0i} &= g_{\mu\nu} U^\mu X_i^\nu = 0. \end{aligned}$$

We may thus write the metric tensor as

$$\mathbf{g} = -\mathbf{d}t^2 + g_{ij}(t,x)\mathbf{d}x^i\mathbf{d}x^j. \tag{1.35}$$

Note that, for expediency, we abbreviate the spatial coordinates (x^1, x^2, x^3) to just x. A coordinate system for which the metric takes the form (1.35) is termed a synchronous coordinate system since only for such coordinates is it possible to physically synchronize clocks [17]. It must be emphasized that although the synchronous coordinate system is itself a local construction the congruence Γ_t and foliation $\{\Sigma_t\}$ are global.

1.4.2 The Field Equations

The space-time metric (1.35) induces the Reimannian metric $\mathbf{h} = \mathbf{g} + \mathbf{d}t^2$ on each hypersurface Σ_t with components $g_{ij}(t,x)$ in the local coordinate frame.† The evolution of g_{ij} with time depends on the kinematic properties of the congruence Γ_t. These are characterized by the geodesic equation and the relativistic strain tensor

$$K_{\mu\nu} = U_{\rho;\sigma} h^\rho{}_\mu h^\sigma{}_\nu$$

†Henceforth, roman indices i,j,k... will denote geometrical objects on the 3-D hypersurfaces Σ_t. Indices are raised and lowered via contraction with g_{ij} and its inverse g^{ij}.

which is equivalent to the 3-tensor $K_{ij} = U_{i;j}$ on Σ_t. K_{ij} is a geometric property of Σ_t as an imbedding in M and is called the *extrinsic curvature* or *second fundamental form* of Σ_t. Evaluating K_{ij} explicitly using (1.35) we find

$$K_{ij} = \Gamma^0_{ij} = \tfrac{1}{2}\dot{g}_{ij}. \tag{1.36}$$

Decomposing K_{ij} into its trace and trace-free parts gives

$$K_{ij} = \sigma_{ij} + \frac{1}{3}g_{ij}K \tag{1.37}$$

where $\sigma^i_i = 0$ and $K = U^i_{;i}$. σ_{ij} and K may be interpreted as the shear and expansion, respectively, of Γ_t. Taking the trace of (1.36) we find;

$$K = \tfrac{1}{2}g^{ij}\dot{g}_{ij} = \frac{\dot{v}}{v} \tag{1.38}$$

where $v = \sqrt{\det(g_{ij})}$ is the volume element on Σ_t. We thus see explicitly that K may be interpreted as the rate of expansion of the universe. K is very important since $|K|$ goes to infinity iff a number of nearby geodesics are converging to a point and intersecting [15]. Since Γ_t is smooth (by assumption) the geodesics may not intersect anywhere on M. It follows that if K goes to infinity at a finite time t along a geodesic $\gamma(t) \in \Gamma_t$ then γ is incomplete and a space-time singularity exists. Such a singularity corresponds to our intuitive idea of a singularity being the result of gravitational collapse.

Writing the EFE in the coordinate frame using the metric (1.35) they become*

$$\begin{aligned} R^0_0 &= \dot{K}^i_i + K_{ij}S^{ij} = T^0_0 - \tfrac{1}{2}T \\ R^0_i &= K^k_{i;k} - K^k_{k;i} = T^0_i \\ R^i_j &= {}^{(3)}R^i_j + \frac{1}{v}\frac{\partial}{\partial t}(vK^i_j) = T^i_j - \tfrac{1}{2}\delta^i_j T \end{aligned} \tag{1.39}$$

where K_{ij} has been identified with the time derivative of g_{ij} via (1.36) and ${}^{(3)}R_{ij}$ is the 3 dimensional Ricci tensor on the hypersurfaces Σ_t. Decomposing

*When labeling systems of equations we shall adopt the following convention: The number adjacent to the equations represents the system as a whole. Individual equations are referred to alphabetically from top to bottom. For example, the R^0_0 and R^0_i equations in (1.39) are referred to as (1.39a) and (1.39b) respectively.

K^i_j into its trace and trace free parts, and using the scalar field stress energy tensor (1.7), the equations become

$$\begin{aligned} \dot{K} + \sigma^2 + \tfrac{1}{3}K^2 &= -\dot{\phi}^2 + V(\phi) \\ \sigma^j_{i|j} - \tfrac{2}{3}K_{,i} &= -\dot{\phi}\phi_i \\ \dot{\sigma}^i_j + \tfrac{1}{3}\dot{K}\delta^i_j &= -K\sigma^i_j - (\tfrac{1}{3}K^2 - V(\phi))\delta^i_j + \phi^i\phi_j -{}^{(3)} R^i_j \end{aligned} \tag{1.40}$$

where $\sigma^2 = \sigma^i_j\sigma^j_i \geq 0$. The subscript "|" denotes covariant differentiation with respect to the metric g_{ij} on Σ_t. The evolution equation for the scalar field (1.10) may be written

$$\ddot{\phi} = -K\dot{\phi} - V'(\phi) + g^{ij}\phi_{i|j}. \tag{1.41}$$

The last term on the right hand side of (1.41) may be interpreted as the spatial divergence of the 3-vector ϕ^i on Σ_t. The equations (1.40) and (1.41) provide a complete set of evolution equations for the quantities $(g_{ij}, \phi, K, \sigma^i_j, \dot{\phi})$ on Σ_t. Equations (1.36), (1.40a), the trace free part of (1.40c), and (1.41) provide evolution equations for g_{ij}, K, σ^i_j and $\dot{\phi}$ respectively, while equation (1.40b) plays the role of a constraint equation on each of the hypersurfaces Σ_t. Taking the trace of (1.40c) yields a further dynamical equation for K:

$$\dot{K} = -K^2 + 3V(\phi) + \phi^i\phi_i -{}^{(3)} R. \tag{1.42}$$

Thus K is over determined and must satisfy an additional constraint equation which may be obtained by substituting (1.42) into (1.40a) to eliminate $\dot{K}$.

$$\tfrac{1}{3}K^2 = V(\phi) + \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi^i\phi_i - \tfrac{1}{2}{}^{(3)}R. \tag{1.43}$$

1.5 Dynamics of Scalar Field Cosmologies: Inflation ?

Most of the recent research into the dynamics of scalar field cosmologies has been carried out in the context of the inflationary universe scenario [18, 19, 20]. Scalar fields, by naturally violating the Strong Energy Condition, seem to provide a mechanism whereby a cosmology with arbitrary initial conditions may undergo a period of exponential expansion (inflation) accompanied by rapid isotropisation, thereby (according to the advocates of inflation) explaining the current observed isotropy and homogeneity of the universe. Inflationary cosmologies vary widely in their details but models where inflation arises as an essentially classical phenomena arising via the dynamical evolution of a scalar field cosmology might collectively be termed Chaotic Inflation. This scenario was first proposed by Linde [21].

1.5.1 Constant Potential Models.

We can most easily see how inflation arises in scalar field cosmologies by considering the simple case where the potential is just a constant $V = V_0$ where $V_0 > 0$. The argument which we give below has been adapted from the Cosmological No-Hair theorems for the closely related problem of the asymptotic stability of de Sitter space in cosmologies with a positive cosmological constant [23, 24, 25, 26, 27].

It is well known that large positive curvature can lead to gravitational collapse before any inflationary type behavior occurs [27] so to proceed analytically it is necessary to place some bound on the amount of positive curvature allowed. Jensen and Stein-Schabes [24] imposed the condition that ${}^{(3)}R \leq 0$ for all t. It has been shown by Wald [23] that this is true globally for *all spatially homogeneous cosmologies except Bianchi Type IX*. For general inhomogeneous space-times, however, it is not obvious whether it is reasonable to expect an initially negative curvature to remain so for all time and this condition seems overly restrictive. It shall be sufficient to impose the somewhat weaker constraint

$$ {}^{(3)}R \leq \sigma^2 + \dot{\phi}^2. \tag{1.44} $$

Thus, while we allow the curvature to become positive at some time, we insist that any positive curvature is always dominated by the shear and/or scalar field energy density. However, we shall not require that this be true everywhere on M, only along some neighbourhood G of a geodesic $\gamma \in \Gamma_t$.

It follows immediately from (1.43) that

$$ K^2 \geq 3V_o. \tag{1.45} $$

Thus, an initially expanding universe can never recollapse and, recalling (1.38), the volume element must satisfy

$$ v(t,x) \geq v_0(x)e^{\sqrt{3V_0}t} \tag{1.46} $$

If the neighbourhood $P_0 = \Sigma_0(G)$ has some small physical volume A_0 this will rapidly expand so that $P_t = \Gamma(t, P_0)$ has volume $A(t) \geq A_0 e^{\sqrt{3V_0}t}$. We now show that K^2 rapidly approaches its lower bound $3V_0$. Let $\Delta = K^2 - 3V_0 \geq 0$. Differentiating Δ with respect to time and using (1.40a) gives

$$\begin{aligned}
\dot{\Delta} &= 2K\dot{K} \\
&= 2K(-\tfrac{1}{3}\Delta - \dot{\phi}^2 - \sigma^2) \qquad (1.47) \\
&\leq' -\tfrac{2}{3}K\Delta. \qquad (1.48)
\end{aligned}$$

Integrating the above inequality, remembering (1.38), gives

$$0 \leq \Delta \leq \Delta_0(x)v^{-\frac{2}{3}}. \qquad (1.49)$$

We therefore see that the exponential expansion of v causes Δ to rapidly approach zero. Furthermore, since Δ decays monotonically $\dot{\Delta}$ must also decay exponentially to zero. It then follows from (1.47) and (1.43) that σ^2, $\dot{\phi}$, $\phi^i\phi_i$ and ${}^{(3)}R$ must also exponentially decay to zero. The neighbourhood $P_t \in \Sigma_t$ asymptotically approaches de Sitter space. To see this explicitly note that (1.36) and (1.37) imply that, asymptotically

$$\dot{g}_{ij} = \tfrac{2}{3}\sqrt{3V_0}g_{ij} + O(e^{-\gamma t}) \qquad (1.50)$$

where γ is a positive constant. This yields, upon integration

$$g_{ij}(t,x) = \tilde{g}_{ij}(x)e^{\frac{2}{3}\sqrt{3V_0}(t-t_1)} + O(e^{-\gamma t}) \qquad (1.51)$$

where $\tilde{g}_{ij}$ is the reimannian metric on P_{t_1} and $t > t_1$. If t_1 is chosen sufficiently large then $\tilde{g}_{ij}$ will be approximately flat and isotropic so that we may choose normal coordinates for which $\tilde{g}_{ij}(x) = \delta_{ij} + \epsilon_{ij}(t_1; x)$ on P_{t_1}, where $\epsilon_{ij} \to 0$ as $t_1 \to \infty$. Thus (1.51) asymptotically becomes the usual de Sitter line element. The volume of this de Sitter region exponentially expands resulting in an arbitrarily large region of flat, homogeneous space-time. Except for small perturbations, the behavior of this space-time does not depend on initial conditions but is determined only by the value V_0.

1.5.2 The Positive Curvature Case

The generality or otherwise of the above argument rests on the reasonableness of the condition (1.44) which ultimately relies on any positive curvature decaying faster than σ^2 or $\dot{\phi}^2$. We can see the effect of positive curvature more clearly if we consider the special case of Bianchi type IX space-time for which ${}^{(3)}R \geq 0$ everywhere. It can be shown [23] that

$$ {}^{(3)}R \leq \frac{B}{v^{\frac{2}{3}}} \tag{1.52} $$

where B is a positive constant dependent on the structure constants of the 3D isometry group. If K is initially positive, consider all Bianchi type IX solutions with B sufficiently small that it satisfies

$$ V_0 > \frac{B}{2v_0^{\frac{1}{3}}} \tag{1.53} $$

where v_0 is the initial volume. Then $V_0 - \frac{1}{2}{}^{(3)}R$ is initially greater than 0 and, according to (1.52), will remain so as long as v is increasing with time. But according to (1.43), K^2 (and hence $\dot{v}$) can only become negative if $V_0 - \frac{1}{2}{}^{(3)}R < 0$ which by (1.52) and (1.53) can only happen if v becomes smaller than its initial value. Thus v will expand forever and ${}^{(3)}R$ must asymptotically approach 0. More precisely let

$$ V_0 - \frac{B}{2v_0(x)^{\frac{1}{3}}} \geq \epsilon \tag{1.54} $$

on some neighbourhood $P_0 \in \Sigma_0$ of the initial hypersurface. Then

$$ V_0 - \tfrac{1}{2}{}^{(3)}R \geq \epsilon $$

for all $t > 0$. It follows from (1.43) that

$$ v(t,x) \geq v_0(x)e^{\sqrt{3\epsilon}t}. \tag{1.55} $$

Observe that the inequality (1.48) is independent of the sign of R. It therefore follows from (1.52), (1.54), (1.55) (1.43) and (1.48) that

$$ -3(V_0 - \epsilon)e^{-\sqrt{3\epsilon}t} \leq \Delta \leq \Delta_0 e^{-\frac{2}{3}\sqrt{3\epsilon}t}. \tag{1.56} $$

Thus space-time exponentially approaches the same de Sitter state as . We see that inflationary behavior can occur in the presence of positive curvature, however, this curvature may inhibit the rate of isotropisation.

If the initial condition (1.53) is not satisfied then it is possible that space-time may recollapse before ever coming close to a de Sitter state.

Observe that the larger the potential V_0 the more dominant its influence and the larger the positive curvature allowed before recollapse may occur.

1.5.3 Inflation in More General Models

Notice that as space-time approaches de Sitter space, above, the energy density and pressure of the scalar field approach V_0 and $-V_0$ respectively. The fluid thus asymptotically satisfies the equation of state $\rho = -p$, clearly violating the Strong Energy Condition. The rapid expansion of space-time may be understood physically as the result of a large negative pressure causing a repulsive gravitational interaction.

For an arbitrary positive potential the equation of state $\rho = -p$ will hold approximately whenever $\dot{\phi}^2 << V(\phi)$. Given the above results it is reasonable to expect some kind of de Sitter type expansion to occur under these conditions. In particular, if V' was small, V will vary very slowly, and space-time might be expected to behave, at any given time, as though V were effectively constant and rapidly approach the solution $K^2 = 3V(\phi(t))$. This is the so called slow roll approximation underlying the original chaotic inflation scenario [21, 34, 24]. Although it seems physically plausible it is far from straitforward to demonstrate rigorously that such behavior actually occurs.

Unlike the constant potential case where de Sitter space (which corresponds to $K^2 = 3V_0$) is an exact solution, the solution $K^2 = 3V(\phi(t))$ will not, in general, be an exact solution of the EFE. This means that the inflationary behavior, if it occurs at all, must be more complicated than simple de Sitter expansion. To see this let $\Delta = K^2 - 3V(\phi)$ as before and assume again that (1.44) holds so that Δ is a non-negative function. Differentiating explicitly and using (1.40a) we have

$$\dot{\Delta} \leq -\tfrac{2}{3}K\Delta - 3V'(\phi)\dot{\phi}. \qquad (1.57)$$

The constraint equation (1.43) implies that $K^2 > 3/2\dot{\phi}^2$ so when Δ is large compared to $|V'|$, Δ will rapidly decrease as it does when the potential is constant. However, when Δ becomes small so that $\Delta \sim V'(\phi)$ it is not at all obvious that it will continue to decay to 0. Clearly, more information on the evolution of $\dot{\phi}$ is required.

In order to proceed any further analytically we must make simplifying assumptions. Assume that the energy world lines of the scalar field are geodesics. In this case we may set $U^\mu = U^\mu_E$ and the spatial derivatives of ϕ vanish. We call such a field a homogeneous scalar field. For spatially homogeneous space-times, which are the most often studied in cosmology, this condition can be shown to be always true exactly. If we wish also to consider inhomogeneous space-times then the condition that ϕ be homogeneous places severe restrictions on the spatial derivatives of the metric via (1.40b). However, if the inhomogeneities in ϕ are small compared to kinetic terms it may be a valid approximation to ignore the spatial gradient terms $\phi^i\phi_i$ and $\phi^i_{|i}$ in the dynamical equations and the following analysis may be interpreted accordingly. The effect of large inhomogeneities of the scalar field on inflationary dynamics remains an outstanding problem in cosmology [22].

The dynamical equation for the scalar field becomes, for a homogeneous scalar field

$$\ddot{\phi} + K\dot{\phi} = -V'(\phi) \tag{1.58}$$

Define the non-negative function

$$\Theta = K^2 - 3V(\phi) - \frac{3}{2}\dot{\phi}^2. \tag{1.59}$$

Differentiating (1.59) and using (1.40a) and (1.58) gives

$$\begin{aligned} \dot{\Theta} &= -\tfrac{2}{3}K\Theta - \tfrac{1}{3}K\sigma^2 && (1.60) \\ &\leq -\tfrac{2}{3}K\Theta && (1.61) \end{aligned}$$

which upon integration yields

$$0 \leq \Theta \leq \Theta_o v^{-\frac{2}{3}}. \tag{1.62}$$

Define $V_t = \inf V \circ \phi([0,t])$ to be the smallest value attained by V in the time interval $[0,t]$ then (1.46) may be generalized to

$$v(t,x) \geq v_0(x) e^{\sqrt{3V_t}t}. \tag{1.63}$$

As long as V_t remains large rapid expansion of v will occur leading to the exponential decay of Θ and, by (1.60) and (1.43), subsequent exponential suppression of σ^2 and ${}^{(3)}R$. If V has a non-zero global minimum, $V_0 > 0$, then rapid isotropisation and flattening of space-time will occur. Indeed it can easily be shown that space-time will eventually tend towards de Sitter space as the scalar field eventually settles into a vacuum state.

If V has greatest lower bound 0, however, the occurrence of inflationary type behavior will be dependent on V remaining large for a sufficient period of time for isotropisation to occur.

Let us examine this situation more closely. Suppose the potential initially has value V_0, and at some later time t_f this has decreased to $V_0/2$. Then

$$v(t_f) > v_0 e^N$$

where $N = \int_0^{t_f} \sqrt{3V} dt$. We would say that the cosmology has exhibited inflationary type behavior if N were a large number. Assume that $V'(\phi)$ is small compared to $V(\phi)$ on the time interval $[0, t_f]$ so that $|V'| < \alpha V$ where $\alpha << 1$. Further assume that the Strong Energy Condition is violated at $t = 0$, ie $\dot{\phi}_0^2 < V_0$.

It follows from (1.58) that

$$\frac{d}{dt}(V(\phi) - \dot{\phi}^2) = 2K\dot{\phi}^2 + 3V'(\phi)\dot{\phi}.$$

Using the fact that $|V'| < \alpha V$ and $K^2 > 3V$ we find that if $\dot{\phi}^2 < V$:

$$\frac{d}{dt}(V(\phi) - \dot{\phi}^2) > (2\sqrt{3} - 3\alpha)V^{\frac{3}{2}} > 0.$$

Thus if the Strong Energy Condition is violated initially it must be violated everywhere on $[0, t_f]$ and indeed $V - \dot{\phi}^2$ will continue to increase as long as $\alpha < 1$. Since $\dot{V} = V'\dot{\phi}$ it follows from the definition of N that

$$\begin{aligned} N &> \frac{\sqrt{3}}{\alpha} \int_{V_0/2}^{V_0} V^{-1} dV \\ &= \frac{\sqrt{3}\ln 2}{\alpha} >> 1. \end{aligned}$$

Although this inequality is crude it demonstrates that, for a sufficiently flat potential, violation of the Strong Energy Condition will result in N being a large number (at least for the homogeneous scalar field). I stress that it is the ratio V'/V that is crucial in determining whether a model exhibits inflation, not the absolute value of V'.

We have illustrated the basic mechanism of inflationary dynamics in scalar field cosmologies. We have demonstrated that scalar field cosmologies with constant potentials rapidly approach de Sitter space as long as the spatial curvature does not become too large and positive. We have also seen that similar behavior is plausible for more general models but that it is difficult to demonstrate unequivicaly that inflation arises naturally from a general inhomogeneous set of initial conditions.

It is probable that this question can only be resolved fully by detailed numerical analysis of specific models. One model which has been studied extensively is the massive scalar field. ie, $V(\phi) = \frac{1}{2}m^2\phi^2$. We shall briefly review below some of the major results.

1.5.4 The Massive Scalar Field

Belinski *et al* [28] carried out an elegant global analysis of massive scalar field cosmologies with Friedmann-Robinson-Walker (FRW) space-time and demonstrated that inflation occurred in the overwhelming majority of flat and open FRW solutions even for large initial values of $\dot{\phi}$. For closed space they found there existed a class of solutions which recollapse before $\dot{\phi}$ is able to decay sufficiently for inflation to occur, thereby confirming the ability of positive curvature to prevent inflation. Similar results were obtained by [29, 30] using different techniques. These authors went further and attempted to compare the number of inflating verses non-inflating solutions in closed FRW space-times by constructing an invariant measure in solution space. Their results were ambiguous however, since both classes of solutions were found to have infinite measure.

Several authors have investigated the generality of inflation for massive scalar field solutions in spatially homogeneous anisotropic space-time [22, 31, 32, 33]. Their results have shown that the presence of anisotropy does not prevent inflation and in fact actually helps it since a large shear term can offset the effect of a large positive curvature term [22]. If one specifies spatially homogeneous data on some initial hypersurface then inflation always occurs provided ϕ_0 is greater than order unity and the initial curvature, if positive, is dominated by the other terms on the right hand side of (1.43).

Initial efforts to investigate the effects of inhomogeneity confined attention to small perturbations about homogeneous solutions [34, 35]. There results indicated, not surprisingly, that inflation was stable under such perturbations. Recently, a number of authors have undertaken detailed numerical investigations into the effect of large initial inhomogeneities [36, 37, 38, 39, 40, 41]. They conclude that while inflation can withstand a fair degree of inhomogeneity, a certain amount of average homogeneity on the part of the scalar field is required. In particular, it seems that the average initial value of the scalar field must be large over a radius of several horizon lengths in order for inflation to occur (it is assumed in standard inflationary scenarios that prior to inflation the radius of the particle horizon is of the order of K^{-1} as one would expect from standard model cosmology [16]).

This is a serious concern for the chaotic inflation program which traditionally assumes that the scalar field is essentially random in the pre-inflationary era [21]. Since the sign of the scalar field is arbitrary we would expect its average value to be close to zero over regions much greater than the size of the horizon [20]. It is therefore not clear whether inflation is the final answer in explaining the large scale homogeneity of the universe or whether it simply provides a mechanism for fine tuning correlations that already existed and must be explained by other means, as suggested by Goldwirth [39]. Since the very purpose for which inflation was originally proposed was to explain correlations in the universe over regions greater than the apparent size of the particle horizon [18, 21] it would be ironic if this were indeed the case.

The resolution of this problem requires careful consideration of which initial conditions can reasonably be expected prior to inflation. The traditional approach to this question is to appeal to quantum gravity for the initial conditions [21, 22], but, at present, no consensus as to how the universe might emerge from the quantum era exists and it has been pointed out by Goldwirth and Piran [22] that two different but plausible arguments lead to the opposite conclusions on this issue. It is, perhaps, premature to appeal to a non-existent quantum gravity theory without first obtaining a proper understanding of the global structure of the classical solution space.

A careful investigation of the global behavior of solutions, in the following chapters, indicates that there exists a quite natural class of solutions whose scalar field satisfies the required degree of homogeneity. The apparent smoothness of the initial scalar field may be interpreted as a natural consequence of the spatial gradients being dominated by the kinetic terms as the initial singularity is approached rather than as a manifestation of correlations in the field over distances greater than the horizon.

Indeed, it is possible that this may be an inevitable consequence of the Weak Energy Condition (1.17). The authors who found that anisotropy in the scalar field prevented inflation [36, 38, 39] all used initial conditions which violated the Weak Energy Condition! In order to simplify the problem of calculating consistent initial data these authors chose $\dot{\phi} = 0$ on the initial hypersurface, arguing that a non-zero $\dot{\phi}$ term would rapidly decay anyway. The only initial scalar field configuration which is then consistent with (1.17) is $\phi = const.$ Obviously no problem with initial anisotropy exists here. In general, for arbitrary initial conditions, the weak energy condition will bound gradients of the scalar field, possibly enough to prohibit solutions in which anisotropy prevents inflation. In the absence of detailed numerical simulations taking into account the Weak Energy Condition this issue remains unresolved.

1.5.5 Structural Stability and Inflationary Dynamics

For the case of the constant potential we saw that the dynamics of inflation were completely characterized by a unique de Sitter solution. For more general potentials, however, our arguments provided very little information as to the nature and details of inflationary dynamics. A possible problem with the generality of inflationary behavior in scalar field cosmologies has been raised by Ellis et. al [42]. This concerns the issue of structural stability [43]. They have argued that small changes in the functional form of $V(\phi)$ could lead to discontinuous changes in the qualitative behavior of solutions, most particularly with respect to inflationary behavior. Care must therefore be exercised in extrapolating the results for the constant potential, or indeed the massive scalar field, to more general potentials.

The general results obtained in the following chapters provide a strong indication that scalar field cosmologies are actually quite structurally robust. The topological structure of solution space, at least for simple space-times, seems to depend only on a few basic structural properties of the potential such as number of local maxima and minima. In particular, we shall see that inflation arises as a natural consequence of the existence of an exact, expanding, (particle) horizon free, Robinson-Walker solution which is an attractor in the full solution space of (1.40-1.41). This solution characterizes exactly the dynamics of the inflationary epoch in direct analogy to the way that de-Sitter space characterizes the dynamics of inflation for constant potential models. We demonstrate the existence of such a solution for a very general class of potentials. Not surprisingly, the key factor in determining its existence is the behavior of the ratio V'/V. Furthermore, small changes to the functional form of V'/V seem to result in small changes in the inflationary solution.

In the next section we shall begin our investigation of the global properties of scalar field cosmologies by examining some simple exact solutions.

1.6 Global Behavior of Exact Solutions

It was mentioned at the end of the last section that an understanding of the global structure of classical scalar field cosmologies may help to resolve some important questions concerning the generality of inflation. This in itself would be sufficient motivation for their study, however the asymptotic behavior of scalar field cosmologies at early times is of considerable interest in its own right.

A major motivation for this research was to investigate whether it was possible that scalar field cosmological models, or models conformally related to these models, might give rise to space-times with novel global features.

Since scalar fields violate the Strong Energy Condition the standard singularity theorems [15], which have been widely interpreted as demonstrating that singularities are inevitable for cosmologies with "well behaved" matter, do not apply. It therefore cannot be assumed *a priori* that an initial singularity will exist in a scalar field cosmology, and this issue requires careful consideration.

A related question is the existence of particle horizons. Standard model cosmologies typically possess particle horizons. It follows that widely separated parts of the observable universe can not have had any causal contact since the big bang. This seems at odds with the observed isotropy and homogeneity of the universe. The inflationary universe scenario attempts to solve this problem by blowing the particle horizons out to very large distances. However, as we have seen it is not entirely clear whether it is able to achieve this without itself appealing to a certain amount of initial homogeneity. It would be interesting to know whether realistic cosmologies exist which are totally free of particle horizons.

Horizon free exact solutions with conventional matter do exist, most notably the mixmaster solutions obtained by Misner [44]. These are horizon free Bianchi Type IX perfect fluid solutions. However, detailed calculations have revealed that the class of such solutions for which mixing is sufficient to remove horizons has measure zero [45].

Similarly, it has recently been shown by Woszczyna and Heilar [46] that the set of horizon free FRW perfect fluid cosmologies which obey the Strong

Energy Condition has measure zero.

As with the singularity theorems this result cannot be applied to scalar field cosmologies and generic horizon free solutions may not be ruled out without further investigation.

The Strong Energy Condition is a sufficient but not a necessary condition for the existence of singularities so its violation by scalar field cosmologies does not automatically mean that singularity free solutions exist. We shall now demonstrate, with the aid of simple exact solutions, that singularity free and horizon free scalar field cosmologies exist in abundance. The conventional definition of a space-time singularity in terms of geodesic incompleteness [15] is used throughout.

We shall consider the simple case of spatially flat FRW space-time. In this case the metric may be written

$$\mathbf{g} = -\mathbf{d}t^2 + a(t)^2 \sum_{i=1}^{3} (\mathbf{d}x^i)^2. \tag{1.64}$$

In terms of this metric the spatial curvature ${}^{(3)}R$ and shear tensor σ_{ij} are, of course, equal to zero and the expansion is a function of time related to a by

$$K = 3\frac{\dot{a}}{a}. \tag{1.65}$$

The constraint equations (1.40b) yield immediately

$$\dot{\phi}\phi_i = 0. \tag{1.66}$$

Since the surfaces of constant ϕ are everywhere space-like this equation implies that $\phi_i = 0$, ie. ϕ is a function of time only. Using (1.40a), (1.43) and (1.58) we obtain the system of ordinary differential equations

$$\begin{aligned} \ddot{\phi} &= -K\dot{\phi} - V'(\phi) \\ \dot{K} &= -\frac{1}{3}K^2 - \dot{\phi}^2 + V(\phi) \\ K^2 &= 3V(\phi) + \frac{3}{2}\dot{\phi}^2. \end{aligned} \tag{1.67}$$

These equations completely determine the evolution of ϕ and K given initial values of ϕ, $\dot{\phi}$ and K subject to the constraint (1.67c). The metric scale factor $a(t)$ is then determined, up to a constant, by integration of (1.65). Eliminating V from (1.67b) using (1.67c) we obtain the useful expression

$$\dot{K} = -\tfrac{3}{2}\dot{\phi}^2. \tag{1.68}$$

Substituting this into (1.67c)

$$3V(\phi) = K^2 + \dot{K}. \tag{1.69}$$

Let us imagine that $K(t)$ is a known function of time. Equations (1.68) and (1.69) allow us to immediately write down $\dot{\phi}(t)$ and $V(\phi(t))$ as explicit functions of time. $\dot{\phi}(t)$ may then, in principle, be integrated to obtain $\phi(t)$. If $\phi(t)$ is $1-1$ then it may be inverted to obtain an expression for the potential $V(\phi)$. Thus, for any smooth monotonic decreasing function of time $K(t)$, there always exists a potential $V(\phi)$ for which, at least locally, $K(t)$ is the expansion for an exact Robinson-Walker space-time.

Let us begin with the simplest possible example. Choose $K = -t$. This space-time is non-singular since K is smooth and finite for all t. In fact, integration of (1.65) reveals that $a(t)$ is given by the Gaussian

$$a(t) = e^{-\frac{t^2}{6}}. \tag{1.70}$$

From (1.68) and (1.69) we have

$$\dot{\phi} = \sqrt{\frac{2}{3}}$$

and

$$3V = t^2 - 1.$$

Integrating the first expression we have

$$\phi = \sqrt{\tfrac{2}{3}}t + c \tag{1.71}$$

where c is an arbitrary constant. Inverting this gives the potential

$$V(\phi) = \tfrac{1}{2}(\phi - c)^2 - \frac{1}{3}. \tag{1.72}$$

Thus we have constructed a model for which the singularity free metric

$$\mathbf{g} = -\mathbf{d}t^2 + e^{-\frac{t^2}{3}}\sum_{i=1}^{3}(\mathbf{d}x^i)^2$$

is an exact solution. This model might be considered unphysical due to the fact that the potential becomes negative when $|\phi - c| < \sqrt{\frac{2}{3}}$ (although note that the energy density never becomes negative). However, consider the potential $V = F(\phi)$ where there exists a positive number $\Lambda < \frac{1}{6}$ for which $F \geq \Lambda$ for all ϕ and $F(\phi) = \frac{1}{2}(\phi - c)^2 - \frac{1}{3}$ for $|\phi - c| > 1$. The above metric is an exact solution for large ϕ and, furthermore, $K^2 > 3\Lambda$ so if space-time is initially expanding it cannot recollapse to a singularity. The model $V = F(\phi)$ must therefore possess at least one singularity free expanding solution.

Although the potential $V = F(\phi)$ seems rather contrived note that for large ϕ we have $F \simeq \frac{1}{2}\phi^2$. It therefore becomes, approximately, a massive scalar field with unit mass.

In any case, the next example provides a singularity free solution where the potential is physically interesting.

Choose

$$K = \tfrac{3}{2}e^{-t} + \Lambda,$$

where Λ is some positive constant. This corresponds to a singularity free expanding space-time with metric scale factor

$$a(t) = \exp\left(\tfrac{1}{3}\Lambda - \tfrac{1}{2}e^{-t}\right). \tag{1.73}$$

Using analysis identical to that above we obtain the solution

$$\phi = \pm 2e^{-\frac{t}{2}} + c. \tag{1.74}$$

Setting the integration constant c equal to 0 we obtain the potential

$$V(\phi) = \tfrac{3}{64}\phi^4 + \tfrac{1}{4}(\Lambda - \tfrac{1}{2})\phi^2 + \tfrac{1}{3}\Lambda^2. \tag{1.75}$$

The special case $\Lambda = 0$ corresponds to the singularity free solution previously obtained by Madsen [49]. More interesting however, is the case $\Lambda \geq \frac{1}{4}$ since the potential then is smooth and non-negative for all values of ϕ. In particular, when $\Lambda = \frac{1}{4}$ the potential becomes

$$V(\phi) = \tfrac{3}{2}(\tfrac{3}{2}\phi^2 - 1)^2$$

which is a special case of a real Higgs type field. As is well known, potentials of this form arise naturally in theories with broken symmetries. They are also important in some inflationary scenarios such as "new inflation" [19, 22], and "dynamical relaxation" [50]. It is thus clear that singularity free solutions can indeed exist in simple and familiar models.

The two examples above possess neither singularities nor particle horizons. Examples may also be found which have initial singularities but are horizon free. Consider a space-time which expands according to the power law

$$a(t) = t^p \tag{1.76}$$

where p is a positive constant. A singularity clearly exists at $t = 0$. The condition for the existence of particle horizons is $p < 1$ [16]. therefore for larger values of p space-time is horizon free. The expansion is easily computed to be

$$K = 3pt^{-1}$$

from which we obtain, using (1.68)

$$\phi = \pm\sqrt{2p}\ln(ct) \tag{1.77}$$

where c is a positive constant. The potential which yields these solutions is found using (1.69)

$$V(\phi) = (3p^2 - p)c^2 \exp\left[\mp\left(\frac{2}{p}\right)^{\frac{1}{2}} \phi\right]. \tag{1.78}$$

We may thus construct a model which exhibits any power law behavior we desire. Solutions with $p \geq 1$ are the well known power law inflation cosmologies [47]. Exponential potentials such as these occur naturally in a number of cosmological contexts [48] particularly when one considers models conformally related to scalar-tensor theories. For example. Brans Dicke theory with a positive cosmological constant is conformally equivalent to a scalar field cosmology with an exponential potential [1]. Let us consider such a model with potential

$$V = Ae^{\lambda\phi}. \tag{1.79}$$

The solutions

$$a(t) = t^{\frac{2}{\lambda^2}} \qquad \phi = \pm 2\lambda^{-1}\left(\ln t + \tfrac{1}{2}\ln\left(\frac{\lambda^4 A}{12 - 2\lambda^2}\right)\right) \tag{1.80}$$

are exact solutions for this model. The necessary condition for these solutions to be horizon free is that $0 < \lambda \leq \sqrt{2}$.

Exact solutions for more complicated exponential potentials of the form

$$V(\phi) = Ae^{t} + Be^{-t}$$

have also been found by Scudellaro and Stornaiolo [51] these solutions also display horizon free behavior similar to that of the potential (1.79).

The above examples demonstrate that scalar field cosmologies display a wide range of behaviors and indeed it is possible to cook up a cosmologogical solution with virtually any features we desire. Given this rich catalogue of behavior it might seem unlikely that any general statements can be made concerning their asymptotic behavior.

Caution must be exercised, however, when drawing conclusions based on exact solutions.

Consider the power law solutions of (1.79), above, for example. The parameters λ and A determine c uniquely so that, for a given model, these solutions do not represent the general solution but rather two specific solutions. In order to ascertain whether these solutions display the typical qualitative features of the general solution requires further work. The general behavior of FRW scalar field cosmologies with exponential potentials has been looked at by Burd and Barrow [48] and by Halliwell [13] using qualitative techniques. It turns out that the horizon free power law solutions are attractors and typical trajectories approach them asymptotically as $t \to \infty$. Conversely, as solutions evolve backwards in time they rapidly diverge from the power law solutions. Thus, although the power law solutions provide examples of horizon free cosmologies they do not indicate that horizon free behavior is *typical* of cosmologies with exponential behavior.

In general, exact solutions of scalar field models do not represent general solutions and are therefore limited in the information they provide on the dynamics of the theory. However, there do exist some particularly simple cases where equations (1.67) can be integrated exactly to obtain a general solution. Consider, for example, the special case where $p = \frac{1}{3}$ above. We then have $V(\phi) = 0$ independently of the constant of integration c. The two 1-parameter families of solutions

$$a(t) = t^{\frac{1}{3}} \qquad \phi = \pm\sqrt{\tfrac{2}{3}}\ln(ct) \tag{1.81}$$

provide the general solution for a massless scalar field with flat Robinson-Walker metric. This solution will turn out to play a particularly important role in the analysis that follows in Chapter 2. It possesses an initial space-time singularity at $t = 0$ and particle horizons exist for all observers co-moving with the energy worldlines of the scalar field (ie stationary observers). This is not surprising as a massless scalar field is indistinguishable from a perfect fluid with equation of state $\rho = p$ and consequently may not violate the Strong Energy Condition.

The only other example of an exact general solution that I am aware of is that of the constant potential model $V(\phi) = V_0$. The general solution is

$$a^3 = \sinh(\alpha t) \qquad \phi = \pm\sqrt{\tfrac{2}{3}}\int \frac{\alpha}{\sinh(\alpha t)}dt.$$

where $\alpha = \sqrt{3V_0}$ and ϕ may be any indefinite integral. It is significant that these solutions also have particle horizons and singularities and, in fact, may be approximated to first order by the general solution (1.81) close to the singularity.

If we are to understand the global dynamics of more general models and account for the apparently infinite variety of solutions, a more systematic approach than that provided by the study of exact solutions is required. In the next chapter we shall turn to the theory of dynamical systems in order to achieve this.

Chapter 2

Scalar Fields In Spatially Flat and Isotropic Space-time.

2.1 Introduction

We have already seen that a diverse range of behavior is exhibited by scalar field cosmologies, even when attention is restricted to the relatively simple case of spatially flat FRW space-time. Our first task shall therefore be to systematically investigate the general asymptotic properties of scalar field cosmologies in flat FRW space-time, so as to gain a greater understanding of the origin and nature of this diversity. We shall approach the problem from the dynamical systems point of view. A summary of relevant mathematical results and notation is provided in appendix A.

The dynamical systems approach to FRW scalar field cosmologies has been employed previously to investigate inflation in a number of specific models. It has already been mentioned in the last section Halliwell [13] and Burd and Barrow [48] independently applied dynamical systems techniques to demonstrate that power law inflation was an attractor in the solution space of FRW models with exponential potential.

Of particular interest is the work of a Russian group headed by Belinskii [28] who carried out a genuine global analysis of FRW massive scalar field cosmologies. By using equation (1.67c) in order to write K as a function of ϕ and $\dot{\phi}$ they were able to express (1.67) as a plane autonomous system in the variables $(x, y) = (\phi/\sqrt{6}, \dot{\phi}/(m\sqrt{6}))$ with time parameter $\eta = mt$. Taking advantage of the quadratic form of the potential they managed to study the global properties of the flow by mapping the (x, y) plane onto the interior of the unit disc using the Poincaré compactification procedure [52]. They found that there exist exactly two non-trivial singularity and horizon free solutions which obey the asymptotic equation of state $\rho = -p$ (ie $\rho/p \to -1$ as $t \to -\infty$). These solutions are initially inflating and continue to inflate until $\phi \sim 1$ at which time the scalar field begins to relax into a vacuum state and space-time approaches Minkowski Space. Belinskii *et al* called these solutions the inflationary seperatrices.

All other non-trivial solutions obey the asymptotic equation of state $\rho = p$ (ie $\rho/p \to 1$ as $t \to 0$) and are approximated by the solution (1.81) in the vicinity of the singularity.

Furthermore, typical trajectories rapidly approach the inflationary seperatrices and remain close during the inflationary periods of evolution, after which they also decay to the Minkowski vacuum solution. A phase plane

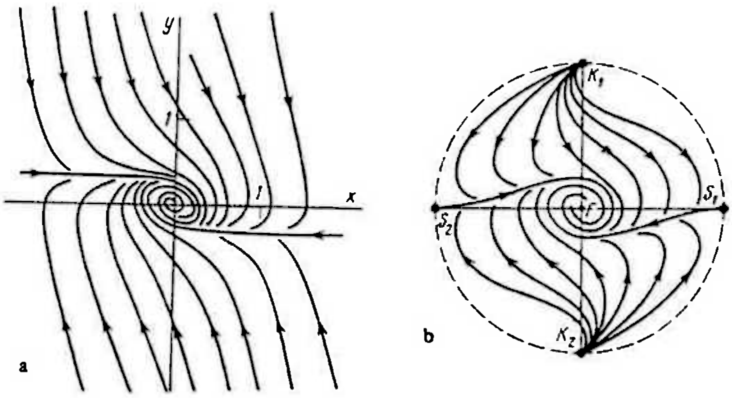

Figure 2.1: Phase plane plot showing orbits for massive scalar field cosmology obtained by Belinskii *et al*; a) using natural coordinates (x, y), and; b) with infinity mapped to the unit circle using the Poincaré transformation.

plot showing the structure of the flow is given in Fig. 2.1a. A better picture of the global structure of solution space is obtained from the flow on the compactified phase space, Fig. 2.1b. Observe that almost every solution has its origin in one of the two non-physical points K_1, K_2. The inflationary seperatrices are the solutions originating from the saddle points S_1, S_2.

These solutions accurately model the dynamics of the cosmology during the inflationary period, essentially independently of initial conditions.

Belinskii and his co-workers also obtained phase portraits for the quartic self interaction ($V = \lambda\phi^4$) and Higgs potential with similar results.

The phase plane structure of the massive scalar field reveals, again, how a model can possess solutions with interesting global features which do not reflect the behavior of the majority of solutions. Such solutions are certainly interesting but it would seem unlikely, without imposing very special boundary conditions, that they are realized in nature. In fact, if we assume that all initial conditions are equally likely to occur then the inflationary seperatrices have probability zero of occurrence. We should therefore be careful to distinguish between features of the model which are generic (have finite probability) and those that are not.

If $\psi_p(t)$ is a solution of a dynamical system intersecting the point p, we shall say that a property of ψ_p is *generic* if there exists a neighbourhood $N(p)$ of p such that all trajectories whose orbits intersect $N(p)$ also possess that property. Furthermore, we shall say that a property of ψ_p is *almost always* true if the union of the orbits of trajectories on which it is untrue has Lebesgue measure zero in phase space.

Besides being non-generic in the above sense, the horizon free solutions found by the Russian group (as typified by the massive scalar field) and the inflating power law solutions of the exponential potential have an important feature in common. Both are exponentially attracting solutions which characterize the dynamics of the inflationary phase of the system. It would seem, therefore, that a close relationship exists between inflation and the global behavior of solutions.

I am aware of no examples in the literature where a horizon free or singularity free solution (for flat FRW space-time) has been demonstrated to be generic. If such solutions do exist for certain classes of models however, then they could lead to interesting and novel cosmologies.

