PARTITIONS INTO LARGE UNEQUAL PARTS

by

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Abstract

Let \( u = (u_j)_{j=1}^\infty \) be a strictly increasing sequence of positive integers and for \( x \geq 1 \) let \( U(x) \) be the number of terms of \( u \) which do not exceed \( x \). For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \) define \( q_u(m, n) \) to be the number of partitions of \( n \) into distinct parts coming from the sequence \( u \) and exceeding \( m \).

In the special case when \( u \) is the sequence of positive integers, the classical function \( q(n) = q_u(0, n) \) and, more recently, the function \( q(m, n) = q_u(m, n) \) have been investigated by several authors. Freiman and Pitman [1] have recently given asymptotic estimates for \( q(m, n) \) as \( n \to \infty \).

In the general case the function \( q_u(m, n) \) has also been studied, mainly for \( m = 0 \). In particular, Roth and Szekeres [2] have given an asymptotic formula for \( q_u(0, n) \) which is widely applicable.

This thesis studies the asymptotic behaviour of \( q_u(m, n) \) as \( n \to \infty \) for sequences such that \( U(x) \sim C_0 x^s (\log x)^{-t} \) as \( x \to \infty \), where \( C_0 > 0 \), \( s > 0 \) and \( t \geq 0 \) are constants. Chapter 1 introduces the problem and provides historical background and Chapter 2 gives auxiliary results.

Chapter 3 presents the main theorem. For \( u \) as above satisfying a suitable further condition, and for given small positive \( \delta \), this gives an asymptotic estimate for \( q_u(m, n) \) which is valid uniformly in \( m \) such that \( 0 \leq m \leq n^{1-\delta} \) as \( n \to \infty \). The result is motivated by probabilistic considerations similar to those of [1] and the proof uses the circle method as in [1].

The next two chapters cover applications of the main theorem. The first part of Chapter 4 shows that the theorem applies to three wide classes of sequences which together include all the specific examples in [2]. The remainder of the chapter shows that under the conditions of the main theorem, for relatively small \( m \), we have, as \( n \to \infty \)

\[
q_u(m, n) \sim 2^{-U(m)} q_u(0, n).
\]

Chapter 5 uses the main theorem to obtain precise results about \( q_u(m, n) \) in the case when \( u \) is the sequence of \( k \)-th powers.

Chapters 6 and 7 are devoted to more detailed study of the case when \( u \) is the sequence of positive
integers. This work extends the results of [1].

References


Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference had been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for photocopying and loan.

SIGNED

DATE 31/3/96
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# Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sets of numbers</strong></td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>The set of integers.</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>The set of positive integers.</td>
</tr>
<tr>
<td>$\mathbb{Q}$</td>
<td>The set of rational numbers.</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>The set of real numbers.</td>
</tr>
<tr>
<td>$(0, \infty)$</td>
<td>The set of positive real numbers.</td>
</tr>
<tr>
<td>$\text{card } A$</td>
<td>The cardinality of the set $A$.</td>
</tr>
<tr>
<td><strong>Variables</strong></td>
<td></td>
</tr>
<tr>
<td>$h, l, m, q$</td>
<td>Non-negative integers</td>
</tr>
<tr>
<td>$j, k, n$</td>
<td>Positive integers.</td>
</tr>
<tr>
<td>$s$</td>
<td>Positive real number $\leq 1$.</td>
</tr>
<tr>
<td>$t$</td>
<td>Non-negative real number.</td>
</tr>
<tr>
<td>$p$</td>
<td>Prime number.</td>
</tr>
<tr>
<td><strong>Orders of magnitude</strong></td>
<td></td>
</tr>
<tr>
<td>$f(n) \sim g(n)$ as $n \to \infty$</td>
<td>$\lim_{n \to \infty} f(n)/g(n) = 1$.</td>
</tr>
<tr>
<td>$f(n) \ll g(n)$ as $n \to \infty$</td>
<td>There is a constant $C &gt; 0$ and a number $n_0$ such that for $n &gt; n_0$, $</td>
</tr>
<tr>
<td>$f(n) = O(g(n))$ as $n \to \infty$</td>
<td>As above.</td>
</tr>
<tr>
<td>$f(n) \ll_{\delta} g(n)$ as $n \to \infty$</td>
<td>There is a constant $C = C(\delta) &gt; 0$ and a number $n_0 = n_0(\delta)$ both depending on $\delta$ such that for $n &gt; n_0$, $</td>
</tr>
<tr>
<td>$f(n) \asymp g(n)$ as $n \to \infty$</td>
<td>Both $f(n) \ll g(n)$ and $f(n) \gg g(n)$ as $n \to \infty$.</td>
</tr>
</tbody>
</table>
Arithmetical symbols

\[ [x] \]  Least integer \( \geq x \).

\[ [x] \]  Greatest integer \( \leq x \).

\[ \{x\} \]  \( x - [x] \).

\[ \| x \| \]  Distance of \( x \) from nearest integer.

\[ e(x) \]  \( e^{2\pi i x} \).

Notation related to the sequence \( u \)

\[ u = (u_j) \]  The infinite sequence of integers \((u_1, u_2, u_3, \ldots)\) otherwise denoted \( (u_j)_{j=1}^{\infty} \).

\[ U(x) \]  The number of members of the sequence \( u \) which are at most \( x \).

\[ q_u(m, n) \]  Number of partitions of \( n \) into unequal parts \( > m \) and coming from \( u \).

\[ p_u(m, n) \]  Number of ordinary partitions of \( n \) into parts \( > m \) and coming from \( u \).

\[ \sum_{a \leq u \leq b} f(u_j) \]  Summation of \( f(u_j) \) over members of the sequence \( u \) lying in the interval \((a, b] \).

Notation related to the sequence of positive integers

\[ \mathbb{N} = (j) \]  The set of positive integers as a sequence.

\[ q(n) \]  \( q_{\mathbb{N}}(0, n) \).

\[ p(n) \]  \( p_{\mathbb{N}}(0, n) \).

\[ q(m, n) \]  \( q_{\mathbb{N}}(m, n) \).

\[ p(m, n) \]  \( p_{\mathbb{N}}(m, n) \).

Notation related to the primes

\[ p_j \]  The \( j \)-th prime number.

\[ \pi(x) \]  The number of primes less than or equal to \( x \).

\[ \sum_{a \leq p \leq b} \]  Summation over primes \( p \) in the interval \((a, b] \).

Other special notation

\[ B_k \]  The \( k \)-th Bernoulli number.

\[ B_k(x) \]  The \( k \)-th Bernoulli polynomial.

\[ \Gamma(z) \]  The Gamma function \( \int_0^\infty t^{z-1} e^{-t} dt \).

\[ \zeta(s) \]  Riemann's zeta function, defined as \( \sum_{1}^{\infty} n^{-s} \).

\[ \varphi \]  The characteristic function corresponding to a random variable.

\[ J_0(z) \]  The \( 0 \)-th order Bessel function.
CHAPTER 1

General introduction

1. Introduction

Let \( u = (u_j)_{j=1}^\infty \) be a given strictly increasing sequence of positive integers. Unless otherwise indicated, all sequences will be indexed by \( \mathbb{N} \), the set of all positive integers, and so we write simply

\[ u = (u_j). \]

Let \( n \) be a positive integer. A partition of \( n \) into parts from \( u \) is an expression of the form

\[ n = u_{j_1} + u_{j_2} + \ldots + u_{j_r}. \]

(1.1)

where

\[ u_{j_1} \leq u_{j_2} \leq \ldots \leq u_{j_r}. \]

Here the \( u_{j_i}'s \) are the parts and the number \( r \) of parts is arbitrary. Such partitions, where the parts are not necessarily distinct, will be called \textit{ordinary partitions}.

This thesis will focus on partitions into \textit{unequal} or \textit{distinct} parts from \( u \), that is, expressions of the form (1.1) where

\[ u_{j_1} < u_{j_2} < \ldots < u_{j_r}. \]

and \( r \) is arbitrary.

We denote the sequence of positive integers (that is, the set \( \mathbb{N} \) viewed as a sequence) by \( \mathbb{N} = (j) \). The classical theory of partitions into positive integers (see, for example, Chapter 9 of Gupta [18]) corresponds to the special case when \( u = \mathbb{N} \).

\textbf{Definition 1.1.} Let \( u = (u_j) \) be a strictly increasing sequence of positive integers and let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Define \( q_u(m, n) \) to be the number of partitions of \( n \) into unequal parts from \( u \) with each part greater than \( m \). Define \( p_u(m, n) \) to be the number of ordinary partitions of \( n \) into parts from \( u \) with each part greater than \( m \).
For the case of partitions into positive integers, we define

\begin{equation}
q(m, n) = q_N(m, n), \quad p(m, n) = p_N(m, n).
\end{equation}

The classical partition functions are the functions \( q(n) \) and \( p(n) \) given by

\begin{equation}
q(n) = q(0, n) = q_N(0, n), \quad p(n) = p(0, n) = p_N(0, n).
\end{equation}

Thus, \( q(n) \) is the number of partitions of \( n \) into distinct positive integers and \( p(n) \) is the number of partitions of \( n \) into positive integers.

Although results for partitions into distinct parts and for ordinary partitions often parallel, some methods appear to work more easily for one and not for the other. This thesis will study partitions into unequal parts, and in particular the asymptotic behaviour of \( q_u(m, n) \) as \( n \to \infty \).

In Section 2 the tool of generating functions will be introduced and some basic lemmas which give an expression for \( q_u(m, n) \) in terms of its associated generating function will be given. In the next six sections, namely, Sections 3 to 8, background on relevant history and methods will be provided, giving more detail on partitions into distinct parts. Apostol [2] provides more information on the estimation of classical partition functions \( q(n) \) and \( p(n) \) described in Section 3. Postnikov [37] gives an excellent introduction to the background to Sections 4 to 8.

In Section 9 and Section 10, I shall introduce the aims, methods and plan of this thesis. Finally, in Section 11, various conventions used throughout this thesis will be introduced.

2. Generating functions and some basic lemmas

We present in this section a lemma which gives an expression for \( q_u(m, n) \). An elegant technique in solving the problem is to construct a function whose power series expansion has \( q_u(m, n) \) as the coefficient of \( z^n \), and this function is called the generating function associated with the partition function \( q_u(m, n) \).

It is well known that (see pages 274–276, §19.3–§19.4 of Hardy & Wright [22]) the generating function of the classical functions \( p(n) \) and \( q(n) \) for \( n \geq 1 \) are the functions given for \( |z| < 1 \) by

\begin{equation}
F(z) = \prod_{j=1}^{\infty} \frac{1}{1 - z^j} = 1 + \sum_{n=1}^{\infty} p(n)z^n.
\end{equation}

and

\begin{equation}
G(z) = \prod_{j=1}^{\infty} (1 + z^j) = 1 + \sum_{n=1}^{\infty} q(n)z^n,
\end{equation}
The above functions are closely related to the Dedekind eta function \( \eta(\tau) \), defined (see, for example, Chapter 3 of Apostol [2]) via

\[
\eta(\tau) = e^{\pi i \tau / 12} \prod_{j=1}^{\infty} (1 - e^{2 \pi i j \tau}) \quad \text{for} \quad \Im(\tau) > 0,
\]

and \( \eta \) transforms under a modular transformation

\[
\tau' = \frac{a \tau + b}{c \tau + d},
\]

(\( a, b, c, d \) are integers such that \( c > 0 \) and \( ad - bc = 1 \)) as described by the functional equation

\[
\eta(\tau) = \eta(\tau')(-i(c \tau + d))^{-1/2} \exp\left(-\pi i \left(\frac{c + d}{12c} + s(-d, c)\right)\right) \quad \Im(\tau) > 0,
\]

where \( s(-d, c) \) is a Dedekind sum (see Theorem 3.4 page 52 of Apostol [2]). It is clear that

\[
F(e^{2\pi i \tau}) = \frac{e^{\pi i \tau / 12}}{\eta(\tau)}
\]

and the functional equation (1.6) for \( \eta \) gives rise to a functional equation for \( F \). By writing the generating function \( G(z) \) (see (1.5)) for \( q(n) \) in the form

\[
G(z) = \frac{F(z)}{F(z^2)},
\]

and using the functional equation for \( F \) we obtain a functional equation for \( G \).

By using Cauchy’s integral formulae for the coefficient of \( z^n \) in (1.4) and (1.5) we obtain the formulae

\[
p(n) = \frac{1}{2\pi i} \int_C F(z)z^{-n-1}dz,
\]

\[
q(n) = \frac{1}{2\pi i} \int_C G(z)z^{-n-1}dz,
\]

where \( C \) is the anticlockwise circle with centre \( O \) and radius \( \rho < 1 \).

By similar methods to those proving (1.4) and (1.5), it can be shown that the generating function of \( q_u(m, n) \) and \( p_u(m, n) \) for general \( u \) are

\[
\prod_{m < u_j < \infty} (1 + z^{u_j}) = 1 + \sum_{n=m+1}^{\infty} q_u(m, n)z^n
\]

and

\[
\prod_{m < u_j < \infty} \frac{1}{1 - z^{u_j}} = 1 + \sum_{n=m+1}^{\infty} p_u(m, n)z^n,
\]

for \( |z| < 1 \). Formula (1.10) leads to the following lemma.
1. General Introduction

Lemma 1.2. Let $\sigma$ be a positive real number. With $q_u(m, n)$ defined as in Definition 1.1 we have

\[(1.11) \quad q_u(m, n) = \frac{1}{2\pi} e^{\sigma n} \prod_{m < u_j < \infty} (1 + e^{-\sigma u_j}) \times \int_{-\pi}^\pi \prod_{m < u_j < \infty} \left( \frac{1}{1 + e^{-\sigma u_j}} + \frac{e^{-\sigma u_j}}{1 + e^{-\sigma u_j}} e^{iu_j \theta} \right) e^{-in\theta} d\theta.\]

Proof. By using Cauchy’s integral formula for the coefficient of $z^n$ in (1.10), we have

\[q_u(m, n) = \frac{1}{2\pi i} \int_C \prod_{m < u_j < \infty} (1 + z u_j) z^{-n-1} dz,\]

where $C$ is the circle of radius $e^{-\sigma}$, parametrised by $z = e^{-\sigma + i\theta}$ for $\theta \in (-\pi, \pi]$, and the result follows.

Alternatively we have the lemma.

Lemma 1.3. Let $\sigma$ be a real number. With $q_u(m, n)$ as in Definition 1.1, we have

\[q_u(m, n) = e^{\sigma n} \prod_{m < u_j \leq n} (1 + e^{-\sigma u_j}) \times \int_{-1/2}^{1/2} \prod_{m < u_j \leq n} \left( \frac{1}{1 + e^{-\sigma u_j}} + \frac{1}{1 + e^{\sigma u_j}} e^{2\pi i u_j} \right) e^{-2\pi i n \alpha} d\alpha.\]

Proof. It is easily seen that $q_u(m, n)$ is the coefficient of $e^{2\pi i n \alpha}$ in the product

\[e^{\sigma n} \prod_{m < u_j \leq n} (1 + e^{-\sigma u_j} e^{2\pi i u_j}).\]

Using the fact that

\[\int_{-1/2}^{1/2} e^{2\pi i k \alpha} d\alpha = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases}\]

we see the result immediately.

This can also be proved by noting that $q_u(m, n)$ is the coefficient of $z^n$ in $\prod_{m < u_j \leq n} (1 + z u_j)$ and using Cauchy’s integral formula for this coefficient.

3. The classical functions $p(n)$ and $q(n)$ and the modular function approach

The first major work in asymptotic partition theory was the important 1918 paper of Hardy and Ramanujan [21] which presented their investigation of the asymptotic behaviour of $p(n)$ and $q(n)$. (This had been preceded by three short papers which had sketched their work).

In this paper, they presented the following theorem, with a full proof of the result for $p(n)$ and a sketch of that for $q(n)$ (these results had also been found independently by Uspensky [43]).

Theorem A. With $p(n)$ and $q(n)$ as in (1.3), as $n \to \infty$,

\[p(n) \sim \frac{e^{\pi^{1/3} \sqrt{2n/3}}}{4^{1/4} n^{3/4}} \quad \text{and} \quad q(n) \sim \frac{e^{\pi^{1/3} \sqrt{2n/3}}}{4^{1/4} n^{3/4}}.\]
Their major result, however, was a theorem giving an asymptotic series expansion of $p(n)$ in terms of Bessel functions of which Theorem A is a consequence.

Hardy and Ramanujan approached the problem of estimating $p(n)$ via the formula (1.8) for $p(n)$ as an integral around the circle $C$ with centre $O$ and radius $r = 1 - \beta/n$ for a constant $\beta$ and large $n$. They dissected $C$ into arcs, each centred at $\rho e^{2\pi i h/k}$ for some rational $h/k$ with $k \leq \sqrt{n}$ and sought to estimate $F(z)$ on each such arc. This work was the starting point of methods which were further developed and applied in partition theory, in the work of Hardy and Littlewood and later Vinogradov and others on Waring's Problem, and later other additive problems. These methods are based on lemmas along the lines of Lemma 1.2 and Lemma 1.3 and are referred to as versions of the circle method (or the Hardy-Littlewood method, see, for example, Vaughan [44]).

In order to prove Theorem A, Hardy and Ramanujan combined the circle method approach with the transformation theory of $\eta$ in (1.6) to obtain good estimates of $F(z)$ on the arc of $C$ corresponding to $h/k$. Their proof of their asymptotic expansion for $p(n)$ required more extensive use of the transformation theory of modular functions.

Hardy and Ramanujan in one of the later sections of their paper stated without proof the asymptotic main term of $p_u(0, n)$ for $u = (j^k)$ and indicated how their work partly leads to the proof. Wright [46] later obtained, using a transformation theory especially developed for the associated generating function, an asymptotic expansion of $p_u(0, n)$ for $u = (j^k)$.

In an earlier paper, Hardy and Ramanujan [20] had also proved that if $u = (p_j)$, the sequence of primes, then

$$\log p_u(0, n) \sim \frac{2\pi}{31/2} \left( \frac{n}{\log n} \right)^{1/2} \text{ as } n \to \infty.$$ 

Modifying the method of Hardy and Ramanujan, Rademacher [38] in 1937 obtained a convergent series for $p(n)$. In 1942 Hua [26] proved the following asymptotic formula for $q(n)$ by exploiting the transformation theory of $G(z)$ in an analogous method to that done by Hardy and Ramanujan.

**Theorem B (Hua).** The function $q(n) = q_u(0, n)$ has the asymptotic expansion

$$q(n) = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \sum_{k|\text{odd}, (h,k)=1} \omega_{h,k} e(-hn/k) \frac{d}{dn} J_0 \left( \frac{ir}{k} \sqrt{\frac{n+1/24}{3}} \right),$$

where

$$J_0(\phi) = \frac{1}{2\pi i} \int_{C} t^{-1} \exp(t - z^2/4t) dt$$
is the 0-th order Bessel function. Here $C$ is a suitable contour encircling the origin once counterclockwise,

\[ \omega_{h,k} = \begin{cases} 
\varepsilon_{h,k} e^{-(h + h')/(24k)}, & k \text{ even,} \\
\varepsilon_{h,k} e^{-(2h - h')/(48k)}, & k \text{ odd,}
\end{cases} \]

and $\varepsilon_{h,k}$ is defined as

\[ e\left( -\left( \frac{h'^2 - 1}{16} \left( \frac{1-hh'}{k} - 1 \right) + \frac{h'(1-hh')}{16k} + \frac{1}{48} \left( k + \frac{1-hh'}{k} \right) \left( hh'^2 - h' - h \right) \right), \right. \]

\[ \left. e\left( \frac{1}{16} (k + (1-hh'))/(h + h' - \sqrt{h' h'}) \right), \quad \text{for } 2 \nmid k, 2 \nmid h, \right. \]

\[ \left. e\left( -\frac{1}{16} (k^2 - 1 - hk + (h + h')(hh' k - (hh' - 1)/k))/3 \right), \quad \text{for } 2 \nmid k \text{ and } 2|h. \right. \]

Here $e(x)$ denotes $e^{2\pi ix}$. \[ \]

It can be checked that Hua's asymptotic expansion implies the result on $q(n)$ in Theorem A.

These ideas have been extended and refined by many authors on the asymptotic theory of partitions
(see, for example, Ahlmvist [1] for a recent example).

However, we cannot expect the generating function (1.10) of $q_u(m, n)$ for a general sequence $u$ to
be closely related to a modular function as in (1.7). For this reason this approach will not be pursued
further here.

4. The Tauberian approach of Ingham

In 1941 Ingham [28] derived the asymptotic behaviour of $p(n)$ and $q(n)$ from a knowledge of the
asymptotic behaviour of the associated generating functions near $z = 1$. His aim was to specify properties
of the generating functions (1.4) and (1.5) of $p(n)$ and $q(n)$ sufficient to deduce Theorem A without
recourse to the sophistication of the transformation theory of modular function. In doing so, he also
obtained asymptotic estimates for $p_u(0, n)$ and $q_u(0, n)$ as well as the associated summatory functions
$\sum_{n \leq x} p_u(0, n)$ and $\sum_{n \leq x} q_u(0, n)$ for a general sequence $u$ satisfying specified conditions.

Ingham first gave a general Tauberian theorem showing that under suitable conditions the asymptotic
behaviour of $A(n)$ as $n \to \infty$ can be deduced from that of

\[ f(s) = \int_0^\infty e^{-sx} dA(x) \quad \text{as} \quad s \to 0, \quad \text{where} \quad A(0) = 0. \]

He applied this with

\[ A(x) = \sum_{n \leq x} p_u(0, n) \]

to obtain his main theorem (Theorem 2 of [28]). By taking $h = 1$ in this theorem we obtain the following
result.
5. CONDITIONS FOR \( q_u(m, n) \) TO BE POSITIVE

**Theorem C.** Let \( u = (u_j) \) be a strictly increasing sequence of positive integers with counting function

\[ U(x) = \{ j : u_j \leq x \} = C_0 x^s + R(x), \]

where

\[ (I) \quad \int_0^x \frac{R(v)}{v} dv = b \log v + c + o(1) \quad \text{as} \quad x \to \infty. \]

Let

\[ \alpha = \frac{s}{1 + s}, \quad M = (C_0 s \Gamma(s + 1) \zeta(s + 1))^{1/s}, \quad M^* = M(1 - 2^{-s})^{1/s}. \]

If \( p_u(0, n) \) is a strictly increasing function of \( n \) then as \( n \to \infty \)

\[ p_u(0, n) \sim \left( \frac{1 - \alpha}{2\pi} \right)^{1/2} e^{c M^{-s\alpha/2} n^{s\alpha/2 - 1} e^{(M^*)^s/n}}. \]

If \( q_u(0, n) \) is a strictly increasing function of \( n \) then as \( n \to \infty \),

\[ q_u(0, n) \sim \left( \frac{1 - \alpha}{2\pi} \right)^{1/2} 2^{-s\alpha/2} n^{s\alpha/2 - 1} e^{(M^*)^s/n}. \]

The estimate for \( q_u(0, n) \) remains valid if Condition (I) is replaced by the weaker condition

\[ (I') \quad \int_0^x \frac{R(v)}{v} dv = b x + o(x) \quad \text{as} \quad x \to \infty. \]

As cases of the above theorem, Ingham obtained Theorem A and also corresponding results for \( p_u(0, n) \) and \( q_u(0, n) \) when \( u \) is the sequence of positive \( k \)-th powers of the integers. Ingham's hypotheses do not include the primes.

In 1956 Bateman and Erdős [4] established necessary and sufficient conditions on \( u \) for the function \( p_u(0, n) \) to be strictly increasing. In the mean time, Auluck and Haselgrove [3] had managed to remove the monotonicity condition for \( p_u(0, n) \) from Theorem C.

In 1950, Bringham [6] obtained (under assumption of the Riemann Hypothesis) an asymptotic formula for a summatory function \( P(n) = \sum_{j \leq n} \nu_0^u(0, j) \), where \( \nu_0^u(0, j) \) is a certain weighted partition function involving primes and powers of primes (Specifically \( \prod_{j=1}^{\infty} (1 - z^n)^{-\Lambda(n)} = \sum_{j=1}^{\infty} \nu_0^u(0, j) z^j \), where \( \Lambda(n) = \log p \) if \( n = p^m \) and \( = 0 \) otherwise) by using a Tauberian theorem of Ingham.

5. Conditions for \( q_u(m, n) \) to be positive

Before studying asymptotic behaviour, it is natural to ask: What conditions must be satisfied by a sequence \( u \) such that every sufficiently large number is representable as the sum of distinct members of \( u \)? Clearly in the case \( u = \mathbb{N} = (j), q_u(0, n) = q(n) > 0 \) and \( p_u(0, n) > 0 \). However, for general sequences it is not obvious whether representations exist.
For ordinary partitions, known results in additive number theory yield a large supply of sequences for which \( p_u(0, n) > 0 \) for large \( n \), see for example Roth and Szekeres [40] and Richmond [39]. Existence of partitions into distinct parts for certain sequences other than \( \mathbb{N} \) follows from the asymptotic results for \( u = (j^k) \) and for general \( u \) discussed in Section 3 and Section 4 above. Cassels [7] looked at the existence question separately for general sequences and proved the following theorem.

**Theorem D (Cassels).** Let \( u \) be a strictly increasing sequence of positive integers such that

\[
(C1) \quad \frac{U(2n) - U(n)}{\log \log n} \to \infty \quad \text{as} \quad n \to \infty
\]

and

\[
(C2) \quad \sum_{0 < u_j < \infty} \|u_j\alpha\|^2 \to \infty \quad \text{as} \quad n \to \infty
\]

for all \( \alpha \in [-1/2, 1/2] \). Then every sufficiently large number is representable as a sum of distinct elements of \( u \).

Examples of sequences that satisfy (C1) and (C2) are \( u = (P(j)) \) and \( u = (P(p_j)) \) where \( P \) is a suitable polynomial.

Cassels’ proof was based on Lemma 1.2. He took \( \sigma \) such that

\[
\sum_{m < u < \infty} \frac{u_j}{1 + e^{u_j}} = n,
\]

in order to remove the first order contribution of \( \alpha \) in the logarithm of the integrand in (1.11). This allowed Cassels to approximate the integrand by a function of the form \( e^{-C\alpha^2} \) and so derive an asymptotic estimate. This method of eliminating the first order contribution of \( \alpha \) to the logarithm of the integrand is commonly referred to as the *saddle point method*, *Laplace’s method* or the *method of steepest descent* (see Section 2 Chapter 7 of Marsden [32] or Section 4.1 Chapter 9 of Gupta [18]).

Using a similar method to Cassels, Szekeres [42] in 1953 had independently published an asymptotic formula for the number of partitions of \( n \) into at most \( k \) parts, all of which come from a fairly general sequence \( u \). He appears to have been among the first to use the approach in the asymptotic theory of partitions (Also Meinardus [33, 34] appears to be among the first. However, his method applies only to ordinary partitions).

### 6. The asymptotic formula of Roth and Szekeres

Ingham’s Tauberian approach to finding asymptotic estimates for \( q_u(0, n) \) required monotonicity of \( q_u(0, n) \). Paul Erdős posed Roth and Szekeres with the problem of proving monotonicity of \( q_u(0, n) \) when
u = (p_j) (p_j is the j-th prime) and motivated by this, Roth and Szekeres [40] answered the question: Find an asymptotic expansion for the number of partitions of a number into distinct members of u, where u is a given sequence satisfying appropriate conditions.

The conditions on the sequence u used by Roth and Szekeres were as follows.

\[ \lim_{n \to \infty} \frac{\log u_n}{\log n} = s > 0, \]

\[ \inf \left( (\log n)^{-1} \sum_{j=1}^{n} \| \alpha u_j \| ^2 \right) \to \infty, \]

where the infimum is taken over \( \alpha \in ((2u_n)^{-1}, 1/2). \)

It can be checked that Hypothesis RS2 implies Hypothesis C2, while Hypothesis C1 implies Hypothesis RS1. Also, Ingham’s Condition (I) implies Hypothesis RS1. Roth and Szekeres showed that Conditions RS1 and RS2 are satisfied by a wide range of sequences u, including sequences of the form \( (P(j)) \) and \( (P(p_j)), \) where \( P \) is a suitable polynomial function in each case, and in particular including the sequences \( u = \mathbb{N}, (p_j) \) and the sequence of k-th powers \( (j^k). \) More detail will be provided in Chapter 4.

The asymptotic formula of Roth and Szekeres for \( q_u(0,n) \) is given in the following theorem.

**Theorem E (Roth & Szekeres).** Let \( u = (u_j) \) be a sequence of positive integers which is strictly increasing for \( j \geq j_0 \) and which satisfies Hypotheses RS1 and RS2 above. Let \( q_u(0,n) \) be as in Definition 1.1 and let \( \delta > 0. \) For any fixed positive integer \( M \geq 2 \) we have as \( n \to \infty \)

\[ q_u(0,n) = \frac{1}{\sqrt{2\pi A_2}} \times \exp \left( \sum_{j=1}^{\infty} \left( \frac{\sigma u_j}{1 + e^{\sigma u_j}} + \log(1 + e^{-\sigma u_j}) \right) \right) \]

\[ \times \left( 1 + \sum_{h=1}^{M-2} D_h + O(n^{-(M-1)/(\alpha+1)+\delta}) \right), \]

where \( \sigma \) is determined from \( n \sum_{j=1}^{\infty} u_j (1 + e^{\sigma u_j})^{-1}, \) and where \( D_h = A_2^{-6h} \sum_{k_1=2}^{\infty} \cdots \sum_{k_{6h}=2}^{\infty} d_{k_1, \ldots, k_{6h}} A_{k_1} A_{k_2} \cdots A_{k_{6h}}, \)

the summation being subject to

\[ k_1 + k_2 + \ldots + k_{6h} = 12h, \]

where the \( d \)'s are certain numerical coefficients, and

\[ A_k = \sum_{0 < u_j < \infty} u_j^k (1 + e^{\sigma u_j})^k, \]

with \( h_k(x) \) being a polynomial of degree \( k-1 \) (the first couple are \( h_1(x) = 1, h_2(x) = x). \)
Note that the notation in the paper of Roth and Szekeres has $g_k$ in place of $h_k$ above. I have altered the notation because later I shall use $g_k$ for $k \geq 1$ defined via $g_k(x) = \sum_{h=1}^{\infty} (-1)^{h+1} h^{k-1} e^{-hx}$, from which it can be shown that

$$A_k = \sum_{0 < u_j < \infty} u_j^k g_k(\sigma u_j),$$

with $g_k(x)$ being the quotient of a polynomial of degree $k - 1$ in $e^x$ divided by $(1 + e^x)^k$ (in fact $g_1(x) = 1/(1 + e^x)$, $g_2(x) = e^x/(1 + e^x)^2$).

Roth and Szekeres showed that $D_j = n^{-j/(1+\epsilon)+\epsilon(1)}$ as $n \to \infty$ and hence for sufficiently large $n$, the $D$'s in (1.12) satisfy

$$D_1 > D_2 > \ldots > D_{M-2} > n^{-(M-1)/(\epsilon+1)+\epsilon}.$$  

Taking $M = 2$ in the above theorem gives the following corollary.

**Corollary 1.4.** With the notation and hypotheses as in the above theorem, as $n \to \infty$

$$q_u(0, n) = \frac{1}{\sqrt{2\pi A_2}} \times \exp \left( \sum_{j=1}^{\infty} \left( \frac{\sigma u_j}{1 + e^{-\sigma u_j}} + \log(1 + e^{-\sigma u_j}) \right) \right) \times \left( 1 + O(n^{-1+\epsilon}) \right).$$

A result of this type, with a single main term plus error will be called an asymptotic estimate, to distinguish from an asymptotic expansion as in the theorem for $M > 2$. The main term is

$$\frac{1}{\sqrt{2\pi A_2}} \times \exp \left( \sum_{j=1}^{\infty} \left( \frac{\sigma u_j}{1 + e^{-\sigma u_j}} + \log(1 + e^{-\sigma u_j}) \right) \right) = \frac{1}{\sqrt{2\pi A_2}} \times e^{\sigma_n} \prod_{j=1}^{\infty} (1 + e^{-\sigma u_j}).$$

For the proof of Theorem E, Roth and Szekeres, like Cassels, used Lemma 1.2 with $\sigma$ chosen as in the saddle point method. In order to estimate the integral on the right hand side of (1.11) they split the range of integration into two parts. The first part was an interval around 0, which I shall, in keeping with the vocabulary of this thesis, call the main interval, corresponding to the main integral, with the remaining part, called the supplementary interval, corresponding to the supplementary integral. Their second hypothesis, RS2, ensures that the supplementary integral is dominated by the main integral.

In their paper, Roth and Szekeres gave a number of applications of Theorem E. They proved, in particular, that if $u$ satisfies RS1 and RS2 then $q_u(0, m)$ is strictly increasing, thus answering the original question.

Although Corollary 1.4 gives an asymptotic estimate it is not explicit in terms of $n$ because of the product

$$e^{\sigma_n} \prod_{j=1}^{\infty} (1 + e^{-\sigma u_j})$$
involving $\sigma$ which needs to be estimated. As will be discussed further in Chapter 5 they gave (without details of proof) an asymptotic estimate of $q_u(0, n)$ for $u = (P(j))$ for a suitable polynomial $P$ of degree $k$.

Roth and Szekeres also gave an asymptotic estimate of the logarithm of the quantity $q_u(0, n)$ when $u$ is the sequence of primes, namely

$$\log q_u(0, n) = \left(\frac{2}{3}\right)^{1/2} \pi \left(\frac{n}{\log n}\right)^{1/2} (1 + O(\log \log n / \log n)) \quad \text{as} \quad n \to \infty.$$ 

There does not appear to have been any subsequent improvement either of this or of the corresponding result for $\log p_u(0, n)$ (with $u = (p_j)$) of Hardy and Ramanujan (see Section 3).

Finally, Roth and Szekeres also provided (without details of proof) an asymptotic formula for $p_u(0, n)$ under the Hypotheses RS1 and RS2', where Hypothesis RS2' is a relaxation of the Hypothesis RS2 and is

$$(RS2') \quad \inf \left( (\log n)^{-1} \sum_{j=1}^{n} \frac{n_j}{u_j^2} ||\alpha u_j||^2 \right) \to \infty,$$

where the infimum is taken over $\alpha \in (1/((2\pi n)^{-1}, 1/2))$.

In 1975, Richmond [39], building on this work gave an asymptotic formula for $p_u(0, n)$ under more general conditions than (RS1) and (RS2'). In 1977, Erdős and Richmond [12] applied this work of Roth and Szekeres to give explicit asymptotic formulae for $p_u(0, n)$ and $q_u(0, n)$ when $u = (\lfloor \theta j \rfloor)$, for almost all irrational numbers $\theta > 1$.

7. Earlier work on partitions into parts greater than $m$

Herzog [23], [24] considered the problem of partitions into distinct parts coming from a sequence $u$ with each part exceeding $m$. Using a Tauberian theorem for the Laplace transform he gave a main term of an asymptotic estimate for the logarithm of both the partition function $q_u(m, n)$ and its associated summatory function.

Over the past few years Erdős, Nicolas and Szalay [13] have investigated the asymptotic behaviour of $q(m, n) = q_N(m, n)$, concentrating on the case when $m$ is relatively small (less than $n^{1/2}$).

**Theorem F.** With $q(m, n)$ as in (1.2) as $n \to \infty$ we have

$$q(m, n) \sim \frac{1}{2^m} q(n + \lfloor m(m - 1)/4 \rfloor)$$

for $m = o(n^{1/4}/(\log n)^{1/3})$.

**Proof.** See Theorem 2 of Erdős, Nicolas and Szalay [13]. □
Their proof involved a Tauberian theorem applied to the logarithm of \( q(m, n) = q_N(m, n) \). Similar results of this kind have been provided for \( p(m, n) \), namely as \( n \to \infty \),

\[
p(m, n) = \left( \frac{n}{\sqrt{6n}} \right)^{m-1} (m-1)! p(n) \left( 1 + O(m/\sqrt{n}) \right),
\]

uniformly for \( 0 \leq m \leq n^{1/4} \), due to Dixmier and Nicolas [9].

8. Results of Freiman and Pitman on \( q(m, n) \)

As discussed in Chapter 3 of Postnikov [37], probabilistic ideas and, in particular, versions of the local limit theorem of probability theory have been successfully applied to additive problems in number theory (see also, for example, Elliott [11] and the references in Postnikov [37]). Freiman developed a method for partition problems which drew on the ideas of local limit theorems, even where no theorem of this type was available, and was based on Lemma 1.3. This method was described and used in a number of papers by Freiman (see, for example, [15]). Recently, this method was used by Freiman and Pitman [16] in their study of the asymptotic behaviour of \( q(m, n) = q_N(m, n) \) as \( n \to \infty \). They proved the following.

**Theorem G (Freiman and Pitman).** For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let \( \sigma \) be defined by \( n = \sum_{m < j \leq n} j/(1 + e^{\sigma j}) \) and let \( A_2 = \sum_{m < j \leq n} j^2 e^{\sigma j}/(1 + e^{\sigma j})^2 \). Then as \( n \to \infty \) we have

\[
q(m, n) = \frac{1}{\sqrt{2\pi A_2}} e^{\sigma n} \prod_{m < j \leq n} (1 + e^{-\sigma j})(1 + E),
\]

where

\[
E = E(m, n) = O((\log n)^{9/2} \max(n^{-1/4}, (m/n)^{1/2})),
\]

uniformly with respect to \( m \) such that \( 0 \leq m \leq K_0 n/\log^9 n \), \( K_0 \) an effective positive constant.

The starting point of the proof of this theorem by Freiman and Pitman was Lemma 1.3 which gives in the case when \( u = N \)

\[
q(m, n) = e^{\sigma n} \prod_{m < j \leq n} (1 + e^{-\sigma j}) \times \int_{-1/2}^{1/2} \varphi(\alpha) e^{-2\pi i m n} d\alpha,
\]

where

\[
\varphi(\alpha) = \prod_{m < j \leq n} \left( \frac{1}{1 + e^{-\sigma j}} + \frac{1}{1 + e^{\sigma j} e^{2\pi i \alpha j}} \right).
\]

Freiman and Pitman used the probabilistic approach of Freiman to motivate their main theorem and justify their choice of \( \sigma \) (which was also the choice which would arise from "steepest descent"). An account of this approach in a more general setting will be given in Chapter 3 after sketching the necessary probabilistic ideas in Chapter 2.
The method by which Freiman and Pitman used (1.13) to estimate $q(m, n)$ for the appropriate $\sigma$ was basically a simple version of the circle method, similar to that used by Cassels and Roth and Szekeres, described in Section 5 and Section 6. However the use of Lemma 1.3 had the advantage that the function $\varphi$ is holomorphic on the whole complex plane and so there were no concerns about convergence, etc. They divided the range of integration into two parts, these being the main interval $\{ \alpha : |\alpha| \leq \alpha_0 \}$ and the supplementary interval $\{ \alpha : \alpha_0 < |\alpha| \leq 1/2 \}$. In order to estimate the supplementary integral they were able to use a simple lower bound for the sum

$$\sum_{m < j \leq n} ||\alpha j||^2$$

in place of (C2) and (RS2).

The main term in Theorem G involves the parameters $\sigma$ and $A_2$ which are not given explicitly in terms of $m$ and $n$. In order to give an explicit estimate of the main term, further estimation of the parameters is needed. Freiman and Pitman derived explicit asymptotic estimates of $q(m, n)$ for both $m = o(n^{1/3})$ and $m$ in specified bands of width at least $o(n^{1/3})$ centred at $n^{(\nu+1)/2}(\nu \log n)^{1/2}$, $1/3 < \nu < 1$. Their theorems will be stated in Chapter 6.

9. Aim and methods of this thesis

The aim of this thesis is to study the asymptotic behaviour of $q_u(m, n)$ as $n \to \infty$ for a reasonably broad class of sequences $u$. For this purpose my objectives were to establish an asymptotic estimate along the lines of Corollary 1.4 and Theorem G which would be valid for a wide range of values of $m$ such that $0 \leq m < n/2$, and to explore applications and possible refinements of such a result. The sequences covered should at least include the $k$-th powers and the primes.

This thesis will draw heavily on the ideas of Roth and Szekeres [40] and Freiman and Pitman [16]. However the present problem involves added difficulties associated with replacing $q_u(0, n)$ by $q_u(m, n)$ in Corollary 1.4 (requiring careful estimation in terms of $m$) and also replacing $N$ by $u$ in Theorem G.

The first step was to identify appropriate conditions on the sequence $u$ to play the role of (RS1) and (RS2) of Section 6. A further definition is needed before formulating the conditions.

DEFINITION 1.5. Given a strictly increasing sequence of positive integers $u = (u_j)$, we define its counting function $U$ to be

$$U(x) = \text{card} \{ u_j : u_j \leq x \}.$$ 

The following conditions turned out to be appropriate:
HYPOTHESIS H. There are constants $C, s, t, C > 0, 0 < s \leq 1, t \geq 0$ such that
\[ U(x) \sim C x^s (\log x)^{-t} \]
as $x \to \infty$.

HYPOTHESIS K. Let $s$ be as in Hypothesis H. For every real number $\lambda \in (1, 2)$ there are positive constants $x_0$ and $K_0$ (which depend only on $\lambda$ and the sequence $u$) such that for every $x > x_0$,
\[ \sum_{x < u_j \leq \lambda x} ||\alpha u_j|| \geq K_0 x^{s(2-\lambda)} \]
whenever $|\alpha| \in (1/(2\lambda x), 1/2)$.

For sequences satisfying Hypotheses H and K and for arbitrary $\delta$ such that $0 < \delta < 1$, the main theorem of this thesis will give an asymptotic estimate of $q_u(m, n)$ which is valid for $0 \leq m \leq n^{1-\delta}$. This result satisfies the initial objectives since, as we shall see in Chapter 4, the sequences $u$ which satisfy Hypotheses H and K certainly include $(j), (p_j), (j^k)$.

The method used will be a generalisation of that of Freiman and Pitman [16] and I shall draw heavily on the ideas of Roth and Szekeres in handling Hypotheses H and K.

From Lemma 1.3 we see that in order to estimate $q_u(m, n)$ we need to focus on estimating the integral which appears in this lemma. This will be done by subdividing the range of integration $[-1/2, 1/2]$ into a major interval $\{\alpha : |\alpha| \leq \alpha_0\}$ and a minor interval $\{\alpha : \alpha_0 < |\alpha| \leq 1/2\}$, where $\alpha_0$ is as yet undetermined. It will be shown that the main contribution comes from the main integral
\[ \int_{\{\alpha : |\alpha| \leq \alpha_0\}} \prod_{m < u_j \leq n} \left( \frac{1}{1 + e^{-\sigma u_j}} + \frac{1}{1 + e^{\sigma u_j} e(\alpha u_j)} \right) e(-an) d\alpha \]
and Hypothesis K will enable the estimation of the supplementary integral
\[ \int_{\{\alpha : \alpha_0 < |\alpha| \leq 1/2\}} \prod_{m < u_j \leq n} \left( \frac{1}{1 + e^{-\sigma u_j}} + \frac{1}{1 + e^{\sigma u_j} e(\alpha u_j)} \right) e(-an) d\alpha. \]

10. Plan of the thesis

In Chapter 2, I shall present auxiliary results. The reader may prefer to omit Chapter 2 initially and refer back as necessary — its purpose is to avoid the interruption of arguments in later chapters by gathering together the necessary auxiliary results, referring to the literature for proofs.

In Chapter 3 I shall introduce the main theorem discussed above via a probabilistic approach, and then prove it. The proof will depend on providing estimates of the main integral in (1.14) in the first half of this chapter and then of the supplementary integral in (1.15) in the remaining part of the chapter.
In Chapter 4 I shall show that certain classes of sequences satisfy Hypotheses H and K and hence the main theorem. In particular it will follow that the theorem applies when \( u = \mathbb{N}, \ u = (p_j) \) and when \( u = (P(j)), \ u = (P(p_j)) \), for the same conditions on the polynomial function \( P \) as given by Roth and Szekeres. In the final section of this chapter the asymptotic relation

\[
q_u(m, n) \sim 2^{-U(m)}q_u(0, n) \quad \text{as} \quad n \to \infty
\]

between \( q_u(m, n) \) and \( q_u(0, n) \) for \( m \) small will be proven.

In Chapter 5 an explicit asymptotic estimate for \( q_u(m, n) \) will be obtained for the case \( u = (j^k) \) with \( 0 \leq m \leq n^{k(4k-1)/(k+1)(4k+2)} \).

The subsequent two chapters will deal with more precise estimation of \( q(m, n) = q_{\mathbb{N}}(m, n) \). In Chapter 6 the integral in (1.13) will be estimated, while in Chapter 7 the product term

\[
e^{\sigma_n} \prod_{m < j \leq n} (1 + e^{-xj})
\]

will be estimated. These estimates will yield refinements of the explicit estimates of \( q(m, n) \) obtained by Freiman and Pitman [16].

11. Conventions

The following conventions are adhered to throughout this thesis. For further notation see Notation on page xiii.

Mathematical notation

We use the lower case Latin (ie. printed in italic type) letters \( h, l, m, q \) to denote non-negative integer variables, while \( j, k, n \) denote positive integer variables. We denote \( e^{2\pi iz} \) by

\[
e(x) = e^{2\pi ix}.
\]

As far as sets of numbers are concerned, we denote the sets of all integers, natural, rational, real and complex numbers by \( \mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) respectively. We express the cardinality of a given set \( A \) in the shorthand form \( \text{card} A \).

Labelling of theorems, lemmas, definitions

Significant results from the literature needed for background purposes in this chapter or in the introductory material for later chapters are labelled A, B, C, etc. throughout this thesis. The work in later chapters does not depend on these results and is self-contained except for dependence on the auxiliary results in Chapter 2.
Items such as definitions, lemmas and theorems other than the background results mentioned above will be labelled consecutively within each chapter. Thus, for example, Definition 2.1, is the first such item in Chapter 2, and Theorem 2.2 is the next. The symbol $\square$ will be used to mark the completion of a proof.
CHAPTER 2

Auxiliary results

1. Introduction

In this chapter I shall present auxiliary material drawn from the literature for later use in this thesis.

Firstly in Section 2 I shall outline ideas from probability theory needed for the probabilistic motivation of the main theorem (which will be discussed in Chapter 3).

In Section 3 some lemmas will be given regarding the order of magnitude of inverse functions. In Section 4 Taylor's Theorem and other basic results of estimation will be presented. Section 5 will give some lemmas on the summation of sequences, both finite and infinite, and Section 6 will present some known results on the estimation of exponential sums.

Finally, I shall provide some necessary results from prime number theory in Section 7.

2. Ideas from probability

For the purposes of this thesis we shall require only basic concepts and results concerning random variables together with local limit theorems. We shall use basic definitions and results as in Kingman and Taylor [29], supplemented by material from Breiman [5] and Moran [35]. The simplified account in Chapter 3 of Halberstam and Roth [19] (which was tailored to the study of number theory problems on sequences) would also provide much of the necessary background. We shall particularly need the following concepts and properties.

A probability space is a triple $(\Omega, \mathcal{F}, P)$ where $\Omega$ is a set of points $\omega$, $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$ (called events), and $P$ is a complete probability measure on $(\Omega, \mathcal{F})$. A random variable $X$ on a given probability space $(\Omega, \mathcal{F}, P)$ is an $\mathcal{F}$-measurable function from $\Omega$ to $\mathbb{R}$ and we denote the probability that the random variable $X$ takes the value $x \in \mathbb{R}$ by

$$\mathbb{P}(X = x) = P\{\omega : X(\omega) = x\}.$$
2. AUXILIARY RESULTS

The expectation (or mean) \( \mathbb{E}(X) \) and variance \( \mathbb{V}(X) \) of the random variable \( X \) are given by

\[
\mathbb{E}(X) = \int_{\Omega} X dP, \quad \mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.
\]

The distribution function of \( X \) is the function \( F \) defined from \( \mathbb{R} \) into \([0, 1] \) by

\[
F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\omega : X(\omega) \leq x).
\]

Recall that a random variable \( X \) is normally distributed with mean \( \mu \) and variance \( \sigma^2 \) if its distribution function is

\[
(2.1) \quad \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt,
\]

and that if, further, \( \mu = 0 \) and \( \sigma = 1 \) then \( X \) is a standard normal variable.

We also need the concept and properties of independence of a sequence of random variables (see, for example, Section 1 of Chapter 3 of Breiman [5]).

For a sequence of probability spaces

\[
((\Omega_i, \mathcal{F}_i, P_i)) \quad i = 1, 2, 3, \ldots,
\]

we shall need the product space

\[(\Omega, \mathcal{F}, P),\]

where \( \Omega = \bigotimes_{i=1}^{\infty} \Omega_i \). For a full treatment see Section 13.1, pages 335-336 of Kingman & Taylor [29] or Breiman [5]. For our purposes Theorem 5, Chapter 3, page 123 of Halberstam & Roth [19] and the special case proved there are sufficient.

