Closed Set Logic in Categories

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Abstract

In this work we investigate two related aspects of a dualisation program for the usual intuitionist logic in categories. The dualisation program has as its end the presentation of closed set, or paraconsistent, logic in place of the usual open set, or intuitionist, logic found in association with toposes. We address ourselves particularly to Brouwerian algebras in categories as the duals of the usual Heyting algebras. The first aspect of the program is that of external or ex-categorial dualisation of logic structures by interpretation of order. This appears in the work as an examination of the notion of a complement classifier. We also use ex-categorial dualisation as a tool to prompt the development of a categorial proof and model theory adequate to the task of modelling theories generated by inconsistency tolerant logics. We make an initial attempt to develop dual logic structures by considering quotient object classifiers in place of subobject classifiers. Ex-categorial dualisation of structure was always meant to act as an indication of the existence of categorial entities that directly satisfy dual descriptions, so the bulk of the work is concerned with the second aspect of the dualisation program: the discovery of logic objects within categories that exhibit paraconsistent algebras in their own right. Our investigation focuses on sheaves for their algebraic properties in relation to base space topologies. We define the notion of a sheaf over the closed sets of a topological space. We find essentially two things. First, logic objects in contravariant sheaf categories contain component Brouwerian algebras but are not generally themselves Brouwerian algebras within their categories. A corollary is that subobject lattices in Grothendieck toposes are Brouwerian algebras (but not naturally so). Second, paraconsistent logic objects do exist. We describe one such within a category of covariant sheaves. As a corollary we find that the original ex-categorial dualisation idea represented by the notion of a complement classifier has an instantiation in categories. Our paraconsistent logic object proves to be the object of a genuine complement classifier.

Statement of Originality and Consent

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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Introduction

When a set of sentences is closed under some consequence relation and under uniform substitution of sentences for atomic sentences we have a sentential logic. A paraconsistent logic is one which allows that sets of sentences contain a sentence and its negation and be closed with respect to the logic's consequence relation without containing every other sentence. The logic is said to tolerate inconsistency. It is rarely remarked that closed set topologies form algebras for exactly this sort of logic while it is well known that their duals, the open set topologies, form the algebras for the logics of Intuitionism. In as much as it is exactly the Intuitionism algebras that are known to occur in and around topos theory it is perhaps surprising that category theory, with its awareness of duality, should have so little to note on the topic of paraconsistency. It is at least true that inconsistency toleration is exactly the right sort of notion to use in the development of any type of machine that requires input so with the emphasis of category theory tending toward useful applications, particularly those computational, it is appropriate that we investigate the underdeveloped area of closed set logic within categories.

The project of this thesis began with the study of closed set topologies as algebras for paraconsistent logics. These were to be developed as the duals of the Intuitionist logics. The background assumption for the usual formalisation of Intuitionism is that any sentence is interpreted on some open set of a topological space. For a sentence S, then, the negated sentence $\neg S$ is the largest open set for which $S \cap \neg S$ is empty. The dual position assumes that any sentence is interpreted on some closed set. We can then interpret a set $\ulcorner S$ in relation to a sentence S by allowing $\ulcorner S$ to be the smallest closed set for which $S \cup \ulcorner S$ contains every other closed set. The operators $\neg$ and $\ulcorner$ are then formally dual; among other theorems we will have that $S \cap \ulcorner S$ need not be empty. This is to say that sentences S and $\ulcorner S$ are such that they cannot both be false but that (given a big enough set of designated values) they may both be true. The background assumption that any sentence is valued on a closed set allows us to avoid the suggestion that $\ulcorner$ is a subcontrary operator rather than negation operator. It follows that the logics that arise as the duals of intuitionist logics are genuinely paraconsistent. With the introduction of Mortensen and Lavers' complement classifiers and complement toposes the idea of dualising logics was linked formally with category and topos theory. The structures that most obviously linked Heyting algebras (and so Intuitionism) and topos theory were the sheaves defined over open set topologies. And so arose the idea of investigating the effect on categorial logic of defining sheaves over topologies of closed sets. The overall aim was and remains one of dualising the logics built into the structures of toposes and categories.

The importance of paraconsistent versions of categorial logic is in terms of categories as semantic objects for assignment functions that determine inconsistent theories: the logic of the category provides the deduction relation under which the modelled theory is closed; inconsistent theories require (and are generated by) a paraconsistent deduction relation. Now clearly, categories are not the only semantic objects that we might use in inconsistency model theory. Equally clearly there has been little or no work on this type of model theory done for categories. Furthermore, category theory is an important mathematical discipline; if we regard paraconsistent logic as significant, then investigation of its place in category theory is mandated. Some note of it has already been made. In the introduction to Lawvere and Schanuel's *Categories in Continuum Physics*, 1986, Lawvere notes in connection with sheaves and categories that a property of complements in algebras of closed sets is that the intersection of a closed set and its complement will not invariably be the least element of the algebra; in other words, using closed set lattices as logical algebras produces logics in which a formula and its negation have, as a rule, truth values with non-zero intersections.

Just above I claimed that paraconsistent logic allows for the existence of inconsistent theories. The idea is this: a theory is a set of sentences closed under a deduction relation; if the deduction relation is paraconsistent, then the presence of inconsistent sentences within the theory need not mean that the theory contain all other sentences and be rendered trivial. We may think of paraconsistent deduction as one that limits the (deductive) impact of contradiction. This is different from being happy to have one's theories loaded with inconsistencies. The paraconsistent logics are, in this light, a way of dealing with problems that arise from otherwise good mathematical and philosophical ideas. A good example is the case of a set theory that adopts unrestricted set abstraction, that is, allows that for any property there is a set of things with that property. Famously, this leads to the existence of paradoxical sets, notably the Russell set, the set that both is and is not a member of itself. But set abstraction is a valuable device so, if there is no theoretically acceptable restriction that we can place on its use, we must tolerate the paradoxical sets; the theory that contains unrestricted set abstraction requires a background logic that is paraconsistent; in that way the details of the ordinary, non-paradoxical sets are not lost in a flood of trivial sentences flowing from the contradiction of, say, the Russell set (provided those ordinary sets are not deductively related to that Russell set). Set theory, in practice, is workable but this is due to the imposition of restrictions that are not in themselves valuable for more than denying the existence of sets like Russell's. Many writers have noted that this seems too ad hoc a solution. And any discussion that leaves us with unrestricted set abstraction also forces on us paraconsistent logics defined as logics that have associated theories that are inconsistent but non-trivial.

The idea that we may be required to accept inconsistency toleration is not necessarily a "lesser of two evils" conclusion. The idea that we must require consistency when we talk about mathematics need not be true; that is, it might just be the case that our meta-language has to be inconsistent. Consider an example not from mathematics but from ordinary (English) language: "This sentence is false". This sentence provides us with a version of the famous Liar paradox: if the sentence is true, then it is false, and if it is false, then it is true. Priest (*In Contradiction*, 1987) and Priest and Routley (*On Paraconsistency*, 1984) tell us that this sentence generates a true contradiction by having semantic conditions that overdetermine its truth value. The sentence has a subject, "This sentence", and a predicate, "is false". In semantic terms the sentence is true if its predicate applies truly to its subject. But just as in the case of the unrestricted set abstraction, this truth making principle (in conjunction with ordinary sentence forming principles) is too strong, it generates contradictory truth values for some sentences. And in the case of this principle there are even fewer satisfactory methods for restricting its use. That the Liar paradox exists is an argument for the relative messiness of language; natural language semantics contains principles that are inconsistent. It may be possible to cure this inconsistency, perhaps by discovering some theoretically adequate restriction on the relevant principles or by reinventing semantics itself, but in any case since there exists inconsistent principles in operation at present and in the foreseeable future, we are called upon to make some philosophical comment. Paraconsistent logic fills the void.

Paraconsistent logics also serve an epistemic purpose. Thinking machines need a method for dealing with inconsistent data. The logical issuses generated by notions of inconsistent databases and decision making on the basis of such databases were discussed notably in Belnap's "A useful four-valued logic", 1977. The issue essentially resides in the question "What is to be done when our thinking machine discovers that it has inconsistent data but still must think?" The problem of inconsistent data is ubiquitous; multiple sources of information can individually be consistent but together be inconsistent; even the idea of a single inconsistent source of information is not unusual. We are called upon to form a method for dealing with such problems, particularly if we are in the business of designing simulations of reasoning in dynamic environments. Plainly a good solution is to be able to mark a datum as "told both" or something of the sort rather than only one of either "true" or "false". But this is inconsistency toleration. It seems, however, preferable to some solution that makes ad hoc choices over which of the contradictory data are true and which are then false.

The notion of containing contradiction is valuable. The fact that it is formally possible offers us the opportunity of understanding old problems in new ways.

There are some things to be said on the nature of the project of this thesis with respect to principal content. It may be suggested that in essence, since the focus is upon algebras, that this is a dissertation on sentential logic with respect to poset theory. In answer to this I suggest that formally speaking the models for the logics considered are indeed posets but that the project came into being by considering the duality of open and closed sets in topology; in the terms of the project the models are topological spaces, and we lose some part of the philosophical content of the thesis if we speak only of posets. There are two questions to address with respect to the suggestion that the thesis is more properly located in poset theory than in topos theory: the first is the straightforward one and is why, if the subject is sentential logic, get involved with topos theory at all, why not just content oneself with posets, or even topological spaces; the second would be why, if the subject is sentential logic, invoke topos theory which is known to provide for formally richer logics, namely quantified logics. I consider both of these questions in what follows.

As to the first question, the issue of invoking topos theory is not one of advantage. The thesis, in major part, is the working out of a hypothesis that topological dualisation of (pre)sheaves, the replacement of closed set notions for open set notions in the definition of the (pre)sheaf notion, will produce structures that can be collected into a category and that that category will exhibit a logic which is paraconsistent, which is to say, dual to the usual topos logics. This hypothesis is first seen in the concluding remarks to chapter 11 of Mortensen's *Inconsistent Mathematics*, 1995. The motivation for an examination of this hypothesis was the expected outcome for the modelling of inconsistent theories on categories. As Mortensen puts it in *Inconsistent Mathematics*, "...deductive theories come with a logic in the background" (p.1); and when the theories are generated by notions of modelling on categories, the background logic is that of the category, which is to say, that of the sets of subobjects of the category. Should the hypothesis have borne the expected fruit, a topos like category with subobject lattices that were Brouwerian algebras, then we would have (the basis of) a theory of categorial semantics for paraconsistent logic and inconsistent theories. The issue is not that the tools of topos theory would act as some aid in the demonstration or otherwise of the hypothesis, it is that the hypothesis was one about category theory. The project of category theory itself is to provide insight into the nature of mathematics essentially by doing it over in a new setting, one with a greater awareness of generalisation and structural issues. To redo inconsistency model theory within toposes required something of mathematics that may or may not be there. That is, it is relatively clear that the notions of "proof only through constructive methods" that come from the Intuitionist position constitute a (reasonably) minimal description of what mathematics and logic are capable of doing. The notions of inconsistency toleration and containment are reasonably likely, if anything, to be part of a sort of maximal description of the reach of mathematics and logic. I'm claiming here that Intuitionism and Paraconsistentism, as philosophical positions, are dual (in a non-technical sense which is nevertheless related to the actual duality of the formal developments of the positions). This is a much grander idea than anything attempted in the thesis; the thesis is a starting point, more technical than philosophical in bent but meant to provide facts with which the philosophical duality idea could be considered anew.

As to the second question, why bother with a theory that provides for quantified logics if the subject matter is only sentential logic, it follows from an understanding of the project as the working out of a hypothesis related, at heart, to the duality of open and closed sets that quantification, while interesting in itself, is a side issue. The concern of the thesis is to develop formal structures for paraconsistent logic exactly by "dualising" existing structures for the logic of Intuitionism in categories. Notice an important point: there are at least three separate formal notions of dualisation (as opposed to duality) at work in the thesis. These are (1) standard categorial dualisation, the replacement of primitive categorial terms by their duals in statements that describe categorial structures; (2) lattice dualisation, the replacement of lattice notions of order with their duals; and (3) topological dualisation, the replacement of topological set notions with their duals. Along with these notions of dualisation there is a notion of the dualisation of the logic structure of a category which means the replacement of Heyting algebra subobject lattices and classifier objects with their "duals", Brouwerian algebra subobject lattices and classifier objects. This is what is meant when I claim that the concern of the thesis is the production by dualisation of categories with paraconsistent logics. This "dualisation" of the logic structures of categories is meant to be effected by some act or combination of acts of the three formal notions of dualisation within the thesis. So we have two points to make: (1) the subject matter of the thesis is category theory rather than sentential logic; and (2) the particular principal concern is with dualisation of logics with the raw material being structures that exhibit Heyting algebras. The project takes on the appearance of a sentential logic treatise since the distinction between intuitionist and paraconsistent logic appears at the propositional level. I claim that it is not so much that I have ignored the quantificational possibilities of topos theory nor so much that I have used too strong a device by invoking topos theory, but that my subject matter, in essence, is Heyting algebras and category theory (and, of course, dualisation of logics) and therefore, topos theory.

The notion of a topos logic as we use it in this work requires some explanation. The usual notion of a topos logic comes from the idea that we may use toposes as semantic objects; this is the idea that we can use the internal structure of toposes to interpret formal languages. In these terms, topos logic is the set of rules of inference that the structure of a topos will support. We consider such systems in chapter fourteen. However, for the bulk of this work we concern ourselves with the structure of subobject classifiers and when we speak of topos logic we will be referring to the internal algebras that arise with respect to these classifiers. Under the usual scheme logical connectives are interpreted with respect to subobject algebras, so there is a measure of justification for our minor misuse of the term "topos logic". We should recognise, too, the difference between Intuitionism, the position on the epistemology of mathematics, and Intuitionistic logic, the logic formalised in terms of Heyting algebras. Generally, whenever we use the word "Intuitionism" and its variants, we will mean Heyting algebra logic.

Now, there are some things to be said of the project of this thesis with respect to method. The various dualisation techniques at the heart of this thesis are mathematically simple. The claim is, however, that, simple or not, these techniques and the working out of the consequences of their use provide some philosophically important insight. To back up this claim we ask the following question: When is it that a mathematically simple technique can give a philosophically important perspective? In answer we say that the technique must lay open an area of mathematics to discussion in terms of a new set of notions that are themselves philosophically significant. An example of a mathematically simple notion is that of the duality between open and closed topological sets. This notion is the basis of a mathematically simple technique: topological dualisation, the replacement of topological set notions in the definition of a mathematical structure with the dual topological set notions. The duality of open and closed sets is philosophically significant for example in the light of two rival empirical hypotheses about the world from the point of view of physics, namely (1) that propositions are only ever true on open sets of points, and (2) that propositions are only ever true on closed sets of points. These hypotheses go to the issue of how we are to think of our claims in physics applying to the world. Examples of how these hypotheses can be understood to come to hold lie in possible claims like one that, from the point of view of the physics of dynamical systems, subparts (that is, sub-bodies) of any body are sets of points of that body and the set of subparts is (isomorphic to) some topological space. This sort of claim is mentioned in the introduction to Lawvere and Schanuel [1986]. This idea of the significance of open and closed set duality deriving from the existence of these rival hypotheses need not be restricted to the realm of physics. The forms of hypotheses (1) and (2) apply to any area where there is a notion of one type of thing under discussion and a notion of classes and subclasses of things of that type. Now, open and closed set duality being philosophically significant tends to suggest that topological dualisation will provide philosophical insight should, say, hypothesis(2) be true and we have before us a mathematics that relies on open set structures to describe the world. In any case, allowing that hypotheses (1) and (2) are meaningful suggests that any differences between the logic of open sets and the logic of closed sets (for example, the differing accounts of negation) are philosophically significant. It follows that differences in useful mathematics brought about by topological dualisation have a philosophical significance. It follows too that paraconsistentists have an interest in open-closed dualities since closed set logic is paraconsistent. And it follows that open-closed duality notions are central to the project of my thesis.

A further example of a mathematically simple but philosophically valuable technique is that of lattice dualisation of Heyting algebras. The technique is simple: it can be performed by replacing "less than or equal to" with "greater than or equal to" (and "greatest lower bound" with "least upper bound" and vice versa, but lub and glb are order dependent concepts and can be regarded as dualised when the order is dualised). And the technique, while simple, is significant: it produces algebras that, when taken seriously as logical algebras, produce paraconsistent logics. This is significant in the terms of the philosophical significance of open-closed dualities since an open set topology ordered by set inclusion is a Heyting algebra and a closed set topology ordered by set inclusion is a Heyting algebra dual. Heyting algebra duals were named Brouwerian algebras by McKinsey and Tarski in their "On closed elements in closure algebras", 1946. McKinsey and Tarski did see that the lattice dualisation notion was useful for developing the properties of the algebras but did not see the significance with respect to logic: McKinsey and Tarski allowed that Brouwerian algebras were algebras for the same logics that were found associated with Heyting algebras; that is, McKinsey and Tarski dualised the theoremhood semantics as well as the algebras rather than develop the logics arising from Heyting algebra lattice-duals together with standard theoremhood semantics. I note this in the thesis in chapter 3 when discussing the significance of Brouwerian algebras as productive of paraconsistent logics.

Notions of open-closed dualities are central elements of the project of my thesis. The aim of the project was to find ways to exhibit Brouwerian algebra structures within, or at least for, categories. There are two strands to the origin of this aim. The first strand consisted in the simple fact that sheaves are defined in terms of topologies. The second strand consisted in the well known fact that toposes carry Heyting algebra structures. The two strands come together in the fact that categories of sheaves are toposes. Another way of stating the project aim, then, is that I was investigating the possibility of effecting some kind of dualisation for the logic of toposes by performing some version of a "closed" for "open" swap. In the case of the sheaf categories the swap was literally that, a topological dualisation of the sheaf notion. In the case of the complement classifier discussion the swap in chapters 11 and 12 was not topological but categorial, however the result was discussion of structures defined on closed sets rather than open. A feature of the discussion in chapters 3, 4, 5, 11, and 12 was explicit proof of dual statements of (more or less) familiar facts and theorems of category theory. In all cases this reasoning technique was a tool to further basic discussion. This tool is technically very simple but the details it revealled needed interpretation in the philosophical terms of the thesis, the open-closed dualities.

All of these techniques, topological and lattice dualisation and the working through of explicit categorial dualisations, are simple but they work precisely because they are being applied in situations of relative complexity: a simple change at a fundamental level to a notion of a thing that stands in a relatively complex relationship to other known things can, since the external relationships are (presumably) affected, lead us to understand the changed notion as that of a thing that is quite new. To make this valuable we need to have some new framework of ideas into which the changed thing can be fitted. If we have no such new framework, then the changed thing is merely the thing changed. In terms of the content of the thesis the new framework of ideas are those of a program for the discovery of structures that act as semantic objects for assignment functions that determine inconsistent theories. Topological dualisation is easily understood as the effecting of "a simple change at a fundamental level" to the notion of a sheaf. Explicit categorial dualisation calls for a wider interpretation of the idea of effecting such a change. The principal "change" is merely a categorial dualisation. Now, it is entirely true to say that once a theorem in category theory is demonstrated, then so is its dual. However, what is not true is the idea that once a theorem is demonstrated, we understand the philosophical nature of the structures associated with the dual of the original theorem; that idea amounts to the claim that once a mathematical theorem is demonstrated we have a philosophical understanding of the importance of its content (that is, for example, once we demonstrate that a Heyting algebra exists we suddenly know why we should care that it exists, we are suddenly struck by the worth of the Intuitionist program). But on the other hand philosophical understanding of the importance of a structure is undoubtedly shaped by demonstration of mathematical detail. For these reasons there is merit in explicit demonstration of dual claims to familiar theorems when it is used to provide detail for novel philosophical developments, just as I assert is the intent of chapters 3, 4, 5, and 15. Chapters 11 and 12 can also be understood as being of this nature.

We have considered the philosophical merit of the nature of the project and now we should consider the question of the philosophical merit of the particular content of this thesis. In other words, does the content of the thesis do justice to the aspirations of the thesis? Under the terms of the thesis we are broadly engaged in the task of developing paraconsistent logic within category theory. What in fact we address ourselves to is the existence of Brouwerian algebras in the subobject structure of toposes. Now it is clearly true that the philosophical merit of a concept, say, paraconsistency, is not enough to establish the philosophical merit of a given formal model for that concept, say, Brouwerian algebras in the subobject structure of some category; further argument is needed. We note then that understanding Brouwerian algebras to be algebras for paraconsistent logic is relatively novel in category theory. We have the Mortensen and Lavers discussion in Mortensen's *Inconsistent Mathematics*. Also, as I noted above, Lawvere is aware of the logical implications of using lattices of closed sets as logical algebras. But these authors seem to be largely alone in this area, or at least, largely alone in their interest in closed set logic as something significant in category theory. Goodman ("The logic of contradictions", 1981) and numbers of other authors are aware of the nature of the logic of closed sets in relation to the logic of open sets, however it seems that only Lawvere and, independently, Mortensen and Lavers have discussed this in the context of category theory. We should also note that the actual development of structures within category theory as algebras for paraconsistent logic is extremely novel. So, the notion of a Brouwerian algebra is far from new but what is new is the idea that they should appear in categories (Lawvere's prior discoveries in Springer Verlag 1488 now acknowledged but at the time unknown to me). This makes for an argument for the worth of finding any examples of Brouwerian algebras in categories, and indeed for the worth of finding cases where the expected examples fail to exist.

On the other hand, there is some need to develop philosophical notions by seeing if there is technical room within the existing discipline. This, surely, is what gives the notion of a "contribution to learning" its meaning. Admitedly there is tension between the idea of existing theory being sacrosanct and the idea of new discovery, however how are we to know if existing theory needs an overhaul unless we check first for the workability within the existing scheme of our new ideas? This calls for an initial philosophical investment, but one, surely, that is modified as technical work progresses. The initial philosophical investment in the thesis is in the substance of notions of paraconsistency and of category theory. The technical investment is then the various investigations of the kinds of dualisations possible. So there is an argument to the effect that (1) category theory is important and has a known relationship to logic (viz. model theory with respect to subobject lattices), (2) paraconsistency theory is important, (3) formalisations of paraconsistent logics arrive most expiditiously by lattice dualisation of formalisations for Intutionist logics, therefore (4) seek out Brouwerian algebras in categories in the terms of the usual logic structures known in category theory (viz. subobject lattices). Plainly this is, as above, a further argument for philosophic relevance of these particular formal models, as required; but it exists along with the idea that there is a sense in which technical development modifies philosophical development so that there is at least two notions of the merit of a formal model: the one that gives us a reason to develop it and the one that is an assessment of its impact. This notion of impact is the sort of thing alluded earlier in this discussion were I suggested that the technical results of the thesis were a starting point from which we could consider anew the idea of Intuitionism and Paraconsistentism being philosophically dual. The results give us some context for the discussion of this philosophical duality just as, for example, the development of Heyting algebras provide a formal context for discussion of Intuitionism as a philosophical position. And in any case, surely the fact that there were negative results to be found (the principal ones being the failure of naturalness of the pseudo difference arrows and the failure of categorial dualisation to produce Brouwerian algebras) answers some part of any triviality claim since it demonstrates that not every dualisation results in an instantiation of the features that make the original topological open-closed dualities philosophically valuable.

Finally, a note on a second aspect of our project. This was the concern to formalise within category theory the ability to address an algebra by considering its dual. In intent this part of the project has much in common with the original complement classifier ideas. In terms of constructions we have chosen rather to build the idea of dualising logics into category theory in the same way that we can build in the idea of theories and models. Theories become categories, models become functors, and the ability to address dual logics arises as a language dualisation functor between theories and models. This ends up allowing us to use existing Heyting algebra structures within categories as though the algebras were the dual paraconsistent algebras.

A word on topologies and topological spaces and their properties as logical algebras. Topologies on a set X are collections of subsets of X satisfying certain properties. The set X, in recognition of the physical notion of topology, is usually called a *space*. An *open set topology* Θ for space X is a collection of subsets of X for which the intersection of any two members of Θ is a member of Θ and the union of any subfamily of Θ is a member of Θ, and as well both X and $\emptyset$ are in Θ. The sets of Θ are called *open sets* of X relative to Θ or just open sets. Notable topologies are the *indiscrete* or *trivial* topology that has only X and $\emptyset$ as members. There is also the *discrete* topology which has all subsets of X as members. The various topologies in between these extremes are identified relative to one another as coarser or finer. A topology Θ_1 is *coarser* than a topology Θ_2 if each open set of Θ_1 is an open set of Θ_2; and then, also, topology Θ_2 is said to be *finer* than Θ_1. Topologies can also be defined in terms of neighbourhood systems. A subset U of X is an *(open) neighbourhood* of a point $x \in X$ if U contains an open set V to which x belongs. A subset U is open relative to a topology iff it contains a neighbourhood for each of its points. Open set topologies have associated interior operators. A point x of a subset U of a topological space X is an *interior point* for U iff U is a neighbourhood of x. Thus we have the *interior of* U, denoted $\mathrm{I}(U)$, as the set of all interior points of U. $\mathrm{I}(U)$ turns out to be the largest open subset of U and U is an open set in a topology iff $\mathrm{I}(U) = U$. We speak of interior operations determining open set topologies. Any open set topology Θ on a space X, when ordered by set inclusion, is a Heyting algebra since there is a unit X and a zero $\emptyset$ and since for any $U, V \in \Theta$, we can define the characteristic operator $\Rightarrow$ by

$$U \Rightarrow V = \mathrm{I}\big((X - U) \cup V\big)$$

where I is the interior operator that determines the topology; alternatively and equivalently we can let $U \Rightarrow V$ be the greatest element of $\{W \in \Theta \colon U \cap W \subseteq V\}$.

A closed set topology Ξ on space X can be defined relative to some open set topology Θ or by a set of conditions dual to those that define open set topologies in general. In the first instance, a subset U of a space X is called *closed* iff $X - U$ is open in Θ. In the second instance we say that a set Ξ of subsets of X is a *closed set topology* for X if the union of any two members of Ξ is a member of Ξ and the intersection of any subfamily of Ξ is likewise a member, and as well both X and $\emptyset$ are in Ξ. The usual notions of indiscrete and discrete, finer and coarser apply and we can define a closed neighbourhood in the obvious way. However, where open sets have interior points, closed sets have accumulation points. A point x of a subset U of X is an *accumulation point* of U iff every neighbourhood of x contains points of U other than x. Accumulation points can be called *cluster* or *limit points.* A subset of a topological space is closed iff it contains the set of its accumulation points. Associated with any closed set topology is a closure operator cl where for any $U \subseteq X$, $cl(U)$ is the union of U with its set of accumulation points. A set U is closed relative to a topology iff $cl(U) = U$. We speak of closure operations determining closed set topologies. On any closed set topology Ξ we can define an operator $\dot{-}$ relative to set inclusion so that for any $U, V \in \Xi$,

$$V \dot{-} U = cl\big((X - U) \cap V\big).$$

Alternatively and equivalently let $V \dot{-} U$ be the least element of

$$\{W \in \Xi : U \cup W \subseteq V\}.$$

When Θ is an open set topology on X and Ξ is a closed set topology such that $U \in \Theta$ iff $X - U \in \Xi$, we have a duality relationship between operators $\Rightarrow$ and $\dot{-}$ following from the facts that

$$X - (U \Rightarrow V) = (X - V) \dot{-} (X - U)$$

and that lattices $(\Theta, \subseteq)$ and $(\Xi, \subseteq)$ are dual isomorphs in the sense that

$$U \subseteq V \quad \text{iff} \quad X - V \subseteq X - U.$$

In later chapters we shall formally identify the lattices associated with closed set topologies and characterised by the $\dot{-}$ operator as Brouwerian algebras. For the moment we point out that our interest in such lattices comes from the presence of the derived operator $\ulcorner$ which we name, for want of anything else, "paraconsistent negation". We say that for any U in a closed set topology on X, $\ulcorner U = X \dot{-} U$. This operator satisfies the characterisation of a paraconsistent negation described in Mortensen's *Inconsistent Mathematics* since for any U, V in a closed set topology,

$$U \cup V = X \quad \text{iff} \quad \ulcorner U \subseteq V.$$

Closed set topologies, then, are paraconsistent algebras. In fact they form a significant subclass of those paraconsistent algebras characterised by the existence of a $\ulcorner$ operator. Related to the existence of the $\ulcorner$ operator is the concept of a boundary of a set in a topology. The closure of a set U is in general bigger than U itself and we can describe an operator B by setting $B(U) = cl(U) - U$. Since in general a set U will have accumulation points x such that $x \notin U$, $B(U)$ is in general non-empty. Plainly then $cl(U) \cap cl(X - U)$ is also in general non-empty. This is what gives us our paraconsistent negation. Closed set sheaf categories become interesting now for the fact that the algebra of the base space topology becomes the algebra of the sheaf section structure. By hypothesis, then, collections of sheaf morphisms will reflect this algebra and produce morphism algebras with paraconsistent negations within categories of closed set sheaves. The hypothesis proved to be correct.

Our interest initially was in what is called a spatial topos. Where X is a topological space with topology Θ, the category of continuous local homeomorphisms over X with respect to Θ is a spatial topos. (Continuous local homeomorphisms are defined explicitly in chapters thirteen and seven). The continuous local homeomorphisms are otherwise called sheaf spaces and are characterised by a behaviour of sections condition. Firstly, where $p: A \to X$ is a continuous local homeomorphism with respect to topologies Θ' on A and Θ on X, a *section* of p is some continuous function $s: U \to A$ such that $U \in \Theta$ and $p \cdot s = id_U$. Then we find that wherever p is a sheaf space and $U \in \Theta$ is such that $U = \bigcup\{U_i: i \in I\}$ for a set of $U_i \in \Theta$, we have that if $\{s_i: i \in I\}$ is a set of sections of p over each U_i such that

$$s_i \mid U_i \cap U_j = s_j \mid U_i \cap U_j \quad \text{all } i, j \in I,$$

then there is exactly one section s over U such that

$$s \mid U_i = s_i \quad \text{all } i \in I.$$

Furthermore, as must no doubt be apparent, since sections are defined with respect to elements of the topology on X, algebras of sections are exactly algebras of the relative base space topologies. This turned out to be more compelling than expected as a reason to consider sheaf spaces over closed set topologies, since sections admitted an interpretation as global elements of a sheaf space within the category Top(X) of sheaf spaces over X, and furthermore Top(X) has a classifier object Ω whose global elements are exactly the "truth values" of the "logic" of the category. The hypothesis was that an adequate definition of a sheaf space over the closed sets of a topology would yield a topos whose logic was exactly that of the closed sets of the base space. For a measure of simplicity, we investigated this hypothesis in terms of sheaves. *Sheaves over a topology* Θ are contravariant functors $F: \Theta^{op} \to$ **Set** distinguished from other contravariant functors by exactly the property that defines sheaf spaces: if $U = \bigcup\{U_i : i \in I\}$ in Θ and $\{s_i \in F(U_i) : i \in I\}$ is such that

$$F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j) \quad \text{all } i, j \in I,$$

then there is exactly one $s \in F(U)$ such that

$$F^{U}_{U_i}(s) = s_i \quad \text{all } i \in I.$$

In this context any contravariant functor $\Theta^{op} \to \mathbf{Set}$ not otherwise identified is called a *presheaf.* As such, the sheaves are thought of as a subclass of the presheaves.

In the early stages investigation of closed set sheaf spaces was confused by a simple mistake in interpretation of the requirements of the behaviour of sections condition. Since we were considering closed set topologies we wondered where we would find enough arbitrary collections of closed sets whose union was in fact a closed set. In particular we wondered how we would discern those arbitrary collections whose union was a closed set from those whose union was not. This was a simple mistake since, as we have written here, the condition on the behaviour of sections is in conditional form. The condition applies only if a cover exists. There is no requirement that particular types of cover exist at all.

Discussion of the logical status and nature of classifier objects in presheaf and sheaf categories forms the bulk of this work. We also address ourselves to the question of the equivalence of closed set sheaf spaces and closed set sheaves, and to the logical nature of classifier objects in more general sheaf categories. All of this forms Part III of the present work.

At this stage it is important to point out that during the time of the development and of the writing of the material on classifier objects, it was understood to be of original content; but in fact in 1991 Lawvere reported the existence of his own considerably more general result. The relevant report is Lawvere, F.W., "Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes", in *Category Theory,* Springer Verlag Lecture Notes in Mathematics, 1488, pp.279-281. In the note cited Lawvere writes

> "In any presheaf topos (and more generally any essential subtopos of a presheaf topos), the lattice of all subobjects of any given object is another example of a co-Heyting algebra (as well as a Heyting algebra). The co-Heyting operations are in general not preserved by substitution (inverse image) along maps..." (Lawvere, 1991, p.280).

This covers the results in my chapter 6 on the non-natural transformation $\{\dot{-}_p : p \in \mathbf{P}\}$ for any category $\mathbf{Set}^{\mathbf{P}}$ where $\mathbf{P}$ is a poset. Now a topos of sheaves is a subcategory of some presheaf category. So Lawvere's result contains my own that any Grothendieck topos has an in general non-natural BrA transformation on the subobject classifier object.

My discussion is a great deal more detailed than Lawvere's. Lawvere's discussion, on the other hand, contains enough detail for an expert to recreate the result and in fact has results relating to circumstances where the BrAs are natural and partially natural. The virtue of my discussion is its attempt to outline *why* the BrAs are not in general natural. This fitted in with my initial program for discovering the implications of using closed sets in place of open sets in various constructions, particularly sheaves. The focus of the thesis became that of discovering BrA logic structures and, broadly, that too is the focus of Lawvere's note. However, our method remained that of topological dualisation: the replacement of open sets by closed in the notions of various structures; it is not clear that this is Lawvere's method. Philosophically speaking, the intention with chapters 6, 8, 9, and 10 was to discover semantic objects for paraconsistent logic in categories. The implication of my actual discoveries is that, along with standard categorial dualisation, topological dualisation of sheaves is not an immediate source of natural semantic structures. My emphasis, then, was different from Lawvere's.

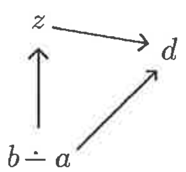
The present work falls naturally into four parts. Chapters one, two, and three form Part I where we describe such preliminary category, topos, and algebraic theory as is needed for the rest of the work. A certain amount of specialist theory on sheaf spaces, categorial j-sheaves, and Grothendieck toposes is saved until it is needed in the relevant chapters. Chapter three is particularly important. There we give a detailed development of those algebras we describe as dual to Intuitionism's Heyting algebras. We consider the nature of the logics that arise from these algebras, and develop a notion of dual logics. Part II is formed by chapters four, five, and six. With chapter four we provide an assessment of the categorial dualisation project in terms of the notion of a complement classifier. In chapter five we investigate straightforward dualisation of subobject logic structures by considering quotient object classifiers. Our conclusions are that if we are to proceed with the project it should be in terms of the development of extra operators for subobject lattices in standard categories. A preliminary attempt is considered in chapter six. Part III takes up where Part II finishes. Here we retain the idea that we are in search of extra operators for subobject lattices. With chapter seven we provide a brief history of the sheaf structure. This acts to motivate the hypothesis that sheaves on closed set topologies will provide us with paraconsistent logic objects for subobject lattices. With chapter eight we detail the generalisation of sheaf spaces to sheaves over categories and from there to j-sheaves in toposes. We describe the appropriate logic and categorial structure of closed set sheaf categories. The notion of j-sheaves allows us to demonstrate that categories of sheaves over closed sets exist and have subobject classifiers. Chapter eight as it appears here is a slightly revised version of that written for Mortensen's *Inconsistent Mathematics*, [1995]. It appears there as chapter twelve. Part III continues with chapter nine. There we find that the sheaf structure carries the algebras of closed sets of the base space into the subobject structures only in part. With chapter ten we generalise the result to Grothendieck toposes and are able to show that subobject lattices in Grothendieck toposes are in fact Brouwerian algebras, which is to say paraconsistent algebras, but not naturally so; they do not yield Brouwerian logic objects within the category. The results of chapters nine and ten (and six) are formally subsumed by the Lawvere [1991] result. The difference is that in the present work we demonstrate the detail of the result. This was independently developed and, in fact, not shown in Lawvere's work. With Chapter eleven we describe a genuine Brouwerian logic object in a category of all covariant functors over a closed set topology. This is the result that shows us that there is a genuine place in category theory for the consideration of paraconsistent logic. With chapter twelve we elaborate on the nature of the object discovered in chapter eleven. We find that it is a classifier object for a category of covariant sheaves. The object in fact provides a genuine complement classifier for the sheaf category in which it exists. In chapter thirteen we finish Part III by considering the viability of the closed set sheaves as semantic objects for paraconsistent logics. We describe a partial equivalence result for closed set sheaves and closed set sheaf spaces. This chapter is a revised version of James,W., "Sheaf spaces on finite closed sets" in Logique et Analyse, *Contemporary Logical Research in Australia*, 1996. Part IV contains the last two chapters of the present work. In chapter fourteen we are interested to use the duality of algebras described in chapter three to our advantage. We develop a dualisation of the usual notion of a category as an object on which to interpret theories. This dualisation allows us to develop the concept of a refutation system, as opposed to a deduction system. We present this as a means of understanding the notion of inconsistency toleration in a logic. The principal contribution is a description of how to model inconsistent theories in categories. With chapter fifteen we mark a beginning of an interest in further logic structures within categories. We consider an aspect of monoids in categories as algebras for relevant logics.

A notion that has played a part in the conceptual development of this work is that of co-exponentiation. This is just the dual of exponentiaition. In that exponential objects in a topos play a part in the fact that subobject algebras are Heyting (this is revealed by part of the working of the Fundamental Theorem of Topoi), we speculate that properly developed co-exponential objects will if not give us Brouwerian subobject algebras, then at least some structure on which to properly interpret paraconsistent logics.

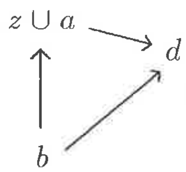
Co-exponentiation and its hypothesised relation to subobject algebras can be described as follows: the condition that $\mathrm{Sub}(d)$ be a BrA is suggestive of the existence of an adjunction. Consider: for $\mathrm{Sub}(d)$ to be a BrA we require that for any $b \rightarrowtail d$, $a \rightarrowtail d$, $z \rightarrowtail d \in \mathrm{Sub}(d)$ there exist $b \dot{-} a \rightarrowtail d \in \mathrm{Sub}(d)$ such that

$$b \dot{-} a \rightarrowtail d \leq z \rightarrowtail d \quad \text{iff} \quad b \rightarrowtail d \leq z \cup a \rightarrowtail d.$$

We can represent this condition in diagram form so that



commuting iff



commuting.

Given, at least, that unions of subobjects are something like categorial colimits, the vertical arrows suggest the condition that for any $b, a \in \mathcal{C}$ there exists an object $b \dot{-} a \in \mathcal{C}$, and for any $a, b, z \in \mathcal{C}$ there exists a bijection of morphisms

$$\mathcal{C}(b \dot{-} a, z) \cong \mathcal{C}(b, z + a).$$

This is a claim that any coproduct functor $(- + a): \mathcal{C} \to \mathcal{C}$ has a left adjoint. We may represent this adjoint as $(- \dot{-} a): \mathcal{C} \to \mathcal{C}$. Note that for a category to have exponentiation any product functor $(- \times a)$ must have a right adjoint. So if $\mathcal{C}$ has exponentiation, then for $\mathcal{C}^{op}$ any $(- + a)$ functor has a left adjoint. Therefore we call $(- \dot{-} a)$ the co-exponentiation functor. Any closed set topology poset category has co-exponentiation since $(- + a)$ becomes exactly $(- \cup a)$.

Part I:

PRELIMINARIES

CHAPTER 1: BASIC CATEGORY THEORY

Introduction: This first chapter is an exposition of the basic notions of category theory. There are two reasons for including this chapter. The first reason has to do with part of the intended readership of this document, namely logicians and philosophers. Since it is broadly true that logicians and philosophers are unacquainted with the detail of category theory, it is appropriate that the thesis contain an exposition of category theory in enough detail that a reader may follow the discussion in the later, more technical, chapters. The second reason for including this first chapter is completeness. The thesis can function as a largely self contained argument for the various propositions and results established in later chapters.

1. Categories and Morphisms

A *category* $\mathcal{C}$ is a collection of items called *objects* together with a collection of items called *arrows* satisfying an existence of associative composition axiom and an existence of identities axiom (both axioms are given below). Such arrows as exist within the category are understood as being between objects in that associated with each $\mathcal{C}$-arrow will be a *domain* and *codomain* both of which are $\mathcal{C}$-objects. These arrows, like functions, have a direction: they are from the domain to the codomain. We represent an arrow g for which the domain is object a and the codomain is object b by $g: a \to b$ or by $a \xrightarrow{g} b$. If g is understood we may use just $a \to b$. We will use $\mathrm{dom}(g)$ to denote the domain of g and use $\mathrm{cod}(g)$ for the codomain. Arrows are also called *morphisms*. Collections of arrows will be called *hom-sets*.

Suppose a collection of objects and a collection of arrows. Let us allow that no arrow in our collection has a domain or a codomain that is not in our object collection. The arrow collection is closed under *(binary) composition* if whenever there is an arrow f and an arrow g such that cod(f)=dom(g), there is also an arrow k in the collection with dom(k)=dom(f) and cod(k)=cod(g), and which is identical to the arrow made when f is followed by g; for example, if we suppose arrows $f: a \rightarrow b$ and $g: b \rightarrow c$, then k would be the arrow $a \xrightarrow{f} b \xrightarrow{g} c$. In general there will be many arrows $a \rightarrow b \rightarrow c$ and the sense to be made of the notion "f followed by g" depends on the nature of f and g as entities. A useful example to prompt intuitions is the usual notion of composition of functions. Following the conventions of functional composition the arrow k that is the arrow of "f followed by g" is denoted $g \cdot f$. We call $g \cdot f$ a *composite (of f and g)*.

Our collection of arrows is closed under *associative (binary) composition* if it is closed under (binary) composition and furthermore, when f, g, h are arrows of the collection, if $h \cdot g \cdot f$ is defined, then $(h \cdot g) \cdot f = h \cdot (g \cdot f)$. Such a collection of arrows is said to satisfy the *existence of associative composition* axiom of categories.

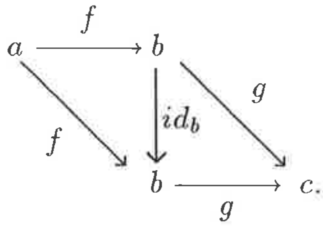
We recognise special arrows called *identities with respect to composition* or just *identities*. These are arrows with identical domain and codomain, though note that not all arrows with identical domain and codomain are identities. To be an identity an arrow must have two properties with respect to the collection of arrows within which they exist. We say that an arrow f is an identity with respect to a collection of arrows if whenever $g \cdot f$ is defined, it is the same arrow as g, and in addition if $f \cdot h$ is defined, then it is the same arrow as h.

Recall that we supposed a collection of arrows and a collection of objects. The arrows were to have no domain nor codomain that was not a member of the object collection. Suppose that we allow that the collection of arrows is closed under a composition operation. We say that the collection of arrows satisfies the *existence of identities* axiom with respect to that composition and the object collection if for any object b, there is some arrow, denoted id_b, in the collection that is an identity with respect to the composition operation. If that composition operation is associative, then our collection of objects together with our collection of arrows is a category.

Owing to what are commonly perceived to be foundational difficulties associated with the practice of category theory there is a distinction made between small and large categories. A category is *small* if its collection of objects and its collection of arrows are both sets. A category is *large* if its object and arrows collections are both classes. As MacLane in *Categories for the Working Mathematician*, [1971] notes the practice of category theory calls upon us to consider such things as a category of *all* mathematical entities of some type. In particular, we will routinely be wanting to consider categories of all set-based entites of particular types, for example, all groups or all topological spaces or all monoids, and this amounts to applying a naive comprehension principle: given a property, form a category of all sets with that property. Within set theory the naive comprehension principles are famous for generating paradoxical sets. This is usually understood to be inappropriate at least within set theory, and is likely to be inappropriate within, at least mathematical, category theory. There have been various responses to this problem ranging from the naive ("a category is a category, not a set") to the paraconsistent ("if we must found category theory on set theory, why not use an inconsistency tolerant logic under which it is possible for, say, the category of all categories to be both a member and a non-member of itself"). While we are interested in paraconsistent categories our concern is not so much with foundations as with internal logic structures. Accordingly we accept the usual solution from set theory and make an in principle distinction between sets and classes. We will make no special assumption about what a class is other than to say that it is a 'collection' that is not a set. As to what a set is, we hope likewise to avoid commitment by noting that there are available various formulations for set theory. Intuitively our practice is to allow naive comprehension for as long as it does not get us into trouble. To some extent this is the aim of all formulations of set theory.

A useful further notion is that of a *locally small* category, which requires only that the collection of arrows between any two objects be a set. The adjective 'small' is sometimes applied to the intuitive notion of a set or collection. The intuitive or naive notion of a set covers both the formal notion of set and of class, so a *small set* will mean a set as defined by some appropriate system.

The concept of a *commuting diagram* is a valuable and basic one within category theory. These are diagrammatic representations of equations that feature arrows and operations on arrows. So, for example, the equations that describe the nature of identities with respect to composition are represented in the diagram



When the equations $id_b \cdot f = f$ and $g \cdot id_b = g$ hold, the diagram is said to be commuting.

Remark: An example of a category is SET, the collection of all sets together with the collection of all functions between sets. Note that SET is a large category. We will denote by **Set** the restriction of SET to all small sets and functions between small sets. **Set**, too, is a large category, but constitutes a useful restriction of SET in that it does not contain elements that can cause difficulties for such mathematics as we may attempt. Another example of a large category is GRP, the category of all groups with all group homomorphisms. **Grp** will be the category of all small groups. TOP is the category of topological spaces and continuous functions between topological spaces. **Top** is the category of all small topological spaces.

In some sense categories of mathematical entities are universes of mathematical discourse. We can, it is suggested, identify a set, say, and all of its useful properties by ascertaining its relative position within the category **Set**. A central intuition in the development of the notion of categories was that it is possible to establish all that it is mathematically necessary to know about an entity by establishing that entity's appropriate, which is to say arrow, relationship to entities of the same type. As a simple example consider that in the normal language associated with set theory we can look at a set and say that it contains another set which we call a subset. However, in the general language of categories it is not so much that the set contains a subset as there exists a particular type of **Set**-morphism, an inclusion function, between two technically separate **Set**-objects. The feature of category theory that has sustained it through this perhaps tortuous usurping of set theory is the generality of its constructs over broader mathematical theory and the insight this can afford.

Terminal and Initial Objects:

A *terminal object* or terminator in a category $\mathcal{C}$ is an object, denoted by 1, such that for every $\mathcal{C}$-object a there is exactly one $\mathcal{C}$-arrow $a \to 1$. The dual is an *initial object* denoted by $\emptyset$. The initial object is an object such that for every object $a \in \mathcal{C}$ there is exactly one $\mathcal{C}$-arrow $\emptyset \to a$. Note that we speak (loosely) of *the* terminal and *the* initial objects. In fact there may be many such objects within a given category. The point however is that all terminal objects, if they exist at all, will be isomorphic within the category and likewise that, if they exist at all, all initial objects will be isomorphic. To speak of *the* terminal object is to use the idea that, *within a category*, an isomorph is as good as the real thing. In general we will be able to identify the canonical construction for an object or structure, but within a given category any isomorph will behave in exactly the same manner and be just as useful as the "original".

Monos, Epis, Isos:

We can identify useful or interesting structures in categories by generalising definitions from more well known areas of mathematics and, in particular, from set theory. A *monomorphism* is the categorial generalisation of an injective function. An arrow $f : b \to c$ is a monomorphism, or *monic*, in a category $\mathcal{C}$, if whenever we have a pair of parallel $\mathcal{C}$-arrows g, h such that the following diagram commutes

$$a \underset{h}{\overset{g}{\rightrightarrows}} b \xrightarrow{f} c$$

that is, $f \cdot g = f \cdot h$, then we have that $g = h$. A monic f is denoted by $b \rightarrowtail c$. Two facts about monics that we make frequent use of are that if f and g are monic, then so is the composite $f \cdot g$, and that if the composite $f \cdot g$ is monic, then so is g.

An *epimorphism* is the categorial generalisation of a surjective function. An arrow $f : b \to c$ is an epimorphism, or *epic*, in a category $\mathcal{C}$, if whenever we have a pair of parallel $\mathcal{C}$-arrows i, j such that the following diagram commutes

$$b \xrightarrow{f} c \underset{j}{\overset{i}{\rightrightarrows}} d$$

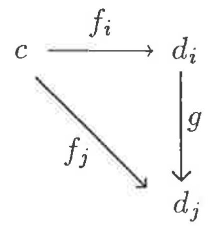
that is, $i \cdot f = j \cdot f$, then we have that $i = j$. An epic f is denoted by $b \twoheadrightarrow c$. The two facts we have about monics dualise (in a sense that we will describe later), so that if f and g are both epic arrows, then so is the composite $f \cdot g$, and that if the composite $f \cdot g$ is epic, then so is f.

An *isomorphism* is the categorial generalisation of a bijective function. An arrow $f : b \to c$ is an isomorphism, or *iso*, in a category $\mathcal{C}$ if it has an *inverse*, that is, there is a $\mathcal{C}$-arrow $f^{-1} : c \to b$ such that $f \cdot f^{-1} = id_c$ and $f^{-1} \cdot f = id_b$. An iso f is denoted by $b \cong c$. The objects b, c of an iso arrow are, within the category, isomorphic and are called *iso objects*.

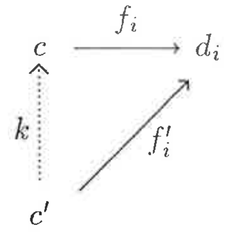
An isomorphism is always both epic and monic. However it is not always true that epic and monic arrows are well behaved. For example it is not true that in all categories an arrow that is both epic and monic is an isomorphism. This indicates only that there are more categories than are (intuitively) the image of SET. The desire to identify categories for which intuitively set theoretic notions like epimorphism, monomorphism, and set membership behaved as they did for SET was at least part of the motivation for developing the theory of toposes. But we are ahead of ourselves.

Limits and Colimits:

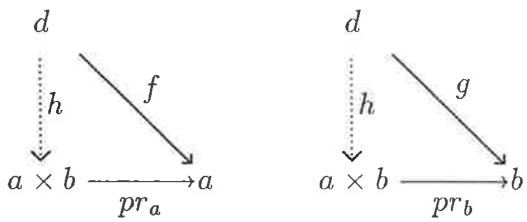
A natural concern arising from contemplation of the notion of a commuting diagram has to do with the existence of limits and their duals, the colimits. It is frequently useful to be aware of the existence or otherwise of a limiting (or colimiting) example of a commuting diagram. Such things feature heavily in the usual development of what we might call the mathematics of such entities as we can collect into categories. A simple example is that of the product and coproduct structures. Most theories of the broadly mathematical type – set theory, group theory, the theory of vector spaces, and so on – have particular notions of product and coproduct. What these notions of product (or coproduct) share is a property of existence as a limit (or colimit). In general, limits within a category are described in terms of cones and diagrams. A *diagram* D in a category $\mathcal{C}$ is any collection of $\mathcal{C}$-objects d_i together with any collection of $\mathcal{C}$-arrows g between those objects. A *cone* for D is a $\mathcal{C}$-object c together with $\mathcal{C}$-arrows $c \xrightarrow{f_i} d_i$ for each diagram object d_i such that for any diagram arrow $d_i \xrightarrow{g} d_j$ we have a commuting triangle



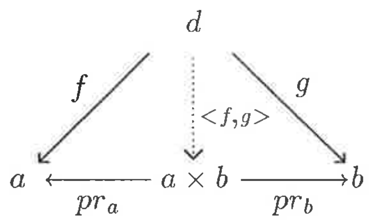
We can denote a cone by $\{c \xrightarrow{f_i} d_i\}$. A *limit for* D is then that cone $\{c \xrightarrow{f_i} d_i\}$ for which where $\{c' \xrightarrow{f_i'} d_i\}$ is any other D-cone, there is exactly one $\mathcal{C}$-arrow $c' \xrightarrow{k} c$ such that for every diagram object d_i we have a commuting triangle



Products arise as limits. Any two objects a, b in a category $\mathcal{C}$ constitute a diagram (admitedly a diagram with no arrows, but a diagram nonetheless). A cone for this diagram is any $\mathcal{C}$-object d together with a pair of arrows $f: d \to a, g: d \to b$. A limiting cone, where it exists, is a $\mathcal{C}$-object c together with a pair of arrows $pr_a: c \to a, pr_b: c \to b$ such that there is exactly one arrow $h: d \to c$ making both



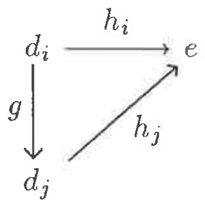
commute in $\mathcal{C}$ whenever $\{a \xleftarrow{f} d \xrightarrow{g} b\}$ is a cone for diagram $\{a, b\}$. And so we have a definition: a *product* of two objects a, b in a category $\mathcal{C}$ is a triple $\langle a \times b, pr_a, pr_b \rangle$ where $a \times b$ is a $\mathcal{C}$-object and pr_a and pr_b are, respectively, $\mathcal{C}$-arrows $a \times b \to a$ and $a \times b \to b$, and for any $\mathcal{C}$-object d and any pair of $\mathcal{C}$-arrows $d \xrightarrow{f} a, d \xrightarrow{g} b$ there is exactly one $\mathcal{C}$-arrow $\langle f, g \rangle : d \to a \times b$ making the following diagram commute.



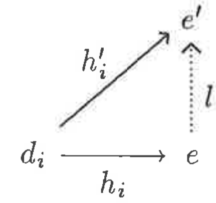
The arrows pr_a and pr_b are called, respectively, the *first* and *second projection maps*. The arrow $\langle f, g \rangle$ is called a *product map*. Notice the convention that morphisms that are unique in making a diagram commute are represented by broken or dotted arrows.

Limits are defined only up to isomorphism. This means that for a diagram D there can be more than one limiting cone, but that if this should be the case, then the limiting cones are isomorphic in the sense that if $\{c \xrightarrow{f_i} d_i : i \in I\}$ and $\{c' \xrightarrow{f'_i} d_i : i \in I\}$ are both limiting cones, then there will exist an isomorphism $k: c \to c'$ such that $f'_i \cdot k = f_i$ for all $i \in I$.

Colimits are described in terms of co-cones. For the diagram D in a category $\mathcal{C}$, a *co-cone* is a $\mathcal{C}$-object e together with $\mathcal{C}$-arrows $d_i \xrightarrow{h} e$ for each diagram object d_i such that for any diagram arrow $d_i \xrightarrow{g} d_j$ we have commuting

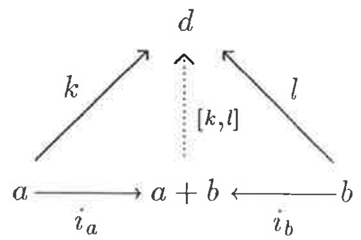


A *colimit for D* is then co-cone $\{d_i \xrightarrow{h_i} e : i \in I\}$ which has the property that if $\{d_i \xrightarrow{h'_i} e' : i \in I\}$ is any other D-co-cone, then there is a unique $\mathcal{C}$-arrow $l : e \to e'$ that makes all triangles



commute in $\mathcal{C}$. Like limits, colimits are defined only up to isomorphism. And just as we can develop the definition of a product in terms of limiting cones, so can we develop the definition of a coproduct in terms of co-cones. In that case, a *coproduct* of two objects a, b in a category $\mathcal{C}$ is a triple $\langle a + b, i_a, i_b \rangle$ where $a + b$ is a $\mathcal{C}$-object, both $i_a : a \to a + b$ and $i_b : b \to a + b$ are $\mathcal{C}$-arrows, and the following is true: for any $\mathcal{C}$-object d and any pair of $\mathcal{C}$-arrows $a \xrightarrow{k} d, b \xrightarrow{l} d$ there is exactly one $\mathcal{C}$-arrow

$[k, l] : a + b \to d$ making the following diagram commute.



The arrows i_a and i_b are called, respectively, the *first* and *second injection maps.* The arrow $[k, l]$ is called a *coproduct map.*

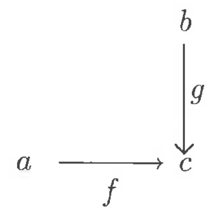
Duality:

There is an important property demonstrated by the notions product and coproduct: they are categorial duals. The same is true of the notions monomorphism and epimorphism. We define this in terms of dual, or "opposite", categories. A category is *dual* or opposite to a category $\mathcal{C}$ if it has the same objects and furthermore there is a distinct arrow $b \to a$ if and only if there is a distinct $\mathcal{C}$-arrow $a \to b$; the dual category is denoted $\mathcal{C}^{op}$ and for $f\colon a \to b$ in $\mathcal{C}$, the corresponding $\mathcal{C}^{op}$-arrow is denoted $f^{op}\colon b \to a$, and whenever $f \cdot g$ is defined in $\mathcal{C}$, $(f \cdot g)^{op}$ is defined in $\mathcal{C}^{op}$ to be a composite $g^{op} \cdot f^{op}$. We have, at least up to isomorphism, that $(\mathcal{C}^{op})^{op}$ is $\mathcal{C}$.

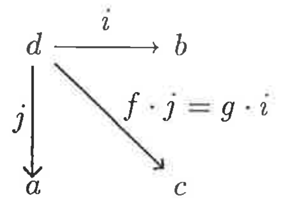
Consider now the definition of a $\mathcal{C}$-monic $f\colon b \rightarrowtail c$. The arrow f is monic if whenever we have a parallel pair of $\mathcal{C}$-arrows $g, h\colon a \rightrightarrows b$ such that $f \cdot g = f \cdot h$, we also have that $g = h$. Now $f^{op}\colon c \to b$ exists in $\mathcal{C}^{op}$ iff f exists in $\mathcal{C}$, and a parallel pair $g^{op}, h^{op}\colon b \rightrightarrows a$ exists in $\mathcal{C}^{op}$ iff g, h exists in $\mathcal{C}$. Since $(\mathcal{C}^{op})^{op}$ is $\mathcal{C}$, we will have that $(f \cdot g)^{op} = (f \cdot h)^{op}$ in $\mathcal{C}^{op}$ iff $f \cdot g = f \cdot h$ in $\mathcal{C}$. Furthermore, since f and g under these conditions are the same arrow in $\mathcal{C}$, their duals, f^{op} and g^{op}, are the same arrow in $\mathcal{C}^{op}$. In other words, f satisfies the monomorphism conditions in $\mathcal{C}$ iff f^{op} satisfies the epimorphism conditions in $\mathcal{C}^{op}$. Much the same discussion will reveal that when f is an epimorphism in $\mathcal{C}$, then and only then would f^{op} be a $\mathcal{C}^{op}$-monomorphism. We say that pairs of constructions are *categorial duals* if when Σ is the statement describing one of the pair in the basic language of categories (that is, using reference only to objects, arrows, composites, domains, and codomains) and Σ' is the statement describing the other in the basic language of categories, then Σ is $(\Sigma')^{op}$ and Σ' is Σ^{op} where for any statement Σ in the basic language of categories, the statement Σ^{op} is obtained by replacing any word "domain" by "codomain", any word "codomain" by "domain", and any equation "$f = g \cdot h$" by "$f^{op} = h^{op} \cdot g^{op}$" (and by correcting dependent grammar as is appropriate).

More Limits and Colimits:

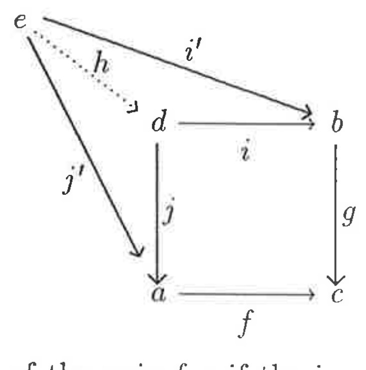
A very common and useful construction within categories is that of the binary limit structure called a pullback. For a pair of arrows



with common codomain, the ***pullback*** is the pair of arrows i, j of the limiting cone



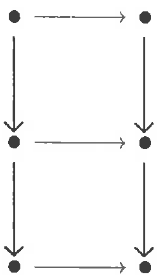
for the diagram $\{f, g\}$. Consider the diagram



The pair i, j is the pullback of the pair f, g if the inner square commutes, that is $g \cdot i = f \cdot j$, and in addition whenever there exists i', j' such that the outer square commutes, there is exactly one $h \colon e \to d$ making the whole diagram commute.

Whenever we have a pullback $\{i, j\}$ of $\{f, g\}$, it is common to say when i and j stand to f and g as they do above, that j is the pullback of g along f and that i is the pullback of f along g. This is in no sense a new definition for the notion of a pullback; it is simply some terminology by which we may identify component arrows in a pullback diagram in their role as parts of that diagram. The binary colimit structure dual to the pullback is called a *pushout*. Similar terminological conventions apply.

Pullback Lemma: *if a diagram of the form*



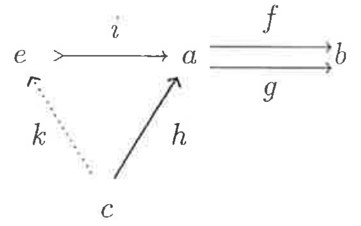
commutes, then

(i) *if the top and bottom squares are pullbacks, then so is the outer rectangle (made from the evident composites), and*

(ii) *if the outer rectangle and the bottom square are pullbacks, then so is the top square.* □

Here we have a demonstration of the usefulness of the notion of duality: once the Pullback Lemma is demonstrated, we can regard the appropriate dual claim, the Pushout Lemma, as equally demonstrated. The reasoning runs as follows: let Σ be a statement in the basic language of categories giving the definition of structure S; let T be a theorem cast in the basic language of categories and on the categorial nature of structures S in any category $\mathcal{C}$; in that case, theorem T^{op} will be on the categorial nature of structures co-S in categories $\mathcal{C}^{op}$ where co-S structures are defined by statement Σ^{op}. Now, if $\mathcal{C}^{op}$ is a category whenever $\mathcal{C}$ is a category and vice-versa, then any category $\mathcal{C}$ is a category $(\mathcal{C}')^{op}$ for some category $\mathcal{C}'$. And it follows that if T is proven for any category $\mathcal{C}$, then so is T^{op}.

Another useful limit notion is that of an equaliser. For any parallel pair $a \underset{g}{\overset{f}{\rightrightarrows}} b$ in a category $\mathcal{C}$, an *equaliser*, if it exists, is a $\mathcal{C}$-arrow $e \xrightarrow{i} a$ such that $f \cdot i = g \cdot i$ and whenever there is some $\mathcal{C}$-arrow $c \xrightarrow{h} a$ such that $f \cdot h = g \cdot h$, there is exactly one $\mathcal{C}$-arrow $c \xrightarrow{k} e$ such that $i \cdot k = h$. For fixed f, g, i, if there is a diagram as follows for any h, then i is an equaliser of f and g.



Co-equalisers are the dual notion. It is readily shown that any equaliser in a category is monic, and, by duality, that any co-equaliser is epic.

It is worth noting as an independent point that the concept of a monic is a limit notion just as that of an epimorphism is one of a colimit.

A category is said to be *complete* if there exists within the category a limiting cone for every diagram. A category is *co-complete* if there exists within the category a colimit for every diagram. A finite diagram is one with a finite number of objects and arrows. A category is called *finitely complete* if there is a limit for every finite diagram. Dually a category is said to *finitely co-complete*.

Theorem 1.1: *if category $\mathcal{C}$ has a terminal object and has a pullback for every pair of arrows with common codomain, then $\mathcal{C}$ is finitely complete.* □

And dually,

Theorem 1.2: *if category $\mathcal{C}$ has an initial object and has a pushout for every pair of arrows with common domain, then $\mathcal{C}$ is finitely co-complete.* □

Functors and Natural Transformations:

An important feature of category theory is the constructions on and between categories. The basic device is the functor. A *functor* is a morphism of categories that preserves identities and composition structure. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between category $\mathcal{C}$ and category $\mathcal{D}$ can be thought of as a pair of assignment functions and so be $F = \{F_{ob}, F_{ar}\}$. For any $\mathcal{C}$-object a, $F(a) = F_{ob}(a)$ which is some $\mathcal{D}$-object. For any $\mathcal{C}$-arrow f, $F(f)$ is $F_{ar}(f)$ which is some $\mathcal{D}$-morphism. There are two types of functor, the *contravariant* and the *covariant*, and the difference is in terms of the action of F_{ar}. For a $\mathcal{C}$-arrow $a \xrightarrow{f} b$, the image $F_{ar}(f)$ under a covariant F is some arrow $F_{ob}(a) \rightarrow F_{ob}(b)$. A contravariant F will map f to some arrow $F_{ob}(b) \rightarrow F_{ob}(a)$. In other words, covariant functors preserve morphism direction while contravariant functors reverse it. What further distinguishes functors from simple assignment functions is their preservation of categorial composition structure. Any functor is required to preserve identities so that, for any $\mathcal{C}$-object a,

$$F(id_a) = id_{F(a)}.$$

Any functor is required to preserve composition structure in the sense that for any $\mathcal{C}$-arrows g, h if $g \cdot h$ is defined in $\mathcal{C}$, then for covariant F,

$$F(g \cdot h) = F(g) \cdot F(h)$$

while for contravariant F

$$F(g \cdot h) = F(h) \cdot F(g).$$

In other words, covariant functors preserve composition while contravariant functors preserve and reverse it.

