AN IMPROVED CONVEXITY MAXIMUM PRINCIPLE AND SOME APPLICATIONS

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CONTENTS

SUMMARY

Concavity-like geometric properties are derived for the solutions of various classes of boundary value problem. The derivations of these results are all obtained by transforming a basic convexity maximum principle which is an improved version of one published by N. Korevaar.

In order to present the results in a coherent manner, the concept of α-concavity is introduced. When α is a positive real number, a non-negative function is said to be α-concave when its αth power, u^{α}, is concave. This "scale of concavity-like properties" is then extended in a natural way to all extended real numbers α in $[-\infty,+\infty]$. The larger the number α is, the stronger the property. If a function is not α-concave, then an α-convexity function can be constructed to measure how far the function deviates from α-concavity. The α-convexity function is equal to zero if the function is α-concave, and positive otherwise. In terms of these concepts, then, the basic convexity maximum principle states essentially that if a function u on a bounded convex domain in $\mathbb{R}^n$ for some $n \geq 2$ is such that the negative of its Laplacian, $-\Delta u$, is (-1)-concave, then the 1-convexity function for u cannot attain a positive maximum in the interior of the domain. Korevaar derived the same result from the assumption that $-\Delta u$ is 1-concave, and herein lies the improvement.

Next, the basic result is transformed in order to obtain a maximum principle for the α-convexity function ~~α-concavity~~ for $0 \leq \alpha \leq 1$. But in order to use these maximum principles, it is necessary to combine them with some knowledge of the behaviour of a function near the boundary of its domain. If it can be shown that a positive maximum of the α-convexity function cannot occur at the boundary, then there is nowhere else for the maximum to be

attained. From this results a proof of α-concavity.

This method is applied to three kinds of boundary value problem on bounded convex domains. In the first, $-\Delta u$ is equal to a function of the domain variable x in $\mathbb{R}^n$. In the second, $-\Delta u$ is equal to a function only of the dependent variable u itself. And in the third, the equation $\Delta u = \exp(u)$ is considered with infinite boundary data - that is, Liouville's problem.

In the first case, it is found that if $-\Delta u$ is β-concave for some $\beta \geq 1$, then u is $(\beta/(1+2\beta))$-concave (if u equals zero on the boundary). In the second, it is shown that if $-\Delta u = u^\gamma$, and u equals zero on the boundary of the domain, then u is $(\frac{1}{2}(1-\gamma))$-concave, for $0 < \gamma < 1$. Thirdly, it is shown that the solution of Liouville's problem in $\mathbb{R}^2$ is convex. The sharpness of the result for the first boundary value problem is shown by an example, and various other examples place limits on the possible extension of other results.

Finally, some calculations on specific functions give concrete meaning to the concepts dealt with in the analysis of general problems.

CLAIM OF ORIGINALITY

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

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1. INTRODUCTION

This thesis presents some new results concerning the "α-concavity" of solutions to boundary value problems. The introduction explains what α-concavity is, briefly summarises the relevant literature, and outlines the contents of the thesis.

Nearly all of the progress in establishing concavity-like properties of solutions to boundary value problems has occurred since 1970. These various properties may be readily compared with each other when expressed in terms of a concavity scale used by Brascamp and Lieb ([1], p.373). A simplification of that scale will be used in this thesis. It provides a clear way of presenting results as well as an effective analytical tool: Let α be an extended real number. A non-negative function u on a convex subset Ω of any Euclidean space $\mathbb{R}^n$ will be said to be <u>α-concave</u> for $\alpha = +\infty$ when u is constant, for $0 < \alpha < +\infty$ when u^α is concave, for $\alpha = 0$ when log u is concave, for $-\infty < \alpha < 0$ when u^α is convex, and for $\alpha = -\infty$ when the upper level sets of u, $\{x \in \Omega;\ u(x) > t\}$, are convex for all real constants t. (Here, log u, and u^α for $-\infty < \alpha < 0$, are taken to mean $-\infty$ and $+\infty$ respectively when $u = 0$, and the usual extended definitions of convexity and concavity are then applied to the extended real valued functions that result.)

α-concavity is monotonic with respect to α in the sense that for any extended numbers α and β such that $\alpha \geqslant \beta$, if u is α-concave then u is β-concave. (This and other properties of α-concavity will be demonstrated in the next section.) Thus any function u which is at least $(-\infty)$-concave is associated with a unique maximum number, $\alpha(u)$ say, such that u is β-concave for all β less than $\alpha = \alpha(u)$, and in fact u is then α-concave for this value of α. It seems that not many physical boundary value problems have solutions that are actually concave (that is, 1-concave), whereas the number of problems known to have α-concave solutions for α less than 1 has grown impressively in the last decade.

In the literature, a 0-concave function is usually referred to as "log concave", and a $(-\infty)$-concave function is often said to be "quasi-concave". Here, functions which are α-concave for positive α will be broadly referred to as "power concave".

The first α-concavity result of which the author is aware is that of Gabriel ([3]), who showed that the Green's function for the Laplacian on a bounded convex domain (that is, non-empty open set) in $\mathbb{R}^3$ is $(-\infty)$-concave. In 1971, Makar-Limanov ([9]) demonstrated, using inequalities peculiar to $\mathbb{R}^2$, the $(-\infty)$-concavity of the solution of $\Delta u+1 = 0$ subject to zero Dirichlet data on the boundary of a bounded convex domain in $\mathbb{R}^2$. In fact, it is clear from the proof that u is $\frac{1}{2}$-concave. In 1976, Brascamp and Lieb ([1]) used the closure of the 0-concave functions under convolution to show that the semigroups corresponding to a class of parabolic equations preserve the 0-concavity of the initial data, from which it is easily deduced that the fundamental solution of $\Delta u+\lambda u = 0$ on a bounded convex domain in $\mathbb{R}^n$ for $n \geqslant 2$ is 0-concave.

In 1977, Lewis ([8]) demonstrated $(-\infty)$-concavity for the potential function of a convex ring in $\mathbb{R}^n$. That is, given two bounded convex domains in $\mathbb{R}^n$ such that the closure of one is included in the other, if u denotes the function which is harmonic in the ring between their two boundaries, equal to 1 on the inner boundary and inside it, and equal to 0 on the outer boundary, then u is $(-\infty)$-concave. The method used somewhat resembled a maximum principle.

In the early 1980s, Korevaar ([6,7]) and Caffarelli and Spruck ([2]) used a maximum principle to prove 0-concavity and 1-concavity for solutions of a rather general class of boundary value problems of the type $a_{ij}(\nabla u)u_{ij}+b(x,u,\nabla u) = 0$ subject to some sort of boundary condition on a bounded convex domain in $\mathbb{R}^n$ for $n \geq 2$. Korevaar applied this to capillary surfaces to demonstrate 1-convexity of the solution (that is, 1-concavity for the negative of the solution), and Caffarelli and Spruck demonstrated the existence of a $(-\infty)$-concave solution to a plasma problem: $\Delta u+(u-k)^+ = 0$ subject to zero Dirichlet data on the boundary of a bounded convex domain in $\mathbb{R}^n$, where t^+ means $(|t|+t)/2$.

The present author developed a method based somewhat on that of Lewis to show that the solution of Poisson's equation, $\Delta u + f(x) = 0$, subject to zero Dirichlet data on the boundary of a bounded convex domain in $\mathbb{R}^n$ for $n \geq 2$ is $(\beta/(1+2\beta))$-concave if f is a non-negative β-concave function for some $\beta \geq 1$ ([5], see appendix). In the limit as $\beta \to \infty$ this gives the generalisation of the Makar-Limanov result to $\mathbb{R}^n$. Shortly before writing this thesis, however, the author discovered an improvement (Theorem 1 of this thesis) to Korevaar's convexity maximum principle ([6] Theorem 1.2, or [7], Theorem 1.3). This permitted both a simplification of the derivation of the Poisson's equation result and a greater applicability to non-linear problems than could be obtained with the method used in [5]. The difference between Korevaar's principle and the improved principle lies essentially in the fact that whereas he requires the 1-concavity of the function b with respect to the joint variable (x,u), the improved principle requires only (-1)-concavity. (It should be pointed out that Korevaar's concepts are upside-down relative to the author's, and that for the sake of uniformity the liberty has been taken of calling his concavity principle a convexity principle, with similar changes for his other definitions.)

The results of the present work are derived almost entirely from Theorem 1 (section 3), which is a rather abstract convexity maximum principle. By means of a simple transformation, this becomes an α-convexity maximum principle for $0 \leqslant \alpha \leqslant 1$, stated in Theorem 2 (section 4). This is equivalent to Korevaar's principle for $\alpha = 0$, and significantly better for $0 < \alpha < 1$, because for such α his principle leads to no result at all. It appears that the unimproved principle only gives results for $\alpha = 0$ or 1, which are precisely the cases used by Korevaar.

In section 5, the maximum principles of the previous two sections are applied to boundary value problems. That is, they are combined with assumptions on the behaviour of the solution near the boundary to give an assertion of α-concavity for some α. Theorem 3 applies Theorem 1 directly; Theorem 4 applies Theorem 2 in the case $0 < \alpha \leqslant 1$; and Theorem 5 applies Theorem 2 in the case $\alpha = 0$, which is not original, as indicated above. Theorem 5 is thus given partly for completeness, to demonstrate what the consequences of the method are, but partly also because the analysis of boundary behaviour in this case rests quite heavily on Lemma 6, a proof of which appears in [7] and [2], ~~but is sketchy in the former and somewhat incomplete in the latter. It was thought desirable, therefore, to provide an explicit proof~~ It was thought desirable to provide here a more detailed proof, however, with a view to facilitating its possible generalisation in the future and providing a more secure basis for Lemma 11, which also depends on it.

Section 6 applies the abstract results of section 5 to various classes of boundary value problem by making the calculations necessary for each specific case. Since the approach presented in this thesis was discovered by the author only shortly before its writing, the results in this section are by no means exhaustive. The approach has only just begun to be applied, and the near future should see some success with other kinds of boundary value problems. Theorem 7 shows how the author's result in [5] can be derived by better means, using Lemma 8 to extend the result to non-smooth force functions. Theorem 9 spells out the requirements for an equation of the type $\Delta u+h(u) = 0$, and shows that the special case $\Delta u+ku^{\gamma} = 0$, with zero Dirichlet data on the boundary, for $0 < \gamma < 1$ and $k > 0$, implies that u is $(\frac{1}{2}(1-\gamma))$-concave. This bridges the gap between the Makar-Limanov type of result for $\Delta u+k = 0$ ($\gamma = 0$ and u is $\frac{1}{2}$-concave), and the Brascamp and Lieb type of result for $\Delta u+ku = 0$ ($\gamma = 1$ and u is 0-concave). Theorem 10 is the result for $\alpha = 0$ analogous to Theorem 9. It was obtained by Korevaar, and is given here merely for completeness.

The solution of Liouville's problem, which satisfies $\Delta u = \exp(u)$ in a bounded domain in $\mathbb{R}^2$ and $u \to +\infty$ at the boundary, is shown to be convex when the domain is convex by Lemma 11 and Theorem 12. The conjecture that -u would be $(-\infty)$-concave under these circumstances was communicated to the author by G. Keady, and a quick application of Theorem 1 showed that it would in fact be 1-concave. When the concavity number of the "force" function for -u is calculated, it turns out to be equal to 0, exactly half way between 1, which is required by Korevaar's principle, and -1, which the improved principle requires. This result has the consequence that under appropriate conditions, the path of a simple point vortex, under the influence only of an otherwise irrotational velocity field in a convex set, encloses a convex set.

The seemingly random set of results in section 6 is due to the recent origin of the method used, and the author is sure that there are many more applications to be found to, for instance, capillary surfaces, minimal surfaces, surfaces of constant mean curvature, parabolic equations, plasma-type problems, Green's functions, potential functions (and generalised potential functions) on convex discs, and elasto-plastic deformations of cylinders.

Section 7 demonstrates the sharpness of many of the earlier results. Theorem 14 shows that the number $\beta/(1+2\beta)$ appearing in Theorem 7 is sharp. Theorem 8 shows that 0-concavity is the best possible for the equation $\Delta u+\lambda u = 0$. Theorem 16 shows that Theorem 7 can not be extended down to $\beta = 0$. Finally, Theorem 18 says that if a domain has a flat portion, then the solution to $\Delta u+1 = 0$ cannot be concave.

Section 8 contains explicit calculations of $\alpha(u)$ for the solution of $\Delta u+4 = 0$ with zero Dirichlet data on an equilateral triangle, and for the Green's function for Δ on a sphere in $\mathbb{R}^n$, with pole at the centre of the sphere.

The appendix contains an article by the author which will not be published for some time. The research reported therein was undertaken for the present Ph.D. candidature, and although the methods used are superseded in many ways by those of the main body of the thesis, they are of interest in that they give added insight into the mechanisms whereby the "force" function, $f = -\Delta u$, and the boundary data combine to produce an α-concave solution.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let $n \geq 2$. A subset Ω of $\mathbb{R}^n$ will be called a domain when it is open and non-empty. When Ω is a domain and k is a non-negative integer or $+\infty$, $C(\Omega)$, $C(\overline{\Omega})$, $C^k(\Omega)$ and $C^k(\overline{\Omega})$ will denote respectively the set of continuous functions on Ω, the set of continuous functions on $\overline{\Omega}$, the set of functions in $C(\Omega)$ whose derivatives of order less than or equal to k are continuous, and the set of functions in $C(\overline{\Omega})$ whose derivatives in Ω of order less than or equal to k have continuous extensions to $\overline{\Omega}$.

If Ω is a bounded convex domain in $\mathbb{R}^n$ and $u: \overline{\Omega} \to \mathbb{R}$ is a bounded function, then the convexity function c for u on $\overline{\Omega}$ is defined on $\overline{\Omega}\times\overline{\Omega}\times[0,1]$ by:

$$c(y,z,\lambda) = (1-\lambda)u(y) + \lambda u(z) - u((1-\lambda)y+\lambda z)$$
$$\text{for } y,z \in \overline{\Omega},\ \lambda \in [0,1].$$

Write $\overline{c} = \sup\{c(y,z,\lambda);\ (y,z,\lambda) \in \overline{\Omega}\times\overline{\Omega}\times[0,1]\}$. $\overline{c}$ is a real number since c is bounded. Also, $\overline{c} \geq 0$ (which follows, for instance, by putting $\lambda = 0$), and $\overline{c} = 0$ if and only if u is concave in $\overline{\Omega}$. The convexity maximum principle referred to in the title of this thesis is a maximum principle for c.

A useful extension of the idea of the convexity of the reciprocal of a function, from positive functions to general real functions, is as follows: If S is a convex set, then $b: S \to \mathbb{R}$ will be said to be harmonic concave when, for all $(y,z,\lambda) \in S\times S\times[0,1]$,

$$b((1-\lambda)y + \lambda z) \geq b(y)b(z)((1-\lambda)b(z) + \lambda b(y))^{-1} \text{ if } (1-\lambda)b(z)+\lambda b(y) > 0$$
$$\text{and } b((1-\lambda)y + \lambda z) \geq 0 \text{ if } b(y) = b(z) = 0.$$

It is readily seen that a positive function b is harmonic concave if and only if $1/b$ is convex (that is, b is (-1)-concave). But it is also true that any concave function, whether it is non-negative or not, is harmonic concave, since if b is concave and $(1-\lambda)b(z)+\lambda b(y) > 0$ then

$$\begin{aligned} & b((1-\lambda)y+\lambda z) - b(y)b(z)((1-\lambda)b(z)+\lambda b(y))^{-1} \\ & \geqslant (1-\lambda)b(y)+\lambda b(z) - b(y)b(z)((1-\lambda)b(z)+\lambda b(y))^{-1} \\ & = \lambda(1-\lambda)(b(y)-b(z))^2((1-\lambda)b(z)+\lambda b(y))^{-1} \\ & \geqslant 0. \end{aligned}$$

The inequality for $b(y) = b(z) = 0$ is clearly satisfied by any concave function b. Harmonic concavity is used in preference to (-1)-concavity in Theorem 1 to allow the convexity maximum principle which is central to this thesis to be stated in the fullest possible generality, although this extra generality will not be used here.

A definition of α-concavity has been given in the introduction, but since the concept is so fundamental to this thesis, it is appropriate to summarise here some of its basic properties. They are given only brief justification since they are mostly elementary, and of value principally for general orientation and for some calculations concerning the counterexamples. Property 4 is probably the most useful for applications, since equation (2.1) provides a simple way of checking whether a function is α-concave.