Discrete random variables

We shall almost exclusively consider random variables assuming non-negative integral values. For convenience I summarise some basic concepts for this case.

For a given probability space \((\Omega, \mathcal{F}, P)\) consider a random variable \( X \) (with domain \( \Omega \)) taking values in \( \{0\} \cup \mathbb{N} \). For each integer \( j \geq 0 \) let

\[
p_j = \mathbb{P}(X = j).
\]

For any real valued function \( f \) with domain \( \{0\} \cup \mathbb{N} \) we have

\[
\mathbb{E}(f) = \sum_{j=0}^{\infty} p_j f(j).
\]

In particular for \( k \geq 1 \) the \( k \)-th moment of \( X \) is

\[
\rho_k = \mathbb{E}(X^k) = \sum_{h=0}^{\infty} p_h h^k,
\]
and we note that the first moment $\rho_1$ is the mean of $X$. The moment generating function of $X$ is

$$\mathbb{E}(e^{tX}) = 1 + \sum_{k=1}^{\infty} \frac{\rho_k t^k}{k!},$$

provided the moments are finite and the series converges.

The characteristic function of $X$ is defined here by

$$(2.2) \quad \varphi(\alpha) = \mathbb{E}(e^{2\pi i \alpha X}) = \sum_{k=0}^{\infty} p_k e^{2\pi i k \alpha}$$

for $\alpha \in \mathbb{R}$. Note that $\varphi(0) = 1$ and that $\varphi$ is periodic with period 1, which is convenient for number theory — it is a rescaling of the classical characteristic function $\mathbb{E}(e^{itX})$ and has corresponding properties. In particular for $n \geq 0$,

$$(2.3) \quad \int_{-1/2}^{1/2} e^{-2\pi \alpha n} \varphi(\alpha) d\alpha = \mathbb{P}(X = n),$$

and the characteristic function of a finite sum of independent random variables is the product of their individual characteristic functions.

Cumulants and moments

If the moments $\rho_1, \rho_2, \ldots, \rho_{2h}$ are all finite then (see Moran [35] Section 6.7, page 266)

$$\varphi(\alpha) = 1 + \sum_{k=1}^{2h} \rho_k \frac{(2\pi i \alpha)^k}{k!} + R_1(\alpha)$$

where $R_1(\alpha) = o(\alpha^{2h+1})$ as $\alpha \to 0$. It follows (see Moran [35]) that (taking the branch of the logarithm which ensures that $\log$ has imaginary part lying in the interval $(-\pi, \pi)$ and using $\log \varphi(0) = 0$)

$$\log \varphi(\alpha) = \sum_{k=1}^{2h} A_k \frac{(2\pi i \alpha)^k}{k!} + R_2(\alpha)$$

where $R_2(\alpha) = o(\alpha^{2h})$ as $\alpha \to 0$ and where the first part is the $2h$-th Taylor polynomial for $\varphi(\alpha)$ near 0 and we have that

$$A_1 = \rho_1 = \mathbb{E}(X)$$

$$A_2 = \rho_2 - \rho_1^2 = \mathbb{V}(X).$$

We call the $A_k$'s (if they are defined) the cumulants of the random variable $X$ or of its distribution and they are given by the expression

$$\log \left( \frac{k!}{(2\pi i)^k} \int_{\alpha=0}^{\alpha} \varphi(\alpha) \frac{d^k}{d\alpha^k} \varphi(\alpha) \bigg|_{\alpha=0} \right).$$

Under suitable conditions (which should hold in our case) the cumulants all exist and we have

$$\log \varphi(\alpha) = \sum_{k=1}^{\infty} A_k \frac{(2\pi i \alpha)^k}{k!}.$$
2. AUXILIARY RESULTS

Remark 2.1. Note that the cumulants of a finite sum of independent random variables are the sums of their corresponding individual cumulants.

Limit theorems

Consider a sequence \((X_j)_{j=1}^{\infty}\) of independent identically distributed random variables on a given probability space \((\Omega, \mathcal{F}, P)\) with mean \(\mu\) and variance \(\sigma\). We shall be interested in the behaviour of the sum

\[ S_k = X_1 + X_2 + \ldots + X_k \quad \text{as} \quad k \to \infty. \]

The classical result on the topic is the Central Limit Theorem below which shows that the distribution function of \(Z_k = (S_k - k\mu)/(\sigma k^{1/2})\) converges to that of a standard normal variable (obtained from (2.1) by taking \(\mu = 0\) and \(\sigma = 1\)). We note that (using independence)

\[ \mathbb{E}(Z_k) = 0, \quad \mathbb{V}(Z_k) = 1. \]

**Theorem 2.2 (Central Limit Theorem).** Let \((X_j)_{j=1}^{\infty}\) be a sequence of independent and identically distributed random variables with finite mean \(\mu\) and finite variance \(\sigma^2\). Define

\[ Z_k = ((X_1 + X_2 + \ldots + X_k) - k\mu)/(\sigma k^{1/2}). \]

Then as \(k \to \infty\),

\[ \mathbb{P}(Z_k \leq z) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx. \]

**Proof.** See for example, Theorem 13.8, page 348, Section 13.4 of Kingman and Taylor [29]. \(\square\)

Limit theorems in this spirit have many applications in number theory (see Elliott [11]). However, for our purposes we need a Local Limit Theorem, which is a theorem regarding the asymptotic behaviour of \(\mathbb{P}(S_k \in I)\) as \(k \to \infty\) for finite intervals \(I\), as distinct from the behaviour of the distribution function of \(Z_k\). If the \(X_j\)'s take only integral values then \(\mathbb{P}(S_k = n) = \mathbb{P}(S_k \in (n-1, n])\) and the following theorem is relevant.

**Theorem 2.3 (Local Limit Theorem).** Let \((X_j)_{j=1}^{\infty}\) be a sequence of independent identically distributed integer-valued random variables with finite mean \(\mu\) and finite variance \(\sigma^2 > 0\). Let

\[ S_k = X_1 + X_2 + \ldots + X_k \]

and let

\[ D = \gcd\{d : \exists i, j \text{ such that } \mathbb{P}(X_i - X_j = d) > 0\}. \]
Then a necessary and sufficient condition that
\[
\sigma\sqrt{k}\mathbb{P}(S_k = n) - \frac{1}{\sqrt{2\pi}}e^{-(n-k\mu)^2/(2k\sigma^2)} \to 0 \quad \text{as} \quad k \to \infty
\]
holds uniformly with respect to \(n\) in the interval \(-\infty < n < \infty\) is that \(D = 1\).

**Proof.** See theorem in Section 49, page 233 of Gnedenko and Kolmogorov [17]. \(\square\)

As remarked in Section 49, page 235 of Gnedenko and Kolmogorov [17], the condition \(D = 1\) is necessary because if \(D > 1\) then the possible values of \(S_k\) will contain gaps due to the difference between two consecutive possible values of the sum \(S_k\) being \(\geq D\).

For related results in this spirit see Section 2.3 of Postnikov [37] (and references mentioned there) and Breimann [5] (Section 10.4 and notes on Chapter 10).

The following lemma guarantees the existence of an appropriate probability space and a sequence of random variables corresponding to the probabilistic motivation in Chapter 3.

**Lemma 2.4.** Let \((a_j)\) and \((b_j)\) be sequences of non-negative integers and let \((p_j)\) be a sequence of real numbers such that \(0 \leq p_j \leq 1\) for all \(j \geq 1\). There exists a probability space \((\Omega, \mathcal{F}, P)\) and a sequence of independent random variables \((X_j)\) taking non-negative integral values and with domain \(\Omega\) such that for all \(j \geq 1\),
\[
\mathbb{P}(X_j = a_j) = p_j, \quad \mathbb{P}(X_j = b_j) = 1 - p_j.
\]

**Proof.** For each \(j\) consider the finite probability space \((\Omega_j, \mathcal{F}_j, P_j)\) given by
\[
\Omega_j = \{a_j, b_j\}, \quad \mathcal{F}_j = \{\emptyset, \Omega_j, \{a_j\}, \{b_j\}\}, \quad P_j(a_j) = p_j, \quad P_j(b_j) = 1 - p_j.
\]

Let \((\Omega, \mathcal{F}, P)\) be the smallest complete probability space containing the product space of the \(\{(\Omega_j, \mathcal{F}_j, P_j)\}\)'s. Then
\[
\Omega = \bigotimes_{j=1}^{\infty} \Omega_j
\]
and each element of \(\Omega\) is a sequence \(\omega = (\omega_j)\) such that \(\omega_j \in \Omega_j\) for each \(j\). Let \(X_j\) be defined by
\[
X_j(\omega) = \omega_j \quad (= j\text{-th component of } \omega).
\]

It is easily checked (by using the definition of the product space) that \((X_j)\) is a sequence of independent random variables and (2.5) holds. \(\square\)

**Note:** We note that in the above situation, for any integer \(c \geq 0\)
\[
P(X_j = c) = 0 \quad \text{if} \quad c \neq a_j, b_j.
\]
3. Order and asymptotic behaviour

We adopt the following conventions. Given real valued functions $f$ and $g$ on $\mathbb{N}$ with $g(n)$ non-negative for large $n$, we say that

$$f(n) \ll g(n) \quad \text{as} \quad n \to \infty$$

or

$$f(n) = O(g(n)) \quad \text{as} \quad n \to \infty$$

if there is a constant $C > 0$ and there is a number $n_0 > 0$ such that for every $n > n_0$,

$$|f(n)| \leq Cg(n).$$

The constants $C$ and $n_0$ are referred to as the implied constants.

Continuing on, $g(n) \gg f(n)$ as $n \to \infty$ if $f(n) \ll g(n)$ as $n \to \infty$. We say that $f(n) = o(g(n))$ as $n \to \infty$ if as $n \to \infty$, $f(n)/g(n) \to 0$. We write $f(n) \sim g(n)$ as $n \to \infty$ if $\lim_{n \to \infty} f(n)/g(n) = 1$ and $f(n) \asymp g(n)$ as $n \to \infty$ if both $f(n) \ll g(n)$ and $f(n) \gg g(n)$ as $n \to \infty$. A subscript attached to any of these symbols denotes that the implied constants depend on the subscripted variable(s). For example, $f(n) \ll g(n)$ denotes the fact that there are constants $C = C(\delta) > 0$ and $n_0 = n_0(\delta) > 0$ depending only on $\delta$ such that for $n > n_0$, $|f(n)| \leq Cg(n)$.

For real valued functions $f$ and $g$ on $\mathbb{R}$, the meaning of $f(x) \ll g(x)$, $f(x) = O(g(x))$, etc. as $x \to \infty$ is similar to that described above for $f$ and $g$ on $\mathbb{N}$.

We commence with a lemma relating the growth of the sequence $u$ with its counting function $U(x)$.

**Lemma 2.5.** Let $u = (u_j)$ be a strictly increasing sequence of positive integers and let $U(x)$ be its counting function as defined by Definition 1.5. We have for numbers $s$ and $t$ such that $0 < s \leq 1$, $t \geq 0$,

$$U(x) \asymp x^s (\log x)^{-t} \quad \text{as} \quad x \to \infty$$

if and only if

$$u_j \asymp j^{1/s} (\log j)^{1/t} \quad \text{as} \quad j \to \infty,$$

where the implied constants may depend only on the sequence $u$.

**Proof.** ((2.6) $\implies$ (2.7)) Suppose (2.6) holds. Then there are positive constants $B_1$ and $B_2$ such that for $x > 1$,

$$B_1 x^s (\log x)^{-t} \leq U(x) \leq B_2 x^s (\log x)^{-t}.$$
Taking logarithms of the expressions in this inequality yields,

\[ \log B_1 + s \log x - t \log \log z \leq \log U(x) \leq \log B_2 + s \log x - t \log \log z. \]

Hence

\[ \log x \ll \log U(x) \ll \log x \]

and we have that

\[ U(x) \asymp z^s (\log U(x))^{-t} \quad \text{as} \quad x \to \infty. \]

Result (2.7) follows by taking \( z = u_j \), noting that \( U(u_j) = j \) and letting \( j \to \infty \).

((2.7) \implies (2.6)) Suppose (2.7) holds. Given \( x \), let \( j = U(x) \), so that

\[ u_j \leq x < u_{j+1}. \]

By (2.7) and the inequality (2.8) satisfied by \( x \) we deduce that \( x \asymp j^{1/s} (\log j)^{1/t} \) as \( j \to \infty \). As done in the first part of this proof we take logarithms to give \( \log x \asymp \log j \) and consequently we obtain \( x \asymp j^{1/s} (\log j)^{1/t} \). After some rearrangement of this magnitudinal equality and substitution of \( U(x) \) for \( j \) we arrive at (2.6). \qed

The following lemma provides the relationship between the order of magnitude of the inverse of a function with the function's order of magnitude.

**Lemma 2.6.** Let \( f \) be a strictly increasing real valued function with domain \([0, \infty)\) and let \( f^{-1} \) be its inverse. For real numbers \( a \) and \( b \) such that \( a > 0 \) we have

\[ f(z) \asymp z^a (\log z)^b \quad \text{as} \quad z \to \infty \]

if and only if

\[ f^{-1}(y) \asymp y^{1/a} (\log y)^{-b/a} \quad \text{as} \quad y \to \infty, \]

where the implied constants depend only on \( a \) and \( b \).

**Proof.** The proof is similar to the proof of Lemma 2.5. \qed

Finally in this section, we present a lemma concerning the order of magnitude of the incomplete gamma function.

**Lemma 2.7.** Let \( z \) be a fixed real number \( \geq 1 \). Let \( \Gamma_z(x) \) be the incomplete gamma function defined by

\[ \Gamma_z(x) = \int_x^\infty t^{x-1} e^{-t} dt. \]
Then for $x \geq 2(z - 1)$ we have

$$\Gamma_x(z) \asymp e^{-x} x^{z-1}.$$  

**Proof.** Note that

$$\frac{d}{dt} x^{t-1} e^{-t} = (-t^{z-1} + (z-1)t^{z-2}) e^{-t}$$

and so for $x \geq 2(z - 1)$ (which implies that $1 - (z-1)t^{-1} \geq 1/2$ for $t \geq x$),

$$x^{t-1} e^{-x} = \int_x^\infty (t^{z-1} - (z-1)t^{z-2}) e^{-t} dt \leq \int_x^\infty t^{z-1} e^{-t} dt \leq 2 \int_x^\infty (t^{z-1} - (z-1)t^{z-2}) e^{-t} dt = 2x^{z-1} e^{-x},$$

which is the result. □

### 4. Some basic estimates involving integrals

We shall use the following version of Taylor’s Theorem with the remainder in integral form.

**Theorem 2.8 (Taylor’s Theorem).** Let $k$ be a positive integer and let $J$ be a real open interval containing 0. Let $f$ be a complex valued function which is $k$ times continuously differentiable on $J$. Then for $\alpha$ in $J$

$$f(\alpha) = f(0) + \sum_{r=1}^{k-1} \frac{\alpha^r}{r!} f^{(r)}(0) + \frac{\alpha^k}{(k-1)!} \int_0^1 (1-v)^{k-1} f^{(k)}(av) dv.$$  

**Proof.** This follows, for example, from the integral form of the remainder on page 346 of Spivak [41]. □

We give a couple of lemmas related to the moments of a normal distribution.

**Lemma 2.9.** Let $h$ be a non-negative integer and let $z$ be a positive real number. Then as $z \to \infty$,

$$\int_z^\infty e^{-t^2/2} t^h dt \ll z^{h/2} e^{-z^2/2}$$

where the implied constants depend only on $h$.

**Proof.** Let

$$I_h(z) = \int_z^\infty e^{-t^2/2} t^h dt.$$  

We have that

$$I_1(z) = e^{-z^2/2}.$$  

$$I_h(z) = \left[ -t^{h-1} e^{-t^2/2} \right]_z^\infty + (h-1) \int_z^\infty e^{-t^2/2} t^{h-2} dt$$

$$= z^{h-1} e^{-z^2/2} + (h-1) I_{h-2}(z).$$

Hence for $h$ odd,

$$I_h(z) = e^{-z^2/2} (z^{h-1} + (h-1)z^{h-3} + (h-1)(h-3)z^{h-5} + \ldots + (h-1)(h-3)(h-5) \ldots 2)$$
and the lemma follows for \( h \) odd. For \( h \) even, we note that \( I_{h}(z) \ll I_{h+1}(z) \) and the argument for \( h \) odd gives the result. \( \square \)

**Lemma 2.10.** Let \( h \) be a non-negative integer. Then

\[
\int_{-\infty}^{\infty} e^{-\beta^2/2}|\beta|^h d\beta = \begin{cases} \frac{\Gamma\left(\frac{h+1}{2}\right)}{\sqrt{2\pi}} & \text{if } h \text{ is even}, \\ 2^{(h+1)/2}((h-1)/2)! & \text{if } h \text{ is odd}. \end{cases}
\]

**Proof.** Let \( h \) be a non-negative integer. Let

\[ I_h = \int_{-\infty}^{\infty} e^{-\beta^2/2}|\beta|^h d\beta. \]

We know that \( I_0 = \sqrt{2\pi} \). Also

\[ I_1 = 2 \int_{0}^{\infty} e^{-\beta^2/2} d\beta = 2 \left[ -e^{-\beta^2/2} \right]_{0}^{\infty} = 2. \]

Integration by parts gives the recurrence relation

\[ I_h = (h-1)I_{h-2}, \]

from which the result follows. \( \square \)

5. General summation lemmas

We adopt the following convention on summation. The capital Greek letters \( \Sigma \) and \( \Pi \) are used to indicate repeated summation and multiplication. For example,

\[ \sum_{a < p \leq b} \log p \]

denotes the sum of \( \log p \) over prime numbers \( p \) lying in the half-open interval \( (a, b] \) while

\[ \prod_{a < u_j \leq b} (1 + e^{-\sigma u_j}) \]

denotes the product of \( 1 + e^{-\sigma u_j} \) over the set \( \{ j : a < u_j \leq b \} \).

We use a couple of lemmas needed for estimating finite sums and summing series.

**Lemma 2.11 (Ingham's Lemma).** Let \( \lambda = (\lambda_j) \) be a strictly increasing sequence of real numbers such that \( \lambda_j \to \infty \) and let

\[ C(x) = \sum_{\lambda_j \leq x} c_j, \]

where the \( c_j \) may be real or complex. Then, if \( X \geq \lambda_1 \) and \( f \) has a continuous derivative, we have

\[ \sum_{\lambda_j \leq X} c_j f(\lambda_j) = -\int_{\lambda_1}^{X} C(x)f'(x)dx + C(X)f(X). \]
If, further, $C(X)f(X) \to 0$ as $X \to \infty$, then
\[ \sum_{j=1}^{\infty} c_j f(\lambda_j) = -\int_{\lambda_1}^{\infty} C(x)f'(x)dx, \]
provided that either side is convergent.

**Proof.** See Theorem A, page 18 of Ingham [27]. \( \square \)

**Lemma 2.12 (Euler–Maclaurin Summation Formula).** Let $M$ be a non-negative integer. Let $a$ and $b$ be real numbers such that $a < b$ and let $f$ be a real-valued function with continuous $2M + 1$-th derivative in the interval $[a, b]$. Then if \{x\} denotes the fractional part of the real number $x$,
\[ \sum_{a<j \leq b} f(j) = \int_a^b f(x)dx - \left[ (\{x\} - 1/2)f(x) \right]_a^b + \sum_{j=1}^{M} \frac{B_{2j}}{(2j)!} \left[ f^{(2j-1)}(x) \right]_a^b + \frac{1}{(2M + 1)!} \int_a^b B_{2M+1}(\{x\})f^{(2M+1)}(x)dx, \]
where $B_n(x)$ is the $n$-th Bernoulli polynomial and $B_n$ is the $n$-th Bernoulli number.

**Proof.** See Section 6.2, page 106 of Edwards [10]. \( \square \)

As a corollary we have the following version of Euler's Formula.

**Corollary 2.13 (Euler’s Formula).** Let $f$ be a continuously differentiable real valued function which is decreasing on the interval $[a, b]$. Then
\[ \left| \sum_{a<j \leq b} f(j) - \int_a^b f(x)dx \right| \leq 4|f(a)|. \]
Modifying the Euler–Maclaurin Summation Formula gives the following corollary.

**Corollary 2.14.** Let $c$ be a positive real number and let $k$ be a positive integer. Let $f$ be a positive real-valued continuously differentiable function $(0, \infty)$ such that
\[ \sum_{j^k > c} f(j^k), \int_c^\infty f(y)y^{1/k-1}dy, \text{ and } \int_c^\infty ((\sqrt[k]{y}) - 1/2)f'(y)dy \]
converge. Then
\[ \sum_{j^k > c} f(j^k) = \frac{1}{k} \int_c^\infty f(y)y^{1/k-1}dy + ((\sqrt[k]{c}) - 1/2)f(c) + \int_c^\infty ((\sqrt[k]{y}) - 1/2)f'(y)dy. \]

**Proof.** We apply Lemma 2.12 to the sum
\[ \sum_{a<j \leq b} g(j) \]
to give
\[ \sum_{a<j \leq b} g(j) = \int_a^b g(t)dt - \left[ (\{t\} - 1/2)g(t) \right]_a^b + \int_a^b ((t) - 1/2)g'(t)dt. \]
Writing \( f(t^k) \) in place of \( g(t) \) and making the change of variables \( y = t^k \) in the right hand side of the above expression gives

\[
\sum_{a^k \leq y \leq b^k} f(y) = \frac{1}{k} \int_{a^k}^{b^k} f(y) \left( y^{1/k} - 1 \right) dy + \left[ \left( y \right) - \frac{1}{2} \right] f(y) dy + \int_{a^k}^{b^k} \left( y^{1/k} \right)^2 dy.
\]

Letting \( b \to \infty \) and replacing \( a^k \) by \( c \) throughout the right hand side of the above expression gives the result. \( \square \)

The following lemma gives an estimate of the integral of the product of a non-negative decreasing function with an oscillating function. This lemma is used for estimating the remainder term in the Euler-Maclaurin Summation Formula (as suggested at the end of Section 6.2 of Edwards [10]).

**Lemma 2.15.** For real \( a \) and \( b \) such that \( a < b \), suppose the real valued function \( f \) is continuous and non-negative on the interval \([a, b]\). Suppose further that there exists \( c \) in \((a, b)\) such that \( f \) is increasing on \([a, c]\) and decreasing on \([c, b]\). Then

\[
\int_a^b \left( (t) - 1/2 \right) dt \leq \int_a^b f(t) dt \leq \int_a^b f(t) dt.
\]

**Proof.** By the Second Mean Value Theorem there exist \( z_0 \) and \( z_1 \) such that \( a \leq z_0 \leq c \leq z_1 \leq b \) and

\[
\int_a^c \left( (t) - 1/2 \right) dt = f(c) \int_a^c \left( (t) - 1/2 \right) dt,
\]

\[
\int_c^b \left( (t) - 1/2 \right) dt = f(c) \int_c^b \left( (t) - 1/2 \right) dt.
\]

From the graph of \( y = (t) - 1/2 \) we see that for any finite interval \( I \)

\[
\int_I \left( (t) - 1/2 \right) dt \leq \int_0^{1/2} \left( (t) - 1/2 \right) dt = \frac{1}{8}.
\]

Hence adding the above integrals over \([a, c]\) and \([c, b]\) gives the result. \( \square \)

6. Exponential sums

Exponential sums, otherwise referred to as trigonometric sums, arise in the estimation of the supplementary integral as in (1.15). Hence these sums are relevant to the situations that will be encountered throughout the thesis. Exponential sums over a variety of sequences are required and the lemmas contained in this section relate to polynomial sequences, prime numbers and as a particular case of polynomial sequences, the integers.

Prior to stating some lemmas on exponential sums, we prove an inequality involving the nearest integer function which is implicit in the work of Roth and Szekeres [40] in proving a lower bound on exponential sums over the primes and \( k \)-th powers.
LEMMMA 2.16. For given real $Q > 1$ and integral $h, q, n$ such that

$$1 < q \leq Q, \quad (h, q) = 1, \quad 1 \leq n \leq Q,$$

let $\alpha$ be a real number such that

$$\left| \alpha - \frac{h}{q} \right| \leq \frac{1}{2qQ}.$$ 

Then (with $\|x\|$ = distance of $x$ from nearest integer) we have

$$(2.9) \quad \|n\alpha\| \geq \frac{1}{2} \|hn/q\|.$$ 

PROOF. We have immediately that

$$(2.10) \quad \left| n\alpha - n\frac{h}{q} \right| \leq \frac{n}{2qQ} \leq \frac{1}{2q}.$$ 

Consider the following three cases.

Case (1): $0 < \{nh/q\} \leq 1/2$.

Using $q \geq 2$ we have

$$\{n\alpha\} \leq \{nh/q\} + |n\alpha - nh/q| \leq \frac{1}{2} + \frac{1}{2q} \leq \frac{3}{4} \leq 1 - \frac{1}{2}\{nh/q\}$$

and by (2.10)

$$\{n\alpha\} \geq \{nh/q\} - \frac{1}{2q} \geq \frac{1}{2}\{nh/q\}.$$ 

Since $\{nh/q\} = \|nh/q\|$, (2.9) follows from the above inequalities.

Case (2): $1/2 < \{nh/q\} < 1$.

Then $\{nh/q\} \geq 1/2 + 1/2q$. Also

$$\{n\alpha\} \geq \{nh/q\} - \frac{1}{2q} \geq \frac{1}{2} \geq 1 - \{nh/q\} = \|nh/q\|$$

and

$$\{n\alpha\} \leq \{nh/q\} + \frac{1}{2q} = 1 - \|nh/q\| + \frac{1}{2q} \leq 1 - \|nh/q\| + \frac{1}{2}\|nh/q\| = 1 - \frac{1}{2}\|nh/q\|.$$ 

Hence we have (2.9).

Case (3): $\{nh/q\} = 0$.

The inequality (2.9) is trivial in this case. □

The following lemma is used in bounding the supplementary integral when $u = (P(j))$, $P$ a suitable polynomial.
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**Lemma 2.17 (Hua's Lemma).** Let

\[ f(\alpha) = \sum_{x=1}^{N} e(P(x)\alpha), \]

where \( P \) is a polynomial of degree \( k \) and \( N \) is an integer. Let \( s \geq 2^k \) and let \( \delta > 0 \). Then as \( N \to \infty \),

\[ \int_{0}^{1} |f(\alpha)|^s d\alpha \ll_{s,k,\delta} N^{s-k+\delta}, \]

where the implied constant depends only on \( s, k \) and \( \delta \).

**Proof.** See Lemma 2.5, page 12 of Vaughan [44]. \( \square \)

In the particular case dealing with the sequence of positive integers, we shall require an estimate of the supplementary integral as given in (1.15). In order to do this, we shall use the following lemma.

**Lemma 2.18 (Freiman and Pitman).** For \( |\alpha| \leq \frac{1}{2} \) and any positive integers \( m \) and \( k \) such that \( k \geq 2 \) we have

\[ \sum_{j=m}^{m+k-1} 2\sin^2 \pi\alpha j \geq \frac{k}{2} \min(1, (ak)^2). \]

**Proof.** See Lemma 8 of Freiman and Pitman [16] (the proof is short and elementary). \( \square \)

7. Prime numbers

The theorems of this section will be used in Chapter 4 in connection with sequences of the form \( P(p_j) \) where \( p_j \) is the \( j \)-th prime and \( P \) a suitable polynomial. Some of these theorems are standard theorems which are to be found in most number theory text books while others are found in books specialising in additive number theory.

**Theorem 2.19 (Prime Number Theorem).** Let \( \pi(x) \) denote the number of prime numbers \( \leq x \). Then as \( x \to \infty \),

\[ \pi(x) \sim x / \log x. \]

**Proof.** See, for example Section 8, Theorem 12, page 36 of Ingham [27]. \( \square \)

**Theorem 2.20 (Dirichlet's Theorem on Primes in Arithmetical Progressions).** Let \( L \) be a given positive constant. For positive integers \( a, q \) such that \( (a,q) = 1 \), let

\[ \pi(x; a, k) = \text{card} \{ p : p \text{ prime, } p \leq x \& p \equiv a \pmod{k} \}. \]

Then for all \( a \) such that \( (a, q) = 1 \) and all \( q \) such that \( 1 \leq q \leq (\log x)^L \), as \( x \to \infty \),

\[ \pi(x; a, q) = \frac{1}{\varphi(q)} \sum_{2 \leq j \leq q} \frac{1}{\log j} + O \left( x \exp\left(-\sqrt{\log x/200}\right) \right), \]
uniformly with respect to \( a \) and \( q \) such that \( (a, q) = 1 \) and \( 1 \leq q \leq (\log x)^L \), where the implied constants depend only on \( L \).

**Proof.** See, for example, Estermann [14], Theorem 55, page 51. \( \Box \)

As a corollary of this we have the following.

**Corollary 2.21.** With the notation of the preceding lemma,

\[
\pi(x; a, q) \sim \frac{1}{\varphi(q)} \frac{z}{\log x} \quad \text{as} \quad x \to \infty,
\]

uniformly with respect to \( a \) and \( q \) such that \( (a, q) = 1 \) and \( 1 \leq q \leq (\log x)^L \).

We shall use Hua's Lemma for primes in showing the applicability of the general theorem on estimating \( g_u(m, n) \) in Chapter 3 to the sequence \( u = (P(p_j)) \), \( P \) a suitable polynomial.

**Lemma 2.22 (Hua's Lemma for Primes).** Let

\[
f(\alpha) = \sum_{2 \leq p \leq N} e(P(p)\alpha)
\]

where \( P \) is a polynomial of degree \( k \), \( N \) is an integer and \( p \) denotes a prime number. Let \( s \geq 2^k + 4 \).

Then as \( N \to \infty \),

\[
\int_0^1 |f(\alpha)|^s d\alpha \ll N^{s-k},
\]

where the implied constant depends only on \( s \) and \( k \).

**Proof.** See Hua [25]. \( \Box \)

This concludes the results needed for our subsequent development.
CHAPTER 3

Estimate of $q_u(m, n)$ for general sequence

1. Introduction

Throughout this chapter we consider a given strictly increasing sequence of positive integers, $u = (u_j)$. Recall from Definition 1.1 that for integers $m$ and $n$ such that $0 \leq m < n/2$, $q_u(m, n)$ is the number of ways of writing $n$ in the form

$$n = u_{j_1} + u_{j_2} + \ldots + u_{j_r},$$

where

$$m < u_{j_1} < u_{j_2} < \ldots < u_{j_r},$$

with $r$ arbitrary.

The purpose of this chapter is to introduce and prove the main theorem of this thesis. For a given $\delta > 0$, this theorem will provide an asymptotic estimate of $q_u(m, n)$ as $n \to \infty$ valid for $m$ in the range $0 \leq m \leq n^{1-\delta}$, provided the sequence $u$ satisfies appropriate hypotheses (the Hypotheses H and K which were briefly discussed in Section 9 of Chapter 1). As discussed in Chapter 1, this theorem generalises both Theorem 1 of Roth and Szekeres [40], given as Theorem E in Section 6 of Chapter 1, and the main theorem of Freiman and Pitman [16], given as Theorem G in Section 8 of Chapter 1.

In Section 2, I shall give a probabilistic motivation for the main theorem based on the approach of Freiman and Pitman [16]. In Section 3, I shall discuss the hypotheses on the sequence $u$ and then in Section 4, I shall state the main theorem. In Section 5, I shall outline the proof of the main theorem and describe the remaining sections of the chapter.

2. Probabilistic motivation

In this section we use the background material on probability theory in Section 2 of Chapter 2.
Let \( u \) be a strictly increasing sequence of positive integers. For each positive integer \( j \) define
\[
\begin{align*}
    p_{1j} &= (1 + e^{-\sigma u_j})^{-1} \\
    p_{2j} &= (1 + e^{\sigma u_j})^{-1} = 1 - p_{1j},
\end{align*}
\]
with \( \sigma \) being a real parameter. In view of Lemma 2.4, there exists a sequence \( (X_j)_1^\infty \) of independent non-negative integer-valued random variables such that
\[
P(X_j = x) = \begin{cases} 
    p_{1j} & \text{for } x = 0; \\
    p_{2j} & \text{for } x = u_j.
\end{cases}
\]

Consider the random variable \( Y = Y_{m,n} \) defined by
\[
Y = \sum_{m < u_j \leq n} X_j.
\]
We have \( E(Y) = \sum_{m < u_j \leq n} p_{2j} u_j \) and because of the independence of the \( X_j \)'s we also have \( \mathbb{V}(Y) = \sum_{m < u_j \leq n} p_{1j} p_{2j} u_j^2 \) and the characteristic function \( \varphi \) of the random variable \( Y \) is given by
\[
\varphi(\alpha) = \prod_{m < u_j \leq n} \varphi_j(\alpha),
\]
where
\[
\varphi_j(\alpha) = p_{1j} + p_{2j} e^{(\alpha u_j)}
\]
is the characteristic function of the random variable \( X_j \).

We denote the first and second cumulants of the random variable \( Y \) by
\[
A_1 = \sum_{m < u_j \leq n} f_1(u_j) = \sum_{m < u_j \leq n} \frac{u_j}{1 + \exp(\sigma u_j)}
\]
and
\[
A_2 = \sum_{m < u_j \leq n} f_2(u_j) = \sum_{m < u_j \leq n} \frac{u_j^2 \exp(\sigma u_j)}{(1 + \exp(\sigma u_j))^2},
\]
where
\[
f_1(x) = \frac{x}{1 + e^{\alpha x}} \quad \text{and} \quad f_2(x) = \frac{x^2 e^{\alpha x}}{(1 + e^{\alpha x})^2}.
\]
We remark that \( Y \) has expectation \( A_1 \) and variance \( A_2 \). Also the third moment of the random variable \( Y \) is denoted by
\[
\rho_3 = \sum_{m < u_j \leq n} \frac{u_j^3}{(1 + \exp(\sigma u_j))}.
\]

We now present two lemmas which provide formulae for the cumulants \( A_k \) corresponding to the random variable \( Y \) (These lemmas will not be needed for the main theorem).
LEMMA 3.1. Let \( k \) be a positive integer and let \( \sigma \) be a positive number. Let \( f_k(z) \) be defined as

\[
(3.9) \quad f_k(z) = \sum_{l=1}^{\infty} (-1)^{l+1} l^{k-1} e^{-l \sigma z}.
\]

Then the \( k \)-th cumulant \( A_k \) of the random variable \( Y \) defined in (3.2) is equal to

\[
\sum_{m < u_j \leq n} u_j^k f_k(u_j).
\]

PROOF. Let \( \varphi_j(\alpha) \) be the characteristic function of the random variable \( X_j \). We observe that

\[
\log \varphi_j(\alpha) = \log(1 + e^{-\alpha u_j / e(\alpha u_j)}) - \log(1 + e^{-\alpha u_j})
\]

and we expand \( \log(1 + z) \) as a power series about \( z = 0 \) so that we have

\[
\log \varphi_j(\alpha) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} e^{-\alpha u_j} e(\alpha u_j) - \log(1 + e^{-\alpha u_j}).
\]

We now expand \( e(\alpha u_j) \) as a power series about \( \alpha = 0 \) in the right hand side of the above equation and interchange the orders of summation to give

\[
\sum_{k=0}^{\infty} \left( \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} e^{-\alpha u_j} \frac{1}{k!} (\alpha u_j)^k \right) (2 \pi i \alpha)^k - \log(1 + e^{-\alpha u_j}).
\]

We can cancel the constant term to yield

\[
\log \varphi_j(\alpha) = \sum_{k=0}^{\infty} \left( \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} e^{-\alpha u_j} \frac{1}{k!} \alpha^{k-1} u_j^k \right) (2 \pi i \alpha)^k.
\]

Thus the \( k \)-th cumulant of \( X_j \) is \( A_{k,j} = \sum_{l=1}^{\infty} (-1)^{l+1} e^{-\alpha u_j} \frac{1}{k!} \alpha^{k-1} u_j^k \) and by Remark 2.1 we have

\[
A_k = \sum_{m < u_j \leq n} \sum_{l=1}^{\infty} (-1)^{l+1} e^{-\alpha u_j} \frac{1}{k!} \alpha^{k-1} u_j^k
\]

as required. \( \square \)

LEMMA 3.2. Let \( k \) be a positive integer. Then the \( k \)-th cumulant of the random variable \( Y \) defined in (3.2) can be written as

\[
A_k = \sum_{r=1}^{k} C(k, r) \sum_{m < u_j \leq n} \frac{u_j^r}{(1 + e^{\alpha u_j})^r},
\]

where

\[
C(k, r) = \frac{1}{r} \sum_{l=1}^{r} \binom{r}{l} (-1)^{l+1} l^k.
\]

PROOF. It is easily shown that

\[
f_k(z) = \sum_{r=1}^{k} C(k, r) \frac{1}{(1 + e^{\alpha z})^r},
\]

and the result follows. \( \square \)
Returning to the motivation of the main theorem, from Lemma 1.3 we have

\[ q_u(m, n) = e^{an} \prod_{m < u_j \leq n} (1 + e^{-\sigma u_j}) \times \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(\alpha)e(-\alpha n)da, \]

with \( \varphi(\alpha) \) as in (3.3), \( \sigma \) arbitrary. By (2.3) in Chapter 2 we have

\[ \mathbb{P}(Y = n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(\alpha)e(-\alpha n)da. \]

If an asymptotic result along the lines of that in the Local Limit Theorem 2.3 held in the present situation, then using (2.4) and putting \( Y \) in place of \( S_k \), we might hope that as \( n \to \infty \),

\[ \mathbb{P}(Y = n) = \frac{1}{\sqrt{2\pi \mathbb{V}(Y)}} e^{-(n-\mathbb{E}(Y))^2/2\mathbb{V}(Y)} + o(1/\sqrt{\mathbb{V}(Y)}). \]

If further we had \( \mathbb{E}(Y) = n \), then, using \( \mathbb{V}(Y) = A_2 \) we would have

\[ \mathbb{P}(Y = n) = \frac{1}{\sqrt{2\pi A_2}} + o(1/\sqrt{A_2}), \]

and since \( A_2 \to \infty \) as \( n \to \infty \) (as will evident from some lemmas estimating \( A_2 \) in forthcoming sections of this chapter) this implies

\[ \mathbb{P}(Y = n) \sim \frac{1}{\sqrt{2\pi A_2}} \quad \text{as} \quad n \to \infty. \]

We can ensure that \( \mathbb{E}(Y) = n \) by choosing \( \sigma \) so that

\[ n = \sum_{m < u_j \leq n} \frac{u_j}{1 + \exp(\sigma u_j)}. \]

We show that such a choice of \( \sigma \) is possible later on.

This leads to the conjecture that, under suitable conditions on the sequence \( u \) and on \( m \), the result (3.11) will hold and hence, by (3.10), as \( n \to \infty \),

\[ q_u(m, n) \sim e^{an} \prod_{m < u_j \leq n} (1 + e^{-\sigma u_j}) \times \frac{1}{\sqrt{2\pi A_2}}, \]

with \( \sigma \) as in (3.12). This is the basis of the main theorem (Theorem 3.4 below) which under appropriate conditions will give an asymptotic estimate of \( q_u(m, n) \) with the right hand side of (3.13) as its main term.

3. Statement of hypotheses

As mentioned in Chapter 1, we shall need two hypotheses on the strictly increasing sequence of positive integers \( u \). The first, Hypothesis H, which will be used frequently throughout, is as follows.

**HYPOTHESIS H.** There are real constants \( s, t, C_0 \) satisfying \( 0 < s \leq 1, t \geq 0, C_0 > 0 \) such that,

\[ U(x) \sim C_0 x^s (\log x)^{-1} \quad \text{as} \quad x \to \infty. \]
3. STATEMENT OF HYPOTHESES

REMARK 3.3. It can be shown, using the ideas of the proof of Lemma 2.5, that (3.14) is equivalent to

\[ u_j \sim (sC_0)^{-1/s} j^{1/s} (\log j)^{1/s} \quad \text{as} \quad j \to \infty. \]

Because of this we call \(1/s\) the growth exponent of the sequence \(u\).

We note also that Hypothesis H implies the existence of a positive constant \(C_1\) depending only on \(u\) such that

\[ U(x) \leq C_1 x^s (\log x)^{-1} \quad \text{for all} \quad x \geq 2. \]

We note that by the Prime Number Theorem (Theorem 2.19), the sequence of primes \(u = (p_j)\) satisfies Hypothesis H with \(s = t = 1\). Also by Remark 3.3 the sequence \(u = (j^k)\) of positive \(k\)-th powers satisfies (3.15) and hence satisfies Hypothesis H with \(s = 1/k\).

It is clear that any sequence satisfying Hypothesis H must satisfy (RS1) given by Roth and Szekeres (see Section 6 of Chapter 1).

The second, Hypothesis K, is a further hypothesis on sequences which already satisfy H and will be used only in estimating the supplementary integral \(S\) in (3.19).

**HYPOTHESIS K.** Let \(s\) be as in Hypothesis H. For every real number \(\lambda \in (1, 2)\), there are positive constants \(x_0\) and \(K_0\) (which depend only on \(\lambda\) and the sequence \(u\)) such that for every \(x > x_0\),

\[ \sum_{x < u_j \leq \lambda x} \|\alpha u_j\|^2 > K_0 x^{s(2-\lambda)} \]

whenever \(|\alpha| \in (1/(2\lambda x), 1/2)\).

Here \(\|\|\) denotes the distance from the nearest integer and because of the inequality \(2\|x\| \leq \sin \pi x \leq \pi \|x\|\) we observe that Hypothesis K is equivalent to the condition that for every \(\lambda \in (1, 2)\) and for every \(x > x_0\),

\[ \sum_{x < u_j \leq \lambda x} \sin^2 \pi \alpha u_j > K'_0 x^{s(2-\lambda)} \]

whenever \(|\alpha| \in (1/(2\lambda x), 1/2)\), for some positive constants \(x_0, K'_0\) depending only on \(\lambda\) and \(u\).

It is easily shown that any sequence satisfying Hypothesis K must satisfy (RS2) given by Roth and Szekeres (see Section 6 of Chapter 1).

In Chapter 4 we shall investigate Hypotheses H and K further and consider examples of sequences which satisfy both (H) and (K). In particular we will show that \(u = (j^k)\) and \(u = (p_j)\) satisfy (K) and hence satisfy both conditions.
4. Statement of main theorem

We are now in a position to state the main theorem of this chapter.

**Theorem 3.4.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypotheses H and K. Let \( \delta \) be a positive constant such that

\[
0 < \delta < 1.
\]

Let \( q_u(m, n) \) be as in Definition 1.1 and let \( \sigma \) and \( A_2 \) be as in be as in (3.12) and (3.5). Then as \( n \to \infty \),

\[
q_u(m, n) = \exp(\sigma n) \prod_{m < u_{j} \leq n} (1 + \exp(-\sigma u_{j})) \times \frac{1}{\sqrt{2\pi A_2}}
\]

\[
\times (1 + O_u(n^{-\delta/(2\delta+2)}(\log n)^{-\delta/(2\delta+2)}) + O_{Ku}((m/n)^{16/33})),
\]

for \( 0 \leq m \leq n^{1-\delta} \).

The conventions mentioned in Section 3 of Chapter 2 will be adopted henceforth in this chapter with the additional convention that because all implied constants in this chapter will depend on the sequence \( u \), we shall neglect writing \( u \) as a subscript in the notation \( O, \gg \) or \( \ll \). All other variables on which the implied constants depend will be indicated by the subscript.

5. Outline of proof of main theorem

By (3.10), in order to prove Theorem 3.4, we must estimate the integral

\[
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n)d\alpha.
\]

We write

\[
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n)d\alpha = M + S,
\]

where

\[
M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha)e(-\alpha n)d\alpha
\]

is the main integral and

\[
S = \int_{\{\alpha: \alpha_0 \leq |\alpha| \leq 1/2\}} \varphi(\alpha)e(-\alpha n)d\alpha
\]

is the supplementary integral, and where \( \alpha_0 \) is a number in the interval \((0, 1/2)\) which is to be chosen appropriately.
6. PRELIMINARY LEMMAS

After commencing with some preliminary lemmas in Section 6 I shall give in Section 7 some lemmas on the size of the characteristic function \( \varphi \) and in Section 8 shall use these to estimate the main integral \( M \):

\[
M = \frac{1}{\sqrt{2\pi A_2}} \left( 1 + E_1(\alpha_0) \right),
\]

with an explicit upper bound for the error \( E_1 \).

In Section 9 I shall give estimates of the variance \( A_2 \) and related quantities and in Section 10 shall give an upper bound for the supplementary integral \( S \).

The climax of this chapter will be reached in Section 11 where the proof of Theorem 3.4 will be completed.

Finally in Section 12 some modifications to the main theorem will be presented along with some suggestions for different hypotheses on the sequence \( u \).

6. Preliminary lemmas

We commence with the following lemma which gives an upper and lower bound on a sum connected with the sum in Hypothesis H. This lemma is used in the proof of subsequent lemmas.

**Lemma 3.5.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H. Then there is a positive integer \( n_0 = n_0(u) \) depending only on \( u \) such that for \( r > n_0 \),

\[
2(1 - 2^{-r}) C_0 r^s (\log r)^{-t} \geq U(r) - U(r/2) \geq \frac{1}{2} (1 - 2^{-r}) C_0 r^s (\log r)^{-t}.
\]

**Proof.** By Hypothesis H, as \( r \to \infty \),

\[
U(r) \sim C_0 r^s (\log r)^{-t}, \quad U(r/2) \sim 2^{-r} C_0 r^s (\log r)^{-t}.
\]

So we have

\[
U(r) - U(r/2) \sim (1 - 2^{-r}) C_0 r^s (\log r)^{-t}
\]

and the result follows easily. \( \square \)

The following lemma guarantees the existence of the parameter \( \sigma \) defined in (3.12).

**Lemma 3.6.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H. For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let

\[
A_1(x) = \sum_{m < u_j \leq n} \frac{u_j}{1 + e^{\sigma u_j}}.
\]
Then there is a positive number $n_0$ depending only on the sequence $u$ such that for every number $n > n_0$ a unique real number $\sigma = \sigma(m, n, u)$ exists such that $A_1(\sigma) = n$, with $0 < \sigma < 1/4$.

**Proof.** Let

$$A_1(x) = \sum_{m < u_j < n} u_j/(1 + \exp(xu_j)).$$

Then $A_1(x)$ is a continuous function of $x$ which is strictly decreasing on $R$. Note that applying some elementary inequalities, Lemma 3.5 and Hypothesis $H$ gives

$$A_1(0) = \frac{1}{2} \sum_{m < u_j < n} u_j \geq \frac{1}{2} \sum_{n/2 < u_j < n} u_j \geq \frac{1}{4} n \sum_{n/2 < u_j < n} 1 \geq \frac{C_0}{8} n^{1+\epsilon} (\log n)^{-\epsilon} (1 - 2^{-\epsilon}).$$

Also

$$A_1(1/4) \leq \sum_{m < u_j < n} u_j e^{-u_j/4} \leq \sum_{k=1}^{\infty} ke^{-k/4} = e^{-1/4}/(1 - e^{-1/4})^2.$$

Thus for sufficiently large $n$, say that for $n > n_0$ with $n_0$ some positive number, $A_1(0) > n > A_1(1/4)$. By the Intermediate Value Theorem, there is a unique $\sigma$ in the interval $(0, 1/4)$ such that $A_1(\sigma) = n. \quad \Box$

Thus we confirm our definition of $\sigma$ in (3.12) as the unique real number which satisfies

$$(3.20) \quad A_1 = n.$$

We now state a lemma which gives bounds on $\sigma$.

**Lemma 3.7.** Let $u$ be a strictly increasing sequence of positive integers satisfying Hypothesis $H$, with the constants $C_0$, $s$ and $t$ as in Hypothesis $H$. Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be the unique number given by (3.12). Then there is a positive number $n_0$ depending only on the sequence $u$ such that for $n > n_0$ the following statements hold.

(i) We have

$$0 < \sigma \ll n^{-1/(1+s)}.$$

(ii) There is a positive constant $K_1 = K_1(s, t, C_0)$ depending only on $s$, $t$ and $C_0$ such that

$$\sigma > \frac{1}{2r} \log \left( \frac{r^{1+t} (\log r)^{-t}}{n} (2^r - 1) C_0/4 \right),$$

for $r \in \mathbb{Z}$ satisfying

$$\max \left( m, K_1 n^{1/(1+s)} (\log n)^{1/(1+s)} \right) \leq r \leq n/2.$$

**Proof.** (i) We have that $\sigma > 0$ from Lemma 3.6. We now prove the second inequality of (i). We have

$$n = \sum_{m < u_j < n} \frac{u_j}{1 + e^{\sigma u_j}} < \sum_{j=1}^{\infty} u_j e^{-\sigma u_j},$$
where the series converges since \( \sigma > 0 \). We note that by (3.16), since \( t \geq 0 \), we have \( U(x) \leq C_1 x^t \) for all \( x \geq 2 \). Applying Lemma 2.11 (with \( f(x) = x e^{-\sigma x} \)) to the sum on the right above and noting that \(-f'(x) < \sigma x e^{-\sigma x}\), we obtain

\[
n < \int_2^\infty U(x) \sigma x e^{-\sigma x} dx \ll \int_2^\infty \sigma x^{t+1} e^{-\sigma x} dx.
\]

Using the change of variables \( \sigma x = y \) on the last integral we see that

\[
n \ll \left( \frac{1}{\sigma} \right)^{t+1} \int_1^\infty y^{t+1} e^{-y} dy \leq \left( \frac{1}{\sigma} \right)^{t+1} \Gamma(t+2) \ll \left( \frac{1}{\sigma} \right)^{t+1},
\]

and hence \( \sigma \ll n^{-1/(1+t)} \).

