



# The Seiberg-Witten Invariant on non-Kähler Complex Surfaces

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# Abstract

The goal of this thesis is to calculate the Seiberg-Witten invariant for complex surfaces,  $X$ , with odd first betti number, that is complex surfaces that do not admit a Kähler metric. It is well known that the Seiberg-Witten equations have a simple, readily solved form when the metric on  $X$  is Kähler. Under the more general assumption that the metric on  $X$  is hermitian, we show that the Seiberg-Witten equations differ from the equations on a Kähler surface by a real one-form (corresponding to the trace of the torsion of the canonical hermitian connection) acting via Clifford multiplication in the Dirac operator. This “extra” term makes the analysis of the Seiberg-Witten equations on an arbitrary hermitian surface somewhat more difficult than the Kähler case. Rather than trying to solve the equations directly, we study how perturbing the Dirac operator by a real one-form effects the Seiberg-Witten moduli space. We show that this perturbed moduli space is compact and cobordant with the “standard” Seiberg-Witten moduli space. Apart from its subsequent applications, this result is aesthetically pleasing, as it proves that the Seiberg-Witten invariant depends only on the first order part of the Dirac operator. Our study of the moduli space means for the purpose of calculating the Seiberg-Witten invariant we can ignore the torsion term present in the Dirac operator when the metric on  $X$  is not Kähler. This simplification enables us to extend many of the results known for Kähler surfaces to a general hermitian surface. In particular it enables us to determine the basic classes and the Seiberg-Witten invariants for all complex surfaces with odd first betti number. Finally we use these results to study the effect of a diffeomorphism on the exceptional curves on a complex surface with odd first betti number.

# Declaration

This work contains no material that has been accepted for the award of any other degree or diploma in any university; and to the best of my knowledge contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, to be available for loan and photocopying.

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# Summary of Results

The main goal of this thesis is to calculate the Seiberg-Witten invariant for non-Kähler complex surfaces. In achieving this goal we also obtain a number of results of a more general nature. A brief summary of the established results relevant to this thesis are presented in §0.1, and in §0.2 the results proved in this thesis are summarised.

## 0.1. A Brief Summary of Established Results.

The Seiberg-Witten invariants were introduced in late 1994 [W] as a consequence of developments in  $N = 2$  super-symmetric Yang-Mills theory [SW1, SW2]. The power of these new invariants was made apparent by the ease with which it established many results suggested by, but inaccessible to, Donaldson Theory. For example a proof of the so called ‘Thom Conjecture’ [KM3] was established contemporaneously with the announcement of the new invariants. Also the vanishing of the Donaldson invariant on manifolds with positive scalar curvature had long been suspected [FS2], but for the Seiberg-Witten invariant this appears as an almost trivial consequence [W].

In Donaldson theory Kronheimer and Mrowka [KM1, KM2] introduced the important notion of a four-manifold of simple type where the Donaldson series [D2] satisfies a type of recurrence relation. They proved that for such manifolds the Donaldson series is determined by a finite set of “basic classes”  $\kappa_i \in H_2(X, \mathbf{Z})$  and an associated set of rational numbers  $a_i$ . In fact all known four-manifolds are of simple type and it is still an open question whether this is true for all four-manifolds. Importantly for us the simple type condition translates easily into Seiberg-Witten theory—a manifold is of (SW-) simple type if the only non-empty Seiberg-Witten moduli spaces are zero-dimensional [W, FS1]. When Witten introduced the invariants

he conjectured that the two notions of simple type coincide, and for manifolds of simple type the set of basic classes are the same, where a Seiberg-Witten basic class is defined as the homology class associated to a line bundle with a non-vanishing Seiberg-Witten invariant. He also conjectured a precise relationship between the Donaldson series and the Seiberg-Witten invariant for manifolds of simple type—from the results of Kronheimer and Mrowka this amounts to conjecturing that not only are the basic classes  $\kappa_i$  the same but their associated rational numbers  $a_i$  bear an exact relationship with the Seiberg-Witten invariant. This conjecture has been confirmed for simply connected elliptic surfaces [FS1], and inroads toward a general proof have been made by [PT].

### The Seiberg-Witten invariant and Kähler Surfaces.

On a Kähler surface  $X$  the Seiberg-Witten equations have a particularly simple form. Witten solved the equations on a Kähler surface [W] and showed that the moduli space can be described completely in terms of the holomorphic structure on  $X$ . Importantly for the case where  $p_g(X) > 0$  he also gave an explicit algorithm for calculating the invariant and determining the basic classes on  $X$ .

**Theorem 0.1.1.** [W] *All Kähler surfaces with  $b^+ > 1$  are of simple type. The Seiberg-Witten invariant associated to the canonical class  $\mathbf{K}_X$  is  $\pm 1$ , and all other basic classes  $L$  satisfy,  $-\deg \mathbf{K}_X \leq \deg L \leq \deg \mathbf{K}_X$ .*

The simple form of the invariants lead quickly to the confirmation of some outstanding problems in algebraic geometry [FM1], the ease with which these results follow from the new theory [FM3, Bs] is further testimony to the power of these new invariants. The results relevant to the work in this paper are:

**Theorem 0.1.2.** [FM3, Bs] *Let  $X$  and  $X'$  be minimal Kähler surfaces with Kodaira dimension at least zero. Suppose that  $\tilde{X}$  and  $\tilde{X}'$  are blowups of  $X$  and  $X'$  at  $m$  and  $n$  distinct points respectively, with associated exceptional curves  $\xi_1, \dots, \xi_m$  and  $\xi'_1, \dots, \xi'_n$ . If  $\phi : \tilde{X} \rightarrow \tilde{X}'$  is an orientation preserving diffeomorphism then  $m = n$  and,*

- (i) *for each  $i$  there is a  $j$  such that  $\phi^*[\xi'_i] = \pm[\xi_j]$ .*
- (ii) *Also  $\phi^*[\mathbf{K}'_X] = \pm[\mathbf{K}_X]$ , where  $\mathbf{K}'_X$  ( $\mathbf{K}_X$ ) denotes the pullback to  $\tilde{X}'$  ( $\tilde{X}$ ) of the canonical class of  $X'$  ( $X$ ).*

Using a general blowup formula [FS1, FM3], connected sums with more general negative definite manifolds can be considered. First suppose that  $N$  is a negative definite manifold such

that  $H_1(N, \mathbf{Z}) = 0$  (thus  $H^2(N, \mathbf{Z})$  is torsion free by the universal coefficient theorem). Then  $H^2(N, \mathbf{Z})$  has a unique basis [D1, FM3]  $\{n_1, \dots, n_k\}$  such that  $n_i^2 = -1$  and  $n_i \cdot n_j = 0$  for  $i \neq j$ .

**Theorem 0.1.3.** [FM3, Bs] *Let  $X$  be a minimal Kähler surface with  $\text{kod } X \geq 0$ , and suppose  $\tilde{X}$  is a blowup of  $X$  at  $p$  distinct points. Let  $N$  be a closed oriented negative definite as described above. If there is an orientation preserving diffeomorphism  $\phi : \tilde{X} \rightarrow M \# N$ , for some four-manifold  $M$ , then for every  $i$ ,  $\phi^*(n_i) = \pm[\xi_j]$  for some exceptional curve  $\xi_j$  on  $\tilde{X}$ .*

In general one can replace  $\tilde{X}$  with a more arbitrary manifold  $M' \# N'$ —however the equality in Theorem 0.1.3. must be replaced with equality mod torsion, that is,  $\phi^*(n_i) \equiv n'_j \pmod{\text{torsion}}$  [FM2, FM3].

### Symplectic Manifolds.

Taubes' results [T1, T2, T3, T4] for symplectic manifolds extend the results for Kähler manifolds. Taubes first proved that the invariant for the canonical  $spin^c$  structure (induced from the almost complex structure defined by the symplectic form) is, as in the Kähler case, equal to  $\pm 1$ . He then announced that the basic classes satisfy a bound analogous to the bound for a Kähler metric, but most striking was his proof [T3, T4] of an equivalence between the Gromov invariants [Gr] defined using pseudo-holomorphic curves and the Seiberg-Witten invariants. When combined with Gromov's results this yields many important consequences on the structure of symplectic manifolds. For full statements of the consequences of this equivalence the reader is referred to Taubes' original papers, here we will merely state the basic structure of the invariants which parallels that of complex surfaces.

**Theorem 0.1.4.** [T1, T2, T3, T4] *All symplectic manifolds with  $b^+ > 1$  are of simple type. The Seiberg-Witten invariant associated to the canonical  $spin^c$  structure  $K$  is  $\pm 1$ . Any basic class satisfies,  $-K \cdot \omega \leq L \cdot \omega \leq K \cdot \omega$ , where  $\omega$  denotes the symplectic form.*

## 0.2. Summary of results proved in this thesis.

The goal of this study is to understand the Seiberg-Witten equations and to calculate the invariant on a general hermitian surface, in particular a hermitian surface whose first betti number is odd. The first step in such a process is to determine what the Seiberg-Witten equations look like for such a surface. This is done via a somewhat tedious calculation in Chapter 2. Here it is found that the Dirac operator for a general hermitian metric differs from the operator associated to a Kähler metric by a real one-form (corresponding to the torsion of the canonical hermitian connection) acting via Clifford multiplication. It is unlikely that this result is not known—its absence from the literature is probably a consequence of its intractability compared to the case when the metric is Kähler.

To simplify the equations obtained in Chapter 2 a new set of equations is defined in Chapter 3. These equations differ from the “standard” equations by an arbitrary real one-form acting via Clifford multiplication in the Dirac equation. It is then shown that solutions to these equations satisfy a  $C^0$  bound. After this bound is obtained the compactness of the moduli space can be proved in exactly the same fashion as for the standard equations.

Chapter 4 contains the fundamental result that enables us to use these new equations to calculate the Seiberg-Witten invariant—that is the independence of the moduli space from the perturbing one-form in the Dirac equation. Irrespective of its subsequent applications, this is an aesthetically pleasing result—it proves that the Seiberg-Witten invariant is independent of the particular “Dirac operator” used in the equations, where we are taking the broadest definition of a Dirac operator as one whose square is a generalised Laplacian [BGV §3.3].

In the first three sections of Chapter 5 we apply the fundamental result of Chapter 4 to reproduce a version of Theorems (0.1.1) and (0.1.4) for an arbitrary hermitian surface. The only restriction placed on the hermitian metric is that its associated Kähler form is  $\bar{\partial}\partial$ -closed. This restriction is minimal since a theorem of Gauduchon [G] states that every hermitian metric has a conformal rescaling so that its associated Kähler form is  $\bar{\partial}\partial$ -closed.

Finally in the concluding two sections of Chapter 5 we return to our original goal, the calculation of the Seiberg-Witten invariant on a non-Kähler surface with  $b^+ > 1$ . It should be noted that classical methods classify such surfaces up to diffeomorphism by their fundamental groups [Ue] so in this direction one cannot hope for improvement. Classical methods also imply that the exceptional curves are preserved in diffeomorphisms between non-Kähler complex

surfaces [FM3]. Using Seiberg-Witten theory we can give a direct proof of this fact, as well as the more general assertion corresponding to Theorem (0.1.3), a result conjectured in [FM3].

Since completing this thesis, I have learned of the existence of a preprint by Olivier Biquard [Bi] covering much of the same material. However I have not had the opportunity to see this preprint and do not know any of his main results.

# Chapter 1

## The Seiberg-Witten Equations

This Chapter contains a brief introduction to the Seiberg-Witten equations as defined in [W]. Essentially this amounts to a presentation of a few facts about  $spin^c$  structures over four-manifolds. There is extensive literature on the subject of spin structures, the book by Lawson and Michelsohn [LM] is a particularly good general reference (see also [BGV, BBW]). Since the advent of Seiberg-Witten theory a number of papers ([Ak, KM3, M, OT, Sz] to name just a few) have offered various explanations of the aspects of  $spin^c$  structures that bear a direct relation to Seiberg-Witten theory. For this reason our treatment of the general theory will be as brief and direct as possible, with particular reference to the equations on a Kähler surface. This example serves to both clarify the general theory and motivate our work on a general hermitian surface.

### 1.1. $Spin^c$ Structures.

In all that follows  $X$  will denote a compact oriented Riemannian four-manifold, thus the tangent bundle  $TX$  has structure group  $SO(4)$ . The group  $Spin^c(4)$  is defined,

$$Spin^c(4) = Spin(4) \times U(1)/\{\pm 1\}.$$

Recall that in dimension 4 we have the isomorphisms,

$$SO(4) \simeq SU(2) \times SU(2)/\{\pm 1\}$$

$$Spin(4) \simeq SU(2) \times SU(2),$$

so we have the following exact sequences,

$$0 \longrightarrow U(1) \longrightarrow Spin^c(4) \longrightarrow SO(4) \longrightarrow 0 \quad (1.1.1)$$

$$0 \longrightarrow Spin(4) \longrightarrow Spin^c(4) \longrightarrow U(1) \longrightarrow 0 \quad (1.1.2)$$

A lift of the principle  $SO(4)$  associated to the tangent bundle  $TX$  to a principle  $Spin^c(4)$  bundle via sequence (1.1.1) is called a  $spin^c$  structure over  $X$ . Importantly for us the obstruction to this lift vanishes on all four-manifolds [HH, Mas], thus every four-manifold has a  $spin^c$  structure, and for every  $spin^c$  structure the sequence (1.1.2) defines a  $U(1)$  bundle  $L$ . To such a  $spin^c$  structure one can also associate a pair of  $U(2)$  vector bundles  $W^+$  and  $W^-$ , defined via representations of the group  $Spin^c(4)$  into  $U(2)$ .

For our purposes it is more convenient to regard a  $spin^c$  structure as a pair of  $U(2)$  vector bundles  $W^+$  and  $W^-$ , such that  $\det W^+ = \det W^- = L$  and a homomorphism, Clifford multiplication,

$$c(\cdot) : T^*X \longrightarrow Hom(W^\pm, W^\mp), \quad (1.1.3)$$

satisfying  $c(v) \circ c(v) = -|v|^2 \text{Id}$ . It is easily confirmed [Ak, LM] that this definition is equivalent to that in the previous paragraph.

Locally the bundles  $W^\pm$  can be written [LM],  $W^\pm = S^\pm \otimes L^{1/2}$ , where  $S^\pm$  are locally defined bundles of spinors and  $L^{1/2}$  is a local square root of the determinant line bundle  $L$ . The bundles  $S^\pm$  and  $L^{1/2}$  exist globally if and only if the manifold  $X$  is a  $spin$  manifold, that is if and only if the second Stiefel-Whitney class  $w_2(X)$  vanishes. The existence of  $spin^c$  structures on all 4-manifolds means that though  $S^\pm$  and the square root  $L^{1/2}$  of  $L$  may not exist globally, their product  $S^\pm \otimes L^{1/2}$  exists, that is their obstructions “cancel” [LM]. More concretely this obstruction is exactly whether  $w_2(X) \in H_2(X, \mathbf{Z}_2)$  lifts to an integral class.

Let  $\Gamma(W^\pm)$  denote the space of smooth sections of the bundles  $W^\pm$ , a section of  $W^+$  ( $W^-$ ) will be called a positive (negative) spinor. The spin connection,

$$\nabla_A^\pm : \Gamma(W^\pm) \longrightarrow \Gamma(W^\pm) \otimes T^*X,$$

is defined by combining the Levi-Civita connection on  $S^\pm$  with a *choice* of connection  $A$  on  $L$ , which induces a connection on  $L^{1/2}$ . Even though the bundles  $S^\pm$  and  $L^{1/2}$  may not exist

globally, the spin connection is well defined [LM] on their product. The composition of the spin connection with Clifford multiplication (1.1.3) defines the Dirac operator,

$$\mathcal{D}_A \equiv c \circ \nabla_A^\pm : \Gamma(W^\pm) \longrightarrow \Gamma(W^\mp). \quad (1.1.4)$$

Note that we have dropped the superscript  $\pm$  from the Dirac operator. Usually we will be interested in the operator acting on positive spinors and the context will make it clear when we are dealing with the operator that acts on negative spinors. Observe that if we fix a base connection  $A_0$  on  $L$  and write the connection  $A$  as  $A = A_0 + ia$  for some one-form  $a$ , then the corresponding Dirac operators are related by,

$$\mathcal{D}_A = \mathcal{D}_{A_0} + \frac{i}{2} a \cdot. \quad (1.1.5)$$

The factor of a half coming from the local description of the bundles  $W^\pm = S^\pm \otimes L^{1/2}$ . Note also that we are also using the shorthand notation,  $a \cdot$ , for Clifford multiplication by the one-form  $a$ .

Clifford multiplication can be extended to the entire exterior (Clifford) algebra  $\Lambda^*(X)$ . For example if  $u \wedge v$  is a decomposable element of  $\Lambda^2(X)$  then,

$$c(u \wedge v) = c(u) \circ c(v) \in Hom(W^\pm, W^\pm), \quad (1.1.6)$$

linearity and the obvious extension of (1.1.6) to higher degree forms defines Clifford multiplication on the whole algebra. The Hodge  $*$ -operator splits the two-forms  $\Lambda^2(X, \mathbf{R}) = \Lambda^+(X, \mathbf{R}) \oplus \Lambda^-(X, \mathbf{R})$ , and (1.1.6) induces isomorphisms between the bundles  $\Lambda^\pm(X)$  and the bundles  $su(W^\pm)$ , of trace free skew hermitian endomorphisms of  $W^\pm$  [Ak, M, KM3]. If we complexify the above definition of Clifford multiplication we get,

$$c(\cdot) : T^*X \otimes \mathbf{C} \longrightarrow Hom(W^\pm, W^\mp),$$

which satisfies  $c(v)^2 = -|v|^2$  on the real subspace. We also get the isomorphism  $\Lambda^+(X, \mathbf{R}) \otimes \mathbf{C} \simeq sl(W^+)$ , the bundle of trace free endomorphisms. The real subspace of  $\Lambda^+(X, \mathbf{R}) \otimes \mathbf{C}$  is still naturally identified with  $su(W^+)$ .

Using the natural identification  $Hom(W^+, W^+) \simeq W^+ \otimes (W^+)^*$ , consider the map defined for  $\Psi \in \Gamma(W^+)$ ,

$$\tilde{q}(\Psi) = \Psi \otimes \Psi^* - \frac{|\Psi|^2}{2} \text{Id},$$

clearly  $\tilde{q}(\Psi) \in sl(W^+)$ . In fact  $\tilde{q}(\Psi) \in isu(W^+)$ , the trace free self-adjoint endomorphisms. This can easily be seen by representing  $\Psi$  locally as  $\Psi = (\alpha, \beta)$  then,

$$\tilde{q}(\Psi) = \begin{pmatrix} \frac{|\alpha|^2 - |\beta|^2}{2} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \frac{|\beta|^2 - |\alpha|^2}{2} \end{pmatrix}. \quad (1.1.7)$$

Using the isomorphism  $\Lambda^+(X, \mathbf{R}) \simeq su(W^+)$  we can identify  $\tilde{q}(\Psi)$  with a purely imaginary self-dual form, which we will denote by  $q(\Psi) \in \Lambda^+(X, i\mathbf{R})$ .

## The Seiberg-Witten Equations.

Let  $W^+, W^-$  be a  $spin^c$  structure on the compact four-manifold  $X$  with determinant line bundle  $\det(W^+) = \det(W^-) = L$ . The Seiberg-Witten equations are for a pair  $(A, \Psi)$  consisting of a unitary connection  $A$  on  $L$  and a section  $\Psi \in \Gamma(W^+)$ , of the positive spinor bundle which satisfy,

$$\begin{aligned} \not{D}_A \Psi &= 0 \\ F_A^+(L) &= q(\Psi). \end{aligned} \tag{1.1.8}$$

Where  $F_A^+(L)$  denotes the self-dual part of the curvature of  $A$ . Recall that the curvature of a unitary connection on a line bundle is a pure imaginary two-form and so the equations are well defined. We will defer the discussion of the gauge group, moduli space and the perturbed equations until Chapter 3 when we introduce the generalised equations.

## 1.2. The Equations on a Kähler Surface.

The case when the four-manifold is a complex surface with a Kähler metric sheds light on the above definitions, as well as providing motivation for our later work when we consider a complex surface with a general hermitian metric.

Any manifold with an almost complex structure has a canonical  $spin^c$  structure given by,  $W_c^+ = \Lambda^0(X) \oplus \Lambda^{0,2}(X)$ ,  $W_c^- = \Lambda^{0,1}(X)$ , with  $\det W_c^+ = \det W_c^- = \Lambda^{0,2}(X) = \mathbf{K}_x^{-1}$ , where  $\mathbf{K}_x$  denotes the canonical line bundle of the almost complex structure. Clifford multiplication on a positive spinor,  $\Psi = (f, s) \in \Lambda^0(X) \oplus \Lambda^{0,2}(X)$ , by an element  $v = v^{1,0} + v^{0,1} \in T^{1,0}X \oplus T^{0,1}X = T^*X \otimes \mathbf{C}$  is given by [BGV §3.6],

$$v \cdot \Psi = \sqrt{2}(v^{0,1}f - *(v^{1,0} \wedge s)),$$

and on negative spinors  $\phi \in \Lambda^{0,1}(X)$ ,

$$v \cdot \phi = \sqrt{2}(v^{0,1} \wedge \phi - *(v^{1,0} \wedge *\phi)).$$

It is easy to see that if  $v^{1,0} = \bar{v}^{0,1}$  that is,  $v$  is real, then  $(v \cdot)v \cdot = -|v|^2$ , and so is consistent with the definition in equation (1.1.3).

Let  $\omega$  denote the Kähler form associated to the Kähler metric on  $X$ , then the complex self-dual two-forms can be decomposed orthogonally as [DK §2.1],  $\Lambda^0(X)\omega \oplus \Lambda^{0,2}(X) \oplus \Lambda^{2,0}(X)$ . If

$\eta = \eta^0\omega + \eta^{0,2} + \eta^{2,0}$  is a real self-dual two form then clearly,  $\eta^0$  is a real function and  $\eta^{0,2} = \bar{\eta}^{2,0}$ . If  $\Psi = (f, s)$  is a positive spinor the pure imaginary two-form  $q(\Psi)$  is given by (see (1.1.7)),

$$q(\Psi) = \frac{i\omega}{2}(|f|^2 - |s|^2) + \bar{f}s - f\bar{s}. \quad (1.2.1)$$

On a Kähler manifold the Levi-Civita and canonical hermitian connection coincide, and if we make the *choice* of the canonical hermitian connection on the determinant line bundle the Dirac operator is exactly [H],

$$\not{D} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*). \quad (1.2.2)$$

Any *spin*<sup>c</sup> structure on  $X$  can be written in terms of the canonical *spin*<sup>c</sup> structure by tensoring it with a line bundle  $E$ . Thus we will write a *spin*<sup>c</sup> structure on a complex manifold as  $W_E^+ = W_c^+ \otimes E = \Lambda^0(X, E) \oplus \Lambda^{0,2}(X, E)$ ,  $W_E^- = W_c^- \otimes E = \Lambda^{0,1}(X, E)$  and  $\det W_E^+ = 2E - \mathbf{K}_X$ . If  $A$  is a connection on  $E$  the Dirac operator naturally induced from (1.2.2) is exactly,

$$\not{D}_A = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*).$$

In this picture the Seiberg-Witten equations are regarded as equations for a spinor  $\Psi = (f, s) \in \Lambda^0(X, E) \oplus \Lambda^{0,2}(X, E)$  and a connection  $A$  on  $E$  satisfying (note that in this picture the equations on the canonical *spin*<sup>c</sup> structure is for a connection on the trivial line bundle),

$$\begin{aligned} \bar{\partial}_A f + \bar{\partial}_A^* s &= 0 \\ F_{A'}^+(2E - \mathbf{K}_X)^{1,1} &= \frac{i\omega}{2}(|f|^2 - |s|^2) \\ F_{A'}(2E - \mathbf{K}_X)^{0,2} &= \bar{f}s \\ F_{A'}(2E - \mathbf{K}_X)^{2,0} &= -f\bar{s}, \end{aligned} \quad (1.2.3)$$

where we have used (1.2.1) and  $A'$  denotes the connection induced on the determinant line bundle  $2E - \mathbf{K}_X$  from the connection  $A$  on  $E$  and the canonical hermitian connection on  $\mathbf{K}_X$ . Formulating the equations in this way makes it easier to analyse the equations for all *spin*<sup>c</sup> structures at once, as well as avoiding the need to use objects that may only be locally defined. Without going into details here (which will be covered in Chapter 5) the solutions to these equations can easily be shown to exist only for certain holomorphic line bundles  $E$ , and the solutions to the equations can be completely described in terms of holomorphic objects associated to such a line bundle [W].

Apart from providing a concrete presentation of the abstract notions of the first section of this Chapter, the ease of analysis of the equations on a Kähler surface should be thought of as

an ideal situation which a large part of our subsequent work will be directed at realising. The first step in the realisation of this ideal for an arbitrary hermitian surface is to get an explicit description of the Dirac operator associated to such a surface. This is the subject of Chapter 2.

