BAER STRUCTURES, UNITALS
AND
ASSOCIATED FINITE GEOMETRIES

by

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Abstract

In this thesis we study the representation of finite translation planes in projective spaces introduced by André [1]. This theory was also developed by Bruck and Bose [21, 22] in a distinct but equivalent form. Throughout this thesis we refer to this representation as the Bruck and Bose representation or simply Bruck-Bose. Of particular importance is the representation of Baer subplanes of translation planes $\pi_{q^2}$ of order $q^2$; the importance is due to the crucial role Baer subplanes have in the characterisation of various substruc-tures, including unitalis and maximal arcs, of projective planes, as will be evident in the text.

In Chapter 1 we present the necessary preliminary material required for the later chapters. In particular we present in detail the Bruck and Bose representation [21, 22] of the Desarguesian plane $PG(2, q^h)$ and the associated coordinatisation.

In Chapter 2 we begin by reviewing the known results concerning the representation of Baer subplanes of $PG(2, q^2)$ in the Bruck and Bose representation in $PG(4, q)$. We provide a new proof of the result of Vincenti [90] and Bose, Freeman and Glynn [19], that the non-affine Baer subplanes of $PG(2, q^2)$ are represented in Bruck-Bose by certain ruled cubic surfaces in $PG(4, q)$ which we term Baer ruled cubic surfaces. We characterise Baer ruled cubic surfaces in $PG(4, q)$ for a general fixed Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$. We determine that non-degenerate conics in Baer subplanes of $PG(2, q^2)$ are represented in Bruck-Bose by normal rational curves; a normal rational curve which arises in this way is of order 2, 3 or 4 and is therefore properly contained in a plane, hyperplane or no hyperplane of $PG(4, q)$ respectively. We apply these results to prove the existence of certain $(q^2 + 1)$–caps in $PG(4, q)$ which are not contained in any hyperplane of $PG(4, q)$ and which contain many normal rational curves of order 4. Further properties of these caps are determined in Chapter 3. We also include a discussion of the ruled cubic surface obtained as the projection from a point $P$ of the Veronese Surface in $PG(5, q)$ onto a hyperplane not containing $P$; in this setting we determine some alternative proofs for our results and prove some extensions.

In Chapter 3 we investigate the Bruck and Bose representation in $PG(n, q)$ with $n > 4$. We prove various results concerning the regular $(h - 1)$–spreads of $PG(2h - 1, q)$ which determine the Bruck and Bose representation of $PG(2, q^h)$ in $PG(2h, q)$, treating the
case $h = 4$ in greater detail. In particular, we prove the existence of induced spreads and show how the induced spreads are closely related to Bruck and Bose representation of the Baer substructures of $PG(2, q^4)$. To obtain further properties of the higher dimensional Bruck-Bose representation of the non-affine Baer substructures of the Desarguesian plane, we make use of the Bose representation [18] of $PG(2, q^2)$. In this chapter, we also prove results concerning the Bruck and Bose representation of non-degenerate conics in $PG(2, q^2)$ and we discuss the relationship between these results and the Bruck-Bose representation of non-affine Baer sublines of $PG(2, q^4)$ in $PG(8, q)$.

In Chapter 4 we investigate Baer subplanes and Buenkenhout-Metz unitalis in $PG(2, q^2)$. In particular we improve the known results by showing that in $PG(2, q^2)$, with $q > 13$, a Baer subplane and a Buenkenhout-Metz unital with elliptic quadric as base have at least 1 point and at most $2q + 1$ distinct points in their intersection. Our method of proof makes use of the Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$ and the properties of a certain irreducible sextic curve in $PG(4, q)$. We also prove that the non-classical Buenkenhout-Metz unitalis, with an elliptic quadric base, in $PG(2, q^2)$ are inherited from the classical unitalis in $PG(2, q^2)$ by a certain procedure of swapping regular 1-spreads of $PG(3, q)$ in the Bruck and Bose representation of $PG(2, q^2)$.

In Chapter 5 we prove that a unital in $PG(2, q^2)$ is a Buenkenhout-Metz unital if and only if there exists a point $T$ of the unital such that each secant line of the unital through $T$ intersects the unital in a Baer subline. This is an improvement of the characterisation of Lefèvre-Percsy [56] and an improvement of the characterisation of Casse, O'Keefe and Penttila [26] for the cases $q > 3$.

In the final chapter we investigate the relationships between Thas maximal arcs, the generalized quadrangle $T_5(O)$ and egglike inversive planes. This work was motivated by the approach of Barwick and O'Keefe [13] in investigating the relationship between Buenkenhout-Metz unitalis and inversive planes (see also [6, Section 5.] and [92]). We attempt to characterise the Thas maximal arcs in those translation planes where they exist using two configurational properties; we do not succeed in this, but prove a characterisation of Thas maximal arcs in $PG(2, q^2)$ for certain values of $q$. 