This issue has implications to all theories which are conformally equivalent to scalar field theories since the causal structure of space-time is invariant under conformal transformations. If a horizon free solution of the vacuum R^2 gravity theory (as has been found by Frenkel and Brecher [53]) were generic, for example, it would imply the existence of a corresponding generic horizon free scalar field cosmology. The potential V for such effective theories may be highly non-trivial and need not even be expressible in analytic form [11].

For this reason, it is desirable that any attempt to investigate whether it is possible to construct models with generic horizon free or singularity free solutions (or to rule them out) should make as few assumptions about the specific analytic form of the potential as possible, and concentrate instead on its qualitative features.

We begin below by proving some general results on the dynamics of the scalar field.

2.2 Preliminary Results.

The basic equations governing the system, (1.67), may be expressed as a dynamical system in the variables $(K, \phi, \dot{\phi})$ as follows:

$$\begin{aligned} \frac{d\phi}{dt} &= \dot{\phi} \\ \frac{d\dot{\phi}}{dt} &= -K\dot{\phi} - V'(\phi) \\ \frac{dK}{dt} &= -\tfrac{1}{3}K^2 - \dot{\phi}^2 + V(\phi) \end{aligned} \tag{2.1}$$

subject to the constraint

$$K^2 = 3V(\phi) + \frac{3}{2}\dot{\phi}^2. \tag{2.2}$$

The constraint equation actually makes (2.1c) redundant since K can be expressed as a function of ϕ and $\dot{\phi}$. We shall refrain from making such a coordinate choice at this stage, however. Later on we shall choose certain pairs of variables as coordinates for particular applications.

Note that for non-negative potentials the sign of K does not change on solutions of (2.1). In order to see this put $K = 0$ in (2.1) then $V(\phi)$, $\dot{\phi}$ and $\dot{K}$ must also vanish. Since V is non-negative, it follows in turn that V' and therefore $\ddot{\phi}$ are also zero. $K = 0$ thus represents a unique steady state solution and no non-trivial solution can pass through it. Since we are interested in expanding universes in this discussion we will restrict attention to solutions where K is positive. The physical expanding solutions of (2.1) may be represented as trajectories in the 2-dimensional phase space

$$\Omega = \{(K, \phi, \dot{\phi}) : K \geq 0, K^2 = 3V(\phi) + \tfrac{3}{2}\dot{\phi}^2\} \subset R^3.$$

A singularity occurs when $K \to \pm\infty$ in a finite proper time. Eliminating V from (2.1b) using (2.1c) gives

$$\dot{K} = -\frac{3}{2}\dot{\phi}^2 \leq 0. \tag{2.3}$$

Thus K is a monotonic non-increasing function of t and is strictly decreasing everywhere except where $\dot{\phi} = 0$. This suggests that K is likely to approach infinity asymptotically in the past. Whether or not this actually happens or whether it occurs within a finite time interval will depend on the behavior of $\dot{\phi}$. We now prove a theorem concerning the asymptotic behavior of the scalar field.

Theorem 2.1 *Assume that V is C^r with $r \geq 3$. Let $p = (K_0, \phi_0, \dot{\phi}_0)$ be a point in Ω and let $O^-(p)$ be the past orbit of p under (2.1). Assume $\phi_0 \neq 0$. Then $\dot{\phi}^2 + \phi^2$ is almost always unbounded on $O^-(p)$.*

Proof:

Firstly, we may identify Ω with R^2 as follows. Let

$$x = \phi, \qquad y = \dot{\phi}. \tag{2.4}$$

Substituting x and y into (2.1) and eliminating K gives the 2-dimensional autonomous system of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -V'(x) - \left(3V(x) + \tfrac{3}{2}y^2\right)^{\frac{1}{2}} y. \end{aligned} \tag{2.5}$$

(2.5) defines a C^{r-1} vector field on R^2. K may be written as a smooth function of the phase space variables,

$$K(x,y) = \sqrt{\frac{3}{2}}(y^2 + 2V(x))^{\frac{1}{2}} \tag{2.6}$$

Explicit differentiation of (2.6) and substitution of (2.5) gives the directional derivative of K along the flow.

$$\dot{K} = -\frac{3}{2}y^2 \tag{2.7}$$

Let $p \in R^2$ be some point for which $O^-(p)$ is contained in a compact set $U \subset R^2$ (ie $\phi^2 + \dot{\phi}^2$ is bounded on $O^-(p)$). Then according to (2.6) and (2.7) K is a bounded monotonic function of time on $O^-(p)$ and must therefore approach a limit as $t \to -\infty$. This implies that $\dot{K}$ tends to zero and hence,

$$\lim_{t \to -\infty} y = 0. \tag{2.8}$$

By Theorem A.1 there exists an invariant set $\alpha(p) \subset U$ which is the α-limit set of p. Furthermore, (2.8) implies that $y = 0$ on $\alpha(p)$. Inspection of (2.5) reveals that the only possible invariant sets lying on the line $y = 0$ are equilibrium points $(x_0, 0)$ satisfying $V'(x_0) = 0$.

In order to prove the theorem we need only investigate the stability of equilibrium points of (2.5). Furthermore, it is only necessary to consider equilibrium points for which $V(x_0) = V_0 > 0$. The reason for this is as follows: Assume $\alpha(p) = (x_0, 0)$ where $V_0 = 0$. Then by (2.6), $K(\alpha(p)) = 0$. It follows from (2.7) and the non-negativity of K that $K = 0$ for all t and hence, that V and $\dot{\phi}$ are also zero for all t. Thus $\psi_p(t)$ corresponds to a trivial scalar field cosmology and can be neglected.

Let $p = (x_0, y_0)$, be an equilibrium point of (2.5) satisfying

$$y_0 = 0 \qquad V'(x_0) = 0 \qquad V(x_0) = V_0 > 0 \tag{2.9}$$

The equilibrium points are steady state solutions and correspond to maximally symmetric solutions of Einstein's equations. In this case, with $V_0 > 0$, p represents a vacuum de Sitter solution with scale factor

$$a(t) = e^{\sqrt{\frac{V_0}{3}}t}.$$

We can classify the equilibrium points by computing the eigenvalues of the matrix of derivatives of the vector field (2.1) at p (see appendix). We find

$$J_p = \begin{pmatrix} 0 & 1 \\ -V''(x_0) & -\sqrt{3V_0} \end{pmatrix} \tag{2.10}$$

The eigenvalues of J_p are:

$$\lambda = \frac{1}{2}\left(-\sqrt{3V_0} \pm \sqrt{3V_0 - 4V''(x_0)}\right) \tag{2.11}$$

The sign of the eigenvalues depends on whether $V''(x_0)$ is positive, negative or zero. We consider each of these possibilities separately below.

i) $V''(x_0) < 0$:

This case corresponds to x_0 being a local maximum of V. Equation (2.11) implies that the eigenvalues of J_p are non-zero and of opposite sign. Thus p is a saddle and exactly two trajectories will be past asymptotic to p (Fig 2.2). It is interesting that these solutions correspond physically to unstable de-Sitter solutions, similar to those discussed by Barrow [54]. They asymptotically approach de Sitter space in the past and possess neither singularities or particle horizons.

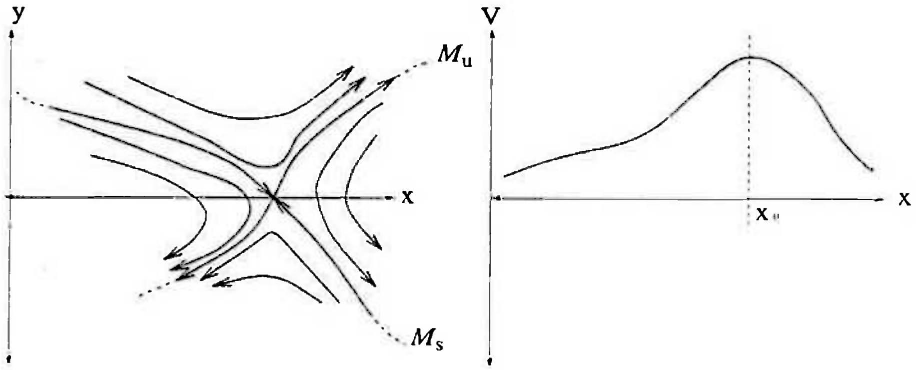

Figure 2.2: A local maximum of V will result in an unstable equilibrium at $(x_0, 0)$. M_s and M_u indicate local stable and unstable manifolds respectively. Exactly two solutions lie on the unstable manifold.

ii) $V''(x_0) > 0$:

This case occurs when x_0 is a local minimum of V. Since $V''(x_0)$ and $V(x_0)$ are positive, the eigenvalues of J_p are positive definite and p is clearly a stable sink. No trajectories will be past asymptotic to p in this case (Fig.2).

iii) $V''(x_0) = 0$:

This case is slightly more subtle than the previous two cases. It could correspond to a degenerate maximum or minimum of V or it could mean that x_0 lies in a region where V is constant. In the latter case p will not be an

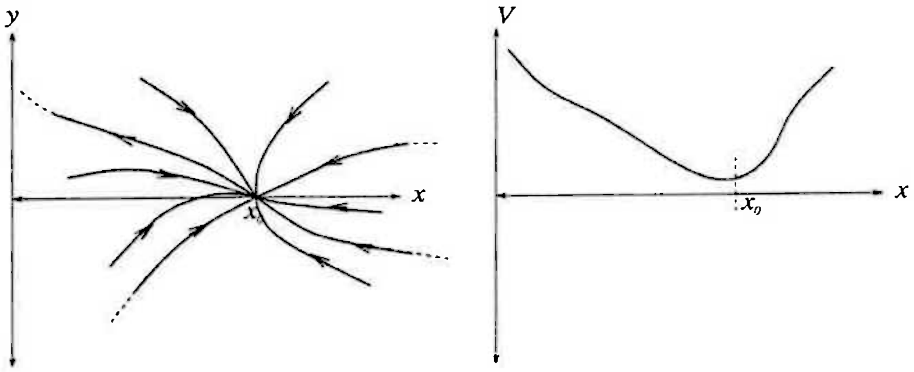

Figure 2.3: A local minimum of V will result in a stable equilibrium at $(x_0, 0)$. Such a point corresponds physically to a stable vacuum de-Sitter solution.

isolated equilibrium point but will be a point in a 1-dimensional equilibrium set.

The eigenvalues of J_p are 0 and $-\sqrt{3V_0}$ with corresponding eigenvectors pointing along the x-axis and the line $y = x_0 - x$ respectively. By the Center Manifold Theorem (Theorem A.5) there exists a 1-dimensional stable manifold tangent to the line $y = x_0 - x$ at p and a 1-dimensional center manifold tangent to the x-axis at p. It follows from Theorem A.7 (note Remark 2) that the center manifold is locally attracting and any solutions past asymptotic to p must lie on the center manifold itself.

Since the center manifold is 1-dimensional there is a maximum of two such solutions for each degenerate equilibrium point p(Fig. 2.4).

If p is an isolated equilibrium point then p may be a saddle (Fig. 2.4a) or a source (Fig. 2.4b).

If p lies on the the interior of an equilibrium set E then there is a neighbourhood of p for which the invariant set E is itself a center manifold. Therefore each point of the center manifold is an equilibrium point and no solutions on the center manifold will approach p asymptotically in the past other than the trivial solution lying on p itself. If p lies on the boundary of E then clearly there may be at most one solution past asymptotic to p (Fig. 2.4c). Therefore, any one dimensional equilibrium set E may contain the α-limit of at most two trajectories.

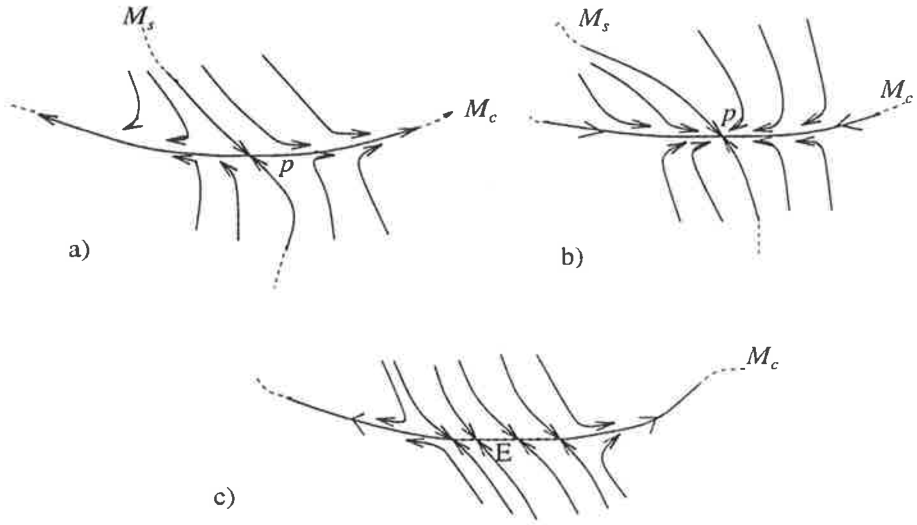

Figure 2.4: Behavior of the flow in the neighbourhood of degenerate equilibria of (2.5). M_S and M_c denote the stable and center manifolds respectively. At most 2 solutions are past asymptotic to the equilibrium set.

We have now considered all possible cases and have shown that any connected equilibrium set can contain the α-limit sets of at most two solutions. It follows that a countable number of trajectories have α-limit sets in the interior of the (x, y) plane. Thus, $x^2 + y^2$ is unbounded on almost all orbits. □

The significance of Theorem 2.1 to the behavior of the gravitational field is illustrated by the following corollary.

Corollary 2.2 *If the potential V is unbounded as $\phi \rightarrow \pm\infty$, then $K(t)$ diverges monotonically on almost all solution curves of (2.1).*

Proof:

The result follows trivially from Theorem 1, equation 2.1c and equation 2.3.□

Theorem 2.1 has the physical interpretation that the energy density of the scalar field almost always diverges as we follow the solution back in time. This does not necessarily imply that the scalar field itself diverges. By way of illustration, suppose we have $\phi(t) = t^{\frac{1}{2}}$ then the energy density of the scalar field diverges as $t \to 0$ but the scalar field itself actually goes to zero. The next theorem asserts that such solutions do not exist, ie if $\phi^2 + \dot{\phi}^2$ goes to infinity then ϕ must also go to infinity.

Theorem 2.3 *Assume V is C^3. Let $p = (\phi_0, \dot{\phi}_0, K_0)$ be a point in Ω and let $O^-(p)$ be the past orbit of p under (2.1). If $\phi^2 + \dot{\phi}^2$ is unbounded on $O^-(p)$ then ϕ^2 is unbounded on $O^-(p)$.*

Proof:

If K is bounded on $O^-(p)$ then by (2.2), $\dot{\phi}$ must also be bounded. It follows immediately that if $\phi^2 + \dot{\phi}^2$ is unbounded, ϕ^2 must be unbounded and the theorem is proved.

Assume that K is unbounded on $O^-(p)$. Then there exists $t_i \in (0, \infty]$ such that

$$\lim_{t \to -t_i} K(\psi_p(t))^{-1} = 0 \tag{2.12}$$

where $\psi_p(t)$ is the unique solution of (2.1) passing through p.

Since K^{-1} is monotonic increasing with time, we can replace K and $\dot{\phi}$ with the new set of coordinates

$$x = \frac{1}{K} \qquad y = \sqrt{\tfrac{3}{2}}\frac{\dot{\phi}}{K} \tag{2.13}$$

and introduce a new time coordinate

$$\tau = \ln v(t) + \tau_0 \tag{2.14}$$

where $v = a^3$ is the spatial volume element and τ_0 is some constant. τ is well defined since v is strictly increasing. Furthermore, v goes to zero, if and only if K goes to infinity [16] so (2.14) implies that $K \to \infty$ as $\tau \to -\infty$. Differentiating (2.14) we find (recalling that $K = \dot{v}/v$)

$$\frac{d}{dt} = K\frac{d}{d\tau}. \tag{2.15}$$

In terms of these new coordinates the dynamical equations for the system become

$$\begin{aligned}\frac{dx}{d\tau} &= y^2x \\ \frac{dy}{d\tau} &= -y - \sqrt{\tfrac{3}{2}}x^2V'(\phi) + y^3 \\ \frac{d\phi}{d\tau} &= \sqrt{\tfrac{2}{3}}y\end{aligned} \tag{2.16}$$

subject to the constraint

$$y^2 + 3x^2V(\phi) = 1. \tag{2.17}$$

The unphysical set $\Sigma = \{(x, y, \phi) : x = 0, y^2 \leq 1\}$ represents the infinity of K. By (2.12), ψ_p must approach Σ as $\tau \to -\infty$. Σ is an invariant manifold of (2.16) and, on Σ, (2.16) defines the plane autonomous system

$$\begin{aligned}\frac{dy}{d\tau} &= -y + y^3 \\ \frac{d\phi}{d\tau} &= \sqrt{\tfrac{2}{3}}y.\end{aligned} \tag{2.18}$$

The line $y = 0$ is an equilibrium set of (2.18). Elsewhere we can divide (2.18a) by (2.18b) to obtain

$$\frac{dy}{d\phi} = -\sqrt{\tfrac{3}{2}}(1 - y^2)$$

which can be integrated exactly to obtain the set of unphysical of orbits

$$\begin{aligned}\Gamma_{\phi_0} &= \left\{(y, \phi) : \phi = \frac{1}{\sqrt{6}}\ln\left(\frac{1-y}{1+y}\right) + \phi_0,\ \ 0 < y^2 < 1\right\} \\ \Gamma_{\pm} &= \{(y, \phi) : y = \pm 1\}\end{aligned} \tag{2.19}$$

on Σ where ϕ_0 is a constant parameter. For each $\phi_0 \in (-\infty, \infty)$ there exists an equilibrium point $P_{\phi_0} = (0, \phi_0)$ and a pair orbits, Γ_{ϕ_0} which approach ϕ_0 as $y \to 0$. Fig. 2.5 sketches some typical orbits on Σ, the arrows represent the direction of the flow on the corresponding orbit.

Let us now assume that ϕ is bounded and demonstrate a contradiction.

Since ϕ is bounded, the physical orbit $O^-(p)$ is contained in a compact region of phase space and it follows from theorem A.1 that ψ_p must approach some union of orbits in a bounded subset of Σ as $\tau \to -\infty$. From inspection of (2.19) it is clear that Γ_{ϕ_0} and $\Gamma_\pm$ are unbounded and therefore $\psi_p(\tau)$ must asymptotically approach some equilibrium point P_{ϕ_0} on the ϕ-axis (This is intuitively obvious from Fig 2.5).

Since ϕ is bounded, the potential V must also be bounded, thus equation (2.17) together with equation (2.12) imply that

$$\lim_{\tau \to -\infty} y(\tau) = \pm 1.$$

This is a contradiction since it implies that P_{ϕ_0} cannot possibly be a limit point of $\psi_p(\tau)$. Thus the theorem is proved. □

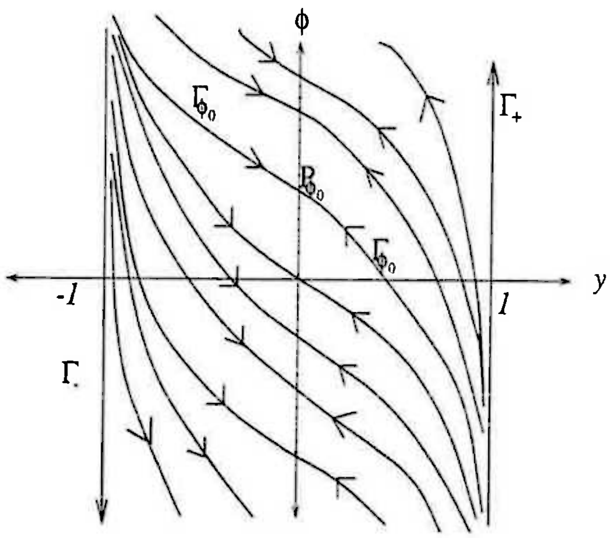

Figure 2.5: Typical orbits on Σ. The only bounded orbits are the equilibria P_{ϕ_0} on the ϕ-axis.

2.3 Conditions on the Potential

2.3.1 Strongly Exponential Dominated Potentials

Theorem 2.3 makes it clear that in order to investigate the generic asymptotic behavior of the system (2.1) we need only to concentrate on the behavior of the system close to $\phi = \pm\infty$. However, studying the equations in the limit $\phi \to \infty$ is not a trivial exercise.

In the next section we shall proceed by mapping infinity to a regular boundary. In order to ensure that the equations are regular at infinity additional smoothness assumptions on $V(\phi)$ will be necessary. In general V diverges as $\phi \to \infty$ so it will be necessary to re-express the equations in such a way that V either does not appear explicitly or is smoothed out by some function which goes to zero faster than V goes to infinity. In order to achieve the latter it is necessary to impose a bound on the rate at which V diverges.

Following natural terminology we will say that a potential is exponential dominated (at infinity) if $\lim_{\phi\to\infty} e^{-\lambda\phi}V(\phi) = 0$ for all $\lambda > 0$.

We shall impose a slightly stronger condition:

Definition 2.1 *Let $V : R \to R$ be a C^2 non-negative potential. Let there exist some $\phi_0 > 0$ for which $V'(\phi) \neq 0$ for all $\phi > \phi_0$. Define $W_V : [\phi_0, \infty) \to R$,*

$$W_V(\phi) = \frac{V'(\phi)}{V(\phi)}.$$

We will say that V is Strongly Exponential Dominated (SED) at infinity if

$$\lim_{\phi\to\infty} W_V(\phi) = 0. \tag{2.20}$$

Not surprisingly, all SED functions are exponential dominated.

Theorem 2.4 *let V be SED at infinity then, for all $\lambda > 0$,*

$$\lim_{\phi\to\infty} e^{-\lambda\phi}V(\phi) = 0. \tag{2.21}$$

proof:

Let $\lambda > 0$. By (2.20) there exists $\phi_0 > 0$ such that for all $\phi > \phi_0$,

$$\frac{V'(\phi)}{V(\phi)} < \frac{\lambda}{2}$$

which implies that

$$0 < V(\phi) < ce^{\frac{\lambda}{2}\phi}$$

where c is some constant. It follows that

$$0 < e^{-\lambda\phi}V(\phi) < ce^{-\frac{\lambda}{2}\phi}.$$

Taking the limit as $\phi \to \infty$ we have the result.□

The converse of the above theorem does not hold. In fact, for oscillating functions such as $\sin^2(\phi)$ (which is clearly exponential dominated) W_V does not even have a limit. The condition that $V' \neq 0$ is included in the definition to exclude such functions from our considerations, as well as to simplify certain later analysis. Note that V is not required to be monotonic, but may not oscillate perpetually.

Another example of exponential dominated functions which do not satisfy the conditions of Definition 2.1 are functions which decay exponentially fast to zero.

As it turns out it is easy to show that if a potential V decays to zero as $\phi \to \infty$, then any solution for which ϕ is unbounded above must asymptotically (in the past) approach the equation of state $\rho = p$. In order to see this reintroduce the coordinates defined in the proof of Theorem 2.3 and consider equation (2.17). Since $x = 1/K$ is bounded in the past and $V \to 0$ as $\phi \to 0$ it is possible to make $\sup_{\phi>\phi_0}(y^2 - 1)$ arbitrarily small by choosing $\phi_0 = \phi(\tau_0)$ sufficiently large. Furthermore, such a ϕ_0 can be chosen so that $y \equiv \dfrac{d\phi}{d\tau} < 0$. In order for ϕ to become less than ϕ_0 to the past of τ_0 it is necessary for y to pass through zero which cannot happen since $\sup_{\phi>\phi_0}(y^2 - 1)$ decreases monotonically as ϕ_0 increases. It follows that $\phi \to \infty$ and, hence, that $y^2 \to 1$. This implies that $\dot{\phi}^2/V(\phi) \to \infty$ (from (2.2) and the definition of y (2.13)) which may be equivalently expressed as $\rho/p \to 1$.

Since theses solutions satisfy the Strong Energy Condition they must possess an initial singularity and physical intuition would suggest that they are approximated asymptotically by a massless scalar field solution (1.81) which satisfies the equation of state $\rho = p$ exactly.

The models whose asymptotic behavior is of primary interest are those which have the gross shape of a potential well (diverging at $\pm\infty$ and having a minimum somewhere near $\phi = 0$). It is when the potential becomes large that we expect the Strong Energy Condition to be violated. Roughly speaking, SED potentials contain all differentiable non-negative functions which diverge (or decay) in a relatively nice way at a slower than exponential rate.

There are a number of reasons why such potentials represent a natural and interesting class of models. Firstly, in standard field theories it is usual to consider polynomial potentials and all such potentials, as well as any potential which is qualitatively similar to a polynomial are SED. Secondly, as has already been alluded to in chapter 1, the behavior of the ratio V'/V has direct dynamical consequences and, in particular, there seems to be a close relationship between the emergence of inflationary behavior and the ratio V'/V.

Nevertheless, models with exponential (or steeper) potentials are not without interest. As has already been mentioned, they appear naturally in the context of scalar tensor theories and other non-standard theories of gravity. The dynamics of such models turns out to be quite different from that of SED potentials and will be discussed in Chapter 3.

2.3.2 Smoothness Conditions

Suppose I were to make a coordinate transformation $z = f(\phi)$ where f is smooth and strictly decreasing for sufficiently large ϕ, and $\lim_{\phi\to\infty} f(\phi) = 0$ so that $\phi = \infty$ is mapped onto the origin of z. If we define the function

$$\bar{W}_V(z) = W_V(f^{-1}(z))$$

then since f is smooth it follows from the Inverse Function Theorem that $\bar{W}_V$ is continuous and differentiable on the range of f. $\bar{W}_V$ can be extended to the origin of z by defining

$$\bar{W}_V(0) = \lim_{\phi\to\infty} W_V(\phi).$$

If V is SED at infinity then (2.20) ensures that $\bar{W}_V$ is continuous at the origin and $\bar{W}_V(0) = 0$. The differentiability of $\bar{W}_V$ at the origin, however, is not assured and in general will depend on the choice of f as well as the properties of V itself. This suggests a useful means of classifying the smoothness of an SED function at infinity. The idea is that a function is SED of class k at infinity if there exists some smooth transformation $z = f(\phi)$ for which $\bar{W}_V$ is

C^k in some neighbourhood of the origin. Furthermore, it is desirable the the function f be sufficiently well behaved that its first derivative with respect to ϕ be expressible as a C^k function of z.

Before making this definition more precise we take note of the following convention, which will be assumed henceforth. If we have some coordinate transformation $z = f(\phi)$ which maps a neighbourhood of infinity to a neighbourhood of the origin, then if g is a function of ϕ, $\bar{g}$ is the function of z whose domain is the range of f plus the origin, which takes the values;

$$\bar{g}(z) = \begin{cases} g(f^{-1}(z)) & , \quad z > 0 \\ \lim_{\phi\to\infty} g(\phi) & , \quad z = 0 \end{cases}$$

$\bar{g}$ should be thought of as the extension of g onto infinity. With this in mind we classify the smoothness of an SED potential at infinity according to the following definition.

Definition 2.2 *A C^k potential V is class k SED at infinity if it is SED at infinity and if there exists $\phi_0 > 0$ and a coordinate transformation $z = f(\phi)$ which maps the interval $[\phi_0, \infty)$ onto $(0, \epsilon]$, where $\epsilon = f(\phi_0)$ and $\lim_{\phi\to\infty} f = 0$, with the following additional properties:*

i) *f is C^{k+1} and strictly decreasing.*
ii) *the functions $\bar{W_V}\ (z)$ and $\bar{f'}\ (z)$ are C^k on the closed interval $[0, \epsilon]$.*
iii) $\dfrac{d\bar{W_V}}{dz}(0) = \dfrac{d\bar{f'}}{dz}(0) = 0.$

We designate the set of all class k SED at infinity functions $\mathcal{E}^k_+$. The third condition in the above definition, that the first derivatives of $\bar{W_V}$ and $\bar{f'}$ vanish at the origin, has been imposed for reasons that will become clear later. It does not effect the generality of the definition since if the first derivatives exist at the origin they can always be made zero by choosing the new coordinate $z' = z^{\frac{1}{2}}$.

Table 2.3.2 lists a number of examples of $\mathcal{E}^\infty_+$ potentials together with convenient coordinate transformations. The first entry on the table is an arbitrary power law potential. In fact it is not difficult to verify that *all* non-negative polynomial potentials are $\mathcal{E}^\infty_+$.

For our purposes it will be sufficient that we require potentials to be $\mathcal{E}^2_+$. It is important to note that if W_V is C^2 it does not necessarily follow that

$V(\phi)$	$W_V(\phi)$	$z = f(\phi)$	$\bar{W}_V\ (z)$	$\bar{f}'\ (z)$
$\left\|\frac{\lambda}{n}\right\| \phi^n$, for $\phi > 0$	$n\phi^{-1}$	$\phi^{-\frac{1}{2}}$	nz^2	$-\frac{1}{2}z^3$
$2e^{\lambda\sqrt{\phi}}$, for $\phi > 0$	$\lambda\phi^{-\frac{1}{2}}$	$\phi^{-\frac{1}{4}}$	λz^2	$-\frac{1}{4}z^5$
$\ln\phi$, for $\phi > 1$	$(\phi\ln\phi)^{-1}$	$(\ln\phi)^{-1}$	$ze^{-\frac{1}{z}}$	$-ze^{-\frac{2}{z}}$
$\phi^2\ln\phi$, for $\phi > 1$	$2\phi^{-1} + (\phi\ln\phi)^{-1}$	$(\ln\phi)^{-1}$	$(2+z)e^{-\frac{1}{z}}$	$-ze^{-\frac{2}{z}}$

Table 2.1: Simple SED potentials. n and λ are arbitrary constants.

$V \in \mathcal{E}_+^2$. In particular, if the derivatives of W_V are oscillatory then these oscillations will tend to pile up as $z \to 0$ and may lead to discontinuities in the derivatives or $\bar{W}_V$. The following theorem shows that if W_V is C^2, V will be class 2 SED whenever the second derivative of W_V decays to zero sufficiently rapidly.

Theorem 2.5 *Let V be SED at infinity and let $W_V(\phi)$ be C^2 for sufficiently large ϕ. If there exists $\epsilon > 0$ such that*

$$\lim_{\phi\to\infty} \phi^{2+\epsilon} W_V''(\phi) = 0 \tag{2.22}$$

then V is class 2 SED at infinity.

Proof:

Let $z = f(\phi)$. If f is continuous and C^3 and f' is strictly negative, the inverse function theorem implies that $\bar{W}_V$ and $\bar{f}'$ are C^2 on the range of f. Thus, it is sufficient to prove that $\bar{W}_V$ and $\bar{f}'$ are C^2 at the origin. It follows from the definition of f and (2.20) that $\bar{W}_V$ and $\bar{f}'$ are continuous at the origin. The derivatives at the origin, will thus be equal to the limit of the derivatives as $z \to 0$ if these limits exist. Thus, by the inverse function theorem

$$\frac{d\bar{W}_V}{dz}(0) = \lim_{\phi\to\infty} \frac{W_V'(\phi)}{f'(\phi)} \tag{2.23}$$

and, if this limit exists,

$$\frac{d^2\bar{W}_V}{dz^2}(0) = \lim_{\phi\to\infty} \left(\frac{W_V''(\phi)}{f'(\phi)^2} - \frac{W_V'(\phi)f''(\phi)}{f'(\phi)^3} \right). \tag{2.24}$$

We shall show that for an appropriate choice of f the above limits vanish. Integrating (2.22) and keeping in mind (2.20), we have;

$$\lim_{\phi\to\infty} \phi^{1+\epsilon} W'_V(\phi) = 0. \tag{2.25}$$

Choose

$$f = \phi^{-\frac{\epsilon}{n}}, \qquad n > Max[\epsilon, 4]. \tag{2.26}$$

Then differentiating twice;

$$\begin{aligned} f'(\phi) &= -\frac{\epsilon}{n}\phi^{-(1+\frac{\epsilon}{n})} \\ f''(\phi) &= \frac{\epsilon}{n}\left(1+\frac{\epsilon}{n}\right)\phi^{-(2+\frac{\epsilon}{n})}. \end{aligned} \tag{2.27}$$

It follows immediately from (2.27) and (2.26) that

$$\bar{f}'(z) = -\frac{\epsilon}{n} z^{1+\frac{n}{\epsilon}} \tag{2.28}$$

which is everywhere C^2 with vanishing first derivative at the origin, provided $n > \epsilon$, and thus satisfies the conditions of the definition.

Substituting (2.27) into the right hand side of (2.23) and (2.24) and taking the limit, making use of (2.22) and (2.25) we have

$$\frac{d\bar{W}_V}{dz}(0) = \frac{d^2\bar{W}_V}{dz^2}(0) = 0 \tag{2.29}$$

which is the required result. □

Theorem 2.5 provides a simple and fairly general criterium for a potential to be $\mathcal{E}^2_+$ and in fact all of the potentials listed in table 2.3.2 satisfy the conditions of Theorem 2.5, as well as all non-negative polynomials and other less "nice" functions such as $V(\phi) = 2\phi + \sin\phi$. Note also, that W_V is C^2 whenever V is C^3 for large ϕ.

One undesirable feature of theorem 2.5, however, is that by imposing an upper limit on the magnitude of the W''_V which is less than $O(\phi^{-2})$ it excludes SED potentials for which W_V decays to zero very slowly (so that it cannot be bounded by a power law) but which nevertheless are very smooth at infinity. Consider, for example, the potential

$$V(\phi) = e^{\frac{\phi}{\ln \phi}}$$

which might be thought of as being "nearly" exponential. Taking the derivative we can obtain;

$$W_V(\phi) = \frac{1}{\ln \phi} - \frac{1}{(\ln \phi)^2}$$

which obviously tends to 0 as $\phi \to \infty$. It is easy to see that there exists no $\epsilon > 0$ which can bound the second derivative in the sense of theorem 2.5. Nevertheless $V \in \mathcal{E}_+^\infty$ as can easily be shown by verifying directly that the transformation

$$z = \frac{1}{(\ln \phi)^{\frac{1}{2}}}$$

satisfies the conditions of Definition 2.2. Although potentials such as the one above may not appear particularly physical, Definition 2.2 was imposed as classification of smoothness, not an additional bound on the rate at which functions diverge. The following theorem provides a condition which does not bound the rate of divergence of the function (beyond that it be slower than exponential, of course). For convenience in dealing with limits I will make use of the extended real number line $[-\infty, \infty]$ in the statement and proof of the theorem and use the notation $\lim_{\phi\to\infty} g = \infty$ for a diverging function g.

Theorem 2.6 *Let V be sub exponential at infinity and let $W_V(\phi)$ be C^3 for sufficiently large ϕ. Then V is class 2 SED at infinity if, for all $\alpha \geq 0$, there exists $L_\alpha \in [-\infty, \infty]$ (which may equal infinity) such that*

$$\lim_{\phi\to\infty} \phi^\alpha \frac{d^3 W_V}{d\phi^3} = L_\alpha. \tag{2.30}$$

This theorem places restrictions on "oscillations" of the third derivative of W_V. It essentially states that for any α, β greater than 0, $W_V'''(\phi)$ cannot oscillate between $\phi^{-\alpha}$ and $\phi^{-\beta}$ as $\phi \to \infty$.

Proof:

As with Theorem 2.5 it will be sufficient to show that the limits as $z \to 0$ of the first two derivatives of $\bar{W}_V$ and $\bar{f}'$ exist and are equal to zero for some $z = f(\phi)$.

Putting α equal to zero and integrating (2.30) three times and comparing with (2.20) yields immediately

$$\lim_{\phi\to\infty} W_V'(\phi) = \lim_{\phi\to\infty} W_V''(\phi) = \lim_{\phi\to\infty} W_V'''(\phi) = 0. \tag{2.31}$$

If we put $\alpha = 3$, (2.30) becomes

$$\lim_{\phi\to\infty} \phi^3 W_V'''(\phi) = L_3 \tag{2.32}$$

It follows from l'Hôpital's rule that

$$\begin{aligned}\lim_{\phi\to\infty} \phi^2 W_V''(\phi) &= -3L_3\\ \lim_{\phi\to\infty} \phi W_V'(\phi) &= 6L_3.\end{aligned}$$

Integrating the last expression and comparing with (2.20) gives $L_3 = 0$. We thus have;

$$\lim_{\phi\to\infty} \phi^n \frac{d^n W_V}{d\phi^n} = 0, \qquad n = 1, 2, 3. \tag{2.33}$$

(2.33) implies that $L_\alpha = 0$ for all $\alpha \leq 3$. Assume that there exists some $\alpha_1 > 3$ for which $L_{\alpha_1} = 0$ then l'Hôpital's rule implies

$$\lim_{\phi\to\infty} \phi^2 W_V''(\phi) = 0.$$

It follows from Theorem 2.5 that V is class 2 SED at infinity and the theorem is proved. Let us assume the converse: ie, there exists no $\alpha_1 > 3$ for which $L_{\alpha_1} = 0$. It follows from (2.30) that for all $\epsilon > 0$, $L_{3+\epsilon} = \infty$. If we have $\epsilon, \delta > 0$ then (2.30) and (2.33) imply there exists $\phi_0 > 0$ such that for $\phi > \phi_0$

$$\frac{\epsilon}{\phi^{3+\delta}} < |W_V'''(\phi)| < \frac{\epsilon}{\phi^3}. \tag{2.34}$$

Which yields upon integration, keeping in mind (2.33), for sufficiently large ϕ;

$$\frac{\epsilon}{\phi^{2+\delta}} < |W_V''(\phi)| < \frac{\epsilon}{\phi^2} \tag{2.35}$$

$$\frac{\epsilon}{\phi^{1+\delta}} < |W_V'(\phi)| < \frac{\epsilon}{\phi}. \tag{2.36}$$

Choose

$$f = W_V^{\frac{1}{2}}.$$

We immediately have $\bar{W}_V\ (z) = z^2$, which is C^∞ at $z = 0$. It remains to show that $\bar{f}'$ is C^2 at the origin. Using the continuity of $\bar{f}'$ and the Inverse Function Theorem gives

$$\frac{d\bar{f}'}{dz}(0) = \lim_{\phi\to\infty} \frac{f''(\phi)}{f'(\phi)} \tag{2.37}$$

and

$$\frac{d^2\bar{f}'}{dz^2}(0) = \lim_{\phi\to\infty} \left(\frac{f'''(\phi)}{f'(\phi)} - \frac{f''(\phi)^2}{f'(\phi)^3}\right). \tag{2.38}$$

Explicitly differentiating f with respect to z gives

$$\begin{aligned} f' &= \frac{1}{2}W_V^{-\frac{1}{2}}W_V' \\ f'' &= -\frac{1}{4}W_V^{-\frac{3}{2}}W_V'^2 + \frac{1}{2}W_V^{-\frac{1}{2}}W_V'' \\ f''' &= \frac{3}{8}W_V^{-\frac{5}{2}}W_V'^3 - \frac{1}{2}W_V^{-\frac{3}{2}}W_V'W_V'' \\ &\quad -\frac{1}{4}W_V^{-\frac{3}{2}}W_V''^2 + \frac{1}{2}W_V^{-\frac{1}{2}}W_V''' \end{aligned} \tag{2.39}$$

Substituting these expressions into (2.37) and (2.38) and making use of the inequalities (2.34-2.36) we find after a tedious but trivial calculation

$$\frac{d\bar{f}'}{dz}(0) = \frac{d^2\bar{f}'}{dz^2}(0) = 0 \tag{2.40}$$

as required. □

The purpose Theorems 2.5 and 2.6 was to convince the reader that $\mathcal{E}_+^2$ is a natural and general class of functions which provides a suitable characterization of well behaved, slower than exponential potentials. In the next section we shall investigate the general past asymptotic behavior of these models. As a final note before proceeding, observe that the condition $\frac{d\bar{f}'}{dz}(0) = 0$ in

Definition 2.2 may be expressed equivalently (using the chain rule and inverse function theorem) as

$$\lim_{\phi\to\infty} \frac{f''}{f'} = 0.$$

It follows that $1/f'$ is SED and hence by Theorem 2.4 that it is exponential dominated. This implies in turn that $1/f$ is also exponential dominated. The function f must therefore obey the following condition: for all $\epsilon > 0$.

$$\lim_{\phi\to\infty} \frac{e^{\epsilon\phi}}{f'} = \lim_{\phi\to\infty} \frac{e^{\epsilon\phi}}{f} = 0. \tag{2.41}$$

2.4 The System Close To $\phi = \infty$.

According to Theorems 2.1 and 2.3, $|\phi|$ diverges in the past of almost all solutions of (2.1). Let us now analyze the system in the region of phase space where ϕ is very large and positive. We shall assume that $V \in \mathcal{E}^2_+$. Define the set $\Sigma_\epsilon \subset R^3$:

$$\Sigma_\epsilon = \{(K, \phi, \dot\phi) : K > 0, \phi > \epsilon^{-1}\} \tag{2.42}$$

where $\epsilon \in R > 0$. If ϵ is chosen sufficiently small, the following coordinate transformation can be made on Σ_ϵ (c.f. Theorem 3):

$$x = \frac{1}{K} \qquad y = \sqrt{\tfrac{3}{2}}\frac{\dot\phi}{K} \qquad z = f(\phi) \tag{2.43}$$

where $f(\phi)$ tends to 0 as $\phi \to \infty$ and satisfies conditions i)-iii) of definition 2.2 with $k = 2$.

Reintroducing the time coordinate τ defined in Equation (2.14), the equations on Σ_ϵ become

$$\begin{aligned} \frac{dx}{d\tau} &= y^2 x \\ \frac{dy}{d\tau} &= -y - \sqrt{\tfrac{3}{2}}x^2\bar{V}'(z) + y^3 \\ \frac{dz}{d\tau} &= \sqrt{\tfrac{2}{3}}\bar{f}'(z)y \end{aligned} \tag{2.44}$$

In terms of these new coordinates, Σ_ϵ can be re-expressed as

$$\Sigma_\epsilon = \{(x, y, z) : x > 0, 0 < z < f(\epsilon^{-1})\}. \tag{2.45}$$

The physical trajectories of (2.44) on Σ_ϵ are those that satisfy the constraint equation

$$y^2 + 3x^2\bar{V}(z) = 1. \tag{2.46}$$

Let $\Omega_\epsilon = \{(x, y, z) \in \Sigma_\epsilon : y^2 + 3x^2\bar{V}(z) = 1\}$. Then Ω_ϵ is the intersection of Σ_ϵ with the physical phase space Ω. It is a 2D invariant manifold of (2.44) imbedded in Σ_ϵ. Restricted to Ω_ϵ the system (2.44) reduces to the 2D dynamical system

$$\begin{aligned}\frac{dy}{d\tau} &= -(y + \tfrac{1}{\sqrt{6}}\bar{W}_V(z))(1 - y^2)\\ \frac{dz}{d\tau} &= \sqrt{\tfrac{2}{3}}\bar{f}'(z)y\end{aligned} \tag{2.47}$$

on the bound, open projection $\Omega_\epsilon^* = \{0 < z < f(\epsilon^{-1}), -1 < y < 1\}$ of Ω_ϵ onto R^2. For convenience I will drop the asterix and simply refer to this set as Ω_ϵ. Equation (2.47a) is obtained by substituting the constraint equation (2.46) into the second term on the right hand side of (2.44b). The variable x can now be treated as a function on Ω_ϵ defined by equation (2.46). Equation (2.44a) gives the directional derivative of x along the flow generated by (2.47).

From the definition of z, (2.43), it is clear that the line $z = 0$ (which lies outside of Ω_ϵ) represents the infinity of ϕ. Thus, by Theorem 2.3, almost all solutions must come arbitrarily close to either this set or its counterpart at $\phi = -\infty$.