Any contravariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ can be understood to be a covariant functor $\overline{F}: \mathcal{C}^{op} \rightarrow \mathcal{D}$ where F_{ob} and $\overline{F}_{ob}$ are the same functions, but for any $\mathcal{C}^{op}$-arrow f^{op} we let $\overline{F}_{ar}(f^{op}) = F_{ar}(f)$. Recall that $f^{op}: b \rightarrow a$ is in $\mathcal{C}^{op}$ iff $f: a \rightarrow b$ is in $\mathcal{C}$, and $F_{ob}(f)$ is a map $F(b) \rightarrow F(a)$. Given this, it is possible to ignore the notion of contravariance and speak only of covariant functors with no loss of generality. Since, however, in following chapters we will be dealing in large part with categorial sheaf theory we will maintain the distinction. We will however adopt the convention of representing contravariant $\mathcal{C} \to \mathcal{D}$ as covariant $\mathcal{C}^{op} \to \mathcal{D}$. It is worth noting a philosophical point: contravariance and covariance remain distinct notions; the elimination of contravariance arrives solely as a result of the isomorphism of categories of covariant functors $\mathcal{C}^{op} \to \mathcal{D}$ and categories of contravariant functors $\mathcal{C} \to \mathcal{D}$. (We shall shortly explain how functor categories exist).

Composition of functors is readily defined as composition of the associated assignment functions so that for $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$, the composite $G \cdot F$ is the pair

$$\{G_{ob} \cdot F_{ob}, G_{ar} \cdot F_{ar}\}.$$

Given a category of categories we can identify those functors that satisfy the usual epi-, mono-, and iso-morphism definitions. Other useful characterisations of functor types include the full, the faithful, and the embedding notions. A (covariant) functor $F: \mathcal{C} \to \mathcal{D}$ is said to be *full* if surjective on hom-sets; that is, F is full if any $g: F(a) \to F(b)$ in $\mathcal{D}$ is $F(f)$ for some $f: a \to b$ in $\mathcal{C}$. A functor is *faithful* if injective on hom-sets; that is, if $F(f) = F(g)$ in $\mathcal{D}$, then $f = g$ in $\mathcal{C}$. A functor is called an *embedding* if the arrow function is injective in the sense that for each arrow g in $\mathcal{D}$ there is at most one arrow f in $\mathcal{C}$ such that $F_{ar}(f) = g$. Note that the definitions of faithful functors and embeddings are not necessarily equivalent. Useful notions also include the hom and representable functors.

A *hom functor* for a (small) category $\mathcal{C}$ is a functor that maps objects of the category to sets of morphisms of the category. For object $a \in \mathcal{C}$ the *covariant hom functor* is functor $\hom_{\mathcal{C}}(a, -) : \mathcal{C} \to \mathbf{Set}$ which maps any object $b \in \mathcal{C}$ to $\hom_{\mathcal{C}}(a, b)$, the collection of $\mathcal{C}$-morphisms $a \to b$, and maps any $\mathcal{C}$-arrow $b \xrightarrow{f} b'$ to $\hom_{\mathcal{C}}(a, f) : \hom_{\mathcal{C}}(a, b) \to \hom_{\mathcal{C}}(a, b')$, the composition function given by

$$(a \to b) \mapsto (a \to b \xrightarrow{f} b').$$

For object $a \in \mathcal{C}$ the *contravariant hom functor* is a functor $\hom_{\mathcal{C}}(-, a) : \mathcal{C}^{op} \to \mathbf{Set}$ that maps any object $b \in \mathcal{C}$ to $\hom_{\mathcal{C}}(b, a)$ and maps any $\mathcal{C}$-arrow $b \xrightarrow{f} b'$ to $\hom_{\mathcal{C}}(f, a) : \hom_{\mathcal{C}}(b', a) \to \hom_{\mathcal{C}}(b, a)$, the function given by

$$(b' \to a) \mapsto (b \xrightarrow{f} b' \to a).$$

Note that we can also use h_a or $\mathcal{C}(a, -)$ to denote functor $\hom_{\mathcal{C}}(a, -)$, and that we can use h^a or $\mathcal{C}(-, a)$ in place of $\hom_{\mathcal{C}}(-, a)$. We will frequently use $\mathcal{C}(a, b)$ or $\hom(a, b)$ with $\mathcal{C}$ understood to denote the collection of $\mathcal{C}$-morphisms $a \to b$. It is useful to note that

$$\hom_{\mathcal{C}}(-, a) = \hom_{\mathcal{C}^{op}}(a, -).$$

In keeping with the powerful idea behind category theory that we may make categories of any mathematical entity, we will define morphisms between functors. These are the natural transformations.

A natural transformation is a morphism of functors that have common domain and common codomain. For functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* τ, denoted $F \xrightarrow{\cdot} G$, from F to G is a collection of $\mathcal{D}$-arrows $\tau_a : F(a) \to G(a)$ for all objects $a \in \mathcal{C}$ that are required to respect the arrow structure of the domain categories as translated by the functors involved; this means that if $f : a \to b$ is an arrow in $\mathcal{C}$, then the following diagram is required to commute in $\mathcal{D}$

$$\begin{array}{ccccc} a & \qquad & F(a) & \xrightarrow{\tau_a} & G(a) \\ {\scriptstyle f}\downarrow & & {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ b & & F(b) & \xrightarrow[\tau_b]{} & G(b) \end{array}$$

When this diagram commutes for any such f with domain a, the map τ_a from $F(a)$ to $G(a)$ is said to be *natural* in a. If F, G are both contravariant functors, then $\{\tau_a : a \in \mathcal{C}\}$ is a natural transformation if for any $f : a \to b$ in $\mathcal{C}$, the following diagram commutes in $\mathcal{D}$

$$
\begin{array}{ccccc}
b & & F(b) & \xrightarrow{\tau_b} & G(b) \\
f \big\uparrow & & F(f) \big\downarrow & & \big\downarrow G(f) \\
a & & F(a) & \xrightarrow[\tau_a]{} & G(a)
\end{array}
$$

Note that the fact that we can, in principle, replace any contravariant functor with a covariant functor means that, in principle, we can have natural transformations between functors of different variance.

That natural transformations are composable amounts to the claim that the components of the natural transformation are composable; that is, for functors $F, G, H: \mathcal{C} \to \mathcal{D}$ and natural transformations $\tau: F \dot{\to} G$ and $\sigma: G \dot{\to} H$, the composite $\sigma \cdot \tau$ is given by components $\sigma_a \cdot \tau_a$ which are composites of $\mathcal{D}$-arrows $F(a) \to G(a)$ and $G(a) \to H(a)$. Plainly this provides a composition operation suitable for the definition of a category. We denote by $\mathcal{D}^{\mathcal{C}}$ the category of all functors $\mathcal{C} \to \mathcal{D}$. We can apply the usual definitions of epi-, mono-, and iso-morphism; it is useful to note that a natural transformation τ is monic in $\mathcal{D}^{\mathcal{C}}$ if each component τ_a is monic in $\mathcal{D}$. A recognised sub-type of the natural transformations are the natural isomorphisms. A *natural isomorphism* $\tau: \mathcal{C} \to \mathcal{D}$ is a natural transformation where for every object $a \in \mathcal{C}$, component τ_a is an isomorphism in $\mathcal{D}$. The natural isomorphisms $\tau: \mathcal{C} \to \mathcal{D}$ are exactly the isomorphisms in $\mathcal{D}^{\mathcal{C}}$.

For a small category $\mathcal{C}$, a *representation* of a functor $K : \mathcal{C} \to \mathbf{Set}$ is a pair $\langle r, \Psi \rangle$ where r is an $\mathcal{C}$-object and

$$\Psi : \hom_{\mathcal{C}}(r, -) \cong K$$

is a natural isomorphism. The object r is called the *representing object.* The functor K is said to be *representable* when such a representation exists. It follows that contravariant functors are representable if isomorphic to some $\hom_{\mathcal{C}}(-, r)$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* when there is a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $G \cdot F \cong id_{\mathcal{C}}$ and $F \cdot G \cong id_{\mathcal{D}}$.

For any two categories $\mathcal{C}$ and $\mathcal{D}$ the *product category*, denoted $\mathcal{C} \times \mathcal{D}$, has as objects all pairs $\langle c, d\rangle$ where $c \in \mathcal{C}$, $d \in \mathcal{D}$. A $(\mathcal{C} \times \mathcal{D})$-arrow is a pair $\langle f, g\rangle$ where f is a $\mathcal{C}$-arrow and g is a $\mathcal{D}$-arrow. The composite $\langle f, g\rangle \cdot \langle f', g'\rangle$ is defined to be $\langle f \cdot f', g \cdot g'\rangle$ and exists whenever $f \cdot f'$ exists in $\mathcal{C}$ and $g \cdot g'$ exists in $\mathcal{D}$. With this information in hand we can consider the Yoneda Lemma.

2. Yoneda

The Yoneda Lemma demonstrates a translation of structure; the internal behaviour of a functor G is manifest as a relationship between functors. Via this lemma we can perform the traditional task of category theory of abstracting away from an element based description of mathematical entities.

The *Yoneda Lemma* asserts that when $\mathcal{D}$ is a category with small hom-sets then, for any covariant functor $G: \mathcal{D} \to \mathbf{Set}$ and $\mathcal{D}$-object a, there is a bijection between the elements of $G(a)$ and the set of natural transformations from $\hom(a, -)$ to G. The bijection in question

$$\mathcal{Y}: \mathrm{Nat}[\hom(a, -), G] \to G(a)$$

is given by

$$\delta \mapsto \delta_a(id_a)$$

where δ_a is the a-component of natural transformation $\delta: \hom(a, -) \to G$. The inverse of $\mathcal{Y}$, denoted $\mathcal{Y}'$, is given by

$$G(a) \ni x \mapsto \xi = \{\xi_b: b \in \mathcal{D}\},$$

where ξ is a natural transformation $\hom(a, -) \to G$ such that for all $b \in \mathcal{D}$ and all $f \in \hom(a, b)$, we have $\xi_b(x) = G(f)(x)$. As a corollary, which we obtain by substituting $\hom(b, -)$ for G, we have that for $a, b \in \mathcal{D}$, each natural transformation

$\hom(a, -) \dot{\rightarrow} \hom(b, -)$ has the form of contravariant $\hom(h, -)$ for a unique $\mathcal{D}$-arrow $h: b \rightarrow a$.

For contravariant functors G, the Yoneda Lemma describes the bijection

$$\mathcal{Y}: \mathrm{Nat}[\hom(-, a), G] \rightarrow G(a)$$

and has as a corollary the claim that for objects $a, b \in \mathcal{D}$, each natural transformation $\delta: \hom(-, a) \dot{\rightarrow} \hom(-, b)$ has the covariant form $\hom(-, h)$ for a unique $\mathcal{D}$-arrow $h: a \rightarrow b$. By Yoneda, that arrow h is $\delta_a(id_a)$.

The Yoneda Lemma can be rewritten in terms of two functors from $\mathbf{Set}^{\mathcal{D}} \times \mathcal{D}$ to $\mathbf{Set}$. These are the functors E and N. E is the evaluation functor $\langle G, a \rangle \mapsto G(a)$ and N is what we can call the nat-trans functor $\langle G, a \rangle \mapsto \mathrm{Nat}[\hom(a, -), G]$. In these terms, the claim that $\mathcal{Y}$ is a bijection becomes the claim of a natural isomorphism $N \dot{\rightarrow} E$; this includes the extra claim that $\mathcal{Y}$ is natural in both a and G.

The *contravariant Yoneda functor* $Y: \mathcal{D}^{op} \rightarrow \mathbf{Set}^{\mathcal{D}}$ is defined by

$$a \mapsto \hom(a, -)$$

and

$$f \mapsto \hom(f, -).$$

Now, we know that Y acts bijectively on hom-sets (loosely, for sets of arrows in $\mathcal{D}$ there are isomorphic sets in $\mathbf{Set}^{\mathcal{D}}$ picked out by Y) since it follows from the Yoneda Lemma that for $c, d \in \mathcal{D}$, we have

$$\hom(c, d) \cong \mathbf{Set}^{\mathcal{D}}(\hom(c, -), \hom(d, -)),$$

Also, arrow $\delta \in \mathbf{Set}^{\mathcal{D}}$ is $Y(f)$ for some f in $\mathcal{D}$ only if δ is $\hom(f, -)$ or isomorphic to $\hom(f, -)$. So, suppose some arrow g in $\mathcal{D}$ such that $Y(g) = \delta$; that is, $Y(g) = Y(f)$. Since Y acts bijectively on hom-sets, if g and f have the same domain and codomain, then $g = f$. It is a further fact about Y that it is injective on objects; that is, for any object $F \in \mathbf{Set}^{\mathcal{D}}$, there is at most one $c \in \mathcal{D}$ such that $Y(c) = F$. It follows, then, that g and f must have the same domain and codomain. In other words, Y embeds $\mathcal{D}$ in $\mathbf{Set}^{\mathcal{D}}$, and, in fact, does so isomorphically. The Yoneda functor is otherwise known as the *Yoneda embedding* and is full and faithful.

The *dual* or *covariant Yoneda functor* $Y' \colon \mathcal{D} \to \mathbf{Set}^{\mathcal{D}^{op}}$, given by

$$a \mapsto \hom(-, a)$$

and

$$f \mapsto \hom(-, f),$$

is an embedding for the same reasons; it thereby allows us to regard $\mathcal{D}$ as a full subcategory of presheaf category $\mathbf{Set}^{\mathcal{D}^{op}}$.

3. Adjoints

Adjoints are a means of describing universal properties that generalises the type of discussion we have engaged in when we described limits and colimits in terms of cones and co-cones. Universal properties in category theory are properties of diagrams; in other circumstances we might call these sorts of properties fundamental or perhaps archetypal. Amongst a collection of diagrams of (loosely) the same type, one diagram (and its isomorphs) is universal with respect to that type if the other diagrams factor uniquely through it; that is, given the diagram that is universal with respect to the type and given another diagram of the type, a bigger diagram can be made with unique arrows from the second diagram to the first. The idea of diagram types here is vague but is meant to invoke the idea of a collection of diagrams that have the same shape; they do not necessarily have the same arrows nor the same objects, but they do have the same number of objects and of arrows and the objects and arrows stand in the same relationships. A concrete example of a diagram with a universal property is that of a limiting cone C for a diagram

D. Amongst all the cones for diagram D, cone C is universal (with respect to the property "is a cone for diagram D") because any other cone factors uniquely through C in exactly the same sense that we gave when we originally defined the notion of a limit for D. Among diagrams of the same "shape", the diagram with the universal property is an exemplar, indeed a construction, of some property of mathematical entities, for example, "is a limit", "is a product", or "is a subobject classifier". There is, therefore, some significance to establishing a general treatment for the existence and description of universal properties.

Adjoints are functors in an adjunction. An *adjunction* between two (covariant) functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is a bijection

$$\varphi = \varphi_{x,a} : \hom_{\mathcal{D}}(F(x), a) \cong \hom_{\mathcal{C}}(x, G(a))$$

which is natural in $x \in \mathcal{C}$ and $a \in \mathcal{D}$. The bijection being natural in x and a means that individual bijections $\varphi_{x,a}$ are the components of a natural transformation between the following hom bifunctors:

$$\hom_{\mathcal{D}}(F(-), -) \quad \text{and} \quad \hom_{\mathcal{C}}(-, G(-)).$$

These bifunctors are not in principle more complicated than the usual hom functors, though note that they are both, as one would expect, contravariant in the first variable and covariant in the second. The object functions of both functors are readily described: for any $x \in \mathcal{C}$ and any $a \in \mathcal{D}$

$$\left(\hom_{\mathcal{D}}(F(-), -)\right)_{ob} : \langle x, a \rangle \mapsto \hom_{\mathcal{D}}(F(x), a)$$

and

$$\left(\hom_{\mathcal{C}}(-, G(-))\right)_{ob} : \langle x, a \rangle \mapsto \hom_{\mathcal{C}}(x, G(a)).$$

We describe the arrow functions in two stages. Consider $\hom_{\mathcal{D}}(F(-), -)$. For any $x \in \mathcal{C}$ and any $\mathcal{D}$-arrow $k: a \to b$, $\hom_{\mathcal{D}}(F(x), k)$ is the usual composition function

$$F(x) \to a \mapsto F(x) \to a \xrightarrow{k} b.$$

Also, for any $\mathcal{C}$-arrow $h: x \to y$ and any $a \in \mathcal{D}$, $\hom_{\mathcal{D}}(F(h), a)$ is

$$F(y) \to a \mapsto F(x) \xrightarrow{F(h)} F(y) \to a.$$

So, for any $h: x \to y$ in $\mathcal{C}$ and any $k: a \to b$ in $\mathcal{D}$, $\big(\hom_{\mathcal{D}}(F(-), -)\big)_{ar}$ is a map

$$\hom_{\mathcal{D}}(F(y), a) \to \hom_{\mathcal{D}}(F(x), a) \to \hom_{\mathcal{D}}(F(x), b)$$

given by

$$F(y) \to a \mapsto F(x) \xrightarrow{F(h)} F(y) \to a \xrightarrow{k} b.$$

Similarly we can arrive at the arrow function for $\hom_{\mathcal{C}}(-, G(-))$. Recall that $\hom_{\mathcal{C}}(-, G(-))$ is contravariant in the first variable and covariant in the second; then, for any $h: x \to y$ in $\mathcal{C}$ and any $k: a \to b$ in $\mathcal{D}$, $\big(\hom_{\mathcal{C}}(-, G(-))\big)_{ar}$ is a map

$$\hom_{\mathcal{C}}(y, G(a)) \to \hom_{\mathcal{C}}(x, G(a)) \to \hom_{\mathcal{C}}(x, G(b))$$

given by

$$y \to G(a) \mapsto x \xrightarrow{h} y \to G(a) \xrightarrow{G(k)} G(b).$$

We are now in a position to recognise that an adjunction φ exists if and only if for all objects $x \in \mathcal{C}$, $a \in \mathcal{D}$ and all arrows $h: x \to y$ in $\mathcal{C}$ and $k: a \to b$ in $\mathcal{D}$, the following two diagrams commute

$$\begin{array}{ccccc}
a & & \hom_{\mathcal{D}}(F(x), a) & \xrightarrow{\varphi_{x,a}} & \hom_{\mathcal{C}}(x, G(a)) \\
k \big\downarrow & & \hom_{\mathcal{D}}(F(x), k) \big\downarrow & & \big\downarrow \hom_{\mathcal{C}}(x, G(k)) \\
b & & \hom_{\mathcal{D}}(F(x), b) & \xrightarrow[\varphi_{x,b}]{} & \hom_{\mathcal{C}}(x, G(b))
\end{array}$$

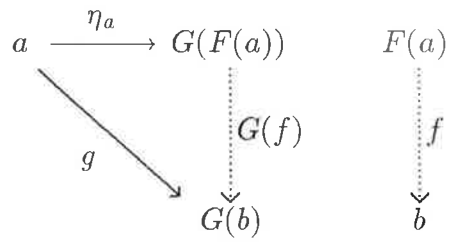
and

$$\begin{array}{ccccc}
y & & \hom_{\mathcal{D}}(F(y), a) & \xrightarrow{\varphi_{y,a}} & \hom_{\mathcal{C}}(y, G(a)) \\
h \big\uparrow & & \hom_{\mathcal{D}}(F(h), a) \big\downarrow & & \big\downarrow \hom_{\mathcal{C}}(h, G(a)) \\
x & & \hom_{\mathcal{D}}(F(x), a) & \xrightarrow[\varphi_{x,a}]{} & \hom_{\mathcal{C}}(x, G(a))
\end{array}.$$

Diagrams of the first sort are the claim that φ is natural in a. Diagrams of the second sort are the claim that φ is natural in x.

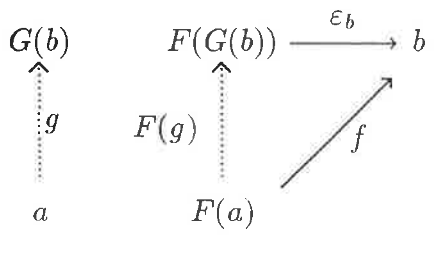
When we have an adjunction φ the functor F is called the *left adjoint.* In its role as left adjoint of G, F is denoted by $F \dashv G$. Functor G is called the *right adjoint* and denoted by $G \vdash F$.

Adjoints are expressible in terms of their units and co-units. The *unit* of an adjunction φ is a natural transformation $\eta : id_{\mathcal{C}} \to F \cdot G$ given by components $\eta_a = \varphi(id_{F(a)})$; component η_a is that element of $\hom_{\mathcal{C}}(a, G(F(a)))$ that is the image under $\varphi_{a,F(a)}$ of $id_{F(a)}$. The unit has the property that for any object $a \in \mathcal{C}$ and any $\mathcal{C}$-arrow $a \xrightarrow{g} G(b)$ there is exactly one $\mathcal{D}$-arrow $F(a) \xrightarrow{f} b$ such that



commutes in $\mathcal{C}$.

Dually, we have the co-unit of an adjunction. The *co-unit* is a natural transformation $\varepsilon\colon F \cdot G \to id_{\mathcal{D}}$ given by components $\varepsilon_b = \varphi^{-1}_{G(b),g}(id_{G(b)})$. The co-unit has the property that for any $b \in \mathcal{D}$ and any $\mathcal{D}$-arrow $F(a) \xrightarrow{f} b$ there is exactly one $\mathcal{C}$-arrow $a \xrightarrow{g} G(b)$ such that



commutes in $\mathcal{D}$.

For functors F and G the claim that natural transformations η and ε exist is exactly the claim that bijection φ exists. It is via the unit and the counit that adjunctions reveal universal constructions.

With this brief summary of the basic categorial concepts we will employ, we proceed to the notion of a topos. In the next chapter we summarise basic definitions and results from topos theory. These are needed as a preliminary to later chapters, where we proceed to establish further results connecting the theory of sheaves, toposes, and paraconsistent logic.

CHAPTER 2: BASIC TOPOS THEORY

Introduction: This second chapter is an exposition of the very basic notions of topos theory and topos logic. The reasons for including this chapter are very much the same as those for including chapter 1, but in this case they apply with more force. The bulk of the specific categories discussed in the course of the thesis are toposes and the structures discussed in the course of the thesis are exactly those that are at the heart of topos logic. Chapter 2 also contains a section on the basic features of the technical device called image factorisation. This is simple exposition and allows us to make simple uses of the device in later technical chapters without comment.

1. Toposes

A topos is a category with some extra structure. The search for a definition of a topos was originally motivated (in part) by the need to identify categories that were sufficiently like SET that various generalised set-theoretic notions like mono- and epi-morphism were well behaved in the sense that they maintained their analogy as constructs with their original set based counterparts. The original toposes were what we now call Grothendieck toposes, that is, categories of sheaves over sites. Since these categories had many of the necessary features the name was appropriated by Lawvere to describe the more general structure that is the elementary topos.

The notion of a topos is now standard within category theory and is so well developed that it is appropriate to speak of toposes as being a subject matter in their own right. Our exposition here is a summary of some standard facts and constructions within this subject matter.

An *elementary topos* is a category $\mathcal{E}$ that

(1) is finitely complete,

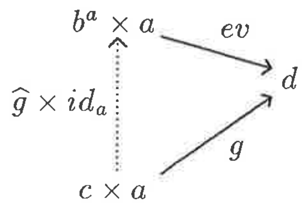
(2) is finitely co-complete,

(3) has exponentiation, and

(4) has a subobject classifier.

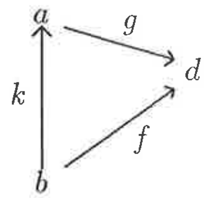
We have seen how to understand the notions of completeness and co-completeness. The third property, that of exponentiation, is the generalisation of the set notion of the existence of exponential objects B^A. (A set based object B^A is ordinarily understood to be the set of all functions from set A to set B). The subobject classifier is the generalisation to categories of the notion of subsets and, in particular, subsets as described by characteristic functions. We will proceed to define these notions in more detail, and we follow this with an exposition of the idea of logic in a topos. We finish this chapter with a brief description of the technical device of image factorisation.

A category $\mathcal{C}$ has *exponentiation* if it at least has products and if for any $\mathcal{C}$-objects a, b there is a $\mathcal{C}$-object b^a and a $\mathcal{C}$-arrow $ev : b^a \times a \to b$ such that for any $\mathcal{C}$-object c and any $\mathcal{C}$-arrow $g : c \times a \to b$, there is a unique $\mathcal{C}$-arrow $\widehat{g} : c \to b^a$ making the following diagram commute



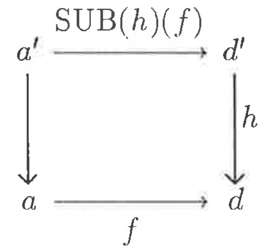
Equivalently, a category $\mathcal{C}$ has exponentiation if for every object $a \in \mathcal{C}$ there is a right product functor $(- \times a) : \mathcal{C} \to \mathcal{C}$ which has a right adjoint. A right product functor $(- \times a)$ is given by $b \mapsto b \times a$ and $(b \xrightarrow{f} c) \mapsto (f \times id_a : b \times a \to c \times a)$. The right adjoint to this functor will be $(-)^a : \mathcal{C} \to \mathcal{C}$. The arrow ev will be the co-unit of the adjunction. Objects b^a are called *exponential objects.*

The subobject classifier is the focal point for what can be understood as the logic of the categorial structure. We can describe lattices of subsets of a set under set inclusion and, in seeking a generalisation of this for categories, we find subobjects and their classifier. In that subsets can be understood in terms of inclusions and the lattices of subsets reworked as lattices of functions, we address ourselves, in general categories, to monics. For an object d in a category $\mathcal{C}$, let Monic(d) be the collection of all $\mathcal{C}$-arrows that are monic with d as codomain. We can define a pre-order (reflexive, transitive order) on Monic(d) so that for $f, g \in$ Monic(d) we say $f \subseteq g$ iff there is a $\mathcal{C}$-arrow k making the following diagram commute



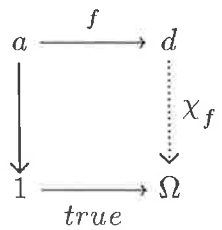
ie, iff there is some k such that $f = g \cdot k$. Since f is monic, so is k. This pre-order will not in general be a partial order (reflexive, transistive, and anti-symmetric) since there will, in general, be isomorphic, but non identical, monics: for such a pair f, g we will have $f \subseteq g$ and $g \subseteq f$, but not $f = g$. We can, however, establish a partial order on Monic(d) under the obvious equivalence relation. We will say that $f, g \in$ Monic(d) are in the same equivalence class if and only if we have both $f \subseteq g$ and $g \subseteq f$. Such an equivalence class is called a *subobject of d*. The collection Monic(d) partitioned under this equvalence relation is denoted Sub(d). It will be usual in what follows to blur the distinction between subobject and representing morphism. In the relevant areas this is the standard practice and follows from the fact that, for such constructions as we consider, all members of a given subobject behave as though identical; in fact, members of the same subobject are usually not identical as arrows, but such differences as exist are not relevant in the usual context of subobject evaluation. This is another example of the pervasive feature of category theory that, in context, an isomorph is as good as the real thing.

The subobject construction for a small category $\mathcal{C}$ can be presented as a contravariant functor SUB : $\mathcal{C}^{op} \to$ **Set** which takes objects d of $\mathcal{C}$ to sets Sub(d), and for $\mathcal{C}$-morphisms $h : d' \to d$, produces a function SUB(h) : Sub$(d) \to$ Sub(d') which takes subobject (representative) $f \in$ Sub(d) to the pullback of f along h



Note that, as is plainly required, any pair of monics in the same subobject will determine the same subobject when pulled back. When the subobject functor is representable in a category $\mathcal{C}$, the object that represents it is usually denoted by Ω. In fact, SUB being representable is equivalent to the existence within $\mathcal{C}$ of a subobject classifier. Object Ω is called the *classifier object.*

A *subobject classifier* for a category $\mathcal{C}$ is a morphism $true: 1 \to \Omega$ which is from the terminal object 1 to an object Ω; in addition the arrow has the property that for any $\mathcal{C}$-monic $f : a \rightarrowtail d$, there is exactly one $\mathcal{C}$-arrow $d \to \Omega$, denoted χ_f, that makes the following diagram a pullback



The maps χ_f are called *classifying maps.* In **Set** these are the characteristic functions. When the subobject classifier exists in a category the assignment of χ_f to f for all arrows f in Sub(d) establishes a bijection

$$\mathrm{Sub}(d) \cong \hom(d, \Omega).$$

The forward looking reader will see that the next point of interest will be the attempt to transfer any algebraic operators that exist for $\mathrm{Sub}(d)$ to $\hom(d, \Omega)$ and from there to the structure of Ω itself.

Remark: When discussing categories, and in particular the structure of toposes, there is a distinction to be drawn between internal and external constructions. Recall that a topos is a category and as such is a collection of objects and arrows. A construction for a category is called *internal* only if that construction is an object or an arrow of the category. Any other structure will be called *external* to the category. As an example, we have just noted that in a topos $\mathcal{E}$ the collection $\mathrm{Sub}_{\mathcal{E}}(d)$ will be isomorphic to $\mathcal{E}(d, \Omega)$. There is in general no reason to believe that $\mathrm{Sub}_{\mathcal{E}}(d)$ exists *as a collection* within the topos, that is, there is no reason in general to believe that the collection $\mathrm{Sub}_{\mathcal{E}}(d)$ is an object or an arrow of $\mathcal{E}$. This makes $\mathrm{Sub}_{\mathcal{E}}(d)$ an external construction. We know however that $\mathcal{E}(d, \Omega)$ is always represented within a topos by object Ω^d. We say then that $\mathrm{Sub}_{\mathcal{E}}(d)$ is the external version of Ω^d; while Ω^d, being an $\mathcal{E}$-object, is the internal version of $\mathrm{Sub}_{\mathcal{E}}(d)$.

2. Topos Logic

The logic objects of a topos are the classifier objects Ω. These are focal points for "topos logic" in the same way that two element sets are the focus (or locus) of logic within set theory. Objects Ω are developed as algebras within a topos by developing those natural operators that exist on each $\mathrm{Sub}(d)$, and, in essence, transfering these operators to Ω. The technical device for this transference is the Yoneda lemma for contravariant functors. The contravariant functor in question is SUB. As an indication of how this works recall the bijection $\mathrm{Sub}(d) \cong \hom(d, \Omega)$ and so the existence of operators on $\hom(d, \Omega)$ whenever there are operators on $\mathrm{Sub}(d, \Omega)$; there is an isomorphism

$$\hom(d, \Omega) \times \hom(d, \Omega) \cong \hom(d, \Omega \times \Omega),$$

and, in that operators on all Sub(d) are to be called natural if they correspond to a natural transformation $\hom(d, \Omega \times \Omega) \to \hom(d, \Omega)$, the Yoneda lemma guarantees us unique maps $\Omega \times \Omega \to \Omega$ for each set of natural operators on lattices Sub(d).

For any object d in a topos $\mathcal{E}$, Sub(d) ordered by subobject inclusion is a bounded, distributive lattice. For $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ in Sub(d), we have the greatest lower bound, $f \cap g : a \cap b \rightarrowtail d$, given by the pullback of f along g,

$$\begin{array}{ccc} a \cap b & \longrightarrow & b \\ \downarrow & & \downarrow g \\ a & \xrightarrow[f]{} & d \end{array}$$

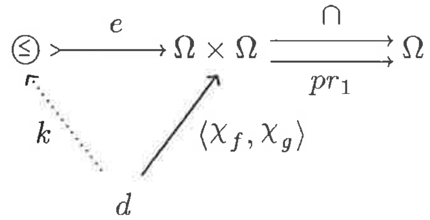
and the least upper bound $f \cup g : a \cup b \rightarrowtail d$ given by the image factorisation of the coproduct map $[f, g]$. We will say more about image factorisation shortly. The unit is the identity morphism on d. The zero is the unique map from the initial object to d. Furthermore, operations $\cap$ and $\cup$ are natural in d meaning that for any $d' \xrightarrow{k} d$,

$$\mathrm{SUB}(k)(f \cup g) = \mathrm{SUB}(k)(f) \cup \mathrm{SUB}(k)(g)$$

and

$$\mathrm{SUB}(k)(f \cap g) = \mathrm{SUB}(k)(f) \cap \mathrm{SUB}(k)(g).$$

Via the Yoneda Lemma, then, we have maps $\cup, \cap : \Omega \times \Omega \to \Omega$. Zeros and units are natural in the same sense. We can define a natural order object, $\textcircled{$\leq$}$, by equaliser



We use this to order sets $\mathcal{E}(d, \Omega)$ and find, as we would expect, that

$$f \subseteq g \quad \text{iff} \quad \chi_f \leq \chi_g$$

where $\subseteq$ is subobject inclusion and $\chi_f \leq \chi_g$ is defined to hold iff $\langle \chi_f, \chi_g \rangle$ factors through e (that is, k exists as in the diagram above).

Since e is monic, we can define its character map and as we would expect this is the intuitionist operator $\Rightarrow : \Omega \times \Omega \to \Omega$. The object Ω is then revealed as an intuitionist or Heyting algebra (HA). The arrow $\Rightarrow$ is used to define the characteristic Heyting algebra operator $\models\!\!\Rightarrow$ for each $\mathcal{E}(d, \Omega)$ and as a result for each $\mathrm{Sub}_{\mathcal{E}}(d)$, and then subobject lattices are revealed likewise as HAs.

Since all details of the just described algebraic structure of Ω will exist as objects and arrows of the topos, the object Ω is described as an *internal* algebra. The use of the term "internal" is something of an extension of the previously offered definition, but does not seem to breach the spirit of that definition: whether it does or not depends on whether or not one requires that the HA in question be understood as the *set* of Ω together with operator arrows and order object.

An alternative method of specifying the logical algebras of any topos is available. This new method produces the same structure but without (obvious) reference to the Yoneda lemma. Given a subobject classifier *true*

(1) $\cap : \Omega \times \Omega \to \Omega$ is the classifying map for the product map $\langle true, true \rangle$;

(2) $\cup : \Omega \times \Omega \to \Omega$ is the classifying map of the image of map

$$[\langle true_\Omega, id_\Omega \rangle, \langle id_\Omega, true_\Omega \rangle]$$

where $true_\Omega$ is the map $\Omega \to 1 \xrightarrow{true} \Omega$;

(3) $\Rightarrow : \Omega \times \Omega \to \Omega$ is the classifying map of the equaliser

$$e : \circledcirc\!\!\!\!\!\leq\ \rightarrowtail \Omega \times \Omega \underset{pr_1}{\overset{\cap}{\rightrightarrows}} \Omega;$$

(4) $\neg : \Omega \to \Omega$ is the classifying map of the arrow $false : 1 \to \Omega$ which is the classifying map of the unique arrow $\emptyset \to 1$ from the initial object to the terminal object.

3. Image Factorisation

Lastly, a summary of some facts about image or epi-monic factorisation in a topos. We follow the Goldblatt [1984] presentation. In a topos $\mathcal{E}$, for any arrow $a \xrightarrow{f} b$ we can form the pushout of f along f

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ b & \xrightarrow[p]{} & r \end{array}$$

We can also form the equaliser of the pushout $b \underset{q}{\overset{p}{\rightrightarrows}} r$. We denote this equaliser by $im\ f\colon f(a) \to b$. Since $q \cdot f = p \cdot f$, there is a unique $f^*\colon a \to f(a)$ making

$$\begin{array}{ccccc} f(a) & \xrightarrow{im\ f} & b & \underset{q}{\overset{p}{\rightrightarrows}} & r \\ {\scriptstyle f^*}\nwarrow & & \nearrow{\scriptstyle f} & & \\ & a & & & \end{array}$$

commute. It happens that $im\ f$ is the smallest subobject through which f factors and that f^* is epic. This is the construction within a topos of what is called the *epi-monic* or *image factorisation* of the arrow f. In general, the image factorisation of an arrow f is the production of an epic f^* and a monic $im\ f$ of which the composite $im\ f \cdot f^*$ is f, and of which it can be said that whenever there is an epic g and a monic h such that $h \cdot g = f$, there is exactly one k making the diagram

$$\begin{array}{ccc} f(a) & \xrightarrow{im\ f} & b \\ {\scriptstyle f^*}\uparrow & \searrow{\scriptstyle k} & \uparrow{\scriptstyle h} \\ a & \xrightarrow[g]{} & c \end{array}$$

commute. It follows from the availability of the construction that epi-monic factorisations exist for any arrow within a topos. The notion may also be considered in more general categories however in this text we will not need it.

The idea of image factorisation is the generalisation to categories of the well-known idea that any set function $A \xrightarrow{f} B$ can be factored into a surjection $f^*: A \twoheadrightarrow f(A)$ followed by an injection $f(A) \hookrightarrow B$ where $f(A) = \{f(x): x \in A\}$ and $f^*(x) = f(x)$, all $x \in A$.

This completes our presentation of such basic features of category and topos theory as we need for later discussion. We will, in later chapters, introduce other structures and constructions which, given the level of scrutiny they have been subjected to over the years, should count as basic to category and topos theory, but which, given our aims, do not count as introductory. Accordingly we leave our exposition of Grothendieck toposes and sheaves until Part III and particularly chapter eight. With the next chapter we introduce such aspects of the study of logic, and particularly paraconsistent and intuitionist logic, that we will need for the rest of this work.

CHAPTER 3: THE HA DUAL

Introduction: Heyting algebras are known to occur within and around the structure of toposes and in particular to occur as subobject lattices. In anticipation of categorial constructions exhibiting dualised lattices, we consider the property of Heyting algebra duals that they are algebras for paraconsistent logics. These are logics that non-trivially allow that a sentence and its negation have overlapping truth values by which we mean that the conjunction of a sentence and its negation has some value other than false without the logic containing all other sentences. The text from which we take our initial understanding of a paraconsistent logic and logics in general is Mortensen's *Inconsistent Mathematics*, [1995].

In a later chapter we give a detailed presentation of a system of rules of inference which, following the conventions of the literature, we call a logic. That collection of rules is better called a deduction system. In what follows we lay out what we mean when we use the term logic in its technical sense. It should become apparent that deduction systems and logics are closely related. We shall give a description of what counts as a paraconsistent logic and a paraconsistent logical algebra in the sense that we will use throughout the rest of this text. The systems we concern ourselves with are essentially those that can be built using the closed sets of topological spaces.

This chapter has four sections. The first is largely expository. Much of what is presented in this section is well known. I provide a formal logical language, a notion of a Heyting algebra (HA), a notion of a Brouwerian algebra (BrA), and a demonstration that the HA and BrA notions are dual. Since the bulk of the thesis is on the existence of algebras of this type, there is a need to be reasonably detailed in the setting up of terminology and the presentation of definitions. I am concerned, too, to be able to make it clear that while a Boolean algebra is both a HA and a BrA, an algebra may be both HA and BrA without being Boolean. In the first section of the present chapter this is merely suggested but it follows from the results of chapters 6, 9, and 10. In particular, since in chapter 10 I demonstrate that any subobject lattice in any Grothendieck topos is both HA and BrA, then if any lattice that is both HA and BrA has to be Boolean, then all Grothendieck toposes must be Boolean toposes; and since there are Grothendieck toposes that are not Boolean, there are lattices that are both BrA and HA without being Boolean. Alternatively note that in part two of section one of chapter 2 I demonstrate that any $\bigcap$-complete lattice with a unit is a BrA so, by duality, any $\bigcup$-complete lattice with a zero is a HA, and so if any lattice that is both HA and BrA must be Boolean, then any bounded, complete lattice must be Boolean. In any case, the most we can say of HA–negation ($\neg$) in relation to BrA–negation ($\ulcorner$) on the same lattice $(\mathcal{L}, \sqsubseteq)$ is that $\ulcorner a \sqsubseteq \neg a$ for any $a \in \mathcal{L}$. For $(\mathcal{L}, \sqsubseteq)$ to be Boolean we require that $\ulcorner a = \neg a$.

The second section reproduces Mortensen's *Inconsistent Mathematics* [1995] notion of a paraconsistent algebra. I demonstrate that BrA's are paraconsistent algebras in this sense. I reproduce some of Mortensen's results on the algebraic nature of paraconsistent algebras. Given the philosophical significance of paraconsistent logic and the relative novelty of paraconsistency as a topic in category theory, the section serves the purpose of making the reader aware of some of the significance of finding BrAs in mathematical structures.

Section three exists to provide us with a notion of connection between logics defined with respect to dual algebras. Notions of duality for operators on the algebras are used to formalise a language dualisation and this is used to provide a notion of dual valuations for (dual) logical languages. With these notions we can provide some illumination for the nature of logics defined with respect to BrAs. This way we can give some philosophical sense to the phrase "paraconsistent logic" as I use it (and its variations) in the thesis. The fourth section continues this kind of investigation with respect to slightly different notions of valuation. I also give some discussion of dualisation of theoremhood semantics. Again this contributes to a philosophical understanding of the kind of logic for which we will be seeking categorial semantics.

1. Langugages, Logics, and Dual Algebras

1.1: A *language* of a logic is a collection of atomic terms, term forming operators, predicates or relations, sentential operators, variables (for terms and for sentences), and quantifiers. Atomic terms refer to individuals and can be considered names. Term forming operators are functions. We use the standard notions of predicates, variables and quantifiers. Sentential operators are connectives that make sentences from sentences. We concern ourselves with the principal connectives $\sim$ (not), $\neg$ (intuitionist negation), $\ulcorner$ (paraconsistent negation), & (and), $\vee$ (or), $\Rightarrow$ (intuitionist implication), $\dot{-}$ (pseudo difference). We frequently use $\wedge$ in place of &. Just as frequently we will use lattice operators $\sqcap$ and $\sqcup$ in place of & and $\vee$ given that we use lattices to interpret these languages. On the sense of the $\dot{-}$ connective Goodman, in "The logic of contradiction",[1981], has suggested the name *but not*. Our preference will be for the name *pseudo difference* after the name of the characteristic operator of the Brouwerian algebras that we will use to interpret the connective.

Formulae are defined by induction so that

(1)(a) if f is an n-place term forming operator and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a formula;

(b) if R is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a formula;

(2) if ψ and φ are formulae, then so are $\sim\varphi$, $\neg\varphi$, $\ulcorner\varphi$, $\varphi\&\psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, and $\varphi \dot{-} \psi$;

(3) if φ is a formula and v is a variable, then $\exists v\varphi$ and $\forall v\varphi$ are formulae.

The formulae described by (1)(a) and (b) are *atomic*. The formulae described by (2) and (3) are *logically compound*.

Sentences are formulae with no variables outside the scope of a quantifier or, in other words, with no free variables. *Uniform substitution* for sentence variables is the process of making new formulae from old by replacing all instances of a sentence variable in a given formulae with a given sentence or formula. We also have the notion of uniform substitution of terms for term variables. A *rule of inference* is a specification that given any sentence of a particular form, we may derive some further sentence of a further particular form. Finally, a *logic* is a set of sentences of a given language closed under uniform substitution, and closed under a collection of rules of inference, also called a *consequence relation*.

The semantics of a language are provided by some interpretation function that associated well formed sentences with some more or less arbitrary value. One or more of these values will be *designated*, and if on all valuations of a given sort a sentence receives a designated value, then that sentence will be called a *theorem*. We can specify a logic by describing a system of valuations and collecting together all theorems. Semantic consequence relations are typically related to valuations by the condition that if the value of a sentence φ is less than that of a sentence ψ in an appropriate sense, such as that of a lattice order, then ψ can be thought of as a *consequence* of φ. Given the right sort of valuation system, a collection of theorems will be a logic.

One can alternatively specify a logic by listing some rules of inference along with some subset of theorems which when closed under the rules of inference yield the complete set of theorems. This is the axiomatic presentation and the theorems of the given subset are the *axioms*.

A *logical algebra* is some systematic algebraic method for determining the valuation of a logically compound sentence given valuations for that sentence's constituent parts. Certain types of lattice with the right sort of operators can be understood as logical algebras. The elements of the lattice are used as sentence values, and the lattice operators interpret the connectives. Certain elements of the lattice will be distinguished as relating to theoremhood. Again the logic is the collection of theorems.

With this chapter we will be concerned with developing logics in terms of the Heyting algebras and their duals. We will shortly define both types of algebra and indicate the sense of duality we are using. In the meantime, to prompt intuitions, we note that any topology of open sets is a Heyting algebra, and any Heyting algebra can be understood as some topology of open sets on some space. Likewise, the Heyting algebra duals are the closed set topologies.

Formally, a *Heyting algebra* is a relative pseudo complemented lattice with a zero. To explain, we note that lattices are a subclass of the posets. A *poset* or partially ordered set is a set P together with a reflexive, transitive, and antisymmetric binary relation R defined on P. Relation R is *reflexive* when, for all $p \in \mathrm{P}$, $\langle p, p\rangle \in R$; R is *transitive* when, for all $p, q, r \in \mathrm{P}$, if $\langle p, q\rangle \in R$ and $\langle q, r\rangle \in R$, then $\langle p, r\rangle \in R$; and R is *antisymmetric* when, for all $p, q \in \mathrm{P}$, if $\langle p, q\rangle \in R$ and $\langle q, p\rangle \in R$, then $p = q$. Such a relation R is commonly called a partial ordering and is commonly denoted by $\sqsubseteq$ or some variant. We will write "$p \sqsubseteq q$" in place of "$\langle p, q\rangle \in R$". For any $p, q \in \mathrm{P}$, the *least upper bound* (lub) or *join* for p and q is an element of P, denoted $p \sqcup q$, such that $p \sqsubseteq p \sqcup q$ and $q \sqsubseteq p \sqcup q$, and if there is some $z \in \mathrm{P}$ such that both $p \sqsubseteq z$ and $q \sqsubseteq z$, then $p \sqcup q \sqsubseteq z$. It will follow that $p \sqcup (q \sqcup r) = (p \sqcup q) \sqcup r$ and that $p \sqcup q = q \sqcup p$. A *greatest lower bound* (glb) or *meet* for p and q is an element of P, denoted $p \sqcap q$, and is such that $p \sqcap q \sqsubseteq p$ and $p \sqcap q \sqsubseteq q$, and if there is some $z \in \mathrm{P}$ such that $z \sqsubseteq p$ and $z \sqsubseteq q$, then $z \sqsubseteq p \sqcap q$. We will have that $p \sqcap (q \sqcap r) = (p \sqcap q) \sqcap r$ and that $p \sqcap q = q \sqcap p$. A *lattice* is a poset

$(\mathrm{P}, \sqsubseteq)$ where, for every $p, q \in \mathrm{P}$, there exists in P a lub and a glb with respect to $\sqsubseteq$. A lattice has a *zero* if there is some element $\emptyset \in \mathrm{P}$ such that, for all $p \in \mathrm{P}$, $\emptyset \sqsubseteq p$. A lattice has a *unit* if there is some element $1 \in \mathrm{P}$ such that, for all $p \in \mathrm{P}$, $p \sqsubseteq 1$. A lattice with a zero and a unit is called *bounded*. A lattice is *distributive* if, for all $p, q, r \in \mathrm{P}$, both

$$p \sqcap (q \sqcup r) = (p \sqcap q) \sqcup (p \sqcap r)$$

and

$$p \sqcup (q \sqcap r) = (p \sqcup q) \sqcap (p \sqcup r).$$

In a bounded lattice an element p has a *meet complement* if there is some $q \in \mathrm{P}$ such that $p \sqcap q = \emptyset$. The element p has a *join complement* if there is some q such that $p \sqcup q = 1$. An element q is a *(Boolean) complement* if it is both a meet and a join complement for some p.

We change our symbols slightly and let $\mathcal{L}$ be a lattice. For lattice elements a, b, there is a *pseudo complement of a relative to b* in the lattice if there is some $c \in \mathcal{L}$ such that

$$\text{for all } x \in \mathcal{L}, \quad x \sqsubseteq c \quad \text{iff} \quad a \sqcap x \sqsubseteq b.$$

This element c is denoted by $a \Rightarrow b$. A lattice is said to *have relative pseudo complements* or *be relative pseudo complemented* (rpc) if $a \Rightarrow b \in \mathcal{L}$ for all $a, b \in \mathcal{L}$. An rpc lattice will always have a unit since for any $a, x \in \mathcal{L}$, $x \sqsubseteq a \Rightarrow a$. If the lattice has a zero, $\emptyset$, we can define a complement operator $\neg$ by allowing that for any $a \in \mathcal{L}$, $\neg a \underset{df}{=} a \Rightarrow \emptyset$. In fact, $\neg$ is a meet complement operator. When our lattice is an open set topology ordered by set inclusion, $\neg a$ proves to be $\bigcup\{c \in \mathcal{L}\colon a \cap c = \emptyset\}$; that is, $\neg a$ is the greatest element $c \in \mathcal{L}$ such that $a \cap c = \emptyset$.

The class of all rpc lattices with zeros are used to characterise the intuitionistic propositional logic IL – see for example McKinsey and Tarski [1946] and the references there identified. Consider a language as defined above but restricted to sentential operators $\sim, \&, \vee, \Rightarrow$. Consider the set of all well formed sentences of this language. An IL-valuation, v, is a function from the set of sentences to some rpc lattice with a zero so that for sentence S

(1) if S is atomic, $v(S)$ is some element of the lattice;

(2) if S is $\sim S_1$ where S_1 is a sentence, then $v(S) = \neg v(S_1)$;

(3) if S is $S_1 \& S_2$ where S_1 and S_2 are sentences, then $v(S) = v(S_1) \sqcap v(S_2)$;

(4) if S is $S_1 \vee S_2$, then $v(S) = v(S_1) \sqcup v(S_2)$;

(5) if S is $S_1 \Rightarrow S_2$, then $v(S) = v(S_1) \Rightarrow v(S_2)$.

If, for all rpc lattices with zeros and all possible valuations v on those lattices, we have $v(S) = 1$ for sentence S, then that sentence is a theorem of IL. The logic IL was developed by Arend Heyting; the IL-characteristic algebras, the rpc lattices with zeros, have since become commonly identified as Heyting algebras.

There is an alternative and equivalent presentation of IL in terms of axioms – see for example Heyting's *Intuitionism*, [1966]. There are eleven axioms and one rule of inference:

I. $\alpha \Rightarrow (\alpha \& \alpha)$

II. $(\alpha \& \beta) \Rightarrow (\beta \& \alpha)$

III. $(\alpha \Rightarrow \beta) \Rightarrow ((\alpha \& \gamma) \Rightarrow (\beta \& \gamma))$

IV. $((\alpha \Rightarrow \beta) \& (\beta \Rightarrow \gamma)) \Rightarrow (\alpha \Rightarrow \gamma)$

V. $\beta \Rightarrow (\alpha \Rightarrow \beta)$

VI. $(\alpha \& (\alpha \Rightarrow \beta)) \Rightarrow \beta$

VII. $\alpha \Rightarrow (\alpha \vee \beta)$

VIII. $(\alpha \vee \beta) \Rightarrow (\beta \vee \alpha)$

IX. $((\alpha \Rightarrow \gamma) \& (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma)$

X. $\neg \alpha \Rightarrow (\alpha \Rightarrow \beta)$

XI. $((\alpha \Rightarrow \beta) \& (\alpha \Rightarrow \neg \beta)) \Rightarrow \neg \alpha$

Rule of inference – *Detachment*: From α and $\alpha \Rightarrow \beta$, derive β.

Essentially, IL has all the axioms of classical logic barring $(\alpha \vee \neg\alpha)$. When we come to discuss paraconsistent logic further we shall avoid the axiomatic presentation in that more variation on the specification of a logic is available simply by changing the number and sort of valuations we consider for algebras. In particular we shall consider the logics derived from individual algebras.

1.2: The *dual* of a lattice $\mathcal{L}$ is a lattice $\mathcal{L}^{op}$ which has the same collection of elements, and has a reversed lattice order in the sense that $a \sqsubseteq_{\mathcal{L}} b$ iff $b \sqsubseteq_{\mathcal{L}^{op}} a$ where $\sqsubseteq_{\mathcal{L}}$ is the order on $\mathcal{L}$ and $\sqsubseteq_{\mathcal{L}^{op}}$ is the order on $\mathcal{L}^{op}$. We will sometimes use the phrases *lattice dual* or *lattice dualisation* to indicate this notion of reversing or reversed lattice orders. We will also speak of the *dual lattice* of $\mathcal{L}$ and mean $\mathcal{L}^{op}$. We shall also allow that any lattice $\mathcal{L}'$ isomorphic to $\mathcal{L}^{op}$ be identified as the dual of $\mathcal{L}$. From the point of view of lattice theory (or at least within the category of lattices) an isomorph is as good as the real thing. We shall distinguish that $\mathcal{L}^{op}$ for which $a \in \mathcal{L}$ iff $a \in \mathcal{L}^{op}$ by the title *the canonical dual* of $\mathcal{L}$.

This notion of lattice dualisation is exactly categorial in the sense that since any lattice $\mathcal{L}$ forms a poset category $\mathbf{L}$ in which the objects are lattice elements and the morphisms are the lattice ordering relationships, the lattice $\mathcal{L}^{op}$ forms the poset category $\mathbf{L}^{op}$ which is the categorial dual of $\mathbf{L}$. However we maintain the terminology of lattice dualisation, since we will later want to identify structures within categories that are in fact lattice duals but are not derived by dualisation within the category.

McKinsey and Tarski in their "On closed elements in closure algebras", [1946], used the notion of a Brouwerian algebra to discuss algebras of closed sets. Using the terminology of Rasiowa and Sikorski's *Mathematics of Metamathematics*, [1963], we understand a Brouwerian algebra to be a pseudo differenced lattice with a unit. A lattice $\mathcal{L}$ is *pseudo differenced* if for all $a, b \in \mathcal{L}$, there exists an element $b \dot{-} a \in \mathcal{L}$

of which it is true that

$$\text{for all } x \in \mathcal{L}, \quad b \dot{-} a \sqsubseteq x \quad \text{iff} \quad b \sqsubseteq a \sqcup x.$$

Any pseudo differenced lattice has a zero since for all $a, x \in \mathcal{L}$, $a \dot{-} a \sqsubseteq x$. When $\mathcal{L}$ has a unit, 1, we can define a complement operator $\ulcorner$ by allowing that for any $a \in \mathcal{L}$, $\ulcorner a \underset{df}{=} 1 \dot{-} a$. In fact $\ulcorner$ is a join complement operator.

When our lattice is a closed set topology ordered by set inclusion, $\ulcorner a$ proves to be $\bigcap\{c \in \mathcal{L}: a \cup c = 1\}$; that is, $\ulcorner a$ is the least element $c \in \mathcal{L}$ such that $a \cup c = 1$. Since the lattice elements are closed sets, that least element exists and is unique.

There is a significant relationship between the Brouwerian algebras (the BrAs) and the Heyting algebras (the HAs).

Theorem 1.2.1: *The lattice dual of a Heyting algebra is a Brouwerian algebra and vice versa.*

Proof: let $(\mathrm{H}, \sqsubseteq)$ be a Heyting algebra. Define $(\mathrm{H}, \sqsubseteq_{op})$ so that for $c, d \in \mathrm{H}$, $c \sqsubseteq d$ iff $d \sqsubseteq_{op} c$. Plainly $(\mathrm{H}, \sqsubseteq)$ and $(\mathrm{H}, \sqsubseteq_{op})$ are lattice duals.

Consider some $z \in \mathrm{H}$. If z is the lub for some $a, b \in \mathrm{H}$ with respect to $\sqsubseteq$, it is the glb for a, b with respect to $\sqsubseteq_{op}$. Also, if z is a glb with respect to $\sqsubseteq$, it is a lub with respect to $\sqsubseteq_{op}$.

Now suppose z is $a \Rightarrow b$ for some $a, b \in \mathrm{H}$. We know, then, that for any $x \in \mathrm{H}$, $x \sqsubseteq z$ iff $a \sqcap x \sqsubseteq b$. If we use $\sqcup_{op}$ to denoted lub with respect to $\sqsubseteq_{op}$, then we can say not only that $x \sqsubseteq z$ iff $z \sqsubseteq_{op} x$, but that $a \sqcap x \sqsubseteq b$ iff $b \sqsubseteq_{op} a \sqcup_{op} x$. In other words, for any $x \in \mathrm{H}$,

$$z \sqsubseteq_{op} x \quad \text{iff} \quad b \sqsubseteq_{op} (a \sqcup_{op} x).$$

So with respect to $\sqsubseteq_{op}$, element z is the pseudo difference $b \dot{-} a$.

Now, $a \Rightarrow b$ is defined for every $a, b \in (\mathrm{H}, \sqsubseteq)$ so $b \dot{-} a$ is defined for every $a, b \in (\mathrm{H}, \sqsubseteq_{op})$. This makes $(\mathrm{H}, \sqsubseteq_{op})$ an rpc lattice. Furthermore, since $(\mathrm{H}, \sqsubseteq)$ has a zero, $(\mathrm{H}, \sqsubseteq_{op})$ has a unit, and it follows that $(\mathrm{H}, \sqsubseteq_{op})$ is a Brouwerian algebra. The proof that where $(\mathrm{H}, \sqsubseteq)$ is a BrA, $(\mathrm{H}, \sqsubseteq_{op})$ is a HA is performed in the same way. □

It follows as a corollary that any topology of closed sets is a BrA, since such a topology ordered by set inclusion is the lattice dual of some topology of open sets ordered by set inclusion. We can, however, usefully prove this corollary directly. Let X be a topological space with a topology Ξ of closed sets. Any closed set topology ordered by set inclusion is a BrA since for any $A, B \in \Xi$ we can define $B \dot{-} A = cl((X - A) \cap B)$, where cl is the closure operator that defines the topology. Alternatively, and equivalently, we allow that $B \dot{-} A$ is the smallest element of Ξ containing $((X - A) \cap B)$. Since Ξ is a closed set topology, it is closed with respect to intersections and the smallest superset of $((X - A) \cap B)$ will exist.

Theorem 1.2.2: *Any closed set topology is a* BrA.

Proof: suppose any $B, A, Z \in \Xi$ which is a closed set topology on X.

Suppose $(X - A) \cap B \subseteq Z$. Then

$$\begin{aligned} ((X - A) \cap B) \cup A &\subseteq Z \cup A, \\ ((X - A) \cup A) \cap (B \cup A) &\subseteq Z \cup A, \\ B \cup A &\subseteq Z \cup A, \\ B &\subseteq Z \cup A. \end{aligned}$$

And on the other hand, if $B \subseteq Z \cup A$, then

$$\begin{aligned} B \cap (X - A) &\subseteq (Z \cup A) \cap (X - A), \\ &\subseteq (Z \cap (X - A)) \cup (A \cap (X - A)), \\ &\subseteq Z \cap (X - A), \\ B \cap (X - A) &\subseteq Z. \end{aligned}$$

So, $(X - A) \cap B \subseteq Z$ iff $B \subseteq Z \cup A$.

Since there will be a smallest such Z, there is an element in Ξ that satisfies the definition of $B \dot{-} A$ for any $B, A \in \Xi$. This makes $(\Xi, \subseteq)$ a pseudo differenced lattice. Plainly, too, the lattice has a unit, since there is a largest closed set namely X. □

Remark: Notice the historical point that Brouwerian algebras were investigated in relation to IL. Using the zero of a lattice to indicate theoremhood, it was discovered that Brouwerian algebras characterise IL. Obviously, since we want BrAs to do a different algebraic job, we will not be using these theoremhood semantics. The guiding insight for our project is that to produce a paraconsistent algebra all one has to do is reverse the lattice order on a Heyting algebra. Initially, then, we make no dualisation of the standard scheme that theoremhood is associated with the lattice unit. This is natural enough since theoremhood semantics are not formally part of the notion of algebra.

1.3: One feature of the BrA operators as strictly dual to those of the Heyting algebras is that $\dot{-}$ is not a good implication operator. The operators $\sqcap, \sqcup, \ulcorner$ can be interpreted as conjunction, disjunction, and (as we shall see) paraconsistent negation. The operator $\dot{-}$, however, suffers from the condition that $b \sqsubseteq a$ iff $b \dot{-} a = \emptyset$. For the present we take no position on implication other than to note a solution suggested for just such a problem in chapter eleven of Mortensen [1995]. The solution is to define a simple implication operator and add it to the stock of operators. For a lattice $\mathcal{L}$, the simple implication operator, denoted $\rightarrow$, is defined so that

$$\text{for } a, b \in \mathcal{L}, \quad a \rightarrow b = \begin{cases} 1 & \text{iff } a \sqsubseteq b \\ \emptyset & \text{otherwise.} \end{cases}$$

A BrA together with $\rightarrow$ will be denoted BrA$^{\rightarrow}$. As an alternative to the introduction of $\rightarrow$ we could rely on the metalinguistic $\models$ related to $\sqsubseteq$ on the lattice. Specifically, for sentences A,B we assert A$\models$B iff $v(\mathrm{A}) \sqsubseteq v(\mathrm{B})$, all valuations v.

1.4: Boolean algebras will from time to time become part of our discussion. These algebras are defined to be those bounded, distributive lattices that have a complement operator that describes both a meet and join complement. Any Boolean algebra is both a Heyting algebra and a Brouwerian algebra. To see this suppose a Boolean algebra BA with a complement operator denoted by $-$. For any $a, b \in$ BA, there is a pseudo complement $a \Rightarrow b$, of a relative to b in BA, namely $a \Rightarrow b = -a \sqcup b$. There is also a pseudo difference $b \dot{-} a$, namely $b \dot{-} a = -a \sqcap b$

Of interest to us in later chapters will be the circumstances under which an algebra that is both Heyting and Brouwerian is also Boolean. It is not immediate that a lattice $\mathcal{L}$ that is both BrA and HA ends up being Boolean; it is possible that for some $a \in \mathcal{L}$, both $\neg a$ and $\ulcorner a$ exist in $\mathcal{L}$ without coinciding, that is without it being true that $\ulcorner a = \neg a$. When such complements do not coincide, there is no claim that $\ulcorner$ is more than a meet complement nor that $\neg$ is more than a join complement unless one or both of these claims were true to begin with. But when complements do coincide for all lattice elements, we plainly have a bounded, distributive lattice with a Boolean complement operator, and so a Boolean algebra.

2. Paraconsistent Algebras

With this section we define a notion of paraconsistent algebra. The definition we give is exactly that found in Mortensen [1995]. Also found in Mortensen's [1995] are those properties of a paraconsistent algebra that we give here as P-theorems one to six. We have introduced some minor modifications to the proofs of these theorems. We find as a straightforward consequence of the definition that any Brouwerian algebra is also a paraconsistent algebra.

A *paraconsistent algebra* is a structure $\langle \mathcal{L}; \sqsubseteq, \sqcap, \sqcup, \ulcorner, 1, \emptyset \rangle$ where $\mathcal{L}$ is a lattice ordered by $\sqsubseteq$ with meet and join operators $\sqcap$ and $\sqcup$ respectively; the lattice has a zero $\emptyset$ and a unit 1; the lattice also has a complement operator, $\ulcorner$, defined with respect to $\sqsubseteq$ so as to satisfy condition P that for all $a, b \in \mathcal{L}$,

$$a \sqcup b = 1 \quad \text{iff} \quad \ulcorner a \sqsubseteq b.$$

It follows as a result of this condition that

(1) $a \sqcup \ulcorner a = 1$; (2) $\ulcorner\ulcorner a \sqsubseteq a$; (3) $\ulcorner(a \sqcup b) \sqsubseteq \ulcorner a \sqcap \ulcorner b$;

(4) $\ulcorner(a \sqcap b) = \ulcorner a \sqcup \ulcorner b$; (5) $\ulcorner(a \sqcap \ulcorner a) = 1$; (6) in general, $a \sqcap \ulcorner a \neq \emptyset$.

For a lattice $\mathcal{L}$ to be a paraconsistent algebra, it will be sufficient that it be a BrA.

Theorem 2.1: *any* BrA $< \mathcal{L}; \sqsubseteq, \sqcap, \sqcup, \dot{-}, \ulcorner, 1 >$ *satisfies condition* P.

Proof: by definition, $1 \dot{-} a \sqsubseteq b$ iff $1 \sqsubseteq a \sqcup b$. So, $\ulcorner a \sqsubseteq b$ iff $a \sqcup b = 1$.□

We now verify that results (1)-(6) apply given condition P. Presume a paraconsistent algebra $\mathcal{L}$ and $a, b \in \mathcal{L}$. Notice in particular that we are not restricting this part of the discussion to BrAs.

P-Theorem 1: $a \sqcup \ulcorner a = 1$.

Proof: under condition P, $\ulcorner a \sqsubseteq \ulcorner a$ iff $a \sqcup \ulcorner a = 1$. □

P-Theorem 2: $\ulcorner\ulcorner a \sqsubseteq a$.

Proof: since $\mathcal{L}$ is a distributive lattice, we always have that $a \sqcup b = b \sqcup a$, and in particular we always have that $a \sqcup \ulcorner a = \ulcorner a \sqcup a$. Then, by P-Th.1, $\ulcorner a \sqcup a = 1$, and by condition P, $\ulcorner\ulcorner a \sqsubseteq a$. □

P-Theorem 3: $\ulcorner(a \sqcup b) \sqsubseteq \ulcorner a \sqcap \ulcorner b$.

Proof: Using P-Th.1 and the properties of distributive lattices,

$$\begin{aligned} 1 &= 1 \sqcup a \\ &= (b \sqcup \ulcorner b) \sqcup a && \text{(P-Th.1)} \\ &= b \sqcup (\ulcorner b \sqcup a) \\ &= b \sqcup (1 \sqcap (\ulcorner b \sqcup a)) \end{aligned}$$

$$
\begin{aligned}
&= 1 \sqcup a \\
&= b \sqcup ((a \sqcup \ulcorner a) \sqcap (\ulcorner b \sqcup a)) \qquad \text{(P-Th.1)} \\
&= b \sqcup ((\ulcorner a \sqcap \ulcorner b) \sqcup a) \\
&= (a \sqcup b) \sqcup (\ulcorner a \sqcap \ulcorner b).
\end{aligned}
$$

So, by condition P, $\ulcorner(a \sqcup b) \sqsubseteq \ulcorner a \sqcap \ulcorner b$. □

P-Theorem 4: $\ulcorner(a \sqcap b) = \ulcorner a \sqcup \ulcorner b$.