(ii) Let \( r \in \mathbb{Z} \) such that \( m \leq r \leq n/2 \). Then because

\[
\frac{1}{1 + e^\sigma} > \frac{1}{2} e^{-\sigma},
\]

we obtain

\[
n = \sum_{m < u_j \leq n} \frac{u_j}{1 + e^\sigma u_j}
\]

\[
> \frac{1}{2} \sum_{r < u_j \leq 2r} u_j e^{-\sigma u_j}
\]

\[
> \frac{1}{2} r e^{-2\sigma r} (U(2r) - U(r)).
\]

By Lemma 3.5 there is \( n_0 > 0 \) such that for \( r > n_0 \),

\[
n > \frac{1}{4} C_0 (2^r - 1) r^{t+1} (\log r)^{-t} e^{-2\sigma r},
\]

and rearranging gives that for \( r > n_0 \),

\[
\sigma > \frac{1}{2r} \log \left( \frac{(2^r - 1) C_0}{4} \times \frac{r^{t+1} (\log r)^{-t}}{n} \right).
\]

This inequality for \( \sigma \) will improve upon \( \sigma > 0 \) provided that

\[
(2^r - 1) \frac{C_0}{4} \times \frac{r^{t+1} (\log r)^{-t}}{n} > 1.
\]

An argument similar to that in the proof of Lemma 2.5 gives us that there is a positive number \( K_1 \) depending only on the sequence \( u \) such that if

\[
r > K_1 n^{1/(1+t)} (\log n)^{1/(1+t)}
\]

then

\[
(2^r - 1) \frac{C_0}{4} \times \frac{r^{t+1} (\log r)^{-t}}{n} > 1.
\]

This completes the proof of (ii). \( \square \)

From this lemma we have some corollaries.
Corollary 3.8 (to Lemma 3.7). Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Let \( \sigma \) be the unique number given by (3.12). Then as \( n \to \infty \),
\[
\sigma n \to \infty \quad \text{and} \quad \frac{1}{\sigma} \to \infty.
\]

Proof. Taking the reciprocal of the expressions in the inequality of Lemma 3.7(i) gives \( 1/\sigma \to \infty \).

In part (ii) Lemma 3.7, taking \( r = [n/2] \) gives \( \sigma n \to \infty \). \( \Box \)

Substituting for \( u \) the sequence of positive \( k \)-th powers (and also the values \( s = 1/k, t = 0, C_0 = 1 \)) in the proof of Lemma 3.7 gives the following corollary.

Corollary 3.9. Let \( k \) be a positive integer and let \( u = (j^k) \). Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (3.12). Then there is a positive number \( n_0 = n_0(k) \) depending only on \( k \) such that for \( n > n_0 \) the following holds.

(i) \( 0 < \sigma \leq n^{-k/(k+1)} \).

(ii) There is a positive constant \( K_1 = K_1(k) \) depending only on \( k \) such that
\[
\sigma > \frac{1}{2r} \log \left( \frac{r^{r+1} + \log n}{n} \right) - \frac{1}{2r} \log (2^{1/k} - 1)/4 \right)
\]
for \( r \in \mathbb{Z} \) satisfying
\[
\max \left( m, K_1 n^{k/(1+k)} \right) \leq r \leq n/2.
\]

Substituting for \( u \) the sequence of positive integers (and also the values \( s = 1, t = 0, C_0 = 1 \)) in the proof of Lemma 3.7 and arguing a little more carefully gives the following corollary which is Lemma 2 of Freiman and Pitman [16].

Corollary 3.10. Let \( u \) be the sequence of positive integers and for integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let \( \sigma \) be as in (3.12). Then there is an absolute constant \( n_0 > 0 \) such that for \( n > n_0 \) the following hold.

(i) \( \sigma \leq 2/\sqrt{n} \).

(ii) Let \( r \) be an integer such that
\[
\max (m, 2\sqrt{n}) \leq r < n/2.
\]

Then \( \sigma > r^{-1} \log (r/2\sqrt{n}) \).
7. Estimation of characteristic function

The purpose of this section is to estimate the behaviour of the characteristic function defined in (3.3). The characteristic function occurs in the integrand of the main and supplementary integrals, as defined in (3.16) and (3.19), and hence is essential to the estimation of these integrals. We proceed to estimate \( \varphi(\alpha) \) by examining its logarithm.

**Lemma 3.11.** Let \( u \) be a strictly increasing sequence of positive integers. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) and \( A_2 \) be as in (3.12) and (3.6). Let \( \alpha \) be a real number in the interval \((-1/2, 1/2)\) and let \( \varphi(\alpha) \) be as in (3.3). Suppose that

\[
(3.21) \quad \text{either } |\alpha| < \sigma/(2\pi \log 3) \text{ or } \sigma m > \log 3.
\]

Then we have

\[
\varphi(\alpha) = e(\alpha n)e^{-2\pi^2 \sigma^2 A_2 + R(\alpha)},
\]

where

\[
(3.22) \quad |R(\alpha)| \leq \frac{5}{3}|2\pi \alpha|^3 \rho_3
\]

and \( \rho_3 \) is the third moment as defined in (3.8).

**Proof.** We know from (3.3) and (3.4) that

\[
(3.23) \quad \log \varphi(\alpha) = \sum_{m < u_j \leq n} \log(p_{1j} + p_{2j} e(\alpha u_j))
\]

where \( p_{1j}, p_{2j} \) are as in (3.1). Let

\[
F_j(x) = \log(p_{1j} + p_{2j} e(x u_j))
\]

so that (3.23) becomes

\[
\log \varphi(\alpha) = \sum_{m < u_j \leq n} F_j(\alpha).
\]

Employing Theorem 2.8 (with \( k = 3 \) and \( f = F_j \)) gives

\[
(3.24) \quad F_j(\alpha) = F_j(0) + \alpha F_j'(0) + \alpha^2 F_j''(0)/2 + \frac{\alpha^3}{6} \int_0^1 (1 - v)^2 F_j'''(v \alpha) dv.
\]

Now \( F_j(0) = \log(p_{1j} + p_{2j}) = \log 1 = 0 \). Also it is easily checked that

\[
(3.25) \quad F_j'(0) = 2\pi i u_j p_{2j} = 2\pi i f_1(u_j),
\]

\[
(3.26) \quad F_j''(0) = (2\pi i u_j)^2 (p_{2j} - p_{2j}^2) = (2\pi i)^2 f_2(u_j),
\]
and

\[ F_j'(x) = (2 \pi u_j)^3 \left( \frac{1}{1 + \exp(\sigma u_j) e(-u_j x)} - \frac{3}{(1 + \exp(\sigma u_j) e(-u_j x))^3} + \frac{2}{(1 + \exp(\sigma u_j) e(-u_j x))^2} \right). \]

We shall estimate the integral in (3.24), namely

\[ \int_0^1 (1 - v)^2 F_j''(v \alpha) dv, \]

by first estimating an integral of the form

\[ \int_0^1 \frac{(1 - v)^2}{(1 + \exp(\sigma u_j) e(-u_j v \alpha))} dv \]

where \( l \in \mathbb{N} \) (we will be taking only \( l = 1, 2, 3 \)). We observe that

\[ 1 + \exp(\sigma u_j) e(-u_j v \alpha) = \exp(\sigma u_j) e(-u_j v \alpha)(1 + \exp(-\sigma u_j)) \left( 1 + \frac{2i \sin(\pi a u_j) e(c u_j / 2)}{1 + \exp(\sigma u_j)} \right). \]

Further we observe the following.

(a) If \( \sigma u_j > \log 3 \) then \( 2/(1 + \exp(\sigma u_j)) < 1/2. \)

(b) If \( \sigma u_j \leq \log 3 \) then by (3.21),

\[ |\sin(\pi a u_j)| \leq \pi |a| u_j \leq \frac{\pi |a| u_j}{\sigma} < \frac{1}{2}. \]

By virtue of the two preceding observations,

\[ \left| \frac{2i \sin(\pi a u_j) e(c u_j / 2)}{1 + \exp(\sigma u_j)} \right| < \frac{1}{2} \]

and so

\[ |1 + \exp(\sigma u_j) e(-u_j v \alpha)| > (1 + \exp(\sigma u_j)) \left( 1 - \left| \frac{2i \sin(\pi a u_j) e(c u_j / 2)}{1 + \exp(\sigma u_j)} \right| \right) > \frac{1}{2} (1 + \exp(\sigma u_j)). \]

Consequently for a positive integer \( l \),

\[ \left| \int_0^1 \frac{(1 - v)^2}{(1 + \exp(\sigma u_j) e(-u_j v \alpha))} dv \right| < \int_0^1 \frac{|1 - v|^2}{(1 + \exp(\sigma u_j))^l} \ dv = \frac{1}{3} 2^l (1 + \exp(\sigma u_j))^{-l} \]

and using (3.27) we deduce that

\[ \frac{1}{6} \left| \int_0^1 (1 - v)^2 F_j''(v \alpha) dv \right| \leq \frac{(2 \pi u_j)^3}{6} (1 + \exp(\sigma u_j))^{-1} \left( \frac{2}{3} + \frac{3}{4} + \frac{8}{3} \right) = \frac{5}{3} (2 \pi u_j)^3 / (1 + \exp(\sigma u_j)). \]

It follows from (3.24), (3.25), (3.26) and (3.28) that

\[ F_j(\alpha) = 2 \pi i a f_1(u_j) + (2 \pi i a)^2 f_2(u_j) / 2 + R_j(\alpha), \]

where

\[ |R_j(\alpha)| \leq \frac{5}{3} |2 \pi i a|^3 u_j^3 / (1 + e^{\sigma u_j}) \]
and where \( f_1(x) \) and \( f_2(x) \) are defined as in (3.7). Summing both sides of (3.29) over \( u_j \in (m, n] \) and using (3.5), (3.6) and (3.12) gives the result. \( \Box \)

**Lemma 3.12.** Let \( u \) be a strictly increasing sequence of positive integers. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (3.12). Let \( \varphi(\alpha) \) be the characteristic function as in (3.3). There is a positive number \( n_0 \) depending only on \( u \) such that for \( n > n_0 \) and for all real numbers \( \alpha \),

\[
|\varphi(\alpha)| \leq \exp\left(-2 \sum_{m < u_j \leq n} e^{-\sigma u_j}||\alpha u_j||^2\right).
\]

**Proof.** From (3.3) and (3.4) we have that for any real number \( \alpha \)

\[
|\varphi(\alpha)|^2 = \prod_{m < u_j \leq n} |(p_{1j} + p_{2j}e^{\alpha u_j})|^2
= \prod_{m < u_j \leq n} (p_{1j}^2 + p_{2j}^2 + 2p_{1j}p_{2j} \cos(2\pi \alpha u_j)).
\]

Using the double angle identity \( \cos 2x = 1 - 2\sin^2 x \) and the inequality for \( x \in [0, 1], 0 \leq 1 - x \leq \exp(-x) \),

\[
|\varphi(\alpha)|^2 = \prod_{m < u_j \leq n} (1 - 4p_{1j}p_{2j} \sin^2(\pi \alpha u_j))
\leq \prod_{m < u_j \leq n} \exp(-4p_{1j}p_{2j} \sin^2(\pi \alpha u_j)).
\]

Now from (3.1) and positivity of \( \sigma \) by Lemma 3.6, we have

\[
p_{1j}p_{2j} = \frac{\exp(-\sigma u_j)}{(1 + \exp(-\sigma u_j))^2} > \frac{1}{4} \exp(-\sigma u_j).
\]

Thus, since \( |\sin \pi x| \geq 2||x|| \)

\[
|\varphi(\alpha)|^2 \leq \prod_{m < u_j \leq n} \exp(-4\exp(-\sigma u_j)||\alpha u_j||^2),
= \exp\left(-4 \sum_{m < u_j \leq n} \exp(-\sigma u_j) ||\alpha u_j||^2\right)
\]
and the result follows. \( \Box \)

8. Estimation of the main integral

Let \( \alpha_0 \) be a number in the interval \((0, 1/2)\) which will be chosen appropriately later on. Lemma 3.11 suggests that the integrand of the main integral is approximated by the probability density function of a normal random variable with variance \( A_2 \). By using a suitable choice of \( \alpha_0 \) together with Lemma 3.12 and Hypothesis K it will be possible to show that the supplementary integral is dominated by the main integral.
DEFINITION 3.13. For $0 \leq z < \infty$ and $c > 0$ define $I(z, c)$ by

$$I(z, c) = \int_{-z}^{z} \exp(-cx^2)dx.$$  

It is apparent that $I(z, c)$ is closely related to the error function

$$\text{erf}(z) = \int_{-\infty}^{z} \exp(-x^2/2)dx$$

in that the identity

$$I(z, c) = \sqrt{2c} \text{erf}(\sqrt{2cz}) - \text{erf}(-\sqrt{2cz})$$

holds.

We shall show that the main integral $M$ is approximated by $I(\infty, 2\pi^2 A_2)$ fairly accurately. To do this, we require two lemmas.

LEMMA 3.14. For a given strictly increasing sequence $u$ of positive integers and for given integers $m$ and $n$ such that $0 \leq m < n/2$, let $\sigma$, $A_2$, $\rho_3$ be as in (3.12), (3.6), (3.8). Suppose that a given real number $\alpha_0$ in the interval $(0, 1/2)$ satisfies the conditions

(3.30) \quad \text{either} \quad \alpha_0 < \sigma/(2\pi \log 3) \quad \text{or} \quad \sigma m > \log 3,

(3.31) \quad \frac{5}{3} (2\pi \alpha_0)^3 \rho_3 < \frac{1}{2}.

Then there is a positive number $n_0$ depending only on $u$ such that for $n > n_0$,

$$M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha)e(-\alpha \alpha)d\alpha$$

satisfies

$$M = I(\alpha_0, 2\pi^2 A_2) + O\left(\frac{\rho_3}{A_2^2}\right)$$

where the implied constants depend only on $u$ and where $I(z, c)$ is as in Definition 3.13.

PROOF. Substituting the expression for the characteristic function $\varphi(\alpha)$ given in Lemma 3.11 into the expression for the main integral given in (3.18) gives

$$M = \int_{-\alpha_0}^{\infty} \exp(-2\pi^2 \alpha^2 A_2 + R(\alpha))d\alpha.$$  

Combining (3.22) with our assumption (3.31) ensures that $|R(\alpha)| < 1/2$ and hence

$$|\exp(R(\alpha)) - 1| \leq \sum_{t=1}^{\infty} \frac{|R(\alpha)|^t}{t!} \leq \frac{|R(\alpha)|}{1 - |R(\alpha)|} < 2|R(\alpha)|.$$

Using (3.22) again gives

$$|\exp(R(\alpha)) - 1| \leq \frac{5}{3} |2\pi \alpha|^{3} \rho_3.$$
Hence

\[ |M - I(\alpha_0, 2\pi^2 A_2)| = \int_{-\alpha_0}^{\alpha_0} \exp(-2\pi^2 \alpha^2 A_2) (\exp(R(\alpha)) - 1) \, d\alpha \]

\[ \leq \int_{-\alpha_0}^{\alpha_0} \exp(-2\pi^2 \alpha^2 A_2) 2 \cdot \frac{5}{3} |2\pi \alpha|^3 \rho_3 \, d\alpha. \]

Writing \( \beta_0 = 2\pi \sqrt{A_2} \alpha_0 \) and using the change of variable \( \beta = 2\pi \sqrt{A_2} \alpha \), we see that the integral on the right is

\[ \frac{\rho_3}{A_2^{3/2}} \frac{1}{2\pi \sqrt{A_2}} \int_{-\beta_0}^{\beta_0} \exp(-\beta^2/2) |\beta|^3 \, d\beta \]

\[ \ll \frac{\rho_3}{A_2^{3/2}}. \]

This completes the proof of the lemma. \( \square \)

**Lemma 3.15.** For a given strictly increasing sequence \( u \) of positive integers and for given integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \) let \( \sigma, A_2, \rho_3 \) be as in (3.12), (3.6), (3.8). Suppose that a given real number \( \alpha_0 \) in the interval \((0, 1/2)\) satisfies the conditions (3.30), (3.31) and further the condition

(3.32)

\[ \alpha_0 > \frac{1}{2\pi \sqrt{A_2}}. \]

Then

\[ I(\infty, 2\pi^2 A_2) - I(\alpha_0, 2\pi^2 A_2) \ll \frac{1}{\sqrt{A_2}} \exp(-2\pi^2 \alpha_0^2 A_2), \]

where the implied constants depend only on \( u \).

**Proof.** We have

\[ I(\infty, 2\pi^2 A_2) - I(\alpha_0, 2\pi^2 A_2) = \left( \int_{\alpha_0}^{\infty} + \int_{-\infty}^{-\alpha_0} \right) \exp(-2\pi^2 \alpha^2 A_2) \, d\alpha \]

\[ = 2 \int_{\alpha_0}^{\infty} \exp(-2\pi^2 \alpha^2 A_2) \, d\alpha. \]

Writing \( \beta_0 = 2\pi \sqrt{A_2} \alpha_0 \) and using the change of variable \( \beta = 2\pi \sqrt{A_2} \alpha \), we see that the integral on the right is

\[ \frac{1}{\pi \sqrt{A_2}} \int_{-\beta_0}^{\beta_0} \exp(-\beta^2/2) \, d\beta \ll \frac{1}{\pi \sqrt{A_2}} \int_{-\beta_0}^{\beta_0} \beta \exp(-\beta^2/2) \, d\beta \]

\[ = \frac{1}{\pi \sqrt{A_2}} \exp(-\beta_0^2/2) \]

\[ = \frac{1}{\pi \sqrt{A_2}} \exp(-2\pi^2 A_2 \alpha_0^2), \]

where the inequality follows from the condition \( \alpha_0 > 1/(2\pi \sqrt{A_2}) \), and this completes the proof. \( \square \)

Combining Lemma 3.14 and Lemma 3.15 yields our final estimate of \( M \).
Corollary 3.16 (to Lemma 3.15). For a given strictly increasing sequence \( u \) of positive integers and for given integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \) let \( \sigma, A_2, \rho_3 \) be as in (3.12), (3.6), (3.8). Suppose that a given real number \( \alpha_0 \) in the interval \((0, 1/2)\) satisfies the conditions (3.30), (3.31), (3.32). Then there is a positive number \( n_0 \) depending only on \( u \) such that for \( n > n_0 \) the main integral

\[
M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha)e(-\alpha n)d\alpha
\]

satisfies

\[
M = \frac{1}{\sqrt{2\pi A_2}}(1 + E_1(\alpha_0))
\]

where

\[
E_1(\alpha_0) \ll \frac{\rho_3}{A_2^{3/2}} + \exp(-2\pi^2 A_2 \alpha_0^2)
\]

and the implied constants depend only on \( u \).

Proof. Write \( E_1(\alpha_0) = \sqrt{2\pi A_2}(M - I(\infty, 2\pi^2 A_2)). \) We know that \( I(\infty, 2\pi^2 A_2) = 1/\sqrt{2\pi A_2}. \) Also combining the triangle inequality with Lemma 3.14 and Lemma 3.15 gives

\[
|M - I(\infty, 2\pi^2 A_2)| \leq |M - I(\alpha_0, 2\pi^2 A_2)| + |I(\infty, 2\pi^2 A_2) - I(\alpha_0, 2\pi^2 A_2)|
\]

\[
\ll \frac{\rho_3}{A_2^{3/2}} + \frac{1}{\sqrt{A_2}} \exp(-2\pi^2 \alpha_0^2 A_2).
\]

It will turn out that there is a choice of \( \alpha_0 \) which is compatible with the conditions (3.30), (3.31), (3.32).

9. Estimation of cumulants

In view of the above corollary, we require an estimate of the second cumulant \( A_2 \) and the quantity \( \rho_3 \). In this section we estimate the cumulants \( A_1, A_2 \) and the related quantities \( \sigma \) and \( \rho_3 \). We start with some general estimates of integrals and sums.

Lemma 3.17. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (3.12). Let \( a, b \) be positive real numbers and let \( r \in \mathbb{N} \) such that

\[
\frac{\max(1, 2a)}{\sigma} \leq r \leq n.
\]

Then

\[
\int_r^\infty x^a (\log x)^{-b} \exp(-\sigma x)dx \ll \frac{1}{a}(\log \frac{1}{a})^{-b} \exp(-\sigma r)
\]

where the implied constants depend only on the real number \( a. \)
PROOF. By using the change of variables $y = \sigma x$ we obtain
\[ \int_{r}^{\infty} x^{a} (\log x)^{-b} \exp(-\sigma x) dx = \sigma^{-a-1} \int_{0}^{\infty} y^{a} (\log \frac{y}{\sigma})^{-b} \exp(-y) dy \]
\[ = \sigma^{-a-1} \int_{0}^{\infty} y^{a} \left( \frac{1}{\log y + \frac{1}{\sigma}} \right)^{b} \exp(-y) dy. \]
Since $\sigma r \geq 1$ the right hand side is
\[ \leq \sigma^{-a-1} \int_{0}^{\infty} y^{a} (\log \frac{1}{\sigma})^{-b} \exp(-y) dy \]
\[ = \sigma^{-a-1} (\log \frac{1}{\sigma})^{-b} \int_{0}^{\infty} y^{a} \exp(-y) dy. \]
The result follows by noting that $\sigma r \geq 2a$ and applying Lemma 2.7 to $\int_{0}^{\infty} y^{a} \exp(-y) dy$. \qed

**Lemma 3.18.** Let $a, b$ be positive real numbers. Then for a real number $z > 1$,
\[ J = J(z; a, b) = \int_{2}^{\infty} x^{a} (\log x)^{-b} \exp(-x/z) dx \ll \frac{z^{a+1}}{(a+1)(\log z)^{-b}}. \]

**Proof.** Using the change of variables $y = x/z$ and the fact that $z$ is positive we have
\[ J = z^{a+1} \int_{0}^{\infty} y^{a} (\log yz)^{-b} \exp(-y) dy. \]

Case 1: Suppose $z > 2$. We write
\[ J = z^{a+1} \left( \int_{2/z}^{1} + \int_{1}^{\infty} \right) y^{a} (\log yz)^{-b} \exp(-y) dy. \]
Now since $\log y > 0$ for $y \in (1, \infty)$ and since $\log z > 0$,
\[ \int_{1}^{\infty} y^{a} (\log y + \log z)^{-b} \exp(-y) dy \leq \int_{1}^{\infty} y^{a} (\log z)^{-b} \exp(-y) dy \]
\[ < (\log z)^{-b} \int_{0}^{\infty} y^{a} e^{-y} dy \]
\[ = \Gamma(a+1)(\log z)^{-b}. \]
The integral over the interval $(2/z, 1)$ is
\[ \int_{2/z}^{1} y^{a} (\log yz)^{-b} \exp(-y) dy = \int_{2/z}^{1} y^{a} (\log z)^{-b} \left( 1 - \frac{\log 1/y}{\log z} \right)^{-b} \exp(-y) dy. \]
Note that for $2/z < y < 1$, we have upon taking reciprocals,
\[ 1 < 1/y < z/2. \]
Then taking logarithms gives
\[ 0 < \log 1/y < \log z - \log 2 \]
and multiplying both sides by $(\log 1/y)/(\log 2 \log z)$ we conclude that
\[ 1 < 1 + \log 1/y \left( \frac{1}{\log 2} - \frac{1}{\log z} \right) = \frac{(\log 1/y)^2}{\log 2 \log z} = \left( 1 + \frac{\log 1/y}{\log 2} \right) \times \left( 1 - \frac{\log 1/y}{\log z} \right), \]
that is
\[ \left(1 - \frac{\log 1/y}{\log z}\right)^{-1} < 1 + \frac{\log 1/y}{\log 2}. \]

Thus
\[ \int_{2/z}^{1} y^{a}(\log z)^{-b} \left(1 - \frac{\log 1/y}{\log z}\right)^{-b} \exp(-y)dy < \int_{2/z}^{1} y^{a}(\log z)^{-b} \left(1 + \frac{\log 1/y}{\log 2}\right)^{b} \exp(-y)dy. \]

We note that the integral
\[ (3.34) \quad \int_{2/z}^{1} y^{a} \left(1 + \frac{\log 1/y}{\log 2}\right)^{b} \exp(-y)dy \leq \int_{0}^{1} y^{a} \left(1 + \frac{\log 1/y}{\log 2}\right)^{b} \exp(-y)dy \]
which is \( \ll 1 \), since the integrand \( \to 0 \) as \( y \to 0 \). The result follows form (3.33) and (3.34).

Case 2: Suppose \( 1 < z \leq 2 \). Then
\[ J < z^{a+1} \int_{1}^{\infty} y^{a}(\log yz)^{-b} e^{-y}dy \]
and the result follows from (3.33). \( \square \)

The following lemma is used in the proof of a subsequent lemma. Its straightforward proof is omitted here.

**Lemma 3.19.** Let \( f, g \) and, for a positive real number \( w \), \( h_w \) be measurable functions from \((0, \infty)\) to \((0, \infty)\) such the integrals
\[ \int_{0}^{\infty} f(x)h_w(x)dx, \quad \int_{0}^{\infty} g(x)h_w(x)dx \]
are finite. Suppose that as \( x \to \infty, f(x) \sim g(x) \). Then as \( z \to \infty, \)
\[ \int_{z}^{\infty} f(x)h_w(x)dx \sim \int_{z}^{\infty} g(x)h_w(x)dx, \]
uniformly in \( w \).

The following lemma will enable us to give upper bounds for \( \rho_1, \rho_2, \rho_3, \ldots \) for all \( m \) in the range \( 0 \leq m < n/2 \) and to estimate the size of these quantities for large \( m \), that is for \( m \geq 1/\sigma \).

**Lemma 3.20.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H with \( C_0, s \) and \( t \) being as in Hypothesis H. Let \( \lambda \) be a real number in the interval \((1, 2)\). Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (3.12). Let \( k \) be a fixed positive integer. Then the following estimates hold.

(i) As \( n \to \infty, \)
\[ \sum_{0 < u_j \leq n} u_j^{k} e^{-\sigma u_j} \ll \left(\frac{1}{\sigma}\right)^{\frac{k+1}{k}} \left(\log 1/\sigma\right)^{-t} \]
where the implied constants depend only on \( k \) and the sequence \( u \).
(ii) Let $r$ be a positive integer such that

$$2(k + s)/\sigma \leq r \leq n/\lambda.$$ 

Then as $n \to \infty$,

$$r^{k+s}(\log r)^{-t}e^{-\sigma r} \ll \sum_{k \leq u_j \leq r} u_j^k e^{-\sigma u_j} \ll r^{k+s}(\log r)^{-t}e^{-\sigma r}.$$ 

PROOF. (i): We have that

$$(3.35) \quad \sum_{0 \leq u_j \leq n} u_j^k e^{-\sigma u_j} < O(1) + \sum_{2 < u_j < \infty} u_j^k e^{-\sigma u_j}.$$ 

Let $\phi(x) = x^k e^{-\sigma x}$, and hence $\phi'(x) = (kx^{k-1} - \sigma x^k)e^{-\sigma x}$. Then, since $U(x)\phi(x) \to 0$ as $x \to \infty$,

Lemma 2.11 gives

$$\sum_{2 < u_j < \infty} \phi(u_j) = -\int_{2}^{\infty} U(x)\phi'(x)dx - U(2)\phi(2).$$

We immediately have the inequality

$$(3.36) \quad -\int_{2}^{\infty} U(x)\phi'(x)dx - U(2)\phi(2) = -\int_{2}^{\infty} U(x)(kx^{k-1} - \sigma x^k)e^{-\sigma x}dx - U(2)\phi(2) \ll \int_{2}^{\infty} U(x)\sigma x^k e^{-\sigma x}dx.$$ 

By Hypothesis H, there is a positive number $C_1$, depending only on $u$, such that for $x \geq 2$, $U(x) \leq C_1 x^r (\log x)^{-t}$. Thus

$$\int_{2}^{\infty} U(x)x^k e^{-\sigma x}dx \ll \int_{2}^{\infty} x^{r+k}(\log x)^{-t}e^{-\sigma x}dx.$$ 

By Lemma 3.18 (with $z = 1/\sigma$; this is $> 1$ by Lemma 3.6)

$$\int_{2}^{\infty} x^{r+k}(\log x)^{-t}e^{-\sigma x}dx \ll \left(\frac{1}{\sigma}\right)^{r+k} (\log \frac{1}{\sigma})^{-t},$$

and so

$$(3.37) \quad \int_{2}^{\infty} U(x)\sigma x^k e^{-\sigma x}dx \ll \left(\frac{1}{\sigma}\right)^{r+k} (\log \frac{1}{\sigma})^{-t}.$$ 

where the implied constant depends only on $k$ and the sequence $u$. The result follows immediately from

(3.35), (3.36) and (3.37).

(ii): By Lemma 2.11

$$\sum_{r < u_j < \infty} u_j^k e^{-\sigma u_j} = -\int_{r}^{\infty} U(x)\phi'(x)dx - \phi(r)U(r)$$

where $\phi(x) = x^k e^{-\sigma x}$. Thus

$$\sum_{r < u_j \leq n} u_j^k e^{-\sigma u_j} \leq -\int_{r}^{\infty} U(x)kx^{k-1}e^{-\sigma x}dx + \int_{r}^{\infty} U(x)\sigma x^k e^{-\sigma x}dx - \phi(r)U(r)$$

$$< \int_{r}^{\infty} U(x)\sigma x^k e^{-\sigma x}dx$$
3. ESTIMATE OF $i_n(m,n)$ FOR GENERAL SEQUENCE

Now as $n \to \infty$, we have that $r \to \infty$ because $r \gg 1/\sigma$ and Corollary 3.8 gives us that as $n \to \infty, 1/\sigma \to \infty$. Hence Hypothesis H and Lemma 3.19 give us that as $n \to \infty$

$$\int_r^\infty U(z)\sigma z^k e^{-\sigma z}dz \sim \int_r^\infty C_0 \sigma z^{k+1}(\log z)^{-t} e^{-\sigma z}dz$$

which in view of Lemma 3.17 gives

$$\int_r^\infty U(z)\sigma z^k e^{-\sigma z}dz \ll (\log \frac{1}{\sigma})^{-t} r^{k+1} \exp(-\sigma r).$$

This completes the proof of the second inequality of (ii).

We know that $\lambda r \leq n$ because $r \leq n/\lambda$ so that we may write

$$\sum_{r \leq u_j \leq \lambda r} u_j^k e^{-\sigma u_j} \geq \sum_{r \leq u_j \leq \lambda r} u_j^k e^{-\sigma u_j}.$$  

Now for $r < u_j \leq \lambda r$ we have that $u_j^k > r^k$ and $\exp(-\sigma u_j) \geq \exp(-\sigma r)$ and so the right hand side above is

$$> r^k e^{-\sigma r} \sum_{r \leq u_j \leq \lambda r} 1.$$  

Also by Hypothesis H we have that as $r \to \infty$

$$\sum_{r \leq u_j \leq \lambda r} 1 = U(\lambda r) - U(r) \sim C_0 (\lambda^t - 1) r^t (\log r)^{-t}.$$  

Thus we have that

$$\sum_{r \leq u_j \leq \lambda r} u_j^k e^{-\sigma u_j} \gg e^{-\sigma r} r^t (\log r)^{-t},$$

which gives the first inequality in (ii). \(\square\)

The following lemma gives upper bounds for $A_1$, $A_2$ and $\rho_3$ in terms of $\sigma$ which are valid for all $m$, together with estimates of these quantities which are valid when $m$ is "large".

**COROLLARY 3.21 (TO LEMMA 3.20).** Let $u$ be a strictly increasing sequence of positive integers satisfying Hypothesis H and let $s$ and $t$ be as in Hypothesis H. Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (3.12). Let $A_1$, $A_2$ and $\rho_3$ be defined as in (3.5), (3.6), and (3.8). Then we have the following estimates.

(i) As $n \to \infty$,

$$n = A_1 \ll \sigma^{-1} (\log 1/\sigma)^{-1}, \quad A_2 \ll \sigma^{-2} (\log 1/\sigma)^{-1}, \quad \rho_3 \ll \sigma^{-3} (\log 1/\sigma)^{-1},$$

where the implied constants depend on $u$ only.

(ii) Let $\lambda$ be a given real number in the interval $(1,2)$. Then as $m \to \infty$,

$$m^{1+t}(\log m)^{-t} e^{-\sigma m} \ll A_1 \ll m^{1+t}(\log m)^{-t} e^{-\sigma m}.$$
9. ESTIMATION OF CUMULANTS

\[ m^{3+s}(\log m)^{-t}e^{-\sigma m^{\lambda}} \leq A_2 \leq m^{3+s}(\log m)^{-t}e^{-\sigma m}, \]

\[ m^{3+s}(\log m)^{-t}e^{-\sigma m^{\lambda}} \leq \rho_3 \leq m^{3+s}(\log m)^{-t}e^{-\sigma m}, \]

for \( 2(3+s)/\sigma \leq m \leq n/2 \).

**Proof.** (i) Since

\[ A_1 < \sum_{m \leq u_j \leq n} u_j e^{-\sigma u_j}, \quad A_2 < \sum_{m \leq u_j \leq n} u_j^2 e^{-\sigma u_j}, \quad \rho_3 < \sum_{m \leq u_j \leq n} u_j^2 e^{-\sigma u_j}, \]

we immediately have the required inequalities upon application of Lemma 3.20(i).

(ii) Combining Lemma 3.20(ii) with the observations that as \( n \to \infty \),

\[ A_1 \asymp \sum_{m \leq u_j \leq n} u_j e^{-\sigma u_j}, \quad A_2 \asymp \sum_{m \leq u_j \leq n} u_j^2 e^{-\sigma u_j}, \quad \rho_3 \asymp \sum_{m \leq u_j \leq n} u_j^2 e^{-\sigma u_j} \]

for \( 1/\sigma \ll m \leq n/2 \), where the implied constants depend only on \( u \), gives the inequalities for \( 2(3+s)/\sigma \leq m \leq n/2 \). \( \Box \)

The following lemma gives lower bounds for the quantities \( A_1, A_2 \) and \( \rho_3 \) in terms of the quantity \( \sigma \) when \( m \) is “small”.

**Lemma 3.22.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H and let \( s \) and \( t \) be as in Hypothesis H. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (3.12). Let \( A_1, A_2 \) and \( \rho_3 \) be defined as in (3.5), (3.6), and (3.8). Let \( K \geq 1 \) be a constant. Then as \( n \to \infty \),

\[ A_1 \gg \left( \frac{1}{\sigma} \right)^{s+1}(\log 1/\sigma)^{-t}, \quad A_2 \gg \left( \frac{1}{\sigma} \right)^{s+2}(\log 1/\sigma)^{-t}, \quad \rho_3 \gg \left( \frac{1}{\sigma} \right)^{s+3}(\log 1/\sigma)^{-t}, \]

for \( 0 \leq m \leq K/\sigma \), where the implied constants depend only on \( K \) and the sequence \( u \).

**Proof.** We prove the first inequality and note that the proof of the remaining two inequalities follow a similar line of reasoning. Using (3.5) and noting that \( m \leq K/\sigma \) and \( (1 + \exp(\sigma u_j))^{-1} > \exp(-\sigma u_j)/2 \) gives

\[ A_1 = \sum_{m \leq u_j \leq n} \frac{u_j}{1 + e^{\sigma u_j}} > \frac{1}{2} \sum_{K/\sigma < u_j \leq n} u_j e^{-\sigma u_j}. \]

By Lemma 3.20(ii) with \( \lambda = 3/2 \) and \( r = [K/\sigma] \) this is

\[ \gg \left( \frac{K}{\sigma} \right)^{s+1}(\log K/\sigma)^{-t}e^{-\sigma(K/\sigma)^{3/2}} \]

\[ \gg \left( \frac{1}{\sigma} \right)^{s+1}(\log 1/\sigma)^{-t}, \]

where the implied constant depends only on \( K \) and the sequence \( u \). \( \Box \)
With the following lemma, we are well on the way to obtaining explicit estimates of the quantities \( A_2 \) and \( \rho_3 \).

**Lemma 3.23.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H and let \( s \) and \( t \) be as in Hypothesis H. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (3.12). Let \( A_1 \), \( A_2 \) and \( \rho_3 \) be defined as in (3.5), (3.6), and (3.8). Let \( \lambda \) be a real number in the interval \( (1, 2) \). Then as \( n \to \infty \),

\[
m \ll \frac{A_2}{A_1} \ll \frac{\rho_3}{\lambda} \ll me^{\sigma m(\lambda - 1)},
\]

for \((6 + 2s)/\sigma \leq m \leq n/2\), where the third implied constant depends only on \( \lambda \) and the sequence \( u \).

**Proof.** For a positive integer \( k \), let

\[
\rho_k = \sum_{m < u_j \leq n} \frac{u_j^k}{1 + e^{\sigma u_j}}.
\]

It is easily seen from (3.6) that

\[
\frac{1}{2} \rho_2 < A_2 < \rho_2.
\]

(3.38)

We now show that

\[
m < \frac{\rho_2}{\rho_1} \leq \frac{\rho_3}{\rho_2} \ll me^{\sigma m(\lambda - 1)},
\]

(3.39)

where the implied constant depends only on \( \lambda \) and the sequence \( u \). In view of (3.38) it is sufficient to show (3.39) in order to prove the lemma.

Firstly,

\[
m \rho_1 = \sum_{m < u_j \leq n} \frac{m u_j}{1 + e^{\sigma u_j}} < \sum_{m < u_j \leq n} \frac{u_j^2}{1 + e^{\sigma u_j}} = \rho_2.
\]

(3.40)

Secondly, we use the Cauchy–Schwarz Inequality to yield

\[
\rho_2^2 = \left( \sum_{m < u_j \leq n} \frac{u_j^{1/2}}{(1 + \exp(\sigma u_j))^{1/2}} \frac{u_j^{3/2}}{(1 + \exp(\sigma u_j))^{3/2}} \right)^2
\]

\[
\leq \sum_{m < u_j \leq n} \frac{u_j}{1 + \exp(\sigma u_j)} \times \sum_{m < u_j \leq n} \frac{u_j^3}{1 + \exp(\sigma u_j)}
\]

\[
= \rho_1 \rho_3.
\]

(3.41)

Finally, to show the third inequality in (3.39) we appeal to Corollary 3.21(ii) to give as \( n \to \infty \),

\[
\frac{\rho_3}{\rho_2} \ll \frac{\rho_3}{A_2} \ll me^{\sigma m(\lambda - 1)},
\]

for \((6 + 2s)/\sigma \leq m \) and this completes the proof of the lemma. \( \Box \)
The following lemma provides an estimate of the quantity $\exp(\sigma m)$ which in tandem with Lemma 3.23 is of use in the estimation of the cumulant $A_2$ and the quantity $\rho_3$.

**Lemma 3.24.** Let $u$ be a strictly increasing sequence of positive integers satisfying Hypothesis H and let $s$ and $t$ be as in Hypothesis H. Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (3.12). Let $A_1$, $A_2$ and $\rho_3$ be defined as in (3.5), (3.6), and (3.8) respectively. Let $\lambda$ be a real number in the interval $(1, 2)$. Then as $n \to \infty$,

$$\left( \frac{m^{1+s}(\log m)^{-t}}{n} \right)^{1/\lambda} \leq e^{\sigma m} \leq \frac{m^{1+s}(\log m)^{-t}}{n},$$

for $(6 + 2s)/\sigma \leq m \leq n/2$, where the implied constants depend only on $\lambda$ and the sequence $u$.

**Proof.** Corollary 3.21(ii) gives us that as $n \to \infty$ for $m \geq (6 + 2s)/\sigma$,

$$m^{1+s}(\log m)^{-t}e^{-\sigma m} \leq A_1 \leq m^{1+s}(\log m)^{-t}e^{-\sigma m}.$$ 

By our choice of $\sigma$ in (3.20) we have $A_1 = n$ and so we have immediately that

$$m^{1+s}(\log m)^{-t}e^{-\sigma m} \leq n \leq m^{1+s}(\log m)^{-t}e^{-\sigma m}.$$ 

Rearranging the inequalities give the result. $\square$

Finally, we present the following explicit estimates of the quantities $A_2$ and $\rho_3$ which combine to cover all $m$ in the range $0 \leq m \leq n/2$.

**Lemma 3.25.** Let $u$ be a strictly increasing sequence of positive integers satisfying Hypothesis H and let $s$ and $t$ be as in Hypothesis H. Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (3.12). Let $A_1$, $A_2$ and $\rho_3$ be defined as in (3.5), (3.6), and (3.8). Let $\lambda$ be a real number in the interval $(1, 2)$.

(i) Let $K \geq 1$ be a constant. For $0 \leq m \leq K/\sigma$ we have that as $n \to \infty$, $\sigma^{-1} \asymp n(\log n)^{t/(t+1)}$,

$$A_2 \asymp n^{(s+2)/(1+s)}(\log n)^{t/(1+s)} \cdot \rho_3 \asymp n^{(s+2)/(1+s)}(\log n)^{t/(1+s)}$$

and

$$\frac{\rho_3}{A_2} \asymp n^{-s/(2s+2)}(\log n)^{-t/(2s+2)},$$

where the implied constants depend only on $K$ and $u$.

(ii) For $(6 + 2s)/\sigma \leq m \leq n/2$, as $n \to \infty$, we have

$$\frac{1}{2}m^2n^2 < A_2 \asymp m^{1+(1+s)(\lambda-1)n^{2-\lambda}}, \quad \rho_3 \asymp m^{2+2(\lambda-1)(1+s)n^{3-2\lambda}}, \quad \frac{\rho_3}{A_2} \asymp \frac{m^{1/2+(1+s)(\lambda-1)/2}}{n^{\lambda/2}},$$

where the implied constants depend only on $u$ and $\lambda$. 

PROOF. (i) For \(0 \leq m \leq K/\sigma\) we appeal to Corollary 3.21(i) and Lemma 3.22 to give
\[
n = A_1 \asymp \frac{1}{K} \left( \frac{1}{\sigma} \right)^{s+1} (\log 1/\sigma)^{-t}, \quad A_2 \asymp \frac{1}{K} \left( \frac{1}{\sigma} \right)^{s+2} (\log 1/\sigma)^{-t}, \quad \rho_3 \asymp \frac{1}{K} \left( \frac{1}{\sigma} \right)^{s+3} (\log 1/\sigma)^{-t},
\]
where the implied constants depend only on \(K\) and \(u\). We can express \(1/\sigma\) in terms of \(n\) by employing Lemma 2.6 (and the fact that \(1/\sigma \to \infty\) as \(n \to \infty\)) to invert the expression for the magnitude of \(A_1\), giving
\[
\frac{1}{\sigma} \asymp n^{1/(s+1)} (\log n)^{t/(s+1)}
\]
from which we have
\[
A_2 \asymp n^{(s+2)/(1+\sqrt{s})} (\log n)^{t/(1+\sqrt{s})} \quad \text{and} \quad \rho_3 \asymp n^{(s+3)/(s+1)} (\log n)^{t/(s+1)},
\]
where the implied constants depend only on \(K\) and \(u\). The remainder of the first part of the lemma follows easily.

(ii) Combining (3.20), (3.38) and (3.39) gives the inequalities
\[
A_2 = A_1 \times \frac{A_2}{A_1} > A_1 \times \frac{\rho_2}{\rho_1} > \frac{1}{2} mn.
\]
Also Lemma 3.23 gives
\[
A_2 = A_1 \times \frac{A_2}{A_1} \ll n \times m e^{m (\lambda - 1)} \quad \text{and} \quad \rho_3 = A_1 \times \frac{A_2}{A_1} \times \frac{\rho_3}{\lambda} \ll n \times \left( m e^{m (\lambda - 1)} \right)^2.
\]
Lemma 3.24 gives
\[
e^{\lambda m} \ll \frac{m^{1+s} (\log m)^{-t}}{n} \ll \frac{m^{1+s}}{n},
\]
so that
\[
A_3 \ll n^{-\lambda} m^{1+s} (\lambda - 1) \quad \text{and} \quad \rho_3 \ll n^{-\lambda} m^{2(1+\sqrt{s}) (\lambda - 1)},
\]
from which the result follows. \(\square\)

10. Estimation of supplementary integral

We estimate the supplementary integral as given in (3.19).

**Lemma 3.26.** Let \(\lambda\) be a real number in the interval \((1, 2)\). Let \(u\) be a strictly increasing sequence of positive integers satisfying Hypothesis H and Hypothesis K. Let \(m\) and \(n\) be integers such that \(0 \leq m < n/2\) and let \(\sigma\) be as in (3.12) and let \(A_1, A_2\) and \(\rho_3\) be defined as in (3.5), (3.6), and (3.8). Let \(\alpha_0\) be some number in the interval \((0, 1/2)\) and let
\[
S = \int_{\|\alpha\| \leq |\alpha| \leq 1/2} \varphi(\alpha) e(-\alpha n) d\alpha
\]
be as in (3.19).
(i) Let $K \geq 1$ be a constant. There is a constant $n_0 > 0$ depending only on $\mu$, $\lambda$ and $K$ and there is a constant $R_1 > 0$ depending only on $K$, $\lambda$ and $\mu$ such that for $n > n_0$, if $0 \leq m \leq K/\sigma$ then
\[ |S| \leq \exp(-R_1 \min(\alpha_0^2 \sigma^{-2}, 1) n^{(2-\lambda)/(\tau+1)}). \]

(ii) There is a constant $n_0 > 0$ depending only on $\lambda$ and $\mu$ and there is a constant $R_2 > 0$ depending only on $\lambda$ and $\mu$ such that for $n > n_0$, if $(6 + 2s)/\sigma \leq m \leq n/2$ then
\[ |S| \leq \exp(-R_2 \min(\alpha_0^2 \sigma^{-2}, 1) n^{\lambda} n^{2s-2\lambda s-\lambda}). \]

**Proof.** From the expression for the supplementary integral in (3.19) we have
\[ |S| \leq \sup_{\{\alpha, \alpha_0 \leq |\alpha| \leq 1/2\}} |\varphi(\alpha)|. \]

From Lemma 3.12 we have an estimate of the size of $\varphi$ and hence an upper bound for $|S|$. This is the idea of the proof.

(i) Suppose $0 \leq m \leq K/\sigma$. Then, since $n \gg 2K/\sigma$ by Corollary 3.8,
\[ \sum_{m < u_j \leq n} e^{-\alpha u_j} ||\alpha u_j||^2 > \sum_{K/\sigma < u_j \leq \lambda K/\sigma} e^{-\alpha u_j} ||\alpha u_j||^2 > e^{-\lambda K} \sum_{K/\sigma < u_j \leq \lambda K/\sigma} ||\alpha u_j||^2. \]

We consider two cases. Firstly, if $\alpha_0 < |\alpha| < \sigma/(2\lambda K)$ then using a lower bound on $U(\lambda K/\sigma) - U(K/\sigma)$ (similar to that provided in Lemma 3.5 for the case $\lambda = 2$) and Hypothesis H we obtain
\[ \sum_{K/\sigma < u_j \leq \lambda K/\sigma} ||\alpha u_j||^2 \geq \sum_{K/\sigma < u_j \leq \lambda K/\sigma} \alpha^2 u_j^2 \geq \alpha_0^2 \sigma^{-2-s}(\log 1/\sigma)^{-t}. \]

Lemma 3.7(i) gives $1/\sigma \gg n^{1/(\tau+1)}$ and hence, since $1 < \lambda < 2$, the above expression is
\[ \geq \alpha_0^2 \sigma^{-2-s} n^{(2-\lambda)/(\tau+1)}. \]

Secondly, if $\sigma/(2\lambda K) < |\alpha| < 1/2$ then Hypothesis K gives
\[ \sum_{K/\sigma < u_j \leq \lambda K/\sigma} ||\alpha u_j||^2 \geq \left(\frac{1}{\sigma^2}\right)^{(2-\lambda)} K \gg n^{(2-\lambda)/(\tau+1)}. \]

Hence from Lemma 3.12 we have the result (i).