# Chapter 2

## The Seiberg-Witten Equations on a Hermitian Surface

In this chapter we establish the form of the Seiberg-Witten equations on a general hermitian surface  $(X, h)$ . As was seen in Chapter 1, if  $h$  is a Kähler metric then the equations have a particularly simple form. If the metric  $h$  is not Kähler then the curvature part of (1.2.3) remains the same, but the Dirac operator will have the form,  $\bar{\partial}_A + \bar{\partial}_A^* + \tau$ , where  $\tau \in \text{Hom}(W^+, W^-)$  [H]. So determining the Seiberg-Witten equations amounts to determining this homomorphism.

**The Spin Connection.** Any complex surface has a canonical  $spin^c$  structure with positive and negative spin bundles given by,

$$W^+ = \Lambda^0(X) \oplus \Lambda^{0,2}(X)$$

$$W^- = \Lambda^{0,1}(X),$$

its determinant line bundle is  $\det W^+ = \det W^- = \Lambda^{0,2}(X)$ .

The spin connection on  $W^+ \otimes W^-$  is characterised by the fact that it agrees with the Levi-Civita connection up to a choice of connection on the determinant line bundle. To make use of this characterisation we use the canonical isomorphism,  $V \simeq V^* \otimes \det V$ , valid for any two dimensional complex vector space. Using the splitting of the complexified tangent bundle

$TX \otimes \mathbf{C} \simeq T^{0,1}X \oplus T^{1,0}X$ , this implies,

$$\begin{aligned} T^{0,1}X \oplus T^{1,0}X &\simeq T^{0,1}X \otimes \Lambda^{0,2}(X) \oplus T^{1,0}X \\ &\simeq T^{0,1}X \otimes (\Lambda^0(X) \oplus \Lambda^{0,2}(X)) \\ T^{0,1}X \oplus T^{1,0}X &\simeq (W^-)^* \otimes W^+. \end{aligned} \quad (2.1)$$

Using the hermitian metric  $h^{a\bar{b}}$  the isomorphism in (2.1) can be realised for  $g^{\bar{a}} \otimes (f, s_{\bar{c}\bar{b}}) \in (W^-)^* \otimes W^+$  as,

$$g^{\bar{a}} \otimes (f, s_{\bar{c}\bar{b}}) \rightarrow g^{\bar{a}}f + h^{a\bar{d}}g^{\bar{a}}s_{\bar{a}\bar{d}} \in T^{0,1}X \oplus T^{1,0}X \quad (2.2)$$

If we denote the hermitian connection by a semi-colon we can write the spin connection as,

$$\nabla^-(g^{\bar{a}}) = g^{\bar{a}}{}_{;\gamma} + B^{\bar{a}}{}_{\bar{b}\gamma}g^{\bar{b}} \quad (2.3a)$$

$$\nabla^+(f, s_{\bar{a}\bar{d}}) = (f)_{;\gamma} + a_{\gamma}f + b^{\bar{e}\bar{f}}{}_{\gamma}s_{\bar{e}\bar{f}}s_{\bar{a}\bar{d};\gamma} + c_{\bar{a}\bar{d}\gamma}f + e_{\gamma}s_{\bar{a}\bar{d}} \quad (2.3b)$$

where  $\gamma = (c, \bar{c})$ . The cross terms  $c$  and  $b$  appear because the Levi-Civita connection is not parallel with respect to the splitting of the complexified tangent space  $T_{\mathbf{C}}X \simeq T^{0,1}X \oplus T^{1,0}X$  (it is parallel if the metric  $h$  on  $X$  is Kähler). The first step to determining the Dirac operator consists of getting explicit expressions for  $B$ ,  $a$ ,  $b$ ,  $c$  and  $e$  in (2.3).

Using the isomorphism (2.2) the spin connection (2.3) induces a connection,  $\nabla_{W^- \otimes W^+}$ , on the tangent bundle,

$$\begin{aligned} \nabla_{W^- \otimes W^+}(g \otimes (f, s)) &= \nabla^-(g) \otimes (f, s) + g \otimes \nabla^+(f, s) \\ &= (g^{\bar{a}}f)_{;\gamma} + (h^{a\bar{d}}g^{\bar{a}}s_{\bar{a}\bar{d}})_{;\gamma} \\ &\quad + B^{\bar{a}}{}_{\bar{b}\gamma}h^{a\bar{d}}g^{\bar{b}}s_{\bar{a}\bar{d}} + c_{\bar{a}\bar{d}\gamma}h^{a\bar{d}}g^{\bar{a}}f + e_{\gamma}h^{a\bar{d}}g^{\bar{a}}s_{\bar{a}\bar{d}} \end{aligned} \quad (2.4a)$$

$$+ B^{\bar{a}}{}_{\bar{b}\bar{c}}g^{\bar{b}}f + a_{\bar{c}}g^{\bar{a}}f + b^{\bar{e}\bar{f}}{}_{\bar{c}}s_{\bar{e}\bar{f}}g^{\bar{a}} \quad (2.4b)$$

$$+ B^{\bar{a}}{}_{\bar{b}\bar{c}}h^{a\bar{d}}g^{\bar{b}}s_{\bar{a}\bar{d}} + c_{\bar{a}\bar{d}\bar{c}}h^{a\bar{d}}g^{\bar{a}}f + e_{\bar{c}}h^{a\bar{d}}g^{\bar{a}}s_{\bar{a}\bar{d}} \quad (2.4c)$$

$$+ B^{\bar{a}}{}_{\bar{b}\bar{c}}g^{\bar{b}}f + a_{\bar{c}}g^{\bar{a}}f + b^{\bar{e}\bar{f}}{}_{\bar{c}}g^{\bar{a}}s_{\bar{e}\bar{f}} \quad (2.4d)$$

Note that a choice of connection on the determinant line bundle amounts to a choice of  $a_{\gamma} + e_{\gamma}$ , and the Levi-Civita connection will determine the components  $B$ ,  $a$ ,  $b$ ,  $c$  and  $e$  up to such a choice. The choice  $a_{\gamma} + e_{\gamma} = 0$  amounts to choosing the canonical hermitian connection on the determinant line.

**The Levi-Civita Connection.** Using the standard formulae [N] for the components of the Levi-Civita connection  $\nabla_L$  and the canonical hermitian connection  $\nabla_H$  we can write

$$\nabla_L = \nabla_H + A$$

where  $A$  is given by,

$$\begin{aligned}
A^a_{bc} &= \frac{1}{2} h^{a\bar{e}} (h_{c\bar{e},b} - h_{b\bar{e},c}) \\
A^a_{\bar{b}c} &= \frac{1}{2} h^{a\bar{e}} (h_{\bar{e}c,\bar{b}} - h_{\bar{b}c,\bar{e}}) \\
A^a_{b\bar{c}} &= \frac{1}{2} h^{a\bar{e}} (h_{\bar{e}b,\bar{c}} - h_{\bar{c}b,\bar{e}}) \\
A^{\bar{a}}_{bc} &= \frac{1}{2} h^{\bar{a}e} (h_{e\bar{b},c} - h_{c\bar{b},e}) \\
A^{\bar{a}}_{b\bar{c}} &= \frac{1}{2} h^{\bar{a}e} (h_{e\bar{c},b} - h_{b\bar{c},e}) \\
A^{\bar{a}}_{\bar{b}c} &= \frac{1}{2} h^{\bar{a}e} (h_{\bar{e}c,\bar{b}} - h_{\bar{b}e,\bar{c}}) \\
A^a_{\bar{b}\bar{c}} &= A^{\bar{a}}_{bc} = 0
\end{aligned} \tag{2.5}$$

If  $h$  is a Kähler metric then  $h_{\bar{e}e,\bar{b}} = h_{\bar{b}e,\bar{e}}$  so  $A$  vanishes in this case as expected.

To allow comparison with (2.4) consider,

$$\begin{aligned}
\nabla_L (g^{\bar{a}} f + h^{a\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}}) &= (h^{a\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}})_{;\gamma} + (g^{\bar{a}} f)_{;\gamma} \\
&\quad + A^a_{bc} h^{b\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}} + A^a_{\bar{b}c} g^{\bar{b}} f
\end{aligned} \tag{2.6a}$$

$$+ A^{\bar{a}}_{\bar{b}\bar{c}} g^{\bar{b}} f + A^{\bar{a}}_{b\bar{c}} h^{b\bar{d}} g^{\bar{b}} s_{\bar{b}\bar{d}} \tag{2.6b}$$

$$+ A^a_{b\bar{c}} h^{b\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}} \tag{2.6c}$$

$$+ A^{\bar{a}}_{\bar{b}c} g^{\bar{b}} f \tag{2.6d}$$

Equations (2.4) and (2.6) have been arranged so as to allow easy comparison of terms. As explained above (2.4) and (2.6) are equivalent up to a choice for  $a_\gamma + e_\gamma$ . So equating the coefficients of the same type in (2.4) and (2.6) yields,

$$A^a_{bc} h^{b\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}} = B^{\bar{a}}_{\bar{b}c} h^{a\bar{d}} g^{\bar{b}} s_{\bar{a}\bar{d}} + e_c h^{a\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}} \tag{2.7a}$$

$$A^a_{\bar{b}c} g^{\bar{b}} f = c_{\bar{a}\bar{d}c} h^{a\bar{d}} g^{\bar{a}} f \tag{2.7b}$$

$$A^{\bar{a}}_{\bar{b}\bar{c}} g^{\bar{b}} f = B^{\bar{a}}_{\bar{b}\bar{c}} g^{\bar{b}} f + a_{\bar{c}} g^{\bar{a}} f \tag{2.7c}$$

$$A^{\bar{a}}_{b\bar{c}} h^{b\bar{d}} g^{\bar{b}} s_{\bar{b}\bar{d}} = b^{\bar{e}\bar{f}} c_{\bar{g}} g^{\bar{a}} s_{\bar{e}\bar{f}} \tag{2.7d}$$

$$A^a_{b\bar{c}} h^{b\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}} = B^{\bar{a}}_{\bar{b}\bar{c}} h^{a\bar{d}} g^{\bar{b}} s_{\bar{a}\bar{d}} + e_{\bar{c}} h^{a\bar{d}} g^{\bar{a}} s_{\bar{a}\bar{d}} \tag{2.7e}$$

$$0 = c_{\bar{a}\bar{d}\bar{c}} h^{a\bar{d}} g^{\bar{a}} f \tag{2.7f}$$

$$A^{\bar{a}}_{\bar{b}c} g^{\bar{b}} f = B^{\bar{a}}_{\bar{b}c} g^{\bar{b}} f + a_c g^{\bar{a}} f \tag{2.7g}$$

$$0 = b^{\bar{e}\bar{f}} c_{\bar{g}} g^{\bar{a}} s_{\bar{e}\bar{f}} \tag{2.7h}$$

The equations (2.7b), (2.7c), (2.7d), (2.7f), (2.7g) and (2.7h) immediately simplify to give the following equations,

$$A_{\bar{a}\bar{b}c} = c_{\bar{b}\bar{a}c} \tag{2.8a}$$

$$A^{\bar{a}}_{\bar{b}\bar{c}} = B^{\bar{a}}_{\bar{b}\bar{c}} + a_{\bar{c}}\delta_{\bar{b}}^{\bar{a}} \quad (2.8b)$$

$$\frac{1}{2}A^{\bar{a}}_{\bar{b}\bar{c}}h^{b\bar{d}}s_{\bar{a}\bar{d}} = b^{\bar{e}\bar{f}}_{\bar{c}}s_{\bar{e}\bar{f}} \quad (2.8c)$$

$$c_{\bar{a}\bar{d}\bar{c}} = 0 \quad (2.8d)$$

$$A^{\bar{a}}_{\bar{b}c} = B^{\bar{a}}_{\bar{b}c} + a_c\delta_{\bar{b}}^{\bar{a}} \quad (2.8e)$$

$$b^{\bar{e}\bar{f}}_c = 0 \quad (2.8f)$$

where the factor of  $1/2$  in (2.8c) comes from taking the trace over  $\delta_{\bar{b}}^{\bar{a}}$ . In (2.7c) the factor of  $s_{\bar{e}\bar{f}}$  has been left there purely for later convenience. We still haven't used equations (2.7a) and (2.7e). First notice that the factor of  $s_{\bar{a}\bar{d}}$  can be cancelled immediately, and if we then replace the  $B^{\bar{a}}_{\bar{b}c}$  term in (2.7a) with the expression obtained in (2.7e) we get,

$$\begin{aligned} A^a_{bc}h^{b\bar{d}}g^{\bar{a}} &= (A^{\bar{a}}_{\bar{b}c} - a_c\delta_{\bar{b}}^{\bar{a}})h^{a\bar{d}}g^{\bar{b}} + e_ch^{a\bar{d}}g^{\bar{a}} \\ A^a_{ac}\delta_{\bar{b}}^{\bar{a}}g^{\bar{b}} &= (2A^{\bar{a}}_{\bar{b}c} - 2a_c\delta_{\bar{b}}^{\bar{a}})g^{\bar{b}} + 2e_c\delta_{\bar{b}}^{\bar{a}}g^{\bar{b}} \\ 2A^a_{ac} &= 2A^{\bar{a}}_{\bar{a}c} - 4a_c + 4e_c \\ e_c - a_c &= \frac{1}{2}(A^a_{ac} - A^{\bar{a}}_{\bar{a}c}) \end{aligned} \quad (2.8g)$$

A similar argument yields,

$$e_{\bar{c}} - a_{\bar{c}} = \frac{1}{2}(A^a_{a\bar{c}} - A^{\bar{a}}_{\bar{a}\bar{c}}) \quad (2.8h)$$

Note that the expressions in (2.8) completely determine the components of the spin connection up to a choice for  $a_\gamma + e_\gamma$ . Such a choice fixes  $a_\gamma$  and  $e_\gamma$  via (2.8g) and (2.8h), and once these are fixed (2.7a)...(2.7f) fix the other components of the spin connection.

**The Dirac Operator.** The Dirac operator  $\not{D}: \Gamma(W^+) \rightarrow \Gamma(W^-)$  is defined by the composition of Clifford multiplication with the spin connection  $\nabla^+$ . Clifford multiplication on  $\Gamma(W^+) = \Lambda^0(X) \oplus \Lambda^{0,2}(X)$  is defined by,

$$\begin{aligned} c: (T^{*0,1}X \oplus T^{*1,0}X) \otimes \Gamma(W^+) &\longrightarrow \Gamma(W^-) \\ c(v^{0,1}, v^{1,0})(f, s) &= \epsilon(v^{0,1})f - \iota(v^{1,0})s, \end{aligned} \quad (2.9)$$

where  $\epsilon$  denotes exterior multiplication and  $\iota$  denotes contraction with the metric. Note that we have dropped the factor of  $\sqrt{2}$  from the definition in §1.2, but this will make no difference to our final result.

Using (2.9) on the expression for  $\nabla^+(f, s) \in (\Lambda^{0,1}(X) \oplus \Lambda^{1,0}(X)) \otimes \Gamma(W^+)$  in (2.3b) shows that the Dirac operator is,

$$\begin{aligned} \not{D}(f, s) &= c \circ \nabla^+(f, s_{\bar{a}\bar{b}}) \\ &= f_{;\bar{c}} + a_{\bar{c}}f + b^{\bar{e}\bar{f}}_{\bar{c}}s_{\bar{e}\bar{f}} - h^{c\bar{a}}(s_{\bar{a}\bar{c};c} + c_{\bar{a}\bar{c}c}f + e_cs_{\bar{a}\bar{c}}) \end{aligned}$$

After substituting the relevant terms from (2.7), this expression becomes,

$$\mathcal{D}(f, s) = f_{;\bar{\varepsilon}} + (a_{\bar{\varepsilon}} - A_{\bar{\varepsilon}}^c c) f - h^{c\bar{a}} s_{\bar{a}\bar{\varepsilon};c} + h^{c\bar{a}} \left( \frac{1}{2} A_{cb\bar{\varepsilon}} h^{b\bar{d}} s_{\bar{a}\bar{d}} - e_c s_{\bar{a}\bar{\varepsilon}} \right) \quad (2.10)$$

The term  $A_{cb\bar{\varepsilon}} h^{b\bar{d}} s_{\bar{a}\bar{d}}$  in (2.10) has to be simplified to get a useable form of the Dirac operator.

**Claim 2.11.**

$$\frac{1}{2} A_{cb\bar{\varepsilon}} h^{b\bar{d}} s_{\bar{a}\bar{d}} = -s_{\bar{a}\bar{\varepsilon}} A^{\bar{d}}_{\bar{d}c}$$

**Proof:** We will need the following identities involving the totally antisymmetric tensor  $\varepsilon^{\bar{a}\bar{b}}$ . Given  $s_{\bar{a}\bar{b}}$  known to be skew and  $f_{\bar{a}\bar{b}}$  not necessarily skew then,

$$\begin{aligned} s_{\bar{a}\bar{b}} &= \frac{1}{2} \varepsilon_{\bar{a}\bar{b}} \varepsilon^{\bar{p}\bar{q}} s_{\bar{p}\bar{q}} \\ f_{\bar{a}\bar{b}} &= f_{\bar{b}\bar{a}} + \varepsilon_{\bar{a}\bar{b}} \varepsilon^{\bar{p}\bar{q}} f_{\bar{p}\bar{q}} \end{aligned}$$

We will also need to use some of the symmetries (which can be confirmed from (2.5)) of the Levi-Civita connection namely,

$$\begin{aligned} A_{bc\bar{a}} &= -A_{cb\bar{a}} \\ A^{\bar{a}}_{\bar{b}\bar{a}} &= A^{\bar{a}}_{\bar{a}\bar{b}} \end{aligned}$$

Note that our connection matrix  $A$  is taken with respect to the hermitian connection so it does not have the same symmetries as the ‘usual’ Levi-Civita connection matrix. Now for the proof of the claim,

$$\begin{aligned} \frac{1}{2} A_{cb\bar{\varepsilon}} h^{b\bar{d}} s_{\bar{a}\bar{d}} &= -\frac{1}{2} A_{bc\bar{\varepsilon}} h^{b\bar{d}} s_{\bar{a}\bar{d}} \\ &= -\frac{1}{2} A^{\bar{d}}_{c\bar{\varepsilon}} s_{\bar{a}\bar{d}} \\ &= -\frac{1}{2} \left( \frac{1}{2} \varepsilon_{\bar{a}\bar{d}} \varepsilon^{\bar{p}\bar{q}} s_{\bar{p}\bar{q}} A^{\bar{d}}_{c\bar{\varepsilon}} \right) \\ &= -\frac{1}{2} \left( \frac{1}{2} \varepsilon_{\bar{a}\bar{\varepsilon}} \varepsilon^{\bar{p}\bar{q}} s_{\bar{p}\bar{q}} A^{\bar{d}}_{c\bar{d}} + \frac{1}{2} \varepsilon_{\bar{d}\bar{\varepsilon}} \varepsilon^{\bar{r}\bar{s}} \varepsilon_{\bar{a}\bar{r}} \varepsilon^{\bar{p}\bar{q}} s_{\bar{p}\bar{q}} A^{\bar{d}}_{c\bar{s}} \right) \end{aligned}$$

Which gives,

$$\frac{1}{2} A_{cb\bar{\varepsilon}} h^{b\bar{d}} s_{\bar{a}\bar{d}} = -\frac{1}{2} (s_{\bar{a}\bar{\varepsilon}} A^{\bar{d}}_{\bar{d}c} - s_{\bar{d}\bar{\varepsilon}} A^{\bar{d}}_{c\bar{a}}) \quad (2.12)$$

A closer inspection of the second term on the RHS of (2.12) yields,

$$\begin{aligned} h^{\bar{a}c} s_{\bar{d}\bar{\varepsilon}} A^{\bar{d}}_{c\bar{a}} &= h^{\bar{a}c} h^{\bar{d}b} s_{\bar{d}\bar{\varepsilon}} A_{bc\bar{a}} \\ &= -h^{\bar{a}c} h^{\bar{d}b} s_{\bar{d}\bar{\varepsilon}} A_{cb\bar{a}} \\ &= -h^{\bar{d}b} s_{\bar{d}\bar{\varepsilon}} A^{\bar{a}}_{\bar{b}\bar{a}} \end{aligned}$$

The claim is proved after changing the dummy indices and substituting back into (2.12).

△

Returning to the Dirac operator we now have,

$$\mathcal{D}(f, s) = f_{;\bar{c}} + (a_{\bar{c}} + A^a{}_{a\bar{c}})f - h^{c\bar{a}}s_{\bar{a}\bar{c};c} - h^{c\bar{a}}(A^{\bar{a}}{}_{\bar{d}c} + e_c)s_{\bar{a}\bar{c}}.$$

If we use the equivalence between contraction with the metric and the Hodge  $*$ -operator, and define  $\beta = A^a{}_{a\bar{c}}$  ( $\beta$  is the trace of the torsion of the hermitian connection, also known as the Lee form [V]). Also choose  $a_\gamma + e_\gamma = 0$ , then equations (2.8g) and (2.8h) yield,

$$\begin{aligned} a_{\bar{c}} &= \frac{1}{4}(A^{\bar{a}}{}_{\bar{a}\bar{c}} - A^a{}_{a\bar{c}}) = -\frac{1}{2}\beta \\ e_c &= \frac{1}{4}(A^a{}_{ac} - A^{\bar{a}}{}_{\bar{a}c}) = -\frac{1}{2}\bar{\beta} \end{aligned}$$

Thus we can write the Dirac operator in coordinate free form as,

$$\mathcal{D}(f, s) = \bar{\partial}f + \bar{\partial}^*s + \Omega \cdot (f, s) \quad (2.13)$$

where  $\Omega = 1/2(\beta + \bar{\beta})$  is a real one-form acting via Clifford multiplication.

Any other  $spin^c$  structure can be obtained from the canonical  $spin^c$  structure by tensoring it with a complex line bundle  $E$  (this will be explained in greater detail in Chapter 5). In this way the Seiberg-Witten equations on a Hermitian surface  $(X, h)$  become an equation for a connection  $A$  on the bundle  $E$  and a section  $(f, s)$  of  $\Lambda^0(X, E) \oplus \Lambda^{0,2}(X, E)$ ,

$$\begin{aligned} \bar{\partial}_A f + \bar{\partial}_A^* s + \Omega \cdot (f, s) &= 0 \\ F_A^+(2E - \mathbf{K}_X)^{1,1} &= \frac{i\omega}{2}(|f|^2 - |s|^2) \\ F_{A'}(2E - \mathbf{K}_X)^{0,2} &= \bar{f}s \\ F_{A'}(2E - \mathbf{K}_X)^{2,0} &= -f\bar{s} \end{aligned} \quad (2.14)$$

where as in (2.13)  $A'$  denotes the connection on  $2E - \mathbf{K}_X$  induced from the natural hermitian connection on  $\mathbf{K}_X$  and the connection  $A$  on  $E$ . The torsion term  $\Omega$  makes the analysis of these equations somewhat more difficult than the Kähler case where  $\Omega = 0$ . The approach we take is to study the effect of perturbing the Dirac operator by a real one-form acting by Clifford multiplication. Understanding such a perturbation requires a detailed study of the Seiberg-Witten moduli space, which is the subject of the next two chapters.

# Chapter 3

## A Generalisation of the Seiberg-Witten Equations

In the previous chapter we saw that the Seiberg-Witten equation on a general Hermitian surface had a torsion term in the Dirac equation that made analysis of the equations difficult in comparison to the case where our surface has a Kähler metric. The goal of the next two chapters is to reduce the study of the Seiberg-Witten equations on a complex surface to a system of equations as easy to analyse as those on a Kähler surface. To this end we define a new set of equations that differ from the “standard” Seiberg-Witten equations by a perturbation in the Dirac operator in (1.1.8). In the remainder of Chapter 3 and also in Chapter 4 we study the properties of the moduli space of these “new” equations. As for the standard equations the moduli space is easily shown to be compact. However, because of our perturbation of the Dirac equation, we need to show the independence of the moduli space from this perturbation before we can claim to be able to calculate the Seiberg-Witten invariant from these new equations. In Chapter 4 we prove this independence. Note that for a similar reason, the idea of perturbing the Dirac operator also appears in [MOY] where the three dimensional Seiberg-Witten equations are studied on Seifert-fibered spaces.

The presentation in the next two chapters will be as self-contained as possible, but it

should be kept in mind that most of the results are just suitable generalisations of standard Seiberg-Witten theory. In fact all the techniques find their roots in Donaldson ASD theory, the beauty of Seiberg-Witten theory lies in how simply these techniques can be applied.

### 3.1 The Generalised Equations.

Let  $X$  be a smooth compact four dimensional manifold with  $b^+(X) > 0$ . Let  $W_L$  be a  $spin^c$  structure on  $X$  with determinant line bundle  $\det(W_L^+) = L$ . With a fixed real one-form  $v \in \Lambda^1(\mathbf{R})$  consider the following equations for a unitary connection  $A \in \mathcal{A}(L)$  on  $L$  and a positive spinor  $\Psi \in \Gamma(W_L^+)$ :

$$(\not{D}_A + v \cdot) \Psi = 0 \tag{3.1.1a}$$

$$F_A^+(L) = q(\Psi) \tag{3.1.1b}$$

where  $v$  acts by Clifford multiplication and the other terms are defined in Chapter 1. We will still refer to these equations as the ‘‘Seiberg-Witten equations’’ but when comparing equation (3.1.1) with equation (1.1.8) we will refer to the former as the ‘‘new’’ or ‘‘generalised’’ equations and the latter as the ‘‘old’’ or ‘‘standard’’ equations.

