PROBLEM SOLVING

IN

HIGH-SCHOOL ALGEBRA:

A THEORY-BASED APPROACH

TO

CLASSROOM PRACTICE

by

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ABSTRACT

This study examines, develops and presents one possibility for a new view of classroom algebra. It is seen as a contribution to identifying and devising classroom approaches that reflect current knowledge about human learning and problem solving.

The study is based in a cognitive view of instruction. This has involved the construction of a framework from a synthesis of theoretical perspectives that are specifying and describing how knowledge and problem-solving skills in mathematics may be developed. The strength of this theoretical approach is seen in its generality, and in its usefulness in guiding, supporting and predicting effective classroom practice.

A book **Talking Maths** is presented as an interpretation of the instructional framework, and an attempt to apply it to classroom algebra. A trial implementation of the course is reported, and factors that are seen as affecting the testing of theoretically-based programs are elicited.

Statement:

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text.

I consent to the thesis being made available for photocopying and loan if it is accepted for the award of the degree for which it is submitted.

Dated: 31st March, 1989 Signed:

Eleanor M. Long

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Chapter I

CHAPTER I

CLASSROOM LEARNING AND TEACHING IN MATHEMATICS

The difficulties faced by many students in learning mathematics are well documented, and the demands involved in teaching the subject effectively are widely recognised. Indeed, the highly-regarded Cockcroft Report "Mathematics Counts" (1982) states in plain terms "Mathematics is a difficult subject both to teach and to learn" (paragraph 228).

The situation with reference to algebra

For algebra, the problem is exacerbated further. While basic arithmetical, measurement and geometrical concepts are generally regarded to be essential knowledge for competency in everyday life, algebra frequently creates the perception of being without meaning or purpose (personal observation; also Hale, 1981; Williams, 1988). The mathematics of number and space deals with abstractions, but the concepts and procedures in general use readily relate to experience. In these domains, people use notation which, typically, is familiar, performing operations that lead to needed information. The fundamental building blocks of algebra, on the other hand, are abstractions of abstractions, well distanced from their original conception. The concepts and procedures are identified, analysed and developed in terms of relationships within and between ideas that have particular meanings within the algebraic context. The ideas are represented by symbols, forming a notational system that allows consistent, precise and efficient manipulation, but is nonetheless complex and demanding to manage.

However, it is the rigorous, abstract nature of the algebraic language and the concise, unambiguous form of its communication that gives it its power and general applicability. Algebra is governed by structures of principles that stimulate hypotheses, support predictions and enable validation of an argument. Algebraic thinking and methods can be used widely in other areas of mathematics, the sciences and the social sciences as a problem-solving tool. The hierarchically-organised abstractions within algebra are a vehicle for generalisation and theory construction. Thus its power for problem solving in a variety of technological and scientific domains makes it a significant and useful area of study.

Algebra in the classroom

In the day-to-day high-school classroom the value of algebra learning needs to be seen in terms of goals that are more immediate and directly relevant to students. For some students the appeal of algebra is intrinsic; its puzzle/problem-solving nature provides fascination and satisfaction. More often than not, though, this seems to be far from the case. Personal observation and statements collected from students during classroom work spanning two decades suggest that, for many, the experience of algebra is an isolated and meaningless experience, often characterised by boredom, frustration, confusion and anxiety. Furthermore, it seems that the situation may not be improving (Booth, 1984a; Williams, 1988). The failure to achieve in algebra that often results becomes a stumbling block to further progress in maths and maths-related activities and studies. Comments offered to me by pre-service primary-teacher students indicate that "mental-blocks" of the subject have been retained since school days, and fear of maths is carried into adult-hood (also Cockcroft, 1982). Yet it is encouraging to note personally that many of those same high-school and adult students have, with support, been able to find meaning and enjoyment in algebra. This novel experience is usually accompanied by a profound sense of relief to discover that they are able to experience its fascination and master a little of its mystery.

A conviction that underlies this study is that algebra may be made accessible and meaningful to many more students than is currently the case, and that it may be used with confidence by them as a means of solving problems. This study aims to specify theoretically-based teaching approaches and suggest instructional support that are directed towards these goals. However, while the task of designing methods that may contribute to more effective classroom practice is compelling, it is also complex and perverse. A means that is possible within the scope of this study is the production of a classroom book that presents theory-based approaches to algebra learning and problem solving. The use of this algebra book in several classrooms is discussed and the outcomes reviewed in a later chapter. It is considered that this trial implementation provides information that is useful for further attempts to introduce course/methodology innovation to classrooms.

ALGEBRA - SPECIAL PROBLEMS FOR STUDENTS AND TEACHERS

A theory-based instructional process is seen as widely applicable in guiding and predicting classroom practice. However, it is also necessary to identify needs that are specific to the context of study. Thus, a preliminary investigation was conducted in order to examine the special problems presented by algebra for both learners and teachers. The inquiry involved observation and recorded interviews of students and teachers in a number of upper primary and secondary classrooms in several South Australian schools, and an analysis of students' responses collected in task-based interviews during a teaching block. Relevant literature was also examined.

Preliminary investigation involving students

An initial investigation examined classroom algebra learning and student performance in one class each of year levels 8-11 (ages 12-16) in a co-educational school of 500 students. Student performance in tests focusing on transformations and relationships in arithmetic - plus some use of generalised number - was analysed for years 6 and 7 in a 200-student primary school on the same campus. At each level students were asked to complete written tests that required demonstration of the mathematical processes used in attempting a solution; the problems were chosen in order to analyse the use of algebraic processes rather than to measure performance in course-work learning. Thus the problems ranged in level from those based on ideas that had been taught to those that looked at students' intuitive thinking on more advanced work. Test papers were analysed with reference to the approach taken, rather than on the basis of the answer given. (See Appendix 1).

At this stage of the study, year 6 children were eliminated from the sample, as their test results indicated a level of performance that was deemed outside this study's emphasis on the development of algebraic thinking. Two criteria were used in making this decision: most importantly a level of procedural competence in tolerating what Collis (1981) calls 'lack of closure', plus a demonstrated ability to perform a simple operation with letter symbols. Solutions commonly presented as $7 + 18 = \boxed{48} + 23$ ($7 + 18 + 23 = 48$) or $7 + 18 = \boxed{25} + 23 = 48$*, and lack of use of more sophisticated procedures via the commutative and distributive laws, indicated that students were reluctant to resist closure. Problems requiring simplification of an algebraic expression proved confusing to all children who were tested: some assigned numbers to the letters, others wrote down that they did not understand what to do. Test results for the year levels 7-11 however showed evidence more generally of algebraic thinking and informed use of algebraic notation. Differences in levels

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of performance were more easily distinguished by procedural approaches than by use of notation. As a follow-up to this testing, three students from each of the years 7-11 classes chosen for follow-up interviews were selected on the basis of performances that demonstrated

(1) flexibility in solution strategy and high tolerance to 'lack of closure'

(2) inflexibility in solution strategy and low tolerance to 'lack of closure'

(3) an approach somewhere within that range.

The selections were also checked for gender balance.

Interviews

Students were presented with problems appropriate to their year level. The tasks were designed to investigate both spontaneous solutions to work not taught and approaches to the work that had been covered in class. I wrote and spoke each problem statement, answering questions when required, prompting only in order to clarify a situation, not to guide the solution process. Students were asked to "think aloud" as they wrote their solutions, and these comments were tape-recorded. The solution strategies and recorded comments were later analysed in terms of performance errors.

On the basis of these interviews I decided to focus on years 9 and 10. Within these levels consistent error patterns emerged: errors were demonstrated more consistently than in the year 8 responses, and were easier to isolate than in the more complex year 11 problem solutions. Thus, in most cases it was possible to identify likely root causes of the errors, providing a good basis for an instructionally-oriented study. Errors were classified as involving (a) over-generalisation (b) lack of understanding of algebraic concepts (c) confusion with algebraic symbols and procedures and (d) failure to make a link between concepts and related procedures - suggesting that procedures had been rote-learned.

Classroom Study

A period of intensive work with the year 9 and 10 classes followed. For a period of several weeks I replaced their regular teacher, teaching a 'block' of algebra to each class under normal classroom conditions. Informal discussions about algebra and algebra learning also took place. During this period of time students' learning and performance were examined and teaching approaches were tested (with 3 lessons video-recorded for further analysis and study) in order to provide starting points for the proposed classroom book in algebra. Students' perceptions of algebra were also examined.

Alegbra-students' perceptions

Rather than seeing algebra as a powerful problem-solving tool, it seems that students tend to see algebra as serving no useful purpose, and irrelevant to life outside the classroom. "I don't see what you could use it for" was a typical comment. Some students see algebra as a challenge, though, and intuitively recognise its flexibility as a thinking tool. A year 8 student put it this way: "It's much more fun doing $x + 5 = 12$ than $7 + 5 = 12$". However it seems to be that relatively few students make sense of its structure of concepts and procedures, or recognise its capacity for meaningful communication and applicability. This disassociation from anything relevant or familiar gives it a 'non-sensical' character. Some students are able to talk about algebra in fairly lucid, if somewhat prosaic, terms such as "the letters represent numbers that you can add, subtract, multiply or divide" or "you have a box instead of a number, and then you put a letter in there." But for many students it is an unpredictable collection of symbols and algorithms, signifying nothing in particular; relationships and links with a known world are not recognised or constructed. Learning frequently appears to be approached in rote-memory fashion; topics and their ideas and procedures are considered to be isolated pieces of knowledge that require photographic memory for recall, along with 'hit or miss' responses - and a keen eye for spotting the trick!

On the basis of these findings I decided that the proposed classroom book should put algebra in context, emphasising the development and interrelationship of concepts and procedures, and integrating them with other areas of mathematical thinking. I also wanted students to recognise that algebra can be a meaningful activity, that its origins are in world experiences, and that it can be applied powerfully to problems in the real world. In particular I considered it important to present algebra as an efficient means to represent problems, and as a tool that is used to solve them.

Classroom survey of student errors: the transition from arithmetic to algebra

This study's investigation of students' verbal and written responses to problems in upper-primary arithmetic and junior high school algebra found that many students begin their study of algebra unprepared for its abstract concepts, its precise symbolism, and procedures focusing on relations and transformations. Analyses of children's approaches to arithmetic at years 6 and 7 levels indicated that they rarely focus on number relations and transformations; arithmetic seems to be seen as a 'number-crunching' exercise, described by Easley (1975) as "symbol pushing". Children commonly believe that arithmetic consists of reproducing algorithms that are directed towards a single numerical answer. Indeed my experience with teaching pre-service primary-teacher students indicates that most anticipate that their maths lessons will consist of precisely that. Students' beliefs about school maths are deeply ingrained and difficult to change.

Collis has drawn attention to what he terms students' intolerance of "lack of closure" (Collis, 1981), noting that informal approaches frequently fail to consider the need for rigour and precision, and the reasoning involved in formal methods of arithmetic are often not fully appreciated. Furthermore, my evidence supported Erlwanger's (1975) observation that many students do not see their answers as rationally linked to the problem itself. Other responses from students involved imprecise and informal uses of symbols, for example, the equals sign given a 'manufacturing' characteristic (Kieran, 1981), and inappropriate approaches to conventions such as the use of brackets and the ordering of operations (Booth, 1982, 1984b). Typically, children's arithmetical knowledge appears to be fragmented, and lacking meaning and developmental potential ; thus, children's misconceptions and inadequacies in this domain appears to be a critical factor in the difficulties faced in algebra.

Student errors in algebra : research and observation

Research in high school algebra has frequently been concerned with trying to identify children's difficulties in the subject so that instructional questions might be addressed. My analysis of student responses to algebra problems suggests a common and wide-spread use of "child-methods" (Booth, 1981, 1984a, 1984b) involving the over-generalisation of prior arithmetical experiences (formal and informal) that inadequately characterise algebra. The Chelsea College based study of high school students' strategies and errors in algebra (Booth 1984a) found that students' algebraic miconceptions are frequently characterised by an arithmetical framework of knowledge. Working within this "alternative framework" (Driver and Easley, 1978; Booth, 1981; Easley, 1984; Driver and Oldham, 1986), students apply informal and inadequate arithmetical procedures and rules to algebra. They are working within a structure that for them has not focused on the precise recording of statements of formal mathematical method. In addition this naive perspective does not reflect explicit awareness of the meaning of algebraic symbolism. For example, the Chelsea study revealed categories of misunderstandings of the letter symbols used in algebra. Students often characterise letters as objects, or inappropriately assign them specific values, or even ignore them (Kuchemann, 1984). My analysis of students' written responses on algebraic tasks also revealed examples of confusion in the use of symbols. Some students, for example, 'simplified' $3 + 2t$ to $5t$, xy^2 was confused with $(xy)^2$ and b^2 with $2b$, $y + y$ was equated with y^2, and $2y + 3y$ was seen as equal $5y^2$. These errors could have been interpreted as 'careless mistakes'. However, follow-up interviews with the students concerned generally showed that the errors had developed from an approach that was rational and systematic, but based on false or inadequate thinking. My interviews also indicated that few students think explicitly about what algebraic letters mean; many are bemused when pressed for an interpretation. Perhaps this is not surprising, as informal conversations with pre-service and in-service maths teachers have often revealed a similar avoidance and/or bewilderment.

Matz's (1980) study of students' errors in high school algebra found similar instances of unsuccessful attempts to adapt informal arithmetical knowledge to the more demanding structural requirements of algebra. She reported a systematic pattern of over-generalisation that involved reasonable but inadequate use of previously-acquired knowledge structures. Using an information-processing approach, Matz developed a theory to account for the persistence and predictability of errors. She posited the possession of deep level rules as a necessary factor in students' ability to use, modify or restructure knowledge appropriately and adequately for the solution of problems; rules in their partially-developed form tend to be applied too widely and without appropriate constraints. Bernard and Bright (1985) similarly used a cognitive perspective to explain errors in linear equations. They saw the occurrence of over-generalisation and binary confusion* to be predictable, and therefore potentially preventable through instructional intervention.

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*'Binary confusion' is a term used by Davis, Jockusch and McKnight (1978) to denote the interference between two rules. Examples of binary confusion provided by Matz, 1981, p. 98, are;

$$a/b + c/d \text{ becomes } (a+c)/(b+d)$$

$$a/(b+c) \text{ becomes } a/b + a/c$$

$$a/b + c/d \text{ becomes } ad \div bc$$

Petitto's (1979) study also illustrated the implications of children's use of informal or intuitive procedures. He cites a case - similar to one noted in my own analysis of task-based interviews - of an unsuccessful problem solver's inability to link explicitly intuitive ideas to the formal techniques needed for solution. His conjecture that the two must be coordinated in a functional way clearly has implications for the instruction of students who do not spontaneously recognise and construct links between informal and formal knowledge.

Rosnick and Clement (1980) however point to the ingrained and resilient nature of students' misconceptions, giving evidence to show that reversal errors are not easy to correct through classroom instruction. While a student may change a procedure to demonstrate a correct solution, this does not necessarily mean that there has been a shift in the student's knowledge structure. Davis (1983) has likened this characteristic to dandelion roots (cf. Matz's "deep level rules") that remain firmly in place after superficial weeding. Rachlin's classroom-oriented investigation (Rachlin and Wagner, 1983) is grappling with the thorny problem of designing instruction that deals adequately with students' recurring problems in algebra. Qualitative analyses of students' performance on selected algebra tasks suggests that instructional intervention needs to emphasise both conceptual knowledge and procedural skills and processes. These findings are reflected in a curriculum development project that focuses on a problem-solving approach to algebra learning.

Studies involving algebra word problems have revealed other indications of students clinging to naive, misconceived beliefs. Instruction that does not take into account the need for what has been variously termed deep-level rules (Davis, 1983), a shift in knowledge structure, or increased flexibility in approach, may merely provide students with a new terminology for expressing their erroneous beliefs (Reed, 1984). For example, Reed's cognitively-oriented study showed that students tended to simplify the information given to fit their own competency in calculating averages, then failed to recognise the equivalence of problems with the same problem structure (cf. Krutetskii's, 1976, "incapable" students). Paige and Simon (1966) and Chaiklin (1984) have considered the variations of difficulty and approach imposed by the particular wording of a problem, the latter drawing attention to the cognitive processes involved in converting words ("verbal rules") into symbols and procedures that lead to solution of a problem.

<u>Preliminary study: instructional issues</u>

In some of the studies cited above, teaching experiments have been conducted on the basis of information revealed in the investigation of children's errors. Clearly the identification of errors and conjecture about their root causes provides an important source of information for devising instructional approaches. It was also considered that a classroom investigation involving local schools could provide further information to guide an instructional perspective. Classroom teaching was observed, and the teachers interviewed, at five high schools in Adelaide, situated in a range of socio-economic areas.

Algebra: instructional inquiry

The teachers observed and interviewed covered a wide range of experience, qualifications and perceived capability and commitment to teaching maths. All agreed that the work was exacting, demanding and often frustrating, though most also spoke of its satisfaction and joys. Commonly there was a belief that "some students can do algebra and some can't", with the implication (which was sometimes stated directly) that "those who can't might as well give up". Teachers generally seemed pessimistic about the possibilities for improvement in achievement in algebra for these students, seeing themselves as ill-equipped to handle learning problems. In all but one classroom of those observed, instruction was tied closely to text-book exercises, and the individual's reproduction of algorithms by far the most common classroom activity. On the results of this survey it seems that instructional difficulties need to be addressed with reference to teacher knowledge (of algebra, and appropriate methodologies) and teacher attitude.

Instructional problems

It was interesting to observe that there was often a strong relationship between the problems experienced by students and what I perceived as limitations in the instructional approaches used. Although not surprising, it was striking to note that almost no references were made to meaning and context in algebra lessons. Letter-symbols were used as if the notions they represented were patently obvious to all, and manipulations followed procedures that were demonstrated through often rigidly-perceived formulations that were never justified or rationalised. The text-books used in the classrooms typically followed the same superficial approach. Instruction displayed a particularly 'classroom' character, with conversation at an abstract level that seemed to be unrelated to anything real. Quite often the students' prior experiences and constructions were not taken into account, nor used as a basis for knowledge extension. The practice of teaching algebraic knowledge 'in isolation' leads to concepts being taught as facts and rules to be rote-learned, and procedures and strategies taught as algorithms to be memorised. Furthermore, conceptual and procedural knowledge seem not to be integrated explicitly, or exploited for their problem-solving potential.

Finally, assessment procedures are seen as unnecessarily limiting and threatening. In the classrooms observed, errors were typically treated with a finality (denoting failure) that overlooked their potential for constructive remediation and generation of more adequate approaches. Alternative assessment approaches - such as formative assessment with its immediate feedback facilitating correction in a non-threatening atmosphere - were hardly used.

These observations suggest that professional development should be considered as an integral part of the classroom implementation of an algebra course. An important focus of the trial will be the ways that teachers choose to use the book and the course in their classrooms.

<u>Summary of instructional issues</u>

Students' experience with algebra is typically characterised as fraught with difficulties. Analyses of student errors and their sources seem to be a reasonable and productive basis on which to build our knowledge about classroom instruction in algebra. Certainly these analyses provide rich insights into the way that instruction might provide intervention and remedial support. Studies that focus on variables in tasks and contexts show how performance and thinking behaviour might be affected. Usually the tasks or areas of inquiry are well-defined, and many include details of mathematical behaviour displayed by students in solving problems in the algebraic task or area of study. In order to give instructional advice they identify differences in students' thinking in a specific task or area, and this information indicates approaches that a teacher may take when difficulties in the topic are encountered. But while the specific, highly-focused nature of these findings is recognised, they lack an interrelated and comprehensive underpinning for the wider needs of classroom teaching of coursework algebra. The findings are useful for relevant situations; but as collected information they hardly provide the coherence, interpretation, and powers of prediction that are needed to guide a school algebra program.

What is required is a comprehensive approach that is supported by a theoretical perspective that can give consistent guidance and provide an explanatory underpinning. A search is needed to ascertain whether such a theory is available, and to investigate the possibility that such a framework might be constructed.

SEEKING A THEORETICAL PERSPECTIVE IN COGNITIVE PSYCHOLOGY

The current approach in psychology that focuses on the mind - about what happens internally - is significant and illuminating for maths teachers. It is probably accurate to state that maths teachers and educational researchers working within this area share a fundamental and enduring concern for improving mathematics learning and instruction. It is believed that the cognitive approach to psychology has the potential to further this cause by offering evidence that has a direct relevance to learning, and therefore to teaching. In particular, cognitive psychology addresses skills that are involved in task performance, seeking to chart the means by which individuals use already-constructed knowledge to approach problems. The orientation stresses similarities in performance behaviour across tasks and across individuals, using a language that facilitates descriptions of common thinking processes. These analyses shed light on how students think about mathematics;

and the theories developed can provide maths educators with a powerful and productive theoretical base for developing ways to enhance students' understanding of mathematical concepts and processes, and for improving their skill in performance.

Maths Education and Cognitive Education: a collaboration

A growing collaboration between mathematics educators and cognitive psychologists and mathematics educators (Resnick, 1983; Schoenfeld, 1987; Lester, 1988) highlights commonalities in interests and applicability within the two fields of research. Studies in mathematics education have traditionally been concerned with building up local theories of mathematical learning. Information is derived from rational analysis of mathematical tasks (eg, Resnick 1976; Resnick and Ford, 1981), and empirically from mathematical behaviour demonstrated on specific tasks (eg, Ginsburg, 1983). The focus, typically, is on variabilities in performance, and the rich and often complex sets of data recorded are often used to underpin an instructional methodology in that particular topic area. Cognitive psychology is providing a body of experimental methods to investigate thinking processes in a variety of content domains, yielding information that shows commonalities in thinking processes, and clarifying how these processes are used to develop knowledge.

By merging the two perspectives, it seems that we may have the best of both worlds; the maths education perspective provides detailed mathematical knowledge and intuition born of familiarity with and understanding of the subject, while the cognitive analysis discerns and clarifies a pattern of performance that can be generalised to students' behaviour in the mathematical domain. Cognitive psychology brings with it a research tradition that is concerned with theory building, and, more recently, with the rigorous and close analysis of protocols of cognitive tasks. This complements the instructional insights developed by maths educators through a close relationship with the classroom situation.

In the specific domain of high school algebra, the claim that there is a match between 'local' theories available and the growing body of cognitively-based evidence from psychology is made cautiously. However, the contention is that a synthesis is possible, and that it will be a useful starting point for developing effective instructional approaches for classroom use.

Developing an integration between cognitive psychology and instruction

Sternberg (1986) has provided a useful checklist summary to identify problem areas in translating cognitive principles to classroom practice. He maintains that key trouble spots are likely to occur at the level of theory, the level of teacher and student ability, and the level of teacher and student motivation.

This last point raises a thorny issue; in an instructional study based on cognitive theory, problems of motivation are not likely to be specifically addressed. Yet without sufficient teacher and student motivation the most promising instructional program can fail. Clearly, then, some motivational means must be built into the program that is implemented at classroom level. Appropriate provisions for reasonable ability differences must also be made at the programming stage. These provisions refer to both students and teachers; the program must reasonably match the learning abilities of the students involved, and it must also deal with, in an adequate and reasonable way, the demands on teachers in terms of knowledge and pedagogical skill. Clearly, ability factors must be incorporated into the instructional theory if they are to be dealt with adequately at the classroom level. Finally, the theory must account comprehensively for practical implementation of the particular course.

Towards a cognitive theory of instruction

As Sternberg (1986) points out, two theories must be developed: one of cognition and one of instruction. Furthermore, the theories of learning and teaching must be sensitive enough to deal adequately with individual needs, but also be strong and dynamic enough to cope with the demands of a whole class group of learners.

For guidance in working with individual needs of students we are able to look for information from studies in maths education. Within the domain of algebra we are able to find data that describe varying levels of students' performance in word problems, linear equations, simple transformation processes and algebraic symbolism. The often detailed analyses of such task performance behaviour provides insights into the nature of the specific algebraic learning involved and indicates possible sources of errors. Studies that use mathematics and other structured knowledge-rich domains as a context for analysing the cognitive processes involved in problem solving performance yield data that link these ideas together. Thus a framework of knowledge can be constructed, allowing information to be seen in a related sense. For example, adequate and inadequate thinking may be seen on a continuum of thought and therefore comparative rather than being considered as categorically separate, similarities in cognitive processes can be identified within varying algebraic procedures, and the development of concepts can be approached in ways that recognise commonalities in the thinking processes required. The meshing of these two disciplines, then, provides an interrelationship that stimulates links between theory and practice, and a framework for supporting practical classroom ideas.

Finally, the underpinning cognitive framework must distinguish between student competence and student performance - what the student may be capable of doing, and what the student actually does in a particular situation. The distinctions need to be made explicit, and the links constructed. The theory developed must not only give a description of competence, but it must also give an explanatory account of how cognitive knowledge is acquired and changes in competence take place. Furthermore, performance must be charted and its acquisition and development clarified.

The main aim of this study, therefore, is to articulate a cognitive framework of instruction - to propose a cognitively-based set of psychological principles that account for classroom instruction in algebra, and to demonstrate through a classroom book how that translation might be effected. Follow-up tasks will involve the implementation of the algebra program in several classrooms, and a monitoring of its efficacy as an approach to algebra instruction in a classroom setting.

PROBLEM SOLVING: THE FOCUS FOR A STUDY IN CLASSROOM ALGEBRA

In an editorial in Journal for Research in Mathematics Education, Kilpatrick (1987b) identifies as significant goals for classroom practice and performance the skills of "questioning, examining, investigating, reasoning and solving problems", adding that "group student interaction" should be investigated as a possible means for facilitating proficiency in these areas. This is a statement of advocacy for an approach that sees mathematics as a context and framework for developing and applying cognitive skills in flexible, problem-solving situations. It is an approach that is consistent with instructional trends in a variety of subject domains where problem-solving attitudes are seen as important preparatory and functional means for living with technological complexity and change (Tuma and Reif, 1980).

Kilpatrick's comments highlight the differences between what might be called a 'rote-learning' focus or 'student-passive' approach in mathematics instruction, and the sort of skills, activities and expectations that characterise classroom problem solving. It indicates a contrast between instruction that emphasises performance outcomes at the expense of the processes involved, with one that recognises thinking behaviour as critical in the problem-solving activity. While understanding is clearly necessary for students to be engaged meaningfully in these processing activities, it does not overlook the significance of a desired product of proficient performance.

Historically, the tendency has been for performance skill and understanding to be seen as competing goals for instruction; the approach taken in this study sees them rather as complementary aims, and necessarily integrated in instructional methodologies (Resnick and Ford, 1981). Problem solving thus provides an appropriate context and perspective for an instructionally-oriented study that seeks to make mathematics a meaningful, accessible and readily applied area of knowledge for students.

However, while there may be general agreement that the development of problem-solving skills should feature prominently in educational contexts, there is, as yet, "no single homogeneous set of skills...identified as important" to assist instructional practice (Greeno,1980b, p.8). Research conducted within the mathematics education community has taken on a more systematic approach, but it is still largely a-theoretical in nature (Lester, 1983, p.231) and far from articulating specific classroom approaches for mathematics curricula. Thus the need is for a framework that can identify critical areas for instructional attention and provide a theoretical base to support a coherent approach to implementation of problem-solving activities in mathematics classrooms.