PROPERTY 1: Let Ω be a convex set in $\mathbb{R}^n$ for some $n \geqslant 1$. A function u on Ω is α-concave if and only if for all y and z in Ω and $\lambda \in [0,1]$, $u(x) \geqslant g_\alpha(\lambda,u(y),u(z))$, where $x = (1-\lambda)y + \lambda z$, and $g_\alpha(\lambda,s,t)$ is defined for $\lambda \in [0,1]$ and $s, t \geqslant 0$ by:

$$g_\alpha(\lambda,s,t) = \begin{cases} \max(s,t) & \text{for } \alpha = +\infty \\ ((1-\lambda)s^\alpha + \lambda t^\alpha)^{1/\alpha} & \text{for } 0 < \alpha < +\infty \\ s^{1-\lambda}t^\lambda & \text{for } \alpha = 0 \\ st((1-\lambda)t^{-\alpha} + \lambda s^{-\alpha})^{1/\alpha} & \text{for } -\infty < \alpha < 0 \\ \min(s,t) & \text{for } \alpha = -\infty \end{cases}$$

where 0^0 is taken to mean 0. This equivalent definition for α-concavity, and the fact that $g_\alpha(\lambda,s,t)$ is monotone increasing with respect to α, are discussed by Brascamp and Lieb ([1], p.373), with minor differences due to the slightly restricted form of definition used here. Similarly it is straightforward to show that for each $\lambda \in (0,1)$ and $s,\ t \geqslant 0$, $g_.(\lambda,s,t) \in C([-\infty,+\infty]) \cap C^\infty(\mathbb{R})$ with respect to the usual two-point compactification topology on $[-\infty,+\infty]$ (but not analytic when only one of s and t is equal to zero, since $g_\alpha(\lambda,0,t) = \lambda^{1/\alpha}t$ for $\alpha > 0$, and equals zero for $\alpha \leqslant 0$).

PROPERTY 2: If α and β are extended real numbers such that $\alpha \geqslant \beta$ and u is α-concave, then u is β-concave. This monotonicity property of α-concavity follows from the monotonicity of g_α with respect to α.

PROPERTY 3: If $\alpha \in \mathbb{R}$ or $\alpha = +\infty$, and a function u is β-concave for all $\beta < \alpha$, then u is α-concave. This continuity property of α-concavity follows from the continuity of g_α with respect to α. Hence for every $(-\infty)$-concave function there is a (unique) $\alpha \in [-\infty,+\infty]$ such that u is β-concave for all $\beta \leqslant \alpha$ and not β-concave for $\beta > \alpha$. This α will be denoted by $\alpha(u)$ and termed the <u>concavity number</u> of u. In the light of this definition, all results on concavity-like properties of solutions of boundary value problems may be regarded as estimates of the concavity number of the solution.

PROPERTY 4: A positive C^2 function on a convex domain Ω is α-concave for $\alpha \in (-\infty,+\infty)$ if and only if

$$(2.1) \qquad u(x)u_{\theta\theta}(x) + (\alpha-1)u_\theta(x)^2 \leqslant 0$$

for all $x \in \Omega$ and $\theta \in S$, where $S = \{\phi \in \mathbb{R}^n;\ |\phi| = 1\}$ is the set of "directions" in $\mathbb{R}^n$, and $u_\theta(x)$ and $u_{\theta\theta}(x)$ denote the first and second derivatives of u with respect to x in the direction θ - namely, $\theta_i \partial u/\partial x_i$ and $\theta_i \theta_j \partial^2 u/\partial x_i \partial x_j$. (The summation convention will apply to all roman subscripts unless the context indicates otherwise.)

To prove (2.1) requires little more than the observations that a C^2 function v in a convex domain is convex if and only if $v_{\theta\theta}$ is everywhere non-negative for all θ, and that

$$(u^\alpha)_{\theta\theta} = \alpha u^{\alpha-2}(uu_{\theta\theta} + (\alpha-1)u_\theta{}^2) \text{ for } \alpha \neq 0$$
$$\text{and } (\log u)_{\theta\theta} = u^{-2}(uu_{\theta\theta} - u_\theta{}^2).$$

PROPERTY 5: It follows from property 4 that if u is a positive C^2 function on a convex domain Ω, and $\alpha \in (-\infty,+\infty)$, then u is α-concave if and only if

$$(2.2) \qquad \alpha \leqslant 1 - \sup\{u(x)u_{\theta\theta}(x)u_\theta(x)^{-2};\ x \in \Omega,\ \theta \in S \text{ and } u_\theta(x) \neq 0\}.$$

In fact, this is also true for $\alpha = +\infty$, since the set in (2.2) is empty if and only if u is constant, in which case the supremum of the set is $-\infty$, and the right hand side equals $+\infty$. Hence for any $(-\infty)$-concave positive C^2 function,

$$(2.3) \qquad \alpha(u) = 1 - \sup\{u(x)u_{\theta\theta}(x)u_\theta(x)^{-2};\ x \in \Omega,\ \theta \in S \text{ and } u_\theta(x) \neq 0\}.$$

This provides an effective means of calculating $\alpha(u)$ (for examples, see Section 8), and exposes the essential simplicity of the definition of

α-concavity.

PROPERTY 6: For any $\alpha \geq 1$, the set of α-concave functions on a convex set Ω is a convex cone. That is, for $\alpha \geq 1$, the α-concave functions are closed under addition and positive scalar multiplication. This follows from a well known extension of Minkowski's inequality which states that if one defines

$$\|x\|_p = (\sum_{i=1}^{m} x_i{}^p)^{1/p} \text{ for } x \in \mathbb{R}^m,$$

$$\text{then } \|(1-\lambda)x_1 + \lambda x_2\|_p \geq (1-\lambda)\|x_1\|_p + \lambda\|x_2\|_p$$

for all $p \in (0,1]$, $\lambda \in [0,1]$ and x_1, $x_2 \in \mathbb{R}^m$. From this, putting $m = 2$, $x_1 = (f(y),g(y))$ and $x_2 = (f(z),g(z))$, one obtains

$$\begin{aligned}
&(1-\lambda)(f(y)^p+g(y)^p)^{1/p} + \lambda(f(z)^p+g(z)^p)^{1/p} \\
&= (1-\lambda)\|x_1\|_p + \lambda\|x_2\|_p \\
&\leq \|(1-\lambda)x_1 + \lambda x_2\|_p \\
&= (((1-\lambda)f(y)+\lambda f(z))^p + ((1-\lambda)g(y)+\lambda g(z))^p)^{1/p} \\
&\leq (f((1-\lambda)y + \lambda z)^p + g((1-\lambda)y + \lambda z)^p)^{1/p}
\end{aligned}$$

whenever f and g are non-negative concave functions. But this inequality means precisely that $(f^p + g^p)^{1/p}$ is concave. Writing $p = \alpha^{-1}$ for $\alpha \in [1,\infty)$ and replacing f^p and g^p with f and g respectively then shows that f + g is α-concave whenever f and g are α-concave. Property 6 readily follows.

PROPERTY 7: If α and $\beta \in [0,+\infty]$, f is α-concave, and g is β-concave, then the pointwise product f.g is γ-concave, where $\gamma \in [0,+\infty]$ satisfies $\gamma^{-1} = \alpha^{-1} + \beta^{-1}$, with the understanding that $0^{-1} = +\infty$ and $(+\infty)^{-1} = 0$. The result is obvious when α or β is infinite, and clear when α and β are both zero. In the other cases $\alpha+\beta \in (0,+\infty)$, and by defining

$\lambda = \alpha/(\alpha+\beta)$ one sees that it will be sufficient to show that $f^{1-\lambda}.g^{\lambda}$ is concave whenever f and g are concave and $\lambda \in [0,1]$. But $f^{1-\lambda}.g^{\lambda}$ is the pointwise limit of $((1-\lambda)f^p + \lambda g^p)^{1/p}$ as $p \to 0^+$, these being concave functions by property 6. Hence $f^{1-\lambda}.g^{\lambda}$ is concave and property 7 follows.

PROPERTY 8: For any α, if f is a pointwise limit of a sequence of non-negative α-concave functions, then f is α-concave. This follows from property 1 and the continuity of $g_{\alpha}(\lambda,s,t)$ with respect to s and t.

3. AN IMPROVED CONVEXITY MAXIMUM PRINCIPLE

This section states and proves a convexity maximum principle which differs little in appearance from one proved by Korevaar ([6], Theorem 1.2). The improvement lies entirely in assumption (ii), which specifies harmonic concavity rather than concavity. The results of this thesis are, however, the direct consequences of this improvement.

THEOREM 1: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$u: \Omega \to \mathbb{R}$, $u \in C^2(\Omega)$.

For $x \in \Omega$, u satisfies the equation

$$a_{ij}(Du(x))u_{ij}(x) + b(x,u(x),Du(x)) = 0,$$

where for all $p \in \mathbb{R}^n$, $(a_{ij}(p))$ is a real symmetric positive semi-definite matrix, and the summation convention for repeated indices is observed.

(i) For all $x \in \Omega$ and $p \in \mathbb{R}^n$, $b(x,\cdot,p)$ is strictly decreasing.

(ii) For all $p \in \mathbb{R}^n$, $b(\cdot,\cdot,p) : \Omega \times \mathbb{R} \to \mathbb{R}$ is harmonic concave.

Assertion:

If $\bar{c} > 0$, then $\bar{c}$ is not attained in $\Omega\times\Omega\times[0,1]$.

PROOF: c is the convexity function defined in the previous section. Suppose its supremum, $\bar{c}$, is positive, and that $\bar{c}$ is attained at $(y,z,\lambda) \in \Omega\times\Omega\times[0,1]$ - that is, $c(y,z,\lambda) = \bar{c}$ at some such (λ,y,z). Then clearly $y \neq z$ and $\lambda \in (0,1)$. Let $D = (D_y,D_z,D_\lambda)$ denote the gradient on $\Omega\times\Omega\times(0,1)$, where $D_y = (\partial/\partial y_1,\ldots\partial/\partial y_n)$ and so forth, and write D^2 for the corresponding Hessian. Then $Dc(y,z,\lambda) = 0$, and $D^2c(y,z,\lambda)$ is negative semi-definite. Thus, writing $x = (1-\lambda)y + \lambda z$, we obtain

$$0 = D_y c = (1-\lambda)Du(y) - (1-\lambda)Du(x),$$

so that $Du(y) = Du(x)$. Similarly, $Du(z) = Du(x)$, and hence (a_{ij}) has the same value, A say, at each of x,y and z. Define the $(2n+1)\times(2n+1)$ matrix B by:

$$B = \begin{bmatrix} s^2A & stA & 0 \\ stA & t^2A & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $s,t \in \mathbb{R}$. Then B is positive semi-definite because A is, and so $Tr(BD^2c)$ ~~$(= Tr(B)Tr(D^2c))$~~ ≤ 0. That is:

(3.1) $$\alpha s^2 + 2\beta st + \gamma t^2 \leq 0$$

where $\alpha = Tr(AD_y{}^2c)$, $\beta = Tr(AD_yD_zc)$ and $\gamma = Tr(AD_z{}^2c)$. For $w = x$, y and z, put $Q_w = a_{ij}u_{ij}(w)$. Then

$$\alpha = (1-\lambda)Q_y - (1-\lambda)^2Q_x$$
$$\beta = -\lambda(1-\lambda)Q_x$$
$$\text{and } \gamma = \lambda Q_z - \lambda^2 Q_x.$$

The non-positivity of the quadratic form in (3.1) implies that $\alpha \leq 0$, $\gamma \leq 0$ and $\beta^2 - \alpha\gamma \leq 0$. That is:

(3.2) $$Q_x \geq (1-\lambda)^{-1}Q_y$$

(3.3) $$Q_x \geq \lambda^{-1}Q_z$$

$$\text{and } 0 \geq \lambda^2(1-\lambda)^2Q_x{}^2 - ((1-\lambda)^2Q_x - (1-\lambda)Q_y)(\lambda^2Q_x - \lambda Q_z)$$
$$= \lambda(1-\lambda)((1-\lambda)Q_xQ_z + \lambda Q_xQ_y - Q_yQ_z).$$

Since $\lambda(1-\lambda) > 0$, this can be written as:

(3.4) $$Q_x((1-\lambda)Q_z + \lambda Q_y) \leq Q_yQ_z.$$

If $(1-\lambda)Q_z + \lambda Q_y \geq 0$, then by (3.2),

$$Q_x((1-\lambda)Q_z + \lambda Q_y) \geq Q_y Q_z + \lambda(1-\lambda)^{-1}Q_y^2,$$

which, taken together with (3.4), implies that $Q_y = 0$. Similarly, $Q_z = 0$. This allows us to conclude that either $(1-\lambda)Q_z + \lambda Q_y < 0$ or $Q_y = Q_z = 0$. In the former case, (3.4) gives

$$(3.5) \qquad Q_x \geq Q_y Q_z((1-\lambda)Q_z + \lambda Q_y)^{-1},$$

whereas when $Q_y = Q_z = 0$, (3.2) (or (3.3)) gives $Q_x \geq 0$. But $Q_w = -b(w,u(w),Du(w))$ for $w = x$, y and z. So

$$\begin{aligned} Q_x &= -b(x,u(x),Du(x)) \\ &< -b((1-\lambda)y + \lambda z,(1-\lambda)u(y) + \lambda u(z),Du(x)) \\ &\qquad\qquad \text{(by (i), as } u(x) < (1-\lambda)u(y) + \lambda u(z)) \\ &\leq \begin{cases} Q_y Q_z((1-\lambda)Q_z + \lambda Q_y)^{-1} & \text{if } (1-\lambda)Q_z + \lambda Q_y < 0 \\ 0 & \text{if } Q_y = Q_z = 0 \qquad \text{(by (ii)).} \end{cases} \end{aligned}$$

This inequality contradicts (3.5). So $\overline{c}$ cannot be attained in $\Omega\times\Omega\times[0,1]$, and the theorem is verified. □

4. A MAXIMUM PRINCIPLE FOR POWER CONVEXITY AND LOG CONVEXITY

Theorem 1 specifies conditions under which a convexity maximum principle may be obtained for a function u. Theorem 2 gives the corresponding conditions for a convexity maximum principle for u^{α} and for log u - in other words, an α-convexity maximum principle for $\alpha \in [0,1]$. This will be used later in proving α-concavity results.

For a C^2 function $b: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, let b_t and b_{tt} denote the first and second partial derivatives of b(x,t) with respect to t, and for $\theta \in S = \{\phi \in \mathbb{R}^n;\ |\phi| = 1\}$ write b_θ and $b_{\theta\theta}$ to denote the quantities $\theta_i \partial b/\partial x_i$ and $\theta_i \theta_j \partial^2 b/\partial x_i \partial x_j$ respectively - that is, the first and second partial derivatives of b(x,t) with respect to x in the direction θ. (The summation convention will not apply to the subscripts t and θ.)

THEOREM 2: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$u: \Omega \to \mathbb{R}$, $u \in C^2(\Omega)$, $u > 0$.

For $x \in \Omega$, u satisfies

$$\Delta u(x) + b(x, u(x)) = 0,$$

where $b: \Omega \times (0,\infty) \to (0,\infty)$ is a C^2 function, and $b(x,t) > 0$ for all $x \in \Omega$ and $t > 0$.

$\alpha \in [0,1]$.

Conditions (i) to (vi) hold for all $t > 0$:

(i) $(1-\alpha)b - tb_t > 0$ (or $t^{\alpha-1}b(x,t)$ is strictly decreasing with respect to t).

(ii) $(1-2\alpha)(1-3\alpha)b + (5\alpha-1)tb_t + t^2 b_{tt} \leq 0$. (That is, for $\alpha = 0$, $e^t b(x, e^{-t})$ is concave with respect to t, and for $\alpha \in (0,1]$, $t^{(1-2\alpha)/\alpha} b(x, t^{-1/\alpha})$ is concave with respect to t.)

Conditions (iii) to (vi) hold for all $\theta \in S$:

(iii) $b_{\theta\theta} \leq 0$

(iv) $K_0 \geq 0$

(v) $K_2 \geq 0$

(vi) $K_1 \geq 0$ or $K_0K_2b - K_1{}^2 \geq 0$

where
$$K_0 = (1-\alpha)b.((1-3\alpha)b_\theta{}^2-(1-2\alpha)bb_{\theta\theta}) + t(1-\alpha)(3bb_tb_{\theta\theta}-2b_tb_\theta{}^2-2bb_\theta b_{t\theta}) + t^2(bb_{tt}b_{\theta\theta}-2b_t{}^2b_{\theta\theta}-2b_{tt}b_\theta{}^2+4b_tb_\theta b_{t\theta}-bb_{t\theta}{}^2)$$
$$K_1 = \alpha b.((1-3\alpha)b_\theta{}^2-(1-2\alpha)bb_{\theta\theta}) + t((1+\alpha)bb_tb_{\theta\theta}-2\alpha bb_\theta b_{t\theta}-(1-\alpha)b_tb_\theta{}^2) + t^2(bb_{tt}b_{\theta\theta}-b_{tt}b_\theta{}^2-b_t{}^2b_{\theta\theta}+2b_tb_\theta b_{t\theta}-bb_{t\theta}{}^2)$$
and
$$K_2 = (1-3\alpha)((1-2\alpha)bb_{\theta\theta}-(1-3\alpha)b_\theta{}^2) + t((5\alpha-1)b_tb_{\theta\theta}+2(1-3\alpha)b_\theta b_{t\theta}) + t^2(b_{tt}b_{\theta\theta}-b_{t\theta}{}^2)$$

Assertion:

$c(y,z,\lambda) = (1-\lambda)v(y) + \lambda v(z) - v((1-\lambda)y+\lambda z)$ does not attain a positive maximum in $\Omega\times\Omega\times[0,1]$, where $v = \alpha^{-1}u^\alpha$ when $\alpha\in(0,1]$, and $v = \log(u)$ when $\alpha = 0$.