Proof: by P-Th.2 $\ulcorner\ulcorner(a \sqcap b) \sqsubseteq a \sqcap b$, and by definition of $\sqcap$, $a \sqcap b \sqsubseteq a$. So

$$\ulcorner\ulcorner(a \sqcap b) \sqsubseteq a.$$

By condition P, $\ulcorner(a \sqcap b) \sqcup a = 1$ iff $\ulcorner\ulcorner(a \sqcap b) \sqsubseteq a$ so, $\ulcorner(a \sqcap b) \sqcup a = a \sqcup \ulcorner(a \sqcap b) = 1$ which under condition P means that

$$\ulcorner a \sqsubseteq \ulcorner(a \sqcap b).$$

In the same way we show that $\ulcorner b \sqsubseteq \ulcorner(a \sqcap b)$ (the first step is to note that $a \sqcap b \sqsubseteq b$ so, by P-Th.2, $\ulcorner\ulcorner(a \sqcap b) \sqsubseteq b$). It follows, by definition of $\sqcup$ as a lub operator, that

$$\ulcorner a \sqcup \ulcorner b \sqsubseteq \ulcorner(a \sqcap b).$$

Now, by the properties of a distributive lattice and by P-Th.1,

$$
\begin{aligned}
(a \sqcap b) \sqcup (\ulcorner a \sqcup \ulcorner b) &= \big(a \sqcup (\ulcorner a \sqcup \ulcorner b)\big) \sqcap \big(b \sqcup (\ulcorner a \sqcup \ulcorner b)\big) \\
&= \big((a \sqcup \ulcorner a) \sqcup \ulcorner b\big) \sqcap \big((b \sqcup \ulcorner b) \sqcup \ulcorner a\big) \\
&= (1 \sqcup \ulcorner b) \sqcap (1 \sqcup \ulcorner a) \\
&= 1
\end{aligned}
$$

which under condition P means that

$$\ulcorner(a \sqcap b) \sqsubseteq \ulcorner a \sqcup \ulcorner b.$$

So

$$\ulcorner(a \sqcap b) = \ulcorner a \sqcup \ulcorner b. \qquad \square$$

Notice that in Mortensen [1995] we find only the claim that $\ulcorner(a \sqcap b) \sqsubseteq \ulcorner a \sqcup \ulcorner b$.

P-Theorem 5: $\ulcorner(a \sqcap \ulcorner a) = 1$.

Proof: By P-Ths.4 and 1, $\ulcorner(a \sqcap \ulcorner a) = \ulcorner a \sqcup \ulcorner\ulcorner a = 1$. $\square$

Lemma 2.1: *for a paraconsistent algebra $\mathcal{L}$ to be Boolean, it is a necessary and sufficient condition that $a \sqcap \ulcorner a = \emptyset$ for all $a \in \mathcal{L}$.*

Proof: that the condition is necessary is trivial, so we prove only sufficiency.

$$\begin{aligned}
&\text{Let} \quad a \sqcap \ulcorner a = \emptyset. \quad \text{Then} && \\
(a \sqcap \ulcorner a) \sqcup \ulcorner\ulcorner a &= \emptyset \sqcup \ulcorner\ulcorner a, & \\
(a \sqcup \ulcorner\ulcorner a) \sqcap (\ulcorner a \sqcup \ulcorner\ulcorner a) &= \ulcorner\ulcorner a, & \text{(dist.latt.)} \\
a \sqcup \ulcorner\ulcorner a &= \ulcorner\ulcorner a, & \text{(P-Th.1)} \\
\text{so,} \quad a &\sqsubseteq \ulcorner\ulcorner a. & \\
\text{But} \quad \ulcorner\ulcorner a &\sqsubseteq a & \text{(P-Th.2)} \\
\text{and therefore} \quad \ulcorner\ulcorner a &= a. & \square
\end{aligned}$$

P-Theorem 6: *in general, $a \sqcap \ulcorner a \neq \emptyset$.*

Proof: by definition, any BrA is the lattice dual of some Heyting algebra and vice versa. As such, a BrA is Boolean if and only if its dual is Boolean. Since Heyting algebras are not in general Boolean, neither are BrAs. So in general, for elements of a BrA, and therefore of a paraconsistent algebra, $a \sqcap \ulcorner a \neq \emptyset$. $\square$

Remark: This last theorem encapsulates the idea that the algebras we are considering are indeed paraconsistent, which is to say represent toleration of inconsistencies. While in general it will not be the case that $a \sqcap \ulcorner a = 1$, it happens that, given an interpretation of $\ulcorner$ as a negation operator, sentences and their negations will overlap in value. If nothing else this is a contradiction. We therefore describe $\ulcorner$ as a paraconsistent operator. The forms of explicit contradictions will not appear as theorems of the logics developed from these algebras until we come to describe appropriate duals of the theoremhood semantics. Logics containing $\ulcorner$ and described by the usual theoremhood semantics are interesting for their particular description of the nature of negation. Logics containing explicit contradictions have a different role. This usually is to formalise the claim that there just are real contradictions.

3. Intuitionism's Dual

We have the opportunity to make use of existing results about IL to describe at least one of the logics we can generate with BrAs. Our task here is similar in spirit (if not in detail) to the (unrelated) project of Goodman [1981]. We shall describe a logic that is dual to IL. We call this logic DIL (IL^{op} would be more in keeping with our use of the *op* notation however we wish to avoid the suggestion that DIL is merely derivative; as a logic DIL is formally independent of IL.) We will have an infinite number of sentence letters, four connectives $\sqcap, \sqcup, \dot{-}, \ulcorner$, and parenthesis devices "(" and ")". The usual sentence formation rules apply. It will be useful to establish that there is a bijection between sentences of the DIL language and sentences of the propositional IL language. We do this by noting that atomic sentences are common to the languages and that where and only where there is an operator from the set $\{\sqcap, \sqcup, \Rightarrow, \neg\}$ in an IL-sentence, there are the respective operators from the set $\{\sqcup, \sqcap, \dot{-}, \ulcorner\}$ in some DIL-sentence. Explicitly, for sentence S in IL, we define sentence S^{op} in DIL so that

(1) if S is an atomic sentence, S^{op} is S;

(2) if S is $\neg S_1$ where S_1 is an IL-sentence, then S^{op} is $\ulcorner(S_1^{op})$;

(3) if S is $S_1 \sqcap S_2$ where S_1, S_2 are IL-sentences, then S^{op} is $S_1^{op} \sqcup S_2^{op}$;

(3) if S is $S_1 \sqcup S_2$, then S^{op} is $S_1^{op} \sqcap S_2^{op}$;

(4) if S is $S_1 \Rightarrow S_2$, then S^{op} is $S_2^{op} \dot{-} S_1^{op}$.

Plainly S is an IL-sentence iff S^{op} is a DIL-sentence. This process of "dualising" sentences is one we will use frequently. We have already seen a useful variant of it used in the proof of Theorem 3.1.2.1.

We will maintain a convention that IL-sentences be represented as S and that DIL-sentences be represented as S^{op}. We use a similar notation to denote dual valuations. We shall say that where v is a valuation of a set of sentences $\{S_i : i \in I\}$ on an algebra $\mathcal{L}$, the *dual valuation*, denoted v^{op}, is of the set of sentences $\{S_i^{op} : i \in I\}$ on the dual lattice $\mathcal{L}^{op}$ so that $v(S_i) = v^{op}(S_i^{op})$, all $i \in I$. Note the important point that while $\mathcal{L}$ and $\mathcal{L}^{op}$ have the same (or isomorphic) underlying sets, the elements of that set play different roles with respect to the orders defining $\mathcal{L}$ and $\mathcal{L}^{op}$. For example, if $x \in \mathcal{L}$ is the unit of $\mathcal{L}$, we have $y \sqsubseteq x$ for all $y \in \mathcal{L}$, but we also have $x \sqsubseteq_{op} y$ where $\sqsubseteq_{op}$ is the order that defines $\mathcal{L}^{op}$ over the same set. In other words, the same element x of the underlying set is the unit for $\mathcal{L}$ iff it is the zero for $\mathcal{L}^{op}$. Likewise x is the zero for $\mathcal{L}$ iff x is the unit for $\mathcal{L}^{op}$. A consequence of this is that for dual valuations v, v^{op} and dual sentences S, S^{op}, we have $v(S) = 1$ iff $v^{op}(S^{op}) = \emptyset$, and $v(S) = \emptyset$ iff $v^{op}(S^{op}) = 1$. It is fair to say that the standard usage of the generic 1 and $\emptyset$ for unit and zero can be a little misleading here. To be perfectly explicit about what we want to say, suppose that 1 and $\emptyset$ are the unit and zero of $\mathcal{L}$ and that $1'$ and $\emptyset'$ are the unit and zero of $\mathcal{L}^{op}$; when $\mathcal{L}^{op}$ is the canonical dual of $\mathcal{L}$, $\emptyset = 1'$ and $1 = \emptyset'$. To avoid confusing equations in the rest of this section, we maintain this slightly non-standard usage.

Plainly there is a bijection between valuations v of a set of sentences using connective set $\{\sqcap, \sqcup, \Rightarrow, \neg\}$ and valuations v^{op} of the dual set of sentences using connective set $\{\sqcup, \sqcap, \dot{-}, \ulcorner\}$.

We will say that a well formed sentence S^{op} is a DIL-theorem iff for all valuations v^{op} on all BrAs we have $v^{op}(S^{op}) = 1'$. The sequence of symbols $\models_{\text{DIL}} S^{op}$ will indicate that S^{op} is a DIL-theorem. Recall that an IL-sentence S is an IL-theorem iff for all valuations v on all Heyting algebras we have $v(S) = 1$. The sequence of symbols $\models_{\text{IL}} S$ will indicate that S is an IL-theorem.

Theorem 3.1: *if* $\models_{\text{IL}} S$, *then* $\models_{\text{DIL}} \ulcorner S^{op}$.

Proof: suppose $\models_{\text{IL}} S$. Then for any IL-valuation v we have $v(S) = 1$. By definition, $v(\neg S) = \emptyset$. The sentence dual of $\neg S$ is $\ulcorner S^{op}$. So for the dual valuation v^{op}, we will have $v^{op}(\ulcorner S^{op}) = 1'$. Since there is no Brouwerian algebra that is not the dual of a HA and no DIL-valuation that is not the dual of an IL-valuation, we have $\models_{\text{DIL}} \ulcorner S^{op}$. □

Notice that we cannot in general prove the converse of this theorem. Consider the following counterexample to that converse: for any element a in a BrA, P-Th.5 tells us that $\ulcorner(a \sqcap \ulcorner a) = 1$, and P-Th.6 tells us that in general $a \sqcap \ulcorner a \neq \emptyset$; it follows that $\not\models_{\text{DIL}} \ulcorner(A^{op} \sqcap \ulcorner A^{op})$ for any IL-sentence A, and that for some DIL-valuations v^{op} we have $v^{op}(A^{op} \sqcap \ulcorner A^{op}) \neq \emptyset'$; and so $v(A \sqcup \neg A) \neq 1$, so $\not\models_{\text{IL}} A \sqcup \neg A$. In other words, from $\models_{\text{DIL}} \ulcorner S^{op}$, it does not follow that $\models_{\text{IL}} S$.

Also, consider the converse of Theorem 3.1 in the following form: if not $\models_{\text{IL}} S$, then not $\models_{\text{DIL}} \ulcorner S^{op}$. The condition that $\not\models_{\text{IL}} S$ tells us only that there is some IL-valuation such that $s(S) \neq 1$. To establish that $\not\models_{\text{DIL}} \ulcorner S^{op}$, we need first to establish that $v(\neg S) \neq \emptyset$ (in other words that $v^{op}(\ulcorner S^{op}) \neq 1'$). For any element a in a HA we have that $a \sqcup \neg a \sqsubseteq 1$, so we are in general denied the desired result. We are however guaranteed the result if we assert that v is an IL-valuation on not just any HA but on a HA that is also a Boolean algebra. In other words, if S is not a theorem of classical logic, then $\ulcorner S^{op}$ is not a theorem of DIL. Since classical logic is non-trivial, there must be some sentences that are not DIL-theorems; so, DIL is not trivial as a logic. Another proof of the non-triviality of DIL arises from the following theorem (together with the fact that IL is non-trivial).

Theorem 3.2: $\models_{\text{DIL}} S^{op}$ *iff* $\models_{\text{IL}} \neg S$.

Proof: suppose $\models_{\text{DIL}} S^{op}$. Then for any DIL-valuation, $v^{op}(S^{op}) = 1'$. By definition, $v^{op}(\ulcorner S^{op}) = \emptyset'$. The sentence dual of $\ulcorner S^{op}$ is $\neg S$. So for the dual valuation v, we will have $v(\neg S) = 1$. Since there is no HA that is not the dual of some BrA, and no IL-valuation that is not the dual of some DIL-valuation, we have $\models_{\text{IL}} \neg S$.

Suppose $\models_{\text{IL}} \neg S$. Then for any IL-valuation, $v(\neg S) = 1$. Now, for any element a in a HA, $a \sqcap \neg a = \emptyset$, so if $\neg a = 1$, then $a = \emptyset$. It follows that for any IL-valuation, $v(S) = \emptyset$. It follows, too, that for any DIL-valuation, $v^{op}(S^{op}) = 1'$. So $\models_{\text{DIL}} S^{op}$. □

IL and DIL are both sublogics of classical logic (CL). The algebras that characterise CL are exactly the Boolean algebras used in just the same way as the HAs for IL and the BrAs for DIL. Since the collection of HAs includes the collection of Boolean algebras, any IL-theorem is a theorem of CL; and since the collection of BrAs includes the collection of Boolean algebras, any DIL-theorem is a CL-theorem. That IL and DIL are different logics can be demonstrated with respect to any sentence $A \sqcup \sim A$ where $\sim$ is a generic negation operator; the sentence $A \sqcup \sim A$ in DIL would be $A \sqcup \ulcorner A$, whereas in IL, it would be $A \sqcup \neg A$. For any a in a HA, it is true that $a \sqcap \neg a = \emptyset$, but it is not in general true that $a \sqcup \neg a = 1$, so $\not\models_{\text{IL}} A \sqcup \neg A$; but for any a in a BrA, $a \sqcup \ulcorner a = 1$ by definition, so $\models_{\text{DIL}} A \sqcup \neg A$. DIL and IL are formally different sets of sentences. Note, too, that since both DIL and IL are sublogics of CL, they are both consistent and non-trivial.

Theorem 3.3: *for any well formed* DIL*-sentence* S^{op},

$$\models_{\text{DIL}} \ulcorner(S^{op} \sqcap \ulcorner S^{op}).$$

Proof: this follows from P-Th.5. □

4. Individual Logics and Natural Duals

To characterise a logic we need not address ourselves to all algebras of a particular type. It is an interesting project in itself to develop those logics characterised by individual algebras. To develop an individual BrA logic we settle upon one BrA and consider all valuations of a set of sentences on that one lattice; we assume a consequence relation tied to the order on the lattice (this relation will have that $A \models B$ iff $v^{op}(A) \sqsubseteq_{op} v^{op}(B)$ for all valuations v^{op} on the fixed BrA) and say that the individual BrA logic is that set of DIL-sentences S^{op} such that for any valuation v^{op} on the fixed BrA, $v^{op}(S^{op}) = 1'$. It is non-trivial but straightforward that such a set of sentences is closed under $\models$ and uniform substitution.

It should be easy to see that these individual BrA logics will in general be consistent: for a given BrA if $1 \neq \emptyset$, then whenever S^{op} is a theorem, $\ulcorner S^{op}$ is not a theorem, and whenever $\ulcorner S^{op}$ is a theorem, S^{op} will not be a theorem. Furthermore, in general, these individual logics will not have contradictions as theorems since, for any element a of the appropriate algebra both $a \sqcap \ulcorner a \sqsubseteq a$ and $a \sqcap \ulcorner a \sqsubseteq \ulcorner a$ so that if $(a \sqcap \ulcorner a) = 1$, then both $a = 1$ and $\ulcorner a = 1$, which will be true only if $1 = \emptyset$.

We can introduce a further variation on the characterisation of logics. This calls for the notion of designated values. Up until this point we have given theoremhood semantics that assume just one designated value, namely the unit of a lattice. In the context of individual BrAs, we can allow that there be more than one designated value. To maintain closure under the lattice order based consequence relation, we usually require that any set of designated values be a filter on the fixed BrA. A subset F of the underlying set L of a lattice $\mathcal{L}^{op}$ is a filter on $\mathcal{L}^{op}$ if when $a \in$ F and $a \sqsubseteq_{op} b$, then $b \in$ F; and in addition, if $a \in$ F and $b \in$ F, then $a \sqcap b \in$ F. We need really only require that for $D \subsetneq$ L to be a set of designated values, if $a \in D$ and $a \sqsubseteq_{op} b$, then $b \in D$; this allows, for example, a set of designated values D that contains all lattice elements other than the zero. Our theoremhood semantics, then, whether the set D of designated values is a filter on $\mathcal{L}^{op}$ or not, are that sentence S^{op} is a theorem of the individual logic if on all valuations v^{op} with the fixed BrA as codomain, $v^{op}(S^{op}) \in D$.

In terms of dualising logics associated with individual HAs, a natural dualisation of the usual unit-as-designated-value scheme is to allow the collection of designated values for the HA-dual to be the set of all lattices elements other than the zero. This is suggested in Mortensen [1995] (p.104; see also Mortensen and Leishman, "Computing dual paraconsistent and Intuitionist logics", [1989]). This allows a simple duality between individual HA logics and individual BrA logics: suppose a HA, $\mathcal{L}$, and a logic, I, generated by considering all $\mathcal{L}$-valuations with 1 as the only designated value; let P be the logic generated by all valuations on BrA $\mathcal{L}^{op}$ with $\emptyset$ being the only non-designated value; the relationship between the logics is then expressed by

$$\models_{\mathrm{I}} S \quad \text{iff} \quad \not\models_{\mathrm{P}} S^{op}.$$

This surely is a most natural idea of dual logics. We consider similar logics similarly related when in a later chapter we come to discuss a logic which we call co-GL, the dual of geometric logic. In the meantime we carry forward from this chapter the essential idea that Browerian algebras and an adequate representation of inconsistency toleration in logics go hand in hand.

With the next chapter we begin Part II of the present work. We address ourselves to an existing attempt to describe within toposes such structures as would give rise to BrAs in place of the usual HAs.

Part II:

CATEGORIAL SEMANTICS FOR PARACONSISTENT LOGIC

CHAPTER 4: THE COMPLEMENT CLASSIFIER

Introduction: The notion of a complement classifier was originally introduced in Mortensen's *Inconsistent Mathematics*, [1995], as a tool with which to discuss paraconsistency within topos theory. Essentially one took a topos and reinterpreted its subobject classifier as a complement classifier. This had the effect of dualising the Heyting algebra structure that was the basis of the topos logic: the usual constructions for lub, and glb, *true*, and *false* become constructions for what are essentially their lattice duals. The discovery of what we here call a BrA associated with any topos is strictly ex-categorial, that is, external; an act of interpretation of an existing structure is required. The point however is that the existing structure is itself interpreted. The subobject classifier is a generalisation of a set-theoretic structure associated with characteristic functions; this generalisation, however, subsumes the structure associated with complement characteristic functions. In Mortensen [1995] the reinterpretation of the classifier is motivated by an analogy with the specification of topological spaces. There it is noted that it is as natural to specify such a space by its closed sets as by its open. The claim then seems to be that with respect to algebras $\mathcal{L}$ (based in Ω in a topos), we might just as naturally speak of algebras $\mathcal{L}^{op}$. It would seem that both algebras are just as natural in that they both successfully describe the same subobject structures within the topos. Our view here is different from that just described. Our preference is to find explicit internal constructions that demonstrate BrA properties. We recognise that the ex-categorial dualisation of the subobject classifier does indeed produce (external) BrAs, but we are unhappy with the fact that the dualisation is ex-categorial. Our concern is not that the dualisation is wrong, for it is not. It is a tenet of category theory that elements of objects are less important than (arrow) relationships between objects, so there is no category theoretic objection to renaming the *true* arrow *false*. Our concern is that we do not reveal any new features of the contents of toposes. In particular, as we shall see, the algebras of subobjects remain intuitionistic. These concerns derive from the type of role we assert for the complement classifier. We had hoped to use it as a tool to discover categories with subobject classifiers which, even as standardly construed, gave rise to BrA subobject lattices. It was presumed that such a classifier would reveal something on the nature of objects in the category. In this we have placed a slightly different emphasis on the complement classifier than was originally intended.

In this and the next chapter two negative results are obtained. The first is developed in this chapter and is that a complement classifier is formally indistinguishable from a subobject classifier. It follows that the notion of a complement classifier reveals the possibility of paraconsistent topos logic without acting as a tool to reveal particular paraconsistency structures. We follow this line of thought in chapters eleven and twelve. In chapter eleven we discover a genuine paraconsistency object in a category of covariant functors. Then in chapter twelve we discover that the object determines a complement classifier by being, on the one hand, a paraconsistency object, and on the other hand, the codomain of a subobject classifier. The point is that we needed first to discover that the object was a paraconsistent logic object before we could declare the existence of a complement classifier. In the chapter following the present one we develop the second of our negative results. We aim there at producing paraconsistent algebras by (accepting the usual interpretation of the subobject classifier and) developing a classifier for the duals of the subobjects, the quotient objects. We find that the quotient object classifier usually does not exist in a topos, and that where it does, it does not provide adequate algebraic structure for the development of its domain as a logic object. We leave open the question of such a classifier in more general categories. The search for structures that reveal BrAs on subobject lattices as standardly ordered will occupy the remainder of Parts II and III. The idea of reinterpretation of lattices as investigated here will remain useful: we will use a related technique in chapter fourteen when we come to interpret dual languages on the same algebras.

The present chapter has two motivations. The first is that it was with the complement classifier notion that the project of the thesis originally began. The second is that with chapters 11 and 12 I claim to have produced a genuine complement classifier; some discussion is required within the thesis to motivate the claim that this is a discovery.

The chapter has two sections. In the first section Mortensen's notion of a complement classifier is described. The algebra associated with the complement classifier is described as being a "true" for "false" dualisation of the algebra associated with the subobject classifier structure. As such a complement classifier can be thought of as a subobject classifier under a new interpretation. I provide a motivation for the legitimacy of this re-interpretation by making a case for the claim that the subobject classifier structure as standardly known captures more notions than just that of a subobject classifier. The subobject classifier structure subsumes a structure for which the dualised subobject classifier interpretation is the most natural. I take this to be by way of clarification of the original Mortensen complement classifier notion. My notion, then, of a genuine complement classifier is that of a subobject classifier structure whose natural interpretation is as a complement classifier. My discussion brings out two points: there are no truly natural interpretations of the classifier structure since all interpretations are essentially arbitrary, but there is a philosophical distinction to be made between a truth arrow interpreted as "true" and a truth arrow interpreted as "false". Also, there are some standard conventions on what kind of truth arrow will count as "true" and what will count as "false". Thus I provide a motivation for an interpretation of the structure I describe in chapters 11 and 12.

The second section of chapter 4 can be thought of as the technical version of the discussion in the first section on the idea that the structure of a subobject classifier subsumes the structure of a complement classifier: I demonstrate that a complement classifier and a subobject classifier in the same topos will be isomorphic.

1. The Classifier

Definition: For a category $\mathcal{C}$ a *complement classifier* is a $\mathcal{C}$-arrow $false : 1 \to \Omega$ where for any monic $f : a \rightarrowtail d$ there is one and only one $\mathcal{C}$-arrow $d \to \Omega$, denoted $\overline{\chi}_f$, making the following a pullback in $\mathcal{C}$,

$$\begin{array}{ccc} a & \xrightarrow{f} & d \\ \big\downarrow & & \big\downarrow{\scriptstyle \overline{\chi}_f} \\ 1 & \xrightarrow[false]{} & \Omega \end{array}$$

Using a complement classifier $false$ in place of the subobject classifer $true$ dualises the usual topos logic constructions: where $false$ was the classifying map of $\emptyset \to 1$, the complement classifying map of $\emptyset \to 1$ is best described as a map $true$; in complement classifier terms the usual constructions for $\cap$ and $\cup$ become, respectively, $\cup$ and $\cap$; the construction for the intuitionist $\Rightarrow$ becomes the construction for a $\dot{-}$ arrow. The technique is to say that where, in a topos, we had a feature of the topos logic described by χ_f for some arrow f, we now say that we have a feature described by $\overline{\chi}_{f'}$ where f' is the arrow f with all instances of $true$ replaced by $false$. For example, $\cap$ is usually constructed as $\chi_{\langle true, true \rangle}$, so the replacement feature of the algebra will be described by $\overline{\chi}_{\langle false, false \rangle}$. The arrow $\chi_{\langle true, true \rangle}$ is $\cap$ essentially because the only time a claim $a \cap b$ is true is when a and b are both true. The arrow

$\overline{\chi}_{\langle false, false \rangle}$ will be a binary operation that values a claim $false$ only when its two arguments are valued $false$; in other words, $\overline{\chi}_{\langle false, false \rangle}$ is $\cup$.

As we have discussed, there is no bar to the act of renaming or reinterpreting $true$ as $false$ and then developing the paraconsistent algebra of Ω. It seems possible indeed to go further and assert that there be actual topos-like categories that have genuine complement classifiers, if only because the foregoing construction would seem to supply a relative consistency argument for the existence of such categories. It is appropriate however that we say more than that there is an analogy with reinterpreted toposes, if we wish to claim that such classifiers exist. An important point seems to be that with respect to functor SUB being representable, there is no difference between complement and ordinary classifiers. We are left to pursue intuitive (ex-categorial) requirements to make a distinction. As we see in later chapters, for example chapter eight, it is possible to represent internal poset Ω as an external poset with structure in the ordinary set-theoretic sense; we can examine the external versions of $true$ and $false$ and make intuitive assessments. If the external version of an arrow is the unit, or related appropriately to the unit, of external Ω, it is usual to call that arrow $true$, and if the arrow is the zero of the poset, then it would be usual to call it $false$. These, of course, are essentially arbitrary judgments made intuitive by common usage. However, there remains the possibility that an arrow that we would on this standard call $false$, exists as the universal arrow associated with the representation of SUB. That is, there remains the possibility that there are subobject classifiers that are in fact complement classifiers. In chapter twelve we demonstrate the existence of just such a classifier.

A point worth noting about complement classifiers as reinterpreted subobject classifers is that they really are classifiers of complements in the following sense. When we accept the usual understanding of subobject inclusion we have that, for

$f, g \in \text{Sub}(d)$,

$$f \subseteq g \quad \text{iff} \quad \overline{\chi}_g \leq \overline{\chi}_f$$

where $\subseteq$ is subobject inclusion and $\overline{\chi}_g \leq \overline{\chi}_f$ holds whenever $\langle \overline{\chi}_g, \overline{\chi}_f \rangle$ factors through $e: \textcircled{\leq} \rightarrowtail \Omega \times \Omega$, the equaliser of $\cap$ and pr_1. Recall that $\cap$ is the complement character construction of what is usually $\cup$, so $\langle \overline{\chi}_g, \overline{\chi}_f \rangle$ factoring through e under the ordinary constructions would mean that $\cup \cdot \langle \overline{\chi}_g, \overline{\chi}_f \rangle = \overline{\chi}_g$ which is best understood as $\overline{\chi}_f \leq \overline{\chi}_g$. This actually is the point of the complement character construction: it reverses the usual order on Ω. We note however that while Ω becomes an internal BrA, all lattices Sub(d) remain Heyting algebras. It is reasonable to suppose that a genuine complement classifier would work in the same way. It is possible to re-order Sub(d) in the sense that we say $g \subseteq f$ iff by subobject inclusion $f \subseteq g$. Our intention there would be to claim something like "having re-ordered Sub(d), I am (by some implied dualisation functor) speaking now about some topos-like category with actual BrA Sub(d) lattices under subobject inclusion". This is less than helpful in a subject that usually proceeds by actual construction. There is however some value in this as a starting analogy. This, after all, prompted the search for standard constructions where Sub(d) really was a BrA, and this search turned out to be successful.

2. Complement Classifier vs. Subobject Classifier

Our concern with this section is to establish that whenever a subobject classifier and a complement classifier exist in the same category, they are formally indistinguishable. Let us suppose that we have a topos $\mathcal{E}$ with a subobject classifier $true: 1 \rightarrow \Omega$ and a complment classifier $false': 1 \rightarrow \Omega'$. Goldblatt [1984] offers a proof (pp.81-2) that subobject classifiers are unique up to isomorphism. We can adapt this to show that the arrow $true: 1 \rightarrow \Omega$ and the arrow $false: 1 \rightarrow \Omega$ are, up to isomorphism, the same. Consider the diagram

$$
\begin{array}{ccc}
1 & \xrightarrow{true} & \Omega \\
\downarrow & & \vdots\ \overline{\chi}_{true} \\
1 & \xrightarrow[false']{} & \Omega' \\
\downarrow & & \vdots\ \chi_{false'} \\
1 & \xrightarrow[true]{} & \Omega
\end{array}
$$

The top square and the bottom square are, by hypothesis, pullbacks. By the pullback lemma, then, the outer rectangle is a pullback. By definition of the subobject classifier, there is exactly one arrow making the rectangle a pullback. This arrow must be $\chi_{\top} = id_{\Omega}$. So,

$$\chi_{false'} \cdot \overline{\chi}_{true} = id_{\Omega}.$$

Replacing *true* with *false'* and *false'* with *true* in the foregoing proof, we get

$$\overline{\chi}_{true} \cdot \chi_{false'} = id_{\Omega'}.$$

It follows that $\chi_{false'}$ is an isomorphism between Ω and Ω'. Now since $false' = \overline{\chi}_{true} \cdot true$ and $true = \chi_{false'} \cdot false'$, we have that *false'* and *true* are the same arrow up to isomorphism.

The foregoing result is a consequence of the definition of the complement classifier. The point however is that this is *not* a flaw in that definition. The flaw lies in the definition of the subobject classifier. It encompasses too much. The standard definition of the subobject classifier does not allow, even in **Set**, for a distinction between complement and ordinary classifiers; the flaw is akin to that of asserting that there are no closed sets because their algebra works as a HA when dualised. The conclusion in Mortensen [1995] is that the nature of the classifier as either "subobject" or "complement" is a matter of interpretation. In fact, at the level of generality that allows us to speak of classifier structures in the absence of particular examples, this must be true. However, at the level of particular examples, we may have some basis for distinction in some kind of examination of the external correlates for the (appropriate) internal algebras. Our aim in later chapters will be to avoid having to interpret structures to get dualisation results: we aim to present structures that are dual rather than dual interpretations of the same structure. In fact, we manage to do both in chapters eleven and twelve.

We should retain two ideas. Firstly, that paraconsistent logic is available in ordinary toposes by fiat of straightforward ex-categorial reinterpretation of the classifier structure. Secondly, finding paraconsistent algebras within the structure of a category is a matter of finding lattices that have the right properties under the usual notions of internal order. With the next chapter we detail one such attempt. In our search for paraconsistency semantic objects in categories we aim at finding a classifier for quotient objects on the grounds that quotient objects and subobjects are dual. We find that such structures as are available to us are inadequate to the task of supporting a logic of the kind we desire. Our discussion will be valuable for revealing the need to look less at straightforward categorial duality and more at the representation of dual structures within the same category. The point of the present chapter was that reinterpretation of structure within categories is available as a tool for injecting paraconsistency into categories, but as a tool it is unsubtle in the same way that the interpretation built into the definition of the subobject classifier is unsubtle. Neither of the acts of interpretation allow for a distinction in particular classifier constructions. There seems to be no obvious reason to accept one interpretation over the other. The next chapter is a first attempt to develop dual structures, rather than dual interpretations.

CHAPTER 5: THE QUOTIENT OBJECT CLASSIFIER

Introduction: A quotient object is an equivalence class of epimorphisms with common domain in the same way that subobjects are equivalence classes of monics with common codomain. In our project of dualisation quotient objects come to our attention in that where there are lattices of subobjects in $\mathcal{C}$, there are lattices of quotient objects in $\mathcal{C}^{op}$. The prospect, then, is that where we have HA subobject lattices, we have BrA quotient object lattices. We will find however that this does not naturally hold. If we order subobject collections by subobject inclusion we get Heyting algebras (at least for toposes) and if we order their duals by quotient object inclusion we find that we have exactly the same algebra. This result provided the first clue that the task of obtaining BrA structures in categories was less about categorial dualisation than about straightforward representation of lattice dualities.

In what follows we address ourselves to the definition of a quotient object and the relationship between lattices Sub(d) in a category $\mathcal{C}$, and their duals, the lattices Quo(d) in the dual category $\mathcal{C}^{op}$. We find that such lattices are naturally isomorphic rather than anti-isomorphic. This means that where lattices Sub(d) are HAs, for example in a topos $\mathcal{E}$, lattices Quo(d) are likewise HAs in the dual category. It follows then that in investigating quotient object lattices with a view to finding BrAs we are preferably interested in considering the relationship of lattices Sub(d) and Quo(d) from the same category. In this context we find that a feature of lattices Quo(d), as standardly constructed, is that they are not automatically BrA. This, too, is indicated by our isomorphism rather than anti-isomorphism conclusion. The following consideration is an elaboration. Quo(d) as a lattice with respect to a category $\mathcal{C}$ is constructed so that its dual, Sub(d) in $\mathcal{C}^{op}$, is a Sub(d) lattice also as standardly constructed. It follows, as we will demonstrate below, that Sub(d) in $\mathcal{C}^{op}$, which we denote by $\mathrm{Sub}_{\mathcal{C}^{op}}(d)$, and Quo($d$) in $\mathcal{C}$, which we denote $\mathrm{Quo}_{\mathcal{C}}(d)$, are naturally isomorphic rather than anti-isomorphic; that is, were Quo(d) to be BrA in $\mathcal{C}$, then Sub(d) must be BrA in $\mathcal{C}^{op}$. But Sub(d) is not ever a BrA (other than when Sub(d) is also Boolean) with respect to the standardly defined operators; the operators, if they exist, always yield a HA. It follows that to find a Quo(d) that is BrA, we need to be able to define a new operator. As a consequence we would be defining a new operator for $\mathrm{Sub}_{\mathcal{C}^{op}}(d)$ lattices. This is our clue that the search for a place for paraconsistent logic within the usual logic structures of a category is the search for a new algebra of subobjects. We take up this search in Part III.

In the first section of this chapter we define quotient objects and establish the relationship between subobject lattices for categories $\mathcal{C}$ and quotient object lattices for categories $\mathcal{C}^{op}$. With the second section we define the notion of a quotient object classifier. We find that such classifiers are unlikely to exist for toposes, nor indeed for any category with a strict initial object. For other categories there is no obvious bar to the construction, so we note by analogy with the notion of a complement classifier that there are dual interpretations available for the proposed quotient object classifier.

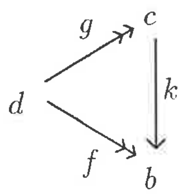
Chapter 5 aims at two conclusions. The first comes from the demonstration in the first section that simple dualisation of the subobject classifier notion does not standardly yield paraconsistent logic for topos duals. In fact the logic does not change through simple dualisation. The result extends to any instance of the dual of a subobject classifier on the assumption that the only known operators for subobject lattices are the ones that make the lattices Heyting algebras. Since the logic does not change in the (simple) dualisation, the task of discovering BrAs in classifier structures is *not* served by simple categorial dualisation of the classifier notion. This argument has at its heart two simple technical results: the first, Proposition 5.1.1, is that $\mathrm{Sub}_{\mathcal{C}}(d) \cong \mathrm{Quo}_{\mathcal{C}^{op}}(d)$; the second, Proposition 5.1.2, is that $\mathrm{Sub}_{\mathcal{C}}(d)$ ordered by subobject inclusion and $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$ ordered by quotient object inclusion are (up to isomorphism) the same ordered set. Arguably Prop.5.1.1 follows immediately from the construction of $\mathrm{Sub}_{\mathcal{C}}(d)$ and the standard notion of categorial duality. The result is perhaps trivial or at least in no need of explicit proof. This is less true for Prop.5.1.2 but arguably still holds (assuming a reader with a firm grasp on duality ideas and consequences). To emphasise this is to misunderstand the project of chapter 5. With chapter 5 I am making an argument for the inadequacy of standard categorial dualisation of the subobject classifier notion for the production of BrAs in a category; my discussion goes beyond the simple demonstration of the isomorphism of algebras $\mathrm{Sub}_{\mathcal{C}}(d)$ and $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$. This first section is more discussive than technical. The consequence of the argument in section 1 is that if we wish to find BrAs in quotient classifier structures we must either find a way for a quotient object classifier to be understood as a sort of complement (quotient object) classifier or we must find new operators for subobject lattices. The second section of chapter 5 deals in part with the first of these two options. This is the second reason for including this chapter in the thesis. In the second section I investigate the notion of a quotient object classifier with a view to discovering whether or not some of the ideas raised in the complement classifier chapter can be applied. This second section has two parts. The second part contains the discussion related to chapter 4 notions. The first part contains a technical result on the existence of quotient object classifiers.

The interpretation of a quotient object classifier as a truth value of some value object is not determined by simple duality unless one decides to allow it to be so. Some discussion is required to bring this out. In particular we are interested to assure ourselves that there is a relatively natural interpretation of the quotient object classifier arrow in **Set** just as there is a relatively natural interpretation of the subobject classifier arrow in **Set**. I do use a technique that amounts to explicit proof of dual claims but I do this to inform a discussion where the issue is interpretation of arrows as truth values. That discussion is not settled by simple categorial dualisation. My conclusion is that complement classifier notions apply to the interpretation of quotient object classifiers if and only if they apply to interpretation of subobject classifier interpretations. With this conclusion I close discussion of one of the options left to us as a consequence of the argument in section 1 of chapter 5. This leaves us with the option of finding new operators for subobject lattices. This I take up in the next chapter. (And since I leave behind the quotient object classifier context, I am free to take up a third option in the search for BrAs, namely, the search for a genuine complement classifier).

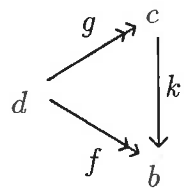
The negative results of this and the previous chapter play a significant role in establishing the nature of the project of the rest of the thesis, in that they demonstrate that not every attempt at dualisation will yield appropriately paraconsistent results.

1. Quotient Object Lattices

Quotient objects are to epimorphisms what subobjects are to monomorphisms. Consider an object d in a category $\mathcal{C}$. Let Epic(d) be the collection of all $\mathcal{C}$-epimorphisms with domain d. We define a preorder (reflexive, transistive ordering) on Epic(d) by allowing that, for $f, g \in$ Epic(d), $f \subseteq g$ iff there is some $\mathcal{C}$-arrow k from the codomain of g to the codomain of f such that the following diagram commutes



That is, $f \subseteq g$ iff there is some k such that $f = k \cdot g$. Since $f = k \cdot g$ and f is epic, we have k as epic. The sense then of the ordering can be seen in **Set**: since k is epic, b is (at least isomorphic to) a subset of c. We say that this ordering is natural since it reflects (in **Set**) a natural idea of an entity being less than or equal to another; that natural idea is the one of subset inclusion. The order we have defined for Epic(d) will not in general be a partial order (reflexive, transistive, antisymmetric ordering) since, in general, there will exist isomorphic epics f and g with the same domain d; that there can be iso but not identical epics means that $f \subseteq g$ and $g \subseteq f$ implies only $f \simeq g$ and not in general $f = g$. We can however establish a partial order on Epic(d) partitioned under the obvious equivalence relationship. We say that epics f and g with the same domain d are in the same equivalence class iff both $f \subseteq g$ and $g \subseteq f$. Such equivalence classes are called *quotient objects of d*. Epic(d) partitoned under this equivalence relation will be denoted Quo(d). The partial order we define on Quo(d) is also denoted by $\subseteq$, and will be called *quotient object inclusion*. We say Quo(d), the collection of all quotient objects for d, is partially ordered so that for $[f], [g] \in$ Quo(d), $[f] \subseteq [g]$ iff for $f : d \twoheadrightarrow b$ and $g : d \twoheadrightarrow c$ there is some $k : c \to b$ such that



commutes. Where the category from which Quo(d) is drawn is not apparent we attach a subscript to Quo that names the category of origin; for example, if Quo(d) is a collection of equivalence classes of epics from category $\mathcal{C}$, we may also denote the collection by $\text{Quo}_{\mathcal{C}}(d)$. We apply the same convention to collections Sub(d).

Proposition 1.1: $\text{Sub}_{\mathcal{C}}(d) \cong \text{Quo}_{\mathcal{C}^{op}}(d)$.

Proof: suppose $[f] \in \text{Sub}_{\mathcal{C}}(d)$. By definition f is monic in $\mathcal{C}$ and has codomain d. It follows that there is an epic f^{op} in $\mathcal{C}^{op}$ with domain d. Now $g \in [f]$ iff g is a

$\mathcal{C}$-monic with codomain d and there is some k_1 and k_2 in $\mathcal{C}$ such that

$$f = g \cdot k_1 \quad \text{and} \quad g = f \cdot k_2.$$

Where such a g exists in $\mathcal{C}$, there exists a $\mathcal{C}^{op}$-epic g^{op} and $\mathcal{C}^{op}$ arrows k_1^{op} and k_2^{op} such that

$$f^{op} = (g \cdot k_1)^{op} = k_1^{op} \cdot g^{op} \quad \text{and} \quad g^{op} = (f \cdot k_2)^{op} = k_2^{op} \cdot f^{op};$$

in other words, $g^{op} \in [f^{op}]$. In fact, the relationship between $\mathcal{C}$ and $\mathcal{C}^{op}$ requires that $g \in [f]$ iff $g^{op} \in [f^{op}]$, and that $\mathcal{C}$-monic f exist iff $\mathcal{C}^{op}$-epic f^{op} exist. So $\mathrm{Sub}_{\mathcal{C}}(d)$ and $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$ are isomorphic collections. □

Proposition 1.2: *for* $[f], [g] \in \mathrm{Sub}_{\mathcal{C}}(d)$ *and* $[f^{op}], [g^{op}] \in \mathrm{Quo}_{\mathcal{C}^{op}}(d)$, *we have*

$$[f] \subseteq_1 [g] \quad \textit{iff} \quad [f^{op}] \subseteq_2 [g^{op}]$$

where $\subseteq_1$ *is subobject inclusion and* $\subseteq_2$ *is quotient object inclusion.*

Proof: $[f] \subseteq_1 [g]$ iff there is some $\mathcal{C}$-arrow k such that $f = g \cdot k$. But this is true iff there is some $\mathcal{C}^{op}$ arrow k^{op} such that $f^{op} = (g \cdot k)^{op} = k^{op} \cdot g^{op}$. And under that circumstance $[f^{op}] \subseteq_2 [g^{op}]$. So if $[f] \subseteq_1 [g]$, then $[f^{op}] \subseteq_2 [g^{op}]$. The converse is established in the same way using the fact that $(\mathcal{C}^{op})^{op}$ is $\mathcal{C}$. □

These propositions establish that $\mathrm{Sub}_{\mathcal{C}}(d)$ and $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$ form isomorphic lattices respectively under subobject and quotient object inclusion.

Since subobject inclusion amd quotient object inclusion both reflect a natural idea of "less than or equal to", that of subset inclusion, we will describe lattices $(\mathrm{Sub}_{calC}(d), \subseteq_1)$ and $(\mathrm{Quo}_{\mathcal{C}^{op}}, \subseteq_2)$ as *naturally* isomorphic. Plainly quotient object inclusion is the not the only ordering possible for the collection $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$, nor is subobject inclusion the only one for $\mathrm{Sub}_{\mathcal{C}}(d)$, but if we change the ordering, then we change the lattice. Knowing then that subobject lattices, as commonly understood, are HAs when drawn from a topos $\mathcal{E}$ obliges us to assert that quotient object lattices, as commonly understood, are HAs when drawn from topos duals $\mathcal{E}^{op}$; the operators that we know to be definable on a lattice $(\mathrm{Quo}(d), \subseteq_2)$ by virtue of the isomorphism with $(\mathrm{Sub}(d), \subseteq_1)$ will always yield a HA rather than a BrA. For collections $\mathrm{Sub}_{\mathcal{C}}(d)$ we can say that $\cap$ exists if $\mathcal{C}$ has pullbacks, $\cup$ exists if $\mathcal{C}$ has image factorisations and coproducts, and $\models\!\!\!\Rightarrow$ exists if $\mathcal{C}$ is something like a topos (this is vague, but for the point we are making we do not need an explicit definition). On the face of it $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$ should be a dual lattice since the conditions under which the lattice operators exist dualise; however, since the order on $\mathrm{Sub}_{\mathcal{C}}(d)$ is defined by what arrows exist, the conditions under which the order exists dualise as well. This is just the point of Proposition 1.2 above. Since the known operators on $\mathrm{Sub}_{\mathcal{C}}(d)$ with respect to the usual ordering are lub, glb, and intuitionist implication, the known operators on $\mathrm{Quo}_{\mathcal{C}^{op}}(d)$ with respect to the usual ordering will be lub, glb, and intuitionist implication. And since any $\mathcal{C}$ is $\mathcal{C}_1^{op}$ for some category $\mathcal{C}_1$, it is a fact that any collection $\mathrm{Quo}_{\mathcal{C}}(d)$ under quotient object inclusion is, at best, a HA. Note well that we do not rule out the possibility of extra operators.

One avenue of investigation that remains open to us is that of interpreted algebras associated with the natural quotient object lattice structure. This is the idea that should a quotient object *classifier* exist, its role as a truth value would be amenable to interpretation for the same reasons that we may vary the interpretation of a subobject classifier.

2. The Functor QUO

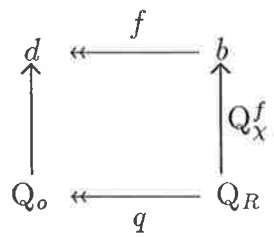
2.1: With this section we define a functor $\text{QUO}: \mathcal{C} \to \mathbf{Set}$ for categories $\mathcal{C}$. The functor is defined to represent the construction of quotient objects in just the way that the functor SUB was defined to represent the construction of subobjects. Since subobjects and quotient objects are dual, the details of the definition of QUO, particularly the details of the definition of $\text{QUO}(f)$ where f is any arrow in $\mathcal{C}$, will be dual to the relevant details of the definition of SUB. It follows that QUO will be a covariant functor.

Our principal concern with this section is to consider the nature, if any, of a quotient object classifier in a category, and since the existence of such a thing is equivalent to functor QUO being representable, we give some consideration to the circumstances under which this occurs. We find firstly that if a quotient object classifier exists, it is an arrow $\text{Q}_R \to \emptyset$ where Q_R is the object of the representation of QUO and $\emptyset$ is an initial object. We find, then, that where a category $\mathcal{C}$ has a strict initial object, for example if $\mathcal{C}$ is a topos, then it is unlikely that a quotient object classifier exists. This does not rule out the existence of such a classifer for categories whose initial objects are not strict.

For a category $\mathcal{C}$, the subobject functor $\text{SUB}: \mathcal{C}^{op} \to \mathbf{Set}$ is a contravariant functor from $\mathcal{C}$ to $\mathbf{Set}$ that takes $\mathcal{C}$-objects a to collections $\text{Sub}(a)$ of subobjects of a, and takes $\mathcal{C}$-arrows $k: a' \to a$ to functions $\text{SUB}(k): \text{Sub}(a) \to \text{Sub}(a')$ where for $[f] \in \text{Sub}(a)$, $\text{Sub}(k)([f])$ is the collection of arrows $g': b \to a'$ where $g' \in \text{Sub}(k)([f])$ iff g' is the pullback in $\mathcal{C}$ of some $g \in [f]$ along k; in fact, $\text{SUB}(k)([f])$ is $[f']$, the subobject of a' represented by f', the pullback of f along k. The construction of SUB relies on the fact that, in any category, if two monics determine the same subobject, then their respective pullbacks along any given arrow in the category also determine the identical subobjects. By duality, this fact gives us that, in any category, if two epics determine the same quotient object, then their respective pushouts along any arrow in the catgeory also determine identical quotient objects. It follows from this relationship bewteen pushouts and epics that we have a ready definition for QUO.

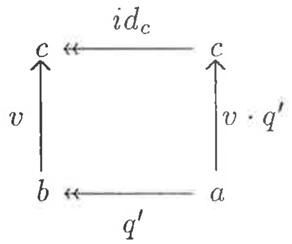
We define functor QUO: $\mathcal{C} \rightarrow$ **Set** to be a covariant functor with an object function that takes and $a \in \mathcal{C}$ to Quo(a), the set of all quotient objects of a, and an arrow function which takes any $\mathcal{C}$-arrow $k: a \rightarrow a'$ to QUO(k): Quo(a) $\rightarrow$ Quo(a') given by Quo(a) $\ni [g] \mapsto$ [pushout of g along k] $\in$ Quo(a').

Suppose now that QUO is representable in a category $\mathcal{C}$ and Q_R is the representing object; that is, suppose some natural isomorphism between QUO and a hom functor hom($Q_R, -$). This is equivalent to the existence of an arrow $q: Q_R \rightarrow Q_o$ in $\mathcal{C}$ of which it is true that for every $b \in \mathcal{C}$ and every $[f] \in$ Quo(b), there is exactly one arrow, $Q^f_\chi: Q_R \rightarrow b$, which we call the "quotient-character of f", such that QUO(Q^f_χ)($[g]$) $= [f]$. Of q this is equivalent to saying that for any epic $f : b \twoheadrightarrow d$ there is exactly one $\mathcal{C}$-arrow, namely Q^f_χ, making the following diagram a pushout



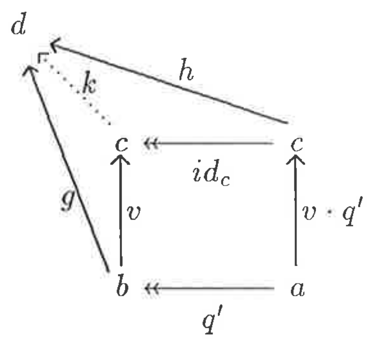
We now give a demonstration on the nature of Q_o. Our discussion here and the following lemma and theorem dualise that found in Barr and Wells, *Toposes, Triples and Theories*, [1985]. There Barr and Wells consider the nature of the classifier arrows whose existence and classifying properties are equivalent to the representability of SUB. They find there that the arrow in question has a terminal object as domain. By exactly dualising their lemma and theorem we find that q has an initial object as codomain; that is, Q_o is an initial object in $\mathcal{C}$.

Lemma 2.1.1: *for any $v : b \to c$ and epic $q' : a \twoheadrightarrow b$, the diagram*



is a pushout.

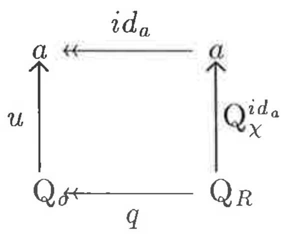
Proof: consider for any g and h the diagram



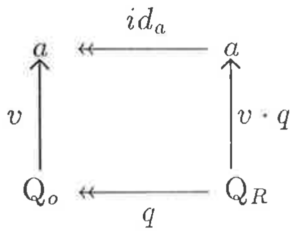
By definition $v \cdot q' = id_c \cdot (v \cdot q')$, so the inner square commutes. Suppose the outer square commutes, that is, suppose $g \cdot q' = h \cdot (v \cdot q')$. Since q' is epic and composition is associative, $g = h \cdot v$. Therefore there is at least one $c \xrightarrow{k} d$ making the whole diagram commute, namely $k = h$. But for the whole diagram to commute it must be the case that $k \cdot id_c = h$. And this is true iff $k = h$. So, there is exactly one k making the whole diagram commute. □

Theorem 2.1.1: *in any category Q_o is an initial object.*

Proof: for any given $a \in \mathcal{C}$, there is at least one map from Q_o to a, namely the map u given by the following pushout



Now suppose some further arrow $v : Q_o \to a$ exists; then, by the lemma, the following is a pushout.



Now, by the nature of q, there is exactly one arrow $Q_R \to a$, namely $Q_X^{id_a}$, making the square a pushout; so $v \cdot q = Q_X^{id_a}$. But we have already seen that $id_a \cdot Q_X^{id_a} = u \cdot q$, so $v \cdot q = u \cdot q$. It follows, by the fact that q is epic, that $v = u$. So, for any $a \in \mathcal{C}$, there is exactly one map from Q_o to a. $\Box$

It is a corollary to this theorem that where the initial object of $\mathcal{C}$ is strict, for example if $\mathcal{C}$ is a topos, that Q_R is an initial object. This follows as a straightforward consequence of the definition of a strict initial object: and initial object $\emptyset$ in a category $\mathcal{C}$ is *strict* if whenever there is an arrow $a \to \emptyset$, then that arrow is an isomorphism. If QUO is representable in a category with a strict initial object, the representing object for QUO must be initial and q must be iso to $id_\emptyset$.

Theorem 2.1.2: QUO *is not in general representable in categories with strict initial objects.*

Proof: if QUO is representable in such a category, then Q_R is an initial object. It follows, then, that for any object a in the category, there is exactly one arrow $Q_R \to a$. This entails that every epic f with domain a is (iso to) the pushout of $id_\emptyset$ along just one arrow $Q_R \to a$. This entails that every object of the category has, at most, one quotient object; in other words, every pair of epics with common domain have isomorphic codomains. This is not impossible, but equally, it cannot hold in general, not even for categories with strict initial objects. Consider any topos with at least two objects, a and b, that are not isomorphic within the topos. Suppose further that there are two epics, the first being $f: a \twoheadrightarrow a$ and the second being $g: a \twoheadrightarrow b$. Objects a and b are not iso in the topos, so g is not an isomorphism.

Arrows f and g are epics with the same domain in a category with a strict initial object, but f and g do not have iso codomains. The extent to which such epics or similar pairs of epics are possible within categories with strict initial objects is some measure of the extent to which QUO is not representable for such categories. We assert that the range of categories with strict initial objects is such that pairs of epics as described are more likely to exist than not; and so assert the theorem demonstrated. □

Notice that when it is true that each object of a category (with strict initial object) has exactly one quotient object, there is a one to one correspondence between quotient objects and arrows with the initial object as domain. Trivially the latter could be called "character" arrows.

2.2: Suppose now that we consider categories that do not have strict initial objects. To support a quotient object classifer, a category requires some initial object so let us consider categories with non-strict initial objects. To assure ourselves that there are categories with non-strict initial objects in which QUO is representable note that when $\mathcal{E}$ is a topos, SUB is representable, so in $\mathcal{E}^{op}$ QUO is representable; and if $\mathcal{E}^{op}$ had a strict initial object, then all maps $1 \to a$ in $\mathcal{E}$, for any $\mathcal{E}$-object a, would be isomorphisms. So let us suppose a category with a non-strict initial object in which QUO is representable. In that case we have a map $\mathrm{Q}_R \to \emptyset$ and no obvious reason to suppose that Q_R is an initial object. It is reasonable then to presume that Q_R is an internal algebra with operator arrows determined by the natural operations on each $\mathrm{Quo}(d) \cong \mathrm{hom}(\mathrm{Q}_R, d)$. The bijection $\mathrm{Quo}(d) \cong \mathrm{hom}(\mathrm{Q}_R, d)$ is described by $[f] \mapsto \mathrm{Q}^f_\chi$. The natural operations on each $\mathrm{Quo}(d)$, that is, the operations that determine components of natural transformations

$$\mathrm{hom}(\mathrm{Q}_R + \mathrm{Q}_R, -) \to \mathrm{hom}(\mathrm{Q}_R, -)$$

are, when $\mathcal{C}$ is the category in question, exactly those whose duals are natural for each $\mathrm{Sub}_{\mathcal{C}^{op}}(d)$. As we have discussed above, the order $\subseteq_2$ on each $\mathrm{Quo}(d)$ together with these operations will yeild HA $\mathrm{Quo}(d)$ provided that we understand $\mathrm{Sub}_{\mathcal{C}^{op}}(d)$ to be a HA. It will follow that we may take Q_R to be an internal HA in the same way that we may take Ω to be an internal HA. The only relatively difficult part of this is that where Ω has operator arrow $\Omega \times \Omega \to \Omega$, Q_R has operator arrows $\mathrm{Q}_R \to \mathrm{Q}_R + \mathrm{Q}_R$.

There is now a distinction to make: this is between the algebraic structure on Q_R and that on each $\mathrm{Quo}(d)$. We have seen that the standard ordering on $\mathrm{Quo}(d)$ produces a HA. We have yet to see how this relates to an order on Q_R. Clearly, we can impose an order on Q_R that directly reflects the order on each $\mathrm{Quo}(d)$. Naturally this would make Q_R an internal HA. However, such an imposition of order amounts to an interpretation of $q\colon \mathrm{Q}_R \to \emptyset$ as a particular truth value, namely as a value *true*. We have yet to see that this is the intuitive interpretation. To do that we must review what we expect the nature of q to be in a know context. Consider **Set** and $\mathbf{Set}^{op}$. any quotient object classifier in a category $\mathcal{C}^{op}$ can be thought of as $true^{op}$ where *true* is a subobject classifier in $\mathcal{C}$. This is just a consequence of the definitions. In that case there is a quotient object classifier in $\mathbf{Set}^{op}$. To get a sense of the arrow $true^{op}$ in $\mathbf{Set}^{op}$ we consider *true* in **Set**. The arrow $true\colon \{\emptyset\} \to \{\emptyset, 1\}$ characterises any inclusion $f\colon a \subseteq b$ by determining an arrow $\chi_f\colon b \to \{\emptyset, 1\}$ described for any $x \in b$ by

$$\chi_f(x) = \begin{cases} 1 & \text{iff } x \in f(a), \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that in $\mathbf{Set}^{op}$ the map $true^{op}$ characterises any superset arrow $f^{op}\colon b \twoheadrightarrow a$ by determining an arrow $(\chi_f)^{op} = \mathrm{Q}_\chi^{f^{op}}\colon \{\emptyset, 1\} \to b$ where for any $x \in b$

$$(\mathrm{Q}_\chi^{f^{op}})^{-1}(x) = \begin{cases} 1 & \text{iff } x \in (f^{op})^{-1}(a), \\ \emptyset & \text{otherwise.} \end{cases}$$

This, and that fact that $\chi_{true} = id_\Omega$ and therefore that $\mathrm{Q}_\chi^{true^{op}} = id_{\mathrm{Q}_R}$, give us reason to believe that whatever interpretation we are given to applying to subobject classifiers is the same one we should give to quotient object classifiers; after all, the usual subobject classifier in **Set** picks out 1 from $\{\emptyset, 1\}$ and is called $true$, and the quotient object classifier $true^{op}$ in $\mathbf{Set}^{op}$ is defined so that $(true^{op})^{-1}(\emptyset) = 1$. It should be apparent thought that just as there is no difficulty in interpreting a subobject classifier as the truth value $false$, there will be no difficulty in interpreting a quotient object classifier, likewise, as a truth value $false$. The effect in both cases will be to dualise the algebraic nature of the usual operator constructions.

We can take two points from this discussion: firstly, straightforward duality of categories does not yield the logic structures we are interested in and, secondly, quotient objects and their lattices are not always immediate candidates as natural logic bearers in the same manner as subobjects. In combining the ideas that come out of this and the last chapter we see that to gain paraconsistent algebras in categories we need a way of building them into the usual relationships between categorial objects. The next chapter demonstrates an attempt to recognise such paraconsistent algebras in a functor category.

CHAPTER 6: A FUNCTOR CATEGORY

Introduction: With this chapter we begin our investigation into the existence of extra operators for lattices of subobjects. Our concern in this chapter is with those categories that are categories of set-valued functors over posets. These are the categories $\mathbf{Set}^{\mathcal{C}}$ of functors $\mathcal{C} \to \mathbf{Set}$ where $\mathcal{C}$ is a poset category. It is a well known result that when $\mathcal{C}$ is small, the category $\mathbf{Set}^{\mathcal{C}}$ is a topos. An example of a small category $\mathcal{C}$ is a poset category $\mathbf{P}$. When we have a partially ordered set $\mathbf{P}$, say the set of all subsets of a set P ordered by set inclusion, we can say that $\mathbf{P}$ determines a category. The category determined by $\mathbf{P}$ is that which has as objects all members p of P, and furthermore has, for any $p, q \in \mathbf{P}$, an arrow $p \to q$ if and only if $p \sqsubseteq q$ where $\sqsubseteq$ is the partial ordering of $\mathbf{P}$. We will use $\mathbf{P}$ to denote both the poset and the category determined by the poset.

For categories $\mathbf{Set}^{\mathcal{C}}$ it is known that structure in $\mathcal{C}$ is related to subobject lattice structure $\mathbf{Set}^{\mathcal{C}}$. We note with interest then that where Ξ is a topology of closed sets, then the lattice $(\Xi, \subseteq)$ of Ξ ordered by set inclusion, is not a HA. In fact $(\Xi, \subseteq)$ is a BrA. Furthermore $(\Xi, \subseteq)$ is a poset. Since the category $\mathbf{Set}^{\Xi}$ must have an algebra $(\Omega, ⓢ)$, we offer the hypothesis that we may use those BrAs that we find in $(\Xi, \subseteq)$ to construct a new operator for logic object Ω. In fact this hypothesis fails, but it is instructive to see why. In what follows we fix poset $\mathbf{P}$ as a set S of subsets p of some set P. The partial ordering for $\mathbf{P}$ is set inclusion. For any topological space X both the lattice of closed sets and the lattice of open sets form posets of this kind. The categories we concern ourselves with, then, are $\mathbf{Set}^{\mathbf{P}}$.

To begin with we had noted in investigation that while the subobject lattice structures in categories $\mathbf{Set}^{\mathbf{P}}$ were related to the algebras and subalgebras of $\mathbf{P}$, they were not directly related in the sense that these algebras could be immediately carried over into functor constructions. The algebras that do reflect the subobject structure of $\mathbf{Set}^{\mathbf{P}}$ can be constructed with reference to $\mathbf{P}$ in the sense that the algebras have cosieves from $\mathbf{P}$ as elements. We will define cosieves shortly but the net result is that while we can vary the algebraic properties of $\mathbf{P}$ (our $\mathbf{P}$ may be based on a closed set topology and so be a BrA, or it may be based on an open set topology and so be a HA), we do not vary the fact that the lattices of cosieves of $\mathbf{P}$ that are used in the construction of the classifier object of $\mathbf{Set}^{\mathbf{P}}$ are always bounded, complete, and distributive. The focus of our investigation moves, then, from the properties of $\mathbf{P}$ to the properties of bounded, complete distributive lattices. Knowing as we do that describing the logic of a topos amounts to describing which arrows $\Omega \times \Omega \to \Omega$ exist within the topos, we varied our original hypothesis on the origin of the extra operators. Any topos $\mathbf{Set}^{\mathbf{P}}$ has an intuitionist $\Rightarrow$ arrow essentially because any bounded, complete distributive lattice supports such an operator. We noted that any such lattice will also support the characteristic operator of a Brouwerian algebra; we speculated that we may use this fact to construct a BrA arrow $\dot{-} : \Omega \times \Omega \to \Omega$ for $\mathbf{Set}^{\mathbf{P}}$ in just the way that the usual $\Rightarrow$ is constructed. The speculation proves in general to be false. It was not that the requisite BrAs do not exist, for they do. In fact we are able to construct a transformation $\Omega \times \Omega \to \Omega$. The problem is that the transformation is not in general a *natural* transformation. This means that the transformation is not in general an arrow in $\mathbf{Set}^{\mathbf{P}}$. In chapter nine we come across the same problem for particular extra arrows in sheaf categories.

What follows is a demonstration that the classifier object Ω of a topos $\mathbf{Set}^{\mathbf{P}}$ plays host to component lattices $\Omega(p)$ for all $p \in \mathrm{P}$ and that along with being HAs, these lattices are BrAs. We follow this with a demonstration that the BrAs $\Omega(p)$ are *not* natural in p and so do *not* produce a BrA operator on Ω itself. So, in as much as this operator does not exist, the slogan "the logic of variable sets is intuitionistic" remains appropriate.

This is the first of the chapters which investigate the possibility of describing new natural operators for subobject algebras. The negative result presented here serves as a prompt for the same hypothesis restricted to particular kinds of toposes investigated in later chapters. With respect to chapter 6, presheaf categories $\mathbf{Set}^{\mathbf{P}}$ were originally chosen for investigation on the grounds that they contained the largest number of things on which we could reasonably perform a topological dualisation, these things being functors from topological spaces. The original discussion was developed so that $\mathbf{P}$ would be a closed set topology. The idea was that where $\mathbf{Set}^{\mathbf{P}}$, with $\mathbf{P}$ an open set topology, has HA subobject lattices, topological dualisation of the objects of $\mathbf{Set}^{\mathbf{P}}$ (that is, the consideration of $\mathbf{Set}^{\mathbf{P}}$ with $\mathbf{P}$ a closed set topology) would produce some new structure for subobject lattices. It was quickly noted that the construction of the classifier object for $\mathbf{Set}^{\mathbf{P}}$ is not especially influenced in terms of the algebraic nature of Ω by the algebraic nature of the poset $\mathbf{P}$. I therefore considered $\mathbf{Set}^{\mathbf{P}}$ where $\mathbf{P}$ is any poset. In fact it would have been relatively easy to extend the discussion to categories $\mathbf{Set}^{\mathcal{C}}$ where $\mathcal{C}$ is some small category. Inasmuch as my concern was for topological dualisation and that the preliminary investigation pointed to a negative result for the existence of natural BrA structures, I decided to focus on the less general categories $\mathbf{Set}^{\mathbf{P}}$.