However, Lester (1983) believes that the building of special theories of problem solving would be a mistaken approach. He suggests two viable alternatives - a search for models within the general problem-solving domain that identify distinctions between mathematical and other problem-solving situations, or a framework that considers problem solving as an essential feature and a natural and potentially productive approach to all mathematics learning (p.255). From this latter perspective, the learning of all mathematical concepts, procedures and strategies may be viewed as problem-solving activities. This second alternative is an approach that is clearly compatible with the aims and purposes of the study so far outlined. This study focuses on classroom algebra learning as a significant cognitive activity involving the knowledge and application of a system of algebraic principles. By approaching the learning and use of algebraic concepts, procedures and strategies through ways that engage students' thinking processes, it will be argued that the relevant activities can constitute problem solving for the students concerned. It is a focus that sees problem solving in terms of students thinking mathematically, often thinking aloud and in a group situation, recording the processes and procedures used, and reporting the results of their thinking. However, such an assertion calls for a clearer statement as to why such activity can be termed problem solving, and it must be supported by a description of what this activity means in a practical classroom context.

To begin with, the term problem solving must be clarified, considering the consensus of opinion within the research context, and then its meaning defined for this study. An overview of problem solving follows in Chapter 2, placing problem solving in a context of across-subject research, and addressing in particular developments that have a direct bearing on classroom practice in mathematics, and specifically in algebra.

Chapter II

CHAPTER II

PROBLEM SOLVING

A key impediment to organising and synthesising research findings in mathematical problem solving has been the various interpretations placed on what constitutes problem-solving behaviour; one cannot rely on a clear-cut common usage.

What is a problem?

In a survey of classical literature on problem solving, Resnick and Glaser (1976) comment that despite the theoretical diversity, in psychological terms, there is a generally shared view that a problem

> "refers to a situation in which an individual is called upon to perform a task not previously encountered and for which externally provided instructions do not specify completely the mode of solution. The particular task, in other words, is new for the individual, although processes or knowledge already available can be called upon for solution." (p.209)

Their definition is clearly task based, compared with Polya's (1957) earlier more general description of problem solving. Capturing the flavour of highly motivated human responses to complex problem situations, Polya sees a problem as a conscious

> "search...for some action appropriate to attain a clearly conceived, but not immediately attainable, aim" (p.117). "To solve a problem" he goes on to say "means to find such action."

Mayer's (1983) summary emphasises the computer-characterised transformation of a problem through successive, goal-directed states. Critically, the transformation activity engages a search for relevant and efficient ways to accomplish the change.

Looking at problems in mathematics from this perspective, Kantowski (1981) specifies

> "a situation that differs from an exercise in that the problem solver does not have a procedure or algorithm which will certainly lead to a solution" (p.113).

In addition, she points out that both the process (or set of behaviours or activities that direct the search for the solution) and the product (the actual solution) are essential components of the problem-solving experience (Kantowski, 1977). Clearly the search and the solution must involve the use of mathematical concepts and principles.

For a study in problem solving in algebra, it seems then that while general procedural considerations are important (that is, the step-by-step processing of goal-directed change), domain-specific characteristics must also be identified. For example, many algebraic problem solutions are characterised by strategies that follow an algorithm, that is, there is a given set procedure available that ensures an appropriate solution. Kantowski's definition clearly emphasises the problem solver's knowledge as critical in determining the status of a problem. The student who reproduces a known solution procedure is not problem solving, but if an algorithm is not known by the student, in his/her eyes it most certainly is a problem, that is, the solution process is not immediately known. Conversely, the most complex of problems (for most people) may be solved in a relatively routine manner by someone who has faced the problem or similar problems many times before. Thus direct or implied reference to a 'problem solver' must form an integral part of an attempt to define a problem.

The problem solver

Clearly, the 'subjective' aspect of a problem is significant; for the individual involved it must constitute a problem, at that particular time. What may be a routine situation for one person may be a problem-solving experience for another. Lester (1983, p.232) calls attention also to the necessary motivational aspects of the problem solver's behaviour; that is, the student must want or need to find a solution, and make an attempt to achieve this. Furthermore, the very fact that a problem involves a procedure that is not readily accessible suggests that the problem solver may experience failure initially, and may reach a solution only with considerable cognitive effort. The difficulties that these aspects add to classroom instruction are obvious.

However, it is the perversity of the instructional problem that has led many researchers to analyse further the nature of problems and the behaviour of humans engaged in a problem-solving task, for it is the identification of these characteristics (and their commonalities or meeting points) that may lead most productively towards finding ways to enhance problem-solving performance.

Nature of problems

Simon's (1973, 1978) investigation of problems distinguished between them on the basis of apparent structure. He has categorised problems as 'well-structured' if a solution requires the information contained in the problem statement and perhaps other information stored in long-term memory. In Simon's information-processing terminology, long-term memory refers to a storage of human memory that contains information about past experiences, including information involving procedural knowledge as well as conceptual knowledge. 'Ill-structured' problems (Simon, 1978) are more complex, and require one to rely more extensively on long-term memory or to go elsewhere for information. Thus they have fewer criteria for determining when the solution has actually been reached. One is likely to encounter problems that are ill-structured in, for example, socio-political and economic domains. Simon (1973), however, points out that ill-structured problems are often solved by being simplified into a series of sub-problems that are well-structured. Greeno (1976, 1980b), citing geometry problems that require construction lines in order to prove a proposition, suggests also that the distinction between well-structured and ill-structured is not always sharply defined.

Structured problems and kinds of thinking involved in problem solving

It is important for this study to pursue some further clarification of Simon's definition of 'structured' problems; school algebra problems usually are included in this category. To assist this, Greeno (1976, 1980) usefully distinguishes between productive thinking and reproductive thinking. Borrowing his terminology from the Gestalt school, he defines reproductive thinking as that which concerns processes of a fairly limited 'recall' nature; he categorises it as thinking that is involved when the solution plan is an algorithm retrieved from long term memory. Frederiksen (1984) aligns the term 'well-structured' with problems that are clearly formulated, for which an algorithm is known, and for which there are readily available criteria for testing the correctness of a solution (p.366). Productive thinking, on the other hand, uses processes that require more extensive skills and resources. Thinking that is productive will be associated with problem-solving procedures that are constructed from propositional or semantic elements, or when the structural properties of the problem representation must be reorganised, or new features must be added. Frederiksen identifies these processes with his 'structured problems requiring thinking' (p.367). Compared with well-structured problems, the additional thinking requirement is that the problem solver must generate some step or procedure in order to reach a solution.

Classroom Implications

This clarification of problem structure on the basis of thinking processes is relevant to an investigation of the problems faced by students in a school classroom. For example, Borasi's (1986) analysis of problems gives a useful overview by drawing attention to factors that affect students' thinking processes. She sees

(1) the formulation of the problem - the explicit definition of the task to be performed

(2) the context of the problem - the situation in which the problem is embedded

(3) the set of solution(s) that could be acceptable for the problem given

and (4) the methods of approach that could be used to reach a solution

as structural commonalities influencing problem solving, and therefore relevant to classroom instruction.

In addition, Borasi's problem classification of

. exercise (involving known algorithms)

. word problem (involving known algorithms)

. puzzle-problem (involving the elaboration of known algorithms)

. proof of a conjecture (involving the elaboration of known algorithms)

. real-life problem (involving the creation of a model)

. problematic situation (involving problem posing)

in terms of structural elements may relate usefully to the algebra problems that form the practical component of this study's application of cognitive psychological theory to classroom teaching.

Classroom problem solving in algebra

For this study, algebra problems may be seen in terms of structural elements, but also ultimately in terms of the degree of sophistication of the problem solver's knowledge at a particular time - possibly both general and specific knowledge. The cognitive processes involved - for that individual - determine how the problem is seen. In school, algebra students are often asked to solve problems that some may view as essentially algorithmic or routine, using strategies that are followed in a reproductive sense. In these problems, students are normally asked to recall the procedures of a previously taught problem and to apply them in an identical type of situation. Variations may occur only in a relatively superficial way, such as a change in quantity or a difference in the ordering of an operation or procedure. These classroom exercises are usually seen in terms of reinforcement and to consolidate students' performance in an area (eg, Brownell, 1935; Larkin, 1981; Greeno, 1987). However, for many students these <u>are</u> problems. Algorithms in school algebra involving

-numbers representing variable measures

-relationships expressed in a variety of forms

-a diversity of structures of relationships

and -different problem solving contexts

are complex to manage. Difficulties in the recall of these processes can be instrumental in transforming a task that may be seen as routine by the teacher into a problem that, for the student, requires productive thinking.

In discussing this particular issue, Greeno suggests that

> "It is seriously misleading to label performance in some situations as problem solving and in other situations in which the same kind of cognitive processes occur as not involving problem solving" (1980b, p.12).

Arguing from a perspective that views knowledge-based performance as virtually indistinguishable from problem solving, he makes the point that

> "when we carefully consider the performance that occurs in more routine situations, we find that the essential characteristics are there also" (p.10).

Thus it would seem that the difference in performance between Borasi's 'exercise' and 'problematic situation' represents essentially a development of cognitive demand for the problem solver, and one that is rooted in an particular knowledge base.

Classroom problem solving in algebra: a definition for this study

For this study then, mathematical problem solving refers to an interaction of thinking processes and the individual's conceptualisation of the problem, in a way that engages the person in productive performance. Simply put, it means to think mathematically; and it is therefore a necessary part of mathematics learning.

The school algebra program associated with the study includesproblems involving combinations of 'routine' concepts and procedures,

-that must be recognised as such (that is, each section of the problem is classified according to a known principle or idea)

and -require selection, ordering and linking of these ideas for solution.

Problem solvers may be required to formulate problems, to dissociate them from context, and recall and select (or possibly devise) appropriate solution procedures and approaches. It is emphasised that there may not be a unique solution or solving procedure (so that strategies used may involve sharing of ideas, decision making, and possibly back-tracking from false or inadequate moves), and that sometimes there may be insufficient information to render a solution possible.

The classroom task

Thus the classroom task is one that involves effective and productive recall and use of procedures, as well as one of enhancing students' ability to think productively and flexibly. In terms of Borasi's processes, the task involves formulating problems, making meaning of the context, searching for known solutions that may be applicable and deciding which of these are most appropriate. In many classrooms it is tacitly assumed that learners develop and enhance these abilities unaided. For some students this is undoubtedly so; their problem-solving approaches are characterised by an ability to relate and generalise mathematical ideas rapidly and broadly, and to use these ideas creatively (Kruteskii, 1976). But thoughtful and effective classroom teaching cannot assume these problem-solving capabilities. Rather, teaching approaches and classroom experiences must be designed to address the development - throughout the class [illegible] - of the integrating, generalising and productive skills that for some are a straightforward matter.

In this study, the practical application of psychological theory is interpreted and demonstrated in a classroom book called 'Talking Maths'. Within an integrated learning and problem-solving framework it will be argued that the development of effective problem-solving approaches and procedures requires knowledge of concepts and related procedures and their interaction, with skilled and informed teacher intervention facilitating and enhancing the process.

To provide an informal basis for a discussion of how problem solving development might occur, origins of the cognitive approach are traced and viewed in context along with earlier alternative perspectives on problem solving in mathematics.

EARLY DEVELOPMENTS IN MATHEMATICAL PROBLEM-SOLVING RESEARCH: A BRIEF HISTORICAL OVERVIEW

An empirical study of thinking and problem solving began early this century with investigations by the Wurzburg Group in Germany. (Mayer 1983) In simple experiments, they set subjects minor tasks (such as asking for a word that the subject associated with one initially offered), and studied the responses given. Their findings relied on subjective introspective accounts of individuals, rather than depending on observable data, and there was no articulated theory to guide their work; nevertheless these early experimental efforts did show that human cognitive processes could be studied.

Associationism and Behaviourism

The failure of these early studies to develop rigorous experimental methods and to formulate a theory led to two different reactions: associationism, and Gestalt psychology. The methodology of the associationists focused on performance behaviours exhibited by problem solvers as they investigated how one element in a problem-solving chain was associated with another. From an associationist view-point, thinking was a 'trial and error' application of pre-existing response tendencies; learning involved the reinforcement of links or associations between problem situations and possible responses. In his seminal book, "The Psychology of Arithmetic", Thorndike (1922) used this notion of mental 'bonds' to formulate a theory of learning. Translated into instructional principles, the theory supported the idea of teaching together bonds that, as his research suggested, were especially compatible. The methodological outcome was an emphasis on drill and practice, an approach to performance achievement that has had a significant and widespread impact on classroom practice.

Associationist principles were adopted also by the behaviourist school, and embedded in a learning theory that stressed the laws of association and conditioning resulting from interaction between a learner and the environment. Gagne (1968, 1974) utilised these laws in an instructional model, through describing "learning hierarchies" drawn from task analysis. The hierarchies constructed focus on observable skills that a learner must be able to perform in order to attain more advanced skills. Gagne argued that an intelligent pre-analysis of the task to be performed (in terms of behavioural objectives) ensure that learners will be guided through events of learning that lead to the "essential incident" (1974, p.26) - that is, when the state of 'not learned' changes to the state of 'learned' - and on to other culminating events of the learning act. Thus problem solving was seen in terms of externally demonstrated behaviour that could be facilitated through instructional objectives.

The Gestalt Approach

In contrast, the Gestaltists saw internal coordination as significant in the problem-solving process. The Gestalt idea of 'insight' stressed the seeing of a situation with 'new eyes'; through a restructuring of knowledge, past experiences could lead to the recognition of a relevant solution to a problem (Wertheimer, 1959). How the problem solver perceived the form of the problem task was of paramount concern. According to the Gestaltists, an innate tendency to organise immediate perceptions according to certain principles meant that problem solvers engaged in transformation processes of initial and subsequent 'Gestalts' (or 'configurations', or 'structures') of the task. Thus they introduced the important concept of 'process' in problem solving, focusing their attention on qualitative observation and 'thinking aloud' protocols (Duncker, 1945). In the cognitive processes involved in problem solving, two kinds of thinking were posited; productive thinking that organises knowledge in new ways, and reproductive thinking that involves the reproduction of old habits or behaviour. A novel solution to a problem is created through productive thinking, whereas reproductive thinking only involves applying past solutions (Wertheimer, 1959).

Stages in problem solving

Recent attempts to investigate how problems are reformulated into smaller problems or sub-goals has its origins in Duncker's protocols. Duncker's definition of a problem is significant in this regard:

"A problem arises when a living creature has a goal but does not know how this goal is reached. Whenever one cannot go from the given situation to the desired situation simply by action, then there has to be recourse to thinking. Such thinking has the task of devising some action which may mediate between the existing and desired situations. Thus a solution of a practical problem must fulfil two demands:

1. its realisation must bring about the goal situation
2. one must be able to arrive at it from the given situation through action."

In an attempt to answer two crucial questions

(1) How does a solution arise from the problem situation?

(2) In what ways is the solution of a problem attained?

Duncker set subjects a problem, recording their verbalised thought processes as they reported them during the solution process. Protocols of typical subjects led Duncker to conclude that problem solving proceeds by stages, with general solutions being reformulated in order to extract a principle - or functional value - of the solution. The final form of the solution in question develops as the principle becomes successively more and more concrete and specific (1945, p. 8). This consideration of internal relations among elements, stressing the coordination of processes (eg, discriminating, recognising, imaging, relating, retaining, recalling) clearly anticipated the now recognised importance of whole and part-whole relationships in understanding and cognition (Riley, Greeno and Heller, 1983).

The significance of prior experience

Drawing on findings in their experimental work, the Gestaltists made some important predictions about the significance of the problem solver's past experience. Luchins (1942) emphasised the negative effects of prior experience, in which thought - "einstellung" or "problem set" - may create "a...blind attitude towards problems; one does not look at a problem on its own merit but is led by a mechanical application of a used method" (p.15). Duncker (1945) likewise investigated how past experience could limit problem-solving productivity. His term 'functional fixedness' illustrates the idea of a persisting input-output relationship that tends to govern new situations. Duncker's empirical evidence indicated that problem solvers tended to retain a previously used (but not necessarily productive) procedure in a new experience, limiting the success of performance. This early work on over-generalisation of procedures is a significant precursor to research in 'alternative frameworks' (eg, Driver & Easley, 1978; Easley, 1984; Driver and Oldham, 1986). A study of learners' limited approaches to problems forms an important component of this study into classroom algebra. A related notion that this study considers is 'error correction' - the idea that problem solvers' errors can be used productively as a basis for developing knowledge and skill in a concept or procedure (eg, Rowell and Dawson, 1979, 1983; Easley, 1984). Duncker observed from empirical work that problem solvers did not always proceed to solutions in simple linear fashion, but often moved from one unsatisfactory attempt back to a whole new line of approach. He concluded, however, that this "retrogression" (1945, p. 13) could still represent a productive phase in the problem solving process. He argued that the necessity for a problem solver to try another approach represented a newly added demand - and therefore a new dimension - to the thinking processes involved. According to Duncker, the significant issue lies not so much in the individual's realisation of the error, but more in the recognition of <u>why</u> the procedure is incorrect or inadequate; the awareness of the ground of the conflict is the means that leads the problem solver to the correct, or at least more adequate, solution procedure. Duncker's work thus provided a beginning focus to three important classroom tasks that feature significantly in the instructional methodology of this study:

- the need to relate new work to knowledge already existing in a student's memory
- recognition of learners' intuitive notions and the role of conflict in learning, and
- the aspect of cognition that involves problem solvers' awareness about the nature of the problem.

A disadvantage in the Gestalt approach is that it was much too vague. Much of the work was based on intuition rather than being precise enough to be tested experimentally. Nevertheless, intuition can be a rich source of ideas, and

Gestalt psychology certainly has contributed some interesting and provocative notions about problem solving that is relevant to school algebra. Indeed, now that information-processing techniques are being used in problem-solving analysis some pertinent Gestalt ideas are being explored further, clarified and tested (Resnick and Ford, 1981).

An emphasis on Meaning

'Thinking' being represented in terms of schemata and their assimilation into mental structures added a new dimension to the Gestalt interpretation of problem solving. While the Gestalt perspective focuses on internal relations among elements of a problem, the schematic view involves an additional (external) process - that of relating the problem to knowledge that exists in an organised form in the problem solver's memory. In these terms, problem solving involves a process of assimilating a new situation to an appropriate and useful set of past experiences, or 'schema'; that is the problem is seen in terms of the individual's understanding of it.

The terms 'assimilation' and 'schema' were drawn originally from the work of Bartlett (1932), in which two fundamental ideas about human mental processes were proposed:

(1) Learning and memory: when new information is acquired, the new material must be assimilated to an existing schema, that is, it must be organised in a way that makes meaning. Information that is learned and retained does not duplicate

what is presented, but is dependent on the schema to which it is assimilated.

(2) Remembering and memory: the act of recall involves an active process in which an existing schema is used to construct information that is consistent with it. This process involving "effort after meaning" (Bartlett, 1932, p.20) indicates that memory tends to be based on general impressions, that is, it is schematic.

Ausubel (1968, 1978, 1980) similarly stressed that new ideas must be related to knowledge that the individual already possesses. His theory of learning posited the possession of "cognitive structures", representing knowledge in memory. If the learner attempts to retain new knowledge by assimilating it into the cognitive structure, this knowledge is made meaningful. Alternatively, an attempt to memorise an idea without relating it to existing knowledge results in this knowledge being remembered rote fashion. In similar vein, Skemp's (1971, 1978) relational and instrumental understanding indicates qualitative distinctions in knowledge structures. Relational understanding indicates a person's ability to know what to do and why, and is evidenced by performance that suggests productive thinking. Instrumental understanding is demonstrated by performance that may reproduce rules and procedures accurately, but the individual is unable to rationalise the strategies followed.

The practical component of this study emphasises that an individual's possession of knowledge that has meaning has a very specific relationship to that person's ability to solve problems successfully. Ausubel defines a problem situation as one in which there is a gap between a present state and a solution state, adding that a necessary first step for the problem solver is to make meaning of the problem-setting propositions. Further steps require the problem solver to transform subsequent propositions - to "mentally manipulate the image representing the meaning of th[e] proposition" (1968, p.70) - in order to reach a solution.

Problem Representation

This emphasis on meaning suggests that problem representation is a critical factor in the problem-solving process: if the individual is to construct a coherent representation of the problem, it must have meaning to that individual. It is evident that the nature of the person's problem representation influences the way that a problem is approached, and possibly the effectiveness of the performance. In the instructional setting, experiences that enable the problem solver to represent the problem in terms of familiar ideas are of significant concern. Learning by analogy is a method that has been used in many mathematics classrooms in an effort to facilitate such problem representations. This involves the mapping of knowledge from one domain over to the target domain, where it is applied to the task in hand. The representations or visual images of the problem structure and the connection between them may be constructed by the learner alone, or with the assistance of peer or teacher interaction. Efforts to develop children's conceptual learning by means of analogy forms a significant part of classroom practice in mathematics in the early years of schooling, and it is therefore investigated as a potential means to facilitate learning in algebra. An overview of relevant literature suggests these possibilities:

(1) Use of concrete materials

Brownell and Moser (1949) investigated the effects of assisting children to 'make meaning' of problems through the use of concrete materials. Children who were taught a subtraction procedure using bundles of sticks (that were rearranged according to the rules of borrowing and grouping into tens) performed better in delayed tests on different problems than those who were taught procedures verbally, and in rote fashion. Dienes (1964a, 1964b) designed materials in an effort to make concrete an extensive range of mathematical procedures and principles, while Bruner's (1968) instructional methodology included an "enactive" mode of representation, achieved by abstracting conceptual elements through the manipulation of appropriate materials. However, it is significant to add that evidence is offered (Lesh, Landau and Hamilton, 1983, p.280, p.293) also to counter the assumption that materials are necessarily a means to facilitate understanding in mathematics. Their findings from a study investigating the approaches used by children in problem-solving tasks suggest that the representational form found by the subjects to be most appropriate differs with the nature of the problem. Indeed the use of concrete aids appeared to hinder the performance in certain problems for some children. Physical representation is less flexible than mental representation, and therefore objects may not always appropriately model mathematical ideas. Other evidence has been offered to indicate that manipulation that is dependent on physical materials does not necessarily 'transfer' into consistent and adequate mathematical thinking (Hart, 1986). It is suggested, rather, that the critical factor in making ideas meaningful is the ability to make structurally significant translations among and within different modes of representation of a problem (Behr, Post, Lesh and Silver, 1983, p.102) - a connection that may involve more sophisticated cognitive processes than some methodology claims would suggest.

(2) The role of imagery

For Bruner, images performed an important bridging function between the concrete and abstract forms of representation. Although he saw this 'iconic' mode of representation as autonomous, he argued that the summarisation of action through images enabled the learner to fill in, complete and extrapolate from the action. Paige and Simon (1966) investigated whether the kind of visual representations used to solve mathematical problems influenced performance. In their study they found that successful solvers were more likely to produce <u>integrated</u>

diagrams, whereas non-solvers' diagrams were presented serially, or displayed information that was different from that given in the problem. They suggest from their findings that combining the strategies of physical representation and verbal translation is probably optimal in solving algebra word problems.

(3) 'Discovery' methods

Much interest has been shown in the idea that when students discover for themselves how to solve a problem (accepting that this 'discovery' may involve teacher guidance), their learning may be embedded in a more stable and substantial context than if they are given solution rules directly. Ausubel (1968) asserted that if the conditions for meaningful learning are met (a meaningful learning set, a logically meaningful learning task, and the availability of relevant established ideas in the learner's cognitive structure), then discovery learning will allow the learner to relate "a potentially meaningful problem-setting proposition to his cognitive structure for the purpose of generating a solution that, in turn, is potentially meaningful" (p.534). 'Discovery' approaches have been used as a basis for 'project' initiatives (eg, Nuffield Scheme) and in mathematics classrooms at junior primary school levels (eg, Biggs, 1972) in an effort to make children's learning more effective; claims of children's sudden insights into mathematical knowledge (the 'aha' phenomenon) have encouraged classroom practitioners to use this approach. However Wittrock's (1966) cautionary reminder that claims for discovery learning have rarely been substantiated by evidence must be recognised in the development of a classroom framework for algebra learning and problem solving. As Carpenter (1986, p.122) points out, while simple addition and subtraction concepts are frequently acquired without explicit instruction, "more complex concepts are not as readily constructed without support."

In this study the possibilities of developing in students the ability to construct integrated representations of algebra problems are explored so that classroom ideas may be suggested. The significance of a matching between the externally represented problem and the internal representation constructed by the individual is recognised as vitally important; thus, the instructional theory must provide guidance and support for learners to make these links. The appropriate use of concrete and pictorial representations in possibility facilitating the construction of these connections is recognised.

However, the need is more particularly for detailed descriptions of the ways that representations of an <u>integrated</u> nature may be constructed in the human mind. We need to address the qualitative distinctions in an individual's representations that made learning meaningful and productive. Furthermore, it is necessary to analyse the nature of representations constructed by would-be problem solvers in order to find commonalities present in the ways that solution procedures are selected and followed. In addition, information is required to provide explicit links between problem representation and successful solution strategies.

With these needs in mind, the discipline of cognitive science is now investigated. This developing area of cognitive psychology describes cognition in information-processing terms; by using the computer modelling techniques of artificial intelligence, cognitive scientists are finding precise ways to simulate (and therefore describe) human cognitive processes. While the aims of artificial intelligence centre on the construction of computer programs that display intelligence, and only possibly consider psychological factors, cognitive science focuses particularly on the ways that learners represent knowledge, and how they use stored information to construct new thinking and ideas. It is this area of research that is seen as particularly relevant to a classroom-based study in problem solving.

CURRENT DIRECTIONS IN PROBLEM SOLVING RESEARCH

The development of information-processing concepts and theories

The beginnings

The contribution of Newell and Simon towards a theory of human problem solving is generally regarded as a landmark in the study of problem-solving behaviour. Their thesis (Newell and Simon, 1972) posited a set of processes (or mechanisms) which produce behaviour in a thinking human. In their studies, data from problem solving protocols provided knowledge about

(1) the demands of the task environment - through their observation of behaviour that met the full expectations of the task situation.

and (2) the psychology of the subject and the nature of integral mechanisms that limit performance - inferred through behaviour that departed from optimal rational (or adaptive) behaviour.

These two aspects were regarded as interrelated and interdependent.

Thus in the attempt to "explain behaviour" (1972, p.9), Newell and Simon saw it as essential to consider three distinct aspects of the task environment:

(1) the environment itself (the 'external representation'). This refers to the actual assignment, and the context in which it is offered - described in terms of the elements and relations that the subject is likely to use in encoding it.

(2) the representation of the task environment used by the subject (the 'problem space'). This is an internal construction of the possibilities involved, as perceived by the subject, using relevant information activated from memory.

(3) the theorist's 'objective' description of the task environment. This involved observing, and describing in formal terms, the behaviour of a motivated individual during a problem solving session.

In summary, behaviour was described as an interaction between an information-processing system, the problem solver and a task environment.

In developing their theory, Newell and Simon provided representations for two large classes of problems:

the set representation, in which the goal is to find a subset (with specified properties) of a given set

and the search representation, involving the transforming of expressions by designated operators.

For each representation the problem solver was viewed as having certain information, while the desired solution of the problem was seen as a request for specified additional information, acquired through cognitive processing mechanisms.

Problem Solving as "Search in a problem space"

(Newell and Simon, 1972, p.809)

Data collected by Newell and Simon from protocols of problem-solving tasks involving cryptarithmetic, simple logic exercises, and the choice of a move in chess, led them to postulate that problem solving could be regarded as a search process through an internally-constructed problem space. This principle was posited as a major invariant of problem-solving behaviour, holding across tasks and across subjects.