In order to study this behavior it is desirable to complete Ω_ϵ and extend (2.47) onto the boundary. If $V \in \mathcal{E}_+^2$ and we have been careful in our choice of f then $\bar{W}_V$ and $\bar{f}'$ will be well defined and C^2 at $z = 0$. It follows immediately from inspection that (2.47) is everywhere C^2 on the closure of Ω_ϵ, $\tilde{\Omega}_\epsilon = \{0 \le z \le f(\epsilon^{-1}), -1 \le y \le 1\}$.

For convenience later, it will be useful to summarize here some of the properties of $\bar{W}_V$ and $\bar{f}'$.

$$\bar{W}_V(0) = \bar{f}'(0) = 0. \tag{2.48}$$

$$\bar{f}'(z) < 0 \text{ for all } z > 0. \tag{2.49}$$

$$\frac{d\bar{W}_V}{dz}(0) = \frac{d\bar{f}'}{dz}(0) = 0. \tag{2.50}$$

The important points to note are that both functions are second order or higher in z and that $\bar{f}'$ is negative on the interior.

Example: *Consider the special case of the power law potential*

$$V(\phi) = \frac{\lambda}{n}\phi^n.$$

where λ and n are positive constants. Introducing the z coordinate $z = \phi^{-\frac{1}{2}}$ and substituting the expressions $\bar{W}_V$ and $\bar{f}'$ given in Table 2.3.2 into equation (2.47) gives the system

$$\begin{aligned}\frac{dy}{d\tau} &= -y + y^3 - \frac{n}{\sqrt{6}}z^2(1-y^2) \\ \frac{dz}{d\tau} &= -\frac{1}{\sqrt{6}}z^3 y.\end{aligned} \tag{2.51}$$

Observe that it is perfectly regular everywhere on $\tilde{\Omega}_\epsilon$ (in fact, everywhere on R^2).

Now that it has been established that (2.47) forms a C^2 dynamical system on the compact set $\tilde{\Omega}_\epsilon$, let us begin to examine the properties of this system. Fig. 2.6 gives a schematic illustration of $\tilde{\Omega}_\epsilon$.

The boundary $\partial\Omega_\epsilon$ is the union of the unphysical sets $\partial\Omega_{\epsilon 1}$, $\partial\Omega_{\epsilon 3}$ and $\partial\Omega_{\epsilon 4}$, as well as the physical set $\partial\Omega_{\epsilon 2}$ on which the physical variables H, ϕ and $\dot{\phi}$ are finite. $\partial\Omega_{\epsilon 1}$, $\partial\Omega_{\epsilon 3}$ and $\partial\Omega_{\epsilon 4}$ are each invariant manifolds of (2.47). $\partial\Omega_{\epsilon 2}$ is the only element of the boundary which is transverse to the flow. It is through this set that physical trajectories enter and leave $\tilde{\Omega}_\epsilon$.

In fact, by (2.49) and (2.47b) it can be seen that the sign of $\frac{dz}{d\tau}$ is opposite to the sign of y. Thus trajectories may enter $\tilde{\Omega}_\epsilon$ only on the right half of the plane and leave only on the left half of the plane.

The solution of (2.47) on $\partial\Omega_{\epsilon 1}$ can be obtained by substituting $z = 0$ into the equations and solving the resultant differential equation

$$\frac{dy}{d\tau} = -y + y^3. \tag{2.52}$$

The direction of the flow is shown on Fig. 2.6. Note that the points $(0, 0)$, $(\sqrt{\frac{2}{3}}, 0)$ and $(-\sqrt{\frac{2}{3}}, 0)$ are equilibrium points. I will refer to them as p_0, p_+ and p_- respectively and will return to them shortly.

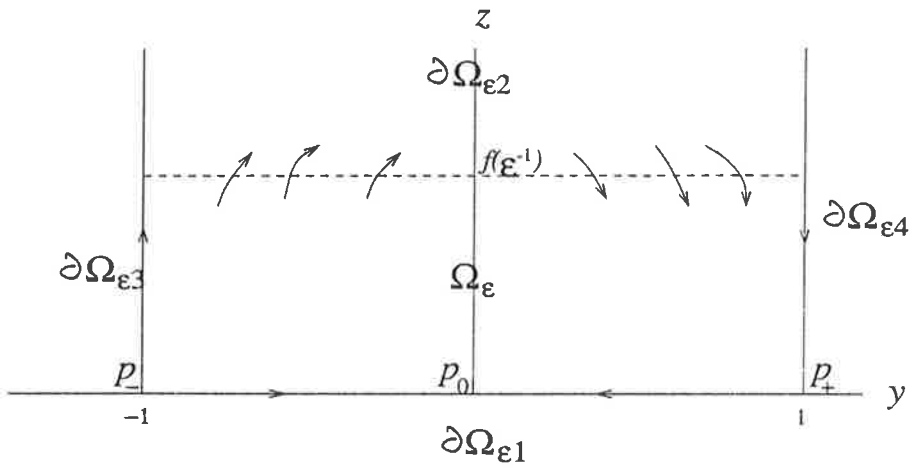

Figure 2.6: Schematic representation of the compact set $\bar{\Omega}_\epsilon$ showing fixed points and the sense of orbits on the boundary.

In order to find the solution of (2.47) on $\partial\Omega_{\epsilon 3}$ and $\partial\Omega_{\epsilon 4}$ it is necessary to solve the differential equation

$$\frac{dz}{d\tau} = \pm \bar{f}'(z)\sqrt{\tfrac{2}{3}}. \tag{2.53}$$

Recalling that $z = f(\phi)$ this is just the equation

$$\frac{d\phi}{d\tau} = \pm\sqrt{\tfrac{2}{3}}$$

which yields

$$\phi = \sqrt{\tfrac{2}{3}}(\pm\tau + \tilde{\phi}) \tag{2.54}$$

where $\tilde{\phi} = \phi(0)$ is a positive constant. Thus

$$z = f\left(\sqrt{\tfrac{2}{3}}(\tilde{\phi} \pm \tau)\right). \tag{2.55}$$

The flow is directed towards the y-axis on $\partial\Omega_{\epsilon 4}$ and away from the y-axis on $\partial\Omega_{\epsilon 3}$ as shown in Fig. 2.6.

2.5 Stability of the Equilibrium Points p_+ and p_-.

Our next step is to classify the equilibrium points of the system. Inspection of (2.47), remembering (2.49) , will satisfy the reader that p_+, p_- and p_0 are the only equilibrium points. Let us first consider p_+ and p_-. From (2.47) we can evaluate the total derivative of the vector field $v = (\frac{dy}{d\tau}, \frac{dz}{d\tau})$ at $p_\pm$ to obtain the matrix

$$J(p_\pm) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.56}$$

The eigenvalues of $J(p_\pm)$ are 2 and 0. Thus, in a neighbourhood of $p_\pm$ there exists a 1-dimensional unstable manifold which is tangent to $\hat{e}_y = (1, 0)$ at $p_\pm$, and a 1-dimensional center manifold which is tangent to $\hat{e}_z = (0, 1)$ at $p_\pm$.

Clearly, from Fig. 2.6, the unstable manifold is just $\partial\Omega_{\epsilon 1}$ (or more precisely, a sufficiently small interval of $\partial\Omega_{\epsilon 1}$ in a neighbourhood of p_+ and p_- respectively). $\partial\Omega_{\epsilon 3}$ is a center manifold of p_+ and $\partial\Omega_{\epsilon 3}$ is a center manifold of p_-.

All solutions in the neighbourhood of p_+ or p_- will rapidly approach their respective center manifolds in the reverse time direction. Consequently, according to Theorem A.8 the stability of p_+ and p_- depends on the asymptotic behavior of the solutions on $\partial\Omega_{\epsilon 3}$ and $\partial\Omega_{\epsilon 4}$ respectively. Thus, by (2.55), p_+ is a saddle (2.7a) and p_- is a source (2.7b).

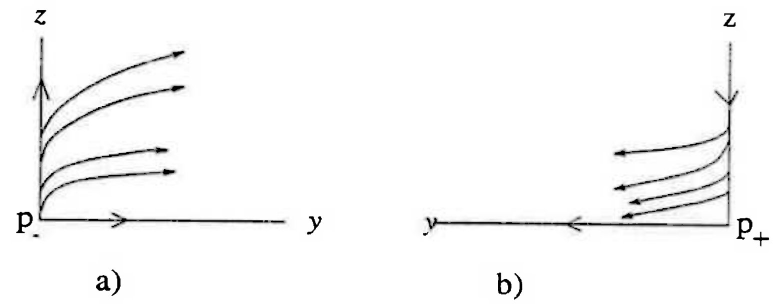

Figure 2.7: The stability of the equilibrium points p_+ and p_-.

Let us concentrate on the source p_- since it is a generic α-limit set of the system. The past asymptotic behavior of solutions close to p_- will be approximated by the exact solution on the center manifold according to Theorem A.8. Thus for τ sufficiently large and negative we may write

$$y = -1 + O(e^{\gamma\tau}) \qquad z = f\left(\sqrt{\tfrac{2}{3}}(\tilde{\phi} - \tau)\right) + O(e^{\gamma\tau}) \qquad (2.57)$$

For some positive constant γ. Note that evaluating the inverse $f^{-1}(z)$ in the above expression and using the mean value theorem and the expressions (2.41) we find that, to first order, ϕ is indeed approximated by (2.54).

Let us now express the asymptotic solution (2.57) fully in terms of the original variables. As $y \to -1$, equation (2.44a) becomes

$$\frac{dx}{d\tau} = x(1 + O(e^{\gamma\tau})). \qquad (2.58)$$

Integrating gives the first order expression

$$x = x_0 e^{\tau}. \qquad (2.59)$$

By (2.15)

$$\begin{aligned} t &= \int x d\tau \\ &= x_0 e^{\tau} + t_i \end{aligned} \qquad (2.60)$$

which implies that $t \to t_i$ as $\tau \to -\infty$. Substituting (2.60) into (2.59) and using (2.43) gives

$$K = (t - t_i)^{-1}. \qquad (2.61)$$

Note that $K \to \infty$ as $t \to t_i$, thus the source p_- does indeed correspond to an initial space-time singularity. Substituting (2.60) and (2.61) into (2.57) (recalling the definition of y) we find

$$\phi = -\sqrt{\tfrac{2}{3}} \ln \frac{(t - t_i)}{c} \qquad \dot{\phi} = -\sqrt{\tfrac{2}{3}}(t - t_i)^{-1} \qquad (2.62)$$

where $c = x_0 e^{\tilde{\phi}}$ is a positive constant.

The asymptotic solution (2.61), (2.62) is the exact solution of the system (2.1) when V is identically zero; ie the massless scalar field. Thus there exists a generic class of cosmologies which, in a sufficiently small neighbourhood of the singularity, behave, approximately, as though the matter content were a massless scalar field.

In order to make the above statement more precise it is necessary to do two things. Firstly, we shall obtain an error estimate for equations (2.61) and (2.62) close to $t = t_i$. Secondly, we shall demonstrate that close to $t = t_i$ there a continuous 1-1 correspondence between massless scalar field cosmologies and cosmologies of the model $V(\phi)$ which are past asymptotic to p_-. We shall assume, without loss of generality, that $t_i = 0$.

Theorem 2.7 *Let $V \in \mathcal{E}^2_+$ be given. There exists a neighbourhood $N(p_-)$ of p_- such that for all $p \in N(p_-)$ the solution $\psi_p(\tau)$ corresponds to a scalar field cosmology which may be written, for sufficiently small t,*

$$\begin{aligned} K &= t^{-1} + O(\epsilon_V(t)) \\ \phi &= -\sqrt{\tfrac{2}{3}}\ln\frac{t}{c} + O(t\epsilon_V(t)) \\ \dot{\phi} &= -\sqrt{\tfrac{2}{3}}t^{-1} + O(\epsilon_V(t)) \end{aligned} \tag{2.63}$$

where

$$\begin{aligned} \epsilon_V(t) &= tV(-\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}) \\ &\to 0 \quad as \quad t \to 0. \end{aligned} \tag{2.64}$$

Proof:

Let us proceed by calculating a second order solution for (2.44a). Any collection of higher order terms which are to be discarded will be denoted by a generic h. By higher order terms I mean, more precisely, that the ratio of h to all undiscarded terms tends to zero as $\tau \to -\infty$. We have

$$\frac{dx}{d\tau} = y^2 x \tag{2.65}$$

where we know that $y^2 \to 1$ as $\tau \to -\infty$ and x approaches the first order solution (2.59). Note also that as an immediate consequence of (2.62), (2.60) and Theorem 2.4,

$$\lim_{\tau\to-\infty} e^{\alpha\tau}\bar{V}(z(\tau)) = \lim_{t\to 0} t^{\alpha}V(\phi(t)) = 0 \;\; \forall\, \alpha > 0. \tag{2.66}$$

From (2.46)

$$\begin{aligned} y^2 &= 1 - 3\bar{V}(z)x^2 \\ &= 1 - 3x_0^2\bar{V}(z)e^{2\tau} + h. \end{aligned} \tag{2.67}$$

Thus,

$$\frac{d\ln x}{d\tau} = 1 - 3x_0^2\bar{V}(z)e^{2\tau} + h \tag{2.68}$$

Since V and z vary slowly compared to e^τ we expect

$$\int \bar{V}(z)e^{2\tau}d\tau \simeq \tfrac{1}{2}\bar{V}(z)e^{2\tau}.$$

Let us now verify that this is indeed the case. Consider the indefinite integral

$$I(\tau) = \int_{\tau_0}^{\tau} \bar{V}(z)e^{2\tau}d\tau$$

where τ_0 will later be allowed to go to $-\infty$. Integrating by parts gives,

$$\begin{aligned} I(\tau) &= \tfrac{1}{2}\bar{V}(z)e^{2\tau}|_{\tau_0}^{\tau} - \tfrac{1}{2}\int_{\tau_0}^{\tau} e^{2\tau}\bar{V}'(z)y\,d\tau & (2.69) \\ &= I_1(\tau) + I_2(\tau) & (2.70) \end{aligned}$$

Consider the second term. Let $\delta(\tau) = \sup_{\tau'<\tau} \frac{1}{2}|\bar{W}_V\ (\tau')|$, then, recalling that $y^2 < 1$;

$$\begin{aligned} |I_2(\tau)| &= \tfrac{1}{2}\left|\int_{\tau_0}^{\tau} e^{2\tau}\bar{V}(z)\bar{W}_V\ (z)y\,d\tau\right| \\ &< \delta(\tau)|I(\tau)| \\ &\leq \delta(\tau)(|I_1(\tau)| + |I_2(\tau)|) \\ &< \frac{\delta(\tau)}{1-\delta(\tau)}|I_1(\tau)| \end{aligned} \tag{2.71}$$

for τ sufficiently small. Letting τ_0 go to $-\infty$ we have for all τ sufficiently negative;

$$I(\tau) = \frac{1}{2}\bar{V}(z)e^{2\tau} + I_2(\tau) \tag{2.72}$$

where

$$|I_2(\tau)| < \frac{1}{2}\delta(\tau)\bar{V}(z)e^{2\tau}. \tag{2.73}$$

This is the required result since

$$\lim_{\tau \to -\infty} \delta(\tau) = \limsup_{z \to 0} \sqrt{\frac{3}{2}} |\bar{W}_V\ (z)| = 0$$

implying that for τ sufficiently negative, the ratio of I_2 to I can be made arbitrarily small.

We may now integrate (2.68), using (2.73) and (2.66) to obtain,

$$\begin{aligned} x &= x_0 \exp\left(\tau - \tfrac{3}{2} x_0^2 \bar{V}(z) e^{2\tau}\right) + h \\ &= x_0 (e^{\tau} - \tfrac{3}{2} x_0^2 \bar{V}(z) e^{3\tau}) + h \end{aligned} \tag{2.74}$$

Integrating again to obtain a second order expression for t, remembering that $t_i = 0$ we get

$$t = x_0 (e^{\tau} - \tfrac{1}{2} x_0^2 \bar{V}(z) e^{3\tau}) + h$$

where the second term on the left hand side has been estimated in precisely the same way as $I(\tau)$ above. This expression may be inverted, to second order, to give

$$x_0 e^{\tau} = t + \frac{\bar{V}(z) t^3}{2} + h. \tag{2.75}$$

substituting this into (2.74) we have

$$x(t) = t - \bar{V}(z) t^3 + h.$$

Using the definition of x, and making use of the binomial expansion we obtain for K;

$$K = t^{-1} + V(\phi) t + h. \tag{2.76}$$

where $\bar{V}$ has been replaced with V in the above expression. In order to obtain an error estimate for ϕ, recall that

$$\frac{d\phi}{d\tau} = \sqrt{\tfrac{2}{3}} y$$

which, using (2.67), becomes

$$\frac{d\phi}{d\tau} = -\sqrt{\tfrac{2}{3}} \left(1 - \tfrac{3}{2} x_0^2 e^{2\tau} V(\phi)\right) + h.$$

Integrating, and replacing τ with proper time t;

$$\phi(t) = -\sqrt{\tfrac{2}{3}}\left(\ln\frac{t}{c} - \tfrac{1}{4}V(\phi)t^2\right) + h. \tag{2.77}$$

Also,

$$\begin{aligned}
\dot{\phi} &= \sqrt{\tfrac{2}{3}}Ky \\
&= -\sqrt{\tfrac{2}{3}}\left(t^{-1} + V(\phi)t\right)\left(1 - \tfrac{3}{2}x_0^2e^{2\tau}V(\phi)\right) + h \\
&= -\sqrt{\tfrac{2}{3}}\left(t^{-1} - \tfrac{1}{2}V(\phi)t\right) + h. \qquad (2.78)
\end{aligned}$$

Clearly, the second term on the right hand side of (2.77) tends to 0 as $t \to 0$. This allows us to Taylor expand V about $\phi = -\sqrt{\frac{2}{3}}\ln\frac{t}{c}$ to obtain,

$$\begin{aligned}
V(\phi(t)) &= V\left(-\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}\right) + bV'\left(-\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}\right)V(\phi)t^2 + h \qquad (2.79) \\
&= V\left(-\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}\right)\left(1 + bW_V\left(-\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}\right)V(\phi)t^2\right) + h
\end{aligned}$$

where b is a constant. Substituting this equation into (2.76) and (2.77) completes the proof. □

Thus we have obtained the required error estimates for our asymptotic solution. Equations (2.63) and (2.107) indicate that for t sufficiently small, equations (2.61) and (2.62) provide an excellent approximation of the physical system since whilst the physical variables diverge the error terms rapidly decay to zero.

Theorem 2.8 *Let $V \in \mathcal{E}^2_+$ be given. For each $c > 0$ there exists a unique cosmology satisfying (2.63). Furthermore, the higher order terms are continuous functions of c and t.*

Proof:

Proving uniqueness is non-trivial. Firstly, recall that for each solution past asymptotic to p_- there exists

$$c = x_0 e^{\dot{\phi}} \tag{2.80}$$

where x_0 and $\tilde{\phi}$ are as defined in (2.59) and (2.57). More precisely we may define

$$x_0 = \lim_{\tau \to -\infty} e^{-\tau} x(\tau) \qquad \tilde{\phi} = \lim_{\tau \to -\infty} \left(\tau + \sqrt{\tfrac{3}{2}} \phi(\tau) \right). \qquad (2.81)$$

Each of these constants depend on the point p at which we take $\tau = 0$ but for a given solution $\psi_p(\tau)$ a translation $\tau \mapsto \tau + \tau_0$ (which moves p along its orbit) leaves c invariant. This reflects the fact that each orbit corresponds to only one physical cosmology.

In order to see this more clearly we make use of the fact that ϕ diverges and is strictly decreasing with τ close to p_-. It follows that we may represent any orbit as a graph $(\phi, y(\phi)$. Substituting (2.81) into (2.80) and remembering that x is a function of state we have:

$$c = \lim_{\phi \to \infty} e^{\sqrt{\frac{3}{2}}\phi} x(\phi, y(\phi)). \qquad (2.82)$$

Thus c depends only on the graph $y(\phi)$ (i.e. the orbit) and is independent of the initial point p.

Let δ_0, δ_1 be small non-negative numbers and consider the surface

$$\Sigma(\delta_0, \delta_1) = \{(y, z) \in \tilde{\Omega}_\epsilon : 0 \leq -1 + y \leq \delta_1, z = \delta_0\}.$$

Clearly $\Sigma(\delta_0, \delta_1)$ is transverse to the flow and, for fixed δ_1, can be made to intersect an arbitrarily large proportion of the orbits asymptotic to p_- by letting $\delta_0 \to 0$.

Our plan of attack shall be as follows. Take $\Sigma(\delta_0, \delta_1)$ as a smooth surface of initial conditions in $\tilde{\Omega}_\epsilon$. We shall demonstrate a continuous 1-1 correspondence between points on $\Sigma(\delta_0, \delta_1)$ and values of c (as defined by (2.80)) on the interval $(0, c_m(\delta_0))$ where $c_m \to 0$ as $\delta_0 \to 0$.

This will immediately prove the first part of the theorem and the second part of the theorem (continuity) follows trivially from the continuity of the flow and the coordinate transformations (2.14), (2.43).

Let $w = y + 1$ then w may be used to label points on $\Sigma(\delta_0, \delta_1)$ and we may view x_0, $\tilde{\phi}$ and c as functions of w on the domain $0 \leq w \leq \delta_1$. We firstly demonstrate that c is continuous and 1-1.

For each $p = (-1 + w, \delta_0) \in \Sigma(\delta_0, \delta_0)$, the orbit $O^-(p)$ intersecting p may be represented by a graph $y(w; \phi)$. Since no orbits may intersect we see that if $w_2 > w_1$ then

$$y(w_2, \phi) - y(w_1, \phi) > 0 \text{ for all } \phi > \phi_0$$

where $\phi_0 = f^{-1}(\delta_0)$ is the initial value of ϕ.

Define the function

$$C(w;\phi) = x(\phi, y(w;\phi))e^{\sqrt{\frac{3}{2}}\phi}$$

then c is, by definition, the limit of C. Taking the derivative of this expression with respect to ϕ and making use of (2.44a,b) we have

$$\frac{dC}{d\phi} = \sqrt{\frac{3}{2}}(y+1)C \tag{2.83}$$

Also, from (2.46) we may obtain an explicit expression for C

$$C(w;\phi)^2 = \frac{e^{\sqrt{6}\phi}}{V(\phi)}(1 - y(w;\phi)^2). \tag{2.84}$$

It is immediately apparent from this expression that c is a continuous since the orbits $y(w;\phi)$ are uniformly continuous in w. Evaluating (2.84) at ϕ_0 we have

$$C(w;\phi_0)^2 = \frac{e^{\sqrt{6}\phi_0}}{3V(\phi_0)}(2w - w^2).$$

Thus $C(w,\phi_0)$ is strictly increasing with w.

Consider two points $p_1, p_2 \in \Sigma(\delta_0, \delta_1)$ with $w_2 > w_1$. For convenience we denote the graphs y and C along the respective orbits by y_1, C_1 and y_2, C_2. Taking the derivative of the difference $C_2(\phi) - C_2(\phi)$ using (2.83) we obtain

$$\frac{d}{d\phi}\left(\ln\frac{C_2(\phi)}{C_1(\phi)}\right) = \sqrt{\frac{3}{2}}(y_2(\phi) - y_1(\phi)) > 0 \tag{2.85}$$

where we have used the fact that $y_2 - y_1 > 0$. Integrating from ϕ_0 to ∞ we obtain

$$\ln\left(\frac{c(w_2)C_1(\phi_0)}{c(w_1)C_2(\phi_0)}\right) > 0.$$

from which it follows that

$$\frac{c(w_2)}{c(w_1)} > \frac{C_2(\phi_0)}{C_1(\phi_0)} > 1$$

since $C(w;\phi_0)$ is strictly increasing. Thus c is strictly increasing with w and hence 1-1.

It remains to show that c is onto.

Note firstly that $\phi_0 \to \infty$ as $\delta_0 \to 0$. Thus, by the definition of c as the limit of C we have for δ_0 sufficiently small;

$$\begin{aligned} c(w)^2 &= C(w;\phi_0)^2 + h \\ &= \frac{e^{\sqrt{6}\phi_0}}{3V(\phi_0)}(2w - w^2) + h, \end{aligned} \tag{2.86}$$

where $\phi_0(\delta_0) \to \infty$ as $\delta_0 \to 0$. Note also that c vanishes identically at $w = 0$. The range of c on the domain $(0, \delta_1)$ is therefore $(0, c_m(\delta_0, \delta_1))$ where

$$c_m(\delta_0, \delta_1) \simeq \frac{e^{\phi_0}}{\sqrt{V(\phi_0)}}(2\delta_1)^{\frac{1}{2}}.$$

Since V is SED this expression goes to infinity as $\phi_0 \to \infty$. This proves the result.□

Theorem 2.8 implies that V is truly dynamically insignificant in the neighbourhood of the singularity p_- in that the family of solutions which asymptotically approach p_- are completely characterized by the solution space of the massless scalar field cosmological model.

2.6 Existence of Inflating Solutions.

Let us now turn our attention to the remaining equilibrium point p_0. This will lead us to a brief digression into inflationary dynamics. Evaluating the derivative of the vector field at this point gives

$$J(p_0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.87}$$

The eigenvalues are -1 and 0. This implies that there locally exists a 1-dimensional center manifold C_0 through p_0, which is tangent to the z-axis at p_0, and a 1-dimensional stable manifold tangent to (and indeed a subset of) $\partial\tilde{\Omega}_{\epsilon 1}$. It follows immediately from Theorem A.7 that any solutions past asymptotic to p_0 must lie on the center manifold; ie the unique solution I_∞ lying on C_0 itself is the only solution which may approach p_0 in the past.

On C_0 we may write y as a function of z. Let $y = g(z)$ where

$$g(0) = \frac{dg}{dz}(0) = 0;$$

ie g is at least second order in z (see appendix). Then I_∞ must satisfy

$$\frac{dy}{d\tau} = \frac{dg}{dz}\frac{dz}{d\tau}.$$

Using (2.47b),

$$\begin{aligned}\frac{dy}{d\tau} &= \sqrt{\tfrac{2}{3}}\frac{dg}{dz}y\bar{f}'(z) \\ &= \sqrt{\tfrac{2}{3}}\frac{dg}{dz}g\bar{f}'(z).\end{aligned} \tag{2.88}$$

Equating with (2.47a) yields

$$\sqrt{\tfrac{2}{3}}\frac{dg}{dz}g\bar{f}'(z) = -(1-g^2)(g + \tfrac{1}{\sqrt{6}}\bar{W}_V(z)). \tag{2.89}$$

Since g and $\bar{f}'$ are both second order in z, there exists a neighbourhood $N_1(0)$ of the origin for which $\left|\frac{dg}{dz}\right|$ and $|\bar{f}'|$ are less than 1. since $g^2 < 1$ on C_0 (2.89) implies that on $N_1(0)$

$$|g + \tfrac{1}{\sqrt{6}}\bar{W}_V(z)| < \sqrt{\tfrac{2}{3}}|g|. \tag{2.90}$$

Note also that since V is SED at infinity, $\bar{W}_V(z)$ must be either strictly positive or strictly negative on C_0 (except at the origin).

Consider first the case $\bar{W}_V(z) < 0$. This will be true when the potential V is decreasing at infinity towards some non-negative limit. By (2.90), $\bar{W}_V(z) < 0$ implies that $g(z) > 0$. It then follows from equation (2.47b) that $\dfrac{dz}{d\tau} < 0$ on C_0. Thus p_0 is a sink as shown in Fig. 2.8a. In this scenario no solutions are past asymptotic to p_o.

Consider now the case $\bar{W}_V(z) > 0$. This is, perhaps, the more interesting scenario since it includes all SED, unbounded potentials. By (2.90), $\bar{W}_V(z) > 0$ implies that $g(z) < 0$. It then follows from equation (2.47b) that $\dfrac{dz}{d\tau} > 0$ on C_0. Thus I_∞ is past asymptotic to p_0 and p_0 is a saddle, Fig. 2.8b.

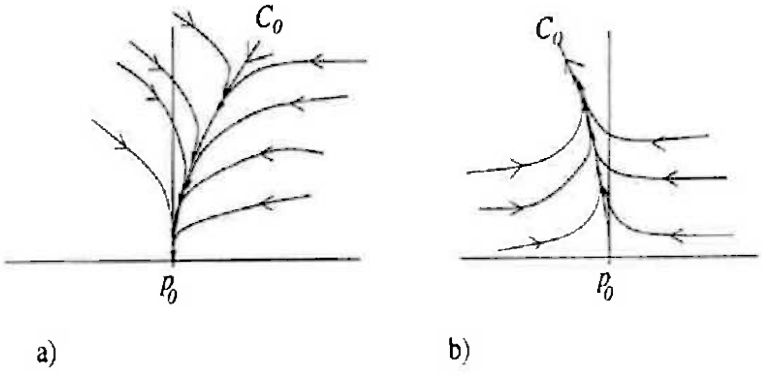

Figure 2.8: p_0 is a sink when $\bar{W}_V < 0$ (a) and a saddle when $\bar{W}_V > 0$ (b). In the latter case there exists a unique solution asymptotic in the past to p_0.

Theorem 2.9 *Let $V \in \mathcal{E}^2_+$. At most one solution is past asymptotic to p_0.*

If $W_V(\phi) > 0$ for ϕ sufficiently large, then there exists exactly one solution asymptotic in the past to p_0. This solution, I_∞, corresponds to a horizon free cosmology.

Proof:

The above discussion has established the existence of I_∞, and the fact that it is past asymptotic to p_0. The uniqueness of I_∞ (and of C_0) is a trivial consequence of Theorem A.7 (see Remark 2) since a given choice of I_∞ must lie on all center manifolds of p_0. It remains to show that it is horizon free. Let proper time t tend to t_i as $\tau \to -\infty$. Particle horizons exist for co-moving observers at time t if and only if the definite integral

$$l(t) = \int_{t_i}^{t} \frac{1}{a} dt \tag{2.91}$$

exists [16]. Changing time parameterization and making use of (2.14) and (2.15), remembering that $v = a^3$ we have

$$l(t) = c \int_{-\infty}^{\tau(t)} e^{-\frac{1}{3}\tau} x d\tau. \tag{2.92}$$

Now, since $I_\infty(\tau) \to p_o$ as $\tau \to -\infty$ it follows that $y \to 0$. Hence for all $\delta > 0$ there exists τ_0 such that $y^2 < \delta$ for all $\tau < \tau_0$. Equation (2.18a) therefore implies that for τ sufficiently negative

$$\frac{dx}{d\tau} < \delta x.$$

Thus, if $\tau < \tau_0$

$$x < x_0 e^{\delta t}.$$

Thus

$$l(t) > l_0(t) + \int_{-\infty}^{\tau_0} e^{-(\frac{1}{3}-\delta)\tau} d\tau.$$

Clearly the integrand diverges as $\tau \to -\infty$ for all $\delta < \frac{1}{3}$ so we have the result.
□

Theorem 2.9 is implies that non-trivial horizon free solutions exist for all models characterized by unbounded $\mathcal{E}_+^2$ potentials. This, to some extent, explains the ease with which we may cook up exact horizon free solutions by a judicious choice of potential. Whether or not I_∞ is singularity free will depend on the behavior of the graph $g(z)$ which, essentially, depends on the function $W_V(\phi)$.

Observe from the proof of Theorem 2.9 that a necessary and sufficient condition for any solution to be horizon free is that $y^2 < \frac{1}{3}$ in the neighbourhood of the past boundary of space-time. This condition corresponds precisely to violation of the Strong Energy Condition as can be verified by writing (1.23) in terms of x, y and z and eliminating x and z via the constraint equation (2.46). In fact as $y \to 0$ the equation of state approaches $\rho = -p$.

Violation of the Strong Energy Condition has been used by Barrow as a definition of inflation [54]. Applying this to the present case, we may say that I_∞ is inflating for all ϕ sufficiently large that $g(z)^2 < \frac{1}{3}$. (This is true in general, for both positive and negative W_V).

The importance of I_∞ lies in the fact that C_0 is an exponential attractor in phase space for sufficiently large ϕ. I_∞ therefore approximates the general inflationary dynamics of the system. For systems with $W_V < 0$ this follows immediately from Theorem A.8 in the appendix, the resulting inflationary dynamics is perpetual (ie it lasts forever) and is closely related to the power law inflation exhibited by exponential potentials. For potentials such as

ϕ^{-1}, which decay to zero, this is rather counter intuitive since the matter approaches the asymptotic equation of state $\rho = -p$ as $V(\phi) \to 0$.

When $\bar{W_V} > 0$, p_0 is unstable and we must be more precise.

Define the function

$$\delta(y, z) = y - g(z).$$

$\delta(p) = 0$ if and only if $p \in C_0$ so δ gives a measure of the distance of a point in Ω_ϵ from C_0.

Theorem 2.10 *Let $V \in \mathcal{E}^2_+$ with $W_V(\phi) > 0$ for ϕ sufficiently large. Let $(y(\tau), z(\tau))$ be a solution of (2.47) with $(y(0), z(0)) = (y_0, z_0)$ and denote $\delta(z_0, y_0)$ by δ_0. For all $0 < \gamma < 1$ there exists a neighbourhood N_γ of p_0 such that for all $(y_0, z_0) \in N_\gamma$;*

$$\delta(\tau) < \delta_0 e^{-\gamma\tau} \ \textit{for all}\ \tau \in I(y_0, z_0), \tag{2.93}$$

where

$$I(y_0, z_0) = \{\tau \leq 0 : (y(\eta), z(\eta)) \in N_\gamma \quad \textit{for all}\ \ 0 \geq \eta \geq \tau\};$$

ie solutions in N_γ exponentially decay onto $h(z)$ as long as they remain in N_γ.

Proof:

By explicit differentiation we have

$$\begin{aligned} \frac{d\delta}{d\tau} &= \frac{dy}{d\tau} - \frac{dg}{dz}\frac{dz}{d\tau} \\ &= \frac{dy}{d\tau} - \sqrt{\tfrac{2}{3}}\frac{dg}{dz}\bar{f}'(z)(\delta + g) \end{aligned} \tag{2.94}$$

where I have used (2.47b) and the fact that $y = \delta + g$. Substituting (2.47b) and (2.89) into the above expression we find, after a little rearrangement

$$\begin{aligned} \frac{d\delta}{d\tau} &= (-1 + 3g(z)^2 + \tfrac{2}{\sqrt{6}}\bar{W_V}\ g(z) - \sqrt{\tfrac{2}{3}}\frac{dg}{dz}\bar{f}'(z))\delta \\ &\quad +(3g(z) + \tfrac{1}{\sqrt{6}}\bar{W_V}\ (z))\delta^2 + \delta^3. \end{aligned} \tag{2.95}$$

Since $g, \bar{f}'$ and $\bar{W_V}$ are all second order in z it is clear that for any $0 < \gamma < 1$ it is possible to choose z and δ sufficiently small that (2.93) holds. This proves the theorem. □

We can make a a considerably stronger statement about the attracting nature of the center manifold C_0 if we concede some generality in our choice of potential. Firstly we make some definitions. Define ϕ_I to be the smallest value of ϕ for which $0 < W_V(\phi) < \sqrt{2}$ for all $\phi > \phi_I$. Define the coordinate transformation $z = f(\phi)$ on the domain $[\phi_I, \infty)$ with $f(\phi_I) = z_I$ and let Ω_I be the set $\Omega_\epsilon : \epsilon^{-1} = \phi_I$. Ω_I is well defined since W_V is well defined and non-zero on this domain.

We shall interpret Ω_I as the region of (positive ϕ) phase space for which V is flat enough for a physically interesting inflationary phase to occur. The following theorem makes this precise.

Theorem 2.11 *Assume that V satisfies the conditions of Theorem 2.10 and, in addition, $\bar{W_V}$ is monotonic increasing on Ω_I. Then the following is true.*

i) *C_0 bisects Ω_I with $g(z)^2 < \frac{1}{3}$ for all $z < z_I$.*

ii) *For all initial points $(y_0, z_0) \in \Omega_I$ there exists a positive constant $\gamma < 1$, depending only on y_0, such that (2.93) holds with $N_\gamma = \Omega_I$ (ie as long as $z(\tau) < z_I$). In particular if $y_0^2 < \frac{1}{3}$ then $\gamma > \frac{2}{3}$.*

Proof:

Assume that $\frac{d\bar{W_V}}{dz} \geq 0$. Since $g(z) \leq 0$ on some neighbourhood $N_1(p_0)$ any neighbourhood $N(p_0) \subset N_1(p_0)$ must contain some value of z for which $\frac{dg}{dz} < 0$. It follows from inspection of (2.89) that such a point must also have $g + \frac{1}{\sqrt{6}}\bar{W_V} > 0$. Now,

$$\frac{d}{dz}(g + \tfrac{1}{\sqrt{6}}\bar{W_V}) = -\frac{\sqrt{3}(1-g^2)}{\sqrt{2}gf'}(g + \tfrac{1}{\sqrt{6}}\bar{W_V}) + \tfrac{1}{\sqrt{6}}\frac{d\bar{W_V}}{dz}.$$

If we set $g + \frac{1}{\sqrt{6}}\bar{W_V} = 0$ then the first term on the right hand side vanishes whilst the second term on the right hand side is clearly non-negative. It follows that if $g + \frac{1}{\sqrt{6}}\bar{W_V} > 0$ for some $z = z_0$ then it must remain non-negative for all $z > z_0$. Since such a z_0 exists on all sufficiently small neighbourhoods of the origin we have

$$g + \tfrac{1}{\sqrt{6}}\bar{W_V} > 0 \text{ for all } z < z_I, \tag{2.96}$$

provided that g exists for all $z < z_I$. This in turn implies, from (2.89), that $\frac{dg}{dz}$ and g are negative functions on the domain of h. Since $g \neq 0$ (2.89) also implies that $\frac{dg}{dz}$ is finite as long as $\bar{W}_V$ remains bounded. It is therefore possible to extend the domain of g to include all $z < z_I$. Since $g < 0$ and $\bar{W}_V < 0$ it follows immediately from (2.96) and the definition of z_I that $g^2 < \frac{1}{3}$ for $z < z_I$ thereby proving the first part of the theorem.

Using (2.89) we may rewrite (2.94) as

$$\begin{aligned} \frac{d\delta}{d\tau} &= \Big[-1 + g(z)^2 + 2g(z)\delta + \delta^2 \\ &\quad + (2g + \frac{(1-g^2)}{h} + \delta)(g(z) + \tfrac{1}{\sqrt{6}}\bar{W}_V(z))\Big]\delta \\ &\equiv a(y,z)\delta. \end{aligned} \tag{2.97}$$

From (2.96), the condition that $-\frac{1}{\sqrt{3}} < g \leq 0$ and the observation that $|\delta| < 1$ it is easily seen that

$$a(y,z) < -1 + g(z)^2 + 2g(z)\delta + \delta^2 = -1 + y^2 < 0.$$

Inspection of (2.47a) reveals that close to $y^2 = 1$, $|y|$ is monotonic decreasing provided $\bar{W}_V(z) < \sqrt{6}$. Therefore, on Ω_I, for each initial value y_0 there exists some positive constant γ such that $a(y,z) < -1 + y^2 < -\gamma$. It follows immediately from (2.97) that

$$\delta(\tau) < \delta_0 e^{-\gamma\tau}.$$

It remains to be shown is that $\gamma > \frac{2}{3}$ for $y_0^2 < \frac{1}{3}$. Set $y^2 < \frac{1}{3}$ then clearly $a(y,z) < \frac{2}{3}$. We need to show that $y_0^2 < \frac{1}{3}$ is sufficient to guarantee that $y(\tau)^2 < \frac{1}{3}$. It is clear from inspection of (2.47a) that this is true for $y > 0$. For $y < 0$ it will be true if $y + \frac{1}{\sqrt{6}}\bar{W}_V$ is always negative at $y = -\frac{1}{\sqrt{3}}$, ie if $\bar{W}_V < \sqrt{2}$. Since this is true on Ω_I by definition, the theorem is proved. □

Theorem 2.11 allows us to uncover the dynamical structure underlying inflation in models for which the potential is not constant. It asserts that for models, such as the massive scalar field, where V'/V decreases monotonically to zero there exists a (locally) unique solution, I_∞, which violates the strong energy condition as long as $W_V < \sqrt{2}$ and which is an exponential attractor for a generic class of solutions. I_∞ is the natural counterpart of de Sitter space which is an attractor in the vacuum Einstein equations with positive cosmological constant (or scalar field models with constant potential).

Inflation may therefore be understood as a manifestation of the existence of such a solution. In particular, a model may be said to exhibit inflationary behavior if and only if it possesses an attracting solution $I(t)$ which violates the Strong Energy Condition on the domain for which it is attractive. For any given solution the inflationary phase of evolution may be defined as the period during which it is approximated (to within some designated error) by such a solution. It is significant that during this period the dynamics of the solution may be modeled by I, effectively independently of the initial conditions.

The fact that I_∞ exists under such general conditions clearly demonstrates that inflationary behavior is a generic feature of scalar field cosmologies.

Unlike de Sitter space it is not possible in general to obtain an exact analytic expression for I_∞. Furthermore, it can not be guaranteed that I_∞ will violate the Strong Energy Condition for all time and consequently Theorem 2.10 is a much weaker form of no-hair theorem than those proven for positive cosmological constants.

For arbitrary initial conditions in Ω_I a solution may not have sufficient time to decay onto C_0 before exiting Ω_I, particularly when $y < 0$ so that $\frac{dz}{d\tau} > 0$. However, $\frac{dz}{d\tau}$ is at least of order z^2. This means that z grows very slowly compared to the rate of decay which is exponential. For any y_m it is always possible to choose a z_m sufficiently small that for all $z_0 < z_m, y_0^2 < y_m^2$ the solution approaches arbitrarily close to C_0 before exiting Ω_I. This ensures that an open set of solutions will undergo inflation.

2.7 Example: Inflation With a Power Law Potential.

The following example illustrates some of the ideas above.