Chapter 6 has two sections: the first section contains the positive result that the subobject classifier has BrAs in its component structure; the second section contains the negative result that the component BrAs do not produce a natural transformation and so do not produce an operator arrow for the subobject algebras of $\mathbf{Set}^{\mathbf{P}}$. The discussion in the second section brings out the point that the failure of naturalness can be closely linked to to the construction of the classifier object and does not appear to be significantly linked to the algebraic nature of $\mathbf{P}$. The hypothesis that drove the research that led to the material of Part III of the thesis was formed partly as a result of this. Since my method was to be one of topological dualisation, I needed to be working with structures defined with respect to topologies and which, when formed into a category, had a subobject classifier whose algebraic properties were relatively closely dependent upon the topologies used to define the original objects. The obvious choice were the sheaves. An extra advantage of working with sheaves was that their categories would be subcategories of categories $\mathbf{Set}^{\mathbf{P}}$; this offered the possibility that whatever was causing the BrAs to be non-natural in $\mathbf{Set}^{\mathbf{P}}$ could be, in a sense, left out, particularly when I came to consider closed set sheaves.

The philosophical significance, then, of chapter 6 is relatively subjective. The objective point of the chapter is that some manipulation (some restriction or extra property) is needed in a category (particularly in a functor category) before we can produce a category with a natural BrA structure. One assessment of the kind of restriction needed led to the discoveries and discussions in chapters 8, 9, and 10.

At the time of research and then of writing the material of this chapter and chapters 9 and 10 was understood to be original. In fact a result subsuming the categorial results in these chapters was reported in 1991 in F.W. Lawvere's "Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes" in *Category Theory*, Springer Verlag Lecture Notes in Mathematics, 1488, pp.279–281.

In the note cited Lawvere writes

> "In any presheaf topos (and more generally any essential subtopos of a presheaf topos), the lattice of all subobjects of any given object is another example of a co-Heyting algebra (as well as a Heyting algebra). The co-Heyting operations are in general not preserved by substitution (inverse image) along maps..." (Lawvere, 1991, p.280).

This covers the results in my chapter 6 on the non-natural transformation $\{\dot{-}_p : p \in \mathbf{P}\}$ for any category $\mathbf{Set}^{\mathbf{P}}$ where $\mathbf{P}$ is a poset. Now a topos of sheaves is a sub-category of some presheaf category. So Lawvere's result contains my own that any Grothendieck topos has an in general non-natural BrA transformation on the subobject classifier object.

My discussion is a great deal more detailed than Lawvere's. Lawvere's discussion, on the other hand, contains enough detail for an expert to recreate the result and in fact has results relating to circumstances where the BrAs are natural and partially natural. The virtue of my discussion is its attempt to outline *why* the BrAs are not in general natural. This fitted in with my initial program for discovering the implications of using closed sets in place of open sets in various constructions, particularly sheaves. The focus of the thesis became that of discovering BrA logic structures and, broadly, that too is the focus of Lawvere's note. However, our method remained that of topological dualisation: the replacement of open sets by closed in the notions of various structures; it is not clear that this is Lawvere's method. Philosophically speaking, the intention with chapters 6, 8, 9, and 10 was to discover semantic objects for paraconsistent logic in categories. The implication of my actual discoveries is that, along with standard categorial dualisation, topological dualisation of sheaves is not an immediate source of natural semantic structures. My emphasis, then, was different from Lawvere's.

1. Component Algebras

When the poset $\mathbf{P}$ is a small category, the category $\mathbf{Set}^{\mathbf{P}}$ is a topos. Topos $\mathbf{Set}^{\mathbf{P}}$ is a particular example of topos $\mathbf{Set}^{\mathcal{C}}$ where $\mathcal{C}$ is an arbitrary small category. The topos structure of $\mathbf{Set}^{\mathcal{C}}$ can be described in terms of $\mathcal{C}$-arrows in collections we call cosieves. Note that while we are largely following Goldblatt's [1984] discusson of categories $\mathbf{Set}^{\mathbf{P}}$, we have in later chapters adopted the dual definition of sieve given in Johnstone's *Topos theory*, [1977], So where Goldblatt has used "sieve" we use "cosieve".

For category $\mathcal{C}$ and fixed $\mathcal{C}$-object a, let

$$\mathrm{S^a} = \{f : \text{for some } \mathcal{C}\text{-object } b, a \xrightarrow{f} b \text{ in } \mathcal{C}\}.$$

A subset S' of S^a satisfies the condition of *closure under left composition* if whenever $b \xrightarrow{g} c$ in $\mathcal{C}$ and $a \xrightarrow{f} b$ in S', then $g \cdot f \in S'$. Any subset of $\mathrm{S^a}$ that is closed under left composition is called a *cosieve*. Such a sieve is sometimes called an *a-cosieve.* It follows that for any $\mathcal{C}$-object a, both $\mathrm{S^a}$ and $\emptyset$ are a-cosieves. For poset $\mathbf{P} = (\mathrm{S}, \sqsubseteq)$, any subset $A \subseteq \mathrm{P}$ is *hereditary* with respect to $\sqsubseteq$ if whenever $p \in A$ and $p \sqsubseteq q$, then $q \in A$. For each $p \in \mathrm{P}$, the set $[p) = \{q : p \sqsubseteq q\}$ is called the *principal hereditary subset of* P *generated by p.* Plainly there is a cosieve S' in category $\mathbf{P}$ iff there is a hereditary subset A in poset $\mathbf{P}$; the bijection being given by:

$$p \to q \in S' \quad \text{iff} \quad q \in A.$$

For this reason we identify cosieves with the appropriate hereditary subsets and develop $\mathbf{Set}^{\mathbf{P}}$ in terms of the latter. The set of all hereditary subsets of P will be denoted $\mathbf{P}^{\oplus}$. The set of all subsets of a given $[p)$ hereditary with respect to $\sqsubseteq$ will be denoted by $[p)^{\oplus}$. Note that if $A \in [p)^{\oplus}$, then $A \in \mathbf{P}^{\oplus}$. $\mathbf{Set}^{\mathbf{P}}$, as a topos, has a classifier object Ω. The standard construction for Ω is the functor $\mathbf{P} \to \mathbf{Set}$ given by:

$$\Omega(p) = [p)^{\oplus};$$

for $p \sqsubseteq q$ in $\mathbf{P}$ the maps $\Omega^p_q \colon \Omega(p) \to \Omega(q)$ are given by $[p)^{\oplus} \ni A \mapsto A \cap [q)$.

Proposition 1.1 (Goldblatt): $\mathbf{P}^{\oplus}$ *ordered by set inclusion forms a bounded, complete distributive lattice.* □

When $\mathbf{P}$ is $(S, \subseteq)$, the lattice $(\mathbf{P}^{\oplus}, \subseteq)$ is bounded by S and $\emptyset$.

Proposition 1.2 (Goldblatt): *for any $p \in S$, $[p)^{\oplus}$ ordered by set inclusion forms a bounded, complete distributive lattice.* □

Lattice $([p)^{\oplus}, \subseteq)$ is bounded by $[p)$ and $\emptyset$.

Theorem 1.1: $(\mathbf{P}^{\oplus}, \subseteq)$ *is a* BrA.

Proof: we recall from chapter three the definition of a BrA as a pseudo-differenced lattice with a unit. We note first of all that when $\mathbf{P}$ is $(S, \subseteq)$, the lattice $(\mathbf{P}^{\oplus}, \subseteq)$ has unit S. Secondly, to show that $(\mathbf{P}^{\oplus}, \subseteq)$ is pseudo-differenced we demonstrate that for any $U, V, X \in \mathbf{P}^{\oplus}$, we have

$$(S - V) \cap U \subseteq X \quad \text{iff} \quad U \subseteq X \cup V.$$

It follows that $(\mathrm{P}^{\oplus}, \subseteq)$ is pseudo-differenced since, for any U, V in a distributive lattice of sets with a unit S, $U \dot{-} V$, if it exists, is the smallest element of the lattice that contains $(S - V) \cap U$. Since $(\mathbf{P}^{\oplus}, \subseteq)$ is complete, the smallest element containing $(S - V) \cap U$ exists and it will always be a subset of any X for which $(S - V) \cap U \subseteq X$. In other words, if we denote by $U \dot{-} V$ the smallest element of $\mathbf{P}^{\oplus}$ containing $(S - V) \cap U$, then the result that $(S - V) \cap U \subseteq X$ iff $U \subseteq X \cup V$ means that $U \dot{-} V \subseteq X$ iff $U \subseteq X \cup V$ for any $U, V, X \in \mathbf{P}^{\oplus}$ making $(\mathbf{P}^{\oplus}, \subseteq)$ a pseudo-differenced lattice.

Suppose that $(S - V) \cap U \subseteq X$. Then,

$$\begin{aligned} ((S - V) \cap U) \cup V &\subseteq X \cup V, \\ ((S - V) \cup V) \cap (U \cup V) &\subseteq X \cup V, \\ U \cup V &\subseteq X \cup V, \\ U &\subseteq X \cup V. \end{aligned}$$

On the other hand, if $U \subseteq X \cup V$, then

$$\begin{aligned} U \cap (S - V) &\subseteq (X \cup V) \cap (S - V), \\ &\subseteq (X \cap (S - V)) \cup (V \cap (S - V)), \\ &\subseteq X \cap (S - V), \\ U \cap (S - V) &\subseteq X. \end{aligned}$$

□

Corollary: *for* $U, V \in \mathbf{P}^{\oplus}$,

$$U \dot{-} V = \{r \in S : q \sqsubseteq r \text{ and } q \in ((S - V) \cap U)\}. \qquad \square$$

It is worth noting that the proof of the theorem relied only on the fact that the lattice $(\mathbf{P}^{\oplus}, \subseteq)$ has a unit and is meet-complete. We have in fact demonstrated that any meet-complete lattice with a unit is a BrA.

Theorem 1.2: *for any* $p \in S$, $\big([p)^{\oplus}, \subseteq\big)$ *is a* BrA. $\square$

The proof of this theorem is essentially identical to that of Th.1.1. If we denote by '$\dot{-}_p$' the pseudo difference operator for $\big([p)^{\oplus}, \subseteq\big)$, we have

Corollary: *for* $s, t \in [p)^{\oplus}$,

$$s \dot{-}_p t = \{r \in [p) : q \sqsubseteq r \text{ and } q \in \big([p) - t\big) \cap s\}. \qquad \square$$

To avoid confusion we note explicitly that when $\big([p) - t\big) \cap s = \emptyset$, it is a set with no members, so there is no $q \in \big([p) - t\big) \cap s$; in other words, if $\big([p) - t\big) \cap s = \emptyset$, then $s \dot{-}_p t = \emptyset$. The same point applies to the formula in the corollary to Theorem 1.1: if $(S - V) \cap U = \emptyset$, then $U \dot{-} V = \emptyset$.

Remark: that each $\Omega(p) = [p)^{\oplus}$ is a bounded, complete and distributive lattice means that each $\Omega(p)$ supports both a BrA operator $\dot{-}_p$ and a HA operator, which we can denote by $\Rightarrow_p$. That there are such HA operators demonstratable in that for any $s, t, x \in [p)^{\oplus}$ we have

$$x \subseteq \big([p) - s\big) \cup t \quad \text{iff} \quad s \cap x \subseteq t.$$

If we define $s \Rightarrow_p t$ to be the largest subset of $\big([p) - s\big) \cup t$, we have

$$x \subseteq s \Rightarrow_p t \quad \text{iff} \quad s \cap x \subseteq t.$$

Since $\big([p)^{\oplus}, \subseteq\big)$ is complete and bounded, that largest subset always exists. It follows that $\big([p)^{\oplus}, \subseteq\big)$ is a relative pseudo complemented lattice with a zero, or in other words, a HA. In fact, the existence of $\Rightarrow_p$ operations for each $[p)^{\oplus}$ corresponds to the existence of the usual intuitionist $\Rightarrow : \Omega \times \Omega \to \Omega$ that makes the logic of $\mathbf{Set}^{\mathbf{P}}$ intuitionist.

The presence of operation $\dot{-}_p$ on each $\Omega(p)$ suggests the possibility of a BrA arrow $\dot{-}$ for $\mathbf{Set}^{\mathbf{P}}$ dual in type to the usual $\Rightarrow$. In fact this does not in general hold. Clearly there is a transformation $\{\dot{-}_p : p \in \mathbf{P}\}$ but the transformation is not in general natural. A feature of the $\Rightarrow$ arrow in $\mathbf{Set}^{\mathbf{P}}$ is that for any $p \subseteq q$ in $\mathbf{P}$, the following diagram commutes in $\mathbf{Set}$.

$$
\begin{array}{ccccc}
p & \qquad & \Omega(p) \times \Omega(p) & \xrightarrow{\Rightarrow_p} & \Omega(p) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \Omega^p_q \times \Omega^p_q} & & \Big\downarrow {\scriptstyle \Omega^p_q} \\
q & & \Omega(q) \times \Omega(q) & \xrightarrow[\Rightarrow_q]{} & \Omega(q)
\end{array}
$$

This is the meaning of the claim that $\Rightarrow$ is more than just a transformation, it is a natural transformation. The arrows of $\mathbf{Set}^{\mathbf{P}}$ are natural transformations between functors $\mathbf{P} \to \mathbf{Set}$, so if the transformation $\dot{-} = \{\dot{-}_p : p \in \mathbf{P}\}$ is not natural, then it does not exist as an arrow in $\mathbf{Set}^{\mathbf{P}}$. In the next section we describe why it is that components $\dot{-}_p$ fail to produce a natural transformation.

2. Operator Arrows

We have discovered that $\Omega(p) = [p)^{\oplus}$ is a BrA with respect to set inclusion. We therefore can define a transformation $\tau : \Omega \times \Omega \to \Omega$ with components τ_p for each $p \in \mathbf{P}$ given by $\dot{-}_p$; that is, for each $p \in \mathbf{P}$ define $\tau_p : \Omega(p) \times \Omega(p) \to \Omega(p)$ so that for $\langle s, t \rangle \in \Omega(p) \times \Omega(p)$,

$$\tau_p\big(\langle s, t \rangle\big) = s \dot{-}_p t.$$

This transformation exists as an arrow in $\mathbf{Set}^{\mathbf{P}}$ only if the transformation is natural.

For τ to be a natural transformation, the following diagram is required to commute in **Set** for all **P**-arrows $p \to q$

$$\begin{array}{ccccc} p & & \Omega(p) \times \Omega(p) & \xrightarrow{\tau_p} & \Omega(p) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \Omega^p_q \times \Omega^p_q} & & \Big\downarrow{\scriptstyle \Omega^p_q} \\ q & & \Omega(q) \times \Omega(q) & \xrightarrow[\tau_q]{} & \Omega(q) \end{array}$$

We need a technical lemma before we can move on to the main demonstration of this section.

Lemma 2.1: *for $p \subseteq q$ and $t \in [p)^{\oplus}$,*

$$[q) - \big(t \cap [q)\big) = \big([p) - t\big) \cap [q).$$

Proof: suppose $x \in [q) - \big(t \cap [q)\big)$. Then $x \in [q)$ and $x \notin t \cap [q)$. But, then we must have that $x \notin t$. Also since $p \subseteq q$, we have $[q) \subseteq [p)$, so $x \in [p)$. Now, since $t \in [p)^{\oplus}$, t is a subset of $[p)$, so $x \in [p) - t$. In other words, $x \in \big([p) - t\big) \cap [q)$. So

$$[q) - \big(t \cap [q)\big) \subseteq \big([p) - t\big) \cap [q).$$

Now suppose that $x \in \big([p) - t\big) \cap [q)$. Then $x \in [q)$. Also, $x \in [p) - t$, so $x \notin t$. If $x \notin t$, then $x \notin t \cap [q)$, but we have already seen that $x \in [q)$, so $x \in [q) - \big(t \cap [q)\big)$. So

$$\big([p) - t\big) \cap [q) \subseteq [q) - \big(t \cap [q)\big). \qquad \square$$

Theorem 2.1: *τ is not in general a natural transformation.*

Proof: the condition that the diagram above commute is the condition that for any $p \subseteq q$ in **P** and any $\langle s, t\rangle \in \Omega(p) \times \Omega(p)$, we have

$$\Omega^p_q\big(\tau_p(\langle s, t\rangle)\big) = \tau_q\big((\Omega^p_q \times \Omega^p_q)(\langle s, t\rangle)\big).$$

Now,

$$\begin{aligned}\Omega_q^p\big(\tau_p(\langle s,t\rangle)\big) &= \Omega_q^p(s \dot{-}_p t)\\ &= (s \dot{-}_p t) \cap [q).\end{aligned}$$

So, by the corollary to Theorem 1.2,

$$\Omega_q^p\big(\tau_p(\langle s,t\rangle)\big) = \big\{r \in [p)\colon v \subseteq r \text{ and } v \in \big([p) - t\big) \cap s\big\} \cap [q).$$

On the other hand we have that

$$\begin{aligned}\tau_q\big((\Omega_q^p \times \Omega_q^p)(\langle s,t\rangle)\big) &= \tau_q\big(\langle s \cap [q), t \cap [q)\rangle\big)\\ &= \big(s \cap [q)\big) \dot{-}_q \big(t \cap [q)\big).\end{aligned}$$

So, again by the corollary to Theorem 1.2, we have

$$\tau_q\big((\Omega_q^p \times \Omega_q^p)(\langle s,t\rangle)\big) = \big\{r \in [q)\colon v \subseteq r \text{ and } v \in \big([q) - \big(t \cap [q)\big)\big) \cap \big(s \cap [q)\big)\big\}.$$

Then, by lemma 2.1,

$$\begin{aligned}\tau_q\big((\Omega_q^p \times \Omega_q^p)(\langle s,t\rangle)\big) &= \big\{r \in [q)\colon v \subseteq r \text{ and } v \in \big(\big([p) - t\big) \cap [q)\big) \cap \big(s \cap [q)\big)\big\}\\ &= \big\{r \in [q)\colon v \subseteq r \text{ and } v \in \big([p) - t\big) \cap s \cap [q)\big\}.\end{aligned}$$

Now, by definition, $s \dot{-}_p t$ is the smallest superset of $\big([p) - t\big) \cap s$, and since there is no guarantee that $\big([p) - t\big) \cap s$ will be a hereditary set, it will in general be smaller than $s \dot{-}_p t$. On occasion, then, there will be some q other than $\emptyset$ such that $p \subseteq q$ and $q \in (s \dot{-}_p t)$ but $q \notin \big([p) - t\big) \cap s$. On such an occasion, since $s \dot{-}_p t$ is a hereditary set,

$$(s \dot{-}_p t) \cap [q) = [q).$$

So,

$$\Omega_q^p\big(\tau_p(\langle s,t\rangle)\big) = [q).$$

But, by the corollary to Theorem 1.2, $q \in (s \dot{-}_p t)$ only if there is some $v \in \big([p) - t\big) \cap s$ such that $v \subseteq q$. Obviously in that case $v \in s$, and since s is hereditary, $q \in s$. But, by hypothesis, $q \notin \big([p) - t\big) \cap s$, so $q \in t$. And t is hereditary, so $[q) \subseteq t$. As a result $\big([p) - t\big) \cap [q) = \emptyset$. It follows then that

$$\tau_q\big((\Omega_q^p \times \Omega_q^p)(\langle s, t\rangle)\big) = \emptyset.$$

And since, by hypothesis, $q \neq \emptyset$, we have $[q) \neq \emptyset$, and therefore

$$\Omega_q^p\big(\tau_p(\langle s, t\rangle)\big) \neq \tau_q\big((\Omega_q^p \times \Omega_q^p)(\langle s, t\rangle)\big).$$

So, at least to the extent that there are non-empty q in $(s \dot{-}_p t)$ such that q is not in $\big([p) - t\big) \cap s$, the transformation τ is not natural. □

It is worth emphasising that the conditions used in Theorem 2.1 do not always hold. We are required, then, to consider how likely it is that they do hold. This will give some meaning to the claim that τ is not *in general* a natural transformation. Theorem 2.1 shows that if there is some $s, t \in [p)^{\oplus}$ for some $p \in \mathbf{P}$ such that $\big([p)-t\big) \cap s$ is a proper subset of $s \dot{-}_p t$, then τ fails to be natural as a transformation. So, the extent to which there are hereditary subsets s, t of $[p)$ for which $\big([p) - t\big) \cap s$ is not hereditary is one measure of the extent to which τ fails. Plainly τ may fail to be natural more often, but it is at least true that τ fails to be natural when such s, t exist. Such s, t exist if it is at least the case that there is $p, y \in \mathbf{P}$ such that $p \subseteq y$ but $p \neq y$. To demonstrate this claim suppose there to such p, y in $\mathbf{P}$. It is at least true that $[y) \in [p)^{\oplus}$ and that $[y) \neq [p)$. Then $[p) - [y)$ contains at least p but not y. As a result $[p) - [y)$ is not hereditary. For the same reason $\big([p) - [y)\big) \cap [p)$ is not hereditary. So, if we let s be $[p)$ and t be $[y)$, we have an example of $s, t \in [p)^{\oplus}$ such that $\big([p) - t\big) \cap s$ is a proper subset of $s \dot{-} pt$. We can say that τ fails to be natural in at least those cases where poset $\mathbf{P}$ has at least two distinct elements p and y such that $p \sqsubseteq y$ where $\sqsubseteq$ is the partial order defining $\mathbf{P}$. This circumstance seems sufficiently common for posets to justify the claim that τ in general fails to be natural.

With the next chapter we offer a brief history of sheaves by way of introduction to the concept. Our concern to establish extra operators for Ω and $\mathrm{Sub}(d)$ generally, becomes focused in later chapters on sheaves for two related reasons. First, sheaves can be defined with respect to topologies and so, in particular, with respect to closed set topologies. As we shall discuss, structure in sheaves varies according to the base space topologies in ways that it does not for presheaves and functors such as we have considered in this chapter. Second, categories of sheaves are the original toposes. As toposes, sheaf categories offer us a structurally rich context in which to develop the issues of paraconsistent topos logic. This idea sustains us until chapter fourteen where we modify it a little and suggest that sheaf categories offer us a structurally rich context in which to develop the issues of paraconsistent model theory.

Part III:

SHEAF CONCEPTS

CHAPTER 7: SHEAVES

a brief history of the structure

Introduction: This chapter is intended to act as an introduction to the notion of a sheaf, particularly for logicians many of whom are unfamiliar with the idea. It announces the definition of a sheaf by considering some of the history of the notion. Our history owes much to John Gray's altogether more comprehensive "Fragments of the history of sheaf theory" as found in Fourman, Mulvey, and Scott's *Applications of Sheaves*, [1979]. However, all sources listed in the bibliography were consulted, and the result is a *reconstruction* of Gray's history, this time with an emphasis on the emergence of the conditions used to define all types of sheaves now known. In particular, it is interesting that the initial account of sheaves privileges closed sets in the base space, which is very much in line with our own paraconsistent notion. But also there were rapid changes in this dominance.

A further point made by the chapter is the relationship between the notion of a sheaf category and the notion of a topos (and so the existence of a subobject classifier). Sheaf categories are a significant subclass of the toposes and so a significant subclass of the categories with subobject classifiers. The move from simple functor categories $\mathbf{Set}^{\mathbf{P}}$ from chapter 6 to the closed set sheaf categories from chapters 8 and 9 is given a further motivation.

In the present day, sheaves exist in at least two forms: on the one hand there are the contravariant functors that are called sheaves and on the other, there are the continuous local homeomorphisms between topological spaces. Both the contravariant functor form and the continuous local homeomorphism forms satisfy essentially the same property. That there are different structures that bear the name sheaf is an acknowledgement that this same "sheaf property" can be described in different contexts. In fact, what we now know as the "sheaf property" is a generalisation. The original description applied only to the continuous local homeomorphisms. In that context the "sheaf property" of a continuous local homeomorphism $p: E \to X$ was a description of the behaviour of particular types of maps to E from members of the topology on X. The relevant maps are the sections of p. Intuitively, the worth of the "sheaf property" in structures was the change of context that it allowed: topological discussion could be recast as discussion of algebras; the sheaf structure would, in a sense, sit above the topological space and the internal structure of the sheaf would vary with topological variations in that base space. Let us, then, describe the sheaf property. A *section*, s, of a sheaf $p: E \to X$ is a map to E from a member, U, of the topology on X such that the map s is continuous and $p \cdot s = id_U$. Such a section is sometimes called a U-section or a section over U. Whenever $U \neq X$, the section s is called a *local section*. Otherwise, s is a *global section*. The sheaf property is a property of sets of sections over covers where a collection $\{U_i: i \in I\}$ of members of the topology on X is a *cover* if its union is also a member of the topology on X. Collection $\{U_i: i \in I\}$ is called a *U-cover* if $\bigcup\{U_i: i \in I\} = U$ and U is a topology element. We then consider collections $\{s_i: s_i \text{ is a } U_i\text{-section}, i \in I\}$. A map $p: E \to X$ has the *sheaf property* if wherever we have a U-cover $\{U_i: i \in I\}$ and a set of U_i-sections such that

$$s_i \mid U_i \cap U_j = s_j \mid U_i \cap U_j, \quad \text{all } i, j \in I,$$

then, there is exactly one section s over U such that

$$s \mid U_i = s_i, \quad \text{all } i \in I.$$

We might think of each section as being like a column. Each section has a base U and reaches up to support E (or, more exactly $s(U) \subseteq E$). The sheaf property is then the idea that columns that overlap in base and support the same "area" of E are part of just one column. Such physical analogies will go astray if we consider them too deeply (different sections can have the same base and support different areas of E), but perhaps the point is made.

Interest in sheaf–like structures was originally cast in terms of the development of theories of co-homology "over" topological spaces. The sense of "over" was that the co-homologies were mapped to closed sets. This was the idea of co-homology with closed support. It happened, then, that the structures regarded as the precursors of sheaves were defined over the closed sets of a topological space. Later, perhaps for more generality, this would change. The writer generally regarded as beginning the interest in sheaf–like structures is Jean Leray. However, we will begin with an earlier writer who had similar ideas.

Alexander:

J.W.Alexander had a notion of a "grating" over a point set. In "Gratings and homology theory", [1947], we find that a grating Γ is a collection of ordered triples $\gamma_i = \langle a_i, b_i, c_i \rangle$ called *cuts* with no two cuts having an element in common. Of interest to us is the notion of a representation (Γ, f, X) of a grating Γ on a point set X called the carrier. A *representation* (Γ, f, X) is a grating Γ together with a carrier X and some function f given by $\Gamma \times X \mapsto \{-1, 0, 1\}$ so that f takes a cut of Γ together with a point of X to -1, 0, or 1. There are three subsets of X defined relative to a representation. These are

$$A_i = \{x \in X : f(\gamma_i, x) = -1\},$$
$$B_i = \{x \in X : f(\gamma_i, x) = 0\},$$
$$C_i = \{x \in X : f(\gamma_i, x) = 1\}.$$

Alexander defines a representation to be *continuous* if X is a topological space and, for any cut γ_i, the sets A_i and C_i are open. Indeed any representation will be continuous since it will be regarded as inducing the coarsest topology on X that makes A_i and C_i open for each γ_i. Each B_i will be a closed set.

The algebraic dimension is introduced via the notion of a chain. First, a *cell* of a grating is any finite sequence of elements of cuts of the grating. Thus we have a cell

$$\mathrm{A} = z_1 z_2 \ldots z_m \quad \text{where} \quad z_i = a_i \text{ or } b_i \text{ or } c_i \text{ for some } \gamma_i.$$

The *type* of cell A is the cell

$$\alpha(\mathrm{A}) = a_1 a_2 \ldots a_m \quad \text{where } a_i \text{ in } \alpha(\mathrm{A}) \text{ only if } z_i \text{ in A}.$$

A *chain* is then a mapping of the set of all grating cells of a fixed type into an arbitrary ring of coefficients without divisors of zero.

We then find defined the notion of a *locus* of a chain. The locus is a subset of the space X of the representation and is determined by application of some union and intersection rules to the subsets A_i, B_i, C_i related to the cuts γ_i from which the elements z_i of the cells of the chain are drawn. According to Alexander the loci of any chain on a grating with a continuous representation will be a closed subset of the carrier set.

Alexander developed various algebraic concepts, including a homology theory, with respect to the chains of gratings. It is from the earlier papers, "On the connectivity ring of a bicompact space" [1936], and "A theory of connectivity in terms of gratings" [1938], that we can develop an idea of the origin and the intentions of Alexander's grating theory. The aim seems to have been to develop ways of remaking topological structure in terms of algebraic structure, and in particular, to make advantageous connections between topological spaces and rings. In Alexander [1938] we find a part statement of the project: "with every grating Γ we are going to associate an abstract ring Π, called the *ring of chains* of Γ." (p.887). At this stage a grating is a collection of ordered pairs $\langle a, c\rangle$ of subspaces a, c of a space x

such that $a \cup c = x$. Associated with any ordered pair is its *barrier* $b = a \cap c$. These developed later into the sets A_i, B_i, C_i relative to a representation.

A similar idea of a ring with support in a topological space was to be important when Leray developed his notion of a faisceau.

Leray:

J.Leray's initial writings in 1944 dealt with constructions he called concrete complexes (Gray tells us these are chain complexes in the modern sense) and special examples he called couvertures. In his "Sur la forme des espaces topologiques et sur les points fixes des répresentations" [1944], Leray defines an *abstract complex* to be a set of variables and what he calls derivatives. Leray also defines a *concrete complex* to be an abstract complex over a space. The abstract complex is "over" the space in the sense that each element of the abstract complex is associated with a non-empty subset of the space. A variation on the notion of a concrete complex is a couverture. As a concrete complex a *couverture* is at least an abstract complex over a space. A concrete complex becomes a couverture if it satisfies, among other properties, the property that it is an abstract complex over a space with a topology such that each element of the abstract complex is associated with a closed set of the space. Leray claims that where "homology" theory was the study of finite closed covers of a space, it can now be the study of couvertures. The principal gain being that we can substitute an algebraic theory for geometric concerns[1].

Between 1944 and 1950 Leray defined and refined another structure that he

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[1] "Jusqu'à présent la théorie de l'homologie a étudié la forme d'un espace topologique en analysant les propriétés de ses recouvrements par un nombre fini d'ensembles fermés; nous allons effectuer cette étude en analysant les propriétés des couvertures de l'espace; nous y gagnons de substituer à une notion de topologie ensembliste une notion bien plus maniable de topologie algébrique", Leray [1944], p.108.

called a faisceau. In this case a closed set topological space is much more immediately associated with a ring structure. We find the final version in "L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue" [1950]. A *faisceau* B on a space X is defined so that

(a) a ring[2] B(F) is associated with each closed subset F of X;

(b) when F_1 is a closed subset of F, there is a homomorphism $B(F) \to B(F_1)$ where the image of $b \in B(F)$ is denoted $F_1 b \in B(F_1)$;

(c) $B(\emptyset) = 0$;

(d) if $F_2 \subset F_1 \subset F$ and $b \in B(F)$, then $F_2(F_1 b) = F_2 b$ which is to say we have two homomorphisms, $B(F) \to B(F_1)$ and $B(F_1) \to B(F_2)$, the composition of which is the homomorphism $B(F) \to B(F_2)$.

A faisceau is called continuous if $B(F) = \lim B(V)$ where $\lim B(V)$ is the direct limit of rings $B(V)$ over closed neighbourhoods V of F.

The next recognized writer on the topic of faisceau was Cartan. In his writings, particularly the Ecole Normale Supérieure Séminaire series, we find faisceau with an altered definition. Now a faisceau is to be defined over open sets of a topological space. The change proved to be a very successful one.

Cartan:

In Cartan's "Cohomologie des groupes, suite spectral, faisceaux" [1950/51], we find credited to Lazard the following definition: where K is a commutative ring with a unit element, a *faisceau* $\mathcal{F}$ *of* K-*modules* on a regular topological space $\mathcal{X}$ is a set F, a set $\mathcal{X}$, and a projection $p : F \to \mathcal{X}$ such that

(1) for all $x \in \mathcal{X}$, $p^{-1}(x) = F_x$ is a K-module;

(2) F has a topology such that

(α) the algebraic operations of F defined by the structure of K-modules F_x

[2] Leray uses "anneau" which Gray translates as "module".

are continuous.

(β) p is a local homeomorphism.

Projection p is a *local homeomorphism* if for any $a \in F$, there is some open $N \subseteq F$ such that $a \in N$, and some open $U \subseteq \mathcal{X}$ such that $p(a) \in U$ where the map $p \mid N$ is a homeomorphism (that is, bijective with both it and its inverse being contionuous).

Sections in the modern sense are defined and called sections. For each open set $X \subset \mathcal{X}$ we denote by $\Gamma(F, X)$ the collection of sections over X. For $s \in \Gamma(F, X)$, if the set $\{x \in X : s(x) \neq \emptyset\}$, is closed, then it is called the *support* of s. If X and Y are two open subsets of $\mathcal{X}$ such that $X \subset Y$, then there is a homomorphism $\Gamma(F, Y) \to \Gamma(F, X)$. If $X \subset Y \subset Z$, then the homomorphism $\Gamma(F, Z) \to \Gamma(F, X)$ is the composition of $\Gamma(F, Z) \to \Gamma(F, Y)$ and $\Gamma(F, Y) \to \Gamma(F, X)$. It is noted that for any $x \in \mathcal{X}$, the k-module F_x that is $p^{-1}(x)$ is the direct limit of sets $\Gamma(F, X)$ where X is an open subset of $\mathcal{X}$ and $x \in X$. Also, a faisceau $\mathcal{F}$ over a space $\mathcal{X}$ can be defined given modules $\mathcal{F}_X$ for each open $X \subset \mathcal{X}$ and a system of homomorphisms $f_{XY} : \mathcal{F}_Y \to \mathcal{F}_X$ for each inclusion $X \subset Y$ such that if $X \subset Y \subset Z$, then $f_{XZ} = f_{XY} \cdot f_{YZ}$. This procedure is now standard.

The concern with, or at least the use of, these structures is still in terms of the performance of co-homological algebra with closed support. In such a project Cartan introduces the notion of Φ-families. These are collections of sets that satisify certain properties and will be important for the notion of faisceau resolution. Cartan regards the introduction of Φ-families as a generalisation of Leray's ideas in that Leray worked with the compact sets of locally compact spaces.

The interesting question is what prompted the change from closed sets to open. We might speculate that there was an attempt to gain greater generality. On the one hand the new faisceau would have complex analysis applications, namely in the study of ideals of germs of holomorphic functions. And on the other hand there is a sense in which the structure is freed of its initial motivation without damage to its intended role. The notion of Φ-resolutions and the developing notion of fin, mou, and flasque faisceau mean that faisceau cohomology theory is still practicable.

In Cartan [1950/51] we see defined the notion of a *carapace*. (Leray also considered structures to which he gave the name carapace. These were different from the notion in Cartan). The presentation of Cartan's definition sees the carapace as a particular type of structure satisfying two conditions given as axioms. If the structure satisfies only the first axiom it is called a "précarapace". Also any carapace determines a faisceau and vice versa.

The next development was to characterise faiseaux in terms of open cover structures of space X. We see this in the writing of Serre.

Serre:

J.-P.Serre in "Faisceaux algébriques cohérents" [1955], defines a *faisceau de groupes abéliens sur* X which has X as a topological space and associated with each $x \in X$ an abelian group $\mathcal{F}_x$. There is also a set $\mathcal{F}$ which is $\bigcup\{\mathcal{F}_x : x \in X\}$. The faisceau is essentially a projection $\Pi : \mathcal{F} \to X$ such that for $f \in \mathcal{F}_x$, $\Pi(f) = x$ and Π is a local homeomorphism. A further condition will hold entailing that collections of sections, now denoted $\Gamma(U, \mathcal{F})$ for $U \subset X$, will be abelian groups. Also, if $U \subset V$ and $s \in \Gamma(V, \mathcal{F})$, then the restriction of s to U is an element of $\Gamma(U, \mathcal{F})$ whereby we have a homomorphism $\rho_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$.

Again it is noted that given an abelian group $\mathcal{F}_U$ for each open $U \subset X$ and a system of homomorphisms $\varphi_U^V : \mathcal{F}_V \to \mathcal{F}_U$ such that $\varphi_U^V \cdot \varphi_V^W = \varphi_U^W$ whenever $U \subset V \subset W$, we can define a faisceau.

It is noted by Serre that given a condition relating the zeros of abelian groups $\mathcal{F}_U$ and $\mathcal{F}_{U_i}$, a faisceau defined via abelian groups $\mathcal{F}_U$ for all open $U \subseteq X$ and homomorphisms φ_U^V for all $U \subseteq V$ will be canonically isomorphic to the faisceau $\mathcal{F}$ with restriction maps ρ_U^V for each $U \subseteq V$ as described above if when $\{U_i : i \in I\}$ is an open cover for some open $U \subseteq X$ and there is a system $\{t_i \in \mathcal{F}_{U_i} : i \in I\}$ such that

$$\varphi^{U_i}_{U_i \cap U_j}(t_i) = \varphi^{U_j}_{U_i \cap U_j}(t_j),$$

then there exists some $t \in \mathcal{F}_U$ such that

$$\varphi^{U}_{U_i}(t) = t_i, \quad \text{all } i \in I.$$

So we see described the "cover condition" or "sheaf property" that is the characteristic of sheaves as we know them today. Notice that this cover condition is a feature of the canonical faisceau, denoted $(\mathcal{F}, \rho^V_U)$, and that Serre's writing suggests that there will be faisceaux $(\mathcal{F}, \varphi^V_U)$ that do not in general satisfy that condition.

Godement:

With Godement's "Topologie algébrique et théorie des faisceaux" [1964] we arrive at the modern sheaf notion presented in terms of categories and functors. Included is the notion of a presheaf. A topology on a space X is understood to be a poset category. A *préfaisceau* is any contravariant functor $\mathcal{F}$ from this category to another. A préfaisceau is a faisceau if it satisfies the axioms:

(F1) for open cover $\{U_i : i \in I\}$ of U in X and $s', s'' \in \mathcal{F}(U)$ if $s' \mid U_i = s'' \mid U_i$ for all $i \in I$, then $s' = s''$;

(F2) for system $\{s_i \in \mathcal{F}(U_i) : i \in I\}$, if the restrictions of s_i and of s_j to $U_i \cap U_j$ are the same for any $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ such that s restricted to U_i is s_i for any $i \in I$.

This notion of faisceau, which we can now call sheaf, is distinguished from Cartan's faisceau which we find described as an *espace étalé* or sheaf space. In Godement [1964] we find a remark[3] to the effect the notion of espace étalé is broader than that of a sheaf.

[3] "La démonstration du Théorème 1.2.1 prouve que tout *préfaisceau* d'ensembles $\mathcal{F}$ définit canoniquement un espace étalé dans X..." Godement [1964], p.112.

Still of concern is cohomology with closed supports. We find presented the tools of faisceaux flasque, mou, and fine.

Grothendieck:

The categorisation of the notion of the sheaf was furthered by the Grothendieck school in the 1960's and 70's with the development of the notion of topology analogs for categories and, by generalisation, the notion of sheaves over arbitrary categories with these "topologies." This began with the thought that a cover for a categorial object U could be represented as a collection of maps $U_i \to U$. The notion of a cover was generalised out of its set theoretic origins by noting that we could consider covering systems rather than particular covers. In that way the defining feature of a cover became its membership in a system of covers rather than the nature of the "union" of the elements of the cover. It was found that the central properties of a set theoretic covering system needed for the expression of the sheaf property were readily recast in a general language of systems of sets of arrows $U_i \to U$. And so arose the notion of a *pretopology* or covering system for an arbitrary category with respect to which we could define sheaves. The notion was refined in the works of various writers and reached one of its final forms with Lawvere's axiomatic development of a categorial topology as a map $j : \Omega \to \Omega$. A detailed representation of these ideas can be found in our chapter eight.

The notion of a topos first arose with the work of Grothendieck and those who followed him. A *Grothendieck topos* was a category of sheaves over a category with a pretopology. Since all Grothendieck toposes have subobject classifiers, they interested Lawvere; he used the term topos when he described the elementary theory of finitely co-complete categories with exponentiation and a subobject classifier. The newest development of the notion of a sheaf came through the work of Lawvere on the notion of a topologies j, and the various Giraud theorems. A sheaf could now be thought of as a distinguish topos object with respect to a topology j. A brief description of these ideas and how we use them forms the introduction to our chapter ten.

The feature of sheaves that interests us most with respect to logic in categories is exactly the one we have been developing throughout this chapter: the passage from topological to algebraic structures. Our algebraic concerns are somewhat simpler than those of the writers who developed the sheaf structures, however the sheaf remains the right tool for carrying topological algebras into categories. In particular, since the structure of a sheaf is influenced significantly by the topological structure of its base space, we would expect to find that the relationships between maps between sheaves over a fixed topological space are influenced significantly by the topological structure of that fixed base space. We would expect to see, then, that the nature of the base space topology will affect the nature of the algebras of subobjects in categories of sheaves. We have a motive, then, to consider, as we do in the rest of Part III, categories of sheaves defined over closed set topologies and the toposes that are categories of such sheaves.

CHAPTER 8: CLOSED SET SHEAVES

Introduction: This chapter exists to demonstrate that the topological dualisation of the usual sheaf notion produces another, but in essence standard, sheaf notion. This is not a radical discovery in that the sheaf notion is already specifiable in the absence of set theoretic topologies; this, however, does not render trivial the actual working out of particular structures in a closed set sheaf category, particularly since these structures will be the subject of original investigation in the next chapter.

Chapter 8 has four sections. The first three sections are expository. They are included for the benefit of that section of the readership that is not familiar with categorial sheaf theory. The fourth section contains the definition of closed set sheaves. In light of the standard material presented in the first three sections, the definition of a sheaf over the closed sets of a closed set topology is no more mysterious than the definition of a sheaf over a category. My emphasis, however, is on the use of the more modern definition of a sheaf to provide a topological dualisation of the more traditional notion of a sheaf, the one that preceded the development of pretopologies and topologies for categories. It is with the topological dual of this more traditional sheaf that we work in the next chapter.

Aside from the historical precedent we outlined in the previous chapter, there are a number of reasons for developing the theory of sheaves over closed sets. First of all, having a base topology of closed sets introduces to the sheaf notion a concept of boundary that does not exist for the open set sheaf notion. One area in which this may work for us is the mathematics of physics where the boundaries of a body are as important as the parts of a body inasmuch as physics concerns itself with the interactions of bodies in a system. Lawvere in the introduction to *Categories* *in Continuum Physics* (F.W.Lawvere and S.H.Schanuel, Springer Lecture Notes in Mathematics, 1174) mentions the speculation that there is a role for a closed set sheaf in thermodynamics as a functor from a category of parts of a body to a category of "abstract thermodynamical state-and-process systems" (p.9). Lawvere recognises the particular properties of closed set topologies that make them interesting to us, namely that as algebras they provide us with a formalisation of what we call a paraconsistent negation. Sheaves are then of interest to us in our project of developing paraconsistent logic in categories for the way in which they transport algebras of a topology into the structure of a category of sheaves over that topology. This "transportation" is most evident in the relationship between the algebras of the base space topology and the algebra that is the classifier object in the category of sheaves over the topology. With this chapter we begin an exploration of various aspects of the relationship between closed set topologies and the sheaves and sheaf-like structures that exist over such topologies. In the present chapter we describe the relationship betwen the algebras in the base space topology and the algebras of the classifier object in the sheaf category. We concern ourselves with establishing that the usual constructions for sheaf categories and subobject classifiers will work when the base space topology is one of closed sets. In the next chapter we detail the specific effects on the classifier algebra of a closed set base space topology. We will find that there are BrA structures within the classifier object itself and that they are drawn from the BrA structures of the base space. We shall find however that this does not translate into the existence of a BrA classifier algebra within the sheaf category. In this aspect, the logic of a closed set sheaf category is not analogous to that of an open set sheaf category: in the open set case, the classifier algebras are Heyting and are determined by HAs in the base space. The next chapter, chapter ten, generalises the discussion of chapters eight and nine by considering categories of sheaves over a finitely complete category. Again we find that there are

BrA structures within the classifier object but that these do not translate into the existence of a BrA classifier algebra in the category. The interest in pursuing the more general sheaf case is firstly in the discussion we can provide of the subobject classifier structure and its relation to subobject lattices and, secondly, in the fine tuning we can give the claim that there is a relationship between the base space structure and the classifier structure in the sheaf category; we fine tune the claim by rediscovering that the base space structure cannot be the sole determinant of classifier algebra structure. With chapter eleven we return to discussion of functors over closed set topologies. Here we make the discovery that we have been waiting for: we can define a covariant functor over a closed set topology that, because it is defined with respect to closed sets, is a paraconsistent logic object in the appropriate category. Chapter twelve sets this discovery in context. We are there able to demonstrate that the discovered object is a classifier object for a category of covariant sheaves. We will see that the best way to interpret the object is as the object of a genuine complement classifier. We have seen in chapter four that the original notion of a complement classifier was best understood as an available and legitimate reinterpretation of the notion of the subobject classifier. With chapter twelve we see that, as foreshadowed in chapter four, genuine complement classifiers exist and that their existence is masked by their isomorphism to subobject classifiers. With chapter thirteen we complete Part III and our discussion of sheaf concepts. We give a limited equivalence of categories result for closed set sheaves and closed set sheaf spaces. Closed set sheaf spaces are of interest to us for the way in which their section structures mirror the algebras of the base space. As such closed set sheaves become objects for paraconsistent semantics.

With the present chapter we examine categorial sheaves over the closed sets of a topological space. The first and second sections contains brief descriptions of some of the existing theory of categorial sheaves. We note that categories of sheaves as standardly understood are toposes. It will be proved in the fourth section that categories of **Set**-valued sheaves over the closed sets of a topological space are toposes in just the same way. As a preliminary to this we have section three in which we discuss subobject classifiers. Since the existence of a subobject classifier and the resulting subobject classifying maps is a defining feature of a topos, we will be obliged to show, contrary to some standard presentations, that there is a construction for the classifying arrows $\mathcal{X}$ of sheaf monics that does not rely on $\bigcup$-completeness of the base space topology. We establish the necessary construction as a corollary to a theorem at the end of section three. With the next chapter we discover that Ω_j, the classifier object for sheaf category $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$, contains BrAs when $\mathcal{C}$ is a closed set topology $\mathcal{T}$ but that these BrAs do not in general yield a BrA arrow $\Omega_j \times \Omega_j \rightarrow \Omega_j$ in the category. The BrAs that Ω_j contains do provide a transformation $\Omega_j \times \Omega \rightarrow \Omega_j$ but it fails in general to be natural. We prove this failure by a counterexample which turns largely on the failure of set theoretic closure operators to distribute over intersections. We will use this counterexample to come to a general conclusion about the conditions needed for a BrA operator arrow $\Omega \times \Omega \rightarrow \Omega$ to exist in any category with a classifier.

1. Presheaves on Categories

With this section we define the notion of a presheaf on a category $\mathcal{C}$. We also describe the structure of a subobject classifier for a category of such presheaves. We give this description in terms of sieves on the base category $\mathcal{C}$. The material discussed in this section is necessary as a preliminary to the next section where we define the notion of a sheaf on a category $\mathcal{C}$.

Definition 1.1: let $\mathcal{C}$ be a small category and let a be an object of $\mathcal{C}$. An *a-sieve* is a set S of $\mathcal{C}$-arrows with codomain a where if $b \xrightarrow{f} a$ is in S and there exists a $\mathcal{C}$-arrow $c \xrightarrow{g} b$, then the composite $f \cdot g\colon c \rightarrow b \rightarrow a$ is in S. The *maximal* *a-sieve* $\{\alpha\colon \mathrm{cod}(\alpha) = a\}$ of all $\mathcal{C}$-arrows with codomain a is denoted $(a]$ or $(id_a]$. The term "sieve" will always mean some a-sieve for some category object a. In some circumstances we will use "sieve" where rightly we should use "a-sieve" for a particular a. We can rely on context to make such usage clear. On occasion, too, we will use the phrase "sieve on a" to mean the same thing as a-sieve.

Let $\mathcal{T}$ be a closed set topology for space X. Let $\mathcal{T}$ also denote the poset category that has as objects all members of topology $\mathcal{T}$ and as arrows all inclusions between members of $\mathcal{T}$. Since the arrows of category $\mathcal{T}$ are inclusions, there can be at most one arrow between any two distinct $\mathcal{T}$-objects U and V. Likewise there is exactly one arrow from any $\mathcal{T}$-object V to itself. It follows that for any object V, a V-sieve can be represented as a set S where $U \in S$ only if $U \in \mathcal{T}$ and $U \subseteq V$. Set S is a V-sieve only if whenever $U \in S$ and $W \subseteq U$ in $\mathcal{T}$, we have $W \in S$. In this form the *maximal V-sieve* is the set $(V] = \{U \in \mathcal{T}\colon U \subseteq V\}$.

Definition 1.2: A *presheaf on* $\mathcal{C}$ is any contravariant functor $\mathcal{C} \to \mathbf{Set}$. Following convention we deal rather with the equivalent covariant functors $\mathcal{C}^{op} \to \mathbf{Set}$. The category of all presheaves on $\mathcal{C}$ is denoted $\mathbf{Set}^{\mathcal{C}^{op}}$.

When $\mathcal{C}$ is a small category it is known that $\mathbf{Set}^{\mathcal{C}^{op}}$ is a topos. We suppose now that, unless otherwise stated, $\mathcal{C}$ is a small category. The terminal object in $\mathbf{Set}^{\mathcal{C}^{op}}$ is the functor 1 given by $\mathcal{C} \ni a \mapsto \{\emptyset\}$ with the obvious restriction maps. The *classifier object* is a presheaf Ω where for object a in $\mathcal{C}$,

$$\Omega(a) = \{\text{all sieves on } a\},$$

and for arrow $b \to a$ in $\mathcal{C}$, the image under Ω of $b \to a$ is $\Omega^a_b : \Omega(a) \to \Omega(b)$ given by

$$\Omega(a) \ni S \mapsto \{c \to b \mid (c \to b \to a) \in S\}.$$

The case of particular interest to us is that of $\mathbf{Set}^{\mathcal{C}^{op}}$ where $\mathcal{C}$ is a poset $\mathcal{T}$. In that case maps Ω^a_b are given by

$$\Omega(a) \ni S \mapsto \{c \subseteq b \mid c \subseteq b \subseteq a \in S\} = S \cap (b].$$

Note that the notation Ω^a_b is a little deficient in that it allows no obvious way to distinguish between the image under Ω of $f: b \to a$ and the image under Ω of some different arrow $g: b \to a$. This is not a difficulty in considering the category $\mathbf{Set}^{\mathcal{T}^{op}}$, however it is significant for $\mathbf{Set}^{\mathcal{C}^{op}}$ is general. In later chapters if we find ourselves required to make the distinction between such images we will resort to "function" style notation where the image under Ω of an arrow f is denoted $\Omega(f)$.

In $\mathbf{Set}^{\mathcal{C}^{op}}$ the *subobject classifier* is a natural transformation $true: 1 \dot{\to} \Omega$ given by components $true_a: \{\emptyset\} \to \Omega(a)$ where $true_a(\emptyset) = (a]$, all $a \in \mathcal{C}$. By definition of the subobject classifier, any $\mathbf{Set}^{\mathcal{C}^{op}}$-monic $\tau: F \rightarrowtail G$ has associated with it a classifying arrow $\chi_\tau: G \to \Omega$. As an arrow in $\mathbf{Set}^{\mathcal{C}^{op}}$, χ_τ is a natural transformation given by the set of components $(\chi_\tau)_a: G(a) \to \Omega(a)$, all $a \in \mathcal{C}$. The arrows χ_τ are constructed as follows: for any object a in $\mathcal{C}$ and any $x \in G(a)$,

$$(\chi_\tau)_a(x) = \{b \to a \mid G^a_b(x) \in \tau_b(F(b))\}$$

where $b \to a$ is a $\mathcal{C}$-arrow, G^a_b is the G-restriction map $G(a) \to G(b)$ defined for that $\mathcal{C}$-arrow, and τ_b is the b-component $F(b) \to G(b)$ of natural transformation τ.

The **Set**-valued sheaves over a category are defined to be **Set**-valued presheaves over a category that satisfy a condition. The condition is essentially the "sheaf property" that we outlined in the previous chapter. In this new context the sheaf property is cast in terms of a covers system, called a pretopology, on the base category. We would then speak of a site over which the sheaves are defined, the site being a category with a pretopology. Over time the original notion of a pretopology has been refined to that of a categorial topology. This notion still has formal links with the original idea of a set-theoretic topology but is now, with respect to the formation of sheaves, considerably more general.

2. Pretopologies and Topologies for Categories

This section follows similar discussions in Johnstone [1977] and in Goldblatt [1984]. We discuss the notions of pretopology and topology for arbitrary categories. We also describe the definition of sheaves in terms of these structures.

A *set theoretic cover* for a member U of a topology $\mathcal{T}$ of topological space X is a set $\{U_i : i \in I\}$ of topology members with the property that $\bigcup\{U_i : i \in I\} = U$. We may call $\{U_i : i \in I\}$ a U-cover. In that $\{U_i : i \in I\}$ contains only topology elements, we will say that it it a *U-cover in $\mathcal{T}$*. A *covering system* for a topology $\mathcal{T}$ is a system which associates with each topology member U the collection of all U-covers in $\mathcal{T}$. The thought that we might generalise the notion of a covering system to categories is based on the awareness that any U-cover $\{U_i : i \in I\}$ can be represented as a set of inclusions $\{U_i \hookrightarrow U : i \in I\}$. The essential property is still that $\bigcup\{U_i : i \in I\} = U$, but we now have the notion of a covering system as a system of sets of arrows from poset category $\mathcal{T}$. There are three properties had by any set theoretic covering system that allow for the expression of the sheaf property: firstly, for any topology element U, the set $\{U\}$ is always a U-cover; secondly, if $\{U_i : i \in I\}$ is a U-cover and $V \subseteq U$ in $\mathcal{T}$, then $\{U_i \cap V : i \in I\}$ is a V-cover; thirdly, if $\{U_i : i \in I\}$ is a U-cover and for each U_i there is some U_i-cover $\{U_{i,k} : k \in K_i\}$, then U is covered by $\{U_{i,k} : k \in K_i, i \in I\}$. Once these three properties are expressed in terms of arrows in the poset category $\mathcal{T}$, we have the basis of the notion of a covering system for a category. All we need do is generalise from collections of arrows of a poset category to collections of arrows of an arbitrary category. In fact we must restrict the generalisation to categories with pullbacks since the generalisation of the second property will involve pullbacks. The generalisation of a set theoretic covering system for categories with pullbacks is called a pretopology.

Definition 2.1: a *pretopology* on a category $\mathcal{C}$ with pullbacks is a system P where for each $\mathcal{C}$-object U there is a set $\mathrm{P}(U)$ of sets $\{U_i \xrightarrow{\alpha_i} U : i \in I\}$ of $\mathcal{C}$-morphisms, and in addition the following conditions are satisfied:

(i) for each $U \in \mathcal{C}$, singleton $\{id_U\} \in \mathrm{P}(U)$;

(ii) if $V \longrightarrow U$ in $\mathcal{C}$ and $\{U_i \xrightarrow{\alpha_i} U : i \in I\} \in \mathrm{P}(U)$, then the pullback family $\{V \times_U U_i \xrightarrow{\pi_i} V : i \in I\}$ is in $\mathrm{P}(V)$;

(iii) if $\{U_i \xrightarrow{\alpha_i} U : i \in I\} \in \mathrm{P}(U)$ and we have $\{V_{i,k} \xrightarrow{\beta_{i,k}} U_i : k \in K_i\} \in \mathrm{P}(U_i)$ for each $i \in I$, then $\{V_{i,k} \xrightarrow{\beta_{i,k}} U_i \xrightarrow{\alpha_i} U : i \in I, k \in K_i\} \in \mathrm{P}(U)$.

In analogy with sheaves over a topological space we have the notion of sheaves over categories with pretopologies. We shall say that any contravariant functor $F: \mathcal{C}^{op} \to \mathbf{Set}$ is a *sheaf* when for each $U \in \mathcal{C}$ and for each $\{U_i \xrightarrow{\alpha_i} U : i \in I\} \in \mathrm{P}(U)$, the following diagram is an equaliser

$$F(U) \xrightarrow{\ e\ } \prod_{i \in I} F(U_i) \overset{d_0}{\underset{d_1}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

where e is the obvious product map and d_0 and d_1 are product arrows respectively of the images under F of the first and second pullback projection maps $U_i \times_U U_j \longrightarrow U_i$ and $U_i \times_U U_j \longrightarrow U_j$, all $i, j \in I$. The projections in question arise in pullbacks

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \alpha_i \\ U_j & \xrightarrow[\alpha_j]{} & U \end{array}$$

of α_j along α_i and α_i along α_j, all $i, j \in I$.

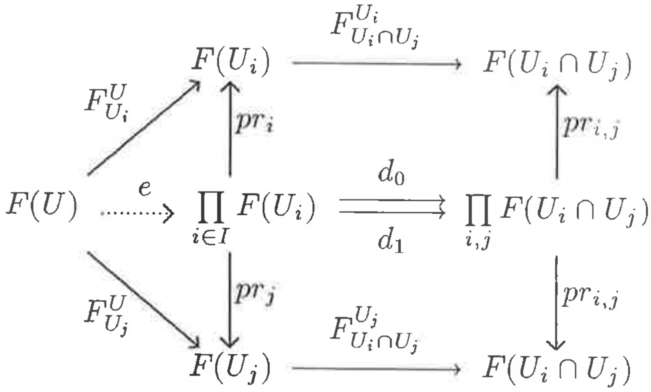
Notice that the notion of a sheaf over a pretopology is exactly that of a sheaf over a topological space when the pretopology in question is the covers system of the topological space. In such a case, the equaliser condition for a sheaf on a pretopology is the condition that whenever $\{U_i : i \in I\}$ is a U-cover in a topology, the diagram

$$F(U) \xrightarrow{\ e\ } \prod_{i \in I} F(U_i) \overset{d_0}{\underset{d_1}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

is an equaliser. But that this diagram is an equaliser is exactly the claim that what we have called the sheaf property holds. Recall from chapter seven the sheaf property: a contravariant functor $F\colon \mathcal{T}^{op} \to \mathbf{Set}$ is a sheaf when for a set of sections $\{s_i \in F(U_i) \colon i \in I\}$ such that

$$F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j), \quad \text{all } i, j \in I,$$

there is exactly one $s \in F(U)$ such that $F^U_{U_i}(s) = s_i$ for all $i \in I$. Now the follwoing diagram shows the construction of e, d_0 and d_1 as product arrows



Arrow d_0 is the product of maps $F^{U_i}_{U_i \cap U_j}$ all $i, j \in I$, and d_1 is the product of maps $F^{U_j}_{U_i \cap U_j}$ all $i, j \in I$. Arrow e is the product of maps $F^U_{U_i}$ all $i \in I$. From here it is straightforward that

Theorem 2.1: *e is an equaliser for d_0 and d_1 iff F satisfies the "sheaf property".*

Proof: let us grant that index set I has n elements. Then we are able to say that for any $s \in F(U)$, $e(s)$ is some n-tuple $\langle s_1, \ldots, s_n \rangle$ where for $1 \leq i \leq n$, $s_i = F^U_{U_i}(s)$. Suppose that for all $s \in F(U)$, we have $d_0(e(s)) = d_1(e(s))$, or, in other words,

$$d_0(\langle s_1, \ldots, s_n \rangle) = d_1(\langle s_1, \ldots, s_n \rangle).$$

This means exactly that

$$F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j), \quad \text{all } i, j \in I.$$

From this it follows that F satisfies the "sheaf property" if there is no $s' \in F(U)$ other than s such that $e(s') = e(s)$. But when e is an equaliser if $e(s') = e(s)$, then $s' = s$. The converse claim is established by the fact that there being no $s' \in F(U)$ other than s such that $e(s') = e(s)$ is exactly what it means for e to be an equaliser.

□

Following this discussion we can say that the usual notion of a sheaf on a topological space is captured by the notion of a sheaf on a category with a pretopology. We lose nothing in the generalisation.

A category together with a pretopology is called a *site*. A category of sheaves defined over a site is called a *Grothendieck topos*. Take note that this is the definition of a site and a Grothendieck topos as originally presented (cf. Artin et al.,[1972]). Since the notion of an adequate set-theoretic covering analog for categories has changed over time so too have the meanings of "site" and "Grothendieck topos". The change in meaning is by way of refinement and as such is not dramatic. However, we must be aware that the possibility for confusion exists. To guard against this we will introduce a method of referring to Grothendieck toposes and sites in such a way as to indicate which level of refinement we are invoking. Ultimately this is necessary only because we have found it easier to develop some parts of our discussion in terms of pretopologies and other parts in terms of the newer notion, which we introduce shortly, of a topology for a category.

A *precanonical pretopology* for a category C is one for which all representable functors are sheaves. A *canonical* pretopology is the precanonical pretopology that includes all other precanonical pretopologies. It is known that canonical pretopologies exist and that for a finitely complete category they are in fact formed by the stable effectively epimorphic families on which notion more is said in chapter ten.

Pretopologies do not in general uniquely determine a category of sheaves. To do that we refine the notion to that of a (categorial) topology.

A *topology on* $\mathcal{C}$ is a system J of sets, $J(U)$, of U-sieves for each $U \in \mathcal{C}$ where sysem J satisfies the following conditions:

(i) for any $U \in \mathcal{C}$, the *maximal* U-sieve $(U] \in J(U)$;

(ii) if $R \in J(U)$ and $V \xrightarrow{f} U$ is a morphism of $\mathcal{C}$, then

$$f^\star(R) = \{W \xrightarrow{\alpha} V : f \cdot \alpha \in R\}$$

is in $J(V)$;

(iii) if $R \in J(U)$ and S is a sieve on U where for each $(V \xrightarrow{f} U) \in R$ we have $f^\star(S)$ in $J(V)$, then $S \in J(U)$.

When J is a topology on $\mathcal{C}$, the sieves in each $J(U)$ are called *covering sieves.* Note that a collection of morphisms with codomain U can be a U-sieve without being a covering sieve on U.

This new categorial analogy of an adequate covering system leads to a new notion of *site* namely that of a category together with a topology. In what follows a pretopology will always be a system P while a topology *on* a category (as opposed to the notion we will shortly encounter of a topology *in* a category) will always be a system J. With this nomenclature we will be able to distinguish between sites $(\mathcal{C}, \mathrm{P})$ defined with respect to pretopologies, and sites $(\mathcal{C}, \mathrm{J})$ defined with respect to topologies on $\mathcal{C}$. We now define a *sheaf on a site* $(\mathcal{C}, \mathrm{J})$ to be any contravariant functor $F: \mathcal{C}^{op} \to \mathbf{Set}$ satisfiying the equaliser condition expressed in terms of covering sieves for U rather than covers. A category of sheaves on a site $(\mathcal{C}, \mathrm{J})$ is called a *Grothendieck topos on a site* $(\mathcal{C}, \mathrm{J})$ and is denoted $sh(\mathcal{C}, \mathrm{J})$.

We note that given a pretopology P we can define a topology J that will given rise to the same sheaves: we say that for any $U \in \mathcal{C}$, covering sieve $R \in J(U)$ iff R contains some pretopology cover $\{\alpha_i : i \in I\} \in \mathrm{P}(U)$. The claim that this topology gives rise to the same sheaves rests on the claim that if a family $\{\alpha_i : i \in I\}$ satisfies the equaliser condition, then a family R that contains $\{\alpha_i : i \in I\}$ will also satisfy that condition. Since we can define topologies J that include pretopologies P in this way, we can say that any Grothendieck topos on a site $(\mathcal{C}, \mathrm{P})$ is a Grothendieck topos on a site $(\mathcal{C}, \mathrm{J})$. The sites $(\mathcal{C}, \mathrm{P})$ and $(\mathcal{C}, \mathrm{J})$ are not (necessarily) the same, but they generate the same Grothendieck toposes. Another way of putting this is that Grothendieck toposes over sites $(\mathcal{C}, \mathrm{P})$ are a subclass of the Grothendieck toposes over sites $(\mathcal{C}, \mathrm{J})$.

Proposition 2.1: *any Grothendieck topos on a site $(\mathcal{C}, \mathrm{J})$ is an elementary topos.*

□

A topology J exists as a presheaf $\mathrm{J}: \mathcal{C}^{op} \to \mathbf{Set}$. This is the presheaf that takes each $\mathcal{C}$-object U to the set of covering U-sieves $\mathrm{J}(U)$, and takes each $\mathcal{C}$-arrow $V \xrightarrow{f} U$ to map $\mathrm{J}(f): \mathrm{J}(U) \to \mathrm{J}(V)$ given by

$$\mathrm{J}(U) \ni R \mapsto f^*(R) = \{W \xrightarrow{\alpha} V : f \cdot \alpha \in R\}.$$

Clearly presheaf J is a sub-functor of Ω. By this we mean that an inclusion $\mathrm{J} \hookrightarrow \Omega$ exists in $\mathbf{Set}^{\mathcal{C}^{op}}$. Since $\mathbf{Set}^{\mathcal{C}^{op}}$ has a subobject classifier, there exists a pullback

$$\begin{array}{ccc} \mathrm{J} & \longrightarrow & \Omega \\ \big\downarrow & & \big\downarrow j \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

where j is the classifying map of $\mathrm{J} \hookrightarrow \Omega$. Maps j of this sort are examples of maps j called elementary topologies.

Definition 2.2: any map $j: \Omega \to \Omega$ in an elementary topos $\mathcal{E}$ is an *(elementary) topology in $\mathcal{E}$* if the following conditions are met:

(i) $j \cdot true = true$;

(ii) $j \cdot j = j$;

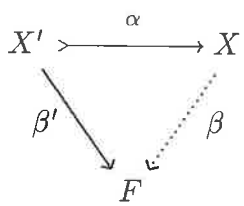
(iii) $\cap \cdot (j \times j) = j \cdot \cap$.