We will also need to consider the following perturbed equations, where the perturbing parameter  $\delta \in \Lambda^+(i\mathbf{R})$  is a pure imaginary self-dual two-form,

$$(\not{D}_A + v \cdot) \Psi = 0 \tag{3.1.2a}$$

$$F_A^+(L) = q(\Psi) + \delta. \tag{3.1.2b}$$

In fact we will be doing all our analysis with these perturbed equations, these have the advantage that reducible solutions are avoided (see Theorem 4.0.1) and for the purposes of this chapter contain (3.1.1) as a special case.

The solutions to (3.1.2) are a subset of  $\mathcal{A}(L) \times \Gamma(W_L^+)$  and the action of the gauge group  $\mathcal{G}(L) = \text{Map}(X, S^1)$  on this space is: if  $g = e^\gamma \in \mathcal{G}(L)$ , where  $\gamma \in \Lambda^0(i\mathbf{R})$  and  $(A, \Psi) \in \mathcal{A}(L) \times \Gamma(W_L^+)$  then,

$$g(A, \Psi) = (g(A), g^{-1}\Psi) = (A + 2d\gamma, e^{-\gamma}\Psi).$$

The factor of two in the gauge action on  $\mathcal{A}(L)$  arises from the fact that the Dirac operator is actually acting on the square root of  $L$  (see Chapter 1).

In order to use the analytical tools of Banach spaces we suppose that:  $A, \Psi$  and  $v$  are of class  $L_k^2$  for some  $k \geq 3$ ; our perturbation  $\delta$  is class  $L_{k-1}^2$ ; and our gauge transformations are  $L_{k+1}^2$ .

Our assumption that  $k \geq 3$  is to make use of the nice Sobolev multiplication,  $L_k^2 \times L_k^2 \rightarrow L_k^2$  in this range. We will see that this is no restriction—the moduli space is independent of this choice of  $k$  and will always consist of smooth objects. Define

$$\mathcal{B}(L) = \mathcal{A}(L)_{L_k^2} \times \Gamma(W_L^+)_{L_k^2} / \mathcal{G}(L)_{L_{k+1}^2},$$

(note the surreptitious dropping of the  $L_k^2$  subscript from  $\mathcal{B}(L)$ ), and the moduli space of solutions to (3.1.2),

$$\begin{aligned} M_{X,g}(L, v, \delta) &= \{(A, \Psi) \in \mathcal{A}(L) \times \Gamma(W_L^+) \mid (\mathcal{D}_A + v \cdot) \Psi = 0, F_A^+(L) = q(\Psi) + \delta\} / \mathcal{G}(L) \\ &\subset \mathcal{B}(L) \end{aligned}$$

### 3.2 Compactness of the Moduli Space.

As in standard Seiberg-Witten theory there is an a priori  $C^0$  bound on solutions to the generalised equations (3.1.2). After this bound is obtained the proof of the compactness is exactly the same for the usual theory. For the purpose of proving the compactness of the moduli space, we assume that our perturbations  $v \in \Lambda^0(R)$  and  $\delta \in \Lambda^+(i\mathbf{R})$  are smooth.

**Lemma 3.2.1.** *At any solution  $(A, \Psi)$  to the generalised equations (3.1.2) we have the following  $C^0$  bound on any solution  $\Psi$ ,*

$$|\Psi|^2 \leq \max(0, c_1(v, s, \delta)),$$

where  $c_1(v, s, \delta)$  is a constant depending only on the scalar curvature,  $s$ , of  $X$  and the perturbation parameters  $(v, \delta)$ . We also have a  $L^2$  bound on  $\nabla_A \Psi$ , that is,

$$\|\nabla_A \Psi\|_{L^2}^2 \leq C,$$

for some constant  $C$  depending only on the scalar curvature and perturbation parameters.

**Proof:** The Dirac operator acts as a derivation over the functions, that is if  $f \in \Lambda^0(X)$  and  $\Psi^\pm \in \Gamma(W_L^\pm)$  then,

$$\mathcal{D}_A^\pm(f\Psi^\pm) = df \cdot \Psi^\pm + f\mathcal{D}_A^\pm \Psi^\pm.$$

When  $v \in \Lambda^1(X)$  acts via Clifford multiplication on  $\Psi \in \Gamma(W_L^+)$  its action is not so simple and we will show below that,

$$\mathcal{D}_A^-(v \cdot \Psi) = \mathcal{D}_A^*(v \cdot \Psi) = (d^*v + dv \cdot) \Psi - v \cdot \mathcal{D}_A \Psi - 2\nabla_{A,Y} \Psi, \quad (3.2.2)$$

where  $Y$  denotes the vector field dual to  $v$  and  $\nabla_{A,Y}$  is the covariant derivative associated to the spin connection  $\nabla_A$  and the vector field  $Y$ .

Proof of (3.2.2): Note that by linearity it suffices to prove (3.2.2) for differential forms of type  $fdg$  where  $f, g \in \Lambda^0(X)$ . Then using the fact that both  $\mathcal{D}_A$  and  $\mathcal{D}_A^*$  act as derivations over the functions,

$$\begin{aligned}
\mathcal{D}_A^*(fdg \cdot \Psi) &= df \cdot dg \cdot \Psi + f\mathcal{D}_A^*(dg \cdot \Psi) \\
&= df \cdot dg \cdot \Psi + f\mathcal{D}_A^*(\mathcal{D}_A(g\Psi) - g\mathcal{D}_A\Psi) \\
&= df \cdot dg \cdot \Psi - fdg \cdot \mathcal{D}_A\Psi + f[\mathcal{D}_A^*\mathcal{D}_A, g]\Psi \\
&= f[\nabla_A^*\nabla_A, g]\Psi + df \cdot dg \cdot \Psi - fdg \cdot \mathcal{D}_A\Psi \tag{3.2.3}
\end{aligned}$$

A quick check (using a local orthonormal frame as in [L, App. II] or for a coordinate free proof see [BGV, Prop. 2.5]) establishes that  $[\nabla_A^*\nabla_A, g]\Psi = d^*dg\Psi - 2(dg, \nabla_A\Psi)$ , and also  $df \cdot dg \cdot \Psi = (df \wedge dg) \cdot \Psi - (f, g)\Psi$ . Substituting these expressions back into (3.2.3) gives,

$$\mathcal{D}_A^*(fdg \cdot \Psi) = \{fd^*dg - (f, g)\}\Psi + d(fdg) \cdot \Psi - fdg \cdot \mathcal{D}_A\Psi - 2(fdg, \nabla_A\Psi).$$

The identity  $fd^*dg - (f, g) = d^*(fdg)$  completes the proof of (3.2.2).

We now return to our proof of Lemma (3.2.1). The Weitzenböch formula on positive spinors is [BGV §3.5, LM],

$$\mathcal{D}_A^*\mathcal{D}_A\Psi = \nabla_A^*\nabla_A\Psi + \frac{s}{4}\Psi + \frac{1}{2}F_A^+ \cdot \Psi.$$

Now suppose that  $(A, \Psi)$  is a solution to equation (3.1.2) then the Weitzenböch formula becomes,

$$\mathcal{D}_A(-v \cdot \Psi) = \nabla_A^*\nabla_A\Psi + \frac{s}{4}\Psi + \frac{1}{2}q(\Psi)\Psi + \delta \cdot \Psi.$$

Rearranging this and taking the pointwise inner-product with  $\Psi$ ,

$$\operatorname{Re}\langle \Psi, \nabla_A^*\nabla_A\Psi \rangle = \operatorname{Re}[\langle \Psi, \mathcal{D}_A(-v \cdot \Psi) \rangle - \frac{s}{4}|\Psi|^2 - \frac{1}{2}\langle \Psi, q(\Psi)\Psi \rangle - \langle \Psi, \delta \cdot \Psi \rangle].$$

Now substituting (3.2.2) and using the identity  $\langle \Psi, q(\Psi)\Psi \rangle = 1/2|\Psi|^4$ , which is evident from (1.1.7) we get,

$$\begin{aligned}
\operatorname{Re}\langle \Psi, \nabla_A^*\nabla_A\Psi \rangle &= \operatorname{Re}[\langle \Psi, (-d^*v - dv \cdot) \Psi \rangle + |v|^2|\Psi|^2 + 2\langle \Psi, \nabla_{A,Y}\Psi \rangle \\
&\quad - \frac{s}{4}|\Psi|^2 - \frac{1}{4}|\Psi|^4 - \langle \Psi, \delta \cdot \Psi \rangle] \\
&\leq c_1(v)|\Psi|^2 + 2|v||\Psi||\nabla_A\Psi| - \frac{s}{4}|\Psi|^2 - \frac{1}{4}|\Psi|^4 + |\delta||\Psi|^2 \\
&\leq c(v, s, \delta)|\Psi|^2 + \frac{1}{2}|\nabla_A\Psi|^2 - \frac{1}{4}|\Psi|^4, \tag{3.2.4}
\end{aligned}$$

where we have used the identity  $2|v||\Psi||\nabla_A\Psi| \leq 2|v|^2|\Psi|^2 + 1/2|\nabla_A\Psi|^2$ , and  $c(v, s, \delta)$  denotes a constant depending only on the scalar curvature and the parameters  $(v, \delta)$ . Let  $\Delta$  denote the Laplacian on functions, at points where  $\Psi$  is a maximum we have,

$$\begin{aligned} 0 &\leq \frac{1}{2}\Delta|\Psi|^2 = \operatorname{Re}\langle\Psi, \nabla_A^*\nabla_A\Psi\rangle - \langle\nabla_A\Psi, \nabla_A\Psi\rangle \\ &\leq c(v, s, \delta)|\Psi|^2 - \frac{1}{4}|\Psi|^4 - \frac{1}{2}|\nabla_A\Psi|^2 \\ &\leq c(v, s, \delta)|\Psi|^2 - \frac{1}{2}|\Psi|^4. \end{aligned}$$

If  $\Psi$  is non-zero at the maximum then  $|\Psi|^2 \leq c(v, s, \delta)$ , and the first inequality is proved. The second inequality follows by substituting the first back into equation (3.2.4) and integrating.

△

Now this  $C^0$  bound has been established, to prove that the moduli space for (3.1.2) is compact we will just repeat the proof in [M, KM3] for the standard Seiberg-Witten theory. The following lemma (3.2.5) gives us the tool to bootstrap  $L_i^2$  bounds on solutions up to  $C^\infty$  bounds. For the remainder of the chapter we will fix a smooth connection  $A_0$  on  $L$  and all our  $L_i^p$  norms will be taken with respect to this connection.

**Lemma 3.2.5.** *Let  $L$  be a complex line bundle over  $X$  and fix a smooth connection  $A_0$  on  $L$ . Then for any  $L_i^2$ ,  $l \geq 1$  connection  $A$  on  $L$  there is a  $L_{i+1}^2$  change of gauge  $g$  such that  $g(A) = A_0 + a$ , where  $a \in \Lambda^1(i\mathbf{R})_{L_i^2}$  satisfies  $d^*a = 0$  and*

$$\|a\|_{L_i^2}^2 \leq c\|F_A^+\|_{L_{i-1}^2}^2 + K,$$

where  $c$  and  $K$  are constants depending only on  $A_0$ ,  $X$  and  $l$ .

**Proof:** The first statement is elementary; a gauge transformation  $g = e^\gamma$ ,  $\gamma \in \Lambda^0(i\mathbf{R})$  acts on  $A = A_0 + a$  by,

$$g(A_0 + a) = A_0 + a + d\gamma.$$

So we want to solve the linear elliptic equation,

$$d^*d\gamma = -d^*a.$$

As  $d^*a$  is  $L^2$  orthogonal to the constant functions elementary linear elliptic theory gives an  $L_{i+1}^2$  solution  $\gamma$  to this equation whenever  $a \in L_i^2$ .

We now prove the bound on solutions. Consider the elliptic complex,

$$\Lambda^0(i\mathbf{R}) \xrightarrow{d} \Lambda^1(i\mathbf{R}) \xrightarrow{d^+} \Lambda^+(i\mathbf{R}),$$

and suppose  $a \in \Lambda^1(i\mathbf{R})_{L^2}$  satisfies  $d^*a = 0$ . The Hodge decomposition of  $a$  can be written for  $z \in \Lambda^0(i\mathbf{R})$  and  $t \in \Lambda^+(i\mathbf{R})$ ,

$$\begin{aligned} a &= h + dz + (d^+)^*t \\ &= h + \rho \end{aligned} \tag{3.2.6}$$

where  $h$  is a harmonic one-form and  $\rho$  is orthogonal to the space of harmonic one-forms. Since  $\rho$  is orthogonal to the harmonic one-forms we have (see p.143 [We]),

$$\|\rho\|_{L^2}^2 \leq c(\|d^*\rho\|_{L^2_{i-1}} + \|d^+\rho\|_{L^2_{i-1}}),$$

for a constant  $c$  depending only on  $X$ ,  $A_0$  and  $l$ . By assumption  $d^*a = 0$  so  $d^*\rho = 0$  implying,

$$\begin{aligned} \|\rho\|_{L^2}^2 &\leq c\|d^+\rho\|_{L^2_{i-1}}^2 = c\|F_A^+ - F_{A_0}^+\|_{L^2_{i-1}}^2 \\ &\leq c'(\|F_A^+\|_{L^2_{i-1}}^2 + \|F_{A_0}^+\|_{L^2_{i-1}}^2) \\ &= c'\|F_A^+\|_{L^2_{i-1}}^2 + K. \end{aligned} \tag{3.2.7}$$

$K$ ,  $c$  and  $c'$  are all constants depending only on  $X$ ,  $l$  and  $A_0$ .

To conclude the proof we must bound  $h$  in the decomposition (3.2.6). Now  $h \in i\mathcal{H} \simeq H^1(X, \mathbf{R})$  and  $H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$  is the compact ‘‘Jacobian’’ torus. So we can write  $h = h_1 + h_0$ , where  $h_1$  lies in the Jacobian torus and so is bounded in  $L^2$  and  $h_0$  represents a class in  $H^1(X, \mathbf{Z})$ . The component group of  $\mathcal{G}(L)$  is isomorphic to  $H^1(X, \mathbf{Z})$  and choosing a gauge from an appropriate component we can arrange that  $h_0$  is zero.

Combining this with (3.2.7) we have now proved,

$$\begin{aligned} \|a\|_{L^2}^2 &= \|h + \rho\|_{L^2}^2 \\ &= \|h_1 + \rho\|_{L^2}^2 \\ &\leq \|h_1\|_{L^2}^2 + c'\|F_A^+\|_{L^2_{i-1}}^2 + K \\ &\leq c'\|F_A^+\|_{L^2_{i-1}}^2 + K'. \end{aligned}$$

Which completes the proof of Lemma (3.2.5).

△

A careful examination of the proof of Lemma 3.2.5 reveals that once we have chosen an  $L^2_{l+1}$  gauge such that  $d^*a = 0$  and  $ih \in H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$  the inequality in (3.2.5) is valid for all  $k \geq l + 1$ .

Note that Lemma (3.2.1) and equation (3.1.2b) immediately gives us a  $L^2$  bound on  $F_A^+$ , and so by Lemma (3.2.5) we have, for an appropriate choice of gauge, an  $L_1^2$  bound on  $a$ . We cannot make immediate use of Lemma (3.2.5) because Sobolev multiplication for  $L_l^2$  works well only in the range  $l \geq 3$  and the somewhat fiddly results in the remainder of this section are directed at improving our bounds on  $a$  and  $\Psi$  into the stable Sobolev range,  $l \geq 3$ . We are essentially repeating Uhlenbeck's [U, FU, DK] results for  $SU(2)$  connections in this simpler linear setting.

**Lemma 3.2.8.** *For any solution to equation (3.1.2) there is a constant  $C$  depending only on  $X$  and  $A_0$  such that,*

$$\|F_A^+\|_{L_1^2} \leq C.$$

Thus Lemma (3.2.5) implies we have an  $L_2^2$  bound on  $a$  for an appropriate choice of gauge.

**Proof:** The curvature equation (3.1.2b) reads,

$$F_A^+ = \Psi \otimes \Psi^* - \frac{1}{2}|\Psi|^2 \text{Id} + \delta.$$

Using the fact that exterior differentiation corresponds to the Levi-Civita connection on differential forms gives,

$$dF_A^+ = \nabla_{LC} F_A^+ = \nabla_A \Psi \otimes \Psi^* + \Psi \otimes \nabla_A \Psi^* - \text{Re}\langle \nabla_A \Psi, \Psi \rangle + d\delta.$$

Lemma 3.2.1 immediately gives us the  $L^2$  bound,

$$\|dF_A^+\|_{L^2} \leq c_0 \tag{3.2.9}$$

Now make the  $L^2$  decomposition  $F_A^+ = H + \rho$  where  $H$  is a harmonic self-dual two form and  $\rho$  is orthogonal to the harmonic two forms. Consider the de-Rham complex,

$$\Lambda^0 \longrightarrow \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \longrightarrow \Lambda^4.$$

The Hodge theory of this complex gives us the following inequality for two forms orthogonal to the space of harmonic forms [We p. 143],

$$\|\rho\|_{L_1^2} \leq c_1(\|d\rho\|_{L^2} + \|d^*\rho\|_{L^2}).$$

In our case  $\rho$  is self dual and  $d^* = - * d *$  so this simplifies to,

$$\begin{aligned} \|\rho\|_{L_1^2} &\leq c_1(\|d\rho\|_{L^2} + \| * d\rho\|_{L^2}) \\ &\leq c_2\|d\rho\|_{L^2} \\ &= c_2\|dF_A^+\|_{L^2} \\ &\leq c_0 c_2, \end{aligned} \tag{3.2.10}$$

where we have substituted (3.2.9) and used the fact that  $dH = 0$ . We already have an  $L^2$  bound on  $F_A^+$ , which gives us an  $L^2$  bound on  $H$ . Elliptic regularity theory now gives us an  $L_1^2$  bound on  $H$ . Combining this bound with (3.2.10) completes the proof of the Lemma.

△

We can now move directly to the bootstrapping argument. The proof of the next theorem requires repeated use of the Sobolev embedding and multiplication theorems which we will state here in the most useful form for our applications. See [P] for a good account of more general results, as well as [FU] and [DK].

**Sobolev Theorems.** *Let  $X$  be a compact four-manifold and  $V$  be a vector bundle over  $X$ . Recall that the  $L_k^p$  sections of  $V$  are sections represented by locally  $L_k^p$  functions in a bundle trivialization. Associate to each  $L_k^p$  the scaling weight  $w(k, p) = k - 4/p$ . The Sobolev theorems then state:*

(i) *If  $w(k, p) \geq w(l, q)$  and  $k > l$  then there is a bounded inclusion map*

$$L_k^p \longrightarrow L_l^q. \quad (3.2.11)$$

*If strict inequality  $w(k, p) > w(l, q)$  holds then the inclusion is compact.*

(ii) *If  $w(k_i, p_i) \geq 0$ ,  $i = 1, 2$  then the multiplication*

$$L_{k_1}^{p_1} \otimes L_{k_2}^{p_2} \longrightarrow L_l^q \quad (3.2.12)$$

*is bounded provided  $l \leq 4/q - \max(w(k_i, p_i))$ .*

(ii') *If  $k \geq l$  and  $k \geq 3$  then multiplication*

$$L_k^2 \otimes L_l^2 \longrightarrow L_l^2 \quad (3.2.13)$$

*is bounded.*

(iii) *There is a compact embedding*

$$L_l^2 \longrightarrow C^{l-3} \quad (3.2.14)$$

*for  $l \geq 3$ .*

**Theorem 3.2.15.** *If  $(A, \Psi) = (A_0 + a, \Psi)$  is a solution to (3.1.2) and a gauge is fixed so that Lemma 3.2.5 holds for  $l = 2$ . Then for every  $l > 2$  there is a constant  $c(l)$  depending only on*

$X$ ,  $l$  and  $A_0$  (recall that our Sobolev norms are being taken with respect to this fixed smooth connection) such that,

$$\|a\|_{L^2} + \|\Psi\|_{L^2} \leq c(l).$$

**Proof:** First suppose that we have shown  $\Psi \in L^2_l$  for some  $l \geq 3$ . The Sobolev multiplication (3.2.13) applied to the curvature equation (3.1.2b) establishes that  $F_A^+ \in L^2_l$ . Lemma 3.2.5 now gives us an  $L^2_{l+1}$  bound on  $a$ . The Dirac equation (3.1.2a) for  $\Psi$  reads,

$$(\mathcal{D}_{A_0} + v)\Psi = -\frac{1}{2}a \cdot \Psi. \quad (3.2.16)$$

Sobolev multiplication (3.2.13) on the right of (3.2.16) implies that  $(\mathcal{D}_{A_0} + v)\Psi \in L^2_l$  and by elliptic regularity  $\Psi \in L^2_{l+1}$ . In light of this induction argument for  $l \geq 3$ , the proof of the theorem reduces to proving  $\Psi \in L^2_3$ .

Lemma (3.2.1) gives us an  $L^2$  bound on  $\nabla_A \Psi$ , thus

$$\begin{aligned} C &\geq \|\nabla_A \Psi\|_{L^2} = \|\nabla_{A_0} \Psi + \frac{1}{2}a\Psi\|_{L^2} \\ &\geq \|\nabla_{A_0} \Psi\|_{L^2} - \frac{1}{2}\|a\Psi\|_{L^2}. \end{aligned} \quad (3.2.17)$$

The  $C^0$  bound on  $\Psi$  along with the  $L^2_2$  bound on  $a$  gives us an  $L^2$  bound on  $a\Psi$ . Rearranging (3.2.17) gives us an  $L^2$  bound on  $\nabla_{A_0} \Psi$ , and thus  $\Psi \in L^2_1$ .

By assumption  $a \in L^2_2$  and by the Sobolev embedding (3.2.11) we get  $a \in L^4$ . This along with the  $C^0$  bound on  $\Psi$  implies there is an  $L^4$  bound on  $(\mathcal{D}_{A_0} + v)\Psi$  as is evident from (3.2.16). Similarly (3.2.11) implies that  $\Psi \in L^4$ , and the elliptic inequality [GT, DK],

$$\|\Psi\|_{L^4_1} \leq c(\|(\mathcal{D}_{A_0} + v)\Psi\|_{L^4} + \|\Psi\|_{L^4}),$$

now proves that  $\Psi \in L^4_1$ .

Once again the Sobolev multiplication (3.2.12)  $L^2_2 \otimes L^4_1 \rightarrow L^3_1$  applied to (3.2.16) implies  $(\mathcal{D}_{A_0} + v)\Psi \in L^3_1$ . Again we use the elliptic inequality

$$\|\Psi\|_{L^3_2} \leq c(\|(\mathcal{D}_{A_0} + v)\Psi\|_{L^3_1} + \|\Psi\|_{L^3}),$$

to give us an  $L^3_2$  bound on  $\|\Psi\|$ .

A final application of (3.2.12)  $L^2_2 \otimes L^3_2 \rightarrow L^2_2$  gives us an  $L^2_2$  bound on  $(\mathcal{D}_{A_0} + v)\Psi$  and the elliptic inequality now gives us an  $L^2_3$  bound. The discussion at the beginning of the proof can now be applied to prove the theorem.

△

Now using (3.2.14) we get as an immediate corollary of Theorem 3.2.15 the compactness of the moduli space.

**Corollary 3.2.18.** *Let  $(A_n, \Psi_n)$  be a sequence of solutions to (3.1.2). There is a subsequence  $(A_{n_i}, \Psi_{n_i})$  and  $L^2_3$  gauge transformations  $\{g_{n_i}\}$  so that  $g_{n_i}(A_{n_i}, \Psi_{n_i})$  converges smoothly to a limit  $(A, \Psi)$ .*

△

# Chapter 4

## Further Properties of the Moduli Space

In this chapter we will exhibit further properties of the moduli space  $M_{X,g}(L, v, \delta)$  that will enable us to define a smooth invariant on  $X$  namely: the free action of the gauge group on the solution space, or equivalently the absence of irreducible solutions;  $M_{X,g}(L, v, \delta)$  is a smooth oriented manifold “independent” of the choice of metric  $g$  and perturbation parameters  $v$  and  $\delta$ , where “independent” means the spaces  $M_{X,g_0}(L, v_0, \delta_0)$  and  $M_{X,g_1}(L, v_1, \delta_1)$  are cobordant up to a suitable (to be determined) genericity of  $g_i$ ,  $v_i$  and  $\delta_i$ ,  $i = 1, 2$ . Compactness is also required, but this was established in Chapter 3.

In Donaldson ASD theory [DK §4.3, FU] one proceeds by showing that for generic metrics the moduli space of solutions to the ASD equations is a smooth manifold free from reducible solutions whenever  $b^+ > 0$ . If  $b^+ > 1$  then there is a smooth cobordism, free from reducible solutions, between any two such generic metrics [DK Cor.4.3.19]. Whilst it can be shown that if  $c_1(L) \neq 0 \in H^2(X, \mathbf{R})$  then for generic metrics the solutions to (3.1.1) are irreducible, but if  $c_1(L)$  is torsion in  $H^2(X, \mathbf{Z})$  this is no longer true. This gives us the main reason for considering the perturbed equations (3.1.2), because for *any* metric  $g$  and generic perturbation  $\delta$  we will show that the moduli space  $M_{X,g}(L, v, \delta)$  is a smooth manifold free from reducible solutions

whenever  $b^+ > 0$ . Although at first glance this “independence” from the metric seems in glaring contrast with Donaldson theory, once one takes note of the fact that the metric determines the space of self-dual forms this result becomes more palatable. In fact to prove the results in Donaldson theory [DK] one deals directly with the space of self-dual forms defined by the metric, and infer from this analysis results on the metric.