Let us return to the special case

$$V(\phi) = \frac{\lambda}{n}\phi^n$$

Then the solutions obey equation (2.51). From Table 2.3.2 we see that $z_I^2 = \sqrt{2}/n$ so Ω_I corresponds to the region of phase space satisfying $\phi > n/\sqrt{2}$. We wish to find an approximate solution for $y = h(z)$. Put

$$y = g(z) = az^2 + O(z^3), \tag{2.98}$$

then differentiating with respect to τ and using (2.51b) gives

$$\frac{dy}{d\tau} = -a^2\sqrt{\tfrac{2}{3}}z^6 + O(z^7). \tag{2.99}$$

Equating (2.99) with (2.51a) and using (2.98):

$$\left(a + \frac{n}{\sqrt{6}}\right) z^2 + O(z^6) = 0.$$

Thus $a = -\frac{n}{\sqrt{6}}$. Repeating the procedure for higher orders we soon find

$$g(z) = -\frac{n}{\sqrt{6}}z^2 + O(z^7). \tag{2.100}$$

We can obtain an approximation to $I_\infty(\tau)$ and thereby solve the inflationary dynamics of the model by substituting (2.100) into (2.51b) with $y = g(z)$ to obtain the approximate differential equation

$$\frac{dz}{d\tau} = \frac{n}{6}z^5.$$

Upon integration this yields

$$z(\tau) = \frac{z_0}{(1 - \frac{2}{3}nz_0^4\tau)^{\frac{1}{4}}} \tag{2.101}$$

where $(z_0, h(z_0))$ is any point on C_0. The approximate solution for y is just

$$y(\tau) = h(z(\tau)) = -\frac{nz_0^2}{\sqrt{6}(1 - \frac{2}{3}nz_0^4\tau)^{\frac{1}{2}}}.$$

We can go further and write I_∞ in terms of the proper time. Rearranging (2.46) we have

$$x^2 = \frac{1}{3\bar{V}(z)}(1-y^2)$$

which, to leading order in z, reduces to

$$x = \sqrt{\frac{n}{3\lambda}}z^n$$

where we have used the fact that $\bar{V}(z) = \lambda/nz^{-2n}$. Using (2.101) and the definition of τ we may integrate x to obtain a first order expression for t as a function of τ.

$$t = \begin{cases} c_1 z_0^{n-4}(1-\frac{2}{3}nz_0^4\tau)^{1-\frac{n}{4}} & n \neq 4 \\ -c_2\ln(1-4az_0\tau) & n=4 \end{cases} \tag{2.102}$$

$$c_1 = \frac{\sqrt{3}}{2\sqrt{n\lambda}(\frac{n}{4}-1)} \qquad c_2 = \sqrt{\frac{3}{4\lambda}}.$$

Observe that for $0 < n \leq 4$, $t \to -\infty$ as $\tau \to -\infty$ whereas for $n > 4$, $t \to 0$ as $\tau \to -\infty$. Thus I_∞ represents a singularity free cosmology if and only if $0 < n \leq 4$. For larger values of n it has a singularity but no particle horizon. It is now a straitforward matter to obtain approximate expressions for $\phi(t), \dot{\phi}(t)$ and $K(t)$. For $n \neq 4$ we have

$$\begin{aligned} \phi(t) &= \left(\frac{t}{c_1}\right)^{\frac{2}{4-n}} \\ K(t) &= \sqrt{\frac{3\lambda}{n}}\left(\frac{t}{c_1}\right)^{\frac{n}{4-n}} \\ \dot{\phi}(t)\ (=\sqrt{\tfrac{2}{3}}Ky) &= -\frac{\lambda^3 2}{\sqrt{3n}}\left(\frac{t}{c_1}\right)^{\frac{n-2}{4-n}}. \end{aligned} \tag{2.103}$$

Similar expressions can be found for $n = 4$.

In the special case $n = 2$ the power law potential may be interpreted as a massive scalar field of mass $m^2 = \lambda$ and the solution (2.103) in the above example becomes

$$\phi = -\sqrt{\tfrac{2}{3}}mt \qquad \dot{\phi} = -\sqrt{\tfrac{2}{3}}m \qquad K = -m^2 t \tag{2.104}$$

which agrees with the approximation to the inflationary seperatrix of the massive scalar field originally found by Belinski *et al* [28]. Furthermore, if we put $m = 1$ then the asymptotic solution (2.104) is precisely the the exact singularity free solution to the quadratic potential (1.72) (which approximates a massive scalar field when ϕ is large) introduced in chapter 1 as an example of a singularity free scalar field cosmology. Similarly, it is easy to demonstrate that the asymptotic solution for $n = 4$ with $\lambda = 3/16$ corresponds exactly to the singularity free solutions for the quartic potentials (1.75) discussed in the same section. We are thus able to account for these solutions as inflationary seperatrices in models closely related to but slightly more complicated than ϕ^2 and ϕ^4 power law potentials. The lower order corrections to the potentials allow I_∞ to be exactly integrated. These solutions are guaranteed by Theorem 2.10 to be exponential attractors within their respective models and are therefore of considerable interest as models of inflation, but not, apparently, for their past asymptotic properties. In the next section we complete the proof that all singularity and horizon free solutions of SED potentials are doomed to this same fate.

2.8 Existence and Nature of a Space-Time Singularity

Based on the analysis of the previous sections we are now able to construct a phase portrait of the system (2.47) on $\tilde{\Omega}_\epsilon$: Fig. 2.9 and 2.10 for the cases $\bar{W}_V(z) > 0$ and $\bar{W}_V(z) < 0$ respectively. These phase portraits provide an accurate summary of the qualitative behavior and in particular the topological structure of the system near $\phi = \infty$. In order to complete the analysis of the global system it is, of course, necessary to also look at the behavior near $\phi = -\infty$. This can be done easily by observing that the equations (2.1) are invariant under the transformation

$$(\phi, \dot{\phi}) \longmapsto -(\phi, \dot{\phi}) \qquad V \longmapsto U \tag{2.105}$$

where $U(\phi) = V(-\phi)$. Thus, for a particular potential V, the behavior of solutions of (2.1) near $\phi = -\infty$ is equivalent (except of course for the sign of $\dot{\phi}$) to the behavior of the system near $\phi = \infty$ with the alternative potential U. Provided that U is in $\mathcal{E}^2_+$, the proceeding analysis on $\tilde{\Omega}_\epsilon$ can be applied (for

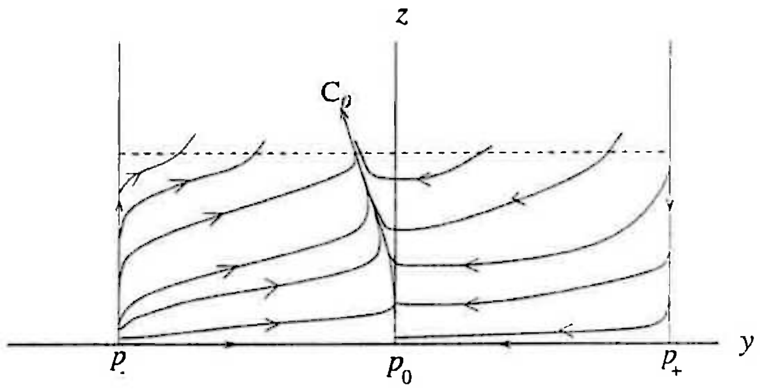

Figure 2.9: Schematic representation of the flow on Ω_ϵ for $\bar{W}_V > 0$.

a suitable choice of ϵ). The correct solution can subsequently be obtained by making the transformation $(\phi, \dot{\phi}) \longmapsto -(\phi, \dot{\phi})$. In what follows, I will refer to a potential V as class k SED at $\pm\infty$ if V and U are both class k sub exponential at infinity and I shall denote the set of all such potentials $\mathcal{E}^k$.

The following theorem concerns the existence and nature of a space-time singularity for solutions of the general system (2.1). It is the main result of this chapter.

Theorem 2.12 *Let V be a $\mathcal{E}^2$ potential, then almost all solutions of (2.1) possess an initial singularity at some finite proper time which may always be chosen as $t = 0$. Furthermore, there exists a continuous 1-1 correspondence between these solutions and the solutions of the massless scalar field as follows: For each $c > 0$ there exists exactly two solutions of (2.1) (one for each choice of sign below) which, sufficiently close to the singularity, may be written*

$$\begin{aligned} K &= t^{-1} + O(\epsilon_V^{\pm}(t)) \\ \phi &= \pm\sqrt{\tfrac{2}{3}}\ln\frac{t}{c} + O(t\epsilon_V^{\pm}(t)) \\ \dot{\phi} &= \pm\sqrt{\tfrac{2}{3}}t^{-1} + O(\epsilon_V^{\pm}(t)) \end{aligned} \tag{2.106}$$

where

$$\epsilon_V^{\pm}(t) = tV(\pm\sqrt{\tfrac{2}{3}}\ln\frac{t}{c})$$

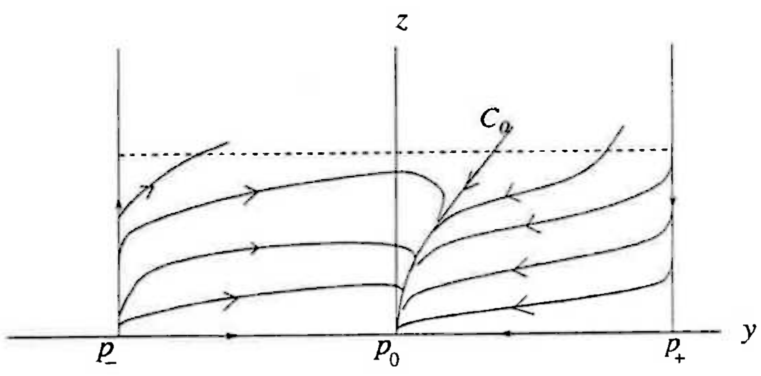

Figure 2.10: Schematic representation of the flow on Ω_ϵ for $\bar{W}_V < 0$.

$$\rightarrow \quad 0 \quad as \quad t \rightarrow 0. \tag{2.107}$$

The class of solutions thereby generated constitute almost all solutions of (2.1).

Proof:

Note first that the two disjoint subsets of the asymptotic solutions (2.106) are mapped onto one another under the transformation $(\phi, \dot{\phi}) \mapsto -(\phi, \dot{\phi})$. It follows from Theorems 2.7 and 2.8 that the theorem will be true if almost all solutions are past asymptotic to p_- or its counterpart at $\phi = -\infty$, which may be constructed via the transformation (2.105) since $V \in \mathcal{E}^2$.

Let $p = (K_0, \phi_0, \dot{\phi}_0)$ be a point in Ω and let $\psi_p : R \rightarrow \Omega$ be the unique solution of (2.1) passing through p, with $p = \psi_p(0)$. Let $O^-(p)$ be the past orbit mapped out by ψ_p. By Theorem 2.3, $|\phi|$ is unbounded on almost all orbits. Assume that $O^-(p)$ is not exceptional in this respect, then ϕ either approaches $+\infty$ or it approaches $-\infty$ (or both), to the past of p.

Assume ϕ approaches $+\infty$. Then the intersection of $O^-(p)$ with Ω_ϵ is non-empty for all $\epsilon > 0$. Since $V \in \mathcal{E}^2$ there exists a $\phi_0 > 0$ on which the conditions of Definition 2.2 hold. Choose $\epsilon < \phi_0^{-1}$ and select the coordinate system (2.43) (with appropriate choice of f) on Ω_{ϵ_1}. Furthermore, ϵ may be chosen sufficiently small that the center manifold C_0 bisects Ω_ϵ as in Fig. 2.9 and 2.9. The function $\bar{W}_V(z)$ is either positive definite or negative definite on Ω_ϵ. Let us consider these two cases separately.

i) $\bar{W}_V > 0$:

Firstly, note that C_0 lies to the left of the z-axis. Let $p_1 \in O^-(p) \cap \Omega_\epsilon$. From Fig. 2.9 it can immediately be seen that if p_1 lies to the left of the center manifold C_0, ψ_p is past asymptotic to p_-.

Assume that ψ_p is not past asymptotic to p_-, then it is clear from Fig. 2.9 that $O^-(p) \cap \Omega_\epsilon$ must lie entirely to the right of C_0. Furthermore, $O^-(p) \cap \Omega_\epsilon$ is the union of a discreet set of line segments, $O_i \subset O^-(p)$, as shown in Fig. 2.11. Associated with each O_i are two points $p_i, q_i \in \partial\Omega_{\epsilon 2}$ and times $t_i < s_i$

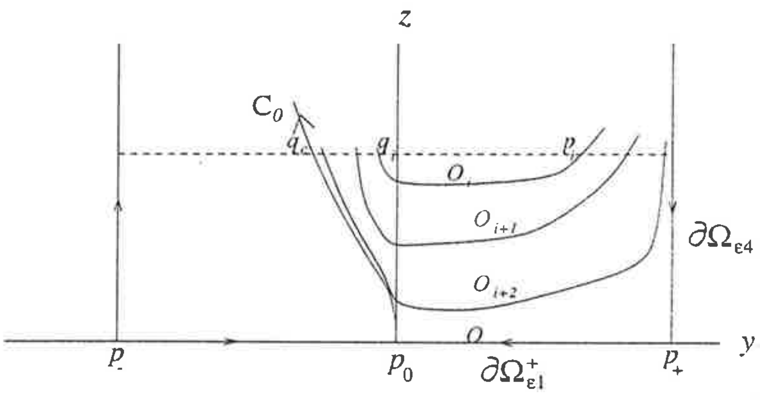

Figure 2.11:

such that $\psi_p(t_i) = p_i$ and $\psi_p(s_i) = q_i$. p_i and q_i represent points where the trajectory enters and leaves Ω_ϵ respectively. By continuity, ϕ is bounded on each individual O_i, thus $\{O_i\}$ must be an infinite sequence. Introduce the time ordering

$$t_1 > t_2 > ..... > t_{i-1} > t_i > ...... \tag{2.108}$$

Then as $i \to \infty$, O_i must asymptotically approach the line segment

$$O = C_0 \cap \Omega_\epsilon + \partial\Omega^+_{\epsilon 1} + \partial\Omega_{\epsilon 4}$$

where $\partial\Omega^+_{\epsilon 1}$ is the segment of $\partial\Omega_{\epsilon 1}$ to the right of the z-axis (See Fig. 2.11). That O is the limit set of $\{O_i\}$ can be seen from the following argument. Consider some $i > 1$ then by (2.46)

$$x(p_i)^2 = c(1 - y_i^2) \tag{2.109}$$

where c is a constant, independent of i, and y_i is just the y coordinate of p_i. Since x is a monotonic increasing function of time, bounded below by 0, it follows from (2.109) and (2.108) that $\{y_i\}$ is a bounded monotonic sequence and that

$$\lim_{i\to\infty} y_i = 1. \tag{2.110}$$

Thus p_i asymptotically approaches $\partial\Omega_{\epsilon 4}$. It follows from continuity, that O is the limit of $\{O_i\}$. We can now see immediately that

$$\lim_{i\to\infty} q_i = q_c \tag{2.111}$$

where q_c is the intersection of C_0 with the boundary, $\partial\Omega_{\epsilon 2}$. Substituting the above limits into (2.46) we find

$$\lim_{i\to\infty} x(p_i) = 0$$

and

$$\lim_{i\to\infty} x(q_i) = x(q_c) > 0.$$

Thus no limit exists for x on O_p^-. But this is a contradiction since x is bounded and monotonic. Therefore, we have shown that $\psi_p(t)$ is past asymptotic to p_- as required by the theorem.

ii) $\bar{W}_V < 0$:

Choose ϵ sufficiently small so that p lies outside of Ω_ϵ. Then, there exists some $t_1 < 0$ for which $\psi_p(t_1) \in \Omega_\epsilon$. Since trajectories may only exit Ω_ϵ to the left of the z-axis and this subset of Ω_ϵ is past invariant (since $\frac{dy}{d\tau} > 0$ along the line $y = 0$) it is clear from Fig. 2.10 that $\psi_p(t)$ must be past asymptotic at the singular point p_- in accordance with the theorem.

We have shown that the theorem holds for orbits on which ϕ is unbounded above. If ϕ is indeed bounded above then by assumption it must be unbounded below. If we make the transformation $\phi' = -\phi$, the proceeding argument holds for the system $(K, \phi', \dot{\phi}')$ and the theorem is proved. □

It is remarkable, given the apparent diversity displayed by exact solutions, that the asymptotic behavior of FRW scalar field cosmologies is, to such a large degree, independent of the form of the potential. We have seen that horizon free and singularity free solutions are abundant in scalar field cosmologies in the sense that they always exist in a large and natural class of cosmologies and rare in the sense that in any given model they are not generic. The general picture that has emerged conforms very closely to the original results of Belinski *et al* [28] for the massive scalar field. If one is to construct generic horizon free solutions then clearly they must arise either in models with very steep potentials or in more complicated space-time structures.

It is possible to generalize Theorem 2.12 somewhat by relaxing the condition that $W_V \to 0$ as $\phi \to \infty$ to the weaker condition that $W_V \to \Lambda$, where Λ is a constant such that $\Lambda < \sqrt{6}$. This includes a large class of exponential potentials and, in fact, contains all exponential potentials which exhibit power law inflation. This added degree of generality has been excluded from the above considerations since its inclusion would have made the discussion considerably more lengthy. The analysis however is essentially the same as for SED potentials. We shall see explicitly that this is true for very simple exponential potentials in the next chapter.

As a final note, it is worth commenting that much of the power of the dynamical systems approach lies in the ease with which more complex cosmological models may be incorporated into the system. The coupling of the scalar field to a perfect fluid with fixed equation of state $\rho = \gamma p$ (with $0 < \gamma < 2$), for example, is a relatively straitforward matter. One finds that the situation is essentially the same as that found above, with the scalar field dominating the asymptotic energy density. Similarly, the coupling of several scalar fields ϕ_i does not seem to significantly alter the above conclusions, although, the phase space structure of a multi-component scalar field cosmology is significantly more complicated and has not been analyzed in detail by the author.

Chapter 3

An Exponentially Steep Potential Well

3.1 Introduction

Although SED potentials clearly represent a natural and general class of models they are by no means the only potentials which are of interest physically. I have mentioned a number of times that scalar-tensor theories and higher order theories of gravity often give rise to effective scalar field cosmologies with exponential type potentials. In fact, since we have no direct experimental evidence of physical fields at energy densities comparable with those of the early universe we really have no idea at all what form physical potentials might take for large values of ϕ (assuming the system can actually be modeled by a quasi-linear action (1.1)).

It is therefore natural to ask whether singularity free or horizon free cosmologies can arise for potentials which are not SED. In particular, we might expect that a very steep potential well could inhibit the divergence of the scalar field, thereby slowing down the gravitational expansion and resulting in singularity or particle horizon avoidance.

I claimed (without proof) at the end of the last chapter that it was possible to extend Theorem 2.12 to include exponential potentials for which $\lim_{\phi\to\pm\infty} < \sqrt{6}$. It is easily demonstrated that it is impossible to go any further than that: Substituting the asymptotic expressions (2.106) into the constraint equation (2.2) one obtains

$$\frac{1}{t^2} = \frac{1}{t^2} + V\left(\pm\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}\right) + h$$

where, as usual, h indicates terms of higher order. In order for this expression to be consistent one requires

$$\lim_{t\to 0} t^2 V\left(\pm\sqrt{\tfrac{2}{3}}\ln\frac{t}{c}\right) = \infty.$$

i.e.

$$\lim_{\phi\to\pm\infty} e^{-\sqrt{6}|\phi|}V(\phi) = 0.$$

In other words, for the massive scalar field to be a consistent asymptotic approximation of a scalar field model the potential must diverge slower than $e^{-\sqrt{6}|\phi|}$. Exponential potentials therefore seem to accommodate a kind of transition between SED type behavior and some other qualitative regime.

In this chapter we examine the behavior of very steep potentials by means of an interesting example.

3.2 The Model and Dynamical Equations.

As in the previous chapter we shall confine attention to spatially flat FRW space-time. The simplest example of a potential which is neither SED at infinity or negative infinity is an exponential well of the form.

$$V(\phi) = ae^{\lambda\phi} + be^{-\mu\phi}.$$

where a, b, λ and μ are positive constants. For simplicity I will confine attention to the case where $a = b = 1$ and $\lambda = \mu$ so that

$$V(\phi) = e^{\lambda\phi} + e^{-\lambda\phi}. \tag{3.1}$$

By increasing λ we shall be able to investigate how the qualitative behavior of the system changes as the potential well becomes increasingly steep. The analysis for the more general potential is similar and the conclusions are essentially the same. Firstly, we introduce the coordinates (2.13) and the usual coordinate time τ (2.14). The field equations with the potential (3.1) are, using (2.16),

$$\begin{aligned}\frac{dx}{d\tau} &= y^2x \\ \frac{dy}{d\tau} &= -y-3\alpha x^2(e^{\sqrt{6}\alpha\phi}-e^{-\sqrt{6}\alpha\phi})+y^3 \\ \frac{d\phi}{d\tau} &= y\end{aligned} \tag{3.2}$$

where $\alpha = \frac{\lambda}{\sqrt{6}}$. The constraint equation becomes

$$y^2+3x^2(e^{\sqrt{6}\alpha\phi}+e^{-\sqrt{6}\alpha\phi}) = 1. \tag{3.3}$$

Let us now define the variables

$$p = \sqrt{3}e^{-\sqrt{\frac{3}{2}}\alpha\phi}x \qquad\qquad q = y. \tag{3.4}$$

then the constraint equation may be written

$$p^2+q^2 = 1-p^2e^{2\sqrt{6}\alpha\phi}. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.2) we obtain

$$\begin{aligned}\frac{dp}{d\tau} &= -\alpha pq+pq^2 \\ \frac{dq}{d\tau} &= q^3+\alpha q^2-q-\alpha+2\alpha p^2.\end{aligned} \tag{3.6}$$

These equations constitute a 2-dimensional dynamical system on the p-q plane. The physical phase space Ω may be represented, according to (3.5), by the imbedding $\Omega = \{(p,q) : p^2+q^2 < 1, p > 0\}$. In other words, all physical trajectories lie on the interior of the unit disc to the right of the q axis. The unphysical boundary $\partial\Omega$ is a closed curve consisting of the union of the smooth arc $\partial\Omega_1 = \{(p,q) : p^2+q^2 = 1, p > 0\}$ and the line segment $\partial\Omega_2 = \{(p,q) : p = 0, |q| \leq 1\}$. The asymmetric appearance of $\partial\Omega$ is a consequence of the positive exponential term in the definition of the coordinate p and is not a physical property of the system itself. In fact it should be pointed out that $\partial\Omega_1$ maps onto $\partial\Omega_2$ under the transformation

$(\phi, \dot{\phi}) \mapsto -(\phi, \dot{\phi})$. $\partial\Omega$ corresponds to the infinity of the expansion K. In order to see this define the function

$$H(p, q) = p^2(1 - p^2 - q^2) \tag{3.7}$$

Clearly H is strictly positive everywhere on Ω and vanishes identically on $\partial\Omega$. From (3.5) and (3.4) we see that

$$H = p^4 e^{2\sqrt{6}\alpha\phi} = 9x^4. \tag{3.8}$$

Thus $\partial\Omega$ is just the set of all points for which $x = 0$ which is by definition the infinity of K. Evaluating the directional derivative of H along the flow using (3.6) we find

$$\frac{dH}{d\tau} = 4Hq^2 \tag{3.9}$$

which is non-negative everywhere on the interior of Ω . In fact the derivative of H with respect to τ is strictly positive everywhere on the interior of Ω except where $q = 0$. Observe also that $\frac{dH}{d\tau} = 0$ on $\partial\Omega$ indicating that the boundary is an invariant manifold (tangent to the flow). No physical trajectories can therefore cross $\partial\Omega$ into the unphysical domain beyond. Since Ω is compact all trajectories must possess an α-limit set which is invariant under the flow. Since H is monotonic, any limit point must have $\frac{dH}{d\tau} = 0$. Therefore, all limit sets must be subsets of either $\partial\Omega$ or the line $q = 0$. From (3.6) the only invariant subset of $q = 0$ is the equilibrium point $\mathbf{p}_d = (\frac{1}{\sqrt{2}}, 0)^\dagger$. However this point is a local maximum of H as is easily verified by evaluating its gradient. Since H is monotonic increasing, no solutions can be past asymptotic to $\mathbf{p}_d$ other than the steady state solution on $\mathbf{p}_d$ itself. It follows that the α-limit sets of all other solutions lie on the boundary $\partial\Omega$.

By the time reverse of the above argument it is clear that all solutions, with the exception of those unphysical solutions lying on $\partial\Omega$ are future asymptotic to $\mathbf{p}_d$.

The future asymptotic set $\mathbf{p}_d$ represents a vacuum de Sitter space-time with constant expansion $K = \sqrt{6}$ and ϕ identically zero. This is consistent with what we would expect for a scalar field cosmology with non-zero vacuum energy.

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†In order to avoid confusion points on the (p, q) plane will be labeled in bold print.

To summarize, it has been established that (almost) all solutions originate near $\partial\Omega$ and subsequently evolve towards the global attractor $\mathbf{p}_d$ (which represents de Sitter space) as $t \to \infty$. Let us now examine the behavior of the system on and near $\partial\Omega$ in more detail.

3.3 The Behavior Close to $\partial\Omega$.

There are a maximum 4 equilibrium points of (3.6) lying on $\partial\Omega$. These are $(\sqrt{(1-\alpha^2)}, \alpha)$, $(0, 1)$, $(0, -1)$ and $(0, -\alpha)$, which I shall label $\mathbf{p}_1$, $\mathbf{p}_+$, $\mathbf{p}_-$ and $\mathbf{p}_2$ respectively. Observe that $\mathbf{p}_1$ and $\mathbf{p}_2$ only exist as distinct equilibrium points on $\partial\Omega$ when $\alpha < 1$. For values of α greater than or equal to one, $\mathbf{p}_\pm$ are the only equilibrium points. In order to investigate the behavior close to $\mathbf{p}_\pm$ equations (15) and (16) can be linearized about these points. The linearized system is:

$$\frac{dp}{d\tau} = (1 \mp \alpha)p \tag{3.10}$$

$$\frac{dq}{d\tau} = 2(1 \pm \alpha)(q \mp 1) \tag{3.11}$$

The solution to the linear system is

$$p = p_0 e^{(1\mp\alpha)\tau} \qquad q = \pm 1 \mp \delta_0 e^{2(1\pm\alpha)\tau} \tag{3.12}$$

where p_0 and δ_0 are positive constants.

3.3.1 The Flow For $\alpha < 1$.

When $\alpha < 1$ both exponential terms have positive coefficients indicating that $\mathbf{p}_\pm$ are sources of the linear system and therefore, by the Hartman-Grobman Theorem (A.3), of the non-linear system also. Since the unphysical solutions on $\partial\Omega$ move away from $\mathbf{p}_\pm$ towards p_1 and p_2 (in the forwards time sense) these points each must possess local stable manifolds, which are subsets of $\partial\Omega$. However all *physical* solutions asymptotically approach $\mathbf{p}_d$ in the future. $\mathbf{p}_1$ and $\mathbf{p}_2$ must therefore be saddles (see appendix). This can be seen more clearly from inspection of the geometry of the flow as illustrated in Fig. 3.1. (An alternative way to verify that $\mathbf{p}_1$ and $\mathbf{p}_2$ are saddles is by linearizing (3.6)

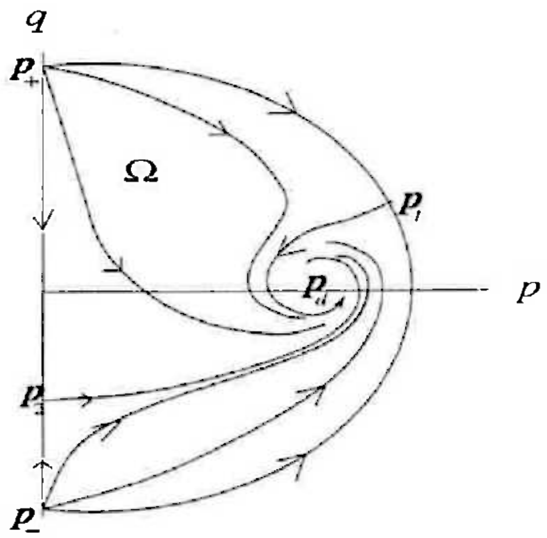

Figure 3.1: Sketch of Ω showing all possible equilibrium points, direction of the flow on the boundary, and some typical trajectories. Trajectories on the boundary are future asymptotic to $\mathbf{p}_1$ and $\mathbf{p}_2$ but all trajectories on the interior approach $\mathbf{p}_d$.

about these points but this exercise shall be left to the reader who remains unconvinced). With the exception of the 2 solutions past asymptotic to $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively, and the steady state solution lying on $\mathbf{p}_d$, all physical trajectories must asymptotically approach either $\mathbf{p}_+$ or $\mathbf{p}_-$ as $\tau \to -\infty$ as indicated in Fig. 3.1.

Using the fact that $1 - q^2$ decays exponentially to zero as $\tau \to -\infty$ we may integrate (3.9) to obtain a first order expression for H:

$$H = H_0 e^{4\tau}$$

and hence

$$x = x_0 e^{\tau}.$$

Using (2.15) we obtain a first order expression for t:

$$t = x_0 e^{\tau}.$$

Observe that $t \to 0$ as $\tau \to -\infty$ indicating that $\mathbf{p}_\pm$ correspond to space-time singularities. Using the definitions of x and q we thus obtain the asymptotic solution for K and $\dot{\phi}$ in the neighbourhood of $t = 0$:

$$K = \frac{1}{t} \qquad \dot{\phi} = \pm\sqrt{\frac{2}{3}}\frac{1}{t} \tag{3.13}$$

and upon integration of $\dot{\phi}$

$$\phi = \pm\sqrt{\frac{2}{3}}\ln\frac{t}{c}. \tag{3.14}$$

This is, of course, the general solution for the massless scalar field. We thus conclude that when $\alpha < 1$, ie $\lambda < \sqrt{6}$, the potential V is not dynamically significant near the singularity. This is precisely the behavior we would have expected from the simple calculation in section 1 for potentials that go to infinity slower than $\exp(\sqrt{6}\phi)$. In fact, the topological structure of the solution space seems to be identical to that for a SED potential well with non-zero vacuum energy. All s solutions emerge from the sources $\mathbf{p}_\pm$ except for the seperatrices I_1 or I_2 which we define as the unique solutions past asymptotic to the saddles $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. By analogy with with the SED case we may infer that these solutions characterize the inflation in the system. (It can be easily verified that the behavior close to $\mathbf{p}_1$ and $\mathbf{p}_2$ is approximated by the power law solutions derived in chapter 1. However this behavior is only inflationary when $\alpha^2 < \frac{1}{3}$ since this condition ensures that the square of the q coordinates of $\mathbf{p}_+$ and $\mathbf{p}_-$ are less than $\frac{1}{3}$ which is necessary and sufficient for violation of the Strong Energy Condition.)

3.3.2 The Flow For $\alpha \geq 1$

Let us now consider what happens when $\alpha \geq 1$. As we demonstrated at the beginning of this chapter, the potential in this case becomes too steep for the solution with $V = 0$ to be consistent as an asymptotic solution. In more physical terms, the gravitational expansion is unable to dominate the scalar field self interaction in the initial expansion phase of the universe. For expediency, we shall assume in what follows that $\alpha > 1$. The special case $\alpha = 1$ differs in the details of analysis, but the qualitative features of the solutions turn out to be essentially the same.

The only equilibrium points possessed by the system when $\alpha > 1$ are $\mathbf{p}_+$, $\mathbf{p}_-$ and the sink $\mathbf{p}_\infty$. Inspection of (3.12) reveals that $\mathbf{p}_\pm$ become hyperbolic saddles in this regime.

The only solution past asymptotic to $\mathbf{p}_-$ is the unphysical trajectory γ_1 which emerges from $\mathbf{p}_-$ at $\tau = -\infty$ and proceeds anticlockwise along $\partial\Omega_1$ (the unit circle), reaching $\mathbf{p}_+$ at $\tau = \infty$. Similarly, the only solution which originates at $\mathbf{p}_+$ is the unphysical trajectory γ_2 which emerges from $\mathbf{p}_+$ at $\tau = -\infty$ and proceeds along $\partial\Omega_2$ (the q-axis), approaching $\mathbf{p}_2$ asymptotically as $\tau \to \infty$. The union of γ_1, γ_2 is a closed heteroclinic cycle γ_L on $\partial\Omega$ which forms a limit cycle for trajectories on the interior of Ω. That is, as $\eta \to -\infty$ a typical trajectory will approach $\partial\Omega$, spiraling clockwise (in the reverse time sense) an infinite number of times. The solution will asymptotically approach the non-physical solutions γ_1 and γ_2 but unlike these will always avoid the equilibrium points $\mathbf{p}_+$ and $\mathbf{p}_-$ and will continue to spiral around $\partial\Omega$ *ad infinitum.* Fig 3.2.

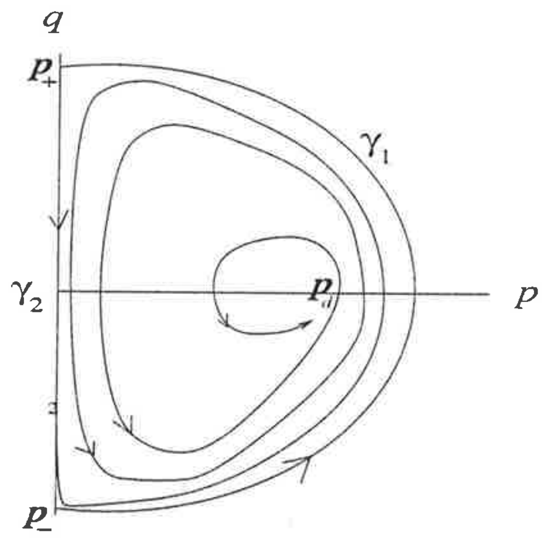

Figure 3.2: Sketch of Ω for $\alpha > 1$ showing the solutions on the boundary and a typical solution on the interior. The points $\mathbf{p}_\pm$ are saddles and therefore unstable, but solutions must approach the boundary so the union of solutions on $\partial\Omega$ forms a limit cycle.

In order to interpret these results recall firstly that, from (3.7) and (3.9), H monotonically decreases to 0 as $\tau \to -\infty$ and hence the expansion K diverges monotonically to infinity. The asymptotic periodic behavior of the trajectories must therefore represent oscillations of the scalar field ϕ. Inspection of (3.5) indicates that $\phi = -\infty$ on the semi-circle $\partial\Omega_1$. It follows from symmetry that $\phi = \infty$ on $\partial\Omega_2$. As the gravitational expansion diverges to infinity (and correspondingly the scale parameter a tends to 0) the scalar field oscillates about $\phi = 0$ with the amplitude of oscillation increasing for each successive cycle, asymptotically diverging to infinity.

No asymptotic equation of state is satisfied by the scalar field. In the "corners" of Ω, close to the points $p_{\pm}$, the equation of state is approximately $\rho = p$ since $\dot{\phi}^2 \simeq \frac{2}{3}K^2$ which implies, via (2.2) that $\dot{\phi}^2 >> V(\phi)$. This corresponds to the period where the scalar field is close to the bottom of the potential well.

Close to the q-axis, on the other hand, the equation of state is approximately $\rho = -p$ since $\dot{\phi} \simeq 0$ whilst $V(\phi) \geq 2$. This corresponds to the scalar field reaching the top of its roll before accelerating down the potential well again. The asymptotic behavior of the space-time may therefore be characterized by an infinite sequence of "stiff" phases, punctuated by de Sitter phases. During the de Sitter phase the matter violates the Strong Energy Condition.

3.4 The Existence of a Singularity and Particle Horizon

Since an infinite number of physical cycles of the scalar field occur for the case $\alpha > 1$, it might reasonably be expected that an infinite interval of proper time must also elapse since the oscillations of the scalar field could be used as a natural physical clock which would measure an infinite time interval to the past of any space-time point. If this were the case then the space-time would be non-singular for all physical solutions. It turns out however, that the oscillations pile up on each other and a space-time singularity is indeed reached after a finite time interval.

In order to show this we must estimate the proper time that elapses to the past of some arbitrary initial point $\mathbf{p}_0$, on the trajectory $\psi_{\mathbf{p}_0}(\tau)$. Since all trajectories are asymptotically tangent to the boundary $\partial\Omega$ it will be sufficient to restrict attention to initial points lying in the set $\Sigma_0 = \{(p,q) : q = 1-\epsilon,\ 0 < p \leq \epsilon\}$ where $\epsilon > 0$ may be chosen arbitrarily small.

Before proving that space-time singularities and particle horizons exist for all solutions it will be convenient to prove the following lemma;

Lemma 3.1 *For all $0 < n < 1$ there exist positive numbers ϵ and p_m such that if $x_0 = (p_0, 1-\epsilon)$ is in Σ_0 and $p_0 < p_m$ then*

$$H(\psi_{\mathbf{p}_0}(\tau)) < H_0 e^{4n\tau} \tag{3.15}$$

to the past of $\mathbf{p}_0$, where $\psi_{x_0}(\tau)$ is the unique trajectory of (3.6) passing through $\mathbf{p}_0$ with $\psi_{\mathbf{p}_0}(0) = \mathbf{p}_0$.

Proof: Integrating (3.9) backwards in time from $\tau = 0$ to some earlier time τ_f we obtain

$$H(\tau_f) = H_0 \exp\left[-4\int_{\tau_f}^{0} q^2 d\tau\right]. \tag{3.16}$$

Let $\Sigma_1 = \{(p,q) \in \Omega : p = \epsilon,\ 1-q \leq \epsilon\}$. As can be seen from Fig. 3.3, the set $\Sigma_1 + \Sigma_0$ encloses a box, Ω_ϵ^+, of area $\simeq \epsilon^2$ in Ω around $\mathbf{p}_+$. Another box, Ω_ϵ^-, can similarly be constructed around $\mathbf{p}_-$ by defining the sets $\Sigma_2 = \{(p,q) \in \Omega : p = \epsilon\ 1+q \leq \epsilon\}$ and $\Sigma_3 = \{(p,q) : q = -1+\epsilon,\ 0 < p \leq \epsilon\}$.

Let I be the time interval $[\tau_f, 0]$, then for a given trajectory $\psi_{\mathbf{p}_0}$ we may write $I = I_c \cup I_b$ where $I_c = \{\tau \in I : \psi_{\mathbf{p}_0}(\tau) \in \Omega_\epsilon^\pm\}$ and $I_b = \{\tau \in I : \psi_{\mathbf{p}_0}(\tau) \notin \Omega_\epsilon^\pm\}$. We thus have

$$\int_{\tau_f}^{0} q^2 d\tau = \int_{I_c} q^2 d\tau + \int_{I_b} q^2 d\tau \tag{3.17}$$

$$> \int_{I_c} q^2 d\tau. \tag{3.18}$$

From the definition of I_c,

$$\int_{I_c} q^2 d\tau = \int_{I_c} (1 - O(\epsilon)) d\tau.$$

Fix n and let m be any number satisfying $n < m < 1$. Then for ϵ sufficiently small we have;

$$\int_{\tau_f}^{0} q^2 d\tau > m \int_{I_c} d\tau. \tag{3.19}$$

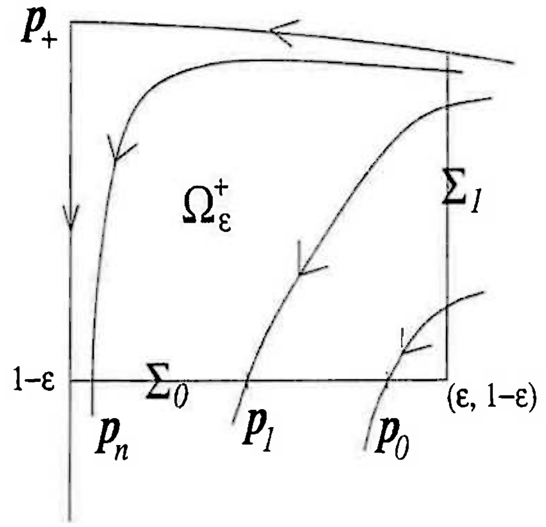

Figure 3.3: The box Ω_ϵ^+. On each successive cycle in the reverse time direction the trajectory intersects the box for a finite period of time coming progressively closer to the boundary of Ω (the arrows indicate the direction of the flow in the *forward* time sense).

In order to complete the proof we must show that by choosing p_0 sufficiently small, the ratio of $\int_{I_c} d\tau$ to $\int_I d\tau$ may be made arbitrarily close to 1.

According to (3.6) p evolves according to the equation

$$\begin{aligned} \frac{d}{d\tau}\ln p &= -\alpha q + q^2 \\ &< 1+\alpha \end{aligned} \tag{3.20}$$

since $|q| < 1$. Therefore if $\tau_2 < \tau_1$ we have

$$\ln\frac{p_1}{p_2} < (1+\alpha)(\tau_1 - \tau_2), \tag{3.21}$$

where $p_1 = p(\tau_1), p_2 = p(\tau_2)$. The parameter time $\Delta\tau$ needed for a point $\mathbf{p}_0 = (p_0, 1-\epsilon) \in \Sigma_0$ to flow through Ω_ϵ^+ and reach Σ_1 must therefore satisfy

$$|\Delta\tau| > \frac{1}{1+\alpha}\ln\left(\frac{\epsilon}{p_0}\right). \tag{3.22}$$

The modulus sign is necessary because we are tracing the trajectory backwards in time so $\Delta\tau$ is negative. As $p_0 \to 0$, $\Delta\tau \to -\infty$. Thus $\Delta\tau$ may be made arbitrarily large by choosing p_0 sufficiently small.

Since $\psi_{\mathbf{p}_0}$ is past asymptotic to the limit cycle γ_l, the intersection of the past orbit $O^-_{\mathbf{p}_0}$ with the line $q = 1 - \epsilon$ (which contains Σ_0) must contain an infinite number of points in addition to $\mathbf{p}_0$ itself. If $\mathbf{p}_1$ is any such point then substituting $q = 1 - \epsilon$ into (3.7) and using the monotonicity of H (3.9) we must have $p_1 < p_0$, where p_1 and p_0 are the p-coordinate values of $\mathbf{p}_1$ and $\mathbf{p}_0$ respectively. In other words $\psi_{\mathbf{p}_0}$ must intersect $\Sigma_{\epsilon 0}$ again after 1 complete cycle of $\partial\Omega$ and the point of intersection $\mathbf{p}_1$ must have a p-coordinate value p_1 which is smaller than p_0. It follows that $\psi_{\mathbf{p}_0}$ passes through the box Ω^+_ϵ on each successive cycle and the time interval $\Delta\tau^+_n$ to traverse Ω^+_ϵ on the n^{th} cycle obeys the inequality;

$$|\Delta\tau^+_n| = \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{p_n}\right) \tag{3.23}$$

$$\geq \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{p_0}\right) \tag{3.24}$$

Similarly, defining $\Delta\tau^-_n$ to be the time taken for $\psi_{\mathbf{p}_0}$ to traverse the box Ω^-_ϵ on the n^{th} cycle we find from (3.21) using an identical argument to that above that

$$|\Delta\tau^-_n| \geq \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{\tilde{p}_n}\right). \tag{3.25}$$

where $\tilde{p}_n$ is the p coordinate of the intersection of $\psi_{\mathbf{p}_0}$ with Σ_3 on the n^{th} cycle (by the n^{th} cycle I mean, precisely, one complete circuit from $(p_n - 1, 1-\epsilon) \in \Sigma_0$ to $(p_n, 1-\epsilon) \in \Sigma_0$). Recalling the definition of H and using the fact that q^2 takes the same value on Σ_0 and Σ_3 we have for any point on Σ_0 *or* Σ_3 that $H^2 = (2\epsilon - \epsilon^2)p^2 + O(\epsilon p^4)$. Since H is monotonic it follows that for ϵ chosen sufficiently small $\tilde{p}_n < p_0$. Therefore,

$$|\Delta\tau^-_n| \geq \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{p_0}\right). \tag{3.26}$$

Let us now consider the parameter time $\Delta\tau$ taken for a point on Σ_1 to reach Σ_2 under (3.6), close to the semi-circular boundary $\partial\Omega_1$. That is , the time taken to flow backwards in time from the top box Ω^+_ϵ to the bottom

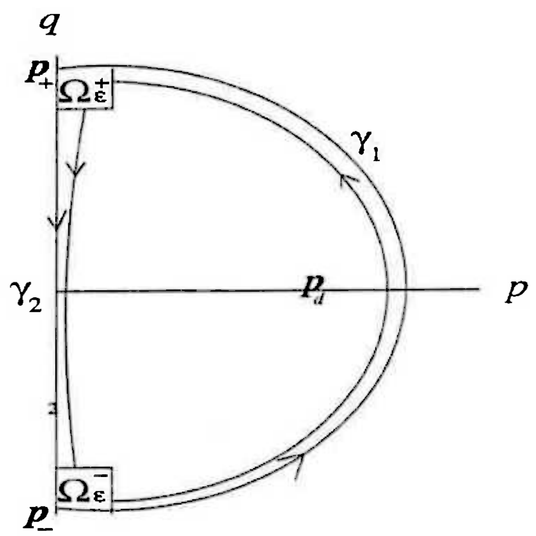

Figure 3.4: The parts of the solution flowing between the boxes Ω_ϵ^+ and Ω_ϵ^- approach the restrictions of γ_1 and γ_2 to finite time intervals.

box Ω_ϵ^- (Fig. 3.4). By continuity, $\Delta\tau$ approaches the (negative) parameter time interval which the asymptotic solution γ_1 takes to map the point $(\epsilon, \sqrt{1-\epsilon^2}) \in \Sigma_1$ to the point $(\epsilon, -\sqrt{1-\epsilon^2}) \in \Sigma_2$. This interval is finite.