The notion of an elementary topology is the final example of the generalisation to categories of the notion of a covering system. The notion of an elementary topology is plainly no longer a system of covers in any literal sense however it does retain the essential feature of such a system that we may express the sheaf property. This is done in terms of j-dense monics in $\mathbf{Set}^{\mathcal{C}^{op}}$.

Definition 2.3: when $j: \Omega \to \Omega$ is an elementary topology in a topos $\mathcal{E}$, let $\mathrm{J} \rightarrowtail \Omega$ denote the monic classified by j. We then say that any $\mathcal{E}$-monic $X' \overset{\alpha}{\rightarrowtail} X$ is *j-dense* if its classifying map, χ_α, factors through $\mathrm{J} \rightarrowtail \Omega$.

The following definition of a sheaf in a topos $\mathcal{E}$ arises as a generalisation of a theorem due to Lawvere.

Definition 2.4: for any topos $\mathcal{E}$ containing an elementary topology j, an object F is a *sheaf with respect to j* or a *j-sheaf* if and only if for any $\mathcal{E}$-arrow $\beta': X' \to F$ and any j-dense monic $\alpha: X' \rightarrowtail X$, there is exactly one $\beta: X \to F$ such that the following diagram commutes.



The category of sheaves identified in this manner is a full subcategory of $\mathcal{E}$ and will be denoted $sh_j(\mathcal{E})$. Note particularly that the terminal object for $\mathcal{E}$ will always be the terminal object for $sh_J(\mathcal{E})$.

Proposition 2.2: *if $\mathcal{E}$ is a topos containing elementary topology j, then $sh_j(\mathcal{E})$ is also a topos.* □

Proposition 2.3: *when* J *is a topology on $\mathcal{C}$ and j is the character map of the inclusion* $\mathrm{J} \to \Omega$ *in* $\mathbf{Set}^{\mathcal{C}^{op}}$*, the Grothendieck topos $sh(\mathcal{C}, \mathrm{J})$ is the topos $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$.*

□

For a proof of this see Johnstone [1977], Example 3.22 and related discusion.

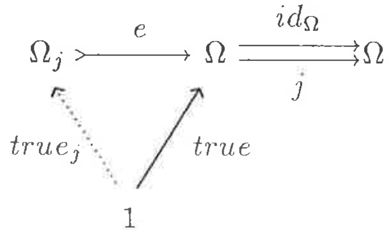
In this section we have discussed three devices by which we may reasonably describe category objects as sheaves. The first device was that of the pretopology. This was a straightforward generalisation to categories of the notion of covers in a topological space. The categories of sheaves we may describe using this device include (equivalents of) categories of the classical sheaves, the continuous local homeomorphisms. The next device was that of a topology for a category. This was a refinement of the pretopology notion. Any category of sheaves over a pretopology can be understood as a category of sheaves over a topology. The next device started out as a refinement of the notion of a topology for a category and became a generalisation. This last device was that of a topology in a category. Sheaves may now be thought of as distinguished topos objects. Equally, we need not now regard the pretopology and topologies–on notions as superfluous. In fact it will be useful in coming sections to have emphasised the relationships between the devices so that we might smoothly pass from one to the next depending upon our technical need. In the next section we will use the device of topologies j to describe the subobject classifier for sheaf categories while in the section following that we will use the device firstly of a pretopology P and then a topology J to justify the notion of sheaves over the closed sets of a topological space.

3. Subobject classifiers in sheaf categories

We have seen in the last section that when sheaves are understood as **Set**-valued contravariant functors, any category of sheaves can be rendered as a j-sheaf category $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ where $\mathcal{C}$ is a small category and j is some topology in $\mathbf{Set}^{\mathcal{C}^{op}}$. In the present section we will use this fact to provide ourselves with a construction for a subobject classifier in any sheaf category. We will also provide a construction for classifying maps for monics. We have two purposes here. Our principal aim is to provide the necessary preliminary detail for our discussion in the next chapter; there our discussion will appeal to the nature of the classifier object Ω_j. Our second purpose for the present section is the verification of the existence of a description of character arrows for monics which, when the sheaf category is $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ for some set theoretic topology $\mathcal{T}$, does not rely on the set theoretic properties of $\mathcal{T}$. In particular, we want to be able to construct character arrows for monics in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ without relying on an assumption of $\bigcup$-completeness in $\mathcal{T}$. When sheaves are defined over topological spaces, it is most often in terms of an open set topology, and these, by definition, are $\bigcup$-complete. However, in the next section, we will be defining sheaves over a closed set topology, so we require a character arrow construction that is, at least, independent of base space topology types. That there is such a construction is demonstrated as a corollary to the main theorem of the present section. This main theorem is on the nature of character arrows in sheaf categories $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ for any small category $\mathcal{C}$.

In section one of this chapter we described subobject classifiers $true: 1 \to \Omega$ for presheaf categories $\mathbf{Set}^{\mathcal{C}^{op}}$. There is a standard construction for a subobject classifier $true_j: 1 \to \Omega_j$ for a sheaf category $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ where j is a topology in $\mathbf{Set}^{\mathcal{C}^{op}}$ (cf. for example, the discussion and references in Goldblatt §14.4, [1984]).

Proposition 3.1: *for any topos $\mathcal{E}$ with a subobject classifier $true: 1 \to \Omega$ and a topology $j: \Omega \to \Omega$, the category $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ has a subobject classifier $true_j: 1 \to \Omega_j$ described by the following equaliser diagram in $\mathcal{E}$ where e is an equaliser and $true_j$ is the unique map making the whole diagram commute.*



□

Corollary: *when $\mathcal{E}$ is a presheaf category $\mathbf{Set}^{\mathcal{C}^{op}}$ for small $\mathcal{C}$, Ω_j is a contravariant functor $\mathcal{C}^{op} \to \mathbf{Set}$ where for any object a in $\mathcal{C}$*

$$\Omega_j(a) = \{S \in \Omega(a) : (id_\Omega)_a(S) = j_a(S)\};$$

and furthermore, for any $b \xrightarrow{f} a$ in $\mathcal{C}$, the maps $(\Omega_j)^a_b$ are functions given by

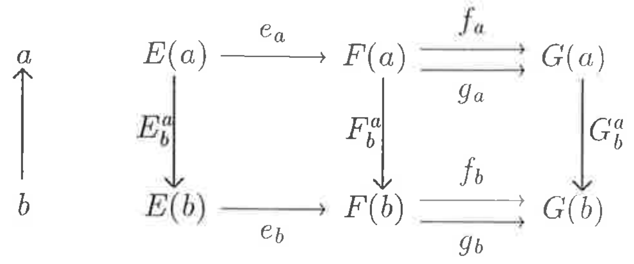
$$\Omega_j(a) \ni S \mapsto \{c \to b \mid (c \to b \xrightarrow{f} a) \in S\} \in \Omega_j(b).$$

In this case we also have that $true_j$ is a natural transformation given by components $(true_j)_a$ such that

$$(true_j)_a(\emptyset) = true_a(\emptyset).$$

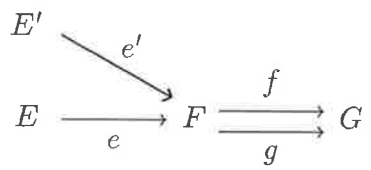
Proof: it is enough to demonstrate that we have natural transformation e that is an equaliser for id_Ω and j whenever we have equalisers e_a for $(id_\Omega)_a$ and j_a all objects $a \in \mathcal{C}$. The corollary is then demonstrated by the fact that the canonical choice for equalisers e_a for $(id_\Omega)_a$ and j_a in **Set** are the inclusions of $\{S \in \Omega(a) : (id_\Omega)_a(S) = j_a(S)\}$ in $\Omega(a)$.

The proof is independent of what natural transformations are equalised so suppose some parallel pair of natural transformations $f, g: F \rightrightarrows G$ between contravariant functors F and G from $\mathcal{C}$ to **Set**. The transformation f is a collection of functions $f_a: F(a) \to G(a)$ for all objects a in $\mathcal{C}$. Likewise, transformation g is a collection of functions $g_a: F(a) \to G(a)$. **Set** has equalisers and we can suppose canonical equalisers $e_a: E(a) \to F(a)$ for each pair for functions f_a and g_a. We can now use the fact that F is a functor to generate a functor E and a natural transformation $e: E \to F$. Let E be the functor $\mathcal{C}^{op} \to \mathbf{Set}$ that takes each object $a \in \mathcal{C}$ to $E(a)$, the domain of equaliser e_a; in addition E takes each $b \to a$ in $\mathcal{C}$ to a map $E^a_b: E(a) \to E(b)$ which is defined so that for any $x \in E(a)$, $E^a_b(x) = F^a_b(x)$. Plainly, if F is a functor, then so is E. Furthermore, for any $b \to a$ in $\mathcal{C}$, the following diagram will commute in **Set**.

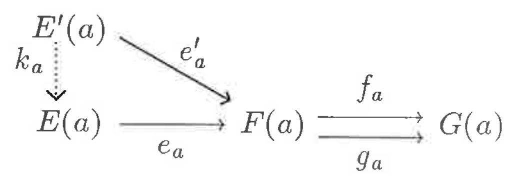


It follows that $\{e_a : a \in \mathcal{C}\}$ constitutes a natural transformation. Let use denote this transformation by e. All that remains is to demonstrate that e is an equaliser of f and g.

We know already that $f \cdot e = g \cdot e$ since $f_a \cdot e_a = g_a \cdot e_a$ for all $a \in \mathcal{C}$. To demonstrate that e is an equaliser we are required to demonstrate that for any natural transformation $e' \colon E' \to F$ such that $f \cdot e' = g \cdot e'$, there is exactly one natural transformation $k \colon E' \to E$ such that $e' = e \cdot k$. Consider the following diagram.



Suppose that $f \cdot e' = g \cdot e'$. We have immediately that for any $a \in \mathcal{C}$, the following diagram commutes.



where k_a is unique in making the triangle commute. If the maps k_a constitute a natural transformation $k \colon E' \to E$, then clearly k will be unique in making $e' = k \cdot e$. To see that maps k_a do constitute a natural transformation suppose any $b \to a$ in $\mathcal{C}$ and consider the following diagram.

$$
\begin{array}{ccccccc}
a & & E'(a) & \xrightarrow{k_a} & E(a) & \xrightarrow{e_a} & F(a) \\
\uparrow & & {\scriptstyle (E')^a_b}\downarrow & & {\scriptstyle E^a_b}\downarrow & & \downarrow{\scriptstyle F^a_b} \\
b & & E'(b) & \xrightarrow[k_b]{} & E(b) & \xrightarrow[e_b]{} & F(b)
\end{array}
$$

By hypothesis $e': E' \to F$ is a natural transformation, so $e'_b \cdot (E')^a_b = F^a_b \cdot e'_a$. But for all $a \in \mathcal{C}$, $e'_a = e_a \cdot k_a$, so

$$e_b \cdot k_b \cdot (E')^a_b = F^a_b \cdot e_a \cdot k_a.$$

Now e is a natural transformation, so $F^a_b \cdot e_a = e_b \cdot E^a_b$, and so

$$e_b \cdot k_b \cdot (E')^a_b = e_b \cdot E^a_b \cdot k_a.$$

By hypothesis e_b is an equaliser, so e_b is monic. Then $k_b \cdot (E')^a_b = E^a_b \cdot k_a$, and k is natural as a transformation. □

That we can describe $true_j$ in terms of $true$ and e indicates that we can describe classifying maps χ^j_τ for $sh_j(\mathcal{E})$-monics τ in terms of e and the classifying maps χ_τ in $\mathcal{E}$. We will demonstrate this shortly, but first we need to demonstrate that wherever τ is monic in $sh_j(\mathcal{E})$, it is also monic in $\mathcal{E}$. Recall that $sh_j(\mathcal{E})$ is a subcategory of $\mathcal{E}$, so it is permissible to describe any map τ in $sh_j(\mathcal{E})$ as being the same map in $\mathcal{E}$. The following proposition establishes the required relationship between $sh_j(\mathcal{E})$-monics and $\mathcal{E}$-monics.

Proposition 3.2 (Lawvere-Tierney): *for any elementary topos $\mathcal{E}$ with topology j, there is a sheafification functor $\mathrm{sh}_j: \mathcal{E} \to sh_j(\mathcal{E})$ that has $\mathrm{sh}_j(b) \cong b$ for each j-sheaf b. This functor preserves all finite limits.* □

(For a proof see, for example, the proof of Theorem 2.61 in Freyd [1972]).

That the sheafification functor preserves all finite limits means that the limit in $\mathcal{E}$ of any finite diagram of j-sheaves is itself a j-sheaf and is the limit in $sh_j(\mathcal{E})$

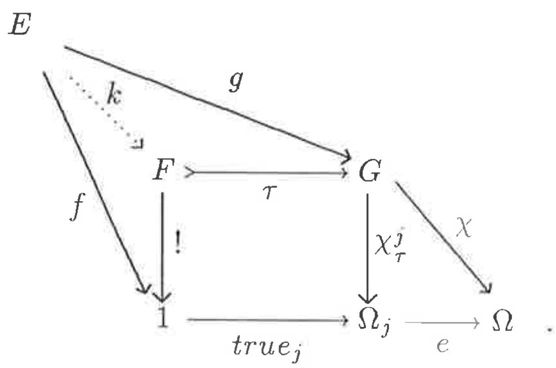
of the same diagram. Plainly since there are no objects nor arrows in $sh_j(\mathcal{E})$ that do not eixst in $\mathcal{E}$, it follows that the limit of a finite diagram in $sh_j(\mathcal{E})$ is a limit of the same diagram in $\mathcal{E}$. A particular consequence is that any pullback in $sh_j(\mathcal{E})$ is a pullback in $\mathcal{E}$ and any pullback in $\mathcal{E}$ of maps between j-sheaves is a pullback in $sh_j(\mathcal{E})$. Now, monics are preserved by any functor that preserves pullbacks since in any category, a map $a \xrightarrow{u} B$ is monic iff the following diagram is a pullback

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle id_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow[u]{} & B \end{array} .$$

(cf. Proposition 21.12, Herrlich and Strecker [1979]). It follows that any $sh_j(\mathcal{E})$-monic is monic in $\mathcal{E}$, and any arrow between j-sheaves that is monic in $\mathcal{E}$ is also monic in $sh_j(\mathcal{E})$. We are guaranteed, then, that when $\tau: F \rightarrowtail G$ is a $sh_j(\mathcal{E})$-monic, τ is monic in $\mathcal{E}$ and so there exists a classifying map χ_τ in $\mathcal{E}$. We let χ^j_τ denote the classifying map for τ in $sh_j(\mathcal{E})$ guaranteed by the existence of $true_j$. We are now in a position to demonstrate the relationship between χ_τ, χ^j_τ, and e, the equaliser of j and id_Ω.

Theorem 3.1: *when $\mathcal{E}$ is a topos with topology j and $\tau: F \rightarrowtail G$ is a $sh_j(\mathcal{E})$-monic, we have $\chi_\tau = e \cdot \chi^j_\tau$.*

Proof: consider the diagram



Let τ be a monic in $sh_j(\mathcal{E})$ and let $\chi = e \cdot \chi^j_\tau$. We will demonstrate that $\chi = \chi_\tau$. In the course of this proof we will refer to parts of the above diagram by (clockwise) vertices. So, for example, the inner square, the pullback diagram for $true_j$ and χ^j_τ, is denoted by $\{F, G, \Omega_j, 1\}$.

To establish that $\chi = \chi_\tau$, it is enough to establish that square $\{F, G, \Omega, 1\}$ made from the evident composites is a pullback. Since $e \cdot true_j = true$, the desired result will follow from the definition of $true$ as a subobject classifier in $\mathcal{E}$.

The square $\{F, G, \Omega, 1\}$ is a pullback only if it satisfies two conditions. First, the square must commute, that is we must have $\chi \cdot \tau = e \cdot true_j \cdot !$. But $\chi = e \cdot \chi^j_\tau$ so the square commutes if $e \cdot \chi^j_\tau \cdot \tau = e \cdot true_j \cdot !$. Now, since $\{F, G, \Omega_j, 1\}$ is a pullback, it at least commutes, so $\chi^j_\tau \cdot \tau = true_j \cdot !$. Plainly, then, $e \cdot \chi^j_\tau \cdot \tau = e \cdot true_j \cdot !$. The second and final condition that $\{F, G, \Omega, 1\}$ must satisfy to qualify as a pullback is that whenever the square $\{E, G, \Omega, 1\}$ made from the evident composites commutes, there is exactly one $E \xrightarrow{k} F$ making the whole diagram commute. Suppose that $\{E, G, \Omega, 1\}$ does commute. This means that $e \cdot \chi^j_\tau \cdot g = e \cdot true_j \cdot f$. But e is an equaliser and therfore monic, so we have $\chi^j_\tau \cdot g = true_j \cdot f$. But in that case, since $\{F, G, \Omega_j, 1\}$ is a pullback, we have exactly one $E \xrightarrow{k} F$ making the whole diagram commute. □

Corollary: *for topos* $\mathbf{Set}^{\mathcal{C}^{op}}$ *with topology* j, *if* $\tau: F \rightarrowtail G$ *is a* $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$*-monic, then for any* $a \in \mathcal{C}$ *and any* $x \in G(a)$,

$$(\chi^j_\tau)_a(x) \simeq (\chi_\tau)_a(x).$$

Proof: for any $a \in \mathcal{C}$, $(\chi_\tau)_a = e_a \cdot (\chi^j_\tau)_a$, and the canonical choice for e_a is an inclusion. □

It follows that in j-sheaf categories $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ we can use the usual construction for maps χ_τ as the construction for maps χ^j_τ. So when $\tau: F \rightarrowtail G$ is a $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$-monic, χ^j_τ is the map $G \to \Omega_j$ where for any a in $\mathcal{C}$ and any $x \in G(a)$

$$(\chi^j_\tau)_a(x) = \{b \to a \mid G^a_b(x) \in \tau_b(F(b))\}$$

where $b \to a$ is some map in $\mathcal{C}$.

4. Closed Set Sheaves

Typically sheaves over topological spaces are defined in terms of the open sets of the base space. The notion of a category of sheaves over a site allows us to define a category of sheaves over the closed sets of a topological space and announce that these categories are toposes.

Suppose some topology $\mathcal{T}$ of the closed sets of some space X. Any topology is partially ordered by set inclusion, so any topology forms a poset category. Let $(\mathcal{T}, \subseteq)$, or when no confusion will result let $\mathcal{T}$, denote the poset category of topology $\mathcal{T}$ ordered by set inclusion. To define a sheaf over the closed sets of X we need only define a pretopology P for poset category $\mathcal{T}$. To do this we note that any sieve in $\mathcal{T}$ will be some $R = \{U_i \xrightarrow{\alpha_i} U : i \in I\}$ where U and U_i for all $i \in I$ are topology elements and each α_i is an inclusion. Now since there can be at most one inclusion between any two topology elements, we can understand R to be a family $\{\mathrm{dom}(\alpha_i) : \alpha_i \in R\}$ of sets. On this understanding the defining conditions for a pretopology P become those originally true of set-theoretic covering systems, namely

(i) for each $U \in \mathcal{T}$, $\{U\} \in \mathrm{P}(U)$;

(ii) if $V \subseteq U$ in $\mathcal{T}$ and $\{U_i : i \in I\} \in \mathrm{P}(U)$, then $\{V \cap U_i : i \in I\} \in \mathrm{P}(V)$;

(iii) if $\{U_i : i \in I\} \in \mathrm{P}(U)$ and we have $\{V_{i,k} : k \in K_i\} \in \mathrm{P}(U_i)$ for each $i \in I$, then $\{V_{i,k} : k \in K_i, i \in I\} \in \mathrm{P}(U)$.

It follows that there is a pretopology P for $\mathcal{T}$ described by the covers system C for X which has for each $U \in \mathcal{T}$ a set

$$\mathrm{C}(U) = \big\{\{U_i : i \in I\} \mid U = \bigcup\{U_i : i \in I\}\big\}$$

of sets where each $U_i \in \mathcal{T}$. While this is not the only pretopology we could describe for $\mathcal{T}$, it is distinguished in that it is the canonical pretopology. A proof that this pretopology is canonical for $\mathcal{T}$ can be performed in terms of a notion to be introduced in chapter ten. There we note the standard result that canonical pretopologies for finitely complete categories are formed by the stable effectively epimorphic families. It is straightforward to show that poset category $\mathcal{T}$ is finitely complete and the the family $\{U_x \to U: x \in X\}$ is effectively epimorphic iff $\bigcup\{U_x: x \in X\} = U$. Furthermore these families are easily proven stable with respect to being effectively epimorphic when pulled back. It is exactly this pretopology which will interest us in the next chapter.

Having defined a pretopology P for category $\mathcal{T}$, it is straightforward that we have a category of sheaves over $\mathcal{T}$. It is worth remarking that the canonical pretopology for $\mathcal{T}$ is essentially just a covering system of the sort we would be familiar with from the task of defining sheaves over open set topologies. The only difference is that we are now considering closed set covers. It is worth remembering, too, that the equaliser condition we use to identify those presheaves that are sheaves is essentially just the familiar "sheaf property". There is an intuitive sense, then, to the claim that (some) sheaves over closed sets just are sheaves in the traditional sense. We finish this section with a statement of the theorem we have essentially already proven.

Theorem 4.1: *any category of sheaves over the closed sets of a topological space is a topos.* □

Of interest to us in the next chapter will be the particular nature of the classifier objects in closed set sheaf categories. We find that a base topology of closed sets does introduce BrAs into the structure of Ω_j but not in a way that yeilds Ω_j as a BrA object in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$. The BrAs that do exist will be best understood as algebras of sections given that Ω_j as a sheaf has an equivalent construction as a continuous local homeomorphism over $\mathcal{T}$. As such we retain our interest in Ω_j as an (ex-categorial) object of paraconsistent semantics rather than as a paraconsistent logic object in a category. We address the equivalence of closed set sheaves and closed set sheaf spaces in chapter thirteen. For discussion of an actual paraconsistent logic object, see chapters eleven and twelve.

CHAPTER 9: BROUWERIAN ALGEBRAS

in

CLOSED SET SHEAVES

Introduction: This chapter follows directly from chapter 8. In chapter 9 I give a technical discussion of an aspect of the nature of the classifier object as a logical algebra in a closed set sheaf category. This chapter is the one that tests and refutes the hypothesis that the topological dualisation of the sheaf notion enables us to describe new natural operators for subobject lattices in sheaf categories. The chapter has two sections. The first section shows that at the component level there are BrA operations to be found in the subobject classifier structure (as we see in the next chapter this means that subobject lattices in the category in question are BrAs). The second section shows that these BrA operations are not natural in the sense of being productive of a natural transformation within the category that would make the classifier object itself a BrA. The second section contains a discussion of why the component BrA operations do not produce a natural transformation. This discussion is given in terms of closure operations that define closed set topologies. The discussion, then, in this chapter is significant in two ways. Firstly it is a refutation of the hypothesis that topological dualisation of the sheaf notion will produce structures which when collected into categories yield natural BrA subobject lattices. Secondly it is the beginning of the discussion on why BrAs do not, in general, exist in toposes in a way completely analogous to the existence of HAs. In this, then, our discussion here is quite different from that in Lawvere [1991] in which paper Lawvere announced a result that subsumes the non-naturalness result of this chapter.

This chapter contains a technical discussion of the algebraic nature of Ω_j in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ where $\mathcal{T}$ is a closed set topology and j is the canonical topology in $\mathbf{Set}^{\mathcal{T}^{op}}$. We will find that for each $V \in \mathcal{T}$, the set $\Omega_j(V)$ ordered by set inclusion is a subalgebra of $(\mathcal{T}, \subseteq)$ and as such is a BrA. We will use $\dot{-}_V$ to denote the BrA operator on each $\Omega_j(V)$. The collection of functions $\{\dot{-}_V : V \in \mathcal{T}\}$ is considered. We shall find that the collection does not constitute a natural transformation $\Omega_j \times \Omega_j \to \Omega_j$ in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$. The existence of collection $\{\dot{-}_V : V \in \mathcal{T}\}$ will be placed in context by the discussion we give in chapter thirteen of sheaves as sheaf spaces. Each $\Omega_j(V)$ can be understood as an algebra of sections of a sheaf space and the existence of $\dot{-}_V$ reflects the fact that this algebra is closely tied to base and stalk space topologies.

In the last section of the last chapter we described the canonical pretopology P that exists for the poset category $\mathcal{T}$. From section two of the last chapter we know that P can be refined to a topology J for $\mathcal{T}$ that will give rise to the same sheaves. We know, too, that J exists as a subfunctor of Ω in $\mathbf{Set}^{\mathcal{T}^{op}}$ and that the classifying map, j, for this inclusion is a topology on $\mathbf{Set}^{\mathcal{T}^{op}}$. It is this j that we use to determine the sheaf category $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ that we consider in this chapter. We know from section three of the last chapter that we can describe the classifier object Ω_j for $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$. It will be found in the first section of the present chapter that for each $V \in \mathcal{T}$, the set $\Omega_j(V)$ is isomorphic to the set $(V]$ and as such, under set inclusion, is a subalgebra of $(\mathcal{T}, \subseteq)$. It follows that we can define a pseudo difference operation $\dot{-}_V$ for each $\Omega_j(V)$. This means that we have a transformation $\{\dot{-}_V : V \in \mathcal{T}\}$. In the second section of the present chapter we show that this transformation is not in general natural. As such the transformation is not in general an arrow in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$.

1. Component Algebras of the Classifier Object

Suppose the pretopology P on closed set poset category $\mathcal{T}$ where for each $U \in \mathcal{T}$,

$$\mathrm{P}(U) = \big\{\{U_i: i \in I\} \mid U = \bigcup\{U_i: i \in I\}\big\}.$$

Let J be the topology on $\mathcal{T}$ defined so that

$$R \in \mathrm{J}(U) \quad \text{iff} \quad R \text{ contains some } \{U_i: i \in I\} \in \mathrm{P}(U).$$

Recall that Ω for $\mathbf{Set}^{\mathcal{T}^{op}}$ has that

$$\Omega(U) = \{\text{all sieves on } U\}$$

so there exists an inclusion $\mathrm{J} \hookrightarrow \Omega$. Let j be the character map of this inclusion. It follows that for all $V \in \mathcal{T}$ and all $S \in \Omega(V)$,

$$j_V(S) = \{U \subseteq V: \Omega_U^V(S) \in \mathrm{J}(U)\}.$$

Lemma 1.1: *for any $V \in \mathcal{T}$ and any $S \in \Omega(V)$, we have $S \subseteq j_V(S)$.*

Proof: if $V \in S$, then $(V] \subseteq S$, in which case it would follow that for any $U \subseteq V$, $\Omega_U^V(S) = S \cap (V] = (V]$. So, by condition (1) of topologies J, if $V \in S$, then $V \in j_V(S)$. □

Now, we say that a V-sieve S is not *maximal* if it is not some $(U]$ for some $U \subseteq V$ in $\mathcal{T}$. A maximal V-sieve $(U]$ is said to have exactly one top element, namely U. A ***top element*** in a V-sieve S is some $W \in S$ such that for all $Z \in S$, it is *not* the case that W is a proper subset of Z.

Lemma 1.2: *if V-sieve S is not maximal, then $S \neq j_V(S)$.*

Proof: if S is not maximal, it will have at least two distinct top elements. Let W and W' be two distinct top elements in S. It follows that $W \cup W' \notin S$. However, $W \cup W' \in j_V(S)$ if $\Omega_{W \cup W'}^V(S) \in \mathrm{J}(W \cup W')$. Now

$$\begin{aligned}\Omega_{W \cup W'}^V(S) &= S \cap (W \cup W'] \\ &= (W] \cup (W']\end{aligned}$$

so, $\bigcup(\Omega^V_{W\cup W'}(S)) = W \cup W'$. This means that $\Omega^V_{W\cup W'}(S)$ is a cover for $W \cup W'$, or in other words, $\Omega^V_{W\cup W'}(S) \in \mathrm{J}(W \cup W')$. □

Lemma 1.3: *if V-sieve S is maximal, then $S = j_V(S)$.*

Proof: suppose S is $(U]$ for some $U \subseteq V$ in $\mathcal{T}$. Suppose that $W \subseteq V$ and $W \notin (U]$, that is, $W \nsubseteq U$. Now $\Omega^V_W(S) = (U] \cap (W] = (U \cap W]$. Also, since $W \nsubseteq U$, we have $U \cap W \neq W$. It follows that $\bigcup(\Omega^V_W(S)) \neq W$, so $\Omega^V_W(S) \notin \mathrm{J}(W)$ and $W \notin j_V(S)$. This gives us $j_V(S) \subseteq S$ which together with lemma 1.1 gives us $S = j_V(S)$. □

Theorem 1.1: *for any $V \in \mathcal{T}$ and any $S \in \Omega(V)$, $S = j_V(S)$ iff S is maximal.*

Proof: lemmas 1.2 and 1.3. □

It follows from the fact that it is the domain of an equaliser of id_Ω and j that we can describe Ω_j as a functor $\mathcal{T}^{op} \to \mathbf{Set}$ such that for all $V \in \mathcal{T}$,

$$\Omega_j(V) = \{(W]: W \subseteq V \text{ in } \mathcal{T}\};$$

and when $U \subseteq V$ in $\mathcal{T}$, restriction maps $(\Omega_j)^V_U$ are given by

$$(\Omega_j)^V_U((W]) = (W] \cap (U].$$

To simplify our discussion note an isomorphism: let Ω'_j be a functor $\mathcal{T}^{op} \to \mathbf{Set}$ where for all $V \in \mathcal{T}$

$$\Omega'(V) = (V];$$

and for all $U \subseteq V$ in $\mathcal{T}$, maps $(\Omega'_j)^V_U$ are defined so that for any $W \in (V]$,

$$(\Omega'_j)^V_U(W) = W \cap U.$$

Theorem 1.2: *Ω_j and Ω'_j are iso in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$.*

Proof: the theorem is demonstrated by describing a bijection

$$\varphi_V: \Omega_j(V) \to \Omega'_J(V)$$

for each $V \in \mathcal{T}$ and showing that $\{\varphi_V : V \in \mathcal{T}\}$ constitutes a natural transformation. If we define φ_V so that for any $(W] \in \Omega_j(V)$,

$$\varphi_V((W]) = W,$$

it is plain that φ_V is a bijection. That $\{\varphi_V : V \in \mathcal{T}\}$ constitutes a natural transformation is the claim that

$$\begin{array}{ccccc} V & \quad & \Omega_j(V) & \xrightarrow{\varphi_V} & \Omega'_j(V) \\ \uparrow & & (\Omega_j)^V_U \downarrow & & \downarrow (\Omega'_j)^V_U \\ U & & \Omega_j(U) & \xrightarrow[\varphi_U]{} & \Omega'_j(U) \end{array}$$

commutes for any $U \subseteq V$ in $\mathcal{T}$. To establish that the squares in question commute observe that for any $(W] \in \Omega_j(V)$

$$\begin{aligned} (\Omega'_j)^V_U\Big(\varphi_V((W])\Big) &= (\Omega'_j)^V_U(W) \\ &= W \cap U \end{aligned}$$

and that

$$\begin{aligned} \varphi_U\Big((\Omega_j)^V_U((W])\Big) &= \varphi_U((W] \cap (U]) \\ &= \varphi_U((W \cap U]) \\ &= W \cap U. \end{aligned}$$

□

Since Ω_j and Ω'_j are iso in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$, we can use them interchangeably in the $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ context.

Theorem 1.3: *for any $V \in \mathcal{T}$, the set $\Omega'_j(V)$ under set inclusion is a* BrA.

Proof: any $W, W' \in \Omega'_j(V)$ are closed sets in $\mathcal{T}$ and are such that $W, W' \subseteq V$. It follows that $W \cap W', W \cup W' \subseteq V$ and are closed sets in $\mathcal{T}$. This means that $\Omega'_j(V)$ is a lattice under set inclusion.

That $\Omega'_j(V)$ is a BrA follows from the fact that $(\mathcal{T}, \subseteq)$ is a BrA. We have that, for any $S, T, Z \in \mathcal{T}$,

$$S \dot{-} T \subseteq Z \quad \text{iff} \quad S \subseteq T \cup Z$$

where $\dot{-}$ is the BrA operation on $(\mathcal{T}, \subseteq)$. Now, if $S, T \subseteq V$, then $S \dot{-} T \subseteq V$, since we always have that $S \dot{-} T \subseteq S$. Plainly, if we define $\dot{-}_V$ on $\Omega'_j(V)$ such that for any $S, T \in \Omega'_j(V)$,

$$S \dot{-}_V T \underset{df}{=} S \dot{-} T,$$

then $\dot{-}_V$ is a BrA operation on $\Omega'_j(V)$. □

It follows from the existence of φ_V that $\Omega_j(V)$ is a BrA under set inclusion.

2. Component Algebras and Natural Transformations

In this section we demonstrate that the collection $\{\dot{-}_V : V \in \mathcal{T}\}$ of functions in general fails to be a natural transformation. We demonstrate this in terms of Ω'_j. The claim that $\{\dot{-}_V : V \in \mathcal{T}\}$ constitutes a natural transformation in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ is the claim that the squares

$$\begin{array}{ccccc}
V & & \Omega'_j(V) \times \Omega'_j(V) & \xrightarrow{\ \dot{-}_V\ } & \Omega'_j(V) \\
\uparrow & & {\scriptstyle (\Omega'_j \times \Omega'_j)^U_V} \downarrow & & \downarrow {\scriptstyle (\Omega'_j)^U_V} \\
U & & \Omega'_j(U) \times \Omega'_j(U) & \xrightarrow[\ \dot{-}_U\ ]{} & \Omega'_j(U)
\end{array}$$

commute for all $U \subseteq V$ in $\mathcal{T}$. Observe that for any $\langle S, T \rangle \in \Omega'_j(V) \times \Omega'_j(V)$

$$\begin{aligned}
(\Omega'_j)^V_U\Big(\dot{-}_V(\langle S, T \rangle)\Big) &= (\Omega'_j)^V_U(S \dot{-} T) \\
&= (S \dot{-} T) \cap U
\end{aligned}$$

and

$$\begin{aligned}
\dot{-}_U\Big((\Omega'_j \times \Omega'_j)^V_U(\langle S, T \rangle)\Big) &= \dot{-}_U(\langle S \cap U, T \cap U \rangle) \\
&= (S \cap U) \dot{-} (T \cap U).
\end{aligned}$$

Now, if cl is the closure operation that determines the closed set topology $\mathcal{T}$ on X, we can describe $S \dot{-} T$ as the set $cl\big((X-T)\cap S\big)$ and $(S\cap U)\dot{-}(T\cap U)$ as the set $cl\Big(\big(X-(T\cap U)\big)\cap S\cap U\Big)$.

There are two cases to consider. The first is where $(\mathcal{T}, \subseteq)$ is a Boolean algebra and the second is where it is not.

Theorem 2.1: *when $(\mathcal{T}, \subseteq)$ is a Boolean algebra, $\{\dot{-}_V : V \in \mathcal{T}\}$ is a natural transformation.*

Proof: Suppose that $(\mathcal{T}, \subseteq)$ is Boolean. This means in particular that for any $T \in \Omega'_j(V)$, $cl(X-T) = X-T$. And then we have that

$$\begin{aligned}(S \dot{-} T)\cap U &= cl\big((X-T)\cap S\big)\cap U\\ &= (X-T)\cap S\cap U\end{aligned}$$

and

$$\begin{aligned}(S\cap U)\dot{-}(T\cap U) &= cl\Big(\big(X-(T\cap U)\big)\cap S\cap U\Big)\\ &= X-(T\cap U)\cap S\cap U;\end{aligned}$$

and since $X-(T\cap U)\cap U = (X-T)\cap U$, we have

$$(S\dot{-}T)\cap U = (S\cap U)\dot{-}(T\cap U). \qquad \square$$

So in the Boolean case we do have a natural transformation. However since $(\mathcal{T}, \subseteq)$ is Boolean, the natural transformation $\{\dot{-}_V : V \in \mathcal{T}\}$ is relatively trivial from our point of view. Its existence does not change the fact that Ω'_j will be a Boolean algebra. That Ω'_j will be a Boolean algebra follows from the fact that each $\Omega'_j(V)$ will be a Boolean algebra.

Theorem 2.2: *when $(\mathcal{T}, \subseteq)$ is not a Boolean algebra, $\{\dot{-}_V : V \in \mathcal{T}\}$ is not a natural transformation.*

Proof: Suppose that $(\mathcal{T}, \subseteq)$ is not Boolean. Suppose further that $V = X$. Since $(\mathcal{T}, \subseteq)$ is not Boolean, there must be at least one $T \in \mathcal{T}$ such that $cl(X-T) \neq X-T$.

For some such T let U be the $b(X-T)$, the boundary of $X-T$. If we let S be X, then

$$(S \dot{-} T) \cap U = cl((X-T) \cap S) \cap U$$
$$= cl((X-T) \cap X) \cap b(X-T)$$
$$= cl(X-T) \cap b(X-T)$$
$$= b(X-T)$$

and

$$(S \cap U) \dot{-} (T \cap U) = cl\Big((X - (T \cap U)) \cap S \cap U\Big)$$
$$= cl\Big((X - (T \cap b(X-T))) \cap X \cap b(X-T)\Big)$$
$$= cl\Big(X - \big(b(X-T)\big) \cap b(X-T)\Big)$$
$$= cl(\emptyset)$$
$$= \emptyset.$$

And since by hypothesis $b(X-T) \neq \emptyset$, the square does not commute. □

Another way to look at this result is to consider the cases where both V and S are X. Then

$$(S \dot{-} T) \cap U = cl(X-T) \cap U$$

and

$$(S \cap U) \dot{-} (T \cap U) = cl\Big((X - (T \cap U)) \cap U\Big).$$

And since $\big(X - (T \cap U)\big) \cap U = (X-T) \cap U$, the claim that $\{\dot{-}_V : V \in \mathcal{T}\}$ is a natural transformation is, in part, the claim that for any $T, U \in \mathcal{T}$,

$$cl(X-T) \cap U = cl((X-T) \cap U).$$

This is the claim that set theoretic closure operations distribute over intersection. This is known to be false in general. We can say that in those cases where it is true, we have a natural transformation $\{\dot{-}_V : V \in \mathcal{T}\}$, and in those cases where it is false, we do not.

The next chapter is a discussion of the properties of Ω_j in the more general sheaf categories $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ where $\mathcal{C}$ is a small category. We will find once again that for each $a \in \mathcal{C}$, $\Omega_j(a)$ is a BrA under set inclusion but that this will not in general yeild a BrA Ω_j in the category. Since the categories $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ are numbered among the categories $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ we would expect a general failure of naturalness for collections of BrA operations on sets $\Omega_j(a)$; the interest in including the next chapter is the fact that we can show subobject lattices from any Grothendieck topos to be Brouwerian algebras under the usual subobject inclusion ordering.

CHAPTER 10: GROTHENDIECK TOPOSES

Introduction: This chapter generalises the results and discussion of the previous chapter. Where chapter 9 considers the subobject classifier in a category of closed set sheaves chapter 10 considers the subobject classifier in (by isomorphism) any Grothendieck topos. The result in chapter 10 on component BrAs in the classifier structure and their failure to yield a natural transformation is a generalisation of the results in chapter 9 inasmuch as the chapter 10 results contain the chapter 9 results. However, the method employed in chapter 10 to produce the results is recognisably different from that used in chapter 9 and does not derive from a topological dualisation. It follows, then, that from the point of view of my project, the discussions, technical and otherwise, are distinct. This is my reason for separating them into two chapters. The significance of chapter 10 is in the fact that it generalises chapter 9.

With this chapter we offer a part generalisation of the discussion in chapters eight and nine. We concern ourselves here with a discussion of subobject classifier objects in categories of sheaves over sites. We will use the fact that any Grothendieck topos is equivalent to the category of sheaves over itself with the canonical pretopology to give a description of the classifier object as a functor Ω_j of subobject lattices. Further, we will use the fact that subobject lattices in Grothendieck toposes are complete and distributive to show that for any object a in the base category, the image $\Omega_j(a)$ is a BrA under set inclusion. As we would expect from the discussion in chapter nine we will also find that these BrAs do not yield a natural transformation in the sheaf category itself; so, in general, it will not follow from the existence of BrAs $\Omega_j(a)$, that Ω_j is a BrA. The discussion in this chapter will add an extra dimension to that in chapters eight and nine by virtue of the fact that we will be able to use the failure of naturalness in a of the particular $\Omega_j(a)$ BrAs here to offer some informed speculation on what it would be for the subobject logic of a category to be paraconsistent.

In section one we will recast our earlier discussion of pretopologies in terms of stable effectively epimorphic families. In section two we will use an equivalence theorem from Makkai and Reyes [1977] to give a description of the classifier object of a Grothendieck topos in terms of subobject lattices. In section three we will use the completeness and distributivity of such lattices in Grothendieck toposes to show that there are BrAs within the structure of the classifier object. We go on to show that these algebras do not yield an extra operator on the classifer algebra within the category.

We make some significant use of the features common to both j-sheaf theory and pretopology theory and we will accomodate this by using a concept of a ***topology defined by (pretopology) saturation.*** If cov is a pretopology for a category $\mathcal{C}$, then we will say that J is a topology for $\mathcal{C}$ defined by saturation of cov if for any $a \in \mathcal{C}$,

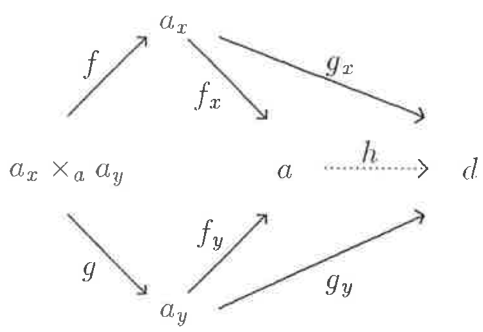
$$R \in \mathrm{J}(a) \quad \text{iff} \quad R \text{ contains some } S \in cov(a).$$

The topology j that is the character of the inclusion $\mathrm{J} \hookrightarrow \Omega$ in $\mathbf{Set}^{\mathcal{C}^{op}}$ will also be called a topology defined by saturation of cov. It is readily shown that $sh_j(\mathbf{Set}^{\mathcal{C}^{op}})$ and $sh(\mathcal{C}, cov)$ are equivalent.

1. Pretopologies and Sheaves revisited

For a category $\mathcal{C}$, a collection $C = \{a_x \xrightarrow{f_x} a \mid x \in X\}$ of $\mathcal{C}$-arrows is called an *epimorphic family* if given any pair of parallel arrows $i, j : a \rightrightarrows b$ such that $i \cdot f_x = j \cdot f_x$ for all $x \in X$, then we have $i = j$. Suppose we have an epimorphic family C. Suppose further that for some $\mathcal{C}$ object d there is a family

$D = \{a_x \xrightarrow{g_x} d: x \in X\}$ of $\mathcal{C}$-arrows such that for any $x, y \in X$ the outer square of the following diagram commutes where $\{f, g\}$ is the pullback of $\{f_x, f_y\}$



If it happens that for any such family D there is exactly one $\mathcal{C}$-arrow $h: a \to d$ such that for all $x \in X$, we have $h \cdot f_x = g_x$, then C is called an *effectively epimorphic family*. Any effectively epimorphic family is an epimorphic family but in general the converse does not hold.

We speak (loosely) of the pullback of a family $C = \{a_x \xrightarrow{f_x} a: x \in X\}$ along a $\mathcal{C}$-morphism $k: a' \to a$. This is a family C' of $\mathcal{C}$-morphisms with the property that $g_x \in C'$ iff g_x is the pullback of f_x along k for some $f_x \in C$. An effectively epimorphic family C is called *stable* if for any $\mathcal{C}$-morphism k, the pullback of C along k is also effectively epimorphic. It is known (cf. for example Goldblatt §16.2, [1984]) that the stable effectively epimorphic families of a finitely complete category form a canonical pretopology for that category; it is also known that for any Grothendieck topos, the stable effectively epimorphic families are exactly the epimorphic families.

A useful theorem follows from the above understanding of pretopologies. The theorem can be found as one part of Theorem 1.4.3, Makkai and Reyes [1977], and is about the nature of what is called the canonical functor for categories of sheaves over sites. To describe the theorem we first introduce some notation and develop the notion of the canonical functor. For this chapter a category of sheaves over a site $\mathbf{C} = (\mathcal{C}, cov)$ we will be denoted by $sh(\mathbf{C})$. When $\mathcal{D}$ is a locally small category, that is when $\hom(a, b)$ is a set for any objects $a, b \in \mathcal{D}$, and $\mathbf{D} = (\mathcal{D}, cov)$ is a site, the *canonical functor*

$$E_{\mathbf{D}}: \mathcal{D} \to sh(\mathbf{D})$$

is the composite of the dual Yoneda functor $\mathcal{Y}': \mathcal{D} \to \mathbf{Set}^{\mathcal{D}^{op}}$ and the sheafification functor $\mathrm{sh}: \mathbf{Set}^{\mathcal{D}^{op}} \to sh(\mathbf{D})$. Consider the Grothendieck topos $sh(\mathbf{C})$ of sheaves over any site $\mathbf{C} = (\mathcal{C}, cov)$. Let COV be the canonical pretopology on category $sh(\mathbf{C})$. Then

Proposition 1.1: *$E_{sh(\mathbf{C})}: sh(\mathbf{C}) \to sh_{COV}(sh(\mathbf{C}))$ is an equivalence of categories and is essentially just (that is, isomorphic to) the dual Yoneda embedding.* □

2. Subobject Classifiers in Grothendieck Toposes

In this section we will be addressing ourselves to the nature of classifier objects in categories of sheaves over Grothendieck toposes with canonical pretopologies. The proposition at the end of the last section tells us that any Grothendieck topos is equivalent to the category of sheaves over itself with the canonical pretopology. It follows that any discussion we make in this section will apply (by isomorphism) to classifier objects in any Grothendieck topos. A feature of this discussion will be the complexity of notation so we first of all will consider some simplifications.

Let $\mathcal{C}$ be a category and let cov be a pretopology for $\mathcal{C}$. Let $\mathbf{C} = (\mathcal{C}, cov)$ denote the site that is $\mathcal{C}$ together with cov. We will use $st(\mathcal{C})$ to denote the category of presheaves $\mathcal{C}^{op} \to \mathbf{Set}$. Previously we have used the notation $\mathbf{Set}^{\mathcal{C}^{op}}$ to denoted $st(\mathcal{C})$. The notation "$st(\mathcal{C})$" can be read "the category of stacks on $\mathcal{C}$". A stack on $\mathcal{C}$ is the same thing as a presheaf on $\mathcal{C}$. We will use $sh_{cov}(\mathcal{C})$ or $sh(\mathbf{C})$ to denote the category of sheaves on $\mathcal{C}$ with respect to cov. The category that we concern ourselves with in this section is $sh_{COV}(sh(\mathbf{C}))$ where COV is the canonical pretopolgy on $sh(\mathbf{C})$. For relative simplicity we assume COV and denote the subject of our discussion by $sh(sh(\mathbf{C}))$. Our discussion so far has been of sheaves with respect to pretopologies since these are the terms of the Makkai and Reyes theorem. However, we will want to make use of the tools of j-sheaf theory to define the classifier object for $sh(sh(\mathbf{C}))$. To do this we let j be the topology in $sh(sh(\mathbf{C}))$ defined by the saturation of COV. In terms of the notation used in previous chapters j is a map $\Omega \to \Omega$ in the category $\mathbf{Set}^{(sh(\mathbf{C}))^{op}}$ with Ω as the classifier object. The category $sh(sh(\mathbf{C}))$ is equivalent to $sh_j(\mathbf{Set}^{(sh(\mathbf{C}))^{op}})$. We will prefer the simpler $sh(sh(\mathbf{C}))$ for the name of the category. We will use Ω_j to denote the classifier object of $sh(sh(\mathbf{C}))$. It will be important to keep in mind that when we write "Ω" we will be referring to the classifier object of the presheaf category $st(sh(\mathbf{C}))$.

The reason for going to the trouble of considering $sh(sh(\mathbf{C}))$ is the description it will afford of Ω_j in terms of subobject lattices in $sh(\mathbf{C})$. We will find, as one would expect, that for any object a in $sh(\mathbf{C})$, $\Omega_j(a)$ ordered by set inclusion is essentially $\text{Sub}_{sh(\mathbf{C})}(a)$ ordered by subobject inclusion. Since any Grothendieck topos over a site is bi-complete, any $\text{Sub}_{sh(\mathbf{C})}(a)$ is a complete lattice. Together with the fact that the lattice is distributive we have that we can define a BrA operation on $\Omega_j(a)$ (and therefore BrA operations on subobject lattices in $sh(\mathbf{C})$). We will find that the BrA operations $\dot{-}_a$ for each $\Omega_j(a)$ do not constitute a BrA operation on Ω_j; but in any case we will have a description of BrA operations on each $\text{Sub}_{sh(\mathbf{C})}(a)$ for any a in $sh(\mathbf{C})$ and a new description of the $sh(\mathbf{C})$ classifier object.

Since $sh(sh(\mathbf{C}))$ is at least an elementary topos we can describe its classifier with the usual equaliser diagram in $st(sh(\mathbf{C}))$ for id_Ω and j

$$\Omega_{j'} \overset{e}{\rightarrowtail} \Omega \underset{j}{\overset{id_\Omega}{\rightrightarrows}} \Omega$$

We have, then, that for any $a \in sh(\mathbf{C})$,

$$\Omega_j(a) = \{S \in \Omega(a) \colon j_a(S) = (id_\Omega)_a(S)\}$$

$$= \{S \colon j_a(S) = S\}.$$

Recall that $\Omega(a)$ is the collection of all a-sieves on $sh(\mathbf{C})$ and that

$$j_a(S) = \{b \to a \mid \Omega^a_b(S) \in \mathrm{J}(b)\}.$$

We saw in chapter eight that where j is the canonical topology on $\mathbf{Set}^{\mathcal{T}^{op}}$, we have for any $V \in \mathcal{T}$ and any V-sieve S, $j_V(S) = S$ iff $S = (U]$ for some $U \subseteq V$ in $\mathcal{T}$. We will demonstrate a similar result here. To describe it we introduce some notation. For a map $b \xrightarrow{f} a$, let $(f]$ or $(b \xrightarrow{f} a]$ or, when f is understood, $(b \to a]$, be the sieve that contains all arrows $c \xrightarrow{g} a$ that factor through f. This then is the sieve such that for any $c \xrightarrow{g} a$, $g \in (b \xrightarrow{f} a]$ iff there is some $c \xrightarrow{h} b$ such that $g = f \cdot h$. On occasion we will use $(a]$ to denote $(id_a]$. In what follows sieves $(b \to a]$ are always sieves of $sh(\mathbf{C})$-arrows. We also introduce the concept of a top element for these sieves. For sieve S of arrows, the arrow $b \xrightarrow{f} a$ is a *top element* if there is no $(c \xrightarrow{g} a) \in S$ through which f factors other than itself or an isomorph. In that case there is no $(c \xrightarrow{g} a) \in S$ and no $b \xrightarrow{h} c$ such that $f = g \cdot h$ other than when $g = f$ or $g \simeq f$. Note that where f and g are isomorphic arrows the sieves $(f]$ and $(g]$ are identical.

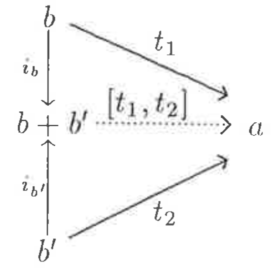
We will show through a series of lemmas that for any $a \in sh(\mathbf{C})$ and any $S \in \Omega(a)$, $j_a(S) = S$ iff $S = (b \rightarrowtail a]$ for some $sh(\mathbf{C})$-monic $b \rightarrowtail a$. It will follow that

$$\Omega_j(a) = \{(f]: f \in \mathrm{Sub}_{sh(\mathbf{C})}(a)\}.$$

We will say that $S \in \Omega(a)$ is *maximal* iff it is $(b \to a]$ for some $b \to a$ in $sh(\mathbf{C})$. If $S \in \Omega(a)$ is not maximal, it must have at least two distinct top elements. These top elements will be distinct in the sense that they are neither identical nor isomorphic in $sh(\mathbf{C})$.

Lemma 2.1: *for any $a \in sh(\mathbf{C})$, if $S \in \Omega(a)$ is not maximal, then $j_a(S) \neq S$.*

Proof: if S is not maximal, it has at least two distinct top elements. Let these elements be $b \xrightarrow{t_1} a$ and $b' \xrightarrow{t_2} a$. Since $sh(\mathbf{C})$ is a topos, the map $[t_1, t_2]: b + b' \to a$ exists in $sh(\mathbf{C})$. Consider the diagram

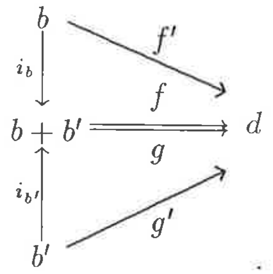


Plainly, both t_1 and t_2 factor through $[t_1, t_2]$. It follows from this and the hypothesis that t_1 and t_2 are top elements that $[t_1, t_2] \notin S$. So we will have demonstrated the lemma if we show that $[t_1, t_2] \in j_a(S)$.

Now $\Omega([t_1, t_2])(S) = \{c \xrightarrow{f} b + b' \mid [t_1, t_2] \cdot f \in S\}$ so we have

$$\{i_b, i_{b'}\} \subseteq \Omega([t_1, t_2])(S).$$

It follows that to demonstrate $\Omega([t_1, t_2])(S) \in \mathrm{J}(b+b')$ and therefore to demonstrate that $[t_1, t_2] \in j_a(S)$, we need only demonstrate that $\{i_b, i_{b'}\}$ is an epimorphic family. Consider the diagram



Suppose some parallel pair $b + b' \overset{f}{\underset{g}{\rightrightarrows}} d$ in $sh(\mathbf{C})$ such that

$$f \cdot i_b = g \cdot i_b \quad \text{and} \quad f \cdot i_{b'} = g \cdot i_{b'}.$$

Plainly, if $f' = f \cdot i_b$ and $g' = g \cdot i_{b'}$, then the whole diagram commutes. But by definition of coproduct there is exactly one arrow $b + b' \to d$ that makes the whole diagram commute, namely $[f', g']$. So $f = g$ and $\{i_b, i_{b'}\}$ is an epimorphic family.

□

Corollary: *for any* $a \in sh(\mathbf{C})$ *and any* $S \in \Omega(a)$, $j_a(S)$ *is maximal.*

Proof: by lemma 1.2, if $j_a(S)$ were not maximal, then $j_a(j_a(S)) \neq j_a(S)$ which would contradict a defining feature of topologies that they are idempotent. □

It follows that a necessary condition for $j_a(S) = S$ is that S be $(b \to a]$ for some $b \to a$ in $sh(\mathbf{C})$. It is relevant, then, to note that, since $sh(\mathbf{C})$ is a topos, any $sh(\mathbf{C})$ arrow $g{:}\, b \to a$ has an epimonic factorisation. This in part means that there exists in $sh(\mathbf{C})$ and epic $g^*{:}\, b \twoheadrightarrow g(b)$ and a monic $im\ g{:}\, g(b) \rightarrowtail a$ such that $g = im\ g \cdot g^*$.

Lemma 2.2: *for* $g{:}\, b \to a$, *if* $g \in S$, *then* $im\ g \in j_a(S)$.

Proof: since S is a sieve, if $g \in S$, then $\Omega(g)(S) = (id_b]$. In that case, it follows by condition (1) of topologies that

$$\Omega(g)(S) \in \mathrm{J}(b).$$

But recall that J is the saturation of the canonical pretopology on $sh(\mathbf{C})$. This means that $\Omega(g)(S) \in \mathrm{J}(b)$ iff $\Omega(g)(S)$ contains an epimorphic family. Let that family be

$$E = \{c_x \xrightarrow{g_x} b \mid x \in X\}.$$

It is plain that if $g_x \in \Omega(g)(S)$, then $g^* \cdot g_x \in \Omega(g)(S)$, so consider the family

$$E^* = \{c_x \xrightarrow{g_x} b \xrightarrow{g^*} g(b) \mid x \in X\}.$$

Suppose some parallel pair $g(b) \underset{j}{\overset{i}{\rightrightarrows}} d$ of $sh(\mathbf{C})$ arrows such that $i \cdot g^* \cdot g_x = j \cdot g^* \cdot g_x$ for all $x \in X$. Since E is an epimorphic family, we have

$$i \cdot g^* = j \cdot g^*$$

and since g^* is epic, we have

$$i = j.$$

It follows that E^* is an epimorphic family; and since $E^* \subseteq \Omega(im\ g)(S)$, we have that $\Omega(im\ g)(S) \in \mathrm{J}\big(g(b)\big)$. As a result $im\ g \in j_a(S)$. □

Corollary: *for any $S \in \Omega(a)$, $j_a(S)$ is $(g]$ for some $sh(\mathbf{C})$-monic g.*

Proof: by the corollary to lemma 2.1, any $j_a(S)$ is $(b \xrightarrow{h} a]$ for some h in $sh(\mathbf{C})$. Now, by the above lemma, $im\ h \in j_a(j_a(S))$. But $j_a(j_a(S)) = j_a(S)$ and h factors through $im\ h$. It follows that h must, up to isomorphism, be $im\ h$; that is, h must be monic. □

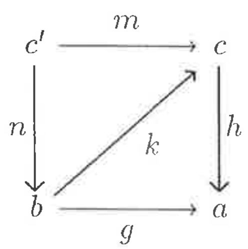
A necessary condition, then, for $j_a(S) = S$ is that S be $(b \overset{g}{\rightarrowtail} a]$ for some $sh(\mathbf{C})$-monic g.

Lemma 2.3: *if sieve $(b \xrightarrow{k} c]$ contains an epimorphic family, then k is an epimorphism.*

Proof: let $E = \{c_x \xrightarrow{k_x} b \xrightarrow{k} c \mid x \in X\}$ be the epimorphic family in $(b \xrightarrow{k} c]$. Suppose a parallel pair $c \underset{j}{\overset{i}{\rightrightarrows}} d$ such that $i \cdot k = j \cdot k$. It follows that $i \cdot k \cdot k_x = j \cdot k \cdot k_x$ for all $x \in X$. Since E is an epimorphic family, $i = j$. □

Theorem 2.1: *for any $a \in sh(\mathbf{C})$ and any $S \in \Omega(a)$, $j_a(S) = S$ iff S is $(b \overset{g}{\rightarrowtail} a]$ for some $sh(\mathbf{C})$-monic g.*

Proof: since we have the corollary to lemma 2.2, we need prove only that if S is $(b \overset{g}{\rightarrowtail} a]$ for some $sh(\mathbf{C})$-monic g, then $j_a(S) = S$. Suppose then that S is $(b \overset{g}{\rightarrowtail} a]$ as described. From the corollary to lemma 2.2 we know that $j_a(S)$ must be $(c \overset{h}{\rightarrowtail} a]$ for some $sh(\mathbf{C})$-monic h. It is straightforward that $S \subseteq j_a(S)$, so there must be some $k\colon b \to c$ such that $h \cdot k = g$. Consider the following diagram.



Plainly, for any $sh(\mathbf{C})$ arrow n with codomain b, $h \cdot k \cdot n = g \cdot n$, so

$$(k] \subseteq \Omega(h)(S).$$

Now suppose that $m \in \Omega(h)(S)$. This requires that $h \cdot m$ factor through g, or in other words, that there is some n such that the outer square commutes. Now, we know that the bottom right triangle commutes, so we have that

$$h \cdot k \cdot n = h \cdot m$$

and, since h is monic, we have that

$$k \cdot n = m$$

so, $m \in (k]$. As a result

$$\Omega(h)(S) = (k].$$

Now, by hypothesis, $h \in j_a(S)$; this means that $\Omega(h)(S) \in \mathrm{J}(c)$. It follows that $\Omega(h)(S) = (k]$ must contain an epimorphic family. In that case, by lemma 2.3, k is an epimorphism. But $g = h \cdot k$ so k is monic. In all, since k is an arrow in a topos, k is an isomorphism. In other words, $j_a(S) = S$. □

Corollary 1: *for any $a \in sh(\mathbf{C})$, $\Omega_j(a) \cong \mathrm{Sub}_{sh(\mathbf{C})}(a)$.*

Proof: from the definition of a subobject as an equivalence class of monics, if $[f] \in \mathrm{Sub}_{sh(\mathbf{C})}(a)$ and $g, h \in [f]$, then $(g] = (h]$. Likewise, if $(g] = (h]$ for monics f, g then f and g determine the same subobject. The rest of the demonstration is straightforward from Theorem 2.1. □

Corollary 2: *$\Omega_j(a)$ ordered by set inclusion and $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ ordered by subobject inclusion are isomorphic lattices.*

Proof: since $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ is a lattice, it is enough to show that $\Omega_j(a)$ and $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ are isomorphic as partially ordered sets.

For $[f], [g] \in \mathrm{Sub}_{sh(\mathbf{C})}(a)$ if $[f] \leq [g]$ where $\leq$ is subobject inclusion, then there must be some k such that $g \cdot k = f$. In that case, $(f] \subseteq (g]$ where $\subseteq$ is set inclusion. Now if $(f] \subseteq (g]$, then $f \in (g]$, so there must again be some k such that $g \cdot k = f$. In that case $[f] \leq [g]$. □

As we would expect, then, when Ω_j is the classifier object for $sh(sh(\mathbf{C}))$ and a is a $sh(\mathbf{C})$-object, $\Omega_j(a)$ is essentially $\mathrm{Sub}_{sh(\mathbf{C})}(a)$.

A complete description of functor Ω_j will include the restriction maps $\Omega_j(k)$ where k is some $sh(\mathbf{C})$-arrow. The following diagram can be expected to commute.

$$
\begin{array}{ccccc}
a & \quad & \Omega_j(a) & \overset{e_a}{\rightarrowtail} & \Omega(a) \\
k\big\uparrow & & \Omega_j(k)\big\downarrow & & \big\downarrow\Omega(k) \\
a' & & \Omega_j(a') & \overset{e_{a'}}{\rightarrowtail} & \Omega(a')
\end{array}
$$

The maps e_a and $e_{a'}$ are equalisers in **Set** and as such can be assumed to be inclusions. The maps $\Omega_j(k)$ is then defined so that for any $S \in \Omega_j(a)$

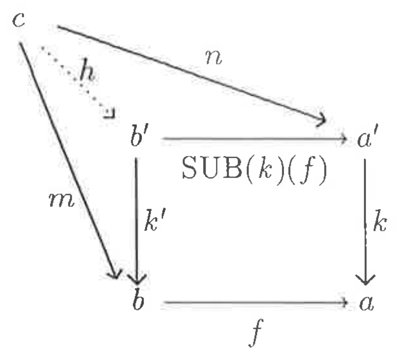
$$\Omega_j(k)(S) = \{b \xrightarrow{\alpha} a' \mid k \cdot \alpha \in S\}.$$

This description of $\Omega_j(k)$ is accurate but not especially informative. It makes no reference to the fact that any $S \in \Omega_j(a)$ is a maximal a-sieve. It happens, in fact, that any $\Omega_j(k)(S)$ is the pullback family of S along k and that where S is $(f]$, $\Omega_j(k)(S)$ is the maximal a'-sieve with the pullback of f along k as top element. We prove this with our next theorem. A point about notation: the pullback of f along k is denoted by $\mathrm{SUB}(k)(f)$ in that the image of k under the subobject functor SUB is $\mathrm{SUB}(k)\colon \mathrm{Sub}(a) \to \mathrm{Sub}(a')$ and for any $f \in \mathrm{Sub}(a)$, $\mathrm{SUB}(k)(f)$ is the subobject determined by the pullback of f along k.

Theorem 2.2: *for any $k: a' \to a$ and $f: b \rightarrowtail a$ in $sh(\mathbf{C})$,*

$$\Omega_j(k)\big((f]\big) = \big(\mathrm{SUB}(k)(f)\big].$$

Proof: consider the following diagram.



Any $n \in \Omega_j(k)\big((f]\big)$ iff $k \cdot n = f \cdot m$ for some m, but in that case the outer square commutes and, since the inner square is a pullback, there is a unique h making the whole diagram commute; in particular

$$\mathrm{SUB}(k)(f) \cdot h = n.$$

So, $n \in \big(\mathrm{SUB}(k)(f)\big]$ if $n \in \Omega_j(k)\big((f]\big)$. On the other hand, if $n \in \big(\mathrm{SUB}(k)(f)\big]$, then there must be some h such that $n = \mathrm{SUB}(k)(f) \cdot h$ in which case define $m = k' \cdot h$ and note that, since the inner square commutes, the outer square will commute making $n \in \Omega_j(k)\big((f]\big)$. □

The discussion in this section is useful for two reasons: firstly we have drawn the link between classifier objects and subobject lattices; and secondly we have presented what amounts to a representation theorem for subobject lattices (in Grothendieck toposes) as lattices of sets. This allows us to discuss the algebraic nature of subobject lattices in relatively simple terms. With the next section we consider the BrA nature of such subobject lattices.

3. Brouwerian Algebras in the Classifier Object

With this section we show that for any object a in $sh(\mathbf{C})$, the set $\Omega_j(a)$ ordered by set inclusion is a BrA. This follows from the fact that in any Grothendieck topos $sh(\mathbf{C})$, the collection $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ ordered by subobject inclusion is a complete, distributive, and bounded lattice. One straightforward consequence of this demonstration is the $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ is a BrA. We will use $\dot{-}_a$ to denote the BrA operation on each $\Omega_j(a)$. We will consider the collection $\{\dot{-}_a : a \in sh(\mathbf{C})\}$. Technically, this collection of functions constitutes a transformation $\Omega_j \times \Omega_j \to \Omega_j$ but we will find that this transformation is not in general natural. That the transformation is not natural means that it is not an arrow in $sh(sh(\mathbf{C}))$; so, while each $\Omega_j(a)$ is a BrA, the object Ω_j is not.