Before proceeding with the main analysis we prove an elementary, but important result on reducible solutions, which ensures the free action of the gauge group on the set of solutions to (3.1.2) for generic perturbations  $\delta$ .

**Lemma 4.0.1.** *Let  $X$  be a smooth four manifold with  $b^+ > 0$  and let  $g$  be an arbitrary metric on  $X$ . There is an open dense subset  $U^+ \subset \Lambda^+(i\mathbf{R})$  such that if  $\delta \in U^+$  the moduli space  $M_{X,g}(L, v, \delta)$  consists only of irreducible points. If  $b^+ > 1$  and  $h : [0, 1] \rightarrow \Lambda^+(i\mathbf{R})$  is a smooth path in  $\Lambda^+(i\mathbf{R})$  there is an arbitrarily small perturbation of  $h$  whose image lies in  $U^+$ .*

**Proof:** A reducible solution to (3.1.2) satisfies  $F_A^+ = \delta$ . Let  $P$  denote the  $L^2$  projection of  $\Lambda^+(i\mathbf{R})$  onto the  $b^+$  dimensional subspace of (pure imaginary) self-dual harmonic forms. Thus reducible solutions exist only if  $P(2\pi/i.c(L)) = P(\delta)$ , and this is clearly not the case if  $\delta$  is chosen from an open dense subset  $U^+$  of  $\Lambda^+(i\mathbf{R})$ . The set of ‘bad’ points is a  $b^+$  codimensional subspace of  $\Lambda^+(i\mathbf{R})$ , so if  $b^+ > 1$  generic paths lie in  $U^+$ .

△

#### 4.1. The Elliptic Complex.

We now turn our attention to a local description of the moduli space—this amounts to a study of the elliptic complex associated to a solution. Once again the techniques are the same as for ASD theory, first studied in [AHS] (see also [L, DK, FU]), although in our case the application of these techniques is simpler.

First we must calculate the linearization of the gauge action and the equations (3.1.2). Recall that the gauge action for an element  $g = e^\gamma \in \mathcal{G}(L)$  is,

$$g(A, \Psi) = (A + 2d\gamma, e^{-\gamma}).$$

The linearization at  $(A, \Psi)$  of this action is given by,

$$Dg_{(A, \Psi)} : \Lambda^0(i\mathbf{R}) \longrightarrow \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+)$$

$$Dg_{(A, \Psi)}(f) = (2df, -f\Psi) \tag{4.1.1}$$

Equation (3.1.2) can be thought of as the zero set of the map,

$$F : \mathcal{A}(L) \oplus \Gamma(W_L^+) \longrightarrow \Lambda^+(i\mathbf{R}) \oplus \Gamma(W_L^-)$$

$$F(A, \Psi) = (F_A^+(L) - q(\Psi) - \delta, (\not{D}_A + v)\Psi) \quad (4.1.2)$$

The linearization of this map at a solution  $(A, \Psi)$  is,

$$DF_{(A, \Psi)} : \begin{array}{ccc} \Lambda^1(i\mathbf{R}) & & \Lambda^+(i\mathbf{R}) \\ \oplus & \longrightarrow & \oplus \\ \Gamma(W_L^+) & & \Gamma(W_L^-) \end{array}$$

$$DF_{(A, \Psi)}(a, \phi) = \begin{pmatrix} d^+ a & -q(\Psi, \phi) - q(\phi, \Psi) \\ \frac{1}{2} a \cdot \Psi & (\not{D}_A + v)\phi \end{pmatrix} \quad (4.1.3)$$

Where  $q(\Psi, \phi) = \Psi \otimes \phi^* - 1/2 \langle \Psi, \phi \rangle \text{Id}$ . Using the same local representation as in (1.1.7) one can confirm that  $q(\Psi, \phi) + q(\phi, \Psi)$  is a traceless self-adjoint endomorphism so can be identified with a pure imaginary self-dual two-form (see §1.1).

We incorporate the linearizations of the gauge action and equations into a single system of equations, which we will prove in Lemma 4.1.5 is an elliptic complex whenever  $(A, \Psi)$  is a solution to equation 3.1.2. It is important that  $(A, \Psi)$  is a solution for otherwise the system is not a complex.

$$0 \longrightarrow \Lambda^0(i\mathbf{R}) \xrightarrow{Dg_{(A, \Psi)}} \begin{array}{c} \Lambda^1(i\mathbf{R}) \\ \oplus \\ \Gamma(W_L^+) \end{array} \xrightarrow{DF_{(A, \Psi)}} \begin{array}{c} \Lambda^+(i\mathbf{R}) \\ \oplus \\ \Gamma(W_L^-) \end{array} \longrightarrow 0 \quad (4.1.4)$$

**Lemma 4.1.5.** *The system of equations (4.1.4) forms an elliptic complex at any solution to equation (3.1.2). The index of the complex is*

$$-\frac{1}{4}(c(L))^2 - (2\chi + 3\sigma)$$

where  $\chi$  and  $\sigma$  denote the Euler number and signature of  $X$  respectively.

**Proof:** The leading terms of both  $Dg$  and  $DF$  are both elliptic operators, so it remains to show that (4.1.4) forms a complex. Suppose  $f \in \Lambda^0(i\mathbf{R})$  then,

$$\begin{aligned} DF_{(A, \Psi)} \circ Dg(f) &= (2d^+ df + q(\Psi, f\Psi) + q(f\Psi, \Psi), df \cdot \Psi - (\not{D}_A + v)f\Psi) \\ &= (\Psi \otimes \bar{f}\Psi^* + f\Psi \otimes \Psi^* - \frac{1}{2}(\langle f\Psi, \Psi \rangle + \langle \Psi, f\Psi \rangle)\text{Id}, df \cdot \Psi - df \cdot \Psi - (\not{D}_A + v)\Psi) \\ &= 0 \end{aligned}$$

The final step follows from the facts that  $f$  is pure imaginary and  $\Psi$  is a solution to the Dirac equation (3.1.2a).

To calculate the index of the complex note that the highest order terms give the following two complexes,

$$\begin{aligned} 0 \longrightarrow \Lambda^0(i\mathbf{R}) \xrightarrow{2d} \Lambda^1(i\mathbf{R}) \xrightarrow{d^+} \Lambda^+(i\mathbf{R}) \longrightarrow 0 \\ 0 \longrightarrow \Gamma(W_L^+) \xrightarrow{\not{D}_A} \Gamma(W_L^-) \longrightarrow 0 \end{aligned}$$

The index of the first is  $1 - b_1 + b^+ = 1/2(\chi + \sigma)$ , and the real index of the second can be calculated using the Atiyah-Singer index theorem and equals  $1/4(\sigma - c(L)^2)$ . Their sum  $1/4(2\chi + 3\sigma - c(L)^2)$  is the index of (4.1.4).

△

The cohomology groups of the complex are:

$$\mathbf{H}_{(A,\Psi)}^0 = \{f \in \Lambda^0(i\mathbf{R}) : Dg_{(A,\Psi)}(f) = 0\} \quad (4.1.6)$$

$$\mathbf{H}_{(A,\Psi)}^1 = \{(a, \psi) \in \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+) : Dg_{(A,\Psi)}^*(a, \psi) = 0, DF_{(A,\Psi)}(a, \psi) = 0\} \quad (4.1.7)$$

$$\mathbf{H}_{(A,\Psi)}^2 = \{(\eta, \xi) \in \Lambda^+(i\mathbf{R}) \oplus \Gamma(W_L^-) : DF_{(A,\Psi)}^*(\eta, \xi) = 0\} \quad (4.1.8)$$

We will denote the dimensions of these spaces by  $h^i$ ,  $i = 0, 1, 2$ , and from Lemma (4.1.5),  $h^0 - h^1 + h^2 = -1/4(c(L)^2 - (2\chi + 3\sigma))$ .

It is easy to see that  $H_{(A,\Psi)}^0 \neq 0$  if and only if  $(A, \Psi)$  is a reducible solution. If  $(A, \Psi)$  is reducible then  $f = ic$  where  $c \in \mathbf{R}$  is a non-trivial solution to  $Dg_{(A,\Psi)}(f) = 0$ . Conversely solutions to  $Dg_{(A,\Psi)}(f) = 0$  are necessarily of the form  $f = ic : c \in \mathbf{R}$ , since  $df = 0$ , but  $f\Psi = 0$  as well, so either  $f = 0$  or  $\Psi = 0$ .

Since we are assuming that our self-dual perturbation  $\delta$  is chosen from the subset  $U^+$  of Lemma (4.0.1) we have no irreducible solutions. Thus  $H_{(A,\Psi)}^0 = 0$  for every solution  $(A, \Psi)$ . For such a  $\delta$  we have  $h^2 - h^1 = 1/4(c(L)^2 - (2\chi + 3\sigma)) \equiv d(L)$ , where the last equality is a definition.

## 4.2 A Slice Theorem.

Our first goal in this chapter is to prove that the moduli space  $M_{X,g}(L, v, \delta)$  is a manifold for generic  $\delta$ . The first step in this process is to show that the quotient  $\mathcal{B}^*(L) = \mathcal{C}^*(L)/\mathcal{G}(L)$  is a Hausdorff manifold, where  $\mathcal{C}^*(L) \subset \mathcal{A}(L)_{L_k^2} \times \Gamma(W_L^+)_{L_k^2}$  is the subset consisting of irreducible

points. We will do this by constructing “slices” for the gauge action (see [FU]). Note that the action of  $\mathcal{G}(L)$  on  $\mathcal{C}^*(L)$  is free.

Let  $(A, \Psi) \in \mathcal{C}^*(L)$ , we will say that an element  $(B, \phi) \in \mathcal{C}^*(L)$  is in SW-gauge relative to  $(A, \Psi)$  if,

$$Dg_{(A, \Psi)}^*(B - A, \phi - \Psi) = 0,$$

where the adjoint  $Dg_{(A, \Psi)}^*$  of  $Dg_{(A, \Psi)}$  is taken with respect to  $L^2$  inner products on  $\Lambda^1(i\mathbb{R}) \oplus \Gamma(W_L^+)$ . A quick check establishes that,

$$Dg_{(A, \Psi)}^*(a, \psi) = 2d^*a + i\text{Im} \langle \Psi, \psi \rangle \quad (4.2.1)$$

and

$$Dg_{(A, \Psi)}^* Dg_{(A, \Psi)} f = 4d^*df - f|\Psi|^2. \quad (4.2.2)$$

In [FU] it is shown that the following theorem is equivalent to proving that  $\mathcal{B}^*(L)$  is a manifold and that its tangent space at  $(A, \Psi)$  can be represented by,

$$T_{(A, \Psi)}\mathcal{B}^*(L) = \{(a, \phi) \in \Lambda^1(i\mathbb{R}) \times \Gamma(W_L^+) \mid Dg_{(A, \Psi)}^*(a, \phi) = 0\}.$$

**Theorem 4.2.3.** *There are constants  $c, \epsilon > 0$  such that if  $(B, \phi) \in \mathcal{C}^*(L)$  and  $(a, \xi) = (A - B, \Psi - \phi)$  satisfies,*

$$\|a\|_{L_k^2} + \|\xi\|_{L_k^2} < c,$$

*then there is a unique  $\gamma \in \Lambda^0(i\mathbb{R})$  with  $\|\gamma\|_{L_{k+1}^2} < \epsilon$ , such that  $g(A, \phi)$ , where  $g = e^\gamma$ , is in SW-gauge relative to  $(A, \Psi) \in \mathcal{C}^*(L)$ .*

**Proof:** With  $g = e^\gamma$  we have,

$$\begin{aligned} g(B, \phi) &= g[(A, \Psi) + (a, \xi)] \\ &= (A, \Psi) - (a + 2d\gamma, e^{-\gamma}\Psi + e^{-\gamma}\xi - \Psi). \end{aligned}$$

Thus proving  $g(B, \phi)$  is in SW-gauge amounts to solving the equation,

$$G(\gamma, a, \xi) = Dg_{(A, \Psi)}^*(a + 2d\gamma, e^{-\gamma}\Psi + e^{-\gamma}\xi - \Psi) = 0.$$

Using (4.2.1) and (4.2.2) this translates to,

$$G(\gamma, a, \xi) = 4d^*d\gamma + 2d^*a + i\text{Im} \langle \Psi, e^{-\gamma}\Psi + e^{-\gamma}\xi \rangle.$$

If the differential of  $G$  restricted to the first variable is onto the image space,  $\text{Im } Dg_{(A, \Psi)}^*$ , then the Implicit Function Theorem will give us a unique small solution  $\gamma$  for sufficiently small  $(a, \xi)$ . Evaluated at  $0 = (0, 0, 0)$  the derivative of  $G$  is,

$$\begin{aligned} DG_0(\chi, b, \eta) &= 4d^*d\chi + 2d^*b + i\text{Im} \langle \Psi, \eta \rangle - \chi|\Psi|^2 \\ &= Dg_{(A, \Psi)}^* Dg_{(A, \Psi)}\chi + Dg_{(a, \Psi)}^*(b, \eta). \end{aligned}$$

Standard elliptic theory implies that the partial derivative  $Dg_{(A, \Psi)}^* Dg_{(A, \Psi)}$  is a bijection onto the space  $\text{Im } Dg_{(A, \Psi)}^*$ . The Implicit Function Theorem applies and completes the proof of the Theorem. △

Note that elliptic regularity would imply that this solution is smooth if  $(A, \Psi)$  and  $(B, \phi)$  are smooth. We also need to show that the manifold  $\mathcal{B}^*(L)$  is Hausdorff.

**Theorem 4.2.4.** *Given two convergent sequences  $(A_n, \Psi_n) \rightarrow (A, \Psi)$  and  $(B_n, \phi_n) \rightarrow (B, \phi)$ , such that  $(B_n, \phi_n) = g_n(A_n, \Psi_n)$ , there is a subsequence of  $g_n$  that converges to  $g$  and  $(A, \Psi) = g(B, \phi)$ .*

In [FU] it is shown that this is equivalent to  $\mathcal{B}^*(L)$  being Hausdorff.

**Proof:** The hypothesis implies that  $\{A_n\}$  and  $\{g_n(A_n)\} = \{B_n\}$  are bounded. Writing  $g_n = e^{\gamma_n}$  we have,

$$2d\gamma_n = B_n - A_n$$

and so  $d\gamma_n$  is bounded in  $L_k^2$ ,  $(A_n, B_n \in L_k^2)$  and  $\gamma_n$  is bounded in  $L_k^2$ . Extracting a convergent subsequence proves the theorem. △

### 4.3 The Parametrized Moduli Space

Recall that we are assuming that  $\mathcal{B}(L) = (\mathcal{A}(L) \times \Gamma(W_L^+))/\mathcal{G}(L)$  is of class  $L_k^2$ , and our perturbation  $\delta \in \Lambda^+(i\mathbb{R})$  is class  $L_{k-1}^2$  for some large  $k$ . This enables us to use Banach space results such as the Implicit Function Theorem and the Sard-Smale theorems. It is important to note that in this section the one-form ‘ $v$ ’ is regarded as *fixed*.

We define the parametrized moduli space,

$$\begin{aligned}\mathcal{M}_{X,g}(L, v) &= \bigcup_{\delta \in \Lambda^+(i\mathbf{R})} M_{X,g}(L, v, \delta) \times \{\delta\} \\ &\subset \mathcal{B}(L) \times \Lambda^+(i\mathbf{R}).\end{aligned}\tag{4.3.1}$$

We will denote the subset of  $\mathcal{M}_{X,g}(L, v)$  consisting of irreducible solutions by  $\mathcal{M}_{X,g}^*(L, v)$ .

**Lemma 4.3.2.**  $\mathcal{M}_{X,g}^*(L, v)$  is a smooth manifold.

**Proof:** Consider the map,

$$\begin{aligned}\tilde{F} : \mathcal{A}(L) \times \Gamma(W_L^+) \times \Lambda^+(i\mathbf{R}) &\longrightarrow \Lambda^+(i\mathbf{R}) \times \Gamma(W_L^-) \\ \tilde{F}(A, \Psi, \delta) &= (F_A^+(L) - q(\Psi) - \delta, (\mathcal{D}_A + v)\Psi),\end{aligned}$$

this is essentially the same as (4.1.2) except that the perturbation,  $\delta$ , is now regarded as a variable. The first step in the proof of the Lemma is to show that the linearization of  $\tilde{F}$  is surjective at an irreducible solution, then the Implicit Function Theorem will imply that the zero set,  $\tilde{Z} - \{\text{reducible solutions}\}$ , of  $\tilde{F}$  is a manifold. Let  $(A, \Psi)$  be an irreducible solution.

The linearization of  $\tilde{F}$  at  $(A, \Psi, \delta)$  is,

$$\begin{aligned}D\tilde{F}_{(A, \Psi, \delta)} : \Lambda^1(i\mathbf{R}) \times \Gamma(W_L^+) \times \Lambda^+(i\mathbf{R}) &\longrightarrow \Lambda^+(i\mathbf{R}) \times \Gamma(W_L^-) \\ D\tilde{F}_{(A, \Psi, \delta)}(a, \psi, \epsilon) &= (d^+a - q(\Psi, \psi) - q(\psi, \Psi) - \epsilon, (\mathcal{D}_A + v)\psi + \frac{1}{2}a \cdot \Psi)\end{aligned}$$

We decompose  $D\tilde{F}_{(A, \Psi, \delta)} = DF_{(A, \Psi)}^1 + DF_{(\delta)}^2$  where,

$$DF_{(A, \Psi)}^1(a, \psi) = (d^+a - q(\Psi, \psi) - q(\psi, \Psi), (\mathcal{D}_A + v)\psi + \frac{1}{2}a \cdot \Psi)$$

and

$$DF_{(\delta)}^2(\epsilon) = -\epsilon.$$

Clearly  $DF_{(\delta)}^2$  is onto  $\Lambda^+(i\mathbf{R})$ , so to prove surjectivity of  $D\tilde{F}_{(A, \Psi, \delta)}$  all that remains to show is that  $DF_{(A, \Psi)}^1$  is onto  $\Gamma(W_L^-)$ .

Suppose that  $\xi$  is  $L^2$ -orthogonal to the image of  $DF_{(A, \Psi)}^1$  that is,

$$\langle \xi, (\mathcal{D}_A + v)\psi + \frac{1}{2}a \cdot \Psi \rangle = 0 \quad \text{for all } a, \psi\tag{4.3.3}$$

Setting  $a$  to zero implies,

$$\langle \xi, (\mathcal{D}_A + v)\psi \rangle = 0 \quad \text{for all } \psi.\tag{4.3.4}$$

Setting  $\psi$  to zero in (4.3.3) gives,

$$\langle \xi, \frac{1}{2}a \cdot \Psi \rangle = 0 \quad \text{for all } a\tag{4.3.5}$$

Equation (4.3.4) implies that  $\xi$  is in the kernel of the adjoint of  $\mathcal{D}_A + v$  and by elliptic regularity cannot vanish on an open set unless it is identically zero. By assumption  $\Psi$  is a non-zero solution to the Dirac equation, so cannot vanish on open sets. Equation (4.3.5) cannot be satisfied for all  $a$ , if both  $\xi$  and  $\Psi$  are non-zero. We conclude that  $\xi = 0$  and so  $D\tilde{F}_{(A,\Psi,\delta)}$  is surjective.

To prove  $\mathcal{M}_{X,g}^*(L, v)$  is a manifold we restrict the domain of  $\tilde{F}$  to values in SW-gauge with respect to  $(A, \Psi)$ . That is,

$$\tilde{F}_0 : \{(A + a, \Psi + \psi) \in \mathcal{A}(L) \times \Gamma(W_L^+) : Dg_{(A,\Psi)}^*(a, \psi) = 0\} \times \Lambda^+(i\mathbf{R}) \longrightarrow \Lambda^+(i\mathbf{R}) \times \Gamma(W_L^-)$$

Define  $DF_0^1$  and  $DF_0^2$  by  $DF_0^1 + DF_0^2 = DF_0$  analogously with  $DF^1$  and  $DF^2$  of  $D\tilde{F}$ . Consider the elliptic complex (4.1.4),

$$0 \longrightarrow \Lambda^0(i\mathbf{R}) \xrightarrow{Dg_{(A,\Psi)}} \begin{array}{c} \Lambda^1(i\mathbf{R}) \\ \oplus \\ \Gamma(W_L^+) \end{array} \xrightarrow{DF_{(A,\Psi)}^1} \begin{array}{c} \Lambda^+(i\mathbf{R}) \\ \oplus \\ \Gamma(W_L^-) \end{array} \longrightarrow 0$$

Since this forms a complex  $DF_{(A,\Psi)}^1|_{\text{Im } Dg} = 0$ . Since the Hodge theory of the complex implies  $\ker Dg_{(A,\Psi)}^*$  is orthogonal to  $\text{Im } Dg_{(A,\Psi)}^*$  and  $DF_{0,(A,\Psi)}^1 = DF_{(A,\Psi)}^1|_{\ker Dg^*}$  we conclude that  $DF_{0,(A,\Psi)}^1 = DF_{(A,\Psi)}^1$ . Clearly we also have  $DF_{0,(A,\Psi)}^2 = DF_{(A,\Psi)}^2$ . Thus  $DF'$  is surjective and as before we conclude using the Implicit Function Theorem that  $\mathcal{M}_{X,g}^*(L, v)$  is a manifold. △

We now want to show that the moduli spaces  $M_{X,g}(L, v, \delta)$  are smooth manifolds. Recall the Sard-Smale theorem [S, DK]

**Sard-Smale Theorem 4.3.6.** *If  $\pi : P \rightarrow Q$  is a proper Fredholm map between paracompact Banach manifolds, the regular values of  $\pi$  form an open dense set in  $Q$ . For every such regular value  $q \in Q$  the set  $\pi^{-1}(q) \subset P$  forms a smooth submanifold with dimension equal to the index of  $\pi$ .*

**Lemma 4.3.6.** *The projection  $\Pi_1^v : \mathcal{M}_{X,g}^*(L, v) \rightarrow \Lambda^+(i\mathbf{R})$  is a proper Fredholm surjection with index  $d(L) = 1/4(c(L)^2 - (2\chi + 3\sigma))$ .*

**Proof:** Recall that a map is Fredholm if its differential has a closed range and finite dimensional kernel and cokernel [BB, FU].

If we use the SW-gauge to take slices for the gauge action, the tangent space to  $\mathcal{M}_{X,g}^*(L, v)$  is given by,

$$T_{(A, \Psi, \delta)} \mathcal{M}_{X, g}^*(L, v) = \{(a, \psi, \epsilon) \in \Lambda^1(i\mathbf{R}) \times \Gamma(W_L^+) \times \Lambda^+(i\mathbf{R}) : Dg_{(A, \Psi)}^*(a, \psi) = 0, \\ (d^+ a - q(\Psi, \psi) - q(\psi, \Psi) - \epsilon, (\not{D}_A + v)\psi + \frac{1}{2}a \cdot \Psi) = 0\}.$$

Let  $D\Pi_{1, (A, \Psi, \delta)}^v$  denote the differential of  $\Pi_1^v$  evaluated at the point  $(A, \Psi, \delta)$ .  $\Pi$  is just projection so clearly,  $D\Pi_{1, (A, \Psi, \delta)}^v(a, \psi, \epsilon) = \epsilon = -DF_{(\delta)}^2(\epsilon)$ , where  $DF_{(\delta)}^2$  is defined in Lemma 4.3.2. Thus

$$\ker D\Pi_{1, (A, \Psi, \delta)}^v = \{(a, \psi, \epsilon) \in \Lambda^1(i\mathbf{R}) \times \Gamma(W_L^+) \times \Lambda^+(i\mathbf{R}) : \epsilon = 0, \\ Dg_{(A, \Psi)}^*(a, \psi) = 0, DF_{(A, \Psi)}^1(a, \psi) = 0\}. \quad (4.3.7)$$

Inspection of (4.1.7) yields  $\dim \ker D\Pi_{1, (A, \Psi, \delta)}^v = h^1$ .

Also it is clear that,

$$\text{Im } D\Pi_{1, (A, \Psi, \delta)}^v = \{\epsilon \in \Lambda^+(i\mathbf{R}) : \epsilon = d^+ a - q(\Psi, \psi) - q(\psi, \Psi), Dg_{(A, \Psi)}^*(a, \psi) = 0\},$$

for  $(a, \psi) \in \Lambda^1(i\mathbf{R}) \times \Gamma(W_L^+)$ . This can be written more succinctly as

$$\text{Im } D\Pi_{1, (A, \Psi, \delta)}^v = P_+ \circ \text{Im } DF_{(A, \Psi)}^1 \Big|_{Dg_{(A, \Psi)}^*(a, \psi)=0} \quad (4.3.8)$$

where  $P_+ \circ \text{Im } DF^1$  denotes the projection of  $DF^1$  onto the  $\Lambda^+(i\mathbf{R})$  part of the range. Using the Hodge decompositions of the elliptic complex (4.1.4) we can in fact ignore the restriction in (4.3.8) that is,

$$\text{Im } D\Pi_{1, (A, \Psi, \delta)}^v = P_+ \circ \text{Im } DF_{(A, \Psi)}^1.$$

Also from the Hodge decomposition of (4.1.4) we have,

$$\Gamma(W_L^+) \oplus \Lambda^+(i\mathbf{R}) = \text{Im } DF^1 \oplus \ker(DF^1)^*,$$

and so  $\text{coker } DF_{(A, \Psi)}^1 \simeq \ker(DF^1)^*$  has dimension  $h^2$ . We know that  $DF^1$  is onto the  $\Gamma(W_L^+)$  variable so we may as well write,

$$\text{coker } DF_{(A, \Psi)}^1 = \frac{\Lambda^+(i\mathbf{R})}{P_+ \circ \text{Im } DF^1},$$

which is exactly  $\text{coker } D\Pi_{1, (A, \Psi)}^v$ .