Therefore, the time interval to go from Σ_1 to Σ_2 must approach a finite (negative) limit. Its modulus must therefore possess a finite upper bound τ_1.

Let $\Delta\tau_n^1$ be the time taken for ψ_{x_0} to go from Σ_1 to Σ_2 on the n^{th} cycle. Then we have

$$|\Delta\tau_n^1| < \tau_1 \tag{3.27}$$

Similarly, if $\Delta\tau_n^2$ is the time taken for ψ_{x_0} to go backwards in time from Σ_3 to Σ_0 on the n^{th} cycle we have

$$|\Delta\tau_n^2| < \tau_2 \tag{3.28}$$

For some finite number τ_2.

Now, Since ψ_{x_0} is incident on Σ_0 and hence, initially flows through Ω_ϵ^+, it follows from the inequalities (3.24), (3.26), (3.27) and (3.28) that

$$\frac{\int_{I_c} d\tau}{\int_{I_b} d\tau} > \frac{1}{\tau_\epsilon(\alpha+1)} \ln\left(\frac{\epsilon}{p_0}\right)$$

where $\tau_\epsilon = \max(\tau_1, \tau_2)$. Choose

$$p_0 = \epsilon \exp\left[-\frac{\tau_\epsilon(\alpha+1)}{(\frac{m}{n}-1)}\right].$$

then we have

$$\begin{aligned} \int_I d\tau &= \int_{I_c} d\tau \left(1 + \frac{\int_{I_m} d\tau}{\int_{I_c} d\tau}\right) \\ -\tau_f &< \frac{m}{n} \int_{I_c} d\tau. \end{aligned} \tag{3.29}$$

Combining (3.29), (3.19) and (3.16) gives

$$H(\tau_f) < H_0 e^{4n\tau_f}$$

which is the required result.□

Recall that we say a scalar field cosmology is non-trivial if there exists some space-time point for which $\dot{\phi}$ is non-zero.

Theorem 3.2 *If $(g_{\mu\nu}, \phi)$ is a non-trivial scalar field cosmology with potential (3.1) and if $g_{\mu\nu}$ is spatially flat and isotropic, then $g_{\mu\nu}$ possesses an initial space-time singularity.*

Proof:

By Lemma 3.1 there exists ϵ and p_m such that all trajectories of (3.6) incident on Σ_0 with $p_0 < p_m$ satisfy

$$H(\tau) < H_0 e^{2\tau}$$

for all $\tau < 0$. It will be sufficient to show that these trajectories reach the boundary in finite proper time. The proper time taken to reach the boundary is given by

$$\Delta t = \int_{-\infty}^{0} x d\tau \tag{3.30}$$

$$< x_0 \int_{-\infty}^{0} e^{\tau/2} d\tau \tag{3.31}$$

$$= 2x_0 \tag{3.32}$$

Since this is finite, we have the result.□

Theorem 3.3 *If $(g_{\mu\nu}, \phi)$ is a non-trivial scalar field cosmology with potential (3.1) and if $g_{\mu\nu}$ is spatially flat and isotropic, then particle horizons exist for all isotropic observers (observers whose worldlines are tangent to the timelike Killing vector field U^μ).*

Proof: A particle horizon exists for an isotropic observer at time t if the integral

$$l = \int_0^t \frac{1}{a} dt$$

exists and is finite. By the definition of τ;

$$a = e^{\frac{2}{3}\tau}$$

and

$$\frac{dt}{d\tau} = x$$

Therefore,

$$l = \int_{-\infty}^{\tau} e^{-\frac{2}{3}\tau} x d\tau.$$

By Lemma 3.1 there exists ϵ and p_m such that all trajectories of (3.6) incident on Σ_0 with $p_0 < p_m$ satisfy

$$H(\tau) < H_0 e^{\frac{5}{6}\tau} \tag{3.33}$$

for all $\tau < 0$. If l is finite at $\tau = 0$ it will be finite for all τ. It will therefore be sufficient to show that trajectories for which (3.33) holds possess a horizon at $\tau = 0$. Using (3.33) we have

$$l \leq x_0 \int_{-\infty}^{0} e^{\frac{\tau}{6}} d\tau \tag{3.34}$$

$$= 6x_0 \tag{3.35}$$

Since this is always finite we have the result. □

Note also that as $x_0 \to 0$, l must also approach 0 indicating that the horizon length shrinks to zero as $t \to 0$.

The physical meaning of the Lemma 3.1 becomes clearer when we recall that $\tau = \ln v$ and, by (3.8) and the definition of x $H = 9K^{-4}$. Lemma 3.1 may thus be interpreted as saying that for all $0 < n < 1$ there exists some $A > 0$ such that

$$K > Av^{-n}.$$

It follows that for all $p > 1$ there exists v_0 such that

$$v(t) > v_0 t^p \tag{3.36}$$

Thus, in the neighbourhood of the singularity the volume element expands faster than any power law with power greater than one. In order to avoid a particle horizon it must expand slower than t^3.

What about the case $p = 1$? Consider the function $e^{-4\tau}H$. Taking the derivative of this function with respect to τ, using (2.7)

$$\frac{d}{d\tau}e^{-4\tau}H = -4e^{-4\tau}H(1-q^2)$$

Thus for $\tau < 0$ we have

$$\ln\left(e^{-4\tau}H\right) = c + 4\int_\tau^0 (1-q^2)d\tau$$

where c is a constant. The integral on the right hand side tends to infinity as $\tau \to -\infty$ since each solution spends a finitely large amount of τ-time with, say, $q^2 < \epsilon$ on *each* cycle (of which there are an infinite number).

Thus,

$$\lim_{\tau \to -\infty} e^{-4\tau}H = \infty.$$

This translates to a corresponding limit for v and t, as above:

$$\lim_{t\to 0} t^{-1}v = 0. \tag{3.37}$$

Comparing this expression with (3.36) we see that v expands slower than t but faster than any power law t^p with $p > 1$. In this respect the behavior of the gravitational field near the singularity is quite subtle and unusual since it can not be adequately modeled by a power law. Note also that it is clearly not admissible to neglect the dynamical effect of the potential when considering the gravitational field near the singularity.

3.5 Discussion

This model is interesting because there exist two possible time scales which seem natural in a physical sense, the affine parameter time t and the period of oscillation of the scalar field, characterized by some coordinate η say. The timelike geodesics are incomplete with respect to t but complete with respect η.

By convention we say that a space-time singularity exists if geodesics are incomplete with respect to their affine parameter since this describes a situation where a freely falling observer would reach the edge of space-time in finite proper time. However, the physical interpretation of t as the proper time is not really meaningful on a neighbourhood of a boundary point of space-time since no normal coordinates can be constructed at such a point (normal coordinates can be constructed at a point arbitrarily close to a singularity but may never be extended to the singularity itself).

On the other hand, η can be interpreted as the natural time scale associated with the matter content of the universe (including all clocks and astronauts since scalar matter is the only matter entering into this model) and therefore might be a more meaningful measure of the time taken to reach the singularity. One could imagine that if more complex (possibly quantum) cosmological models could be shown to display a similar oscillatory behavior for *all* physical fields then one would be led to an interpretation whereby a singularity exists but can never be reached in a finite amount of time by any physical object.

I believe that the qualitative features displayed by the above model, including the oscillatory behavior and existence of a singularity and and particle horizon, are characteristic of models which have very steep potential wells. Preliminary investigations of the steeper than exponential potential $V(\phi) = e^{\lambda\phi^2}$ and the "hard wall" potential $V(\phi) = \dfrac{1}{\lambda - \phi^2}$ on the domain $\phi^2 < \lambda$ have been carried out by the author revealing no significant departure from the qualitative behavior of the exponential potential well [9].

As a final comment, before returning to SED potentials in the next chapter, I note that asymptotic oscillatory behavior of the scalar field in this model is reminiscent of the oscillatory behavior of the components of the shear tensor displayed by Bianchi type IX perfect fluid cosmologies (and perhaps generic inhomogeneous vacuum and perfect fluid cosmologies). This behavior is associated with the existence of dynamical chaos in the solution space of these cosmologies [55]. It would, I believe, be very interesting to investigate the dynamics of exponential potential well models in Bianchi type IX space-time.

Chapter 4

Bianchi Type I Scalar Field Cosmologies

4.1 Introduction

So far, for simplicity, we have confined attention to spatially isotropic space-times. FRW space-times provide an excellent model of the present universe but tend to be unstable in the past and are therefore unlikely to represent realistic space-time models of the very early universe, particularly if an initial space-time singularity exists. The reason for this, as we shall see explicitly below for scalar field cosmologies, is the tendency of the components of the shear tensor to diverge in the approach to a singularity.

The simplest anisotropic space-time is Bianchi type I (BI). This is defined as a space-time possessing a 3D, abelian, group of isometries whose orbits are space-like hypersurfaces which foliate space-time. In addition to providing simple models from which to investigate the effect of shear on the evolution of the system BI cosmologies often characterize many of the important features of more general anisotropic and inhomogeneous cosmologies, particularly close to the initial singularity ([56, 57, 58])(We shall say more about this in the next chapter.). Bianchi type I is therefore a more reasonable space-time model of the early universe than FRW and represents a first step in a more general analysis.

For general scalar field cosmologies there is, of course, no guarantee that BI will characterize the dynamical behavior near the singularity or even that a singularity will exist. Nevertheless, the results of this chapter and Chapter 5 will lead us to conclude that subject to certain generic conditions typical solutions of $\mathcal{E}^2$ models possess many of the asymptotic features of Bianchi type I cosmologies.

4.2 Preliminary Results

4.2.1 Setting Up an Autonomous System.

It is easy to show (see, for example, [59, 60]) that the existence of a 3D abelian isometry group with spacelike orbits is equivalent to the existence of a synchronous coordinate system (t, x) in which the metric takes the form

$$\mathbf{g} = -\mathrm{d}t^2 + g_{ij}(t)\mathrm{d}x^i\mathrm{d}x^j \tag{4.1}$$

where the index i runs from 1 to 3 and the hypersurfaces, Σ_t, of constant time are the orbits of the 3D isometry group. Clearly each Σ_t is a flat, homogeneous manifold with Reimannian metric $g_{ij}(t)$. Since g_{ij} is independent of x the Ricci tensor ${}^{(3)}R_{ij}$ on Σ_t vanishes. Substituting (4.1) into the field equations (1.40b-c), (1.41) and (1.43) we see that BI solutions must satisfy each of the following equations.

$$\begin{aligned} 0 &= \dot{\phi}\phi_i & (4.2)\\ \dot{\sigma}^i_j + \tfrac{1}{3}\dot{K}\delta^i_j &= -K\sigma^i_j - (\tfrac{1}{3}K^2 - V(\phi))\delta^i_j + \phi^i\phi_j & (4.3)\\ \ddot{\phi} &= -K\dot{\phi} - V'(\phi) + g^{ij}\phi_{i|j} & (4.4)\\ \tfrac{1}{3}K^2 &= V(\phi) + \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi^i\phi_i. & (4.5)\end{aligned}$$

Where the expansion K and shear σ^i_j are functions of time. Since ϕ^μ is non-spacelike (4.2) implies that ϕ must also be a function of t only. The spatial gradient terms in (4.4), (4.5) and (4.3) therefore vanish identically.(4.4) may then be obtained from (4.3) and (4.5) so we see that the EFE for the above metric are equivalent to the system of equations

$$\begin{aligned} \dot{\sigma}^i_j + \tfrac{1}{3}\dot{K}\delta^i_j &= -K\sigma^i_j - (\tfrac{1}{3}K^2 - V(\phi))\delta^i_j\\ \tfrac{1}{3}K^2 &= V(\phi) + \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\dot{\phi}^2. \end{aligned} \tag{4.6}$$

where K, σ^i_j and ϕ are functions of time only.

Taking the trace of (4.6a) yields

$$\dot{K} + K^2 = 3V(\phi). \tag{4.7}$$

Thus the shear must satisfy

$$\frac{d}{dt}(\sigma^i_j) = -K\sigma^i_j \tag{4.8}$$

which can be integrated immediately (recalling that $K = \dot{v}/v$) to yield

$$\sigma^i_j(t) = \frac{A^i_j}{v(t)}. \tag{4.9}$$

Where A^i_j is a constant, trace-free, 3x3 matrix. Consider some spatial section Σ_0 corresponding to time $t = t_0$. Since Σ_0 is flat it is always possible to choose a cartesian coordinate system on Σ_0 so that $g_{ij}(t_0) = \delta_{ij}$. Recall from (1.37) and (1.36)

$$\sigma^i_j + \tfrac{1}{3}\delta^i_j K = \tfrac{1}{2}g^{ik}\dot{g}_{kj} \tag{4.10}$$

Thus

$$\sigma^i_j(t_0) + \delta^i_j K(t_0) = \tfrac{1}{2}\delta^{ik}\dot{g}_{kj}(t_0) = \tfrac{1}{2}\dot{g}_{ij}(t_0). \tag{4.11}$$

(4.11) and (4.9) imply that A^i_j is a symmetric matrix on Σ_0. Thus there exists a cartesian coordinate system on Σ_0 in which A^i_j is diagonal (ie with coordinate curves tangent to the eigenvectors of the matrix A^i_j in R^3). Extending this coordinate system onto M (4.9) implies that σ^i_j will be diagonal for all time. It then follows from (4.10) and(4.11) that g_{ij} is diagonal for all time. We thus have

$$g_{ij} = \text{diag}(g_1, g_2, g_3). \tag{4.12}$$

So that

$$K_{ij} = \tfrac{1}{2}\text{diag}(\dot{g}_1, \dot{g}_2, \dot{g}_3)$$

and

$$g^{ij} = \text{diag}\left(\frac{1}{g_1}, \frac{1}{g_2}, \frac{1}{g_3}\right).$$

It follows from (4.10) and (4.9) that for each g_i

$$\frac{\dot{g}_i}{g_i} = \frac{2}{3}\frac{\dot{v}}{v} + \frac{2\alpha_i}{v} \tag{4.13}$$

where the α_i are the components of the diagonal trace-free matrix A^i_j and must therefore satisfy

$$\sum_{i=1}^{3} \alpha_i = 0. \tag{4.14}$$

Integrating (4.13) we obtain

$$g_i(t) = v(t)^{\frac{2}{3}} \exp\left(2\alpha_i F(t)\right) \tag{4.15}$$

where

$$\frac{dF(t)}{dt} = \frac{1}{v(t)}$$

The metric is thus completely determined by the three numbers α_i and the volume element $v(t)$ (up to a constant which can be removed by a coordinate transformation). Furthermore, taking the square of (4.9) we have

$$\sigma^2 = \frac{\alpha^2}{v^2} \tag{4.16}$$

where

$$\alpha^2 = \sum_{i=1}^{3} \alpha_i^2.$$

We may now take as our state variables the set $(K, \sigma, \dot{\phi}, \phi)$, then the evolution of the system will be governed by the set of ordinary differential equations

$$\begin{aligned} \ddot{\phi} &= -K\dot{\phi} - V'(\phi) \\ \dot{K} &= -\tfrac{3}{2}\dot{\phi}^2 - \tfrac{3}{2}\sigma^2 \\ \dot{\sigma} &= -\sigma K \end{aligned} \tag{4.17}$$

subject to the constraint

$$K^2 = \tfrac{3}{2}\sigma^2 + \tfrac{3}{2}\dot{\phi}^2 + 3V(\phi). \tag{4.18}$$

Equation (4.17b) is obtained from (4.7) by substitution of the constraint (4.18) to eliminate $V(\phi)$. Equation (4.17c) follows from explicit differentiation of (4.16). Note that when $\sigma = 0$ these equations are just the equations for the flat FRW space-time (2.1). Indeed, flat FRW space-time is a special case of Bianchi type 1 space-time, corresponding to $\alpha^2 = 0$. If α^2 is non-zero, then σ is a strictly positive function of time. Conversely, if σ is non-zero at some time t_0, then α must be non-zero and σ is positive for all time.

4.2.2 Elementary Properties of The Flow

All physical (expanding) solutions of (4.17) lie in the 3D phase space $BI = \{(K, \sigma, \dot{\phi}, \phi) : K \geq 0, \sigma \geq 0, K^2 - \frac{3}{2}\sigma^2 - \frac{3}{2}\dot{\phi}^2 - 3V(\phi) = 0\}$ imbedded in R^4. The flat FRW phase space, Ω, is the intersection of BI with the $\sigma = 0$ hypersurface in R^4. It can immediately be demonstrated that all singularity free Bianchi type 1 solutions reside in Ω.

Theorem 4.1 *Let $p_0 = (K_0, \sigma_0, \dot{\phi}_0, \phi_0) \in BI$ and suppose $\sigma_0 > 0$. Then the flow Ψ_t generated by (4.17) maps K_0 and σ_0 to infinity in a finite time interval, to the past of p_0.*

Proof:

The theorem is an immediate consequence of the constraint equation (4.18). Since all the terms on the right hand side are non-negative we may write

$$K^2 \geq \tfrac{3}{2}\sigma^2.$$

It follows from (4.16) that

$$\dot{v} \geq \sqrt{\tfrac{3}{2}}\alpha.$$

Integrating with respect to time from some time $t < 0$ to 0, gives

$$v(0) - v(t) \geq -\sqrt{\tfrac{3}{2}}\alpha t$$

which implies that

$$v(t) \leq v(0) + \sqrt{\tfrac{3}{2}}\alpha t.$$

Since $\sigma_0 > 0$, α must be nonzero. Thus $v(t)$ must pass through zero at some some finite negative time $|t_i| \leq \frac{2v(0)}{\alpha}$. Equation (4.16) then implies that $\sigma(t_i) = \infty$. The constraint equation (4.18) subsequently implies that $K(t_i) = \infty$. This completes the proof. □

Corollary 4.2 *All equilibrium points of (4.17) on BI lie on the invariant manifold Ω. There exist no periodic orbits or limit cycles of (4.17) on BI.*

Proof:

Theorem (4.1) implies that K will be unbounded on any solution for which $\alpha > 0$. Since equilibrium points and limit cycles are, by definition, solutions which are maximal in a bounded region of phase space, it follows that they must lie on the set $\alpha = 0$; ie Ω. We have already demonstrated in the proof of Theorem 2.1 that no limit cycles exist exist on Ω, thus we have the result. □

4.2.3 The Massless Scalar Field

In the special case $V = 0$ it is possible to exactly integrate (4.17). Since this solution will be important to us later it will prove useful to derive it now. Setting $V' = 0$ in (4.17) and rearranging slightly

$$\ddot{\phi} = -\frac{\dot{v}}{v}\dot{\phi}$$

which may be integrated to obtain

$$\dot{\phi} = \frac{\beta}{v} \tag{4.19}$$

for some constant β. Substituting (4.19) and (4.16) into the constraint equation (4.18) we have

$$\frac{\dot{v}^2}{v^2} = \frac{3(\alpha^2 + \beta^2)}{2v^2}$$

and hence, upon integration,

$$v = \sqrt{\tfrac{3}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}} t. \tag{4.20}$$

For a given physical metric the metric components g_i are only unique up to an arbitrary constant coefficient, since a coordinate transformation $x^i \mapsto cx^i$ will not change the form of (4.1). It follows that f , also, is only unique up to a constant coefficient. We are therefore free to choose

$$\tfrac{3}{2}(\alpha^2 + \beta^2) = 1$$

and we may may introduce the parameter θ on the domain $0 \leq \theta \leq \pi$ such that $\alpha = \sqrt{\frac{2}{3}}\cos\theta$, $\beta = \sqrt{\frac{2}{3}}\sin\theta$. After integration of (4.19) to obtain ϕ we obtain the general solution

$$v = t \qquad \sigma = \sqrt{\tfrac{2}{3}}\sin\theta t^{-1} \qquad \phi = \sqrt{\tfrac{2}{3}}\cos\theta \ln\frac{t}{c} \tag{4.21}$$

for which

$$K = t^{-1} \qquad \dot{\phi} = \sqrt{\tfrac{2}{3}}\cos\theta t^{-1}. \tag{4.22}$$

Observe that the special cases $\theta = 0$ and $\theta = \pi$ which give $\sigma = 0$ generate the general solution for the massless scalar field in flat FRW space-time (1.81). The case $\theta = \pi/2$ corresponds to the Kasner vacuum solution ([61]). The general case with arbitrary θ also corresponds to a well known solution originally obtained by Jacob ([62]). We shall defer discussion of the physical properties of the above solution until after the next section. Firstly we shall show, for an arbitrary $\mathcal{E}^2$ potential V, that (4.21-4.22) approximates the asymptotic behavior of almost all solutions.

4.3 Behavior of Solutions Near the Singularity.

In this section we shall be interested in the behavior of solutions close to the initial singularity. The approach that shall be followed is just a natural generalization of our analysis for the FRW case. Firstly, since we know that K and σ diverge let us introduce the coordinates

$$x = \frac{1}{K} \qquad S = \sqrt{\tfrac{3}{2}}\frac{\sigma}{K} \qquad y = \sqrt{\tfrac{3}{2}}\frac{\dot{\phi}}{K} \tag{4.23}$$

and the time coordinate τ satisfying

$$\frac{d}{dt} = K\frac{d}{d\tau}. \tag{4.24}$$

defined as in (2.14). The dynamical equations (4.17) now become

$$\begin{aligned}
\frac{dx}{d\tau} &= x(y^2 + S^2) \\
\frac{dS}{d\tau} &= -S + S^3 + Sy^2 \\
\frac{dy}{d\tau} &= -y + y^3 + yS^2 - \sqrt{\tfrac{3}{2}}x^2V'(\phi) \\
\frac{d\phi}{d\tau} &= \sqrt{\tfrac{2}{3}}y
\end{aligned} \tag{4.25}$$

The constraint equation becomes

$$y^2 + S^2 + 3x^2V(\phi) = 1 \tag{4.26}$$

4.3.1 Solutions For Which $|\phi|$ Remains Bounded

It is natural to ask whether ϕ remains bounded in the approach to the singularity. It turns out that when $\sigma \neq 0$ there exist solutions with "vacuum dominated singularities" where the scalar field (and indeed the energy density) remains finite. Let us consider these solutions first.

Theorem 4.3 *The set of all past orbits of (4.25) on which ϕ is bounded has measure zero in phase space.*

Proof:

Using (4.26), S can be eliminated from (4.25) yielding the 3-dimensional unconstrained system of equations

$$\begin{aligned} \frac{dx}{d\tau} &= x - 3x^3V(\phi) \\ \frac{dy}{d\tau} &= -3x^2V(\phi) - x^2V'(\phi) \\ \frac{d\phi}{d\tau} &= \sqrt{\tfrac{2}{3}}y \end{aligned} \qquad (4.27)$$

on the subset of R^3, $BI^* = \{(x, y, \phi) : x \geq 0, y^2 \leq 1)\}$. BI^* is just a projection of BI onto R^3 with the addition of the unphysical boundary set $\Sigma = \{(\phi, x, y) \in BI^* : x = 0)\}$. Σ is an invariant manifold of (4.27) and restriction of (4.27) to this set yields the trivial system of equations

$$\begin{aligned} \frac{d\phi}{d\tau} &= \sqrt{\tfrac{2}{3}}y \\ \frac{dy}{d\tau} &= 0 \end{aligned} \qquad (4.28)$$

on Σ. Clearly the y-axis is an equilibrium set. On all other trajectories $\phi \to \pm\infty$ as $\tau \to -\infty$.

Let p be an arbitrary point in BI^* corresponding to initial conditions of (4.17) with non-zero shear and assume ϕ is bounded on the past orbit $O^-(p)$ of p under (4.27). By Theorem 4.1, the trajectory $\psi_p(\tau)$ passing through p must approach Σ asymptotically as $\tau \to -\infty$. Since ϕ is bounded, $O^-(p)$ is contained in a compact subset of BI^*. It follows from (4.28) and the continuity of the flow that ψ_p must be past asymptotic to some point $p_0 = (\phi_0, 0, 0)$ on the y-axis (Appendix Theorem A.1 gives a rigorous justification).

Now consider the total derivative of the vector field (4.27) at p_0.

$$J(p_0) = \begin{pmatrix} 1 & & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 \end{pmatrix} \qquad (4.29)$$

the eigenvalues are 1 and 0. There exists 1-dimensional unstable manifold tangent to the x-axis at p_0 and a 2D center manifold which is easily seen to be just Σ. Since p_0 is unstable on the center manifold Σ it follows from Theorem A.8 that p_0 is unstable in BI^*. In fact it is clear that there exists exactly 1 solution of (4.27) past asymptotic to p_0, namely the unique solution lying on the unstable manifold. Since p_0 is an arbitrary point on the y-axis, it follows that there exists a one parameter family of curves each past asymptotic to some such point p_0. The union of these curves will form a 2D invariant manifold, BI_f, intersecting the y-axis of BI^*. Since BI_f is of lower dimension than BI it clearly has measure 0 in phase space. □

4.3.2 The System Close to $\phi = \infty$

Now that we have established that $|\phi|$ diverges on almost all solutions we can analyze the behavior near the singularity in much the same way as for the FRW case. Assume that $V \in \mathcal{E}^2$ and define $BI_\epsilon = \{(\phi, x, y, z) \in BI : \phi > \epsilon^{-1}\} \subset BI$. Where ϵ is chosen sufficiently small for W_V to be well defined. The set $\Omega_\epsilon \subset \Omega$ defined the previous chapter is the intersection of BI_ϵ with the $z = 0$ hypersurface. We now make the coordinate transformation $(x, S, y, \phi) \mapsto (x, S, y, z))$ where

$$z = f(\phi)$$

and f is a smooth function satisfying the conditions of Definition 2.2. Substituting (4.26) into (4.25c) to eliminate x, and taking (S, y, z) as our coordinates on BI_ϵ we obtain the 3-dimensional system of equations on BI_ϵ:

$$\frac{dS}{d\tau} = -S + S^3 + Sy^2 \qquad (4.30)$$

$$\frac{dy}{d\tau} = -y + y^3 + yS^2 - \tfrac{1}{\sqrt{6}}\bar{W}_V(z)(1 - y^2 - S^2) \qquad (4.31)$$

$$\frac{dz}{d\tau} = \sqrt{\tfrac{2}{3}}\bar{f}'(z)y$$

This set of equations forms a C^2 dynamical system on the closure of BI_ϵ, $\tilde{BI}_\epsilon = \{(y, S, z) : 0 \le w \le f(\epsilon^{-1}),\, y^2 + z^2 \le 1, z \ge 0\}$. This set is illustrated in Fig. 4.1. The boundary consists of the Robinson-Walker submanifold

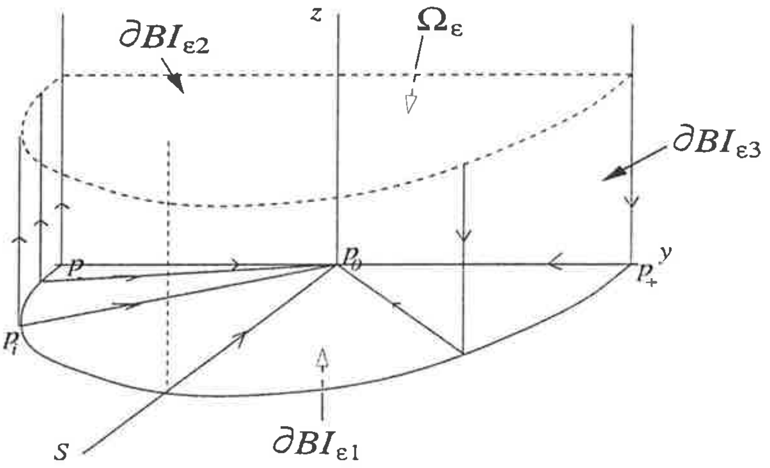

Figure 4.1: BI_ϵ.

Ω_ϵ, the semi-cylindrical mantle $\partial BI_{\epsilon 3}$, defined by the algebraic condition $y^2 + S^2 = 1$, and the caps $\partial BI_{\epsilon 1}$, $\partial BI_{\epsilon 2}$ which satisfy the constraints $z = 0$ and $z = f(\epsilon^{-1})$ respectively.

$\partial BI_{\epsilon 2}$ is the only component of the boundary transverse to the flow Ψ_τ generated by (4.25). As for the FRW case, solutions leave BI_ϵ through $\partial BI_{\epsilon 3}$ to the left of the $y = 0$ plane and enter through this set to the right of the $y = 0$ plane.

The flow on Ω_ϵ has already been analyzed in detail in the last chapter and includes, in particular, the 1-dimensional center manifold C_0 intersecting the origin.

The flow on $\partial BI_{\epsilon 3}$ may be obtained by substituting the constraint $y^2 + S^2 = 1$ into (4.31) to obtain the trivial set of differential equations

$$\frac{dS}{d\tau} = 0$$
$$\frac{dy}{d\tau} = 0 \qquad (4.32)$$

$$\frac{dz}{d\tau} = \sqrt{\tfrac{2}{3}}\bar{f}'(z)y.$$

The orbits are just the vertical lines on $\partial BI_{\epsilon 3}$ (Fig. 4.1). Equation (4.32c) is equivalent to

$$\frac{d\phi}{d\tau} = \sqrt{\tfrac{2}{3}}y$$

which, since y is constant, can be integrated exactly to yield

$$\phi = \sqrt{\tfrac{2}{3}}y_i(\tau - \tilde{\phi}) \tag{4.33}$$

or

$$z = f(\sqrt{\tfrac{2}{3}}y_i(\tau - \tilde{\phi})). \tag{4.34}$$

Where $-1 < y_i < 1$. Note that the direction of the flow depends on the sign of y_i and, in particular, $z \to 0$ as $\tau \to -\infty$ iff $y_i < 0$.

The unphysical set $\partial BI_{\epsilon 1}$ corresponds to $\phi = \infty$. The flow on $\partial BI_{\epsilon 1}$ can be found by setting z to zero, to obtain the 2D system of equations

$$\begin{aligned} \frac{dS}{d\tau} &= -S(1 - y^2 - S^2) \\ \frac{dy}{d\tau} &= -y(1 - y^2 - S^2). \end{aligned} \tag{4.35}$$

Clearly the semi-circle $\Upsilon = \partial BI_{\epsilon 1} \cap \partial BI_{\epsilon 3}$ is an equilibrium set. It includes, as its endpoints, the equilibrium points $p_\pm$. The only other equilibrium point on $\partial BI_{\epsilon 1}$ is the origin p_0.

The equations (4.35) become particularly simple in polar coordinates. Let

$$S = r\sin\theta \qquad\qquad y = r\cos\theta \tag{4.36}$$

then in terms of r and θ the equations become

$$\begin{aligned} \frac{dr}{d\tau} &= -r(1 - r^2) \\ \frac{d\theta}{d\tau} &= 0 \end{aligned} \tag{4.37}$$

which can be integrated exactly to yield

$$r = (1 + ce^{2\tau})^{-\frac{1}{2}} \qquad\qquad \theta = \theta_0. \tag{4.38}$$

where c and θ_0 are constants of integration. Thus, solutions on $\partial BI_{\epsilon 1}$ are radial lines, emanating from the the semi-circle Υ at $\tau \to -\infty$ and approaching the origin as $\tau \to \infty$.

4.3.3 Stability of Equilibrium Sets and Identification of the Generic Source

We now classify the equilibrium points of $\tilde{BI}_\epsilon$. In addition to Υ and p_0, the only other equilibrium points in $\tilde{BI}_\epsilon$ lie on $\partial BI_{\epsilon 3}$ along the line $y = 0$. These are precisely the equilibrium points which are the past asymptotes of the 2-dimensional set BI_f, discussed above, and will not be considered further. Let us consider first the origin p_0. The matrix of derivatives of (4.31) at p_0 is given by

$$J(p_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.39}$$

the eigenvalues are 0, -1 and -1 with corresponding eigenvectors $\hat{e}_w$, $\hat{e}_y$ and $\hat{e}_z$ respectively. It follows from the Center Manifold Theorem (A.5) and (4.38) that $\partial BI_{\epsilon 1}$ is a 2-dimensional stable manifold and there exists a 1-dimensional center manifold, which is, of course, the curve C_0 lying on Ω_ϵ. Any solutions in a sufficiently small neighbourhood of p_0 will exponentially decay onto the the center manifold C_0. The behavior of solutions on C_0 has already been discussed in the last chapter.

The other equilibrium set is the semi-circle Υ. Let $p_i = (\sin\theta_i, \cos\theta_i, 0) \in \Upsilon$, where θ_i is an arbitrary angle between 0 and π. The matrix of derivatives of (4.31) at p_i is

$$J(p_i) = \begin{pmatrix} 2\sin^2\theta_i & 2\cos\theta_i\sin\theta_i & 0 \\ 2\cos\theta_i\sin\theta_i & 2\cos^2\theta_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.40}$$

The eigenvalues of this matrix can be computed to be 0 and 1 with eigenvectors $\hat{e}_z$ and $\hat{e}_\theta$ spanning the eigenspace of the zero eigenvalue; and the eigenvector $\hat{e}_r$ spanning the eigenspace of the eigenvalue 1. $\hat{e}_\theta$ and $\hat{e}_r$ are the unit vectors tangent and normal, respectively, to the circle Υ at p_i. It follows that the half-cylinder $\partial BI_{\epsilon 3}$ is a center manifold of p_i and the radial line from p_i to the origin lying on $\partial BI_{\epsilon 2}$ is an unstable manifold.

According to Theorem A.8 the (negative) stability of p_i in BI_ϵ will be determined by its stability in $\partial BI_{\epsilon 3}$. From equation (4.34) we see that Υ can be treated as the union of three qualitatively distinct segments,
$\Upsilon_+ = \{p_i \in \Upsilon : 0 \leq \theta_i < \frac{\pi}{2}\}$, $\Upsilon_- = \{p_i \in \Upsilon : \frac{\pi}{2} < \theta_i \leq \pi\}$ and

$\Upsilon_0 = \{p_i \in \Upsilon : \theta_i = \frac{\pi}{2}\}$. Points in Υ_+ are unstable and no (physical) trajectories of (4.31) are past asymptotic to these sets. Υ_- and Υ_0, on the other hand, are negatively stable.

If N_i is a sufficiently small neighbourhood of $p_i \in \Upsilon_- \cup \Upsilon_0$ and $p \in N_i$ equation (4.34) and Theorem A.8 imply there exists $\gamma > 0$ such that ψ_p may be written

$$\begin{aligned} S(\tau) &= \sin\theta + O(e^{\gamma\tau}) \\ y(\tau) &= \cos\theta + O(e^{\gamma\tau}) \\ z(\tau) &= f(\sqrt{\tfrac{2}{3}}\cos\theta(\tau - \tilde{\phi})) + O(e^{\gamma\tau}) \end{aligned} \tag{4.41}$$

where θ here is a constant parameter in some interval of θ_i. Note that for any given $p \in N_i$ the α-limit point $\alpha(p) = (\sin\theta, \cos\theta, 0)$ does not necessarily correspond to p_i, although its distance from p_i is bounded by the "radius" of N_i. Furthermore, when $\theta = \frac{\pi}{2}$ z does not decay to zero. It follows that no trajectories are actually asymptotic to Υ_0.

Thus Υ_- is the unique generic source of the system and Theorem A.8ii ensures that each $p \in \Upsilon_-$ is the α-limit of a continuous two dimensional sheet or orbits tangent to $\hat{e}_z$ and $\hat{e}_r$ at p and projecting into BI_ϵ. The union of these sheets constitutes a continuous foliation on some neighbourhood of Υ_-.

4.4 Identification With Massless Scalar Field Solutions

We may identify each solution close to Υ_- with a solution of the massless scalar field as follows. In a sufficiently small neighbourhood of Υ_- equation (4.25b) becomes

$$\frac{dx}{d\tau} = x(1 + O(e^{\gamma\tau})).$$

We may now integrate (2.15) (remembering that $K = \dfrac{1}{x}$) to obtain the first order expressions

$$t = x_0 e^{\tau} \tag{4.42}$$

and

$$K = t^{-1}. \tag{4.43}$$

where t has been chosen so that it approaches 0 as $\tau \to -\infty$. Substituting (4.42) into (4.41c) gives the asymptotic solution for ϕ

$$\phi = \sqrt{\tfrac{2}{3}} \cos\theta \ln \frac{t}{c}. \tag{4.44}$$

where θ and $c = x_0 e^{\bar{\phi}}$ are positive constants with $\frac{\pi}{2} < \theta \leq \pi$. As for the FRW case c takes the same value for all initial points along a given orbit as is required for consistency since the origin of the coordinate t is fixed. The first order solution for σ is obtained from (4.23) and the fact that $\lim_{t\to 0} S = \sin\theta$. We find,

$$\sigma = \sqrt{\tfrac{2}{3}} \sin\theta t^{-1}. \tag{4.45}$$

Similarly for $\dot{\phi}$ we obtain

$$\dot{\phi} = \sqrt{\tfrac{2}{3}} \cos\theta t^{-1}. \tag{4.46}$$

Comparing the asymptotic solution, (4.43-4.46) with the exact solution for the massless scalar field (4.21-4.22) we see that it does indeed correspond.

The following is a generalization of Theorem 2.7.

Theorem 4.4 *Let $V \in \mathcal{E}^2_+$ be given. There exists a neighbourhood N of Υ_- such that each orbit intersecting N in the interior of BI corresponds to a solution of (4.17) for which K, σ, ϕ and $\dot{\phi}$ may be written, for t close to zero,*

$$\begin{aligned} K &= t^{-1} + O(\epsilon_V(\theta_i;t)) \\ \sigma &= \sin\theta_i t^{-1} + O(\epsilon_V(\theta_i;t)) \\ \phi &= \sqrt{\tfrac{2}{3}} \cos\theta_i \ln\frac{t}{c} + O(t\epsilon_V(\theta_i;t)) \\ \dot{\phi} &= \sqrt{\tfrac{2}{3}} \cos\theta t^{-1} + O(\epsilon_V(\theta;t)) \end{aligned} \tag{4.47}$$

where,

$$\begin{aligned} \epsilon_V(\theta_i;t) &= tV(-\sqrt{\tfrac{2}{3}} \cos\theta_i \ln\tfrac{t}{c}) \\ &\to 0 \;\; as \;\; t \to 0. \end{aligned} \tag{4.48}$$

Proof:

For the most part, the proof the same as that of Theorem 2.7 and is a straitforward calculation. It will be sufficient to provide an outline. Using the first order approximation to x together with equation (4.26)we obtain

$$y^2 + S^2 = 1 - 3x_0^2 V(\phi) e^{2\tau} + h \tag{4.49}$$

Substituting (4.49) into (4.25b) allows us to obtain a second order expression for x and subsequently t and K in identical fashion to the FRW case. We find, as before

$$x_0 e^{\tau} = t + \frac{V(\phi)t^3}{2} + h. \tag{4.50}$$

and

$$K = t^{-1} + V(\phi)t + h. \tag{4.51}$$

Now recall that by definition $v \propto e^{\tau}$. It follows from equation (4.16) that $\sigma \propto e^{-\tau}$ and hence, since we know that $\sqrt{\frac{3}{2}}\sigma/K \to \sin\theta$ as $\tau \to -\infty$:

$$\sigma = \sqrt{\tfrac{2}{3}}\sin\theta(t^{-1} - \tfrac{1}{2}V(\phi)t) + h.$$

Plugging the above expressions back into (4.49) remembering the definition of S we find

$$y^2 = \tfrac{2}{3}cos^2\theta + O(e^{2\tau}V(\phi))$$

The error bound for y^2 allows us to immediately obtain an estimate for $\dot{\phi}$ and, upon integration of y with respect to τ, an estimate for ϕ and subsequently for V performing identical calculations to the FRW case thereby completing the proof.□

It now remains to verify that the correspondence between solutions of the above type and massless scalar field solutions is indeed 1-1, thereby generalizing Theorem 2.8. This turns out to be significantly more complicated than for the FRW case.

It shall be convenient to work in cylindrical polar coordinates (r, θ) defined as in (4.36). θ here is a coordinate and should not be confused with the parameter θ used in the asymptotic expressions above.

We shall begin by following the same approach as previously. Define the initial 2-surface

$$\Sigma(\delta_0, \delta_1) = \left\{(r, \theta, z) : z = z_0, 0 < 1 - r < \delta_1, \frac{\pi}{2} < \theta < \pi\right\}.$$

We know from the above discussion that for each θ_i between $\pi/2$ and π there exists a continuous 1-parameter family of orbits for which the equilibrium point $(1, \theta_i, 0)$ is a limit point. This family of orbits will intersect Σ along a unique continuous curve $\Sigma_i(\delta_0, \delta_1)$. Furthermore, it is clear that Σ_i depends continuously on θ_i.

Since θ_i appears as an explicit parameter in the asymptotic massless scalar field solution of a particular orbit it will be sufficient to show that, for each θ_i, the remaining parameter c is continuous and 1-1 as a map from $\Sigma_i(\delta_0, \delta_1)$ onto the interval $(0, c_m(\delta_0, \delta_1))$ where $c_m \to \infty$ as $\delta_0 \to 0$. (it then follows that there is a continuous 1-1 correspondence between exact orbits of the system and the pair (c, θ_i) which continuously parameterize the first order solutions).

As with the FRW case we shall parameterize $\Sigma_i(\delta_0, \delta_1)$ with the variable $w = 1 - r$. The reason that the proof becomes complicated is the presence of the angular variable θ. We shall shortly prove a lemma which enables us to control the θ dependence of solution curves. Before doing so it will be convenient to introduce a new parameterization of the curves. Since $\phi \to \infty$ monotonically (sufficiently close to Υ_-) as $\tau \to -\infty$ we may represent each orbit as a unique graph $(\phi, r(\phi), \theta(\phi))$. The ϕ dependence of r and θ is determined by the equations

$$\begin{aligned} \frac{dr}{d\phi} &= -(1-r^2)\left(\frac{1}{\cos\theta} + \frac{W_V(\phi)}{\sqrt{6}r}\right) \\ \frac{d}{d\phi}(\ln(\sin\theta)) &= W_V(\phi)\frac{1-r^2}{r^2} \end{aligned} \tag{4.52}$$

Note that these equations are non-autonomous since ϕ appears explicitly on the right hand side (thereby eliminating z). For each solution curve belonging to a particular orbit τ can be treated as a function of ϕ satisfying

$$\frac{d\tau}{d\phi} = \sqrt{\frac{3}{2}}y^{-1}. \tag{4.53}$$

Indeed, from (4.47), (4.42) and the fact that $c = x_0 e^{\tilde{\phi}}$ we have

$$\tau = \frac{\sqrt{3}}{\sqrt{2}\cos\theta_i}\phi + \tilde{\phi} + h. \tag{4.54}$$

In what follows the reader should keep in mind that $r \to 1$ and $\theta \to \theta_i$ as $\phi \to \infty$, independently of the initial point w. Furthermore, the initial value of ϕ, $\phi_0 = f^{-1}(\delta_0)$ tends to ∞ as $\delta_0 \to 0$.