REMARK: The equivalence of the parenthesised conditions in assumption (ii) to the accompanying inequality ~~are~~ is an immediate consequence of following calculations:

$$\partial^2/\partial t^2\ (t^{(1-2\alpha)/\alpha}b(x,t^{-1/\alpha}))$$
$$= \alpha^{-2}t^{(1-4\alpha)/\alpha}((1-2\alpha)(1-3\alpha)b(x,\tau) + (5\alpha-1)\tau b_\tau(x,\tau) + \tau^2b_{\tau\tau}(x,\tau)),$$

where $\tau = t^{-1/\alpha}$ for $\alpha \neq 0$, and

$$\partial^2/\partial t^2\ (e^tb(x,e^{-t})) = e^t(b(x,\tau) - \tau b_\tau(x,\tau) + \tau^2b_{\tau\tau}(x,\tau)),$$

where $\tau = e^{-t}$.

PROOF: Theorem 1 will be applied to v. It will show that c can not attain a positive maximum in $\Omega\times\Omega\times[0,1]$ under some conditions on

$\beta = -\Delta v$. The proof of Theorem 2 consists in expressing these conditions in terms of $b = -\Delta u$.

Write $t = u(x)$ and $s = v(x)$, and define the function g so that $u = g \circ v$. That is, define $g(s) = (\alpha s)^{1/\alpha}$ when $\alpha \in (0,1]$, and $g(s) = \exp(s)$ when $\alpha = 0$. Then $\Delta u = g'\Delta v + g''|\nabla v|^2$, where g' and g'' denote the first and second derivatives of g, so that for $x \in \Omega$, $\Delta v = (g')^{-1}(-b-g''|v|^2)$. Hence

$$\Delta v(x) + \beta(x,v(x),Dv(x)) = 0,$$

where $\beta(x,s,p) = g'(s)^{-1}b(x,g(s)) + g''(s)g'(s)^{-1}|p|^2$. For g as defined, $\beta = t^{\alpha-1}b + (1-\alpha)t^{-\alpha}p^2$, and so

$$\beta_s = b_t - (1-\alpha)t^{-1}b - \alpha(1-\alpha)t^{-2\alpha}p^2,$$

where β_s, β_{ss}, β_θ and $\beta_{\theta\theta}$ will denote derivatives of $\beta(x,s,p)$ analogous to those defined above for $b(x,t,p)$ with respect to t and θ.

It follows from (i), and the assumption that $\alpha \in [0,1]$, that $\beta_s < 0$, so that condition (i) of Theorem 1 is satisfied with v and β in place of u and b. Alternatively, the condition that β be decreasing with respect to s may be met by requiring merely that $t^{\alpha-1}b(x,t)$ be decreasing with respect to t. The only condition of Theorem 1 remaining to be proved now is (ii) - that is, the harmonic concavity of β jointly with respect to the variables $x \in \Omega$ and $s > 0$.

For each $p \in \mathbb{R}^n$, $\beta(\cdot,\cdot,p)\colon \Omega\times(0,\infty) \to (0,\infty)$ is a positive C^2 function. So β is harmonic concave if and only if $1/\beta$ is convex. But a twice differentiable function on an open convex subset of $\mathbb{R}^{n+1}$ is convex if and only if its second partial derivatives are non-negative in all directions, ϕ say, in $\mathbb{R}^{n+1}$. When ϕ is expressed as $(\mu\theta,\nu)$, where $\theta \in \mathbb{R}^n$, and μ and ν are suitable real numbers, the condition $(1/\beta)_{\phi\phi} \geq 0$ for all ϕ is seen to be equivalent to the following set of simultaneous inequalities:

(4.1) $$(1/\beta)_{ss} \geq 0$$

(4.2) $$(1/\beta)_{\theta\theta} \geq 0 \text{ for all } \theta$$

(4.3) $$(1/\beta)_{ss}(1/\beta)_{\theta\theta} - (1/\beta)_{s\theta}^2 \geq 0 \text{ for all } \theta,$$

where the subscripts s and θ denote partial differentations as defined above.

Now $(1/\beta)_{ss} = \beta^{-3}(2\beta_s{}^2-\beta\beta_{ss})$ is non-negative if and only if $2\beta_s{}^2-\beta\beta_{ss}$ is non-negative. Treating (4.2) and (4.3) similarly, we replace (4.1),(4.2) and (4.3) with the following equivalent set of inequalities:

(4.4) $$2\beta_s{}^2-\beta\beta_{ss} \geq 0$$

(4.5) $$2\beta_\theta{}^2-\beta\beta_{\theta\theta} \geq 0 \text{ for all } \theta$$

(4.6) $$(2\beta_s{}^2-\beta\beta_{ss})(2\beta_\theta{}^2-\beta\beta_{\theta\theta})-(2\beta_s\beta_\theta-\beta\beta_{s\theta})^2 \geq 0 \text{ for all } \theta.$$

To calculate and use the derivatives β_s, β_θ, β_{ss}, $\beta_{\theta\theta}$ and $\beta_{s\theta}$, it is convenient to write $\beta(x,s,p) = F(x,s) + G(s)p^2$, where

$$F(x,s) = (\alpha s)^{(\alpha-1)/\alpha}b(x,g(s)) = t^{\alpha-1}b(x,t)$$
$$\text{and } G(s) = (1-\alpha)(\alpha s)^{-1} = (1-\alpha)t^{-\alpha}.$$

Despite strenuous efforts, the author has been unable to simplify the calculations for conditions (4.4),(4.5) and (4.6) beyond what has been achieved in the following:

$$\beta = F + Gp^2$$

$$\beta_s = F_s + G_s p^2$$

$$\beta_{ss} = F_{ss} + G_{ss}p^2$$

$\beta_\theta = F_\theta$

$\beta_{\theta\theta} = F_{\theta\theta}$

$\beta_{s\theta} = F_{s\theta}$

$dt/ds = t^{1-\alpha}$

$F = t^{-1+\alpha}b$

$F_s = t^{-1}(-(1-\alpha)b+tb_t)$

$F_{ss} = t^{-1-\alpha}((1-\alpha)b-(1-\alpha)tb_t+t^2b_{tt})$

$F_\theta = t^{-1+\alpha}b_\theta$

$F_{\theta\theta} = t^{-1+\alpha}b_{\theta\theta}$

$F_{s\theta} = t^{-1}(-(1-\alpha)b_\theta+tb_{t\theta})$

$G = (1-\alpha)t^{-\alpha}$

$G_s = -\alpha(1-\alpha)t^{-2\alpha}$

$G_{ss} = 2\alpha^2(1-\alpha)t^{-3\alpha}$

It may be noted at this point that if the transformation g had been used with Korevaar's convexity maximum principle as a starting point, then $\beta_{ss} \leq 0$ would have been required, which in turn would have required $G_{ss} \leq 0$ and therefore $\alpha = 0$ or 1, which are in fact precisely the values of α used by Korevaar in applications. But now, back to work.

$$2\beta_s^2-\beta\beta_{ss} = 2F_s^2-FF_{ss} + p^2(4F_sG_s-FG_{ss}-F_{ss}G) + p^4(2G_s^2-GG_{ss})$$

$$\begin{aligned}2G_s^2-GG_{ss} &= 2\alpha^2(1-\alpha)^2t^{-4\alpha} - (1-\alpha)t^{-\alpha}.2\alpha^2(1-\alpha)t^{-3\alpha}\\ &= 0\end{aligned}$$

$$\begin{aligned}2F_s^2-FF_{ss} &= 2t^{-2}(-(1-\alpha)b+tb_t)^2 - t^{-1+\alpha}b.t^{-1-\alpha}((1-\alpha)b-(1-\alpha)tb_t+t^2b_{tt})\\ &= t^{-2}((1-\alpha)(1-2\alpha)b^2 - 3(1-\alpha)tbb_t + t^2(2b_t^2-bb_{tt}))\\ &= -t^{-2}b.((1-2\alpha)(1-3\alpha)b+(5\alpha-1)tb_t+t^2b_{tt}) + 2t^{-2}((1-2\alpha)b-tb_t)^2\\ &\geq 0 \text{ by (ii)}\end{aligned}$$

$$4F_sG_s - FG_{ss} - F_{ss}G = 4t^{-1}(-(1-\alpha)b + tb_t).(-\alpha(1-\alpha)t^{-2\alpha})$$
$$- t^{-1+\alpha}b.(2\alpha^2(1-\alpha)t^{-3\alpha})$$
$$- t^{-1-\alpha}((1-\alpha)b - (1-\alpha)tb_t + t^2b_{tt}).(1-\alpha)t^{-\alpha}$$
$$= -(1-\alpha)t^{-1-2\alpha}((1-2\alpha)(1-3\alpha)b + (5\alpha-1)tb_t + t^2b_{tt})$$
$$\geqslant 0 \text{ by (ii).}$$

Thus $2\beta_s{}^2 - \beta\beta_{ss} \geqslant 0$.

$$2\beta_\theta{}^2 - \beta\beta_{\theta\theta} = 2F_\theta{}^2 - FF_{\theta\theta} + (-GF_{\theta\theta})p^2$$

$$2F_\theta{}^2 - FF_{\theta\theta} = t^{2\alpha-2}(2b_\theta{}^2 - bb_{\theta\theta})$$
$$\geqslant 0 \text{ by (iii).}$$

$$-GF_{\theta\theta} = -(1-\alpha)t^{-1}b_{\theta\theta}$$
$$\geqslant 0 \text{ by (iii).}$$

So $2\beta_\theta{}^2 - \beta\beta_{\theta\theta} \geqslant 0$.

$$2\beta_s\beta_\theta - \beta\beta_{s\theta} = 2F_sF_\theta - FF_{s\theta} + p^2(2G_sF_\theta - GF_{s\theta})$$

$$2F_sF_\theta - FF_{s\theta} = 2t^{-1}(-(1-\alpha)b + tb_t).t^{-1+\alpha}b_\theta - t^{-1+\alpha}b.t^{-1}(-(1-\alpha)b_\theta + tb_{t\theta})$$
$$= t^{-2+\alpha}(-(1-\alpha)bb_\theta + 2tb_tb_\theta - tbb_{t\theta})$$

$$2G_sF_\theta - GF_{s\theta} = -2\alpha(1-\alpha)t^{-2\alpha}.t^{-1+\alpha}b_\theta - (1-\alpha)t^{-\alpha}.t^{-1}(-(1-\alpha)b_\theta + tb_{t\theta})$$
$$= (1-\alpha)t^{-1-\alpha}((1-3\alpha)b_\theta - tb_{t\theta}).$$

Hence $(2\beta_s{}^2 - \beta\beta_{ss})(2\beta_\theta{}^2 - \beta\beta_{\theta\theta}) - (2\beta_s\beta_\theta - \beta\beta_{s\theta})^2 = q_0 + q_1p^2 + q_2p^4$, where:

$$q_0 = (2F_s{}^2 - FF_{ss})(2F_\theta{}^2 - FF_{\theta\theta}) - (2F_sF_\theta - FF_{s\theta})^2$$

$$q_1 = (2F_s{}^2 - FF_{ss})(-GF_{\theta\theta}) + (4F_sG_s - FG_{ss} - F_{ss}G)(2F_\theta{}^2 - FF_{\theta\theta})$$
$$- 2(2F_sF_\theta - FF_{s\theta})(2G_sF_\theta - GF_{s\theta})$$

$$q_2 = (4F_sG_s - FG_{ss} - F_{ss}G)(-GF_{\theta\theta}) - (2G_sF_\theta - GF_{s\theta})^2$$

Now $q_0 + q_1p^2 + q_2p^4 \geqslant 0$ for all p if and only if $q_0 \geqslant 0$, $q_2 \geqslant 0$ and either $q_1 \geqslant 0$ or $4q_0q_2 - q_1{}^2 \geqslant 0$.

$$q_0 = t^{-2}((1-\alpha)(1-2\alpha)b^2 - 3(1-\alpha)tbb_t + t^2(2b_t{}^2 - bb_{tt})).t^{-2+2\alpha}(2b_\theta{}^2 - bb_{\theta\theta})$$
$$-t^{-4+2\alpha}(-(1-\alpha)bb_\theta + 2tb_tb_\theta - tbb_{t\theta})^2$$
$$= bt^{-4+2\alpha}K_0 \quad \geqslant 0 \text{ by (iv).}$$

$$q_2 = -(1-\alpha)t^{-1-2\alpha}((1-2\alpha)(1-3\alpha)b+(5\alpha-1)tb_t+t^2b_{tt}).(-(1-\alpha)t^{-1}b_{\theta\theta})$$
$$-(1-\alpha)^2t^{-2-2\alpha}((1-3\alpha)b_\theta-tb_{t\theta})^2$$
$$= (1-\alpha)^2t^{-2-2\alpha}K_2 \quad \geqslant 0 \text{ by (v).}$$

$$q_1 = t^{-2}((1-\alpha)(1-2\alpha)b^2 - 3(1-\alpha)tbb_t + t^2(2b_t{}^2-bb_{tt})).(-(1-\alpha)t^{-1}b_{\theta\theta})$$
$$-(1-\alpha)t^{-1-2\alpha}((1-2\alpha)(1-3\alpha)b + (5\alpha-1)tb_t + t^2b_{tt}).t^{-2+2\alpha}(2b_\theta{}^2-bb_{\theta\theta})$$
$$-2t^{-2+\alpha}(-(1-\alpha)bb_\theta+2tb_tb_\theta-tbb_{t\theta}).(1-\alpha)t^{-1-\alpha}((1-3\alpha)b_\theta-tb_{t\theta})$$
$$= 2(1-\alpha)t^{-3}K_1.$$

Hence $4q_0q_2-q_1{}^2 = 4bt^{-4+2\alpha}K_0.(1-\alpha)^2t^{-2-2\alpha}K_2 - 4(1-\alpha)^2t^{-6}K_1{}^2$
$= 4t^{-6}(1-\alpha)^2(K_0K_2b-K_1{}^2).$

Thus by (vi), either $q_1 \geqslant 0$ or $4q_0q_2-q_1{}^2 \geqslant 0$, so that (4.6) is satisfied. This completes the proof of Theorem 2. □

5. POWER CONCAVITY AND LOG CONCAVITY THEOREMS

In this section, the maximum principles of the previous two sections are combined with boundary data to obtain proofs of α-concavity.

THEOREM 3: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

(i) $u \in C(\overline{\Omega})$.

(ii) $u(z)-u(y) < \limsup_{t \to 0^+} t^{-1}(u(y+t(z-y))-u(y))$ for all $y \in \partial\Omega$ and $z \in \overline{\Omega}$ such that $[y,z]$ (the straight line segment joining y to z) is not a subset of $\partial\Omega$.

The restriction of u to Ω satisfies all of the assumptions of Theorem 1.

Assertion:

u is concave.

PROOF: By (i), c is continuous on the compact set $\overline{\Omega}\times\overline{\Omega}\times[0,1]$, and so $\overline{c}$ must be attained at some $(y,z,\lambda) \in \overline{\Omega}\times\overline{\Omega}\times[0,1]$. Suppose $\overline{c} > 0$. Then by Theorem 1, $(y,z,\lambda) \notin \Omega\times\Omega\times[0,1]$. But $c(y,z,\lambda) = 0$ when $\lambda = 0$ or 1. The remaining possibility is that (y,z,λ) is in $\partial\Omega\times\overline{\Omega}\times(0,1)$ (or $\overline{\Omega}\times\partial\Omega\times(0,1)$).