We have established that  $\text{index } \Pi_1^v = h^1 - h^2$ . Since  $(A, \Psi)$  is an irreducible solution,  $h^0 = 0$ , and so  $\text{index } \Pi_1^v = d(L)$  as claimed.

△

Using the fact that finite intersections of open dense sets are open and dense, we can combine the Smale-Sard theorem applied to the projection  $\Pi_1^v$  and Lemma (4.0.1) to immediately conclude,

**Corollary 4.3.9.** *Let  $X$  be a smooth compact four dimensional manifold with  $b^+ > 0$ . There is an open dense subset  $U_v^+ \subset \Lambda^+(i\mathbf{R})$  consisting of regular values of the projection  $\Pi_1^v$  such that if  $\delta \in U_v^+$  then  $M_{X,g}(L, v, \delta)$  consists only of irreducible points. Thus  $M_{X,g}(L, v, \delta)$  is a smooth, compact  $d(L)$  dimensional manifold. If  $b^+ > 1$  then generic paths in  $\Lambda^+(i\mathbf{R})$  lie in  $U_v^+$ .*

△

#### 4.4 The Totally Parametrized Moduli Space.

Our next step is to show that the moduli spaces defined using different “ $v$ ’s” differ by a compact cobordism. This is the fundamental result that enables us to calculate the Seiberg-Witten invariant using equation (3.1.2) for any choice of “ $v$ ”. Many of the arguments in this section are the same as in §4.3 so some proofs will be kept brief.

First we define the “totally parametrized moduli space”, where the term “totally” is used to distinguish this space from (4.3.1).

$$\begin{aligned} \mathcal{M}_{X,g}(L) &= \bigcup_{\substack{\delta \in \Lambda^+(i\mathbf{R}) \\ v \in \Lambda^1(\mathbf{R})}} M_{X,g}(L, v, \delta) \times \{v\} \times \{\delta\} \\ &\subset \mathcal{B}(L) \times \Lambda^1(\mathbf{R})_{L_k^2} \times \Lambda^+(i\mathbf{R})_{L_{k-1}^2}. \end{aligned} \quad (4.4.1)$$

Denote the subset of the totally parametrized moduli space consisting of irreducible solutions by  $\mathcal{M}_{X,g}^*(L)$ . We prove that  $\mathcal{M}_{X,g}^*(L)$  is a smooth manifold in exactly the same way as (4.3.2).

**Lemma 4.4.2.**  *$\mathcal{M}_{X,g}^*(L)$  is a smooth manifold.*

**Proof:** As in Lemma (4.3.2) we linearize the map,

$$\begin{aligned} \hat{F} : \mathcal{A}(L) \oplus \Gamma(W_L^+) \oplus \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) &\longrightarrow \Lambda^+(i\mathbf{R}) \oplus \Gamma(W_L^-) \\ \hat{F}(A, \Psi, v_0, \delta) &= (F_A^+(L) - q(\Psi) - \delta, (\mathcal{D}_A + v_0)\Psi) \end{aligned}$$

at an irreducible point  $(A, \Psi, v_0, \delta)$ . Note that the only difference between this map and  $\tilde{F}$  of Lemma (4.3.2) is that now we are regarding  $v_0$  as a variable.

$$\begin{aligned} D\hat{F}_{(A, \Psi, v_0, \delta)} : \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+) \oplus \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) &\longrightarrow \Lambda^+(i\mathbf{R}) \oplus \Gamma(W_L^-) \\ D\hat{F}_{(A, \Psi, v_0, \delta)}(a, \psi, v, \epsilon) &= (d^+a - q(\Psi, \psi) - q(\psi, \Psi) - \epsilon, (\mathcal{D}_A + v_0)\psi + (v + \frac{1}{2}a) \cdot \Psi). \end{aligned}$$

If we suppress the  $\Lambda^1(\mathbf{R})$  variable  $v$  we get exactly the same map,  $D\tilde{F}$  as in Lemma (4.3.2) which we know to be surjective. Thus  $D\hat{F}$  is surjective and the Implicit Function Theorem

implies that the zero set of  $\hat{F}$  is a manifold. We can construct slices of the gauge action using the SW-gauge exactly as in Lemma (4.3.2) and thus  $\mathcal{M}_{X,g}^*(L)$  is a manifold.

△

Lemma (4.3.6) can be generalised to the totally parametrized moduli space.

**Lemma 4.4.3.** *The projection  $\Pi_2 : \mathcal{M}_{X,g}^*(L) \rightarrow \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R})$  is a proper Fredholm surjection with index  $d(L)$ .*

**Proof:** The proof of this theorem varies only slightly to that of (4.3.6), so the arguments will not be repeated. However an explicit description of  $\ker D\Pi_{2,(A,\Psi,v_0,\delta)}$  will be of subsequent use so we will give it here. As in (4.3.6) we can represent the tangent space to  $\mathcal{M}_{X,g}^*(L)$  as,

$$T_{(A,\Psi,v_0,\delta)}\mathcal{M}_{X,g}^*(L) = \{(a, \psi, v, \epsilon) \in \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+) \oplus \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) : \\ Dg_{(A,\Psi)}^*(a, \psi) = 0, d^+a - q(\Psi, \psi) - q(\psi, \Psi) - \epsilon = 0, \not{D}_A + v_0)\psi + (\frac{1}{2}a + v) \cdot \Psi = 0\}$$

Calculating the linearization of the projection gives  $D\Pi_{2,(A,\Psi,v_0,\delta)}(a, \psi, v, \epsilon) = (v, \epsilon)$ . Thus we can write the kernel of this map at an irreducible point as,

$$\ker D\Pi_{2,(A,\Psi,v_0,\delta)} = \{(a, \psi, v, \epsilon) \in \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+) \oplus \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) : \epsilon = v = 0, \\ Dg_{(A,\Psi)}^*(a, \psi) = 0, 0 = d^+a - q(\Psi, \psi) - q(\psi, \Psi), (\not{D}_A + v_0)\psi + \frac{1}{2}a \cdot \Psi\} \quad (4.4.4)$$

Clearly this has dimension  $h^1$ , the proof that the cokernel has dimension  $h^2$  proceeds in the same vein as (4.3.6).

△

Once again the Sard-Smale Theorem gives us,

**Corollary 4.4.5.** *Let  $X$  be a compact, smooth four-manifold with  $b^+ > 0$ . Then there is an open dense subset  $V \subset \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R})$  consisting of regular values of  $\Pi_2$ , such that if  $(v, \delta) \in V$  then the moduli space  $M_{X,g}(L, v, \delta)$  consists only of irreducible points. If  $b^+ > 1$  then  $V$  is connected.*

We are now in a position to prove the the fundamental theorem of the chapter, that is the “independence” of the moduli space from the perturbation  $v$  in the Dirac equation (3.1.2). This follows from the Fredholm transversality theorem [DK Prop. 4.3.10].

**Fredholm Transversality Theorem 4.4.6.** *Suppose  $\pi : P \rightarrow Q$  is Fredholm map between paracompact Banach manifolds and  $\gamma : [0, 1] \rightarrow Q$  is a smooth path in  $Q$ , which is transverse*

to  $\pi$  at its end-points. Then there is a path  $\gamma' : [0, 1] \rightarrow Q$  arbitrarily close to  $\gamma$ , such that  $\gamma'$  is transverse to  $\pi$  and  $\gamma'(0) = \gamma(0)$  and  $\gamma'(1) = \gamma(1)$ .

**Theorem 4.4.7.** *Suppose  $X$  is a compact four dimensional manifold with  $b^+ > 0$ . Then there is a compact cobordism between  $M_{X,g}(L, v_0, \delta)$  and  $M_{X,g}(L, v_1, \delta)$  for arbitrary  $v_0, v_1 \in \Lambda^1(\mathbf{R})$  and generic  $\delta \in \Lambda^+(i\mathbf{R})$ .*

**Proof:** The proof consists in setting up the problem so that we can apply the Fredholm Transversality Theorem. A comparison of (4.3.7) and (4.4.4) yields

$$\dim \ker(D\Pi_{1,(A,\Psi,\delta)}^v) = \dim \ker(D\Pi_{2,(A,\Psi,v,\delta)}), \quad (4.4.8)$$

so if  $\delta$  is a regular value of  $\Pi_1^v$  then  $(v, \delta)$  is a regular value of  $\Pi_2$ .

Consider the intersection  $U_{v_0}^+ \cap U_{v_1}^+$ , where  $U_{v_i}^+, i = 1, 2$  is as in in Corollary (4.3.9). Since both  $U_{v_0}^+$  and  $U_{v_1}^+$  are open and dense in  $\Lambda^+(i\mathbf{R})$  their intersection is also open and dense. Equation (4.4.8) implies that if  $\delta \in U_{v_0}^+ \cap U_{v_1}^+$  then both  $(v_0, \delta)$  and  $(v_1, \delta)$  are regular values of  $\Pi_2$ . For such a  $\delta$  define the path,

$$\begin{aligned} \gamma : [0, 1] &\longrightarrow \Lambda^1(\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) \\ \gamma(t) &= [(1-t)v_0 + tv_1, \delta] \equiv [v(t), \delta(t)], \end{aligned}$$

between  $(v_0, \delta)$  and  $(v_1, \delta)$ . By the Fredholm Transversality Theorem there is a path  $\gamma'(t) = [v'(t), \delta'(t)]$  arbitrarily close to  $\gamma$ , and with the same endpoints. We can in fact assume that  $v'(t) = v(t)$ , but this is irrelevant for our purposes. The set  $\Pi_2^{-1}(\gamma'(t))$  gives us the required cobordism, if we note that for  $\delta'(t)$  sufficiently close to  $\delta$  then  $\Pi_2^{-1}(\gamma'(t))$  will consist only of irreducible points.

△

Contained in Theorem 4.4.7 is the existence of a cobordism between the moduli space of solutions to equation (3.1.2) and the usual Seiberg-Witten equations (1.1.8), that is equation (3.1.2) with  $v$  set to zero. Apart from its usefulness as a tool for calculating the Seiberg-Witten invariant, as we will see in Chapter 5, this result is also aesthetically pleasing—it proves the independence of the Seiberg-Witten invariant from the particular “Dirac operator” used to define it.

## 4.5 The Seiberg-Witten Invariant.

Before an invariant can be defined, we must prove that the moduli space  $M_{X,g}(L, v, \delta)$  is orientable and any two moduli spaces  $M_{X,g_i}(L, v, \delta_i)$ ,  $i = 1, 2$ , defined using generic metric-perturbation pairs  $(g_i, \delta_i)$ ,  $i = 1, 2$ , are cobordant. Note that independence from the “ $v$ ” parameter has been proved in Theorem (4.4.7) so this parameter will be largely ignored in the following discussion. In fact because of this independence from “ $v$ ” we could deduce the orientability and independence from  $(g_i, \delta_i)$  from the fact that these properties are known for the usual Seiberg-Witten moduli space,  $M_{X,g}(L, 0, \delta)$  (see [W, M, FS2]). Nevertheless, for the sake of completeness, proofs of these facts will be presented here, though they bear no difference to the proofs in the papers referenced above which deal with the standard Seiberg-Witten equations.

### Orientability of the Moduli Space.

In §4.2 we saw that the tangent space to the moduli space at the point  $[(A, \Psi)]$  is given by the kernel of the elliptic operator,

$$P_{(A,\Psi)} = Dg_{(A,\Psi)}^* + DF_{(A,\Psi)} : \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+) \longrightarrow \Lambda^0(i\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) \oplus \Gamma(W_L^-),$$

where,

$$Dg_{(A,\Psi)}^*(a, \psi) = 2d^*a + i\text{Im} \langle \Psi, \psi \rangle$$

and,

$$DF_{(A,\Psi)}(a, \psi) = (d^+a - q(\Psi, \psi) - q(\psi, \Psi), (\not{D}_A + v)\psi + \frac{1}{2}a \cdot \Psi).$$

A trivialization of the determinant of the tangent bundle,  $\det(\ker P)$ , corresponds to an orientation of the moduli space  $M_{X,g}(L, v, \delta)$ . If  $[(A, \Psi)] \in M_{X,g}(L, v, \delta)$  then  $\text{coker } P = 0$  since we are assuming that  $\delta$  is chosen so that the points in  $M_{X,g}(L, v, \delta)$  are smooth and irreducible. Because of this it suffices to trivialize the determinant line,  $\det(\ker P) \otimes (\det \text{coker } P)^{-1}$ , of  $P$  defined over the space  $\mathcal{B}^*(L)$ —the space of all irreducible pairs  $(A, \Psi)$  mod gauge.

Any operator homotopic to  $P$  defines an isomorphic determinant line. So to prove  $P$  has a trivial line it suffices to trivialize the determinant of the operator,

$$Q_{(A,\Psi)} = 2d^* + d^+ + \not{D}_A : \Lambda^1(i\mathbf{R}) \oplus \Gamma(W_L^+) \longrightarrow \Lambda^0(i\mathbf{R}) \oplus \Lambda^+(i\mathbf{R}) \oplus \Gamma(W_L^-).$$

The determinant of  $Q_{(A,\Psi)}$  can be decomposed,

$$\det Q_{(A,\Psi)} = \det(2d^* + d^+) \otimes \det \not{D}_A.$$

The Dirac operator is complex linear so  $\det \mathcal{D}_A$  has a natural orientation coming from the complex structure. The operator  $(2d^* + d^+)$  is independent of  $(A, \Psi)$  so is naturally trivial. Furthermore,

$$\begin{aligned} \det(2d^* + d^+) &= \det(\ker(2d^* + d^+)) \otimes (\det \operatorname{coker}(2d^* + d^+))^{-1} \\ &= \det(H^1(X, i\mathbf{R})) \otimes \det(H^{2,+}(X, i\mathbf{R}) \oplus H^0(X, i\mathbf{R}))^{-1}, \end{aligned}$$

so choosing an orientation for  $H^{2,+}(X, i\mathbf{R})$ ,  $H^1(X, i\mathbf{R})$  and  $H^0(X, i\mathbf{R})$  determines an orientation of the moduli space. Note that this is the same information required to orient the ASD instanton moduli space (Theorem 7.1.39 in [DK]).

### Independence of $M_{X,g}(L, v, \delta)$ from $g$ and $\delta$ .

Before we can prove this result we need to introduce a framework within which the problem is manageable. First note that we are assuming the parameter  $v \in \Lambda^1(\mathbf{R})$  is fixed.

The problem at hand is: Given  $(g_0, \delta_0)$  and  $(g_1, \delta_1)$ , where both of the  $\delta_i, i = 0, 1$  satisfy Corollary (4.3.9) for their respective metrics  $g_i, i = 0, 1$ , construct a cobordism between the moduli spaces  $M_{X,g_0}(L, v, \delta_0)$  and  $M_{X,g_1}(L, v, \delta_1)$ . We need to introduce the restriction  $b^+ > 1$  so that a generic path joining  $(g_0, \delta_0)$  to  $(g_1, \delta_1)$  won't contain reducible points (see Corollary 4.3.9.). Following [DK §4.3.3. and §1.1.5.] we introduce the following framework that will enable us to identify the spaces of self-dual two forms  $\Lambda_g^+(i\mathbf{R})$  defined using different metrics  $g$ , and refer the reader to this source for a more complete discussion. First fix a reference metric  $g_0$  which defines the splitting  $\Lambda^2(i\mathbf{R}) = \Lambda_{g_0}^+(i\mathbf{R}) \oplus \Lambda_{g_0}^-(i\mathbf{R})$  and to each metric  $g$  we can associate a map,  $m_g : \Lambda_{g_0}^-(i\mathbf{R}) \rightarrow \Lambda_{g_0}^+(i\mathbf{R})$  (c.f. [DK §1.1.5] enabling us to identify  $\Lambda_g^+(i\mathbf{R})$  with  $\Lambda_{g_0}^+(i\mathbf{R})$ ). Explicitly this is done as follows: write  $\delta \in \Lambda_g^+(i\mathbf{R})$  as  $\delta = \delta_0^+ + \delta_0^- \in \Lambda_{g_0}^+(i\mathbf{R}) \oplus \Lambda_{g_0}^-(i\mathbf{R})$ , with respect to the reference splitting. We identify  $\delta \in \Lambda_g^+(i\mathbf{R})$  with  $\delta' = \delta_0^+ + m_g \delta_0^- \in \Lambda_{g_0}^+(i\mathbf{R})$ .

Exhibiting the cobordism is now a simple application of the Fredholm Transversality Theorem (4.4.6).

**Theorem 4.5.1.** *Suppose  $X$  is a closed compact four manifold with  $b^+ > 1$ . If  $(g_0, \delta_0)$  and  $(g_1, \delta_1)$  are two metric-perturbation pairs satisfying Corollary (4.3.9), then for a dense set of paths  $\gamma = (g(t), \delta(t))$  between  $(g_0, \delta_0)$  and  $(g_1, \delta_1)$  the space*

$$W_\gamma = \{([A], [\Psi], t) \subset \mathcal{B}(L) \times [0, 1] \mid ([A], [\Psi]) \in M_{X,g(t)}(L, v, \delta(t))\}$$

*consists only of irreducible points and gives a smooth cobordism between  $M_{X,g_0}(L, v, \delta_0)$  and  $M_{X,g_1}(L, v, \delta_1)$ .*

**Proof:** Take any path  $\gamma(t) = (g(t), \delta(t))$  such that  $\gamma(0) = (g_0, \delta_0)$  and  $\gamma(1) = (g_1, \delta_1)$ . We will now fix the path  $g(t)$  and prove the theorem for generic perturbations of the path  $\delta(t)$ . By Lemma 4.0.1 along with our assumption that  $b^+ > 1$  we can assume that the path  $\gamma(t) = (g(t), \delta(t))$  has no points  $t_i$  where the moduli space  $M_{X, g(t_i)}(L, v, \delta(t_i))$  has reducible solutions, and this is also true for sufficiently small perturbations of the path  $\delta(t)$ .

Following the discussion before the statement of the theorem we fix a reference space  $\Lambda_{g_0}^+(i\mathbf{R})$ , and to each  $g(t)$  associate a map,  $m_t : \Lambda_{g_0}^-(i\mathbf{R}) \rightarrow \Lambda_{g_0}^+(i\mathbf{R})$ . This enables us to identify the path  $\delta(t) \subset \Lambda_{g(t)}^+(i\mathbf{R})$  with  $\delta'(t) = \delta_0^+(t) + m_t \delta_0^-(t) \subset \Lambda_{g_0}^+(i\mathbf{R})$ .

Similarly we can identify the parametrized moduli spaces,

$$\mathcal{M}_{X, g(t)}(L, v) \subset \mathcal{B}(L) \times \Lambda_{g(t)}^+(i\mathbf{R})$$

with

$$[\tilde{\mathcal{M}}_{X, g(t)}(L, v), t] \subset \mathcal{B}(L) \times \Lambda_{g_0}^+(i\mathbf{R}) \times [0, 1].$$

Exactly as before we have a Fredholm map,  $\pi : [\tilde{\mathcal{M}}_{X, g(t)}^*(L, v), t] \rightarrow \Lambda_{g_0}^+(i\mathbf{R}) \times [0, 1]$ , and a path  $\delta' : [0, 1] \rightarrow \Lambda_{g_0}^+(i\mathbf{R}) \times [0, 1]$ . By assumption the path  $\delta'$  is transverse to  $\pi$  at 0 and 1. The Fredholm Transversality Theorem implies that we can find a new path  $\delta'' : [0, 1] \rightarrow \Lambda_{g_0}^+(i\mathbf{R}) \times [0, 1]$  arbitrarily close to  $\delta'$ , such that  $\delta''$  is transverse to  $\pi$ . For  $\delta''$  sufficiently close to  $\delta'$  there are no reducible points in any of the moduli spaces  $M_{X, g(t)}(L, v, \delta''(t))$ . The space

$$\pi^{-1}(\delta''(t)) \equiv W_{\gamma''} = \{([A], [\Psi], t) \in \mathcal{B}(L) \times [0, 1] \mid ([A], [\Psi]) \in M_{X, g(t)}(L, v, \delta''(t))\},$$

gives the required cobordism between  $M_{X, g_0}(L, v, \delta_0)$  and  $M_{X, g_1}(L, v, \delta_1)$ .

△

**The Seiberg-Witten Invariant.** With these technical details out of the way we can now define a smooth invariant for closed compact four-manifolds with  $b^+ > 1$  using the equations (3.1.2), for an arbitrary choice of ‘ $v$ ’.

First we need to list a few properties of  $spin^c$  structures. If  $H^2(X, \mathbf{Z})$  has no two torsion then the  $spin^c$  structure  $W_L^c$  is completely determined by the class of the determinant line,  $L \in H^2(X, \mathbf{Z})$ . Also there is a canonical way to identify two  $spin^c$  structures  $W_L^c$  and  $W_{L'}^c$ , on  $X$  via an element  $E \in H^2(X, \mathbf{Z})$ . Such an  $E$  relates the two bundles  $W_L^+$  and  $W_{L'}^+$ , via  $W_{L'}^+ = W_L^+ \otimes E$ . Thus the best way to think of the Seiberg-Witten invariant is as an integer,  $SW_X(E)$ , assigned to each class  $E \in H^2(X, \mathbf{Z})$  with respect to a fixed  $spin^c$  structure  $W_L^c$ .

However not all manifolds have an obvious choice for this fixed  $spin^c$  structure, so it is often convenient to think of the invariant as associated with the class of the determinant line bundle  $L$ . Nevertheless it will be clear from the context which picture we are using.

If the virtual dimension of the moduli space  $d(L)$  is strictly less than zero, then the Seiberg-Witten invariant is defined to be zero.

In the case where the virtual dimension of the moduli space  $d(L)$  equals zero the moduli space generically consists of a finite set of points. To each of these points one can associate a sign, coming from the determinant of the operator  $T$  defined above. The sum of these points counted with sign is an integer, the Seiberg-Witten invariant  $SW_X(L)$  associated to the  $spin^c$  structure  $W_L^c$  with determinant  $L$ .

If the  $d(L) > 0$  the invariant can be defined by a suitable integration over the moduli space [FS1, FS2, M]. We will not give details here because in our application the moduli space will always have virtual dimension equal to zero. Indeed there are no known cases where these “higher” dimensional invariants are non-trivial (at least for the case when  $b^+ > 1$  [MS]), and there is a conjecture that this is always the case.

Although implicit in the above discussion we will state the following result which contains the equivalence between the Seiberg-Witten invariant as defined in [W], and the invariant defined using the equation 3.1.2.

**Theorem 4.5.2.** *Suppose  $X$  is a closed compact four manifold with  $b^+ > 1$ . Let  $W_L^c$  be a  $spin^c$  structure over  $X$  with determinant line bundle  $L$ . The integer  $SW_X(L)$ , defined above using equation (3.1.2), is independent of the perturbation  $v$  in the Dirac equation (3.1.2a). Moreover  $SW_X(L)$  is independent from the choice of Riemannian metric  $g$  on  $X$  and generic perturbation  $\delta \in \Lambda_g^+(i\mathbf{R})$  in equation (3.1.2b). Thus  $SW_X(L)$  is a diffeomorphism invariant and equal to the invariant defined in [W].*

# Chapter 5

## The Invariant for Compact

## Non-Kähler Complex Surfaces

In this chapter we return to our original goal, the study of the Seiberg-Witten equations on a general hermitian surface. Using the results of Chapters 3 and 4 we simplify this study from equation (2.14) obtained in Chapter 2, to a system of equations analogous to the Seiberg-Witten equations over a Kähler surface (1.2.3). This enables us in §5.1, §5.2 and §5.3 to generalise the results obtained by Witten [W] for Kähler surfaces to an arbitrary hermitian surface. Contained in this generalisation is a solution to the vortex equation (§5.2) on a complex surface with an arbitrary hermitian metric.

Section 5.4 sees us restrict attention to the case where the first betti number,  $b_1$ , of our surface is odd, or equivalently the case where the surface does not admit a Kähler metric [BPV]. After listing the relevant properties of such surfaces with  $b^+ > 1$  we calculate the Seiberg-Witten invariant on these surfaces. The knowledge gained in §5.4 of the basic classes for non-Kähler complex surfaces enables us to generalise Theorems (0.1.1), (0.1.2) and (0.1.3) to non-Kähler complex surfaces.

### 5.1. The Generalised Equation on a Hermitian Surface $(X, h)$ .

Let  $(X, h)$  be a compact complex surface with  $b^+ > 1$  and  $h$  a hermitian metric on  $X$ , such that the Kähler form,  $\omega$ , associated to  $h$  satisfies  $\bar{\partial}\partial\omega = 0$ . A theorem of Gauduchon [G] states that every hermitian metric has a conformal rescaling so that its associated Kähler form is  $\bar{\partial}\partial$ -closed, so our assumption on  $h$  puts no restriction on  $X$  or choice of hermitian metric.