Lemma 4.5 *Assume that* $V \in \mathcal{E}^2_+$. *Let* w_1 *and* w_2 *label two points on* $\Sigma_i(\delta_0, \delta_1)$ *and let* $(r_1(\phi), \theta_1(\phi))$ *and* $(r_2(\phi), \theta_2(\phi))$ *be the solutions of (4.52) intersecting these points. For all* $\delta > 0$ *there exists* $\phi_1 > \phi_0$ *such that that, for all* $\phi \geq \phi_1$

$$|\cos\theta_2 - \cos\theta_1| < \delta|r_2 - r_1|.$$

Proof:

The proof is divided into three steps. In the first part we show that, for ϕ sufficiently large, $|\cos\theta_1 - \cos\theta_2| < \delta I(r_1, r_2; \phi)$ where

$$I(r_1, r_2; \phi) = \int_\phi^\infty |r_2 - r_1| d\phi.$$

In Step 2 we show that it possible to choose an arbitrarily large ϕ_1 for which $I(r_1, r_2; \phi_1) < \cos\theta_i/(\sqrt{6} - \epsilon)|r_2(\phi_1) - r_1(\phi_1)|$ for arbitrary positive ϵ. In Step 3 we show that this condition must remain true for all $\phi > \phi_1$ thereby completing the proof.

Step 1:

Applying (4.52b) to each of the orbits at a particular value of ϕ and subtracting we find

$$\frac{d}{d\phi} \ln\left(\frac{\sin\theta_2}{\sin\theta_1}\right) = \frac{W_V(\phi)(r_1^2 - r_2^2)}{r_2^2 r_1^2} \tag{4.55}$$

Since $W_V \to 0$ as $\phi \to \infty$ it is possible, for all $\epsilon > 0$, to choose ϕ_1 such that for all $\phi \geq \phi_0$,

$$\left|\frac{d}{d\phi} \ln\left(\frac{\sin\theta_2}{\sin\theta_1}\right)\right| < \epsilon|r_2 - r_1|$$

Integrating from ϕ to ∞ gives

$$\left|\ln\left(\frac{\sin\theta_2}{\sin\theta_1}\right)\right| < \epsilon I(r_1, r_2; \phi)$$

which may be expanded in a straitforward manner to yield

$$|\sin\theta_2 - \sin\theta_1| < \sin\theta_1 \epsilon I(r_1, r_2; \theta)$$

where ϵ is understood to be arbitrary. Since $\tan\theta \simeq \tan\theta_i$ for ϕ sufficiently large, it can be bounded by choosing ϕ_1 sufficiently small. It then follows that $\cos\theta$ is Lipschitz in $\sin\theta$ so for all $\delta > 0$ we may choose ϵ such that

$$|\cos\theta_2 - \cos\theta_1| < \delta I(r_1, r_2; \phi) \tag{4.56}$$

This completes Step 1 of the proof.
Step 2:
From (4.49) we have

$$1 - r^2 = 3x_0^2 V(\phi) e^{\tau} + h.$$

Also, from (4.54)

$$\tau = \frac{\sqrt{3}}{\sqrt{2}\cos\theta_i}\phi + \check{\phi} + h. \tag{4.57}$$

therefore,

$$1 - r^2 < aV(\phi)\exp\left(\frac{\sqrt{6}}{\cos\theta_i}\phi\right).$$

for some positive constant a. Since $V(\phi)$ is SED it follows that for all $\epsilon > 0$ there exists $A > 0$ such that

$$1 - r < A\exp\left(\frac{\sqrt{6}-\epsilon}{\cos\theta_i}\phi\right) \tag{4.58}$$

for all $\phi > \phi_1$. Since $|r_2 - r_1| < |r_2 - 1| + |r_1 - 1|$ the above inequality provides the following bound for $I(r_1, r_2; \phi)$:

$$I(r_1, r_2; \phi) < B\exp\left(\frac{\sqrt{6}-\epsilon}{\cos\theta_i}\phi\right) \tag{4.59}$$

for some positive constant B. This implies that there exists some $\phi_2 \geq \phi_1$ such that

$$I(r_1, r_2; \phi_2) < \frac{|\cos\theta_i|}{\sqrt{6}-2\epsilon}|r_2(\phi_2) - r_1(\phi_2)|. \tag{4.60}$$

In order to see this assume the converse. i.e. assume that

$$I(r_1, r_2; \phi) \geq \frac{|\cos\theta_i|}{\sqrt{6}-2\epsilon}|r_2(\phi) - r_1(\phi)|.$$

for all $\phi > \phi_1$. By the definition of I this implies

$$\left|\frac{dI}{d\phi}\right| < \frac{\sqrt{6}-2\epsilon}{|\cos\theta_i|}I$$

for all $\phi > \phi_1$. This implies there exists some constant C such that

$$I > C \exp\left(\frac{\sqrt{6}-2\epsilon}{\cos\theta_i}\phi\right)$$

which contradicts (4.59). Therefore (4.60) must hold. Since the inequalities involving ϕ_1 hold equally well for any $\phi > \phi_1$ we may choose, without loss of generality, $\phi_1 = \phi_2$. This completes Step 2 of the proof.

Step 3:

Using (4.52a) we obtain the ϕ dependence of $r_2 - r_1$:

$$\begin{aligned}
\frac{d}{d\phi}(r_2 - r_1) &= -\frac{(1-r_2^2)}{\cos\theta_2} + \frac{(1-r_1^2)}{\cos\theta_1} - W_V(\phi)\frac{(r_1 - r_1 r_2^2 - r_2 + r_1^2 r_2)}{\sqrt{6} r_1 r_2} \\
&= \left[\frac{(r_2+r_1)}{\cos\theta_1} + W_V(\phi)\frac{(1+r_1 r_2)}{\sqrt{6} r_1 r_2}\right](r_2 - r_1) \\
&\quad + \frac{(1-r_2)^2}{\cos\theta_1\cos\theta_2}(\cos\theta_2 - \cos\theta_1). \qquad (4.61)
\end{aligned}$$

Now, according to (4.56) $|\cos\theta_2 - \cos\theta_1|$ is bounded by I for sufficiently large ϕ. Furthermore, if ϕ is chosen sufficiently small W_V may be made arbitrarily small whilst θ and r may be made arbitrarily close to θ_i and 1 respectively. Recalling also that on the domain of θ_i $\cos\theta_i \neq 0$ it is clear that it is possible to choose ϕ_1 sufficiently large that the first term in the square brackets in (4.61) dominates the second term so that we have the following inequality for $\phi > \phi_1$:

$$\frac{d}{d\phi}|r_2 - r_1| > \frac{1}{\cos\theta_i}\left((\sqrt{6}-3\epsilon)|r_2 - r_1| + \frac{\epsilon}{\cos\theta_i} I(r_2, r_1)\right) \qquad (4.62)$$

for any $\epsilon > 0$. Note that the first term on the right hand side is negative whilst the second is positive since $\cos\theta_i < 0$. The number $\sqrt{6-3\epsilon}$ could equally well been any number greater than 2. The reason for this peculiar choice shall become clear below.

Let us now consider a ϕ_1 at which (4.60) is satisfied with $\phi_2 = \phi_1$. At any such value of ϕ we have

$$\frac{I(r_1, r_2; \phi_1)}{|r_2 - r_1|} < \frac{\cos\theta_i}{\sqrt{6}-2\epsilon}.$$

Taking the derivative with respect to ϕ, using the definition of I gives

$$\frac{d}{d\phi}\left(\frac{I(r_1,r_2;\phi_1)}{|r_2-r_1|}\right) = -1-\frac{I(r_1,r_2;\phi_1)}{|r_2-r_1|^2}\frac{d}{d\phi}|r_2-r_1| \tag{4.63}$$

$$< -1+\frac{\sqrt{6}-(3-(\sqrt{6}-2\epsilon)^{-1})\epsilon}{\sqrt{6}-2\epsilon} \tag{4.64}$$

$$< 0 \ . \tag{4.65}$$

The second line was obtained using (4.60) and (4.62) evaluated at ϕ_1. It follows that (4.60) holds for all $\phi > \phi_1$, i.e.

$$I(r_1,r_2;\phi) < \frac{|\cos\theta_i|}{\sqrt{6}-2\epsilon}|r_2(\phi)-r_1(\phi)|.$$

for all $\phi > \phi_1$. It follows immediately from (4.56) that for all $\delta > 0$ we may choose ϕ_1 such that

$$|\cos\theta_2-\cos\theta_1| < \delta|r_2-r_1| \tag{4.66}$$

for all $\phi > \phi_1 \geq \phi_0$ as required. □

Now that we have θ under control the completion of the proof that c is $1-1$ is essentially the same as the FRW case. Firstly, we introduce a definition of c explicitly as an orbital quantity. Recall that $c = x_0 e^{\tilde{\phi}}$ where

$$x_0 = \lim_{\tau\to-\infty} e^{\tau}x(\tau)$$

and, according to (4.54),

$$\tilde{\phi} = \lim_{\tau\to-\infty} \tau - \frac{\sqrt{3}}{\sqrt{2}\cos\theta_i}\phi.$$

Combining these expressions we may write c as

$$c = \lim_{\phi\to\infty} x\exp\left(-\frac{\sqrt{3}}{\sqrt{2}\cos\theta_i}\phi\right). \tag{4.67}$$

Theorem 4.6 *For δ_1, δ_0 sufficiently small c is a continuous 1-1 function of w. The range of c for fixed δ_1 is $(0, c_m(\delta_0))$ where $c_m(\delta_0) \to \infty$ as $\delta_0 \to 0$.*

Proof:

We firstly show that c is continuous and 1-1. For each initial point w in $\Sigma_i(\delta_0, \delta_1)$ denote the corresponding trajectory $(r(w;\phi), \theta(w;\phi))$.

Firstly we make the following observation: The union of orbits intersecting $\Sigma_i(\delta_0, \delta_1)$ is a continuous 2-surface in BI_ϵ which could be parameterized by ϕ and r since these have non-zero derivative along the orbits (with respect to the parameter τ, say). Therefore, since distinct orbits may never intersect it follows that if $w_2 < w_1$ so that $r(w_2;\phi_0) - r(w_1;\phi_0) > 0$ then

$$r_2 - r_1 > 0 \text{ for all } \phi > \phi_0. \tag{4.68}$$

where r_1 and r_2 are understood as shorthand notation for $r(w_1;\phi)$ and $r(w_2;\phi)$ respectively. Define the function

$$C(r,\phi) = x(r,\phi)\exp\left(-\frac{\sqrt{3}}{\sqrt{2}\cos\theta_i}\phi\right)$$

then from (4.25a,d) the directional derivative of C along the orbits satisfies

$$\frac{dC}{d\phi} = \sqrt{\tfrac{3}{2}}C\left(\frac{r}{\cos\theta} - \frac{1}{\cos\theta_i}\right). \tag{4.69}$$

On a particular orbit we can write C explicitly using (4.26):

$$C(w;\phi)^2 = \frac{\exp\left(-\frac{\sqrt{6}}{\cos\theta_i}\phi\right)}{3V(\phi)}(1 - r(w;\phi)^2) \tag{4.70}$$

Clearly, C approaches $c(w)$ in the limit as $\phi \to \infty$. The continuity of c is immediately apparent since the functions $C(w;\phi)$ are uniformly continuous with respect to w.

Denote by $C_1(\phi)$ and $C_2(\phi)$ the restriçtion of C to the orbits intersecting w_1 and w_2 respectively, where $w_1 > w_2$. It follows immediately from (4.70) and (4.68) that $C_2(\phi) < C_1(\phi)$ for all $\phi > \phi_0$.

The difference $C_2 - C_1$ evolves according to

$$\frac{d}{d\phi}\ln\left(\frac{C_2}{C_1}\right) = \sqrt{\tfrac{3}{2}}\left(\frac{r_2}{\cos\theta_2} - \frac{r_1}{\cos\theta_1}\right) \tag{4.71}$$

Choose $\phi_1 > \phi_0$ so that the conditions of Lemma 4.5 hold with $\delta < \cos\theta_i/4$ and θ and r are bounded very close to 1 and θ_i respectively. Then for all $\phi > \phi_1$

$$\frac{d}{d\phi}\ln\left(\frac{C_2}{C_1}\right) < \sqrt{\tfrac{3}{2}}\left(\frac{(r_2 - r_1)}{2\cos\theta_i}\right) < 0$$

Integrating from ϕ_1 to ∞ we have

$$\frac{c(w_2)C_1(\phi_1)}{c(w_1)C_1(\phi_1)} < 1 \tag{4.72}$$

which implies

$$\frac{c(w_2)}{c(w_1)} < \frac{C_2(\phi_1)}{C_1(\phi_1)} < 1.$$

This proves that c is $1-1$.

Finally, since $\phi_0 \to \infty$ as $\delta_0 \to 0$ and $c = \lim_{\phi\to\infty} C$, by definition, we have for small δ_0

$$c(w)^2 \simeq C(w;\phi_0)^2 = \frac{\exp\left(-\frac{\sqrt{6}}{\cos\theta_i}\phi_0\right)}{3V(\phi_0)}(2w - w^2)$$

Thus on the domain $0 < w \leq \delta_1$ c takes values on the range $(0, c_m)$ where

$$c_m^2 \simeq \frac{\exp\left(-\frac{\sqrt{6}}{\cos\theta_i}\phi_0\right)}{3V(\phi_0)}(2\delta_1 - \delta_1^2)$$

tends to ∞ as $\delta_0 \to 0$. This proves the theorem.□

Theorem 4.6 allows us to conclude that the pair (θ_i, c) uniquely and continuously determines the orbits of (4.17) which have the asymptotic behavior described in Theorem 4.4. Note that this implies that the error terms in (4.47) are continuous functions of c, θ and t.

We now investigate further the asymptotic space-time structure of these solutions.

4.5 Physical Properties of Solutions

4.5.1 Asymptotic Form of the Metric

In order to obtain an asymptotic expression for the space-time metric in the neighbourhood of the singularity Υ_- we first use (4.16) to obtain for the volume element v;

$$v = \frac{\sqrt{3}\alpha t}{\sqrt{2}\sin\theta_i} + O(t^2\epsilon_V(\theta;t)). \tag{4.73}$$

Recalling again that for a given physical metric v is only unique up to a constant coefficient we are free to choose

$$\frac{\sqrt{3}\alpha t}{\sqrt{2}\sin\theta} = 1$$

ie,

$$\sin^2\theta = \frac{3}{2}\alpha^2. \tag{4.74}$$

We thus have

$$v = t + O(t^2\epsilon_V(\theta;t)) \tag{4.75}$$

This corresponds to our choice of gauge in (4.21) and, if the first order solution were exact, would correspond to choosing a cartesian coordinate system on the spatial section Σ_1 corresponding to $t = 1$. Furthermore, it allows us to identify $\frac{2}{3}\sin^2\theta$ with the exact parameter of the system α^2. The metric is determined by both $v(t)$ and the three parameters α_i according to (4.15). Thus to each (θ, c) there corresponds a continuous family of space-time metrics, determined by α_i and satisfying, by (4.15)

$$g_i = t^{\frac{2}{3}+2\alpha_i} + O(t^{1+2\alpha_i}\epsilon_V(\theta;t)) \tag{4.76}$$

where the α_i are subject only to the constraints

$$\sum_{i=1}^{3}\alpha_i = 0 \qquad \sum_{i=1}^{3}\alpha_i^2 = \frac{2}{3}\sin^2\theta. \tag{4.77}$$

Setting ϵ_V equal to zero gives the general space-time solution of the massless scalar field. We may summarize the results of this chapter as follows.

Theorem 4.7 *Consider a scalar field cosmological model with potential $V \in \mathcal{E}^2$. There exists a family of Bianchi Type I cosmologies, constituting almost all Bianchi Type I cosmologies, which have an initial singularity at $t = 0$ for some proper time coordinate t. Furthermore, there exists a continuous 1-1 correspondence between these solutions and the general BI solution for the massless scalar field as follows:*

For each possible diagonal shear matrix $A^i_j = (\alpha_1, \alpha_2, \alpha_3)$ *(as defined by (4.9)) for which* α_i *satisfy the constraints (4.77) with* $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{2}$ *and for each* $c > 0$ *there exists exactly one cosmology of the above class which, sufficiently close to the singularity, may be written*

$$\begin{aligned} g_i &= t^{\frac{2}{3}+2\alpha_i} + O(t^{1+2\alpha_i}\epsilon_V(\theta; t)) \\ \phi &= \sqrt{\tfrac{2}{3}}\cos\theta \ln \tfrac{t}{c} + O(t\epsilon_V(\theta; t)) \end{aligned} \tag{4.78}$$

and, in addition,

$$\begin{aligned} K &= t^{-1} + O(\epsilon_V(\theta_i; t)) \\ \sigma &= \sin\theta_i t^{-1} + O(\epsilon_V(\theta_i; t)) \\ \dot{\phi} &= \sqrt{\tfrac{2}{3}}\cos\theta t^{-1} + O(\epsilon_V(\theta; t)) \end{aligned} \tag{4.79}$$

where

$$\epsilon_V(\theta; t) = tV(\sqrt{\tfrac{2}{3}}\cos\theta \ln \tfrac{t}{c})$$

g_i *are the components of the metric defined by (4.1) and (4.12). The higher order terms depend continuously on* α_i *and* c.

Proof:

If $\alpha^2 = 0$ the result follows immediately from Theorem 2.12. Assume that $\alpha^2 > 0$; ie that the metric is anisotropic. By Theorem 4.1 all such solutions possess an initial singularity.

Consider solutions of the system (4.17). With the exception of the solutions lying on the 2-dimensional manifold Ω_f, $|\phi|$ is unbounded on almost all solutions.

Firstly, consider solutions for which ϕ is unbounded above. If all such solutions are past asymptotic to $\phi = \infty$ then they must have their source at the singular equilibrium set Υ_- since this is the only source at $\phi = \infty$. Taking into account Theorem 4.4, Theorem 4.6 and Equation 4.76 this would be sufficient to prove the theorem for $\frac{\pi}{2} < \theta \leq \pi$.

The possibility remains however that ϕ could oscillate rather than diverging to ∞. This possibility may be discounted as follows: For large ϕ the function $x = 1/K$ vanishes identically on the union of the unphysical boundary sets $\partial BI_{\epsilon 1}$ and $\partial BI_{\epsilon 3}$ (Fig. 4.1). By Theorem 4.1 K diverges and hence $x \to 0$ as $t \to 0$. Since x is continuous it is possible, for a given solution, to choose t sufficiently small that its intersection with Ω_ϵ must be arbitrarily close to the aforementioned boundary. By continuity it must approach one of the trajectories on the boundary (see 4.3.2). Furthermore, in order for ϕ to become arbitrarily large it is necessary that it approaches a boundary solution for which $y < 0$. It follows that it must approach an arbitrarily small neighbourhood of Υ_- and is therefore asymptotic to Υ_-, as required.

Now consider the case where ϕ is unbounded below. We make the observation that, as with the isotropic case, the system of equations (4.17) is invariant under the discrete transformation

$$(\phi, \dot{\phi}) \mapsto -(\phi, \dot{\phi}) \qquad V \mapsto U$$

where we recall that $U(\phi) = V(-\phi)$. Furthermore, solutions of the massless scalar field depend on the parameter θ and reflection through $(\phi, \dot{\phi}) = 0$ maps the family of solutions for which $\theta < \frac{\pi}{2}$ onto the family of solutions with $\theta > \frac{\pi}{2}$ according to $\theta \mapsto \theta + \frac{\pi}{2}$. Since U is class 2 SED at infinity, it follows that (4.78) gives the general first order behavior for all solutions on which $|\phi|$ diverges. This proves the theorem. $\Box$

4.5.2 Singularity Structure

Let us now investigate some of the consequences of the above result. Close to the singularity, the the space-time metric may be approximated by the metric

$$\mathbf{g} = -\mathrm{d}t^2 + \sum_{i=1}^{3} t^{\frac{2}{3}+2\alpha_i}(\mathrm{d}x^i)^2. \tag{4.80}$$

where α_i obey the constraints (4.77). Together with the equation of state $\rho = p$ (also approximately satisfied), this solution, which was first found by Jacobs [62], shall be referred to as a Jacobs stiff fluid cosmology. For each stiff fluid solution there corresponds a 1-parameter family of scalar field solutions with the same asymptotic energy density (and pressure),

$$\rho \simeq \tfrac{1}{2}\dot{\phi}^2 \simeq \tfrac{2}{3}\cos^2\theta t^{-1}$$

These solutions differ in the constant of integration c. For the massless scalar field they are physically indistinguishable (unless we were able to directly measure ϕ) but for the general case their long term behavior might be quite different as they evolve and V becomes dynamically significant.

From the constraints (4.77) it follows that

$$|\alpha_i| < \frac{2}{3}\sin\theta.$$

Furthermore, there must be at least one negative α_i and at least one positive if the sum of all three is to vanish. If $\theta < \frac{\pi}{3}$ then inspection of (4.80) reveals that each of the g_i become zero at the singularity. Thus the space-time must expand in every direction. If $\theta > \frac{\pi}{3}$ then it is possible for 1 or perhaps even 2 of the g_i to diverge resulting in a "cigar" or "pancake" singularity where space collapses along 1 principle direction while expanding along another. If one considered a spherical ball of matter (either scalar "fluid" or non-interacting test particles), comoving with the fluid, then as the singularity were approached it would be distorted into an ellipse which would become infinitely long and thin. The volume of the ellipse would, of course, go to zero at the singularity.

In order to classify the singularity structure of each solution define the new shear parameters,

$$\begin{aligned} \alpha_+ &= \tfrac{3}{2}(\alpha_1 + \alpha_2) \\ \alpha_- &= \tfrac{\sqrt{3}}{2}(\alpha_1 + \alpha_2). \end{aligned} \tag{4.81}$$

Inverting these equations and using the constraint (4.77) we find that

$$\begin{aligned} \alpha_1 &= \tfrac{1}{3}(\alpha_+ + \sqrt{3}\alpha_-) \\ \alpha_2 &= \tfrac{1}{3}(\alpha_+ - \sqrt{3}\alpha_-) \\ \alpha_3 &= -\tfrac{2}{3}\alpha_+ \end{aligned}$$

The line element now becomes

$$\mathbf{g} = -\mathbf{d}t^2 + t^{\frac{2}{3}(1+\alpha_+ + \sqrt{3}\alpha_-)}(\mathbf{d}x^1)^2 + t^{\frac{2}{3}(1+\alpha_+ - \sqrt{3}\alpha_-)}(\mathbf{d}x^2)^2 + t^{\frac{2}{3}(1-2\alpha_+)}(\mathbf{d}x^3)^2. \tag{4.82}$$

α_+ and α_- are related to θ by

$$\alpha_+^2 + \alpha_-^2 = \sin^2\theta \tag{4.83}$$

Thus each Jacobs stiff fluid solution may be represented as a point on the interior of the unit disc in (α_+, α_-) space, (Fig 4.2). To each θ there corre-

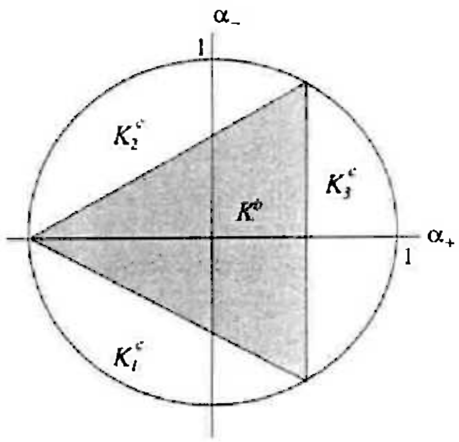

Figure 4.2: The Kasner Disc. Each point (α_+, α_-) may be identified with a Jacobs stiff fluid cosmology. The shaded region K^b corresponds to metrics which collapse along all three principle directions.

sponds a ring of Stiff Fluid solutions at a distance $\sin\theta$ from the origin. The origin itself represents the isotropic case with $\alpha^2 = 0$. The boundary of the disc corresponds to the general Bianchi type 1 vacuum solution, the so-called Kasner solution. For this reason it shall be referred to as the Kasner ring ([63]). The entire disc will be referred to as the Kasner disc, K. Although we have not explicitly proved this it is not difficult to see that the Kasner ring is the general asymptotic space-time of the vacuum dominated solutions lying on the 2D invariant manifold Ω_f (c.f. Theorem 4.3) since these solutions have $\dot{\phi}/K$ and V/K tending to zero as t tends to zero. These are the only anisotropic solutions in BI which do not obey Theorem 4.7. Thus, every anisotropic space-time solution and almost every isotropic space-time solution of the (Bianchi Type 1) field equations for $\mathcal{E}^2$ scalar field cosmologies may be identified some point in K.

From inspection of the line element we obtain the following three algebraic conditions for the collapse of the metric along each of the respective principle directions.

$$\begin{array}{rcl} \alpha_+ + \sqrt{3}\alpha_- & > & -1 \\ \alpha_+ - \sqrt{3}\alpha_- & > & -1 \\ \alpha_+ & < \frac{1}{2} & \end{array} \tag{4.84}$$

We may thus divide the Kasner disc into a number of qualitatively distinct regions (Fig. 4.2). Points occupying the shaded region, K^b satisfy all three inequalities and correspond to "big bang" singularities with collapse in every direction. The unshaded region satisfies two of the constraints and represent "cigar" singularities with expansion along one of the principle directions. The three unshaded sectors, K_i^c are related by cyclic permutations of the coordinates x_i and as such are physically indistinguishable. Points along the boundaries saturate one or more of the inequalities (4.84) and have one or two components of the metric which asymptotically approach a constant non-zero value. It is clear from Fig. 4.2) that such behavior is not generic.

4.6 Summary

In conclusion, we have shown that Bianchi Type I scalar field models with $\mathcal{E}^2$ potentials have a well defined asymptotic structure which is largely independent on the potential. In particular, for a given V, we may identify a class of "good" cosmologies, containing almost all solutions, which is homeomorphic to the exactly integrable set of Bianchi type I massless scalar field cosmologies. Each of these exact cosmologies is a first approximation to a unique cosmology for the model V in the limit as an initial singularity is approached. We may therefore identify the space-time singularity of each solution with the space-time structure of its corresponding exact massless scalar field solution as characterized by points in the Kasner Disc K.

In the next chapter we discuss the extension of these ideas to more general space-times.

Chapter 5

Asymptotically Expansion Dominated Cosmologies.

5.1 Introduction

In this chapter we attempt to put the results of the previous chapters in a more general context. Although we have obtained interesting results for Bianchi type I space-times these models are only really of interest in as far as they provide insight into the behavior of more general solutions of Einstein's equations. We should therefore give careful consideration to the degree to which we may infer results for more general metrics.

The full Einstein equations are extraordinarily complex so in order to make progress we shall abandon (for now) the rigorous dynamical systems approach and proceed instead along more heuristic lines. We aim to provide a plausible picture of the asymptotic structure of (a class of ?) generic solutions.

Our approach shall be based on two "complimentary" observations. Firstly, we have proven in the previous chapter that almost all Bianch Type I scalar field cosmologies possess a space-time singularity and that close to the singularity they are essentially equivalent to massless scalar field cosmologies. Secondly, the massless scalar field is equivalent to a perfect fluid with equation of state $\rho = p$ and the generic behavior of these cosmologies close to the initial singularity is very closely related to their Bianchi Type I solutions.

The reason for this is that, for many space-times obeying a linear equation of state, the expansion K tends to blow up very rapidly as a singularity is approached and to dominate the spatial gradients. Thus, quantities such as ${}^{(3)}R/K^2$ and $(\rho + p)_{,i}/K^2$ tend to approach 0 at the singularity. Provided these spatial gradient terms decay sufficiently rapidly compared to other terms in the EFE they will not be dynamically significant close to the singularity and a first approximation to the metric and matter fields may be obtained by setting them to zero in the field equations. The solutions that one obtains from these simplified equations will have identical time-dependence to Bianchi Type I but are in general inhomogeneous. This is sometimes called the long wavelength approximation in the modern literature since the characteristic length scale of inhomogeneities is large compared to the characteristic scale of temporal evolution K^{-1} [64, 65, 66].

Such an approximation scheme was first employed as a means of classifying singularity structure by Lifshitz an Khalatnikov [56] and, later, made more formal by Eardly, Liang and Sachs [67, 58] with their concept of a *velocity dominated singularity*. However, it has long been known that for the generic linear equation of state $p = (\gamma - 1)\rho$, with $1 \leq \gamma < 2$, velocity dominated singularities are unstable even within the class of spatially homogeneous space-times. For this type of matter Bianchi type I solutions are asymptotic to the Kasner vacuum solutions. However, the Kasner Ring ∂K lacks sufficient degrees of freedom to provide unique initial data for the time evolution of the most general homogeneous solutions. The highest dimensional (Bianchi Type VIII and IX) solutions exhibit chaotic oscillatory behavior near the singularity which is not velocity dominated since it depends manifestly on the presence of spatial curvature [57].

This behavior seems to be generic with respect to the full equations and is now believed to be a template for the general inhomogeneous cosmological solution near the initial singularity [68, 65]).

For the extreme case $\gamma = 2$ corresponding to a "stiff" fluid with equation of state $\rho = p$ the situation is quite different. In this case the general Bianchi Type I solution is the general Jacobs stiff fluid solution and this is a source in the class of spatially homogeneous stiff fluid models (a more precise statement of this fact may be found in [63]). It has been demonstrated that all but a set of measure zero spatially homogeneous irrotational stiff fluid cosmologies are asymptotic in the past to this family of solutions and, therefore, have velocity dominated singularities [63]. Interestingly, the generic asymptotic solution does not have the structure of the Kasner disc K but rather the subset $K^b \subset K$ corresponding, physically, to gravitational expansion along all three principle directions. Thus pancake singularities are not a generic feature of spatially homogeneous stiff fluid cosmologies.

As with more conventional equations of state ($\gamma < 2$) the general homogeneous solution seems to characterize inhomogeneous asymptotic behavior and typical stiff fluid cosmologies are believed to possess velocity dominated singularities [69, 65].

This has been verified explicitly for a number of exact inhomogeneous stiff fluid cosmologies including cylindrically symmetric, plane wave and spherically symmetric solutions [58, 70, 65].

Based on the above considerations it seems that the long wavelength approximation is also likely to be appropriate for the study of singularity structure in scalar field cosmologies. We shall therefore consider below the class of solutions of the EFE for which such an approximation is valid at early times following an analogous approach to that of Eardly Liang and Sachs. No claim is made that this type of asymptotic behavior is exhibited by all (or almost all) scalar field cosmologies but it is argued that it *is* generic and indeed represents a large and natural class of cosmologies.

In the first part of this chapter we define what we call asymptotically expansion dominated cosmologies and investigate some of their properties.

5.2 Definitions

Firstly, we need a precise definition of a first order approximation to a cosmological solution.

Defining convergence is slightly problematical when dealing with the metric, particularly near a singularity. Some components of the metric may diverge while others may tend to zero or be identically zero or diverge at a different rate. It is therefore overly restrictive to require that each individual component of the metric should converge. Nevertheless, a minimum requirement should be that measurements of length made with either metric agree to first order. It is also natural to require that the extrinsic curvature of the surfaces of constant (geodesic) time and the gradient of the scalar field agree to first order (in some sense).

We shall begin by making a few preliminary definitions.

Assume that M is an expanding space-time so that $v(t, x)$ is strictly increasing in time, at least on some time interval $t < t_m$. Consider the geodesic congruence Γ_t. Let us reparameterize this congruence with the now familiar parameter

$$\tau = \ln v(t,x) + \tau_0(x) \tag{5.1}$$

along each curve, where τ_0 is an arbitrary function of x. Call the new congruence Γ_τ to indicate that it is parameterized by τ. Note that we still retain the proper time t as our coordinate time on M. If G is any collection of geodesics in Γ_τ then G sweeps out a subset M_G of M which is the image of G. Call the intersection of Σ_t with M_G, $\Sigma_t(G)$. Let us further assume that v vanishes in the past of all geodesics $\gamma \in G$. This will be true if the expansion K has some positive lower bound to the past of t_m and is a reasonable assumption for a physical cosmological solution, excluding the possibility of a cosmological bounce (which may only occur if the spatial curvature is negative). It follows from (5.1) that $\tau \to -\infty$ as $v \to 0$ and hence that the congruence G is past complete with respect to its parameter time.

Recall that $\mathbf{h}$ is the restriction of the metric $\mathbf{g}$ to the tangent space on the surfaces of constant time Σ_t. Define $\hat{T}(M)$ to be the set of all $\mathbf{X} \in T(M)$ for which $\mathbf{h}(\mathbf{X},\mathbf{X}) = 1$; ie, we have for $\mathbf{X}$ in $\hat{T}(M)$, $g_{ij}X^iX^j = 1$.

Call the pair (h, ϕ) on M_G the *induced cosmology* with respect to the foliation $\{\Sigma_t\}$. Define the set $\Pi(M_G)$ as the set of all differentiable pairs, consisting of a Riemannian metric and a scalar field on the foliation $\{\Sigma_t(G)\}$. If $(\tilde{\mathrm{h}}, \tilde{\phi}) \in \Pi(M_G)$ then there exists a corresponding Lorentz metric $\tilde{\mathrm{g}} = -\mathrm{d}t^2 + \tilde{\mathrm{h}}$ on M_G. $\tilde{\mathrm{g}}$ need not be a solution of the EFE.

We can now give a precise definition to a first approximation of a scalar field cosmology

Definition 5.1 *Let $(\mathrm{h}, \phi) \in \Pi(M_G)$ be defined on the foliation $\{\Sigma_t(G)\}$ of M_G where $G \subset \Gamma_\tau$. We say that $(\tilde{\mathrm{h}}, \tilde{\phi}) \in \Pi(M_G)$ is a past asymptote of (h, ϕ) if, along each geodesic curve $\gamma \in G$, the following holds as $\tau \to -\infty$:*

i) $\mathrm{h}(\mathbf{X},\mathbf{Y}) - \tilde{\mathrm{h}}(\mathbf{X},\mathbf{Y}) \to 0$ *for all* $\mathbf{X},\mathbf{Y} \in \hat{T}(M)$

ii)

$$\frac{\phi - \tilde{\phi}}{\phi} \to 0$$

iii)

$$\frac{K^i_j - \tilde{K}^i_j}{(K^i_j K^j_i)^{\frac{1}{2}}} \longrightarrow 0$$

iv)

$$\frac{\phi^\mu - \tilde{\phi}^\mu}{(-\phi^\rho \phi_\rho)^{\frac{1}{2}}} \longrightarrow 0$$

where

$$\tilde{K}^i_j = \tfrac{1}{2}\tilde{g}^{il}\frac{d\tilde{g}_{lj}}{dt} \qquad\qquad \tilde{\phi}^\mu = \tilde{g}^{\mu\rho}\tilde{\phi}_{,\rho}.$$

Conditions i) and ii) provide a physically sensible criterium for $(\mathbf{h}, \phi)$ to converge to $(\tilde{\mathbf{h}}, \tilde{\phi})$ as $\tau \longrightarrow -\infty$. If we choose unit vectors

$$\hat{\mathbf{X}}_{(i)} = \frac{\mathbf{X}_{(i)}}{\mathbf{h}(\mathbf{X}_{(i)}, \mathbf{X}_{(i)})^{\frac{1}{2}}}$$

where $\mathbf{X}_i$ are the coordinate basis vectors. We see that i) implies

$$\frac{g_{ij} - \tilde{g}_{ij}}{(g_{ii}g_{jj})^{\frac{1}{2}}} \longrightarrow 0 \tag{5.2}$$

Conditions iii) and iv) impose slightly weaker convergence conditions on the extrinsic curvature of the surfaces Σ_t and the normal $\mathbf{U}_E$ to the surfaces of constant ϕ. It is easily verified that these conditions are independent of the choice of spatial coordinates. iii) can be decomposed in terms of the shear and expansion to yield the two convergence relations

$$\frac{K - \tilde{K}}{(K^2 + \sigma^2)^{\frac{1}{2}}} \longrightarrow 0 \qquad\qquad \frac{\sigma^i_j - \tilde{\sigma}^i_j}{(K^2 + \sigma^2)^{\frac{1}{2}}} \longrightarrow 0 \tag{5.3}$$

A couple of comments are in order at this point.

Firstly, it can be verified from Theorem 4.7 that typical *BI* cosmologies have a past asymptote (in the sense of Definition 5.1) which is a massless scalar field cosmology. In fact Theorem 4.7 goes further and insists that a homeomorphism exists between cosmologies and past asymptotes in this case.

Secondly, the first approximation will not unique. In particular, any first approximation of a first approximation of a scalar field cosmology will also be a first approximation to that cosmology.

Thirdly, we have required only pointwise convergence of spatial functions in the above definition so our criteria for one cosmology to approximate another is quite weak. Also, whilst our definition provides for the convergence of the metric and first fundamental form (which is essentially the first time derivative of the metric) it says nothing about how smoothly the spatial derivatives of the metric converge to those of its past asymptote. In particular, the spatial curvature tensors, which depend on the second spatial derivatives of the metric, need not correspond. When one attempts to extend the definition to incorporate some notion of smooth spatial convergence difficulties are encountered which lead to an unacceptably complicated definition. Part of the reason for this difficulty is that the surfaces of constant τ do not correspond to the surfaces Σ_t of constant time.

We now define an asymptotically expansion dominated cosmology.

Definition 5.2 *Consider a scalar field cosmology* $(\mathbf{g}, \phi)$ *on* M. *If there exists a foliation* $\{\Sigma_t\} = M$ *normal to a geodesic congruence* Γ_τ *and some* $G \subset \Gamma_\tau$ *such that*

$$\lim_{\tau \to -\infty} \frac{{}^{(3)}R^i_j}{K^l_m K^m_l} = \lim_{\tau \to -\infty} \frac{\phi^i \phi_j}{K^l_m K^m_l} = 0 \tag{5.4}$$

and there exists a past asymptote $(\breve{\mathbf{h}}, \breve{\phi}) \in \Pi(G)$ *of the induced cosmology* $(\mathbf{h}, \phi)$ *on* M_G *which obeys the equations*

$$\begin{aligned} \dot{\breve{\sigma}}^i_j + \tfrac{1}{3}\dot{\breve{K}}\delta^i_j &= -\breve{K}\breve{\sigma}^i_j - (\tfrac{1}{3}\breve{K}^2 - V(\breve{\phi}))\delta^i_j \\ \tfrac{1}{3}\breve{K}^2 &= V(\breve{\phi}) + \tfrac{1}{2}\breve{\sigma}^2 + \tfrac{1}{2}\dot{\breve{\phi}}^2 . \end{aligned} \tag{5.5}$$

then we say $(\mathbf{g}, \phi)$ *is asymptotically expansion dominated on* M_G.

Observe that (5.5) is essentially just R^0_0 and R^i_j equations of the EFE with all terms of second order in the spatial gradient set to zero. Thus an asymptotically expansion dominated space-time is just one for which, to a first approximation, the dynamical effect of spatial gradients can be neglected asymptotically in the past. In the case where all geodesics in G are past incomplete with respect to proper time t this is closely analogous to the velocity dominated singularity [67]. No spatial constraints have been imposed on $(\tilde{\mathbf{h}}, \tilde{\phi})$, however, the fact they must be a first approximation to a solution of the EFE satisfying the R_i^0 equation (1.40b) and (5.4) means that the spatial dependence is implicitly constrained to some degree.

Taking the time derivative of (5.5b) and substituting (5.5) back into the resulting expression one obtains

$$\ddot{\tilde{\phi}} = -\tilde{K}\dot{\tilde{\phi}} - V'(\tilde{\phi}). \tag{5.6}$$

The equations (5.5) differ from the Bianchi type I field equations (4.6) in that $\tilde{g}_{ij}$ and $\tilde{\phi}$ are, in general, functions of the spatial coordinates as well as time. If they are functions of time only then $(\tilde{\mathbf{h}}, \tilde{\phi})$ will itself be a (Bianchi Type I) solution of the EFE but for inhomogeneous space-times we do not expect this to be true in general.

Consider a particular geodesic $\gamma \in G$. If we specify initial data $\tilde{K}^i_j, \tilde{g}_{ij}, \tilde{\phi}$ and $\dot{\tilde{\phi}}$ at some point $\gamma(\tau_0)$ then (5.5) determines $\tilde{K}^i_j, \tilde{g}_{ij}, \tilde{\phi}$ and $\dot{\tilde{\phi}}$ as functions of time everywhere along the geodesic. Furthermore, the solution $(\tilde{g}_{ij}(t), \tilde{\phi}(t))$, where $t = \gamma^o(\tau)$, must be a solution of (4.6), i.e. it must correspond to a Bianchi Type I cosmology.

Using the continuity of $\tilde{g}_{ij}$ and $\tilde{\phi}$ on the initial surface Σ_{t_0} and the results of the previous chapters it is possible to classify the asymptotic behavior of asymptotically expansion dominated scalar field cosmologies with SED potentials.

5.3 Strongly Expansion Dominated Cosmologies

We know from the work of section 1.6 that there is a great deal of diversity in solutions of (5.5) and hence in the conceivable types of behavior of asymptotically expansion dominated cosmologies. However, much of this diversity disappears if we insist on the stronger condition that the past asymptote be a *generic* solution of (5.5). We call such cosmologies *strongly* expansion dominated. By generic we mean that an arbitrarily small perturbation of the initial conditions will not change the qualitative properties of the solution.

It is reasonable to suppose that the vast majority of asymptotically expansion dominated space-times are strongly expansion dominated. In fact, if there exists a homeomorphism between a (sufficiently large) class of exact expansion dominated cosmologies and their first approximation, as we found for Bianchi type I, then almost all solutions must be strongly expansion dominated since any weakly expansion dominated solutions would be unstable under perturbations of the initial conditions.

5.3.1 Dynamical Insignificance of $V(\phi)$.

For models with SED potentials the condition that space-time be strongly expansion dominated is sufficient to ensure the existence of an initial singularity and the dynamical insignificance of the potential close to the singularity.

In order to see this assume that $V \in \mathcal{E}^2$ and $(\mathbf{g}, \phi)$ is strongly expansion dominated on M_G. Its induced cosmology $(\mathbf{h}, \phi)$ therefore has a past asymptote $(\check{\mathbf{h}}, \check{\phi})$ which is a generic solution of (5.5). Let $\gamma \in G$. The restriction of $(\check{\mathbf{h}}, \check{\phi})$ to γ yields the functions $(\check{K}(t), \check{\sigma}^i_j(t), \check{\phi}(t))$ which may be identified with the expansion, shear and scalar field of a unique BI scalar field cosmology.

Consider the following qualitative conditions which this BI solution may or may not satisfy: i) $\check{\phi}$ diverges in the past; ii) The pressure ($\check{p} = \frac{1}{2}\dot{\check{\phi}}^2 - V(\check{\phi})$) diverges to *positive* infinity.

The class of solutions defined by Theorem 4.7 contains precisely those BI solutions which satisfy both i and ii. Call these $\mathcal{A}$-type solutions and call the remaining BI solutions $\mathcal{B}$-type. Since $\mathcal{B}$-type solutions are unstable, by Theorem 4.7, there always exists an arbitrarily small perturbation of the initial conditions which will result in an $\mathcal{A}$-type solution.

Let $B \subset G$ be the largest *open* set of geodesics such that geodesics γ whose corresponding BI solution is of $\mathcal{B}$-type are dense on B.

Assume that B is non-empty. Then there exists an arbitrarily small, differentiable, perturbation of the initial conditions on $\Sigma_0(G)$ which takes all solutions on B to $\mathcal{A}$-type solutions. This perturbation will clearly change the qualitative properties of $(\check{\mathbf{h}}, \check{\phi})$ in contradiction to the assertion that $(\check{\mathbf{h}}, \check{\phi})$ is generic.