Suppose $y \in \partial\Omega$, $z \in \overline{\Omega}$ and $\lambda \in (0,1)$. If $[y,z]$ is a subset of $\partial\Omega$ then $c(y,z,\lambda) = 0$. Otherwise, by (ii), there exists $t \in (0,\lambda)$ such that

$$u((1-t)y + tz) > (1-t)u(y) + tu(z).$$

Write $y' = (1-t)y + tz$ and $\mu = (\lambda-t)/(1-t)$, so that $(1-\mu)y' + \mu z = (1-\lambda)y + \lambda z$. Then

$$\begin{aligned} c(y',z,\mu) &= (1-\mu)u(y') + \mu u(z) - u((1-\mu)y' + \mu z) \\ &> (1-\mu)(1-t)u(y) + ((1-\mu)t + \mu)u(z) - u((1-\mu)y' + \mu z) \\ &= (1-\lambda)u(y) + \lambda u(z) - u((1-\lambda)y + \lambda z) \\ &= c(y,z,\lambda). \end{aligned}$$

So c does not attain its maximum at this choice of (y,z,λ). The result is the same if $(y,z) \in \overline{\Omega}\times\partial\Omega$. Since all choices of (y,z,λ) lead to contradictions, we conclude that $\overline{c} = 0$. That is, u is concave in $\overline{\Omega}$. □

THEOREM 4: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$\alpha \in (0,1]$.

$u \in C(\overline{\Omega})$, and $u|_{\partial\Omega} = 0$.

$u(z) < \limsup_{t\to 0^+} t^{-1/\alpha}\, u(y+t(z-y))$ for all $y \in \partial\Omega$ and $z \in \Omega$.

The restriction of u to Ω satisfies all of the conditions of Theorem 2 for $\alpha \in (0,1]$.

Assertion:

u^α is concave in $\overline{\Omega}$.

PROOF: This theorem follows immediately from Theorems 2 and 3 by defining $v = u^\alpha$ and applying Theorem 3 to v, since v thus defined satisfies the assumptions of Theorem 1, and $t^{-1}v =$ ~~$(t^{-1/\alpha})\alpha$.~~ $(t^{-1/\alpha}u)^\alpha$ □

THEOREM 5: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$\partial\Omega$ is C^2 and Ω is uniformly convex (that is, the principal curvatures of the boundary are bounded below by a positive constant).

$u \in C^2(\overline{\Omega})$, and $u|_{\partial\Omega} = 0$.

$u_\nu|_{\partial\Omega} > 0$, where ν denotes the interior normal of $\partial\Omega$.

The restriction of u to Ω satisfies the assumptions of Theorem 2 for $\alpha = 0$.

Assertion:

log u is concave in Ω.

PROOF: The case $\alpha = 0$ is much harder than $\alpha \in (0,1]$ because the unboundedness of log u forces us to use limit arguments near the boundary of Ω. Suppose log u is not concave in Ω, and define

$$c(y,z,\lambda) = (1-\lambda)v(y) + \lambda v(z) - v((1-\lambda)y+\lambda z) \tag{5.1}$$

for $(y,z,\lambda) \in \Omega\times\Omega\times[0,1]$, and $\overline{c} = \sup c$, where $v = \log u$. Then $\overline{c} \in (0,\infty]$. So there exists a sequence of points (y_i,z_i,λ_i) such that $\lim_{i\to\infty} c(y_i,z_i,\lambda_i) = \overline{c}$, and (choosing a subsequence if necessary) $\lim_{i\to\infty} (y_i,z_i,\lambda_i) = (y,z,\lambda)$ for some $(y,z,\lambda) \in \overline{\Omega}\times\overline{\Omega}\times[0,1]$. Theorem 2 implies that $(y,z) \notin \Omega\times\Omega$. If $y \in \partial\Omega$, $z \in \Omega$, and $\lambda > 0$, then $\lim_{i\to\infty} c(y_i,z_i,\lambda_i) = -\infty$, which is impossible. So $(y,z,\lambda) \notin \partial\Omega\times\Omega\times(0,1]$. Similarly, $(y,z,\lambda) \notin \Omega\times\partial\Omega\times[0,1)$. The only possibilities remaining to be excluded are that $(y,z,\lambda) \in \partial\Omega\times\Omega\times\{0\}$ (which is similar to $\Omega\times\partial\Omega\times\{1\}$), or $(y,z) \in \partial\Omega\times\partial\Omega$. In the latter case, clearly $y = z$, as Ω is uniformly convex. The following lemma (equivalent to results by Korevaar [7], and Caffarelli and Spruck [2]) removes these two remaining possibilities. As explained in the introduction, Theorem 5 and Lemma 6 are presented here principally for completeness, and are not original. We write $d(y,z)$ for the euclidean distance between y and z, and $d(x)$ for the distance from x to $\partial\Omega$.

LEMMA 6: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$\partial\Omega$ is C^2, and Ω is uniformly convex.

$u:\overline{\Omega} \to \mathbb{R}$, $u|_{\partial\Omega} = 0$, and $u|_{\Omega} > 0$.

$u \in C^2(\overline{\Omega})$.

$u_\nu|_{\partial\Omega} > 0$.

$f:(0,\infty) \to \mathbb{R}$ is a C^2 function satisfying (i) to (v):

(i) $f' > 0$

(ii) $\lim_{t \to 0^+} f'(t) = \infty$

(iii) $f'' < 0$

(iv) $\lim_{t \to 0^+} f(t)/f'(t) = 0$

(v) $\lim_{t \to 0^+} f'(t)/f''(t) = 0.$

$v \in C^2(\Omega)$ is defined by $v = f \circ u$.

Assertions:

If c is defined in terms of v by equation (5.1), then

(i) for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(d(z) > \varepsilon,\ d(y) < \delta \text{ and } \lambda < \delta) \Rightarrow c(y,z,\lambda) \leq 0$$

(ii) for some $\delta > 0$,

$$(d(y),\ d(z),\ d(y,z) < \delta) \Rightarrow c(y,z,\lambda) \leq 0.$$

PROOF: To prove assertion (ii), we define the derivatives of u in the direction $\theta \in S$ by $u_\theta = \theta_i u_i$, $u_{\theta\theta} = \theta_i \theta_j u_{ij}$ and so forth, and commence by showing that for x in a neighbourhood of $\partial\Omega$, $v_{\theta\theta}(x) \leq 0$ for all θ close enough to the tangent plane of the level curve of v passing through x. Then it is shown that for x in a neighbourhood of $\partial\Omega$, $v_{\theta\theta}(x) \leq 0$ for all other directions θ, and assertion (ii) readily follows, since v is then concave in a neighbourhood of $\partial\Omega$.

Consider the metric space $\overline{\Omega} \times S$ with the metric induced by $\mathbb{R}^{2n}$, and define

$$T = \{(x,\theta) \in \overline{\Omega} \times S;\ x \in \partial\Omega \text{ and } \theta \cdot \nu(x) = 0\}.$$

Then T is a compact subset of $\overline{\Omega} \times S$, as $\partial\Omega$ is compact and ν is continuous on $\partial\Omega$.

Section 5

Because $u_\nu|_{\partial\Omega} > 0$ and Du is continuous on $\bar{\Omega}$, there is a $\delta_0 > 0$ such that for y satisfying $d(y,\partial\Omega) < \delta_0$, $|Du(y)|$ is bounded below by a positive constant, so that ν may be extended to such y as a continuous function by writing $\nu(y) = Du(y)/|Du(y)|$. For $\delta \in (0,\delta_0)$, define

$$G(\delta) = \{(y,\phi) \in \bar{\Omega}\times S;\ d(y,\partial\Omega) < \delta \text{ and } \phi\cdot\nu(y) < \delta\}.$$

Then $G(\delta)$ is an open neighbourhood in $\bar{\Omega}\times S$ of T.

By the compactness of T and the continuity of ν, for each δ there is a $\delta_1 > 0$ such that whenever $d(y,\partial\Omega) < \delta_1$, there exists $x \in \partial\Omega$ for which $d(\nu(y),\nu(x)) < \delta$. Consequently, for $(y,\phi) \in G(\min(\delta_1,\delta))$ there exists $(x,\theta) \in T$ such that

$$\begin{aligned} d(\phi,\theta) &\leq \phi\cdot\nu(x) + d(\nu(y),\nu(x)) \\ &< 2\delta, \end{aligned}$$

and therefore $d((y,\phi),T) < \sqrt{5}.\delta$. Thus every neighbourhood in $\bar{\Omega}\times S$ of T contains a set of the form $G(\delta)$ for some $\delta > 0$.

For $x \in \partial\Omega$, and $\theta \in S$ such that $\theta\cdot\nu(x) = 0$ (that is, such that θ is a tangential vector at x), write $k_\theta(x)$ for the curvature of $\partial\Omega$ at x in the direction θ. Then an often-made calculation shows that $u_{\theta\theta}(x) = -k_\theta(x)u_\nu(x)$. From the uniform convexity of $\partial\Omega$ and the fact that $u_\nu|_{\partial\Omega} > 0$ and $u_\nu \in C(\partial\Omega,\mathbb{R}^n)$, it follows that $u_{\theta\theta}(x) < 0$ for all $x \in \partial\Omega$ and $\theta \in S$ such that $\theta\cdot\nu(x) = 0$. That is, $u_{\theta\theta}(x)$ is negative on T. But the function on $\bar{\Omega}\times S$ which maps (y,ϕ) to $u_{\phi\phi}(y)$ is continuous (as $u \in C^2(\bar{\Omega})$). So for some $\delta > 0$, $u_{\phi\phi}(y)$ is negative for (y,ϕ) in $G(\delta)$.

In Ω, $v_{ij} = f'(u)u_{ij} + f''(u)u_iu_j$, so that

$$v_{\theta\theta} = f'(u)u_{\theta\theta} + f''(u){u_\theta}^2. \tag{5.2}$$

It then follows from the properties (i) and (iii) of f, that $v_{\phi\phi}(y) < 0$

for $(y,\phi) \in G(\delta)$.

Next consider $G'(\delta)$, defined by:

$$G'(\delta) = \{(y,\phi) \in \overline{\Omega} \times S;\ d(y,\partial\Omega) < \delta \text{ and } \phi \cdot \nu(y) \geq \delta\}.$$

It follows from (5.2) and property (v) of f that for some $\delta > 0$, $v_{\phi\phi}(y)$ is negative for (y,ϕ) in $G'(\delta)$, because $u_{\phi\phi}$ is bounded, and $u_\phi(y) = |Du(y)|\phi \cdot \nu(y)$ is bounded below on $G'(\delta)$ by a positive constant. The results for $G(\delta)$ and $G'(\delta)$ taken together imply that $v_{\phi\phi}(y) < 0$ for y in some neighbourhood in $\overline{\Omega}$ of $\partial\Omega$, from which assertion (ii) readily follows (by halving δ and noting that then v is concave at all points of the line segment joining y to z).

To prove assertion (i), (~~the necessity of which appears to have been overlooked~~ which was not necessary in [2] (since λ was assumed constant in their convexity maximum principle) and is met by a different method in [7]~~)~~, we let y and z be points in Ω and note that

$$c(y,z,\lambda) = (1-\lambda)(v(y)-v(y+\lambda(z-y))) - \lambda v((1-\lambda)y+\lambda z) + \lambda v(z).$$

By the mean value theorem there exists $t \in (0,\lambda)$ such that

$$v(y)-v(y+\lambda(z-y)) = -\lambda(z-y) \cdot Dv(y+t(z-y)).$$

The proof of assertion (ii) shows that for small enough $d(y,\partial\Omega)$ and λ, v is concave (and hence Dv is monotonic) along the line segment joining y to $y+\lambda(z-y)$. So

$$(z-y) \cdot Dv(y+\lambda(z-y)) \leq (z-y) \cdot Dv(y+t(z-y)).$$

Hence: $v(y)-v(y+\lambda(z-y)) \leq -\lambda(z-y) \cdot Dv(y+\lambda(z-y))$

$$= -\lambda f'(u(y+\lambda(z-y)))\,(z-y) \cdot Du(y+\lambda(z-y)).$$

In view of the positivity of u_ν on $\partial\Omega$, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $(z-y) \cdot Du(y+\lambda(z-y))$ is bounded below by a positive constant, μ say, on the set of (y,z,λ) such that $d(y,\partial\Omega) < \delta$, $\lambda < \delta$ and $d(z,\partial\Omega) > \varepsilon$. Then putting $x = (1-\lambda)y+\lambda z$, we have:

$$c(y,z,\lambda) \leq -\lambda\big(f'(u(x)).\big((1-\lambda)\mu+f(u(x))/f'(u(x)) \big) + v(z) \big),$$

from which assertion (i) can be deduced by applying assumptions (iv) and then (ii). □

Continuing with the proof of Theorem 5, we note that $f(u)=\log(u)$ satisfies the conditions of Lemma 6, and so eliminates the remaining possibilities for the limit point (y,z,λ). So log u is concave. □

It should be remarked here that f defined by $f(t) = t^{\alpha}$ for $\alpha > 0$ satisfies assumptions (i) to (v) of Lemma 6, thereby providing an alternative way of dealing with boundary behaviour for Theorem 4.

6. APPLICATIONS TO BOUNDARY VALUE PROBLEMS

The general concavity theorems of the previous section are applied in this section to particular boundary value problems by making some simple calculations. Theorem 7 was proved by the present author in [5] by a method different to that given here, but the technique used there leads to weaker generalisations to non-linear problems and involves harder calculations. Theorem 7 was earlier proved for $n = 2$ and constant f by Makar-Limanov ([9]). Lemma 8 enables the result of Theorem 7 to be extended from C^2 β-concave functions f to general β-concave functions. Theorem 9 gives conditions which will imply α-concavity when $b(u) = -\Delta u$ is independent of the variable $x \in \mathbb{R}^n$. Theorem 12 applies Theorem 1 and Lemma 6 to Liouville's problem.

THEOREM 7: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$u: \overline{\Omega} \to \mathbb{R}$, $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, $u|_{\partial\Omega} = 0$.

$\alpha \in [\frac{1}{3}, \frac{1}{2})$.

For $x \in \Omega$, u satisfies

$$\Delta u(x) + f(x) = 0, \tag{6.1}$$

where $f: \Omega \to \mathbb{R}$ is a positive β-concave function for $\beta = \alpha/(1-2\alpha)$. (Hence $\alpha = \beta/(1+2\beta)$ and $\beta \geq 1$).

Assertions:

(i) u is α-concave in $\overline{\Omega}$.

(ii) If f is a positive constant in Ω, then $\sqrt{u}$ is concave. (That is, if f is $(+\infty)$-concave, then u is $\frac{1}{2}$-concave.)

REMARK 1: The boundary value problem in Theorem 7 has a solution for all f, since by Jensen's inequality, if f^β is concave then f is concave, and f is therefore bounded and locally Lipschitz continuous in Ω, so that by Schauder theory ([4] Thm. 6.13, p.101) a C^2 solution u exists which satisfies the boundary condition.

REMARK 2: Any non-negative function f on Ω which is β-concave for some $\beta \geq 1$ is either identically zero in Ω or else positive at all points of Ω.

PROOF: Assertion (ii) follows readily from (i), since if f is a positive constant then f is β-concave for all $\beta \geq 1$, and so by part (i), u is α-concave for all $\alpha < \frac{1}{2}$. Then by property 3 of α-concavity, u is $\frac{1}{2}$-concave.

To prove assertion ~~(ii)~~ (i), we first suppose f to be a C^2 function. Then to apply Theorem 4 we only need to show that b defined by $b(x,t) = f(x)$ satisfies assumptions (i) to (vi) of Theorem 2 and the last assumption of Theorem 4.

$(1-\alpha)b - tb_t = (1-\alpha)f(x)$ is positive because $\alpha < 1$ and $f(x) > 0$, so that (i) is satisfied. (ii) is clearly satisfied whenever $\alpha \in [\frac{1}{3},\frac{1}{2}]$, since b does not depend on t. (iii) holds because f is concave. (iv), (v) and (vi) require only that $(1-3\alpha)b_\theta{}^2 - (1-2\alpha)bb_{\theta\theta}$ be non-negative. But the β-concavity of f implies (see property 4) that

$$0 \geq ff_{\theta\theta} + (\beta-1)f_\theta{}^2$$
$$= -(1-2\alpha)^{-1}((1-3\alpha)f_\theta{}^2 - (1-2\alpha)ff_{\theta\theta}),$$

since $\beta-1 = -(1-3\alpha)/(1-2\alpha)$. Hence the conditions of Theorem 2 are satisfied for all $\alpha \in [\frac{1}{3},\frac{1}{2})$. For Theorem 4 we must show:

$$\limsup_{t\to 0^+} t^{-1/\alpha}u(y+t(z-y)) > u(z) \text{ for all } y \in \partial\Omega \text{ and } z \in \Omega.$$

But as was shown in [5] (Lemma 2.2) by means of a sub-barrier argument, if f is bounded below by a positive constant then for such y and z, there exists $k > 0$ such that $u(y+t(z-y)) \geq kt^2$ for all small enough $t > 0$. Thus

$$t^{-1/\alpha}u(y+t(z-y)) \geq kt^{-(1-2\alpha)/\alpha},$$

which becomes arbitrarily large for small $t > 0$ when $\alpha \in [\frac{1}{3},\frac{1}{2})$, and the conditions of Theorem 4 are all satisfied. So u is α-concave for any f bounded below by a positive constant. But property 6 in section 2 shows that if f is altered by the addition of a positive constant, f is still β-concave, and the solution corresponding to this altered f, and the same boundary data, converges to u as the added constant tends to zero. So u is α-concave even when f is not bounded below by a positive constant. In conclusion, then, u is α-concave if f is a C^2 β-concave function.