Using this metric we can define the degree of a holomorphic line bundle  $L$  on  $X$  [B]. Let  $L$  be a holomorphic line bundle with holomorphic structure given by  $\bar{\partial}_A$ , then the degree of  $L$  with respect to  $\omega$  is,

$$\deg(L) = \deg(L, \omega) = \frac{i}{2\pi} \int_X F_{A'}(L) \wedge \omega,$$

where  $F_{A'}(L)$  is the curvature of any holomorphic connection on  $L$  compatible with the structure defined by  $\bar{\partial}_A$ . Since any two such connections on  $L$  differ by a  $\bar{\partial}\partial$ -exact form, integration by parts shows that  $\deg(L, \omega)$  is independent of the hermitian connection on  $L$ . However  $\deg(L, \omega)$  is a topological invariant if and only if  $X$  admits a Kähler metric [B], that is if  $b_1(X)$  is even.

Recall from §1.2 that every complex manifold (indeed every almost complex manifold) has a canonical *spin<sup>c</sup>* structure  $W^c$  with positive and negative spin bundles given by,

$$W_c^+ = \Lambda^0(X) \oplus \Lambda^{(0,2)}(X) = \mathbf{I} \oplus \mathbf{K}_X^{-1}$$

$$W_c^- = \Lambda^{(0,1)}(X),$$

where  $\mathbf{I}$  and  $\mathbf{K}_X \simeq \Lambda^{(2,0)}(X)$  are the trivial and canonical line bundles on  $X$  respectively. Also recall that for  $f = f^{0,1} + f^{1,0} \in T^{*0,1}(X) \oplus T^{*1,0}(X)$  and a positive spinor  $\Psi^+ = (\alpha, \beta) \in \Gamma(W^+)$  Clifford multiplication is given by

$$f \cdot \Psi^+ = f \cdot (\alpha, \beta) = \sqrt{2}(f^{0,1}\alpha - *(f^{1,0} \wedge \beta))$$

On negative spinors  $\gamma \in \Gamma(W^-)$  Clifford multiplication by  $f$  is

$$f \cdot \gamma = \sqrt{2}(f^{0,1} \wedge \gamma - *(f^{1,0} \wedge *\gamma)).$$

One easily checks that if  $f$  is real, that is  $f^{0,1} = \bar{f}^{1,0}$ , then  $f \cdot f \cdot \phi = -|f|^2\phi$  for any spinor  $\phi$ . Choosing the canonical hermitian connection on  $\det W_c^+ = \mathbf{K}_X^{-1}$  the Dirac operator on  $\Gamma(W_c^+)$  is given by (2.13),

$$\mathcal{D} = \bar{\partial} + \bar{\partial}^* + \Omega,$$

where  $\Omega$  is a real one-form depending on the hermitian metric  $h$  ( $\Omega = 0$  if and only if  $h$  is Kähler).

Every other  $spin^c$  structure  $W_E$  can be written in terms of the canonical  $spin^c$  structure  $W^c$  by tensoring it with a complex line bundle  $E$ . The positive and negative spin bundles will be denoted by  $W_E^\pm$  and have the form,

$$W_E^+ = (\mathbf{I} \oplus \mathbf{K}_x) \otimes E = E \oplus (E \otimes \mathbf{K}_x^{-1})$$

$$W_E^- = \Lambda^{0,1}(X, E).$$

A connection  $A$  on  $E$  and the canonical hermitian connection on  $\mathbf{K}_x$  determine a connection  $A'$  on  $\det(W_E^+) = 2E - \mathbf{K}_x \equiv L$ , where we are using the additive notation for line bundles. With this choice the Dirac operator on  $W_E^+$  becomes,

$$\mathcal{D}_A = \bar{\partial}_A + \bar{\partial}_A^* + \Omega.$$

We will abuse notation slightly and represent the Chern class associated to a line bundle  $L$  by “ $L$ ”  $\in H^2(X, \mathbf{Z})$ . The line bundle  $L = \det(W_E^+)$  is a characteristic class, that is its intersection pairing,  $L \cdot T$  with any line bundle  $T$  satisfies  $L \cdot T \equiv T^2, \text{ mod } 2$ . Furthermore we will call the line bundle  $L$  a basic class if the Seiberg-Witten invariant associated to the  $spin^c$  structure associated to  $L$  does not vanish.

The work of the previous two chapters shows that in order to determine the basic classes, and Seiberg-Witten invariants of any four-manifold with  $b^+ > 1$  we can study the equations (3.1.1) and (3.1.2) with any choice of  $v$ . We make the choice  $v = -\Omega$  and the equations on a hermitian surface  $(X, h)$  with  $spin^c$  structure  $W_E$  become (see 2.14),

$$\bar{\partial}_A f + \bar{\partial}_A^* s = 0 \tag{5.1.1a}$$

$$F_{A'}^+(L)^{1,1} = \frac{i\omega}{2}(|f|^2 - |s|^2) \tag{5.1.1b}$$

$$F_{A'}(L)^{0,2} = \bar{f}s \tag{5.1.1c}$$

$$F_{A'}(L)^{2,0} = -f\bar{s} \tag{5.1.1c'}$$

where  $f$  is a section of  $E$  and  $s$  is a section of  $E - \mathbf{K}_x$ , and  $A'$  is defined above. With these equations we generalise the results obtained in [W] for Kähler surfaces to the case where  $(X, h)$  is any compact complex surface with  $b^+ > 1$  and whose metric  $h$  has been conformally rescaled so that its associated Kähler form is  $\bar{\partial}\partial$ -closed.

**Theorem 5.1.2.** *Let  $(X, h)$  be a compact complex surface with  $b^+ > 1$  and hermitian metric  $h$  that has been conformally rescaled so that its associated Kähler form  $\omega$  is  $\bar{\partial}\partial$ -closed. Then the following conditions are necessary and sufficient for the line bundle  $E$  to have a solution to*

the equations (5.1.1), and determine a moduli space with virtual dimension  $d(L) \geq 0$ .

(i)  $L^2 \geq \mathbf{K}_x^2$ . By definition  $L = 2E - \mathbf{K}_x$ , so this condition is equivalent to  $E^2 \geq E \cdot \mathbf{K}_x$ .

(ii)  $\bar{\partial}_A^2 = 0$ . That is  $A$  must determine a holomorphic structure  $\mathcal{E}$  on  $E$ .

(iii) For an irreducible solution we require that either  $H^0(X, \mathcal{E}) \neq 0$  and  $0 \leq \deg \mathcal{E} < \frac{1}{2} \deg \mathbf{K}_x$ , or  $H^0(X, \mathbf{K}_x - \mathcal{E}) \neq 0$  and  $\frac{1}{2} \deg \mathbf{K}_x < \deg \mathcal{E} \leq \deg \mathbf{K}_x$ . For a reducible solution we require that  $2 \deg \mathcal{E} = \deg \mathbf{K}_x$  and  $A'$  determines an anti-self-dual connection on  $L$ .

**Proof:** On a complex surface  $\mathbf{K}_x^2 = 2\chi + 3\sigma$  so we can write virtual dimension of the moduli space  $d(L) = 1/4(L^2 - \mathbf{K}_x^2)$ . Clearly (i) is necessary for a non-empty moduli space.

To prove the necessity of (ii) note the following identities that hold at any solution to equation (5.1.1),

$$\bar{\partial}_A^2 = F_A(E)^{0,2} = \frac{1}{2} F_{A'}(L)^{0,2} = \frac{1}{2} \bar{f}s. \quad (5.1.3)$$

These identities are a consequence of the connection  $A'$  on  $L = 2E - \mathbf{K}_x$  being induced from the connection  $A$  on  $E$ , and the canonical connection on  $\mathbf{K}_x$  which has curvature of type  $(1, 1)$ . Suppose  $(A, f + s) \in \mathcal{A}(E) \times \Gamma(W_E^+)$  is a solution to the equation (5.1.1). Applying the operator  $\bar{\partial}_A$  to (5.1.1a) gives,

$$\bar{\partial}_A^2 f + \bar{\partial}_A \bar{\partial}_A^* s = 0.$$

If we take the pointwise inner product with  $s$  and use (5.1.3) this equation becomes,

$$\frac{1}{2} \langle s, \bar{f}fs \rangle + \langle s, \bar{\partial}_A \bar{\partial}_A^* s \rangle = 0. \quad (5.1.4)$$

Now integrate (5.1.4) over  $X$  to obtain the following equality which is true at any solution to (5.1.1),

$$\|f\|^2 \|s\|^2 + \|\bar{\partial}_A^* s\|^2 = 0. \quad (5.1.5)$$

Both terms in (5.1.5) are manifestly non-negative and we conclude that either  $f$  or  $s$  is zero at a solution to (5.1.1). Either way we see from (5.1.1c) that  $F_A(E)^{0,2} = 0$ , which proves the necessity of (ii).

Now suppose that  $\mathcal{E}$  is a holomorphic structure on  $E$ . The sign of  $2 \deg \mathcal{E} - \deg \mathbf{K}_x$  will determine which of  $f$  or  $s$  vanishes as well as the necessity of (iii). At a solution  $(A, f + s) \in \mathcal{A}(E) \times \Gamma(W_E^+)$  we have,

$$\begin{aligned} \deg(2\mathcal{E} - \mathbf{K}_x) &= \frac{i}{2\pi} \int_X F_A^+(L) \wedge \omega \\ &= \frac{i}{2} \frac{i}{2\pi} \int_X (|f|^2 - |s|^2) \omega \wedge \omega \\ \deg(2\mathcal{E} - \mathbf{K}_x) &= \frac{1}{2\pi} \int_X (|s|^2 - |f|^2) dV \end{aligned} \quad (5.1.6)$$

From (5.1.6) it is evident that if  $\deg(2\mathcal{E} - \mathbf{K}_X) \leq 0$  then  $s = 0$ , and if  $\deg(2\mathcal{E} - \mathbf{K}_X) \geq 0$  then  $f = 0$ . Note that there is a problem with reducible solutions when  $\deg(2\mathcal{E} - \mathbf{K}_X) = 0$ , which is why we will consider the perturbed equations in §5.3. Nevertheless it is clear that reducible solutions exist if  $2\deg \mathcal{E} = \deg \mathbf{K}_X$  and  $L = 2\mathcal{E} - \mathbf{K}_X$  has an anti-self-dual connection.

If  $\deg L = \deg(2\mathcal{E} - \mathbf{K}_X) < 0$  then an irreducible solution  $f$  is a section of the line bundle  $\mathcal{E}$ , implying that  $H^0(X, \mathcal{E}) \neq 0$ . Furthermore equation (5.1.1a) implies that  $\bar{\partial}_A f = 0$ , so  $\mathcal{E}$  has holomorphic sections, which implies  $\deg \mathcal{E} \geq 0$  [B]. Combining this with the restraint on the degree of  $L = 2\mathcal{E} - \mathbf{K}_X$  gives us the inequality

$$0 \leq \deg \mathcal{E} < \frac{1}{2} \deg \mathbf{K}_X,$$

proving the necessity of the first statement in (iii).

If on the other hand  $\deg(2\mathcal{E} - \mathbf{K}_X) > 0$  an irreducible solution to the equations consists of a non-zero anti-holomorphic section  $s$  of  $\mathcal{E} - \mathbf{K}_X$ . This implies that  $H^0(X, \mathcal{E} - \mathbf{K}_X) \neq 0$  and  $\deg(\mathcal{E} - \mathbf{K}_X) \leq 0$ . Again combining these inequalities yields,

$$\frac{1}{2} \deg \mathbf{K}_X < \deg \mathcal{E} \leq \deg \mathbf{K}_X,$$

and the second statement in (iii) is proved.

We now prove the sufficiency of the conditions in the Theorem. Note that we have covered the reducible case where  $\deg(2\mathcal{E} - \mathbf{K}_X) = 0$ . So we will assume from now on that  $\deg(2\mathcal{E} - \mathbf{K}_X) \neq 0$ .

First suppose that  $\deg(2\mathcal{E} - \mathbf{K}_X) < 0$ , and the conditions of the Theorem are satisfied. The system of equations we are trying to solve for the line bundle  $\mathcal{E}$ , corresponds to finding a connection  $A$  on  $\mathcal{E}$  and a section  $f$  of  $\mathcal{E}$  (we know  $H^0(X, \mathcal{E}) \neq 0$ ) satisfying,

$$\begin{aligned} \bar{\partial}_A^2 &= 0 \\ \bar{\partial}_A f &= 0 \\ F_A^+(2\mathcal{E} - \mathbf{K}_X) &= 2F_A^+(\mathcal{E}) - F^+(\mathbf{K}_X) = \frac{i\omega}{2}|f|^2, \end{aligned} \tag{5.1.7}$$

subject to the condition that  $2\deg \mathcal{E} - \deg \mathbf{K}_X < 0$ . This system of equations has the form of the ‘‘generalised vortex equations.’’ These equations have been studied extensively, and solved, for the case where the metric  $h$  on  $X$  is Kähler, see for example [Br, OT]. The proof that a solution to (5.1.7) exists for a general hermitian surface is a considerable diversion and will be proved in §5.2.

Now suppose that  $\deg 2\mathcal{E} - \mathbf{K}_x > 0$ . Make the definition  $\mathcal{E}' = \mathbf{K}_x - \mathcal{E}$ . Then  $F_A^+(2\mathcal{E}' - \mathbf{K}_x) = -F_A^+(2\mathcal{E} - \mathbf{K}_x)$ . We are looking for a connection  $A$  on  $E'$  and a section  $s$  of  $E'$  satisfying,

$$\begin{aligned}\bar{\partial}_A^2 &= 0 \\ \bar{\partial}_A s &= 0 \\ 2F_A^+(\mathcal{E}') - F^+(\mathbf{K}_x) &= \frac{i\omega}{2}|s|^2,\end{aligned}\tag{5.1.8}$$

where  $2\deg \mathcal{E}' - \deg \mathbf{K}_x < 0$ . This equation (5.1.8) is completely equivalent to (5.1.7), which will be solved in §5.2. Subject to this proof, Theorem (5.1.2) is completed. △

## 5.2. The Vortex Equations on a Hermitian Surface.

To solve (5.1.7) we use the fact that for any holomorphic line bundle  $\mathcal{E}$ , there is a unique connection  $B$ , the metric connection, compatible with both the holomorphic structure and hermitian metric  $H$ . Using this fact we can redirect our search for a pair  $(\bar{\partial}_A, f)$  satisfying (5.1.7), to a search for a hermitian metric  $H'$  whose metric connection  $B'$  satisfies (5.1.7) for a given holomorphic structure and section. Though the technique is standard, the paper [Br] offers a particularly detailed account of the equivalence between the two approaches (see also our discussion of the unperturbed moduli space following the proof of Theorem 5.2.4).

With the above understood suppose  $\mathcal{E}$  is a holomorphic line bundle with a hermitian metric  $H$  and metric connection  $B$ . Any other hermitian metric  $H'$  on  $\mathcal{E}$  can be expressed in terms of  $H$  by,  $H' = e^\lambda H$  for  $\lambda \in C^\infty(X, \mathbf{R})$ . The curvature of the connection  $B'$  compatible with  $H'$  and the holomorphic structure on  $\mathcal{E}$  can be written,

$$F_{B'}(\mathcal{E}) = F_B(\mathcal{E}) + \bar{\partial}\partial\lambda.$$

Also if  $f$  is a non-zero holomorphic section then  $|f|_{H'}^2 = |f|_H^2 e^\lambda$ . We are looking for a metric  $H'$  whose metric connection  $B'$  satisfies,

$$F_{B'}^+(\mathcal{E}) - \frac{1}{2}F^+(\mathbf{K}_x) = \frac{i\omega}{4}|f|_{H'}^2,$$

for the prescribed holomorphic section  $f$ . In terms of  $H$  this is written,

$$F_B^+(\mathcal{E}) + (\bar{\partial}\partial\lambda)^+ - \frac{1}{2}F^+(\mathbf{K}_x) = \frac{i\omega}{4}|f|_H^2 e^\lambda.\tag{5.2.1}$$

Define  $\Lambda$  to be the  $L^2$  adjoint of  $\cdot \wedge \omega$ . Then (5.2.1) is equivalent to,

$$\begin{aligned} \Lambda F_B(\mathcal{E}) + \Lambda \bar{\partial} \partial \lambda - \frac{1}{2} \Lambda F(\mathbf{K}_x) &= \frac{i}{2} |f|_H^2 e^\lambda \\ i \Lambda \bar{\partial} \partial \lambda + \frac{1}{2} |f|_H^2 e^\lambda - \left( \frac{1}{2} i \Lambda F(\mathbf{K}_x) - i \Lambda F_B(\mathcal{E}) \right) &= 0 \\ P \lambda + \frac{1}{2} |f|_H^2 e^\lambda - C &= 0 \end{aligned} \quad (5.2.2)$$

Where we have defined  $P = i \Lambda \bar{\partial} \partial$  and  $C = (\frac{1}{2} i \Lambda F(\mathbf{K}_x) - i \Lambda F_B(\mathcal{E}))$ . Note that the restriction  $\deg(2\mathcal{E} - \mathbf{K}_x) < 0$  in (5.1.7), means that  $\int_X C > 0$ .

It is convenient to write (5.2.2) in a form so that 'C' is constant. Define  $\bar{c} = \int_X C$ , and let  $v \in C^\infty(X, \mathbf{R})$  be the unique solution to,

$$Pv = C - \bar{c}.$$

The fact that such a  $v$  exists relies on our assumption that  $\bar{\partial} \partial \omega = 0$ , since if this is the case then  $\ker P^* = \mathbf{R} [B]$ , and our solution  $v$  follows from the fact that  $\int_X (C - \bar{c}) = 0$  and standard elliptic theory. This observation also proves the existence of reducible solutions to (5.1.1) when  $\deg L = 0$ , because this translates to the case  $f = 0$  and  $\int_X C = 0$  in (5.2.2). Define  $u = \lambda - v$ . Then  $\lambda$  is a solution to (5.2.2) if and only if  $u$  is a solution to,

$$Pu = \left( \frac{1}{2} |f|_H^2 e^v \right) e^u - \bar{c} = 0. \quad (5.2.3)$$

A solution to (5.2.3) is provided by the following Theorem.

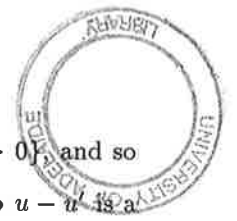
**Theorem 5.2.4.** *Let  $X$  be a compact complex surface whose Kähler form,  $\omega$ , is  $\bar{\partial} \partial$  closed, then if  $c > 0$  the equation,*

$$Pu + h e^u - c = 0 \quad (5.2.5)$$

*has a unique solution  $u$  if  $0 \leq h \in C^\infty(X, \mathbf{R})$  and  $h > 0$  at some point.*

**Proof:** . We first show that if a solution exists then it is unique. Note that if  $\bar{\partial} \partial \omega = 0$  then  $d^* d = \Delta = P + P^*$  on functions  $[B]$ . Let  $u$  and  $u'$  be two smooth solutions to (5.2.5) then,

$$\begin{aligned} 0 &\leq \|d(u - u')\|^2 \\ &= \int_X \langle \Delta(u - u'), u - u' \rangle \\ &= \int_X \langle P(u - u') + P^*(u - u'), u - u' \rangle \\ &= \int_X \langle P(u - u'), u - u' \rangle + \langle u - u', P(u - u') \rangle \\ &= 2 \operatorname{Re} \int_X \langle -h(e^u - e^{u'}), u - u' \rangle. \end{aligned}$$



The expression in the integral is strictly non-positive on the set  $\{x \in X : h(x) > 0\}$ , and so  $u \equiv u'$  on this set. On the set  $\{x \in X : h(x) = 0\}$  we have  $d(u - u') = 0$ , and so  $u - u'$  is a constant function. In fact we must have  $u = u'$  otherwise this would contradict the continuity of  $u$  and  $u'$ .

**Existence.** We follow the proof in [KW] (see also [BW]). It is a classical result [GT], that if  $k \geq c > 0$  is smooth, then the elliptic equation,  $Lu = f$  where  $L = P + k$ , has unique, smooth solutions  $u$  for all smooth  $f$ . Also since  $k \geq 0$  the operator  $L$  satisfies the maximum principle, namely: if  $Lu \geq 0$  then  $u \geq 0$ . We will have cause to use these properties of  $L$  repeatedly. Note that the operator  $P$  has negative symbol so that the statements here differ from those in [GT] by a sign change.

Following [KW] we construct sub and super-solutions,  $u_-$  and  $u_+$  such that  $u_+ > u_-$  and,

$$Pu_- + he^{u_-} - c < 0 \quad (5.2.6)$$

$$Pu_+ + he^{u_+} - c > 0 \quad (5.2.7)$$

Clearly  $u_-$  can be taken to be a sufficiently large negative constant.

To construct  $u_+$ , consider a solution,  $v$ , to the equation  $Pv = \bar{h} - h$  where  $\bar{h} = \int_X h$ . Then choose constants  $a$  and  $b$  large enough so that  $a\bar{h} > c$  and  $e^{av+b} - a > 0$ . Defining  $u_+ = av + b$  we have,

$$\begin{aligned} Pu_+ + he^{u_+} - c &= a(\bar{h} - h) + he^{av+b} - c \\ &= a\bar{h} - c + h(e^{av+b} - a) \\ &> 0 \end{aligned}$$

Now choose a smooth function  $k_1(x)$ , such that  $k_1(x) \geq \max\{1, h(x)\} > 0$ , and then define  $k(x) = k_1(x)e^{u_+}$ . Define  $L = P + k(x)$  and recall from the discussion above that  $L$  satisfies the maximum principle, and has unique smooth solutions  $u$  to the equation  $Lu = f$  for smooth  $f$ .

With  $u_0 = u_-$  we define  $u_{i+1}$  inductively to be the unique solution to,

$$Lu_{i+1} \equiv Pu_{i+1} + ku_{i+1} = -he^{u_i} + c + ku_i.$$

**Claim.**  $\{u_i\}$  is monotonic increasing and  $u_- \leq u_i \leq u_+$ .

**Proof of Claim.** We use induction on  $u_i$ .

$$\begin{aligned} Lu_1 &= -he^{u_-} + c + ku_- \\ &> Pu_- + ku_- \quad \text{by (5.2.6)} \\ &= Lu_- \end{aligned}$$

It follows from the maximum principle that  $u_1 - u_- > 0$ .

Now consider,

$$\begin{aligned}
Lu_+ - Lu_1 &= Pu_+ + ku_+ + he^{u_-} - c - ku_- \\
&> -he^{u_+} + ku_+ + he^{u_-} - ku_- \quad \text{by (5.2.7)} \\
&= h(e^{u_-} - e^{u_+}) + k_1(u_+ - u_-)e^{u_+} \quad \text{as } k = k_1e^{u_+} \\
&> he^{u_+}\{(e^{-(u_+ - u_-)} - 1) + u_+ - u_-\} \quad \text{since } k_1 \geq h \geq 0 \\
&\geq 0.
\end{aligned}$$

The last line follows since the function  $e^{-x} + x - 1$  is positive for  $x > 0$ . Using the maximum principle we conclude that  $u_+ - u_1 \geq 0$  and the claim is true for  $u_1$ .

Now suppose that the claim is true for  $i \leq j$ . Then,

$$\begin{aligned}
Lu_{j+1} - Lu_j &= -he^{u_j} + ku_j + he^{u_{j-1}} - ku_{j-1} \\
&\geq he^{u_+}\{e^{-(u_+ - u_{j-1})} - e^{-(u_+ - u_j)} + (u_+ - u_{j-1}) - (u_+ - u_j)\} \\
&\geq 0.
\end{aligned}$$

The last line follows from the inductive hypothesis and the fact that the function  $e^{-x} + x$  is strictly increasing for  $x \geq 0$ . The maximum principle now implies that  $\{u_i\}$  is monotonic.

To finish the proof of the claim we need to show that  $\{u_i\}$  is bounded by  $u_+$ .

$$\begin{aligned}
Lu_+ - Lu_{j+1} &= Pu_+ + ku_+ + he^{u_j} - c - ku_j \\
&> he^{u_+} + he^{u_j} + k(u_+ - u_j) \quad \text{by (5.2.7)} \\
&\geq he^{u_+}\{e^{-(u_+ - u_j)} - 1 + u_+ - u_j\} \\
&\geq 0.
\end{aligned}$$

Once again the last line follows since the function  $e^{-x} - 1 + x$  is positive when  $x \geq 0$ . One last application of the maximum principle proves the claim by induction.

The theorem now follows from a standard bootstrapping argument. Let  $p > 4 = \dim X$ , then the elliptic inequality for  $u_i$  gives,

$$\begin{aligned}
\|u_i\|_{L_2^p} &\leq C(\|Lu_i\|_{L^p} + \|u_i\|_{L^p}) \\
&= C(\| -he^{u_{i-1}} + c + hu_{i-1}\|_{L^p} + \|u_i\|_{L^p}) \\
&\leq C',
\end{aligned}$$

where the uniform bound on  $u_i$  was used to obtain the last inequality. Thus  $\{u_i\} \in L_2^p$ . The monotonicity of  $u_i$  and the uniform bound now imply  $u_i \rightarrow u \in L_2^p$ . The Sobolev embedding  $L_2^p \rightarrow C^1$  proves  $u \in C^1$ . We can repeat the argument above to prove that  $u \in C^\infty(X, \mathbb{R})$ .