(ii) Suppose $(6 + 2s)/\sigma \leq m \leq n/2$. Then
\[ \sum_{m < u_j \leq n} e^{-\alpha u_j} ||\alpha u_j||^2 > \sum_{m < u_j \leq \lambda m} e^{-\alpha u_j} ||\alpha u_j||^2 > e^{-\lambda \sigma m} \sum_{m < u_j \leq \lambda m} ||\alpha u_j||^2. \]

From Lemma 3.24
\[ e^{\sigma m} \ll \frac{m^{1+s}(\log m)^{-t}}{n} \ll \frac{m^{1+s}}{n}. \]
so that from (3.42)
\[ \sum_{m < u_j \leq \lambda m} e^{-\pi u_j} \|\alpha u_j\|^2 \gtrsim (n/m^{1+\epsilon})^\lambda \sum_{m < u_j \leq \lambda m} \|\alpha u_j\|^2. \]

We consider two cases. Firstly, if \( \alpha_0 < |\alpha| < 1/(2\lambda m) \) then Hypothesis H and an argument along the lines of the proof of Lemma 3.5 give
\[ \sum_{m < u_j \leq \lambda m} \|\alpha u_j\|^2 = \sum_{m < u_j \leq \lambda m} \alpha^2 u_j^2 \]
\[ \gtrsim \alpha_0^2 m^{-2-\epsilon} (\log m)^{-\epsilon} \]
\[ \gtrsim \alpha_0^2 m^{-2} m^{2(2-\lambda)}. \]

Secondly, if \( 1/(2\lambda m) \leq |\alpha| < 1/2 \) then Hypothesis K gives
\[ \sum_{m < u_j \leq \lambda m} \|\alpha u_j\|^2 \gtrsim m^{2(2-\lambda)}. \]

In the light of Lemma 3.12 and the remarks at the commencement of this proof, we have the result (ii). \( \square \)

11. Proof of the main theorem

Preliminaries to commencement of the proof

Let \( \delta > 0 \) be given such that \( 0 < \delta < 1 \) (as in (3.17)). In order to prove Theorem 3.4 we must show that
\[ (3.43) \quad \int_{-1/2}^{1/2} \varphi(\alpha) e(-\alpha n) d\alpha = \frac{1}{\sqrt{2\pi A_2}} (1 + O(n^{-1/2} (\log n)^{-1/2^{(2s+2)})} + O(\{(m/n)^{16/33}\}). \]

For any \( \alpha_0 \) such that \( 0 < \alpha_0 < 1/2 \) we have
\[ \int_{-1/2}^{1/2} \varphi(\alpha) e(-\alpha n) d\alpha = M + S, \]
where, by Lemma 3.16
\[ M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha) e(-\alpha n) d\alpha = \frac{1}{\sqrt{2\pi A_2}} (1 + E_1(\alpha_0)) \]
where
\[ E_1(\alpha_0) \ll \frac{P_3}{A_2^{3/2}} + \exp(-2\pi^2 A_2 \alpha_0^2), \]
provided that the conditions (3.30), (3.31), (3.32) are satisfied, and Lemma 3.26 gives different upper bounds on
\[ S = \int_{\{\alpha: \alpha_0 \leq |\alpha| \leq 1/2\}} \varphi(\alpha) e(-\alpha n) d\alpha, \]
depending on whether or not \( (6 + 2\epsilon)/\sigma < m < n^{1-\epsilon} \).
We divide the main work of the proof in two parts, Part 1 corresponding to the case \( 1 \leq m \leq (6 + 2s)/\sigma \) and Part 2 corresponding to the case \((6 + 2s)/\sigma \leq m \leq n^{1-\delta}\).

**Part 1** The case \(0 \leq m \leq (6 + 2s)/\sigma\).

For this case we shall show that for any \(\lambda\) such that \(1 < \lambda < 2\) we have

\[
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n)d\alpha = \frac{1}{2\pi A_2} (1 + O(\lambda^{-s/(2s+2)}(\log n)^{-\frac{s}{2(2s+2)})}).
\]

We therefore now consider \(\lambda\) such that \(1 < \lambda < 2\), \(\lambda\) fixed throughout this part of the argument and use this value of \(\lambda\) in the various estimates.

In order to apply Lemma 3.16 we require that each of the conditions (3.30), (3.31), (3.32) hold. Thus what requires checking is whether

\[
\sqrt{\frac{3}{10(2\pi)^3} \rho_3^2} > \alpha_0 > \frac{1}{2\pi\sqrt{A_2}}
\]

\[
\sigma/(2\pi\log 3) > \alpha_0.
\]

It suffices to choose \(\alpha_0\) satisfying (3.45) and we can simplify the choice for \(\alpha_0\) by minimising the error terms \(E_1\) and \(E_2\) which are such that

\[
M = \frac{1}{2\pi A_2} \left(1 + E_1(\alpha_0)\right),
\]

\[
S = \frac{E_2(\alpha_0)}{\sqrt{2\pi A_2}}.
\]

Lemma 3.16 gives

\[
E_1(\alpha_0) \ll \frac{\rho_3}{A_2^{3/2}} + \exp(-2\pi^2 A_2 \alpha_0^2).
\]

Lemma 3.26(i) gives

\[
|S| \leq \exp(-R_1 \min(\alpha_0^2 \sigma^{-2}, 1) n^{s(2s+2))/(s+2))},
\]

where \(R_1 = R_1(\lambda)\) depends only on \(\lambda\) and \(u\). It is clear that making \(\alpha_0\) as large as possible will minimise \(E_1\) and \(E_2\).

Now from Lemma 3.25(i) (with the choice \(K = 6 + 2s\))

\[
A_2 \asymp n^{(s+3)/(s+1)}(\log n)^{s/(s+1)},
\]

\[
\rho_3 \asymp n^{(s+3)/(s+1)}(\log n)^{s/(s+1)},
\]

\[
\frac{\rho_3}{A_2^{3/2}} \asymp n^{-s/(2s+2)}(\log n)^{-s/(2s+2)},
\]

where the implied constants depend only on \(u\).
Thus,

\[
\sqrt[3]{\frac{3}{10(2\pi)^3 \rho_3}} \sigma \gg n^{-1/3(3+\varepsilon)}(\log n)^{-\varepsilon/(3(1+\varepsilon))},
\]

(3.51)

\[
\sigma/(2\pi \log 3) \gg n^{-1/3(1+\varepsilon)}(\log n)^{-\varepsilon/(1+\varepsilon)},
\]

and

\[
\frac{1}{2\pi \sqrt{A_2}} \ll n^{-1/(2(3+\varepsilon))}(\log n)^{-\varepsilon/(2(3+\varepsilon))}.
\]

Hence there is a positive constant \(K_4\) (depending only on \(u\)) such that the choice

(3.52)

\[
\alpha_0 = K_4 n^{-1/(1+\varepsilon)}(\log n)^{-\varepsilon/(1+\varepsilon)}
\]

for \(\alpha_0\) is consistent with (3.51) and (3.45).

In the light of (3.50) and the above choice for \(\alpha_0\) in (3.52) we have from (3.47) that

\[
E_1(\alpha_0) \ll n^{-1/(2s+2)}(\log n)^{-\varepsilon/(2s+2)}.
\]

Using the lower bound for \(\sigma^{-1}\) from Lemma 3.25(i) and the choice for \(\alpha_0\) in (3.52) we have from (3.46), (3.48) and the the fact that \(2 - \lambda > 0\), we obtain

\[
E_2(\alpha_0) \ll n^{-1} \ll n^{-1/(2s+2)}(\log n)^{-\varepsilon/(2s+2)},
\]

and the conclusion (3.44) follows.

**Part 2** This is the case where \(n^{1-s} \geq m \geq (6 + 2s)/\sigma\).

For this case we shall show that for any \(\lambda\) such that \(1 < \lambda \leq 1 + \delta/33\) we have

(3.53)

\[
\int_{-1/2}^{1/2} \varphi(a) e(-an) da = \frac{1}{\sqrt{2\pi A_2}} (1 + O_{\lambda}((m/n)^{16/33})).
\]

(The reason for the upper bound \(1 + \delta/33\) will become clear later in the argument). We therefore take \(\lambda\) such that \(1 < \lambda \leq 1 + \delta/33\), fixed throughout this part of the argument, and we note that this implies

(3.54)

\[
\delta \geq 33(\lambda - 1).
\]

As in Part 1, in order to apply Lemma 3.16 we require that each of the conditions (3.30), (3.31), (3.32) hold. Automatically condition (3.32) holds. All that remains to be checked is whether

(3.55)

\[
\sqrt[3]{\frac{3}{10(2\pi)^3 \rho_3}} \sigma_0 > \frac{1}{2\pi \sqrt{A_2}}.
\]

Thus it suffices to choose \(\alpha_0\) satisfying (3.55). It is clear from the bound for \(E_1(\alpha_0)\) in Lemma 3.16 and the bound for \(|S|\) in Lemma 3.26(ii) that making \(\alpha_0\) as large as possible will minimise \(E_1\) and \(E_2\) from (3.46).
11. PROOF OF THE MAIN THEOREM

From Lemma 3.25(ii) we obtain

$$\sqrt[3]{\frac{3}{10(2\pi)^3}} \frac{1}{\rho_3} \propto \frac{m^{-2(1+(\lambda-1)(1+\epsilon))}}{n^{-7/3+2\lambda}}$$

and

$$\frac{1}{2\pi \sqrt{A_2}} < \frac{1}{\sqrt{2\pi}} n^{-1/2} m^{-1/2}.$$ 

Hence there is a constant $K_5 > 0$ depending only on $\lambda$ and $u$ such that the choice for $\alpha_0$, namely

$$\alpha_0 = K_5 m^{-2(1+(\lambda-1)(1+\epsilon))/3} n^{-7/3+2\lambda},$$

is consistent with (3.55). It follows from the bound on $\rho_3/A_2^{3/2}$ in Lemma 3.25(ii) and the choice for $\alpha_0$ in (3.56) that the bound for $E_1(\alpha_0)$ in Lemma 3.16 becomes

$$E_1(\alpha_0) \ll \frac{m^{1+(1+\epsilon)(\lambda-1)/2}}{n^{3/2}} + \exp\left(-2\pi^2 \frac{1}{2} m n \left(m^{-2(1+(\lambda-1)(1+\epsilon))/3} n^{-7/3+2\lambda}\right)^2\right).$$

Since the exponent in the exponential function $\exp$ above is

$$\propto (n/m)^{1/3} \times (n/m^{(1+\epsilon)/3})^{\lambda-1} \equiv \frac{n^{1/3}}{m^{1/3}},$$

we have

$$E_1(\alpha_0) \ll \frac{m^{1+(1+\epsilon)(\lambda-1)/2}}{n^{3/2}}.$$ 

Also using (3.46) and the bound for $S$ in Lemma 3.26(ii) and then substituting (3.56) we obtain

$$E_2(\alpha_0) \ll \frac{m^{1/2+(1+\epsilon)(\lambda-1)/2}}{n^{3/2}} \frac{m^{2(3-7+10\epsilon)(\lambda-1)/3} n^{-14/3+5\lambda}}{\exp(-R_4 m^{1-3-(7+10\epsilon)(\lambda-1)/3} n^{-14/3+5\lambda})}$$

for some positive constant $R_4$ depending only on $\lambda$ and $u$. Because

$$m^{-1/3-(7+10\epsilon)(\lambda-1)/3} n^{-14/3+5\lambda} > n^{\delta/3-32(\lambda-1)/3} \quad \text{for} \quad m < n^{1-\delta},$$

having $\delta \geq 33(\lambda - 1)$ ensures that

$$\exp(-R_4 m^{1-3-(7+10\epsilon)(\lambda-1)/3} n^{-14/3+5\lambda}) \ll n^{-1}$$

and hence that

$$E_2(\alpha_0) \ll \frac{m^{1/2+(1+\epsilon)(\lambda-1)/2}}{n^{3/2}}.$$ 

We can simplify the bound on the error terms $E_1$ and $E_2$ by writing

$$\frac{m^{1/2+(1+\epsilon)(\lambda-1)/2}}{n^{3/2}} = \left(\frac{m}{n}\right)^{16/33} \frac{m^{16/33+(1+\epsilon)(\lambda-1)/2}}{n^{1/3}} \frac{m^{16/33+(1+\epsilon)(\lambda-1)/2}}{n^{1/3}}.$$ 

Using the condition $m \leq n^{1-\delta}$ and that $s \leq 1$, it can be checked that

$$\frac{m^{1/66+(1+\epsilon)(\lambda-1)/2}}{n^{1/66+(\lambda-1)/2}} \ll n^{-s/66+(\lambda-1)/2-s(\lambda-1)/2},$$
and this last expression will be less than unity provided that the exponent of \( n \) is negative. It is sufficient that (3.54) be satisfied. Thus the bounds on the error terms \( E_1 \) and \( E_2 \) are \( O_4((m/n)^{16/33}) \) as required in (3.53).

**Completion of the proof**

If \( 0 \leq m \leq (6 + 2\delta)/\sigma \), we use a particular \( \lambda \) in Part 1, say \( \lambda = 3/2 \), giving

\[
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n)d\alpha = \frac{1}{\sqrt{2\pi A_2}} \left( 1 + O(n^{-1/(2\delta+2)}(\log n)^{-1/(2\delta+2)}) \right).
\]

where the implied constants are absolute. We note that in the proof of Lemma 3.22 the choice \( \lambda = 3/2 \) was made, but could equally have been any other specified \( \lambda \).

If \( m \geq (6 + 2\delta)/\sigma \), we take \( \lambda = 1 + \delta/33 \) in Part 2, giving

\[
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n)d\alpha = \frac{1}{\sqrt{2\pi A_2}} \left( 1 + O((m/n)^{16/33}) \right).
\]

Thus in all cases we obtain (3.43), as required. This completes the proof of the main theorem of this chapter. \( \square \)

**12. Discussion**

**Modifications of main theorem**

Hypothesis K is equivalent to the statement that Hypothesis K(\( \lambda \)) holds for all \( \lambda \) such that \( 1 < \lambda < 2 \), where K(\( \lambda \)) is as follows.

**HYPOTHESIS K(\( \lambda \)).** Let \( \alpha \) be as in Hypothesis H. There are positive constants \( x_0 \) and \( K_0 \) (which depend only on \( \lambda \) and the sequence \( u \)) such that for every \( x > x_0 \),

\[
\sum_{\alpha < u_j \leq \lambda x} ||\alpha u_j||^2 > K_0 x^{(2-\lambda)}
\]

whenever \( |\alpha| \in (1/(2\lambda x), 1/2) \).

While it was convenient to assume K(\( \lambda \)) for all \( \lambda \) such that \( 1 < \lambda < 2 \) so as to allow arbitrarily small \( \delta \) in Theorem 3.4, careful study of the proof and preceding lemmas (especially Lemmas 3.20 and 3.26) shows that Part 1 required K(\( \lambda \)) for just one \( \lambda \) in the interval (1, 2) and Part 2 required K(\( \lambda \)) for \( \lambda - 1 = \delta/33 \). If we start with \( \lambda \) such that \( 1 < \lambda < 34/33 \), take \( \delta = 33(\lambda - 1) \), and use the same \( \lambda \) in both parts of our argument we obtain the following alternative version of the main theorem.

**Theorem 3.27.** Suppose \( \lambda \) is given such that \( 1 < \lambda < \frac{34}{33} \). Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypotheses H and K(\( \lambda \)). Let \( q_u(m, n) \) be as in Definition 1.1 and let \( \sigma \) and
12. DISCUSSION

\( A_2 \) be as in (3.12) and (3.6). Then as \( n \to \infty \),

\[
q_u(m, n) = \exp(\sigma n) \prod_{m < u_j \leq n} (1 + \exp(-\sigma u_j)) \times \frac{1}{\sqrt{2\pi A_2}}
\times (1 + O_{\lambda,u}(n^{-s/(2s+2)}(\log n)^{-t/(2s+2)}) + O_{\lambda,u}((m/n)^{16/33}),
\]

for \( 0 \leq m \leq n^{2s-33\lambda} \).

We note that for \( m = o(n^{1/1+s})(\log n)^{t/(1+s)} \) we have by Lemma 3.25(i) that \( m < (6+2s)/\sigma \). Hence Part 1 of the proof of Theorem 3.4 is relevant and we have as a corollary of Theorem 3.4 the following theorem.

**THEOREM 3.28.** Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypotheses \( H \) and \( K \). Let \( q_u(m, n) \) be as in Definition 1.1 and let \( \sigma \) and \( A_2 \) be as in be as in (3.12) and (3.6). Then as \( n \to \infty \),

\[
q_u(m, n) = \exp(\sigma n) \prod_{m < u_j \leq n} (1 + \exp(-\sigma u_j)) \times \frac{1}{\sqrt{2\pi A_2}}
\times (1 + O_{\lambda,u}(n^{-s/(2s+1)}(\log n)^{-t/(2s+2)})
\]

for \( m = o(n^{1/(1+s)})(\log n)^{t/(1+s)} \).

By modifying the second part of the proof of Theorem 3.4 it is possible to improve the \( O_{\lambda,u}((m/n)^{16/33}) \) error term in both Theorem 3.27 and in the main theorem to yield an error term of the form \( O_{\lambda,u}((m/n)^{1/2-\epsilon}) \) for any \( \epsilon \) satisfying \( 0 < \epsilon < 1/2 \), under appropriate conditions on \( \lambda \).

**Hypotheses on sequence**

The hypotheses of Roth and Szekeres are sufficient to prove a theorem like Theorem 3.4 certainly when \( m = 0 \) and without much modification, when \( m \) is not “too large”. However, it appeared difficult to provide an estimate of \( q_u(m, n) \) for \( m \) “large” under Hypotheses RS1 and RS2 of Roth and Szekeres because the asymptotic estimation of the cumulants when \( m \) is “large” required a knowledge of the behaviour of \( u_j \) more detailed than that deducible from Hypothesis RS1. This was the reason for the tightening of (RS1).

Although not pursued in this thesis, there is the possibility of weakening Hypotheses \( H \) and \( K \) by introducing the concept of regularly varying functions. The function \( C_0 x^t (\log x)^{-t} \) is regularly varying and this suggests that it may be possible replace Hypothesis \( H \) by the condition that \( U(x) \) is regularly varying.
CHAPTER 4

Applications of Chapter 3

1. Introduction

In the first part of this chapter (Sections 2 to 6) I shall consider sequences \( u \) for which the main theorem (Theorem 3.4) or its alternative version (Theorem 3.27) provide an asymptotic estimate of \( q_u(m, n) \). In the last part of the chapter (Section 7) I shall use the main theorem to provide information about the behaviour of the ratio \( q_u(m, n)/q_u(0, n) \) as \( n \to \infty \).

Sequences satisfying Hypotheses H and K or K(\( \lambda \))

We recall the hypotheses used in Chapter 3, namely, Hypotheses H, K(\( \lambda \)) and K on a strictly increasing sequence of positive integers \( u \).

HYPOTHESIS H. There are real constants \( s, t, C_0 \) satisfying \( 0 < s \leq 1, t \geq 0 \), \( C_0 > 0 \) such that as \( x \to \infty \),

\[
U(x) \sim C_0 x^s (\log x)^{-t}.
\]

Here \( U \) is the counting function of \( u \) as in Definition 1.5 and in view of the equivalent statement (3.15), 1/s is called the growth exponent of \( u \).

For a sequence satisfying Hypothesis H, with \( s \) as in Hypothesis H, and for a given \( \lambda \) such that \( 1 < \lambda < 2 \), Hypothesis K(\( \lambda \)) is:

HYPOTHESIS K(\( \lambda \)). There are positive constants \( x_0 > 1 \) and \( K_0 \) (which depend only on \( \lambda \) and the sequence \( u \)) such that for every \( x > x_0 \),

\[
\sum_{z < u_j \leq \lambda x} \|u_j\|^2 > K_0 x^{s(2-\lambda)}
\]

whenever \( |\alpha| \in (1/(2\lambda x), 1/2) \).

For a sequence satisfying Hypothesis H, with \( s \) as in Hypothesis H, Hypothesis K is:

HYPOTHESIS K. Hypothesis K(\( \lambda \)) holds for all \( \lambda \) such that \( 1 < \lambda < 2 \).

In Chapter 3, these hypotheses correspond to the Hypotheses RS1 and RS2 in the work of Roth and
Szekeres [40] on $q_u(0, n)$ described in Section 6 of Chapter 1. Roth and Szekeres presented a wide range of sequences satisfying (RS1) and (RS2), and my work on H, K(λ) and K presented here will draw heavily on their ideas. I shall present three classes of sequences which satisfy H and K or K(λ), together with the corresponding consequences of Theorem 3.4 or Theorem 3.27. We recall from Section 3 of Chapter 3 that Hypothesis H is satisfied by the sequence $(j^k)$ of $k$-th powers, $(p_j)$ of primes, and we shall see, more generally, that Hypothesis H is satisfied by sequences of the form $(P(j))$, $(P(p_j))$, where $P$ is a suitable polynomial function and $p_j$ is the $j$-th prime.

**Plan of Chapter**

In Section 2, I shall introduce some notation and lemmas. In Section 3, I shall give an auxiliary condition, Hypothesis A(λ), which implies Hypothesis K(λ) for suitable λ for sequences $u$ satisfying Hypothesis H with growth exponent $1/α < 3/2$. It will follow immediately from this that $N = (j)$ and $(p_j)$ satisfy Hypothesis K, and as another application I shall show in Section 4 that certain uniformly distributed sequences satisfy Hypotheses H and K.

Roth and Szekeres showed that (RS1) and (RS2) are satisfied by sequences $u = (u_j)$ such that $u_j = P(j)$, where $P$ is a polynomial function which is integer valued on the integers and strictly increasing and positive on $(0, \infty)$ and such that $P$ satisfies a suitable further condition (Hypothesis B). In Section 5, I shall show that such sequences in fact satisfy the stronger condition, Hypothesis K.

Roth and Szekeres also gave a similar result for sequences of the form $u = (u_j)$ with $u_j = P(p_j)$, with $P$ as above satisfying a further condition (Hypothesis C). In Section 6, I shall show that these sequences, too, satisfy H and K.

**Behaviour of the ratio $q_u(m, n) / q_u(0, n)$ for large $n$**

As a final application of Theorem 3.4 in this chapter, I shall show in Section 7 for a given sequence $u$ satisfying Hypotheses H and K that as $n \to \infty$

\[ q_u(m, n) \sim q_u(0, n)2^{-U(m)} \]

for $m = o(n^{1/(1+λ)^2}(\log n)^{(2+λ)/(1+λ)^2})$. In Section 7 of Chapter 1 I mentioned the earlier work of others (see Theorem F, in particular, the work of Erdős [13], and for ordinary partitions, Dixmier [9]) in proving asymptotic relations of this kind for the sequence $u = N$. The proof of this result will involve considering the ratio of the asymptotic estimates of $q_u(m, n)$ and $q_u(0, n)$ provided by Theorem 3.4.
2. Preliminary notation and lemmas

We commence with a definition.

**Definition 4.1.** Let \( u = (u_j) \) be a strictly increasing sequence of positive integers and let \( \lambda \) be given such that \( 1 < \lambda < 2 \). For \( x \geq 1 \) and integers \( w, q, q > 0 \), let

\[
N(x; q, w) = \text{card} \{ u_j : u_j \equiv w \pmod{q} \text{ and } u_j \in (x, \lambda x] \}.
\]

For \( L \) a given positive integer and \( x > 0 \) let

\[
r_L(x; n) = \text{card} \{ (x_1, \ldots, x_L) : n = x_1 + \ldots + x_L, \text{ and } x_i \in u \cap (x, \lambda x] \text{ for } 1 \leq i \leq L \}.
\]

Thus \( r_L(x; n) \) is the number of partitions of the integer \( n \) into \( L \) parts such that each part is a member of the sequence \( u \) and each part lies in the half open interval \( (x, \lambda x] \). The following lemma gives an upper bound on the number of members \( u_j \) of the sequence \( u \) in the interval \( (x, \lambda x] \) which lie in one of \( H \) distinct residue classes modulo \( q \), for a positive integer \( H \).

**Lemma 4.2.** Let \( x \) and \( \lambda \) be real numbers satisfying \( 2\lambda x \geq 1 \) and \( \lambda > 1 \), and let \( L \) be a given positive integer. Let \( q, H, h' \) be integers satisfying \( 1 \leq q \leq 2\lambda x \), \( 1 \leq H < q/2L \), \((h', q) = 1 \). Let \( u = (u_j) \) be a strictly increasing sequence of positive integers. Further let \( N(x; q, w) \) and \( r_L(x; n) \) be as in Definition 4.1.

Then as \( x \to \infty \),

\[
\left( \sum_{w=-H}^{H} N(x; q, h'w) \right)^{2L} \leq \frac{Hx}{q} \sum_{n=1}^{\infty} r_L(x; n)^2.
\]

**Proof.** We have

\[
\left( \sum_{w=-H}^{H} N(x; q, h'w) \right)^{L} = \sum_{u_1=-H}^{H} \cdots \sum_{u_L=-H}^{H} \sum_{j_1}^{H} \cdots \sum_{j_L}^{H}
\]

where summation is over \( j_1, \ldots, j_L \) such that for \( i = 1, \ldots, L \),

\[
u_j, \equiv v_ih' \pmod{q} \text{ and } u_j \in (x, \lambda x].
\]

Since \( 2LH < q \) and since for each term of the multiple sum above,

\[
|v_1 + \ldots + v_L| \leq LH,
\]

we have

\[
\sum_{u_1=-H}^{H} \cdots \sum_{u_L=-H}^{H} \sum_{j_1}^{H} \cdots \sum_{j_L}^{H} \leq \sum_{w=-LH}^{LH} \sum_{j_1}^{H} \cdots \sum_{j_L}^{H}
\]

\[
= \sum_{w=-LH}^{LH} R_L(x; q, w)
\]
where summation on the right hand side above is over \(j_1, \ldots, j_L\) satisfying \(u_{j_1} + \ldots + u_{j_L} \equiv w' \pmod{q}\) and where

\[
R_L(x; q, w) = \sum_{n \equiv w' \pmod{q}}^{\infty} r_L(x; n).
\]

Noting that \(r_L(x; n) = 0\) if either \(n > \lambda x\) or \(n \leq Lx\) means that the series in (4.2) that defines \(R_L(x; q, w)\) contains at most \((\lambda - 1)Lx/q + 1\) nonzero terms. Thus using the Cauchy–Schwarz inequality we obtain

\[
(\sum_{w = -LH}^{LH} R_L(x; q, w))^2 \leq (2LH + 1) \sum_{w = -LH}^{LH} R_L(x; q, w)^2
\]

and

\[
R_L(x; q, w)^2 \leq (1 + (\lambda - 1)xL/q) \sum_{n = 1}^{\infty} r_L(x; n)^2.
\]

Combining the inequalities (4.1), (4.3), (4.4) gives the result. \(\square\)

In order to apply Lemma 4.2 to sequences of the type \(u_j = P(j)\) and \(u_j = P(p_j)\), with \(P\) being a polynomial function, we shall give in Lemma 4.3 and in Lemma 4.4 upper bounds for \(\sum_{n = 1}^{\infty} r_L(x; n)^2\) when \(u_j = P(j)\) and \(u_j = P(p_j)\).

**Lemma 4.3.** Let \(k\) be a fixed positive integer and let \(P\) be a polynomial of degree \(k\) which is integer valued on the integers and which is strictly increasing and positive on \((0, \infty)\). Let \(u\) be the sequence such that \(u_j = P(j)\). Let \(\lambda > 1\) be a real number and let \(L\) be a positive integer such that \(L \geq 2^{k-1}\). Also let \(\epsilon\) be a given (arbitrarily small) positive number and let \(r_L(x; n)\) be as in Definition 4.1. Then as \(x \to \infty\),

\[
\sum_{n = 1}^{\infty} r_L(x; n)^2 \lesssim_{k, L, \lambda, \epsilon} x^{2L/k - 1 + \epsilon}.
\]

**Proof.** Let \(s_L(x; n)\) be defined as

\[
s_L(x; n) = \sum_{j_1} \ldots \sum_{j_L} 1
\]

where summation is over those \(j_1, \ldots, j_L\) such that \(1 \leq u_{j_1} + \ldots + u_{j_L} \leq \lambda x\) and such that \(n = u_{j_1} + \ldots + u_{j_L}\).

Then we have that \(r_L(x; n) < s_L(x; n)\).

Let

\[
g(x; \alpha) = \sum_{1 \leq u_j \leq x} e(au_j).
\]

Then we have that

\[
\sum_{n = 1}^{\infty} s_L(x; n)^2 = \int_0^1 |g(x; \alpha)|^2 d\alpha.
\]
Lemma 2.17 (Hua’s Lemma) gives us that for any positive $\epsilon$,

$$
\int_0^1 |g(x; \alpha)|^{2L} d\alpha \ll \frac{(\lambda x)^{2L/d \cdot 1+\epsilon}}{L, d, \epsilon}.
$$

The result follows immediately. □

The following lemma is for sequences which are polynomial in the prime numbers.

**Lemma 4.4.** Let $k$ be a fixed positive integer and let $P$ be a polynomial of degree $k$ which is integer valued on the integers and which is strictly increasing and positive on $(0, \infty)$. Let $u$ be the sequence such that $u_j = P(p_j)$, where $p_j$ is the $j$-th prime number. Let $\lambda > 1$ be a real number and let $L$ be a positive integer such that $L \geq 2^k + 2$. With $r_L(x; n)$ as in Definition 4.1 we have as $x \to \infty$

$$
\sum_{n=1}^{\infty} r_L(x; n)^2 \ll \frac{x^{2L/k - 1}}{k, L, \lambda}.
$$

**Proof.** Let $s_L(x; n)$ be as in (4.5). Then $s_L(x; n) > r_L(x; n)$. Let $g(x; \alpha)$ be as in (4.6). Then we have that

$$
\sum_{n=1}^{\infty} s_L(x; n)^2 = \int_0^1 |g(x; \alpha)|^{2L} d\alpha.
$$

Lemma 2.22 (Hua’s Lemma for primes) gives us that

$$
\int_0^1 |g(x; \alpha)|^{2L} d\alpha \ll (\lambda x)^{2L/k - 1}.
$$

The result follows immediately. □

The following lemma demonstrates that sequences which are polynomial in the integers or primes satisfy Hypothesis H.

**Lemma 4.5.** Let $k$ be a fixed positive integer and let $P$ be a polynomial of degree $k$ which is integer valued on the integers and which is strictly increasing and positive on $(0, \infty)$. Then the sequences $u = (P(j))$ and $u = (P(p_j))$ satisfy Hypothesis H.

**Proof.** As $x \to \infty$, $P(x) \sim P_k x^k$, where $P_k$ is the coefficient of $x^k$ in $P(x)$. Hence if $u_j = P(j)$ then $u_j \sim P_k j^k$ and by the equivalence of (3.14) and (3.3), Hypothesis H is satisfied with the choice $s = 1/k$ and $t = 0$. Also if $u_j = P(p_j)$ then by Lemma 2.19 (Prime Number Theorem) and an argument similar to that used in the proof of Lemma 2.5 we obtain $u_j \sim P_k j^k (\log j)^k$. The equivalence of (3.14) and (3.3) implies that Hypothesis H is satisfied with $s = 1/k$ and $t = 1$. □

We shall require the following definition to provide a lower bound for the sum

$$
\sum_{x < u_j \leq \lambda x} \|au_j\|^2
$$
in the following lemma.

**Definition 4.6.** For a positive integer $q$ let

$$R_q = \{ [-q/2], \ldots, [-q/2] + q - 1 \}$$

be a complete system of reduced residues modulo $q$. For coprime integers $q$ and $h$ let

$$S(x; q, h) = \sum_{x \in R_q} \frac{v^2}{q^2} N(x; q, h'^x),$$

where $N(x; q, w)$ is as in Definition 4.1 and $h'$ is such that $hh' \equiv 1 \pmod{q}$.

The following lemma will be used to verify that sequences satisfy Hypothesis K (or Hypothesis $K(\lambda)$).

**Lemma 4.7.** Let $x$ and $\lambda$ be real numbers such that $\lambda x \geq 1$ and $\lambda > 1$. Let $\alpha \in (1/(2\lambda x), 1/2)$ and let $u$ be an increasing sequence of positive integers. Let $S(x; q, h)$ be as in Definition 4.6. Then there are coprime integers $h$ and $q$ such that $2 \leq q \leq 2\lambda x$ and

$$\sum_{x < u_j \leq \lambda x} ||\alpha u_j||^2 \geq \frac{1}{4} S(x; q, h).$$

**Proof.** Let $\alpha \in (1/(2\lambda x), 1/2)$. Then (by Dirichlet’s Theorem, see, for example, Theorem 36 in Section 3.8 page 30 of Hardy & Wright [22]) there are coprime integers $h$ and $q$ such that

$$|\alpha - \frac{h}{q}| \leq \frac{1}{2\lambda x} \quad 2 \leq q \leq 2\lambda x.$$

Let $R_q$ be as in Definition 4.6 and for each $u_j \in (x, \lambda x]$ we define $v_j$ as the unique number in the set $R_q$ such that $v_j = hu_j \pmod{q}$. By Lemma 2.16 (with $Q = \lambda x$ and $n = u_j$, $||\alpha u_j|| \geq \frac{1}{2}||hu_j/q|| = \frac{1}{2}|v_j|/q$.

Thus

$$\sum_{x < u_j \leq \lambda x} ||\alpha u_j||^2 \geq \sum_{x < u_j \leq \lambda x} \frac{v_j^2}{4q^2} = \frac{1}{4} \sum_{v \in R_q} \frac{v^2}{q^2} N(x; q, h') = \frac{1}{4} S(x; q, h).$$

\[\square\]

3. Sequences with growth exponent $< 3/2$

Let $\lambda$ be a fixed number such that $1 < \lambda < 2$. In this section I shall show that if a strictly increasing sequence $u$ of positive integers satisfies Hypothesis H with $2/(1 + \lambda) < s \leq 1$ and satisfies an auxiliary hypothesis, then Hypothesis $K(\lambda)$ holds. With $s, t$ as in Hypothesis H and $\lambda$ prescribed as above, the auxiliary hypothesis is as follows.

**Hypothesis A(\lambda).** As $x \to \infty$,

$$\text{card} \{u_j : u_j \in (x, \lambda x], q | u_j\} \geq q^2 x^{2(\lambda - 1)},$$
for all integers \( q \) such that \( 2 \leq q < 20z/(U(\lambda x) - U(x)) \), where the implied constants depend only on \( u \) and \( \lambda \).

Clearly the sequence \( N \) of natural numbers and \( (p_j) \) of primes, and in fact any sequence satisfying Hypothesis \( H \) and containing at least \( x^{2-\lambda}(\log x)^{2\eta} \) primes among its members \( u_j \) satisfying \( x < u_j \leq \lambda x \) satisfies Hypothesis \( A(\lambda) \).

We commence with the following lemma.

**Lemma 4.8.** Let \( S \) be a non-empty set and let \( q \geq 4 \) be an integer. Further let \( B \geq 1 \) be a real number such that \( qB \geq \text{card} \ S \geq 4B \). Let \( V = \lceil \text{card} \ S/2B \rceil - 1 \) and let \( R_q \) be as in Definition 4.6. Suppose that a given function \( f: S \to R_q \) assumes each value in its range at most \( B \) times. Then

\[
\sum_{v \in R_q} v^2 \text{card} \ f^{-1}(v) \geq B \sum_{v=1}^{V} v^2.
\]

**Proof.** Enumerate the elements of the set \( S \) via

\( s_1, s_2, s_3, \ldots, s_{\text{card} \ S} \).

Define the function \( g: S \to R_q \) such that

\[
g(s_i) = \begin{cases} 
[i/\lceil B \rceil], & \text{for } 1 \leq i \leq V\lceil B \rceil, \\
-[i/\lceil B \rceil] + V, & \text{for } V\lceil B \rceil < i \leq 2V\lceil B \rceil, \\
0, & \text{otherwise}.
\end{cases}
\]

Note that \( g \) assumes each of the values in the set \( \{-V, -V + 1, \ldots, -1, 1, \ldots, V\} \) exactly \( \lceil B \rceil \) times and that (because \( (2V + 2)\lceil B \rceil \leq \text{card} \ S \)) \( g \) assumes the value 0 at least \( 2\lceil B \rceil \) times. Also, because \( 4B \leq \text{card} \ S, V \geq 1 \).

Further note that \( V \in R_q \) since \( \text{card} \ S < qB \). It is clear that because \( g \) takes inferior values to \( f \) on \( S \),

\[
\sum_{v \in R_q} v^2 \text{card} \ f^{-1}(v) = \sum_{s \in S} f(s)^2 > \sum_{s \in S} g(s)^2 = 2\lceil B \rceil \sum_{v=1}^{V} v^2.
\]

Finally since \( B \geq 1, 2\lceil B \rceil > B \) and the required inequality follows. \( \square \)

**Lemma 4.9.** Let \( \lambda \) be a fixed real number such that \( 1 < \lambda < 2 \). Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis \( H \) with

\[
2/(1 + \lambda) < s \leq 1,
\]

and satisfying Hypothesis \( A(\lambda) \) above. Then \( u \) satisfies Hypothesis \( K(\lambda) \).
PROOF. Let \( x \) and \( \lambda \) be fixed real numbers such that \( 2 > \lambda > 1 \) and \( \lambda x > 1 \). Let \( \alpha \in (1/2, 1/2) \) be fixed such that \( |\alpha| > (2\lambda x)^{-1} \) and let \( h \) and \( q \) be integers such that

\[
2 \leq q \leq \lfloor 2\lambda x \rfloor \quad \text{and} \quad |\alpha - h/q| \leq (2\lambda x)^{-1}.
\]

Let \( S(x; q, h) \) be as in Definition 4.6. It suffices to show that

\[
S(x; q, h) \geq x^{(2-\lambda)}
\]

because in view of Lemma 4.7 it would follow from (4.8) that

\[
\sum_{x < u_j \leq \lambda x} |\alpha u_j|^2 \geq x^{(2-\lambda)}.
\]

If \( q < 20x/(U(\lambda x) - U(x)) \) then by Hypothesis A(\( \lambda \)) and noting that

\[
\text{card} \{u_j : x < u_j \leq \lambda x, \ q \ |u_j| = \sum_{v=1}^{q-1} N(x; q, h', v),
\]

we obtain

\[
S(x; q, h) \geq \frac{1}{q^2} \sum_{u \in R_q \setminus \{0\}} N(x; q, h') \geq x^{(2-\lambda)}
\]

and this is (4.8).

Hence we now suppose that \( q \geq 20x/(U(\lambda x) - U(x)) \), that is

\[
\frac{1}{2} \frac{U(\lambda x) - U(x)}{5x/q} \geq 2.
\]

We note that \( q \geq 4 \) for \( x \) sufficiently large because \( 20x/(U(\lambda x) - U(x)) > 20x/(U(2x) - U(x)) \) \( \rightarrow \infty \) as \( x \rightarrow \infty \), unless \( s = 1, \ t = 0 \) in which case \( 20x/(U(2x) - U(x)) \rightarrow 20/C_0 \geq 20 \), since \( C_0 \leq 1 \).

We observe the following:

\[
\sum_{w=1}^{q} N(x; q, w) = U(\lambda x) - U(x)
\]

and because \( 1 \leq 2\lambda x/q \) (see (4.7)) and \( \lambda < 2 \),

\[
N(x; q, w) \leq \frac{\lambda x - x}{q} + 1 \leq \frac{(3\lambda - 1)x}{q} < \frac{5x}{q}.
\]

These two observations and Lemma 4.8 (with the choices \( S = \{u_j : x < u_j \leq \lambda x\}, B = 5x/q, f(a) = \text{residue of } a \text{ modulo } q \text{ lying in } R_q \) lead to the inequality

\[
S(x; q, h) = \sum_{u \in R_q} \frac{q^2}{q} N(x; q, h' u) \geq \frac{5x}{q} \sum_{0 < v \leq V} v^2
\]

where

\[
V = \left[ \frac{1}{2} \frac{U(\lambda x) - U(x)}{5x/q} \right] - 1.
\]

Note that this sum is non-empty because it is easily seen from (4.9) that \( V \geq 1 \).
Thus the right hand side of (4.10) is
\[
\asymp \frac{x^2}{q^{-2}V^3}
\]
\[
\asymp \frac{x^2}{q^{-2}} \left( \frac{U(\lambda x) - U(x)}{x/q} \right)^3
\]
and using Hypothesis H and simplifying this is
\[
\asymp \frac{x^2}{\lambda} \left( x^s (\log x)^{-t} \right)^3.
\]
Because \( s > 2/(1 + \lambda) \) we have
\[
x^{-2} \left( x^s (\log x)^{-t} \right)^3 \left( x^{s(2-\lambda)} \right)^{-1} = x^{s+\lambda-2}(\log x)^{-3t} \gg 1
\]
and we have (4.8).

Because the implied constants in the above inequalities are independent of \( h \) and \( q \) (and hence \( \alpha \)) Hypothesis K(\( \lambda \)) has been shown to hold. \( \square \)

In the above lemma, in order that a sequence \( u \) satisfying Hypothesis H have \( s \) such that \( 2/(1 + \lambda) < s \leq 1 \) for some \( \lambda \in (1, 2) \), it is necessary and sufficient that \( s > 2/3 \), that is, the growth exponent \( 1/s < 3/2 \).

Consequences of the above lemma are that \( N \) and \( (p_j) \) satisfy Hypothesis K.

4. Uniformly distributed sequences

We shall only need the basic idea of a sequence which is uniformly distributed over the positive integers. For more detail on this topic, see Chapter 3 page 27 of Niven [36] or Chapter 5 of Kuipers and Niederreiter [30].

**Definition 4.10.** We say that a strictly increasing sequence \( u = (u_j) \) of positive integers is uniformly distributed modulo \( q \) if for all integers \( a \) such that \( 1 \leq a \leq q \),
\[
\lim_{x \to \infty} \frac{\text{card} \{ u_j : u_j \equiv a \mod q \text{ and } u_j \leq x \}}{U(x)} = \frac{1}{q}.
\]
Further, we say that a strictly increasing sequence \( u = (u_j) \) of positive integers is uniformly distributed over the positive integers if it is uniformly distributed modulo \( q \) for every integer \( q \geq 2 \).

It is immediately seen that the sequence \( \mathbb{N} = (j) \) is uniformly distributed over the positive integers.

We have the following result, where \([x]\) denotes the integral part of \( x \).

**Lemma 4.11.** Suppose \( \theta \) is irrational and \( \theta > 1 \). Then the sequence \( u = (u_j) \) given by \( u_j = [j\theta] \) is uniformly distributed over the positive integers.
PROOF. See Theorem 3.3, page 27 of Niven [36]. □

**Proposition 4.12.** Let \( u \) be a strictly increasing sequence of positive integers which is uniformly distributed over the positive integers and which satisfies Hypothesis \( H \) with \( s = 1, \ t = 0 \). Then Hypothesis \( A(\lambda) \) holds for every \( \lambda \) such that

\[
1 < \lambda < 2.
\]

**Proof.** Let \( \lambda \) be a fixed real number in the interval \((1, 2)\). For a given integer \( q \geq 2 \) let

\[
N_q(x) = \text{card} \{ u_j : u_j \in (x, \lambda x] \text{ and } q \nmid u_j \}.
\]

By Hypothesis \( H \) with \( s = 1 \) and \( t = 0 \), as \( x \to \infty \) we have

\[
\frac{20x}{U(\lambda x) - U(x)} \ll \lambda.
\]

Hence in order to show that Hypothesis \( A(\lambda) \) holds, it is sufficient to show that as \( x \to \infty \),

\[
N_q(x) \gg q^2 x^{2-\lambda}
\]

for \( 2 \leq q \leq K \), where \( K = K(\lambda) \) is a given positive constant depending only on \( \lambda \).

Since \( u \) is uniformly distributed in \( \mathbb{N} \), for each \( q \) such that \( 2 \leq q \leq K \), \( K \) as above,

\[
N(x; q, 0) \sim \frac{1}{q} (U(\lambda x) - U(x)),
\]

where \( N(x; q, 0) \) as in Definition 4.1 and hence for each \( q \) such that \( 2 \leq q \leq K \), there is \( x_0(q, \lambda) > 0 \) such that for \( x > x_0(q, \lambda) \),

\[
N(x; q, 0) < \frac{2}{3} (U(\lambda x) - U(x)).
\]

Let

\[
x_0(\lambda) = \max \{ x_0(q, \lambda) : 2 \leq q \leq K \}.
\]

Since

\[
N_q(x) = U(\lambda x) - U(x) - N(x; q, 0),
\]

it follows from (4.13) that for \( x > x_0(\lambda) \) and all \( q \) such that \( 2 \leq q \leq K \) we have

\[
N_q(x) > \frac{1}{3} (U(\lambda x) - U(x)).
\]

Now by Hypothesis \( H \) (with \( s = 1, t = 0 \))

\[
\frac{1}{3} (U(\lambda x) - U(x)) \gg x \gg K^2 x.
\]
Hence for all \( q \) such that \( 2 \leq q \leq K \) we obtain

\[ N_q(x) \gtrsim q^2 x, \]

which implies (4.12) since \( x \geq x^{2-\lambda} \) by (4.11). \( \square \)

If \( u \) is a strictly increasing sequence of positive integers which is uniformly distributed in \( \mathbb{N} \) and satisfies Hypothesis H with \( s = 1, t = 0 \) then by Proposition 4.12 and Lemma 4.9 the sequence \( u \) satisfies both Hypotheses H and K, and so Theorem 3.4 yields an asymptotic estimate for \( q_u(m,n) \).

**Example.** By the above remarks and Lemma 4.11, we see that Theorem 3.4 yields an asymptotic estimate of \( q_u(m,n) \) if \( u = (u_j) \) is given by

\[ u_j = \lfloor \theta j \rfloor \]

where \( \theta > 1 \) and \( \theta \) is irrational.

5. Polynomial sequences

In this section we shall consider a sequence \( u = (u_j) \) given by \( u_j = P(j) \) where \( P \) is a suitable polynomial. In order to ensure that \( u \) is a strictly increasing sequence of positive integers we shall assume that \( P \) is integer-valued on the integers and that \( P \) be positive and strictly increasing on \((0, \infty)\).

Some further condition is needed to ensure that all sufficiently large positive integers can be expressed as sums of distinct terms \( u_j \) since, for example, this would be impossible if all \( u_j \)'s (and hence all sums of \( u_j \)'s) possessed a common divisor greater than unity. Roth and Szekeres showed that the following further condition is sufficient to ensure that \( u \) as above satisfies their Hypothesis RS2.

**Hypothesis B.** For each prime \( p \), there exists a positive integer \( x_0 \) such that \( p \not| P(x_0) \).

I shall show that the same Hypothesis B is sufficient to ensure that \( u \) as above satisfies Hypothesis K (which is stronger than (RS2)).

**Lemma 4.13.** Let \( k \) be a fixed positive integer. Let \( P \) be a polynomial function of degree \( k \) which is integer-valued on the integers and is positive and strictly increasing on \((0, \infty)\), and suppose further that \( P \) satisfies Hypothesis B. Then the sequence \( u = (P(j)) \) satisfies Hypotheses H and K.

**Proof.** From Lemma 4.5 Hypothesis H is satisfied with \( s = 1/k \) and \( t = 0 \).

By Lemma 4.7 in order to prove Hypothesis K we must show that for every \( \lambda \in (1, 2) \) we have

\[ S(x; q, h) \gtrsim x^{(2-\lambda)/k} \]

for every \( q \) and \( h \) such that \( 1 < q \leq 2\lambda x \), \((h, q) = 1\) and \( 1 \leq h \leq q \).
We take \(q\) and \(h\) such that \(2 \leq q \leq 2\lambda x\) and consider two cases.

Case (1): \(1 < q \leq x^{(\lambda - 2)/3k}\).

Since \(P\) satisfies Hypothesis B there is some \(x_0 \in \mathbb{Z}\) and there is some \(w_0 \in \mathbb{Z}\) such that \(P(x_0) \equiv w_0\) (mod \(q\)) with \((w_0, q) = 1\). Thus

\[
N(x; q, w_0) \geq \sum_{x < u_j \leq \lambda x, j \equiv x_0 \text{ (mod } q)} 1 \asymp x^{1/k}/q,
\]

and so

\[
S(x; q, h) \gg x^{1/k}/q^3 \geq x^{(2-\lambda)/k}.
\]

Case (2): \(x^{(\lambda - 1)/3k} < q \leq 2\lambda x\).