In the general case where f is not assumed to be C^2, f must be approximated by C^2 functions. The following lemma establishes the existence of the appropriate approximations.

LEMMA 8: Assumptions:

$n \geq 1$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$f: \Omega \to \mathbb{R}$, $f \geq 0$, $\beta > 0$, and f is β-concave in Ω.

Assertions:

(i) There exists a sequence (f_i) of non-negative β-concave functions in $C(\overline{\Omega}) \cap C^2(\Omega)$, and a number $k > 0$, such that $f_i - f \to 0$ uniformly on relatively compact open subsets of Ω, and $f_i(x) \leq k$ for all $x \in \Omega$ and all i.

(ii) For such a sequence (f_i), if u_i denotes the solution in $C(\overline{\Omega}) \cap C^2(\Omega)$ (which exists and is unique) of the problem

$$\Delta u_i + f_i = 0 \text{ in } \Omega$$

$$\text{and } u_i = 0 \text{ on } \partial\Omega$$

then $u_i - u \to 0$ uniformly in Ω, where u is the solution of (6.1)

PROOF: If this lemma holds for $\beta = 1$, then the result for other values of β follows by substituting f^β for f, since the mapping $z \to z^\beta$ is uniformly continuous on intervals $[0,a]$ for all $a > 0$. So suppose $\beta = 1$. Then the concavity of f implies that for all y in Ω there exists an affine function π_y: $\mathbb{R}^n \to \mathbb{R}$ whose graph is an upper supporting hyperplane for the graph of f at $(y, f(y)) \in \mathbb{R}^{n+1}$. π_y is not unique in general, but given any choice of π_y at each $y \in \Omega$, f satisfies:

$$f(x) = \inf_{y \in \Omega} \pi_y(x) \text{ for } x \in \overline{\Omega}.$$

Define a sequence (g_i) by

$$g_i(x) = \inf_{j \leq i} \pi_j(x) \quad \text{for } x \in \mathbb{R}^n,$$

where (π_j) is the sequence of affine functions corresponding to any sequence (y_i) in Ω such that $\{y_i\}$ is dense in $\overline{\Omega}$. Then define the sequence (f_i) by:

$$f_i = \phi_i * g_i - \inf_{\Omega}(\phi_i * g_i),$$

where ϕ is any non-negative function in $C^2(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n;\ |x| < 1\}$ and integral equal to 1, and ϕ_i is defined in terms of ϕ by $\phi_i(x) = i^n \phi(ix)$ for $x \in \mathbb{R}^n$. The convolution of any concave function with a non-negative function with compact support is concave. So each f_i (restricted to Ω) is a non-negative concave function in $C^2(\Omega)$. Since $g_i - f \to 0$ uniformly on relatively compact open subsets of Ω, and a standard argument regarding mollifiers shows that $f_i - g_i \to 0$

~~similarly~~, ~~and~~ assertion (i) follows when it is noted the sequence (f_i) is uniformly bounded in Ω.

To prove part (ii), it is useful to note that the solution b of $\Delta b+k = 0$ on Ω with zero Dirichlet data is uniformly continuous on $\overline{\Omega}$ and majorises all functions u_i defined in assertion (ii), and the same is true of u. Hence for all $\varepsilon > 0$, there is a relatively compact open subset Ω_ε of Ω such that for all i, $|u_i-u| < \varepsilon/2$ on $\Omega\sim\Omega_\varepsilon$. But by part (i), $|\Delta(u_i-u)| \to 0$ uniformly on Ω_ε, and so for large enough i, (using the barrier b to estimate the supremum of $|u_i-u|$ on Ω_ε in terms of the supremum of $|\Delta(u_i-u)|$ on Ω_ε,) $|u_i-u| < \varepsilon/2$ on Ω. This completes the proof of part (ii). □

The proof of Theorem 7 may now be completed by approximating the function f by the sequence (f_i) of Lemma 8 and using the result obtained above for C^2 β-concave functions to assert that each u_i is α-concave, so that by property 8, u is α-concave. □

The following theorem, dealing as it does with non-linear problems, naturally raises questions of existence, uniqueness and regularity. These are not addressed here, although the case $h(t) = t^\gamma$ for $0 < \gamma < 1$ can be shown to have a non-trivial solution by variational methods. Thus this theorem may be regarded as providing an a priori estimate of the concavity number.

THEOREM 9: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

Ω satisfies an interior sphere condition.

$u: \overline{\Omega} \to \mathbb{R}$, $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, $u|_{\partial\Omega} = 0$, $u|_\Omega > 0$.

$\alpha \in (0,1)$.

For $x \in \Omega$, u satisfies

$$\Delta u(x) + h(u(x)) = 0,$$

where $h: (0,\infty) \to (0,\infty)$ is a C^2 function such that (i) and (ii) hold for all $t > 0$:

(i) $t^{\alpha-1}h(t)$ is strictly decreasing with respect to t, (or $(1-\alpha)h - th_t > 0$)

(ii) $t^{(1-2\alpha)/\alpha}h(t^{-1/\alpha})$ is concave with respect to t, (or $(1-2\alpha)(1-3\alpha)h + (5\alpha-1)th_t + t^2h_{tt} \leq 0$)

Assertions:

u is α-concave in $\overline{\Omega}$.

In particular, if $h(t) = kt^{\gamma}$ for some $k > 0$ and $\gamma \in (0,1)$, then u is $(\frac{1}{2}(1-\gamma))$-concave.

PROOF: All of the conditions of Theorem 2 are met. So Theorem 4 can be applied if the boundary condition,

$$u(z) < \limsup_{t \to 0^+} t^{-1/\alpha}\, u(y+t(z-y)) \text{ for all } y \in \partial\Omega \text{ and } z \in \Omega,$$

is satisfied. But the Hopf maximum principle easily guarantees that the right hand side is infinite when $\alpha \in (0,1)$.

It remains to show that the function $h(t) = kt^{\gamma}$ has the required properties. Substituting $\gamma = 1-2\alpha$ gives:

$t^{\alpha-1}h(t) = t^{-\alpha}$, which is decreasing, and

$t^{(1-2\alpha)/\alpha}h(t^{-1/\alpha}) = 1$, which is a concave function. □

REMARK: The style of maximum principle presented in the appendix to this thesis may be used to obtain Theorem 9 in a restricted form. It leads to the result for $h(t) = kt^{\gamma}$, for instance, but with the condition that $\alpha \in [\frac{1}{3},\frac{1}{2})$ (that is, $\gamma \in (0,\frac{1}{3}]$). In fact, its application is always limited in this way to $\alpha \in [\frac{1}{3},\frac{1}{2})$. This is the reason for which the author has preferred to adopt the methods presented here.

THEOREM 10: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$\partial\Omega$ is C^2 and Ω is uniformly convex. (See Lemma 6.)

$u: \overline{\Omega} \to \mathbb{R}$, $u \in C^2(\overline{\Omega})$, $u|_{\partial\Omega} = 0$, $u|_{\Omega} > 0$.

$u_\nu|_{\partial\Omega} > 0$.

For $x \in \Omega$, u satisfies

$$\Delta u(x) + h(u(x)) = 0,$$

where $h: (0,\infty) \to (0,\infty)$ is a C^2 function such that $e^t h(e^{-t})$ is concave and strictly increasing with respect to t for all $t \in \mathbb{R}$.

Assertion:

u is 0-concave in Ω.

PROOF: Theorem 10 is a straightforward consequence of Theorem 5. (Both Theorem 10 and Theorem 5 are equivalent to results obtained by Korevaar, and are thus only given here for completeness.) □

Lemma 11 and Theorem 12 deal with Liouville's problem. In $\mathbb{R}^2$, the solution to this problem is, under appropriate conditions, the velocity potential for the path of a free vortex under the influence of an otherwise irrotational flow in a simply connected domain ([10]). When the domain is convex, Theorem 12 implies that a free vortex moves in a convex path (that is, a path which bounds a convex set), and that there is a unique interior point at which a free vortex will remain stationary.

LEMMA 11: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$\partial\Omega$ is C^2 and Ω is uniformly convex.

$u: \overline{\Omega} \to \mathbb{R}$, $u|_{\partial\Omega} = 0$, $u|_{\Omega} > 0$.

$u \in C^2(\overline{\Omega})$.

$u_\nu|_{\partial\Omega} > 0$.

For $x \in \Omega$, ϕ satisfies

$$\Delta\phi(x) = \exp(\phi(x)),$$

where ϕ is defined by $\phi = -2\log(u)$.

Assertion:

ϕ is convex.

PROOF: Let $v = -\phi$. Then v satisfies:

$$\Delta v + \exp(-v) = 0 \text{ in } \Omega.$$

After putting $b(t) = \exp(-t)$, and noting that b is strictly decreasing and $1/b = \exp(t)$ is convex, Theorem 1 may be applied to the equation $\Delta v + b(v) = 0$ to show that the convexity function for v does not attain its maximum in $\Omega\times\Omega\times[0,1]$. Then Lemma 6 can be applied to u and v with $f(t) = \log(t)$ to show that the convexity function for v is non-positive in a suitable neighbourhood of the boundary of $\Omega\times\Omega$. Hence v is concave in Ω, and so ϕ is convex in Ω, as promised. □

THEOREM 12: Assumptions:

Ω is a bounded convex domain in $\mathbb{R}^2$.

$\phi \in C^2(\Omega)$.

$\phi(x) \to +\infty$ as $d(x,\partial\Omega) \to 0$.

For $x \in \Omega$, ϕ satisfies

$$\Delta\phi(x) = \exp(\phi(x)).$$

Section 6

Assertion:

ϕ is convex.

REMARK: The existence and uniqueness of a solution to the above problem are well known ([10]).

PROOF: Let $u = \exp(-\phi/2)$. Suppose first that $\partial\Omega$ is an analytic curve and that Ω is uniformly convex. It is known ([10], p.324) that u can be expressed in terms of any conformal mapping g from Ω onto an open unit disc as follows:

$$u(z) = k|g'(z)|^{-1}(1-|g(z)|^2)$$

for all $z \in \Omega$, where $k = 1/\sqrt{8}$, and g' denotes the complex derivative of g. When $\partial\Omega$ is an analytic curve, g has an analytic extension to a neighbourhood of $\overline{\Omega}$, and so $|g'|$ is bounded below by a positive constant on Ω, $g \in C^2(\overline{\Omega})$, and u_ν can be calculated on $\partial\Omega$. In fact, since $|g| = 1$ on $\partial\Omega$, $1-|g| = |g'|d(z,\partial\Omega) + O(d(z,\partial\Omega)^2)$ as $z \to \partial\Omega$, and so, writing $u = k(1+|g|)(1-|g|)/|g'|$, it easily follows that $u_\nu|_{\partial\Omega} = 1/\sqrt{2}$, which is positive. Thus all of the requirements of Lemma 11 are satisfied, and so ϕ is convex.

To deal with non-analytic boundaries, Ω will now be approximated externally by a set Ω' and internally by a set Ω''. It will be shown that an arbitrary bounded convex domain Ω may be covered by a uniformly convex open set Ω' such that $\partial\Omega'$ is an analytic curve, and $d = \sup\{d(x,\Omega);\ x \in \partial\Omega'\}$ is as small as desired. This is straightforward if $\partial\Omega'$ is only required to be C^2. So suppose Ω has been approximated by a set bounded by such a C^2 curve. Then it must be shown that this curve in turn can be approximated by a suitable analytic curve. This will be done in terms of polar coordinates. Assuming without loss of generality that $0 \in \Omega$, the C^2 curve may be represented by a C^2 function $r: [0,2\pi] \to (0,\infty)$ such that the curvature,

$$\kappa = (r^2+2(r')^2-rr'')(r^2+(r')^2)^{-3/2},$$

is uniformly positive, and the derivatives $r^{(i)}$ satisfy

$$r^{(i)}(0) = r^{(i)}(2\pi) \text{ for } i = 0,1 \text{ and } 2. \tag{6.2}$$

The analytic functions on $\mathbb{R}$, restricted to $[0,2\pi]$ and satisfying (6.2), are dense in the subspace of $C^2[0,2\pi]$ satisfying (6.2). To show this, it is sufficient to obtain first an analytic approximation to r which does not necessarily satisfy the endpoint conditions, and then satisfy the $i = 2$ condition by adding a linear function to the second derivative of the approximation. The modulus of the linear function need not be greater than the C^2 norm of the difference between r and the approximation. This altered approximation may be integrated on $[0,2\pi]$ to obtain a function which can similarly be adjusted at the endpoints by the addition of a linear function. When this has been done to satisfy (6.2) for all i, it is only necessary to finish by noting that the integral on a bounded interval of a small function is also small, and that the addition of a linear function to a derivative does not affect higher derivatives.

So an analytic function ρ may be constructed for which κ is uniformly positive, the conditions at 0 and 2π are satisfied, $\rho \geqslant r$, and $|\rho-r|$ is as small as desired. Then ρ represents an appropriate set Ω'. On Ω', a C^2 function ϕ' may be defined by

$$\Delta\phi' = \exp(\phi') \text{ in } \Omega'$$

$$\text{and } \phi'(x) \to +\infty \text{ as } d(x,\partial\Omega') \to 0.$$

By Lemma 11, ϕ' is convex. It may be supposed, by translating Ω if necessary, that Ω covers an open disc with centre 0 and radius $r > 0$. For $x' \in \Omega'$ write $x'' = ax'$, where $a = r/(d+r)$ and d is as defined above. A simple geometrical argument using separating hyperplanes shows that $x'' \in \Omega$, and so Ω'' defined by

$$\Omega'' = a\Omega' = \{ax;\ x \in \Omega'\}$$

is a subset of Ω. Define ϕ'' in Ω'' by

$$\phi''(x) = \phi'(a^{-1}x) - 2\log a \quad \text{for } x \in \Omega''.$$

Then ϕ'' satisfies $\Delta\phi'' = a^{-2}\Delta\phi' = a^{-2}\exp(\phi''+2\log a) = \phi''$ in Ω'', and $\phi''(x) \to +\infty$ as $d(x,\partial\Omega'') \to 0$. A comparison principle for Liouville's equation ([10] Lemma 2) implies that $\phi' \leqslant \phi$ in Ω because $\phi' \leqslant \phi$ on $\partial\Omega$, and similarly $\phi \leqslant \phi''$ in Ω''. So in Ω'' we have $\phi' \leqslant \phi \leqslant \phi''$. But as $d \to 0$, $a \to 1$, so that $\phi''-\phi' \to 0$ pointwise in Ω. Hence ϕ is the pointwise limit of the convex functions ϕ' and so must also be convex. □

The uniqueness of the stationary points of an isolated vortex in an otherwise irrotational flow in a bounded convex domain in $\mathbb{R}^2$ - that is, points where $\Delta\phi = 0$ - follows from the convexity and analyticity of ϕ.

7. COUNTEREXAMPLES TO SOME PLAUSIBLE GENERALISATIONS

This section commences with a proof (Theorem 14) that the number $\alpha = \beta/(1+2\beta)$ in Theorem 7 is sharp as is the number $\alpha = \frac{1}{2}$ when f is constant. Lemma 13 is useful for showing that certain functions are not α-concave.

LEMMA 13: Assumptions:

$n \geq 2$, and Ω is a convex domain in $\mathbb{R}^n$.

$u \in C(\overline{\Omega})$, $u|_{\partial\Omega} = 0$, and $u|_{\Omega} > 0$.

$\alpha > 0$ and u is α-concave in Ω.

$y \in \partial\Omega$ and $z \in \Omega$.

Assertion:

$$\liminf_{t \to 0^+} t^{-1/\alpha} u(y+t(z-y)) > 0.$$

PROOF: For $t \in (0,1)$, the concavity of u^α implies that

$$\begin{aligned} u^\alpha(y+t(z-y)) &\geq (1-t)u^\alpha(y) + tu^\alpha(z) \\ &= tu^\alpha(z), \end{aligned}$$

from which the assertion follows, since $t^{-1/\alpha}u = (t^{-1}u^\alpha)^{-1}$. □

Now let $n \geq 2$ and $x \in \mathbb{R}^n$, and write $x_n = x \cdot e_n$ and $x' = x - x_n e_n$, where $e_n = (0,\ldots0,1)$ is the nth unit vector in the standard basis for $\mathbb{R}^n$. Define an infinite open cone K for $a \in (0,1)$ by $K = \{x \in \mathbb{R}^n;\ |x'| < ax_n\}$.