In response to this proposal, problem-solving research proceeded to model search behaviour and to verify that humans-at least to some degree-solve problems by searching a problem space for the means to reach sub-goals, and finally an end-goal. Evidence confirmed that systems operate serially, rather than in parallel, reflecting the limited capacity of short term memory - considered to be between five and seven chunks or units, based on memory span (Miller, 1957; Simon, 1974).

Means-end analysis and computer simulation

The connection between information processing theories and computers - which probably began with Turing's (1950) question "Can machines think?" - was made concrete by the construction of a system that solved problems only humans had been able to deal with previously. From an early program (called 'The Logic Theorist') Newell and Simon (1972) developed the General Problem Solver - a model that uses computer simulation to represent and analyse characteristics of strategies employed by a problem solver to generate a sequence of moves. The model consists of a programming device, a problem-solving core (consisting of a basic executive, and of heuristic methods that enable the executive to set up, evaluate and attempt goals), and a task environment that provides an interpretation of the workings of the core (Reitman, 1965). The heuristic search system finds differences between current and desired situations, then finds a relevant operator to reduce the difference.

In essence, the technique of computer simulation is an extension of earlier, more pragmatic work by Gagne (1968) in task analysis. Fundamentally, computer simulation requires the cognitive scientist to construct a hierarchy of task processes; the advantages of technological techniques enable a more rigorous and sophisticated prediction (and confirmation) of performance features.

In early studies using computer simulation for representing and analysing characteristics of means-end strategies, the usual domains were puzzles and games. Specific puzzle-problems, such as Hobbits and Orcs (Greeno, 1974; Thomas, 1974), Tower of Hannoi (Simon, 1975), the Water Jug problem (Atwood and Polson, 1976) and Missionaries and Cannibals (Jeffries, Polson, Razran and Atwood, 1977; Simon and Reed, 1976), used a simple application of means-end analysis, involving a series of logical steps that reduced differences between successive situations by a single complex operator called MOVE. The use of a single operator eliminated the need for a large memory search (Simon, 1980).

Production Systems: A formal description model

The information-processing model proposed by Newell and Simon produced

(1) a conceptual framework for problem-solving behaviour

and (2) a formal description of problem-solving behaviour - in terms of production systems.

Thus it became possible to compare detailed data relating to the problem-solving processes of individuals, and also to interpret and situate important research findings.

Within this framework problem-solving behaviour might be described as follows:

> The problem solver tries to comprehend the information in the problem, and to summarise it in a way that is personally meaningful. This requires the individual to analyse the data in the task environment and to match this as closely as possible to an internal representation of the problem, constructed from information present in memory. In this way information is coordinated to form knowledge states, which are situated on the nodes of the search (or problem) space. Information is stored in an organised and related form, so if some elements of the cluster of information are activated, probably all will be activated. Information present in the various knowledge states forms the 'frame of reference' that the problem solver can use to solve the problem. The problem solver makes use of this information in order to analyse the problem, and to determine the initial problem state and goal state(s). Operations are carried out on the initial knowledge state, with successive transformations of the problem into new problem states that eventually lead to the desired goal state.

The operations are determined by successive production rules. Production rules are rules that define and connect specific actions (operations) to conditions specified within the knowledge state. The construction of a production system involves fulfilling an 'if...then...' operation that controls problem-solving behaviour.

An adequate or correct solution is characterised by an adequate search strategy that is guided (or restricted) by the conditions that are derived from the knowledge states. Thus, for an adequate or correct solution the problem solver must possess declarative knowledge (that is, knowledge of concepts, rules, formulae, laws, algorithms) that is relevant to the problem, and sufficient to construct required knowledge states. This makes it possible for sets of conditions (to which the actions must conform) to be specified. If this declarative(conceptual) knowledge is inadequate or confused, it is unlikely that the system will be able to anticipate or derive relevant conditions. This means that the search process necessary for the construction and execution of a solution path - the procedural knowledge - will be inadequate. However, while sufficient pre-requisite declarative knowledge is necessary in the solution process, it does not ensure the specification of a condition for a relevant action or operation. Anderson (1983) considers that declarative facts can influence behaviour, but they only guide problem-solving behaviour if existing sets of productions provide the knowledge to allow a productive interpretation of this information. After the knowledge has been applied in this way a number of times, a set of productions can be compiled that applies the knowledge directly and automatically.

This description raises a number of issues that must be explored further in a study of classroom problem solving in algebra. In the following sections, research into problem solving in knowledge-rich domains (such as algebra) is reported, and recent advanced analyses of information-processing concepts of problem solving will be discussed - with particular reference to the increasing emphasis on the organisation of knowledge as significant in high-level cognitive performance. In particular, the interacting nature of declarative (conceptual) and procedural knowledge will be explored.

Complex Problem Domains

Insights from research in puzzle-type environments provided a useful starting point for the study of problem-solving processes. However, this early research was characterised by evidence of (and investigation into) general strategies in problem solving. The simple structure of puzzles, games and block-worlds adapted very well to means-end analysis, for example. Furthermore, these general strategies occurred in ways that were independent of a specific knowledge base.

Puzzle problems require information only in so far as it is stated in the problem task; they do not demand that the solver has in long-term memory a rich accumulation of information that is acquired over a period of time. Thus, responses generated from simple task problems did not adequately model the kind of strategies that might be needed to solve problems that were less constrained, nor did they consider the possibilities and requirements of solving problems in the context of a rich body of knowledge.

Nevertheless, knowledge gained in simple domains has enabled researchers to extend their work productively into arenas that combine an organised and well-defined structure with a knowledge base that is semantically rich and relatively complex. Larkin's attention to 'formal' domains (1980, 1981), and her studies into competency and the acquisition of skill in these domains, provide analyses that are useful to this investigation. Larkin defines formal domains as those

> "involving a considerable amount of rich semantic knowledge, but characterised by a set of principles logically sufficient to solve problems in the domain" (1981, p.311).

School algebra, along with other areas of pure and applied mathematics, the 'hard' sciences (such as physics and chemistry), and sophisticated games (eg, chess and 'go') provide examples of Larkin's formal domain.

Certainly, problem-solving studies in formal domains provide data that is interesting and pertinent in its educational applications. Investigations in the areas of chess (de Groot, 1966; Chase & Simon, 1973a, 1973b), architecture (Akin, 1980), textbook physics problems (Larkin, 1980, 1981; Chi, Glaser & Rees, 1981) and mathematics (Paige and Simon, 1966; Schoenfeld, 1983) indicate some of the variety of work that examines performance in formal domains. The functional utility of formal domains lies in their relatively taut and transparent structure; although the tasks studied are clearly more complex than the simple constructs of puzzle-type problems, they are made manageable for close scrutiny by a characteristic framework of relatively few general principles. This represents a marked difference from the sort of problems found in other semantically-rich domains (such as history, psychology, geography) where it is not so easy to formulate an unambiguous set of principles. While problems in formal domains are structured enough to allow intensive investigation, they also require that an elaborated store of knowledge must be available in (and accessible from) long-term memory. The typically 'organised' format of these problems, in addition, lends itself to specific research in knowledge organisation. Another interesting aspect of formal domains is that, despite limitations, they nevertheless suggest some of the flavour of real-world problems.

Towards a cognitive description of problem-solving performance in school algebra

In school algebra we are concerned with a domain that is well structured, yet rich in complex knowledge. While the rules and procedures of algebra are strongly defined and tightly governed, the principles formulated require a well developed store of knowledge for their application. It is necessary, therefore, to look further at studies that examine problem-solving behaviour in formal domains. The search seeks particularly a vocabulary of concepts, means of description, and clarification through analyses that together might provide an account appropriate to direct instructional practice. The proposed instructional ideas outlined earlier emphasise productive thinking and classroom approaches that involve flexible and informed practice. Thus the theories examined must suggest what the significant characteristics of problem-solving performance might be, and provide an analysis that is precise enough to describe in what ways these factors can shed light on instructional practice. Ultimately the aim is to link theory to practice in clearly defined and practical terms.

Chapter III

CHAPTER III

RECENT DEVELOPMENTS IN THEORIES OF HUMAN PROBLEM SOLVING

The importance of domain-specific knowledge

The nature of knowledge organisation in high-level cognitive performance is a common theme of recent literature in problem-solving. Many of these studies report investigations of different levels of performance in knowledge-rich formal domains. Computer models are particularly suited to this type of inquiry; by creating programs that model contrasting levels of skill in human problem-solving behaviour, key factors that appear to be similar to human behaviour can be identified and analysed. What researchers in both artificial intelligence and cognitive science are finding is that search techniques, no matter how powerful, are inefficient when it comes to complex problem-solving situations. Certainly, general skills such as matching, planning and searching are applicable as means to facilitate productive thinking, but alone they are insufficient when the tasks are premised on the possession of a body of knowledge. The problem solver uses these general cognitive processes as tools whereby new knowledge is constructed; but it is specific information that the processor uses as a reference or knowledge base for the construction of productive thinking. Thus, the development of problem-solving models must take into account the role of knowledge and its organisation; programs need to model transformation procedures as they engage with a coherent knowledge base.

Evidence that supports this emphasis on knowledge comes from contexts that are complex in content, yet clearly structured. The emerging consensus is that findings in these 'manageable' areas of study are showing the direction of problem solving in a wider set of domains. For example, Greeno's (1980) summary of recent progress in problem-solving research concludes:

> "All problem solving is based on knowledge. A person may not have learned exactly what to do in a specific problem situation, but whatever the person is able to do requires some knowledge."

Simon's (1980) thesis is similar:

> "Research on cognitive skills has taught us...that there is no such thing as expertness without knowledge - extensive and accessible knowledge,

while Reif's (1980) "cognitive engineering" also focuses on specific knowledge requirements.

A useful analysis of the 'knowledge' issue comes from studies examining the characteristics of skilled performance in chess. Investigations that compare players' perceptual responses to given chess positions have led Chase and Chi (1980, pp. 11-12) to assert that a "a large long-term knowledge base underlies skilled performance in several varieties of...domains". They go on to state that "practice is...the best predictor of performance", maintaining that practice within a specific domain can produce relevant conceptual and procedural knowledge. The mathematics educator, Kilpatrick (1985b, pp.7-8) concurs, reminding practitioners that

> "It is easy to underestimate the deep knowledge of

mathematics and extensive experience in solving mathematical problems that underlie proficiency in solving mathematical problems".

Comparing the current emphasis on knowledge with earlier search dominated approaches, he adds decisively that "people do not live by processes alone".

Problem-Solving Models

What are the particular aspects of problem-solving performance that have led cognitive psychologists and maths educators to identify coherently-represented and organised knowledge as critical in problem-solving performance? Programming a computer requires very precise calculations. So, if a programmer is to model the thinking behaviour demonstrated by humans in performing a task (as in cognitive science) or construct an intelligent way of solving a problem (as in artificial intelligence) each step must be identified very clearly; for each step represents a transformation of the original problem to a new problem, with each transformation representing a step closer towards the solution goal. While general goal-oriented search procedures characterise the overall process, successful solution strategies depend very definitely on specific knowledge that can initiate the process and maintain successive transformations towards the goal. Analysing this specific knowledge further, we can now see that two kinds of domain-referenced knowledge are necessary - knowledge that enables the construction of a coherent problem representation (understanding), and knowledge that facilitates successful performance. Clearly, both understanding (which involves conceptual knowledge) and performance (or procedural knowledge) constitute the knowledge base that problem solvers require. Consequently, both kinds of knowledge must be addressed in a discussion of problem-solving models; and in particular, the relationship between the two must be examined. A useful starting point for an overview and analysis of relevant problem-solving theories is to look at the ways that a problem task is begun. Newell and Simon's (1972) seminal work showed that the first step in any problem solving situation is the generation of a problem representation.

Generation of the problem representation

In a study using isomorphs of the Tower of Hanoi problem, Hayes and Simon (1976) investigated how subjects generate problem representations. In their model, called UNDERSTAND, the generation of a problem representation involved two sub-processes: interpreting the language of the instructions and representing the problem space. Problem solving, in turn, involved this understanding process plus a further process of exploring the constructed space in order to try to solve the problem.

In analysing the understanding process, Hayes and Simon found that a subject's representation of a new problem depends significantly on the way that the problem is stated. Subjects tended to adopt a representation that was derived most directly from the instructions given, so the particular wording was able to affect the difficulty of the problem for the problem solver. Their evidence also suggested that prior experiences may affect understanding, and in different ways (eg, syntactically or semantically) for different subjects.

These findings showed that an individual's understanding in the problem-solving process had to be dealt with explicitly in performance models - and therefore in instructional approaches. Until this time, what was known about human understanding was almost completely tacit. While it was quite common for educators to talk about the need to understand a maths problem, they were not able to to specify the means by which this attainment might be possible. Suggestions to read the problem carefully and to think about what algorithm might be appropriate was about all the teacher could do to facilitate student understanding. Therefore the Hayes and Simon study aroused great interest, and a spate of further investigations followed. Simulation models were developed, and they proved to be most effective in providing detailed knowledge about the understanding process (eg, Hinsley, Hayes and Simon, 1977; Clement, 1982; Chaiklin, 1984). For example, understanding of a word problem was shown to involve constructions of representations based on the words in the problem text. In particular, the words are recognised by the problem solver as patterns of information.

Similarly, in the domain of physics, Chi and her colleagues

(Chi, Feltovich and Glaser, 1981; Chi, Glaser and Rees, 1981) developed models that describe how prior knowledge is used in generating problem representations. They have shown that a problem representation may be constructed through encoding recognised features of the problem in a way that is interpretable by the information processing system. The information given in the problem statement is encoded in forms that match elements in the individual's knowledge structure; here again, patterns of information are recognised. It is the recognition of these patterns that allows one's knowledge to be applied to the problem at hand. Clearly, then, there is a need to analyse the patterning process with a view to identifying and clarifying the factors involved in pattern recognition.

Pattern Recognition

Studies that distinguished between the moves made by expert chess players and those that characterise novice behaviour had been the first to alert researchers to the importance of pattern recognition in thinking tasks. Early investigations into the choice of moves in chess led de Groot (1966) to observe that a grandmaster in chess might discover the correct move in a complex position within five seconds of being shown the position, a feat well beyond that accomplished by a less skilled player. Yet, when both grandmasters and novices were asked to recall random configurations of pieces that did not represent chess positions, memory capacity was much the same for both groups. From this evidence, de Groot concluded that perceptual processes - in particular, pattern recognition - formed an important role in problem solving. Continuing research on perception in chess (eg, Chase and Simon, 1973; Simon, 1974; Chase and Chi, 1980) has built up a substantial body of evidence to indicate that an important component of the grandmasters' skill is their ability to recognise a great many configurations of pieces in chess positions, and to associate them with patterns of information about appropriate actions (procedures). Chase and Simon have observed that the essential difference between the chess experts and the amateurs consist in the experts' possession of well-organised cognitive structures regarding the various chess positions. Their recognition, at a glance, of chunks of related pieces enables them to process information in coherent and confident ways. Chase and Chi conclude that "a very important component of the knowledge base is a fast-action pattern-recognition system...that greatly reduces processing load." So this evidence suggests that knowledge that is meaningful can be seen in terms of recognised patterns, or perhaps as a match between the features present in the task and available mental structures, while the performance of a successful move or strategy is facilitated by patterning that is constructed in such a way as to enable a large amount of processing to take place 'at one go'.

Chi, Glaser and Rees (1981) describe this operation as a categorising process. A problem is recognised as one of a type; then the problem is solved by applying existing sub-routines (present in long-term memory) that are known to be applicable for that type of problem. This schematic view is consistent with Nelson's (1988) findings in the language domain. She describes language in terms of categories of event-based knowledge which are continually reorganised to form hierarchies that are interrelated with ever increasing complexity. Performance skill increases with a person's ability to use categories, rather than contextually-bound events, as referents for thinking processes. This increasing abstraction from 'concrete' or 'event-based' situations - and their interrelatedness - leads to representations that support more sophisticated recognition and more powerful processing of information.

Representation of problems in terms of schemata

Indeed, much of the work on representation in psychology has been driven by attempts to construct theories of how people understand language (that is heard or read) by connecting words or sentences to already established schemata. Anderson (1983, p.5) defines a schema as "an abstract structure of information". He goes on to say: "It is abstract in the sense that it summarises information about many particular cases. A schema is structured in the sense that it represents relationships among components." Within these abstract structures, Schank and Abelson (1977) have identified a subset of schemata that is simpler, yet well-structured enough to permit more focused analysis and experimentation. Their concept of 'script' focuses on the understanding of prose dealing with mundane events, that is, the understander is hypothesised as possessing conceptual representations of stereotyped event sequences. Thus a statement may activate a script so that events are seen in relation to familiar contexts rather than in isolation. As a result, the events are understood; the associations are connected meaningfully through causal "enablements" between script events (Ableson, 1981, p.717). Abelson asserts that such a category is "defined not by absolutely critical attributes, but by family resemblances among its members", that is, by relevant scripts. He continues

> "There is a family resemblance among the many realisations of a given script because they tend to share events in common, though not necessarily always the same events. The most prototypical realisations of a script are the ones that embody the most commonly experienced events" (p.725).

This revisionary orientation in category theory can be applied directly to algebra problems of the type presented in school maths courses. This study includes problems that involve classification of the problem according to at least one (and usually more that one) problem type for which an algorithm is known.

Hinsley, Hayes and Simon (1977) have evidence from students' responses to algebra word problems that knowing (not to be confused with rote-learning) means that a conceptual structure is in place for the individual. They observed that when brief sections of algebra word problems were read to subjects, the phrases alone were sufficient to activate in people an entire description of the problem, and even part of its solution. They found that students possessed knowledge structures representing a number of distinct types, and that problems were rapidly classified as belonging to particular categories, in this case through recognising familiar problem verbal phrases. For example, students who were read a few words such as "A river steamer..." were able to categorise the problem quickly, indicating also that an appropriate algebraic formulation was close at hand. It may be deduced that the students had, stored in memory, a formal data representation structure - schematic knowledge - that could be retrieved quickly and efficiently for problem-solving use.

However, the retrieval of mathematical knowledge involves a constraint not usually encountered in other domains. In the internal organisation of schemata are variables for which specific values will be sought from the input data: when present input does not provide enough information to permit certain variables to be filled, it is likely that the variable slot will be replaced by the best available information gleaned from past experience. It is a particular difficulty in mathematics that approximations for variables are not satisfactory: every matching must be complete and precise.

High- and low-level performance in problem solving

An additional factor must be considered. Evidence offered by

Chi, Glaser and Rees (1981) demonstrate that while there are similarities in general problem-solving strategies in high- and low-level performance on mechanics problems, there are crucial differences in the nature of representations of the problems. In the investigations carried out by Chi, Glaser and Rees, novices and experts in physics were required to categorise problems on the basis of similarities in methods of solution. It was found that the novices tended to identify problems on the basis of <u>surface features</u>, while the experts categorised problems on the basis of the <u>fundamental principles</u> of physics that were involved. It was concluded that the greater knowledge and experience of the experts enabled them to represent a problem in terms of a schema that contained both the conceptual and procedural knowledge necessary for solving these particular kinds of problems. The experts' representations of conceptual knowledge included, in addition to literal details of the problem, relevant inferences and abstractions derived from knowledge in long-term memory; their procedural knowledge gave them the ability to recognise how a knowledge structure could be manipulated and the conditions under which it could be used appropriately. Thus, for the experts, the two kinds of knowledge interacted in a way that enabled them to recognise and manipulate available information.

Simon and Simon (1978) have reported further evidence on expert-novice differences. They observed the problem-solving behaviour of two subjects - one possessing both knowledge and experience in physics, and the other with adequate but not extensive knowledge in the area. When the two subjects were faced with identical kinematics problems, the novice problem solver used the more general, weaker strategy of means-end analysis, while the expert seemed rather to translate the problem into a physical representation, which then cued the application of appropriate principles. The expert was thus able to avoid the inefficiency of means-end analysis, and to solve problems with confidence. What was it that enabled the expert to operate so much more proficiently? Simon and Simon concede that the 'practice effect' is important; a skilled subject benefits from the extensive amount of experience working in the area. But they see as most important the <u>organisation</u> of knowledge. They suggest that a coherent body of knowledge, present in the expert's long-term memory, provides the basis for the development of 'intuition' - a somewhat elusive component of skill that often enables an individual to operate effectively with maximum efficiency and minimum effort. Simon and Simon see 'intuition', in respect to a problem, as the possession of a representation that makes explicit "the main direct connections, especially causal connections, of the components of the situation" (p.337). The 'expert' approach to problem solving is seen as involving the construction of a schema that represents essential relations in the situation, then sets up operations that correspond to this representation. In contrast, a novice's representations lack integration, tending to be based on superficial elements of the problem.

Larkin (1980) has called this physical intuition 'qualitative analysis'. Of importance for school mathematics is her suggestion that qualitative analysis - applied before physical equations are retrieved and used - may constitute a representation that is based on concrete physical referents. Paige and Simon (1966) suggested this possibility when they found that subjects changed physically unrealisable situations to invoke solutions that were meaningful, probably by imagining physical referents of the objects mentioned. Simon and Simon (1978) have also reported evidence that physical representations provide a basis for generating equations, and a situation for checking errors. Chi, Glaser and Rees (1981) argue that physical representations provide a concise and global description of a problem and its important features, permitting inferences to be drawn from the problem representation that were not present in the problem statement.

In an experiment requiring physics experts and novices to think aloud while they solved mechanics problems, Larkin (1981) found that the physics principles that experts recalled and used were part of large-scale coherent units of knowledge. She concludes that a very important part of skilled problem solving involves constructing good problem representations, particularly qualitative representations. Riley, Greeno and Heller (1983) reached a similar conclusion in their study of young children's problem solving in arithmetic. Their analysis revealed that children whose conceptual understanding generalises across a range of tasks possess schemata in which conceptual and procedural knowledge are integrated, and act as principles for organising the information in a problem. In summary, it seems that model problem solvers have at their disposal, and are able to activate from memory, "large-scale functional units" - related pieces of information, stored in a coherent form (Larkin, 1980, 1981).

Problem representation in terms of abstract algebraic principles

The above contributions from cognitive science have been paralleled in a number of investigations involving problem solving in algebra. For example, several studies have demonstrated the ability of good problem solvers to construct representations of the essential structure of the problem and to disregard superficial elements. (Krutetskii, 1976; Burton, 1980; Rosnick and Clement, 1980; Clement, 1982). Clement's oft quoted experiments with college students found that the statement "There are six times as many students as professors at this university" invoked the correct equation $S = 6P$ for students who were able to dissociate the problem from intuitive, contradictory representations.

In lengthy and wide-ranging studies of "capable", "average" and "incapable" mathematics students, Krutetskii (1976) found that capable students showed a particular ability to generalise mathematical ideas "on the spot". Krutetskii refers to effective generalisation as a double-edged skill in knowing from past experiences _what_ procedure to apply and _where_ to apply it; it is a process of abstraction that leads the problem solver to recognise the generality behind contextual details, and to see "the deep inner essence of phenomena behind an external design" (p.240). Krutetskii cites examples of students solving algebra problems to reveal the decisive role of generalisation in proficient problem solving. One such example involves a student who, given a formula for the expansion of the perfect square of a binomial, was able to recognise that the expansion of $(c + d + e)(e + c + d)$ involved an identical procedure. Studies by Silver (1979, 1982) and Vergnaud (1983) similarly demonstrate the importance in mathematics of skill that enables problem solvers to recognise different problems as isomorphic because of references to identical mathematical structures or to an identical operational conception.

Problem Solving Strategies

Thus we find that problem solvers who use successful strategies follow a pattern of behaviour that is consistent with recognising and applying the underlying principles of the problem. But how are these strategies applied? Greeno's (1976) observation of students solving congruent triangle problems in geometry led him to conclude that students may engage in a recognition process that is preceded by some planning at an abstract level rather than in a concrete 'problem space'. His findings showed that although the students used means-end strategies, the problem-solving activity consisted more of a search for information - replacing unknowns by known values - rather than a search to reach a particular goal. Similar strategies have been observed by Larkin (1980, 1981; Larkin, McDermott, Simon and Simon, 1980) in subjects solving physics problems. Her computer-implemented model ABLE operates on the principle that learning occurs through the system's enrichment of its knowledge of abstract principles. In comparison with the barely ABLE model, the 'expert' knowledge-development model ABLE is dominated by strategies that use the specific information it has learned rather than by (weaker) general, domain-independent strategies. ABLE's strength is seen in its capacity to recognise patterns of information already in a problem, thus facilitating the construction of a simplified problem. The planning technique involves solving the constructed, abstracted problem - which then acts as a guide to solving the original problem. Furthermore, 'overlearned' pattern-recognition procedures serve to simplify and make more efficient the problem-solving process. This involves combining the selection and application of a principle, allowing the computer model to collect necessary information and use it to generate new information, all in a single step.

The question that arises next is: can this modelled 'behaviour' be translated to instructional advice for a school classroom? What are the factors that are crucial in designing algebra instruction that aims to develop students' productive thinking and enhance their problem-solving performance?

Problem-solving instruction

Davis (1983, p.268) points out that complicated problems in mathematics require that students engage in two separate activities - a planning or search/recognition procedure, and a calculation process. In line with models developed by cognitive scientists he proposes two main requirements in the problem-solving process: there is a need for an understanding of the problem situation, and there must be, within access, the skill necessary to perform the task in a way that is appropriate for and acceptable within the laws of algebra.

Undoubtedly, general content-independent procedural knowledge plays a role in organisation and in selecting and using relevant strategies. However it is clear from the findings discussed in the previous sections that in a specific knowledge area, recognition and calculation processes depend on 'matchings' in or with knowledge structures already existing; that is, the problem must be understood. Clearly then, the relevant mental structures belong to a specific knowledge base; so a critical factor is the problem solver's stored information within a particular area.

Rowell and Dawson (in press) summarise this position, drawing together supporting evidence from studies in artificial intelligence and cognitive science. Furthermore, they offer evidence from tests based on Wason's (1966) problems to demonstrate that students' experience can be a significant factor in solution procedures, and to support a claim that content-specific procedural schemata have the potential to render general procedures unnecessary when problems of a particular type are re-encountered. Rowell and Dawson cite a study in which graduate students improved scores markedly in a problem which retained the same structure as a previously assigned task, but which replaced the abstract elements with thematic variables. Thus, for the thematic problem, students were able to draw successfully on experiential knowledge, enabling them to reason in a way that was not evident when the problem was presented in abstract terms, dissociated from their own experience. A conclusion that is significant for this study is that effective performance involves the application of available knowledge that has its roots in a coherent knowledge base.

Thus, if human knowledge is organised, not around general processes, but with reference to specific content-experience, it is clear that a learner cannot be proficient in algebra without the development of appropriate (algebraic) knowledge structures. For in order to assign meaning to an algebraic situation or idea it is necessary for the student to assimilate it into these knowledge structures. This, in turn, involves reconstructing the situation or idea in terms of available algebraic knowledge - knowledge that may be intuitively used or explicitly conceptualised. Instructional design, therefore, must be premised on students' need to construct coherent (and therefore potentially usable) knowledge within the algebraic domain. Furthermore, it may be necessary to find the means to teach students how to access that knowledge, and to use it in effective and efficient ways.

Thus there seems to be a mutual dependence between algebraic knowledge and problem-solving performance in algebra. The classroom emphasis must thus be on interrelating the concepts and procedures used in algebra. Understanding and performance represent complementary and interactive goals of instruction, not outcomes that must compete for status. Therefore instructional methodologies need to be aimed at underpinning an integrated understanding/performance approach to algebra learning.