Let \(L\) and \(H\) be such that

\[
L = 2^{k-1}, \quad H = \lfloor q^{-1-(\lambda - 1)/3k} (2L)^{-1} \rfloor.
\]

Applying Lemma 4.2 (with \(L\) and \(H\) as in (4.15)) and Lemma 4.3 (with \(c = (\lambda - 1)^2/18k^2\)) gives

\[
\left( \sum_{u = -H}^{H} N(x; q, h') \right)^{2L} \ll \frac{Hz}{q} x^{2L/k-1+c} \ll q^{-1-(\lambda - 1)/3k} x^{2L/k+c} \ll x^{2L/k-\epsilon}.
\]

Thus

\[
\sum_{u = -H}^{H} N(x; q, h') \ll x^{1/k-c/(2L)} = o(x^{1/k}).
\]

Since

\[
\sum_{u \in R_q} N(x; q, w) \asymp x^{1/k},
\]

we have

\[
\sum_{v \in R_q \mid |v| > H} N(x; q, h') \asymp x^{1/k},
\]

from which we have

\[
S(x; q, h) > \frac{H^2}{q^2} \sum_{v \in R_q \mid |v| > H} N(x; q, h').
\]

Using (4.15) the right hand side of this is

\[
\ll q^{-2(\lambda - 1)/3k} x^{1/k} \gg x^{1/k-2(\lambda - 1)/3k}.
\]

Since \(\lambda > 1\) it follows that \(S(x; q, h) \gg x^{(2-\lambda)/k}\).

Thus in either case we conclude that (4.14) holds for every \(\lambda \in (1, 2)\) and so Hypothesis K is satisfied. \(\square\)
6. SEQUENCES WHICH ARE POLYNOMIAL IN PRIMES

The above lemma ensures that a polynomial sequence of the type considered satisfies the hypotheses of Theorem 3.4, and so we have the following consequence immediately.

**Proposition 4.14.** Let \( k \) be a fixed positive integer. Let \( u = P(j) \) with \( P \) being a polynomial function of degree \( k \), integer valued on the integers and strictly increasing and positive on \((0, \infty)\). Suppose that \( P \) satisfies Hypothesis B. Let \( \delta \) be a positive constant such that \( 0 < \delta < 1 \). Let \( \sigma \) and \( A_2 \) be defined by

\[
\begin{align*}
n &= \sum_{m < P(j) \leq n} \frac{P(j)}{1 + c \sigma^m}, \\
A_2 &= \sum_{m < P(j) \leq n} \frac{P(j)^2 e^{\sigma P(j)}}{(1 + c \sigma^m)^2}.
\end{align*}
\]

Then as \( n \to \infty \),

\[
q_n(m, n) = \exp(\sigma n) \prod_{m < P(j) \leq n} \frac{1}{\sqrt{2 \pi A_2}} \left(1 + O(n^{-1/(2k+2)}) + O_1((m/n)^{16/33})\right),
\]

for \( 0 \leq m \leq n^{1-\delta} \), where the implied constants depend only on \( \delta \) and the polynomial function \( P \).

6. Sequences which are polynomial in primes

We consider a sequence \( u \) whose \( j \)-th term is \( u_j = P(p_j) \), where \( p_j \) is the \( j \)-th prime number and where \( P \) is a polynomial which is integer valued on the integers, which is strictly increasing and positive on \((0, \infty)\). In this case Roth and Szekeres [40] used the following hypothesis.

**Hypothesis C.** For each prime \( p \), there exists a positive integer \( x_0 \) such that \( p \nmid x_0 P(x_0) \).

This hypothesis imposes upon the polynomial \( P \) similar restrictions to those imposed by Hypothesis B (as described in the introductory paragraph of Section 5). The additional feature of Hypothesis C is the existence of, not only values in the range of \( P \) but, corresponding values in the domain of \( P \) which lie in non-zero residue classes modulo any prime number. This allows one to lift a solution \( x = x_0 \) of the congruence \( P(x) \equiv w_0 \pmod{p} \) to a solution \( x = p_j \) of the integer equation \( P(x) = u \) for some prime \( p_j \). Roth and Szekeres showed that for \( u \) as above, Hypothesis C ensures that (RS2) holds. I shall show that Hypothesis C ensures the stronger condition, Hypothesis K.

**Lemma 4.15.** Let \( k \) be a fixed positive integer. Let \( P \) be a polynomial of degree \( k \) satisfying Hypothesis C which is integer valued on the integers and is strictly increasing and positive on \((0, \infty)\). Then the sequence \( u = (P(p_j)) \), \( p_j \) being the \( j \)-th prime number, satisfies Hypotheses H and K with \( s = 1/k \) and \( t = 1/k \).

**Proof.** From Lemma 4.5 Hypothesis H is satisfied with \( s = 1/k \) and \( t = 1/k \).

Let \( L_1 \) be such that

\[
L_1 = 2^k + 2.
\]
We consider two cases.

Case (1): \( 2 \leq q \leq (\log x)^{6L_1} \)

By Hypothesis C on \( P \), there is some \( x_0 \in \mathbb{Z} \) and there is some \( w_0 \in \mathbb{Z} \) such that \((x_0w_0, q) = 1\) and \( f(x_0) \equiv w_0 \) (mod \( q \)). Since \((x_0, q) = 1\), there is some prime \( p_0 \) such that \( p_0 \equiv x_0 \) (mod \( q \)). Hence, \( f(p_0) \equiv w_0 \) (mod \( q \)). Let
\[
\pi(X, h, q) = \text{card} \{ p \leq X : p \text{ prime and } p \equiv h \pmod{q} \}.
\]

By Corollary 2.21 (with \( L = 6L_1 \)) we have that as \( X \to \infty \),
\[
\pi(X, h, q) \sim \frac{1}{\phi(q)} \frac{X}{\log X},
\]

uniformly with respect to \( h, q \) such that \((h, q) = 1\) and \( 1 \leq q \leq (\log X)^{6L_1} \). Hence using this and the fact that \( P^{-1}(x) \sim (x/P_k)^{1/k} \) (where \( P_k \) denotes the coefficient of \( x^k \) in \( P(x) \) and where \( P^{-1} \) denotes the inverse function of \( P \)) we obtain
\[
N(x; q, w_0) = \pi(P^{-1}(\lambda x), w_0, q) - \pi(P^{-1}(x), w_0, q) \approx \frac{1}{\lambda_k q \phi(q)} x^{1/k} (\log x)^{-1},
\]

uniformly with respect to \( w_0, q \) such that \( 1 \leq q \leq (\log P^{-1}(x))^{6L_1} \) and hence with respect to \( q \) such that \( 1 \leq q \leq (\log x)^{6L_1} \). Thus
\[
S(x; q, h) \geq \frac{1}{q^2} N(x; q, w_0) \approx \frac{1}{\lambda_k q^2 \phi(q) \log x} \gg x^{1/k} (\log x)^{-15L_1-1}.
\]

Case (2): \((\log x)^{5L_1} < q \leq 2\lambda x\)

Let \( H \) be such that
\[
(4.16) \quad H = \left\lfloor \frac{q}{(\log x)^{4L_1}} \right\rfloor.
\]

Applying Lemma 4.2 (with \( L = L_1 \) and \( H \) as in (4.16)) and Lemma 4.4 gives
\[
\left( \sum_{w \in R_x} N(x; q, h') \right)^{2L_1} \ll \frac{H x^{2L_1/k - 1}}{q \lambda_k} \approx (\log x)^{-4L_1} x^{2L_1/k},
\]

where the implied constants depend only on \( u, \lambda \) and \( k \). Thus
\[
(4.17) \quad \sum_{w \in R_x} N(x; q, h') \ll x^{1/k} (\log x)^{-2} = o(x^{1/k} (\log x)^{-1}).
\]

Since
\[
\sum_{w \in R_x} N(x; q, w)
\]
is just the number of primes \( p \) such that \( x < P(p) \leq \lambda x \) we have by the Prime Number Theorem (Theorem 2.19) that

\[
\sum_{u \in R_x} N(x; q, u) \asymp x^{1/k}(\log x)^{-1},
\]

and it follows from (4.17) that

\[
\sum_{v \in R_x \mid |v| > H} N(x; q, hv) \asymp x^{1/k}(\log x)^{-1}.
\]

Consequently, with \( S(x; q, h) \) as in Definition 4.6,

\[
S(x; q, h) \asymp \frac{H^2}{q^2} \sum_{v \in R_x \mid |v| > H} N(x; q, hv)
\]

and with \( H \) as in (4.16), the right hand side of this is

\[
\asymp (\log x)^{-2L_1} x^{1/k}(\log x)^{-1}.
\]

Thus in either case we conclude that for all \( \lambda \in (1, 2) \)

\[
S(x; q, h) \gg \frac{x^{1/k}}{(\log x)^{1+15L_1}} \gg x^{(2-\lambda)/k},
\]

and so Hypothesis K is satisfied. \( \Box \)

As a corollary of Theorem 3.4 and Lemma 4.15 we have Theorem 4.16.

**Theorem 4.16.** Let \( k \) be a fixed positive integer. Let \( u = (P(p_j)) \), where \( P \) is a polynomial function of degree \( k \) which is integer valued on the integers and which is strictly increasing and positive on \((0, \infty)\).

Let \( \delta \) be a positive constant such that \( 0 < \delta < 1 \). Let \( \sigma \) be defined implicitly by the equation

\[
n = \sum_{m < P(p_j) \leq n} \frac{P(p_j)}{1 + e^{\sigma P(p_j)}}
\]

and let \( A_2 \) be defined as

\[
\sum_{m < P(p_j) \leq n} \frac{P(p_j)^2 e^{\sigma P(p_j)}}{(1 + e^{\sigma P(p_j)})^2}.
\]

Suppose \( P \) satisfies Hypothesis C. Then as \( n \rightarrow \infty \),

\[
q_u(m, n) = \exp(\sigma n) \prod_{m < P(p_j) \leq n} (1 + \exp(-\sigma P(p_j))) \times \frac{1}{\sqrt{2\pi A_2}}
\]

\[
\times \left(1 + O(n^{-1/(2k+2)})(\log n)^{-k/(2k+2)}+O_\delta((m/n)^{16/33})
\right),
\]

for \( 0 \leq m \leq n^{1-\delta} \), where the implied constants depend only on \( \delta \) and the polynomial \( P \).
7. Relationship between \( q_u(m, n) \) and \( q_u(0, n) \) for small \( m \)

In this section an asymptotic relation

\[
q_u(m, n) \sim 2^{-U(m)} q_u(0, n)
\]

as \( n \to \infty \) will be derived for \( m \) in a suitable range. Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypotheses H and K. We define the notation corresponding to the two partition functions \( q_u(m, n) \) and \( q_u(0, n) \). Let \( \sigma_m \) be the unique positive real number such that

\[
\sum_{m < u_j \leq n} \frac{u_j}{1 + e^{\sigma_m u_j}} = n
\]

and let \( \sigma_0 \) be such that

\[
\sum_{0 < u_j \leq n} \frac{u_j}{1 + e^{\sigma_0 u_j}} = n.
\]

Further let

\[
A_{2,m} = \sum_{m < u_j \leq n} \frac{u_j^2 e^{\sigma_m u_j}}{(1 + e^{\sigma_m u_j})^2} \quad \text{and} \quad A_{2,0} = \sum_{0 < u_j \leq n} \frac{u_j^2 e^{\sigma_0 u_j}}{(1 + e^{\sigma_0 u_j})^2}
\]

be the corresponding second cumulants. Also let

\[
A_{0,m} = \sum_{m < u_j \leq n} \log(1 + e^{-\sigma_m u_j}) \quad \text{and} \quad A_{0,0} = \sum_{0 < u_j \leq n} \log(1 + e^{-\sigma_0 u_j}).
\]

We assume that \( m = o(n^{1/(1+\varepsilon)}(\log n)^{1/(1+\varepsilon)}) \). Then from Theorem 3.4 we have

\[
q_u(m, n) = \frac{1}{\sqrt{2\pi A_{2,m}}} e^{\sigma_m n + A_{0,m}} \times (1 + O_u(n^{-1/(2\varepsilon+2)})(\log n)^{-1/(2\varepsilon+2)}))
\]

\[
q_u(0, n) = \frac{1}{\sqrt{2\pi A_{2,0}}} e^{\sigma_0 n + A_{0,0}} \times (1 + O_u(n^{-1/(2\varepsilon+2)})(\log n)^{-1/(2\varepsilon+2)}))
\]

It follows that as \( n \to \infty \),

\[
\frac{q_u(m, n)}{q_u(0, n)} = \sqrt{\frac{A_{2,0}}{A_{2,m}}} e^{(\sigma_m - \sigma_0) n} e^{A_{0,m} - A_{0,0}} \times \left(1 + O_u(n^{-1/(2\varepsilon+2)})(\log n)^{-1/(2\varepsilon+2)}\right)
\]

and we see that examination of the size of the quantities

\[
\frac{A_{2,0}}{A_{2,m}}, \quad \sigma_m - \sigma_0, \quad A_{0,m} - A_{0,0}
\]

is necessary to estimate the ratio \( q_u(m, n)/q_u(0, n) \).
To this end, we define for \( x > 0 \) the functions

\[
F_0(x) = \sum_{0 < u_j < \infty} \log(1 + e^{-xu_j})
\]

\[
F_1(x) = \sum_{0 < u_j < \infty} \frac{u_j}{1 + e^{xu_j}}
\]

\[
F_2(x) = \sum_{0 < u_j < \infty} \frac{u_j^2 e^{xu_j}}{(1 + e^{xu_j})^2}
\]

\[
F_3(x) = \sum_{0 < u_j < \infty} \frac{u_j^3 e^{xu_j}(e^{xu_j} - 1)}{(1 + e^{xu_j})^3}
\]

(4.21)

We have the following lemma on mean values of the above functions.

**Lemma 4.17.** Let \( u \) be a strictly increasing sequence of positive integers and let \( F_0, F_1, F_2, F_3 \) be functions as in (4.21). Then for real numbers \( a \) and \( b \) such that \( 0 < a < b \),

\[
|b - a| < \frac{|F_1(b) - F_1(a)|}{|F_2(b)|}
\]

\[
|F_0(b) - F_0(a)| < |F_1(b) - F_1(a)| \frac{|F_1(a)|}{|F_2(b)|}
\]

\[
|F_2(b) - F_2(a)| < |F_3(b) - F_3(a)| \frac{|F_3(a)|}{|F_2(b)|}
\]

**Proof.** We observe that the functions \( F_0, F_1, F_2, F_3 \) are differentiable (see for example, Corollary 3 §5.3 page 109 of Marsden [31] where it follows that a sum of functions of the nature indicated is differentiable).

We apply the Mean Value Theorem to each of the functions \( F_0, F_1, F_2, F_3 \) on the interval \([a, b]\) to give

\[
\frac{F_i(b) - F_i(a)}{b - a} = F_i'(c_i) \quad i = 0, 1, 2
\]

for some \( c_i \in (a, b) \). But \( F_i'(c_i) = -F_{i+1}(c_i), i = 0, 1, 2 \) and because \( F_i \) is strictly decreasing on \( \mathbb{R} \),

\[-F_{i+1}(a) < F_i'(c_i) < -F_{i+1}(b) < 0.
\]

Hence for \( i = 0, 1, 2 \)

\[
|F_i(b) - F_i(a)| < |b - a|F_{i+1}(a)
\]

and

\[
|b - a| = \left| \frac{F_i(b) - F_i(a)}{F_i'(c_i)} \right| < \left| \frac{F_i(b) - F_i(a)}{F_{i+1}(b)} \right|.
\]

Combining these inequalities gives all the required inequalities. \( \square \)

The following lemma establishes that \( \sigma_m \leq \sigma_0 \) for \( n \) sufficiently large.
LEMMA 4.18. Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H. Let \( \sigma_m \) and \( \sigma_0 \) be as in (4.18) and (4.19). Then there is a number \( n_0 = n_0(u) \) such that for \( n > n_0 \),

\[
\sigma_m \leq \sigma_0.
\]

PROOF. We have

\[
\sum_{m < u_j \leq n} \frac{u_j}{1 + e^{\sigma_m u_j}} = n = \sum_{0 < u_j \leq n} \frac{u_j}{1 + e^{\sigma_0 u_j}} \geq \sum_{m < u_j \leq n} \frac{u_j}{1 + e^{\sigma_0 u_j}},
\]

which implies that \( \sigma_m \leq \sigma_0 \), since \( 1/(1 + e^{au}) \) decreases as \( x \) increases. \( \square \)

LEMMA 4.19. Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H. Let \( F_1(x) \) be as in (4.21) and let \( \sigma_m, \sigma_0 \) be as in (4.18) and (4.19). Then as \( n \to \infty \),

\[
F_1(\sigma_m) - F_1(\sigma_0) \ll mU(m),
\]

for \( m = o(n^{1/(1+s)}(\log n)^{1/(1+s)}) \), where the implied constants depend only on \( u \).

PROOF. Since the inequality is trivial for the case \( m = 0 \), assume that \( m \geq 1 \). Partitioning the summation \( \sum_{0 < u_j < \infty} \) in \( F_1(\sigma_m) \) and \( F_1(\sigma_0) \) and cancelling those summations which equal \( n \) gives

\[
F_1(\sigma_m) - F_1(\sigma_0) = \left( \sum_{0 < u_j \leq m} + \sum_{n < u_j < \infty} \right) \frac{u_j}{1 + e^{\sigma_m u_j}} - \sum_{n < u_j < \infty} \frac{u_j}{1 + e^{\sigma_0 u_j}}.
\]

Applying Lemma 2.11 (Ingham's Lemma) to both the sums

\[
\sum_{n < u_j < \infty} \frac{u_j}{1 + e^{\sigma_m u_j}}, \quad \sum_{n < u_j < \infty} \frac{u_j}{1 + e^{\sigma_0 u_j}}
\]

and employing Hypothesis H in the process gives the inequalities

\[
\sum_{n < u_j < \infty} \frac{u_j}{1 + e^{\sigma_m u_j}} \ll n^{1+s} e^{-\sigma_m n}, \quad \sum_{n < u_j < \infty} \frac{u_j}{1 + e^{\sigma_0 u_j}} \ll n^{1+s} e^{-\sigma_0 n}.
\]

It is clear that

\[
\sum_{0 < u_j \leq m} \frac{u_j}{1 + e^{\sigma_m u_j}} \ll mU(m).
\]

Finally, we note that

\[
n^{1+s} e^{-\sigma_m n} \leq n^{1+s} e^{-\sigma_0 n} \ll mU(m)
\]

and the result follows from the above inequalities. \( \square \)

The following lemma estimates the difference between corresponding values of \( A_0 \).
Lemma 4.20. Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypothesis H. Let \( A_{0,m} \) and \( A_{0,0} \) be as in (4.20). Then as \( n \to \infty \),

\[
A_{0,m} - A_{0,0} = -U(m) \log 2 + O \left( \frac{mU(m)}{n^{1/(1+s)} \log n} \right),
\]

for \( m = o(n^{1/(1+s)} \log n)^{t/(1+s)} \), where the implied constants depend on the sequence \( u \) only.

Proof. We have

\[
A_{0,m} - A_{0,0} = \sum_{0 < u_j \leq m} \log(1 + e^{-\sigma_m u_j}) - \sum_{0 < u_j \leq n} \log(1 + e^{-\sigma_0 u_j})
\]

\[
= -\sum_{0 < u_j \leq m} \log(1 + e^{-\sigma_m u_j}) + \sum_{0 < u_j \leq n} \log(1 + e^{-\sigma_m u_j}) - \sum_{0 < u_j \leq n} \log(1 + e^{-\sigma_0 u_j}).
\]

We note that because \( \sigma_m \leq \sigma_0 \),

\[
0 \leq \sum_{0 < u_j \leq m} \log(1 + e^{-\sigma_m u_j}) - \sum_{0 < u_j \leq n} \log(1 + e^{-\sigma_0 u_j}) < F_0(\sigma_m) - F_0(\sigma_0)
\]

and applying Lemma 4.17 (with \( a = \sigma_m \) and \( b = \sigma_0 \)) gives that this is

\[
\leq \frac{F_1(\sigma_m)}{F_2(\sigma_0)} |F_1(\sigma_m) - F_1(\sigma_0)|.
\]

A modification of the proof of Lemma 3.22 gives

\[
F_1(x) \asymp x^{-\epsilon}(\log \frac{1}{x})^{-\epsilon}, \quad F_2(x) \asymp x^{-\epsilon-2}(\log \frac{1}{x})^{-\epsilon},
\]

and these inequalities give, in tandem with the fact \( \sigma_m, \sigma_0 \asymp n^{-1/(1+s)}(\log n)^{-t/(1+s)} \) (using part (i) of Lemma 3.25 with \( m = o(n^{1/(1+s)}(\log n)^{t/(1+s)}) \) and inverting), that

\[
F_1(\sigma_0) \asymp F_1(\sigma_m) \asymp n, \quad F_2(\sigma_0) \asymp F_2(\sigma_m) \asymp n^{(t+2)/(s+1)}(\log n)^{t/(s+1)}.
\]

Thus

\[
\frac{F_1(\sigma_m)}{F_2(\sigma_0)} |F_1(\sigma_m) - F_1(\sigma_0)| \ll n^{-1/(1+s)}(\log n)^{-t/(1+s)} mU(m)
\]

and this is absorbed into the error term in the lemma.

Finally we estimate the the sum

\[
\sum_{0 < u_j \leq m} \log(1 + e^{-\sigma_m u_j})
\]

by noting that

\[
\log(1 + e^{-\sigma_m u_j}) = \log 2 + O(\sigma_m u_j),
\]

from which we have

\[
\sum_{0 < u_j \leq m} \log(1 + e^{-\sigma_m u_j}) = U(m) \log 2 + O(\sigma_m mU(m)).
\]

Our proof is concluded. \( \Box \)
REMARK 4.21. In a similar fashion to the method of the above proof we can show that under the same conditions of the above lemma,

$$\sigma_0 n - \sigma_m n \ll m U(m)n^{-s/(1+s)}(\log n)^{s/(1+s)}$$

and

$$A_{2,m} - A_{2,0} \ll m U(m)n^{-s/(1+s)}(\log n)^{-s/(1+s)}.$$

It is clear from Remark 4.21 and Hypothesis H that if \( m = o(n^{3/(1+s)})(\log n)^{(2+s)/(1+s)^2} \) then the upper bounds on the differences between corresponding parameters (namely the bound \( m U(m)n^{-s/(1+s)}(\log n)^{-s/(1+s)} \)) are \( o(1) \) as \( n \to \infty \) and in view of this comment we have the theorem below.

THEOREM 4.22. Let \( u \) be a strictly increasing sequence of positive integers satisfying Hypotheses H and K where \( s \) and \( t \) are as in Hypothesis H. Let \( q_u(m, n) \) be defined as in Definition 1.1. Then as \( n \to \infty \)

$$q_u(m, n) \sim 2^{-U(m)}q_u(m, n),$$

for \( m = o(n^{3/(1+s)})(\log n)^{(2+s)/(1+s)^2} \).

In the case when \( u = \mathbb{N} \) we have the following result obtained by taking \( s = 1 \) and \( t = 0 \). This was also obtained by Freiman and Pitman [16] from an application of their main theorem (see Proposition J in Section 1 of Chapter 6). The result had been obtained previously by Erdős, Nicolas and Szalay [13].

COROLLARY 4.23. With \( q(m, n) \) and \( q(n) \) as in (1.2), as \( n \to \infty \),

$$q(m, n) \sim 2^{-m}q(n),$$

for \( m = o(n^{1/4}) \).

In the particular case when \( u \) is the sequence of prime numbers we have the following corollary.

COROLLARY 4.24. Let \( u \) be the sequence of prime numbers. Then as \( n \to \infty \),

$$q_u(m, n) \sim 2^{-\pi(m)}q_u(0, n),$$

for \( m = o(n^{1/4}(\log n)^{3/4}) \).

In the particular case when \( u \) is the sequence of positive \( k \)-th powers of the integers we have the following corollary.
Corollary 4.25. Let $k$ be a fixed positive integer. Let $u$ be the sequence whose $j$-th term is $u_j = j^k$.

Then as $n \to \infty$,

$$q_u(m, n) \sim 2^{-\left\lfloor m^{1/k} \right\rfloor} q_u(0, n),$$

for $m = o\left(n^{k^2/(k+1)^2}\right)$.

In the following chapter I shall consider the sequence of $k$-th powers further.
CHAPTER 5

Partitions into distinct \( k \)-th powers

1. Introduction

If \( u \) is the sequence of positive \( k \)-th powers, that is, \( u = (j^k) \), then by Definition 1.1, \( q_u(m, n) \) denotes the number of partitions of \( n \) into distinct \( k \)-th powers larger than \( m \) and \( p_u(m, n) \) denotes the number of ordinary partitions of \( n \) into distinct \( k \)-th powers larger than \( m \). As mentioned in Chapter 1, the asymptotic behaviour of these functions when \( m = 0 \) has been studied by Hardy and Ramanujan [21], Ingham [28] and Wright [46]. For the case of distinct parts, Ingham gave the following result (which can be obtained by taking \( C_0 = 1, b = -1/2, c = -\frac{1}{2}k \log 2\pi, M = (\Gamma(1 + 1/k)\zeta(1 + 1/k))^k, \alpha = 1/(k + 1) \) and \( s = 1/k \) in Theorem C of Section 4 of Chapter 1).

**Proposition H (Ingham).** Let \( u = (j^k) \), \( k \) a fixed positive integer. Then for \( q_u(0, n) \) as in Definition 1.1, as \( n \to \infty \),

\[
q_u(0, n) \sim 2^{-1} \left( \frac{k}{\pi (k + 1)} \right)^{1/2} \frac{1}{n} \left( \gamma^k n \right)^{1/(2k + 2)} \times \exp \left( (k + 1) \left( \gamma^k n \right)^{1/(k + 1)} \right),
\]

where

\[
(5.1) \quad \gamma = (1 - 2^{-1/k}) \frac{1}{k} \Gamma(1 + 1/k)\zeta(1 + 1/k).
\]

More generally, as an application of their theorem (Theorem E of Section 6 of Chapter 1), Roth and Szekeres [40] gave (without details of proof) the following result pertaining to the case \( u = (P(j)) \) where \( P \) is a fairly general polynomial of degree \( k \).

**Proposition I (Roth and Szekeres).** Let \( k \) be a positive integer and let \( u = (u_j) \) be a sequence of positive integers which is eventually strictly increasing such that

\[
u_j = aj^k + bj^{k-1} + \ldots ,
\]
with \( a \) and \( b \) being integers. Then as \( n \to \infty \),

\[
g_u(0, n) = (1 + O(n^{-1/(1+k)}))2^{-(1+b/ka)} \left( \frac{k}{\pi(k+1)} \right)^{1/2} \frac{1}{n} \left( \frac{\gamma^k n}{a} \right)^{1/(2k+2)} \exp \left( (k+1) \left( \frac{\gamma^k n}{a} \right)^{1/(k+1)} \right),
\]

with \( \gamma \) as in (5.1) and where the implied constants depend only on \( a, b \) and \( k \).

We note that Proposition H is the particular case \( a = 1, b = 0 \) of this result.

If \( u = (j^k) \) and \( 0 \leq m \leq o(n^{k^2/(1+k)^2}) \), then, using Corollary 4.25 together with Proposition H and the fact that \( U(x) = [x^{1/k}] \) (where \( [x^{1/k}] \) denotes the integral part of \( x^{1/k} \)), we see that as \( n \to \infty \),

\[
g_u(m, n) \sim 2^{-1-[m^{1/k}]} \left( \frac{k}{(k+1)\pi} \right)^{1/2} \frac{1}{n} (\gamma^k n)^{1/(2k+2)} \exp \left( (1 + k)(\gamma^k n)^{1/(1+k)} \right)
\]

for \( 0 \leq m = o(n^{k^2/(1+k)^2}) \). A drawback of this asymptotic relation is that it does not provide any information on the influence of \( m \) other than that provided by the factor \( 2^{-[m^{1/k}]} \).1

The aim of this chapter is to derive an asymptotic estimate of \( g_u(m, n) \) for \( u = (j^k) \) as \( n \to \infty \) for \( m \) in the range \( 0 \leq m \leq n^{k^2/(1+k)(4k+2)} \) that goes beyond the estimate (5.2). The exponent of \( n \) in the upper end of this range is chosen to be a number less than \( k/(k+1) \) which is sufficiently large to allow an influence of \( m \) which does not appear in (5.2) to occur in the asymptotic estimate. The result obtained will be as follows.

**Theorem 5.1.** Let \( k \) be a positive integer, let \( u = (j^k) \), and let \( \gamma \) be as in (5.1). Then with \( g_u(m, n) \) as in Definition 1.1, as \( n \to \infty \),

\[
g_u(m, n) = 2^{-1-[m^{1/k}]} \left( \frac{k}{(k+1)\pi} \right)^{1/2} \frac{1}{n} (\gamma^k n)^{1/(2k+2)} \exp \left( (1 + k)(\gamma^k n)^{1/(1+k)} + \frac{1}{2k + 2} m^{(k+1)/k} \right) \times \left( 1 + O(n^{-1/(2+2k)}) \right)
\]

for \( 0 \leq m \leq n^{k^2/(1+k)(4k+2)} \), where the implied constants depend only on \( k \).

2. Plan of chapter

From now on, let \( u = (j^k) \) and let \( g_u(m, n) \) be the corresponding partition function as in Definition 1.1. I shall apply Theorem 3.28 to obtain an estimate of \( g_u(m, n) \). The notation introduced in Chapter 3 is specialised to the case \( u_j = j^k \) as follows.

We have that \( \sigma \in (0, \infty) \) such that

\[
n = \sum_{m < j^k \leq n} \frac{j^k}{1 + \exp(\sigma j^k)}.
\]
We have the cumulants

\[ A_0 = \sum_{m < j^k \leq n} f_0(j^k), \quad A_1 = \sum_{m < j^k \leq n} f_1(j^k) = n, \quad A_2 = \sum_{m < j^k \leq n} f_2(j^k) \]

where for \( x > 0 \),

\[ f_0(x) = \log(1 + e^{-\sigma x}), \quad f_1(x) = \frac{x}{1 + e^{\sigma x}}, \quad f_2(x) = \frac{x^2 e^{\sigma x}}{(1 + e^{\sigma x})^2} \]

From Theorem 3.27 and Lemma 4.13 (with \( P(x) = x^k \)) as \( n \to \infty \),

\[ q_n(m, n) = \exp(\sigma n + A_0) \frac{1}{\sqrt{2\pi A_2}} (1 + O(n^{-1/2 + 2k})) \]

It is clear that the task at hand is to estimate \( A_0, \sigma \) and \( A_2 \).

In Section 3, I shall present some lemmas dealing with the estimation of the cumulants \( A_0, A_1 \) and \( A_2 \) using Euler–Maclaurin Summation. Then in Section 4, I shall make explicit estimates of \( \sigma, A_0 \) and \( A_2 \). In Section 5, I shall use these estimates in combination with Theorem 3.28 to furnish a proof of Theorem 5.1 and consequently obtain Corollary 4.25 as a corollary of this theorem.

3. Using Euler–Maclaurin summation

Euler–Maclaurin Summation is used to provide preliminary estimates of the quantity \( A_0 \) and the cumulants \( A_1, A_2 \).

The following lemma establishes that a certain type of function is unimodal: This will be needed in connection with estimating the integral in the remainder of the Euler–Maclaurin Summation Formula.

**Lemma 5.2.** For given \( a, b, c > 0 \) and for \( x > 0 \), let

\[ f(x) = \frac{x^a}{1 + e^{cx^b}}. \]

Then there is a unique number \( x_0 = x_0(a, b, c) > 0 \) such that \( f \) is strictly increasing on the interval \((0, x_0)\) and strictly decreasing on \((x_0, \infty)\). Further, \( x_0 \) is of the form \( x_0 = (y(a, b)/c)^{1/b} \), where \( y(a, b) \) depends only on \( a \) and \( b \) and \( a/b < y(a, b) < 2a/b \).

**Proof.** It is easily checked that

\[ f'(x) = \frac{x^{a-1}}{(1 + e^{cx^b})^2} \left( a + (a - bcy^b)e^{cx^b} \right). \]

Writing \( g(y) = a + (a - by)e^y \), where \( y = cy^b \), we see that \( f'(x) = 0 \) if and only if \( g(y) = 0 \) and \( f'(x) \) has the same sign as \( g(y) \). We observe that \( g(y) > 0 \) for \( 0 < y < a/b \) while \( g(y) < 0 \) for \( 2a/b < y \). Also \( g'(y) = (a - by - b)e^y \) is negative on the interval \((a/b, 2a/b)\). Consequently, \( g(y) \) has a unique zero.
\[ y_0 = y_0(a,b) \] on the interval \((0, \infty)\) and it satisfies \(a/b < y_0(a,b) < 2a/b\). The conclusion of the lemma follows by considering the corresponding zero \(x_0 = (y_0(a,b)/c)^{1/6}\) of \(f'(x)\). □

We now give a lemma which indicates the order of magnitude of the parameter \(\sigma\).

**Lemma 5.3.** Let \(k\) be a fixed positive integer and let \(m\) and \(n\) be integers such that \(0 \leq m < n/2\). Let \(\sigma\) be as in (5.3). Suppose \(m = o(n^{k/(1+k)})\) as \(n \to \infty\). Then as \(n \to \infty\),

\[
\sigma \asymp n^{-k/(k+1)}.
\]

**Proof.** Part (i) of Corollary 3.9 gives \(\sigma \ll n^{-k/(1+k)}\) while Part (ii) of Corollary 3.9 gives (with the choice \(r = [K_2 n^{k/(1+k)}]\), \(K_2 = K_2(k)\) being chosen sufficiently large, and \(m = o(n^{k/(1+k)})\)) \(2\sigma r > \log(r^{1+k}/n) + O(1)\) and so \(\sigma \ll n^{-k/(1+k)}\). Thus \(\sigma \asymp n^{-k/(1+k)}\). □

This following lemma gives a bound on the error term arising in the Euler–Maclaurin Summation Formula.

**Lemma 5.4.** Let \(k\) be a fixed positive integer. Let \(m\) and \(n\) be integers such that \(0 \leq m < n/2\) and let \(\sigma\) be as in (5.3). For \(r = 0, 1, 2\) let \(f_r(x)\) be as in (5.5). Then for \(m = o(1/\sigma)\), as \(n \to \infty\),

\[
\int_m^\infty ((y^{1/k} - 1/2)f_r(y)dy) \ll \sigma^{-1/k-r},
\]

where \(\{x\}\) denotes the fractional part of \(x\).

**Proof.** Differentiating the expressions in (5.5) gives

\[
f'_0(x) = -\sigma \frac{1}{1 + e^{\sigma x}}, \quad f'_1(x) = \frac{1}{1 + e^{\sigma x}} - \frac{\sigma x e^{\sigma x}}{(1 + e^{\sigma x})^2},
\]

and

\[
f'_2(x) = -\frac{\sigma x^2 e^{\sigma x} (e^{\sigma x} - 1)}{(1 + e^{\sigma x})^3} + \frac{2x e^{\sigma x}}{(1 + e^{\sigma x})^2}.
\]

Using the fact that \(e^{\sigma y}/(1 + e^{\sigma y}) \leq 1\), we see that \(f'_r(y)\) is bounded above by twice a sum of terms of the form

\[
\frac{\sigma^l y^{r+l} - 1}{(1 + e^{\sigma y})^h},
\]

where \(0 \leq l \leq 1, 1 \leq h \leq r + 1, 0 \leq r \leq 2\).

Thus, in looking at the integral in (5.7), we only need consider the integral

\[
\int_m^\infty ((y^{1/k} - 1/2)\sigma^l \frac{y^{r+l-1}}{(1 + e^{\sigma y})^h}dy).
\]

Let

\[
I_{a,b} = \int_m^\infty ((y^{1/k} - 1/2)\frac{y^a}{(1 + \exp(\sigma y))^b}dy).
\]
where \( a, b \) are integers such that \( b \geq 1 \). Then making the transformation \( y = x^k \) in the above integral gives

\[
I_{a,b} = k \int_{m^{1/k}}^\infty ((x) - 1/2)f(x)dx
\]

where

\[
f(x) = \frac{x^{a+k+1}}{(1 + e^{\sigma x^k})^b} \quad (x > 0).
\]

By applying Lemma 5.2 to the \( b \)-th root of \( f(x) \), we see that there is a unique maximum at \( x_0 = \sqrt[1/k]{C(a,b)/\sigma} \) such that \( f \) is strictly increasing on \((0, x_0)\) and strictly decreasing on \((x_0, \infty)\) and, further

\[
x_0 \approx \sigma^{1/k}.
\]

Since \( m = o(\sigma^{-1}) \) by hypothesis, we have that \( m^{1/k} < x_0 \) for \( n \) sufficiently large. By applying Lemma 2.15 to

\[
\int_{m^{1/k}}^x ((x) - 1/2)f(x)dx
\]

where \( m^{1/k} < x < x_0 \), and letting \( x \to \infty \), we see that

\[
I_{a,b} \ll f(x_0) \ll x_0^{a+k+1} \ll (\sigma^{-1})^{a+1-1/k}.
\]

Thus the integral in (5.8) is

\[
\ll \sigma^{r+l-1,1} \ll \sigma^r \sigma^{-r+1-1/k} = \sigma^{r+1/k}.
\]

The result follows. \( \Box \)

The next lemma will estimate the tail of the infinite sum

\[
\sum_{l}^{\infty} f_b(j^k)
\]

for \( b = 0, 1, 2 \), where \( f_b(x) \) is as in (5.5).

**Lemma 5.5.** Let \( k \) be a positive integer. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (5.3). Let \( f_0(x) \), \( f_1(x) \) and \( f_2(x) \) be defined as in (5.5). Then for any given positive number \( a \) we have that as \( n \to \infty \), for \( 0 \leq m = o(n^{k/(1+k)}) \),

\[
\sum_{n < j^k < \infty} f_b(j^k) \ll \sigma^a,
\]

for \( b = 0, 1, 2 \).
5. PARTITIONS INTO DISTINCT $k$-TH POWERS

PROOF. Throughout this proof let $b$ be a fixed number in the set $\{0, 1, 2\}$. We observe from (5.5) that

$$f_0(x) \ll e^{-\sigma x}, f_1(x) \ll xe^{-\sigma x}, f_2(x) \ll x^2e^{-\sigma x}.$$  

Thus

$$\sum_{n < j^k < \infty} f_b(j^k) \ll \sum_{n < j^k < \infty} j^{kb}e^{-\sigma j^k}.$$  

Since $m = o(n^{b/(k+1)})$, we observe that Lemma 5.3 implies that as $n \to \infty$ we have $\sigma \asymp n^{-k/(1+k)}$ and hence

$$\sigma n \asymp n^{1/(1+k)} \quad \text{as} \quad n \to \infty. \quad (5.9)$$

Hence we may assume that $n \geq b/\sigma$. Since $xe^{-\sigma x}$ is decreasing on $(b/\sigma, \infty)$ and $n \geq b/\sigma$, we see that $ye^{-\sigma y}$ is decreasing on $[n^{1/k}, \infty)$. Hence by Corollary 2.15 we obtain

$$\sum_{n < j^k < \infty} j^{kb}e^{-\sigma j^k} = \int_{n^{1/k}}^\infty y^{kb}e^{-\sigma y} \, dy + O(n^{b}e^{-\sigma n}). \quad (5.10)$$

Now upon putting $z = \sigma y$,

$$\int_{n^{1/k}}^\infty y^{kb}e^{-\sigma y} \, dy = \int_{\sigma n}^\infty z^{k}e^{-z} \frac{1}{k} z^{1/k-1} dz$$

and this integral is, by Lemma 2.7 (with the choice $x = \sigma n$, $z = b + 1/k$),

$$\ll b, k e^{-\sigma n} \ll \sigma e^{-\sigma n}. \quad (5.11)$$

From Lemma 5.9, there is a positive constant $C$ such that

$$\exp(-\sigma n) \ll \exp(-C\sigma^{-1/k}) \ll \sigma^{a_0},$$

for any $a_0 \in (0, \infty)$. Thus for any $a \in (0, \infty)$, the expression in (5.11) satisfies

$$\sigma^{-1}n^{b+1/k-1}e^{-\sigma n} \ll \sigma^{a}$$

while the $O$-term in (5.10) satisfies

$$n^{b}e^{-\sigma n} \ll \sigma^{a},$$

giving the result. □

The following lemma expresses the integrals $I_0$ and $I_2$ in terms of $I_1$, where these integrals $I_0, I_1, I_2$ are defined below and arise in the integral term of the Euler–Maclaurin Summation Formula when applied to the sums

$$\sum_{j} f_0(j^k), \quad \sum_{j} f_1(j^k), \quad \sum_{j} f_2(j^k).$$
LEMMA 5.6. Let \( k \) be a positive integer. For \( z \geq 0 \), let
\[
\begin{align*}
g_0(z) &= \log(1 + e^{-z}), \\
g_1(z) &= \frac{1}{1 + e^z}, \\
g_2(z) &= \frac{e^z}{(1 + e^z)^2}.
\end{align*}
\]
For \( r \in \{0, 1, 2\} \) and \( z > 0 \) let
\[
I_r(z) = \int_z^\infty z^{r-1+1/k} g_r(z) \, dz.
\]
Then
\[
I_0(z) = -kz^{1/k} g_0(z) + kI_1(z), \quad I_2(z) = z^{1+1/k} g_1(z) + (1 + 1/k)I_1(z).
\]

**Proof.** We note that \( g'_r(z) = -g_{r+1}(z) \). Integrating \( I_r(z) \) by parts gives
\[
\int_z^\infty z^{r-1+1/k} g_r(z) \, dz = \left[ -z^{r-1+1/k} g_{r-1}(z) \right]_{x=z}^{x=\infty} - \int_z^\infty \left( r - 1 + 1/k \right) z^{r-1+1/k} (-g_{r-1}(z)) \, dz
\]
which is just
\[
I_r(z) = z^{r-1+1/k} g_{r-1}(z) + (r - 1 + 1/k)I_{r-1}(z).
\]
The special cases \( r = 1 \) and \( r = 2 \) of this recurrence formula gives the lemma. \( \square \)

LEMMA 5.7. Let \( k \) be a positive integer. Let \( m \) and \( n \) be integers such that \( 0 \leq m = \theta(n)n^{1/(1+k)} \),
where \( \theta(n) \) is a function of \( n \) which tends to 0 as \( n \to \infty \), and let \( \sigma \) be as in (5.3). Then
\[
(5.13) \quad n = \frac{I_1(\sigma m)}{k\sigma^{1+1/k}} + \left( \frac{m^{1/k}}{2} \right) - \frac{1}{m(1 + e^m)} + O(\sigma^{1-k-1}),
\]
where the implied constants are depend only on \( k \) and where
\[
I_1(z) = \int_z^\infty t^{1/k} \frac{1}{1 + e^t} \, dt.
\]

**Proof.** In view of (5.3) and (5.4) we have
\[
n = A_1 = \sum_{m < j^k < \infty} f_1(j^k) - \sum_{m < j^k < \infty} f_1(j^k),
\]
where \( f_1 \) is as in (5.5). The sum over \( j \) satisfying \( n < j^k < \infty \) is handled by Lemma 5.5 while Lemma 2.14 (with \( c = m \) and \( f = f_1 \)) combined with the estimate of the remainder term given in Lemma 5.4 (with \( r = 1 \)) provide the estimate of the sum over \( m < j^k < \infty \). \( \square \)

LEMMA 5.8. Let \( k \) be a fixed positive integer. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Let \( \sigma \)
and \( A_2 \) be as in (5.3) and (5.4). Then as \( n \to \infty \)
\[
(5.14) \quad A_2 = \frac{m^{1+1/k}}{k\sigma} \frac{1}{1 + e^m} + \frac{k + 1}{k\sigma} \left( n - \left( \frac{m^{1/k}}{2} \right) \frac{m}{1 + e^m} \right) + \left( \frac{m^{1/k}}{2} \right) - \frac{1}{2} \frac{m^2e^m}{(1 + e^m)^2} + O(\sigma^{1-k-2})
\]
for $0 \leq m \leq \theta(n)n^{k/(1+k)}$, where $\theta(n)$ is a function of $n$ which tends to 0 as $n \to \infty$ and where the implied constants depend only on $k$ and $\theta$.

**Proof.** From (5.4) we have

$$A_2 = \sum_{m < j^k < \infty} f_2(j^k) - \sum_{n < j^k < \infty} f_0(j^k),$$

where $f_2$ is as in (5.5). Lemma 5.5 shows that the summation over $n < j^k < \infty$ above is $\ll \sigma^{1/k - 2}$.

Lemma 2.14 and Lemma 5.4 (with $r = 2$) give

$$A_2 = \frac{I_2(\sigma m)}{k\sigma^{2+1/k}} + \left(\{m^{1/k}\} - 1/2\right)m \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2} + O(\sigma^{1/k - 2}).$$

Substituting the expression for $I_2(x)$ from Lemma 5.6 into (5.15) and then into the resulting equation substituting the expression for $I_1(\sigma m)$ obtained by transposing (5.13) gives the result. □

**Lemma 5.9.** Let $k$ be a fixed positive integer. Let $m$ and $n$ be integers such that $0 \leq m < n/2$. Let $\sigma$ and $A_0$ be as in (5.3) and (5.4). Then as $n \to \infty$

$$A_0 = k\sigma n + (-\{m^{1/k}\} - 1/2)\log(1 + e^{-\sigma m}) - k(\{m^{1/k}\} - 1/2)\frac{\sigma m}{1 + e^{\sigma m}} + O(\sigma^{1/k}),$$

for $0 \leq m \leq \theta(n)n^{k/(1+k)}$, where $\theta(n)$ is a function of $n$ which tends to 0 as $n \to \infty$ and where the implied constants depend only on $k$ and $\theta$.

**Proof.** From (5.4) we have

$$A_0 = \sum_{m < j^k < \infty} f_0(j^k) - \sum_{n < j^k < \infty} f_0(j^k),$$

where $f_0(x) = \log(1 + e^{-\sigma x})$ as in (5.5). Lemma 5.5 shows that $\sum_{n < j^k < \infty} f_0(j^k)$ is $O(\sigma^{1/k})$. Lemma 2.14 applied to the sum $\sum_{m < j^k < \infty} f_0(j^k)$ and Lemma 5.4 (with $r = 0$) give

$$A_0 = \frac{I_0(\sigma m)}{k\sigma^{1/k}} + \left(\{m^{1/k}\} - 1/2\right)\log(1 + e^{-\sigma m}) + O(\sigma^{1/k}),$$

where $I_0(z) = \int_{z}^{\infty} x^{1/k-1} \log(1 + e^{-x}) dx$. By Lemma 5.6 we can replace $I_0(z)$ with the expression $-kz^{1/k} \log(1 + e^{-z}) + kI_1(z)$ in the right hand side of the above equation for $A_0$. Then we substitute the expression for $I_1(\sigma m)$ obtained by rearranging (5.13) into the resulting expression to give the result. □

**4. Explicit asymptotic estimates of $\sigma$, $A_2$ and $\sigma n + A_0$**

In this section explicit asymptotic estimates of the quantities $\sigma$, $A_2$ and $\sigma n + A_0$ are found by firstly finding an asymptotic estimate of $\sigma$ (explicit in $m$ and $n$) and secondly by substituting this in (5.14) and (5.16). Lemma 5.3 is employed in giving the explicit estimate of $\sigma$ in the following lemma.
LEMMA 5.10. Let \( k \) be a fixed positive integer and let \( \gamma \) be as in (5.1). Let \( m \) and \( n \) be integers such that \( 0 \leq m \leq n^{k(4k-1)/(k+1)(4k+2)} \) and let \( \sigma \) be as in (5.3). Let \( R \) be the quantity

\[
R = (\gamma/n)^{1/(1+k)}.
\]

Then as \( n \to \infty \),

\[
\sigma = R^k \left( 1 - \frac{k}{2(k+1)^2\gamma} m^{(k+1)/k} R^{k+1} + O(n^{-3/(2k+2)}) \right),
\]

where the implied constants depend only on \( k \).

PROOF. Multiplying both sides of (5.13) by \( 1/\gamma \) gives

\[
n/\gamma = \frac{1}{\sigma^{1+1/k}} \left( \frac{I_1(\sigma m)}{k\gamma} + ((m^{1/k}) - \frac{1}{2} \frac{m^{1+1/k}}{\gamma(1 + e^\sigma m^2)} + O(\sigma^{2/k}) \right).
\]

In order to estimate \( I_1(\sigma m) \) we shall need to evaluate \( I_1(0) \) and we do this first. We have

\[
I_1(0) = \int_0^\infty \frac{z^{1/k}}{1 + e^z} dz.
\]

By interchanging the order of integration and summation we obtain

\[
I_1(0) = \int_0^\infty z^{1/k} e^{-z} \sum_{i=0}^\infty (-1)^i e^{-iz} dz = \sum_{i=0}^\infty (-1)^i \int_0^\infty z^{1/k} e^{-(1+i)z} dz.
\]

The change of variable \( w = (1+i)z \) and the definition of \( \gamma \) in (5.1) then give that

\[
I_1(0) = \sum_{i=0}^\infty \frac{(-1)^i}{(1+i)^{1+1/k}} \Gamma(1 + 1/k) = k\gamma.
\]

We now estimate \( I_1(z) \) by noting that

\[
I_1(z) = I_1(0) - \int_0^z y^{1/k}/(1 + e^y) dy.
\]

Expanding the integrand about \( y = 0 \) and then integrating and using \( I_1(0) = k\gamma \) we obtain

\[
\frac{I_1(\sigma m)}{k\gamma} = 1 - \frac{1}{2(k+1)^2} (\sigma m)^{(k+1)/k} + O((\sigma m)^{2+1/k}).
\]

For \( m = O(n^{k(4k-1)/(k+1)(4k+2)}) \) it can be checked that,

\[
\sigma^{1+1/k} \ll n^{-(7k+2)/(k+1)(4k+2)}, \quad \sigma^{2/k} \gg n^{-2/(k+1)}, \quad (\sigma m)^{2+1/k} \ll n^{-3/(2k+2)}
\]

and all of this allows simplification of the right hand side of (5.19) to be made. Raising both sides of (5.19) to the power \(-1/(1+k)\) and incorporating the simplifications just mentioned gives

\[
(\gamma/n)^{1/(1+k)} = \sigma^{1/k} \left( 1 - \frac{1}{2(k+1)^2\gamma} (\sigma m)^{(k+1)/k} + O(n^{-3/(2k+2)}) \right)^{-1/(1+k)}.
\]

Applying the Binomial Theorem to the right hand side of this equation and transposing yields

\[
\sigma^{1/k} = R \left( 1 - \frac{1}{2(k+1)^2\gamma} (\sigma m)^{(k+1)/k} + O(n^{-3/(2k+2)}) \right),
\]
with \( R = (\gamma/n)^{1/(1+k)} \) as in (5.17).