**The Unperturbed Moduli Space.** Theorem 5.1.2 and the solution to the vortex equation gives us the machinery to completely describe the moduli space associated to the line bundle  $E$  in terms of the holomorphic structure on  $E$ . Firstly, the moduli space is empty unless  $E^2 \geq E \cdot \mathbf{K}_x$ .

Suppose  $\mathcal{E}$  is a holomorphic structure on the complex line bundle  $E$ , such that  $E^2 \geq E \cdot \mathbf{K}_x$  and  $2\deg \mathcal{E} - \deg \mathbf{K}_x < 0$ . Initially we will suppose that the holomorphic structure is unique (true if  $b_1 = 0$ ), and generalise after this initial discussion. The holomorphic line bundle  $\mathcal{E}$  admits solutions if and only if  $0 \leq \deg \mathcal{E} < \deg \mathbf{K}_x$  and  $H^0(X, \mathcal{E}) \neq 0$ . If this is the case then all solutions to (5.1.1) are irreducible, as can be seen from the insolubility of  $Pv = C$  when  $\int_X C > 0$ . Let  $\mathcal{G}_\mathbb{C} = \text{Map}(X, \mathbb{C}^*)$  denote the complexified gauge group, then the points in the moduli space are exactly the pairs  $(\bar{\partial}_A, f)/\mathcal{G}_\mathbb{C}$ , where  $\bar{\partial}_A$  is a holomorphic connection on  $\mathcal{E}$  and  $f \neq 0$  is a holomorphic section of  $\mathcal{E}$ . The action of  $\mathcal{G}_\mathbb{C}$  identifies the pairs  $(\bar{\partial}_{A'}, f')$  and  $(\bar{\partial}_A, f)$  if and only if  $\bar{\partial}_{A'}$  and  $\bar{\partial}_A$  determine isomorphic holomorphic structures [DK], and  $f' = tf$  where  $t \in \mathbb{C}^*$ . So when the holomorphic structure on  $E$  is unique, the moduli space can be identified with  $H^0(X, \mathcal{E})/\mathbb{C}^*$ .

We can check this description as follows: Suppose we are given a pair  $(\bar{\partial}_A, f)$  on the bundle  $\mathcal{E}$  (with reference metric  $H$ ), associated to this pair we found a unique positive function  $h \in C^\infty(X, \mathbf{R})$  so that the metric connection  $B$  associated to the holomorphic structure determined by  $\bar{\partial}_A$  and metric  $Hh$  satisfied,

$$F_B^+(\mathcal{E})_{Hh} = \frac{i\omega}{2} |f|_{Hh}^2.$$

The function  $h$  can be decomposed uniquely (up to a factor of  $e^{i\gamma}$ ,  $\gamma \in C^\infty(X, \mathbf{R})$ ) as  $h = g^*g$  where  $g \in \mathcal{G}_\mathbb{C}$ . It is easy to check that with respect to our original metric  $H$  we have,

$$F_{g(B)}^+(\mathcal{E})_H = \frac{i\omega}{2} |gf|_H^2,$$

( $g(B)$  is in fact the metric connection of  $H$ ), and this solution is uniquely associated to the holomorphic structure determined by  $\bar{\partial}_A$  and the section  $f$ . Taking the  $\mathcal{G}_\mathbb{C}$  orbit of  $(\bar{\partial}_A, f)$  makes the correspondence exact.

If we replace  $\mathcal{E}$  by  $\mathcal{E}' = \mathbf{K}_x - \mathcal{E}$ , the description of the moduli space in the case when  $\deg(2\mathcal{E} - \mathbf{K}_x) > 0$  is exactly the same as above.

When  $\deg(2\mathcal{E} - \mathbf{K}_x) = 0$  the moduli space corresponds the moduli space of ASD connections on  $L$ .

If the holomorphic structure is not unique, the description is not so neat, but can be identified with a union, taken over all the holomorphic structures  $\{\mathcal{E}\}$  on  $E$ , of the spaces  $H^0(X, \mathcal{E})/C^*$ .

### 5.3 The Perturbed Equations.

Although in §5.2 we got an explicit description of the moduli space, this description is inadequate for a number of reasons. Firstly there is a possibility of encountering reducible solutions—which shouldn't happen for generic equations. Secondly the expected dimension of the moduli space and the dimension of the spaces described in §5.2 do not necessarily coincide. For example an elliptic surfaces of class  $IV_0$  and  $VI_0$  in [K3, K4] satisfies  $p_g \geq 1$ , where  $p_g$  is by definition  $\dim H^0(X, \mathbf{K}_X)$ . This means the expected dimension,  $d(L) = 1/4(L^2 - \mathbf{K}_X^2)$ , of the moduli space associated to the canonical bundle  $\mathbf{K}_X$  is zero. However the dimension of the space described in §5.2 is  $p_g - 1 \geq 0$ . So unless  $p_g = 1$  these spaces are clearly non-generic.

So to refine our description of the moduli space, we must consider the perturbed equations. On any complex surface the inequality [BPV Ch. IV §2]  $b^+ \geq 2p_g$  holds, so we can choose a non-zero holomorphic two-form,  $h \in H^0(X, \mathbf{K}_X)$ , as our perturbing parameter. The equations (3.1.2) with the perturbation  $(-\Omega, h - \bar{h})$  become,

$$\bar{\partial}_A f + \bar{\partial}_A^* s = 0 \tag{5.3.1a}$$

$$2F_A^+(E)^{1,1} - F^+(\mathbf{K}_X)^{1,1} = F_{A'}^+(L) = \frac{i\omega}{2}(|f|^2 - |s|^2) \tag{5.3.1b}$$

$$2F_A(E)^{2,0} = F_{A'}(L)^{2,0} = -f\bar{s} + h \tag{5.3.1c}$$

$$2F_A(E)^{0,2} = F_{A'}(L)^{0,2} = \bar{f}s - \bar{h} \tag{5.3.1c'}$$

where  $f$  is a section of  $E$  and  $s$  is a section of  $E - \mathbf{K}_X$  and  $F^+(\mathbf{K}_X)$  denotes the self-dual part of the curvature of the canonical hermitian connection on  $\mathbf{K}_X$ .

**Theorem 5.3.2.** *Let  $X$  be a compact complex surface with  $b^+ > 1$  whose hermitian metric has been conformally rescaled so that its associated Kähler form is  $\bar{\partial}\partial$ -closed. Then the following conditions are necessary and sufficient for a line bundle  $E$  to determine an irreducible basic class  $L = 2E - K$ ,*

- (i)  $E^2 \geq E \cdot \mathbf{K}_X$  (or equivalently  $L^2 \geq \mathbf{K}_X^2$ ).
- (ii)  $E$  has a holomorphic structure  $\mathcal{E}$ .
- (iii)  $0 \leq \deg \mathcal{E} \leq \deg \mathbf{K}_X$ , and both  $H^0(X, \mathcal{E}) \neq 0$  and  $H^0(X, \mathbf{K}_X - \mathcal{E}) \neq 0$ .

(iv) For a fixed generic  $h \in H^0(X, \mathbf{K}_X)$  there is a factorization  $h = f\bar{s}$  where,  $0 \neq f \in H^0(X, \mathcal{E})$  and  $0 \neq \bar{s} \in H^0(X, \mathbf{K}_X - \mathcal{E})$ .

**Proof:** Clearly the condition  $E^2 \geq E \cdot \mathbf{K}_X$  is required for a moduli space of non-negative dimension.

Since the unperturbed moduli space vanishes unless  $c_1(E)$  is type  $(1, 1)$  we can assume that the perturbed moduli space also vanishes unless  $c_1(E)$  is type  $(1, 1)$ . This being so it follows that  $F^{0,2}(E)$  is exact. Using the fact that  $h$  is holomorphic we get the identities,

$$\int_X F_A^{0,2}(E) \wedge h = \int_X F_A^{2,0}(E) \wedge \bar{h} = 0. \quad (5.3.3)$$

Then applying  $\bar{\partial}_A$  to the Dirac equation gives,

$$\bar{\partial}_A^2 f + \bar{\partial}_A \bar{\partial}_A^* s = 0$$

$$\langle s, F_A(E)^{0,2} f \rangle + \langle s, \bar{\partial}_A \bar{\partial}_A^* s \rangle = 0$$

Now integrate over  $X$  and make expedient use of the identities in (5.3.3),

$$\begin{aligned} 0 &= \|\bar{\partial}_A^* s\|^2 + \int_X s \wedge (f\bar{s} - h)\bar{f} \\ 0 &= \|\bar{\partial}_A^* s\|^2 + \int_X (\bar{f}s \wedge f\bar{s} - \bar{f} \wedge h) \\ 0 &= \|\bar{\partial}_A^* s\|^2 + \int_X (\bar{f}s \wedge f\bar{s} - (F_A(E)^{0,2} + \bar{h}) \wedge h) \\ 0 &= \|\bar{\partial}_A^* s\|^2 + \int_X (\bar{f}s \wedge f\bar{s} - \bar{h} \wedge h) \\ 0 &= \|\bar{\partial}_A^* s\|^2 + \int_X \{(\bar{f}s - \bar{h}) \wedge (f\bar{s} - h) + \bar{f}s \wedge h + \bar{h} \wedge f\bar{s} - 2\bar{h} \wedge h\} \\ 0 &= \|\bar{\partial}_A^* s\|^2 + \|\bar{f}s - \bar{h}\|^2 \end{aligned} \quad (5.3.4)$$

Equation (5.3.4) clearly implies that both terms on the right must vanish, thus proving the necessity of (ii), (iii) and (iv). We have now established the necessity of the conditions in the Theorem.

We now prove the sufficiency of these conditions. Suppose we are given a holomorphic structure  $\mathcal{E}$ , determined by the operator  $\bar{\partial}_A$ , on the line bundle  $E$  and a factorization  $f\bar{s} = h$  where  $f \in H^0(X, \mathcal{E})$  and  $\bar{s} \in H^0(X, \mathbf{K}_X - \mathcal{E})$ . Clearly  $\bar{\partial}_A f = \bar{\partial}_A^* s = 0$ . The system of equations we need to solve is for a holomorphic connection  $A'$  on  $\mathcal{E}$  satisfying,

$$2F_{A'}^+(\mathcal{E}) - F^+(\mathbf{K}_X) = \frac{i\omega}{2}(|f|^2 - |s|^2).$$

Following the same steps as at the start of §5.2, this can easily be shown to be equivalent to the equation for  $\lambda \in C^\infty(X, \mathbf{R})$  given by,

$$P\lambda + \frac{1}{2}|f|^2 e^\lambda - \frac{1}{2}|s|^2 e^{-\lambda} - C = 0, \quad (5.3.5)$$

where  $\int_X C = -1/2 \deg(2\mathcal{E} - \mathbf{K}_X)$ , and the operator  $P = i\Lambda\bar{\partial}\partial$  is defined in §5.2. In Theorem 5.3.6 below we prove the existence of solutions to equation (5.3.5) whenever  $f$  and  $s$  are non-trivial. If we assume this result for the moment the proof of Theorem 5.3.2 is complete.

△

**Theorem 5.3.6.** *Let  $X$  be a compact hermitian surface whose Kähler form  $\omega$  is  $\bar{\partial}\partial$ -closed. Suppose also that  $A \geq 0, B \geq 0$  and  $C$  are smooth functions such that  $A$  and  $B$  are not identically zero, then the equation,*

$$P\lambda + Ae^\lambda - Be^{-\lambda} - C = 0 \tag{5.3.7}$$

has a unique smooth solution  $\lambda \in C^\infty(X, \mathbf{R})$ .

**Proof:** The method of proof is similar to Theorem 5.2.4 which, as mentioned there, is due to [KW] and [BW]. The proof will be kept brief to avoid excessive repetition of the arguments in §5.2.

**Uniqueness.** Recall that if  $\bar{\partial}\partial\omega = 0$  then  $d^*d = \Delta = P + P^*$  on functions [B]. Suppose that  $u$  and  $u'$  are two solutions to (5.3.7) then,

$$\begin{aligned} 0 &\leq \|d(u - u')\|^2 \\ &= 2\operatorname{Re} \int_X \langle P(u - u'), u - u' \rangle \\ &= 2\operatorname{Re} \int_X \langle -A(e^u - e^{u'}) + B(e^{-u} - e^{-u'}), u - u' \rangle. \end{aligned}$$

The expression inside the integral is non-positive whenever either  $A > 0$  or  $B > 0$ , and so  $u = u'$  on the set  $\{x \in X : A(x) > 0\} \cup \{x \in X : B(x) > 0\}$ . On the set where both  $A$  and  $B$  vanish we have  $d(u - u') = 0$  and so  $u - u'$  is a constant function, which must be zero to avoid contradicting the continuity of  $u$  and  $u'$ .

**Existence.** As in Theorem (5.2.4) the first step is to construct upper and lower solutions to equation (5.3.7). Let  $v$  be a solution to the equation  $Pv = \bar{A} - A$ , where  $0 < \bar{A} = \int A$ . If we define  $\lambda_+ = cv + N$  where  $c$  and  $N$  are constants then,

$$\begin{aligned} P\lambda_+ + Ae^{\lambda_+} - Be^{-\lambda_+} - C &= c(\bar{A} - A) + Ae^{cv+N} - Be^{-cv-N} - C \\ &= c\bar{A} - C + A(e^{cv+N} - c) - Be^{-cv-N}. \end{aligned}$$

If we choose  $c$  so large that  $c\bar{A} > C$ , and  $N$  big enough that both  $e^{cv+N} - c > 0$  and  $c\bar{A} - C - Be^{-cv-N} > 0$ , then it is clear that  $\lambda_+$  is an upper solution. To construct a lower solution

consider  $w$ , a solution to  $Pw = B - \bar{B}$ , where  $0 < \bar{B} = \int B$ . Then defining  $\lambda_- = nw - M$  for constants  $n$  and  $M$  we have,

$$\begin{aligned} P\lambda_- + Ae^{\lambda_-} - Be^{-\lambda_-} - C &= n(B - \bar{B}) + Ae^{nw-M} - Be^{-nw+M} - C \\ &= -n\bar{B} - C + Ae^{nw-M} + B(n - e^{-nw+M}). \end{aligned}$$

We can choose  $n$  so large that  $-n\bar{B} - C < 0$ , and  $M$  so large that the three inequalities  $n - e^{-nw+M} < 0$ ,  $n\bar{B} - C + Ae^{nw-M} < 0$  and  $\lambda_- < \lambda_+$  are all satisfied. With such choices it is clear that  $\lambda_-$  is a lower solution.

Following the proof of Theorem (5.2.4) we let  $k$  be a smooth function such that,  $k \geq \max(Ae^{\lambda_+} + Be^{-\lambda_-}, 1) > 0$ . Recall from Theorem (5.2.4) and [GT] that for such a  $k$ , the equation,  $Pf + kf = g$ , has a unique smooth solution  $f$ , for any given smooth  $g$ . Also recall (Theorem 5.2.4. and [GT]) that the operator,  $P + k$ , satisfies the maximum principle.

Define a sequence of functions  $\{\lambda_i\}$ , by letting  $\lambda_0 = \lambda_-$ , and inductively defining,  $\lambda_i$  as the unique smooth solution to,

$$P\lambda_i + k\lambda_i = -Ae^{\lambda_{i-1}} + Be^{-\lambda_{i-1}} + C + k\lambda_{i-1}.$$

Exactly as in Theorem (5.2.4) we can make repeated use of the maximum principle to show that  $\{\lambda_i\}$  is monotonic and bounded uniformly above and below by  $\lambda_+$  and  $\lambda_-$  respectively. The elliptic estimates and Sobolev theorems can easily be applied to the  $\lambda_i$  to show that  $\lambda_i \rightarrow \lambda \in C^\infty(X, \mathbf{R})$ . The details are almost identical to Theorem (5.2.4) and will not be repeated.

△

Theorem (5.3.2) provides us with an explicit way of determining which holomorphic bundles  $\mathcal{E}$  have a non-zero Seiberg-Witten invariant [W]. Suppose that the zero set of the section  $h \in H^0(X, \mathbf{K}_X)$  determines a divisor of the form,

$$\mathbf{K}_X = \sum_i m_i F_i,$$

for irreducible curves  $F_i$  of multiplicity  $m_i$ . The holomorphic bundles  $\mathcal{E}$  which factorize  $h$  are exactly those determined by divisors of the form,

$$\mathcal{E} = \sum_i a_i F_i,$$

where  $0 \leq a_i \leq m_i$ . With a bit more information on the curves  $F_i$  one can count the number of different factorizations that determine the same  $\mathcal{E}$ , and this will be the Seiberg-Witten invariant associated to the holomorphic bundle  $\mathcal{E}$ .

#### 5.4 The Invariant for non-Kähler Surfaces with $b^+ > 1$ .

**The minimal case.** The first step is to collect the properties of minimal non-Kähler surfaces with  $b^+ > 1$ . Most of these results were first proved by Kodaira [K1, K2, K3, K4] but the books [BPV] and [FM2] are also invaluable references.

Firstly, all non-Kähler complex surfaces have odd first Betti number  $b_1$  and  $b^+ = 2p_g$ , where  $p_g = \dim H^0(X, \mathbf{K}_X)$  is the geometric genus. Thus if  $b^+ > 1$  then  $p_g > 1$  and hence the Kodaira dimension of  $X$ ,  $kod(X)$  is equal to 0 or 1—the case  $kod(X) = \infty$  is eliminated since this implies that  $p_g = 0$  [BPV Ch.I §7] and the case  $kod(X) = 2$  is eliminated since all such surfaces have  $b_1 \equiv 0 \pmod{2}$  [BPV Ch. VI §1].

**$kod(X) = 0$ .** Then from the Enriques-Kodaira classification of surfaces we see that the only such surfaces with  $b^+ \geq 2$  are the primary Kodaira surfaces [BPV Ch VI §1, Ch V §5]. Their invariants are:  $p_g = 1$ ;  $b_1 = 3$ ;  $e(X) = \chi(X) = 0$ . Furthermore the canonical bundle  $\mathbf{K}_X$  is trivial [BPV Ch V §5] and  $X$  has an elliptic fibration over an elliptic curve with no multiple fibres [FM2].

**$kod(X) = 1$ .** Then  $X$  admits an elliptic fibration,  $\pi : X \rightarrow S$ , where  $S$  is a curve of genus  $g \geq 1$  [FM2]. Also  $e(X) = \chi(X) = 0$ . To see this note that  $X$  can only have fibres whose reduction is smooth [FM2 p.201] (i.e. Kodaira type  $mI_0$ ) and then using the formula for the Euler number given in [BPV Ch.III §11] we see that  $e(X) = 0$ . Since  $12\chi(X) = e(X)$  on an elliptic surface the result follows. If  $f$  is the class of a general fibre and  $\{F_i\}$  the singular fibres then it is a result of Kodaira's [K1, K2] that these are the only irreducible curves on  $X$ . It follows from the above results and the canonical bundle formula [BPV Ch.V §12] that,

$$\mathbf{K}_X = (2g - 2)f + \sum_i (m_i - 1)F_i \quad (5.4.1)$$

where  $m_i > 1$  is the multiplicity of the singular fibre. Importantly for us the general fibre,  $f$ , on such a surface determines a torsion class in  $H^2(X, \mathbf{Z})$  [RL, Bz](using Poincaré duality). In fact  $f$  generates the torsion of the Neron-Severi group [Bz]. This must be taken into account when calculating the Seiberg-Witten invariant since the line bundles determined by divisors of the form  $af + \sum b_i F_i$  are not necessarily topologically distinct when the coefficients 'a' and 'b<sub>i</sub>'

are different. Identifying which divisors are topologically equivalent is crucial for determining the Seiberg-Witten invariant on non-Kähler surfaces. In the paragraph below we will make this identification explicit, but it should be kept in mind that it is simply a combinatorial problem depending on the order of the torsion classes  $f$  and  $F_i$  in  $H^2(X, \mathbf{Z})$ .

Suppose we have a line bundle  $E$  that can be represented by the divisor,

$$E = af + \sum b_i F_i,$$

and also suppose that the order of  $f$  in  $H^2(X, \mathbf{Z})$  is ' $n$ '. Then clearly the divisors,

$$(a + kn)f + \sum b_i F_i,$$

are all representatives of ' $E$ ' in  $H^2(X, \mathbf{Z})$ . Furthermore since  $f$  generates the torsion of the Neron-Severi group [Bz] a multiple fibre  $F_i$  is homologous to some multiple of  $f$ ; this means the coefficients  $a$  and  $b_i$  are not necessarily uniquely determined. With this in mind we will define the number  $x_a(E)$  as the number of ways  $E$  can be represented as a divisor  $af + \sum b_i F_i$  with  $0 \leq b_i < m_i$ . Finally note that the numbers  $x_a(E)$  and  $x_{(a+kn)}(E)$  are equal for all integers  $k$ .

With these subtleties in mind we can use Theorem (5.3.2) and the discussion following it to determine the basic classes  $E \in H^2(X, \mathbf{Z})$  as well as the invariant associated to such a class.

**Theorem 5.4.2.** *Let  $X$  be a minimal non-Kähler complex surface with  $b^+ \geq 2$ .*

(i) *If  $\text{kod}(X) = 0$ , that is if  $X$  is a primary Kodaira surface, then the only line bundle determining a basic class is the trivial bundle  $\mathbf{K}_X$  and*

$$SW_X(\mathbf{K}_X) = 1.$$

(ii) *If  $\text{kod}(X) = 1$  and a general fibre of the elliptic fibration  $X \rightarrow S$  has order ' $n$ ', then the Seiberg-Witten basic classes,  $E$ , can be represented by a divisor of the form,*

$$E = af + \sum_i b_i F_i,$$

where  $0 \leq a \leq \min\{2g - 2, n - 1\}$  and  $0 \leq b_i \leq m_i - 1$ . The invariant for such an  $E$  is,

$$SW(E) = \sum_{0 \leq a < n} x_a(E) \sum_k (-1)^{a+kn} \binom{2g-2}{a+kn}, \quad (5.4.3)$$

where the second sum is taken over all integers  $k$  such that  $0 \leq a + kn \leq 2g - 2$ .

**Proof:** (i) If  $\text{kod}(X) = 0$  then the canonical bundle  $\mathbf{K}_X$  is trivial, and the holomorphic sections are the complex constants, thus  $\text{deg } \mathbf{K}_X = 0$  and  $\mathbf{K}_X$  is the only holomorphic bundle determining a non-zero invariant. Clearly a generic perturbation has only one factorization (up to

multiplication by scalars) so the invariant is  $\pm 1$ . In [W, §4] it is argued that the sign of the invariant is given by  $(-1)^w$  where  $w = h^0(X, \mathbf{K}_X|_{Z(\alpha)})$ , where  $Z(\alpha)$  denotes the zero set of the section  $\alpha$  in our factorization. Clearly in this case  $w = 0$  and part (i) of the Theorem is proved.

(ii) Suppose  $kod(X) = 1$ . By Theorem (5.3.2) and the discussion following it we conclude that the only holomorphic line bundles determining a non-empty moduli space arise from divisors of the form,

$$\mathcal{E} = af + \sum_i b_i F_i,$$

where  $0 \leq a \leq (2g - 2)$  and  $0 \leq b_i \leq (m_i - 1)$ . As explained above not all these divisors are topologically distinct, and so the Seiberg-Witten invariant for a class  $E \in H^2(X, \mathbf{Z})$  will involve contributions from each representation of  $E$  as a divisor of the form,  $af + \sum b_i F_i$  where  $0 \leq a \leq 2g - 2$  and  $0 \leq b_i \leq m_i - 1$ . So our first step is to calculate the contribution from each such representation of  $E$ .

The invariant associated to a divisor  $\mathcal{E} = af + \sum b_i F_i$  is given by the number of different factorizations  $h = \alpha \bar{\beta}$  of a given holomorphic two form  $h \in H^0(X, \mathbf{K}_X)$ , where  $\alpha \in H^0(X, \mathcal{E})$  and  $\bar{\beta} \in H^0(X, \mathbf{K}_X - \mathcal{E})$ . Assume that such an  $h$  is fixed and is given by,

$$h = \mu_1 \dots \mu_{2g-2} \cdot \lambda,$$

where the  $\mu_i$  vanish along a general fibre of  $\pi : X \rightarrow S$ , and  $\lambda$  vanishes on the singular fibres. A factorization  $h = \alpha \bar{\beta}$  where  $\alpha$  determines a divisor of the form (5.4.3) can be written,

$$h = \alpha \bar{\beta} = (\mu_{i_1} \dots \mu_{i_a} \cdot \lambda') (\mu_{j_1} \dots \mu_{j_{2g-2-a}} \cdot \lambda'').$$

Note that the multiple fibres are in the fixed part of  $\mathcal{E}$  so won't contribute to the invariant. The number of different factorizations is exactly,

$$\binom{2g-2}{a}.$$

The sign of the invariant will be given by ([W, §4])  $(-1)^w$  where  $w = h^0(X, \mathcal{E}|_{Z(\alpha)})$ . The fact that  $w = a$  follows because the normal bundle of each general fibre is trivial [BPV, Ch.III Lemma 8.1] and the restriction to the multiple fibres cannot have sections since these bundles are non-trivial [BPV, Ch. III Lemma 8.3] with zero degree.