Understanding and Performance in Mathematics

Together, studies in cognitive science and artificial intelligence provide a language and a body of experimental methods that address both understanding and performance skill in mathematical problem solving. In particular they provide the means for analysing and clarifying how already-constructed knowledge is used to solve problems. By means of psychological descriptions of human problem-solving behaviour, and the development of programs that process information 'intelligently', the acquisition of knowledge and the relationships between different kinds of knowledge may be investigated and explored-with classroom instruction in mind. The provision of increasingly detailed descriptions of knowledge acquisition and its development are seen, within the perspective of this study, to provide a sound basis for developing instructional programs that aim for improved understanding and performance. In this context, the development of conceptual and procedural knowledge must be analysed, and the relationships between the two kinds of knowledge explored.

CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Conceptual Knowledge

In distinguishing between conceptual and procedural knowledge, Hiebert and Lefevre (1986) suggest that

> "Conceptual knowledge is characterised most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information."

They propose that conceptual knowledge develops through the construction of links between pieces of information. Unlike the bonding based on frequency of association as articulated within Associationist theory, these associations are formed through relationships based on meaning.

How these links are formed is described in terms of structures available in human memory. Hiebert and Lefevre suggest two main ways that these connections might be constructed:

(1) Relationships may be created between new information and knowledge stored in memory. Connections between the new material and existing knowledge are recognised, and the new information is assimilated into the appropriate knowledge networks. What is new now becomes part of the existing network, which, in turn, is enriched as a result. Piaget (1960; Vuyk, 1981) developed an 'equilibration' model to account for this process, using the terms assimilation (into the existing network) and accommodation (adjustment of the network to incorporate the new) to describe the means by which knowledge is acquired and developed. The phenomenon has been identified by Brownell and Moser (1949) and Ausubel (1968, 1978) as "meaningful learning", by Skemp (1971, 1978) as "relational understanding", and by Davis (1983, 1984) as mathematical understanding.

(2) Connections may also be recognised and constructed between information already existing in memory structures. Items are given new meaning when (sometimes quite suddenly) connections between them are recognised. This 'aha' phenomenon has long fascinated those who study and are involved in human learning; and it is a means in the construction of knowledge that can well be exploited by classroom teachers. Thoughtful teaching methodologies can

provide experiences and explicit opportunities for learners to be exposed to relationships in existing mathematical knowledge. Such opportunities may also be fostered by curriculum planners who present mathematics in an integrated form - a very different situation from the discrete topic chapters usually favoured by textbook writers.

Hiebert and Lefevre also point out that there are hierarchial distinctions between the links made between pieces of mathematical knowledge. For example, a learner's thinking may occur at a superficially obvious or perceptual level, making connections between items of information that certainly advance understanding, but are limited in extent in that they are tied to the context in which the connections are made.

On the other hand, higher-order connections that are constructed at more abstract levels are freed from the bounds of context. Piaget (1972, 1977; Vuyk, 1981) called this internalised knowledge the development of logico-mathematical structures or reflective abstraction. He considered it to be an abstraction at a level which may be validated by perceptual experience, but was in no way reliant on it. It is seen as freeing the learner to think at a level that transcends perception, enabling the learner to step back and reflect (mentally) on what is known. More recently, schema theorists (eg. Larkin, 1981; Chi, Glaser and Rees, 1981; Nelson, 1988) have described this knowledge in terms of principles that abstract the common features of superficially different information, and organise them into coherent, interrelated chunks. Psychologists working in a mathematical context (eg, Krutetskii, 1978; Riley, Greeno and Heller, 1983;) emphasise this process as an integration of separate elements of a mathematics problem into a significant, ordered structure that enables a learner to see the mathematical terrain from a vantage point, allowing further higher order relationships to be made.

Indeed, the possibility of learners actively viewing their own cognition from a vantage point has aroused a great deal of interest in the mathematics education community. This has been a focus point for a growing number of studies that seek to identify the roles played in problem solving of "an individual's awareness of his or her own cognitive processes" and "the active monitoring and consequent regulation and orchestration of these processes" (Flavell, 1976, p.232). Pioneers in the field, Flavell and Wellman (1977) identified person, task and strategy variables that are likely to influence a person's problem-solving behaviour. These aspects of metamemory involve belief systems, awareness of features of the task, and planning and reviewing skills that interact as "driving" (Schoenfeld, 1982; Silver, 1982) or "guiding" (Lester, 1983) forces in the problem solving process. The possibility of taking a 'meta-cognitive' approach to classroom problem solving will be addressed within the instructional methodology of this study.

Procedural Knowledge

Hiebert and Wearne (1986) identify two distinct parts of knowledge that make up the procedures used in mathematics. One of these is the symbolic representation system which is the means of communication for mathematics (of all levels) and a precise tool through which mathematical thinking may be facilitated. However, acceptable manipulation of symbols does not necessarily imply that the symbols have meaning for the learner. This is a concern that has been recognised and addressed by mathematics educators (eg, Brownell and Moser, 1949; Skemp, 1971, 1976), and it forms an important part of the instructionally-oriented components of this study.

A second type of procedural knowledge refers to the sequence of steps that are followed in order to complete a mathematical task. These steps may be prescribed, as in an algorithm or formula-directed procedure that involves operations on symbols; or they may form the problem representation and strategies that this study ascribes to problem solving.

An important aspect of the knowledge involved in mathematical procedures is that it is hierarchical in nature. Step-by-step manipulations may be seen as sub-procedures that are followed in linear fashion; a combination or chunk of these sub-procedures may however be accessed from memory as one super-procedure that reduces demand in mental processing. This linear sequence of steps lends itself particularly well to an instructional approach that is based on inductive reasoning (Polya, 1954; VanLehn, 1986). This method involves a learner investigating examples, recognising a pattern of commonality within the examples, and generalising. These processes allow knowledge to be integrated into the system. As such it is seen as an appropriate foundation for a curriculum methodology. Ideally, these cognitive activities produce learning that is systematically productive, either by utilising new knowledge or by avoiding errors.

However, as VanLehn (1986) points out, learning by induction is hardly adequate to describe the fine-grained, minute-by-minute learning in a classroom. Many motivational and social factors intervene. In addition VanLehn argues that as a form of inference, inductive reasoning is incomplete. In order to be useful, induction must be constrained by biases that enable the linking process to take place - to enable generalisation from and integration of examples. One such bias is the need to extract the simplest concept that is consistent with the examples given. Another necessary bias is for a vocabulary of primitive features with which to describe the examples. However, natural language provides a poor vehicle for describing mathematical procedures; even with concerted efforts at articulating procedures clearly, ambiguities often remain. Thus the biases and language necessary for inductive reasoning are not communicated easily to students, and misconceptions are acquired that frequently resist eradication. The presence of systematic errors in students' algebra (eg, Matz, 1980; Booth, 1984b) and arithmetic (eg, Brown and Burton, 1978; VanLehn, 1986) indicates that the construction of inadequate or false generalisations through induction is a problem that must be accounted for in an instructional methodology. Indeed, this tendency for young learners to delete, insert or substitute incorrectly forms the essential component of a growing body of research-based information that is variously labelled as "alternative frameworks" (Driver and Easley, 1978; Easley, 1984), "children's ideas" (Booth, 1982; Hart, 1986), children's intuitive thinking, and naive ideas. Classroom methods that seek to deal productively with misconceptions and errors in the formative stages are addressed in the instructional methodology in this study.

Relationships between Conceptual and Procedural Knowledge

Essentially, problem solving in algebra is concerned with conceptual knowledge involving knowing and understanding, and procedural knowledge that is applied to known ideas in order to reach a solution state. Through modelling problem-solving performance cognitive scientists are showing that the coordination of these two kinds of knowledge is important. Indeed, Glaser (1979) believes that "understanding the relationships between these two forms of knowledge...will provide the real payoff" for mathematics learning. Hiebert and Lefevre (1986) agree, emphasising the "crucial, interactive roles" that both play in the development of mathematical competence (p.23). In regard to problem-solving situations, Silver (1986) argues strongly that attempts to distinguish between a person's conceptual and procedural knowledge are ineffectual. He believes that the blurring of boundaries between the two and the extreme importance and complexity of their relationship give good reasons for focusing our attention away from their distinctions, and on to the interaction between the two. He asserts that "it is the relationship between the knowledge types that gives one's knowledge the power of application in a wide variety of settings" (p.183). However, the question still remains; how is the acquisition of skilled problem-solving behaviour - which distinguishes adequate from inadequate performance - explained? In addressing this problem it is useful to ask a second question: To what extent do computer systems model a cognition of learning and performance? In the next section two models that are representative of current approaches will be summarised.

Models of knowledge acquisition and development

In Rumelhart and Norman's (1981) schematic ASN model, knowledge is represented in unitary fashion, allowing the same knowledge to be both declarative (conceptual) and procedural at the same time. The acquisition of declarative facts ("accretion") occurs by way of fitting new items appropriately into existing schemata. Procedural knowledge (involving the creation of a new schema) is generated either through modifying an existing schema ("patterned generation"), or using one that is constructed through a 'best fit' matching (learning by analogy). The blurred distinction between declarative and procedural knowledge, leaves unexplained the automatisation phenomena associated with skilled performance. Furthermore the size and complexity of schemata makes it difficult to model the gradual progression that learning usually typifies (Anderson, 1983, p.39). Anderson (1980, 1983; Neves and Anderson, 1981; Anderson, Greeno, Kline and Neeves, 1981) posits three knowledge-referenced stages in the acquisition of problem-solving expertise:

a declarative stage,

a knowledge compilation stage,

and a procedural stage

during which knowledge is acquired or retrieved, converted into procedures, and used autonomously.

Anderson's ACT system is designed to provide an explicit division between procedural and declarative knowledge. Declarative knowledge is seen as acquired by direct cognitive encoding, potentially to be used in a multiplicity of ways; the representations are concise, and capable of generating many productions from a single statement of facts. However, the system's capacity to encode this knowledge depends on interpretive procedures that are relevant to the particular circumstance. Productions (procedural knowledge) cannot be directly acquired; unlike declarative units that can merely influence behaviour, productions have control over behaviour, and the system must be circumspect in creating them. So it is not possible to add a production in the way that it is possible simply to encode a unit. Rather procedural learning occurs only in executing a skill; one learns by doing (Anderson, 1983, p.215). Knowledge that is repeatedly used in the same way is subject to a stream-lining process, making the knowledge more closely tied to the source situation (that is, it is domain specific) but also more automatic in its application. Anderson refers to this process as 'knowledge compilation', the intermediatory function that eventually allows direct application of the knowledge without the interpretive step. Thus piecemeal applications become unified; procedural skill develops by collapsing multiple steps into one. This implies that procedural knowledge is specific to its conceptual source and available intact for use in further knowledge construction.

The above outlines show that both the ASN and ACT models are premised in knowledge that has been constructed by the system. Thus, for both, 'more circumstances' are fundamental to knowledge development. They are based in programmable (that is, predictable) possibilities and parameters, and as yet are not able to describe adequately the (harmonious) functioning of a system in which novel solutions become available and unforseen constraints arise.

In characterising learning "the discarding of less adequate strategies" (Kuhn and Phelps, 1982, p.40) must also be taken into account. The discussion so far considers learning and problem-solving only in terms of building on to what the learner knows. An adequate instructional theory must also recognise and provide for classroom situations in which students' knowledge constructions are based in limited thinking and misconceptions. Therefore it is necessary to add a further dimension to the framework being built. The task now is to account for classroom experience in which 'what the learner knows' must be <u>changed</u>.

Chapter IV

CHAPTER IV

LEARNING AS CHANGE

In the previous section it was argued that while the computational concepts and procedures of information processing psychology allow us to identify and analyse significant features of cognitive processes, they do not yet provide the means to chart adequately changes in capability and performance. For a study that seeks to develop an explanatory framework for classroom learning and teaching, this exposes a significant weakness: there is not present, as yet, a unifying and coordinating dimension that adequately accounts for the range of cognitive demands in classroom learning and teaching. Processes of induction characterise knowledge extension but they do not account for shifts in the way things may be perceived - that is, disturbances that result in a change in the quality and directions of thought and action.

An inquiry that focuses on classroom learning and practice must be concerned with what brings about change. Particularly, the dynamics of group interaction demand special considerations in regard to human knowledge acquisition and development. We need to know how classroom instruction may be approached and organised in order to bring about qualitative changes in thinking; and this includes making possible the assimilation of new thinking that may be in conflict with ideas already held by the individual.

Boden (1982) calls this "the development of harmonious novelties", in which "genuinely new structures are created out of older ones without any impairment of the overall integration of the system" (p.165). Is some such interpretation of learning and development available elsewhere, even in limited and fragmented form?

Piaget's Genetic Epistemology

The work of Piaget is examined as a possible contribution towards a more adequate understanding of knowledge acquisition and development - with special reference to classroom algebra learning. Piaget used structures to characterise human thinking, maintaining that knowledge involves essentially systems of transformations: humans can know something if (and only if) they can construct and transform it. From an interactionist perspective, Piaget argued that changes that occur in the human mind, enabling it to develop through progressively enriched (more sufficient) states of knowledge, result from actions: actions on things and interaction with people. He saw this development as characterised by increasingly advanced forms of abstract reasoning.

Working within a framework which he called genetic epistemology Piaget used philosophical methods to identify the critical areas for study, applying empirical means to investigate them more fully. In particular, he used rational analysis to identify logical conditions that hold between successively transformed states of knowledge, while empirical methods were used to establish the temporal aspect of knowledge change. Conditions identified as those characterising adult competence were used to guide further investigations into the 'knowing' that necessarily precedes the logical use (requiring 'adult' understanding) of a concept. So we find, for example, on Russellian analysis, that the basic mathematical concept of 'number' (Piaget, 1952) is defined in terms of certain logical notions such as class, correspondence and asymmetrical relationships. Such an inquiry centres empirically on whether or not Russellian analysis fits the facts. The question "Do children understand number?" is transposed into the subordinate questions "Do children understand class, correspondence and asymmetrical relations?"; and they are answered in respect to what Piaget claimed to be an <u>adult</u> capacity for understanding. In addition, Piaget invoked a traditional philosophical method in analysing a concept in terms of its logical conditions, that is, by considering whether the conditions identified (and, for Piaget, also tested) were deemed necessary or sufficient.

Piaget: findings relevant to this study

In order to verify empirically the conditions extracted from his philosophical analysis, Piaget formulated the method of critical exploration. This approach provided a built-in flexibility, unlike the tightly controlled statistically-based procedures that had been traditionally used in psychology. The format of this methodology is governed by subjects' responses to an initial question asked by the researcher; the answers given and the actions observed are recognised as reliable and valid data. From this philosophical/empirical approach, evidence was established for a mechanism that was seen to govern the growth of intelligence. Piaget called this mechanism equilibration. It is characterised as an organising process, involving interaction between the system and the environment. This interaction is described in structural terms - as a representation in which component parts are considered in relation to the whole. For Piaget, structures are humanly constructed, giving them a dynamic, responsive quality. Intelligence (human knowing) therefore develops through people's need to make interactions with the environment meaningful and coherent, that is, via conceptual development. For this to be possible, human knowledge structures reorganise in response to interactions. This reorganisation is an enriching process, generating a more coherent, elaborated structure that is now in a position to meet new situational demands. In this context it can be seen that Piaget's earlier formulations saw adapted action (development of procedural knowledge) as a developmental factor that he did not specifically analyse.

Piaget used the terms 'assimilation' (referring to the adaptation of the new situation to mental structures possessed) and 'accommodation' (the adaptation of mental structures to the demands of the new situation) to signify these twin regulatory/growth processes. Specifically he saw the adaptive interaction of the system and the environment (that is, simultaneous assimilation and accommodation) as being a process such that the system maintains integration within changing structural parameters. For Piaget, this signified a development that is general, that "intervenes in every hereditary or acquired process, and intervenes in their interactions" (1958, p.836). Thus he saw equilibration as an explanatory concept, with the potential to unify all forms of development.

Equilibration is thus offered as a concept to describe cognitive development, while structures form the underlying framework for this account. It may be helpful at this stage, therefore, to clarify further Piaget's understanding of the nature and function of structural change.

Mental Structures and their development in Piagetian terms

Piaget interpreted evidence of increasing sophistication of human thinking as the ability to take into account an increasing number of variables in any situation, and to recognise how transformations in one part of the system will affect others. For example, young children may view a situation or problem in an uncoordinated way, dealing with superficial features of the situation, and as separate entities; their responses are characterised by perceptual approximating and unsystematic judgements. Older children, faced with the same task or situation, are able to respond in a more measured, comprehensive way. They are able to see the problem as a whole, in context, and in terms of its related parts. They are able to consider several features at once, and to relate them through organised perceptual and mental actions. This more advanced level of capability is characterised by transformations involving operations: it involves the ability to make and undo transformations through mental actions and to think in terms of more than one dimension at a time. In his earlier formulations Piaget saw this capability as a turning point developmentally: he claimed that operational thinking was signified by a marked advance in logical thinking. This advance enabled a child to demonstrate, for example, far more sophisticated behaviours in mathematical tasks.

For Piaget, this development culminated in capabilities that he saw as characteristic of formal thinking. This kind of thinking enables an individual to hypothesise and to take into account all logical possibilities inherent in a problem or task. It is thinking that can be dissociated from perceptual reality; that is, to think at increasing levels of abstraction. It is thinking, none-the-less, that has its <u>roots</u> in perceptual activity. This view, thus, cannot be commensurate with, for example, classroom practice that involves passively noting or rote memorising information that is presented to a learner. Learners must necessarily be involved in an active, thinking engagement with the environment, for actions - mental and/or physical - are fundamental to Piaget's understanding of 'thinking', and are therefore an integral part of structural development. Any attempt, therefore, to use Piagetian formulations as a basis to instructional methodologies must emphasise the need for action

and account for the learner's environment as an instrument for both generating ideas and validating predictions.
These factors, initially, are critical in establishing whether or not Piagetian psychology can contribute to a theory of classroom learning and teaching in the comparatively abstract domain of algebra. Can these requirements be tied to notion of equilibration? This is the key factor in this discussion of knowledge acquisition and development, and it is the issue which must now be addressed.

EQUILIBRATION: A FURTHER ANALYSIS

In order to examine further Piaget's assumption that equilibration is the essential mechanism of cognitive development, it is helpful to consider his own development in articulating the concept.

In earlier writings Piaget presented an intuitive formulation of cognition as being a system that is fundamentally self-regulatory or self-referential. Drawing on his study of biological organisation, he posited a system that is organised and stabilised through the interaction of interactions between different parts of the system. The outcome is not a maintenance of the status-quo, but a means or force that provides for direction and establishment of growth. As the system is confronted by disturbances arising either from inner reflection or outside contradictions, compensatory actions restore equilibrium. Piaget asserted that it was the integration of these disturbances into wider and more powerful structures that characterise the growth process.

Initially, Piaget focused on the general, structural features that dominate particular stages within growth; however, in later years he examined the possibilities of describing development in terms of their functional aspects. It was through the concept of equilibration that Piaget articulated and began to demonstrate a focus on processes in micro-formation. Thus, what has been called the "most important...concept" (Lovell, 1979, p.16) and <u>the</u> core concept (Rowell, 1983, p.62), and by Piaget "an essential agent of development" (1980, p.58) equilibration replaced the former concern with stages. The concept was clarified and justified through elaborating on and extending the previously introduced constructs 'abstraction' and 'generalisation.' These processes were seen to be the means by which action schemes were coordinated, and thus assimilated into an individual's knowledge structures. For this study these concepts are analysed with reference to an early number concept, and then extended to encompass more complex algebraic ideas.

<u>A young child's acquisition of the concept 'five'-in Piagetian terms</u>

Very early in a child's life s/he will have experiences in which s/he hears someone saying 'five'. An initial experience is likely to involve someone counting the child's fingers of one hand and/or the toes of one foot - an experience involving hearing, seeing and touching. For Piaget, knowledge is constructed out of these actions; the sensori-motor child (identified by Piaget as characterising the first two years of life) gradually comes to know that objects exist even without immediate perception, and to coordinate object parts into a whole that is recognisable from different perspectives. For the pre-operational child the external world becomes represented through the medium of symbols, primarily through generalisation from examples - so 'five' arises from the child's experiences. Connections that are made (after attempts to manipulate reality are met with frustration) are intuitively regulated rather than governed by mental actions (operations that are internally recognised). Thus, the notion of 'fiveness' lacks permanence; the experience involving five must be acted on physically each time for it to be meaningful, and the (internalised) memory of these counting experiences (now possibly extended to five people sitting on five chairs using five knives/forks/spoons/plates/dishes) is tied to the particular experiences in which the concept is embedded. In a classroom, a teacher will observe that the child needs to re-count five objects that are represented in a fresh format or context; in other words, competence is lacking an operational dimension. (Piaget's argument here has been the subject of much debate and critical review. Indeed, cognitive psychologists (eg, Gelman & Gallistel, 1978; Gelman and Meck, 1986; Hughes, 1986) have cited research evidence to indicate that in such children, the operational dimension <u>is</u> present; that is, children who need to re-count for each experience are nevertheless able to perform addition and subtraction operations on objects. In discussing this issue, Smith (1986, p.204) argues that these findings have no necessary relevance to Piaget's use of competence, which means 'understanding' as construed in the Russellian manner.)

For Piaget, concrete-operational thought empowers the school-age child to assimilate experiences into his/her developing mental structures. Thus the child is able to internalise action, that is, to carry out actions in thought as well as with objects. It is in analysing these processes that Piaget's post-1972 theorising and his development of the terms 'abstraction' and 'generalisation' are particularly pertinent. To continue with the concept of five as an example: a child's generalising of initial experiences of five requires the recognition of a pattern in these experiences. The recognition of a 'fiveness' quality as being the essential commonality is an abstraction process that is mathematically significant, and which can lead to competent (adequate) generalisation. A child's first experiences of 'five' may be limited to hearing the word in connection with objects that can be seen and touched, but further experience (including those arranged by a thoughtful teacher) will involve events (temporally-defined experiences) that indicate that 'five' can also refer to happenings such as five claps, five mouthfuls that are eaten, five minutes of time, and so on. Of significance for the teaching situation is also the inclusion of a wide variety of contexts, objects that are arranged (and counted) in varying formats, as well as a focus on what is not five. These examples present a deliberate attempt to provide children with a richness of experience in the concept of five, in juxtaposition with experiences that specifically do not represent that property. In particular they aim to lead children to the recognition of what it is that represents the essential commonality (in this case the numerical attribute 'fiveness') that characterises the experience, plus a realisation that 'five' can be attributed by extension (generalisation) to other situations in which this same quality is perceived. But what is it in these classroom experiences that lead to this reasoning, this claim that the concept of five is being developed in this way? In Russellian terms, development of a concept requires an understanding of class, correspondence and asymmetrical relations. Logically these processes involve reasoning on the basis of transitivity and class inclusion. So for the classroom teacher, a child's development of the concept of five may begin with a variety of action-experiences in which five objects are counted, or matched with five other objects. Note that counting and matching involves more than the pre-operational 'assimilation through physical action on what can be perceived'; it involves a mental action, and thus a <u>necessary</u> correspondence - achieved, according to Piaget, through concrete-operational thinking. Also significant are experiences that involve recognising less/more than five as being not five but as related to five, with, specifically, the class of 'fourness' as being included in the class of 'fiveness', which in turn is included in the class of 'sixness'.

Equilibration described in terms of abstraction and generalisation

The acquisition of specific knowledge (in this case the development of the concept 'five') may usefully become a focus for testing the applicability of Piaget's new theorising on equilibration to classroom learning. As a starting point it is useful to examine Piaget's distinction between two related kinds of experiences, and the abstractions drawn from them. "Knowledge" Piaget asserts "is abstracted from actions, from the coordination of actions, and not from objects" (1970, p.17) But it is the objects or physical experiences (the things or events that allow manipulation in a specifically mathematical way) that make possible "simple" or empirical abstraction. At first the knowledge is physical - the abstractions are based in individual 'concrete' actions, a 'reading' through perception. In contrast, the more powerful logico-mathematical knowledge is abstracted from the <u>coordination</u> of actions. This abstraction (termed 'reflective abstraction') is rooted not in individual actions but in an organisation of actions; and it is constructed through mathematical thought-actions that purposefully operate and relate through joining, ordering, matching, and establishing intersections - the sort of counting-based experiences suggested in the previous section as being applicable in a classroom.

Reflective abstraction thus provides a framework for empirical abstractions. Relations between empirical abstractions are constructed, allowing influences to be made and coordinated. In turn, this construction of reflective abstraction acts as an internal feedback to enrich progressively the quality of the cognitive structure. The child's interpretation of experimental data thus becomes changed (developed) as a result. Initially, empirical abstraction is the dominant partner in these related abstraction processes; but with growth and progressive cognitive enrichment, reflective abstraction becomes the more important and cognitively influential process. The relative significance of reflective abstraction becomes an increasingly significant factor in mathematical thinking, for within the school curriculum it is in mathematics only that reality (requiring, in the first instance, empirical abstraction) eventually may no longer be taken into account in learning and thought development.

In order to clarify the functional means by which this knowledge growth takes place, Piaget has also distinguished between two related generalisation processes. 'Inductive generalisation' refers to generalisation in so far as it can be tied to evidence, namely, as an extension of what is perceived. In the classroom an investigatory methodology may enable a child to examine situations with reference to the 'fiveness' quality of sets of objects and events. Using empirical methods as means of verification, the degree of generality may be found; that is, the conditions for sufficiency are established. In the case of the concept 'five' this means an internalised recognition that, for example, the last object or event counted corresponds to the number five, or that the set to be tested matches, one-to-one, another set containing five. However, Piaget's logical analysis of concept also requires that conditions are recognised in terms of necessity. This happens when the child conserves the number five. Now there is an operational framework in place that allows inferences to be coordinated; knowledge goes beyond what is observed, enabling the child to deduce from the inductive generalisations made that the number (irrespective of its arrangement or context) <u>must</u> be five. So there is no need for recounting a different arrangement of what is known to contain five objects, no need to wonder if four and one is the same as one and four. Three and one is recognised as necessarily less than four and one, with the classes of one, two, three and four necessarily included in the class of five. These properties are <u>attributed</u> to these numbers. Thus we have an explanation (giving 'the reasons for') which links together successive inductive generalisations. It goes further than extending what has been observed: it <u>accounts</u> for it. This framework which provides the impetus for seeing these possibilities, for understanding things that until now escaped the child's conscious knowing, is called constructive generalisation.

Investigating algebraic concepts

In school algebra we are dealing with numbers: specific numbers and generalised numbers. Thus the analysis of a variable number (in Piagetian terms) focuses also on the logical notions 'class', 'correspondence', and 'asymmetrical relations'. However, the concept of variable may be seen as a higher-order abstraction, well distanced from concrete-referenced experience. Thus it may not be possible to employ the same approach of validating thinking through empirical means and accounting for what has been observed, as in the development of the concept of five. The processes employed must extend to mentally-derived ideas and procedures that may appear to be totally divorced from physical action. However, it will be argued that it is possible to take physical experience into account through a sequence of links between the abstract and the concrete, where possibly the concrete form is a physical embodiment that is isomorphic with the mathematical idea.

Abstraction and Generalisation Processes

A variable may be perceived as the representation of an empty space that can be replaced by a name from a chosen set of names. The representation may be in symbolic form; and in school algebra, the symbol will represent a number. It is important that students develop an understanding of this idea.