From Lemma 5.3 we have \( \sigma \asymp n^{-k/(k+1)} \asymp R^k \). Substituting \( \sigma \asymp R^k \) into the right hand side of (5.20) gives

\[
(5.21) \quad \sigma^{1/k} = R(1 + O(R^{k+1}m^{(k+1)/k}) + O(n^{-3/(2k+2)})).
\]

Substituting (5.21) into the right hand side of (5.20) gives

\[
(5.22) \quad \sigma^{1/k} = R(1 - \frac{1}{2(k+1)^2\gamma}(R^k m)^{(k+1)/k} + O((R^k m)^{(2k+2)/k}) + O(n^{-2/(k+1)})).
\]

We note that when \( m = O(n^{k/(4k-1)(k+1)(4k+2)}) \), \( (R^k m)^{(2k+1)/k} \ll n^{-2/(k+1)} \) and raising both sides of (5.22) to the power \( k \) gives the result. \( \square \)

We have estimated \( \sigma \) over the range \( 0 \leq m \leq n^{k/(4k-1)(k+1)(4k+2)} \) because, as will be seen, this allows for an influence of \( m \) in the estimate of \( \sigma n + A_0 \) which is not apparent when \( m = o(n^{k^2/(1+k)^2}) \).

The following lemma provides an asymptotic estimate of \( A_2 \) which involves only the variable \( n \).

**Lemma 5.11.** Let \( k \) be a fixed positive integer and let \( \gamma \) be as in (5.1). Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (5.3). Then as \( n \to \infty \),

\[
A_2 = \frac{1 + 1/k}{\gamma^{k/(1+k)}n^{2-1/(1+k)}}(1 + O(n^{-3/(4k+2)})),
\]

for \( 0 \leq m \leq n^{k/(4k-1)(k+1)(4k+2)} \) where the implied constants depend only on \( k \).

**Proof.** Using Lemma 5.8 we obtain

\[
A_2 = (1 + 1/k)^{n-k} \sigma(1 - \frac{m^{1/k}}{n} - \frac{m}{1 + e^{\sigma m}} + \frac{m^{1+1/k}}{(k+1)n(1 + e^{\sigma m})} + \frac{k}{k+1}n(\frac{m^{1/k}}{n} - \frac{m}{1 + e^{\sigma m}})^2 + O(\sigma^{1/k-1}n^{-1})).
\]

Keeping in mind throughout the course of this proof that \( m \leq n^{k/(4k-1)(k+1)(4k+2)} \) we have the estimates

\[
\sigma^{1/k-1}n^{-1} \ll n^{-2/(k+1)}, \quad \frac{m^{1+1/k}}{n(1 + e^{\sigma m})} \ll n^{-3/(4k+2)}, \quad \frac{\sigma m^2}{n} \ll n^{-2/(k+1)}
\]

and substituting these into the above equation for \( A_2 \) gives the simplification

\[
A_2 = \frac{k + 1}{k} \sigma \left( 1 + O(n^{-3/(4k+2)}) \right).
\]

Using only the main term of the estimate of \( \sigma \) given in Lemma 5.10 gives the estimate \( \sigma = (\gamma/n)^{k/(1+k)}(1 + O(n^{-3k/(k+1)(2k+1)})) \) and substituting this into the above equation for \( A_2 \) concludes the proof. \( \square \)

The following lemma is the final lemma that is required in obtaining an asymptotic estimate of \( \sigma n + A_0 \) explicit in \( m \) and \( n \).
LEMMA 5.12. Let \( k \) be a fixed positive integer and let \( \gamma \) be as in (5.1). Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (5.3). Then as \( n \to \infty \),

\[
\sigma n + A_0 = (k + 1)\gamma^{k/(k+1)}n^{1/(1+k)} + \frac{1}{2k + 2}m^{(k+1)/k}\left(\gamma/n\right)^{k/(k+1)} - \left(\lfloor m^{1/k} \rfloor + 1/2\right)\log 2 + O(n^{-1/(2k+2)}),
\]

for \( 0 \leq m \leq n^{k/(k+1)}(k+1)/(4k+2) \) where the implied constants depend only on \( k \).

PROOF. In the expression for \( A_0 \) in (5.16), we expand \( \log(1 + e^{-x}) \) to the second order and merge one of the terms into an \( O \)-term, yielding

\[
(5.23) \quad \sigma n + A_0 = (k + 1)\sigma n - \left(\lfloor m^{1/k} \rfloor + 1/2\right)\left(\log 2 - \sigma m/2 + O((\sigma m)^2)\right) + O(\sigma m) + O(\sigma^{1/k}).
\]

For \( 0 \leq m \leq n^{k/(k+1)}(k+1)/(4k+2) \) we have the estimates

\[
[m^{1/k}]\sigma m \ll n^{(k-1)/(k+1)}(4k+2), \quad [m^{1/k}](\sigma m)^2 \ll n^{-1/(2k+2)}, \quad \sigma^{1/k} \asymp n^{-1/(1+k)},
\]

\[
\sigma m \ll n^{-3k/(4k+2)(k+1)},
\]

and incorporating these into (5.23) we obtain

\[
(5.24) \quad \sigma n + A_0 = (1 + k)\sigma n - \left(\lfloor m^{1/k} \rfloor + 1/2\right)\log 2 - [m^{1/k}]\sigma m/2 + O(n^{-1/(2k+2)}).
\]

We have from Lemma 5.10 with some simplification that

\[
n\sigma = n^{1/(1+k)}\gamma^{k/(k+1)} - \frac{k}{2(k+1)^2}\gamma^{k/(k+1)}m^{(k+1)/k}n^{1/(1+k)} + O(n^{-1/(2k+2)})
\]

and

\[
\frac{1}{2}[m^{1/k}]\sigma m = \frac{1}{2}[m^{1/k}]mn^{-k/(k+1)}\gamma^{k/(k+1)} + O(n^{-(k+2)/(2k+1)(k+1)}).
\]

Hence we substitute these estimates into the right hand side of (5.24) to give

\[
\sigma n + A_0 = (k + 1)\gamma^{k/(k+1)}n^{1/(1+k)} + \frac{1}{2k + 2}m^{(k+1)/k}\left(\gamma/n\right)^{k/(k+1)} - \left(\lfloor m^{1/k} \rfloor + 1/2\right)\log 2 + O(n^{-1/(2k+2)}),
\]

which is the desired result. \( \square \)

5. Proof of Theorem 5.1 and some consequences

We prove Theorem 5.1 in this section. We assume that \( 0 \leq m \leq n^{k/(k+1)}(k+1)/(4k+2) \) and note that Lemma 5.11 and Lemma 5.12 apply. Into the right hand side of (5.6) we substitute the estimates of \( A_2 \) and \( \sigma n + A_0 \) provided in Lemma 5.11 and Lemma 5.12 to give

\[
q_u(m, n) = \exp\left((1 + k)\gamma^{k/(k+1)}n^{1/(1+k)} + \frac{1}{2k + 2}m^{(k+1)/k}\left(\gamma/n\right)^{k/(k+1)} - \left(\lfloor m^{1/k} \rfloor + 1/2\right)\log 2 + O(n^{-1/(2k+2)})\right) \times 2^{\left(\lfloor m^{1/k} \rfloor - 1/2\right)} \times \frac{1}{\sqrt{2\pi}} \frac{k}{k + 1} \gamma^{k/(2k+2)}n^{-1+1/(2k+2)} \times (1 + O(n^{-1/(2k+2)})).
\]
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Some minor rearrangement of the right hand side of the above expression for $q_u(m, n)$ concludes the proof of Theorem 5.1. \( \square \)

Putting $m = 0$ in Theorem 5.1 we obtain

$$q_u(0, n) = 2^{-1} \left( \frac{k}{\pi (k+1)} \right)^{1/2} \frac{1}{n} \left( \gamma^k n \right)^{1/(2k+2)} \times \exp\left( (k+1) \left( \gamma^k n \right)^{1/(k+1)} \right) \left( 1 + O(n^{-1/(2k+2)}) \right),$$

which corresponds to the case $a = 1$, $b = 0$ of Proposition I, except that the error is $O(n^{-1/(2k+2)})$ rather than $O(n^{-1/(k+1)})$. It follows that as $n \to \infty$,

$$q_u(0, n) \sim 2^{-1} \left( \frac{k}{\pi (k+1)} \right)^{1/2} \frac{1}{n} \left( \gamma^k n \right)^{1/(2k+2)} \times \exp\left( (k+1) \left( \gamma^k n \right)^{1/(k+1)} \right) \tag{5.25}$$

and so we obtain Ingham's Proposition H.

By taking $0 \leq m = o(n^{k^2/(1+k)^2})$ in the theorem, we obtain the result (5.2) and by combining this result with (5.25) we see that for $0 \leq m = o(n^{k^2/(1+k)^2})$, as $n \to \infty$,

$$q_u(m, n) \sim 2^{-1} m^{1/k} q_u(0, n),$$

that is, we recover Corollary 4.25 to Theorem 4.22. It was the above observation which led to Theorem 4.22.
CHAPTER 6

Partitions into positive integers – estimating the integral

1. Introduction

Throughout this chapter and Chapter 7 we consider partitions into parts drawn from the sequence \( u = \mathbb{N} = \{j\} \), that is, the sequence of positive integers. Thus we take \( u_j = j \) and for \( 0 \leq m < n/2 \) consider the partition function \( q(m, n) = q_n(m, n) \) which gives the number of partitions of \( n \) into distinct integers greater than \( m \).

As mentioned in Chapter 1, the asymptotic behaviour of the classical partition function \( q(n) = q(0, n) \) was studied by Hardy and Ramanujan [21], Ingham [28] and Hua [26] and was later studied for \( m > 0 \) by Erdös, et al [13], Dixmier and [9] (see Section 7 of Chapter 1), and more recently by Freiman and Pitman [16] (see Section 8 of Chapter 1). In particular, by applying their main theorem (see Theorem G of Chapter 1) for small \( m \) and an extension of it for large \( m \), Freiman and Pitman proved the following results (in which minor changes have been made to fit in with notation here — the function they studied was \( q(m - 1, n) \)).

**Proposition J (Freiman and Pitman).** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Then as \( n \to \infty \) we have

\[
q(m, n) \sim \frac{1}{2^{m+\frac{1}{2}}3^{\frac{1}{4}}4^{\frac{3}{4}}n^{\frac{3}{4}}} \exp \left( \pi ((n/3)^{1/2} + m(m + 1)/8\sqrt{3n}) \right),
\]

uniformly with respect to \( m \) such that \( 0 \leq m \leq \theta(n)n^{1/3} \), where \( \theta(n) \) is a sequence of positive numbers that tends to 0, and where all implied constants are effective and independent of \( m \) and \( n \) (but dependent on the function \( \theta(n) \)).

**Proposition K (Freiman and Pitman).** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and for \( X > 0 \) let \( F \) be the function defined by \( F(X) = X^{-2} \int_{X}^{\infty} \frac{\psi(1+e^y)}{y(1+e^y)} \, dy \). Let \( \psi(n) \) be a function that tends
to \( \infty \) as \( n \to \infty \) and that is \( o(n^{1/2}(\log n)^{-10}) \). Then as \( n \to \infty \) we have
\[
q(m, n) \sim \frac{1}{\sqrt{2\pi mn}} \exp \left( \frac{2n}{m+1} F^{-1}(n/(m+1)^2) - (m+1) \log(1 + e^{-F^{-1}(n/(m+1)^2)}) \right),
\]
uniformly with respect to \( m \) such that \( \psi(n)n^{1/2} \leq m \leq K_n \log^{10} n \), where \( K_n \) is the constant in Theorem G, while all other implied constants are effective and independent of \( m \) and \( n \) (but dependent on \( \psi \)).

A consequence of the above proposition is the following.

**Proposition L (Freiman and Pitman).** Suppose \( 1/3 < \nu < 1 \). Then as \( n \to \infty \)
\[
q(m, n) \sim \frac{1}{\sqrt{2\pi mn}} \exp \left( \frac{m}{n^n} \left( 1 + \frac{2}{\nu \log n} \right) \right)
\]
uniformly with respect to \( m \) such that
\[
m = \frac{\nu \log n}{\sqrt{1 + \nu \log n}} n^{(1+\nu)/2} + o(n^{1/2}).
\]

The aim of this chapter and Chapter 7 is to study the case \( u_j = j \) more deeply and hence to extend the results of Proposition J and Proposition L.

**Notation**

Specialising the notation of Section 2 in Chapter 3 to the case \( u_j = j \), we consider the characteristic function

\[
(6.1) \quad \varphi(\alpha) = \prod_{m<j\leq n} \varphi_j(\alpha),
\]
where

\[
(6.2) \quad \varphi_j(\alpha) = p_{1j} + p_{2j} e^{\alpha j}
\]

and

\[
(6.3) \quad p_{1j} = \frac{1}{1 + e^{-\alpha j}}, \quad p_{2j} = \frac{1}{1 + e^{\alpha j}} = 1 - p_{1j}.
\]

Once again we assume that \( \sigma \) is the unique positive real number (whose existence is guaranteed by Lemma 3.6) such that

\[
(6.4) \quad n = \sum_{m<j\leq n} \frac{j}{1 + e^{\sigma j}}.
\]

The cumulants \( A_k \) and moments \( \rho_k \) (see Section 2 of Chapter 2) are given by

\[
(6.5) \quad A_k = \frac{d^k}{dx^k} \left( \sum_{m<j\leq n} \log(p_{1j} + p_{2j} e^{\alpha j}) \right) \bigg|_{\alpha = 0},
\]
\[ \rho_k = \frac{d^k}{dx_k} \left( \prod_{m<j \leq n} (p_{1j} + p_{2j}e^x) \right) \bigg|_{x=0}. \]

In particular I shall use
\[ A_1 = \sum_{m<j \leq n} \frac{j}{1 + \exp(j)} = n, \]
\[ A_2 = \sum_{m<j \leq n} \frac{j^2 \exp(j)}{(1 + \exp(j))^2}, \]
\[ A_3 = \sum_{m<j \leq n} j^3(p_{2j} - 3p_{2j}^2 + 2p_{2j}), \]
\[ A_4 = \sum_{m<j \leq n} j^4(p_{2j} - 7p_{2j}^2 + 12p_{2j}^3 - 6p_{2j}), \]
and
\[ \rho_5 = \sum_{m<j \leq n} \frac{j^5}{1 + e^{oj}}. \]

Lemma 1.3 gives
\[ q(m, n) = e^{\sigma_n} \prod_{m<j \leq n} (1 + e^{-oj}) \int_{-1/2}^{1/2} \varphi(\alpha)e^{-\alpha n}d\alpha. \]

In this chapter I shall estimate the integral
\[ \int_{-1/2}^{1/2} \varphi(\alpha)e^{-\alpha n}d\alpha \]
giving more precise results for this special case than were obtained for a general sequence \( u \) in Chapter 3.

Then in Chapter 7 I shall estimate the product
\[ e^{\sigma_n} \prod_{m<j \leq n} (1 + e^{-oj}) \]
and use both results to obtain two explicit asymptotic expressions for \( q(m, n) \) in terms of \( m \) and \( n \). One of these will be valid for \( m \) in the range \( 0 \leq m \leq \theta(n)n^{1/3} \) and the other for \( m \) in small intervals centered at points \( m^* \) lying in the range \( \psi(n)n^{2/3} \leq m^* \leq \theta(n)n/\log n \) (where \( \theta(n) \) and \( \psi(n) \) are functions of \( n \) such that as \( n \to \infty, \theta(n) \to 0 \) and \( \psi(n) \to \infty \)).

The main results of the present chapter are the following two theorems.

**Theorem 6.1.** For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let \( \sigma, \varphi(\alpha) \) and \( A_2 \) be as in (6.4), (6.1) and (6.8). Let \( \theta(n) \) be a function of \( n \) such that as \( n \to \infty, \theta(n) \to 0 \). Then as \( n \to \infty, \)
\[ \int_{-1/2}^{1/2} \varphi(\alpha)e^{-\alpha n}d\alpha = \frac{1}{\sqrt{2\pi}A_2} \left( 1 - \frac{3\sqrt{3}}{8\pi} n^{-1/2} + o(n^{-1/2}) \right), \]
for \( 0 \leq m \leq \theta(n)\sqrt{n} \), where the implied constants are absolute.
THEOREM 6.2. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Let \( \sigma \) be as in (6.4), let the characteristic function \( \varphi(\alpha) \) be as in (6.1) and the second cumulant \( A_2 \) as in (6.8). Let \( \theta(n) \) and \( \psi(n) \) be functions of \( n \) such that \( \theta(n) \to 0 \) and \( \psi(n) \to \infty \). Then as \( n \to \infty \),

\[
\int_{-1/2}^{1/2} \varphi(\alpha) e(-\alpha n) d\alpha = \frac{1}{\sqrt{2\pi} A_2} \left( 1 - \frac{1}{12} \frac{m}{n} + o(m/n) \right),
\]

for \( \psi(n) \sqrt{n} \leq m \leq \theta(n) n / (\log n)^6 \), where the implied constants are absolute.

Plan of Chapter

I shall estimate the characteristic function \( \varphi(\alpha) \) in Section 2 and this will lead on to an estimation of the main integral in Section 3. Following this I shall provide asymptotic estimates of the cumulants for both \( m \) small and large in Section 4. The supplementary integral will be dealt with in Section 5 and I shall conclude the chapter with proofs of Theorem 6.1 and Theorem 6.2 in Section 6 and Section 7.

2. Estimation of characteristic function

The material in this section follows closely the material in Section 7 of Chapter 3. In this section, we estimate the behaviour of the characteristic function for the special case \( u_j = j \) in a more precise way. The following lemma is a more precise estimate of \( \varphi(\alpha) \) than that given in Lemma 3.11.

**Lemma 6.3.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (6.4). Let \( \alpha \) be a real number in the interval \((-1/2, 1/2)\) and let \( \varphi(\alpha) \) be as in (6.1). Suppose that

\[
(6.15) \quad \text{either } |\alpha| < \sigma/(2\pi \log 3) \text{ or } \sigma m > \log 3.
\]

Then we have

\[
(6.16) \quad \varphi(\alpha) = e(n\alpha) \exp(-2\pi^2 \alpha^2 A_2 + (2\pi i\alpha)^3 A_3 / 6 + (2\pi i\alpha)^4 A_4 / 24 + R(\alpha)),
\]

where \( A_2, A_3, A_4 \) are as in (6.8), (6.9), (6.10) and where \( R(\alpha) \) is such that

\[
(6.17) \quad |R(\alpha)| \leq \frac{11}{3} |2\pi \alpha|^5 \rho_5,
\]

with \( \rho_5 \) as in (6.11).

**Proof.** We argue in a similar manner to that done in the proof of Lemma 3.11. Let

\[
F_j(x) = \log \varphi_j(x)
\]

for each \( j \) satisfying \( m < j \leq n \). Then Lemma 2.8 with \( s = 5 \), \( x = 0 \), \( h = \alpha \) and \( f = F_j \) gives

\[
(6.18) \quad F_j(\alpha) = F_j(0) + \alpha F_j'(0) + \alpha^2 F_j''(0)/2 + \alpha^3 F_j'''(0)/6 + \alpha^4 F_j^{(4)}(0)/24 + R_j(\alpha),
\]
where

\begin{equation}
R_j(\alpha) = \frac{\alpha}{120} \int_0^1 (1 - v)^4 F_j'(v\alpha) dv.
\end{equation}

Now \( F_j(0) = 0 \) and by using (6.2) and (6.3), the higher order derivatives of \( F_j \) are easily calculated and we obtain in particular

\begin{align*}
F_j''(z) &= (2\pi i)^2 (p_{2j}(x) - 15p_{2j}(x)^2 + 50p_{2j}(x)^3 - 60p_{2j}(x)^4 + 24p_{2j}(x)^5) \\
F_j'''(0) &= (2\pi i)^3 (p_{2j} - p_{2j}^2) \\
F_j''''(0) &= (2\pi i)^4 (p_{2j} - 3p_{2j}^2 + 2p_{2j}^3) \\
F_j''''(0) &= (2\pi i)^5 (p_{2j} - 7p_{2j}^2 + 12p_{2j}^3 - 6p_{2j}^4),
\end{align*}

where

\[ p_{2j}(x) = \frac{1}{1 + e^{\sigma x} e^{-jx}}. \]

We estimate the integral occurring in the remainder term \( R_j(\alpha) \) in (6.19) by arguing as in the proof of Lemma 3.11 to give

\[ R_j(\alpha) = \frac{1}{120} \left| \int_0^1 (1 - v)^4 F_j'(v\alpha) dv \right| < \frac{11}{3} |2\pi\alpha j|^5/(1 + \exp(\sigma j)). \]

Thus, using (6.3) we obtain from (6.18) the result

\[ F_j(\alpha) = 2\pi i \alpha f_1(j) + (2\pi i\alpha)^2 f_2(j)/2 + (2\pi i\alpha)^3 f_3(j)/6 + (2\pi i\alpha)^4 f_4(j)/24 + R_j(\alpha) \]

where

\begin{align*}
f_1(x) &= \frac{x}{1 + e^{\sigma x}}, & f_2(x) &= \frac{x^2 e^{\sigma x}}{(1 + e^{\sigma x})^2}, & f_3(x) &= \frac{x^3 e^{\sigma x} (e^{\sigma x} - 1)}{(1 + e^{\sigma x})^3}, \\
f_4(x) &= \frac{x^4 e^{\sigma x} (e^{\sigma x} - 1)(1 - 4e^{\sigma x} + e^{2\sigma x})}{(1 + e^{\sigma x})^4}
\end{align*}

and

\[ |R_j(\alpha)| \leq \frac{11}{3} |2\pi\alpha j|^5 \frac{j^5}{1 + e^{\sigma j}}. \]

Summation over \( j \) satisfying \( m < j \leq n \) and noting that \( A_1 = n \) gives

\[ \log \varphi(\alpha) = 2\pi i\alpha n + (2\pi i\alpha)^2 A_2/2 + (2\pi i\alpha)^3 A_3/6 + (2\pi i\alpha)^4 A_4/24 + R(\alpha), \]

which immediately gives the result. \( \square \)
Lemma 6.4. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (6.4). Let \( \varphi \) be the characteristic function as in (6.1). Then there is a positive absolute constant \( n_0 \) such that for \( n > n_0 \) and for all real numbers \( \alpha \),

\[
|\varphi(\alpha)| \leq \exp \left( -\frac{1}{2} \sum_{m<j \leq n} e^{-\sigma j \sin^2 \pi \alpha j} \right).
\]

Proof. We follow the proof of Lemma 3.12. From (6.1) we have that for any real number \( \alpha \)

\[
|\varphi(\alpha)|^2 = \prod_{m<j \leq n} (1 - 4p_{1j}p_{2j} \sin^2 (\pi \alpha j))
\leq \prod_{m<j \leq n} \exp(-4p_{1j}p_{2j} \sin^2 (\pi \alpha j)).
\]

Noting that from (6.3),
\[
p_{1j}p_{2j} > \frac{1}{4} e^{-\sigma j},
\]
we have

\[
|\varphi(\alpha)|^2 \leq \prod_{m<j \leq n} \exp(-\exp(-\sigma j) \sin^2 \pi \alpha j),
\]

\[
= \exp \left( -\sum_{m<j \leq n} \exp(-\sigma j) \sin^2 \pi \alpha j \right),
\]

which is the result. \( \square \)

3. Estimation of the main integral

This section resembles Section 8 of Chapter 3, but here the main integral is estimated to greater accuracy than done previously in order to give a more refined estimate of the integral in (6.13).

As in Section 8 of Chapter 3, \( \alpha_0 \) is a number in the interval \( (0, 1/2) \) which is to be chosen appropriately later on. Extending the ideas presented in Section 8 of Chapter 3, we express the main integral

\[
(6.20) \quad M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha) e(-\alpha n) d\alpha
\]

as a two term asymptotic expansion.

It is easy to generalise the method of this section to providing a multiterm asymptotic expansion of the main integral in terms of the higher order cumulants. However, this does not prove fruitful unless the product term (6.14) is estimated to the same degree of accuracy. As will be seen later on, estimating the product term to a comparable degree of accuracy involves much tedious detail and so we confine ourselves to a modest two term estimate.
3. ESTIMATION OF THE MAIN INTEGRAL

We shall use the auxiliary integral $J(z, c, f)$ defined, for $z \geq 0, c > 0$ and for a continuous complex-valued function $f$ defined on the real numbers, by

\begin{equation}
J(z, c, f) = \int_{-z}^{z} e^{-cx^2} (1 + f(x) + (f(x))^2/2) dx.
\end{equation}

We note that $J(z, c, f)$ is closely related to the auxiliary integral $I(z, c)$ defined in (3.13) via the identity $I(z, c) = J(z, c, 0)$.

We shall approximate the main integral $M$ in (6.20) sequentially by the auxiliary integrals $J(\alpha_0, 2\pi^2 A_2, T)$, $J(\alpha_0, 2\pi^2 A_2, V)$ and $J(\infty, 2\pi^2 A_2, V)$, where the functions $V$ and $T$ are defined by

\begin{equation}
V(\alpha) = (2\pi i \alpha)^3 A_3/6 + (2\pi i \alpha)^4 A_4/24
\end{equation}

and

\begin{equation}
T(\alpha) = V(\alpha) + R(\alpha) = (\log \varphi(\alpha) - 2\pi i \alpha n + 2\pi^2 \alpha^2 A_2),
\end{equation}

with $R(\alpha)$ as in Lemma 6.3. To do this, we require three lemmas.

**Lemma 6.5.** For integers $m$ and $n$ such that $0 \leq m < n/2$, let $\sigma$ be as in (6.4) and let $A_2, A_3, A_4, \rho_5$ be as in (6.8) through (6.11). Let $M$ be as in (6.20) and let the auxiliary integral $J(z, c, T)$ be as in (6.21), with $T$ as in (6.23). Suppose that a given real number $\alpha_0$ in the interval $(0, 1/2)$ satisfies the conditions

\begin{equation}
either \alpha_0 < \sigma/(2\pi \log 3) or \sigma m > \log 3
\end{equation}

and

\begin{equation}
\frac{(2\pi \alpha_0)^3 A_3}{6} + \frac{(2\pi \alpha_0)^4 A_4}{24} + \frac{11}{3} (2\pi \alpha_0)^5 \rho_5 < \frac{1}{2}.
\end{equation}

Then as $n \to \infty$,

\[M - J(\alpha_0, 2\pi^2 A_2, T) \ll \frac{1}{\sqrt{A_2}} \left( \left( \frac{A_3}{A_2^{3/2}} \right)^3 + \left( \frac{A_4}{A_2^2} \right)^3 + \left( \frac{\rho_5}{A_2^{5/2}} \right)^3 \right),\]

where the implied constants are absolute.

**Proof.** Using (6.16) in (6.20) we obtain

\[M = \int_{-\alpha_0}^{\alpha_0} \exp(-2\pi^2 \alpha^2 A_2 + T(\alpha)) d\alpha,
\]

where $T(\alpha)$ is as in (6.23). Using the fact that $|T(\alpha)| < 1/2$ by virtue of (6.24) and (6.25),

\[|\exp(T(\alpha)) - 1 - T(\alpha) - T(\alpha)^2/2| < |T(\alpha)|^3/3\]
6. PARTITIONS INTO POSITIVE INTEGERS – ESTIMATING THE INTEGRAL

gives

\[ |M - J(\alpha_0, 2\pi^2 A_2, T)| \leq \frac{1}{3} \int_{-\alpha_0}^{\alpha_0} \exp(-2\pi^2 \alpha^2 A_2) |T(\alpha)|^2 d\alpha. \]

Hölder’s Inequality gives

\[ |T(\alpha)| \leq 3^{2/3} \left( \frac{(2\pi)^9}{6^3} |\alpha|^9 A_3^3 + \frac{(2\pi)^{12}}{24^3} |\alpha|^{12} A_4^4 + \frac{(2\pi)^{15}}{3^3} |\alpha|^{15} \rho_5 \right)^{1/3}. \]

Hence

\[ |M - J(\alpha_0, 2\pi^2 A_2, T)| \leq \frac{1}{3} \int_{-\alpha_0}^{\alpha_0} e^{-2\pi^2 \alpha^2 A_2} |T(\alpha)|^2 d\alpha \]

\[ \leq \frac{1}{3} \int_{-\alpha_0}^{\alpha_0} e^{-2\pi^2 \alpha^2 A_2} \left( \frac{(2\pi)^9}{6^3} |\alpha|^9 A_3^3 + \frac{(2\pi)^{12}}{24^3} |\alpha|^{12} A_4^4 + \frac{(2\pi)^{15}}{3^3} |\alpha|^{15} \rho_5 \right) d\alpha. \]

Writing \( \beta_0 = 2\pi \sqrt{A_2} \alpha_0 \) and making the change of variable \( \beta = 2\pi \sqrt{A_2} \alpha \) gives that the above integral is equal to

\[ \frac{3}{2\pi \sqrt{A_2}} \int_{\beta_0}^{\beta_0} e^{-\beta^2/2} \left( \frac{1}{6^3} |\beta|^9 A_3^3 / A_2^{9/2} + \frac{1}{24^3} |\beta|^{12} A_4^4 / A_2^{6} + \frac{113}{3^3} |\beta|^{15} \rho_5 / A_2^{15/2} \right) d\alpha \]

\[ \leq \frac{3}{2\pi \sqrt{A_2}} \int_{-\infty}^{\infty} e^{-\beta^2/2} \left( \frac{1}{6^3} |\beta|^9 A_3^3 / A_2^{9/2} + \frac{1}{24^3} |\beta|^{12} A_4^4 / A_2^{6} + \frac{113}{3^3} |\beta|^{15} \rho_5 / A_2^{15/2} \right) d\alpha \]

\[ \ll \frac{1}{\sqrt{A_2}} \left( A_3^3 / A_2^{9/2} + A_4^4 / A_2^6 + \rho_5^5 / A_2^{15/2} \right), \]

where the implied constants are absolute. This completes the proof of the lemma. \( \square \)

**Lemma 6.6.** Suppose that the hypotheses of Lemma 6.5 are satisfied and suppose further that \( \alpha_0 \) satisfies the condition

(6.26)

\[ \alpha_0 > \frac{1}{2\pi \sqrt{A_2}}. \]

Then as \( n \to \infty \),

\[ |J(\alpha_0, 2\pi^2 A_2, T) - J(\alpha_0, 2\pi^2 A_2, V)| \ll \frac{1}{\sqrt{A_2}} \frac{\rho_5}{A_2^{5/2}}, \]

where \( V \) is as in (6.22) and the implied constants are absolute.

**Proof.** We have

\[ |J(\alpha_0, 2\pi^2 A_2, T) - J(\alpha_0, 2\pi^2 A_2, V)| = \left| \int_{-\alpha_0}^{\alpha_0} e^{-2\pi^2 \alpha^2 A_2} (T(\alpha) - V(\alpha + T(\alpha)^2 - V(\alpha^2)) d\alpha \right| \]

\[ = \left| \int_{-\alpha_0}^{\alpha_0} e^{-2\pi^2 \alpha^2 A_2} R(\alpha)(1 + T(\alpha) + V(\alpha)) d\alpha \right| \]

and since \( |T(\alpha)| \) and \( |V(\alpha)| \) are \( < 1/2 \) by virtue of (6.25), applying (6.17) gives that this is

\[ \leq \int_{-\alpha_0}^{\alpha_0} e^{-2\pi^2 \alpha^2 A_2} \frac{11}{3} |2\pi \alpha|^5 \rho_5/2 d\alpha \]

\[ \leq \frac{22}{3} \int_{-\alpha_0}^{\alpha_0} e^{-2\pi^2 \alpha^2 A_2} |2\pi \alpha|^5 \rho_5 d\alpha. \]

Applying Lemma 2.10 gives us that this is equal to

\[ \frac{352}{3} \frac{1}{2\pi \sqrt{A_2}} \frac{\rho_5}{A_2^{5/2}} \]
and the result is proven. □

**Lemma 6.7.** Suppose that the hypotheses of Lemma 6.6 are satisfied. Let \( \beta_0 = 2\pi\sqrt{A_2}a_0 \). Then as 
\( n \to \infty, \)
\[
J(\alpha_0, 2\pi^2A_2, V) - J(\infty, 2\pi^2A_2, V) \ll \frac{1}{\sqrt{A_2}} \exp(-\beta_0^2/2)(1 + \beta_0 A_4/A_2^2 + \beta_0^2 A_3^3/A_2^3 + \beta_0^3 A_2^4/A_2^4)
\]
where the implied constants are absolute.

**Proof.** We have upon substituting the expression for \( V(\alpha) \) in (6.22) that
\[
|J(\infty, 2\pi^2A_2, V) - J(\alpha_0, 2\pi^2A_2, V)|
= \left| \left( \int_{\alpha_0}^{\infty} e^{-2\pi^2\alpha^2} (1 + V(\alpha) + V(\alpha)^2) d\alpha \right) \right|
= \left| \left( \int_{\alpha_0}^{\infty} e^{-2\pi^2\alpha^2} \left( 1 + \frac{(2\pi i)^4 A_4}{24} + \frac{(2\pi i)^6 A_3^3}{6^2} + \frac{(2\pi i)^8 A_2^4}{24^2} \right) d\alpha \right) \right|
\]
Writing \( \beta_0 = 2\pi\sqrt{A_2}a_0 \) and making the change of variable \( \beta = 2\pi\sqrt{A_2}\alpha \) gives that the above expression is equal to
\[
\frac{1}{\sqrt{A_2}} \int_{\beta_0}^{\infty} e^{-\beta^2/2} \left( 1 + \beta^4 A_4/24A_2^2 - \beta^6 A_3^3/6^2A_2^3 + \beta^8 A_2^4/24^2A_2^4 \right) d\beta
\ll \frac{1}{\sqrt{A_2}} e^{-\beta_0^2/2} \left( 1 + \frac{A_4}{A_2^2} \beta_0^4 + \frac{A_3^3}{A_2^3} \beta_0^6 + \frac{A_2^4}{A_2^4} \beta_0^8 \right)
\]
and in view of Lemma 2.9 this is
\[
\ll \frac{1}{\sqrt{A_2}} e^{-\beta_0^2/2} \left( 1 + \frac{A_4}{A_2^2} \beta_0^4 + \frac{A_3^3}{A_2^3} \beta_0^6 + \frac{A_2^4}{A_2^4} \beta_0^8 \right)
\]
where the implied constants are absolute. Hence the proof is complete. □

Combining Lemma 6.5, Lemma 6.6 and Lemma 6.7 yields our final estimate of the main integral \( M \).

**Corollary 6.8.** For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let \( \sigma \) be as in (6.4) and let \( A_2, A_3, A_4, \rho_6 \) be as in (6.8) through (6.11). Let \( M \) be as in (6.20). Suppose that a given real number \( \alpha_0 \) in the interval \((0, 1/2)\) satisfies the conditions (6.24), (6.25) and (6.26). Let \( \beta_0 = 2\pi\sqrt{A_2}a_0 \). Then as 
\( n \to \infty, \)
\[
M = \frac{1}{\sqrt{2\pi A_2}} (1 + \frac{A_4}{8A_2^2} - \frac{5A_3^3}{24A_2^3} + \frac{35A_2^4}{384A_2^4} + E_1(\alpha_0))
\]
where
\[
E_1(\alpha_0) \ll \left( \frac{A_3}{A_2^2} \right)^3 + \left( \frac{A_4}{A_2^2} \right)^3 + \left( \frac{\rho_6}{A_2^{5/2}} \right)^3 + \left( \frac{\rho_6}{A_2^{5/2}} \right)^3
\]
\[
+ (\exp(-\beta_0^2/2)(1 + \beta_0^4 A_4/A_2^2 + \beta_0^6 A_3^3/A_2^3 + \beta_0^8 A_2^4/A_2^4))
\]
and where the implied constants are absolute.
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PROOF. Write \( E_1(\alpha_0) = \sqrt{2\pi A_2}(M - J(\infty, 2\pi^2 A_2, V)) \), where \( V \) is as in (6.22). It can be checked from Lemma 2.10 that

\[
J(\infty, 2\pi^2 A_2, V) = \frac{1}{\sqrt{2\pi A_2}} \left( 1 + \frac{A_4}{8A_2^2} - \frac{5A_3^2}{24A_2^3} + \frac{35A_2^3}{384A_2^4} \right).
\]

Using the triangle inequality as well as Lemma 6.5, Lemma 6.6 and Lemma 6.7 gives

\[
|M - J(\infty, 2\pi^2 A_2, V)| \ll \frac{\rho_5}{A_2^3} + \frac{1}{\sqrt{A_2}} \left( \left( \frac{A_3}{A_2^{3/2}} \right)^3 + \left( \frac{A_4}{A_2^2} \right)^3 + \left( \frac{\rho_5}{A_2^{5/2}} \right)^3 \right) + \frac{1}{\sqrt{A_2}} (\exp(-\rho_5^2/2)(1 + \rho_5^2 A_4/A_2^2 + \rho_5^4 A_3^2/A_2^3 + \rho_5^6 A_2^3/A_2^4))
\]

and so we have the upper bound for \( E_1(\alpha_0) \), which completes the proof. \( \square \)

4. Estimation of cumulants

In view of the above corollary, we require an estimate of the cumulants \( A_2, A_3 \) and \( A_4 \) and the related quantities \( \sigma \) and \( \rho_5 \). We therefore include in this section some estimates of sums and integrals which are more refined than those presented in Section 9 of Chapter 3.

We start with a couple of lemmas which give the behaviour of the quantity \( \sigma m \) for \( m \) in lower and upper ranges of the interval \([0, n/2] \).

**Lemma 6.9.** For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let \( \sigma \) be as in (6.4). Let \( \theta(n) \) be a function of \( n \) such that as \( n \to \infty \), \( \theta(n) \to 0 \). If \( 0 \leq m \leq \theta(n)\sqrt{n} \) then \( \sigma m \to 0 \) as \( n \to \infty \).

**PROOF.** From Corollary 3.10(i) we have that \( \sigma \ll 1/\sqrt{n} \) as \( n \to \infty \). Consequently,

\[
\sigma m \ll \frac{m}{\sqrt{n}} \ll \theta(n)
\]

and so as \( n \to \infty \), \( \sigma m \to 0 \). \( \square \)

**Lemma 6.10.** For integers \( m \) and \( n \) such that \( 0 \leq m < n/2 \), let \( \sigma \) be as in (6.4). Let \( \psi(n) \) be a function of \( n \) such that as \( n \to \infty \), \( \psi(n) \to \infty \). If \( m \geq \psi(n)\sqrt{n} \) then \( \sigma m \to \infty \) as \( n \to \infty \).

**PROOF.** From (6.4) and the fact that \( 2m \leq n \), we see that

\[
n > \sum_{m < j \leq 2m} \frac{j}{2e^{2m\sigma}} > \sum_{m < j \leq 2m} \frac{m}{2e^{2m\sigma}} = \frac{m^2}{2e^{2m\sigma}},
\]

from which the result follows. \( \square \)

The following lemmas provide for large \( m \) estimates of certain sums related to the cumulants.
LEMMA 6.11. Let \( l, k \) be positive integers such that \( l \leq k \). Let \( m \) and \( n \) be integers such that \( k/\sigma < m < n/2 \), with \( \sigma \) as in (6.4). Then there is an absolute constant \( n_0 > 0 \) such that for \( n > n_0 \),

\[
\left| \sum_{m < j < \infty} \frac{j^k e^{-\sigma j l}}{\sigma^{-1} l^{-1} m^k e^{-\sigma m}} - 1 \right| < \frac{(k + 2)!}{\sigma m}.
\]

PROOF. Using Lemma 2.12 and letting

\[
I(m, \sigma) = \int_m^\infty x^k e^{-\sigma x} \, dx \quad \text{and} \quad E(m, \sigma) = \int_m^\infty \left( x - \frac{1}{2} \right) (k x^{k-1} - \sigma x^k) e^{-\sigma x} \, dx
\]

gives

\[
\sum_{m < j < \infty} j^k e^{-\sigma j l} = I(m, \sigma) - \frac{1}{2} m^k e^{-\sigma m} + E(m, \sigma).
\]

Now it can be checked that

\[
I(m, \sigma) = \frac{1}{\sigma l} m^k e^{-\sigma m} + E_0(m, \sigma)
\]

where

\[
E_0(m, \sigma) = (\sigma l)^{-k-1} \left( \sum_{h=0}^{k-1} (\sigma m)^h \binom{k}{h} (k-h)! \right) e^{-\sigma m}
\]

and satisfies

\[
|E_0(m, \sigma)| < (\sigma l)^{-k-1} (\sigma m)^{k-1} (k+1)! e^{-\sigma m} = (k+1)! \sigma^{-2} m^{k-1} e^{-\sigma m}.
\]

From Corollary 3.10(i) we have that for \( n \) sufficiently large, \( \sigma^{-2} > n/4 \geq m/2 \), and consequently

\[
\frac{1}{2} m^k e^{-\sigma m} < \sigma^{-2} m^{k-1} e^{-\sigma m}.
\]

We have that

\[
|E(m, \sigma)| \leq \int_m^\infty \left( x - 1/2 \right) k x^{k-1} e^{-\sigma x} \, dx + \int_m^\infty \left( x - 1/2 \right) \sigma x^k e^{-\sigma x} \, dx.
\]

Since both the functions \( k x^{k-1} e^{-\sigma x} \) and \( \sigma x^k e^{-\sigma x} \) are positive and decreasing for \( x \geq k/(\sigma l) \) and since \( m \geq k/(\sigma l) \) we have by Lemma 2.15 that

\[
|E(m, \sigma)| < \frac{1}{4} k m^{k-1} e^{-\sigma m} + \frac{1}{4} \sigma m^k e^{-\sigma m}.
\]

The lemma follows from (6.27), (6.28) and (6.29) upon application of the triangle inequality. \( \square \)

We now estimate a sum of the form \( \sum_{m < j \leq n} j^k/(1+e^{\sigma j})^l \). This sum arises in the expression of the \( k \)-th cumulant \( A_k \).
Corollary 6.12. Let \( l, k \) be positive integers such that \( l \leq k \). Let \( m \) and \( n \) be integers such that \((k+1)/\sigma \leq m < n/2\), with \( \sigma \) as in (6.4). Then there is a positive constant \( n_0 = n_0(k,l) \) depending only on \( k \) and \( l \) such that for \( n > n_0 \),

\[
\sum_{m < j \leq n} j^k/(1 + e^{\sigma j})^l = \frac{1}{\sigma l} m^k e^{-\sigma ml}(1 + O(\frac{1}{\sigma m})),
\]

where the implied constants depend only on \( k \) and \( l \).

Proof. Firstly because for a positive integer \( l \) and for \( x > 1 \),

\[
1 - \frac{1}{x^l} < 1,
\]

we have

\[
\sum_{m < j < \infty} j^k e^{-\sigma j}(1 - le^{-\sigma j}) < \sum_{m < j < \infty} j^k (1 + e^{\sigma j})^{-l} < \sum_{m < j < \infty} j^k e^{-\sigma j},
\]

and so

\[
\left| \sum_{m < j < \infty} j^k (1 + e^{\sigma j})^{-l} - \sum_{m < j < \infty} j^k e^{-\sigma j} \right| < l \sum_{m < j < \infty} j^k e^{-\sigma (l+1)j}.
\]

We now provide an estimate of the sum

\[
\sum_{n < j < \infty} j^k (1 + e^{\sigma j})^{-l}.
\]

Lemma 2.11 gives

\[
\sum_{n < j < \infty} j^k (1 + e^{\sigma j})^{-l} = - \int_n^\infty U(x) \phi'(x) dx - \phi(n)U(n)
\]

where \( U(x) = x^k \) and \( \phi(x) = x^k/(1 + e^{\sigma x})^l \). This is

\[
< \int_n^\infty \frac{lx^{k+1}}{e^{\sigma x}} dx < (k+2)! (\sigma)^{-k-1} (\sigma n)^{k+1} e^{-\sigma ln} = (k+2)! n^{k+1} e^{-\sigma ln}
\]

and since the function \( x^{k+1} e^{-\sigma lx} \) is decreasing on the interval \([ (k+1)/(\sigma), \infty) \) this is

\[
\leq (k+2)! (2m)^{k+1} e^{-2\sigma lm}.
\]

Applying the triangle inequality gives

\[
\left| \sum_{m < j < \infty} j^k (1 + e^{\sigma j})^{-l} - (\sigma l)^{-1} m^k e^{-\sigma ml} \right| < \left| \sum_{m < j < \infty} \frac{j^k}{(1 + e^{\sigma j})^l} - \sum_{m < j < \infty} j^k e^{-\sigma lj} \right|
\]

\[
+ \left| \sum_{m < j < \infty} j^k e^{-\sigma lj} - \frac{m^k}{\sigma l} e^{-\sigma ml} \right| + \sum_{n < j < \infty} j^k (1 + e^{\sigma j})^{-l}
\]

and applying (6.30) to the first difference, Lemma 6.11 to the second difference and (6.31) to the third difference gives that the right hand side of the above inequality is

\[
< l \sum_{m < j < \infty} j^k e^{-\sigma (l+1)j} + \frac{(k+2)!}{\sigma lm} \sigma^{-1} l^{-1} m^k e^{-\sigma lm} + (k+2)! (2m)^{k+1} e^{-2\sigma lm}
\]
Applying Lemma 6.11 to the first sum in the above expression gives that this is
\[
< l \frac{m^k}{(l+1)\sigma} e^{-\sigma(l+1)m} \left( 1 + \frac{(k+2)!}{\sigma m} \right) + \frac{(k+2)!}{\sigma m} e^{-\sigma m} + (k+2)!2m^k e^{-2\sigma m} \\
= \frac{m^{k-1}}{\sigma^2} e^{-\sigma m} \left( m e^{-\sigma m} (1 + (k+2)!/(\sigma m)) + (k+2)!/l + 2^{k+1}(k+2)!\sigma^2 m^2 e^{-\sigma m} \right) \\
\leq \frac{1}{l\sigma^2} m^{k-1} e^{-\sigma m}
\]
Thus
\[
\sum_{m<j \leq n} j^k (1 + e^{\sigma j})^{-l} = \frac{1}{\sigma l} m^k e^{-\sigma m} (1 + O\left(\frac{1}{\sigma m}\right)),
\]
where the implied constants depend only on \(l\) and \(k\). □

The following lemma gives an estimate of the cumulants for \(m\) large.

**Lemma 6.13.** Let \(k\) be a positive integer. Let \(m\) and \(n\) be integers such that \((k+1)/\sigma \leq m \leq n/2\), where \(\sigma\) is as in (6.4). Then with \(A_k\) and \(\rho_k\) as in (6.5) and (6.6),
\[
A_k = \sigma^{-1} m^k e^{-\sigma m} (1 + O\left(\frac{1}{\sigma m}\right)),
\]
\[
\rho_k = \sigma^{-1} m^k e^{-\sigma m} (1 + O\left(\frac{1}{\sigma m}\right)),
\]
where the implied constants depend only on \(k\).

**Proof.** As demonstrated in Lemma 3.2 we have for constants \(C(k, r)\) defined via
\[
C(k, r) = \frac{1}{r} \sum_{i=1}^{r} \binom{r}{i} (-1)^{i+1} i^k
\]
that
\[
A_k = \sum_{r=1}^{k} C(k, r) \sum_{m<j \leq n} j^k (1 + e^{\sigma j})^{-r}.
\]
From Corollary 6.12 this is equal to
\[
\sum_{r=1}^{k} C(k, r)(\sigma r)^{-1} m^k e^{-\sigma mr} (1 + O\left(\frac{1}{\sigma m}\right))
\]
\[
= \sigma^{-1} m^k e^{-\sigma m} (1 + O\left(\frac{1}{\sigma m}\right)),
\]
and the result follows for both \(A_k\) and \(\rho_k\). □

As a corollary, the explicit estimates of the cumulants \(A_k\) for \(m\) large follow upon substitution of the estimate for \(A_1\), i.e. \(n = \sigma^{-1} m e^{-\sigma m} (1 + O(1/(\sigma m)))\).