We saw in the last section that $\Omega_j(a)$ and $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ are isomorphic lattices. Following from the bi-completeness of any Grothendieck topos, any $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ is complete as a lattice. It follows that $(\Omega_j(a), \subseteq)$ is complete. It is also the case that any $\mathrm{Sub}_{sh(\mathbf{C})}(a)$ has a unit, namely the identity arrow on a. It follows that $(id_a]$ is the unit for $(\Omega_j(a), \subseteq)$. As we saw in chapter six that a lattice of sets is a BrA will follow from that lattice being meet-complete and having a unit. We now reproduce that demonstration for $(\Omega_j(a), \subseteq)$.

For any $(\Omega_j(a), \subseteq)$ define an operator $\dot{-}_a$ so that for any $(f], (g] \in \Omega_j(a)$,

$$(f] \dot{-}_a (g] = \bigcap \Big\{ (h] : \big((id_a] - (g]\big) \cap (f] \subseteq (h] \Big\}.$$

The set $(id_a] - (g]$ is just the set-theoretic subtraction of $(g]$ from $(id_a]$. It happens, then, that $g' \in (g] - (id_a]$ iff there is no $sh(\mathbf{C})$ arrow i such that $g' = g \cdot i$.

Another, and equivalent, way of describing $(f] \dot{-}_a (g]$ is that it is the smallest $(h] \in \Omega_j(a)$ that contains $\big((id_a] - (g]\big) \cap (f]$. Since $(\Omega_j(a), \subseteq)$ is complete, a smallest such $(h]$ will always exist. Note the exact similarity bewteen the definition of $\dot{-}_a$ and the definition of a BrA operation on a closed set topology.

Theorem 3.1: *$(\Omega_j(a), \subseteq)$ is a* BrA.

Proof: the theorem is demonstrated if for any $x, y, z \in \Omega_j(a)$, we have

$$x \dot{-}_a y \subseteq z \quad \text{iff} \quad x \subseteq y \cup z.$$

Since $x \dot{-}_a y$ is the smallest $z \in \Omega_j(a)$ containing $(1-y) \cap x$ where 1 denotes the unit of $(\Omega_j(a), \subseteq)$, it is enough to demonstrate that

$$(1-y) \cap x \subseteq z \quad \text{iff} \quad x \subseteq y \cup z.$$

We do this in what follows. Note that we will use the fact that $(\Omega_j(a), \subseteq)$ is a distributive lattice. If $(1-y) \cap x \subseteq z$, then

$$\begin{aligned} \big((1-y) \cap x\big) \cup y &\subseteq z \cup y, \\ \big((1-y) \cup y\big) \cap (x \cup y) &\subseteq z \cup y, \\ x \cup y &\subseteq z \cup y, \\ x &\subseteq y \cup z. \end{aligned}$$

On the other hand, if $x \subseteq y \cup z$, then

$$\begin{aligned} x \cap (1-y) &\subseteq (y \cup z) \cap (1-y), \\ &\subseteq \big(y \cap (1-y)\big) \cup \big(z \cap (1-y)\big), \\ &\subseteq z \cap (1-y), \\ (1-y) \cap x &\subseteq z. \end{aligned}$$

□

Note that there appears to be a difference in the definitions of $\dot{-}_a$ for $\Omega_j(a)$ and $\dot{-}_V$ for $\Omega'_j(V)$ where Ω'_j is the classifier object for closed set sheaf category $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$. In chapter nine we described $\dot{-}_V$ for $\Omega'_j(V)$ so that for $S, T \in \Omega'_j(V)$,

$$S \dot{-}_V T \underset{df}{=} S \dot{-} T = cl\big((X - T) \cap S\big)$$

where X is the unit of $(\mathcal{T}, \subseteq)$, $\dot{-}$ is the BrA operation, and cl is the closure operation that determines $\mathcal{T}$ on X. If we follow the method described in this chapter we should have

$$S \dot{-}_V T = cl((V - T) \cap S).$$

However, since $S, T \subseteq V$, we have

$$cl((X - T) \cap S) = cl((V - T) \cap S)$$

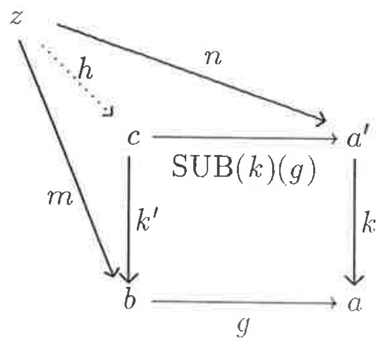
and there is no difference in operator definition. It follows, then, that the failure of $\{\dot{-}_V : V \in \mathcal{T}\}$ to constitute a natural transformation in $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ should indicate a general failure of $\{\dot{-}_a : a \in sh(\mathbf{C})\}$ to constitute a natural transformation for $sh(sh(\mathbf{C}))$. The failure of naturalness for $\{\dot{-}_a : a \in sh(\mathbf{C})\}$, as we shall see, occurs for much the same reason as in the $sh_j(\mathbf{Set}^{\mathcal{T}^{op}})$ case.

We take it to be the case that the successful definition of $\dot{-}_a$ amounts to an understanding of $(\Omega_j(a), \subseteq)$ as a closure algebra. It is at least the case that since $(\Omega_j(a), \subseteq)$ is a BrA, it is isomorphic to some closed set topology. The following lemma will help make it clear that for $\{\dot{-}_a : a \in sh(\mathbf{C})\}$ to be natural, the operators $\dot{-}_a$ must distribute over intersections.

Lemma 3.1: *for $k: a' \to a$ and $g: b \rightarrowtail a$ in $sh(\mathbf{C})$, we have*

$$(\mathrm{SUB}(k)(f)] \simeq (g] \cap (k].$$

Proof: consider the diagram



where the inner square is a pullback. If $f \in (g] \cap (k]$, then there is some map $z \to a$ for which there is an m and an n such that

$$f = g \cdot m = k \cdot n$$

in which case the outer square commutes and there is a unique h making the whole diagram commute; in particular, $\mathrm{SUB}(k)(g) \cdot h = n$ and therefore $n \in \big(\mathrm{SUB}(k)(g)\big]$. Now, if $n \in \big(\mathrm{SUB}(k)(g)\big]$, then there is some h such that $\mathrm{SUB}(k)(g) \cdot h = n$, in which case define m to be $k' \cdot h$. It would follow that the outer square commutes and $k \cdot n \in (g] \cap (k]$. So there is a bijection $\big(\mathrm{SUB}(k)(f)\big] \simeq (g] \cap (k]$ given by

$$n \mapsto k \cdot n. \qquad \square$$

Theorem 3.2: $\{\dot{-}_a : a \in sh(\mathbf{C})\}$ *does not, in general, constitute a natural transformation in* $sh(sh(\mathbf{C}))$.

Proof: the collection of functions is a natural transformation if for any $k: a' \to a$ in $sh(\mathbf{C})$, we have commuting diagrams

$$\begin{array}{ccccc}
a & \quad & \Omega_j(a) \times \Omega_j(a) & \xrightarrow{\ \dot{-}_a\ } & \Omega_j(a) \\
\uparrow k & & \downarrow (\Omega_j \times \Omega_j)(k) & & \downarrow (\Omega_j)(k) \\
a' & & \Omega_j(a') \times \Omega_j(a') & \xrightarrow[\ \dot{-}_{a'}\ ]{} & \Omega_j(a')
\end{array}$$

Now, for any $\langle (f], (g] \rangle \in \Omega_j(a) \times \Omega_j(a)$

$$\begin{aligned}
(\Omega_j)(k)\Big(\dot{-}_a\big(\langle (f], (g] \rangle\big)\Big) &= (\Omega_j)(k)\big((f] \dot{-}_a (g]\big) \\
&\simeq \big((f] \dot{-}_a (g]\big) \cap (k]
\end{aligned}$$

and

$$\begin{aligned}
\dot{-}_a\Big((\Omega_j \times \Omega_j)(k)\big(\langle (f], (g] \rangle\big)\Big) &\simeq \dot{-}_{a'}\big(\langle (f] \cap (k], (g] \cap (k] \rangle\big) \\
&= \big((f] \cap (k]\big) \dot{-}_{a'} \big((g] \cap (k]\big).
\end{aligned}$$

To simplify our discussion let us use "$cl_a(S)$" to mean "the smallest $(h] \in \Omega_j(a)$ such that $S \subseteq (h]$". Then

$$((f] \dot{-}_a (g]) \cap (k] = cl_a\Big(((id_a] - (g]) \cap (f]\Big) \cap (k]$$

and

$$((f] \cap (k]) \dot{-}_{a'} ((g] \cap (k]) = cl_{a'}\Big(\Big((id_{a'}] - ((g] \cap (k])\Big) \cap (f] \cap (k]\Big).$$

Suppose now that $(f]$ is $(id_a]$ and $k \in (g]$. Then

$$((f] \dot{-}_a (g]) \cap (k] = cl_a((id_a] - (g]) \cap (k]$$

and

$$\begin{aligned} ((f] \cap (k]) \dot{-}_{a'} ((g] \cap (k]) &= cl_{a'}\Big(((id_{a'}] - (k]) \cap (k]\Big) \\ &= cl_{a'}(\emptyset) \\ &= \emptyset. \end{aligned}$$

Now we know that

$$(id_a] - (g] \subseteq cl_a((id_a] - (g])$$

so we are not guaranteed that $cl_a((id_a] - (g]) \cap (k]$ is an empty set. In fact, if we suppose that $(k] = (g]$, then, in general, $cl_a((id_a] - (g]) \cap (k]$ is not an empty set. In other words, there will in general be some a and some k in $sh(\mathbf{C})$ such that

$$(\Omega_j)(k)\Big(\dot{-}_a\Big(\langle(f],(g]\rangle\Big)\Big) \neq \dot{-}_{a'}\Big((\Omega_j \times \Omega_j)(k)\Big(\langle(f],(g]\rangle\Big)\Big). \qquad \square$$

In this and in the previous chapter we have discovered BrA structures related to the internal structure of various sheaf categories. The BrAs themselves have all been external in the sense that the underlying sets of the BrAs have not been objects of the categories in question and the BrA operations on those underlying sets have not been categorial arrows of the right sort. With the next chapter we describe a paraconsistent logic object that is wholly internal to a covariant functor category. And with chapter twelve we will see that this paraconsistent logic object is in fact a sheaf.

CHAPTER 11: COVARIANT LOGIC OBJECTS

Introduction: With this chapter we describe a covariant functor $\mathcal{B}$ that exists within the category $\mathbf{Set}^{\mathcal{T}}$ where $\mathcal{T}$ is a closed set topology. The functor $\mathcal{B}$ is shown to be a paraconsistent logic object with the feature that it induces paraconsistent algebras on sets $\hom(d, \mathcal{B})$ for all objects d of $\mathbf{Set}^{\mathcal{T}}$. In the next chapter we show that $\mathcal{B}$ is isomorphic to the classifier object Ω_{cl} for a subcategory $sh_{cl}(\mathbf{Set}^{\mathcal{T}})$ of $\mathbf{Set}^{\mathcal{T}}$. The category $sh_{cl}(\mathbf{Set}^{\mathcal{T}})$ is a category of covariant functors $\mathcal{T} \to \mathbf{Set}$ that are sheaves with respect to what we call a co-topology C on poset category $\mathcal{T}$. Co-topologies C on categories $\mathcal{C}$ are defined by dualisation of topologies J on categories $\mathcal{C}^{op}$.

A *logic object* in a category $\mathcal{C}$ is an object A of $\mathcal{C}$ for which there are $\mathcal{C}$-arrows $A \times A \to A$ and $\mathcal{C}$-arrows $A \to A$ that can be understood as algebraic operations. We call such arrows *operator arrows.* As an example we can point to any classifier object Ω in any topos. The usual arrows $\cap, \cup, \Rightarrow : \Omega \times \Omega \to \Omega$ and $\neg : \Omega \to \Omega$ make Ω an intuitionist logic object within the topos. The notion of a logic object is a generalisation of the usual idea of logical algebras as sets together with truth functions. We may develop the intuitive idea by saying that any classifier object Ω together with the usual truth arrows is a logic object for its home topos in just the same way that any two element set together with the usual truth functions is a logic object for set theory. Logic objects need not be tied to the subobject structure of a category; all that is required is that the right sort of operator arrows exist.

A point to note with respect to understanding the sense of this chapter and the next is brought out in the following discussion. Let X be a topological space, let Θ be the open sets of X, and let $\mathcal{T}$ be the closed sets of X. The sets Θ and $\mathcal{T}$ are isomorphic by bijection $\Theta \ni U \mapsto X - U$. Consider now the posets of Θ ordered by set inclusion and of $\mathcal{T}$ ordered by set inclusion. Since $U \subseteq V$ in Θ iff $X - V \subseteq X - U$ in $\mathcal{T}$, we have posets $(\Theta, \subseteq)$ and $(\mathcal{T}, \subseteq)$ as dual isomorphs. In categorial terms poset category $\mathcal{T}$ is poset category Θ^{op}. So a covariant functor $\mathcal{T} \to \mathbf{Set}$ is essentially a covariant functor $\Theta^{op} \to \mathbf{Set}$ which is essentially a contravariant functor $\Theta \to \mathbf{Set}$, ie., a presheaf on an open set topology. In other words, the categories $\mathbf{Set}^{\mathcal{T}}$ of covariant functors on closed set topology $\mathcal{T}$ are, up to isomorphism, familiar categories of presheaves on open set topologies. Since this isomorphism of categories holds, the discussion and proofs of chapter 11 are formally unnecessary. However, that $\mathcal{B}$ is a paraconsistent logic object in $\mathbf{Set}^{\mathcal{T}}$ remains true. Arguably this is a discovery in itself. It is a hidden feature of the discussion in chapter 11 but the result is established by much the same dualisation of operators as described in chapter 4. The point is that here the dualisation is eminently reasonable. In the absence of the intuition pump of knowing that $\mathcal{B}$ is essentially just the subobject classifier of a familiar open set sheaf category, the most straightforward way of building operators on $\mathcal{B}$ produces the BrA object described.

1. A Paraconsistent Logic Object in a Covariant Functor Category

Consider the category $\mathbf{Set}^{\mathcal{T}}$ of covariant functors $\mathcal{T} \to \mathbf{Set}$ where $\mathcal{T}$ is a closed set topology on a space X. For any $U \in \mathcal{T}$, let $[U)$ be the set of all supersets W of U that are in $\mathcal{T}$. To put this another way, we will say that $W \in [U)$ iff $U \subseteq W$ and $W \in \mathcal{T}$. We now define a functor $\mathcal{B}$ as follows: for each $U \in \mathcal{T}$, let

$$\mathcal{B}(U) = [U);$$

and where $U \subseteq V$ in $\mathcal{T}$ let there be a function $\mathcal{B}^U_V \colon \mathcal{B}(U) \to \mathcal{B}(V)$ given by

$$\mathcal{B}(U) \ni S \mapsto S \cup V.$$

Lemma 1.1: *for $U \subseteq U$ in $\mathcal{T}$, $\mathcal{B}^U_U = id_{\mathcal{B}(U)}$.*

Proof: $S \in \mathcal{B}(U)$ iff $U \subseteq S$, so for any $S \in \mathcal{B}(U)$, $\mathcal{B}^U_U(S) = S \cup U = S$. □

Lemma 1.2: *for any $U \subseteq V \subseteq W$ in $\mathcal{T}$, we have $\mathcal{B}^U_W = \mathcal{B}^V_W \cdot \mathcal{B}^U_V$.*

Proof: for any $S \in \mathcal{B}(U)$,

$$\begin{aligned} \mathcal{B}^V_W(\mathcal{B}^U_V(S)) &= \mathcal{B}^V_W(S \cup V) \\ &= (S \cup V) \cup W \\ &= S \cup W \\ &= \mathcal{B}^U_W(S). \end{aligned}$$

□

Theorem 1.1: *$\mathcal{B}$ is a covariant functor $\mathcal{T} \to$ **Set**.*

Proof: the necessary properties of preservation of identities and composition are demonstrated in lemmas 1.1 and 1.2 above. □

Our next task is to show that $\mathcal{B}$ is a logic object. We do this by demonstrating the existence of distributive lub and glb operator arrows, respectively $\bigcup: \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $\bigcap: \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, in $\mathbf{Set}^{\mathcal{T}}$. The existence of such arrows means that we may call $\mathcal{B}$ a ***distributive lattice object*** in $\mathbf{Set}^{\mathcal{T}}$. Then, to the extent that any distributive lattice is an algebra for a logic, $\mathcal{B}$ is a logic object. In the first instance we demonstrate the existence of two natural transformations, $\bigcup$ and $\bigcap$, which we describe as being lub and glb operators on $\mathcal{B}$. The sense in which these arrows do constitute such operators is in their effect on sets hom$(d, \mathcal{B})$. We will see later that there is a natural definition of an order for sets hom$(d, \mathcal{B})$ arising from the existence of these arrows and that, under this order, each hom$(d, \mathcal{B})$ is a distributive lattice where for any $f, g \in$ hom$(b, \mathcal{B})$, lub(f, g) is given by $\bigcup \cdot \langle f, g \rangle$ and glb(f, g) is given by $\bigcap \cdot \langle f, g \rangle$.

We shall go on to show that $\mathcal{B}$ is a BrA logic object. We will do this by demonstrating the existence of a natural transformation $\dot{-}: \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ which determines BrA operations for the lattices hom$(d, \mathcal{B})$.

Lemma 1.3: *for any $U \in \mathcal{T}$, $\mathcal{B}(U)$ ordered by set inclusion is a bounded, distributive lattice.*

Proof: to show that $\mathcal{B}(U)$ is a distributive lattice under set inclusion it is enough to show that if $S, T \in \mathcal{B}(U)$, then $S \cup T$ and $S \cap T$ are in $\mathcal{B}(U)$ where $\cup, \cap$ are set theoretic union and intersection. Now if $S, T \in \mathcal{B}(U)$, then it is at least true that S, T are closed sets in $\mathcal{T}$ and that $U \subseteq S, T$. It would follow that $S \cup T$ and $S \cap T$ were closed sets, and furthermore that both $U \subseteq S \cup T$ and $U \subseteq S \cap T$. So if $S, T \in \mathcal{B}(U)$, then $S \cup T, S \cap T \in \mathcal{B}(U)$. To show that $(\mathcal{B}(U), \subseteq)$ is bounded, it is enough to point out that both $U \in \mathcal{B}(U)$ and $X \in \mathcal{B}(U)$. □

It follows from this lemma that we have functions

$$\cap_U : \mathcal{B}(U) \times \mathcal{B}(U) \to \mathcal{B}(U) : \langle S, T \rangle \to S \cap T$$

and

$$\cup_U : \mathcal{B}(U) \times \mathcal{B}(U) \to \mathcal{B}(U) : \langle S, T \rangle \to S \cup T$$

for any $U \in \mathcal{T}$. The next two lemmas demonstrate that these functions constitute natural transformations in $\mathbf{Set}^{\mathcal{T}}$.

Lemma 1.4: *the collection of functions $\{\cap_U \mid U \in \mathcal{T}\}$ constitutes a natural transformation.*

Proof: we are required to show that wherever $U \subseteq V$ in $\mathcal{T}$, the following diagram commutes:

$$\begin{array}{ccccc}
U & \quad & \mathcal{B}(U) \times \mathcal{B}(U) & \xrightarrow{\cap_U} & \mathcal{B}(U) \\
\downarrow & & \Big\downarrow (\mathcal{B} \times \mathcal{B})^U_V & & \Big\downarrow \mathcal{B}^U_V \\
V & & \mathcal{B}(V) \times \mathcal{B}(V) & \xrightarrow[\cap_V]{} & \mathcal{B}(V)
\end{array}$$

Now if $U \subseteq V$ in $\mathcal{T}$ and $\langle S, T \rangle \in \mathcal{B}(U) \times \mathcal{B}(U)$, then we have

$$\begin{aligned}
\mathcal{B}^U_V\Big(\cap_U(\langle S, T \rangle)\Big) &= \mathcal{B}^U_V(S \cap T) \\
&= (S \cap T) \cup V
\end{aligned}$$

and

$$\begin{aligned}\cap_V\Big((\mathcal{B}\times\mathcal{B})_V^U(\langle S,T\rangle)\Big) &= \cap_V(\langle S\cup V, T\cup V\rangle)\\ &= (S\cup V)\cap(T\cup V)\\ &= (S\cap T)\cup V.\end{aligned}$$

So

$$\mathcal{B}_V^U\cdot\cap_U = \cap_V\cdot(\mathcal{B}\times\mathcal{B})_V^U.$$ □

We will denote the natural transformation $\{\cap_U \mid U\in\mathcal{T}\}$ by $\bigcap$.

Lemma 1.5: *the collection of functions* $\{\cup_U \mid U\in\mathcal{T}\}$ *constitutes a natural transformation.*

Proof: we are required to show that wherever $U\subseteq V$ in $\mathcal{T}$, the following diagram commutes:

$$\begin{array}{ccccc} U & & \mathcal{B}(U)\times\mathcal{B}(U) & \xrightarrow{\ \cup_U\ } & \mathcal{B}(U)\\ \Big\downarrow & & {\scriptstyle(\mathcal{B}\times\mathcal{B})_V^U}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathcal{B}_V^U}\\ V & & \mathcal{B}(V)\times\mathcal{B}(V) & \xrightarrow[\ \cup_V\ ]{} & \mathcal{B}(V)\end{array}$$

Now if $U\subseteq V$ in $\mathcal{T}$ and $\langle S,T\rangle\in\mathcal{B}(U)\times\mathcal{B}(U)$, then we have

$$\begin{aligned}\mathcal{B}_V^U\Big(\cup_U(\langle S,T\rangle)\Big) &= \mathcal{B}_V^U(S\cup T)\\ &= (S\cup T)\cup V\end{aligned}$$

and

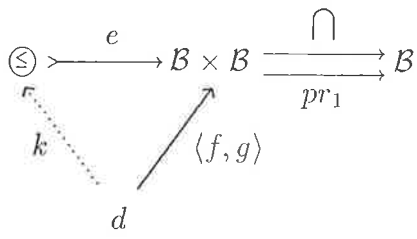
$$\begin{aligned}\cup_V\Big((\mathcal{B}\times\mathcal{B})_V^U(\langle S,T\rangle)\Big) &= \cup_V(\langle S\cup V, T\cup V\rangle)\\ &= (S\cup V)\cup(T\cup V)\\ &= (S\cup T)\cup V.\end{aligned}$$

So

$$\mathcal{B}_V^U\cdot\cup_U = \cup_V\cdot(\mathcal{B}\times\mathcal{B})_V^U.$$ □

We will denote the natural transformation $\{\cup_U \mid U \in \mathcal{T}\}$ by $\bigcup$.

Since $\bigcap$ exists for $\mathcal{B}$ and $\mathbf{Set}^{\mathcal{T}}$ is a topos, we can define a $\mathbf{Set}^{\mathcal{T}}$ object $\circledS$ by equaliser as in the following diagram.



For any $f, g \in \hom(d, \mathcal{B})$, we will say that

$$f \leq g \quad \text{iff} \quad \bigcap \cdot \langle f, g \rangle = pr_1 \cdot \langle f, g \rangle.$$

The sense of this definition is that product map $\langle f, g \rangle$ factors uniquely through e iff $\bigcap \cdot \langle f, g \rangle = pr_1 \cdot \langle f, g \rangle$. Now, since $\bigcap$, pr_1, and any $f, g \in \hom(d, \mathcal{B})$ will be natural transformations, we have $f \leq g$ iff for all $U \in \mathcal{T}$,

$$\cap_U \cdot \langle f_U, g_U \rangle = (pr_1)_U \cdot \langle f_U, g_U \rangle;$$

and we have this iff for all $x \in d(U)$

$$\big(\cap_U \cdot \langle f_U, g_U \rangle\big)(x) = \big((pr_1)_U \cdot \langle f_U, g_U \rangle\big)(x)$$

or, in other words, iff

$$f_U(x) \cap g_U(x) = f_U(x).$$

It is straightforward, then, that $\leq$ is a partial order for each $\hom(d, \mathcal{B})$ and that, with respect to this order, maps $\bigcap \cdot \langle f, g \rangle$ and $\bigcup \cdot \langle f, g \rangle$ are, respectively, glb's and lub's for any $f, g \in \hom(d, \mathcal{B})$. In these terms each $\hom(D, \mathcal{B})$ is a distributive lattice. In fact, these lattices will be bounded. We can define a natural transformation $unit\colon \mathcal{B} \to \mathcal{B}$ as the collection of functions

$$unit_U\colon \mathcal{B}(U) \to \mathcal{B}(U)\colon S \mapsto X$$

and a natural transformation $zero: \mathcal{B} \to \mathcal{B}$ as the collection of functions

$$zero_U: \mathcal{B}(U) \to \mathcal{B}(U): S \mapsto U,$$

and then any $\hom(d, \mathcal{B})$ will be bounded by $unit \cdot f$ and $zero \cdot f$ for any $f \in \hom(d, \mathcal{B})$.

Theorem 1.2: *$\mathcal{B}$ is a bounded, distributive lattice object.*

Proof: the meaning and proof of the theorem are contained in the above discussion and sequence of lemmas. □

The aim of this chapter was to demonstrate that $\mathcal{B}$ is a paraconsistent logic object. We do that now with a sequence of lemmas leading to Theorem 1.3.

Lemma 1.6: *each $\mathcal{B}(U)$ is a* BrA *under set inclusion.*

Proof: associated with any closed set topology $\mathcal{T}$ on space X is a BrA operation $\dot{-}$ given by allowing that for any $S, T \in \mathcal{T}$, $S \dot{-} T = cl((X - T) \cap S)$ where cl is the closure operation that determines the topology $\mathcal{T}$. The BrA operation is characterised by the property that for any $S, T, Z \in \mathcal{T}$

$$S \dot{-} T \subseteq Z \quad \text{iff} \quad S \subseteq T \cup Z.$$

It follows from the definition of $\dot{-}$ that if $U \subseteq S, T$, then, in general, $U \not\subseteq S \dot{-} T$. However, $U \subseteq (S \dot{-} T) \cup U$, and if $U \subseteq Z$, then

$$S \dot{-} T \subseteq Z \quad \text{iff} \quad (S \dot{-} T) \cup U \subseteq Z$$

so, for any $S, T, Z \in \mathcal{B}(U)$

$$(S \dot{-} T) \cup U \subseteq Z \quad \text{iff} \quad S \subseteq T \cup Z.$$

It follows that we can define a BrA operator, $\dot{-}_U$, on $\mathcal{B}(U)$ by stipulating that for any $S, T \in \mathcal{B}(U)$,

$$S \dot{-}_U T \underset{df}{=} (S \dot{-} T) \cup U.$$ □

Lemma 1.7: *for any closed set topology $\mathcal{T}$ on a space X and any $S, T, V \in \mathcal{T}$,*

$$(S \dot{-} T) \cup V = ((S \cup V) \dot{-} (T \cup V)) \cup V.$$

Proof:

$$\begin{aligned}
((S \cup V) \dot{-} (T \cup V)) \cup V &= cl\Big((X - (T \cup V)) \cap (S \cup V)\Big) \cup V \\
&= cl\Big(\big((X - (T \cup V)) \cap (S \cup V\big) \cup V\Big) \\
&= cl\Big(\big((X - (T \cup V)) \cup V\big) \cap ((S \cup V) \cup V)\Big) \\
&= cl\Big(\big(((X - T) \cap (X - V)) \cup V\big) \cap (S \cup V)\Big) \\
&= cl\Big(\big(((X - T) \cup V) \cap ((X - V) \cup V)\big) \cap (S \cup V)\Big) \\
&= cl\Big(((X - T) \cup V) \cap (S \cup V)\Big) \\
&= cl\Big(((X - T) \cap S) \cup V\Big) \\
&= cl((X - T) \cap S) \cup V \\
&= (S \dot{-} T) \cup V.
\end{aligned}$$

□

The equation in the above lemma is in fact true of any BrA. We have restricted the lemma to topological BrAs for the relatively simple proof the closure operation definition of $\dot{-}$ allows.

Lemma 1.8: *the collection $\{\dot{-}_U \mid U \in \mathcal{T}\}$ constitutes a natural transformation.*

Proof: we are required to show that wherever $U \subseteq V$ in $\mathcal{T}$, the following diagram commutes:

$$\begin{array}{ccccc}
U & & \mathcal{B}(U) \times \mathcal{B}(U) & \xrightarrow{\dot{-}_U} & \mathcal{B}(U) \\
\big\downarrow & & {\scriptstyle (\mathcal{B} \times \mathcal{B})^U_V}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{B}^U_V} \\
V & & \mathcal{B}(V) \times \mathcal{B}(V) & \xrightarrow[\dot{-}_V]{} & \mathcal{B}(V)
\end{array}$$

Now, for any $U \subseteq V$ in $\mathcal{T}$ and any $\langle S,T\rangle \in \mathcal{B}(U)\times\mathcal{B}(U)$, we have

$$\begin{aligned}\mathcal{B}_V^U\Big(\dot{-}_U(\langle S,T\rangle)\Big) &= \mathcal{B}_V^U((S \dot{-} T)\cup U)\\ &= \big((S\dot{-}T)\cup U\big)\cup V\\ &= (S\dot{-}T)\cup V\end{aligned}$$

and

$$\begin{aligned}\dot{-}_V\Big((\mathcal{B}\times\mathcal{B})_V^U(\langle S,T\rangle)\Big) &= \dot{-}_V(\langle S\cup V, T\cup V\rangle)\\ &= \big((S\cup V)\dot{-}(T\cup V)\big)\cup V.\end{aligned}$$

It follows from lemma 1.7 that the diagram commutes. □

We will denote the natural transformation $\{\dot{-}_U : U \in \mathcal{T}\}$ by $\dot{-}$. This natural transformation imposes a BrA structure on each $\hom(d,\mathcal{B})$: we say that for any $f,g \in \hom(d,\mathcal{B})$, the pseudo difference of f with respect to g is $\dot{-}\cdot\langle f,g\rangle$.

Theorem 1.3: *$\mathcal{B}$ is a paraconsistent logic object in* $\mathbf{Set}^{\mathcal{T}}$.

Proof: theorem 1.2 together with lemma 1.8. □

So far we have shown that $\mathcal{B}$ exists as a paraconsistent logic object in $\mathbf{Set}^{\mathcal{T}}$. We have said nothing explicit on the relationship of $\mathcal{B}$ to the usual logic structures in a category, the subobject lattices and the classifier objects. In fact, $\mathcal{B}$ is not a classifier object for $\mathbf{Set}^{\mathcal{T}}$. However, it is an isomorph of the classifier object for a particular subcategory of $\mathbf{Set}^{\mathcal{T}}$. As we shall see in the next chapter, it will not follow that the subcategory in question has a paraconsistent subobject structure. The classifier algebra in question is in fact intuitionist but there is an intuitively natural manoeuver that allows us to dualise the order and produce $\mathcal{B}$. It is worth taking careful note of the fact that $\mathcal{B}$ is naturally understood to be a paraconsistent logic object. The isomorphism that we develop in the next chapter of $\mathcal{B}$ to a classifier object does not negate this natural understanding. In the relevant subcategory of $\mathbf{Set}^{\mathcal{T}}$, $\mathcal{B}$ can be thought of as the codomain of a natural complement classifier.

CHAPTER 12: COVARIANT SHEAVES

Introduction: With this chapter we consider the notion of a co-topology C on a category $\mathcal{C}$ and develop the notion of sheaves in covariant functor categories $\mathbf{Set}^{\mathcal{C}}$ with respect to such co-topologies. We demonstrate that the category of such sheaves has a subobject classifier and is finitely complete. The notion of a co-topology C on a category $\mathcal{C}$ will be exactly dual to the notion of a topology J on a category $\mathcal{C}^{op}$. It will follow that we have a notion of a canonical co-topology. We will be particularly interested in the category of sheaves in $\mathbf{Set}^{\mathcal{C}}$ with respect to a canonical co-topology where $\mathcal{C}$ is a topology $\mathcal{T}$. In the first instance we let $\mathcal{T}$ be a closed set topology for a space X. We saw in the last chapter that in such a case the category $\mathbf{Set}^{\mathcal{T}}$ contains a paraconsistent logic object $\mathcal{B}$. We demonstrate that $\mathcal{B}$ and the classifier object for the category of sheaves with respect to the canonical co-topology on $\mathcal{T}$ are isomorphic as objects in $\mathbf{Set}^{\mathcal{T}}$ and are dually isomorphic as logic objects. It follows that the subobject classifier of the sheaf category can be given with $\mathcal{B}$ as codomain and as such can be thought of as a complement classifier. The existence of a well motivated interpretation of the subobject classifier as a complement classifier will be of use to us in later chapters where we discuss the interpretation of paraconsistent theories as categories.

As with chapter 11 we must note an isomorphism of categories that affects an understanding of the sense of the present chapter. In the same way that chapter 11 deals with familiar categories of presheaves on open set topologies, chapter 12 deals, by isomorphism, with familiar categories of sheaves over open sets. Accordingly everything in chapter 12 up to and including the two corollaries of Theorem 12.3.3 is no more than explicit proof of dual claims to familiar facts. This leaves untouched the discussion of $\mathcal{B}$ as the object of a genuine complement classifier. This discussion brings out the fact hidden in chapter 11 that I have produced a BrA object by a method that is equivalent to applying a "false" for "true" dualisation to a familiar subobject classifier. However, $\mathcal{B}$ remains a successful construction. Demonstrating that $\mathcal{B}$ is the object of a subobject classifier is not original but demonstrating that it is the object of a complement classifier must be since complement and subobject classifiers are *philosophically* different notions.

1. Co-topologies

The notion of a categorial topology J as a system of sieves is easily dualised. Let us consider a system C of cosieves with respect to a category $\mathcal{C}$. System C will be a collection of sets $\mathrm{C}(a)$ for each $a \in \mathcal{C}$ where each $\mathrm{C}(a)$ is a set of a-cosieves from $\mathcal{C}$. Recall that an a-cosieve from a category $\mathcal{C}$ is a set R of $\mathcal{C}$-arrows with domain a such that if $a \xrightarrow{f} b$ is in R and $b \xrightarrow{g} c$ is an arrow in $\mathcal{C}$, then $a \xrightarrow{f} b \xrightarrow{g} c$ is in R. Recall, too, the notion of a dual category $\mathcal{C}^{op}$ for any category $\mathcal{C}$. We say of $\mathcal{C}$ and $\mathcal{C}^{op}$ that they have the same collection of objects but that there is an arrow $f: a \to b$ in $\mathcal{C}$ iff there is an arrow $f^{op}: b \to a$ in $\mathcal{C}^{op}$. The notion of dual categories will help us to formalise the notion of the dual of any a-cosieve R for any $a \in \mathcal{C}$. We denote the dual by R^{op} and say that

$$f \in R \quad \text{iff} \quad f^{op} \in R^{op}.$$

Plainly, where R is an a-cosieve from $\mathcal{C}$, R^{op} is an a-sieve from $\mathcal{C}^{op}$. Given a system C of sets of cosieves with respect to a category $\mathcal{C}$, we can define a system C^{op} of sets of sieves with respect to a category $\mathcal{C}^{op}$ by stipulating that

$$R \in \mathrm{C}(a) \quad \text{iff} \quad R^{op} \in \mathrm{C}^{op}(a)$$

for all objects $a \in \mathcal{C}$. We will say that system C is a *co-topology on* $\mathcal{C}$ iff system C^{op} is a categorial topology, in the sense of topologies J, for $\mathcal{C}^{op}$. Thus we have the following definition.

Definition 1.1: a *co-topology* C *for a category* $\mathcal{C}$ is a system $\{C(a) \mid a \in \mathcal{C}\}$ where each $C(a)$ is a set of a-cosieves and in addition

(1) $\{\alpha \mid \mathrm{dom}(\alpha) = a\} \in C(a)$;

(2) if $R \in C(a)$ and $a \xrightarrow{f} b$ in $\mathcal{C}$, then $f^+(R) = \{b \xrightarrow{\alpha} c \mid \alpha \cdot f \in R\}$ is in $C(b)$;

(3) if $R \in C(a)$ and S is an a-cosieve such that for each $a \xrightarrow{f} b$ in R, we have $f^+(S)$ in $C(b)$, then $S \in C(a)$.

The cosieves in each $C(a)$ are called *covering a-cosieves* or just *covering cosieves.*

Any co-topology C for category $\mathcal{C}$ determines a covariant functor $C: \mathcal{C} \to \mathbf{Set}$ as follows: let the image under functor C of any $a \in \mathcal{C}$ be the set $C(a)$ of covering a-cosieves, and let the image under functor C of any $\mathcal{C}$-morphism $a \xrightarrow{f} b$ be $C(f): C(a) \to C(b): S \mapsto f^+(S)$. When no confusion will result we will also use C_b^a to denote $C(f)$.

Lemma 1.1: *for any* $a \in \mathcal{C}$ *and any co-topology* C *on* $\mathcal{C}$, $C(id_a) = id_{C(a)}$.

Proof: for any $S \in C(a)$,

$$C(id_a)(S) = (id_a)^+(S) = \{a \xrightarrow{\alpha} b \mid \alpha \cdot id_a \in S\} = S. \qquad \square$$

Lemma 1.2: *if* $a \xrightarrow{f} b \xrightarrow{g} c$ *in* $\mathcal{C}$, *then* $C(g \cdot f) = C(g) \cdot C(f)$.

Proof: for any $S \in C(a)$, we have

$$\alpha \in C(g)\big(C(f)(S)\big) \text{ iff } \alpha \cdot g \in C(f)(S) \text{ iff } \alpha \cdot g \cdot f \in S.$$

We also have that $C(g \cdot f)(S) = \{c \xrightarrow{\alpha} d \mid \alpha \cdot g \cdot f \in S\}$. $\square$

Theorem 1.1: $C: \mathcal{C} \to \mathbf{Set}$ *is a covariant functor.*

Proof: the necessary properties of preservation of identities and composition are demonstrated in lemmas 1.1 and 1.2 above. $\square$

Recall now that covariant functor category $\mathbf{Set}^{\mathcal{C}}$ has a subobject classifier $true: 1 \to \Omega$ where Ω is a covariant functor $\mathcal{C} \to \mathbf{Set}$ such that for any $a \in \mathcal{C}$,

$$\Omega(a) = \{\text{all } a\text{-cosieves}\}:$$

and for any $a \xrightarrow{f} b$ in $\mathcal{C}$,

$$\Omega(f): \Omega(a) \to \Omega(b): S \mapsto \{b \xrightarrow{\alpha} c \mid \alpha \cdot f \in S\}.$$

The classifier $true: 1 \to \Omega$ is a natural transformation $\{true_a: a \in \mathcal{C}\}$ with $true_a(\emptyset) = [id_a)$, all $a \in \mathcal{C}$. Now, it is plain that there is an inclusion $\mathrm{C} \hookrightarrow \Omega$. Since $\mathbf{Set}^{\mathcal{C}}$ has a subobject classifier, there is a character arrow of this inclusion. We use cl to denote this character arrow. Recall that if $\tau: F \rightarrowtail G$ is a $\mathbf{Set}^{\mathcal{C}}$-monic, then χ_τ is constructed pointwise by allowing that for all $a \in \mathcal{C}$ and all $x \in G(a)$,

$$(\chi_\tau)_a(x) = \big\{a \xrightarrow{f} b \mid \Omega(f)(x) \in \tau_b(F(b))\big\}.$$

So we say that cl is a natural transformation $\{cl_a: a \in \mathcal{C}\}$ such that for any $R \in \Omega(a)$,

$$cl_a(R) = \{a \xrightarrow{f} b \mid \Omega(f)(R) \in \mathrm{C}(b)\}.$$

To date we have been able to show that co-topologies cl have at least two of the properties that characterise elementary topologies. These are the properties that $cl \cdot true = true$ and $cl \cdot cl = cl$. We give the demonstration of these properties in the following sequence of theorems and lemmas.

Theorem 1.2: $cl \cdot true = true$.

Proof: for any $a \in \mathcal{C}$,

$$\begin{aligned} cl_a\big(true_a(\emptyset)\big) &= cl_a\big([id_a)\big) \\ &= \big\{a \xrightarrow{f} b \mid \Omega(f)\big([id_a)\big) \in \mathrm{C}(b)\big\}. \end{aligned}$$

Now, $\Omega(f)\big([id_a)\big) = \big\{a \xrightarrow{\alpha} b \mid \alpha \cdot f \in [id_a)\big\}$. But $\alpha \cdot f \in [id_a)$ iff $\alpha \in [id_b)$, so

$$\Omega(f)\big([id_a)\big) = [id_b).$$

It follows by condition (1) of co-topologies that $\Omega(f)\big([id_a)\big) \in \mathrm{C}(b)$ for any $f \in [id_a)$. And, since if $f \notin [id_a)$, then $f \notin cl_a\big([id_a)\big)$, we have $cl_a\big([id_a)\big) = [id_a) = true_a(\emptyset)$. In other words, for any $a \in \mathcal{C}$,

$$cl_a(true_a(\emptyset)) = true_a(\emptyset). \qquad \square$$

Lemma 1.3: *for $R, R' \in \Omega(a)$, if $R \subseteq R'$ and $R \in \mathrm{C}(a)$, then $R' \in \mathrm{C}(a)$.*

Proof: suppose $R, R' \in \Omega(a)$ such that $R \subseteq R'$. Now, if $a \xrightarrow{f} b \in R$, then $a \xrightarrow{f} b \in R'$, and, since R is a cosieve, $f^{+}(R') = [id_b)$. Then by condition (1) of co-topologies, $f^{+}(R') \in \mathrm{C}(b)$. But this is true for any $f \in R$, so by condition (3) of co-topologies, $R' \in \mathrm{C}(a)$. □

Lemma 1.4: *for $R \in \Omega(a)$, $R \subseteq cl_a(R)$.*

Proof: if $a \xrightarrow{f} b$ in R, then $\Omega(f)(R) = \{b \xrightarrow{\alpha} c \mid \alpha \cdot f \in R\} = [id_b)$. □

Lemma 1.5: *for $R \in \Omega(a)$ and any $a \xrightarrow{f} b$ in $\mathcal{C}$, $\Omega(f)(R) \subseteq \Omega(f)(cl_a(R))$.*

Proof: $\alpha \in \Omega(f)(R)$ iff $\alpha \cdot f \in R$. But if $\alpha \cdot f \in R$, then, by Lemma 1.4, $\alpha \cdot f \in cl_a(R)$, from which it follows that $\alpha \in \Omega(f)(cl_a(R))$. □

Theorem 1.3: *$cl \cdot cl = cl$.*

Proof: by the definition of cl, we can demonstrate that the theorem if we can show that for any $a \in \mathcal{C}$, any $R \in \Omega(a)$, and any $a \xrightarrow{f} b$ in $\mathcal{C}$,

$$\Omega(f)(R) \in \mathrm{C}(b) \quad \text{iff} \quad \Omega(f)\big(cl_a(R)\big) \in \mathrm{C}(b).$$

This would show that $f \in cl_a(R)$ iff $f \in cl_a(cl_a(R))$.

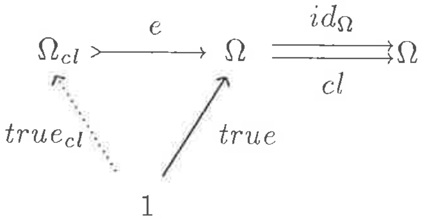
So, suppose some $a \in \mathcal{C}$, some $R \in \Omega(a)$, and some $a \xrightarrow{f} b$ in $\mathcal{C}$. Suppose further that $\Omega(f)(R) \in \mathrm{C}(b)$. It follows by Lemmas 1.5 and 1.3 that

$$\Omega(f)\big(cl_a(R)\big) \in \mathrm{C}(b).$$

Suppose now that $\Omega(f)\big(cl_a(R)\big) \in \mathrm{C}(b)$. Now if $b \xrightarrow{\alpha} c$ is in $\Omega(f)\big(cl_a(R)\big)$, then $\alpha \cdot f \in cl_a(R)$, which means that $\Omega(\alpha \cdot f)(R) \in \mathrm{C}(c)$. But $\Omega(\alpha \cdot f)(R) = \Omega(\alpha)\big(\Omega(f)(R)\big) = \alpha^{+}\big(\Omega(f)(R)\big)$. So, by condition(3) of co-topologies,

$$\Omega(f)(R) \in \mathrm{C}(b).$$ □

Since $\mathbf{Set}^{\mathcal{C}}$ is a topos, it has equalisers and we can define an object Ω_{cl} as in the following diagram where e is an equaliser.



The existence of map $true_{cl}$ follows from Theorem 1.2 and the universal property of equalisers.

2. Categorial Co-topologies on Closed Set Topologies

Since topologies J and co-topologies are exactly dual, we have a notion of canonicity for co-topologies. We say that a co-topology C on $\mathcal{C}$ is canonical iff C^{op} is the canonical topology on $\mathcal{C}^{op}$. We will show that when C is a canonical co-topology on a closed set topology $\mathcal{T}$, functors Ω_{cl} and $\mathcal{B}$ are isomorphic in $\mathbf{Set}^{\mathcal{T}}$.

Proposition 2.1: *the canonical co-topology for any topology $\mathcal{T}$ is that co-topology C where for any $U \in \mathcal{T}$, $R \in \mathrm{C}(U)$ iff $R = \{U \xrightarrow{\alpha_i} U_i \mid i \in I\}$ with*

$$\bigcap\{U_i : i \in I\} = U. \qquad \square$$

Suppose $\mathcal{T}$ is a closed set topology and C is the canonical co-topology on $\mathcal{T}$. Let cl be the character map in $\mathbf{Set}^{\mathcal{T}}$ of the inclusion $\mathrm{C} \hookrightarrow \Omega$. For any two objects U and U_i in the poset category $\mathcal{T}$, there can be at most one arrow $U \xrightarrow{\alpha} U_i$ and if it exists, it will be an inclusion, so let us identify cosieves $R = \{U \xrightarrow{\alpha_i} U_i \mid i \in I\}$ with sets of closed sets $R = \{U_i \mid i \in I\}$. Under this identification any co-topology C becomes a system $\{\mathrm{C}(U): U \in \mathcal{T}\}$ where each $\mathrm{C}(U)$ is a set of U-cosieves and in addition

(1) $[U) \in \mathrm{C}(U)$;

(2) if $R \in \mathrm{C}(U)$ and $f: U \subseteq V$ in $\mathcal{T}$, then $f^+(R) = R \cap [V)$ is in $\mathrm{C}(V)$;

(3) if $R \in \mathrm{C}(U)$ and S is a U-cosieve such that for each $f: U \subseteq V$ in R, we have $f^+(S) \in \mathrm{C}(V)$, then $S \in \mathrm{C}(U)$.

Furthermore, we can describe the classifier object of $\mathbf{Set}^{\mathcal{T}}$ as a functor Ω where for any $U \in \mathcal{T}$,

$$\Omega(U) = \{\text{all } U\text{-cosieves}\}$$

and for any $f: U \subseteq V$ in $\mathcal{T}$, the map $\Omega(f): \Omega(U) \to \Omega(V)$, also denoted Ω_V^U, is given by

$$\Omega(U) \ni S \mapsto S \cap [V).$$

It follows, too, that cl becomes a natural transformation $\{cl_U: U \in \mathcal{T}\}$ such that for any $R \in \Omega(U)$,

$$cl_U(R) = \{W \mid \Omega_W^U(R) \in \mathrm{C}(W)\}.$$

The following lemma and theorem sequence leads us to the theorem that $\mathcal{B}$ and Ω_{cl} are isomorphic in $\mathbf{Set}^{\mathcal{T}}$.

Lemma 2.1: *if* $U \subseteq W, V$, *then* $[W) \cap [V) = [W \cup V)$.

Proof: if $x \in [W) \cap [V)$, then $x \in [W)$ and $x \in [V)$ in which case $W \subseteq x$ and $V \subseteq x$. But then $W \cup V \subseteq x$, so $x \in [W \cup V)$. If $x \in [W \cup V)$, then $W \cup V \subseteq x$, so $W \subseteq x$ and $V \subseteq x$. In that case $x \in [W)$ and $x \in [V)$, so $x \in [W) \cap [V)$. □

Theorem 2.1: *for any* $U \in \mathcal{T}$ *and any* $R \in \Omega(U)$, $cl_U(R) = R$ *iff* $R = [W)$ *for some* $U \subseteq W$ *in* $\mathcal{T}$.

Proof: suppose $R \neq [W)$. Then R is some U-cosieve with at least two bottom elements; that is, R contains at least two distinct elements Y and Y' neither of which have proper subsets in R. Since Y and Y' are distinct bottom elements, $Y \cap Y' \notin R$. To show that $cl_U(R) \neq R$, we show that $Y \cap Y' \in cl_U(R)$. Now, if $Y, Y' \in R$, then $U \subseteq Y \cap Y'$ so $\Omega_{Y \cap Y'}^U(R)$ exists. Since Y and Y' are bottom elements

$$\Omega_{Y \cap Y'}^U(R) = R \cap [Y \cap Y') = [Y) \cup [Y').$$

Now, since $\bigcap([Y) \cup [Y')) = Y \cap Y'$, we have $\Omega_{Y \cap Y'}^U(R) \in \mathrm{C}(Y \cap Y')$ and therefore $Y \cap Y' \in cl_U(R)$.

Suppose now that $R = [W)$. Then $cl_U(R) \neq R$ only if there is some $Z \in \Omega(U)$ such that $Z \in cl_U(R)$ and $W \not\subseteq Z$. Now for $Z \in cl_U(R)$ we require that $\Omega_Z^U(R)$ be in C(Z) which is to say we require that $\bigcap(\Omega_Z^U(R)) = Z$. In the present case $\Omega_Z^U(R) = [W) \cap [Z)$ which, by lemma 2.1, is $[W \cup Z)$ so $\bigcap(\Omega_Z^U(R)) = W \cup Z$. But if $W \not\subseteq Z$, then $W \cup Z \neq Z$, so $cl_U(R) = R$. □

Corollary: *in* $\mathbf{Set}^{\mathcal{T}}$, Ω_{cl} *is the functor where for* $U \in \mathcal{T}$.

$$\Omega_{cl}(U) = \{[W): U \subseteq W \text{ in } \mathcal{T}\}$$

and for $U \subseteq V$ *in* $\mathcal{T}$,

$$(\Omega_{cl})_V^U : \Omega_{cl}(U) \to \Omega_{cl}(V) : [W) \mapsto [W) \cap [V).$$ □

Theorem 2.2: $\mathcal{B}$ *and* Ω_{cl} *are isomorphic objects in* $\mathbf{Set}^{\mathcal{T}}$.

Proof: the theorem calls for the demonstration that there is a natural isomorphism between $\mathcal{B}$ and Ω_{cl} in $\mathbf{Set}^{\mathcal{T}}$. To this end we note that for any $U \in \mathcal{T}$ the function $\varphi_U : \Omega_{cl}(U) \to \mathcal{B}(U)$ given by $[W) \mapsto W$ is a bijection since $[W) \in \Omega_{cl}(U)$ iff W is a closed superset of U, just as $W \in \mathcal{B}(U)$ iff W is a closed superset of U. Now, if the collection $\{\varphi_U : U \in \mathcal{T}\}$ constitutes a natural transformation, the fact that each φ_U is a bijection will make $\{\varphi_U : U \in \mathcal{T}\}$ a natural isomorphism.

The collection $\{\varphi_U : U \in \mathcal{T}\}$ is a natural transformation if for any $U \subseteq V$ in $\mathcal{T}$, the following diagram commutes.

$$\begin{array}{ccc} \Omega_{cl}(U) & \xrightarrow{\varphi_U} & \mathcal{B}(U) \\ {\scriptstyle (\Omega_{cl})_V^U}\downarrow & & \downarrow{\scriptstyle \mathcal{B}_V^U} \\ \Omega_{cl}(V) & \xrightarrow[\varphi_V]{} & \mathcal{B}(V) \end{array}$$

Now for any $[W) \in \Omega_{cl}(U)$ we have

$$\begin{aligned} \mathcal{B}_V^U(\varphi([W))) &= \mathcal{B}_V^U(W) \\ &= W \cup V \end{aligned}$$

and

$$\begin{aligned}\varphi_V\big((\Omega_{cl})^U_V([W])\big) &= \varphi_V([W)\cap[V)) \\ &= \varphi_V([W\cup V)) \qquad \text{(Lemma 2.1)} \\ &= W\cup V.\end{aligned}$$

So, as required,

$$\varphi_V\cdot(\Omega_{cl})^U_V = \mathcal{B}^U_V\cdot\varphi_U. \qquad \square$$

Our next claim will be that $\mathcal{B}$ and Ω_{cl} are dually isomorphic as logic objects. In part this requires of us the claim that Ω_{cl} has an existing logic object structure. We delay that until the next section where we demonstrate that Ω_{cl} is the classifier object of a subcategory $sh_{cl}(\mathbf{Set}^{\mathcal{T}})$ of $\mathbf{Set}^{\mathcal{T}}$. The dual isomorphism of logic objects is directly analogous to the idea of dual isomorphism or anti isomorphism of lattices. Two lattices $(\mathcal{L}_1,\sqsubseteq_1)$ and $(\mathcal{L}_2,\sqsubseteq_2)$ are dually isomorphic if there is an isomorphism $\phi\colon\mathcal{L}_1\to\mathcal{L}_2$ of the underlying sets and, in addition, if for any $a,b\in\mathcal{L}_1$

$$a\sqsubseteq_1 b \quad \text{iff} \quad \phi(b)\sqsubseteq_2\phi(a).$$

The dual isomorphism of $\mathcal{B}$ and Ω_{cl} as logic objects follows from the fact that for any $U\in\mathcal{T}$ and any $V,W\in\mathcal{B}(U)$

$$V\subseteq W \quad \text{iff} \quad [W)\subseteq[V).$$

We have seen that $\mathcal{B}$ is a logic object essentially because any $\mathcal{B}(U)$ is a bounded, distributive lattice under set inclusion. It follows from the above bicondition and the known isomorphism of $\mathcal{B}$ and Ω_{cl} in $\mathbf{Set}^{\mathcal{T}}$ that each $\mathcal{B}(U)$ and $\Omega_{cl}(U)$ are dually isomorphic as lattices under set inclusion. From this follows the dual isomorphism of $\mathcal{B}$ and Ω_{cl} as logic objects. In the next section we will see that Ω_{cl} is the classifier object for a subcategory $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ of functor category $\mathbf{Set}^{\mathcal{C}}$. It is worth noting that the usual idea of a classifier object as an algebra makes Ω_{cl} a logic object whose structure, in the case of $\mathbf{Set}^{\mathcal{T}}$, is exactly described by saying that it is dually isomorphic to $\mathcal{B}$.

Theorem 2.3: *$\mathcal{B}$ and Ω_{cl} are dually isomorphic as logic objects inasmuch as Ω_{cl} is a classifier algebra.* □

The proof of this theorem is indicated in the above discussion.

3. Sheaves on Co-topologies

We have seen that where C is a co-topology on a category $\mathcal{C}$ we can define a map $cl : \Omega \to \Omega$ and then an object Ω_{cl} and a map $true_{cl}: 1 \to \Omega_{cl}$ in $\mathbf{Set}^{\mathcal{C}}$. With this section we demonstrate that there is a subcategory $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ of $\mathbf{Set}^{\mathcal{C}}$ of which $true_{cl}$ is the subobject classifier. This will be a category of what we will call cl-sheaves. These are to be distinguished objects of $\mathbf{Set}^{\mathcal{C}}$ identified with respect to map cl in just the same way as j-sheaves are identified with respect to topologies j. Notice that while co-topologies C and topologies J are duals we are not proposing to claim $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ to be a category co-sheaves (or sheaf duals). In what follows we will define objects of $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ to be covariant functors with a particular property with respect to cl-dense monics in $\mathbf{Set}^{\mathcal{C}}$. The property in question will be exactly the one used to identify contravariant functors as sheaves. In the case of sheaves the property is cast in terms of j-dense monics where j is a topology. We take it, then, that for our $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ objects to be co-sheaves, our notion of cl-denseness must be dual to the usual notion of j-denseness. It, however, is not. The two notions of denseness of monics are, in categorial terms, the same.

Assume that C is a co-topology on a category $\mathcal{C}$ and that cl is the character map for $\mathrm{C} \hookrightarrow \Omega$ in $\mathbf{Set}^{\mathcal{C}}$. In what follows all arrows are $\mathbf{Set}^{\mathcal{C}}$-arrows. Suppose a monic $\sigma: X' \rightarrowtail X$ with a character map χ_σ.

Definition 3.1: $\sigma: X' \rightarrowtail X$ is *cl-closed* if

$$\chi_\sigma = cl \cdot \chi_\sigma.$$

Since all monics in a subobject have the same character arrow, a subobject $[\sigma]$ will be called cl-closed if σ is cl-closed.

Definition 3.2: $\sigma: X' \rightarrowtail X$ is *cl-dense* if

$$cl \cdot \chi_\sigma = \chi_{id_X}.$$

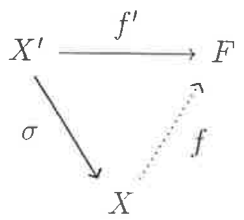
In other words, if $cl \cdot \chi_\sigma$ is the character map for the identity arrow on X, then σ is cl-dense. Where cl-dense σ is the representative morphisms of a subobject $[\sigma]$, we say that $[\sigma]$ is a *cl-dense subobject.*

Since cl is an arrow $\Omega \rightarrow \Omega$, it imposes local operators cl on each $\mathrm{Sub}(d)$ in $\mathbf{Set}^{\mathcal{C}}$ that are natural in d. For any $f \in \mathrm{Sub}(d)$ we define $\mathrm{cl}_d(f)$ to be that subobject classified by $cl \cdot \chi_f$. The idea that these operators are natural in d means exactly that for any $k: d' \rightarrow d$ and any $f \in \mathrm{Sub}(d)$,

$$\mathrm{cl}_{d'}(\mathrm{SUB}(k)(f)) = \mathrm{SUB}(k)(\mathrm{cl}_d(f)).$$

The notions of cl-closed and cl-dense monics can be cast in terms of these operators. A monic $\sigma: X' \rightarrowtail X$ is cl-closed iff $\mathrm{cl}_X(\sigma) \simeq \sigma$, and cl-dense iff $\mathrm{cl}_X(\sigma) \simeq id_X$.

Definition 3.3: we will say that any object F of $\mathbf{Set}^{\mathcal{C}}$ is a *cl-sheaf* iff given any cl-dense monic $\sigma: X' \rightarrowtail X$ and any $f': X' \rightarrow F$, there exists exactly one $f: X \rightarrow F$ such that



commutes. The category of cl-sheaves in $\mathbf{Set}^{\mathcal{C}}$ will be denoted $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$.

Notice that with respect to topologies j this definition exists as a theorem that says that where j is a topology on some topos $\mathcal{E}$, an object $a \in \mathcal{E}$ is an object $a \in sh_j(\mathcal{E})$ iff it has the above property with respect to j-dense monics. This is taken to be a theorem in that the category $sh_j(\mathcal{E})$ is regarded as already specified in terms of the usual equaliser definition of j-sheaves. Plainly though, the theorem may be used as a definition.

We next aim to demonstrate that Ω_{cl} in $\mathbf{Set}^{\mathcal{C}}$ is a cl-sheaf. To do this we need the following technical lemmas.

Lemma 3.1: *there is a 1-1 correspondence between cl-closed subobjects* $[\sigma]$ *and maps* $X \xrightarrow{f} \Omega_{cl}$.

Proof: if $[\sigma]$ is a cl-closed subobject of X, then $\chi_\sigma = cl \cdot \chi_\sigma$. It follows by the properties of e as the equaliser of cl and id_Ω that there is exactly one $f: X \to \Omega_{cl}$ such that $\chi_\sigma = e \cdot f$. Suppose now some arrow $f: X \to \Omega_{cl}$. By composition there will be an arrow $X \xrightarrow{f} \Omega_{cl} \xrightarrow{e} \Omega$ which, since it has codomain Ω, is the character arrow of some monic $\sigma: X' \rightarrowtail X$; that is, $e \cdot f = \chi_\sigma$ for some monic σ. Since e is the equaliser of cl and id_Ω, we have $cl \cdot e = e$, and so we have $cl \cdot e \cdot f = e \cdot f$. From this it follows that σ, and the subobject it determines, are cl-closed. We complete the proof by noting that if $e \cdot f$ is the character arrow for some further subobject $[\phi]$, then, by definition of character arrows, $[\sigma] = [\phi]$. □

Lemma 3.2: *the pullback of a cl-dense subobject is a cl-dense subobject.*

Proof: let $\sigma: X' \rightarrowtail X$ be a cl-dense monic. Now, for any $k: Y \to X$ in $\mathbf{Set}^{\mathcal{C}}$, $\mathrm{SUB}(k)(\sigma)$ is the pullback of σ along k, so if $\mathrm{SUB}(k)(\sigma)$ is a cl-dense monic, then the lemma is demonstrated. But $\mathrm{SUB}(k)(\sigma)$ is cl-dense iff $\mathrm{cl}_Y(\mathrm{SUB}(k)(\sigma)) \simeq id_Y$. Now, we know that

$$\mathrm{cl}_Y(\mathrm{SUB}(k)(\sigma)) \simeq \mathrm{SUB}(k)(\mathrm{cl}_X(\sigma))$$

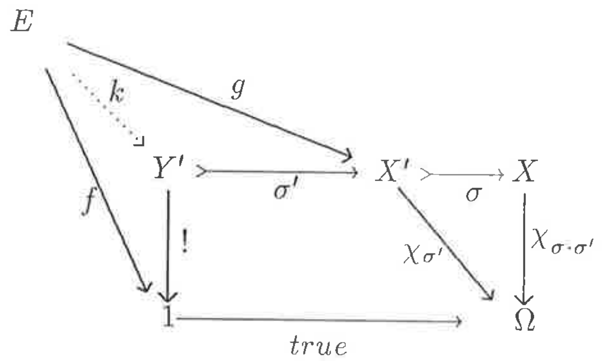
and since σ is cl-dense, $\mathrm{cl}_X(\sigma) \simeq id_X$, so

$$\begin{aligned}\mathrm{cl}_Y(\mathrm{SUB}(k)(\sigma)) &\simeq \mathrm{SUB}(k)(id_X)\\ &\simeq id_Y.\end{aligned}$$

□

Lemma 3.3: *for monics $\sigma': Y' \rightarrowtail X'$ and $\sigma: X' \rightarrowtail X$, we have $\chi_{\sigma'} = \chi_{(\sigma\cdot\sigma')} \cdot \sigma$.*

Proof: consider the diagram



By the properties of the subobject classifier, the lemma is demonstrated if we can show that $\chi_{(\sigma\cdot\sigma')} \cdot \sigma$ makes the square $\{Y', X', \Omega, 1\}$ a pullback. We are required, then, to show that the square $\{Y', X', X, \Omega, 1\}$ commutes and that whenever the outer square $\{E, X', X, \Omega, 1\}$ commutes, there is exactly one $k: E \to Y'$ such that the whole diagram barring $\chi_{\sigma'}$ commutes. Firstly, the square $\{Y', X', X, \Omega, 1\}$ is a pullback so it at least commutes. Secondly, suppose some f and g such that

$$\chi_{(\sigma\cdot\sigma')} \cdot \sigma \cdot f = true \cdot f;$$

that is, suppose that $\{E, X', X, \Omega, 1\}$ commutes. We have already noted that the square $\{Y', X', X, \Omega, 1\}$ is a pullback, so if the square $\{E, X', X, \Omega, 1\}$ commutes, then there is exactly one k such that most of the diagram barring $\chi_{\sigma'}$ commutes. We only say that most of the diagram commutes since we have not yet established that $g = \sigma' \cdot k$. However, we have that $\sigma \cdot \sigma' \cdot k = \sigma \cdot g$. Then, since σ is monic, we have $\sigma' \cdot k = g$. In all, if the outer square $\{E, X', X, \Omega, 1\}$ commutes, then there is exactly one k such that the whole diagram barring $\chi_{\sigma'}$ commutes. □

Lemma 3.4: *for any $f, g \in \mathrm{Sub}(d)$, if $\chi_f \leq \chi_g$, then $cl \cdot \chi_f \leq cl \cdot \chi_g$.*

Proof: the lemma requires us to demonstrate that for all $a \in C$ and all $R \in \Omega(a)$,

$$\text{if} \quad (\chi_f)_a(R) \subseteq (\chi_g)_a(R), \quad \text{then} \quad cl_a((\chi_f)_a(R)) \subseteq cl_a((\chi_g)_a(R))$$

so suppose that $(\chi_f)_a(R) \subseteq (\chi_g)_a(R)$. From this we have that for any $z\colon a \to b$ and any $\alpha\colon b \to c$, if $\alpha \cdot z \in (\chi_f)_a(R)$, then $\alpha \cdot z \in (\chi_g)_a(R)$. It follows that if $\alpha \in \Omega(z)\big((\chi_f)_a(R)\big)$, then $\alpha \in \Omega(z)\big((\chi_g)_a(R)\big)$, or in other words

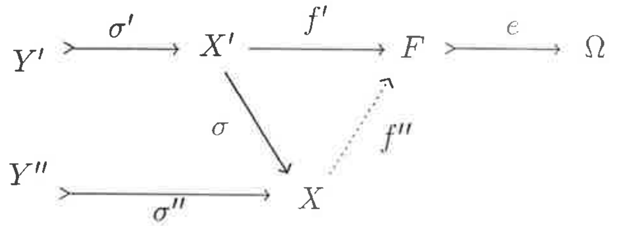
$$\Omega(z)\big((\chi_f)_a(R)\big) \subseteq \Omega(z)\big((\chi_g)_a(R)\big).$$

Now $z \in cl_a\big((\chi_f)_a(R)\big)$ iff $\Omega(z)\big((\chi_f)_a(R)\big) \in \mathrm{C}(b)$. So, by lemma 1.3,

$$\text{if} \quad z \in cl_a\big((\chi_f)_a(R)\big), \quad \text{then} \quad z \in cl_a\big((\chi_g)_a(R)\big). \qquad \square$$

Theorem 3.1: *Ω_{cl} is a cl-sheaf.*

Proof: suppose some cl-dense monic $\sigma\colon X' \rightarrowtail X$ and some map $f'\colon X' \to \Omega_{cl}$. It follows from lemma 3.1 that there is some cl-closed monic $\sigma'\colon Y' \rightarrowtail X'$ such that $\chi_{\sigma'} = e \cdot f'$. Let $\sigma''\colon Y'' \rightarrowtail X$ be the monic classified by $cl \cdot \chi_{\sigma \cdot \sigma'}$. By definition σ'' is cl-closed so there is some $f''\colon X \to \Omega_{cl}$ such that $\chi_{\sigma''} = e \cdot f''$. We have the following diagram.