Empirical abstraction refers to ideas that are derived from physical experience. Students who are being introduced to the concept 'variable' - usually in the first or second year of secondary schooling - typically will find an experiential familiarity with the concept if their attention is drawn to it. So a classroom introduction to 'variable' will make use of analogy, focusing on prior active experience. While this involvement may not produce a physically-observed reading of the facts, experiences may nevertheless involve perceptually-referenced action. So students' attention to 'variable' may be focused through the means of group games and activities in which participants are required to use variable names in a number of familiar contexts. Experiences then are extended to the use of numerical variables (which form a significant subset of the set of variables), and the distinction between a variable number and non-variable (specific) number. A further distinction (through familiar experiences) is made between the use of the symbol for a variable in variable situations, and those situations in which the symbol refers to a number to be evaluated, a specific unknown, or a generalised number.

Experiences involving empirical abstraction must be linked in various ways in order to construct the framework that characterises reflective abstraction. Thus, variable numbers must be acted on mentally in concrete-referenced experiences involving joining, ordering, matching, and establishing intersections. Investigations (and the verification through evidence) may, for example, focus on the use of the commutative and associative laws of addition. A rule involving two first degree variables, for example, x+y=10, may be investigated concretely using various groupings of objects. Students are asked to construct a range of different addition bonds using natural numbers, through the manipulation of objects, or pictorially or graphically (or possibly symbolically) represented quantities. The addition of two variable numbers according to this (or similar linear rules) may then be extended, through investigation, to integers and then to rational numbers. Graphical representation on a cartesian plane of pairs of variable numbers that satisfy a given rule presents a further possibility to link together and to coordinate perceptually-derived evidence. These experiences are also used as a means of empirical verification, and to establish the extent to which consistent generalisations may be made.

But, in Piagetian terms, there is a further recognition that goes beyond the application of evidence to reality, namely, the explanation of why things are as they are. This is a constructive generalisation that makes possible a new way of seeing things - an approach that gives an account of how they _must_ be. In the case of relations between first degree variables, a constructive generalisation enables the student to attribute to reality facts that are known through a _coordination_ of inductive generalisations, originally made through experiences. Thus a student recognises that equations of the form $x+y=10$ necessarily involve a relationship of linear dependence (which may be represented graphically as a straight line), and that all numbers that satisfy this relationship are governed by this principle.

Further investigative experiences (approached typically through the representational sequence: objects, graphs, symbols) are directed towards incorporating this knowledge into a framework (constructed through the coordination of inferences drawn from other experiences with mathematical functions) that creates possibilities for further integration, resulting in a more adequate knowing. Then experiences are used also as perceptual reference for generalisation by induction (experiences applied to reality), which leads in turn to the generalisation through which, Piaget claims, reality is constructed and explained.

ANALYSIS OF THE EQUILIBRATION MODEL AS A BASIS FOR INSTRUCTION

Of practical value to classroom teachers are Piaget-inspired studies that seek to deal specifically with the creation of 'new possibilities' in knowledge. The reference here is particularly to learning that involves change in knowledge - in addition to developing and enhancing what students already know. The equilibration model is used as a basis for designing instructional approaches that are consonant with the way that cognitive change is seen to occur. Thus, typically, teaching strategies employ as a starting point knowledge that the students consider to be relevant. The sequence proceeds through a step-by-step process in which knowledge gaps are recognised, and then compensated for in the construction of further knowledge. Thus progress is generated by a disturbance in the students' interpretation of reality - a situation that may be provoked through planned teacher intervention (eg, Karmiloff-Smith and Inhelder, 1975; Case, 1978; Nussbaum and Novick, 1981; Rowell and Dawson, 1981, 1983, 1985; Easley, 1984). Teaching methodologies are thus designed to encourage students to confront their thinking with reality (possibly physical evidence) and the ideas of others, creating a state of conflict that must be resolved in order to restore (temporary) equilibrium.

However, bringing students into conflict with what they believe to be so is not necessarily sufficient to change their thinking (eg, Karmiloff-Smith and Inhelder, 1975; Driver and Easley, 1978; 1975; Easley, 1984). Typically, counter-examples are considered to be no more than anomalies that are irrelevant to the beliefs or theories-in-action that guide behaviour (Karmiloff-Smith and Inhelder, 1975). A parallel may be drawn to the historical study of scientific development: evidence suggests that progress takes place, not through falsification of ideas, but by the emergence of a better theory (eg, Kuhn, 1962; Lakatos, 1976). Thus instructional strategies within the equilibration framework seek to model (at the individual level) the pattern of response and adjustment for scientists in their collective quest for more adequate explanations of phenomena.

Instructional suggestions

Studies by Novick and Nussbaum (1981) and Rowell and Dawson (1983, 1985) suggest ways in which this approach might be followed in classroom teaching in science. Novick and Nussbaum describe a sequence in which students are faced with a critical situation requiring interpretation. Contributions by members of the class present a conflict situation in which competing suggestions are presented and defended. Rowell and Dawson provide a model in which the 'better theory' is dealt with quite explicitly. They insert into the instructional sequence a new theory (a more adequate explanation) that can be constructed, and accepted by students, before old ideas need be rejected. The new theory is presented as a powerful alternative to ideas already held.

These studies provide useful suggestions for similar situations in classroom algebra. Very importantly, they indicate how the conflict methodology can be adapted to teaching from a syllabus, and in class-group situations. In addition, they use conflict (including student errors) as a positive and constructive part of the learning process, reducing the threat usually associated with the giving up of cherished or long-held beliefs.

Accordingly, Rowell and Dawson (1985) suggest a sequence of instructional activities involving:

(1) assessing students' prior knowledge in respect to the proposed new learning

(2) gauging 'gaps' in student knowledge that can be accentuated into potential imbalances

(3) monitoring potential imbalances to assess whether they have become actual disturbances in the students' knowledge systems

(4) recognising that the restoring of balance is likely to follow a multi-step process of imbalance followed by temporary equilibrium

(5) possible teacher intervention to promote a desired direction, that is, towards more adequate constructions.

Rowell and Dawson also draw attention to the need for theoretical knowledge on the part of teachers. Firstly, it must be understood that although teachers may intervene in and guide the construction of knowledge, equilibration is an internal process, and it cannot be taught. Secondly, teachers should recognise that over-generalisations are easier to change than false constructions.

Summary

Piaget's concept of equilibration has been presented in terms of a self regulatory system that describes cognitive functioning.

It is presented as a concept that clarifies, through the detailing of processes, how knowledge is constructed from action-experience, and progressively abstracted and coordinated through successive generalisations. Particularly it is seen as a mechanism that suggests how qualitative changes in thinking might take place; that is, it is concerned with a learner's theory-building, and the means by which new possibilities are recognised and integrated into the system, and limited or false notions discarded. As such, it provides a powerful framework to support the design of instructional approaches that aim at addressing this need for qualitative change in thinking - the change that makes possible thinking and action at a higher cognitive level.

Chapter V

CHAPTER V

DEVELOPING A COGNITIVE FRAMEWORK FOR INSTRUCTION

In the previous two chapters there has been an attempt to elucidate the critical growth points within two major perspectives of human learning and problem solving. Information-processing analyses provide rich insights into the ways that humans use already-constructed knowledge in the development of new ideas and ways of thinking. Detailed patterns of information about skills involved in thinking performance are providing the means to describe more fully how this competence might be developed. The increasing specificity of these descriptions gives maths educators fine-grained tools that enable a more precise analysis of thinking behaviour in mathematical domains. The computational approach in programming demands that ideas and procedures are modelled in a formal manner; this requires clear stipulation of the ways that knowledge must be organised in order to be processed productively by the system. Thus we have a characterisation of knowledge that is rich and dynamic, with an emphasis on its organisation and interaction. Of particular value to mathematics is the detailing of procedural and conceptual knowledge and their role in the construction of a solution to a problem task. The ways that these strands of knowledge interact in this process suggest a particularly productive area of research (eg, Resnick and Ford, 1981; Hiebert and Lefevre, 1986; Silver, 1986).

However, it has been suggested that qualitative changes in thinking are not at present adequately modelled by computational concepts and procedures, and it is my intention in this study to investigate the possibility of complementing the detailed accounts of knowledge development within information processing theories with Piaget's equilibration model, which is seen as a process that accounts for growth. Piaget has suggested a mechanism that makes possible a re-conceptualisation of how things are seen: a 'seeing things with new eyes', which implies/recognises that false perspectives have been discarded, and limited or inappropriate ones re-positioned. Points of reference have undergone a <u>shift</u> to enable new possibilities to be incorporated into the system in an harmonious way. The claim is that this re-structuring process can be understood within the framework of constructive generalisation.

The question, then, is whether there can be a merging or a synthesis that allows each perspective of learning and problem solving to make a contribution that is mutually beneficial, and possibly more productive in educational application than what could be provided separately.

<u>Literature survey on a possible merger</u>

Suggestions that the merging of information processing approaches and Piaget's recent functional orientation is not only possible, but potentially productive, have been made in a number of contexts.

Groen and Kieran (1983) suggest that Papert's construct of a micro-world is compatible with the logico-mathematical structures that Piaget saw as being fundamental in the development of mathematical thinking at the formal level. They suggest that there is a similarity in the state-independent nature of the transformations as defined within both theories. Their position is specified through using Papert's turtle geometry as an example of a micro-world, which, in turn, is seen as a mini-domain of Piagetian mathematics. In a discussion about Piaget and school mathematics, Groen and Kieran recognise the considerable gap between Piagetian tasks - designed to establish general notions about intellectual structures - and the specific nature of tasks in curriculum-driven school mathematics. However, from a psychological perspective they see the more unconventional logo geometry as a mathematics that is more closely matched to the structures of mind. They add, too, that program inventing encourages the students to think about their own actions, besides providing them with immediate feedback that shows up errors. Here the dialectical nature of equilibration is paralleled in the way that counter-examples and anomalies are recognised, yielding information that may lead to major modifications in the learner's perspective of how things must be.

In their search for a Piagetian mathematics, Groen and Kieran present a Piagetian mathematics that is based on the procedural notions developed in the post-1972 Genevan school, which they see as more amenable to information processing analysis. Within both frameworks there is a focus on process and structure, and the resolution of dialectical tension between the two. Information processing theories provide detailed knowledge on functioning (within structures) that is ill-defined within the Genevan perspective. Piagetian concepts, however, give a clearer global picture to provide parameters and a framework for this functioning. The two together, claim Groen and Kieran, potentially form a construct that is more complete than the two in isolation.

Boden (1979, 1982) presents the possibility of productive interaction between Piagetian and information processing contributions from a primarily philosophical perspective. While highly critical of Piaget's elaboration of the equilibration concept and the general applicability he attributed to it, she nevertheless sees value in Piaget's identification of processes that are seen to characterise a shift to a higher level of thinking. The real potential in Piaget's construct of equilibration, Boden believes, lies not in his account of it, but in the questions raised by it. Boden's contention is that Piaget has identified important problems in our present understanding of cognition that need to be addressed; and she suggests that computational concepts may be able to provide a precision that Piaget failed to achieve. Her rationale for suggesting that the two approaches are compatible, and possibly complementary, lies primarily in their common reference to the cybernetic notion of feedback.

Computer programs that enable researchers to examine thought processes and memory structures employed in intelligent behaviour use rule-governed goal-directed routines that are flexibly interrelated. Programs are designed so that they recognise and respond to symbols that represent information of both a semantic and syntactic kind. They are able to deal with symbols in a way that integrates information into the system, using transformation procedures that have the potential to reflect both assimilation and accommodation processes. Boden suggests that there are benefits to be gained from the formalising of this self-regulatory process in computational terms. Piaget formalised equilibration in terms of algebraic structures, but this model failed to detail procedures for integrating information into human cognitive structures. Therefore Boden believes that computational models that are constructed to address equilibration processes may enhance our understanding of human learning and cognitive development.

Rowell and Dawson (in press) have suggested the possibility of integrating Piaget's equilibration model as a general concept of learning, with findings in artificial intelligence and cognitive science that highlight specific characteristics of learners. They use the concept of 'knowledge' as a reference point for investigating possible similarities and meeting points between the two perspectives. In particular, they see a common conceptual basis in Piaget's focus in the functioning of structures and the schematists' emphasis on processes that have structure (eg, Schank and Abelson, 1977; Abelson, 1981;

Rumelhart and Norman, 1981). Rowell and Dawson argue that Piaget's post-1970 elucidation of generalisation processes signify the importance he attached to factual knowledge as "an indispensible feature of the construction of reality". They cite the learner's reference to reality within a familiar context as a clear indication of Piaget's new attention to specific knowledge in describing knowledge construction. This focus corresponds with the context-specific nature of, for example, the sub-schema construct 'script' (eg, Abelson, 1981) as proposed within schema theory.

The "integrated theory" proposed by Rowell and Dawson is essentially Piagetian, with the equilibration model forming the hard core of the theory (Rowell, 1983), in the Lakatosian manner of theory construction. The emphasis is on theory building through a cycle of conjecture, proof by perceptually-referenced evidence, and refutation by counter-examples (Rowell and Dawson, 1983). Directed towards educational application, Rowell and Dawson address the task of building a theory of knowledge acquisition and growth that is appropriate to science learning in classrooms. They have given evidence to show that both Piagetian and schema theorists have articulated thinking processes in terms of inductive reasoning - in the sense of extension or generalisation from examples, and their interaction into the system. Both focus on these examples as being determined by an individual's reality within a specific domain.

However, Piaget is concerned with an additional generalisation process: he incorporates into his model a growth factor enabling a higher level action on reality. This, in turn, allows not only a more extensive reading of what is experienced, but it also promotes the drawing of inferences that add to a causal understanding. Rowell and Dawson argue that the theory of equilibration explains the growth of "an increasing number and quality of inductions" (emphasis mine) a phenomenon that, they assert, the inductive/assimilation model in schema theory fails to address. Instruction, therefore, must be concerned with something more than accumulated knowledge; and it must also take into account the mental restructuring that makes possible qualitatively-changed conceptions of the environment, while maintaining an internal stability that does not impair the functioning of the system.

A similar position is taken by Brown (1988). Indeed, Brown proposes that a synthesis of cognitive science and Piagetian psychology has the potential for "a complete theory of mind" (p.60). Referring to the recent fundamental shift in Genevan research efforts (signalling a focus on procedural aspects of knowledge construction), Brown suggests that these new formulations of knowledge construction should lead to a more fruitful cooperation between Piagetian epistemology and cognitive science. Brown supports his position by reporting a brief overview of Cellerier's (1979) attempt at synthesising the two perspectives. In order to develop his synthesis, Cellerier (according to Brown) focuses initially on fundamental commonalities, and a particular difference in the two views. Both, contends Cellerier, focus on the cognitive construction of knowledge and the self-regulating characteristic of intelligence. However where the Genevan view sees successive knowledge transformations as originating in action, information processing theories are usually premised on constructed knowledge as determining reality for the individual, and therefore as a precurser to her/his action. As a result, Piagetians have been concerned primarily with learning in general developmental terms, while cognitive scientists seek to specify detailed characteristics of learners. The Piagetian 'epistemic transformation' (how knowledge is constructed out of action) assigns to knowledge an independence from the experience which generated it, whereas the cognitive science 'pragmatic transformation' (how action is constructed out of knowledge) gives knowledge a quality that is very closely related (through direct and successive abstractions) to the context and time in which the learning experience takes place. However, Cellerier claims that the two types of transformations should be seen for their commonalities rather than for their differences: in both, the assimilation/accommodation processes are central, and these processes can be programmed as production rules (see Chapter 3). Cellerier's synthesis involves describing both views in terms of each other: this he demonstrates by taking Newell and Simon's (1972) General Problem Solver model and modifying it to incorporate the epistemic transformation. From the cognitive science perspective, a learner's need to assign meaning to a situation (so that it can be assimilated into existing knowledge structures) is described by Cellerier as a mental reconstruction of the situation in terms of explicitly conceptualised or intuitively used operations of knowledge. The accommodation process, during which the system sets up a goal-means structure in order to act on the environment (the problem task) in a way that changes (reconstructs) it, requires a form of knowledge that enables the learner to choose appropriate/desirable action and to monitor its implementation. This metacognitive knowledge is termed by Cellerier "axiological knowledge". In complex situations it is instrumental in setting up a structure that must incorporate a sequence of transformations that at the same time reduces as much as possible the distance traced to reach the goal (see Larkin et al., 1981, in Chapter 3). The Piagetian description focuses conversely on knowledge structures that are constructed out of active reality. These action schemes (procedures that are remembered) are coordinated through abstraction and generalisation processes (elucidated in the previous chapter). This process sets up within the system a capacity for incorporating new possibilities into reality - a characteristic that Piaget attributes to mental operations. The coordination of the action schemes is managed by a control system that seeks to maximise growth (improved change) while maintaining stability (equilibrium) within the system. It involves compensating for a gap in knowledge by means of a mechanism that is not dependent on knowledge specifically, but is an outcome of operations of a logico-mathematical kind - that is, operations that can be manipulated (mentally) in ways that enable great flexibility (eg, operations that can be reversed). Knowledge structures thus developed enable a dissociation from specifics of time and experience, allowing the individual to see from an explanatory viewpoint how things <u>must</u> be.

Attempt at a merger

In Chapter 4, Piaget's equilibration was elaborated in terms of the generation and validation of knowledge through action in specific areas of mathematics. Change to a higher level of action, as modelled by constructive generalisation, represents theory building in terms of an evolutionary cycle that has started in action-experience. Arguments have been presented to suggest that this account may also provide an explanatory framework for learning in mathematics.

Reference has also been made to information-processing analyses that give a detailed and precise description of the processes involved in developing mathematical knowledge. This perspective highlights problem solving as a process of selecting, transforming and organising mathematical concepts, procedures and principles with reference to already constructed knowledge. Problem-solving activity that leads to successful performance requires knowledge that has meaning for the learner.

While the focus of each framework may differ - one is concerned with theory-construction while the other emphasises task performance - the cycles are not incompatible. The global structural considerations that formerly dominated Piaget's theory have been shifted aside in order to study more closely micro-genetic functioning. Clearly, this focus is consistent with the cognitive science concern for processing of knowledge present in memory. Both theories are concerned with processes within structure; and both involve self-regulatory mechanisms that are directed towards a satisfactory resolution.

My aim in proposing a merger of these frameworks is to provide guidance and support for learning and teaching in a curriculum-driven area of knowledge. I am suggesting that this synthesis has the potential to direct instructional approaches that might lead to more effective classroom practice. I believe that a merger of information processing and Genevan perspectives may provide a tool to detail a problem-solving approach to algebra, and in addition address the problem of change to a higher level of knowing.

Merging at a point that is seen as useful for algebra instruction

For a teacher, a prime task is to support the student's transition to a more advanced state of cognitive activity. This is the problem that Piaget addressed, and which he formulated through his theory of equilibration. In his elaboration of constructive generalisation - the framework in which successive inductive generalisations are coordinated - he articulated a model to account for change, that is, a shift in the way that the individual sees things. This is theory construction, the means by which an individual generates possibilities and integrates them into mental structures: the means by which a higher level of knowing is reached.

In the case of algebra, this attainment must involve not only more advanced levels of understanding, but also more sophisticated and refined performance strategies. Both conceptual and procedural knowledge are needed for dealing with problem tasks. Cognitive science has demonstrated that problems are solved by building mental representations of the problems and selecting solution procedures that are appropriate to them. Rich descriptions of the processes involved in the representation and retrieval of concepts and procedures indicate that indeed a key factor in successful problem solving is the connecting of procedures with their conceptual underpinnings (Hiebert and Lefevre, 1986; Silver 1986). Some studies (eg, Anderson, 1983, 1980; Riley, Greeno and Heller, 1983; Rumelhart and Norman, 1982) have begun to address the problem of how this interaction might be modelled. At this stage, it is certain that understanding and success are inter-related and inter-dependent. What is significant (but not yet clearly determined) is the role of metacognitive behaviours in this process (see Chapter 3).

In a collaborative work with Voyat (Piaget and Voyat, 1979), Piaget directs his attention beyond the development of concepts to an analysis of change as it is represented within the procedural aspects of knowledge development. The problem faced by Piaget was the recognition that developmentally determined 'stages' of intellectual growth do not alone explain adapted action. Thus he turned to investigating the function of dialectical processes and their role in the development of knowledge.

In doing so, Piaget presents a conceptual/procedural framework that parallels that proposed within cognitive science. Presentative schemata (or concepts) are seen as strongly perceptual, easily abstracted and generalised, and retaining their individuality even within a broader category. For example, the concept 'five' is developed through a recognition of commonalities in and generalisation from appropriate numerical situations; and it does not lose its identity when classified more generally as a natural number, or as an integer, or as a rational number. Procedural schemata are described by Piaget as consisting of a sequence of goal-directed actions. They are difficult to abstract from their context and retain their individuality only temporarily and to a limited degree. Adding three to five, for example, involves a procedure that cannot exist without utilising particular concepts; however when the addition process is performed (the action defined by the goal) the procedure is completed. These schemata are preserved only in so far as the number bond 'five and three are eight' can be evoked in terms of presentative schemata; that is, the knowledge is now conceptual. What synthesises the two forms of schemata, in

Piagetian terms, are a third kind - operational schemata. These schemata drive goal-defined action in the same sense as procedural schemata, but they are also dependent on general structural development within the individual.

The nature of the interplay between presentative and procedural schemata, and the role of operational schemata, may be understood more clearly through an elaboration of two underlying systems. System I is aimed at understanding. This represents an individual's reality - which can be perceptually or mentally constructed. It therefore includes operations that are recognised as concepts, such as the number bond 'five and three are eight'. It is the system to which equilibration applies; and therefore is the system that models an individual's re-conceptualisation of how things must be. Through processes of equilibration a child progresses from, for example, 'five and three are... five, six, seven, eight' to 'five and three must be eight' (which can be memorised meaningfully). But how does this happen? The function of System II is to integrate procedural and operational schemata in a way that aims at and brings about success. Clearly this refers to problem solving activity involving procedures, and operations in action, such as the adding of three to five. The linking mechanism here is induction (generalisation) not equilibration. Indeed System II does not involve resolution but continual goal-directed activity. Successive procedures are compared with those shown to be successful (adequate) in previous, related situations; they are partially abstracted, and then extended to bring about and open up new possibilities. This involves a learner's recognition of a 'knowledge gap', a situation in which s/he exhibits monitoring and managerial (metacognitive) behaviours.

Importantly, within this process, errors (eg, over generalisations) are seen as possibilities; therefore errors are potentially productive, a means towards attaining a higher level of action. Thus System II is a tool for compensatory construction. It is never a system in equilibrium; it has the role of bringing about a compensatory enrichment in understanding in System I.

An empirical study by Karmiloff-Smith and Inhelder (1975) gives a good example of these processes at work. They report children's actions in developing the concept of balance, focusing on the interplay between children's actions and their implicit ideas about how a fulcrum can be put into a state of balance. Karmiloff-Smith and Inhelder observe that children hold on to their "theories-in-action" for as long as possible, despite a sequence of evidence-through-action that counters their assumptions. Eventually, however, these counter-examples impact on the children's consciousness, through "organisation and reorganisation..., lengthening of sequences...[and] repetition of their actions (p.304); that is through the construction of a better theory. Thus, generalisations are made which can be applied to new situations, giving rise to "discoveries that will regulate the theories, just as the theories have a regulating effect on the action sequences." (p.304). Understanding and succeeding may be seen to interact as a "cross catalytic looping process...through the operation of increasing equilibration" (Rowell, 1988).

Thus within both perspectives discussed, there are significant meeting points. For both essentially focus on an individual's understanding and its interaction with processes that direct towards and make possible successful task performance.

Instructional implications

The Genevan shift towards investigating procedural notions and processes provides an important link between Piagetian and information-processing perspectives, at a point that is significant for mathematics (specifically algebra) instruction. Despite the obvious complexity of the relationships among elements of conceptual and procedural knowledge, some pointers are emerging that may be of advantage to learning and problem solving in mathematics. It is suggested that both Piagetian and information processing theories have the potential to contribute significantly to instructional design, and that the two in synthesis is more complete than each in isolation.

Piaget's revised position and direction is encapsuled by Brown in his suggestion that the Genevan focus on processes will be furthered "by studying the microgenetic invention of action strategies in specific contexts" (1986, p.63). Thus, there is a clear recognition of the importance of specificity in content in learning, as the cognitive scientists insist. Furthermore, each perspective emphasises the role of both conceptual and procedural knowledge in dealing with problem tasks. Both provide descriptions of the development of these two strands of knowledge, turning their focus more recently onto ways in which they might interact. In addition, the Piagetian study of the dialectical processes involved in knowledge construction emphasises the potential value of errors-in-action as a critical factor in the shift to a more advanced level of knowing.

Thus we have in hand a framework of knowledge development that has the potential to guide and support algebra-learning. The task ahead is to give that knowledge educational application, specifically application for a school classroom. This requires a translation of theoretical concerns in learning to those concerned with implementation in an instructional setting.

A classroom book for algebra learning.

In the implementation of a cognitive framework for classroom instruction, two groups of learners must be considered: the students studying algebra according to the program devised, and the teachers implementing a psychologically-based theory in a school classroom. Thus, both the algebra program and the theoretical guidelines must be presented clearly and in a way that matches the needs and practical concerns of the users.

The presentation of a new algebra course in written form is no problem if the work is to be set out in textbook format. However, a book that focuses substantially on problem exercises cannot be considered as suitable for a course that aims to promote a thinking, active, productive (not reproductive) approach to algebra. More appropriate is a classroom book that presents an algebra program as a development of knowledge, promoted by learning and teaching in ways that are compatible with the theory proposed. In this presentation classroom activities and approaches will need to be detailed, and the framework for algebra learning outlined. An attempt to meet these considerable demands in ways that guide effective classroom practice is presented in a classroom book **Talking Maths**. It includes topics normally covered in algebra at the year 9 (and possibly year 10) levels in South Australian schools, and so is considered suitable for study by these classes.

Chapter VI

CHAPTER VI

AN ATTEMPT AT TRANSLATING THEORY INTO PRACTICE*

A framework has been presented for guiding and supporting classroom learning and problem solving in algebra. It represents a synthesis of two perspectives within cognitive psychology: the information processing approach with its analyses of problem-solving processes, and the Piagetian concept of equilibration as a way to describe development in performance capability. Despite differences in orientation, significant commonalities have been identified in the two approaches. In particular, a mutual concern with conceptual and procedural knowledge (and their interaction in the problem-solving process) is identified as being significant in the learning and teaching of algebra.

If this framework is to underpin an approach to classroom algebra, then instructional issues must be addressed, particularly in the development of conceptual and procedural knowledge and their interaction. Secondly, the factor of social interaction in the classroom must be considered. So far, learning and problem-solving performance have been seen in relationship to individuals: can the the framework presented provide a comparable support for the teaching of groups of individuals in the classroom setting?

*(page numbers in this chapter refer to Appendix 3)

CLASSROOM ALGEBRAIC PROBLEM SOLVING

In line with Newell and Simon's (1972) General Problem Solver model, cognitive science traditionally emphasises two steps in the problem solving process:

Representation (understanding the problem) - The problem as presented is encoded in a way that allows it to be represented in the mind. This representation includes the given state, goal state, and operators that can be used for solution. A 'problem space' is built upon the individual's understanding of the problem.

Solution (searching the problem space) - The problem solver searches for an appropriate path through the problem space. The problem solver keeps applying operators to problem states until the goal is achieved.