THEOREM 14: Assumptions:

Ω is a bounded convex domain in $\mathbb{R}^n$ and a subset of K.

$0 \in \partial\Omega$ and $e_n \in \Omega$.

$\beta \geq 1$.

(7.1) $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$\Delta u(x) + f(x) = 0 \text{ for } x \in \Omega,$$

$$\text{and } u(x) = 0 \text{ for } x \in \partial\Omega,$$

where $f(x) = k_1 x_n^q - k_2 x_n^{q-2}|x'|^2$ for $x \in \Omega$, $q = \beta^{-1}$,

$k_1 = 2(n-1) - a^2(q+1)(q+2)$, and $k_2 = q(1-q)$.

(7.2) $a^2 \leq (n-1)/(2+q^2)$.

Assertions:

(i) f is non-negative and f is β-concave.

(ii) u is not α-concave when $\alpha > \beta/(1+2\beta)$.

(iii) For $q = 0$, f is $(+\infty)$-concave (that is, constant), and u is not α-concave for $\alpha > \frac{1}{2}$.

REMARK: Problem (7.1) has a unique solution, as f is a bounded locally Lipschitz function in Ω.

PROOF: (i) For $x \in K$, $|x'|^2 \leq a^2 x_n^2$, and so by (7.2)

$$\begin{aligned} f(x) &\geq x_n^q(2(n-1)-a^2(q+1)(q+2)-q(1-q)a^2) \\ &\geq 2(n-1)(1-q)^2 x_n^q/(2+q^2) \\ &\geq 0. \end{aligned}$$

If $q = 1$ then $f^\beta = f = 2(n-1-3a^2)x_n$, which is a non-negative concave function, as required for $\beta = 1$. Suppose that $q \in (0,1)$. Then f^β is concave if and only if $(k_2^{-1}f)^\beta$ is concave. So without loss of generality k_1 and k_2 may be replaced by $k = k_1 k_2^{-1}$ and 1 respectively. k_1 is non-negative for $q \leq 1$, and so k is also non-negative. It follows from property 4 (section 2) that f is β-concave in Ω if and only if

$$qff_{\theta\theta} + (1-q)f_\theta^2 \leq 0 \quad \text{in} \quad \Omega,$$

for all directions θ, since $\beta-1 = q^{-1}(1-q)$ and $q > 0$. This quantity may be calculated as follows:

$f(x) = x_n^{q-2}(kx_n^2-|x'|^2).$

$f_\theta(x) = x_n^{q-3}(kqx_n^2\theta_n-(q-2)|x'|^2\theta_n-2x_nx'\cdot\theta).$

$f_{\theta\theta}(x)=x_n^{q-4}(kq(q-1)x_n^2\theta_n^2-4(q-2)x_n\theta_nx'\cdot\theta-(q-2)(q-3)\theta_n^2|x'|^2-2x_n^2|\theta'|^2).$

Hence $\frac{1}{2}x_n^{6-2q}(qff_{\theta\theta}+(1-q)f_\theta^2) = (kqx_n^2+(q-2)|x'|^2)(2x_n\theta_nx'\cdot\theta-\theta_n^2|x'|^2)$
$+ x_n^2(2(1-q)(x'\cdot\theta)^2+q|x'|^2|\theta'|^2-kqx_n^2|\theta'|^2).$

But $x'\cdot\theta \leq |x'||\theta'|$, $x_n \geq 0$, and $\theta_n \leq |\theta_n|$, and so by virtue of (7.2),

$$\begin{aligned}(kqx_n^2+(q-2)|x'|^2) &= (k_1x_n^2-(2-q)(1-q)|x'|^2)/(1-q)\\ &\geq x_n^2(2(n-1)-a^2(q+1)(q+2)-(2-q)(1-q)a^2)/(1-q)\\ &\geq (n-1)x_n^2(2-((q+1)(q+2)+(2-q)(1-q))/(2+q^2))/(1-q)\\ &= 0.\end{aligned}$$

Hence $\frac{1}{2}x_n^{6-2q}(qff_{\theta\theta}+(1-q)f_\theta^2) \leq -(kqx_n^2+(q-2)|x'|^2)(|\theta_n||x'|-|\theta'|x_n)^2$, which is non-positive, so that f is indeed β-concave.

(ii) Define a function b: $\overline{K} \to \mathbb{R}$ by $b(x) = x_n^q(a^2x_n^2-|x'|^2)$. Then $b(x) \geq 0$ for all $x\in\overline{K}$, and direct calculation shows that for $x\in K$, $\Delta b(x)+f(x) = 0$. Since Ω is a subset of K, $b(x) \geq 0$ for all $x\in\partial\Omega$. But $\Delta b = \Delta u$ in Ω. So the comparison principle for Δ on Ω implies that $u(x) \leq b(x)$ for all $x\in\overline{\Omega}$. For $t\in(0,1]$, let $x = te_n$. Then $x\in\Omega$ and so $u(x) \leq b(x) = a^2t^{q+2}$. Hence

$$\limsup_{t\to0^+} t^{-1/\alpha}u(x) = 0$$

if $-\alpha^{-1}+q+2 > 0$; that is, if $\alpha > (q+2)^{-1} = \beta/(1+2\beta)$. Then Lemma 13 with $y = 0$ and $z = e_n$ shows that u is not α-concave for such α.

(iii) Putting $q = 0$ in the expressions for f and b makes f a non-negative constant, $2(n-1-a^2)$, while $b(x) = a^2x_n^2-|x'|^2$, so that u^α is not concave when $\alpha > \frac{1}{2}$, using the same kind of argument as in the proof of part (ii). □

Section 7

Whereas Theorem 14 part (iii) showed that $\frac{1}{2}$-concavity for the equation $\Delta u+k = 0$ is sharp, Theorem 15 will show that the 0-concavity proved by Brascamp and Lieb for the fundamental solution for a clamped membrane on a convex set (satisfying $\Delta u+\lambda u = 0$) is sharp. These are the cases $\gamma = 0$ and 1 of the equation $\Delta u+ku^\gamma = 0$, which naturally raises the question of whether the number $\frac{1}{2}(1-\gamma)$ in Theorem 9 is sharp - a question that has not yet been looked into by the author.

THEOREM 15: Assumption:

$\alpha > 0$.

Assertion:

There exists a bounded convex domain Ω in $\mathbb{R}^2$, a number $\lambda > 0$, and a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\Delta u(x) + \lambda u(x) = 0 \text{ for } x \in \Omega$$

$$\text{and } u(x) = 0 \text{ for } x \in \partial\Omega$$

and u is positive in Ω, but not α-concave.

PROOF: It is sufficient to consider the higher order eigenfunctions of the Laplacian on the unit circle, restricted to suitable internodal domains. Let $\theta(x) = \cos^{-1}(x_2|x|^{-1})$ for $x \neq 0$, and $\theta(0) = 0$. It is well known that the function $u: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u(x) = J_m(z_m|x|)\cos(m\theta(x)),$$

for positive integers m, where z_m denotes the least positive zero of the Bessel function J_m, satisfies

$$\Delta u(x) + z_m^2 u(x) = 0 \text{ for all } x \in \mathbb{R}^2.$$

$\Omega = \{x \in \mathbb{R}^2;\ 0 < |x| < 1 \text{ and } m\theta < \pi/2\}$ is a bounded convex domain,

$u(x) = 0$ on $\partial\Omega$ and $u(x) > 0$ for $x \in \Omega$. But J_m has a zero of order m at 0, and so by the same reasoning as that used to prove Theorem 14 part (ii), u is not α-concave for $\alpha > 1/m$, and the theorem follows by taking $m \to \infty$. □

It is reasonable to ask whether the restriction $\alpha \geq \frac{1}{3}$ (that is, $\beta \geq 1$) in Theorem 7 can be removed. Theorem 16 shows that α cannot be extended down to $\alpha = 0$ (that is, $\beta = 0$) in Theorem 7. The question of whether the inequality $\beta \geq 1$ is sharp is thus not answered here.

THEOREM 16: Assumption:

$n \geq 2$.

Assertion:

For some numbers α and β such that $\beta > 0$ and $\alpha = \beta/(1+2\beta)$, there exists a bounded convex domain Ω in $\mathbb{R}^n$ and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $u(x) = 0$ for $x \in \partial\Omega$, $f = -\Delta u$ is non-negative and f is β-concave in Ω, and u is not α-concave.

PROOF: Let $\Omega = \{x \in \mathbb{R}^n;\ |x| < 1\}$, and for $a > 0$ define f on Ω by $f(x) = \exp(-a|x|^2)$ for $x \in \Omega$. Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be the solution (which exists and is unique) of

$$\Delta u(x) + f(x) = 0 \text{ for } x \in \Omega$$
$$\text{and } u(x) = 0 \text{ for } x \in \partial\Omega$$

Then f is non-negative and

$$\begin{aligned} f^{-2}(ff_{\theta\theta} + (\beta-1)f_\theta^{\,2}) &= 4a^2(x.\theta)^2 - 2a|\theta|^2 + (\beta-1).4a^2(x.\theta)^2 \\ &= 4a^2\beta(x.\theta)^2 - 2a|\theta|^2 \\ &\leq 2a(2a\beta|x|^2|\theta|^2 - |\theta|^2) \end{aligned}$$

$$= 2a(2a\beta|x|^2 - 1)$$
$$\leqslant 0 \text{ in } \Omega \text{ if } \beta \leqslant (2a)^{-1}.$$

But as $a \to \infty$, $a^{n/2}f$ converges in the weak topology of the space of Radon measures on Ω to a positive multiple of the Dirac delta distribution with support at the origin. Hence $a^{n/2}u$ converges pointwise to a positive multiple of the Green's function g for Δ on Ω with pole at 0, namely the function defined by $g(x) = -\log(|x|)$ for $n = 2$, and $g(x) = |x|^{2-n} - 1$ for $n \geqslant 3$. However, for all $n \geqslant 2$, log g is not a log concave function. In fact, a routine application of property 4 (see section 8) shows that $\alpha(g) = -\infty$ for $n = 2$, and $\alpha(g) = -(n-2)^{-1}$ for $n \geqslant 3$. Hence for some $a > 0$, u is not $\alpha(g)$-concave, and hence not α-concave for ~~all $\alpha > 0$~~ $\alpha = \beta/(1+2\beta)$. The proof is completed by setting $\beta = (2a)^{-1}$ for this value of a. □

The research reported in this thesis originated in an attempt to prove the concavity of the solution to

$$\Delta u(x) + k = 0 \quad \text{for } x \text{ in } \Omega$$
$$\text{and } u(x) = 0 \text{ for } x \text{ in } \partial\Omega$$

for positive constants k and bounded convex domains Ω. Theorem 14, part (iii), shows that when Ω has a sharp enough vertex, $\alpha(u) = \frac{1}{2}$. However, when Ω is an ellipsoid it is well-known that u is a concave quadratic function of x, so that $\alpha(u) = 1$. If the boundary of an ellipsoid undergoes a small perturbation in the $C^{2,\alpha}$ sense, an application of standard Schauder theory enables one to make a $C^{2,\alpha}$ estimate of the resulting perturbation of the solution to the above boundary value problem in terms of a $C^{2,\alpha}$ bound on the perturbation of the boundary. (The application is straightforward, but takes many pages and contains no surprises. The reader is therefore spared an account of this application.) Since the solution on an ellipsoid has strictly negative directional derivatives, this implies that for Ω close to an ellipsoid in the $C^{2,\alpha}$ sense, the solution to the above boundary value problem is concave. This suggests the possibility of guaranteeing the concavity of the solution by means of a bound on the curvature of the boundary. The necessity of an upper bound is indicated by a consideration of the upper level sets close to the boundary for the counterexample in Theorem 14 part (iii). The necessity of bounding the curvature from below in some sense is indicated by Theorem 18, which says essentially that if $\partial\Omega$ is locally a portion of a hyperplane at some point, and Ω is bounded, then the solution to the above problem is not concave in some neighbourhood of that portion. This theorem also implies, incidentally, that u is not concave for some arbitrarily small C^1 deviations of Ω from an ellipsoid, since if an arbitrarily small flat slice is removed from Ω, and $\partial\Omega$ is smoothed near the edges of the cut, Ω will be arbitrarily close to an ellipsoid in the C^1 sense and yet u will not be concave. Theorem 18 is preceded by an elementary algebraic result.

LEMMA 17: Assumptions:

$n \geq 2$, and $A = (a_{ij})$ is an $n \times n$ symmetric matrix.

$a_{nn} = 1$, and for all $i \neq n$, $a_{ii} = 0$.

Assertion:

A is positive semi-definite if and only if all off-diagonal entries of A are equal to zero (that is, $a_{ij} = 0$ for all $i \neq j$).

PROOF: $a_{ij}x_ix_j = a_{nn}x_n^2 + 2a_{nj}x_nx'_j + a_{ij}x'_ix'_j$, for all $x \in \mathbb{R}^n$, where x' has been defined earlier as the projection of x onto $\mathbb{R}^{n-1}$. Suppose

$a_{nj} \neq 0$ for some $j \neq n$, and put $x = te_j + e_n$ for $t \in \mathbb{R}$, where e_i denotes the ith unit vector in the standard basis for $\mathbb{R}^n$. Then $a_{ij}x_ix_j = 1 + 2ta_{nj}$, which is negative for some t, so that A is not positive semi-definite. So suppose that $a_{nj} = 0$ for all $j \neq n$ and $a_{ij} \neq 0$ for some i and j with $i \neq j$ and $i \neq n \neq j$. For real s and t, put $x = se_i + te_j$. Then $a_{ij}x_ix_j = 2sta_{ij}$, which is negative for some choice of s and t. This completes the proof. □

THEOREM 18: Assumptions:

$n \geq 2$, and Ω is a bounded convex domain in $\mathbb{R}^n$.

$0 \in \partial\Omega$, and for some open ball B with centre 0, $B \cap \Omega = \{x \in B;\ x_n > 0\}$.

$u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$\Delta u(x) + 1 = 0 \text{ for } x \text{ in } \Omega$$

$$\text{and } u(x) = 0 \text{ for } x \text{ in } \partial\Omega.$$

Assertion:

u is not concave in $B \cap \Omega$.

PROOF: Kellogg's theorem implies that $u \in C^\infty(B \cap \overline{\Omega})$, since Δu is C^∞. Thus all derivatives of u are defined on $B \cap \partial\Omega$. Let $x' \in B \cap \partial\Omega$, and define the n×n matrix $A = (a_{ij})$ by $a_{ij} = -u_{ij}(x')$. Then for $i \neq n$, $a_{ii} = 0$, since $u = 0$ on $\partial\Omega$, and so $a_{nn} = 1$, as $\Delta u + 1 = 0$.

Suppose now that u is concave in Ω. Then the Hessian (u_{ij}) of u is negative semi-definite in Ω and therefore on $\partial\Omega$. From Lemma 17 it then follows that $a_{ij} = 0$ for $i \neq j$. In particular, $u_{nj}(x') = 0$ for all $j \neq n$, so that $u_n(x')$ is constant for $x' \in B \cap \partial\Omega$. Denote this (non-negative) constant by c. Then both Dirichlet and Neumann data are specified on $B \cap \partial\Omega$, thereby uniquely determining the function u, which must be the function given by $u(x) = -\frac{1}{2}x_n^2 + cx_n$ for x in Ω. But this contradicts the boundary data for the rest of $\partial\Omega$, and the theorem is thus proved. $\square$

8. CONCAVITY CALCULATIONS FOR MISCELLANEOUS EXAMPLES

This section commences with a discussion in detail of the α-concavity properties of the solution to $\Delta u+4 = 0$ with zero Dirichlet data on an equilateral triangle. This example neatly demonstrates Theorem 14, part (iii), and Theorem 18. Then the corresponding calculations are made for the Green's function for Δ on a sphere in $\mathbb{R}^n$ with pole at the centre of the sphere. These are used for Theorem 16 and also indicate limits to any future results on the α-concavity of the Green's function.