We will have proven the theorem if we show that f'' makes the inner triangle commute and is unique in doing so. We show first that f'' makes the triangle commute.

From lemma 3.3 we have that $\chi_{\sigma'} = \chi_{(\sigma \cdot \sigma')} \cdot \sigma$, so we have that

$$cl \cdot \chi_{\sigma'} = cl \cdot \chi_{(\sigma \cdot \sigma')} \cdot \sigma.$$

But σ' is cl-closed and $\chi_{\sigma''} = cl \cdot \chi_{(\sigma \cdot \sigma')}$, so

$$\chi_{\sigma'} = \chi_{\sigma''} \cdot \sigma.$$

Furthermore $\chi_{\sigma'} = e \cdot f'$ and $\chi_{\sigma''} = e \cdot f''$, so

$$e \cdot f = e \cdot f'' \cdot \sigma.$$

Now, e is an equaliser and therefore monic. It follows that

$$f' = f'' \cdot \sigma$$

and the triangle commutes. We show now that f'' is unique in making the triangle commute.

Suppose that there is some further $f \colon X \to \Omega_{cl}$ such that $f' = f \cdot \sigma$. Since f exists, there is some cl-closed $\alpha \colon Y \rightarrowtail X$ such that $\chi_\alpha = e \cdot f$. Our aim is to show that α and $\sigma'' = \mathrm{cl}_X(\sigma \cdot \sigma')$ determine the same subobject. It will follow, by lemma 3.1, that $f = f''$ as required. Consider the following diagram where $\{i, h\}$ is the pullback of $\{\alpha, \sigma\}$.

$$\begin{array}{ccc}
Z & \overset{h}{\rightarrowtail} & X' \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \sigma} \\
Y & \overset{\alpha}{\rightarrowtail} & X \\
\Big\downarrow & & \downarrow{\scriptstyle f} \\
 & & \Omega_{cl} \\
 & & \downarrow{\scriptstyle e} \\
1 & \underset{true}{\longrightarrow} & \Omega
\end{array}$$

Since the top and bottom squares are pullbacks, the outer rectangle is a pullback making $\chi_h = e \cdot f \cdot \sigma$. But, by hypothesis, $\sigma \cdot f = f'$, so

$$\chi_h = e \cdot f'.$$

But, further, $e \cdot f' = \chi_{\sigma'}$, so

$$\chi_h = \chi_{\sigma'}.$$

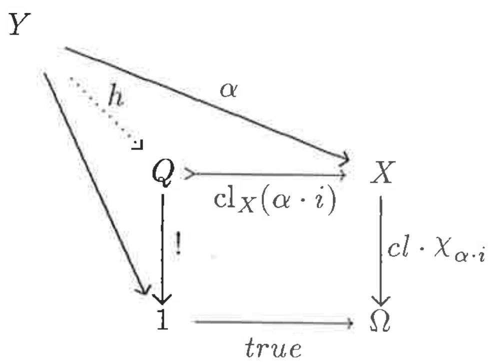
In other words, $h \simeq \sigma'$. Now, since the top square is a commutes, $\alpha \cdot i \simeq \sigma \cdot \sigma'$. It follows that $\mathrm{cl}_X(\sigma \cdot \sigma') \simeq \mathrm{cl}_X(\alpha \cdot i)$. Therefore our proof is complete if we demonstrate that

$$\mathrm{cl}_X(\alpha \cdot i) \simeq \alpha.$$

We have seen that $\alpha \cdot i \simeq \sigma \cdot \sigma'$. It follows that $\sigma \cdot \sigma' \subseteq \alpha$ where $\subseteq$ is subobject inclusion. From this we have that $\chi_{(\sigma \cdot \sigma')} \leq \chi_\alpha$. Then, from lemma 3.4, we have that $cl \cdot \chi_{(\sigma \cdot \sigma')} \leq cl \cdot \chi_\alpha$. Recall that α is cl-closed, so we have that

$$cl \cdot \chi_{(\sigma \cdot \sigma')} \leq \chi_\alpha.$$

This gives us that $\mathrm{cl}_X(\sigma \cdot \sigma') \subseteq \alpha$. Now, consider the following diagram.



The inner square is a pullback by definition. To demonstrate that the outer square commutes note that from lemma 3.3 we have that $\chi_i = \chi_{(\alpha \cdot i)} \cdot \alpha$ and that from lemma 3.2 we have that i is cl-dense. Since i is cl-dense

$$cl \cdot \chi_i = \chi_{id_Y}$$

and χ_{id_Y} is the map $Y \xrightarrow{!} 1 \xrightarrow{true} \Omega$ otherwise denoted $true_Y$. So

$$cl \cdot \chi_{(\alpha \cdot i)} \cdot \alpha = true \cdot !.$$

It follows that there is a unique h that makes the whole diagram commute. In particular we have

$$\alpha = \mathrm{cl}_X(\alpha \cdot i) \cdot h$$

from which we have that $\alpha \subseteq \mathrm{cl}_X(\alpha \cdot i)$. Recall that $\alpha \cdot i = \sigma \cdot h \simeq \sigma \cdot \sigma'$ so we have that

$$\alpha \subseteq \mathrm{cl}_X(\sigma \cdot \sigma'). \qquad \square$$

To demonstrate that Ω_{cl} is the classifier object for $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ we need first demonstrate that $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ and $\mathbf{Set}^{\mathcal{C}}$ agree on limits of finite diagrams of cl-sheaves. It will follow from this first, that $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ is finitely complete and second, that $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ and $\mathbf{Set}^{\mathcal{C}}$ agree on monics between cl-sheaves. The classifier result follows from the universal propoerties of e as an equaliser.

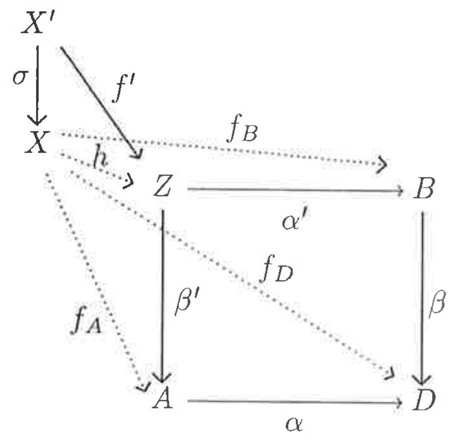
The demonstration that $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ and $\mathbf{Set}^{\mathcal{C}}$ agree on limits of finite diagrams of cl-sheaves comes in two parts: first we show that the terminal object in $\mathbf{Set}^{\mathcal{C}}$ is a cl-sheaf and, second, we show that the pullback in $\mathbf{Set}^{\mathcal{C}}$ of a diagram of cl-sheaves is a cl-sheaf. This suffices as a demonstration of agreement on finite limits since any category is finitely complete if it has pullbacks and a terminal object.

Theorem 3.2: *the terminal object in* $\mathbf{Set}^{\mathcal{C}}$ *is a cl-sheaf.*

Proof: for any $X' \xrightarrow{f'} 1$ and any cl-dense monic $\sigma: X' \rightarrowtail X$, there is exactly one map $f: X \to 1$ such that $f' = f \cdot \sigma$, namely the unique map $X \to 1$ guaranteed by the definition of 1 as a terminal. That $f' = f \cdot \sigma$ follows from the fact that $\hom(X', 1)$ must contain exactly one member. $\square$

Theorem 3.3: *a pullback in* $\mathbf{Set}^{\mathcal{C}}$ *of cl-sheaves is a cl-sheaf.*

Proof: consider the diagram



Suppose that the center square $\{Z, B, D, A\}$ is a pullback and that A, B, D are cl-sheaves. Suppose that σ is a cl-dense monic and that some arrow $f': X' \to Z$ exists. Since D is a cl-sheaf, there is a unique f_D such that

$$f_D \cdot \sigma = \beta \cdot \alpha' \cdot f' = \alpha \cdot \beta' \cdot f'.$$

Now, A is a cl-sheaf, so there is unique f_A such that $\beta' \cdot f' = f_A \cdot \sigma$. But then

$$\alpha \cdot \beta' \cdot f' = \alpha \cdot f_A \cdot \sigma,$$

so, by uniqueness of f_D, $f_D = \alpha \cdot f_A$. Likewise we show that $f_D = \beta \cdot f_B$ since β is a cl-sheaf. So,

$$\alpha \cdot f_A = \beta \cdot f_B.$$

But this together with the fact that the inner square is a pullback means that there is a unique $h: X \to Z$ such that the whole subdiagram $\{X, Z, B, D, A\}$ commutes.

Now, the fact that $\alpha \cdot f_A = \beta \cdot f_B$ means that

$$\alpha \cdot f_A \cdot \sigma = \beta \cdot f_B \cdot \sigma$$

and, again, by the fact that the inner square $\{Z, B, D. A\}$ is a pullback, there will be exactly one $f: X' \to Z$ making the whole diagram, barring the $\{X', X, Z\}$ triangle, commute. But f' is one such arrow and so is $h \cdot \sigma$. It follows that

$$f' = h \cdot \sigma. \qquad \square$$

Corollary 1: *$sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ is finitely complete and agrees with $\mathbf{Set}^{\mathcal{C}}$ on limits of finite diagrams of cl-sheaves.*

Proof: the present theorem together with theorem 3.2 together with the standard result that pullbacks and terminal objects imply all finite limits. □

Corollary 2: *$true_{cl}$ is the subobject classifier for $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$.*

Proof: by the first corollary to Theorem 3.3, $sh_{cl}(\mathbf{Set}^{\mathcal{C}})$ and $\mathbf{Set}^{\mathcal{C}}$ agree on monics between cl-sheaves. The present corollary follows from this and the universal and monomorphic properties of e as an equaliser. The proof is a variation on that of Theorem 8.3.1. □

The above corollary is interesting in the light of Theorem 2.2 and the claim of dual isomorphism for logic objects $\mathcal{B}$ and Ω_{cl}. In the case where $\mathcal{T}$ is a closed set topology, the natural isomorphism $\varphi: \mathcal{B} \cong \Omega_{cl}$ in $\mathbf{Set}^{\mathcal{T}}$ means that $\varphi \cdot true_{cl}: 1 \to \mathcal{B}$ must be a classifier for $sh_{cl}(\mathbf{Set}^{\mathcal{T}})$. In fact, since $true_{cl}: 1 \to \Omega_{cl}$ os a natural transformation $\{(true_{cl})_U : U \in \mathcal{T}\}$ where $(true_{cl})_U(\emptyset) = [U)$ the map $\varphi \cdot true_{cl}$ is given by components $\varphi_U \cdot (true_{cl})_U$ where

$$\varphi_U\big((true_{cl})_U(\emptyset)\big) = U.$$

Since $U \subseteq Z$ for all $Z \in \mathcal{B}(U)$, the map $\varphi \cdot true_{cl}$, in terms of the natural algebra on $\mathcal{B}$, is better thought of as a truth value $false$. When this is taken into account we see that $\varphi \cdot true_{cl}$ functions as a complement classifier in $sh_{cl}(\mathbf{Set}^{\mathcal{T}})$. By this we mean that if $\varphi \cdot true_{cl}$ is used to construct operator arrows on $\mathcal{B}$ in just the way that $true_{cl}$ is used with respect to Ω_{cl}, then the algebra $\mathcal{B}$ is the (type) dual of the algebra Ω_{cl}. (In fact we get the algebra that we have described in chapter eleven). Consider, for example, the usual construction of the glb operator arrow. On Ω_{cl} the glb arrow is constructed as the $\varphi \cdot true_{cl}$-character of the product map $\langle true_{cl}, true_{cl} \rangle$. This generalises the idea that the truth function *and*: $\{\emptyset, 1\} \times \{\emptyset, 1\} \to \{\emptyset, 1\}$ has $and(x, y) = 1$ iff $x = 1$ and $y = 1$. Now, if we consider the character map of $\langle \varphi \cdot true_{cl}, \varphi \cdot true_{cl} \rangle$ recalling that $\varphi \cdot true_{cl}$ is intuitively the truth value $false$, we have an arrow that is a generalisation of the truth function $f: \{\emptyset. 1\} \times \{\emptyset, 1\} \to \{\emptyset, 1\}$ which has $f(x, y) = \emptyset$ iff $x = \emptyset$ and $y = \emptyset$; in other words, the $\varphi \cdot true_{cl}$-character map of $\langle \varphi \cdot true_{cl}, \varphi \cdot true_{cl} \rangle$ is a lub operator arrow. The same discussion applies to the usual constructions of $\bigcup$, Ⓢ, $\Rightarrow$ in just the way we have described in chapter four and was originally described in chapter eleven of Mortensen [1995]. As we suggested in chapter four, if a complement classifier was to exist, it would be categorially indistinguishable from a subobject classifier; this is exactly the case for $\varphi \cdot true_{cl}$. The thing that makes $\varphi \cdot true_{cl}$ a complement classifier rather than a subobject classifier is a (relatively) intuitive assessment of the nature of $\varphi \cdot true_{cl}$ as a truth value. This assessment, even in the case of bona fide subobject classifiers, is never strictly categorial. It is based on the convention that the unit of a lattice interprets $\top$, or truth, while the zero of the lattice interprets $\perp$, or falsehood. If we accept that convention, as we do in the usual subobject classifier case, then surely we accept it in the case of $\varphi \cdot true_{cl}$. This makes $\varphi \cdot true_{cl}$ a genuine complement classifier.

The next chapter marks the end of Part III and concludes our discussion of sheaf concepts. The discussion there will be somewhat different from the foregoing sheaf discussion in that we focus our attention on sheaf spaces rather than functors. In fact our concern is a generalisation of that which we have exhibited in the last five chapters. In the last five chapters we have been concerned to discuss the nature of particular sheaves, namely sheaves that are classifier objects, as logic objects and, more broadly, as objects of paraconsistent semantics. In the next chapter we describe an equivalence result for categories of sheaf spaces and categories of sheaves. Sheaf spaces are of interest for something like the reason sheaves were invented: they transport algebras from the base space into the section structure of the sheaf space projection itself. An equivalence result between sheaves and sheaf spaces over closed sets has the effect of making closed set sheaves generally interesting as objects for paraconsistent semantics.

CHAPTER 13:

SHEAF SPACES ON FINITE CLOSED SETS

Introduction: A sheaf space is a continuous local homeomorphism between topological spaces. It is known that a sheaf over an open set topology will give rise to a sheaf space and vice versa, and it is usual to note that the category of all sheaves on an open set topology Θ for a space X is equivalent to the category of all sheaf spaces over X with the same topology. With this chapter we modify the notion of local homeomorphism to deal with closed sets and verify that at least a restricted class of closed set sheaves are equivalent to "closed set sheaf spaces".

The notion of a closed set (pre)sheaf is a particular example of the notion of a (pre)sheaf over a category and as such is uncontroversial. A sheaf space over closed sets will be defined in exactly the same way as the usual sheaf spaces. However in the absence of a general theory allowing us to forgo open set topologies, we could be accused of misusing the "sheaf space" name. Our claim is that since the categorial notion of a sheaf has proven amenable to dualisation in terms of being defined over closed sets rather than open without loss of the defining features of a sheaf, we can make a similarly conservative dualisation for the more traditional notion of a sheaf here called a sheaf space. We would then be in a position to develop the features of a sheaf space that make it attractive to a mathematical logician mindful of the new tool of closed sets in the base space.

There is some expectation that an adequate description of a closed set sheaf space can be put to use in terms of Davey's representation constructions. Davey in his "Sheaf spaces and sheaves of universal algebra" [1973], describes a general method for converting a subdirect product representation of an algebra to a representation of an algebra of global sections of a sheaf space. We note that Davey's construction is given in terms of open set sheaf spaces. One of our guiding speculations has been that mathematical and logical objects arising from inconsistent but non-trivial theories could be collected into categories that were structurally different in some recognisable way from categories of objects from consistent theories. A milder version of this speculation has that with the right sort of objects we would have categories that exhibit paraconsistent algebras either as objects or as morphism structures. One way to investigate these speculations is to address the nature of categories of sheaf spaces with each sheaf space containing a representation of some paraconsistent algebra. Such an investigation would seem to be most fruitful if our sheaf spaces were defined over closed sets. In the first instance, though, we must set about discovering the viability of the notion. In particular, for the extension of the discussion into more general category theory, we will want to know how closely the theory of closed set sheaves and sheaf spaces mirrors the theory of open set sheaves and sheaf spaces. To that end we consider an equivalence of categories result for closed set sheaves and sheaf spaces.

Another of our motivations for considering such a result is simpler. We have been interested throughout Part III in the nature of sheaves, and particularly sheaves of closed sets as logic objects in categories. We have paid closest attention to classifier objects in these categories. There are two things we would like to do: first, find a way to extend our discussion to other objects in the sheaf category, and second, move some way toward finding out if there can be categories with paraconsistent subobject structures. The first of these tasks is handled by moving to sheaf equivalent sheaf spaces in that such structures do what the original sheaves do: they transport base space algebras into the section structure of the sheaf space. The second of our two tasks is only touched upon with this chapter. We have seen in chapters nine and ten that paraconsistent subobject structures exist but fail to be natural in a way that can be expressed by saying set theoretic closure operations fail to distribute over intersections. The move from sheaves to sheaf equivalent sheaf spaces changes the context of discussion and allows us to speak of subobject lattices directly in terms of the continuous maps that exist between objects. We expect this, one way or the other, to be profitable but we follow it no further in the present work.

With this chapter we present a somewhat restricted revision of the standard constructions for the presheaf to sheaf space functor L and the sheaf space to sheaf functor Γ that can deal with structures on closed sets rather than open. While we propose to proceed along the usual line of development, we shall at times be required to alter the usual proofs to accomodate the new nature of the stalk and base space topologies.

It will be advantageous to restrict the functor constructions to presheaves and sheaf spaces over finite spaces X with topologies $\mathcal{T}$ where any member of $\mathcal{T}$ is a finite subset of X. That is to say the usual construction will not in general work for closed sets without some restriction of this sort. We make significant use of this restriction and we note that it is not entirely arbitrary. Where $\mathcal{C}$ is a small finitely complete category and *cov* is a finite pretopology in the sense that for all objects $c \in \mathcal{C}$, $cov(c)$ is finite, then the site $\mathbf{C} = (\mathcal{C}, cov)$ is called *finitary* and the category $sh(\mathbf{C})$ is a *coherent topos*. In fact any coherent topos is equivalent to some $sh(\mathbf{C})$ for finitary $\mathbf{C}$. Such toposes are significant as classifying toposes for algebraic geometry (see, for example, Johnstone [1977] and Makkai and Reyes [1977]). We note that if every member of a topology $\mathcal{T}$ is finite, then the canonical pretopology *cov* for poset category $\mathcal{T}$ yields a finitary site $(\mathcal{T}, cov)$ and so a coherent topos $sh(\mathcal{T}, cov)$ of closed set sheaves.

We will also be required to restrict our constructions to presheaves F where for any closed U, the set $F(U)$ is finite. This is in response to what seems to be a deep feature of the consistent construction of sheaf space morphisms from presheaf morphisms: given a presheaf morphism $f: F \longrightarrow F'$ it is possible to describe a function $\mathrm{L}f$ from constructed sheaf space $(\mathrm{L}F, p_F)$ to $(\mathrm{L}F', p_{F'})$. but to prove that function continuous in general we will be required to accept arbitrary unions in the topology on space $\mathrm{L}F$. Notice too that the usual construction of sheaf ΓE from sheaf space (E, p) may not guarantee finite sets $\Gamma E(U)$. The particular implication for us is that while we can describe a functor Γ from the category of all sheaf spaces over X to the category of sheaves over X, it will not in general compose with the functor L restricted to sheaves. To avoid this problem the domain of our Γ will be restricted to sheaf spaces (E, p) where E is finite. These restrictions are somewhat ad hoc but only from the point of view of creating a more general "sheafification" theory.

Note well that the above restrictions apply only for the particular construction of functors $\underline{\mathrm{L}}$ and $\underline{\Gamma}$ included here. There should be no conclusion that this indicates which presheaves and sheaf spaces can exist on closed sets.

There is a criticism to be dealt with here. It is that the finiteness assumption for the relevant topologies renders the material of this chapter philosophically trivial in that finite closed set topologies (and finite open set topologies) are not really distinguishable from finite ditributive lattices. In answer to that criticism we note that the first order of business for this chapter is to produce the desired equivalence of categories result; the second order of business, following on from the first, is to note that, all other things being equal, the equivalence of sheaves on closed sets and sheaf spaces on closed sets can be performed *only* for finite closed sets. There may be some restriction on the nature of the topological space that allows the construction for non-finite closed sets but without some such restriction, the non-finite construction cannot go ahead. This is pointed out a number of times during the discussion. The significance of the result is then that sheaf and sheaf space theory on closed sets is similar but importantly different from the same theory on open sets. If the emphasis of the chapter were solely on achieving the equivalence result, then the chapter would be of no special significance since under the finiteness assumption the result is covered by the usual equivalence theorem for sheaves and sheaf spaces on open sets (open and closed set topologies are formally distinct only in their respective treatment of non-finite collections of topology elements). However in the chapter I give some discussion of the possibility of achieving the result with respect to non-finite closed sets. With respect to at least one point of the ordinary construction I was able to show that the topological dual of that construction is not formally constructible without the restriction to finiteness (all other things being equal).

We will adopt the following conventions: $\mathcal{T}$ is always a closed set topology of finite subsets of some finite X; $\underline{presh}(X, \mathcal{T})$ is the name for the category of closed set presheaves over topological space X where for any closed $U \subseteq X$ and any presheaf F the set $F(U)$ is finite; $\underline{sh}(X, \mathcal{T})$ is the category of sheaves in $\underline{presh}(X, \mathcal{T})$; $\underline{sheafsp}(X, \mathcal{T})$ is the category of sheaf spaces (E, p) over X where E is finite. Any category name given without an underbar should be taken to refer to the unrestricted categories in question. $\underline{\mathrm{L}}$ will be a functor $\underline{presh}(X, \mathcal{T}) \to \underline{sheafsp}(X, \mathcal{T})$. $\underline{\Gamma}$ will be a functor $\underline{sheafsp}(X, \mathcal{T}) \to \underline{sh}(X, \mathcal{T})$.

We shall end by discovering that $\underline{sh}(X, \mathcal{T})$ is equivalent to $\underline{sheafsp}(X, \mathcal{T})$.

Our discussion below owes much to the demonstration of the equivalence of open set sheaves to open set sheaf spaces found in Tennison's *Sheaf Theory* [1975] and is a later draft, but essentially the same as, James, W., "Sheaf spaces on finite closed sets" in *Contemporary Logical Research in Australia*, Logique et Analyse, [1996].

1. Sheaves and Sheaf Spaces

With this section we define the notions of closed set sheaf and closed set sheaf space that we will use. Notice that what we will be calling a sheaf in this chapter is indeed a sheaf according to the usage of the word accepted in chapters eight and ten. The sheaves in the present chapter are exactly those functors identified with respect to the canonical topology j in $\mathbf{Set}^{\mathcal{T}^{op}}$ where $\mathcal{T}$ is the closed set topology.

Once we have given the necessary definitions we will give some technical lemmas on the nature of sections in a closed set sheaf space. These will be used in the next two sections where we show that we can construct closed set sheaf spaces from closed set sheaves and vice versa. The last section contains the demonstration that the existence of these particular constructions implies an equivalence of categories.

Definition 1.1: Presheaves are contravariant functors. When $\mathcal{T}$ is a topology for a space X, any contravariant functor $F: \mathcal{T}^{op} \to \mathbf{Set}$ is called a *presheaf on* $\mathcal{T}$ or, if $\mathcal{T}$ is understood, a *presheaf over* X. A *sheaf on* $\mathcal{T}$ is any presheaf F that satisfies the following condition: if $U \in \mathcal{T}$ and there is some $\{U_i: i \in I\}$ with each $U_i \in \mathcal{T}$ and $\bigcup\{U_i: i \in I\} = U$, then whenever we have $\{s_\in F(U_i): i \in I\}$ such that

$$F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j)$$

for all $i, j \in I$, there is exactly one $s \in F(U)$ such that

$$F^{U}_{U_i}(s) = s_i$$

all $i \in I$. For closed set topologies $\mathcal{T}$, a sheaf on $\mathcal{T}$ is called a *closed set sheaf.*

Definition 1.2: For any presheaf F on $\mathcal{T}$ over X and any $x \in X$, the *stalk* F_x *of* F *at* x is defined to be the direct limit of the system of sets $F(U)$ where $x \in U$ and arrows F^U_V where $x \in U \subseteq V$.

We can construct a stalk F_x for F as follows. Fix $x \in X$. Let Z be the disjoint union of all $F(U)$ where $x \in U$. We define an equivalence relation $\sim_x$ on Z by saying that if $x \in U, V$ and $u \in F(U)$ and $v \in F(V)$, then

$$u \sim_x v$$

iff

there is some $W \in \mathcal{T}$ such that $x \in W \subseteq U \cap V$ and

$$F_W^U(u) = F_W^V(v).$$

Then F_x is $Z/\sim_x$ together with maps $F(U) \to F_x$ which are $F(U) \hookrightarrow Z \to (Z/\sim_x)$ given by $s \mapsto s_x$ with each s_x being the equivalence class for $s \in F(U)$ under $\sim_x$.

It is useful to note that for any $s_x, t_x \in F_x$ where $s \in F(U)$ and $t \in F(V)$, we have $s_x = t_x$ iff there is some $W \subseteq U \cap V$ such that $x \in W$ and $F_W^U(s) = F_W^V(t)$.

Morphisms of presheaves induce what we will call *stalk morphisms*. Suppose $f: F \to F'$ between presheaves F and F'. Recall that f is a natural transformation $\{f_U: U \in \mathcal{T}\}$. Then for each $x \in X$ there are stalk morphisms $f_x: F_x \to F'_x$ given by

$$s_x \mapsto \big(f_U(s)\big)_x$$

where $x \in U$. For composite presheaf morphisms $F \xrightarrow{f} F' \xrightarrow{g} F''$, we have $(g \cdot f)_x = g_x \cdot f_x$.

Definition 1.3: A map $p: E \to X$ between topological spaces E and X is *continuous* iff the inverse of each open set in X is open in E. Equivalently, the map is continuous iff the inverse of each closed set is closed (Th.3.1, Kelley [1955]).

Definition 1.4: A map $p: E \to X$ is a *homeomorphism* if it is a bijection and both it and its inverse are continuous.

Definition 1.5: A map $p: E \to X$ is a *local homeomorphism* if for any $e \in E$, there is some homeomorphism $p|N: N \to U$ such that both N and U are open and $e \in N$,

$p(e) \in U$. In essence a map is a local homeomorphism if it is a homeomorphism when restricted or "localised" to an open subset of its domain. Plainly, we can describe a similar property of maps in terms of closed sets. Replace all occurences of "open" with "closed" in the definition of a local homeomorphism and a homeomorphism and we have the definition of a *closed set local homeomorphism*.

Definition 1.6: A *closed set sheaf space on* X is a closed set local homeomorphism $p: E \to X$ that is continuous between topological spaces E and X. When X is understood we use (E, p) to denote the sheaf space. For what follows all homeomorphisms are defined with respect to closed sets.

Definition 1.7: For sheaf spaces over the same X a *sheaf space morphism*

$$g: (E, p) \to (E', p')$$

is a continuous map $g: E \to E'$ such that $p = p' \cdot g$.

Definition 1.8: For closed set sheaf space $p: E \to X$ and closed subset U of X, a *closed set section of* p *over* U, or just U*-section of* p, is a continuous map $s: U \to E$ such that $p \cdot s = id_U$. The collection of all sections over U is denoted $\Gamma E(U)$. The notation recalling functor Γ is deliberate.

Definition 1.9: Any collection β of sets will be called a *basis for a closed set topology* Ξ on a space $X = \bigcup \beta$ when we have that $b \in \Xi$ iff b is a finite union of members of β. Any collection α is a *subbasis for closed set topology* Ξ if the collection of all intersections of members of α is a basis for Ξ. Plainly, any collection α can be used as a subbasis for a topology on $\bigcup \alpha$.

Lemma 1.1: *any homeomorphism* $p|N: N \to U$ *guaranteed by* p *as local homeomorphism gives rise to a section* $(p|N)^{-1}: U \to E$.

Proof: $(p|N)^{-1}$ is by definition continuous and plainly $p \cdot (p|N)^{-1} = id_U$. □

Definition 1.9: A map $s: U \to E$ between topological spaces is a *closed map* iff the image in E of each closed set in U is closed.

Definition 1.10: For any e in topological space E, a set M is a *closed neighbourhood of* e if there is some $N \subseteq M$ such that $e \in N$ and N is closed in E. In what follows when we speak of neighbourhoods of e we will mean closed sets M such that $e \in M$.

Lemma 1.2: *any section $s: U \to E$ of sheaf space $p: E \to X$ is a closed map.*

Proof: recall that we have supposed any topology $\mathcal{T}$ on X to contain only finite subsets of X. By hypothesis, then, U is finite. It follows that $s(U)$ is finite. Now, for any $e \in s(U)$ the definition of p as a sheaf space assures us of closed sets $M \subseteq E$ and $V \subseteq X$ such that $e \in M$, $p(e) \in V$, and $p\,|\,M: M \to V$ is a homeomorphism. It follows that $(p|M)(M)$ is closed in X. Therefore $(p|M)(M) \cap U$ is closed in X. Now, s is a section, so $(p|M)$ must map $M \cap s(U)$ bijectively to $(p|M)(M) \cap U$ and since $(p\,|\,M)$ is continuous, $M \cap s(U)$ is closed in E. Choose one $(p\,|\,M)$ for each $e \in s(U)$ and $s(U)$ becomes the finite union of the associated sets $M \cap s(U)$. □

Lemma 1.3: *any section $s: U \to E$ is a homeomorphism $s: U \to s(U)$.*

Proof: since $p \cdot s = id_U$ the map $s: U \to s(U): x \mapsto s(x)$ has a bijective inverse $p|s(U)$. The section s is continuous. Also s is a closed map so, given p as continuous, the map $p|s(U)$ is continuous (see Lemma 4.1). □

Lemma 1.4: *the collection of sets formed by the images of all sections s over all closed sets U of the sheaf space (E, p) is a basis for the topology on E when E is finite.*

Proof: recall that we have suggested that any base space X is finite, so any closed subset U of X is finite. It follows that any $s(U)$ is a finite subset of E. Now, Let M be any closed subset of E. For any $e \in M$ there is some closed neighbourhood $N \subseteq E$ such that a homeomorphism $p|N$ exists. Finite intersections of closed sets are closed sets, so the set $M \cap N$ is closed in E; and since $p|N$ is a homeomorphism we have a section $s = \big(p|(M \cap N)\big)^{-1}$ over closed set $(p|N)(M \cap N) = p(M \cap N)$. Plainly $e \in s\big(p(M \cap N)\big) \subseteq M$. Since the space E is finite, the subset M must be finite, so choose one s for each $e \in M$ as described above and then the set M can be described as the finite union of sets $s\big(p(M \cap N)\big)$. Since E is itself a member of the topology, it follows that E is some finite union of sets $s(U)$. Since also any $s(U) \subseteq E$ the space E is the union of all $s\big(p(E \cap N)\big)$. □

2. From Presheaves to Sheaf Spaces

In this section we first describe the construction of a sheaf space $(\underline{\mathrm{L}}, p_F)$ given a presheaf F. Secondly, we describe the construction of a sheaf space morphism $\underline{\mathrm{L}}f$ from $(\underline{\mathrm{L}}F, p_F)$ to $(\underline{\mathrm{L}}F', p_{F'})$ given a presheaf morphism $f: F \to F'$. We finish this section with a demonstration that these constructions describe a functor

$$\underline{\mathrm{L}}: \underline{presh}(X, \mathcal{T}) \to \underline{sheafsp}(X, \mathcal{T}).$$

Suppose a presheaf $F: \mathcal{T}^{op} \to$ **Set** from $\underline{presh}(X, \mathcal{T})$ where $\mathcal{T}$ is a closed set topology on X. We will construct a topological space $\underline{\mathrm{L}}F$ and a map $p_F: \underline{\mathrm{L}}F \to X$. We will go on to demonstrate that $(\underline{\mathrm{L}}F, p_F)$ is a sheaf space.

Construction 2.1: let $\underline{\mathrm{L}}F$ be the disjoint union of the stalks F_x of F for all $x \in X$. Since any element of F_x is some s_x determined by some $s \in F(U)$ where $x \in U$, we may, wherever U is closed in X and $s \in F(U)$, define a map $\hat{s}: U \to \underline{\mathrm{L}}F$ by

$$U \ni x \mapsto s_x.$$

With respect to such maps each F_x is the union of all sets $\hat{s}(U)$ where $x \in U \in \mathcal{T}$ and $s \in F(U)$. It follows that we may topologise $\underline{\mathrm{L}}F$ by accepting the collection of sets $\hat{s}(U)$ for all $U \in \mathcal{T}$ and all $s \in F(U)$ as a closed set subbasis. In fact, we may demonstrate that the sets $\hat{s}(U)$ form a closed set basis. We do this with the following two lemmas.

Lemma 2.1: *if β is a collection of finite sets and α is a non-finite subcollection of β, $\bigcap \alpha = \bigcap \alpha'$ for some finite subcollection α' of α.*

Proof: if any of the members of α are disjoint, the lemma is proven. Otherwise, choose some $b \in \alpha$. We define α' in terms of b. Firstly, let $b \in \alpha'$. Then, note that for any $x \in b$, if $x \notin \bigcap \alpha$, then there is some $b' \in \alpha$ such that $x \notin b'$. For any such $x \in b$ choose exactly one such $b' \in \alpha$ and let $b' \in \alpha'$. Let no other members of β be members of α'. Since b is finite, α' must be finite; and, by virtue of its definition, $\bigcap \alpha' = \bigcap \alpha$. □

Lemma 2.2: *the collection of sets $\hat{s}(U)$ for all$U \in \mathcal{T}$ and all $s \in F(U)$ is a closed set basis for $\underline{\mathrm{L}}F$.*

Proof: any collection β of sets identified as a closed set subbasis for a topology is a closed set basis for the same topology if any arbitrary intersection of members of β is a finite union of members of β. Now, since by hypothesis any $U \in \mathcal{T}$ is finite, any $\hat{s}(U)$ is finite. By lemma 2.1, then, we need demonstrate only that any finite intersection of sets $\hat{s}(U)$ is a finite union of sets $\hat{s}(U)$. We demonstrate this if we show that the intersection of two sets $\hat{s}(U)$ is a finite union of sets $\hat{s}(U)$.

Let $\hat{s}(U)$ be defined for some $s \in F(U)$ and let $\hat{t}(V)$ be defined for some $t \in F(V)$. if $e \in \hat{s}(U) \cap \hat{t}(V)$, then $e = s_x = t_x$ for some $x \in U \cap V$. But in that case there must be some $W \in \mathcal{T}$ such that $x \in W$ and

$$F^U_W(s) = F^V_W(t).$$

Let r be that element of $F(W)$ picked out by $F^U_W(s)$ and $F^V_W(t)$. Plainly, for all $x \in W$, $r_x = s_x = t_x$; that is, for all $x \in W$

$$F^W_W(r) = F^U_W(s) = F^V_W(t).$$

It follows that $\hat{r}(W) \subseteq \hat{s}(U) \cap \hat{t}(V)$. Since this is true for any $e \in \hat{s}(U) \cap \hat{t}(V)$ and $\hat{s}(U) \cap \hat{t}(V)$ is finite, the set $\hat{s}(U) \cap \hat{t}(V)$ can be described as the finite union of the sets $\hat{r}(W)$. □

Construction 2.2: let $p_F: \underline{L}F \to X$ be defined so that $(p_F)^{-1}(x) = F_x$ for all $x \in X$.

To show that $(\underline{L}F, p_F)$ is a sheaf space we must show that p_F is a continuous local homeomorphism.

Lemma 2.3: *the map $p_F: \underline{L}F \to X$ is continuous with respect to the topology on $\underline{L}F$.*

Proof: for any closed $U \subseteq X$, we have $(p_F)^{-1}(U)$ as the disjoint union of all F_x for $x \in U$. It follows that $(p_F)^{-1}(U)$ is the collection of points s_x for all $x \in U$ and all $V \in \mathcal{T}$ such that $x \in V$ and $s \in F(V)$. But note that for any $s \in F(V)$, $F^V_{U \cap V}(s) = F^{U \cap V}_{U \cap V}(s \mid U \cap V)$ which means that $s_x = (s \mid U \cap V)_x$ for all $x \in U \cap V$. It follows that we may describe $(p_F)^{-1}(U)$ as the union of sets $\hat{s}(V)$ where $s \in F(V)$ and $V \subseteq U$ in $\mathcal{T}$. Now, by hypothesis U is finite, so there are only a finite number of $V \in \mathcal{T}$ such that $V \subseteq U$. Furthermore, $F \in \underline{presh}(X, \mathcal{T})$ so any $F(V)$ is finite. It follows that $(p_F)^{-1}(U)$ is a finite union of closed sets $\hat{s}(U)$. □

Lemma 2.4: *$p: \underline{L}F \to X$ is a local homeomorphism.*

Proof: any $e \in \underline{L}F$ will have some closed neighbourhood $\hat{s}(U)$. The maps $p_F \,|\, \hat{s}(U)$ and $\hat{s}$ are bijective inverses. Since p_F is continuous, $p_F \,|\, \hat{s}(U)$ is continuous. It follows from the construction of the topology on $\underline{L}F$ that $p_F \mid \hat{s}(U)$ is a closed map and since $\hat{s}$ is its inverse, $\hat{s}$ is continuous (see Lemma 4.1). □

Theorem 2.1: *if $F \in \underline{presh}(X, \mathcal{T})$, then $(\underline{L}F, p_F) \in \underline{sheafsp}(X, \mathcal{T})$.*

Proof: lemmas 2.3 and 2.4 together with the fact that, as constructed, $\underline{L}F$ is finite. □

Suppose a morphism $f: F \to F'$ in $\underline{presh}(X, \mathcal{T})$. We construct a map

$$\underline{\mathrm{L}}f: (\underline{\mathrm{L}}F, p_F) \to (\underline{\mathrm{L}}F', p_{F'})$$

and go on to demonstrate that $\underline{\mathrm{L}}f$ is a $\underline{sheafsp}(X, \mathcal{T})$ morphism.

Construction 2.3: recall that presheaf morphisms $f: F \to F'$ induce stalk morphisms $f_x: F_x \to F'_x$ for all $x \in X$ given by $F_x \ni s_x \mapsto \big(f_U(s)\big)_x$ where $x \in U$. Define $\underline{\mathrm{L}}f: (\underline{\mathrm{L}}F, p_F) \to (\underline{\mathrm{L}}F', p_{F'})$ so that for any $s_x \in \underline{\mathrm{L}}F$,

$$(\underline{\mathrm{L}}f)(s_x) = \big(f_U(s)\big)_x.$$

Lemma 2.5: $p_F = p_{F'} \cdot \underline{\mathrm{L}}f$.

Proof: for any $s_x \in \underline{\mathrm{L}}F$, $p_F(s_x) = x$ while

$$\begin{aligned} p_{F'}\big(\underline{\mathrm{L}}f(s_x)\big) &= p_{F'}\Big(\big(f_U(s)\big)_x\Big) \\ &= x. \end{aligned}$$

□

Lemma 2.6: *$\underline{\mathrm{L}}f$ is continuous.*

Proof: any member of the basis of the topology on $\underline{\mathrm{L}}F'$ is $\widehat{s'}(U)$ for some $s' \in F'(U)$ and some $U \in \mathcal{T}$. To demonstrate the lemma it is enough to demonstrate that $(\underline{\mathrm{L}}f)^{-1}\big(\widehat{s'}(U)\big)$ is a closed set in $\underline{\mathrm{L}}F$. But for any $e \in \widehat{s'}(U)$, $e = (s')_x$ and

$$(\underline{\mathrm{L}}f)^{-1}\big((s')_x\big) = \{s \in F(U) : f_U(s) = s'\};$$

and since by definition of $\underline{\mathrm{L}}f$, we have $(\underline{\mathrm{L}}f)\big(\hat{s}(U)\big) = \widehat{f_U(s)}(U)$, we have

$$(\underline{\mathrm{L}}f)^{-1}\big(\widehat{s'}(U)\big) = \bigcup\{\hat{s}(U) : s \in F(U) \text{ and } f_U(s) = s'\}.$$

Recall that $F \in \underline{presh}(X, \mathcal{T})$, so $F(U)$ is finite. It follows that $(\underline{\mathrm{L}}f)^{-1}\big(\widehat{s'}(U)\big)$ is a finite union of closed sets in $\underline{\mathrm{L}}F$. □

It should be apparent that if we do not restrict the size of $F(U)$ it is possible that there be a non-finite number of $s \in F(U)$ for which $f_U(s) = s'$, in which case we would need some extra hypothesis about the topology on $\underline{\mathrm{L}}F$, an alternative topology, or another construction for $\underline{\mathrm{L}}f$.

Theorem 2.2: *if f is a $\underline{presh}(X, \mathcal{T})$ morphism, then $\underline{\mathrm{L}}f$ is a $\underline{sheafsp}(X, \mathcal{T})$ morphism.*

Proof: lemmas 2.5 and 2.6. □

Theorem 2.3: *the construction of sheaf space $(\underline{\mathrm{L}}F, p_F)$ from presheaf F and sheaf space morphism $\underline{\mathrm{L}}f$ from sheaf morphism f determine a functor*

$$\underline{\mathrm{L}}: \underline{presh}(X, \mathcal{T}) \to \underline{sheafsp}(X, \mathcal{T}).$$

Proof: the lemma is demonstrated if we show that the $\underline{\mathrm{L}}$ morphism construction preserves identities and composition. Suppose a $\underline{presh}(X, \mathcal{T})$ identity map $id_F: F \to F$. The $\underline{\mathrm{L}}$ morphism construction preserves identities if

$$\underline{\mathrm{L}}(id_F) = id_{(\underline{\mathrm{L}}F, p_F)}.$$

Now, for any $s_x \in \underline{\mathrm{L}}F$, we have

$$\underline{\mathrm{L}}(id_F)(s_x) = \big((id_U)(s)\big)_x = s_x$$

while

$$id_{(\underline{\mathrm{L}}F, p_F)}(s_x) = (id_{\underline{\mathrm{L}}F})(s_x) = s_x.$$

Suppose a $\underline{presh}(X, \mathcal{T})$ composite $F \xrightarrow{f} F' \xrightarrow{g} F''$. The $\underline{\mathrm{L}}$ morphism construction preserves composition if

$$\underline{\mathrm{L}}(g \cdot f) = \underline{\mathrm{L}}g \cdot \underline{\mathrm{L}}f.$$

Now, for any $s_x \in \underline{\mathrm{L}}F$,

$$\underline{\mathrm{L}}(g \cdot f)(s_x) = (g \cdot f)_x(s_x) = (g_x \cdot f_x)(s_x)$$

while

$$(\underline{L}g \cdot \underline{L}f)(s_x) = \underline{L}g\big(\underline{L}f(s_x)\big)$$
$$= g_x\big(f_x(s_x)\big)$$
$$= (g_x \cdot f_x)(s_x). \qquad \square$$

3. From Sheaf Spaces to Sheaves

In this section we describe the construction of a sheaf ΓE given a sheaf space (E, p) on X. We describe the construction of a sheaf morphism $\Gamma g \colon \Gamma E \to \Gamma E'$ given a sheaf space morphism $g \colon (E, p) \to (E', p')$. We finish this section with a demonstration that these constructions describe a functor

$$\Gamma \colon sheafsp(X, \mathcal{T}) \to sh(X, \mathcal{T}).$$

This functor is readily restricted to a functor

$$\underline{\Gamma} \colon \underline{sheafsp}(X, \mathcal{T}) \to \underline{sh}(X, \mathcal{T})$$

and in the next section we demonstrate the main result of this chapter that $\underline{L}$ restricted to sheaves and $\underline{\Gamma}$ are an equivalence of categories.

Suppose a sheaf space (E, p) on X. We construct a functor $\Gamma E \colon \mathcal{T}^{op} \to \mathbf{Set}$ and go on to demonstrate that ΓE is a sheaf.

Construction 3.1: define $\Gamma E \colon \mathcal{T}^{op} \to \mathbf{Set}$ by allowing that for any closed $U \subseteq X$

$$\Gamma E(U) = \{\text{all sections } s \text{ of } p \text{ over } U\};$$

and when $V \subseteq U$ in $\mathcal{T}$, the restriction map $(\Gamma E)^U_V$ is given by

$$s \mapsto s \mid V.$$

Theorem 3.1: *ΓE is a sheaf.*

Proof: suppose closed set U and set $\{U_i: i \in I\}$ of closed sets such that $U = \bigcap\{U_i: i \in I\}$. Suppose that we have $\{s_i \in (\Gamma E)(U_i): i \in I\}$ such that

$$(\Gamma E)^{U_i}_{U_i \cap U_j}(s_i) = (\Gamma E)^{U_j}_{U_i \cap U_j}(s_j)$$

all $i, j \in I$. This allows us to define a map $s: U \to E$ by setting $s(x) = s_i(x)$ whenever $x \in U_i$. The theorem is demonstrated if we show that $s \in \Gamma E(U)$ and in unique in making $(\Gamma E)^U_{U_i}(s) = s_i$ all $i \in I$. First of all, $s \in \Gamma E(U)$ if s is continuous and $p \cdot s = id_U$. The fact that $p \cdot s = id_U$ follows directly from the fact that $p \cdot s_i = id_{U_i}$ all $i \in I$. Now, by definition, for any closed $N \subseteq E$

$$s^{-1}(N \cap s(U)) = \bigcup\{s_i^{-1}(N \cap s_i(U_i)): i \in I\};$$

and since each s_i is continuous and, by lemma 1.2, a closed map, $s^{-1}(N \cap s(U))$ is a union of closed sets. Now, I may not be finite, but each $s_i^{-1}(N \cap s_i(U_i))$ must be a subset of U_i which, by hypothesis, is finite, so $s^{-1}(N \cap s(U))$ can be represented as the union of some finite subset of $\{s_i^{-1}(N \cap s_i(U_i)): i \in I\}$. It follows that s is continuous.

Now, it follows by definition of s that $(\Gamma E)^U_{U_i}(s) = s_i$ all $i \in I$ but suppose there is some further $s' \in \Gamma E(U)$ such that $(\Gamma E)^U_{U_i}(s') = s_i$ all $i \in I$. The fact that $\{U_i: i \in I\}$ covers U and that each s_i is a bijective function requires that $s = s'$.

□

Suppose a sheaf space morphism $g: (E, p) \to (E', p')$. We construct maps

$$(\Gamma g)_U: \Gamma E(U) \to \Gamma E'(U)$$

for each closed $U \in \mathcal{T}$ and go on to demonstrate that $\{(\Gamma g)_U: U \in \mathcal{T}\}$ constitutes a natural transformation $\Gamma g: \Gamma E \to \Gamma E'$.

Construction 3.2: for any closed $U \subseteq X$ define $(\Gamma g)_U : \Gamma E(U) \to \Gamma E'(U)$ by

$$\Gamma E(U) \ni s \mapsto g \cdot s.$$

That $g \cdot s \in \Gamma E'(U)$ follows from that fact that both g and s are continuous and from the fact that $p \cdot s = id_U$ together with $p' \cdot g = p$ means that $p' \cdot g \cdot s = id_U$.

Theorem 3.2: $\{(\Gamma g)_U : U \in \mathcal{T}\}$ *constitutes a natural transformation.*

Proof: the theorem is demonstrated if whenever $V \subseteq U$ in $\mathcal{T}$, the following diagram commutes.

$$\begin{array}{ccccc}
U & \qquad & \Gamma E(U) & \xrightarrow{(\Gamma g)_U} & \Gamma E'(U) \\
\uparrow & & {\scriptstyle (\Gamma E)^U_V}\downarrow & & \downarrow{\scriptstyle (\Gamma E')^U_V} \\
V & & \Gamma E(V) & \xrightarrow[(\Gamma g)_V]{} & \Gamma E'(V)
\end{array}$$

But this holds since for any $s \in \Gamma E(V)$

$$\begin{aligned}
(\Gamma E')^U_V\big((\Gamma g)_U(s)\big) &= (\Gamma E')^U_V(g \cdot s) \\
&= (g \cdot s) \mid V
\end{aligned}$$

and

$$\begin{aligned}
(\Gamma g)_V\big((\Gamma E)^U_V(s)\big) &= (\Gamma g)_V(s \mid V) \\
&= g \cdot (s \mid V).
\end{aligned}$$

Since g and s are functions

$$(g \cdot s) \mid V = g \cdot (s \mid V). \qquad \square$$

Theorem 3.3: *the construction of sheaf ΓE from sheaf space (E, p) and sheaf morphism Γg from sheaf space morphism g determine a functor*

$$\Gamma: sheafsp(X, \mathcal{T}) \to sh(X, \mathcal{T}).$$

Proof: the theorem is demonstrated if we show that the Γ morphism construction preserves identities and compositions. So, suppose a $sheafsp(X, \mathcal{T})$ identity $id_{(E,p)}: (E, p) \to (E, p)$. Identities are preserved if

$$\Gamma(id_{(E,p)}) = id_{\Gamma E}.$$

This holds only when, for all $U \in \mathcal{T}$,

$$\big(\Gamma(id_{(E,p)})\big)_U = (id_{\Gamma E})_U.$$

But, for any $s \in (\Gamma E)(U)$,

$$\begin{aligned}\big(\Gamma(id_{(E,p)})\big)_U(s) &= id_E \cdot s \\ &= s \\ &= (id_{\Gamma E})_U(s).\end{aligned}$$

Suppose now a $sheafsp(X, \mathcal{T})$ composite $(E, p) \xrightarrow{f} (E', p') \xrightarrow{g} (E'', p'')$. The Γ morphism construction preserves composition if

$$\Gamma(g \cdot f) = \Gamma g \cdot \Gamma f.$$

This holds only when, for all $U \in \mathcal{T}$,

$$\big(\Gamma(g \cdot f)\big)_U = (\Gamma g)_U \cdot (\Gamma f)_U.$$

But, for any $s \in (\Gamma E)(U)$

$$\big(\Gamma(g \cdot f)\big)_U(s) = g \cdot f \cdot s$$

while

$$\begin{aligned}(\Gamma g)_U \cdot (\Gamma f)_U(s) &= (\Gamma g)_U\big((\Gamma f)_U(s)\big)\\ &= (\Gamma g)_U(f \cdot s)\\ &= g \cdot (f \cdot s).\end{aligned}$$ □

It remains true that in producing a sheaf ΓE from sheaf space (E, p) we have accepted and used a restricted topology on the base space X, but note that we have required no restriction on E. Later we shall have need of a restricted domain Γ. Plainly if we restrict the domain to sheaf spaces (E, p) where E is finite, we can define a functor

$$\underline{\Gamma}: \underline{sheafsp}(X, \mathcal{T}) \to \underline{sh}(X, \mathcal{T}).$$

4. Equivalence of Categories

With this section we demonstrate that the functor $\underline{L}$ restricted to sheaves and the functor $\underline{\Gamma}$ are an equivalence of categories for $\underline{sh}(X, \mathcal{T})$ and $\underline{sheafsp}(X, \mathcal{T})$. This is demonstrated by showing a natural isomorphism

$$\underline{\Gamma} \cdot \big(\underline{L} \mid \underline{sh}(X, \mathcal{T})\big) \cong id_{\underline{sheafsp}(X,\mathcal{T})}$$

and a natural isomorphism

$$\big(\underline{L} \mid \underline{sh}(X, \mathcal{T})\big) \cdot \underline{\Gamma} \cong id_{\underline{sh}(X,\mathcal{T})}.$$

The demonstration proceeds in three parts. We first show that for any sheaf space (E, p) in $\underline{sheafsp}(X, \mathcal{T})$, there is a $\underline{sheafsp}(X, \mathcal{T})$ isomorphism

$$k_E: (E, p) \to (\underline{L}\underline{\Gamma}E, p_{\underline{\Gamma}E}).$$

Secondly we show that for any sheaf F in $\underline{presh}(X, \mathcal{T})$, there is a $\underline{presh}(X, \mathcal{T})$ isomorphism

$$h_F: F \to \underline{\Gamma}\underline{L}F.$$

Thirdly we show that these isomorphisms constitute the required natural transformations. To begin with, we give a technical lemma that is needed for the next lemma.

Lemma 4.1: *for topological spaces X and Y and inverse maps $k: X \to Y$ and $k': Y \to X$, if both k and k are closed maps, then both k' and k are continuous.*

Proof: suppose k is not continuous; that is, suppose there is at least one closed $U \subseteq Y$ such that $k^{-1}(U)$ is not closed in X. But k' is k^{-1} and k' is a closed map. This means that wherever U is closed in Y, $k'(U)$, and therefore $k^{-1}(U)$, is closed in X. Map k is continuous. The same proof applies for k' when k is a closed map.

□

Lemma 4.2: *for any (E, p) in $\underline{sheafsp}(X, \mathcal{T})$ there is an isomorphism*

$$k_E: (E, p) \to (\underline{\mathrm{L}}\underline{\Gamma} E, p_{\underline{\Gamma} E}).$$

Proof: a sheaf space isomorphism is a continuous isomorphism $k: E \to \underline{\mathrm{L}}\underline{\Gamma} E$ such that $p = p_{\underline{\Gamma} E} \cdot k$. We construct two maps, $k: E \to \underline{\mathrm{L}}\underline{\Gamma} E$ and $k': \underline{\mathrm{L}}\underline{\Gamma} E \to E$, and show them to be bijective inverses and both closed maps. This, together with lemma 4.1, gives us continuous isomorphism k. We show also for our constructed k that $p = p_{\underline{\Gamma} E} \cdot k$.

Consider any $e \in E$. Let $s: U \to E$ and $s': U' \to E$ be any two sections of p such that $e \in s(U)$ and $e \in s'(U')$. By definition $s \in \underline{\Gamma} E(U)$ and $s' \in \underline{\Gamma} E(U')$. Now, s and s' have overlapping images; that is, $s(U) \cap s'(U')$ is not empty. Consider then the set $p\big(s(U) \cap s'(U')\big)$. It will be the case that

$$s \,|\, p\big(s(U) \cap s'(U')\big) = s' \,|\, p\big(s(U) \cap s'(U')\big)$$

only if, for any $x \in p\big(s(U) \cap s'(U')\big)$, $s(x) = s'(x)$. Note that we must have both $s(x)$ and $s'(x)$ in $s(U) \cap s'(U')$ since s and $p \,|\, s(U)$ are bijective inverses as are s'

and $p\,|\,s'(U')$. But since s and $p\,|\,s(U)$ are bijective inverses we have that for any b and b' in $s(U)$, if $\big(p\,|\,s(U)\big)(b) = \big(p\,|\,s(U)\big)(b')$, then $b = b'$. The same holds with respect to s' and any $b, b' \in s'(U')$. It follows that for any $x \in p\big(s(U) \cap s'(U')\big)$, $s(x) = s'(x)$, since $p(s(x)) = p(s'(x)) = x$. In other words, s and s' are identical when restricted to $p\big(s(U) \cap s'(U')\big)$. Now, by lemma 1.2, both s and s' are closed maps, so $s(U) \cap s'(U')$ is closed in E. And since both s and s' are continuous, $s^{-1}\big(s(U) \cap s'(U')\big)$ as well as $(s')^{-1}\big(s(U) \cap s'(U')\big)$ are closed in X. But

$$s^{-1}\big(s(U) \cap s'(U')\big) = (s')^{-1}\big(s(U) \cap s'(U')\big) = p\big(s(U) \cap s'(U')\big),$$

so there is a closed set in X, namely $p\big(s(U) \cap s'(U')\big)$, restricted to which s and s' are identical. In other words, there is a $W \in \mathcal{T}$ such that

$$(\underline{\Gamma}E)^U_W(s) = (\underline{\Gamma}E)^{U'}_W(s').$$

It also happens that, since $e \in s(U) \cap s'(U')$, $p(e) \in W$. It follows that where s is some section of p with $e \in \mathrm{cod}(s)$, then $s \sim_{p(e)} s'$ for all other sections s' of p with $e \in \mathrm{cod}(s')$. Therefore, we may define a map $k\colon E \to \underline{\mathrm{L}\Gamma}E$ by

$$e \mapsto s_{p(e)}$$

where s is any section of p with $e \in \mathrm{cod}(s)$.

We now show that k is injective. Suppose $e, e' \in E$ such that $e \neq e'$. If $e \in p^{-1}(x)$ and $e' \in p^{-1}(y)$ such that $x \neq y$, then it is automatically the case that $k(e) \neq k(e')$ since $\underline{\mathrm{L}\Gamma}E$ is the disjoint union of stalks $(\underline{\Gamma}E)_x$ all $x \in X$. Suppose, then, that $e, e' \in p^{-1}(x)$. Suppose sections $s\colon U \to E$ and $t\colon V \to E$ such that $e \in s(U)$ and $e' \in t(V)$. By definition, $s(x) = e$ and $t(x) = e'$, so $s(x) \neq t(x)$. It follows that there can be no $W \in \mathcal{T}$ such that $x \in W$ and

$$(\underline{\Gamma}E)^U_W(s) = (\underline{\Gamma}E)^V_W(t).$$

It follows, then, that $s_{p(e)} \neq t_{p(e')}$; or in other words, $k(e) \neq k(e')$.

We now show that k is a closed map. By lemma 1.4, k is a closed map if for any section $s: U \to E$ of p, $k(s(U))$ is closed in $\underline{L\Gamma E}$. Now, by definition of k, we have that $k(e) = s_{p(e)}$ for any $e \in s(U)$, so

$$k(s(U)) = \{s_{p(e)}: e \in s(U)\}.$$

Now, $p(s(U)) = U$, so

$$k(s(U)) = \{s_{p(e)}: p(e) \in U\}$$

but this is the set $\hat{s}(U)$. By construction 2.1, then, $k(s(U))$ is closed in $\underline{L\Gamma E}$.

We now define a map $k': \underline{L\Gamma E} \to E$ which we show to be an injective closed map. We show that k' is the inverse of k. Since both k and k' are, then, known to be injective, both k and k' are seen to be bijections.

Any element of $\underline{L\Gamma E}$ is s_x for some $s \in (\underline{\Gamma}E)(U)$ with $x \in U$. Now, for any $s' \in (\underline{\Gamma}E)(U')$ we have $s \sim_x s'$, and therefore $s_x = s'_x$, only if there is some $W \subseteq U \cap U'$ with $x \in W$ over which s and s' agree. In particular, if $s \sim_x s'$, then $s(x) = s'(x)$. It follows that we may define a map $k': \underline{L\Gamma E} \to E$ by

$$s_x \mapsto s(x).$$

We now show that k' is injective. Suppose sections $s \in (\underline{\Gamma}E)(U)$ and $t \in (\underline{\Gamma}E)(V)$ such that $x \in U, V$. By definition of sections we have that

$$s \mid p(s(U) \cap t(V)) = t \mid p(s(U) \cap t(V))$$

with $p(s(U) \cap t(V))$ being a closed set. Now, if $s(x) = t(x)$, then there is some W, namely $p(s(U) \cap t(V))$, such that

$$(\underline{\Gamma}E)^U_W(s) = (\underline{\Gamma}E)^V_W(t)$$

with $x \in W \subseteq U \cap V$; in other words, $s \sim_x t$, so $s_x = t_x$.

We now show that k' is a closed map. By lemma 2.2, it is enough to show that $k\big(\hat{s}(U)\big)$ for any $s \in \underline{\Gamma}E(U)$, is closed in E. But

$$\begin{aligned} k\big(\hat{s}(U)\big) &= \{s(x) \colon x \in U\} \\ &= s(U). \end{aligned}$$

So, by lemma 1.4, $k\big(\hat{s}(U)\big)$ is closed in E.

We now show that k' is the inverse of k. This is straightforward. For any $e \in E$, $k(e) = s_{p(e)}$ where $e \in \mathrm{cod}(s)$. Recall that k is injective. It follows that $k^{-1}\big(k(e)\big) = e$. Now $k'\big(k(e)\big) = s\big(p(e)\big)$, and since s is a section, $s\big(p(e)\big) = e$. Furthermore, for any $s_x \in \underline{\mathrm{L}}\underline{\Gamma}E$, $k'(s_x) = s(x)$ and, since obviously $s(x) \in \mathrm{cod}(s)$, $k\big(k'(s_x)\big) = s_{p(s(x))} = s_x$.

It follows that $k \colon E \to \underline{\mathrm{L}}\underline{\Gamma}E$ is a continuous isomorphism. To complete the demonstration of the lemma observe that for any $e \in E$ if $p(e) = x$, then $k(e)$ is $s_{p(e)}$ which is s_x for some s with $e \in \mathrm{cod}(s)$, and, by definition $p_{\underline{\Gamma}E}\big(k(e)\big) = x$. It follows that

$$p = p_{\underline{\Gamma}E} \cdot k. \qquad \square$$

Lemma 4.3: *for any sheaf F in $\underline{presh}(X, \mathcal{T})$ there is an isomorphism*

$$h_F \colon F \to \underline{\Gamma}\underline{\mathrm{L}}F.$$

Proof: for any $U \in \mathcal{T}$ define a function $(h_F)_U \colon F(U) \to \underline{\Gamma}\underline{\mathrm{L}}F(U)$ so that for any $s \in F(U)$

$$(h_F)_U(s) = \hat{s}.$$

The lemma is demonstrated if we show that each $(h_F)_U$ is a bijection and that for any $U \subseteq V$ in $\mathcal{T}$, the following diagram commutes.

$$\begin{array}{ccccc} U & & F(U) & \xrightarrow{(h_F)_U} & (\underline{\Gamma}\underline{\mathrm{L}}F)(U) \\ \uparrow & & \Big\downarrow F^U_V & & \Big\downarrow (\underline{\Gamma}\underline{\mathrm{L}}F)^U_V \\ V & & F(V) & \xrightarrow[(h_F)_V]{} & (\underline{\Gamma}\underline{\mathrm{L}}F)(V). \end{array}$$

First, we demonstrate that the diagram commutes. For any $s \in F(U)$

$$\begin{aligned}(\Gamma\underline{\mathrm{L}}F)^U_V((h_F)_U(s)) &= (\Gamma\underline{\mathrm{L}}F)^U_V(\hat{s}) \\ &= \hat{s} \mid V\end{aligned}$$

and

$$\begin{aligned}(h_F)_V(F^U_V(s)) &= (h_F)_V(s \mid V) \\ &= \widehat{s \mid V}.\end{aligned}$$

Now, $F^U_V(s) = F^V_V(s \mid V)$, so $s \sim_x (s \mid V)$ and $s_x = (s \mid V)_x$ for any $x \in V$. It follows that

$$\hat{s} \mid V = \widehat{s \mid V}.$$

We now demonstrate that any $(h_F)_U$ is injective. Any section $\hat{s} \in \Gamma\underline{\mathrm{L}}F(U)$ is given by $\hat{s}(x) = s_x$ for all $x \in U$, so for $\hat{s}, \hat{t} \in \Gamma\underline{\mathrm{L}}E(U)$ we have $\hat{s} = \hat{t}$ only if $s_x = t_x$ all $x \in U$. But, in that case, for each $x \in U$ there must be some $W \in \mathcal{T}$ such that $x \in W$ and $F^U_W(s) = F^U_W(t)$. It follows that U is covered by these sets W and, when F is a sheaf, $s = t$.