These steps involve processes similar to the Systems I and II (understanding and succeeding) proposed by Piaget. These meeting points will therefore provide a framework of reference in which to address issues in problem-solving instruction in algebra. The design of instructional approaches for classroom algebra will centre firstly on these problem representation/understanding and problem solution/succeeding aspects, noting particularly the interaction of conceptual and procedural knowledge during these processes. Social aspects of classroom implementation that are addressed within this framework will be highlighted, and interwoven into the discussion. Although neither theory provides a social perspective of cognition, both attend to significant aspects of social interaction that can be identified in designing appropriate classroom practice.

UNDERSTANDING

The initial step in Polya's (1957) descriptive model of problem solving is to 'understand the problem'. In information processing terms this means to interpret what the problem is (or means) and to construct an adequate representation of it (see Chapter 3). Typically, the problem solver is seen as initiating a representation based on an interpretation of the language of the instructions; meaning is attached to a situation. This is gradually elaborated and refined until a representation that is adequate for solution is attained. A number of studies (e.g., Chi, Feltovich and Glaser 1981; Riley, Greeno and Heller; 1983) have indicated that expert problem solvers spend considerably more time in developing meaningful representations of problems than those who are novices in the task area. 'Qualitative analysis' of the problem involves finding meaning in the situation and marshalling resources before taking action (page 7). This is the expert mode. Novices instead tend to rush into quantitative (computational) procedures and premature closure; they focus on finding a solution rather than spending adequate time in making sense of what the problem is all about.

Schema theory has provided further useful information about instructionally relevant aspects of generating and constructing adequate problem representations. A critical factor in successful problem solving is the retrieval and use of clusters of related information. Davis (1983, 1984) has suggested that these chunks or 'frames' may explain many aspects of algebra performance (also Matz, 1981). Students tend to select solution methods on the basis of recognised features of a problem. Sometimes, as Davis points out, this means that a student 'latches in' to a solution method that is not applicable because recognition is based on superficial similarities. The student incorrectly categorises the problem according to surface features or rules, unaware of (or ignoring) the "deep-level rules" (Davis 1983, p.276) that form the inner structure of the problem. In contrast, the categorisation of problems on the basis of their fundamental principles (eg, Chi, Feltovich and Glaser, 1981; Silver, 1986) marks a good problem solver. This approach characterises an algebra student who sees a new problem not as a novel experience but in terms of a task in which algebraic structures are recognised (Krutetskii, 1976).

For Piaget, 'understanding' refers to a construction process in which the learner finds meaning in a situation. Two kinds of knowledge are needed for understanding: that which is derived from the properties of objects manipulated, and that which is derived from actions and the coordination of actions on the objects handled. Through the coordinating processes (such as sorting and classifying, matching, and ordering of ideas), actions become internalised. The resulting logico-mathematical knowledge makes handling unnecessary; the learner knows how things must be (see Chapter 4 for the role of constructive generalisation in this process). However, it is also this logico-mathematical knowledge (that is, knowledge at a level of reflective abstraction: see Chapter 4) that enables the student to experiment, to organise the experimental activity for maximum learning, and to understand the outcome of experiments (Sinclair and Sinclair, 1986, p.63).

Instructionally this must be translated into experience in which students may directly observe/handle materials both as a means for discovery (reinvention) about their properties and as a checking procedure for hypotheses made about likely outcomes. Throughout the activity, language is a vital tool for clarification, exchange of ideas, and the monitoring of inferences and deductions made. Indeed, the Piagetian position is that language and social interaction are essential aids in the meaningful construction of new knowledge. Clearly, the implication for instruction is that settings and formats for teaching and learning must encourage and support the use of discussion and group work (class- or small-group) in the classroom.

TEACHING FOR UNDERSTANDING

In classrooms, both teachers and students talk a lot about 'understanding'. "Do you understand?", "I don't understand", "I'm beginning to understand": these comments are made frequently as teacher and students engage in a learning process. 'Understanding', on most occasions, is used intuitively to denote an awareness of knowing the meaning of something.

In algebra learning, to understand means that new algebraic information is recognised with reference to (or in relation to) something that is already known; that is, an appropriate connection is made between the unknown and the known (Krutetski, 1976; Skemp, 1977; Davis, 1984; also see Appendix 3, p.4). In Piagetian terms, this is a process of assimilating new material into appropriate knowledge structures that accommodate to these demands, resulting in knowledge growth. It is essential, therefore, that as a prime consideration, instruction aims to develop students' understanding. Ways of doing this effectively must be devised within a constructivist perspective.

How 'Understanding' is emphasised in Talking Maths

Ideally (what teachers would like!) students would have already in place coherent knowledge structures that are relevant to new knowledge being introduced, so that appropriate mappings can be made. However, the actual situation is more likely to be that students' relevant prior knowledge may be fragmented and/or infused with inconsistencies (Davis, 1983, 1984). So what can a teacher do to assist algebra students to construct a relevant and coherent knowledge base? Gick and Holyoak (1983) substantiate with evidence their advice that it is not enough for a teacher to simply 'explain the principle' involved; the principle (or schema) must be presented in terms specific to a domain in order for it to be applied to a new situation. In other words, students do not learn by having a general notion explained to them; rather they learn from examples that are analogous to (have the same structure as) the idea that is new (Cheng and Holyoak, 1985; Cheng, Holyoak, Nisbett, and Oliver, 1986). In a complex domain such as algebra, this is likely to be no easy matter. Frequently a teacher's knowledge is implicit (both in regard to how the idea is understood, and what procedures may be appropriate to the situation), making clear communication difficult. Furthermore, the use of analogy is rarely characterised by large-scale coherent mappings between domains (Keane, 1985). Thus, it is necessary for teachers to tease apart the processes involved in the formation of a piece of algebraic knowledge in order to provide students with correspondences between examples in a step-by-step fashion. In this manner partial understandings are gradually built onto, integrating prior and new knowledge in a coherent way.

These learning needs are addressed, and appropriate processes encouraged, in **Talking Maths** in significant ways. Firstly, the abstract ideas involved in algebra are communicated explicitly through references that are very familiar to high school students. This approach seeks to foster the development of conceptual knowledge as summarised in chapter 3: it reflects a process that is rooted in prior experience and made coherent through the construction of relevant connections. Secondly, the choice of algebra content and the format of its presentation strongly reflects the schematic position outlined. As a beginning exercise in the selection of content for a classroom book, a hierarchy within an area of algebra was constructed (see Appendix 2). Starting with the notion of 'ordered pairs', a network of related concepts and procedures was structured to present a 'chunk' of algebraic knowledge. In the absence of research evidence that demonstrates how algebraic ideas (within the chosen area of knowledge) may be linked for maximum benefit in learning, the hierarchy was based on connections and commonalities in algebraic thinking and ideas attending particularly to perceived prerequisite concepts and skills. This task was aided considerably by extensive personal experience in teaching this work, including observing students' learning responses and assessing their performance. Thus the hierarchy is an attempt to blend conceptual and procedural knowledge in a network that optimises the students' opportunities to make appropriate connections.

This approach is distinctly different from courses which present algebraic 'topics', primarily in symbolic form without attaching meaning explicitly to the symbols used. **Talking Maths** represents an integrated, developmental program in which meaning, continuity, step-by-step progression and the growth of algebraic thinking are emphasised. Introduced through and founded on the fundamental concept 'variable', the program unfolds by developing and extending initial ideas and ways of thinking, unifying them through both content and process. This means that the algebraic knowledge presented represents a progressive extension and elaboration of algebraic concepts and procedures, while the mathematical processes involved support and coordinate their development.

'TALKING MATHS' : THE CONTENT

The 'Talking Maths' program begins by introducing the concept **'variable'**. This does not mean that it represents students' first contact with and initiation into algebra. Early algebra experiences (judging by text books commonly used in South Australia, for example, Haese, Harris, Haese, Webber and Danielsen, 1983; Lynch, Parr, Picking and Keating, 1986; Henry, Ebos and Robinson, 1983) give considerable attention to the use of letter symbols that represent some specificity within the domain of numerical values. Consequently this course is designed for students who have already have this experience in algebra. It assumes that students will be familiar with letter symbols in contexts such as

- assigning the letter a numerical value (evaluating)
- recognising the letter as a specific unknown (solving an equation)
- recognising that the letter may take more than one value (for example, plotting points on a graph).

These categories represent a progression of cognitive demand on the part of students (Kuchemann, 1984). In the first instance (in which the letter symbol is assigned a specific value), students have only to substitute a number for a value. The operation involved (evaluating) is a transformation procedure involving computation (that is, some form of counting); closure is reached rapidly. In other words, apart from the realisation that a specific number can be substituted for a letter, students are required only to perform an arithmetical calculation. The second category also deals with specific values. In this instance, students are faced with what might be called a straightforward 'puzzle' situation. A number exists, but it is unknown to the students. Its value can be found through a search procedure that may be known as an algorithm. Not that this algorithm should be considered as straightforward for many students even in simple problems.

Although a problem such as 'find the value of a in 3a-2 = 13' requires only basic arithmetical operations (for each side, add 2 and then divide by 3) there is considerable cognitive demand involved in (1) recognising how balance may be maintained in an equation (2) ordering the operations and (3) maintaining the use of a letter symbol for a number right up until the final operation when the solution is found. The third use of letter symbols involves evaluation, requiring an arithmetical procedure as in the first case. However, there is a move now to something more abstract than what has been encountered in previous examples; although specific values are computed, the context is now a 'data base'. Students are concerned not with just one value, but with a sequence of values within a defined domain. Implicitly, the notions of variable and function are being introduced; so while the procedure involved is straightforward computation, the concepts linked to it are possibly far removed from the students' known world of mathematics. Potentially, this can be a situation in which an algorithm may be used to attain a correct solution through manipulating numbers, (represented by letters) without meaning. Thus, in the introduction of 'variable' in **Talking Maths** special attention is paid to developing a sound understanding of the concept, so that associated skills and procedures are used with meaning. Indeed the program begins at this point in an effort to address a need that textbooks in algebra usually ignore. In my experience with students who are bewildered by algebra, difficulties usually involve (and may be rooted in) an overly narrow interpretation of the 'variable' concept (also Rosnick and Clement, 1980).

Letter as variable

In Kuchemann's hierarchy of letter use, letter as variable is considered to make the greatest cognitive demand on students. In understanding the concept of variable, the ideas of

- a letter representing a range of unspecified values

and - a systematic relationship existing between two such sets of values

demand that students 'distance' themselves from perceptual references, and demonstrate tolerance to lack of closure. In **Talking Maths** 'letter as a variable' is presented initially in a sequence of transformation procedures that enable students to become familiar with appropriate format, symbolism and processes. For example, it is essential that students break away from the often all-too-ingrained tendency to 'close' a procedure prematurely by assigning a variable a particular number, or by combining variables in ways that are appropriate only to numbers that are specifically known, such as considering the sum of 2a and 3b to be equivalent to 5ab. The rush to reach a numerical 'answer' (which students often perceive to be the purpose of school arithmetic) is counter to the fundamental idea of transformation, and inappropriate to its procedural format. A transformation involves an operation that enables an idea to be seen from a new perspective; and that is the notion that should be emphasised. In this context the idea is represented through algebraic symbolism, and the transformation procedure allows the idea to be presented in a different format that may be more appropriate to a given problem-solving situation.

An approach to arithmetic that emphasises structure

In **Talking Maths** students' knowledge in arithmetic is frequently used as a starting point for the development of new thinking in algebra. It is essential, however, that students' prior arithmetical knowledge provides an appropriate base for a study that uses structure as a means to represent, integrate and communicate mathematical ideas. The traditional view of school arithmetic as the mastery (and continued practice!) of basic computational skills is severely limited, and is inadequate as a foundation for algebra. Students who know arithmetic only as an unwieldy collection of isolated facts and rules that must duly be memorised are in no position to appreciate and come to terms with the processes and relationships involved in algebra. Expert problem solvers work, not from superficial elements of the problem, but from the principles or deep-level rules that characterise its processes. Thus the approach taken in arithmetic must emphasise the common structure that unifies processes involving integers (positive and negative), rational numbers (fractions, decimals and percentages) and their extension to numbers that are unspecified (variables). Although a detailed exposition of how such an approach might be implemented in primary school arithmetic is beyond the scope of this study, **Talking Maths** gives some indications of the way that structural considerations may be emphasised. For example on pages 32, 33, arithmetic is used to validate (or otherwise) students' hypotheses regarding the distributive law. This link between arithmetic and algebra is emphasised through recognising that the product 3 x 10 can be seen also in terms of a product 3 x (7+3) - or using any other numbers with the sum or difference of 10 - and a geometrical representation involving appropriate rectangles. A teaching sequence using this approach to teach multiplication of two-digit numbers to primary school children is outlined in Long (1983). The sequence models an approach proposed by Bruner (1968) in which instruction follows through a development of representational modes. Representation in the enactive mode involves children's investigations of square and rectangular numbers, extending to numbers of two digits, using tiles or flip blocks (flat, square-shaped counters, red on one side and yellow on the other). Through manipulation of the materials and discussion of outcomes in terms of commonalities and recurring patterns, children are directed to the characteristics that emphasise the processes involved in two-digit multiplication. For example, the number of flip blocks in a rectangle three by seven may be found by counting one by one, or more efficiently by threes or by sevens, or still more efficiently by multiplying three by seven. The number of flip blocks in a rectangle thirteen by seventeen may be found in the same ways; however, to make the efficient multiplication method more manageable the rectangle is partitioned into four sections three by seven, three by ten, ten by seven and ten by ten, and then the sum is found. The same concepts and procedures are then represented in the iconic mode (assisted by the use of squared paper), and finally linked to representations using symbols. Similar connections are made on pages 110, 111 between form and representation of linear equations. Pages 54, 55 also indicate another example in which pattern-making reveals commonalities that unify processes involving positive and negative integers, giving rules for directed numbers purpose and meaning.

Thus, students' transition from arithmetic to algebra is seen as beginning with, and also depending on, the study of arithmetic as related number systems that evolve in logical, structured ways, and are characterised by commonalities in concepts and processes. This structural perspective leads appropriately to the early study of algebra as a generalisation of arithmetic, as well as seeing algebra as a logical framework of ideas. Thus, there is a decided move away from the routinely manipulative skills that have been (and sometimes still are) the focus of many textbook presentations of algebra. Although **Talking Maths** is concerned with a level of skill proficiency that allows students to gain insights into the nature of algebra within a particular area, the emphasis is on algebra as a means of representation and communication, and on algebraic processing as a tool for problem solving. The aim is to present algebra as a language that is meant to be understood, and as an area of knowledge that has purpose and applicability.

Relations

A brief introduction to 'relations' and its significance mathematically continues the emphasis on algebra as a means of meaningful and ordered communication. In this section, algebraic expressions of the type that are reorganised and/or simplified through procedures such as addition or subtraction of like terms, expansion, and factorisation (pages 23-34) are revisited in other contexts and presentation formats. Here the focus is not so much on the procedures to be performed, but rather on the concepts underlying the expressions. Relations integrate concepts. Thus through considering the use of relations as connecting, unifying processes, students are given the opportunity to develop insights into the nature of the ideas that are communicated in algebraic expressions. In addition there is an emphasis on students' need to attach meaning to the symbols and symbolic means of expression used. A colleague who taught this section to a class of low-achieving Year 10 students (during the time that I was conducting the preliminary study) reported that her class were astonished (and fascinated) to find connections between language and algebraic symbols; the class group activities suggested on pages 42, 43 introduced entirely new experiences to them. Sensitive and knowledgeable handling of the section by this teacher developed in her students an appreciation of the language-symbol links. In contrast, two classes in the **Talking Maths** project (Chapter 7) who were left to work through this section without teacher intervention reported difficulties in being able to do the activities. It is possible that they had not fully recognised the point of the study on relations, or made sense of the problems. Also important is the need for students to recognise that meanings of relations are sharply defined within contexts, and that liberties cannot be taken with their use; indeed it is their very precision that is the essence of algebraic communication. Pages 45, 46 focus on this aspect, and while most students have found these activities to be very demanding, some have found an especial delight and fascination in solving them.

Directed numbers

The language of algebra cannot be understood or communicated effectively without flexible knowledge and use of negative as well as positive integers. My experience in teaching high school algebra has indicated that many students approach calculations involving directed numbers in ways that lack consistency. Particularly evident is the confusion between <u>adding</u> directed numbers and multiplying them. While the latter operation is usually acquired quickly through the application of rules, my observation and questioning of students suggests that few use these procedural rules with a sense of meaning. Thus, as soon as practice exercises that focus specifically on multiplication are replaced by use of directed numbers in a wider context, confusion results. Many students are unable to differentiate between addition and multiplication of directed numbers in algebraic expressions, and the result is that multiplication rules are applied whether or not they are appropriate to the situation.

The section on directed numbers (pages 47-63) addresses these perceived difficulties. Addition of directed numbers is introduced first and consolidated before multiplication procedures are applied to problems, and the distinction between the two is made explicit. In addition, the investigative approach used to determine the rules used for multiplication of directed numbers steers students away from the 'rule-given' approach to one where students evolve appropriate procedures through reasoning and discussion based on understanding.

Functions

Functions and functional thinking are vitally important elements in the language of algebra. The concepts of function has its roots in phenomenologically based relationships between quantities. An early definition given by Euler referred to the essential dependence of one variable quantity upon another variable quantity, and it is considered that this notion provides a useful aspect of thinking about functional relationships (Freudenthal, 1983). However it is the set-theoretical notion of correspondence (to show that certain objects are assigned certain other objects) that is the essence of the function concept, and the one that is developed in this section.

In many school textbooks, 'function' as a concept takes second place to procedures involving functions. Observed classroom teaching (typically following a textbook course) indicates that the idea is often introduced at a formalised, abstract level and in a context that emphasises manipulation of symbolic notation. Indeed, my impression is that an air of mystery and esoteric authority frequently accompanies students' introduction to functions; their applicability in real-world situations is not demonstrated, and an opportunity to build new knowledge from prior experience is missed. Students interviewed in the preliminary study were generally confused, even mystified about functions. A response from a Year 11 student was typical : when asked what he could tell me about functions his reply was "I think we did them last term." For this student, functions were text-book exercises to be performed by manipulating symbols in a prescribed way; clearly they were without meaning or purpose. Nor did he realise that he had been working with functions before Year 11, term 1 - through arithmetical operations, geometrical transformations (such as flips, slides and turns), representing algebraic expressions in tables and graphs, and in other input-output activities such as using a calculator or playing computer games. His responses showed that he was unaware of the unifying nature of functions and their power as a problem-solving tool in situations that involve relationships between quantities. However, situations similar to the one described above do not justify the omission of functions from high school mathematics. When inappropriate teaching methods influence negatively students' classroom experiences in an area of knowledge, there is reason to ask whether the knowledge serves a useful purpose and whether it can be taught more appropriately, before assuming that the only solution is to drop it from high school courses for most students.

The concept and use of function forms an integral part of this algebra course because the study of functions is considered to be a useful and significant part of mathematics learning. Functions are a tool for problem solving - a means to represent and interpret many situations (both those occurring in the real world, and those created by mathematical thinking) involving relationships between quantities. The creation of graphing packages and spreadsheets in computer technology means that there is a much readier access to working with functions than was formerly the case, and a much wider applicability. Functions represented by these means (and particularly through interactive packages) can form a particularly relevant and useful part of mathematics learning. Although these means are outside the scope of this course, the introduction to functions in **Talking Maths** is seen as a beginning to working with functions in flexible, meaningful ways.

Thus, in **Talking Maths**, functions (pages 64-81) are presented in terms of a concept that has meaning and relevance to the real world, and as a study that is developed from the students' prior experiences. It is formally introduced at a lower year level than what was observed in the preliminary study to be normal practice in South Australian schools. This is so that students are given the opportunity to understand the concept over a period of time in which associated procedures develop from simple, relatively concrete ideas towards those that are more complex and abstract. The aim is for students to develop, through a step-by-step process of concept attainment, an elaborated understanding of functions. This involves an increasing appreciation of the pervasiveness of functions and their power in representing situations that are complex in ways that are more manageable. This approach follows the theoretical framework of this study which sees knowledge development as an interaction of generalisation and abstraction processes, rooted in prior experience. More advanced knowledge is characterised by increasing abstraction and a greater power and applicability in relevant situations. The development of procedural knowledge focuses on applying ideas that have meaning for the students. Procedures begin with familiar experiences in pattern making (both in numbers and geometrical representations), focusing on the prediction of outcomes through analysing and interpreting a pattern of events. Thus, the emphasis is on solving problems through modelling them as functions. Conceptual knowledge is developed in context with and interrelated with the development of appropriate symbols and methods - in an interaction which is seen to promote productive thinking in task situations.

Linear functions and their representation

Linear functions were chosen for special attention in **Talking Maths** for two related reasons. My teaching experience and interviews with students in the preliminary study indicated that students' familiarity with straight line representation is extensive, but that it lacks the coordination needed for systematic study. This makes linear functions a good focus for the building of new knowledge from prior experience. Secondly, representations of linear functions in tabular, graphical and symbolic forms provide a simple base for making connections between different means of representation and for recognising how to represent sets of data in ways that are most appropriate to a given problem-solving situation.

Within this section, linear functions are presented as one kind of function to be investigated, with a view to developing more extensive functional thinking and knowledge. In order to achieve this (to more than an informal level) the criteria that define the representation of a particular straight line in a cartesian plane (gradient and axes intercepts) are examined in some detail. These investigations begin with looking at gradients of straight lines from several different perspectives.

Gradients

Although straight lines and their gradients describe many situations and circumstances in the immediate world, my classroom experience suggests that students rarely recognise that familiarity. When students consider representation of linear functions on cartesian planes, straight lines frequently become divorced from any other context or situation: it becomes an esoteric study unrelated to the world outside the classroom.

During the preliminary study I taught (in a 7-week contract position) a Year 11 course that was designed specifically to familiarise students with mathematics in their day-to-day world. During the section on gradients it became apparent to me that while lines and gradients are present in many everyday or familiar settings, the students had not recognised them in <u>their</u> world, or considered them to be something that could be investigated mathematically.

In **Talking Maths** some instances of real-world modelling of linear functions are demonstrated in contexts that, in my experience, are meaningful to high school students. The emphasis here is not on drawing the graph but on interpreting it. My interviews with students in the preliminary study and my examination of textbooks normally used by them suggests that the students have had a great deal of practice in plotting points and drawing graphs (though maybe fewer experiences in systematically collecting and representing data), but they rarely have been given the opportunity to interpret graphical representations of real-life situations. The section on linear functions thus seeks to emphasise graphical representation as a means for problem solving and communication. In particular, comparisons between different gradients indicate the significance of this factor in linear modelling. Problems on pages 93-98 give students the opportunity to compare the ratios represented, describe trends, estimate and predict, and to develop argument through discussion.

Assigning gradients a number, developing a rule for calculating gradients, and calculating gradients are also included in this section. During the preliminary study I observed and questioned students at Years 9, 10 and 11 levels about numerical descriptions of gradients. Many were not able to look at a surface within or outside the classroom and suggest an appropriate number that might be ascribed to its gradient. Students were usually not aware of the relation between vertical and horizontal displacements, nor of the fact that the ratio of displacements described the gradient of the particular line. For most students, a formula for gradient was memorised as given and applied as prescribed; what it meant was not really considered or seen as important. Thus, in part of this section (pages 88-92), student investigations are aimed towards developing competence in describing gradients numerically - that is, in acquiring basic concepts and procedures, and seeing them as connected. This includes recognising commonalities in gradients (such as in the gradients of parallel lines, that is, translations) and relationships formed (as in gradients that are perpendicular, or when one is a reflection or rotation of another).

Rules for Linear Graphs (Linear Equations)

A vital aspect in making sense of mathematics is making connections between different modes of representation - in this case between the graphical and symbolic forms. This section of work takes on a more analytical approach, in which the gradient and position (on a cartesian plane) of a straight line is connected explicitly to the rule (or equation) that represents it. The teachers of the year 9 classes that took part in the **Talking Maths** project considered the section as being too advanced for most of their students. However, I discussed parts of the section with a small group of students (considered to be "fast workers") who graphed and grappled with the ideas involved with a good deal of enthusiasm.

In the preliminary study I taught this section of work to a high-achieving year 10 class. As a class they were taken aback by the links that they discovered between 'the rule' and 'the gradient and position' of straight lines; there had been absolutely no expectation of connections, so my class investigations were designed so that the symbolic and graphical links literally 'jumped out' at them. Thus, this section in **Talking Maths** seeks to make relevant and meaningful the rules (or equations) that represent symbolically straight lines drawn on a cartesian plane. The focus is on using these rules to find commonalities and connections that can provide students with the resources for independent problem solving. For example, these activities aim to facilitate quick sketching of straight-line graphs by looking for numbers in the rule that refer to gradient and y-intercept. This predictive power is important for working successfully with problems that are solved by curve fitting. Competence in this area also provides an essential foundation for problem solving in linear programming. An introduction to straight lines as boundaries of areas also begins a conceptualisation of the processes involved in this area of work. Thus although this final section of **Talking Maths** begins to explore linear functions in more abstract, analytical ways, the investigations are seen as important experiences in developing relevant problem-solving abilities, plus a foundation for generalisations to higher degree functions.

DEVELOPING UNDERSTANDING : METHODOLOGY

While the content of **Talking Maths** is developed in a way to assist students to make structural (deep-level) connections between what is known and what is to be learned, instructional approaches also address the importance of understanding the algebraic ideas that form a basis to the study.

Two major algebraic concepts are developed in this course: **variable** and **function**. If students are to work confidently and competently with these abstract ideas, they must have a sound grasp of what they are and the purpose of their use. My preliminary investigations showed that few students have really thought about what a variable is; and the conclusion that I drew is that it has not been discussed or brought to their attention in most classes. Typical responses to my questions about what students knew about variables were fairly incoherent comments about 'x and y'. I also spoke to a group of mathematics graduates who were studying to become mathematics teachers. None was able to articulate the meaning of variable. Granted it is a difficult concept to describe; but only one had thought about discussing 'variable' with school students. He said that his own teacher had made an attempt to describe what a variable is when he was in year 8.

Developing concepts : variable and function

Both cognitive scientists and Piaget demonstrate a constructivist approach to learning that considers as crucial the sound development of concepts. In this, the construction of relationships is critical: from an information processing perspective, concept development refers to the formation of a connected web of pieces of information, while the Piagetian view focuses on the links between experiences, made through action, to form a network of knowledge.

In practice, precise distinctions between the two perspectives are less important than recognising commonalities that can support educational practice (see Chapter 5). In **Talking Maths**, the development of the concepts 'variable' and 'function' is assisted through experiences involving what students already know and can do, that is, their prior knowledge. These experiences are presented as activities (specifically group activities: but this aspect will be dealt with later). Through active involvement with familiar ideas - that is, through contact with examples that are progressively integrated into the system - students are able to use what is known as evidence that both predicts, and substantiates conjectures.

These examples individually involve the use of analogy as a point of reference. In this process structural elements within one mode of representation are mapped over to a target domain and is applied appropriately (see Chapter 2: Problem representation). VanLehn (1986) gives evidence from cognitive science to suggest that the drawing of analogies engenders conceptual knowledge, but is far less appropriate to the development of procedural knowledge. In developing the concepts of 'variable' and 'function' in **Talking Maths**, analogies are initially used to engender meaning in examples, which in turn generate the twin processes of generalisation and abstraction in the growth of conceptual knowledge. However, in this course 'what is known' (the examples) does not necessarily mean 'what is perceptually referenced'. This needs some explanation. In Chapter 4 it was argued that while Piagetian theory sees conceptual knowledge as necessarily derived from and validated by physical experience, relevant evidence may also be provided through activities that are strongly connected to prior concrete experience. Thus, while students are asked to work with ideas that are necessarily abstract, activities have been designed to enable students to root progressively constructed abstractions in previous experiences involving action on their environment.

