Generalized Quadrangles
and Associated Structures

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Abstract

Our aim in this thesis has been to consider questions concerning the relationship between a Generalized Quadrangle (GQ) and various substructures, with a view to proving characterisation and classification results. We also lay the groundwork for new GQ construction methods, although no new GQs are constructed here.

In Chapter 1 we introduce preliminary concepts and results required for the rest of the thesis, involving graphs, quadrics, geometries, GQs, algebraic topology on a simplicial complex and covers of geometries.

Chapter 2 contains a detailed investigation of the ovoid $K_1(\sigma)$ of $Q(4,q)$ constructed by Kantor in [30], including construction of non-dual rosettes of $Q(4,q)$ containing only $K_1(\sigma)$ ovoids and rosettes containing both $K_1(\sigma)$ ovoids and elliptic quadric ovoids.

In Chapter 3 we show that if $S$ is a GQ of order $(q,q^2)$ and $S'$ is a subquadrangle of order $a$ doubly subed in $S$, then the subbounded ovoid/rosette structure is a Semi-Partial Geometry (SPG). A new SPG is constructed from a GQ of Kantor ([31]) and a $Q(4,q)$ subquadrangle. For a $q$-class GQ $S$, $q$ even, Payne constructed a family of subquadranges $S_\alpha$ of order $q$ ([55]). We derive the algebraic conditions under which $S_\alpha$ is doubly subed in $S$, and hence gives an SPG.

In Chapter 4 it is shown that if $q$ is even a non-classical GQ of order $(q,q^2)$ containing a subquadrangle isomorphic to $Q(4,q)$ implies the existence of a new ovoid of $PG(3,q)$. Also, by a homology calculation, it is shown that if $S$ is a GQ of order $(q,q^2)$, $q$ odd, such that $S$ contains a $Q(4,q)$ subquadrangle, with each ovoid of $Q(4,q)$ subed by $S$ an elliptic quadric ovoid, then $S$ is isomorphic to $Q(5,q)$.

In Chapter 5 we show a GQ $S$ of order $s$ with a regular point $(\infty)$ gives rise to a cover of the affine plane constructed from $S$ and $(\infty)$, as in [45, 1.3.1]. Given an affine plane $\pi$ of order $s$ and an $s$-fold cover of $\pi$, satisfying special conditions we construct a GQ of order $s$ with a regular point. If the cover of $\pi$ is algebraic the condition on the cover is interpreted in cohomological terms; we investigate these for the remainder of the Chapter 5.