We thus conclude that B is empty and the BI solution associated with $\gamma \in G$ is of type $\mathcal{A}$ for *almost all* γ.

Let us (for now) choose G so that it does not include any "bad"($\mathcal{B}$-type) geodesics and choose a coordinate system so that $\gamma = (t, x_0)$ and $\check{g}_{ij}(t, x_0)$

is diagonal (this can always be done since for fixed x the equations (5.5) are identical to the BI field equations). Clearly, it may be identified directly with the diagonal BI metric (4.12). We may therefore write

$$\begin{aligned} \tilde{g}_{ij}(t,x_0) &= \text{diag}[(t-\bar{t}(x_0))^{\frac{2}{3}+2\alpha_i(x_0)} + h_i(\theta(x_0), c(x_0), t-\bar{t}(x_0))] \qquad (5.7) \\ \tilde{\phi}(t,x_0) &= \sqrt{\tfrac{2}{3}}\cos\theta(x_0)\ln(t-\bar{t}(x_0)) + c(x_0) + h_\phi(\theta(x_0), c(x_0), t-\bar{t}(x_0)) \end{aligned}$$

where $\alpha_i(x_0)$ and $\theta(x_0)$ obey the constraints (4.77) and each of the constants $c(x_0)$, $\bar{t}(x_0)$, $\alpha_i(x_0)$ and $\theta(x_0)$ are unique. The functions h_i and h_ϕ are higher order terms which tend to 0 as $t-\bar{t}\to 0$. Note that they are continuous in θ and c and differentiable in $t-\bar{t}$. Note also that corresponding approximations for $K^i_j(t,x_0)$ and $\dot{\phi}(t,x_0)$ can also be written down using (4.79). These agree with the expressions obtained by taking the time derivative of the first order terms above.

Equations (5.7) immediately imply that for all strongly expansion dominated cosmologies the pressure and density of the scalar field asymptotically obey the stiff equation of state $\rho = p$. Furthermore, all geodesics $\gamma \in G$ are past incomplete with respect to proper time: ie there exists an initial space-time singularity. Note that I have put the time of the singularity as being $t = \bar{t}(x_0)$ rather than $t = 0$ since the gauge choice which places the singularity at $t = 0$ can not, in general, be made simultaneously on all $\gamma \in G$. ie the singularity may occur at different times in different places.

By constructing the above approximation along each γ in the congruence we obtain a unique set of functions $c(x)$, $\alpha_i(x)$ and $\bar{t}(x)$ (unique up to a constant) and an atlas of spatial coordinates corresponding to diagonalization of the metric at each point. These characterize the asymptotic behavior of the cosmology and allow us to construct a unique first approximation, $(\hat{\mathbf{h}}, \hat{\phi})$, to $(\mathbf{h}, \phi)$ which is an exact solution of the system

$$\begin{aligned} \dot{\hat{\sigma}}^i_j + \tfrac{1}{3}\dot{\hat{K}}\delta^i_j &= -\hat{K}\hat{\sigma}^i_j - \tfrac{1}{3}\hat{K}^2\delta^i_j \\ \tfrac{1}{3}\hat{K}^2 &= \tfrac{1}{2}\hat{\sigma}^2 + \tfrac{1}{2}\dot{\hat{\phi}}^2 \qquad (5.8) \end{aligned}$$

In order that this solution be a past asymptote in the strict sense we need to assume a certain amount of regularity in the asymptotic structure. If the above functions and atlas are C^k then, using common terminology, we say that $(\mathbf{h}, \phi)$ possesses a C^k singularity. We assume below that the singularity is at least C^1.

Let $(\hat{\mathbf{h}}, \hat{\phi}) \in \Pi(M_G)$ be the unique pair constructed by retaining the lower order terms to $(\tilde{\mathbf{h}}, \tilde{\phi})$ in (5.7). (in an arbitrary coordinate system it will not retain its simple form but clearly remains uniquely defined). It is obvious that $(\hat{\mathbf{h}}, \hat{\phi})$ must obey (5.8). Furthermore, it will clearly be a past asymptote of the physical cosmology $(\mathbf{h}, \phi)$ provided it is a past asymptote to $(\tilde{\mathbf{h}}, \tilde{\phi})$. We now verify that this is indeed so.

Let $\gamma = (t, x_0)$ and choose coordinates, as above, such that $\tilde{\mathbf{h}}$ and $\hat{\mathbf{h}}$ are diagonal at x_0. It is clear from (5.8) that the convergence conditions i and ii of Definition 5.1 are satisfied. Furthermore, it is obvious from (4.79) that condition iii is satisfied. There is however, a technical subtlety in demonstrating condition iv due to the presence of spatial gradients. We need to show that the gradient of the error term in (5.8b), which retains its form under a coordinate transformation, is small compared to $(\tilde{\phi}^{\mu}\tilde{\phi}_{\mu})^{\frac{1}{2}}$. Since θ, c and $\bar{t}$ are C^1 functions $\hat{\phi}$ and h_ϕ are differentiable with respect to x and t. We therefore have

$$\begin{aligned} \dot{\tilde{\phi}}(t,x) &= \sqrt{\frac{2}{3}}\frac{\cos\theta(x)}{(t-\bar{t}(x))} + h_{\phi,0} \\ \tilde{\phi}_i(t,x) &= \sqrt{\frac{2}{3}}\frac{\bar{t}_{,i}\cos\theta(x)}{(t-\bar{t}(x))} + -\sqrt{\frac{2}{3}}\sin\theta\theta_{,i}(x)\ln(t-\bar{t}(x)) + c_{,i}(x) + h_{\phi,i}\,. \end{aligned} \tag{5.9}$$

From (4.79) we know that $h_{\phi,0} = \dfrac{\partial h_\phi}{\partial(t-\bar{t})} \to 0$ as $t-\bar{t} \to 0$ so the time derivatives of $\tilde{\phi}$ and $\hat{\phi}$ agree to first order.

Now, recall that h_ϕ depends continuously on c and θ. Consider two BI cosmologies whose θ differs by an amount $\Delta\theta$ and whose value of h_ϕ at time t differs by Δh. As $t-\bar{t} \to 0$, $\frac{\Delta h}{\Delta\theta} \to 0$ since h_ϕ is continuous in θ and vanishes identically at $t-\bar{t}=0$. Similarly, if c is varied by a small amount Δc then $\frac{\Delta h}{\Delta c} \to 0$. It follows, from the continuity of h_ϕ that $h_{\phi,i}$ is either bounded or, if it diverges, is dominated by the term $\dfrac{\partial h_\phi}{\partial(t-\bar{t})}\bar{t}_{,i}(x)$ for small $t-\bar{t}$. But it has already been established above that the derivative of h_ϕ with respect to $t-\bar{t} \to 0$ is bounded. Therefore $h_{\phi,i}$ remains bounded as $t-\bar{t} \to 0$.

Now, the dominant term in the difference $\tilde{\phi}^i - \hat{\phi}^i$ is,

$$\hat{g}^{ij}h_{\phi,j} = (t-\bar{t})^{-\frac{2}{3}-2\alpha_i}h_{\phi,j} < O((t-\bar{t})^{-2}),$$

since $\alpha_i < \frac{2}{3}\sin\theta$. But $\tilde{\phi}^\mu\tilde{\phi}_\mu$ diverges as $(t-\bar{t})^{-2}$ implying that

$$\lim_{t-\bar{t}\to 0} \frac{\hat{g}^{ij}h_{\phi;j}}{\tilde{\phi}^\mu\tilde{\phi}_\mu} = 0$$

This proves that condition iv of Definition 5.1 is satisfied by $\hat{\phi}$ and hence that $(\hat{\mathbf{h}}, \hat{\phi})$ is a first approximation of $(\mathbf{h}, \phi)$.

To summarize the above discussion, it has been shown that subject to certain genericity and regularity conditions the dynamical influence of the potential on scalar field cosmologies can be neglected (almost everywhere) sufficiently far in the past, provided the characteristic length of spatial gradients is sufficiently small compared to the characteristic magnitude of velocity terms. This is a direct consequence of Theorem 4.7 and provides an appropriate context within which we may interpret the significance of that result to realistic cosmologies. It also highlights the limitations of Theorem 4.7 since if curvature, for example, plays a significant role in the asymptotic past then it is not really reasonable to expect the potential to be insignificant in the asymptotic past. Nevertheless these results indicate strongly that the dynamical influence of the scalar field is not in itself sufficient to avoid a singularity and must work in concert with large curvature and/or inhomogeneity if singularity avoidance is to be achieved. In section 5.4 we shall look at the dynamical effect of curvature in spatially homogeneous models. Firstly, we investigate further the general properties of strongly expansion dominated cosmologies.

5.3.2 Singularity Structure

We now investigate the structure of the singularity in more detail. Firstly, we obtain the general solution to (5.8). From (5.7) we see that for each x there exists a coordinate system for which

$$\begin{aligned} \hat{g}_{ij}(t,x) &= \delta_{ik}(t-\bar{t}(x))^{\frac{2}{3}+2\alpha_k(x)} \\ \hat{\phi}(t,x) &= \sqrt{\tfrac{2}{3}}\cos\theta(x)\ln\left(t-\bar{t}(x)\right)+c(x) \end{aligned}$$

This expression may be made fully covariant on the 3-surface $\Sigma_t(G)$ by identifying $\mathrm{diag}(\alpha_i(x))$ with a 3-tensor $B^i_j(x)$ and δ_{ij} with a 3-tensor $A_{ij}(x)$. Using the constraints (4.77) and the fact that $t^\alpha = \exp(\alpha \ln t)$ we now have

$$\hat{g}_{ij}(t,x) = A_{ik}(x_0)(t-\bar{t}(x))^{\frac{2}{3}} \exp\left[2B^k_j(x)\ln(t-\bar{t}(x_0))\right] \quad (5.10)$$

$$\hat{\phi}(t,x) = \pm\sqrt{1-B^2(x)}\ln\left[t-\bar{t}(x)\right]+c(x) \quad (5.11)$$

where $B^2 = B^i_j B^j_i$. This is a fully covariant expression and holds independently of any particular choice of coordinate system on $\Sigma_t(G)$. It follows that it holds everywhere on $\Sigma_t(G)$ and is the general solution to (5.8). Differentiating (5.10) with respect to time we find

$$\hat{K}^i_j(t,x) = (\delta^i_j + B^i_j(x))(t-\bar{t}(x))^{-1} \equiv C^i_j(x)(t-\bar{t}(x))^{-1} \quad (5.12)$$

from which it follows that

$$\hat{\sigma}^i_j(t,x) = B^i_j(x)(t-\bar{t}(x))^{-1} \quad (5.13)$$

and

$$\hat{K}(t,x) = (t-\bar{t}(x))^{-1}. \quad (5.14)$$

Note that A^i_j is symmetric and positive definite and is thus a Riemannian metric on the surfaces $\Sigma_t(G)$ of constant time. Also, B^i_j is symmetric and trace free and must obey the constraint (4.77)

$$B^i_j B^j_i = \tfrac{2}{3}\sin^2\theta < \tfrac{2}{3} \quad (5.15)$$

It is therefore possible to identify the space-time singularity with a three dimensional manifold S_G with positive definite metric A_{ij} and extrinsic curvature $C_{ij} = A_{ik}C^k_j$. Two scalar functions, $\bar{t}(x)$ and $c(x)$, defined on S_G complete the unique characterization of the asymptotic behavior of the cosmology.

A singularity with this structure corresponds to a semi-Kasner like singularity in the classification scheme of Eardly *et al* [67]. It is the generic velocity dominated singularity for an irrotational perfect fluid with equation of state $\rho = p$ (although the function $c(x)$ does not appear in the description of the stiff fluid) and is the most general type of singularity known for this type of matter. It is possible to explicitly calculate the Ricci tensor ${}^{(3)}\hat{R}^i_j$ and demonstrate that the first order solution is consistent with the curvature condition in (5.4) [65]. We shall address the consistency of the spatial gradient condition in a moment.

For each strongly expansion dominated cosmology the singularity structure is unique and, when $M_G = M$, it is natural to ask whether the singularity structure uniquely determines the space-time as we have shown to be the case for Bianchi type I. The full resolution of this question is beyond the scope of the present discussion and must await future research. However, not all solutions of (5.8) can correspond to first approximations of physical cosmologies. The absence of spatial gradients in the equations (5.5) mean that the singularity as we have defined it actually has more degrees of freedom than solutions of Einstein's equations so that extra constraints on A_{ij} and B^i_j are required.

One possible way that this could be achieved is to insist that the first approximation $(\hat{\mathbf{h}}, \hat{\phi})$ respect the R^0_i equation in the EFE (1.40b). This extra constraint, together with (5.8), has been used to generate the first order terms in the long wavelength approximation scheme of Deruelle and Langlois [65]. They have recently demonstrated this to be a valid approximation scheme for a generic class of stiff fluid cosmologies near the big bang, suggesting that it is also likely to be valid for scalar fields. There is a certain appeal to such a scheme since if we specify A_{ij} and B^i_j the constraints (1.40b) and (1.40b) uniquely determine $\hat{\phi}_i$ which, as shall be shown explicitly below, determines $\bar{t}_{,i}$ and $c_{,i}$. Thus c is determined up to a constant of integration and $\bar{t}$ up to a, physically irrelevant, constant. The asymptotic solution in this case has the same number of degrees of freedom as are needed to specify arbitrary initial conditions for the exact EFE on a hypersurface of constant time Σ_t.

The semi-Kasner like structure of the singularity becomes more manifest if we define the vector basis $\mathbf{e}_a(x)$ on Σ_t where for each x $\mathbf{e}_a(x)$ are the tangent vectors to the coordinate system in which the metric takes the particularly simple form (5.7) at x. It follows immediately that

$$\hat{\mathbf{h}} = \sum_{a=1}^{3} (t - \bar{t}(x))^{\frac{2}{3}+2\alpha_a(x)} \mathbf{e}_a \otimes \mathbf{e}_a. \tag{5.16}$$

This expression shall be useful below when we prove the existence of particle horizons.

Lets now look at the properties of $\hat{\phi}$. Differentiation of (5.10b) gives the timelike and spatial components of the gradient of $\hat{\phi}$

$$\dot{\hat{\phi}}(t, x) = \pm \frac{\sqrt{1 - B^2(x)}}{(t - \bar{t}(x))} \tag{5.17}$$

$$\hat{\phi}_i(t,x) = \pm\frac{\bar{t}_{,i}\sqrt{1-B^2(x)}}{(t-\bar{t}(x))} \mp \frac{B_{,i}(x)\ln[t-\bar{t}(x)]}{\sqrt{1-B^2(x)}} + c_{,i}(x) \quad (5.18)$$

If $\bar{t}_{,i}$ is non-zero the first term on the right hand side of (5.18) will dominate as $t \to \bar{t}$. Equation (5.4) implies that $\phi^i\phi_i/K^2 \to 0$ as $t \to \bar{t}$ and hence, using (5.14) and Definition 5.1 we obtain a constraint on the function $\bar{t}(x)$

$$\hat{g}^{ij}\bar{t}_{,i}\bar{t}_{,j} \to 0. \quad (5.19)$$

This equation immediately implies that the singularity is spacelike. That is, the normal to the surface $t = \bar{t}$ is everywhere timelike. Secondly, if we choose spatial coordinates in the principle directions along the curve (t, x_0) where x_0 is fixed so that $B^i_j(t,x_0) = \mathrm{diag}(\alpha_i)$ and $A^i_j(t,x_0) = \delta_{ij}$, then whenever $\alpha_i > -\frac{1}{3}$,the corresponding metric component g_{ii} will go to zero and g^{ii} will diverge. This can only be consistent with (5.19) if $\bar{t}_{,i}(t,x_0)$ vanishes identically. In particular, if $B^2 < \frac{1}{3}$ then space-time expands in all directions and the gradient of $\bar{t}$ is zero. We just proved the following theorem.

Theorem 5.1 *Let $B^2 < \frac{1}{3}$ on some $H \subset S_G$. Then the bang time $\bar{t}$ is constant on H.*

If $H = S_G$ in the above theorem then we are free to set $\bar{t} = 0$ on S_G and the general solution (5.10) is simplified considerably

$$\hat{g}_{ij}(t,x) = A_{ik}(x)t^{\frac{2}{3}}\exp\left[2B^k_j(x)\ln t\right] \quad (5.20)$$

$$\hat{\phi}(t,x) = \sqrt{1-B^2(x_0)}\ln t + c(x) \quad (5.21)$$

The results of the next section, where we analyze in some detail the very important example of spatially homogeneous space-times of class A, suggest that this is in fact the generic first approximation for strongly expansion dominated space-times and that solutions for which g_{ij} diverges along one or more principle axes are pathological. Firstly, to conclude this section it is shown that particle horizons exist for all (sufficiently regular) strongly expansion dominated cosmologies.

5.3.3 Existence of Particle Horizons

Theorem 5.2 *Let $V \in \mathcal{E}^2$ and let $(\mathbf{g}, \phi)$ be a strongly expansion dominated cosmology on M with a C^1 singularity. If $p \in M$ then all past directed non-*

spacelike curves from p intersect the singularity a finite spatial coordinate distance r from p. Furthermore, if $p = (t, x)$ then $r \to 0$ as $t - \bar{t}(x) \to 0$.

Proof:

Since a spacelike foliation exists on M it will be sufficient to show that non-spacelike curves intersect the surface $t - \bar{t}(x) = 0$ whenever they become sufficiently close, i.e. we can choose p arbitrarily close to the singularity. Furthermore, since the chronological past of p is bounded by its null geodesics [15] the theorem will follow if it holds for all null geodesics.

Choose the spatial coordinates to take their origin at p so $p = (t_0, 0)$. Consider a past directed null geodesic $(t(\tau), x(\tau))$, parameterized by τ with tangent vector ξ. Since ξ is null we thus have

$$0 = \mathbf{g}(\xi, \xi) = -\left(\frac{dt}{d\tau}\right)^2 + \mathbf{h}(\xi, \xi). \tag{5.22}$$

If $t_0 - \bar{t}(0)$ is sufficiently small we therefore have, using (5.16),

$$\left(\frac{dt}{d\tau}\right)^2 = \sum_{a=1}^{3}(t - \bar{t}(x))^{\frac{2}{3}+2\alpha_a(x)}\pi_a(\tau)^2 + h \tag{5.23}$$

where $\pi_a = \mathbf{e}_a(\xi)$ are the spatial projections of ξ at $x(\tau)$ and h indicates terms of higher order.

Now, $|\alpha_i| \leq \frac{2}{3}\sin\theta < \frac{2}{3}$. Therefore there exists a number $\delta > 0$ such that, for $(t - \bar{t}(x)) < 1$,

$$(t - \bar{t}(x))^{\frac{2}{3}+2\alpha_a(x)} \geq (t - \bar{t}(x))^{2-2\delta}.$$

It therefore follows that

$$\sum_{a=1}^{3} \pi_a^2 \leq b(t - \bar{t}(x))^{-2+2\delta}\left(\frac{dt}{d\tau}\right)^2 \tag{5.24}$$

for some positive constant b.

Recall that $\frac{dt}{d\tau} = K^{-1}$ by definition and hence, by (5.14)

$$\frac{dt}{d\tau} = (t - \bar{t}) + h \tag{5.25}$$

which yields, upon integration,

$$t - \bar{t}(x) = a(x)e^{\tau} + h \tag{5.26}$$

where $a(x)$ is a positive function. Hence,

$$\frac{dt}{d\tau} = a(x)e^{\tau} + h \tag{5.27}$$

We thus have a first order relation between the parameter time and proper time. Substituting (5.26) and (5.27) into (5.24) yields the inequality

$$\sum_{a=1}^{3} \pi_a^2 \leq ba(x)e^{2\delta\tau} \tag{5.28}$$

To see that this implies the theorem, let $H \subset R^3$ be our coordinate patch on Σ_t and consider the two positive definite metrics

$$\begin{aligned} \mathbf{m} &= \sum_{a=1}^{3} \mathbf{e}_a \otimes \mathbf{e}_a \\ \mathbf{r}^2 &= \sum_{i=1}^{3} \mathbf{X}_i \otimes \mathbf{X}_i \end{aligned}$$

where $\mathbf{X}_i$ are the coordinate basis vectors. For each $x \in H$ continuity and linearity dictate that there exists a positive constant $D(x)$ such that for all $\mathbf{Y} \in T_x$:

$$\mathbf{r}^2(\mathbf{Y}, \mathbf{Y}) \leq D\mathbf{m}(\mathbf{Y}, \mathbf{Y}).$$

Taking $\mathbf{Y}$ to be the spatial projection of ξ on each Σ_t it follows that

$$\left(\frac{dr}{dt}\right)^2 \leq D(x(\tau)) \sum_{a=1}^{3} \pi_a^2 \leq ba(x)D(x)e^{2\delta\tau} \tag{5.29}$$

where $r = \sum_i (x^i)^2$ is the coordinate radius. Let $ba(0)D(0) = 2E^2$ then there exists a neighbourhood $N_\epsilon \subset H = \{(t, x) : |x| < \epsilon\}$ of the spatial origin such that for $x(\tau) \in N_\epsilon$:

$$\left(\frac{dr}{dt}\right)^2 < E^2 e^{2\delta\tau}. \tag{5.30}$$

Integrating this inequality from τ_0 to some earlier time τ_1 yields

$$|r(\tau_1)| < \frac{E}{\delta}\left(e^{\delta\tau_0} - e^{\delta\tau_1}\right) \tag{5.31}$$

Choose τ_0 sufficiently large and negative so that the right hand side is less than ϵ irrespective of the value of τ_1 and let τ_1 go to $-\infty$. We then have

$$r < \frac{E}{\delta}e^{\delta\tau_0} < \epsilon \tag{5.32}$$

Thus r remains finite and tends to zero as $\tau_0 \to -\infty$. The coordinate radius of the chronological past of p is bounded by the above inequality and we have the result. □

5.4 Example: Bianchi Models of class A

The validity of the procedure followed in the previous sections is largely dependent on whether it is reasonable to expect the spatial gradients to be small compared to velocity terms asymptotically in the past.

With respect to inhomogeneities of the scalar field it follows from the Weak Energy Condition that $\phi^i\phi_i < \dot{\phi}^2$. It therefore seems reasonable, although certainly not inevitable, that $\dot{\phi}^2$ should diverge faster than $\phi^i\phi_i$ and dominate at early times. Certainly the converse can not occur.

The Ricci Tensor ${}^{(3)}R^i_j$ is more problematical but some insight into the effect of curvature can be gained from spatially homogeneous cosmologies. These models illustrate the effect of spatial curvature and non-trivial spatial geometry on asymptotic behavior whilst still being mathematically tractable. The experience of the linear equation of state would suggest that the generic behavior of spatially homogeneous models provides a good indication of the generic inhomogeneous behavior. It is hoped, furthermore, that they provide a good model of scalar field cosmologies whose inhomogeneities are not too large.

Bianchi cosmologies take on a particularly simple form in scalar field cosmologies due to the fact that, as has been pointed out by Ellis *et al* [42], the surfaces of constant ϕ must coincide with the orbits of the 3D isometry group (at least in the generic case). The gradient of the scalar field, and hence the energy worldlines of the fluid, are therefore normal to the orbits of the isometry group. Spatially homogeneous models which satisfy this property are called Orthogonal Spatially Homogeneous (OSH) models.

The OSH models fall into two Classes; A and B. We shall consider only those of type A, corresponding to Bianchi types I, II, VI_0, VII_0, VIII and IX. In particular these include the highest dimensional OSH models, those corresponding to Bianchi types VIII and IX. Models of class B, Bianchi types V, VI_h and VII_h can be analyzed using similar techniques and conclusions will be essentially the same.

The approach adopted below owes considerably to that of Wainwright and Hsu [63] for linear equation of state perfect fluids.

5.4.1 The Equations of Motion.

Since the normal curves to the orbits of the isometry group are geodesics, we can choose these orbits as our surfaces of constant time Σ_t. An immediate consequence of this is that the scalar fields are functions of time only. Rather than working in coordinate bases it will be convenient to introduce an orthonormal frame $\{\mathbf{U}, \mathbf{e}_a\}$ where the index a runs from 1 to 3. The $\mathbf{e}_a$ obviously form an orthonormal basis on the hypersurfaces Σ_t with respect to which the components of the spatial metric $\mathbf{h}$ are δ_{ab}. Choose $\mathbf{e}_a$ so that it commutes with the Killing vectors ξ_a. Furthermore, models of class A are precisely those for which is possible to choose such a basis so that $\mathbf{e}_a$ are non-rotating, ie the Fermi derivative $e^{\mu}_{a} e_{a\mu;\nu} U^{\nu} = 0$. In this case the commutation relations for the frame become

$$[\mathbf{e}_a, \mathbf{U}] = K^b_a(t)\mathbf{e}_b \tag{5.33}$$

$$[\mathbf{e}_a, \mathbf{e}_b] = \epsilon^c_{ab} n^d_c(t)\mathbf{e}_d \tag{5.34}$$

Where K^a_b are the components of the the extrinsic curvature of Σ_t in the orthonormal basis. These commutator relations are completely standard and further details can be found in Ellis and MacCallum [71]. We thus have

$$K_{ab} = \sigma_{ab} + \tfrac{1}{3}K\delta_{ab} \tag{5.35}$$

Note that tensor components in the orthonormal basis will always be labeled with the characters a, b, c etc. Indices are raised by means of δ^{ab}. The symmetric three tensors n_{ab} are related to the structure constants of the isometry group and determine the Bianchi type of the model. They also determine the spatial Ricci tensor according to the equation [71]

$$^{(3)}R_{ab} = 2n_a^c n_{cb} - n_c^c n_{ab} - \delta_{ab}(n_{cd}n^{cd} - \tfrac{1}{2}(n_c^c)^2) \qquad (5.36)$$

It will be convenient to split the Ricci tensor into its trace and trace free parts, so we define the trace free curvature,

$$^{(3)}R_{ab} = Q_{ab} + \tfrac{1}{3}{}^{(3)}R\delta_{ab} \qquad (5.37)$$

Rewriting the R_0^0 and R_j^i equations of the EFE in the orthonormal basis immediately provides evolution equations

$$\begin{aligned} \dot{K} &= -\sigma^2 - \tfrac{1}{3}K^2 - \dot{\phi}^2 + V(\phi) \qquad (5.38) \\ \dot{\sigma}_{ab} &= -K\sigma_{ab} - Q_{ab} \qquad (5.39) \end{aligned}$$

and the constraint equation

$$\tfrac{1}{3}K^2 = V(\phi) + \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\dot{\phi}^2 + -\tfrac{1}{2}{}^{(3)}R. \qquad (5.40)$$

The R_i^0 term in the EFE reduces to the algebraic constraint

$$\sigma_b^a n_{ca} = \sigma_c^a n_{ba} \qquad (5.41)$$

in the orthonormal basis. The evolution equations for n_{ab} are obtained from the Jacobi identities for the commutators of the orthonormal frame which yield

$$\dot{n}_{ab} = -\tfrac{1}{3}Kn_{ab} + 2\sigma_{(a}^c n_{b)c} \qquad (5.42)$$

By rotating the basis vectors $\mathbf{e}_a$ on the initial hypersurface Σ_0 it is possible to simultaneously diagonalize σ_{ab} and n_{ab} without violating (5.41). The evolution equations (5.39) and (5.42) ensure that this diagonalization is preserved for all time. The system of equations is completed by the evolution equation for the scalar field which, since the scalar field is homogeneous, is just

$$\ddot{\phi} = -K\dot{\phi} - V'(\phi) \qquad (5.43)$$

Equations (5.38)-(5.43) constitute a 7-dimensional autonomous system of ordinary differential equations for the variables $(K, \sigma_{ab}, n_{ab}, \phi, \dot{\phi})$. The full analysis of this system would take a considerable amount of time and effort.

Rather, we shall be content to look for asymptotically expansion dominated solutions.

We shall proceed in much the same way as we did for the Bianchi type I case. Firstly define the dimensionless shear variables

$$S_{ab} = \sqrt{\tfrac{3}{2}}\frac{\sigma_{ab}}{K} \tag{5.44}$$

and dimensionless spatial curvature variables

$$N_{ab} = \frac{n_{ab}}{K}. \tag{5.45}$$

Q_{ab} and ${}^{(3)}R$ are quadratic in n_{ab} so define

$$q_{ab} = \sqrt{\tfrac{3}{2}}\frac{q_{ab}}{K^2} \qquad\qquad r = -\frac{3}{2}\frac{{}^{(3)}R}{K^2}. \tag{5.46}$$

q_{ab} and r are thus functions of N_{ab} given by

$$r = \tfrac{2}{3}N^{ab}N_{ab} - \tfrac{1}{3}(N^a_a)^2 \tag{5.47}$$

$$q_{ab} = \sqrt{\tfrac{3}{2}}(2N^c_a N_{cb} - N^c_c N_{ab} + \tfrac{1}{2}r\delta_{ab}). \tag{5.48}$$

As usual we define the dimensionless scalar field gradient

$$y = \sqrt{\tfrac{3}{2}}\frac{\dot{\phi}}{K} \equiv \sqrt{\tfrac{3}{2}}\frac{d\phi}{d\tau} \tag{5.49}$$

Since we are interested in the asymptotic behavior it will be convenient to write the evolution equations as differential equations with respect to the geodesic parameter time τ. In order for τ to be well defined it is necessary that the expansion K be strictly positive. By inspection of (5.40) we see that a sufficient condition for this to be true is that the function $\Theta = \sigma^2 -{}^{(3)} R$ be strictly positive. This is precisely the function Θ defined in (1.59) and since $\phi_i = 0$ we therefore know from (1.61) that

$$\dot{\Theta} \leq -\tfrac{2}{3}K\Theta.$$

It follows that if $\Theta(t_0) > 0$ then $\Theta > 0$ for all $t < t_0$. We are thus free to restrict our considerations to the region of phase space H^+ satisfying

$$S^2 + r > 0. \tag{5.50}$$

If we specify an initial state p in H^+ then the past orbit O_p^- of p under the above system of equations will be contained in H^+. The past evolution of the system is thus well defined on H^+.

Restricting attention to H^+ is equivalent to imposing the condition that $\sigma^2 > -{}^{(3)}R$ at some time during the evolution of the universe. It turns out that this condition is sufficient to guarantee that almost all solutions are strongly expansion dominated. If this condition is violated then the strong positive curvature can actually cause a cosmological bounce to occur resulting in non-singular, even periodic cosmologies. We shall briefly discuss solutions of this type later.

Let us now continue with our analysis. Our aim is to eliminate K from the evolution equations, so it will be convenient to introduce the function

$$L(y, S_{ab}, N_{ab}) = \frac{d}{dt}\left(\frac{1}{k}\right) = y^2 + S^2 + \tfrac{1}{3}r \tag{5.51}$$

where $S^2 = S^a_b S^b_a$. The above expression was obtained by substituting the constraint equation (5.40) into (5.38) to eliminate V.

We may now derive evolution equations for S_{ab} and N_{ab}. Using (5.42), (5.39) and (5.51) and recalling that $\frac{dt}{d\tau} = K^{-1}$ we find

$$\frac{dS_{ab}}{d\tau} = (L-1)S_{ab} - q_{ab} \tag{5.52}$$

$$\frac{dN_{ab}}{d\tau} = (L - \tfrac{1}{3})N_{ab} + 2\sqrt{\tfrac{2}{3}}S^c_{(a}N_{b)c} \tag{5.53}$$

Choosing the basis which diagonalizes S_{ab} and N_{ab} we may write

$$S_{ab} = \mathrm{diag}(S_1, S_2, S_2) \qquad N_{ab} = \mathrm{diag}(N_1, N_2, N_3) \tag{5.54}$$

and since S_{ab} is trace free, we introduce the shear variables,

$$S_+ = \sqrt{\tfrac{3}{2}}(S_{11} + S_{22}) \qquad S_- = \tfrac{1}{\sqrt{2}}(S_{11} - S_{22}) \tag{5.55}$$

Note that

$$S^2 = S_+^2 + S_-^2 \tag{5.56}$$

Writing the evolution equations explicitly in terms of the the new variables we obtain

$$\frac{dS_+}{d\tau} = (L-1)S_+ - q_+$$

$$\frac{dS_-}{d\tau} = (L-1)S_- - q_-$$
$$\frac{dN_1}{d\tau} = \tfrac{1}{3}(3L - 1 + 2S_+ + 2\sqrt{3}S_-)N_1 \tag{5.57}$$
$$\frac{dN_2}{d\tau} = \tfrac{1}{3}(3L - 1 + 2S_+ - 2\sqrt{3}S_-)N_2$$
$$\frac{dN_3}{d\tau} = \tfrac{1}{3}(3L - 1 - 4S_+)N_3$$

where q_+ and q_- are the free parameters associated with the trace free diagonal matrix q_{ab} and are defined according too

$$q_+ = \sqrt{\tfrac{3}{2}}(q_{11} + q_{22}) \qquad q_- = \tfrac{1}{\sqrt{2}}(q_{11} - q_{22}) \tag{5.58}$$

Written explicitly in terms of N_1, N_2 and N_3, the functions r, q_+ and q_- are

$$q_+ = \tfrac{1}{2}[(N_1 - N_2)^2 - N_3(2N_3 - N_1 - N_2)] \tag{5.59}$$
$$q_- = \tfrac{1}{2}(N_2 - N_1)(N_3 - N_1 - N_2) \tag{5.60}$$
$$r = \tfrac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 - N_3N_1)] \tag{5.61}$$

In order to complete the system we require evolution equations for y and ϕ. We shall assume that $V \in \mathcal{E}^2$. Since we are interested in the behavior when ϕ is large we may assume that $W_V(\phi)$ is well defined and that there exists some coordinate

$$z = f(\phi) \tag{5.62}$$

which obeys the properties required in Definition 2.2. We may now write

$$\frac{dy}{d\tau} = (-1 + L)y - \tfrac{1}{3}\bar{W}_V\,(z)(1 - y^2 - S^2 - r) \tag{5.63}$$
$$\frac{dz}{d\tau} = -y\bar{f}'\,(z) \tag{5.64}$$

This system of equations will be valid on some domain $\phi > \phi_o$. Equations (5.57) and (5.64) provide a complete set of differential equations for the variables $(N_1, N_2, N_3, S_+, S_-, y, z)$. Note that (5.57) hold everywhere on H^+, whereas (5.64) are well defined only for $\phi > \phi_0$. The expansion K has been uncoupled from the system and may be treated as a function of the state variables, determined by

$$K^2 = \frac{3\bar{V}(z)}{1 - y^2 - S^2 - r} \tag{5.65}$$

and having directional derivative on the solution curves

$$\frac{dK}{d\tau} = -LK. \tag{5.66}$$

Equation (5.65) implies that physical solutions occupy the region of phase space

$$y^2 + S_+^2 + S_-^2 + r < 1. \tag{5.67}$$

5.4.2 The Bianchi Invariant Sets.

At this point, I shall make some comments concerning the relationship between the above system of equations and the classification of Bianchi. Equations (5.57) imply that The hypersurfaces $N_a = 0$ are invariant manifolds for each $a = 1, 2, 3$. It Follows that phase space may be partitioned into the union of invariant subsets defined by all possible combinations of the conditions

$$N_a > 0 \qquad N_a < 0 \qquad N_a = 0 \tag{5.68}$$

for $a = 1, 2, 3$. 27 such subsets cover phase space but not all are physically distinct. The evolution equations are invariant under the discrete transformations

$$(N_1, N_2, N_3) \mapsto \epsilon P(N_1, N_2, N_3) \qquad (S_1, S_2, S_3) \mapsto P(S_1, S_2, S_3) \tag{5.69}$$

where P is any permutation of the three objects and $\epsilon = +1$ for P even and -1 for P odd. These transformations correspond to all possible permutations of the spatial basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. This leaves us with 6 physically distinct possibilities for N_a. These are the Bianchi invariant subsets and each corresponds to a particular Bianchi type as detailed in Table 5.4.2. Each Bianchi invariant subset (except BI)is the union of several disjoint components in phase space corresponding to the various permutations of the conditions (5.68). For example the Bianchi type IX invariant set is the union of the disjoint union of the sets $\mathrm{B}^+\mathrm{IX}$ and $\mathrm{B}^-\mathrm{IX}$ corresponding to N_a all positive and all negative respectively.

When all the components of N_a are zero we have the Bianchi type I cosmologies and indeed it is easy to verify that the evolution equations reduce to (4.31) in this case when one combines (5.57a-b) to obtain an evolution

Notation	Restrictions on the N_a	d	N
BI	All zero	4	1
BII	One non-zero	5	6
BVI_0	Two non-zero with opposite sign	6	6
$BVII_0$	Two non-zero with the same sign	6	6
BVIII	All non-zero differing in sign	7	6
BIX	All non-zero with the same sign	7	2

Table 5.1: The Bianchi Invariant sets. d is the dimension of the set in phase space and N is the number of disjoint components. These components map onto each other under the discreet symmetries above

equation for S so that the system is expressed in terms of quantities which are scalars on the surfaces of constant time Σ_t.

The highest dimensional Bianchi Invariant sets are those corresponding to types $VIII$ and IX. These are the only sets which occupy non-zero volume in phase space and must thus be considered the generic OSH models.

5.4.3 Geometry of the Flow for $z = 0$.

Our task now is to identify solutions which are asymptotically expansion dominated. Since we know that ϕ diverges to the past of such solutions consider the unphysical invariant hypersurface $z = 0$ corresponding to $\phi = \infty$. Call this set ∂H_0. Restricted to ∂H_0 the system becomes

$$\begin{aligned}
\frac{dS_+}{d\tau} &= (L-1)S_+ - q_+ \\
\frac{dS_-}{d\tau} &= (L-1)S_- - q_- \\
\frac{dN_1}{d\tau} &= \tfrac{1}{3}(3L - 1 + 2S_+ 2\sqrt{3}S_-)N_1 \\
\frac{dN_2}{d\tau} &= \tfrac{1}{3}(3L - 1 + 2S_+ - 2\sqrt{3}S_-)N_2 \\
\frac{dN_3}{d\tau} &= \tfrac{1}{3}(3L - 1 - 4S_+)N_3 \\
\frac{dy}{d\tau} &= (-1+L)y
\end{aligned} \tag{5.70}$$

We shall now briefly analyze the system on ∂H_0. Since ∂H_0 has compact closure all solutions must possess an invariant α-limit set. Consider the function $P = 1 - y^2 - S_+^2 - S_-^2 - r$. Then $0 \le P \le 1$ on the closure of ∂H_0. Taking the time derivative of (5.65) and using (5.66) we find

$$\frac{dP}{d\tau} = LP \tag{5.71}$$

The gradient of P is thus strictly positive except when $P = 0$ or $L = 0$. However, since P is monotonic increasing all α-limit sets must have $\frac{dP}{d\tau} = 0$. The only point on ∂H_0 where L vanishes is the origin and this point is a global maximum of P, we have $P(0) = 1$. Since P can never decrease, no solutions are past asymptotic to this point except the steady state solution lying at the origin itself. It follows that all other solutions on ∂H_0 asymptotically approach the hypersurface $P = 0$ as $\tau \to -\infty$ (in dynamical systems language P is a *Liapunov function*).

Furthermore, observe that the variable y satisfies

$$\frac{dy^2}{d\tau} = (-1 + L)y^2$$

on ∂H_0. By (5.67) $0 \le L \le 1$ on the closure of ∂H_0. By an identical argument to that used above for P we see that all α-limit sets must satisfy either or both of the conditions $L = 1$ or $y = 0$. However if a limit set satisfies $y = 0$ it will be unstable (as a past limit) since if y^2 is perturbed slightly it can only increase in the past and may therefore not asymptotically tend to 0. We therefore arrive at the conclusion that with the possible exception of a set of measure zero, all solutions on the set ∂H_0 must asymptotically (in the past) approach a limit set lying on the intersection of the hypersurfaces $P = 0$ and $L = 1$. This is precisely the Bianchi type I hypersurface E_0 which we define by the condition

$$N_a = 0 \qquad y^2 + S_+^2 + S_-^2 = 1. \tag{5.72}$$

We now determine the subset of E_0 which is the source in ∂H_0. E_0 corresponds to the unit sphere on the Bianchi Type I subset of ∂H_0. It is therefore natural to introduce polar coordinates θ, φ defined on the intervals $[0, \pi]$ and $[0, 2\pi]$ respectively. We may then write points on E_0 as

$$p(\theta, \varphi) = (0, 0, 0, \alpha_+, \alpha_-, \cos\theta, 0)$$

where

$$\alpha_+ = \sin\theta\cos\varphi \qquad \alpha_- = \sin\theta\sin\varphi. \tag{5.73}$$

Substituting this into the field equations we find that E_0 is an equilibrium set. Now E_0 lies on the boundary of the Bianchi invariant set BI which we analyzed in detail in Chapter 4. Inspection reveals that E_0 corresponds to the semi-circular equilibrium set Υ (Fig. 4.1). The correspondence between the 2D sphere E_0 and the semi-circle Υ follows because the ratio S_+/S_- is a constant of motion on BI so the the two shear variables may be replaced by the single variable S. Each point p on Υ corresponds to a ring of points on E_0 at fixed θ satisfying $S^2 = \alpha_+^2 + \alpha_-^2 = \cos^2\theta$. Where θ is precisely the angle labeling p in Υ.

It follows from Fig. 4.1 and (4.37) that each point $p(\theta,\varphi)$ possesses a 1-dimensional unstable manifold in $\partial BI \cap \partial H_0$ spanned by the radial "eigenvector" $\mathbf{e}_r$ normal to E_0 at $p(\theta,\varphi)$. The union of these unstable manifolds corresponds to all radial lines from the origin to the surface of the unit sphere and covers $\partial BI \cap \partial H_0$. (Here ∂BI is the boundary of BI). Therefore an arbitrary BI solution is equally likely to be past asymptotic to any point on E_0. Note that any individual point or 1-dimensional curve on E_0 will be a past asymptote to at most a set of solutions of measure zero in BI.

In order to determine whether a given point $p(\theta,\varphi)$ is negatively stable with respect to trajectories which have *non-zero* N_a we we need only consider the linearization of equations (5.57c-e)close to $p(\theta,\varphi)$. We find:

$$\begin{aligned} \frac{dN_1}{d\tau} &= \beta_1 N_1 \\ \frac{dN_2}{d\tau} &= \beta_2 N_2 \\ \frac{dN_3}{d\tau} &= \beta_3 N_3 \end{aligned} \tag{5.74}$$

where

$$\begin{aligned} \beta_1 &= \tfrac{2}{3}(1+\alpha_+ + \sqrt{3}\alpha_-) \\ \beta_2 &= \tfrac{2}{3}(1+\alpha_+ - \sqrt{3}\alpha_-) \\ \beta_3 &= \tfrac{2}{3}(1-2\alpha_+) \end{aligned} \tag{5.75}$$

Inspection of (5.75) reveals that $p(\theta,\varphi)$ is negatively stable with respect to perturbations of N_a in ∂H_0 if the following three conditions are simultaneously satisfied

$$\begin{aligned} 1+\alpha_+ + \sqrt{3}\alpha_- &> 0 \\ 1+\alpha_+ - \sqrt{3}\alpha_- &> 0 \\ 1-2\alpha_+ &> 0. \end{aligned} \tag{5.76}$$

If (θ, φ) does not satisfy all three of the conditions (5.76) then $p(\theta, \varphi)$ will be a saddle and can not be considered a generic past asymptote. Strictly speaking the case where 1 or more of the β_a vanish requires further analysis to assess stability. However, solutions past asymptotic to these points will constitute a set of measure zero and need not be considered further (the rigorous proof of this statement actually follows from Theorem A.8 since E_0 is a Center Manifold).