Concept of variable

The development of the concept 'variable' follows this sequence in the course:

(1) The idea of variable is shown to be embedded in familiar arithmetical knowledge.
(2) Activities involving their every-day world serve to elaborate students' understanding of the use of variables.
(3) Symbolic representation of variables is tied to this experience-based understanding.

(4) 'Variable', in symbolic form, is used in a variety of procedures to establish its role in problem solving and the accepted format of presentation.

Thus, 'variable' is presented in ways that attempt to show the essential characteristics of the concept, through perceptually-referenced experiences, and in different contexts and modes of representation. Through these means, students are given the opportunity to construct an understanding of 'variable' that is strongly connected, and embedded in rich and meaningful experiences.

Concept of function

A similar sequence of activities is suggested to develop students' understanding of 'function'.

(1) Functions and functional thinking are presented as familiar and relevant aspects of the way that humans seek to organise and interpret the world. A brief reference to the historical development of the concept of function indicates to students the evolving nature of mathematical thinking.

(2) Activities and uses in everyday life serve as examples in students' understanding of the idea of function. Students are led to realise that while the word 'function' may be assigned a variety of different mathematical acts it is characterised by an essential (and unique) idea of correspondence.

(3) Activities involving functions are presented within familiar contexts and through known procedures.

(4) Graphical and symbolic representations of functions are tied to this experience-based understanding.

(5) 'Functions', represented in graphical and symbolic forms, are used in a variety of procedures to establish their role and purpose in problem solving.

As in the development of 'variable', the instructional approach is to develop an understanding of 'function' that is based on investigation within a variety of explicitly-related settings and representations. The aim is to promote sound conceptual development for key ideas that are used in problem-solving procedures.

SUCCEEDING: and its interaction with UNDERSTANDING

The need for learners to understand the algebra they are studying - to make sense of the algebraic language as a concise and precise means of communication - is emphasised throughout **Talking Maths**. It is recognised that an essential part of problem solving within the domain of algebra is to understand what the problem is about. This discussion began earlier with an overview of the algebraic content of the course, and its presentation as a meaningful, coherent and relevant body of knowledge. The development of knowledge throughout the course focuses on the connections that can be made between the known and the unknown, and the commonalities and relationships that are made evident through the selected sequence of ideas and skills. Thus the focus so far has been on assisting students to construct knowledge representations that are coordinated and well elaborated.

The importance of making meaning in algebra, however, is not limited to understanding the language of algebra and the nature of the problems. It also includes doing: performing algebra competently and finding solutions to problems. The theoretical discussions in earlier chapters indicate clearly that conceptual and procedural knowledge are necessarily interactive in the problem-solving process: understanding and succeeding are inextricably intertwined in students' successful performance.

Thus **Talking Maths** is not limited to a presentation of content, developed to emphasise and encourage understanding. An additional feature of this hand-book is a methodology for classroom instruction: a coordinated program of approaches, methods and teaching ideas designed to guide, support and interpret algebra learning and problem solving in a high school classroom. This adds a new dimension to the presentation of a school algebra course. The processes and means of learning are seen as critical factors, along with the content to be learned. The knowledge considered important for students consists not only of what is to be learned, but also how it may be learned. Certainly the methods and approaches to learning are essential knowledge for the teachers involved in algebra instruction; but they are also a means of creating awareness on the part of students about learning processes, suggesting how their learning may possibly be enhanced.

The instructional approaches suggested in **Talking Maths** represent an interpretation of how educational application may be derived from psychological theory. With the complexities and uncertainties of classroom practice in mind, a synthesis of epistemologies has been constructed in order to predict, support and guide algebra learning and problem solving more adequately than, it is argued, a singular approach could provide.

There is no assumption, though, that psychological theory, however comprehensively constructed, can match precisely the needs and concerns of educational practice: the task is too complicated for that. However, as has been argued in earlier chapters, a collaboration between research programs - variously focused on the acquisition and development of knowledge, and its application to classroom mathematics - is considered to be the best resource educators have to hand.

TALKING MATHS: PROBLEM SOLVING IN THE CLASSROOM

The complex interaction of conceptual and procedural knowledge that is seen to be an essential feature of the problem-solving process (eg, Piaget and Voyat, 1979; Silver, 1986) is addressed in the methodology contained in **Talking Maths** in several ways. It is recognised that there is no straightforward approach to dealing with this issue; however it is considered that the synthesis of psychological theories constructed in this study provides direction and prediction for developing appropriate classroom methodologies that emphasise performance with understanding.

A constructivist approach

Knowledge acquisition from a constructivist perspective focuses on students' building up of knowledge through an active, step-by-step processes. Essential components of learning and problem-solving processes, according to the theoretical base presented in this study, involve the students' prior knowledge/experiences, with new knowledge/actions interpreted and constructed through personally-referenced meanings. Furthermore, the knowledge/actions outcome is a further part of this development. The teacher's task, then is to promote this development in students; and in algebra, this means solving algebraic problems. Procedural knowledge is a necessary part of this competency; and, as demonstrated, this skill is developed in interaction with the problem solver's understanding of the task. In Piaget's later theorising (Piaget & Voyat, 1979) and within the cognitive science framework (VanLehn, 1986; see also Chapter 3), 'induction' the term used for the generalisation and integration of examples through successive abstractions) is proposed as a means for developing procedural knowledge in meaningful ways. In the case of algebraic procedures, an 'example' is an <u>execution</u> of a procedure; and learning occurs when the student generalises this action and through abstraction incorporates it into his/her procedural knowledge. The theoretical perspective developed for this study demonstrates that these generalisation processes and the recognition of structural (abstract) commonalities involves an active construction of knowledge.

Classroom instruction that reflects this view of learning will naturally differ considerably from approaches where the teacher transmits prescriptive (as a rule!) procedural information to a passive class. Significantly the focus will be on active student involvement (in constructing knowledge), aided by discussion and instructional approaches that support understanding. Induction (that is, inductive reasoning) is a key method used in **Talking Maths**, especially when new ideas are being introduced. In the book it is called **investigation**.

Students' 'investigation' in Talking Maths

An investigatory approach to learning encourages students' involvement in mathematical thinking, building up through many experiences and language the notation and rules that make up algebra. It enables students to discover and to invent during the process of abstracting the essential mathematical characteristics of the concept and/or procedure. The interactive, step-by-step nature of the sequence provides the learner with manageable and continuous feedback on his/her performance.

The investigatory approach used in this course involves, typically:

(1) the examining of an initial idea - in the light of something already known,

(2) further exploration of a sequence of similar ideas', and

(3) constructive feedback on students' thinking and performance-attempts to provide information for further exploration.

Essential to the method is discussion in which students' suggestions, inventions and attempts are

(1) valued as valid contributions

(2) seen as starting points for further learning

(3) used as a basis for inventing new procedures and

(4) allowed to develop in a supportive, non-threatening environment.

A feature of the 'investigation' method is the extent and the nature of assessment that it allows. In many classrooms, assessment is a formal, entirely summative, written test, administered at the end of a topic section. The feedback to students tends to be in terms of a grading or mark, along with labels attached to the learner and his/her performance. The opportunity to use student errors in productive, constructive ways is missed, and the mistakes made tend to be seen as an indictment on the students' capabilities and/or efforts rather than as a legitimate part of the learning process.

From a constructivist perspective, assessment should be, like learning, seen as a <u>process</u>. Summative assessment has its place in educational activity, but formative assessment is a far more appropriate model for day-to-day use in classrooms where, it is anticipated, all students will be engaged in an ongoing learning process. Formative assessment has enormous advantages in that much of it takes place in verbal, interactive forms. Responses to questions can be given quickly, and the student's understanding of the response (or question) can be gauged/monitored by the teacher and other students. Errors and misconceptions can be identified and used as a basis for building adequate conceptualisations, instead of being left in place to distort thinking and approaches.

Quizzes (see page 22)

In **Talking Maths** student and teacher investigations (eg, pages 23, 24, 34, 36, 48, 49, 60, 61, 71-73, 80, 81, 103-105, 109, 110, 115-117, 120, 121) are often initiated through the use of quizzes. This approach to learning by induction was developed originally with a year 4 class of very active 8-9 year olds: children who responded well to constant attention and stimulation in learning. During the preliminary study I adapted the idea successfully to 12-15+ learning. The method is simple: the teacher works with the students as a fellow learner, except for one crucial difference - the teacher knows the work and is therefore aware of optimum, reasonable, and unacceptable directions for learning. The role of 'fellow learner' is not difficult to take in quiz sessions; the teacher is learning during them - watching, and learning about the students' knowledge, their intuitive approaches, and their learning needs. Far from being a routine or passive procedure, the quiz method is a lively, active, interesting and often exciting approach to learning for students and teacher. It creates a state of expectancy, demands involvement on the part of all students, and gives a high return of achievement and satisfaction. Because the teacher is observing/listening to the students as they discuss new ideas, s/he is able to match the learning activities to perceived levels of student understanding, and other learning needs. For example, in this method the teacher is made aware of how demanding the new work is for the students, giving an indication of how to regulate the introduction of more advanced ideas. It provides a diagnostic tool too. Students' responses sometimes indicate difficulties that are too extensive to be dealt with in a whole-class group; in such cases, I've found students to be quite satisfied to wait if I say "we'll form a small group to look at this problem again when the whole-class session is over." Students know then that a special session that meets their needs and pace is given high priority in the teacher's program. Another positive feature of the 'quiz' method, is that the interaction method tells the teacher in no uncertain terms when the students have 'had enough' and that the concentration needed for demanding learning is not being maintained.

Theoretical basis for the quiz method

The approach outlined above suggests an unproblematic situation, with learning taking place in accordance with what a student knows. From an information-processing perspective the sequence might be seen as involving the extension and elaboration of existing knowledge, directed towards mental organisation that focuses on the essential structure of the problems. This increasing abstraction assists the development of appropriate procedures for solving problems. The work on expansions on page 31, for example, starts with already constructed arithmetical knowledge involving a well-known number bond (addition or subtraction of numbers under 10), and multiplication (or division) of numbers familiar from the 'tables'. Extension of this knowledge is facilitated in the second quiz by repeating the previous quiz in structure, but with one of the specific numbers replaced by a variable for each problem. The aim is to use the familiar problems as examples, and in the process to generalise procedures using specific numbers to those using non-specific numbers. The induction process continues in quizzes 3, 4, 5 and 6 as previously constructed knowledge is used in problems which are novel only in the format in which they are presented. Thus, the same ideas and procedures are re-presented in ways to develop elaborated meanings, and competence in performance.

In Piagetian equilibration terms, the mathematical activity that students engage in during these generalisation and integration processes stimulates the construction of meaning. Students' mental structures are modified in this experience, enabling enriched understanding that enables activity on further related ideas. Specifically, the action-experience that students engage in in quiz 1 provides examples (that can be observed through physical activity if necessary) to make meaning of the situation. Through providing a number of identically-structured examples, the aim is for students to coordinate empirical abstractions so that thinking focuses on the essence of the problem. Quizzes 2-6 provide activities designed to activate in students a knowledge imbalance or knowledge 'gap', in this case, an awareness of not knowing how to do the problems faced. The introduction of more advanced procedures is sequenced in order to assist abstraction processes through the inductive generalisation. However, as pointed out in Chapter 4, students cannot be taught equilibration: their own experience, and their interpretation of it, must form the compensatory constructions needed for attainment to a higher state of knowledge. The quizzes presented in this book reflect the typical progress of students who took part in the preliminary study. Clearly teacher monitoring and intervention must allow for differences between classes and individuals.

CONFLICT

While the non-problematic situation described above usefully summarises much of the learning that might take place in an algebra classroom, it does not give an account of learning that is not optimally developed from what is known. What is known sometimes provides the starting point for a false or inadequate construction of knowledge; so what is known must be changed.

In this study it has been asserted that while the information processing description of extension and elaboration of knowledge is not sufficient for situations requiring knowledge change, equilibration theory has the potential to do so. Instructional approaches designed to translate this conflict theory into practice are cited in Chapter 4; in this study class-based approaches are clearly necessary.

Implementation of 'Conflict' teaching

In **Talking Maths** - a program that represents day-to-day curriculum-driven classroom algebra - a sequence following Rowell and Dawson's (1983) model has been adopted. This model takes into account the criteria suggested by Rowell and Dawson (1985) (outlined in Chapter 4) as applicable to equilibration theory. Of particular importance is the presentation of a powerful alternative theory that is made available to students in a challenging but non-threatening way. The teaching strategies suggested by Rowell and Dawson involve:

(1) finding out students' intuitive solutions to a given problem

(2) accepting all solutions as reasonable

(3) retaining students' solutions - possibly by means of a 'paper memory' as Rowell and Dawson suggest in their 1985 study

(4) teaching an adequate solution, using ideas that form part of the students' knowledge

(5) comparing all solutions, and reaching a satisfactory conclusion via classroom consensus.

In adapting these strategies to the program presented in **Talking Maths**, it was important to use a methodology that could be used routinely by teachers whenever conflict methods seemed applicable. Thus the methodology is not seen as an instructional feature that is used only when major imbalances occur in students' knowledge, but as a procedure that can be easily adopted whenever students' generalisations are preventing their adequate construction of new knowledge.

During the preliminary study I monitored students' learning with a view to selecting sections within the algebra program where conflict teaching might be employed most profitably. Insights gained from my own teaching experiences also assisted significantly in my planning of appropriate teaching intervention. Clearly, conflict teaching is not a needed feature in much algebra instruction that takes place in a classroom. Students' intuitive solutions and ways of thinking do not always need to be changed, and of course, the situation varies from student to student. So, in **Talking Maths**, conflict teaching is suggested in situations where, it seemed to me, false- and/or over-generalisations were most likely to occur. The use of the strategy, however, is left to the discretion of teachers implementing the program.

Conflict in 'Talking Maths'

A typical 'conflict' sequence is outlined for teachers' use on pages 30, 31, 32. The suggested approach emphasises

(1) students' prior knowledge, especially their intuitive solutions to a given problem

(2) the acceptance of all solutions as 'reasonable'

(3) discussion to promote awareness of the nature of the solutions

(4) the teaching of an adequate solution that is based on students' prior knowledge and which provides concrete-referenced evidence for adequacy of solution.

(5) consolidation of adequate solutions (and the giving up of inadequate solutions) through large- and small-group discussion and practice.

The particular teaching sequence outlined on page 32 aims to develop students' understanding and competence in the distributive law. My school experience suggests that the distributive law is often seen by teachers as 'routine' and 'non-problematic'. Furthermore, it is taught usually as a procedure to be memorised, and thus many students manipulate the symbols involved without attaching meaning to them. So, while students may be able to manage a straightforward operation involving integers, for example, 'rewrite 3(x + 5)', they do not recognise that 'simplify $\frac{3x + 6}{3}$ involves the same principles.

Furthermore, a limited understanding of the law leads students to believe that, for example, (x + 3) (x + 5) may be rewritten as $x^2 + 15$. During the preliminary study this misconception was demonstrated graphically in a video lesson that I conducted with a mid-level-achieving class of 13-14 year olds (Year 9). Students were asked to record on paper their intuitive responses to several problems of the type (a + b) (c + d), using both specific and variable numbers. During open class discussion of possible solutions to the problem, only one student suggested a sum of four products (ac + bc + ad + bd). A teaching sequence followed in which I provided a solution based on reference to the area of the facade of a rectangular building (a + b) by (c + d), that could clearly be seen as the sum of four rectangles. Discussion indicated that there was a general consensus that this solution provided a better alternative to that which most of the students had suggested previously. Furthermore they were quite confident in the procedures involved, that is, the calculating areas of rectangles, and adding parts to make a total. Assessment during and at the end of a section on this and related work showed an excellent retention of the procedure, and in varied contexts.

Although the situation described above considered work that is not included in the **Talking Maths** course, the problems on pages 32-36 involve the same principle. So it seems good reason to promote a conflict situation in this introduction to the distributive law, with the aim of developing a framework of knowledge which is adequate for the task in hand.

Conflict strategy

The teaching strategy is initiated through students' involvement in familiar, related activities. In particular, students' relative confidence in arithmetic provides a starting point for the sequence. Students' intuitive solutions to problems are presented in 'open forum' in which ideas are discussed in constructive ways. My experience suggests that it is likely in this particular example that the 'better theory' will be provided by students; thus, teacher intervention most appropriately can provide evidence (arithmetical and diagrammatic) to demonstrate the worth of the 'adequate' solutions. Practice of the procedure is suggested as a sequence of quizzes, in which the distributive law is presented in different contexts, and with a step-by-step increase in cognitive demand.

Similar approaches are suggested in the introduction to directed numbers and equations of straight line graphs, and in investigations involving special gradients and gradient relationships. In the preliminary study, conflict strategy was used (in my judgement, successfully) with two classes in introducing directed numbers (mixed ability year 9: 13-14 year olds) and with one high achieving year 10 class (14-15 year olds) investigating linear equations. One of the year 9 lessons is reported below.

<u>Introduction of directed numbers : Year 9</u>

Aiming to draw on students' intuitive thinking, I wrote 5-2 = on the board and asked the students to suggest a solution. Not surprisingly, there was complete agreement on 5-2 = 3! Next, I wrote -5 + 2 = on the board and asked everyone to write down a solution. Solutions offered were:

$$- 5 + 2 = -3$$

$$- 5 + 2 = 3$$

$$- 5 + 2 = -7$$

$$- 5 + 2 = 2$$

$$- 5 + 2 = 7$$

- in that order. I wrote all solutions on the board without comment. During the ensuing discussion, students were asked to defend their solutions. Arguments given were:

- 5 + 2 = -3 : take five, add two, is negative three;
and, go down five, go back two; you're at negative three

- 5 + 2 = 3 : five take two is three

- 5 + 2 = -7 : five and two are seven; but there's a negative in front of the five, so it must be <u>negative</u> 7.

- 5 + 2 = 2 : you can't have negative five things, so negative five is the same as having nothing; nothing and two is two.

- 5 + 2 = 7 : five and two are seven.

The solutions were left in place while I used another part of the board to lead an investigation into addition of directed numbers.

Teaching strategy

The teaching strategy followed the outline presented in

Talking Maths, pages 46-49. It is based on the idea that positive and negative numbers have equal status; thus, adding a positive number to a negative of the same number (or adding a negative to a positive) yields no outcome. Thus, just as + 1 - 1 = 0, so also -1 + 1 = 0. Another way of recording this operation: (+ -), emphasises that when a positive is 'paired' with a negative, there is no additional ('left-over') outcome.

Class Investigation

To embed the approach in context, I asked students to suggest an end point if: they climbed up 1 stair, then climbed

down 1 stair: recorded	+ 1 - 1 = ?
down 1 stair, up 1 stair:	- 1 + 1 = ?
up 2 stairs, down 2 stairs:	+ 2 + -2 = ?
down 2 stairs, up 2 stairs:	- 2 + 2 = ?
up 3 stairs, down 2 stairs:	+ 3 - 2 = ?
down 2 stairs, up 3 stairs:	- 2 + 3 = ?
down 3 stairs, up 2 stairs:	- 3 + 2 = ?

Also, for example, -3 + 2 was recorded as: (- +)

(- +)

\-

(yielding an outcome of -1)

(If necessary, connecting the idea to a banking transaction can present an additional reference to prior knowledge. For example, if \$1 is put into a bank account, and \$1 is drawn from the same account, there is no change in the original total. Similarly if \$1 is drawn from the account, and then \$1 is put into the account there again is no change. These transactions can be recorded mathematically as $+1-1=0$ and $-1+1=0$.)

Reaching a consensus

In a class group, we now returned to the original problem $-5+2=?$, recorded on the board with the solution suggestions. Through discussion the students decided which procedure adequately solved the problem, and why.

SOCIAL ASPECTS OF CLASSROOM IMPLEMENTATION

Social aspects of the framework presented for classroom algebra learning have been evident in the methodologies outlined in the previous section. Clearly, social interaction must form some part of classroom instruction: but how important is this aspect and what forms can it take within the theoretical framework?

Importance of social interaction : theoretical perspective

The title of the classroom program **Talking Maths** indicates that communication is considered, from this study's perspective, to be an integral part of algebra learning. For one thing, algebra is a language - a form of communication - and that suggests immediately that social interaction is an essential part of developing understanding and competence in using it. However, from the point of view of the instructional framework proposed, exchange between learners is more than a means of communication - it is considered to be a vital element in the process of elaboration of (and possibly change from) known ideas, that is, the construction of new knowledge.

Thus, from a constructivist perspective, students who are learning need to co-operate, that is, operate together in an active building of new ideas. Cooperation in this sense does not necessarily mean a good-natured sharing of activities. Rather, it refers to situations in which students and teacher strive for progress in demanding tasks, arguing for strategies, and making decisions in relationship to the thinking and ideas of others. It includes looking at one's ideas critically, seeing possibilities in the suggestions of others, and being persuaded to alter some thinking that is shown to be inadequate or false. Particularly, the Piagetian perspective of learning as change essentially involves the experience of disturbance; and a sense of that disturbance is most likely brought about by interactions with others in a common task.

In such situations, of course, it is necessary that the participants are in a position to make informed decisions; there is likely to be little cognitive progress if ideas are merely stirred around. Thus, although the constructivist framework developed for this study may support social equality, in classroom practice it is necessary for the teacher to be knowledgeable in mathematics and learning theories in order to promote maximum effect and efficiency in learning developments. This does not mean, though, that theory may not be compatible with practice; in this chapter, learning situations have been suggested that give students' contributions equal status, and avoid placing them (or their ideas) in any position of threat. Thus, while students may be challenged on the basis of their existing beliefs, strategies are available to work towards re-conceptualisations that (a) use their prior knowledge and (b) are discussed frankly in a constructive, supportive environment.

A further aspect of the instructional theory considers the need for students to investigate, analyse and review systematically the procedures they are adopting and the knowledge that they are acquiring. This self-awareness, and the self-regulatory aspect of problem solving can be considered within the area of metacognitive decision making in the classroom (eg, Baird and Mitchell, 1986; Baird, 1986; Lester and Garofolo, 1982; Schoenfeld; 1983, 1985; Long, 1986; see also Chapter 3). This approach emphasises social interaction as a basis and motivation for students taking control over their own learning. Taking a metacognitive approach in the classroom may involve students in

(a) understanding better their own thought processes
(b) monitoring problem-solving behaviour in ways that lead to more effective performance
(c) changing their beliefs about mathematics (Schoenfeld, 1985). These are processes involving self-consciousness,

through questions, reflection, self-criticism and so on. They involve internal dialogue, which, as Kitchener (1981) points out, arises from external dialogue. For it is through defending one's ideas to others, evaluating approaches, seeing the other person's point of view, and sharing thinking tasks that students learn to look more closely at their own thinking and decision making.

Social interaction : practical ideas for the classroom

The two interrelated theoretical aspects discussed above - cooperation and cognitive control in learning - characterise the methods and approaches suggested for classroom algebra in the program **Talking Maths**. Chiefly social interaction is suggested through two main classroom arrangements: class-group discussion and small-group problem-solving activity.

Class-group discussion

This is the chief classroom activity suggested in **Talking Maths**. In the preliminary study, most of the lessons involved a considerable amount of class-group interaction. Two main formats were used:

(1) In introducing new work, I took the role of scribe, coordinator and provocateur. Sometimes, some exposition was necessary, but it was offered in a context of mutual concern and need, rather than transmitting a set of instructions to a passive audience.

(2) Practising (consolidating) of known ideas usually took the form of quiz sessions in which ideas were offered, tried out, evaluated, changed, refined and so on. In these sessions my role was to question, coordinate, monitor and provide feedback (including formative assessment).

These two formats are thus recommended in **Talking Maths** on both theoretical and practical grounds.

The advantages of class-group discussion are numerous. In particular, a teacher who is well-informed theoretically is able to use class sessions as a means to monitor students' responses (and assess needs) according to critical learning features. For example, as I listen to what students have to say I can assess informally their understanding of the concepts and processes involved and monitor their development of relevant procedural knowledge. It is an excellent opportunity to act as role-model, too. Students learn that teachers do not always expect to solve problems instantly and easily, that mulling ideas around is worthwhile, that there is no shame in trying out a strategy that comes to nothing (but that it is good to check-up every so often to see if the strategy seems still worth pursuing), and that the puzzling nature of algebra involves fascination and satisfaction. Furthermore, the class-group method means that the process of learning is shared. Everyone is in it together; and the 'burden' is somewhat lighter because you are not left to struggle on alone. I've found that students become more confident in tentatively suggesting initial ideas in this shared situation because someone else may be able to clarify it or extend on from it. Alone, the student can become submerged in uncertainty, whereas in a group, feedback and assurance are always at hand. From a teacher's management point of view class-group sessions also provide an environment where everyone is involved and concentrating. Quick puzzles followed by class discussion and feedback in a supportive setting gives everyone a reason and opportunity for active learning.

Small group problem solving activities

Two formats for small group activities are also used in **Talking Maths.**

(1) Small groups are seen as an appropriate means for discussion of new concepts, for example, in the introduction to the term 'variable'. Small groups are also ideal for solving problems where the concepts and procedures required are familiar to students; through discussion, suitable representations can be constructed and appropriate strategies identified and tried out.

(2) Forming a group with others who share an immediate concern is seen as an efficient and productive means towards learning development and improved performance in problem solving. Small groups formed for this purpose usually are based on a common task (doing a section of work together) or common need (requiring assistance for a problem).

Although small group work is not easy to manage in a class of 30 or so high-school students, it provides an excellent opportunity for sharing ideas, and for the teacher to attend to students' special needs. Students who are unable (or unwilling) to articulate their concerns in front of the whole class, or who are struggling to formulate in their minds how a confusion has arisen, may find small group work more acceptable. Also, students who do not realise that they don't understand, or who feel vaguely disturbed or somewhat threatened about what they're doing, usually appreciate sharing their ideas with two or three others, including, perhaps, the teacher. Finally, practice in algebra is seen as an important means towards enhanced understanding and improved performance; and in **Talking Maths** practice in small groups is seen as a better alternative to working alone.

Motivating the students

In the final analysis, motivation is one of the prime tasks of a teacher. In general, motivating students to learn algebra is difficult, demanding and frustrating; yet it cannot be underestimated in its importance, because unless the students are motivated, they may not learn at all. Attempting to incorporate motivational elements into a written course is a daunting task: there is no substitute for a teachers' skill and enthusiasm in exciting students to learn. But it is also an element which cannot be disregarded by a writer. Throughout my work in writing **Talking Maths** I was mindful of the need to present the algebra content in ways that are meaningful to students and relevant to their interests: to present algebra as a language in which ideas are related, and meant to be communicated. It is considered that these factors are important motivational elements. Most importantly, high motivation stems from understanding, succeedings, and the knowledge of achievement - which form the framework (and therefore the vision) for this interpretation of classroom practice. However, one other opportunity to create student goodwill presented itself during the preliminary study. I asked a year 12 art student to add some drawings to the worksheets that I was preparing for student practice in the classes I was teaching. The response from the students to the drawings was enthusiastic to a degree that frankly astonished me. Many students claimed that the drawings improved their attitude to maths classes, and an air of expectancy greeted each new batch of worksheets. The drawings provided an unexpected motivational bonus that could not be ignored; and so the format and presentation of **Talking Maths** took shape.