**Corollary 6.14.** Let \(k\) be a positive integer. Let \(m\) and \(n\) be integers such that \((k+1)/\sigma \leq m < n/2\), with \(\sigma\) as in (6.4). Then with \(A_k\) and \(\rho_k\) as in (6.5) and (6.6) we have
\[
A_k = m^{k-1} n \left( 1 + O\left(\frac{1}{\sigma m}\right) \right),
\]
\[ \rho_k = m^{k-1} \left( 1 + O\left( \frac{1}{\sigma m} \right) \right), \]

where the implied constants depend only on \( k \).

The following lemma offers more than that which the corollary of the previous lemma gives because it conclusively provides explicit lower bounds on the ratios of successive cumulants, especially when \( m \) is near \( 6/\sigma \).

**Lemma 6.15.** Let \( m, n \) be integers such that \( 6/\sigma \leq m < n/2 \), with \( \sigma \) as in \((6.4)\). Let \( A_1, A_2, A_3, A_4 \) and \( \rho_5 \) be as in equations \((6.7)\) through \((6.11)\). Then as \( n \to \infty \),

\[ m \ll \frac{A_2}{A_1} \ll \frac{A_3}{A_2} \ll \frac{A_4}{A_3} \ll \frac{\rho_5}{A_4} \ll m, \]

where the implied constants are absolute.

**Proof.** For a positive integer \( k \) let \( \rho_k \) be as in \((6.6)\) and for a positive integer \( l \) let

\[ \rho_{kl} = \sum_{m<j\leq n} \frac{j^k}{(1 + e^{\sigma j})^l}. \]

We note that from \((6.8)\), \( A_2 = \rho_2 - \rho_22 \) and \( \rho_22 < \rho_2(1 + \exp(\sigma m))^2 < (1 + e^3)^{-1} \rho_2 < \rho_2/20 \) so that

\[ \frac{19}{20} \rho_2 = \rho_2 - \frac{1}{20} \rho_2 < \rho_2 - \rho_22 = A_2 < \rho_2. \]

Similarly

\[ \rho_3(1 - 3/20) < A_3 = \rho_3 - 3\rho_32 + 2\rho_33 < \rho_3(1 + 2/(20)^2), \]

\[ \rho_4(1 - 7/20 - 6/(20)^3) < A_4 = \rho_4 - 7\rho_42 + 12\rho_43 - 6\rho_44 < \rho_4(1 + 12/(20)^2). \]

Thus we have

\[ \frac{19}{20} \rho_2 < A_2 < \rho_2, \quad \frac{17}{20} \rho_3 < A_3 < \frac{201}{200} \rho_3, \quad \frac{5}{8} \rho_4 < A_4 < \frac{103}{100} \rho_4. \]

In view of \((6.32)\), in order to prove the lemma it suffices to show that

\[ \frac{m}{\rho_1} \leq \frac{\rho_2}{\rho_1} \leq \frac{\rho_3}{\rho_2} \leq \frac{\rho_4}{\rho_3} \leq \frac{\rho_5}{\rho_4} \ll m, \]

where the implied constants are absolute.

Firstly,

\[ m\rho_1 = \sum_{m<j\leq n} \frac{j}{1 + e^{\sigma j}} < \sum_{m<j\leq n} \frac{u_j}{1 + e^{\sigma u_j}} = \rho_2. \]

Secondly, it is easily shown by the Cauchy–Schwarz Inequality that \( \rho_k^2 \leq \rho_{k-1} \rho_{k+1} \).

Finally, to show the inequality \( \rho_5/\rho_4 \ll m \) in \((6.33)\) we appeal to Corollary 6.14 to give for \( m \geq 6/\sigma \),

\[ \frac{\rho_5}{\rho_4} \ll m. \]
This completes the proof of the lemma. \( \square \)

The following lemma gives a bound on the quantity \( \sigma m \).

**Lemma 6.16.** Let \( m, n \) be integers such that \( 0 \leq m < n/2 \) and for \( \sigma \) as in (6.4), suppose that \( 2/\sigma \leq m \leq n/2 \). Then

\[
\sigma m \leq \log n.
\]

**Proof.** Lemma 6.13 gives us that

\[
A_1 < \frac{1}{\sigma} me^{-\sigma m},
\]

when \( 2/\sigma \leq m \). By our choice of \( \sigma \) in (6.4) we have \( A_1 = n \) and so we have immediately that \( n e^{-\sigma m} \leq 2m \leq n \) which gives \( e^{\sigma m} \leq n \), a rearrangement of the result. \( \square \)

The following lemma provides an idea of the order of magnitude of the cumulants over the entire range of \( m \).

**Lemma 6.17.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (6.4). Let \( A_1, A_2, A_3, A_4 \) and \( \rho_5 \) be as in equations (6.7) through (6.11).

(i) For \( 0 \leq m \leq 6/\sigma \) we have

\[
A_2 \asymp n^{3/2}, \quad A_3 \asymp n^2, \quad A_4 \asymp n^{5/2}, \quad \rho_5 \asymp n^3.
\]

where the implied constants are absolute.

(ii) For \( 6/\sigma \leq m \leq n/2 \),

\[
A_2 \asymp m n, \quad A_3 \asymp m^2 n, \quad A_4 \asymp m^3 n, \quad \rho_5 \asymp m^4 n,
\]

where the implied constants are absolute.

**Proof.** (i) For \( 0 \leq m \leq 6/\sigma \) we appeal to Corollary 6.12 and Lemma 6.13 to give

\[
A_1 \asymp \left( \frac{1}{\sigma} \right)^2, \quad A_2 \asymp \left( \frac{1}{\sigma} \right)^3, \quad A_3 \asymp \left( \frac{1}{\sigma} \right)^4, \quad A_4 \asymp \left( \frac{1}{\sigma} \right)^5, \quad \rho_5 \asymp \left( \frac{1}{\sigma} \right)^6,
\]

where the implied constants are absolute. Since \( A_1 \asymp 1/\sigma^2 \) and since \( A_1 = n \) from our choice of \( \sigma \) in (6.4) we have that \( 1/\sigma \asymp 1/\sqrt{n} \), from which we obtain (6.35) with the implied constants absolute.

(ii) Combining the fact that \( A_1 = n \) (by our choice of \( \sigma \) in (6.4)) with Lemma 6.15 we obtain

\[
A_2 = A_1 \times \frac{A_2}{A_1} \asymp mn, \quad A_3 = A_1 \times \frac{A_2}{A_1} \times \frac{A_3}{A_2} \asymp nm^2,
\]

and similarly for \( A_4 \) and \( \rho_5 \). Hence we have (6.36). \( \square \)
We use the formula for $A_k$ given in Lemma 3.1 to prove the following proposition regarding asymptotic estimates of the cumulants for small $m$.

**Proposition 6.18.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$. Let $k$ be a positive integer and let $A_k$ be the $k$-th cumulant as in (6.5). Then as $n \to \infty$,

$$A_k \sim k! \left( \frac{\sqrt{12}}{\pi} \right)^{k-1} n^{(k+1)/2},$$

for $0 \leq m \leq \theta(n) \sqrt{n}$, where $\theta(n)$ is a function of $n$ such that as $n \to \infty$, $\theta(n) \to 0$.

**Proof.** From Lemma 3.1,

$$A_k = \sum_{m < j \leq n} f_k(j),$$

where $f_k(x)$ is as in (3.9). We consider the sum

$$A_k^{(1)} = \sum_{m < j < \infty} f_k(j).$$

Lemma 2.12 gives

$$A_k^{(1)} = \int_m^\infty f_k(x)dx - \frac{1}{2} f_k(m) + \int_m^\infty B_1(\{x\}) f_k(x)dx.$$

Note that from Lemma 5.5,

$$\int_m^\infty B_1(\{x\}) f_k(x)dx < C_2 e^{-(k-1)},$$

for some constant $C_2$ depending only on $k$.

We observe that for some positive constant $C_0$,

$$\left| \int_0^m f_k(x)dx \right| < C_0 \int_0^m x^k e^{-\sigma x}dx < C_0 m^{k+1} e^{-\sigma m}.$$

Now from the definition of $f_k(x)$ in (3.9) we have

$$\int_0^\infty f_k(x)dx = \int_0^\infty \sum_{l=0}^\infty (-1)^l (\sigma l - k-1) x^{k-1} e^{-\sigma x} dx$$

and interchanging the order of integration and summation gives us that this is equal to

$$\sum_{l=0}^\infty \frac{(-1)^l \sigma l - k-1}{l!} \int_0^\infty y^k e^{-\sigma y}dy$$

$$= \sum_{l=0}^\infty \sigma^{-k-1} l^{-2} \Gamma(k+1)(-1)^{l+1}$$

$$= k! \pi^2 \sigma^{-k-1}/12.$$

From Lemma 5.5 we have that

$$A_k^{(2)} = \sum_{m < j < \infty} f_k(j) \ll \sigma.$$
5. Estimation of Supplementary Integral

Thus we have that

\[(6.37)\quad A_k \sim k! \pi^2 \sigma^{-k-1}/12.\]

Since \(A_1 = n\), by our choice of \(\sigma\) in (6.4), we have for the case \(k = 1\) that \(\sigma^{-1} \sim \sqrt{12n}/\pi\) and substituting this asymptotic estimate of \(1/\sigma\) into (6.37) gives the result. \(\Box\)

This concludes our estimation of the cumulants.

5. Estimation of supplementary integral

We shall now estimate the supplementary integral \(S\) defined by

\[(6.38)\quad S = \int_{\{x: a_0 < |a| < 1/2\}} \varphi(a) e(-an) \, da,

where \(\varphi(a)\) is as in (6.1). The estimation of \(S\) in this section is more precise that the corresponding estimation of the supplementary integral in Section 10 of Chapter 3 because the relevant trigonometric sum is easier to estimate when \(u_j = j\) (see Lemma 2.18).

**Lemma 6.19.** Let \(m\) and \(n\) be integers such that \(0 \leq m < n/2\) and let \(\sigma\) be as in (6.4). Let \(a_0\) be some number in the interval \((0, 1/2)\) and let \(S\) be as in (6.38).

(i) Let \(0 \leq m \leq 6/\sigma\). Then there is an absolute constant \(R_1 > 0\) such that

\[|S| \leq \exp(-R_1 \min(\sigma^2, 1)n^{1/2}).\]

(ii) Let \(6/\sigma \leq m \leq n/2\). Then there is an absolute constant \(R_2 > 0\) such that

\[(6.39)\quad |S| \leq \exp(-R_2 \min(\sigma^2 m^2/(\log n)^2, 1)n/m).\]

**Proof.** From (6.38) we have

\[|S| \leq \sup_{\{x: a_0 < |a| < 1/2\}} |\varphi(a)|.

Lemma 6.4 provides an upper bound for \(|\varphi(a)|\) and hence for \(|S|\) and so we focus on estimating the trigonometric sum occurring in Lemma 6.4. We have from Lemma 2.18 (with \(k = \lfloor 1/\sigma \rfloor\)) that

\[(6.40)\quad G(\alpha) = \sum_{m < j \leq n} e^{-\sigma j^2} \sin^2 \pi a_j \geq \sum_{m < j \leq m + \lfloor 1/\sigma \rfloor} e^{-\sigma j^2} \sin^2 \pi a_j \geq e^{-\sigma m} e^{-\sigma \lfloor 1/\sigma \rfloor} \min(1, \sigma^2 [1/\sigma]^2) [1/\sigma]^2/2.

We consider two cases.

Firstly, suppose \(0 \leq m \leq 6/\sigma\). Then (6.40) gives

\[G(\alpha) \geq e^{-3/4} \min(1, \sigma^2 n/4) \sqrt{n}/4,

by Corollary 3.10 (i).
which gives (i).

Secondly, suppose $6/\sigma \leq m \leq n/2$. Then

$$G(\alpha) > e^{-\sigma m} e^{-1} \min(1, \alpha_3^2 m^2/(2\sigma m)^2)/(4\sigma).$$

Using the bound for $\sigma m$ in Lemma 6.16 gives us that the above is

$$> \frac{1}{\sigma} e^{-\sigma m} \times \frac{1}{4em} \min(1, \alpha_3^2 m^2/(2\log n)^2)$$

and from the estimate in Lemma 6.13 for $A_1$ this is

$$> R_2 n \min(1, \alpha_3^2 m^2/(\log n)^2),$$

for some positive constant $R_2$. Hence we have the result (ii). \(\square\)

6. Proof of Theorem 6.1 (small m)

Throughout this section, $\theta(n)$ will denote a function of $n$ such that $\theta(n) \to 0$ as $n \to \infty$. Suppose that $0 \leq m \leq \theta(n) \sqrt{n}$. We need to show that

$$\int_{-1/2}^{1/2} \varphi(\alpha) e(-\alpha n) d\alpha = \frac{1}{\sqrt{2\pi A_2}} \left( 1 - \frac{3\sqrt{3}}{8\pi} n^{-1/2} + o(n^{-1/2}) \right),$$

as $n \to \infty$, where the implied constants are absolute.

Write

$$\int_{-1/2}^{1/2} \varphi(\alpha) e(-\alpha n) d\alpha = M + S,$$

where

$$M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha) e(-\alpha n) d\alpha$$

is the main integral,

$$S = \int_{\{\alpha: \alpha_0 < |\alpha| \leq 1/2\}} \varphi(\alpha) e(-\alpha n) d\alpha$$

is the supplementary integral, and $\alpha_0$ still to be chosen.

If $\alpha_0$ can be chosen so that each of the conditions (6.24), (6.25) and (6.26) hold, then Corollary 6.8 will give

$$M = \frac{1}{\sqrt{2\pi A_2}} \left( 1 + \frac{A_4}{8 A_2^2} - \frac{5 A_2^2}{24 A_2} + E_1(\alpha_0) \right),$$

where

$$E_1(\alpha_0) \ll \left( \left( \frac{A_3}{A_2^{3/2}} \right)^3 + \left( \frac{A_4}{A_2^3} \right)^3 + \left( \frac{2 \rho_5 / A_2^{3/2}}{A_2^{3/2}} \right)^3 + \frac{\rho_5 / A_2^{5/2}}{A_2^{5/2}} \right)$$

$$+ \left( \frac{1 + \beta_5 A_4/A_2^2 + \beta_6 A_2^3/A_2^2 + \beta_6 A_2^5/A_2^2}{A_2^2} \right),$$

(6.43)
and where \( \beta_0 = 2\pi\sqrt{A_2} \alpha_0 \), and Lemma 6.19(i) will give

\[
S = \frac{E_2(\alpha_0)}{\sqrt{2\pi A_2}},
\]

where

\[
E_2(\alpha_0) \ll \sqrt{A_2} \exp(-R_1 \min(\alpha_0^2 n, 1) n^{1/2}),
\]

with \( R_1 \) as in Lemma 6.19(i).

It is clear that making \( \alpha_0 \) as large as possible and satisfying

\[
\frac{1}{2\pi\sqrt{A_2}} < \alpha_0 < \min \left( \frac{1}{2\pi\sqrt{A_2}}, \frac{1}{2\pi\sqrt{A_3}}, \frac{1}{2\pi\sqrt{4A_4}}, \frac{1}{2\pi\sqrt{22\rho_0}} \sigma/(2\pi \log 3) \right)
\]

will minimise \( E_1 \) and \( E_2 \). In view of Lemma 6.17(i), (6.45) becomes

\[
n^{-3} \ll \alpha_0 \ll \min(n^{-2/3}, n^{-5/6}, n^{-3/5}, n^{-1/2}) = n^{-3/3}.
\]

There is an absolute constant \( K_0 \) such that the choice \( \alpha_0 = K_0 n^{-2/3} \) is consistent with (6.46). With this choice for \( \alpha_0 \) we have \( \beta_0 = 2\pi\sqrt{A_2} K_0 n^{-2/3} \propto n^{1/12} \).

We now estimate \( E_1, E_2 \) and the terms in (6.42). Applying Lemma 6.17(i) to the right hand side of the inequality (6.43) yields

\[
E_1(\alpha_0) \ll n^{-3/4} + n^{-3/2} + n^{-9/4} + n^{-3/4} + e^{-\beta_0^2/2}(1 + \beta_0 n^{-1/2} + \gamma_0 n^{-1/2} + \delta_0 n^{-1}).
\]

It follows from (6.47) and the above choice for \( \alpha_0 \) that

\[
E_1(\alpha_0) \ll (n^{-3/4} + e^{-K_1 n^{1/12}})
\]

and for some constant \( K_2 > 0, \)

\[
E_2(\alpha_0) \ll n^{3/4} \exp(-K_2 n^{1/6}) \ll n^{-3/4}.
\]

To complete the proof of the lemma we give an asymptotic estimate of

\[
\frac{A_4}{8A_2^2} - \frac{5A_2^2}{24A_3^2} \sim \frac{24(\sqrt{12}/\pi)^3 n^{5/2}}{8(2(\sqrt{12}/\pi) n^{3/2})^2} - \frac{5(\sqrt{12}/\pi)^2 n^{3/2}}{24(2(\sqrt{12}/\pi) n^{3/2})^2} = \frac{3\sqrt{3}}{8\pi} n^{-1/2}.
\]

Lemma 6.18 gives

\[
A_2 \sim 2(\sqrt{12}/\pi)n^{3/2}, \quad A_3 \sim 6(\sqrt{12}/\pi)^2 n^2, \quad A_4 \sim 24(\sqrt{12}/\pi)^3 n^{5/2},
\]

and so

\[
\frac{A_4}{8A_2^2} - \frac{5A_2^2}{24A_3^2} \sim \frac{24(\sqrt{12}/\pi)^3 n^{5/2}}{8(2(\sqrt{12}/\pi) n^{3/2})^2} - \frac{5(\sqrt{12}/\pi)^2 n^{3/2}}{24(2(\sqrt{12}/\pi) n^{3/2})^2} = \frac{3\sqrt{3}}{8\pi} n^{-1/2}.
\]
The result (6.41) follows from (6.42) and the estimates (6.48), (6.49) and (6.50). □

7. Proof of Theorem 6.2 (large \( m \))

Throughout this section, \( \theta(n) \) will denote a function of \( n \) such that \( \theta(n) \to 0 \) as \( n \to \infty \) and \( \psi(n) \) will denote a function of \( n \) such that \( \psi(n) \to \infty \) as \( n \to \infty \). Suppose that

\[
\psi(n) \sqrt{n} \leq m \leq \theta(n) \frac{n}{\log n}.
\]

We need to show that

\[
(6.51) \quad \int_{-1/2}^{1/2} \varphi(\alpha) e^{-\alpha n} d\alpha = \frac{1}{\sqrt{2\pi A_2}} \left( 1 - \frac{1}{12} \frac{m}{n} + o\left(\frac{m}{n}\right) \right),
\]

as \( n \to \infty \), where the implied constants are absolute. The proof is similar to the proof of Theorem 6.1 in Section 6.

As done in the proof of Theorem 6.1 in Section 6, write

\[
\int_{-1/2}^{1/2} \varphi(\alpha) e^{-\alpha n} d\alpha = M + S,
\]

where

\[
M = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha) e^{-\alpha n} d\alpha
\]

is the main integral, and

\[
S = \int_{|\alpha| \leq |\alpha| \leq 1/2} \varphi(\alpha) e^{-\alpha n} d\alpha
\]

is the supplementary integral, and \( \alpha_0 \) still to be chosen.

If \( \alpha_0 \) can be chosen so that each of the conditions (6.24), (6.25) and (6.26) hold, then Corollary 6.8 will give

\[
(6.52) \quad M = \frac{1}{\sqrt{2\pi A_2}} \left( 1 + \frac{A_4}{8A_2^2} - \frac{5A_3^2}{24A_2^3} + E_1(\alpha_0) \right),
\]

where \( E_1(\alpha_0) \) is as in (6.43), and Lemma 6.19(i) will give

\[
S = \frac{E_2(\alpha_0)}{\sqrt{2\pi A_2}},
\]

where

\[
(6.53) \quad E_2(\alpha_0) \ll \sqrt{A_2} \exp(-R_2 \min(\alpha_2 m^2/(\log n)^2, 1)n/m)).
\]

It is clear that making \( \alpha_0 \) as large as possible and satisfying

\[
(6.54) \quad \frac{1}{2\pi \sqrt{A_2}} < \alpha_0 < \min\left(\frac{1}{2\pi \sqrt{A_3}}, \frac{1}{2\pi \sqrt{4A_4}}, \frac{1}{2\pi \sqrt{22A_5}}\right),
\]
will minimise $E_1$ and $E_2$. (For $m$ in the range $\psi \sqrt{n} \leq m \leq \theta(n)n/(\log n)^9$, Condition (6.24) automatically holds.) Using the estimates

$$A_2 \asymp mn, \quad A_3 \asymp m^2 n, \quad A_4 \asymp m^3 n, \quad \rho_6 \asymp m^4 n$$

from Lemma 6.17(ii), (6.54) becomes

$$(mn)^{-1/2} \ll \alpha_0 \ll \min(m^{-2/3}n^{-1/3}, m^{-3/4}n^{-1/4}, m^{-4/5}n^{-1/5}) \ll m^{-2/3}n^{-1/3}. \quad (6.56)$$

There is an absolute constant $K_0$ such that the choice $\alpha_0 = K_0 m^{-2/3}n^{-1/3}$ is consistent with (6.56).

With this choice for $\alpha_0$ we have

$$\beta_0 = 2\pi \sqrt{A_2} K_0 m^{-2/3}n^{-1/3} \propto m^{-1/6}n^{1/6}. \quad (6.57)$$

Again using the estimates in (6.55), (6.43) becomes

$$E_1(\alpha_0) \ll (m/n)^{3/2} + (m/n)^3 + (m/n)^{9/2} + (m/n)^{3/2}$$

$$+ e^{-\beta_0^2/2}(1 + \beta_0^2(m/n) + \beta_0^6(m/n) + \beta_0^8(m/n)^2). \quad (6.58)$$

It follows from (6.57) and the above choice for $\alpha_0$ that

$$E_1(\alpha_0) \ll (m/n)^{3/2} + e^{-K_2(n/m)^{1/6}}$$

$$\ll (m/n)^{3/2} \quad (6.59)$$

and for some constant $K_4 > 0$,

$$E_2(\alpha_0) \ll (mn)^{1/2} \exp(-K_4(n/m)^{1/3} / (\log n)^2). \quad (6.60)$$

We note that if $m = o(n/(\log n)^9)$,

$$(mn)^{1/2} \exp(-K_4(n/m)^{1/3} / (\log n)^2) \ll (m/n)^{3/2}. \quad (6.61)$$

To complete the proof of the lemma we give an asymptotic estimate of

$$\frac{A_4}{8A_2^2} - \frac{5A_2^3}{24A_2^3},$$

which occurs in (6.52). Using Corollary 6.14 we obtain

$$\frac{A_4}{8A_2^2} - \frac{5A_2^3}{24A_2^3} \sim \frac{1}{8n} - \frac{5}{24n} = -\frac{1}{12n}. \quad (6.62)$$

The result (6.51) follows from (6.58), (6.59) and (6.60). □
CHAPTER 7

Partitions into positive integers — estimating the product term

1. Introduction

In this chapter I shall continue the work of Chapter 6 and give asymptotic estimates of \( q(m, n) \) (explicit in \( m \) and \( n \)) for \( m \) in the ranges \( 0 \leq m = o(n^{1/3}) \) and, roughly speaking, \( m = \nu^{1/2}n^{(1+\nu)/2}(\log n)^{1/2}(1 + o(1)) \) for \( 1/3 < \nu < 1 \). These estimates will be refinements of the corresponding results of Freiman and Pitman [16] (given as Propositions J and L of Section 1 of Chapter 6).

We continue to use the notation introduced in Section 1 of Chapter 6 — including, in particular, \( \sigma \) as given by (6.4) and the cumulants \( A_k \) \((k \geq 1)\). We recall from (6.12)

\[
q(m, n) = e^{\sigma n} \prod_{m < j \leq n} (1 + e^{-\sigma j}) \int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n) d\alpha.
\]

In Chapter 6 the integral

\[
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n) d\alpha
\]

was estimated in terms of \( m, n \) and \( A_2 \). This chapter complements the Chapter 6 by providing an estimate of the product term

(7.1) \[ e^{\sigma n} \prod_{m < j \leq n} (1 + e^{-\sigma j}) \]

and, to a degree greater than done in Chapter 6, an estimate of the second cumulant \( A_2 \).

As done in Chapter 3, I write the logarithm of the product as

(7.2) \[ A_0 = \log \prod_{m < j \leq n} (1 + e^{-\sigma j}). \]

This chapter is divided into two parts: The first part caters for \( m \) small, while the second part deals with the case \( m \) large. Different approaches to estimating the product term are used in each part.

As will be seen, estimating the product term is more difficult than estimating the integral in the previous chapter.
In order to find an asymptotic formula for $\sigma n + A_0$, and hence $q(m, n)$, I shall commence with some preliminary lemmas which assist in the estimation of $\sigma$, $A_0$ and $A_2$. Following the preliminary lemmas, explicit estimates of $\sigma$, $A_0$ and $A_2$ will be made which will lead to the estimate of $q(m, n)$.

I shall commence with some preliminary lemmas for $m$ small in Section 2. In Section 3 I shall give an explicit estimate of $\sigma$ for $m = o(n^{1/3})$. In Section 4 explicit estimates of the second cumulant and the logarithm of the product term for small $m$ will be given. Finally in Section 5 the proof of an asymptotic estimate of $q(m, n)$ as $n \to \infty$ for $m$ in the range $0 \leq m \leq \theta(n)n^{1/3}$, where $\theta(n)$ is a function of $n$ such that $\theta(n) \to 0$ as $n \to \infty$, will be presented. The result obtained will be as follows.

**Theorem 7.1.** With $q(m, n)$ as in (1.2) as $n \to \infty$,

$$q(m, n) = 2^{-m} - 3^{1/4} \frac{m}{n^{3/4}} \exp \left( \frac{\pi m^2}{3n} + \frac{\pi^2 m^3}{8n^{3/2}} - \frac{\pi^3 m^4}{288n^2} \right) \times$$

$$\left( 1 - \frac{3\sqrt{3}}{8\sqrt{n}} - \frac{3m^2}{16n} + \frac{\pi m}{8\sqrt{3m^2}} - \frac{\pi^2 m^2}{384n} + \frac{\pi}{48\sqrt{3m}} \right. \left. \frac{m^4}{128\sqrt{3m^2}} + \frac{\pi m^5}{2153n^3} + \left( \frac{1}{960} + \frac{1}{2\pi^2} \right) \frac{\pi^4 m^6}{144n^2} + o(n^{-1/2}) \right),$$

for $0 \leq m \leq \theta(n)n^{1/3}$, where $\theta(n)$ is a function of $n$ such that $\theta(n) \to 0$ as $n \to \infty$ and where the implied constants depend only on the function $\theta$.

I shall then proceed with the case $m$ large, with the introduction of some preliminary lemmas in Section 7. Then in Section 8 an estimate of $\sigma$ will be achieved. Explicit estimates of the second cumulant $A_2$ and the logarithm of the product term, namely $\sigma n + A_0$, will be given in Section 9, from which will follow a proof of an asymptotic estimate of $q(m, n)$ as $n \to \infty$ for $m$ in specified bands throughout the range $\psi(n)/\sqrt{n} \leq m \leq \theta(n)n/\log(n)$, where $\theta(n)$ is a function of $n$ such that $\theta(n) \to 0$ as $n \to \infty$ and $\psi(n)$ is a function of $n$ such that $\psi(n) \to \infty$ as $n \to \infty$. The result obtained will be as follows.

**Theorem 7.2.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$. Suppose that as $n \to \infty$,

$$\left| m - \sqrt{n/F} \right| = o((\log n)^{1/2} n^{(3\nu - 1)/2}),$$

where

$$F = \left( \frac{1}{\nu \log n} + \frac{1}{\nu^2 \log^2 n} \right)n^{-\nu} - \left( \frac{1}{2\nu \log n} + \frac{1}{4\nu^2 \log^2 n} \right)n^{-2\nu}.$$
and $1/3 < \nu < 1$. Then with $q(m,n)$ as in (1.2) as $n \to \infty$,
\[
q(m,n) = \exp \left( m \left( 1 + \frac{2}{\nu \log n} \right) n^{-\nu} - m \left( \frac{1}{2} + \frac{1}{2\nu \log n} \right) n^{-2\nu} \right) \\
\times \frac{1}{(2\pi)^{1/2}} n^{-3/2} n^{\nu/2} (\nu \log n)^{1/2} \\
\times (1 + 2/(\nu \log n) + 2/(\nu \log n)^2 - (1 + 1/(\nu \log n) + 1/(2(\nu \log n)^2))n^{-\nu})^{-1/2} \\
\times (1 - \frac{1}{12} (\nu \log n)^{1/2} n^{(\nu-1)/2} + o((\log n)^{1/2} n^{(\nu-1)/2})),
\]
where the implied constants are absolute.

In Section II, I shall then obtain Theorem L (see Section I of Chapter 6) as a corollary of Theorem 7.2.

2. Preliminary lemmas for small $m$

We firstly give a couple of preliminary lemmas concerning derivatives and integrals involving the function $f_k$ defined for $x > 0$ via

\[
f_k(x) = \sum_{m<j \leq n} j^k g_k(\sigma j),
\]
where
\[
g_k(x) = \sum_{l=1}^{\infty} (-1)^{l+1} x^{k-1} e^{-lx}.
\]
(cf (5.5).)

**Lemma 7.3.** Let $k$ be a non-negative integer and let $f_k$ be as in (7.3), with $g_k$ as in (7.4). Let $\sigma$ as in (6.4) and $x \in (0, \infty)$. Then we have

\[
f_k'(x) = kx^{k-1} g_k(\sigma x) - \sigma x^k g_{k+1}(\sigma x),
\]

(7.5)

\[
f_k''(x) = 2 \binom{k}{2} x^{k-2} g_k(\sigma x) - 2 \binom{k}{1} x^{k-1} \sigma g_{k+1}(\sigma x) + \sigma^2 x^k g_{k+2}(\sigma x),
\]

(7.6)

\[
f_k'''(x) = 6 \binom{k}{3} x^{k-3} g_k(\sigma x) - 6 \binom{k}{2} \sigma x^{k-2} g_{k+1}(\sigma x) + 3 \binom{k}{1} \sigma^2 x^{k-1} g_{k+2}(\sigma x) - \sigma^3 x^k g_{k+3}(\sigma x).
\]

(7.7)

**Proof.** Fix the nonnegative integer $k$. Firstly we have that

\[
\frac{d}{dx} g_k(\sigma x) = (-\sigma)^j g_{k+j}(\sigma x).
\]

(7.8)

We are in a position to show something more general than that stated in the lemma without an excess of detail, namely that

\[
f_k^{(j)}(x) = \sum_{l=0}^{j} \binom{k}{l} (-1)^{j-l} (j-l)! (-\sigma)^{j+l-i} g_{k+l}(\sigma x).
\]

(7.9)
Employing the rule of Leibnitz for the multiple derivatives of a product we have
\[
f_k^{(j)}(z) = \frac{d^j}{dz^j} (z^k g_k(\sigma z)) = \sum_{l=0}^{j} \binom{j}{l} \frac{d^{j-l}}{dz^{j-l}} (z^k) \frac{d^l}{dz^l} g_k(\sigma z)
= \sum_{l=0}^{j} \binom{j}{l} \binom{k-j+l}{j-l} (j-l)! z^{k-j+l-1} (-\sigma^l)^{k+l}(\sigma z).
\]

The following lemma is analogous to Lemma 5.6.

**Lemma 7.4.** For a nonnegative integer \(k\), let \(g_k(z)\) be as in (7.4) and let
\[
I_k(z) = \int_{z}^{\infty} z^k g_k(x) dx.
\]

Then
\[
I_0(z) = -z g_0(z) + I_1(z) \quad \text{and} \quad I_2(z) = z g_1(z) + 2I_1(z).
\]

**Proof.** We note that \(g'_k(z) = -g_{k+1}(z)\). Integrating \(I_k(z)\) by parts gives
\[
\int_{z}^{\infty} z^k g_k(x) dx = \left. [-z^k g_{k-1}(x)] \right|_{x=z}^{\infty} - \int_{z}^{\infty} k z^{k-1} (-g_{k-1}(x)) dx,
\]
which is just
\[
I_k(z) = z^k g_{k-1}(z) + k I_{k-1}(z).
\]
This recurrence formula gives for \(k = 1\) and \(k = 2\),
\[
I_1(z) = z g_0(z) + I_0(z) \quad \text{and} \quad I_2(z) = z g_1(z) + 2I_1(z)
\]
from which the lemma follows. \(\square\)

The final three lemmas of this section pertain to a preliminary estimate of the cumulants using Euler-Maclaurin Summation.

**Lemma 7.5.** Let \(m\) and \(n\) be integers such that \(0 \leq m < n/2\). Let \(\sigma\) be as in (6.4) and let \(I_{l}(z)\) be as in Lemma 7.4. Then as \(n \to \infty\),

\[
(7.10) \quad n = \frac{I_{1}(\sigma m)}{\sigma^2} - \frac{1}{2} m \frac{1}{1+e^{-m}} - \frac{B_2}{2} \left( \frac{1}{1+e^{-m}} - \sigma m \frac{e^{-m}}{(1+e^{-m})^2} \right) + O(\sigma^2),
\]

for \(0 \leq m \leq \theta(n)/\sqrt{n}\), where \(\theta(n) \to 0\) as \(n \to \infty\) and where the implied constants depend only upon the function \(\theta\).

**Proof.** The proof is exactly the same as the proof of Lemma 5.7 except that it uses the Euler-Maclaurin Summation Formula in Lemma 2.12 to the third degree rather than the first. \(\square\)

We estimate the second cumulant in a similar way to that done in Lemma 5.8.
Lemma 7.6. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Let \( \sigma \) be as in (6.4) and let \( A_2 \) be as in (6.8). Then as \( n \to \infty \),

\[
A_2 = \left( \frac{2n}{\sigma} + \frac{m^2}{\sigma} \frac{1}{1 + e^{\sigma m}} + \frac{m}{\sigma} \frac{1}{1 + e^{\sigma m}} - \frac{1}{2} m^2 \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2} \right) + O(\sigma^{-1}),
\]

for \( 0 \leq m \leq \theta(n) \sqrt{n} \), where \( \theta(n) \to 0 \) as \( n \to \infty \) and where the implied constants depend only upon the function \( \theta \).

Proof. This lemma follows as a corollary of Lemma 5.8. \( \Box \)

Finally, we estimate \( A_0 \) in a similar manner to that done for \( A_2 \).

Lemma 7.7. Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Let \( \sigma \) be as in (6.4) and let \( g_3(x) \) be as in (7.4). Let \( A_2 \) be as in (6.8). Then as \( n \to \infty \),

\[
A_0 = \sigma n - (m + \frac{1}{2}) \log(1 + e^{-\sigma m}) + \frac{1}{2} \sigma m \frac{1}{1 + e^{\sigma m}} + B_2 \sigma \frac{1}{1 + e^{\sigma m}} - \frac{B_2}{2} \sigma^3 m \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2} + O(\sigma^3),
\]

for \( 0 \leq m \leq \theta(n) \sqrt{n} \), where \( \theta(n) \to 0 \) as \( n \to \infty \) and where the implied constants depend only upon the function \( \theta \).

Proof. The proof is similar to the proof of Lemma 5.9. The exceptions are outlined as follows.

Lemma 2.12 gives (with \( f_0(x) = \log(1 + e^{-x}) \))

\[
\sum_{j \geq m} f_0(j) = \int_m^\infty f_0(x) \, dx - \frac{1}{2} f_0(m) - \frac{B_2}{2} f_0(m) + \frac{1}{3!} \int_m^\infty B_3([x]) f_0''(x) \, dx
\]

and a modification of Lemma 5.4 to handle third derivatives gives

\[
\int_m^\infty B_3([x]) f_0''(x) \, dx \ll \sigma^3.
\]

Differentiating \( f_0(x) \) with respect to \( x \) gives

\[
f_0'(x) = -\frac{\sigma}{1 + e^{\sigma x}}.
\]

All of (7.13), (7.14), (7.15) combine to yield

\[
A_0 = -E_0 + \frac{I_0(\sigma m)}{\sigma} - \frac{1}{2} \log(1 + e^{-\sigma m}) + \frac{B_2}{2} \sigma \frac{1}{1 + e^{\sigma m}} + O(\sigma^3),
\]

where \( E_0 = \sum_{n \leq m \leq \infty} f_0(j) \) and where \( I_0(x) \) is as in Lemma 7.4. Substituting the expression for \( I_0(x) \) given by Lemma 7.4 into the right hand side of (7.16) gives

\[
A_0 = -E_0 + \frac{I_1(\sigma m)}{\sigma} - (m + \frac{1}{2}) \log(1 + e^{-\sigma m}) + \frac{B_2}{2} \sigma \frac{1}{1 + e^{\sigma m}} + O(\sigma^3).
\]
3. An asymptotic estimate of $\sigma$ for small $m$

We commence with a lemma which indicates the order of magnitude of the parameter $\sigma$ when $m = o(n^{1/2})$.

**Lemma 7.8.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$. Let $\sigma$ be as in (6.4). Then as $n \to \infty$,

$$\sigma \asymp 1/\sqrt{n},$$

for $0 \leq m \leq \theta(n)\sqrt{n}$, where $\theta(n) \to 0$ as $n \to \infty$ and where the implied constants depend only on the function $\theta$.

**Proof.** We use Corollary 3.10. Part (i) gives $\sigma \leq 2/\sqrt{n}$ while Part (ii) gives (with the choice $r = 3\sqrt{n}$ and $m \leq \theta(n)n^{1/2}$) that $\sigma r > \log(r/(2\sqrt{n}))$ and so $\sigma > \log(3/2).1/(3\sqrt{n})$. Thus $\sigma \asymp 1/\sqrt{n}$. \qed

We estimate $\sigma$ to an accuracy of $o(n^{-3/2})$ in the following lemma. The reason for this is that $\sigma n$ need only be estimated within $o(n^{-1/2})$ since this is the size of the error in the estimate of the integral in the previous chapter.

**Lemma 7.9.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$. Let $\sigma$ be as in (6.4). Then as $n \to \infty$,

$$\sigma = R(1 + \sum_{k=2}^{5} U_k(m) R^k + o(n^{-1})), \tag{7.18}$$

for $0 \leq m \leq \theta(n)\sqrt{n}$, where $\theta(n) \to 0$ as $n \to \infty$, $R = \pi/\sqrt{12n}$, and for $2 \leq k \leq 5$, $U_k(x)$ is a polynomial of degree $k$ in $x$. The implied constants depend only on the function $\theta$ and the polynomials are as follows.

$$U_2(m) = -\frac{3}{2\pi^2}m^2 - \frac{3}{2\pi^2}m - \frac{1}{4\pi^2}, \quad U_3(m) = \frac{m^3}{2\pi^2} + \frac{3m^2}{4\pi^2}, \quad U_4(m) = \frac{27m^4}{8\pi^4},$$

$$U_5(m) = \left(-\frac{1}{40\pi^2} - \frac{3}{\pi^4}\right)m^5.$$

**Proof.** Multiplying both sides of (7.10) by $12/\pi^2$ gives

$$12n/\pi^2 = \frac{12}{\pi^2\sigma^2} \left( I_1(\sigma m) - \frac{1}{2} m \sigma^2 \frac{1}{1 + e^{\sigma m}} - \frac{B_2}{2} \sigma^2 \frac{1}{1 + e^{\sigma m}} + \frac{B_2}{2} \sigma^3 m g_2(\sigma m) + O(\sigma^4) \right).$$

Raising both sides of this equation to the power $-1/2$ and noting that $\sigma \asymp 1/\sqrt{n}$ from Lemma 7.8 gives

$$R = \sigma \left(1 - \frac{12}{\pi^2} \int_0^{\sigma m} \frac{x}{1 + \exp(x)} \, dx - \frac{6}{\pi^2} m - \frac{\sigma^2}{1 + \exp(\sigma m)} \right)^{-1/2} \tag{7.19}$$

$$+ \frac{1}{\pi^2} \left(-\frac{\sigma^2}{1 + \exp(\sigma m)} + \frac{\sigma^3 m \exp(\sigma m)}{(1 + \exp(\sigma m))^2} + O(n^{-1/2}) \right)^{-1/2},$$
where $R = \pi/\sqrt{12n}$. We have the following truncated power series expansions which are convergent for all $x \in \mathbb{R}$:

$$\frac{-12}{\pi^2} \int_0^x \frac{z}{1 + \exp(z)} \, dz = \frac{12}{\pi^2} \left( -\frac{1}{4} x^2 + \frac{1}{12} x^3 - \frac{1}{240} x^5 + O(x^7) \right),$$

$$\frac{1}{1 + \exp(z)} = \frac{1}{2} - \frac{1}{4} x^2 + O(x^3).$$

For $m = o(n^{1/3})$ we have

$$(\sigma m)^6 = o(n^{-1}), \quad m\sigma^2 (\sigma m)^2 = o(n^{-1}), \quad \sigma^2 (\sigma m) = o(n^{-7/6}), \quad m\sigma^3 = o(n^{-7/5}).$$

Hence,

$$1 - \frac{12}{\pi^2} \int_0^{\sigma m} \frac{z}{1 + \exp(z)} \, dz - \frac{6}{\pi^2} \frac{\sigma^2}{m^{1/2}} \frac{\sigma^2}{1 + \exp(\sigma m)} + \frac{1}{\pi^2} \left( -\frac{\sigma^2}{1 + \exp(\sigma m)} + \frac{\sigma^3 m \exp(\sigma m)}{(1 + \exp(\sigma m))^2} \right)$$

$$= 1 + \frac{12}{\pi^2} \left( -\frac{1}{4} (\sigma m)^2 + \frac{1}{12} (\sigma m)^3 - \frac{1}{240} (\sigma m)^5 + O((\sigma m)^7) \right)$$

$$- \frac{6m\sigma^2}{\pi^2} \left( \frac{1}{2} - \frac{1}{4} \sigma m + O((\sigma m)^3) \right) - \frac{\sigma^2}{\pi^2} \left( \frac{1}{2} + O(\sigma m) \right) + O(\sigma^3 m)$$

$$= 1 + \left( -\frac{3}{\pi^2} m^2 - \frac{3}{\pi^2} m - \frac{1}{2\pi^2} \right) \sigma^2 + \left( \frac{1}{\pi^2} m^3 + \frac{3}{2\pi^2} m^2 \right) \sigma^3 - \frac{1}{20\pi^2} m^5 \sigma^5 + o(n^{-1})$$

and we can write this as

$$1 + \sum_{k=2}^{5} Q_k(m) \sigma^k + o(n^{-1}),$$

where

$$Q_2(m) = -\frac{3}{\pi^2} m^2 - \frac{3}{\pi^2} m - \frac{1}{2\pi^2}, \quad Q_3(m) = \frac{m^3}{\pi^2} + \frac{3}{2\pi^2} m, \quad Q_4(m) = 0, \quad Q_5(m) = \frac{m^5}{20\pi^2}.$$

In the light of the preceding observations and (7.19) we arrive at

$$\sigma = R \left( 1 + \sum_{k=2}^{5} Q_k(m) \sigma^k + o(n^{-1}) \right)^{1/2}.$$ 

Since $(\sum_{k=2}^{5} Q_k(m) \sigma^k)^3 = o(n^{-1})$, we expand $(1 + \sum_{k=2}^{5} Q_k(m) \sigma^k + o(n^{-1}))^{1/2}$ to 2 terms using the binomial theorem to give

$$\sigma = R \left( 1 + \frac{1}{2} \sum_{k=2}^{5} Q_k(m) \sigma^k + \left( \frac{1}{2} \right)^2 (\sum_{k=2}^{5} Q_k(m) \sigma^k)^2 + o(n^{-1}) \right).$$

But

$$\left( \sum_{k=2}^{5} Q_k(m) \sigma^k \right)^2 = (Q_2 \sigma^2 + Q_3 \sigma^3 + Q_5 \sigma^5)^2 = Q_2^2 \sigma^4 + 2Q_2Q_3 \sigma^5 + o(n^{-1}) = \frac{9m^4}{\pi^4} \sigma^4 - \frac{6m^5}{\pi^4} \sigma^5 + o(n^{-1}).$$

Hence

$$\sigma = R \left( 1 + \frac{1}{2} \sum_{k=2}^{5} Q_k \sigma^k - \frac{9m^4}{8\pi^4} \sigma^4 + \frac{3m^5}{4\pi^4} \sigma^5 + o(n^{-1}) \right),$$

$$= R \left( 1 + \sum_{k=2}^{5} P_k(m) \sigma^k + o(n^{-1}) \right),$$

(7.20)
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where for \( 2 \leq k \leq 5 \), \( P_k(m) \) is a polynomial in \( m \) of degree \( k \). We list the following expressions for \( P_k(x) \), \( 2 \leq k \leq 5 \).

\[
P_k(m) = \left( -\frac{1}{40\pi^2} + \frac{3}{4\pi^4} \right) m^5, \quad P_4(m) = -\frac{9}{8\pi^4} m^4, \quad P_3(m) = \frac{m^3}{2\pi^2} + \frac{3m^2}{4\pi^2}, \quad P_2(m) = \frac{m^2}{2\pi^2} + \frac{3m}{2\pi^2} - \frac{3m}{2\pi^2}.
\]

Let \( P(x, y) = \sum_{k=2}^{5} P_k(x) y^k \).

From Lemma 7.8 we have \( \sigma \asymp 1/\sqrt{n} \), that is to say,

\[
(7.21) \quad \sigma \asymp R.
\]

We employ the method of bootstrapping to express \( \sigma \) in terms of \( m \), \( R \), whereby information about the solution \( \sigma \) is repeatedly fed back into the equation (7.20) defining \( \sigma \) and in so doing, the approximate solution is refined at each iteration (see, for example, Sections 2.4 and 2.5 of de Bruijn [8]).

Substituting (7.21) into the right hand side of (7.20) and noting that \( \sigma \asymp n^{-1/2} \), gives

\[
(7.22) \quad \sigma = R(1 + O(R^2m^2) + o(n^{-1})).
\]

Substituting (7.22) into the right hand side of (7.20) gives

\[
\sigma = R(1 + P(m, R(1 + O(R^2m^2) + o(n^{-1}))) + o(n^{-1}))
\]

\[
= R(1 + P_2(m)R^2 + P_3(m)R^3 + O(R^4m^4) + o(n^{-1})).
\]

Substituting (7.23) into the right hand side of (7.20) gives

\[
\sigma = R(1 + P(m, R(1 + P_2(m)R^2 + P_3(m)R^3 + O(R^4m^4) + o(n^{-1}))) + o(n^{-1}))
\]

\[
= R(1 + P_2(m)R^2 + P_3(m)R^3 + (2P_2(m) + P_4(m))R^4 + (5P_2(m)P_3(m) + P_5(m))R^5 + O(R^6m^6) + o(n^{-1})).
\]

The error term \( O(R^6m^6) \) in (7.24) is \( o(n^{-1}) \) for \( m = o(n^{1/3}) \). We note that

\[
2P_2^2 + P_4 = 2 \left( -\frac{1}{4\pi^2} + \frac{3m^2}{2\pi^2} + \frac{3m}{2\pi^2} \right)^2 - \frac{9m^4}{8\pi^4}
\]

\[
= \frac{27m^4}{8\pi^4} + \frac{9m^3}{\pi^4} + \frac{6m^2}{\pi^4} + \frac{3m}{2\pi^4} + \frac{1}{8\pi^4}
\]

and so

\[
(2P_2^2 + P_4)R^4 = \frac{27m^4}{8\pi^4} R^4 + o(n^{-1}).
\]

Also we note that

\[
(5P_2P_3 + P_5)R^5 = (-15m^5/(4\pi^4) + 3m^5/(4\pi^4) - m^5/(40\pi^2))R^5 + o(n^{-7/6})
\]

\[
= -(3/\pi^4 + 1/(40\pi^2))m^5 R^5 + o(n^{-7/6}).
\]

Thus

\[
\sigma = R(1 + \sum_{k=2}^{5} U_k(m)R^k + o(n^{-1})
\]
where

\[ U_2(m) = P_2(m), \quad U_3(m) = P_3(m), \quad U_4(m) = \frac{27}{8\pi^4} m^4, \quad U_5(m) = -\left(\frac{3}{\pi^4} + \frac{1}{40\pi^2}\right) m^5. \]

\[ \square \]

4. An asymptotic estimate of the parameters for small \( m \)

Firstly we estimate the second cumulant.