Define a bounded convex domain Ω and a function $u: \overline{\Omega} \to \mathbb{R}$ by

$$\Omega = \{(x,y) \in \mathbb{R}^2;\ -2/9 + |y\sqrt{3}| < x < 1/9\}$$

$$\text{and } u(x,y) = 4/243 - (x^2+y^2) - 3x(x^2-3y^2)$$

$$= 4/243 - r^2 - 3r^3\cos(3\theta),$$

where $r = (x^2+y^2)^{1/2}$ and $\theta = \mathrm{sign}(y).\cos^{-1}(x/r)$. (Ambiguities in the definition of θ are irrelevant for these purposes.) Then Ω is an equilateral triangle with sides of length $1/\sqrt{3}$, $u = 0$ on $\partial\Omega$, $u \in C^\infty(\overline{\Omega})$ and u satisfies $\Delta u + 4 = 0$ in Ω. In order to calculate the number $\alpha(u) = 1 - \sup uu_{\theta\theta}u_\theta^{-2}$ (see property 5), it is convenient to take the supremum over all directions θ for a fixed point x first and then take the supremum of this quantity over x in Ω. This intermediate calculation gives a pointwise definition of $\alpha(u)$, which may be written as $\alpha(u,x)$, the concavity number of u at x. Then the concavity number for u on any convex subset of Ω may be evaluated by simply calculating the supremum of $\alpha(u,x)$ for x in this subset. Although the quantity $\alpha(u,x)$ has the virtue of simultaneously checking a given function for all of the α-concavity properties for $\alpha \in (-\infty,+\infty]$, the author has unfortunately not been able to use it as an analytical tool on classes of functions given as solutions of boundary value problems (that is, functions not given explicitly). It does not seem to obey any kind of useful maximum principle, unlike the quantity $u_{\theta\theta}$, which is harmonic for all $\theta \in S$ if Δu is constant, and subharmonic if Δu is convex (that is, $\Delta u + f(x) = 0$, where f is concave), or $\sup\{u_{\theta\theta}(x);\ \theta \in S\}$, which is subharmonic if Δu is convex. (These observations follow from the equation $\Delta(u_{\theta\theta}) = (\Delta u)_{\theta\theta}$ and the fact that the supremum of a set of subharmonic functions is itself subharmonic.)

Now returning to the case of the equilateral triangle, it is necessary to calculate the supremum of $uu_{\theta\theta}u_\theta^{-2}$ over $\theta \in S$. For general $n \geq 2$, let A denote the $n \times n$ matrix $(u_{ij}(x))$ - that is, the Hessian of u at x - and write b for $\nabla u(x)$, the gradient of u at x. Then a convenient quantity to maximise is $Q(\theta) = u_{\theta\theta}u_\theta^{-2}$ when u is positive. This can be written as $Q(\theta) = (a_{ij}\theta_i\theta_j)(b_k b_l \theta_k \theta_l)^{-1}$, whose derivative with respect to θ_k (regarding Q as a function on the whole space $\mathbb{R}^n$) is

$$\partial Q/\partial\theta_k = 2(a_{ik}b_j\theta_i\theta_j - a_{ij}b_k\theta_i\theta_j)(b.\theta)^{-3},$$

whenever $b.\theta \neq 0$. This derivative vanishes for any θ such that $A\theta$ is a multiple of b, since if $A\theta = tb$ for some real t, then $a_{ik}\theta_i = tb_k$ and $a_{ij}\theta_i = tb_j$, as A is symmetric. (Note that for any non-zero real number t and non-zero vector θ, $Q(t\theta) = Q(\theta)$.) It is straightforward to show that these are the only values of θ (when $b.\theta \neq 0$) for which the derivative vanishes for all k. Indeed the vanishing of the derivative implies that $y_k b_j \theta_j - b_k y_j \theta_j = 0$, where $y = A\theta$. On the assumption that $b.\theta$ is non-zero, this equation implies that $y = b(y.\theta)(b.\theta)^{-1}$, a multiple of b. Thus the supremum of Q on S occurs either for θ such that $A\theta$ is a multiple of b, or in the limit as $b.\theta \to 0$. For the function in question, $a_{11} = -2-18x$, $a_{12} = 18y$, and $a_{22} = -2+18x$. So det A is equal to $4(1-81r^2)$, which vanishes only when x lies on the incircle of Ω. Elsewhere A^{-1} exists, and by putting $\theta = A^{-1}b$ one obtains $Q = (b^tA^{-1}b)^{-1}$ which, after a long calculation, becomes $(\det A)/(-243r^2u(x,y))$.

The case $b.\theta \to 0$ is easily taken care of in $\mathbb{R}^2$, for such θ must converge to a multiple of $b^\perp = (b_2,-b_1)$. But $(b^\perp)^tAb^\perp$ is equal to $(\det A).(b^tA^{-1}b)$, which is the negative quantity $-243r^2u(x,y)$ given above. So $Q \to -\infty$ as $b.\theta \to 0$. When det A vanishes, A has eigenvalues 0 and -2. But $b.\theta \neq 0$ when θ is an eigenvector of A with eigenvalue 0, because the only direction in which $b.\theta$ equals 0 is that of $b^\perp$, and $(b^\perp)^tAb^t$ is negative, as stated above. So on the incircle, sup Q = 0. Hence one obtains sup $Q = (\det A)/(-243r^2u)$ throughout Ω.

The final outcome of these calculations is that $\alpha(u,x) = 1-u(x)\sup Q$ is equal to $\frac{1}{3}(1 + 2/(81r^2))$. Thus $\alpha(u,x)$ decreases from $+\infty$ at the circumcentre of the triangle (that is, $r = 0$) to $\frac{1}{2}$ at the vertices (that is, $r = 2/9$), and hence $\alpha(u) = \frac{1}{2}$. It may be noted that $\alpha(u,x) \geq 1$ for x inside and on the incircle (that is, $r \leq 1/9$), so that u is ~~convex~~ concave in this local sense on the boundary at one and only one point on each side, which agrees well with Theorem 18, which implies that u can not be concave in any neighbourhood of any point on a flat portion of the boundary.

The comment should be made here that if u is known to be quasiconcave, then $\theta A\theta \leq 0$ whenever $b.\theta = 0$, since $u_{\theta\theta} \leq 0$ for all directions θ which are tangential to the level curves of u. An even greater simplification of the calculations occurs when it is known that $\nabla u = 0$ only at the maximum of u, and the level sets of u are uniformly convex, for then $\theta A\theta < 0$ whenever $b.\theta = 0$ and so the case $b.\theta \to 0$ need not be considered at all. Then the result is simply $\alpha(u,x) = u/(b^tAb)$.

Calculations can now be made for $\alpha(g)$, where g denotes the Green's function of the unit sphere, as defined in section 7. It follows from Study's theorem that the Green's function is quasiconcave (that is, $(-\infty)$-concave) for general bounded convex domains in $\mathbb{R}^2$, and Gabriel ([3]) demonstrated the same result in $\mathbb{R}^3$. It will now be shown that the number $-\infty$ is sharp in $\mathbb{R}^2$, because $\alpha(g) = -\infty$ for $n = 2$. Also, any attempts to find α-concavity results for Green's functions when $n \geqslant 3$ are limited by the fact that $\alpha(g) = -(n-2)^{-1}$.

Write r for $|x|$. Then in $\mathbb{R}^2$, $g = -\log r$, $g_\theta = -r^{-2}(x.\theta)$, and $g_{\theta\theta} = -r^{-4}(2(x.\theta)^2-r^2|\theta|^2)$, so that $gg_{\theta\theta}g_\theta{}^{-2} = -\log(r)(2-r^2(x.\theta)^{-2})$. Hence $\alpha(g,x) = 1 + \log(r)$. Unlike the torsion function on the equilateral triangle dealt with above, this function has a local concavity number which decreases with proximity to the maximum of the function. In fact, $\alpha(g,x) \to -\infty$ as $r \to 0$, and so the function is at most $(-\infty)$-concave.

The same calculations for $n \geqslant 3$ reveal that

$$gg_{\theta\theta}g_\theta{}^{-2} = (n-2)^{-1}(1-r^{n-2})(n-r^2|x.\theta|^{-2}).$$

Hence $\alpha(g,x) = ((n-1)r^{n-2}-1)/(n-2)$, which once again is equal to 1 on the boundary, and decreases towards the centre. In this case, though, $\alpha(g)=-1/(n-2)$, a fact which could have been found by a simple inspection of the function.

APPENDIX

This appendix contains a preprint of an article submitted to the Australian Journal of Mathematics (series A) in December 1983. The results of this article are obtained in the text of the thesis by a different method, except for Lemma 2.2, which is cited in the thesis.

THE CONCAVITY PROPERTIES OF SOLUTIONS OF THE LINEAR CLAMPED MEMBRANE PROBLEM ON A CONVEX SET

Alan U. Kennington

Abstract

Suppose u is the solution of the clamped membrane problem: $-\Delta u = f(x)$ on a bounded convex domain Ω in $\mathbb{R}^n$, and $u = 0$ on the boundary of Ω, where f is a non-negative function of x in Ω such that f^β is concave (that is, $-(f^\beta)$ is convex) for some $\beta \geqslant 1$. Then it is shown that u^α is concave in Ω, where $\alpha = \beta/(1+2\beta)$, and that if f is constant, $u^{\frac{1}{2}}$ is concave. Hence whenever f is non-negative and concave, the level surfaces of u enclose convex sets.

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Short title: Clamped membrane problem.

1. Introduction

The principal results obtained in this paper are the following:

Theorem 1.1: Let Ω be a bounded convex domain (that is, non-empty open set) in $\mathbb{R}^n$ for some $n \geqslant 2$, and suppose f is a non-negative function defined on Ω such that f^β is concave for some $\beta \geqslant 1$. If u is continuous in $\bar{\Omega}$, twice continuously differentiable in Ω, and satisfies

(1.1a) $-\Delta u(x) = f(x)$ for x in Ω

(1.1b) $u(x) = 0$ for x in $\partial\Omega$

then u^α is concave in Ω, where $\alpha = \beta/(1+2\beta)$.

Corollary 1.2: Under the assumptions of theorem 1.1, if f is constant, then $u^{\frac{1}{2}}$ is concave.

It should be remarked that under the conditions of Theorem 1.1, f is concave and therefore Lipschitz continuous, so that there exists a unique solution to problem (1.1) which is continuous in $\bar{\Omega}$ and twice continuously differentiable in Ω. Moreover, this solution is non-negative.

Makar-Limanov [9] showed that when $n = 2$ and f is constant, the upper level sets, $\{x \in \Omega;\ u(x) > c\}$, of u are convex. The proof given shows that in fact, $u^{\frac{1}{2}}$ is concave. The method used does not readily extend to $n \geqslant 3$.

The method used in the present paper to prove theorem 1.1 is essentially adapted from one used by Lewis [8] to prove the convexity of upper level sets for a different problem. From the solution u of (1.1), a function v is constructed so that u and v are different whenever u^α is not concave. Then maximum principle arguments are applied to the difference between u and v to arrive at a contradiction if u^α is assumed not concave. The natural extension of this method to the non-linear clamped membrane problem is envisaged as the topic of a later paper.

The author is grateful to J.H. Michael for supplying the simplified barrier function b used to considerably shorten the proof of lemma 2.2.

2. Proof of Theorem 1.1 and Corollary 1.2

For positive numbers α, we construct the function v on $\bar{\Omega}$ from the solution u of problem (1.1) by defining

$$v(x) = \sup((1-\lambda)u(y)^\alpha + \lambda u(z)^\alpha)^{1/\alpha}$$

for x in $\bar{\Omega}$, where the supremum is over all y and z in $\bar{\Omega}$, and $\lambda \in [0,1]$, such that $x = (1-\lambda)y + \lambda z$. It is clear (by putting $y=z=x$) that $v(x) \geqslant u(x)$ for all x, and that v is identically equal to u if and only if u^α is concave.

Now suppose that u^α is not concave. Then for some ε satisfying $0 < \varepsilon < 1$, the function w on $\bar{\Omega}$ defined by

$$w = (1-\varepsilon)v - u$$

has a positive supremum. It will be shown that this supremum is attained at some x_0 in Ω (by lemma 2.1), and that the supremum implicit in the definition of $v(x_0)$ is attained at some pair (y_0, z_0) in $\Omega \times \Omega$ (by lemma 2.2). A calculation will then lead to a contradiction to the concavity of f^β at the triple (x_0, y_0, z_0).

Lemma 2.1: Let Ω be a bounded convex domain in $\mathbb{R}^n$, u be any continuous function on $\bar{\Omega}$, and v be the function constructed from u as above. Then

(i) for all x in $\bar{\Omega}$, there exist y and z in $\bar{\Omega}$ and $\lambda \in [0,1]$ such that

$$v(x) = ((1-\lambda)u(y)^\alpha + \lambda u(z)^\alpha)^{1/\alpha}$$

and

$$x = (1-\lambda)y + \lambda z$$

and

(ii) v is upper semi-continuous in $\bar{\Omega}$.

Proof: It is sufficient to treat the case $\alpha = 1$, as the result for general $\alpha > 0$ follows by substituting u^α for u.

(i) Let x be in $\bar{\Omega}$. From the definition of $v(x)$, there are sequences $(y_i)_{i=1}^\infty$ and $(z_i)_{i=1}^\infty$ in $\bar{\Omega}$, and a sequence $(\lambda_i)_{i=1}^\infty$ in $[0,1]$, such that

$$v(x) = \lim_{i \to \infty} ((1-\lambda_i)u(y_i) + \lambda_i u(z_i))$$

and

$$x = (1-\lambda_i)y_i + \lambda_i z_i \quad \text{for all } i.$$

Choosing subsequences if necessary, it may be assumed that $y_i \to y$, $z_i \to z$ and $\lambda_i \to \lambda$ as $i \to \infty$, for some y and z

in $\bar{\Omega}$ and $\lambda \in [0,1]$. Then $x = (1-\lambda)y+\lambda z$, and since u is continuous, $v(x) = (1-\lambda)u(y)+\lambda u(z)$.

(ii) Let x be in $\bar{\Omega}$. From the definition of $\limsup_{p \to x} v(p)$, where p is restricted to be in $\bar{\Omega}$, a sequence $(x_i)_{i=1}^{\infty}$ can be constructed in $\bar{\Omega}$ so that

$$\lim_{i \to \infty} v(x_i) = \limsup_{p \to x} v(p)$$

and

$$\lim_{i \to \infty} x_i = x.$$

Applying part (i) of this lemma to each x_i, sequences (y_i) and (z_i) in $\bar{\Omega}$, and (λ_i) in $[0,1]$ can be constructed so that

$$v(x_i) = (1-\lambda_i)u(y_i) + \lambda_i u(z_i)$$

and

$$x_i = (1-\lambda_i)y_i + \lambda_i z_i.$$

Choosing subsequences if necessary, it may be assumed that $y_i \to y$, $z_i \to z$ and $\lambda_i \to \lambda$ as $i \to \infty$, for some y and z in $\bar{\Omega}$ and $\lambda \in [0,1]$. Then $x = (1-\lambda)y + \lambda z$, and as u is continuous,

$$\begin{aligned} v(x) &\geqslant (1-\lambda)u(y) + \lambda u(z) \\ &= \lim_{i \to \infty}((1-\lambda_i)u(y_i) + \lambda_i u(z_i)) \\ &= \limsup_{p \to x} v(p). \end{aligned}$$

That is, v is upper semi-continuous at x. □

Lemma 2.1(ii) implies that the (positive) supremum of w is attained at some point, x_0 say, of $\bar{\Omega}$. But $w(x) = 0$ for

$x \in \partial\Omega$. So $x_0 \in \Omega$. By lemma 2.1 (i), there are points y_0 and z_0 in $\bar{\Omega}$, and λ in $[0,1]$, such that

$$v(x_0) = ((1-\lambda)u(y_0)^\alpha + \lambda u(z_0)^\alpha)^{1/\alpha}$$

and

$$x_0 = (1-\lambda)y_0 + \lambda z_0.$$

Now suppose that $y_0 \in \partial\Omega$, and put $y = (1-\delta)y_0 + \delta z_0$ whenever $0 \leqslant \delta \leqslant \lambda$. Then $x_0 = (1-\mu)y + \mu z_0$, where $\mu = (\lambda-\delta)/(1-\delta) \in [0,1]$, and so

$$v(x_0) \geqslant ((1-\mu)u(y)^\alpha + \mu u(z_0)^\alpha)^{1/\alpha}.$$

But $v(x_0) = \lambda^{1/\alpha}u(z_0)$, since $u(y_0) = 0$. A brief calculation then shows that

(2.1) $$u(y) \leqslant \delta^{1/\alpha}u(z_0).$$

Under appropriate conditions on α and f, this inequality is shown to be impossible.

Lemma 2.2: Let Ω be a bounded convex domain in $\mathbb{R}^n$ for some $n \geqslant 2$, and suppose u is a function which is continuous in $\bar{\Omega}$, twice differentiable in Ω, and satisfies $-\Delta u \geqslant c$ in Ω for some $c > 0$, and $u = 0$ on $\partial\Omega$. Then for all p in $\partial\Omega$ and q in Ω, there are positive numbers r and k such that

$$u((1-t)p + tq) \geqslant kt^2$$

whenever $0 \leqslant t < r$.