We have shown that each representation of  $E$  as a divisor  $af + \sum b_i F_i$  contributes an amount,

$$(-1)^a \binom{2g-2}{a},$$

to the invariant for  $E$ . According to the discussion before the theorem, the sums in (5.4.3) account for all the representations of  $E$  as a divisor  $af + \sum b_i F_i$  with  $0 \leq a \leq 2g - 2$  and  $0 \leq b_i \leq m_i - 1$ . This completes the proof of Theorem (5.4.2). △

Although implicitly assumed in the above count of solutions (such factorizations are isolated) the fact that the moduli spaces are zero-dimensional for minimal non-Kähler surfaces can also be seen more directly. All of the above surfaces have  $K_x^2 = 0$ , and so the dimension of the moduli space is therefore equal to  $1/4L^2$  for a holomorphic bundle  $L$ . On non-Kähler surfaces all holomorphic bundles  $\mathcal{E}$  satisfy  $\mathcal{E}^2 \leq 0$  [BPV, Ch. IV §2], and so the non empty moduli spaces must all be zero dimensional.

**Corollary 5.4.4.** *All minimal non-Kähler complex surfaces are of simple type, that is the only non-trivial moduli spaces are zero dimensional.* △

This result confirms for the minimal non-Kähler case, the conjecture in [W] that all four-manifolds are of simple type.

**The non-minimal case.** The results for minimal surfaces easily extend to the case where  $X$  not necessarily minimal. The references [BPV, Ch. I §9] and [GH, Ch.1 §4] contain all the relevant details on “blow-ups” required for this subsection. Let  $\tilde{X} \xrightarrow{\pi} X$  be the blow-up of  $X$  ( $X$  is not necessarily minimal) at a single point  $x$ , and let  $\xi = \pi^{-1}(x)$  be the exceptional divisor. If  $\omega$  is a  $\bar{\partial}\partial$ -closed Kähler form on  $X$  then we can construct a  $\bar{\partial}\partial$ -closed Kähler form  $\tilde{\omega} = \pi^*\omega - \epsilon\xi$  for sufficiently small  $\epsilon$  [B]. Note that the canonical class  $\tilde{K}_x \in H^2(\tilde{X}, \mathbf{Z})$  is given by  $\tilde{K}_x = \pi^*K_x + \xi$  and  $H^2(\tilde{X}, \mathbf{Z}) \simeq H^2(X, \mathbf{Z}) \oplus \mathbf{Z}\{\xi\}$  [BPV, Ch. I Theorem 9.1].

**Theorem 5.4.5.** *Let  $X$  be a complex surface, not necessarily minimal, with  $b_1$  odd and  $b^+ > 1$ . Let  $\tilde{X}$  be the blow-up of  $X$  at a single point with  $\bar{\partial}\partial$ -closed Kähler form  $\tilde{\omega}$  and exceptional curve  $\xi$  as above, then*

- (i) *The holomorphic line bundles  $\tilde{\mathcal{E}}$  that determine non-zero invariants are exactly the classes  $\pi^*\mathcal{E}$  or  $\pi^*\mathcal{E} + \xi$  where  $\mathcal{E}$  admits non-zero invariants on  $X$ .*
- (ii) *The invariant for such a class  $\tilde{\mathcal{E}} = \pi^*\mathcal{E}$  or  $\tilde{\mathcal{E}} = \pi^*\mathcal{E} + \xi$  satisfies,*

$$SW_{\tilde{X}}(\tilde{\mathcal{E}}) = SW_X(\mathcal{E}).$$

**Proof:** (i) Suppose  $\mathcal{E}$  is a holomorphic line bundle on  $X$  determining a non-zero invariant. First we will show that the classes  $\tilde{\mathcal{E}} = \pi^* \mathcal{E}$  and  $\tilde{\mathcal{E}}' = \pi^* \mathcal{E} + \xi$  satisfy the conditions of Theorem (5.3.2) for  $\tilde{X}$ . By assumption  $\mathcal{E}$  satisfies Theorem (5.3.2) so we have,

$$\tilde{\mathcal{E}}^2 = \mathcal{E}^2 \geq \mathcal{E} \cdot \mathbf{K}_X = \tilde{\mathcal{E}} \cdot (\mathbf{K}_X + \xi) = \tilde{\mathcal{E}} \cdot \tilde{\mathbf{K}}_X.$$

Also,

$$\tilde{\mathcal{E}}'^2 = \mathcal{E}^2 - 1 \geq \mathcal{E} \cdot \mathbf{K}_X - 1 = \tilde{\mathcal{E}}' \cdot \tilde{\mathbf{K}}_X.$$

Similarly we have,

$$0 \leq \tilde{\mathcal{E}} \cdot \tilde{\omega} = \mathcal{E} \cdot \omega \leq \mathbf{K}_X \cdot \omega \leq \mathbf{K}_X \cdot \omega + \epsilon = \tilde{\mathbf{K}}_X \cdot \tilde{\omega}$$

It follows just as easily that  $0 \leq \tilde{\mathcal{E}}' \cdot \tilde{\omega} \leq \tilde{\mathbf{K}}_X \cdot \tilde{\omega}$ . We have shown that  $\tilde{\mathcal{E}}$  and  $\tilde{\mathcal{E}}'$  satisfy Theorem (5.3.2) if  $\mathcal{E}$  does, now we must show that these are the only such classes that determine non-zero invariants.

Any holomorphic bundle on  $\tilde{X}$  can be written in the form  $\tilde{\mathcal{E}} = \pi^* \mathcal{E} + a\xi$  for some holomorphic bundle  $\mathcal{E}$  on  $X$  and integer  $a$ . We need to show that  $\mathcal{E}$  satisfies Theorem (5.3.2). Since  $\tilde{\mathcal{E}}$  satisfies Theorem (5.3.2),

$$\tilde{\mathcal{E}}^2 = \mathcal{E}^2 - a^2 \geq \tilde{\mathcal{E}} \cdot \tilde{\mathbf{K}}_X = \mathcal{E} \cdot \mathbf{K}_X - a,$$

so clearly  $\mathcal{E}^2 \geq \mathcal{E} \cdot \mathbf{K}_X$  and  $\mathcal{E}$  satisfies (i) of Theorem (5.3.2). Suppose that  $\mathcal{E} \cdot \omega < 0$  then,

$$\tilde{\mathcal{E}} \cdot \tilde{\omega} = \mathcal{E} \cdot \omega + a\epsilon$$

and for sufficiently small  $\epsilon$  we contradict the fact that  $0 \leq \tilde{\mathcal{E}} \cdot \tilde{\omega}$ . Similarly supposing  $\mathcal{E} \cdot \omega > \mathbf{K}_X \cdot \omega$  implies,

$$\tilde{\mathcal{E}} \cdot \tilde{\omega} = \mathcal{E} \cdot \omega + a\epsilon \leq \tilde{\mathbf{K}}_X \cdot \tilde{\omega} = \mathbf{K}_X \cdot \omega + \epsilon,$$

and for small enough  $\epsilon$  we contradict the fact that  $\tilde{\mathcal{E}} \cdot \tilde{\omega} \leq \tilde{\mathbf{K}}_X \cdot \tilde{\omega}$ .

By assumption  $H^0(\tilde{X}, \tilde{\mathcal{E}}) \neq 0$ , thus a section in  $H^0(\tilde{X}, \tilde{\mathcal{E}})$  determines a section in  $H^0(X - \{\text{pt}\}, \mathcal{E})$ . By Hartogs Theorem [BM] this extends across the point to determine a section in  $H^0(X, \mathcal{E})$ . The proof that  $H^0(X, \mathbf{K}_X - \mathcal{E}) \neq 0$  is similar. We have proved that  $\mathcal{E}$  satisfies the conditions of Theorem (5.3.2) and so determines a non-zero invariant.

To finish the proof of (i) we need to show that  $a = 0$  or  $a = 1$ , which we do by induction on the number of blow-ups.

If  $X$  is minimal then  $\mathcal{E}^2 = \mathcal{E} \cdot \mathbf{K}_X$  for all holomorphic line bundles determining non-zero invariants (this is the simple type condition). Suppose inductively that  $\mathcal{E}^2 = \mathcal{E} \cdot \mathbf{K}_X$  for all such line bundles on surfaces  $X$  with  $n$  or fewer blow-ups. Let  $\tilde{X}$  be a blow-up of the surface  $X$  with  $n$  blow-ups. We have proved above that the line bundles on  $\tilde{X}$  determining non-zero invariants are of the form  $\tilde{\mathcal{E}} = \pi^* \mathcal{E} + a\xi$  where  $\mathcal{E}$  gives non-zero invariants on  $X$  and  $a$  is an integer. Since  $\tilde{\mathcal{E}}$  satisfies Theorem (5.3.2),

$$\tilde{\mathcal{E}} = \mathcal{E}^2 - a^2 \geq \tilde{\mathcal{E}} \cdot \tilde{\mathbf{K}}_X = \mathcal{E} \cdot \mathbf{K}_X - a.$$

The inductive hypothesis is that  $\mathcal{E}^2 = \mathcal{E} \cdot \mathbf{K}_X$  so  $a^2 \leq a$ . So  $a = 0$  or  $a = 1$  and part (i) of the Theorem is proved.

(ii) This is actually a consequence of a general blow-up formula [FS] or more directly an examination of the proof of Theorem (5.4.2) shows that the presence of exceptional curves will not effect the count of factorizations.

△

Contained in the proof of part (i) Theorem (5.4.5) is a direct proof of the fact that non-Kähler complex surfaces, minimal or not, are of simple type.

**Corollary 5.4.6.** *All non-Kähler surfaces are of simple type.*

△

## 5.5. An Application of the Invariants.

With the results of the previous sections we can generalise the results in [FM3] to the non-Kähler case. For this purpose we need to use the ‘basic classes’ which are independent from a reference  $spin^c$  structure, and so are a diffeomorphism invariant. The line bundles ‘ $E$ ’ used above are defined relative to the canonical  $spin^c$  structure and so cannot be said to be preserved under diffeomorphisms. Nevertheless there is an exact relationship between the two pictures which will be exploited (in both directions) in the following Theorems. In the following paragraph summarises the relationship between the two pictures.

Each complex line bundle  $E$  that determines a non-zero invariant also determines a basic class—which by definition is the class of the determinant line bundle  $L = 2E - \mathbf{K}_X$  of the  $spin^c$

structure determined by  $E$ . These basic classes are characteristic that is,  $L \cdot \eta \equiv \eta^2 \pmod{2}$ , for all  $\eta \in H^2(X, \mathbf{Z})$ . The symmetry— $\mathbf{K}_X - E$  determines solutions if and only if  $E$  does— translates to  $L$  is a basic class if and only if  $-L$  is a basic class. It is worth keeping both pictures in mind, since although the basic classes are diffeomorphism invariants and have more manageable symmetries, they only determine  $spin^c$  structures mod two torsion. With this in mind, Corollary (5.4.5) in the terminology of basic classes reads: the basic classes  $\tilde{L}$  on  $\tilde{X}$  are exactly the classes  $L \pm \xi$  for the basic classes  $L$  on  $X$ .

With these formalities out of the way we can use the results of §5.4 to extend the arguments in [FM3] to the case of non-Kähler surfaces.

**Theorem 5.5.1.** *Let  $X$  be a minimal non-Kähler complex surface with  $b^+ > 1$ , and suppose  $\tilde{X}$  is a blow-up of  $X$  at  $l$  distinct points with  $\{\xi_1, \dots, \xi_l\}$  the corresponding exceptional curves. Let  $N$  be a closed negative definite 4-manifold with  $H_1(N, \mathbf{Z}) = 0$  and suppose  $\{n_1, \dots, n_k\}$  is a basis for  $H^2(N, \mathbf{Z})$  such that  $n_i^2 = -1$  for all  $i$ , and  $n_i \cdot n_j = 0$  for  $i \neq j$ . If there is an orientation preserving diffeomorphism  $\phi : \tilde{X} \rightarrow M \# N$  for some  $M$ , then for every  $i$ ,  $n_i = \pm[\xi_j]$  for some  $j$ .*

**Proof:** The blow-up formula for basic classes ([FS2], see also [FM3]) imply that the basic classes of  $\tilde{X}$  are of the form  $P + \sum_{i=1}^k \pm n_i$ , where  $P$  is a basic class on  $M$  (note that we are suppressing the pullback  $\phi^*$  for notational convenience). Given an  $n \in \{n_j\}$  we can write all the basic classes of  $\tilde{X}$  in the form,  $\pm n \pm L$  for certain classes  $L$ . In particular given such an  $L$ , then  $\pm n \pm L$  are all basic classes.

We can recover  $n$  from the set of basic classes on  $\tilde{X}$  as the difference of two classes of square  $-1$ . Thus,

$$n = \frac{L_1 - L_2}{2} \pm \xi_i,$$

for some basic classes  $L_1, L_2 \in H^2(X, \mathbf{Z})$  for  $X$  (recall that the basic classes for  $X$  all have square 0). Define  $T = (L_1 - L_2)/2$  and after renumbering and a possible sign change we can suppose  $n = T + \xi_1$ .

Let  $K$  be an arbitrary basic class on  $X$ . Since  $K + \xi_1 + \sum_{i>1} \xi_i$  is a basic class on  $\tilde{X}$ , then, using the property of the basic classes discussed in the first paragraph of the proof, either,

$$K + \xi_1 + \sum_{i>1} \xi_i = (T + \xi_1) + L, \tag{5.5.2}$$

or

$$K + \xi_1 + \sum_{i>1} \xi_i = -(T + \xi_1) + L, \tag{5.5.3}$$

for some class  $L$ .

Suppose (5.5.2) is true, then  $L = K + T + 2\xi_1 + \sum_{i>1} \xi_i$  and this would imply,

$$T + \xi_1 + L = K + 2T + 3\xi_1 + \sum_{i>1} \xi_i$$

is also a basic class on  $\tilde{X}$ . However this class defines a moduli space with virtual dimension  $-4$  and so clearly cannot define a basic class. Thus (5.5.3) is true and so  $L = K - T + \sum_{i>1} \xi_i$ . This means,

$$-(T + \xi_1) + L = K - 2T - \xi_1 + \sum_{i>1} \xi_i,$$

is also a basic class on  $\tilde{X}$ . It follows that  $K - 2T$  is a basic class on  $X$  whenever  $K$  is a basic class for  $X$ .

Since  $\mathbf{K}_X$  is a basic class on  $X$  this means  $\mathbf{K}_X - 2T$  is also a basic class. Define  $E$  by  $2E - \mathbf{K}_X = \mathbf{K}_X - 2T$ , that is  $E$  defines the  $spin^c$  structure associated to the  $spin^c$  structure with determinant line  $\mathbf{K}_X - 2T$ . Thus both  $E = \mathbf{K}_X - T$  and  $\mathbf{K}_X - E = T$  determine basic classes. From Theorem (5.3.2)  $T$  must be effective with a non-zero holomorphic section.

But  $-\mathbf{K}_X$  is also a basic class and by defining  $E'$  by  $2E' - \mathbf{K}_X = -\mathbf{K}_X - 2T$  we get  $E' = -T$  is also a basic class. The only way that both  $T$  and  $-T$  can determine effective divisors is if  $T = 0$ , thus  $n = \xi_1$  and the theorem is proved.

△

Theorem 5.5.1. obviously contains as a special case,

**Theorem 5.5.4.** *Let  $X$  and  $X'$  be minimal non-Kähler surfaces with Kodaira dimension at least zero. Suppose that  $\tilde{X}$  and  $\tilde{X}'$  are blowups of  $X$  and  $X'$  at  $m$  and  $n$  distinct points respectively, with associated exceptional curves  $\xi_1, \dots, \xi_m$  and  $\xi'_1, \dots, \xi'_n$ . If  $\phi : \tilde{X} \rightarrow \tilde{X}'$  is an orientation preserving diffeomorphism then  $m = n$  and for each  $i$  there is a  $j$  such that  $\phi^*[\xi'_i] = \pm[\xi_j]$ .*

△

# Bibliography

- [AHS] Atiyah, M. F., Hitchin, N. J., Singer, I. M. "Self-Duality in Four-Dimensional Riemannian Geometry." *Proceedings of the Royal Society of London. A.* **362** (1978), 425–462.
- [Ak] Akbulut, S. "Lectures on Seiberg-Witten Invariants." *Proceedings of 4th Gökova Geometry Topology Conference 1995*. Turkey (1996).
- [B] Buchdahl, N. P. "Hermitian-Einstein Connections and Stable Vector Bundles Over Compact Complex Surfaces." *Mathematische Annalen* **280** (1988), 625–648.
- [BB] Booß, B., Bleeker, D. D. "Topology and Analysis: The Atiyah-Singer Index Formula and Gauge-Theoretic Physics." Springer-Verlag; New York (1985)
- [BBW] Booß, B., Wojciechowski, K.P. "Elliptic Boundary Problems for Dirac Operators." Birkhauser; Boston (1993)
- [BGV] Berline, N., Getzler, E., Vergne, M. "Heat Kernels and Dirac Operators." Springer-Verlag; Berlin, New York (1992)
- [Bi] Biquard, O. "Les équations de Seiberg-Witten sur une surface complexe non kählérienne." Preprint.
- [BM] Bochner, S., Martin, W.T. "Several Complex Variables." Princeton University Press (1948)
- [BPV] Barth, W., Peters, C., Van de Ven, A. "Compact Complex Surfaces." *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 4*. Springer-Verlag; New York (1984)
- [Br] Bradlow, S. B. "Vortices in Holomorphic Line Bundles over Closed Kähler Manifolds." *Communications in Mathematical Physics* **135** (1990), 1–17.

- [Bs] Brussee, R. “The Canonical Class and the  $C^\infty$ -Properties of Kähler Surfaces.” Preprint.
- [BW] Bryan, J. A., “The Multi-Monopole Equations for Kähler Surfaces.” *Proceedings of 4th Gökova Geometry-Topology Conference 1995*. Turkey (1996)
- [Bz] Brînzănescu, V. “Neron-Severi group for nonalgebraic elliptic surfaces II: non-kählerian case.” *Manuscripta Mathematica* **84** (1994), 415–420.
- [D1] Donaldson, S. K. “The Orientation of Yang-Mills Moduli Spaces and 4-manifold Topology.” *Journal of Differential Geometry* **26** (1987), 397–428.
- [D2] ———, “Polynomial Invariants for Smooth 4-manifolds.” *Topology* **29** (1990), 257–315.
- [DK] Donaldson, S. K., Kronheimer, P. B. “The Geometry of Four-Manifolds.” Clarendon, Oxford (1990)
- [FM1] Friedman, R., Morgan, J. W. “Algebraic Surfaces and 4-manifolds: Some Conjectures and Speculations.” *Bulletin of the American Mathematical Society* **18** (1990), 1–19.
- [FM2] ———, “Smooth Four-Manifolds and Complex Surfaces.” *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 27*. Springer-Verlag, New York (1994)
- [FM3] ———, “Algebraic Surfaces and Seiberg-Witten Invariants.” Preprint.
- [FS1] Fintushel, R., Stern, R. J. “Rational Blowdowns of Smooth 4-Manifolds.” Preprint.
- [FS2] ———, “Immersed Spheres in 4-Manifolds and the Immersed Thom Conjecture.” *Turkish Journal of Mathematics* **19** (1995), 145–157.
- [FU] Freed, D., Uhlenbeck, K. K. “Instantons and Four-Manifolds.” Springer-Verlag; New York (1984)
- [G] Gauduchon, P. “Le Théorème de l'excentricité nulle.” *Comptes Rendus Acad. Sci. Paris* **285** (1977), 387–390.
- [GH] Griffiths, P., Harris, J. “Principles of Algebraic Geometry.” Wiley, New York (1978)
- [Gr] Gromov, M. “Pseudo-Holomorphic curves in Symplectic manifolds.” *Inventiones Mathematicae* **82** (1985), 307–347.
- [GT] Gilbarg, D., Trudinger, N. S. “Elliptic Partial Differential Equations of Second Order.” Springer-Verlag, Berlin (1983)
- [H] Hitchin, N. “Harmonic Spinors.” *Advances in Mathematics* **14** (1974), 1–55.

- [HH] Hirzebruch, F., Hopf, H. “Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten.” *Mathematische Annalen* **136** (1958), 156–172.
- [K1] Kodaira, K. “On Compact Analytic Surfaces.” *Analytic Functions*. Princeton University Press, Princeton, New Jersey 121–135 (1960)
- [K2] ———, “On Compact Analytic Surfaces, I.” *Annals of Mathematics* **71** (1960), 111–152.
- [K3] ———, “On the Structure of Compact Complex Analytic Surfaces.” Lecture Notes 1964. In “Collected Works,” Princeton University Press, Princeton, New Jersey (1975)
- [K4] ———, “On the Structure of Compact Complex Analytic Surfaces, I.” *American Journal of Mathematics* **86** (1964), 751–798.
- [KM1] Kronheimer, P. B., Mrowka, T. S. “Recurrence relations and asymptotics for four-manifold invariants.” *Bulletin of the American Mathematical Society* **30** (1994), 215–221.
- [KM2] ———, “Embedded Surfaces and the Structure of Donaldson’s Polynomial invariants.” *Journal of Differential Geometry* **41** (1995), 573–734.
- [KM3] ———, “The Genus of Embedded Surfaces in the Projective Plane.” *Mathematical Research Letters* **1** (1994), 797–808.
- [KW] Kazdan, J. L., Warner, F. W. “Curvature Functions for Compact 2-Manifolds.” *Annals of Mathematics* **2 99** (1974), 14–47.
- [L] Lawson, H. B. “The Theory of Gauge Fields in Four Dimensions.” *CBMS Regional Conference Series in Mathematics*. American Mathematical Society. Providence, RI (1985)
- [LM] Lawson, H. B., Michelsohn, M. “Spin Geometry.” Princeton University Press, Princeton, New Jersey (1989)
- [M] Morgan, J. “The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds.” Math. Notes, 44. Princeton University Press; Princeton, NJ (1996)
- [Mas] Massey, W. S. “On the Stiefel-Whitney classes of a manifold II.” *Proceedings of the American Mathematical Society* **13** (1962), 938–942.
- [MOY] Mrowka, Ozsvath, Yau “Seiberg-Witten monopoles on Seifert fibered spaces.” MSRI preprint 1996-093.
- [MS] McDuff, D., Salamon, D. “A survey of symplectic 4-manifolds with  $b^+ = 1$ .” *Proceedings of 4th Gökova Geometry-Topology Conference 1995*. Turkey (1996).

- [N] Nakahara, M. “Geometry, Topology and Physics.” Bristol, Great Britain (1991)
- [OT] Okonek, C., Teleman, A. “The Coupled Seiberg-Witten Equations, Vortices, and Moduli Spaces of Stable Pairs.” *International Journal of Mathematics* **6** (1995), 893–910.
- [P] Palais, R. S. “Foundations of Global Non-Linear Analysis.” Benjamin, New York (1968)
- [PT] Pidstrigach, V., Tyurin, A. “Localisation of the Donaldson Invariants along Seiberg-Witten classes.” Preprint.
- [RL] Harvey, R., Lawson, H. B. Jr. “An intrinsic characterization of Kähler manifolds.” *Inventiones Mathematicae* **74** (1983), 169–198.
- [S] Smale, S. “An infinite dimensional version of Sard’s Theorem.” *American Journal of Mathematics* **87** (1965), 861–866.
- [Sc] Schoen, R. “Conformal Deformation of a Riemannian Metric to Constant Scalar Curvature.” *Journal of Differential Geometry* **20** (1984), 479–495.
- [SW1] Sieberg, N., Witten, E. “Electromagnetic duality, monopole condensation and confinement in  $N = 2$  supersymmetric Yang-Mills Theory.” *Nuclear Physics* **B426** (1994), 19–52.
- [SW2] ———, “Monopoles, duality and chiral symmetry breaking in  $N = 2$  supersymmetric QCD.” *Nuclear Physics* **B431** (1994), 581–640.
- [Sz] Szabó, Z. “Notes on Seiberg-Witten Invariants.” Lecture Notes (1995)
- [T1] Taubes, C. H. “The Seiberg-Witten Invariants and Symplectic Forms.” *Mathematical Research Letters* **1** (1994), 809–822.
- [T2] ———, “More Constraints on Symplectic Manifolds from Seiberg-Witten Invariants.” *Mathematical Research Letters* **2** (1995), 9–14.
- [T3] ———, “The Seiberg-Witten and Gromov Invariants.” *Mathematical Research Letters* **2** (1995), 221–238.
- [T4] ———, “SW $\Rightarrow$ Gr: from the Seiberg-Witten Equations to Pseudo-Holomorphic Curves.” *Journal of the American Mathematical Society* **9** (1996), 845–918.
- [U] Uhlenbeck, K. K. “Connections with  $L^p$  bounds on curvature.” *Communications in Mathematical Physics* **83** (1982), 31–42.
- [Ue] Ue, M. “On the diffeomorphism types of elliptic surfaces with multiple fibers.” *Inventiones Mathematicae* **84** (1986), 633–643.

- [V] Vaisman, I. "Complex 2-Dimensional Hermitian Manifolds." *Colloquia Mathematica Societatis János Bolyai 46. Topics in Differential Geometry*. Debrecen (Hungary) (1984)
- [W] Witten, E. "Monopoles and 4-Manifolds." *Mathematical Research Letters* **1** (1994), 769–796.
- [We] Wells, R. O. "Differential Analysis on Complex Manifolds." Springer-Verlag; New York (1980)