**Lemma 7.10.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \), and let \( \sigma \) be as in (6.4), with \( A_2 \) as in (6.8). Then as \( n \to \infty \),

\[ A_2 = \frac{4\sqrt{3}}{\pi} n^{3/2} \times \left(1 + \frac{3m^2}{8n} + o(n^{-1/2})\right), \]

for \( 0 \leq m \leq \theta(n)n^{1/3} \), where \( \theta(n) \) is a function of \( n \) such that \( \theta(n) \to 0 \) as \( n \to \infty \) and where the implied constants depend only on \( \theta \).

**Proof.** Lemma 7.6 gives, for \( m = o(n^{1/2}) \),

\[ A_2 = 2\frac{n}{\sigma} + \frac{m^2}{\sigma} + \frac{1}{1 + e^{\sigma m}} + \frac{m}{\sigma} \frac{1}{1 + e^{\sigma m}} - \frac{1}{2} m^2 g_2(\sigma m) + o(n^{1/2}) = \frac{\pi^2}{6R^2\sigma} + \frac{m^2}{\sigma} \frac{1}{1 + e^{\sigma m}} + \frac{m}{\sigma} g_1(\sigma m) - \frac{m^2}{2} g_2(\sigma m) + o(n^{1/2}). \]

We rewrite this in the form

\[ A_2 = \frac{\pi^2}{6R^2} \times \frac{R}{\sigma} \times \left(1 + \frac{6R^2}{\pi^2} m^2 \frac{1}{1 + e^{\sigma m}} + \frac{6R^2}{\pi^2} m^2 \frac{1}{1 + e^{\sigma m}} - \frac{3R^2}{\pi^2} m^2 g_2(\sigma m) + o(n^{-1})\right). \]

We note that

\[ \frac{6R^2 m^2}{\pi^2} \frac{1}{1 + e^{\sigma m}} = \frac{6R^2 m^2}{\pi^2} (1/2 + O(\sigma m)), \]

and for \( m = o(n^{1/3}) \),

\[ \frac{6R^2 m}{\pi^2} \frac{1}{1 + e^{\sigma m}} = o(n^{-2/3}), \]

\[ \frac{3R^2 \sigma m^2}{\pi^2} \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2} = o(n^{-5/6}), \]

giving for \( m = o(n^{1/3}) \),

\[ A_2 = \frac{\pi^2}{6R^2} \times \frac{R}{\sigma} \times \left(1 + \frac{3R^2 m^2}{\pi^2} + o(n^{-1/2})\right). \]

From (7.18) we are able to write the expression for \( \sigma/R \) with an error of \( o(n^{-1/2}) \), that is for \( m = o(n^{1/3}) \),

\[ \frac{\sigma}{R} = 1 - \frac{3}{2\pi^2} m^2 R^2 + o(n^{-1/2}). \]
Taking reciprocals of both sides and incorporating the resulting expression for \( R/\sigma \) into the above expression for \( A_2 \) gives

\[
A_2 = \frac{\pi^2}{6R^3} \times \left( 1 + \frac{9m^2R^3}{2n^2} + o(n^{-1/2}) \right).
\]

The lemma follows upon substitution of \( R = \pi/\sqrt{12n} \).

We now estimate the logarithm of the product term in (6.14), namely the quantity \( \sigma n + A_0 \).

**Lemma 7.11.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) and \( A_0 \) be as in (6.4) and (7.2). Then as \( n \to \infty \),

\[
\sigma n + A_0 = \frac{\pi^2 \sigma}{6R^2} - (m + 1/2) \log(1 + e^{-\sigma m}) + \frac{\sigma m}{2} \frac{1}{1 + e^{\sigma m}} + B_2 \frac{\sigma}{1 + e^{\sigma m}} - \frac{B_2}{2} \frac{\sigma^2 m}{(1 + e^{\sigma m})^2} + O(\sigma^3)
\]

and substituting \( n = \pi^2 R^2/12 \) throughout the right hand side gives

\[
\sigma n + A_0 = \frac{\pi^2 \sigma}{6R^2} - (m + 1/2) \log(1 + e^{-\sigma m}) + \frac{\sigma m}{2} \frac{1}{1 + e^{\sigma m}} + \frac{1}{6} \frac{\sigma}{1 + e^{\sigma m}} - \frac{1}{12} \frac{\sigma^2 m}{(1 + e^{\sigma m})^2} + O(n^{-3/2}).
\]

(7.25)

We write the right hand side of (7.25) with an error \( o(n^{-1/2}) \). Now for \( m = o(n^{1/3}) \),

\[
\frac{\pi^2 \sigma}{6R^2} = \frac{\pi^2}{6R} \times \frac{\sigma}{R} = \frac{\pi^2}{6R} \times (1 + U_2 R^2 + U_3 R^3 + U_4 R^4 + U_5 R^5 + o(n^{-1})),
\]

\[
m \log(1 + e^{-\sigma m}) = m(\ln 2 - (\sigma m)/2 + (\sigma m)^2/8 - (\sigma m)^4/192 + O((\sigma m)^6))
\]

\[
= m \ln 2 - m^2 \sigma/2 + m^3 \sigma^2/8 - m^5 \sigma^4/192 + o(n^{-2/3}),
\]

\[
\log(1 + e^{-\sigma m}) = \ln 2 - \sigma m/2 + (\sigma m)^2/8 + O(n^{-2/3}),
\]

\[
\frac{\sigma}{1 + e^{\sigma m}} = \frac{\sigma}{2} + o(n^{-2/3}), \quad \frac{\sigma m}{(1 + e^{\sigma m})^2} = o(n^{-2/3}),
\]

\[
\frac{1}{2} \sigma m = (\sigma m)/4 - (\sigma m)^2/8 + o(n^{-2/3}).
\]

We insert all these estimates into the right hand side of (7.25) to give for \( m = o(n^{1/3}) \),

\[
\sigma n + A_0 = \frac{\pi^2}{6R} + U_2 n^2 R^2 + U_3 n^2 R^3 + U_4 n^2 R^4 + U_5 n^2 R^5 + o(n^{-1/2})
\]

\[
- m \ln 2 + m^2 \sigma/2 - m^3 \sigma^2/8 + m^5 \sigma^4/192
\]

\[
- \frac{1}{2} \ln 2 + (\sigma m)/4 - (\sigma m)^2/16 + (\sigma m)/4 - (\sigma m)^2/8 + \frac{15}{2}.
\]

(7.26)
We substitute into (7.26) the estimates (valid for \( m = o(n^{1/3}) \))

\[
\sigma = R + o(n^{-5/6}), \quad \sigma m = Rm + o(n^{-1/2}), \quad (\sigma m)^2 = R^2m^2 + o(n^{-2/3}),
\]
\[
m^2\sigma = m^2R + U_2m^2R^3/2 + U_3m^2R^4 + o(n^{-1/2}),
\]
\[
m^3\sigma^2 = m^3R^2 + 2U_2m^3R^4 + o(n^{-1/2}), \quad m^5\sigma^4 = m^5R^4 + o(n^{-2/3}),
\]
yielding

\[
\sigma n + A_0 = \frac{\pi^2}{6R} + U_2\frac{\pi^2}{6}R + U_3\frac{\pi^2}{6}R^2 + U_4\frac{\pi^2}{6}R^3 + U_5\frac{\pi^2}{6}R^4 + o(n^{-1/2})
\]
\[- m \ln 2 + m^2R^2/2 + U_2m^2R^3/2 + U_3m^2R^4/2
\]
\[- m^3R^2/8 - U_2m^3R^4/4 + m^5R^4/192 - \frac{1}{2} \ln 2
\]
\[- + Rm/2 - 3R^2m^2/16 + R/12 + o(n^{-1/2}).
\]

Using \( m = o(n^{1/3}) \) and \( R = \pi/\sqrt{12n} \) simplifies the right hand side of the above equation to

\[
\frac{\pi^2}{6R} - \frac{3}{16\pi^2}m^4R^3 + (1/960 + 1/(8\pi^2))m^5R^4 + o(n^{-1/2}).
\]

\( \square \)

This concludes our estimation of the parameters and thus we are in a position to prove our result immediately.

## 5. Proof of Theorem 7.1

We prove Theorem 7.1 in this section as follows. Into (6.12) we substitute the estimates of the integral and the product term provided in Theorem 6.1 and Lemma 7.11 to give as \( n \to \infty \),

\[
q(m, n) = \exp(\sigma n + A_0)\frac{1}{\sqrt{2\pi A_2}}\left(1 - \frac{3\sqrt{3}}{8\pi}n^{-1/2} + o(n^{-1/2})\right),
\]

for \( m = o(n^{1/2}) \). From Lemma 7.6, as \( n \to \infty \),

\[
\frac{1}{\sqrt{2\pi A_2}} = 2^{-3/2}2^{-1/4}n^{-3/4}\left(1 + \frac{3m^2}{8n} + o(n^{-1/2})\right)^{-1/2} = 2^{-3/2}2^{-1/4}n^{-3/4}\left(1 - \frac{3m^2}{16n} + o(n^{-1/2})\right),
\]

for \( m = o(n^{1/2}) \). Substituting this and the estimate of \( \sigma n + A_0 \) from Lemma 7.11, valid for \( m = o(n^{1/3}) \), into (7.27) gives as \( n \to \infty \),

\[
q(m, n) = \exp\left(\frac{\pi^2}{6R} - (m + 1/2)\ln 2 + (m^2/4 + m/4 + 1/24)R + (-m^3/24 - m^2/16)R^2 \right.
\]
\[
- \frac{3}{16\pi^2}m^4R^3 + (1/960 + 1/(2\pi^2))m^5R^4 \times
\]
\[
2^{-3/2}2^{-1/4}n^{-3/4}\left(1 - 3\sqrt{3}n^{-1/2}/(8\pi) - \frac{3m^2}{16n} + o(n^{-1/2})\right),
\]

for \( m = o(n^{1/3}) \). Theorem 7.1 follows upon substitution of \( R = \pi/\sqrt{12n} \).
6. Comparison with other results for small $m$

By taking $m = 0$ in Theorem 7.1 we obtain the following corollary.

**Corollary 7.12.** As $n \to \infty$,

$$q(n) = 2^{\frac{1}{2}}3^{-\frac{3}{4}}n^{-\frac{3}{4}}\exp(\pi \sqrt{n/3}) \times \left(1 + \frac{\pi}{48\sqrt{3\sqrt{n}}} - \frac{3\sqrt{3}}{8\pi\sqrt{n}} + o(n^{-1/2})\right),$$

where the implied constants are absolute.

We now look at how this fits in with Hua's result, Theorem B of Section 3 of Chapter 1.

We note that (see Section 17.7, page 372 of Whittaker & Watson [45])

$$\frac{d}{dz} J_0(iz) = J_1(iz)/i$$

and when $-3\pi/2 < \arg z < \pi/2$ as $z \to \infty$,

$$i^{-n} J_n(iz) = \frac{e^z}{\sqrt{2\pi z}} \left(1 + \sum_{r=1}^{\infty} (-1)^r \frac{\left(4n^2 - 1^2\right)\left(4n^2 - 3^2\right)\ldots\left(4n^2 - (2n-1)^2\right)}{r!2^{2r}z^{2r}}\right)$$

$$+ \frac{e^{-(n+1/2)\pi i e^{-z}}}{\sqrt{2\pi z}} \left(1 + \sum_{r=1}^{\infty} \frac{\left(4n^2 - 1^2\right)\left(4n^2 - 3^2\right)\ldots\left(4n^2 - (2n-1)^2\right)}{r!2^{2r}z^{2r}}\right).$$

Using these results it can be deduced from Theorem B that as $n \to \infty$,

$$q(n) = 2^{\frac{1}{2}}3^{-\frac{3}{4}}n^{-\frac{3}{4}}\exp(\pi \sqrt{n/3}) \times \left(1 + \frac{\pi}{48\sqrt{3\sqrt{n}}} - \frac{3\sqrt{3}}{8\pi\sqrt{n}} + O(n^{-1})\right).$$

We see that Corollary 7.12 is the same as this apart from having a larger error term $o(n^{-1/2})$ instead of $O(n^{-1})$.

By taking $m = o(n^{1/2})$ in Theorem 7.1 and enlarging the error term to $o(1)$ we deduce the following corollary.

**Corollary 7.13.** With $q(m, n)$ as in (1.2) as $n \to \infty$,

$$q(m, n) \sim 2^{-m-2}3^{-\frac{3}{4}}n^{-\frac{3}{4}}\exp\left(\frac{m}{3} + \frac{\pi m^2}{8\sqrt{3n}}\right),$$

for $m = o(n^{1/3})$, where the implied constants are absolute.

We see this is equivalent to the result of Freiman and Pitman stated as Proposition J in Section 1 of Chapter 6.

Finally, we take $k = 1$ in Theorem 5.1 to yield the following corollary.

**Corollary 7.14.** With $q(m, n)$ as in (1.2) as $n \to \infty$,

$$q(m, n) = 2^{-m-2}3^{-\frac{3}{4}}n^{-\frac{3}{4}}\exp\left(\frac{m}{3} + \frac{\pi m^2}{8\sqrt{3n}}\right)(1 + O(n^{-1/4})),$$

for $0 \leq m \leq n^{1/4}$, where the implied constants are absolute.
We see that this is consistent with Theorem 7.1 when the range of \( m \) is restricted to \( 0 \leq m \leq n^{1/4} \).

7. Preliminary lemmas for large \( m \)

We estimate the quantities \( A_0, A_1, A_2 \) for large \( m \) to a greater degree of accuracy than that done in Chapter 6. The following lemma gives a bound on the remainder term in the Euler–Maclaurin Summation for the sum \( \sum_j f_k(j) \) when \( n/2 > m \geq \psi(n)\sqrt{n} \) with \( \psi(n) \to \infty \) as \( n \to \infty \).

**Lemma 7.15.** Let \( k \in \mathbb{N} \). Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( f_k(x) \) be as in (5.5), with \( \sigma \) as in (6.4). Let \( B_3(x) = x^3 + x^2 + x \) be the third Bernoulli polynomial. Then as \( n \to \infty \),

\[
\int_{m}^{\infty} B_3([x]) f_k''(x) dx \ll \sigma^3 m^k e^{-\sigma m},
\]

for \( \psi(n)\sqrt{n} \leq m < n/2 \), where \( \psi(n) \) is a function of \( n \) such that \( \psi(n) \to \infty \) as \( n \to \infty \), and where the implied constants depend only on \( k \) and \( \psi \).

**Proof.** The proof is similar to the proof of Lemma 5.4. In brief, we consider an integral (in connection with the integral in (7.28)) of the form

\[
I_{a,b} = \int_{m}^{\infty} B_3([x]) \frac{x^a}{(1 + \exp(\sigma x))^b} dx,
\]

where \( a, b \) are integers such that \( b \geq 1 \) and observe (by Lemma 5.2) that the function

\[
\frac{x^a}{(1 + \exp(\sigma x))^b}
\]

is monotone decreasing on the interval \([m, \infty)\) for \( m \geq \psi(n)\sqrt{n} \). Noting that \( B_3([x]) \) is periodic with period 1 and alternates in sign on the interval \([0, 1]\) allows a result similar to Lemma 2.15 to be applied to the integral \( I_{a,b} \), yielding

\[
I_{a,b} \ll \frac{m^a}{(1 + \exp(\sigma m))^b} \times m^a e^{-\sigma m b}.
\]

Thus for integers \( l \) and \( h \) such that \( 0 \leq l \leq 3, 1 \leq h \leq k + l \)

\[
\sigma^l I_{k+l-3,h} \ll \sigma^l m^{k+l-3} e^{-\sigma m h},
\]

(see proof of Lemma 5.4 for the connection between \( \sigma^l I_{k+l-3,h} \) and the integral in (7.28)) and the result follows by choosing \( l = 3 \) and \( h = 1 \) which maximises the right hand side of the above inequality. \( \square \)

We now estimate the tail of the series \( \sum_j f_k(j) \).
Lemma 7.16. Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (6.4). Let $k$ be a positive integer and let $f_k(j)$ be as in (5.5). Then as $n \to \infty$,

$$\sum_{n<j<n} f_k(j) \ll m^{k+1}e^{-2\sigma m}.$$  

for $m \geq \psi(n)^{\sqrt{n}}$, where $\psi(n)$ is a function of $n$ which tends to $\infty$ as $n \to \infty$ and where the implied constants depend only on $k$ and $\psi$.

**Proof.** By the proof of Lemma 3.2 $f_k(j)$ can be written as a linear combination, with bounded coefficients, of the functions

$$\frac{1}{1+e^{\sigma j}}, \quad \frac{1}{(1+e^{\sigma j})^2}, \quad \cdots, \quad \frac{1}{(1+e^{\sigma j})^k}.$$  

Therefore it suffices to estimate the sum

$$\sum_{n<j<n} j^k(1+e^{\sigma j})^{-l}.$$  

Employing Lemma 2.11 with $\phi(z) = z^k/(1+e^{\sigma z})^l$ and $U(z) = z$ gives

$$(7.29) \quad \sum_{n<j<n} j^k(1+e^{\sigma j})^{-l} = - \int_n^\infty U(z)\phi'(z)dz - \phi(n)U(n).$$  

Because

$$\phi'(z) = \frac{kz^{k-1}}{(1+e^{\sigma z})^l} - l\sigma z^k \left( \frac{1}{(1+e^{\sigma z})^l} - \frac{1}{(1+e^{\sigma z})^{l+1}} \right)$$

the right hand side of (7.29) is

$$\leq \int_n^\infty \frac{l\sigma z^{k+1}}{(1+e^{\sigma z})^l}dz$$

$$< \int_n^\infty \frac{l\sigma z^{k+1}}{e^{\sigma z}}dz.$$  

Making the substitution $y = \sigma z$ gives us that this is equal to

$$\sigma^{k-1} \int_{\sigma n}^\infty y^{k+1}e^{-y}dy$$

$$< (k+2)! l\sigma^{k-1} (\sigma n)^{k+1} e^{-\sigma n}$$

$$= (k+2)! n^{k+1} e^{-\sigma n}.$$  

Since $z^{k+1}e^{-\sigma z}$ is decreasing on the interval $[(k+1)/(l\sigma), \infty)$ this is

$$\leq (k+2)! (2m)^{k+1} e^{-2\sigma m}.$$  

When $l = 1$ this upper bound in maximised and we have the result.  \[ \square \]

We define an auxiliary function $F_m(z)$ as

$$(7.30) \quad F_m(z) = z^{-2}I_1(z) - \left( \frac{1}{2m} + \frac{1}{12m^2} \right) \frac{1}{1+e^z} + \frac{1}{12m^2z} \frac{e^z}{(1+e^z)^2}.$$
Freiman and Pitman [16] used an auxiliary function $F(x)$ not involving $m$ in connection with estimating $\sigma$ for large $m$. By including $m$ in the auxiliary function $F_m$ defined here, an improvement on the result of Freiman and Pitman will be made.

We provide in the next lemma, an expression for $n$ (and hence the first cumulant $A_1$) in terms of $F_m(x)$, $m$ and the quantity $\sigma m$, with an error term.

**Lemma 7.17.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (6.4). Let the auxiliary function $F_m$ be as in (7.30). Then as $n \to \infty$,

$$n = m^2 F_m(\sigma m) + O(\sigma^3 m e^{-\sigma m} + m^2 e^{-2\sigma m}), \tag{7.31}$$

for $\psi(n)\sqrt{n} \leq m$, $\psi(n)$ being a function of $n$ which tends to $\infty$ as $n \to \infty$, and where the implied constants depend only on $\psi$.

**Proof.** We argue in a manner as done for the proof of Lemma 7.5 except that we apply two different lemmas in the process: We apply Lemma 7.15 to give (with $M = 1$ and $k = 1$)

$$\int_{m}^{\infty} B_3([y]) f_m''(x) dx \ll \sigma^3 m e^{-\sigma m}, \tag{7.32}$$

and we apply Lemma 7.16 to give

$$\sum_{n < j < \infty} f_1(j) \ll m^2 e^{-\sigma m}.$$

Finally, we observe that the non-error terms in the resulting expression for $A_1$ can be written as $m^2 F_m(\sigma m)$ and the proof is complete. $\square$

We express the second cumulant $A_2$ in terms of the derivative of the auxiliary function $F_m$, $m$ and $\sigma m$, with an error term.

**Lemma 7.18.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (6.4). Let $A_2$ be defined as in (6.8). Let $I_1(x)$ be the integral as in Lemma 7.4 and let the auxiliary function $F_m$ be defined as in (7.30). Then

$$- F_m''(x) = x^{-1} \frac{1}{1 + e^{x}} + 2x^{-3} I_1(x) - \left( \frac{1}{2m} + \frac{1}{6m^2} \right) \frac{e^{x}}{(1 + e^{x})^2} + \frac{1}{12m^2} \frac{e^{x}(e^{x} - 1)}{(1 + e^{x})^3}, \tag{7.33}$$

and we have as $n \to \infty$,

$$A_2 = -m^3 F_m'(\sigma m) + O(\sigma^3 m^2 e^{-\sigma m} + m^3 e^{-2\sigma m}), \tag{7.34}$$

for $\psi(n)\sqrt{n} \leq m < n/2$, $\psi(n)$ being a function of $n$ which tends to $\infty$ as $n \to \infty$, where the implied constants depend only on $\psi$. 

7. PARTITIONS INTO POSITIVE INTEGERS — ESTIMATING THE PRODUCT TERM

PROOF. Using (6.8) we write

\[ A_2 = \sum_{j > m} f_2(j) - \sum_{n < j < \infty} f_3(j). \]

Lemma 2.12 gives

\[ \sum_{j > m} f_2(j) = \int_m^\infty f_2(x)dx - \frac{1}{2}f_2(m) - \frac{B_2}{2}f_2(m) + \frac{1}{3!}\int_m^\infty B_3'(\{x\})f_2''(x)dx. \]

A modification of Lemma 7.15 to handle third derivatives rather than first derivatives gives (with \( k = 2 \))

\[ \int_m^\infty B_3'(\{x\})f_2''''(x)dx \ll \sigma^3 m^2 e^{-\sigma m}. \]

Also we have from (7.3) that

\[ f_2'(x) = 2x \frac{e^x}{(1 + e^x)^2} - \sigma x^2 \frac{e^x(e^x - 1)}{(1 + e^x)^3}. \]

Lemma 7.16 shows that \( \sum_{n < j < \infty} f_2(j) \ll m^3 e^{-2\sigma m}. \)

All of the above statements combine to yield

\begin{equation}
A_2 = \frac{I_2(\sigma m)}{\sigma^3} - \frac{1}{2} m^2 \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2} - \frac{B_2}{2} \frac{m^2}{(1 + e^{\sigma m})^2} - \sigma m \frac{e^{\sigma m} - 1}{(1 + e^{\sigma m})^3} + O(\sigma^3 m^2 e^{-\sigma m} + m^3 e^{-2\sigma m}),
\end{equation}

where \( I_2(x) \) is as in Lemma 7.4. By substituting the expression for \( I_2(x) \) in Lemma 7.4, we can simplify (7.35) to

\begin{equation}
A_2 = \frac{m^2}{\sigma} \frac{1}{1 + e^{\sigma m}} + 2 \frac{I_1(\sigma m)}{\sigma^3} - \frac{m^2}{2} \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2} - B_2 \frac{m^2}{(1 + e^{\sigma m})^2}
+ \sigma^3 m^2 e^{-\sigma m} + \sigma m^3 e^{-2\sigma m}).
\end{equation}

Noting that the non-error terms in (7.36) can be written as \(-m^3 F_m'(\sigma m)\) gives the result. □

The following lemma assists in providing an expression for the logarithm of the product term, denoted \( \sigma n + A_0 \). The quantity \( A_0 \) is expressed in terms of the auxiliary function \( H_m(x) \) with some error terms.

We define the auxiliary function \( H_m(x) \) as

\begin{equation}
H_m(x) = x^{-1} I_1(x) - \left(1 + \frac{1}{2m}\right) \log(1 + e^{-x}) + \frac{1}{12m^2} x^2 \frac{1}{1 + e^x}. \end{equation}

\textbf{Lemma 7.19.} Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (6.4). Let \( A_0 \) be defined as in (7.2) and let \( H_m(x) \) be as in (7.37). Then as \( n \to \infty \),

\begin{equation}
A_0 = m H_m(\sigma m) + O(\sigma^3 e^{-\sigma m}) + O(me^{-2\sigma m}),
\end{equation}

for \( \psi(n) \sqrt{n} \leq m \leq n/2 \) with \( \psi(n) \to \infty \) as \( n \to \infty \), where the implied constants depend only on \( \psi \) and where \( I_1(x) \) is as in Lemma 7.4.
proof. From (7.2) we have

\[ A_0 = \sum_{m<j<\infty} f_0(j) - \sum_{n<j<\infty} f_0(j). \]

and we proceed in an analogous manner to that done in the proof of Lemma 7.18 to arrive at

\[ A_0 = \frac{I_1(\sigma m)}{\sigma} - (m + \frac{1}{2}) \log(1 + e^{-\sigma m}) + \frac{B_2}{2} \frac{1}{1 + e^{\sigma m}} + O(\sigma^2 e^{-\sigma m} + O(me^{-2\sigma m}). \]

Noting that the non-error terms of the expression in the right hand side above can be replaced by \(H_m(\sigma m)\) completes the proof. \(\square\)

The following lemma gives both an idea of the asymptotic behaviour and the local behaviour of \(F_m(x)\) and \(G_m(x)\), where \(F_m\) is given by (7.30) and

\[ G_m(x) = 2x^{-1} I_1(x) - \left(1 + \frac{1}{2m}\right) \log(1 + e^{-x}) - \frac{1}{2m} x \frac{1}{1 + e^x} + \frac{1}{12m^2 x ^2 \left(1 + e^x\right)^2}. \]

**Lemma 7.20.** Let \(F_m\) and \(G_m\) be as in (7.30) and (7.39). Then the following statements hold true, with all implied constants being absolute (and hence independent of \(a, b, m\) and \(x\)).

(i) Let \(a\) and \(b\) be positive real numbers such that

\[ |b - a| < a/2, \quad F_m^{-1}(a) \geq 1, \quad F_m^{-1}(b) \geq 1. \]

Then

\[ |F_m^{-1}(b) - F_m^{-1}(a)| \ll \frac{1}{a} |b - a|. \]

(ii) Let \(a\) and \(b\) be positive real numbers such that \(|b - a| < a/2\). Then

\[ |G_m(b) - G_m(a)| \ll a |F_m(b) - F_m(a)|, \]

\[ |F_m'(b) - F_m'(a)| \ll |F_m(b) - F_m(a)|. \]

(iii) For \(x \geq 1,\)

\[ F_m(x) = (x^{-1} + x^{-2} - (12m^2)^{-1} - (2m)^{-1} + x(12m^2)^{-1}) e^{-x} \]

\[ + (-2^{-1} x^{-1} - 4^{-1} x^{-2} + (12m^2)^{-1} + (2m)^{-1} - x(6m^2)^{-1}) e^{-2x} \]

\[ + O((x^{-1} + m^{-1} + x^2 m^{-2}) e^{-3x}), \]

\[ G_m(x) = (1 + 2x^{-1} - (2m)^{-1} - x(2m)^{-1} + x^2 (12m^2)^{-1}) e^{-x} \]

\[ + (-1/2 - (2x)^{-1} + (4m)^{-1} + x(2m)^{-1} - x^2 (6m^2)^{-1}) e^{-2x} \]

\[ + O((1 + x/m + x^2 /m^2) e^{-3x}) \]

\[ = x F_m(x) \frac{x + 2}{x + 1} + O(x e^{-2x}), \]
\[ F'_m(x) = (-2/x^2 - 2/x - 1/x + 2^{-1}m^{-1} + 6^{-1}m^{-2} - x(12m^2)^{-1})e^{-x} \\
+ (x^{-2} + 2^{-1}x^{-3} + 1/x - 1/m - 3^{-1}/m^2 + x(3m^2)^{-1})e^{-2x} \\
+ O((1/x^2 + 1/x + 1/m + x/m^2)e^{-3x}). \]

**Proof.** (i) By the mean value theorem

\[ F_m^{-1}(b) - F_m^{-1}(a) = (b - a)/F'_m(F_m^{-1}(c)) \]

for some \( c \) between \( a \) and \( b \). It is easily checked that \( F'_m(x) > 0 \) for \( x > 0 \) (so that \( F'_m \) is strictly monotonic) and that for \( x \geq 1 \),

\[ F_m(x) \ll |F'_m(x)| \ll F(x). \]

Therefore

\[ |F'_m(F_m^{-1}(c))| \gg F_m(F_m^{-1}(c)) = c \gg a, \]

and the required inequality follows.

(ii) By Cauchy's mean value theorem

\[ G(b) - G(a) = (F(b) - F(a))G'(c)/F'(c), \]

for some \( c \) between \( a \) and \( b \).

It is easily checked that for \( x > 0 \) we have \( G'(x) = xF'(x) \).

Hence

\[ G'(c)/F'(c) = c \ll a. \]

(iii) All the results follow from the following estimates for \( x > 0 \).

\[ I_1(x) = (x + 1)e^{-x} - (x/2 + 1/4)e^{-2x} + O(xe^{-3x}), \]

\[ \log(1 + e^{-x}) = e^{-x} - \frac{1}{2} e^{-2x} + O(e^{-3x}), \]

\[ \frac{1}{1 + e^{-x}} = e^{-x} - e^{-2x} + O(e^{-3x}), \]

\[ \frac{e^x}{(1 + e^{-x})^2} = e^{-x} - 2e^{-2x} + O(e^{-3x}), \]

\[ \frac{e^x(e^x - 1)}{(1 + e^{-x})^3} = e^{-x} - 4e^{-2x} + O(e^{-3x}). \]

\( \square \)

We are able to use this lemma in the estimation of \( \sigma \), the second cumulant and the estimation of the logarithm of the product term.
8. An asymptotic estimate of \( \sigma \) for large \( m \)

We commence with a lemma which indicates the order of magnitude of the parameter \( \sigma \).

**Lemma 7.21.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Let \( \sigma \) be as in (6.4). Then as \( n \to \infty \),

\[
\sigma \sim \frac{1}{m} \log(m^2/n),
\]

for \( \psi(n) \sqrt{n} \leq m \leq n/2 \), where \( \psi(n) \) is a function of \( n \) such that \( \psi(n) \to \infty \) as \( n \to \infty \).

**Proof.** From the equation

\[
A_1 = \frac{1}{m} e^{-\sigma m} (1 + O((\sigma m)^{-1}))
\]

we have that

\[
n = \frac{1}{m} e^{-\sigma m} (1 + O((\sigma m)^{-1}))
\]

and dividing both sides by \( m^2 \), taking reciprocals and then logarithms gives

\[
\log(m^2/n) = \log(\sigma m) + \sigma m + O((\sigma m)^{-1}) = \sigma m \left(1 + \frac{\log(\sigma m)}{\sigma m} + O((\sigma m)^{-2})\right).
\]

Since \( \sigma m \to \infty \) as \( n \to \infty \) for \( m \geq \psi(n) \sqrt{n} \) (see Lemma 6.10) the required asymptotic relation follows. \( \square \)

The following lemma enables \( \sigma \) to be estimated with a considerably greater degree of accuracy than the previous lemma.

**Lemma 7.22.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \) and let \( \sigma \) be as in (6.4). Let \( F_m(x) \) be as in (7.30). Then as \( n \to \infty \),

\[
(7.40) \quad \sigma = \frac{1}{m} F_m^{-1}(n/m^2) + O(\sigma^3 n^{-1} e^{-\sigma m} + n^{-1} e^{-2\sigma m}),
\]

for \( \psi(n) \sqrt{n} \leq m \leq n/2 \), where \( \psi(n) \) is a function of \( n \) such that \( \psi(n) \to \infty \) as \( n \to \infty \) and where the implied constants depend on \( \psi \) only.

**Proof.** Using (6.4) gives

\[
(7.41) \quad n = \frac{I_1(\sigma m)}{\sigma^2} - \frac{1}{2} m^{-1} - \frac{1}{1 + e^{\sigma m}} - \frac{B_2}{2} \left(\frac{1}{1 + e^{\sigma m}} - \sigma m \frac{e^{\sigma m}}{(1 + e^{\sigma m})^2}\right) + O(\sigma^3 m e^{-\sigma m} + m^2 e^{-2\sigma m}),
\]

where \( I_1(x) = \int_{-\infty}^{\infty} \frac{e^x}{1 + \exp(x)} \, dx \) and \( B_2 = \frac{1}{6} \). Dividing both sides of (7.41) by \( m^2 \) and letting \( X = \sigma m \) gives

\[
\frac{n}{m^2} = \frac{I_1(X)}{X^2} - \frac{1}{2m} \frac{1}{1 + e^X} - \frac{1}{12m^2} \left(\frac{1}{X} - X \frac{e^X}{(1 + e^X)^2}\right) + O(\sigma^3 e^{-\sigma m} / m + e^{-2\sigma m}).
\]
Then with $F_m(x)$ as in (7.30)

\begin{equation}
\frac{n}{m^2} = F_m(\sigma m) + O(\sigma^3 e^{-\sigma m}/m + e^{-2\sigma m}).
\end{equation}

But from Lemma 7.20(i)

$$|F_m^{-1}(n/m^2) - F_m^{-1}(F_m(\sigma m))| \ll \frac{1}{n/m^2} |n/m^2 - F_m(\sigma m)|,$$

and the result follows. □

9. Towards asymptotic estimates of parameters for large $m$

The following is a corollary to Lemma 7.18.

**Corollary 7.23.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$. Let $\sigma$ be as in (6.4). Then with $F_m'(x)$ as in (7.33) we have as $n \to \infty$,

$$A_2 = -m^2 F_m(F_m^{-1}(n/m^2)) + O(\sigma^3 m^2 e^{-\sigma m} + m^3 e^{-2\sigma m}),$$

for $\psi(n)\sqrt{n} \leq m \leq n/2$, where $\psi(n)$ is a function of $n$ such that $\psi(n) \to \infty$ as $n \to \infty$ and where the implied constants depend only on $\psi$.

**Proof.** From (7.40) we have

$$\sigma m - F_m^{-1}(n/m^2) \ll \sigma^3 mn^{-1} e^{-\sigma m} + m^2 n^{-1} e^{-2\sigma m}.$$  

Using this equation, Lemma 7.20(ii) and Lemma 7.18 in combination with the inequality

$$|A_2 + F_m'(F_m^{-1}(n/m^2))| \leq |A_2 + F_m'(\sigma m)| + |F_m'(\sigma m) - F_m'(F_m^{-1}(n/m^2))|,$$

gives the result. □

We now estimate the logarithm of the product term, that is, $\sigma n + A_0$.

**Lemma 7.24.** Let $m$ and $n$ be integers such that $0 \leq m < n/2$ and let $\sigma$ be as in (6.4). Let $F_m$ and $G_m$ be as in (7.30) and (7.39). Then as $n \to \infty$,

$$\sigma n + A_0 = mG_m(F_m^{-1}(n/m^2)) + O(\sigma^4 m e^{-\sigma m} + \sigma m^2 e^{-2\sigma m}),$$

for $\psi(n)\sqrt{n} \leq m \leq n/2$, where $\psi(n)$ is a function of $n$ such that $\psi(n) \to \infty$ as $n \to \infty$ and where the implied constants depend only on $\psi$. 


10. PROOF OF THEOREM 7.2

PROOF. Multiplying the expression for $n$ in Lemma 7.17 by $\sigma$, adding this to the expression for $A_0$ in Lemma 7.19 and observing that $G_m(x) = xF_m(x) + H_m(x)$ gives

\begin{equation}
(7.43) \quad \sigma n + A_0 = mG_m(\sigma m) + O(\sigma^4 m e^{-\sigma m}) + O(\sigma m^2 e^{-2\sigma m}).
\end{equation}

Using the bound on $\sigma m - F_m^{-1}(n/m^2)$ from (7.40), using Lemma 7.20(ii) and combining (7.43) with the inequality

\[ |\sigma n + A_0 - mG_m(F_m^{-1}(n/m^2))| \leq |\sigma n + A_0 - mG_m(\sigma m)| + m|G_m(\sigma m) - G_m(F_m^{-1}(n/m^2))|, \]

gives the result. \( \square \)

10. Proof of Theorem 7.2

We prove Theorem 7.2 in this section. We assume from now on that $\psi(n) \sqrt{n} \leq m \leq n/(\log n)^9$. Theorem 6.2 combined with (6.12) give us $n \to \infty$,

\begin{equation}
(7.44) \quad q(m, n) = \exp(\sigma n + A_0)\frac{1}{\sqrt{2\pi A_2}}(1 - \frac{1}{12} \frac{m}{n} + O(m/n)).
\end{equation}

We shall use Lemma 7.24 to estimate $\exp(\sigma n + A_0)$ and Corollary 7.23 to estimate $A_2$. We shall subdivide the proof into three parts and conclude the proof by combining the results from each part. The precision to which each of the quantities is estimated is dictated by the stipulation that the term $-\frac{1}{12} m/n$ in (7.44) contribute to the estimate of $q(m, n)$.

Suppose that

\begin{equation}
(7.45) \quad \left| m - \sqrt{\frac{n}{F}} \right| \leq E,
\end{equation}

where

\begin{equation}
(7.46) \quad F = \left( \frac{1}{\nu \log n} + \frac{1}{\nu^2 \log^2 n} \right) n^{-\nu} - \left( \frac{1}{2\nu \log n} + \frac{1}{4\nu^2 \log^2 n} \right) n^{-2\nu},
\end{equation}

$\nu$ is such that $0 < \nu < 1$, and where $E$ is as yet an undetermined function of $n$ aside from requiring that it be $o(\sqrt{n/F})$. We note that, with $F_m$ as in (7.30) and $F$ as in (7.46),

\[ F_m(\nu \log n) = F + O((\log n)^{-1} n^{-2\nu}) \]

and that by factorising $n/F - m^2$ as a difference of two squares and using (7.45) gives

\[ \frac{n}{m^2} - F = \frac{F}{m^2}(n/F - m^2) \ll \frac{F}{m} E. \]

We deduce from (7.45) that

\begin{equation}
(7.47) \quad m \sim (\nu \log n)^{1/2} n^{(1+\nu)/2},
\end{equation}
and in order that the term \(-1/3 m/n\) in the right hand side of (7.44) contribute, we need to estimate quantities with an error \(o(m/n) \approx o((\log n)^{1/2} n^{\nu - 1/2})\).

**Estimating \(\sigma m\)**

With the condition (7.45) on the value of \(m\), we have the following lemma.

**Lemma 7.25.** Let \(m\) and \(n\) be integers such that \(0 \leq m < n/2\). Then for \(m\) as in (7.45), where \(F\) is as in (7.46) and \(E = o(\sqrt{n/F})\), we have as \(n \to \infty\), \(\sigma m - \nu \log n = o(1)\).

**Proof.** By the triangle inequality we have

\[
|\sigma m - \nu \log n| \leq |\sigma m - F_m^{-1}(n/m^2)| + |F_m^{-1}(n/m^2) - \nu \log n|.
\]

We apply Lemma 7.20(i) with \(a = n/m^2\) and \(b = F_m(\sigma m)\) to give

\[
\sigma m - F_m^{-1}(n/m^2) \ll \frac{m^2}{n} |F_m(\sigma m) - n/m^2|.
\]

From (7.42) this is

\[
\ll \frac{m^2}{n} (\sigma^3 e^{-\sigma m} / m + e^{-2\sigma m}).
\]

Applying Lemma 7.20(i) with \(a = n/m^2\) and \(b = F_m(\nu \log n)\) gives

\[
|F_m^{-1}(n/m^2) - \nu \log n| \ll \frac{m^2}{n} |n/m^2 - F_m(\nu \log n)|
\]

and thus

\[
|\sigma m - \nu \log n| \ll \sigma^3 m n^{-1} e^{-\sigma m} + m^2 n^{-1} e^{-2\sigma m} + \frac{1}{n/m^2} |n/m^2 - F_m(\nu \log n)|
\]

\[
\ll \sigma^3 m n^{-1} e^{-\sigma m} + m^2 n^{-1} e^{-2\sigma m} + \frac{1}{n/m^2} |E n^{-\nu}/(m \log n) + n^{-3\nu}/\log n|
\]

\[= o(1).\]

\(\square\)

**Estimating \(\sigma n + A_0\)**

We estimate \(\sigma n + A_0\) with \(m\) as in (7.45). Now Lemma 7.20(ii) gives

\[
m G_m(F_m^{-1}(n/m^2)) - m G_m(\nu \log n) \ll m \nu \log n \times |n/m^2 - F_m(\nu \log n)|
\]

\[
\ll E n^{-\nu} + m n^{-3\nu}.
\]

This inequality combined with Lemma 7.24 yields

\[
\sigma n + A_0 = m G_m(\nu \log n) + O(En^{-\nu} + mn^{-3\nu}) + O(\sigma^4 me^{-\sigma m} + \sigma m^2 e^{-2\sigma m}).
\]
From Lemma 7.20(iii)

\[ mG_m(\nu \log n) = m \left( 1 + 2/(\nu \log n) - \frac{1}{2m} \frac{\nu \log n}{2m} + \frac{\nu^2 \log^2 n}{12m^2} \right) n^{-\nu} \]

\[ + m \left( -\frac{1}{2} - \frac{1}{2\nu \log n} + \frac{1}{4m} + \frac{\nu \log n}{2m} - \frac{\nu^2 \log^2 n}{6m^2} \right) n^{-2\nu} + O(mn^{-3\nu}). \]

Using the estimate of \( \sigma m \) in Lemma 7.25 we obtain

\[
\sigma n + A_0 = m \left( 1 + 2/(\nu \log n) - \frac{1}{2m} \frac{\nu \log n}{2m} + \frac{\nu^2 \log^2 n}{12m^2} \right) n^{-\nu} \\
+ m \left( -\frac{1}{2} - \frac{1}{2\nu \log n} + \frac{1}{4m} + \frac{\nu \log n}{2m} - \frac{\nu^2 \log^2 n}{6m^2} \right) n^{-2\nu} \\
+ O(En^{-\nu} + mn^{-3\nu}) + O((\log n)^4 n^{-\nu} m^{-3} + (\log n)^3 n^{-2\nu} m^{-1}).
\] (7.48)

We use (7.47) to refine the error estimates in (7.48) to give

\[
\sigma n + A_0 = m(1 + 2/(\nu \log n)) n^{-\nu} - \frac{m}{2} (1 + 1/(\nu \log n)) n^{-2\nu} \\
+ O(En^{-\nu} + (\log n)^{1/2} n^{(1-5\nu)/2} + (\log n)^{3/2} n^{-(3+5\nu)/2} + (\log n)^3 n^{-(1+5\nu)/2}).
\] (7.49)

In order that all the error terms in the above expression be \( o((\log n)^{1/2} n^{(\nu-1)/2}) \) it is sufficient that \( \nu > 1/3 \) and that \( E = o((\log n)^{1/2} n^{3\nu-1/2}) \).

**Estimating \( A_2 \)**

We estimate \( A_2 \). Lemma 7.20(ii) gives

\[
F_m(F_m^{-1}(n/m^2)) - F_m(\nu \log n) \ll |n/m^2 - F_m(\nu \log n)| \\
\ll m \log n En^{-\nu} + m^2 \nu n^{-3\nu} \log n.
\]

From this inequality, Lemma 7.20(iii) and Corollary 7.23 we have

\[
A_2 = m^3 \left( (\nu \log n)^{-1} + 2(\nu \log n)^{-2} + 2(\nu \log n)^{-3} \right) n^{-\nu} \\
- m^3 \left( (\nu \log n)^{-1} + (\nu \log n)^{-2} + \frac{1}{2} (\nu \log n)^{-3} \right) n^{-2\nu} \\
+ O(m^3 n^{-3\nu} / \log n)
\]

and taking out a factor of \( m^2 n^{-\nu}(\nu \log n)^{-1} \), gives

\[
A_2 = \frac{m^3}{\nu \log n} \left( 1 + \frac{2}{\nu \log n} + \frac{2}{(\nu \log n)^2} - \left( 1 + \frac{1}{\nu \log n} + \frac{1}{(2(\nu \log n)^2)} \right) n^{-\nu} + O(n^{-2\nu}) \right).
\] (7.50)

**Conclusion of proof**

From (7.50) we have that as \( n \to \infty \),

\[
\frac{1}{\sqrt{2\pi A_2}} = \frac{1}{(2\pi)^{1/2} m^{-3/2} n^{\nu/2}(\nu \log n)^{1/2}} \\
\times \left( 1 + 2/(\nu \log n) + 2/(\nu \log n)^2 - (1 + 1/(\nu \log n) + 1/(2(\nu \log n)^2)) n^{-\nu} \right)^{-1/2} \\
\times (1 + O(n^{-2\nu})).
\]
Also, using (7.47) we obtain, as \( n \to \infty \),
\[
1 - \frac{1}{12} m/n = 1 - \frac{1}{12} (\nu \log n)^{1/2} n^{(\nu-1)/2} + o((\log n)^{1/2} n^{(\nu-1)/2}).
\]

Hence from Theorem 6.2 and the above two equations we have as \( n \to \infty \),
\[
(7.51)
\int_{-1/2}^{1/2} \varphi(\alpha)e(-\alpha n) d\alpha = \frac{m^{-3/2}n^{\nu/2}(\nu \log n)^{1/2}}{(2\pi)^{1/2}} \times (1 + 2/(\nu \log n) + 2/(\nu \log n)^2 - (1 + 1/(\nu \log n) + 1/(2(\nu \log n)^2))n^{-\nu})^{-1/2} \times (1 - \frac{1}{12} (\nu \log n)^{1/2} n^{(\nu-1)/2} + o((\log n)^{1/2} n^{(\nu-1)/2})).
\]

The theorem follows immediately by combining (7.49) with (7.51) in (7.44) to give: For \( \nu > 1/3 \),
\[
E = o((\log n)^{1/2} n^{(3\nu-1)/2})
\text{ and } m \text{ as in (7.45), as } n \to \infty,
\]
\[
q(m, n) = \exp \left( m \left( 1 + 2/(\nu \log n) \right) n^{-\nu} - m \left( \frac{1}{2} + \frac{1}{2\nu \log n} \right) n^{-2\nu} \right) \times \frac{1}{(2\pi)^{1/2}} m^{-3/2} n^{\nu/2} (\nu \log n)^{1/2} \times (1 + 2/(\nu \log n) + 2/(\nu \log n)^2 - (1 + 1/(\nu \log n) + 1/(2(\nu \log n)^2))n^{-\nu})^{-1/2} \times (1 - \frac{1}{12} (\nu \log n)^{1/2} n^{(\nu-1)/2} + o((\log n)^{1/2} n^{(\nu-1)/2})).
\]

11. Comparison with result of Freiman and Pitman for large \( m \)

By removing the error terms in Theorem 7.2 and settling for an asymptotic main term we obtain the following corollary.

**Corollary 7.26.** Let \( m \) and \( n \) be integers such that \( 0 \leq m < n/2 \). Suppose that as \( n \to \infty \),
\[
\left| m - \sqrt{n/F} \right| = o((\log n)^{1/2} n^{(3\nu-1)/2}),
\]

where
\[
F = \left( \frac{1}{\nu \log n} + \frac{1}{\nu^2 \log^2 n} \right) n^{-\nu} - \left( \frac{1}{2\nu \log n} + \frac{1}{4\nu^2 \log^2 n} \right) n^{-2\nu},
\]

and \( 1/3 < \nu < 1 \). Then with \( q(m, n) \) as in (1.2) as \( n \to \infty \),
\[
q(m, n) \sim \exp \left( m \left( 1 + \frac{2}{\nu \log n} \right) n^{-\nu} \right) \times \frac{1}{(2\pi)^{1/2}} n^{-(3\nu+1)/4} (\nu \log n)^{-1/4}.
\]

This is consistent with the result obtained by Freiman and Pitman, stated in Proposition L, although it is valid for \( m \) in smaller bandwidths than the \( o(n^\nu) \) bandwidth of Freiman and Pitman.
12. Concluding remarks

In Theorems 7.1 and 7.2 I have given explicit estimates for \( q(m, n) \) for \( m \) both small and large. Each constitute a refinement of the corresponding results Proposition J and Proposition L of Freiman and Pitman (see Section 1 of Chapter 6). The starting point of these refinements was the more precise estimates of the integral (6.13) in Chapter 6. However the estimation of the product (7.1) in this chapter has turned out to be considerably more difficult, and it could be expected that similar or greater difficulties would arise in the case of \( q(m, n) \) for a general sequence \( u \).

For small \( m \) the range in Theorem 7.1 was \( m = o(n^{1/3}) \). Enlarging this range to \( m = o(n^{1/2-\delta}) \), for some positive number \( \delta < 1/6 \), would yield an asymptotic expression for \( q(m, n) \) with considerably more terms than in that given in Theorem 7.1 if the error term were to remain as it is.

For large \( m \), I believe that further improvement on the estimate of \( q(m, n) \) in Theorem 7.2 could be achieved by extending the ideas of Section 10, and in particular by using a more precise version of Lemma 7.20.
References