Proof: After making an appropriate translation, rotation and dilation if necessary, it may be supposed that $p = 0$ and

$q = e_n = (0,0,\ldots 1)$. For points x in $\mathbb{R}^n$, write $x_n = x.e_n$ and $x' = x - x_n e_n$. Since Ω is convex and $q \in \Omega$, the (cone-shaped) domain

$$G = \{x \in \mathbb{R}^n ; |x'| < ax_n \text{ and } 0 < x_n < 1\}$$

is a subset of Ω for some a satisfying $0 < a < 1$. For x in $\bar{G}$, define $b(x)$ by

$$b(x) = (a^2 x_n^2 - |x'|^2)(1-x_n)^2.$$

Then $\Delta b(x) = -2(n-1)(1-x_n)^2 - 2|x'|^2 + 2a^2(6x_n^2 - 6x_n + 1) \geqslant -2n-1$ for x in G, and $b(x) = 0$ for x in ∂G.

Hence by virtue of the comparison principle for Δ on G, we conclude that for $x \in G$,

$$u(x) \geqslant \frac{c}{2n+1} b(x).$$

But $b((1-t)p + tq) \geqslant \tfrac{1}{2}a^2 t^2$ when $0 \leqslant t \leqslant \tfrac{1}{2}$. Putting $r = \tfrac{1}{2}$ and $k = \tfrac{1}{2}a^2 c/(2n+1)$ then verifies the lemma. □

If we now assume (temporarily) that f is bounded below by a positive constant, and put $p = y_0$, $q = x_0$ and $t = \delta/\lambda$ in lemma 2.2, it is seen that for small enough δ, ~~equation~~ inequality (2.1) is contradicted whenever $\alpha < \tfrac{1}{2}$. Thus y_0, and similarly z_0, is in Ω. So the first and second derivatives of u are defined at y_0 and z_0, whereas v need not necessarily possess these derivatives at x_0. To overcome this difficulty, it is convenient to define upper and lower directional derivatives.

For any real function h on a domain Ω, at any point x in Ω, in any direction $\theta \in S_n = \{\phi \in \mathbb{R}^n ; |\phi| = 1\}$, define

$$h_\theta^+(x) = \lim_{t \to o^+} \sup t^{-1}(h(x+t\theta)-h(x))$$

$$h_\theta^-(x) = \lim_{t \to o^+} \inf t^{-1}(h(x+t\theta)-h(x))$$

and $h_\theta(x) = h_\theta^-(x) = h_\theta^+(x)$ if equality holds.

Similarly, if $h_\theta(x)$ exists, define

$$h_{\theta\theta}^{+\prime}(x) = \lim_{t \to o^+} \sup 2t^{-2}(h(x+t\theta)-h(x)-th_\theta(x))$$

$$h_{\theta\theta}^-(x) = \lim_{t \to o^+} \inf 2t^{-2}(h(x+t\theta)-h(x)-th_\theta(x))$$

and $h_{\theta\theta}(x) = h_{\theta\theta}^+(x) = h_{\theta\theta}^-(x)$ if equality holds.

Using these definitions, it is now possible to analyse the first and second-order behaviour of v near x_0. For arbitrary $\theta \in S_n$, for small enough positive t, put $y = y_0 + t\theta$ and $x = (1-\lambda)y + \lambda z_0 = x_0 + (1-\lambda)t\theta$. Then by the definition of $v(x)$,

$$v(x)^\alpha \geqslant (1-\lambda)u(y)^\alpha + \lambda u(z_0)^\alpha$$

so that

$$v(x)^\alpha - v(x_0)^\alpha \geqslant (1-\lambda)(\alpha u(y_0)^{\alpha-1}u_\theta(y_0)t + o(t)) \quad \text{as} \quad t \to o^+,$$

whereas for a suitable set of positive values of t, (a set whose closure includes 0,)

$$v(x)^\alpha - v(x_0)^\alpha \leqslant \alpha(1-\lambda)v(x_0)^{\alpha-1}v_\theta^-(x_0)t + o(t) \quad \text{as} \quad t \to o^+.$$

Combining these two inequalities, dividing by $\alpha(1-\lambda)t$, and taking the limit as $t \to o^+$, one obtains

$$v(x_0)^{\alpha-1}v_\theta^-(x_0) \geqslant u(y_0)^{\alpha-1}u_\theta(y_0) .$$

So $v_\theta^-(x_0) \geqslant (v(x_0)/u(y_0))^{1-\alpha}u_\theta(y_0)$.

But since $w(x_0)$ is a local maximum of w, $w_\theta^+(x_0) \leqslant 0$, and so $v_\theta^+(x_0) \leq (1-\varepsilon)^{-1} u_\theta(x_0)$.

Hence

$$(v(x_0)/u(y_0))^{1-\alpha} u_\theta(y_0) \leq v_\theta^-(x_0) \leq v_\theta^+(x_0) \leq (1-\varepsilon)^{-1} u_\theta(x_0).$$

If these calculations are repeated with $-\theta$ in place of θ, one obtains

$$\begin{aligned}(v(x_0)/u(y_0))^{1-\alpha} u_\theta(y_0) &= -(v(x_0)/u(y_0))^{1-\alpha} u_{-\theta}(y_0) \\ &\geq -v_{-\theta}^-(x_0) \\ &\geq -v_{-\theta}^+(x_0) \\ &\geq -(1-\varepsilon)^{-1} u_{-\theta}(x_0) \\ &= (1-\varepsilon)^{-1} u_\theta(x_0).\end{aligned}$$

Clearly equality holds throughout, and so v is differentiable at x_0, and $v_\theta(x_0) = \rho_1^{1-\alpha} u_\theta(y_0)$, where $\rho_1 = v(x_0)/u(y_0)$. Similarly $v_\theta(x_0) = \rho_2^{1-\alpha} u_\theta(z_0)$, where $\rho_2 = v(x_0)/u(z_0)$. As a result, $v_{\theta\theta}^+(x_0)$ and $v_{\theta\theta}^-(x_0)$ are defined, and the second-order behaviour of v near x_0 may be analysed.

Let $\theta \in S_n$, and for small enough positive t put $y = y_0 + t\theta$, $z = z_0 + \rho^\alpha t\theta$, and $x = (1-\lambda)y + \lambda z = x_0 + \rho_1^\alpha t\theta$, where $\rho = \rho_1/\rho_2$. From the definition of $v(x)$,

$$\begin{aligned}v(x)^\alpha - v(x_0)^\alpha &\geq (1-\lambda)(u(y)^\alpha - u(y_0)^\alpha) + \lambda(u(z)^\alpha - u(z_0)^\alpha) \\ &= t((1-\lambda)\alpha u(y_0)^{\alpha-1} u_\theta(y_0) + \lambda\alpha u(z_0)^{\alpha-1} u_\theta(z_0)\rho^\alpha)\end{aligned}$$

$$+ \tfrac{1}{2}t^2((1-\lambda)\alpha(\alpha-1)u(y_0)^{\alpha-2} u_\theta(y_0)^2 + (1-\lambda)\alpha u(y_0)^{\alpha-1} u_{\theta\theta}(y_0)$$

$$+\lambda\alpha(\alpha-1)u(z_0)^{\alpha-2} u_\theta(z_0)^2 \rho^{2\alpha} + \lambda\alpha u(z_0)^{\alpha-1} u_{\theta\theta}(z_0)\rho^{2\alpha}) + o(t^2)$$

as $t \to 0^+$.

For a suitable set of values of t,

$$v(x)^{\alpha} - v(x_0)^{\alpha} \leqslant t\alpha v(x_0)^{\alpha-1} v_{\theta}(x_0)\rho_1{}^{\alpha}$$

$$+ \tfrac{1}{2}t^2(\alpha(\alpha-1)v(x_0)^{\alpha-2}v_{\theta}(x_0)^2\rho_1^{2\alpha} + \alpha v(x_0)^{\alpha-1}v_{\theta\theta}^{-}(x_0)\rho_1^{2\alpha}) + o(t^2)$$

as $t \to o^+$.

After combining these two inequalities, cancelling first-order terms in t, dividing by $\tfrac{1}{2}t^2\alpha v(x_0)^{\alpha-1}\rho_1^{2\alpha}$, and taking the limit as $t \to o^+$, the result is

$$(\alpha-1)v(x_0)^{-1}v_{\theta}(x_0)^2 + v_{\theta\theta}^{-}(x_0) \geqslant$$

$$(1-\lambda)v(x_0)^{1-\alpha}u(y_0)^{\alpha-1}((\alpha-1)u(y_0)^{-1}u_{\theta}(y_0)^2 + u_{\theta\theta}(y_0))\rho_1^{-2\alpha}$$

$$+ \lambda v(x_0)^{1-\alpha}u(z_0)^{\alpha-1}((\alpha-1)u(z_0)^{-1}u_{\theta}(z_0)^2 + u_{\theta\theta}(z_0))\rho^{2\alpha}\rho_1^{-2\alpha}$$

$$= (1-\lambda)\rho_1^{1-3\alpha}u_{\theta\theta}(y_0) + \lambda\rho_2^{1-3\alpha}u_{\theta\theta}(z_0)$$

$$+ (\alpha-1)v(x_0)^{1-\alpha}((1-\lambda)u(y_0)^{\alpha-2}u_{\theta}(y_0)^2\rho_1^{-2\alpha}$$

$$+ \lambda u(z_0)^{\alpha-2}u_{\theta}(z_0)^2\rho_2^{-2\alpha}).$$

But $(1-\lambda)u(y_0)^{\alpha-2}u_{\theta}(y_0)^2\rho_1^{-2\alpha} + \lambda u(z_0)^{\alpha-2}u_{\theta}(z_0)^2\rho_2^{-2\alpha}$

$$= v_{\theta}(x_0)^2 v(x_0)^{\alpha-2}((1-\lambda)\rho_1^{-\alpha} + \lambda\rho_2^{-\alpha}) = v_{\theta}(x_0)^2 v(x_0)^{\alpha-2}.$$

Hence (due to the careful choice of x, y and z) the above inequality becomes simply

$$v_{\theta\theta}^{-}(x_0) \geqslant (1-\lambda)\rho_1^{1-3\alpha}u_{\theta\theta}(y_0) + \lambda\rho_2^{1-3\alpha}u_{\theta\theta}(z_0).$$

Now choose any n orthogonal directions θ, and denote the sum of $v_{\theta\theta}^{+}(x_0)$ (respectively $v_{\theta\theta}^{-}(x_0)$) over these n directions by $\Delta^{+}v(x_0)$ (respectively $\Delta^{-}v(x_0)$). Then

$$\Delta^- v(x_0) \geq (1-\lambda)\rho_1^{1-3\alpha}\,\Delta u(y_0) + \lambda\rho_2^{1-3\alpha}\Delta u(z_0) = -h(\rho),$$

where

$$h(\rho) = (1-\lambda)\rho_1^{1-3\alpha}\, f(y_0) + \lambda\rho_2^{1-3\alpha}\, f(z_0)$$

$$= (1-\lambda)(1-\lambda+\lambda\rho^\alpha)^{(1-3\alpha)/\alpha} f(y_0) + \lambda((1-\lambda)\rho^{-\alpha}+\lambda)^{(1-3\alpha)/\alpha}\, f(z_0).$$

h is a differentiable function of the positive variable ρ, and is equal to the constant $(1-\lambda)f(y_0)+\lambda f(z_0)$ when $\alpha = \frac{1}{3}$. When $\alpha > 1/3$,

$$\lim_{\rho\to 0^+} h(\rho) = (1-\lambda)^{(1-2\alpha)/\alpha}\, f(y_0)$$

and

$$\lim_{\rho\to\infty} h(\rho) = \lambda^{(1-2\alpha)/\alpha} f(z_0).$$

To find the stationary points of h, note that

$$\frac{d\rho_1}{d\rho} = \lambda\rho_2^{1-\alpha} \text{ and } \frac{d\rho_2}{d\rho} = -(1-\lambda)\rho_1^{-\alpha-1}\rho_2^2, \text{ so that}$$

$$\frac{dh}{d\rho} = (1-3\alpha)\lambda(1-\lambda)\rho_1^{-3\alpha}\rho_2^{1-\alpha}(f(y_0)-\rho_1^{2\alpha-1}\rho_2^{1-2\alpha}f(z_0)),$$ which is zero only at $\rho = \rho' = (f(y_0)/f(z_0))^{1/(2\alpha-1)}$.

$$h(\rho') = ((1-\lambda)f(y_0)^{\alpha/(1-2\alpha)} + \lambda f(z_0)^{\alpha/(1-2\alpha)})^{(1-2\alpha)/\alpha} \leq f(x_0)$$

since $\alpha/(1-2\alpha) = \beta$ and f^β is concave. Thus $h(\rho) \leq f(x_0)$ for all $\rho > 0$ when $1/3 \leq \alpha < 1/2$. (The reason that the proof of theorem 1.1 does not readily extend to $\beta < 1$ is the unsatisfactory behaviour of h when $\alpha < 1/3$.) So $\Delta^- v(x_0) \geq -f(x_0)$.

But the definition of x_0 implies that for each θ, $w^+_{\theta\theta}(x_0) \leq 0$, so that $\Delta^+ v(x_0) \leq (1-\varepsilon)^{-1}\Delta u(x_0)$. Summarising the second-order inequalities obtained,

$$(1-\varepsilon)^{-1} f(x_0) = -(1-\varepsilon)^{-1}\Delta u(x_0) \leq -\Delta^+ v(x_0) \leq -\Delta^- v(x_0) \leq f(x_0).$$

Since $f(x_0) > 0$, this is impossible. So u^α must have been concave.

Returning to the case where f is not necessarily bounded below on Ω by a positive constant, let $c > 0$ and consider the solution u_c to the following modification of problem (1.1):

$$-\Delta u_c' = f + c \quad \text{in } \Omega$$

$$u_c = 0 \qquad \text{on } \partial\Omega.$$

The function $(f+c)^\beta$ is shown to be concave in Ω by the following lemma.

Lemma 2.3: Suppose $r,s,t,c \geqslant 0$, $\beta \geqslant 1$ and $\lambda \in [0,1]$.
If $r \geqslant ((1-\lambda)s^\beta+\lambda t^\beta)^{1/\beta}$ then $r+c \geqslant ((1-\lambda)(s+c)^\beta+\lambda(t+c)^\beta)^{1/\beta}$.

Proof: It is sufficient to show that

$$((1-\lambda)s^\beta+\lambda t^\beta)^{1/\beta} \geqslant ((1-\lambda)(s+c)^\beta+\lambda(t+c)^\beta)^{1/\beta} - c.$$

Equality holds for $c = 0$, and the derivative of the right hand side with respect to c is

$$((1-\lambda)(s+c)^{\beta-1}+\lambda(t+c)^{\beta-1})((1-\lambda)(s+c)^\beta+\lambda(t+c)^\beta)^{(1-\beta)/\beta} - 1.$$

By Jensen's inequality, as $0 \leqslant (\beta-1)/\beta \leqslant 1$,

$$(1-\lambda)(s+c)^{\beta-1}+\lambda(t+c)^{\beta-1} \leqslant ((1-\lambda)(s+c)^\beta+\lambda(t+c)^\beta)^{(\beta-1)/\beta}.$$

So the right hand side derivative is non-positive, thus proving the lemma. □

Putting $r = f(x)$, $s = f(y)$ and $t = f(z)$ in lemma 2.3 whenever $x = (1-\lambda)y + \lambda z$, the concavity of $(f+c)^\beta$ follows from that of f^β, and hence u_c^α is concave for all $c > 0$.

But $u_c \to u$ uniformly in $\bar{\Omega}$ as $c \to o^+$. So u^α is concave, and the proof of the theorem is complete. □

To prove corollary 1.2, note that if f is constant then f^β is concave for all $\beta \geq 1$, so that u^α is concave for all α satisfying $1/3 \leq \alpha < 1/2$. As $\alpha \to \frac{1}{2}^-$, $u^\alpha \to u^{\frac{1}{2}}$ uniformly on compact subsets of Ω. So $u^{\frac{1}{2}}$ is concave. □

3. Remarks

(i) The concavity of a positive power of u implies (by Jensen's inequality) the concavity of any lower positive power of u, and also the convexity of its upper level sets. The power $\frac{1}{2}$ in corollary 1.2 is sharp in the sense that for each $n \geq 2$ there is a set Ω for which u^γ is not concave when $\gamma > \frac{1}{2}$. A simple example of this is the equilateral triangle in $\mathbb{R}^2$.

(ii) A local interpretation of the concavity of u^α if u is twice differentiable is that for all directions θ, $uu_{\theta\theta} \leq (1-\alpha)u_\theta^2$.

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