We now demonstrate that any $(h_F)_U$ is surjective. This is the demonstration that for any $e \in \Gamma\underline{\mathrm{L}}F(U)$, there is some $s \in F(U)$ such that $e = \hat{s}$. If $e \in \Gamma\underline{\mathrm{L}}F(U)$, then e is some section $U \to \underline{\mathrm{L}}F$ of p_F. As such, e is a closed map making $e(U)$ closed in $\underline{\mathrm{L}}F$. It follows that $e(U)$ is some finite union

$$\bigcup\{\widehat{s_i}(U_i) : i \in I\}$$

where $\widehat{s_i} \in \Gamma\underline{\mathrm{L}}F(U_i)$ all $i \in I$. It follows from the definition of p_F that

$$\bigcup\{U_i : i \in I\} = U.$$

Suppose now that for some $x \in U_i \cap U_j$, we have $\widehat{s_i}(x) \neq \widehat{s_j}(x)$. Since e is a section, $p|e(U)$ is the bijective inverse of e, so

$$(p|e(U))(\widehat{s_i}(x)) \neq (p|e(U))(\widehat{s_j}(x)).$$

But, this is the claim that $p(\widehat{s_i}(x)) \neq p(\widehat{s_j}(x))$ and since $\widehat{s_i}$ and $\widehat{s_j}$ are sections, it must be that $p(\widehat{s_i}(x)) = p(\widehat{s_j}(x)) = x$. In other words, we can characterise e by $e(x) = \widehat{s_i}(x)$ for all $x \in U_i$ any $i \in I$. Another way of putting this is that

$$\widehat{s_i} \mid U_i \cap U_j = \widehat{s_j} \mid U_i \cap U_j, \quad \text{all } i, j \in I.$$

It follows, when F is a sheaf, that

$$F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j) \quad \text{all } i, j \in I.$$

Now, since the sets U_i cover U, if F is a sheaf, there is exactly one $s \in F(U)$ for which $F^U_{U_i}(s) = s_i$ all $i \in I$. So, for any U_i and any $x \in U_i$, we have

$$F^U_{U_i}(s) = F^{U_i}_{U_i}(s_i),$$

which is to say, $(s_i)_x = s_x$. It follows that e is the map $\hat{s}: U \to \underline{\mathrm{L}}F$ determined by that $s \in F(U)$ for which $F^U_{U_i}(s) = s_i$, all $i \in I$. Since there is exactly one such s, $(h_F)_U$ is surjective. □

Theorem 4.1: *functors $\underline{\Gamma}$ and $\underline{\mathrm{L}} \mid (\underline{sh}(X, \mathcal{T}))$ are an equivalence of categories.*

Proof: for this proof let $\underline{\mathrm{L}'}$ be the functor $\underline{\mathrm{L}}$ restricted to the sheaves of $\underline{presh}(X, \mathcal{T})$. Functors $\underline{\mathrm{L}'}$ and $\underline{\Gamma}$ are an equivalence of categories if there are natural isomorphims

$$\underline{\mathrm{L}'\Gamma} \cong id_{\underline{sheafsp}(X,\mathcal{T})} \quad \text{and} \quad \underline{\Gamma \mathrm{L}'} \cong id_{\underline{sh}(X,\mathcal{T})}.$$

From lemmas 4.2 and 4.3 we have the isomorphisms $k_E: (E, p) \to \underline{\mathrm{L}\Gamma}E$ for sheaf space (E, p) and $h_F: F \to \underline{\Gamma \mathrm{L}}F$ for sheaf F. The present theorem is established if

$$\{k_E: (E, p) \text{ in } \underline{sheafsp}(X, \mathcal{T})\} \quad \text{and} \quad \{h_F: F \text{ in } \underline{sh}(X, \mathcal{T})\}$$

constitute natural transformations.

The collection $\{h_F: F \text{ in } \underline{sh}(X, \mathcal{T})\}$ constitutes a natural transformation if whenever $f: F \to F'$ is a $\underline{sh}(X, \mathcal{T})$ morphism, the following diagram commutes.

$$\begin{array}{ccc} F & \xrightarrow{h_F} & \underline{\Gamma}\underline{\mathrm{L}}F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \underline{\Gamma}\underline{\mathrm{L}}(f)} \\ F' & \xrightarrow[h_{F'}]{} & \underline{\Gamma}\underline{\mathrm{L}}F'. \end{array}$$

The diagram commutes if the component diagrams for each $U \in \mathcal{T}$ commute. Now, for any $s \in F(U)$

$$\begin{aligned} \big(\underline{\Gamma}\underline{\mathrm{L}}(f)\big)_U\big((h_F)_U(s)\big) &= \big(\underline{\Gamma}\underline{\mathrm{L}}(f)\big)_U(\hat{s}) \\ &= (\underline{\mathrm{L}}f) \cdot \hat{s} \end{aligned}$$

while

$$(h_{F'})_U\big(f_U(s)\big) = \widehat{f_U(s)}.$$

Now, $(\underline{\mathrm{L}}f) \cdot \hat{s}$ is a map $U \to \underline{\mathrm{L}}F'$ where for any $x \in U$, we have $\big((\underline{\mathrm{L}}f) \cdot \hat{s}\big)(x) = \big(f_U(s)\big)_x$. In other words,

$$(\underline{\mathrm{L}}f) \cdot \hat{s} = \widehat{f_U(s)}.$$

The collection $\{k_E\colon (E,p) \text{ in } \underline{sheafsp}(X,\mathcal{T})\}$ constitutes a natural transformation if, whenever $g\colon (E,p) \to (E',p')$ is a $\underline{sheafsp}(X,\mathcal{T})$ morphism, the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{k_E} & \underline{\mathrm{L}}\underline{\Gamma}E \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \underline{\mathrm{L}}\underline{\Gamma}(g)} \\ E' & \xrightarrow[k_{E'}]{} & \underline{\mathrm{L}}\underline{\Gamma}E' \end{array}$$

Now, for any $e \in E$, $k_E(e) = s_{p(e)}$ where s is some section of p with $e \in \mathrm{cod}(s)$. Assume s is a section over U. Now, for any $s_x \in \underline{\mathrm{L}}\underline{\Gamma}F$ with $s \in (\underline{\Gamma}E)(U)$ and $x \in U$

$$\begin{aligned} \big(\underline{\mathrm{L}}\underline{\Gamma}(g)\big)(s_x) &= (\underline{\Gamma}g)_x(s_x) \\ &= \big((\underline{\Gamma}g)_U(s)\big)_x. \end{aligned}$$

It follows, then, that

$$\begin{aligned} \big(\underline{\mathrm{L}}\underline{\Gamma}(g)\big)\big(k_E(s)\big) &= \big((\underline{\Gamma}g)_U(s)\big)_{p(e)} \\ &= (g \cdot s)_{p(e)}. \end{aligned}$$

We also have the following

$$k_{E'}\big(g(e)\big) = s'_{p'(g(e))}$$

where s' is some section of p' with $g(e) \in \mathrm{cod}(s')$. Now, both g and s are continuous, so $g \cdot s$ is continuous. Furthermore, since $p = p' \cdot g$, we have $p' \cdot g \cdot s = id_U$. And plainly, $g(e) \in (g \cdot s)(U)$ when $e \in s(U)$. It follows that $g \cdot s$ is a section of p' with $g(e) \in \mathrm{cod}(g \cdot s)$. It follows, too, that

$$s'_{p'(g(e))} = (g \cdot s)_{p'(g(e))}$$

and since $p'(g(e)) = p(e)$, we have

$$\big(\underline{\mathrm{L}\Gamma}(g)\big)\big(k_E(s)\big) = k_{E'}\big(g(e)\big)$$

as required. □

In summary:

A restricted class of sheaves over closed sets is provably equivalent to a restricted class of sheaf spaces over closed sets.

This chapter marks the end of Part III and our discussion of the properties of sheaves as objects of paraconsistent semantics. With the next chapter we begin a discussion of categories themselves as objects on which we may model theories.

Part IV:

THEORIES and RELEVANCE

CHAPTER 14:

INCONSISTENT THEORIES IN CATEGORIES

Introduction: With this chapter we continue our interest in categories and the semantics of paraconsistent logics. In particular we will be concerned to show how we may describe categories as semantic objects for inconsistent theories. In the first instance we develop the usual notion of a category as a suitable semantics object for a theory in a many sorted language. This is the idea that sorts in a language can be modelled by objects of a category and then formulae in such languages can be modelled by soubobject of particular sort models. We then speak of a model for a theory as a functor from the language to the category. The ability to model inconsistent theories then becomes, at least, the ability to describe an adequate notion of sets of designated values (if our semantic objects have more than one designated value per lattice, then we have the possibility of a formula and its negation receiving (different) designated values). Alternatively we can seek out categories with BrA subobject algebras and use these as models. This idea is usefully combined with the first, but is itself lacking to some extent in that those BrA subobject algebras we have discovered to date lack the pleasing categorial property of naturalness with respect to other such algebras in the same category. A third idea, and one that we shall pursue, calls for the use of lattice dualisation. We have a standard result that any subobject lattice in a topos is a HA, so it follows that we have something of a plethora of opportunities to produce BrAs by dualisation. There is, however, a sense that using dualised subobject lattices in this way is not quite the same as straightforwardly modelling a theory in a category. We will therefore use the tool of language, rather than lattice. dualisation. This is the idea that we model formulae in a language on a structure by allowing that a formula φ recieve a value that, in a standard model, would be given to the dual formula φ^{op} where φ and φ^{op} are dual in the sense that wherever there is an *and* connective in φ there is an *or* connective in φ^{op}, and so on. We will give a complete description of what constitutes a language dualisation in section two of the present chapter however the claim that a language dualisation on a standard model amounts to the same thing as a lattice dualisation in the model should be understandable.

This chapter is an explicit statement of the effect of the application of a language dualisation to the notion of topos logic as a sequent system. The kind of dualisation here called language dualisation is different from both "false" for "true" dualisation discussed in chapter 4 and ordinary categorial dualisation. Language dualisation applies to formulae in a logical language and is motivated by standard notions of algebraic duality. So a language dualisation of a formula φ produces a formula φ^{op} where $\wedge$ replaces any instance of $\vee$ (and vice versa) and $\dot{-}$ replaces any instance of $\Rightarrow$ (and vice versa). Given this notion of language dualisation I develop the details of modelling formulae in toposes $\mathcal{E}$ by allowing that where a standard model assigns some topos arrow or object to formula φ, we assign that object or arrow to φ^{op}. This is analogous to using open set topologies as semantic objects for closed set logics (by including a dualisation function in the interpretation function). In the absence of what we might call a co-topos, a topos-like category with natural BrA structures, some manipulation of the sort described in chapter 14 is needed if we are to model inconsistent theories in categories in the same way that we model such consistent theories as we do in toposes. At least some explicit working out of the details of such manipulated notions is needed since the idea that what we are working with is a deduction system is affected by these manipulations. Explicitly, by applying a language dualisation to a sequent system and models for languages we produce a system that preserves falsehood rather than truth. So the philosophical significance of chapter 14 is twofold: firstly I have produced a method of modelling inconsistent theories in toposes, and secondly I have made explicit the nature of the inference system that goes with these models.

A consequence of this notion of language dualisation is that, given a language $\mathcal{L}$ and its dual $\mathcal{L}^{op}$, any model of a theory T in language $\mathcal{L}$ amounts to a model of the dual theory T^{op} in $\mathcal{L}^{op}$ and vice versa. Furthermore there are significant consequences in terms of proof theories. Given a notion of language dualisation for models and some proof theory for theories T in language $\mathcal{L}$, we can develop what we may call a dual proof theory for theories T^{op} in $\mathcal{L}^{op}$. The idea is that straightforward language dualisation of a proof system produces a disproof system. With respect to categories as semantic objects the standard proof system is a Gentzen system called geometric logic, or GL. In section three we will apply our language dualisation to GL and produce a system that we call co-GL and which is best understood as being a system that from falsehoods derives falsehoods. This leads us to the principal claim of this chapter: that we can model inconsistent theories by providing what we call refutation models. This is the idea that inconsistent theories can be characterised by collections of formulae that are undeniably false. Consider, for example, the theory of classical arithmetic. Let $\overline{P}$ be the collection of all well formed formulae that are false in standard models for classical arithmetic. Let $\overline{P}_1$ be a proper subset of $\overline{P}$ and allow that only $\overline{P}_1$ sentences are undeniably false. Among the various models of a theory T under which all of $\overline{P}_1$ are false, there will be some (non-classical) models for which only the sentences of $\overline{P}_1$ are false. We may describe T as an inconsistent theory characterised by falsehoods $\overline{P}_1$ and wherever T has a model that falsifies all of $\overline{P}_1$, we say that T is modelled by a refutation model of $\overline{P}_1$. The principal idea behind refutation models and disproof systems like co-GL is that a set of falsehoods closed under falsehood preservation rules allows for the claim that all other well formed sentences are true (or at least designated) even if, classically, some of them would be false (or undesignated). This seems a natural philosophical dual of the Intuitionist principle that the lack of a proof of the truth of a sentence is not a proof of its falsehood.

The use of categories as semantic objects for many sorted languages is not solely a model theoretic exercise. The categorisation of the task is extended by the construction of categories $\mathcal{C}_{\mathrm{T}}$ with respect to theories T and the demonstration that models of T in Grothendieck toposes $\mathcal{E}$ correspond to continuous morphisms between sites $(\mathcal{C}_{\mathrm{T}}, cov)$ and $\mathcal{E}$. Our discussion in this chapter will not go this far. The language dualisation program is wholly amenable to expression in this form but is not wholly motivated. The usual reason for developing the notion of a theory modelled in a category is to begin discussing category theory itself in terms of the language of models. Then it is possible to describe categories in terms of formulae of the language that hold in those categories. The language dualisation program actually hampers such discussion in that both a formula and its dual would describe the same feature of a category but the dual would be a description at one remove for having been dualised. The point of language dualisation of models in this chapter is to demonstrate the types of concerns we will have when and if we find categories with natural BrA subobject structures and start to use them as standard semantic objects for standard (inconsistent) theories. In that such a situation would require some proof system other than the classical or the intuitionist a chapter such as the present one has a useful role: by dualisation of existing category based proof systems we can discover such systems as will be useful in non-standard settings.

With section one we give a formalisation of a many sorted language and briefly describe the details of interpretation of the language within a topos. With section two we develop the disproof system co-GL and its relation to inconsistent theories.

1. Many Sorted Languages

An *elementary language* consists of primitive symbols together with a collection of variables. Such languages are called single-sorted when the variables range over elements of what is intuitively one sort or type, namely, those from just one interpreting set. *Many-sorted* languages are those whose variables require interpretation in terms of more than one set or structure. These are the languages used to formalise, for example, scalar multiplication of vectors.

Let $\mathcal{S}$ be a class whose members will be called *sorts*. An $\mathcal{S}$-sorted language $\mathcal{L}$ will require one denumerable set of variables V_a for each $a \in \mathcal{S}$ such that if $a, b \in \mathcal{S}$ and $a \neq b$, then V_a and V_b are disjoint. When $v \in \mathrm{V}_a$ we write $v{:}\,a$. For our $\mathcal{S}$-sorted language there will be a basic connective alphabet of

(i) propositional connectives: $\bigwedge, \bigvee, \neg, \ulcorner, \dot{-}, \Rightarrow$;

(ii) quantifiers: $\forall, \exists$;

(iii) identity: $\approx$.

We also include parenthesis devices) and (.

Furthermore there shall be

(iv) individual constants $\mathbf{c}$ that are matched with sorts. The sort of $\mathbf{c}$ is denoted by $\mathbf{c}{:}\,a$;

(v) relation symbols $\mathbf{R}$ that are assigned a natural number n, called its *number of places*, and a sequence of sorts $\langle a_1, ..., a_n \rangle$. This is denoted by $\mathbf{R}{:}\,\langle a_1, ..., a_n \rangle$;

(vi) operation symbols $\mathbf{g}$ that have a *number of places* n and a sequence of sorts $\langle a_1, ..., a_n, a_{n+1} \rangle$. This is denoted by $\mathbf{g}{:}\,\langle a_1, ..., a_n \rangle \rightarrow a_{n+1}$.

An $\mathcal{S}$-sorted language $\mathcal{L}$ is then a collection of sorted variables together with a collection of operation, relation, and logical symbols, and individual constants.

Terms of a language are expressions within it denoting individuals. For a many-sorted language terms are always terms of a given sort. For $\mathcal{L}$ the *terms* are the variables, constants, and operation statements.

The *atomic formulae* include the identity expressions and the relation statements. Identity expresions are of the form $t \approx u$ where t, u are terms of the same sort. Relation statements are $\mathbf{R}(t_1, ..., t_n)$ where for $\mathbf{R}$: $\langle a_1, ..., a_n \rangle$ the terms $t_1, ..., t_n$ are of sorts $a_1, ..., a_n$ respectively.

General formulae are built inductively following the rules

(i) any atomic formula is a formula;

(ii) if φ and ψ are formulae, then so are $(\varphi \Rightarrow \psi), (\varphi \dot{-} \psi), (\neg\varphi)$ and $(\ulcorner\varphi)$;

(iii) if Θ is a set of formulae, then $\bigwedge \Theta, \bigvee \Theta$ are formulae:

(iv) if φ is a formula and v a variable, then $(\forall v)\varphi$ and $(\exists v)\varphi$ are formulae.

There is a definitional distinction to be made between free and bound variables which we can blur and say just that a variable is *bound* in a formula if it falls within the scope of a quantifier and otherwise variables in formulae are *free*. A *sentence* of the language is a formula in which any variable is bound. Any formula containing at least one free variable is called *open*. Sentences and formulae will be denoted by Greek letters and, for example, $\varphi(v)$ will denote an open formula φ with free v.

We include the special formulae $\top$ and $\bot$ which will denote respectively empty conjunction and empty disjunction.

Interpretation in a topos:

Interpretation comes in two parts. First we give a direct interpretation of sorts and symbols, and then we give interpretations of terms and formulae with respect to sequences of variables. Where $\mathcal{E}$ is a topos an $\mathcal{E}$-model for an $\mathcal{S}$-sorted language $\mathcal{L}$ is a function $\mathcal{U}$ with domain $\mathcal{S} \cup \mathcal{L}$ such that

(i) for each sort $a \in \mathcal{S}$, we let $\mathcal{U}(a)$ be an $\mathcal{E}$-object;

(ii) for each $\mathbf{g}: \langle a_1, ..., a_n \rangle \to a_{n+1}$, we let $\mathcal{U}(\mathbf{g})$ be an $\mathcal{E}$-arrow

$$\mathcal{U}(a_1) \times \ldots \times \mathcal{U}(a_n) \to \mathcal{U}(a_{n+1});$$

(iii) for each $\mathbf{R}: \langle a_1, ..., a_n \rangle$, we let $\mathcal{U}(\mathbf{R})$ be a subobject of $\mathcal{U}(a_1) \times \ldots \times \mathcal{U}(a_n)$;

(iv) for each $\mathbf{c}: a$, we let $\mathcal{U}(\mathbf{c})$ be an arrow $1 \to \mathcal{U}(a)$.

The second part of an interpretation calls for interpretation with respect to sequences $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$ of distinct variables. A word on the notation that follows. When $\mathbf{v} = \langle v_1, \ldots, v_m \rangle$ is a sequence of distinct variables with $v_i: a_i$, we use $\mathcal{U}(\mathbf{v})$ to denote the product $\mathcal{U}(a_i) \times \ldots \times \mathcal{U}(a_m)$. This is slightly misleading since $\mathcal{U}(\mathbf{v})$ is not meant as an interpretation of $\mathbf{v}$ so much as an interpretation of the associated sequence of sorts. However the notation is conventional so we maintain it. Also we will be using apparently distinct functions $\mathcal{U}^{\mathbf{v}}$ to interpret terms and formulae with respect to distinct $\mathbf{v}$. In fact the superscripted $\mathbf{v}$ is no more than a reminder: for example, the symbols "$\mathcal{U}^{\mathbf{v}}(\varphi)$" mean "the interpretation under $\mathcal{U}$ of φ with respect to sequence of variables $\mathbf{v}$". Again, the notation is conventional and we maintain it. Terms and formulae are interpreted with respect to sequences of variables that are *appropriate*. For a term t a sequence $\mathbf{v}$ is *appropriate to t* if it contains at least all the variables in t. For a formula φ, a sequence $\mathbf{v}$ is *appropriate to* φ if it contains at least all the *free* variables in φ.

Terms: for a term t where $t: a$ suppose that sequence $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ with $v_i: b_i$ contains at least all the variables that occur in t. Define $\mathcal{U}(\mathbf{v})$ to be the product $\mathcal{U}(b_1) \times \ldots \times \mathcal{U}(b_n)$. Then we define the interpretation under $\mathcal{U}$ of t with respect to $\mathbf{v}$ to be a map

$$\mathcal{U}^{\mathbf{v}}(t): \mathcal{U}(\mathbf{v}) \to \mathcal{U}(a)$$

such that

(i) if t is variable $v_i: a$, then $\mathcal{U}^{\mathbf{v}}(t)$ is the projection $\mathcal{U}(\mathbf{v}) \to \mathcal{U}(a)$;

(ii) if t is $\mathbf{c}$, then $\mathcal{U}^{\mathbf{v}}(t)$ is the composite $\mathcal{U}(\mathbf{v}) \to 1 \xrightarrow{\mathcal{U}(\mathbf{c})} \mathcal{U}(a)$;

(iii) if t is $\mathbf{g}(t_1, \ldots, t_m)$ where $\mathbf{g}: \langle a_1, \ldots, a_m \rangle \to a$, then $\mathcal{U}^{\mathbf{v}}(t)$ is the composite

$$\mathcal{U}(\mathbf{v}) \xrightarrow{\langle \mathcal{U}^{\mathbf{v}}(t_1), \ldots, \mathcal{U}^{\mathbf{v}}(t_m) \rangle} \mathcal{U}(a_1) \times \ldots \times \mathcal{U}(a_m) \xrightarrow{\mathcal{U}(\mathbf{g})} \mathcal{U}(a).$$

Note that $\langle \mathcal{U}^{\mathbf{v}}(t_1), \ldots, \mathcal{U}^{\mathbf{v}}(t_m) \rangle$ is a product map.

Formulae: for a formula φ of $\mathcal{L}$ suppose that sequence $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ with $v_i: b_i$ contains at least all the free variables of φ. Define $\mathcal{U}(\mathbf{v})$ to be $\mathcal{U}(b_1) \times \ldots \times \mathcal{U}(b_n)$. Then we define the interpretation under $\mathcal{U}$ of φ with respect to $\mathbf{v}$ to be a subobject of $\mathcal{U}(\mathbf{v})$; in addition, the subobject is denoted $\mathcal{U}^{\mathbf{v}}(\varphi)$ and satisfies the rules that

(1) $\mathcal{U}^{\mathbf{v}}(\top)$ is maximum subobject $id_{\mathcal{U}(\mathbf{v})}: \mathcal{U}(\mathbf{v}) \to \mathcal{U}(\mathbf{v})$;

(2) $\mathcal{U}^{\mathbf{v}}(\bot)$ is minimum subobject $\emptyset \to \mathcal{U}(\mathbf{v})$;

(3) $\mathcal{U}^{\mathbf{v}}(t \approx u)$ and for terms t, u, both of sort a, are equalisers of

$$\mathcal{U}(\mathbf{v}) \underset{\mathcal{U}^{\mathbf{v}}(u)}{\overset{\mathcal{U}^{\mathbf{v}}(t)}{\rightrightarrows}} \mathcal{U}(a);$$

(4) $\mathcal{U}^{\mathbf{v}}(\mathbf{R}(t_1, \ldots, t_m))$ for $\mathbf{R}: \langle a_1, \ldots, a_m \rangle$ is the pullback

$$\begin{array}{ccc} d' & \xrightarrow{\mathcal{U}^{\mathbf{v}}(\varphi)} & \mathcal{U}(\mathbf{v}) \\ \downarrow & & \downarrow \langle \mathcal{U}(\mathbf{v})(t_1), \ldots, \mathcal{U}(\mathbf{v})(t_m) \rangle \\ d & \xrightarrow[\mathcal{U}(\mathbf{R})]{} & \prod_j^m \mathcal{U}(a_j) \end{array}$$

of $\langle \mathcal{U}^{\mathbf{v}}(t_1), \ldots, \mathcal{U}^{\mathbf{v}}(t_m) \rangle$ along map $\mathcal{U}(\mathbf{R})$;

(5) connectives $\bigwedge, \bigvee, \neg, \ulcorner, \Rightarrow, \dot{-}$ are interpreted as the usual operations $\bigwedge$, $\bigvee$, $\neg$, $\ulcorner$, $\Rightarrow$, $\dot{-}$ operations on $\mathrm{Sub}(\mathcal{U}(\mathbf{v}))$ provided that those operations exist for $\mathcal{E}$;

(6) quantifiers $\forall, \exists$ are interpreted by functors $\forall_f, \exists_f: \mathrm{Sub}(\mathrm{dom} f) \to \mathrm{Sub}(\mathrm{cod} f)$ as follows: suppose that for formula $(\forall w)\psi$ or $(\exists w)\psi$ all free variables of ψ appear in the sequence $\mathbf{v}, w$. Consider projection map $pr: \mathcal{U}(\mathbf{v}, w) \to \mathcal{U}(\mathbf{v})$ and let

$$\mathcal{U}^{\mathbf{v}}(\exists w \psi) = \exists_{pr}(\mathcal{U}^{\mathbf{v},w}(\psi))$$

and

$$\mathcal{U}^{\mathbf{v}}(\forall w\psi) = \forall_{pr}(\mathcal{U}^{\mathbf{v},w}(\psi)).$$

The notion of interpretation of a language in a topos can be extended to interpretation in arbitrary categories by allowing that the interpretation of a language $\mathcal{L}$ on a category $\mathcal{C}$ is a function $\mathcal{U}: \mathcal{S} \cup \mathcal{L} \to \mathcal{C}$ that satisfies all those properties of an interpretation in a topos that $\mathcal{C}$ has the structure to support.

Truth in a model:

A formula φ of a language $\mathcal{L}$ will be said to *hold in a model* $\mathcal{U}$ if, when $\mathbf{v}$ contains all and only the free variables of φ, we have

$$\mathcal{U}^{\mathbf{v}}(\varphi) = \mathcal{U}^{\mathbf{v}}(\top).$$

We will denote this with

$$\mathcal{U} \models_{1} \varphi.$$

To accomodate a broader notion of "holding in a model" we introduce the concept of an object of designated values. An *object of designated values* will exist in the first instance only in a category with a subobject classifier and will be an object D of the category for which there is an inclusion $D \hookrightarrow \Omega$ where Ω is the classifier object. The notion is best described in terms of some functor category $\mathbf{Set}^{\mathcal{C}}$. In that case, for each $a \in \mathcal{C}$, we have $\Omega(a)$ as a set of sets. Furthermore we have $D(a) \subseteq \Omega(a)$. So D works as a designated values object in the sense that we allow each $D(a)$ to be a set of designated values in $\Omega(a)$. We say for any $f \in \text{Sub}(d)$ that f is *designated* iff for each $a \in \mathcal{C}$ and each $x \in d(a)$,

$$(X_f)_a(x) \in D(a).$$

Plainly, some designated value objects will be more intuitive than others. Consider for example an object D in $\mathbf{Set}^{\mathcal{C}}$ given by $D(a) = [id_a)$ all $a \in \mathcal{C}$. For such an object

$f \in \mathrm{Sub}(d)$ is designated iff $f \simeq id_d$. Such an object describes exactly those values in any $\mathrm{Sub}(d)$ that are most commonly regarded as designated, namely the units of the lattices $(\mathrm{Sub}(d), \leq)$. At the other extreme is an object D given by $D(a)$ for all $a \in \mathcal{C}$ where $D(a)$ is $\Omega(a)$ wothout the zero of $(\Omega(a), \subseteq)$. This object designates all $f \in \mathrm{Sub}(d)$ other than $\emptyset \rightarrowtail d$. This extreme will be formally interesting in later discussion when we come to dualise models via language dualisation, so take note of it here. We will say of a formula φ of language $\mathcal{L}$ that φ is *not undeniably refuted* by model $\mathcal{U}$ if where $\mathbf{v}$ contains exactly the free variables of φ

$$\mathcal{U}^{\mathbf{v}}(\varphi) \neq \mathcal{U}^{\mathbf{v}}(\bot).$$

We will denote this by

$$\mathcal{U} \models_{2} \varphi.$$

Notice as a final point that the reason we use objects included in Ω as a representation of designated values is to solve the problem of coordinating the sets of designated values on each $\mathrm{Sub}(\mathcal{U}(\mathbf{v}))$. Since in general there is more than one $\mathrm{Sub}(\mathcal{U}(\mathbf{v}))$ under consideration, we require some formal link between designated sets of subobjects if only that we may have a means of abstracting our discussion from particular subobject structure in particular categories.

2. Geometric logic, Sites, and Language Algebras

With this section we define two dual fragments, $\mathcal{L}^g$ and $\mathcal{L}^{g^{op}}$, of the language $\mathcal{L}$. The relationship of duality is used to define dual theories and then dual models for these dual theories. There is a proof theory, called geometric logic or GL, associated with the fragment $\mathcal{L}^g$. We use the notion of language duality to define a dual system which we call co-GL. The system co-GL decomes a type of proof theory for $\mathcal{L}^{g^{op}}$. We will use the notion of dual models for theories to develop the idea that co-GL is a disproof. This is the idea that we use co-GL to derive falsehoods from falsehoods. Our development of GL is based upon the discussions in Goldblatt [1984] and Makkai and Reyes [1977].

Definition 2.1: For a language $\mathcal{L}$ a *fragment* $\mathcal{L}'$ is a subclass of the class of all formulae of $\mathcal{L}$ that is closed under the inclusion of subformulae and substitution. So if $\phi \in \mathcal{L}'$ and ψ is a well-formed subformulae of ϕ, then $\psi \in \mathcal{L}'$; and if t is a term of $\mathcal{L}$ and v is a free variable in ϕ, then $\phi(v/t) \in \mathcal{L}'$.

Definition 2.2: A formula φ of a language $\mathcal{L}$ as described in section one is called *positive existential* if, in addition to atomic formulae, it contains no logical symbols other than $\top, \bot, \wedge, \vee$, and $\exists$ where $\wedge, \vee$ are finite $\bigwedge, \bigvee$. The collection of positive existential formulae of $\mathcal{L}$ is denoted $\mathcal{L}^g$. A formula φ is called *coherent* or *geometric* if it is formula $\phi \Rightarrow \psi$ where $\phi, \psi \in \mathcal{L}^g$. All formulae φ in $\mathcal{L}^g$ can be called coherent in that each φ can be identified with $\top \Rightarrow \varphi$. As a result we refer to $\mathcal{L}^g$ as the coherent or geometric fragment of $\mathcal{L}$.

Dually,

Definition 2.3: A formula φ of $\mathcal{L}$ is called *co-positive-existential* or *negative universal* if, in addition to atomic formulae, it contains no logical symbols other than $\bot, \top, \vee, \wedge$, and $\forall$. The collection of negative universal formulae of $\mathcal{L}$ is denoted $\mathcal{L}^{g^{op}}$. A formula φ is called *co-coherent* of *co-geometric* if it is a formula $\phi \dot{-} \psi$ for $\phi, \psi \in \mathcal{L}^{g^{op}}$. All φ in $\mathcal{L}^{g^{op}}$ can be called co-coherent in that each φ can be identified with $\varphi \dot{-} \bot$. As a result we refer to $\mathcal{L}^{g^{op}}$ as the co-coherent or co-geometric fragment of $\mathcal{L}$.

Definition 2.4: A *sentence* of a language $\mathcal{L}$ is a formula φ of $\mathcal{L}$ with no unbound variables.

Definition 2.5: A *theory* of a language $\mathcal{L}$ is a set of sentences of the language $\mathcal{L}$ closed under a consequence relation and satisfying the property that if φ and ψ

are sentences of the theory, then so is $\varphi \wedge \psi$. When all sentences of the theory are sentences in $\mathcal{L}^g$, then the theory is called coherent or geometric. When all sentences of the theory are in $\mathcal{L}^{g^{op}}$, the theory is called co-coherent or co-geometric.

Languages and Models:

There is a straightforward duality relationship to be drawn between $\mathcal{L}^g$ and $\mathcal{L}^{g^{op}}$. It can be described by defining formula φ^{op} for any $\varphi \in \mathcal{L}^g$ as follows

(1) if φ is atomic formula ψ, then φ^{op} is ψ;

(2) if φ is $\psi_1 \wedge \psi_2$, then φ^{op} is $\psi_1^{op} \vee \psi_2^{op}$;

(3) if φ is $\psi_1 \vee \psi_2$, then φ^{op} is $\psi_1^{op} \wedge \psi_2^{op}$;

(4) if φ is $\psi_1 \Rightarrow \psi_2$, then φ^{op} is $\psi_2^{op} \dot{-} \psi_1^{op}$;

(5) if φ is $\exists w \psi$, then φ^{op} is $\forall w \psi^{op}$.

Recall that sorts, terms, and atomic formulae are common to $\mathcal{L}^g$ and $\mathcal{L}^{g^{op}}$ and it is apparent that

$$\varphi \in \mathcal{L}^g \quad \text{iff} \quad \varphi^{op} \in \mathcal{L}^{g^{op}}$$

and that we may define a duality function

$$l : \mathcal{S} \cup \mathcal{L}^g \rightarrow \mathcal{S} \cup \mathcal{L}^{g^{op}}$$

where if $a \in \mathcal{S}$, $l\ (a) = a$; and if $\varphi \in \mathcal{L}^g$, then $l\ (\varphi) = \varphi^{op}$. As a point of nomenclature, since l is a bijection, we will frequently use symbol l to represent both the function and its inverse.

We can use this duality function to make plain a relationship of duality between models for $\mathcal{L}^{g^{op}}$ and models for $\mathcal{L}^g$. Given a model $\mathcal{U}: \mathcal{S} \cup \mathcal{L}^g \rightarrow \mathcal{E}$ where $\mathcal{E}$ is any topos, we can define a model $\mathcal{U}_{op}: \mathcal{S} \cup \mathcal{L}^{g^{op}} \rightarrow \mathcal{E}$ by

$$\mathcal{U}_{op}: \mathcal{S} \cup \mathcal{L}^{g^{op}} \xrightarrow{l} \mathcal{S} \cup \mathcal{L}^g \xrightarrow{\mathcal{U}} \mathcal{E}.$$

Plainly, when x is a sort, or an operation or relation symbol, $\mathcal{U}_{op}(x) = \mathcal{U}(x)$. Also, when t is a term and $\mathbf{v}$ contains at least the variables in t

$$\mathcal{U}_{op}^{\mathbf{v}}(t) = \mathcal{U}^{\mathbf{v}}(t).$$

And when $\varphi \in \mathcal{L}^{g^{op}}$ and $\mathbf{v}$ contains at least all the free variables of φ

$$\mathcal{U}^{\mathbf{v}}_{op}(\varphi) = \mathcal{U}^{\mathbf{v}}(\varphi^{op}).$$

Given the relationship of duality between $\mathcal{L}^g$ and $\mathcal{L}^{g^{op}}$, it should be apparent that

$$\mathcal{U} \text{ exists iff } \mathcal{U}_{op} \text{ exists.}$$

In fact, the relationship between $\mathcal{U}$ and $\mathcal{U}_{op}$ is usefully described by the following diagram

$$\begin{array}{ccc} \mathcal{S} \cup \mathcal{L}^{g^{op}}_{\infty} & \longrightarrow & \mathcal{E}^{op} \\ {\scriptstyle l}\downarrow & \searrow^{\mathcal{U}_{op}} & \downarrow{\scriptstyle d_{\mathcal{E}}} \\ \mathcal{S} \cup \mathcal{L}^{g}_{\infty} & \underset{\mathcal{U}}{\longrightarrow} & \mathcal{E} \end{array}$$

where the left triangle is known to commute and the outer square is defined to be commuting with $d_{\mathcal{E}}$ being the usual categorial dualisation functor. The point to note is that $\mathcal{U}$ and $\mathcal{U}_{op}$ share exactly the same range but that while $\mathcal{U}$ would interpret $\wedge, \vee, \exists$ by $\bigcap, \bigcup, \exists_f$ in $\mathcal{E}$, the model $\mathcal{U}_{op}$ would have $\bigcap, \bigcup, \exists_f$ interpreting respectively $\vee, \wedge, \forall$. In effect, where $\mathcal{U}$ targets the lattice structures of $\mathcal{E}$, model $\mathcal{U}_{op}$ targets their duals.

The notable examples under the dualisation are the formulae $\top$ and $\bot$. Since $\top$ is defined to be empty conjunction and $\bot$ is empty disjunction, we have that $\mathcal{U}^{\mathbf{v}}_{op}(\top) = \mathcal{U}^{\mathbf{v}}(\bot)$ and $\mathcal{U}^{\mathbf{v}}_{op}(\bot) = \mathcal{U}^{\mathbf{v}}(\top)$. It follows that if models $\mathcal{U}$ designate only $\mathcal{U}^{\mathbf{v}}(\top)$ of each Sub(d) and if we allow all subobject other than the $\mathcal{U}^{\mathbf{v}}_{op}(\bot)$ to be designated under $\mathcal{U}_{op}$, then we have the following relationship between formulae that hold in models of the coherent language and formulae that hold in models of the co-coherent language.

$$\mathcal{U} \models_1 \varphi \quad \text{iff} \quad \mathcal{U}_{op} \not\models_2 \varphi^{op}.$$

In general, so long as models $\mathcal{U}_{op}$ do not designate the $\mathcal{U}^{\mathbf{v}}_{op}(\bot)$ of Sub$(\mathcal{U}(\mathbf{v}))$, we have that

$$\text{if} \quad \mathcal{U} \models_1 \varphi, \quad \text{then,} \quad \mathcal{U}_{op} \not\models \varphi^{op}.$$

Notice a slightly confusing feature of the dualisation that wherever t is a term and $\mathbf{v}$ is a sequence of variables containing exactly those free in $t \approx t$, we have

$$\mathcal{U}^{\mathbf{v}}(t \approx t) = \mathcal{U}^{\mathbf{v}}_{op}(t \approx t)$$

and under the usual interpretations $\mathcal{U}$ we would have that

$$\mathcal{U}^{\mathbf{v}}(t \approx t) = \mathcal{U}^{\mathbf{v}}(\top)$$

so we have

$$\mathcal{U}^{\mathbf{v}}_{op}(t \approx t) = \mathcal{U}^{\mathbf{v}}_{op}(\bot).$$

The only serious problem with this is that it may be misread as meaning that under $\mathcal{U}_{op}$, equations involving identical terms will fail. This is a misreading in that the only interpretation of $\approx$ to be made is the one provided by $\mathcal{U}_{op}$. The symbol $\approx$ is standard for identity but under $\mathcal{U}_{op}$, the two place relation $\approx$ does *not* behave as identity and so should not be considered as such. In fact, under $\mathcal{U}_{op}$, $\approx$ behaves as non-identity. It is necessary that we bear this in mind when we come to consider co-GL.

Proof theory:

We will consider two finite sequent systems. The first is the standard system of geometric logic called GL presented by Makkai and Reyes [1977]. The second one we will call co-GL and will be the dual of GL in the same sense that $\mathcal{L}^g$ and $\mathcal{L}^{g^{op}}$ are dual languages. We describe co-GL by applying the language dualisation function l to all the axioms and rules of GL. Obviously this does not create a new deduction system as such; some interpretation is required. Consider first the axioms of GL. These are formulae φ such that for any topos $\mathcal{E}$ and any $\mathcal{L}^g$-model $\mathcal{U}$, it is expected that

$$\mathcal{U} \models_1 \varphi.$$

Assuming that any $\mathcal{L}^{g^{op}}$-model $\mathcal{U}_{op}$ in $\mathcal{E}$ does not designate $\mathcal{U}_{op}^{\vee}(\perp)$ in any subobject lattice it follows that

$$\mathcal{U}_{op} \not\models \varphi^{op}.$$

So, if φ is an axiom for GL, then $l\ (\varphi)\varphi^{op}$ is an axiomatic falsehood, or perhaps absurdity, for co-GL.

Consider now the rules of GL. They will have the form

$$\frac{\{\Theta_i : i \in I\}}{\Theta}$$

and mean that from $\{\Theta_i : i \in I\}$ derive Θ. That these are rules means that whenever $\mathcal{U}$ is an $\mathcal{L}^g$-model in $\mathcal{E}$, it is expected that if

$$\mathcal{U} \models_1 \Theta_i \quad \text{all } i \in I,$$

then

$$\mathcal{U} \models_1 \Theta.$$

Putting this another way we have that if

$$\mathcal{U}^{\vee}(\Theta_i) = \mathcal{U}^{\vee}(\top)$$

for all $i \in I$, then

$$\mathcal{U}^{\vee}(\Theta) = \mathcal{U}^{\vee}(\top).$$

Applying the language dualisation to both formulae and models we have that whenever

$$\mathcal{U}^{\vee}_{op}(\Theta^{op}_i) = \mathcal{U}^{\vee}_{op}(\bot)$$

for all $i \in I$, then

$$\mathcal{U}^{\vee}_{op}(\Theta^{op}) = \mathcal{U}^{\vee}_{op}(\bot).$$

It follows that if all subobjects other than those $\mathcal{U}^{\vee}_{op}(\bot)$ are designated under all $\mathcal{L}^{g^{op}}$-models $\mathcal{U}_{op}$, then the dualised GL rules of inference are rules of falsehood, or absurdity, preservation for co-GL. For these reasons we call co-GL a system of refutation or disproof.

We consider the finite systems since there are well known completeness and soundness results for finite GL. We will describe GL first.

GL: A *sequent* will be an expression $\Gamma \Rightarrow \psi$ where Γ is a finite set of formulae. When ψ and all formulae in Γ are $\mathcal{L}^g$ formulae, the sequent is *geometric*. A sequent is not a formula but can easily be re-written as one if required: Γ should be re-written as the finite conjunction of all its members. In what follows the union $\Gamma \cup \Delta$ will be written Γ, Δ.

Axioms of identity:

$v \approx v$,

$v \approx w \Rightarrow w \approx v$,

$v \approx w, \varphi \Rightarrow \varphi(v/w)$,

where v and w are variables of the same sort and φ is atomic.

Axiom A1:

$\Gamma \Rightarrow \psi$, if $\psi \in \Gamma$.

Rules of inference:

The rules have the form

$$\frac{\{\Theta_i : i \in I\}}{\Theta}$$

meaning from $\{\Theta_i : i \in I\}$ derive Θ.

$$(R\bigwedge_1) \qquad \frac{\Delta, \bigwedge\Gamma, \varphi \Rightarrow \psi}{\Delta, \bigwedge\Gamma \Rightarrow \psi} \qquad \text{if } \varphi \in \Gamma;$$

$$(R\bigwedge_2) \qquad \frac{\Delta, \bigwedge\Gamma \Rightarrow \psi}{\Delta \Rightarrow \psi} \qquad \text{if } \Gamma \subseteq \Delta;$$

$$(R\bigvee_1) \qquad \frac{\Delta, \varphi, \bigvee\Gamma \Rightarrow \psi}{\Delta, \varphi \Rightarrow \psi} \qquad \text{if } \varphi \in \Gamma$$

and all free variables occurring in Γ also occur free in the conclusion;

$$(R\bigvee_2) \qquad \frac{\{\Delta, \bigvee\Gamma, \varphi \Rightarrow \psi : \varphi \in \Gamma\}}{\Delta, \bigvee\Gamma \Rightarrow \psi} \qquad ;$$

$$(R\exists_1) \qquad \frac{\Delta, \varphi(v/t), \exists w\varphi(v/w) \Rightarrow \psi}{\Delta, \varphi(v/t) \Rightarrow \psi} \qquad ;$$

$$(R\exists_2) \qquad \frac{\Delta, \exists w\varphi(v/w), \varphi \Rightarrow \psi}{\Delta, \exists w\varphi(v/w) \Rightarrow \psi}$$

if v does not occur free in the conclusion;

$$(R\mathrm{T}) \qquad \frac{\Delta, \Gamma(t_1, \ldots, t_n), \varphi(t_1, \ldots, t_n) \Rightarrow \psi}{\Delta, \Gamma(t_1, \ldots, t_n) \Rightarrow \psi}$$

provided that all free variables in the premiss occur free in the conclusion and that for some $v_1, \ldots, v_n$ the sequent $\Gamma(v_1, \ldots, v_n) \Rightarrow \varphi(v_1, \ldots, v_n)$ belongs to theory T together with the GL axioms.

When there is a finite sequence of geometric sequents ending in Θ with all members of the sequence being axioms, members of the geometric theory T, or following from earlier members of the sequence by application of the GL rules, then the sequent Θ is said to be *GL-derivable* from T. We denote the existence of such a sequence by

$$\mathrm{T} \vdash_1 \Theta.$$

There are known to be soundness and (classical) completeness theorems (respectively, Theorem 3.2.8 and Corollary 5.2.3, in Makkai and Reyes [1977]) for system GL. Let T be some collection of $\mathcal{L}^g$ formulae.

Soundness Theorem: *if* $\mathrm{T} \vdash_1 \Theta$ *and* $\mathcal{U}$ *is a* T*-model in topos* $\mathcal{E}$, *then* $\mathcal{U} \models_1 \Theta$. □

Classical Completeness Theorem: *if* $\mathrm{T} \nvdash_1 \Theta$, *then there is a* **Set***-model* $\mathcal{U}$ *such that* $\mathcal{U} \models_1 \mathrm{T}$ *and* $\mathcal{U} \not\models_1 \Theta$. □

Given what we know about the duality of $\mathcal{L}^g$ and $\mathcal{L}^{g^{op}}$ we can dualise GL and produce a system of co-geometric logic which we shall call co-GL. As we have discussed, co-GL will be a system of axiomatic falsehoods and falsehood preserving rules.

co-GL: The specification of co-GL is basically the same as that for GL except that all dualisations by l from $\mathcal{L}^g$-formulae to $\mathcal{L}^{g^{op}}$-formulae apply. co-GL will be a system of axiomatic falsehoods (for want of a better phrase - we mean formulae used as axioms but meant to be false) and falsehood preserving rules. A co-GL *sequent* will be an expression $\psi^{op} \dot{-} \Gamma^{op}$ where Γ^{op} is a finite set of formulae. When ψ^{op} and all formulae in Γ^{op} are $\mathcal{L}^{g^{op}}$ formulae, the sequent is *co-geometric*. A sequent is not a formula but can easily be re-written as one if required: Γ^{op} should be re-written as the finite *disjunction* of all its members. In what follows the *intersection* $\Gamma^{op} \cap \Delta^{op}$ will be written Γ^{op}, Δ^{op}.

Axiomatic falsehoods of *non*-identity:

$v \approx v;$

$w \approx v \dot{-} v \approx w;$

$v \approx w, \varphi^{op}(v/w) \dot{-} \varphi^{op}$

where v and w are variables of the same sort and φ^{op} is atomic. To assure oneself of the actual falsehood of this third "axiom" recall our earlier discussion to the effect that $\approx$ will behave as $\not\approx$ and note that $v \approx w$ and $\varphi^{op}(v/w) \dot{-} \varphi^{op}$ are conjoined; the meaning of the axiom is, then, that when v is identical to w, formula $\varphi^{op}(v/w) \dot{-} \varphi^{op}$ is false.

Axiom(atic falsehood) $A^{op}1$:

$\psi^{op} \dot{-} \Gamma^{op}, \quad \text{if} \quad \psi^{op} \in \Gamma^{op}.$

Rules of falsehood preservation:

The rules have the form

$$\frac{\{\Theta_i^{op} : i \in I\}}{\Theta^{op}}$$

meaning from the falsehood of $\bigwedge\{\Theta_i^{op} : i \in I\}$ derive the falsehood of Θ^{op}.

$$(R^{op}\bigvee_1) \qquad \frac{\Delta^{op}, \bigvee\Gamma^{op}, \psi^{op} \dot{-} \varphi^{op}}{\Delta^{op}, \psi^{op} \dot{-} \bigvee\Gamma^{op}} \qquad \text{if } \varphi^{op} \in \Gamma^{op};$$

$$(R^{op}\bigvee_2) \qquad \frac{\Delta^{op}, \psi^{op} \dot{-} \bigvee\Gamma^{op}}{\psi^{op} \dot{-} \Delta^{op}} \qquad \text{if } \Gamma^{op} \subseteq \Delta^{op};$$

$$(R^{op}\bigwedge_1) \qquad \frac{\Delta^{op}, \varphi^{op}, \psi^{op} \dot{-} \bigwedge\Gamma^{op}}{\Delta^{op}, \psi^{op} \dot{-} \varphi^{op}} \qquad \text{if } \varphi^{op} \in \Gamma^{op}$$

and all free variables occurring in Γ^{op} also occur free in the conclusion;

$$(R^{op}\bigwedge_2) \qquad \frac{\{\Delta^{op}, \bigwedge\Gamma^{op}, \psi^{op} \dot{-} \varphi^{op} : \varphi^{op} \in \Gamma^{op}\}}{\Delta^{op}, \psi^{op} \dot{-} \bigwedge\Gamma^{op}} \qquad ;$$

$$(R^{op}\forall_1) \qquad \frac{\Delta^{op}, \varphi^{op}(v/t), \psi^{op} \dot{-} \forall w \varphi^{op}(v/w)}{\Delta^{op}, \psi^{op} \dot{-} \varphi^{op}(v/t)} \quad ;$$

$$(R^{op}\forall_2) \qquad \frac{\Delta^{op}, \forall w \varphi^{op}(v/w), \psi^{op} \dot{-} \varphi^{op}}{\Delta^{op}, \psi^{op} \dot{-} \forall w \varphi^{op}(v/w)}$$

if v does not occur free in the conclusion;

$$(R^{op}\mathrm{T}^{op}) \qquad \frac{\Delta^{op}, \Gamma^{op}(t_1, \ldots, t_n), \psi^{op} \dot{-} \varphi^{op}(t_1, \ldots, t_n)}{\Delta^{op}, \psi^{op} \dot{-} \Gamma^{op}(t_1, \ldots, t_n)}$$

provided that all free variables in the premiss occur free in the conclusion and that for some $v_1, \ldots, v_n$, the sequent $\varphi^{op}(v_1, \ldots, v_n) \dot{-} \Gamma^{op}(v_1, \ldots, v_n)$ belongs to T^{op} together with the co-GL axioms.

When there is a finite sequence of co-geometric sequents ending in Θ in which all members of the sequence are axiomatic falsehoods of co-GL, formulae of some co-geometric theory T, or are consequences of earlier formulae in the sequence by the co-GL rules of flasehood preservation, then we say that Θ is *co-GL derivable* from T. We denote this by

$$\mathrm{T} \vdash_2 \Theta.$$

Theorem 2.1: *if* T *is a geometric theory and* T^{op} *is defined by allowing* $\varphi^{op} \in \mathrm{T}^{op}$ *iff* $\varphi \in \mathrm{T}$, *then we have*

$$\mathrm{T}^{op} \vdash_2 \Theta^{op} \quad \textit{iff} \quad \mathrm{T} \vdash_1 \Theta.$$

Proof: the result follows by definition of the dualisation on $\mathcal{L}^g$ and on GL. □

Dualisation also provides us with two special non-triviality theorems. To properly describe them, some definitions are in order.

Definition 2.6: Where $\mathcal{L}$ and $\mathcal{L}^{op}$ are dual languages so that where φ is in $\mathcal{L}$, the dual in $\mathcal{L}^{op}$ is denoted φ^{op}, we say that a set T^{op} of sentences φ^{op} of $\mathcal{L}^{op}$ is a *co-theory* iff T described by

$$\varphi \in \mathrm{T} \quad \text{iff} \quad \varphi^{op} \in \mathrm{T}^{op}$$

is a theory.

Definition 2.7: For any set of sentences, T, of a language $\mathcal{L}$. if $\mathcal{U}$ is a model for $\mathcal{L}$ and

$$\mathcal{U} \models \varphi$$

for all $\varphi \in \mathrm{T}$, then $\mathcal{U}$ is a T-*model* or a *model for* T.

Definition 2.8: For any set of sentences T of a language $\mathcal{L}$. if $\mathcal{U}$ is a model for $\mathcal{L}$ and

$$\mathcal{U} \not\models \varphi$$

for all $\varphi \in \mathrm{T}$, then $\mathcal{U}$ is called a T *refutation* or a *refutation model for* T.

Theorem 2.2: *for any set of $\mathcal{L}^{g^{op}}$ formulae T^{op}, if $\mathrm{T}^{op} \vdash_2 \Theta^{op}$ and $\mathcal{U}_{op}$ is a refutation model for T^{op} in a topos $\mathcal{E}$, then we have*

$$\mathcal{U}_{op} \not\models_2 \Theta^{op}.$$

Proof: by duality, since by the Soundness theorem for GL, for any T, if $\mathrm{T} \vdash_2 \Theta$ and $\mathcal{U}$ is a model for T in a topos $\mathcal{E}$, then $\mathcal{U} \models_1 \Theta$. □

Theorem 2.3: *for any set of $\mathcal{L}^{g^{op}}$ formulae T^{op}, if $\mathrm{T}^{op} \nvdash_2 \Theta^{op}$, then there is a refutation model $\mathcal{U}_{op}$ for T^{op} in* **Set** *such that $\mathcal{U}_{op} \models_2 \Theta^{op}$.*

Proof: by duality since by the Completeness theorem for GL if T is a set of $\mathcal{L}^g$ formulae and $\mathrm{T} \nvdash_1 \Theta$, then there is a model $\mathcal{U}$ for T in **Set** such that $\mathcal{U} \models_1 \mathrm{T}$ and $\mathcal{U} \not\models_1 \Theta$. □

In the light of these last two theorems we can we can see how we may use the concept of a refutation model with respect to a refutation system like co-GL to model inconsistent theories. First of all note that for any negation inconsistent theory T, that is any theory T that contains φ and $\neg\varphi = \varphi \Rightarrow \bot$, the co-theory T^{op} is negation inconsistent. This is because if $\varphi \in T$, then $\varphi^{op} \in T^{op}$, and if $\neg\varphi \in T$, then $(\neg\varphi)^{op} = (\varphi \Rightarrow \bot)^{op} = (\top \dot{-} \varphi^{op}) = \ulcorner(\varphi^{op})$ is in T^{op}. Furthermore, we can demonstrate that co-GL is consistent in the sense that

$$\text{if} \quad T^{op} \vdash_2 \Theta^{op}, \quad \text{then} \quad T^{op} \nvdash_2 \ulcorner(\Theta^{op}).$$

This follows directly from the fact that GL is consistent together with Theorem 2.1. So, to model inconsistent theories T using co-GL methods we need only consider models for subsets of consistent co-theories. If T' is a proper subset of some co-theory T^{op} such that for some φ^{op}, neither φ^{op} nor $\ulcorner(\varphi^{op})$ are in T', then, generally, there will be at least one refutation model $\mathcal{U}_{op}$ for T^{op} such that

$$\text{both} \quad \mathcal{U}_{op} \models_2 \varphi^{op} \quad \text{and} \quad \mathcal{U}_{op} \models_2 \ulcorner(\varphi^{op}).$$

All that is required is that

$$\text{neither} \quad T' \vdash_2 \varphi^{op} \quad \text{nor} \quad T' \vdash_2 \ulcorner(\varphi^{op}).$$

We regard falsehood preservation systems as reasonable tools for use with inconsistency tolerant logics. After all, the acceptance of BrAs as algebras for paraconsistent logics amounts to the claim that inconsistency is not meant to mean that some sentence is both true and false; the claim of inconsistency is that some sentence and its negation are both not false. Toleration of inconsistency would seem to mean something like avoiding the proliferation of inconsistent claims; one way to do this would be to make consequence or inference an issue only of undeniable falsehood.

CHAPTER 15: THE OMEGA MONOID

Introduction: This is the final technical chapter of the thesis. It is a reproduction for categories of a theorem for sets due to Faith and found in C. Faith [1973]. It is included for its demonstration of the difficulty for the discovery of relevant logic algebras that are based in toposes in the same way as the Heyting algebras. This represents both an added layer of meaning for the original Faith result and a signal of further interest in discovering the nature of categories as semantic objects for non-classicial logics generally.

Throughout this work our interest has been in the categorial expressions of paraconsistent logic. With this chapter we give a preliminary result for the investigation of a broader range of non-classical logics. Our interest here is with Anderson and Belnap's relevant logic as described in *Entailment* [1975]. The algebras for this logic are the De Morgan monoids. Now, for any object a in a category $\mathcal{C}$, there is always a monoid $\hom_{\mathcal{C}}(a, a)$ where composition is the multiplication operation. This gives us an opportunity to discuss relevant logics within category theory.

In the context of this text we have two constraints on our investigation of monoids. First, we are interested to see structures related to Ω objects or at least with subobject lattices. Second, we are considering De Morgan monoids which means we will require that our monoids have some lattice structure. It will be shown in this chapter that, even before consideration of the required relationship between the lattice order and the multiplication operation, the requirement that a De Morgan monoid be commutative imposes the restriction that where c is the categorial object around which we define our monoid there must be at most one arrow $1 \to c$, that is, exactly one global element. This is a significant limitation compared with the usual treatment of Heyting algebras Ω in which algebraic elements correspond to the global elements. In passing we note the possibility of revitalising this investigation by consideration of such structures as sheaves of monoids and sheaves defined over monoids.

1. De Morgan Monoids

The following definitions are taken from Anderson and Belnap [1975].

Definitions 1.1: A *semi-group* $(\mathrm{S}, \otimes)$ is a non-empty set S closed under $\otimes$, an associative binary operation. When $a \otimes b = b \otimes a$ for all $a, b \in \mathrm{S}$, the semi-group is *commutative*. If $t \in \mathrm{S}$ such that $t \otimes a = a \otimes t = a$, then t is an *identity* for the semi-group. A semi-group with identity is called a *monoid*. If S is a lattice and $a \otimes (b \cup c) = (a \otimes b) \cup (a \otimes c)$ for all $a, b, c \in S$, then the semi-group is *lattice ordered*.

A *De Morgan lattice* is a lattice $(\mathrm{S}, \leq, -)$ where "$-$" is a unary operator for which $-(-a) = a$ and $a \leq b$ implies $-b \leq -a$.

A *De Morgan monoid* is a structure $(\mathrm{S}, \otimes, \leq, -)$ where $(\mathrm{S}, \otimes, \leq)$ is a lattice ordered, commutative semi-group, $(\mathrm{S}, \leq, -)$ is a De Morgan lattice, and the following two conditions are satisfied:

$$(a \otimes b) \leq c \quad \text{iff} \quad b \otimes (-c) \leq -a \quad \text{iff} \quad (-c) \otimes a \leq -b$$

$$a \leq a \otimes a.$$

When $\mathcal{C}$ is an arbitrary category, the collection of endomorphisms $\mathcal{C}(c, c)$ for any $c \in \mathcal{C}$ is a set closed under composition of arrows. This set can be represented as a single object category having object c and morphisms $\mathcal{C}(c, c)$. This category is called a *strict monoidal category* or a *categorial monoid* with composition as the multiplication operation. When Ω is the truth value object of an arbitrary topos $\mathcal{E}$, the set $\mathcal{E}(\Omega, \Omega)$, which we shall call an Ω-monoid, derives lattice structure from the lattice $(\mathrm{Sub}(\Omega), \subseteq)$ where $\subseteq$ is subobject inclusion. By definition of Ω as a truth value object, we have a bijection $\mathrm{Sub}(\Omega) \cong \mathcal{E}(\Omega, \Omega)$ described by

$$\mathrm{Sub}(\Omega) \ni f \mapsto \chi_f \in \mathcal{E}(\Omega, \Omega)$$

which allows us to define an order $\leq$ on $\mathcal{E}(\Omega, \Omega)$ in terms of subobject inclusion on $\mathrm{Sub}(\Omega)$. The fact that $\mathrm{Sub}(d)$ is a HA gives us $(\mathcal{E}(\Omega, \Omega), \leq)$ as a HA.

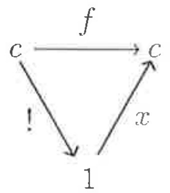
Proposition 1.1: *for any object c in (small) category $\mathcal{C}$, the set $\mathcal{C}(c, c)$ of all $\mathcal{C}$-morphisms $c \to c$ is a monoid with respect to composition.*

Proof: $\mathcal{C}(c, c)$ is closed under composition since the composition of any two morphisms $c \to c$ is a morphism $c \to c$. Composition is by definition an associative binary operation on morphisms. Furthermore, we have the identity arrow $id_c : c \to c$ for which we have $id_c \cdot f = f \cdot id_c = f$, any $f \in \mathcal{C}(c, c)$. So $(\mathcal{C}(c.c).\cdot)$ is a semi-group with identity. □

The following results draw on a result in Faith's *Algebra: Rings, Modules and Categories I* [1973] where it is shown that *Maps* X, the semigroup with respect to composition of all functions from non-empty set X to X, is comutative if and only if X is a singleton. Assume a small category $\mathcal{C}$ with a terminal object 1.

Definition 1.2: For any object $c \in \mathcal{C}$, a (*global*) *element* of c is a map $1 \to c$.

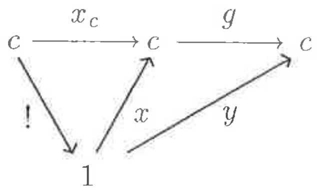
Formally analogous to the notion of a constant endomorphic function is a map $f : c \to c$ that factors through a global element of c. This is a map f for which there is a commuting diagram of the following sort,



We will denote by x_c a map $c \to c$ that factors through element $x : 1 \to c$.

Lemma 1.1: *for any $g, x_c \in \mathcal{C}(c,c)$, we have $g \cdot x_c = y_c$ for some $y_c \in \mathcal{C}(c,c)$.*

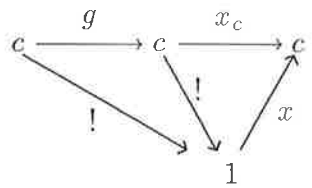
Proof: consider the diagram



Define y to be the arrow such that $y = g \cdot x$. Now $x_c = x \cdot !$, so $g \cdot x_c = g \cdot x \cdot ! = y \cdot !$ which by definition is y_c. □

Lemma 1.2: *for any $g, x_c \in \mathcal{C}(c,c)$, we have $x_c \cdot g = x_c$.*

Proof: consider the diagram



We have $x_c \cdot g = x \cdot ! \cdot g$. But, by definition of the terminal object, we have $! = ! \cdot g$. So, $x_c \cdot g = x \cdot ! = x_c$. □

Notice a particular corollary that any x_c is idempotent. As proof, let $g = x_c$ in the lemma. We now prove the main result of this chapter.

Theorem 1.1: *in a category $\mathcal{C}$ with a terminal object the monoid $(\mathcal{C}(c,c), \cdot)$ is commutative only if there is at most one map $1 \to c$.*

Proof: suppose at least two distinct maps $1 \xrightarrow{x} c$ and $1 \xrightarrow{y} c$. Since $x \cdot !$ and $y \cdot !$ are both constant morphisms, that is they factor through 1, when $x \neq y$, then $x \cdot ! \neq y!$. So, for distinct x, y we have distinct x_c and y_c. It follows, by lemma 1.2, that for any $g \in \mathcal{C}(c,c)$,

$$x_c \cdot g \neq y_c \cdot g.$$

Now, $x_c \in \mathcal{C}(c,c)$, so

$$x_c \cdot x_c \neq y_c \cdot x_c.$$

But, again by lemma 1.2, $x_c \cdot x_c = x_c$, so

$$x_c \neq y_c \cdot x_c.$$

Now, if the monoid is commutative, $y_c \cdot x_c = x_c \cdot y_c$, so we should have

$$x_c \neq x_c \cdot y_c.$$

But, by lemma 1.2, $x_c = x_c \cdot y_c$ since $y_c \in \mathcal{C}(c, c)$. It follows that $(\mathcal{C}(c, c), \cdot)$ cannot be commutative unless $\mathcal{C}(c, c)$ contains exactly one element. □

In particular, this means that the Ω-monoid where it exists for a category will not be commutative, and hence not be De Morgan, unless it is at least true that there is exactly one truth value $1 \to \Omega$. Notice in finishing that the reason for recasting Faith's result for *Maps* X in categorial language is that Faith's discussion is in terms of sets and functions and, unlike sets, categorial objects are not necessarily completely determined by their global elements.

In seeking out commutative monoids on the structure of categories we are not left without resources.

Definition 1.3: For a semi-group S and a non-empty subset X of the underlying set,

$$\textit{center } X = \{a \in S \mid a \otimes x = x \otimes a \text{ for all } x \in X\}.$$

Clearly, *center* X is a commutative sub-semi-group of S. In the case of small $\mathcal{C}(c, c)$ the identity arrow on c is always an element of *center* $\mathcal{C}(c, c)$. Two further topics of interest will then be under what conditions is *center* $\mathcal{E}(\Omega, \Omega)$ more than just $\{id_\Omega\}$, and will *center* $\mathcal{E}(\Omega, \Omega)$ have any structure related to the lattice on $\mathcal{E}(\Omega, \Omega)$. This direction will not be pursued in this work.

CHAPTER 16: CONCLUSIONS

In this work we have investigated two aspects of a dualisation program for logic in categories. The first aspect was that of external or ex-categorial dualisation of logic structures by reinterpretation of order structures. This was the core of the original dualisation program that featured the notion of a complement classifier. This type of dualisation featured heavily in our discussion in chapter fourteen where we considered modelling theories in categories. The principal contribution of that chapter was the notion that we could use this type of dualisation to prompt the construction of a deduction system suited to the notion of inconsistency toleration. The bulk of this work, however, was given over to investigation of the other aspect of the dualisation program: the attempt to describe internal logic objects that exhibit paraconsistent algebras in their own right. We were led to this type of investigation by the discovery that straightforward categorial duals of ordinary subobject logic structures would not produce logic structures that were dual in the logical sense. This was the import of chapters four and five. Our investigations focused on sheaves for their properties in relation to base space topologies. We found essentially two things. First, logic objects in sheaf categories contain component BrAs but are not generally themselves BrAs within their categories. This was the import of chapters eight and nine. An interesting corollary of this investigation was that subobject lattices in Grothendieck toposes are indeed BrAs (but not naturally so). Second, we discovered that the original dualisation idea contained in the complement classifier notion has an instantiation in categories. With chapter eleven and twelve we found a genuine complement classifier in a category of covariant sheaves.

A barrier to the discovery of BrA logic objects with respect to subobject structure in categories seems to be the failure of naturalness of BrA operations by virtue essentially, of the failure of closure operations to distribute over intersections. An interest for the future is in seeing if this remains a problem in categories that are co-exponentiated (in the way of closed set topologies) as opposed to exponentiated in the way of open set topologies and toposes. The notion of co-exponentiated categories is itself of interest with respect to the internal language of categories. We have seen in chapter fourteen one of the ways that we may develop categories in terms of a logical language. The method there located interpretation of logical connectives in the subobject lattices. We expect that a genuinely interesting extension of our investigation will be in the examination of those categorial structures determined by the imposition of a given non-classical logic as a background for the interpretation of the usual formulae that describe categories.

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