Therefore, almost all solutions on the unphysical set ∂H_0 are past asymptotic to the subset $E_s \subset E_0$ obeying (5.76). It follows that if any subset of ∂H_0 is a source for the full system (5.57, 5.64) it must also be a subset of E_s. This implies that if there exists a generic family of physical solutions for which $\lim_{\tau\to-\infty} z = 0$ then these solutions must be past asymptotic to E_s.

5.4.4 Asymptotic Behavior of Physical Trajectories.

Let us now look more closely at the asymptotic behavior of the system near E_s.

Firstly note that the above results imply that for each $p(\theta, \varphi) \in E_s$ there exists a four dimensional unstable manifold in ∂H_0 spanned by the mutually orthogonal vectors $\mathbf{e}_r$, $\mathbf{e}_{N_a}$ for $a = 1, 2, 3$, ($\mathbf{e}_{N_a}$ are the vectors tangent to the N_a coordinate curves).

Define the set ∂BI_1 to be the hypersurface on (the boundary of) H^+ satisfying the conditions (5.72) but with z not necessarily zero; ie ∂BI_1 is the extension of E_0 off of the $z = 0$ hypersurface. Then ∂BI_1 is a three dimensional center manifold for each point on the equilibrium set E_0. This may be verified either by directly evaluating the eigenvalues of the vector field at $p(\theta, \varphi)$ or by identifying ∂BI_1 with the center manifold $\partial BI_{\epsilon 1}$ in Chapter 4.

Since the unstable manifold is orthogonal to the center manifold, the union of the center sub-space and unstable sub-space spans R^7 at $p(\theta, \varphi)$.

It follows from the reduction of dimensions principle (Theorem A.8) that $p(\alpha_+,\alpha_-)$ will be negatively stable if it is negatively stable for the restriction of the flow to the center manifold. This will be true if and only if $y<0$ at $p(\theta,\varphi)$; ie if $\frac{\pi}{2}<\theta\leq\pi$.

Take E_s^- to be the subset of E_s satisfying this condition then almost all solutions for which $z\rightarrow 0$ asymptotically in the past are past asymptotic to some point in E_s^-. Furthermore, from Theorem A.8 there exists a neighbourhood N of E_s and a positive constant γ for which

$$\begin{aligned} S_+ &= \alpha_+ + O(e^{\gamma\tau}) & S_+ &= \alpha_- + O(e^{\gamma\tau}) \\ y &= \cos\theta + O(e^{\gamma\tau}) & z &= f(\sqrt{\frac{2}{3}}\cos\theta\tau+\tilde{\phi}) + O(e^{\gamma\tau}) \qquad (5.77) \\ N_a &= O(e^{\gamma\tau}). \end{aligned}$$

where the lowest order terms just represent the exact solution on the center manifold ∂BI_1. The first order solution for the N_a may be obtained by substituting (5.77) into (5.57c-e) to obtain

$$N_a = A_a e^{\beta_a\tau} \qquad (5.78)$$

The A_a are arbitrary constants which depend continuously on the initial conditions and whose sign is determined by the Bianchi type of the solution.

Noting the symmetry of the system under the transformation $\phi\mapsto-\phi$ we see that similar conclusions may be drawn for solutions which have ϕ tending to negative infinity.

We summarize the above results in the following theorem.

Theorem 5.3 *Consider a class A spatially homogeneous scalar field cosmology which satisfies the following properties: The potential $V\in\mathcal{E}^2$; There exists a time t_0 for which $\sigma^2(t_0)>^{(3)}R(t_0)$; $|\phi|\rightarrow\infty$ as $\tau\rightarrow-\infty$. Then for $|\phi|$ sufficiently large it may be described by the system (5.57)-(5.64) and the following is almost always true: There exist constants θ, φ, c and A_a such that, for τ sufficiently large and negative,*

$$N_a = A_a e^{\beta_a\tau} + h \qquad S_\pm = \alpha_\pm + h \qquad (5.79)$$

$$y = \cos\theta + h \qquad \phi = \sqrt{\tfrac{2}{3}}\cos\theta\tau + h \qquad (5.80)$$

where $\alpha_\pm$ and β_a are functions of θ and φ defined by (5.73) and (5.75) respectively. Furthermore, θ and φ are such that $\alpha_\pm$ satisfy the inequalities

(5.76) (implying that $\beta_a > 0$). The h represent any higher order terms and $\lim_{\tau\to-\infty} h = 0$.

It turns out that the third condition, that $|\phi| \to \infty$, is redundant and the first two conditions are actually sufficient for the theorem to hold. This is because it may be shown that if the first two conditions of the theorem hold ϕ almost always diverges. To see this observe that the function $\frac{1}{K}$ is monotonic increasing and may vanish only (and identically) on the unphysical part of the boundary of H. This is the union of the sets $\phi = \infty$, $\phi = -\infty$ and $P = 0$. Thus all solutions must be past asymptotic to this boundary. However solutions on the set $P = 0$ correspond to massless scalar field cosmologies (ie, from the constraint equation $P = 0$ iff $V = 0$) which, as we know, are equivalent to perfect fluid solutions with equation of state $\rho = p$. These have been analyzed in detail for OSH models by Wainwright and Hsu [63] who have proven that for almost all solutions ρ diverges as $1/t$ as $t \to 0$ which is equivalent to saying that ϕ diverges as $\ln t$. It follows from continuity that ϕ diverges for almost all solutions on H.

5.4.5 Structure of The Singularity

It is intuitively obvious that the generic solutions described in Theorem 5.3 are strongly expansion dominated. In order to make this more more explicit let us first rewrite the asymptotic expression in terms of the proper time t and the familiar quantities K, σ^a_b, n_{ab}, ϕ and $\dot\phi$. It follows from (5.66) that to first order as $\tau \to -\infty$

$$K = K_0 e^{-\tau} + h$$

and hence, recalling that

$$\frac{d\tau}{dt} = K$$

we find

$$t = K_0 e^{\tau} + h$$

where t has been chosen so that it vanishes at the singularity. A straitforward calculation gives

$$K = t^{-1} + h \qquad \sigma^a_b = \mathrm{diag}\left(\alpha_a t^{-1} + h\right) \tag{5.81}$$

$$N_{ab} = \mathrm{diag}\left(B_a t^{\beta_a - 1} + h\right) \tag{5.82}$$

$$\dot\phi = \sqrt{\tfrac{2}{3}}\cos\theta t^{-1} + h \qquad \phi = \sqrt{\tfrac{2}{3}}\cos\theta \ln(t/c) + h \tag{5.83}$$

Where $B_a = A_a/K_0^{\beta_a}$ and α_a are the asymptotic dimensionless shear variables defined by the relations

$$\alpha_+ = \sqrt{\tfrac{3}{2}}(\alpha_1 + \alpha_2) \qquad \alpha_- = \tfrac{1}{\sqrt{2}}(\alpha_1 - \alpha_2) \tag{5.84}$$

Note that the higher order terms are all diagonal. We define the first approximation to the cosmology as follows: By substituting the above expressions into the commutation relations it can be seen that by scaling the basis vectors just slightly we may define a unique basis $\{\hat{\mathbf{e}}_a\}$ on Σ_t satisfying the following commutation relations:

$$\begin{aligned} [\hat{\mathbf{e}}_a, \mathbf{U}] &= \frac{(1+\alpha_a)}{t}\hat{\mathbf{e}}_a \\ [\hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b] &= \epsilon^c_{ab} B_c t^{\beta_c - 1}\hat{\mathbf{e}}_c \end{aligned} \tag{5.85}$$

and such that the components of $\mathbf{h}$ in the basis $\{\hat{\mathbf{e}}_a\}$ are

$$g_{ab} = \delta_{ab} + \epsilon_{ab} \tag{5.86}$$

where ϵ_{ab} is diagonal and $\lim_{t\to 0}\epsilon_{ab} = 0$. Define $\hat{\mathbf{h}}$ to be the metric on Σ_t with components $\hat{g}_{ab} = \delta_{ab}$ in the basis $\{\hat{\mathbf{e}}_a\}$. Since the basis vector $\hat{\mathbf{e}}_a$ is proportional to the corresponding vector $\mathbf{e}_a$ it follows that the components of K^a_b are unaffected by the change of basis. Also, the unphysical spacetime $\hat{\mathbf{g}} = -dt^2 + \hat{\mathbf{h}}$ is of the same Bianchi type as $\mathbf{g}$. By inspection of the commutation relations we have

$$\hat{K} = t^{-1} \qquad \hat{\sigma}^a_b = \operatorname{diag}\left(\alpha_a t^{-1}\right) \tag{5.87}$$

$$\hat{N}_a = B_a t^{\beta_a - 1}. \tag{5.88}$$

Finally, define the first approximation to the scalar field to be

$$\hat{\phi} = \sqrt{\tfrac{2}{3}}\cos\theta \ln(t/c). \tag{5.89}$$

It follows immediately from (5.86), (5.89) and (5.83) that $(\hat{\mathbf{h}}, \hat{\phi})$ obeys conditions i), ii) and iv) of definition (5.1). Condition iii) is slightly more complicated since it is expressed in terms of a coordinate basis on Σ_t and will be obviously invariant only under a change of basis which is stationary; ie a basis on Σ_t which commutes with $\mathbf{U}$. We therefore define the new basis vectors $\hat{\mathbf{f}}_a = t^{-(\frac{1}{3}+\alpha_a)}\hat{\mathbf{e}}_a$ where there is no summation over the indices a. Then it is easily verified that

$$[\hat{\mathbf{f}}_a, \mathbf{U}] = 0.$$

However the components of $\hat{K}^a_b$ and K^a_b are unaffected by this change of basis so it can be immediately verified from (5.87) and (5.81) that iii) holds and hence $(\hat{\mathbf{h}}, \hat{\phi})$ is a past asymptote to $(\mathbf{h}, \phi)$. It is a simple matter to verify directly that $(\hat{\mathbf{h}}, \hat{\phi})$ satisfies (5.8) and hence that $(\mathbf{g}, \phi)$ is strongly asymptotically expansion dominated. Note that in terms of the basis $\{\hat{\mathbf{f}}_a\}$ the first approximation to the metric takes the more familiar form

$$g_{ab} = \mathrm{diag}(t^{\frac{2}{3}+2\alpha_a})$$

The constraints (5.76) ensure that for generic solutions $\alpha_a < 1/3$. Hence, typical OSH models expand along all 3 principal directions and pancake and cigar singularities may be understood as pathological behavior arising in unstable solutions.

In line with the discussion of the last section we may represent the singularity as a 3 surface S_M with Riemannian metric δ_{ab} and extrinsic curvature $C^a_b = \mathrm{diag}(1+\alpha_a)$ in a basis $\{\mathbf{s}_a\}$ satisfying the commutation relations

$$[\mathbf{s}_a, \mathbf{s}_b] = \epsilon^c_{ab} B_c \mathbf{s}_c$$

Note that S_M may be identified with the hypersurface Σ_1 in M with the metric $\hat{\mathbf{h}}$. It is also necessary to specify the constant function c on S_M as an initial condition for the scalar field. The singularity structure is uniquely determined by 6 numbers B_a, α_+, α_- and c. Thus the singularity is isomorphic to the set $R^4 \times K^b$ where K^b is the subset of the Kasner disc satisfying the constraints (5.76). Note that the other regions of the Kasner disc also give rise to OSH cosmologies but such solutions constitute a lower dimensional set in phase space.

Since cigar and pancake singularities are unstable in homogeneous spacetimes it is very difficult to conceive that they could be generic in the larger class of inhomogeneous cosmologies.

Chapter 6

Discussion

Clearly, a full understanding of the generic asymptotic structure of scalar field cosmologies is yet to be achieved. However, the results we have obtained throughout this thesis provide every indication that for models with SED potential the strongly expansion dominated cosmologies represent a large and important class of solutions. The rigorous results obtained for Bianchi Type I space-times also hint that for a given potential this class of solutions may actually be homeomorphic (in some sense) to the set of massless scalar field cosmologies thereby allowing a regular boundary to be attached to solution space via a uniquely characterized singularity structure, as discussed in section 5.3.2.

Important features of these cosmologies include the existence of a space-like semi-Kasner singularity, particle horizons and an asymptotic equation of state $\rho = p$ for the scalar field.

The behavior of the spatially homogeneous models suggests, further, that the generic solutions have big bang type singularities with space-time expanding along all principle directions. Assuming this to be true then, by Theorem 5.1, the time of the singularity can be set to zero everywhere thereby defining a unique cosmological time.

The existence of a particle horizon means that for any $p \in M$ there exist regions of space with which the point p has had no causal contact. This is usually interpreted as implying that the values of physical fields should be completely uncorrelated over sufficiently large distances. One must be cautious, however, because the singularity may impose boundary conditions which break symmetries existing in the field equations themselves. If we insist that the singularity is regular and the scalar field diverges along all timelike geodesics (ie the singularity has a global semi-Kasner structure) then by continuity it must diverge to *either* positive infinity or negative infinity *for all space*. The scalar field therefore does not average to zero over a large number of horizon volumes, as would be expected from the symmetry of the field equations themselves (and is usually taken for granted in inflationary cosmology). Rather, some nonzero average may be expected to persist over all space as a residue of the choice of sign at the singularity. Recalling the discussion of section 1.5.4, this is exactly the type of condition which Goldwirth and Piran [22] found to be necessary for the onset of inflation in massive scalar field cosmologies. This suggests that semi-Kasner singularities represent a natural boundary for the class of cosmologies which undergo inflation.

Conversely, one could turn the above observation around and argue that since semi-Kasner singularities require correlations in the scalar field over distances greater than the horizon length in order to be globally consistent they violate causality and, therefore, can not represent a typical class of solutions. However, the conclusion that the average value of the scalar field will be large over all space-time does not really depend on the detailed structure of semi-Kasner singularities but seems to follow from the straitforward conjecture that; a) there exists a singularity which is spacelike. and; b) the scalar field diverges along all timelike geodesics.

I believe that a timelike singularity is an unlikely scenario given the constraint imposed by the the Weak Energy Condition. that the surfaces of constant ϕ be space-like. Consider, for example, a static space-time with naked, timelike singularity. By symmetry the surfaces of constant ϕ must be timelike thereby violating the Weak Energy Condition.

A more plausible possibility is that there could exist regions of space where the scalar field remains finite at the singularity. Such regions could form boundaries between open regions where the scalar field diverges to $+\infty$ and $-\infty$ respectively. Such a scenario might in principle be accommodated within the framework of strongly expansion dominated cosmologies since these do not rule out the possibility of a measure-zero set of spatial points with $\mathcal{B}$-type asymptotic behavior. However careful inspection of the details of the possible $\mathcal{B}$-type solutions for which the scalar field remains finite (see sections 2.2 and 4.3.1) reveals that such a solution must violate the Weak Energy Condition.

The reason for this is sketched as follows: Assume $\phi(x_0, t)$ remains finite for some x_0. Then there exists an arbitrarily close point x_1 for which the scalar field diverges. Thus, $|\phi_{,i}(x_0, t)| \to \infty$ and it can easily be shown that $g^{ij}\phi_i\phi_j > O(t^{2/3})$. Furthermore, it can also be shown that for the allowed solutions $\dot{\phi}^2 < O(t^2)$. Thus the gradient to the scalar field $\phi^\mu(x_0, t)$ would become space-like close to the singularity in violation of the Weak Energy Condition.

It is not immediately obvious that $\mathcal{B}$-type solutions for which ϕ diverges, such as the inflating solutions I_∞ (see section 2.6), can be ruled out in a similar manner. This raises the intriguing possibility that, amongst the infinite number of spatial points in an inhomogeneous universe, non-generic horizon free and (in some cases) singularity free inflationary attractors could characterize the asymptotic dynamics of a very small number. Even if such a (past) asymptotically inflating solution existed along just a *single* timelike geodesic it could act as a seed causing extremely efficient isotropization and expansion along nearby geodesics and generating our entire observable universe! I do not suggest that this scenario is likely however.

It is not clear to me that the size of the particle horizon is even relevant to these considerations. The real issue is whether or not a particular class of cosmologies is a large and natural one within the solution space of the EFE. This is a purely dynamical question. To illustrate the point imagine we have some cosmology where the volume element v expands, near the singularity, according to some power law. For such solutions the characteristic proper horizon length will be of order K^{-1}. Assume that the energy density is of comparable or lower order to the expansion so that we have, in particular, $\dot{\phi}^2 \sim K^2$. If we insist that the Weak Energy Condition is satisfied we must therefore have

$$\phi^i\phi_i < K^2.$$

If $|\phi| >> 1$ for some x then the above bound means that it will remain large over a characteristic proper length $\Delta X >> K^{-1}$. Thus by imposing a simple dynamical condition we find that the characteristic scale of spatial variation for the scalar field *must* be larger than the size of the horizon. In fact the algebraic constraint equation (1.43) implies that this is true *whenever* ${}^{(3)}R < \sigma^2 + \dot{\phi}^2$. I would therefore suggest that very wrinkly solutions for which ϕ varies rapidly compared to the characteristic size of the particle horizon can probably be ruled out by the Weak Energy Condition together with some kind of bound on the positive curvature (cf no-hair theorems). This requires no special correlations in the initial data at all. The magical boundary conditions imposed by semi-Kasner Singularity structure really have their origins in the detailed dynamical structure of Einstein's equations.

The picture that seems to have emerged, is of a typical universe originating at a semi-Kasner like singularity, rapidly approaching an inflationary attractor and then evolving into some future asymptotic state according to the details of its potential. In Chapter 2 it was proved that such inflationary solutions do indeed exist for a wide class of potentials. However, it remains to to be shown rigorously that they are indeed attractors in the full inhomogeneous solutions space (although the arguments of sections 1.5.3 and 1.5.4 strongly suggest that they are, at least on an appropriate domain).

I have argued that strongly expansion dominated cosmologies are a large and natural class of solutions. Nevertheless it would be foolhardy to suggest that strongly expansion dominated cosmologies represent the only generic class of solutions to the EFE which satisfy the Weak Energy Condition.

In order to obtain expansion dominated Bianchi Type IX solutions, for example, it was necessary to place a bound on the initial value of the spatial curvature relative to the shear tensor. The question that immediately arises is whether there exists solutions for which this condition is never satisfied (solutions whose orbits are contained in the compliment of H^+). Clearly such solutions can not be expansion dominated.

The answer to this question is that such solutions do exist and there exists evidence that in certain cases they may be generic. The simplest Bianchi Type IX space-times are the closed FRW models. These are the spatially isotropic space-times with positive curvature. A number of solutions are known to exist for these metrics which "bounce" before reaching the singularity thereby avoiding it completely.

Such a class of solutions was found for the massive scalar field by Hawking [72] and analyzed in detail by Page [73]. These solutions, which are perpetually bouncing but in general aperiodic, turn out to be non-generic and in fact seem to occupy a fractal set in solutions space [73].

For constant potential models with $V = V_0$ the field equations can be expressed as a 2-dimensional autonomous system and the phase plane structure can easily be analyzed (see for example [74]). One finds that there exists a *generic* class of solutions which are past asymptotic to the anti-de Sitter solution $a = e^{-\sqrt{3V_0}t}$ and future asymptotic to the de Sitter solution $a = e^{\sqrt{3V_0}t}$

undergoing a single cosmological bounce. This type of behavior seems to be generic for models which have non-zero global potentials. Simple numerical experiments carried out by the author for the model $V(\phi) = \frac{1}{2}\phi^2 + V_0$, for example, indicate that bouncing cosmologies occur under a wide range of initial conditions. It is not surprising that these solutions should arise since the set $\dot{a} = 0, \ddot{a} > 0$, representing the bounce point of a trajectory, constitutes a continuous line of initial conditions in phase space. In fact, in general Bianchi type IX phase space, this set is also intersected by a family of orbits occupying a finite phase space volume, suggesting that bouncing cosmologies are generic for certain Bianchi type IX models as well.

The difficulty in assessing the physical significance of bouncing cosmologies lies in comparing the "number" of bouncing solutions to the "number" of solutions with an initial singularity. In order to do this one needs a physically meaningful measure on solution space. Such a measure has been proposed by Gibbons, Hawking and Stewart [30] based on the Liouville measure for canonical gravity constructed using configuration space variables (a, ϕ). The naturalness of this measure as a measure on the set of homogeneous universes really depends on whether the underlying quantum theory is related to this classical canonical phase space and must be viewed as speculative at best. Nevertheless, *very* preliminary results obtained by applying this measure to bouncing cosmologies suggests that they may have vanishingly small measure relative to space-times with singularities and, furthermore, that the measure of bouncing solutions is strongly peaked around solutions whose minimum scale factor a_0 (ie the scale factor at the bounce) approaches infinity, therefore making them unlikely to represent our universe. I hope to explore this issue in more detail in the future.

Taking note of the possible existence of bouncing cosmologies, I would put forward the following conjecture:

Almost all expanding scalar field cosmologies arising from models with (class 2) SED potentials, which obey the Weak Energy Condition and which do not undergo a cosmological bounce are strongly expansion dominated.

The above conjecture would be extremely difficult to prove in general and clearly a good deal more evidence is yet required before it can be considered secure. Much insight will clearly be gained by looking at special cases of inhomogeneous metrics such as spherically symmetric and plane symmetric solutions and attempting to construct counter examples.

By concentrating almost entirely on SED potentials we have really only taken the first step in cataloging the full range of asymptotic behavior exhibited by scalar field cosmologies. Much work remains to be done, particularly since exponential type potentials, which were only briefly discussed above, represent such an important class of models. The results that we have obtained strongly suggest that, whatever the details of their asymptotic structures turn out to be, singularities and particle horizons are an inevitable feature of realistic scalar field cosmologies.

Appendix A

Dynamical Systems

The purpose of this appendix is to introduce some notation and state a number of important theorems which shall be useful in our analysis of scalar field cosmologies. The results can be found in a number of modern texts on dynamical systems and the qualitative theory of nonlinear ODEs. In particular, excellent introductions, including clear discussions of center manifolds and comprehensive lists of major references, are provided by Guckenheimer and Holmes [75] and Wiggins [76].

Consider the autonomous system of ordinary differential equations

$$\frac{dx}{dt} = f(x) \tag{A.1}$$

where $x \in U \subset R^n$, $t \in R^1$ and $f : U \to R^n$ is a C^r vector valued function of x. The coordinate indices have been suppressed for ease of notation but it is understood that $f = (f_1, f_2, .., f_n)$, etc. We say that (A.1) is a C^r dynamical system or vector field on U and call U the phase space *. A (global) solution or trajectory of (A.1) is a map $\psi : I \subset R^1 \to U$ which satisfies

$$\frac{d\psi(t)}{dt} = f(\psi(t)), \tag{A.2}$$

Where I is the largest interval on which ψ can be defined so that it satisfies (A.2). The standard uniqueness theorems for ODEs imply that for each

*The term dynamical system is often reserved for the flow generated by (A.1) but for our purposes this distinction is unimportant.

$x \in U$ there exists a unique solution, $\psi_x(t)$ passing through x satisfying the initial condition $\psi_x(0) = 0$. Furthermore, these solutions are C^r functions of t and x [77, 78]. The set of all solutions defines a dynamical flow Ψ_t which may be interpreted geometrically as a smooth congruence of curves on U. For each t, Ψ_t defines a C^r map $\Psi_t : U \to U$ which transports points in U along the flow; ie $\Psi_t(x) = \psi_x(t)$. The dynamical system (A.1) has the geometric interpretation of a vector field with components $f(x)$ tangent to the flow at each point x. (A.1) thus provides geometrical information about the flow even before it is integrated.

Let $x \in U$ and let ψ_p be defined on the interval $I \subset R$. We define the orbit of x, $O(x)$, to be the image of I under ψ_p; ie $O(x) = \{y = \psi_x(t) : t \in I\}$. If f is C^1 then the existence and uniqueness theorems, and smoothness of the vector field on U, imply that if $O(x)$ is contained in a compact subset of U then $I = (-\infty, \infty)$. Another way of saying this is that local solutions can always be extended unless they blow up to infinity. A proof of this important elementary property is given in Hirsch and Smale [77].

Similarly, we may define the past orbit of x, $O^-(x)$ (sometimes called the negative semi-orbit), to be the image of the non-positive elements of I under ψ_p. If $O^-(x)$ is contained in a compact set then I contains $(-\infty, 0)$; ie $\psi_p(t)$ exists for all $t \leq 0$.

We say that a set $V \in U$ is invariant under the flow if for all $x \in V$ we have $\psi_x(t) \in V$ for all $t \in R$. Similarly we say that V is past invariant if for all $x \in V$ we have $\psi_x(t) \in V$ for all $t \in (-\infty, 0]$.

If an invariant set V is a C^r manifold we call it a C^r invariant manifold. Let $V \in U$ be an m-dimensional C^r invariant manifold defined by a set of constraints

$$g_i(x) = 0, \tag{A.3}$$

with $m < n$. Then the system (A.1) restricted to V by (A.3) can be termed a C^r dynamical system on V (note that V need not be a subset of R^m).

One particularly important type of invariant set is an equilibrium point or fixed point of (A.1). This is any point $x_0 \in U$ for which $f(x_0) = 0$. It is easily seen to be invariant since the solution $x = x_0$ satisfies (A.1) for all $t \in R$.

In our analysis, we are mainly interested in the asymptotic behavior of solutions as $t \to -\infty$, (provided they exist that long). A point $y \in U$ is said to be an α-limit point of x if there exists a sequence $\{t_i\}$ with $t_i \to -\infty$

such that $\lim_{i\to\infty} \psi_x(t_i) = y$. The set of all α-limit points of a point x is termed its α-limit set $\alpha(x)$. The following elementary result is fundamental to asymptotic analysis of dynamical systems.

Theorem A.1 *Let $x \in U$ and let there exist a compact set $V \subset U$ such that $O^{-}(x) \subset V$. Then the following is true;*

i) $\alpha(x)$ is non-empty.

ii) $\alpha(x)$ is a closed.

iii)$\alpha(x)$ is invariant under the flow; ie it is a union of orbits.

iv) $\alpha(x)$ is connected.

Proof:

i) Let $\{t_i\}$ be a sequence with $t_i \to -\infty$. Then $y_i = \psi_p(t_i) \in V$ for all t_i. Since V is compact there exists a convergent subsequence of $\{y_i\}$. The limit point of this sequence is, by definition, an element of $\alpha(x)$.

ii) We need only show that the compliment of $\alpha(p)$ is open. Let $y \notin \alpha(p)$ then it follows from the definition of α-limit points that there must exist a neighbourhood $B(y)$ that is disjoint from the set of points $\psi_x(t) : t < t_o$ for some $t_0 < 0$. Thus $B(y)$ lies outside of α_p and we have the result.

iii) Let $y \in \alpha(x)$ and let $y_1 = \psi_y(s)$ for some $s \in I$ where I is the interval on which ψ_y is defined. Choose a sequence $t_i \to -\infty$ with $\psi_x(t_i) \to y$. Then $\psi_x(t_i + s)$ is defined, provided we choose i sufficiently large that $t_i + s \leq 0$. We have $\psi_x(t_i + s) = \Psi_{t_i+s}(x) = \Psi_s(\psi_x(t_i))$ which converges to y_1 as $i \to \infty$. Thus $\psi_y(s) \in \alpha(x)$ for all $s \in I$.

Now recall that $\alpha(x) \subset V$ by definition. It follows that $O^{-}(y) \subset V$ and hence, that $I = (-\infty, \infty)$. This completes the proof that $\alpha(x)$ is invariant.

iv) Suppose $\alpha(x)$ were not connected. Then we can choose 2 disjoint open sets V_1, V_2 such that $\alpha(x) = \alpha_1(x) \cup \alpha_2(x)$ where $\alpha_1(x) \subset V_1$ and $\alpha_2(x) \subset V_2$. The orbit of x accumulates on points in both α_1 and α_2, It is therefore possible to find an infinite sequence $\{t_i\}$ with $t_i \to -\infty$ for which $\psi_x(t_i) \in K = V - V_1 \cup V_2$. By compactness, we can find a convergent subsequence of $\{t_i\}$ with limit point y lying outside of $V_1 \cup V_2$. Since $y \in \alpha(x)$, by definition, we have a contradiction. □

Theorem A.1 essentially means that if a solution $\psi_x(t)$ is confined to a compact region there exists some invariant set whose solutions approximate $\psi_x(t)$ in the limit as $t \to -\infty$. Obviously, the analogous result also holds for future asymptotic behavior. The following is an immediate corollary of Theorem A.1.

Corollary A.2 *If $V \subset U$ is a compact past invariant set then properties i-iv of TheoremA.1 hold for all $x \in V$.*

A number of important results in dynamical systems theory relate to the behavior of the flow in the neighbourhood of equilibrium points. Suppose we have a fixed point x_0 so that $f(x_0) = 0$ and assume f is C^r with $r \geq 2$. Close to x_0, we may approximate f, to first order, by the linear function $f_L = Df(x_0)(x - x_0)$, where $Df = [\frac{\partial f_i}{\partial f_j}]$ is the total derivative of f. This suggests that the behavior of the flow near x_0 may be approximated by the solutions to the linear dynamical system

$$\frac{d(x - x_0)}{dt} = Df(x_0)(x - x_0) \tag{A.4}$$

which may by integrated exactly to obtain the *linearized flow*

$$\Psi_t^L(x - x_0) = (x - x_0)e^{Df(x_0)t} \tag{A.5}$$

The behavior of the linearized flow is characterized by the eigenvalues and eigenvectors of the matrix $Df(x_0)$. Define the vector spaces E^u, E^s and E^c as the spans of the eigenvectors whose eigenvalues have positive, negative and zero real part respectively.(Note that the union of these vector spaces span R^n). We may identify each of these sets with the hyperplanes intersecting x_0 whose tangent spaces correspond to the respective eigenspaces. It is then simple to demonstrate that the hyperplanes E^u, E^s and E^c are invariant under the linearized flow. We label them the unstable, stable and center subspaces of the system respectively. Points on the unstable subspace are exponentially repelled from x_0 under the linear flow. Points on the stable subspace are exponentially attracted to x_0. The center subspace contains all closed orbits of the linear flow.

We say that an equilibrium point is *hyperbolic* if the matrix $Df(x_0)$ possesses no zero or purely imaginary eigenvalues. It turns out that the condition that the linear flow adequately characterizes the nonlinear flow near x_0 is that it be hyperbolic. We state below two important theorems on the geometry of hyperbolic equilibrium points. The proofs may be found in Hartman [79].

Theorem A.3 (Hartman-Grobman Theorem) *If x_0 is a hyperbolic equilibrium point then there exists a homeomorphism h defined on some neighbourhood $B(x_0)$ of x_0 which takes orbits of Ψ_t to orbits of the linearized flow Ψ_t^L. h preserves the sense of orbits and parameterization by time.*

Before stating the second theorem we make a couple of definitions. Let $M^s(x_0)$ be a positively invariant manifold defined in a neighbourhood of a fixed point x_0. We say $M^s(x_0)$ is a local *stable manifold* of x_0 if for all $x \in M^s(x_0)$ we have $\lim_{t\to\infty} \psi_x(t) = x_0$. Similarly, we say that a negatively invariant set $M^u(x_0)$ is a local *unstable manifold* if for all $x \in M^s(x_0)$ we have $\lim_{t\to-\infty} \psi_x(t) = x_0$.

It is easily verified that the stable and unstable manifolds of the linear flow Ψ_t^L the hyperplanes E^u and E^s respectively.

Theorem A.4 (Stable Manifold Theorem) *Let f be C^r with $r \geq 2$ and let x_0 be a hyperbolic equilibrium point of (A.1). Let E^s and E^u have dimension k and m respectively (note that $k + m = n$ where $U \subset R^n$). Then there exist local C^r stable and unstable manifolds, $M^s(x_0)$ and $M^u(x_0)$ of dimension k and m respectively, for the nonlinear flow Ψ_t. Furthermore, $M^s(x_0)$ and $M^u(x_0)$ are tangent to E^s and E^u respectively at the equilibrium point x_0.*

The upshot of Theorems A.3 and A.4 is that it is safe to linearize the system asymptotically close to hyperbolic equilibrium points. In particular, Theorem A.3 implies that the stability (or instability) of x_0 is the same for the linear and nonlinear flow. We now introduce a more precise classification of the stability of a fixed point.

We say that a fixed point x_0 is a *source* if it possesses an n-dimensional unstable manifold which contains some neighbourhood of x_0. A hyperbolic fixed point is a source iff each of the eigenvalues of $Df(x_0)$ have positive real part.

We say that x_0 is a *sink* if it possesses an n-dimensional stable manifold which contains some neighbourhood of x_0. A hyperbolic fixed point is a sink if each of the eigenvalues of $Df(x_0)$ have negative real part.

We say that x_0 is a *saddle* if it possesses an m-dimensional stable manifold $M^s(x_0)$ and an $(n-m)$-dimensional unstable manifold $M^u(x_0)$ with $1 \leq m < n$ and, in addition, there exists a neighbourhood $B(x_0)$ such that for all $x \in B(x_0) - M^s(x_0) \cup M^u(x_0)$ there exists a $T > 0$ such that $\psi_x(t) \notin B(x_0)$

for all $t \notin (-T, T)$. A hyperbolic fixed point is a saddle whenever $Df(x_0)$ has both positive and negative eigenvalues.

Non-hyperbolic equilibrium points may or may not fall into one of the above stability classes. It will be useful to define a weaker notion of stability. We shall say an equilibrium point is ***negatively stable*** if for any neighbourhood $B(x_0)$ there exists a neighbourhood $B_1(x_0) \subset B(x_0)$ such that for all $x \in B_1(x_0)$, $O^-(x) \subset B(x_0)$. Positively stable solutions are defined analogously. Conversely x_0 is negatively (positively) ***unstable*** if there exists a neighbourhood $B(x_0)$ such that the set of points whose past (future) orbits are contained in $B(x_0)$ has Lebesgue measure zero. An unstable equilibrium is both positively and negatively unstable. Sources and sinks are examples of negatively and positively stable points respectively. Saddles are unstable.

When the matrix $Df(x_0)$ has eigenvalues with zero real part the linearized flow fails to adequately characterize the asymptotic behavior. Nevertheless, certain aspects of the linear theory do carry over to the nonlinear case. We shall state below a number of important theorems which summarize the important asymptotic features of these systems. The proofs may be found in [80], also see [81] for a more transparent discussion with partial proofs. We first define a local center manifold.

If there exists a local invariant manifold for the non-linear flow, tangent to the center subspace E^c at the fixed point x_0, we call it a local *center manifold* of x_0 and denote it $M^c(x_0)$. The following theorem generalizes Theorem A.4 to non-hyperbolic cases and asserts the existence of a center manifold in these cases.

Theorem A.5 (The Center Manifold Theorem) *Let f be C^r with $r \geq 2$ and let x_0 be an equilibrium point of (A.1). Let E^s, E^u and E^c have dimension k, m and s respectively. Then there exist local C^r stable, unstable and center manifolds, $M^s(x_0)$, $M^u(x_0)$ and $M^c(x_0)$ tangent to E^s, E^u and E^c respectively at x_0. Furthermore, $M^s(x_0)$ and $M^u(x_0)$ are unique (but $M^c(x_0)$ need not be).*

The importance of center manifolds, and the reason that the linear approximation is inadequate for their study, lies in the fact that, whilst the asymptotic behavior on $M^u(x_0)$ and $M^s(x_0)$ is exponential, the dynamics on the center manifold is characterized by the nonlinear terms in f and is therefore much slower than exponential. (in fact a center manifold can be defined as the set of all points whose trajectories are exponential dominated in both forward and backward time [80]). The consequence of this is that, if x_0 is positively (or negatively) stable, typical solutions rapidly decay onto (or off of) the center manifold, so that the long term behavior of the system is characterized entirely by the behavior on the center manifold which is manifestly non-linear. We shall make this idea more precise in a theorem below.

We assume below that all non-zero real eigenvalues are positive, the case for negative eigenvalues follows directly by reversing the sense of time.

Let the linear subspaces E^u and E^c (which span R^n in this case) have dimension m and s respectively, with $m + s = n$. It is convenient to choose coordinates for which x_0 lies at the origin and E^u and E^c are tangent to respective coordinate axes.

The matrix $Df(x_0)$ takes the form

$$Df(0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A is an $s \times s$ matrix having eigenvalues with zero real parts, B is an $m \times m$ matrix having eigenvalues with positive real part. The dynamical system (A.1) takes the form

$$\begin{aligned} \frac{dx}{dt} &= Ax + g_1(x,y) \\ \frac{dy}{dt} &= Bx + g_2(x,y), \end{aligned} \tag{A.6}$$

where $(x, y) \in R^s \times R^m$ (ie we identify x and y with points on the center and unstable subspaces respectively). $g_1(x,y)$ and $g_2(x,y)$ are C^r functions satisfying

$$g_2(x,y) = O(\|(x,y)\|^2) \qquad g_1(x,y) = O(\|(x,y)\|^2) \tag{A.7}$$

Since the center manifold is C^r, $r \geq 2$, and tangent to E^c at the origin it may be represented locally as a graph;

$$M^c(0) = \{(x,y) : y = h(x), h(0) = 0, Dh(0) = 0\} \tag{A.8}$$

On the center manifold the system may be reduced to an s dimensional dynamical system as follows:

Theorem A.6 (Dynamics on the center manifold.) *The dynamics of (A.6) restricted to the center manifold is, for x sufficiently small, given by the s dimensional dynamical system*

$$\frac{dx}{dt} = Ax + g_1(x, h(x)). \tag{A.9}$$

The next theorem tells us that the solutions close to the center manifold are repelled as $t \to \infty$ unless they actually lie on the center manifold.

Theorem A.7 *There exists a neighbourhood $B(0)$ of the origin such that if $p \in B(0)$ and $O^+(p) \subset B(0)$ then $p \in M_c(0)$.*

Remarks:

1) It is clear that the point p described in Theorem A.7 must lie on *all* center manifolds of the origin. An immediate consequence of this is the following: If the origin is a positively stable fixed point of the reduced dimension system A.9 the center manifold must be unique.

2) Taking the time reverse of this theorem implies that, in the case where B has all negative eigenvalues (rather than all positive), any solution which is *past* asymptotic to the fixed point at the origin must lie on all center manifolds.

Another important result on the relationship between the asymptotic behavior of the reduced dimensional system and the full dynamical system is the following:

Theorem A.8 (Reduction of Dimensions Principle) *i) If the zero solution for the dynamical system (A.9) is negatively stable (negatively unstable) then the zero solution for the full dynamical system (A.6) is negatively stable (unstable).*

ii) In particular, if the zero solution of (A.9) is negatively stable there exists neighbourhoods $U, V \subset R^s \times R^m$ of the origin and a constant $\gamma > 0$ which depends only on the matrix B such that, for all $(u, z_0) \in U$, there exists a unique $p = (x_0, y_0) \in V$ satisfying $y_0 - h(x_0) = z_0$ such that the solution $\psi_p(t)$ of (A.6) with $\psi_p(0) = p$ may be written

$$x(t) = \phi_u(t) + O\left(e^{\gamma t}\right), \tag{A.10}$$

$$y(t) = h(\phi_u(t)) + O\left(e^{\gamma t}\right), \tag{A.11}$$

where $\phi(t)$ is the unique solution of the reduced system (A.9) with $\phi(0) = u$.

iii) furthermore, the map $S : U \to V$, $S(u, z_0) = (x_0, y_0)$ defined in ii is continuous 1-1 and (for an appropriate choice of V) onto.

Remarks:

1) The above theorem is an extremely powerful result. It tells us that whenever the origin is a negatively stable fixed point of (A.9) the past asymptotic behavior of the full system (for initial conditions close to the origin) is completely determined by the dynamical system (A.9). In particular, as the system evolves backwards in time, each solution approaches a unique solution on the center manifold exponentially rapidly.

2) Conversely, each solution on the center manifold has a continuous family of solutions of (A.6) which it approximates asymptotically. In fact it can be shown that for a given $p \in M_c(0)$ the set of initial conditions whose solutions approach $\psi_p(t)$ is actually a C^r curve in R^n intersecting $M_c(0)$ at p.

3) An important point to note is that if we consider initial conditions at a fixed distance z_0 from the center manifold then (locally) there is a continuous 1-1 map between x_0 and the initial point u on the center manifold.

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Page 83, the right hand side of equation (3.2c) should be $\sqrt{\frac{2}{3}}y$.

Page 105, line 7, "Jacob" should be replaced with "Jacobs".

Page 106, equation (4.27b) should read

$$\frac{dy}{d\tau} = -\sqrt{\frac{3}{2}}x^2V'(\phi) - 3x^2yV(\phi).$$

Page 107, the missing entry in the middle column of the top row of the matrix on the right hand side of equation (4.29) is "0".

Page 114, lines 8, 10 replace "parameterize" with "parametrize". Similarly, it should be " parametrization" on line 14. Similarly " reparameterization" should be replaced with "reparametrization" on **Page 130** line 2.

Page 128, line 14, replace "Lifshitz an Khalatnikov" with "Lifschitz and Khalatnikov". On line 15 replace "Eardly" with "Eardley". This error also occurs on **Page 129** line 15, **Page 138** 8^{th} line from bottom and on **Page 176** reference [67].

Page 136, line 14, there should be a comma following "functions".

Page 164, 4^{th} line from bottom, replace "Where" with "where".

Page 174. In Reference [22] replace "ans" with "and". In references [25] and [26] "Sikilos" should be "Siklos". In reference [28] "Kalantikov" should be "Khalatnikov".

Page 175, references [39] and [41], replace "Goldsworth" with "Goldworth".

Page 176, reference [53], delete "bf". Also, "26" should be in bold print.

Errata

Page 9, Equation (1.25) should read

$$\dot{U}^{\nu}_{E} = U^{\nu}_{E;\rho} U^{\rho}_{E}.$$

Page 17, "reimannian" should be replaced by "Riemannian" in the line following equation (1.51).

Page 18, The sentence following equation (1.56) should read "Thus space-time exponentially approaches de Sitter space."

Page 19, in the last sentence of paragraph 2 "straightforward" is incorrectly spelt "straitforward". This error is repeated on **Page 74** line 15, **Page 80** line 7 from the bottom and **Page 160** line 20.

Page 22, line 2, "unequivicaly" should be replaced by "unequivocally".

Page 28, the last equation on the page should read

$$V(\phi) = \frac{1}{48}\left(\frac{3}{2}\phi^2 - 1\right)^2.$$

Page 33, lines 9, 13 and the bottom line, "separatrices" is incorrectly spelt "seperatrices". This error is repeated on **Page 75** lines 1 and 10 and **Page 87** line 14.

Page 38, the second sentence should begin "Eliminating V from (2.1c) using (2.2) gives..."

Page 42, the line above equation (2.19) should read "which can be integrated exactly to obtain the set of unphysical orbits". In the line 2 below equation (2.19) "pair orbits" should be replaced by "pair of orbits".

Page 45, 7^{th} line from bottom, replace "theses" with "these".

Page 49, 7^{th} line from the bottom, replace "criterium" with "criterion".

Page 57, the last sentence of paragraph 3 should read "$\partial\Omega_{\epsilon 4}$ is a center manifold of p_+ and $\partial\Omega_{\epsilon 3}$ is a center manifold of p_-".

Page 62, 4^{th} line above Theorem 2.8, replace "(2.107)" with "(2.64)".

Page 82, the right hand side of the second equation should read 0 not ∞.