Chapter VII

CHAPTER VII

IMPLEMENTATION OF 'TALKING MATHS'

The classroom book **Talking Maths** is seen to have two main purposes. Firstly, it is designed as a module-style course for students studying high school algebra. The course is not considered to be complete in itself; it is offered as part of an ongoing study in mathematics. However, the contents of the book have been aimed at presenting some ideas that are essential to the study of algebra, and important starting points for extension into other related areas. In addition, particular approaches to algebra learning are advocated. A second purpose of the book is to provide a guide and a resource for teachers of algebra, both in its approach to algebraic content and in the use of appropriate theoretically-based methodologies. **Talking Maths** thus aims to address the task of developing effective classroom practice, specifically that which is informed by new conceptions of learning and thinking performance. Implicit in this aim is the need for change - change from practice that is shown to be out-of-touch with new knowledge, and change to approaches that are seen to reflect recent relevant findings. In this chapter the efficacy of **Talking Maths** as an instrument of change in classroom practice (in algebra) is examined. The review also includes the possibility of changing aspects of the book that fail to meet perceived classroom needs (see Appendix 3).

CLASSROOM INTERPRETATION

In order to assist the development of the **Talking Maths** program I wrote to a number of South Australian high school principals asking for expressions of interest from maths staff to take part in a trial implementation of the course during the first term of 1988 (10 weeks). Of the positive responses received, the offer from Mitcham Girls' High School was considered to be the most suitable. They considered that the methodologies used in **Talking Maths** were consistent with approaches that, they hoped, would increase girls' participation in mathematics and science. Thus it was agreed that five classes of 13-14 year-olds (the entire year 9 level) would take part in the program. This was a most satisfactory situation because it meant that the course could be implemented with students representing a range of achievement in mathematics. The classes involved were divided into five groups, two considered to be high achievers, with three other mixed groups of students ranging from mid-level to low achievement. Maths was time-tabled for one 40-minute period each day.

The book **Talking Maths** contains a program of algebraic knowledge, presented as a developmental sequence of experiences that are considered to be meaningful to high-school students. The emphasis is on problem-solving activities that aim to provide intellectual challenge and to consolidate learning that is basic to many other areas of mathematical study. Included in the book is a program of teaching approaches that is informed through and justified by a theoretically-based framework of learning and problem solving. Ways in which these methodologies can be integrated into the specific content of the **Talking Maths** course are also presented. Ideally, then, it seems that the book could 'stand alone' as a means for program implementation: it contains what to teach and how to teach it. However this study is concerned with more than the theoretical considerations and ideal aspects of program implementation: although practical factors may not be the first to be considered, in the end they are the ones that determine whether the program is accepted or rejected. This final part of the study was thus seen as a trial implementation of the course - an opportunity to observe and report some of the factors that affected this attempt to translate theoretically-based innovation into actual practice. I had implemented the course and teaching approaches during the preliminary study; however it was important to test the validity of the program in the hands of other practitioners. Could this course operate effectively in the classroom with teachers who have not developed the program?

Another aspect to be considered was the extent to which **Talking Maths** could be seen as a means for self-directed implementation by teachers. In other words, could teachers operate the program effectively within the confines of their own school staff-development procedures and resources, or would additional support be deemed necessary? A further aim of the one-term trial at Mitcham was to identify some of the aspects of professional development that were seen as significant in a small-scale implementation.

Thus, it was arranged that I should visit the school regularly during the term, initially primarily as an observer, but also being available for support during the visits as needed. The in-school coordinator assumed the role of 'key person': to be a motivator and guide for teachers, to provide informal on-going monitoring of the project, and to negotiate (on behalf of me and the teachers) any required changes in direction or nature of my involvement.

My involvement in the program

The agreement reached was that I should be at the school on one day per week. All year 9 maths classes were scheduled for the same time-slot, so this allowed me to spend time with one or two classes during each visit. In addition I would be available to talk with teachers during their non-contact periods and at recess and lunch times.

However my involvement in the project took a decided turn in direction very early in the term. The in-school coordinator became ill; and, as it turned out, she was absent for most of the program's implementation. Thus my role of investigator and observer of the program-in-action was necessarily expanded to include the responsibilities of coordinator and 'key person'.

It was not possible to increase the amount of time spent at the school due to my other work commitments. So in response to the situation as it was, I saw my day-a-week visit to the school to be essentially devoted to supporting teachers' implementation of the program, and informally monitoring its progress. It is relevant to mention here that the teachers had not chosen to take part in the project: that decision had been made by the maths coordinator. So, in consultation with the teachers involved, it was considered that my ongoing assistance and professional support should be a top priority.

Essentially, the support was informal in nature, usually taking the form of talking to teachers in the staffroom, individually and in small groups, about the methodologies suggested in the book. In three of the classrooms I gave lessons demonstrating the 'conflict' approach to teaching; and a video (demonstrating the same approach) that I made with a year 9 class at another school was also made available. In addition, I worked with small groups within the classes, usually in response to a teacher's perceived needs. For example, in some groups I concentrated on talking with students about the algebra they were doing - clarifying understandings, talking over approaches that might be suitable, reflecting on things we had just done. Other group activities included re-teaching some work to those who were feeling confused about it, or challenging high achievers with some of the extension work included in the program. These discussions with the teachers gave me valuable feed-back about the content of the book and also its presentation. Changes to several pages in the book have resulted from teachers indicating, for example, that a statement needed to be clarified, or that a section of work became too demanding too quickly.

In response to a perceived need, I also conducted one 'development' session which all teachers involved in the project attended. I had hoped that several sessions would be possible in order to give teachers the opportunity to view the program from a wider, more coherent (and less isolated) perspective. However, other school commitments on the part of the teachers made more extensive structured and programmed support impossible.

Program implementation as Change

Classroom program implementation may be seen as a process involving **change**; indeed it is useful to look at the implementation of **Talking Maths** in the context of the theoretical framework of learning and problem solving developed for this study. Learning in a complex domain is demanding in terms of time, intellectual effort and commitment; and classroom practice provides such a domain. In a school implementation project such as the one reported here it may be difficult to know whether the teachers involved are actually making a commitment to a process of professional development and change. For example, how motivated were they to take part in the project? The teachers involved in the Mitcham trial demonstrated a great deal of goodwill and interest in the project, but as mentioned previously, they did not choose to take part in it. In addition, to what extent were they committed to professional change - even, indeed, if they saw the project as a reason for self-evaluation and possible change? Development and change, as the theoretical framework of the study demonstrates, is a multi-step process that takes time, and it often involves conflict with one's personal beliefs and established practices. 'Change' may be considered too threatening, or argued as 'not necessary'. Indeed, avoiding change may end up taking precedence over good intentions that are found too demanding and time consuming to follow through.

Certainly, these problems seemed apparent in the actual classroom practice that I observed. It appeared that the teachers adhered to the methods suggested in **Talking Maths** to some extent only. Despite my modelling of class-group teaching approaches and methods for initiating discussion among students, the teachers seemed reluctant to put the ideas into practice in a systematic way. In the early stages of the book students worked at roughly the same level, and teachers were able to follow class-group teaching methodologies, as well as implement small-group discussions. Problems in following the approaches began when achievement levels within the class widened. Although guidance in this area is specified in the book, it appeared instead that the teachers progressively allowed students to work through the pages of the book at their own rate and with the minimum of intervention. The problem of varying levels of achievement in a class thus seemed to be resolved in a manner that was at odds with the problem-solving approach outlined. Teachers said that the format and presentation of the book, including concept development of ideas and step-by-step outlines of procedures, gave faster workers a chance to 'move ahead'. Thus, they allowed students to work straight through a set of exercises and then turn immediately to the next page - almost as if algebra learning was seen as existing only (or at least primarily) through doing the problems printed in the book. My observations suggested that the teachers did not regularly involve the students in mathematical discussion that spanned a <u>range</u> of thinking on a particular cluster of ideas. Therefore, guidelines that were offered as starting points for discussion, elaboration, argument and justification were often implemented as routines to be followed through directly. An important intention of the program was to engage the students in thinking mathematically, supported by the stimulation of discussion within both the class-group and small-groups. This assumes active and constructive methods of learning, requiring a change from dependence on prescriptive approaches. I think the teachers recognised this, and they certainly encouraged the students to talk amongst themselves while they worked through the problems. However, I sensed that lessons were frequently times in which the students used the book as a self-teaching manual and set of work-sheets: <u>they</u> approached teachers for help and only when problems were beyond them. The teachers were always very happy for me to participate in lessons on my weekly visits, but mainly I think they preferred me to 'help out' with their routine approaches rather than have me assist in implementing the **Talking Maths** methodologies.

These experiences show clearly the critical role of the teacher as learner and decision maker in program implementation. They also highlight the need for teachers to be well informed in mathematics as well as being more aware of approaches that are likely to enhance learning and improve instruction. In addition, the need to communicate via the written word must be recognised as a significant constraint. As the theoretical framework of this study would predict, not all users of the program will interpret or implement it in the ways that the author intended.

Time constraints, in the day-to-day sense, were also an expected practical difficulty. My own experience as a teacher makes me only too aware of the expanding range of demands made on teachers in their out-of-classroom time. The teachers at Mitcham were continually faced with a long list of duties requiring their attention during their non-contact lessons, lunch breaks and after-school time. Term 1 is a busy time for sporting contests, and the school as a whole is committed to a 'girls' participation in education' program. In addition, there were other subject meetings that the teachers were required to attend. Whilst the project teachers indicated that they appreciated and valued professional discussions with me, it was clear that the time spent was usually at the expense of some other need or commitment. The current educational climate in South Australia is one of ever-increasing expectations of schools and their personnel without a corresponding expansion in resources. This situation militates against innovation in curriculum implementation, and professional development on the part of teachers. It cannot be assumed, therefore that teachers who participate in programs of change will be given adequate time and support in effecting change. Hurried, interrupted conversations are neither appropriate nor sufficient for the demands of professional learning.

Program implementation and staff development as cultural change

In characterising curriculum implementation in terms of its effect on school life, Romberg and Price (1983) draw attention to the need to consider the culture of schools. In particular, they emphasise curriculum development as "an effort to change the belief structures and work habits of the school staff" (p.159). Furthermore, they see the school culture as resistant to change. This resistance is demonstrated in two main ways: the innovation is adopted but not actually used, or it is assimilated into the school in ways that fit the approaches already taken by the teachers rather than in the manner intended by the developer. This tendency to maintain "stability" (p.160) is expressed in practices such as giving new labels to conventional approaches, and reverting to old habits after a brief period of enthusiasm and apparent (but not actual) change.

This framework proposes that two developmental factors need to be addressed: curriculum change and cultural change. The starting points for each and the degree of structuring attempted will affect the outcomes. Romberg and Price have represented these variables on a continuum, ranging from 'ameliorative innovation' that aims to improve the efficiency and effectiveness of some ongoing practice without challenging the values and traditions of the school culture, to 'radical innovation' which requires major changes in established beliefs, values and practices.

This model provides a useful perspective for analysing the small-scale trial innovation implemented at Mitcham. If the **'Talking Maths'** program is looked at within the broad educational context, it may be seen as a contribution towards change in classroom practice that is informed by widely-recognised advances in knowledge about human learning and problem solving. Indeed, **Talking Maths** was developed primarily with this purpose in mind. In this context, the Mitcham project could be viewed as a trial implementation of 'radical innovation': an exercise that attempts to challenge seriously established classroom practice, and replace it with approaches that are considered more appropriate to an optimal learning environment. However, the introduction of radical innovation - as the theoretical framework of this study would suggest - involves long-term, large-scale, complex learning processes on the part of the school community. Thus, the project at Mitcham was seen more in terms of a small-scale trial to test teachers' initial responses to enhancing their knowledge of algebra learning, with a view to improving their classroom effectiveness.

Monitoring of the implementation of 'Talking Maths'

Talking Maths was implemented at Mitcham Girls' High School as part of their commitment to increasing girls' participation in mathematics and science. My commitment is to making algebra more accessible and relevant to students generally through classroom practice that is more effective - and more appropriate to current knowledge about learning and task performance. The two are compatible; but for this study, the project was monitored from the perspective of the framework for algebra learning and problem solving which had been developed. Within this framework, the goals of the project were modest: not too much could be expected from a ten-week implementation. The enhancement of algebra learning and problem solving is very much a long term aim, and dependent on many student, teacher and school variables that cannot be considered in this study. Furthermore, in the context of this study, enhanced learning and problem solving is not seen in terms of easily measured attainment; it refers to outcomes that are less readily discerned and rather more complex in nature. It involves an approach to classroom practice that evolves over time. Also, although the theoretical framework and suggested means for translating it into practice within a particular content area may be articulated specifically, the actual classroom activity is subject to considerably greater variability. On the part of teachers, professional competence, motivation, desire for professional development, level of confidence, level of understanding of the theory, perceived value of the theory, attitude to teaching, attitude to algebra, and skill in classroom management are some of the factors that affect program implementation in its initial stages.

It was considered, therefore, that the outcomes that could conceivably be expected and reasonably estimated from a trial implementation of the project were not seen so much in terms of *actual* change, but rather in terms of preparedness for change and receptiveness towards different ideas and approaches. Although there was no pre-project information on teachers to allow comparisons, it was believed that teachers would recognise to some extent the value of a deliberate attempt to develop a theoretical framework to underpin classroom practice; that the interactive, problem-solving approach suggested would be seen as valuable; that they would consider that students would benefit from the presentation and approach of the course; and that participation in the project would provide useful professional development. A questionnaire was designed to seek teachers' responses to these goals (see Appendix 4).

Teacher interviews

A mathematics teacher who was at Mitcham Girls' High School for the term in which the project was conducted, agreed to interview the teachers involved in the year 9 algebra course. This proved to be a highly satisfactory arrangement as she was 'on hand' to interview teachers at times that suited them in the busy end-of-term (report writing) time. Futhermore, although she was not personally involved in the **Talking Maths** project she was familiar with it, and understood well its aims and purposes. Four teachers were interviewed (according to the questionnaire devised); one teacher was absent from school during the time of interviewing.

The course: implementation of methodology

The teachers saw the course as strongly focused on the development of algebraic concepts. This emphasis on understanding, they felt, was developed through working from students' existing frameworks, through relating ideas to everyday situations, and through making connections with other areas of mathematics such as statistics and coordinate geometry. One teacher commented that she saw the course as aimed at presenting algebra in a different, more enjoyable way. Mostly, she felt, this enjoyment was evident in her class, although towards the end of the book (when the work became harder) students were feeling less positive.

One teacher said her class had "talked a lot about maths" throughout the course, and engaged in arguments about solution methods from time to time. The students had appreciated the relationship of algebra to other areas of knowledge, and had indicated to her that their understanding of graphs and related equations had been enhanced. Some teachers said that their lower-achieving students lacked some of the pre-requisite skills, impeding their progress through the course. They also felt that extra problem practice should be provided in several areas.

All teachers perceived professional support as being a vital part of program innovation. Each one said that my being available in the staffroom and my assistance in the classroom were important and positive factors in the project. One teacher said that he would like to have had even more help in implementing some of the methodologies. Those in the group, however, did consult each other on occasions, but they missed the on-going guidance and support that an in-school coordinator could provide. Clearly, a well-informed 'key person' within the school must be considered as essential in program development that requires methodological change.

Nevertheless all the teachers said that they had tried out approaches that were suggested in **Talking Maths**, and one said that she had used the ideas throughout the project. Another teacher said that although she felt she understood well the aims and purposes of small group work, she found it difficult to manage this sort of class activity. A comment from one teacher indicated that the use of students' intuitive ideas ("asking them what <u>they</u> think") was an ideal way of introducing new work. Overall, though, the teachers said that they had made little real change to their usual teaching approaches.

The course : algebra for year 9

Teachers' views of the algebra course that was chosen varied. They liked the presentation of algebra as a study related to everyday experience, and its connections with future work (eg, analytic geometry) and year 8 algebra (one teacher used the 'variables' presentation with her current year 8 class). They were less happy that some topics, considered to be "year 9 algebra", were left out of the book; some commented that year 8 revision, equations, and expressions and expansions involving second degree variables should have been included. Also, some particular areas, such as 'functions as mappings' and the later graph work, were considered too difficult conceptually. Probably the strongest comment from all the teachers was that there were not enough practice exercises included in the book, while two added that more extension work should have been provided for fast workers.

Nevertheless, it appeared that the teachers could assimilate modifications in content presentation and approaches more easily than differences in methodology. They found the algebra in **Talking Maths** (with its emphasis on understanding) different from and a decided improvement on the textbooks they had used before in algebra. They preferred its focus on the development and coordination of mathematical ideas to their usual practice of considering separate topics: as one teacher said "it gives an overall perspective to algebra". Relating algebra to everyday experience was seen as "a major plus" in developing thinking within a meaningful context. The teachers also appreciated the approaches suggested to developing concepts: the introductions to variables, directed number, functions (as machines), and gradients were mentioned as being different from and/or better than ideas normally used. One teacher found the derivation of the gradient formula confusing and had chosen to teach it "her way".

The course : theoretical basis

All the teachers said that they had read the theoretical (type-written) sections of the **Talking Maths** book, although one had found time only to read the parts that dealt with group work. She said that the small-group approach made sense to her, and she had implemented the idea with her class. Another teacher had appreciated the way that a theoretical framework is able to coordinate teaching approaches and algebraic learning. She could recognise the developmental links emphasised, that is, the use of prior knowledge as a basis to new thinking and the connecting of familiar experience to related abstract ideas. The other two teachers felt that the theoretical framework was presented clearly, but they did not feel that the book dealt adequately with the wide range of student achievement within their class.

Student Responses - as the teachers perceived them.

How did the teachers see student reactions to **Talking Maths**? Certainly the students saw **Talking Maths** as different from their previous algebra course (two teachers were asked "are we really doing algebra?" by their classes), and there were other comments that **Talking Maths** was more enjoyable than usual maths courses. Of course, what it means to "enjoy" a course is not easy to define clearly. By and large, however the students (according to reports) were more motivated and positive about algebra than teachers believed was usual with year 9. Three teachers added that interest had been maintained in the course right up until the last week or so of the term, when enthusiasm had begun to wilt. Another teacher commented that two-thirds of her class enjoyed the course, but that the other one-third are "switched off" from school work, and "don't enjoy anything". Improved student confidence of their student in algebra was noted by two teachers.

Presentation of the book

The teachers said that both they and the students found the presentation of the book agreeably different from the usual, and most appealing. Common comments were that the use of handwriting was very personal and "user friendly" and that it was larger and much more acceptable than typewritten work, and easy to read (except for a confusion between b and 6). The students liked the pictures, they enjoyed using a book that was easy and light to carry around, and they appreciated being able to write in it.

The project as professional development

The teachers summed up their participation in the project as "worthwhile professionally". Specifically, they mentioned that **Talking Maths** had provided up-to-date information about learning and learners, and it had presented specific ways to use this knowledge in classroom teaching and assessment. One teacher said that experience in the program had encouraged her to think a good deal more about how children learn. Another said that it had benefited him professionally "more than any other book had done." Overall, although the teachers had been encouraged to look at algebra topics in fresh ways, they doubted that participation in the project had caused them to make significant changes in their approach to teaching algebra. However, a message recently sent to me from the school indicates that **Talking Maths** is being used as a "useful" resource in algebra classes in Mitcham Girls' High School in 1989.

SUMMARY OF THE STUDY

This study has examined, developed, and presented one possibility for a new view of classroom algebra. It is seen as a contribution to identifying and devising classroom approaches that reflect current knowledge about human learning and problem solving.

The program developed has been initiated and justified theoretically. This has involved the building of a framework from a synthesis of perspectives that are considered to be compatible and complementary. The strength of the theoretical framework constructed is seen in its generality, and its applicability as a guide, support and predictor of effective classroom practice.

A book **Talking Maths** represents an interpretation of the instructional framework presented, and an attempt to apply it to classroom algebra. Within this domain, a trial implementation in one school setting is reported. The review is discussed in terms of staff development within a prevailing school culture, and how some aspects of professional change may be seen within in this context. Factors that are seen as affecting the testing of theoretically-based programs are elicited in order to assist further implementations of this and other courses.

Appendix 1

PRE-TESTS
(Preliminary Study)

Instructions to students:

1. Use the method that seems easiest for you as you do these problems.
2. I'm more interested in the approach you take than in the answer you give. Please write down as much as you can to show how you are thinking about the problem.

Name: Years 6 and 7 4/12/84

$5 \times 17 \times 2$ $=$	$143 \times 9 - 9 \times 43$ $=$
$1+2+3+97+98+99$ $=$	63×99 $=$
$6 \times (5+8)$ $=$	41×103 $=$
$72 \times 25 \times 4$ $=$	$1-4+7$ $=$
$99 \times 7 + 1 \times 7$ $=$	$7+18 = \boxed{} + 23$

Name: Years 6 and 7 4/12/84

$(12 \div 6) \div 2$ $=$	$7 \times (9-6) = b - 42$ $b =$
$12 \div (6 \div 2)$ $=$	Look at this sequence. What comes next? 19, 25, 31, 37, ___
$7 \times (9+6) = \boxed{} + 42$	Look at this sequence. What comes next? 11, 8, 5, 2, ___
$20 \times (6+4) = (20 \times 6) + (20 \times \_\_\_)$	Write this statement in a simpler way: $3a + a$ $=$
$8 \times (\frac{1}{8} + \frac{1}{4})$ $=$	Write this statement in a simpler way: $3a - b + a$ $=$

Name :   Years 8 and 9   11/12/84

$25 \times 9 - 25 \times 5$ $=$	$8 \times (\frac{1}{8} + \frac{1}{4})$ $=$
$143 \times 9 - 9 \times 43$ $=$	$13 \times 12 + 13 \times 8 - 26 \times 6$ $=$
74×103 $=$	$7 \times (9-6) = b - 42$ $b =$
$1{\cdot}4 \times 7 + 1{\cdot}6 \times 7$ $=$	$(c+2) \times 8 = 16 + 16$ $c =$
$20 \times (6+4) = (20 \times 6) + (20 \times \_\_)$	Simplify : $3a - b + a$ $=$

Name : Years 8 and 9 11/12/84

Simplify : $3(a-2) + (a+2)3$ $=$	Expand : $(b-a)^2$
Simplify : $8 - 4(y+2)$ $=$	Expand and simplify : $(b-a)(b+a)$
If $c+d = 7$, what can you say about $c+d+2$?	Simplify : $\frac{x-1}{1-x}$
If $r = s+t$, and $r+s+t = 30$, $r =$	A formula for converting a temperature from °C to °F is $5(F+40) = 9(C+40)$ If the temperature is 5°C, what is that temperature in °F ?
If $u+5 = w$, and $3w = 27$, what can you say about u ?	If $5(F+40) = 9(C+40)$, and $F = 14$ what is the value of C ?

Name: Years 10 and 11 11/12/84

If $r = s+t$, and $r+s+t = 30$, $r =$	If $A = P + PRT$ $R =$
If $u + 5 = W$, and $W = 27$, what can you say about u?	Factorise: $\pi R^2 - \pi r^2$
If $x + y = 8$, what can you say about $x+y-3$?	Factorise: $t^4 - 81$
Expand $\frac{1}{2}(2p-6)^2$	Factorise: $x^2 - (x-y)^2$
Simplify $\frac{1}{4} + \frac{1}{4}(8e - 4f)$	A formula that relates a temperature measured in °F to the same temperature measured in °C is $5(F + 40) = 9(C + 40)$ If the temperature is 14°F what is that in °C?

Name: Years 10 and 11   11/12/84

Simplify:

$$\frac{x-1}{1-x}$$

If $A = P + PRT$

$P =$

Look at these ordered pairs: (3, 7)
(4, 9)
(5, 11)
(6, 13)
Write a rule that applies to every pair.

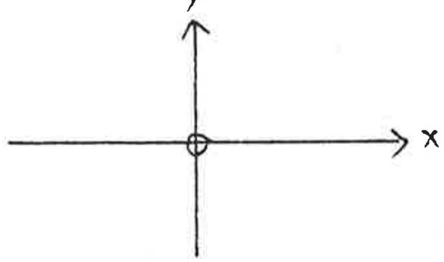
Solve for x:
$$9(x + 40) = 5(x + 40)$$

Look at these ordered pairs: (1, 1)
(2, 4)
(3, 7)
(4, 10)
Write a rule that applies to every pair.

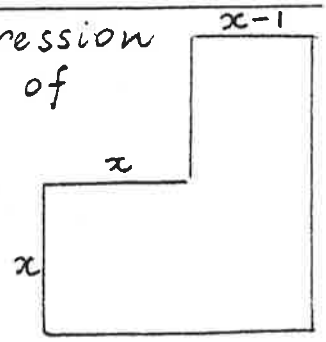
$13 \times 12 + 8 \times 13 - 26 \times 6$

$=$

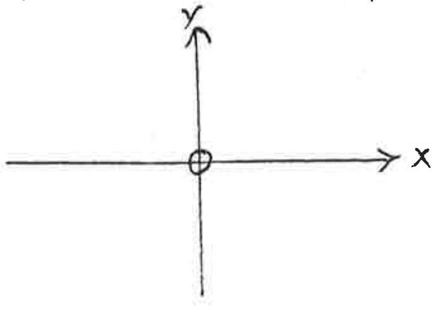
Sketch a graph for this rule: $y = 2x - 3$



Find an expression for the area of this region. Simplify it.

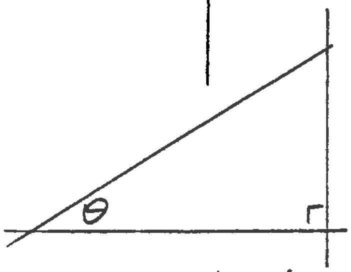


Sketch a region for this rule: $2x - 3 \geqslant 0$

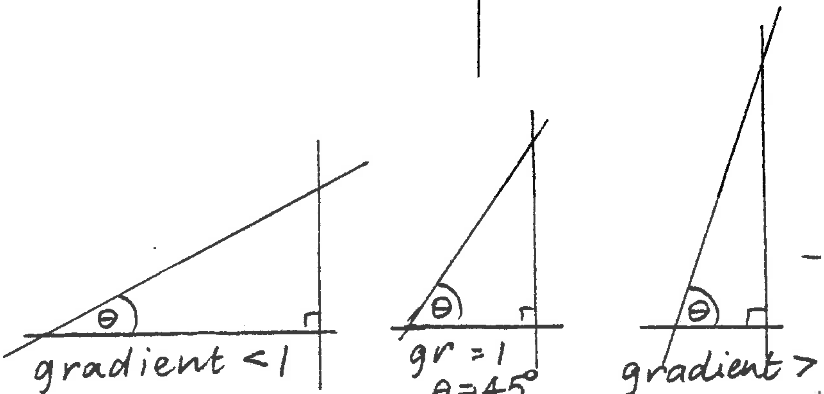


Factorise:
$$e^{2t} - 5e^{t} + 6$$

Appendix 2



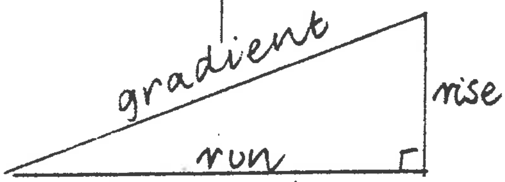
gradient = $\tan\theta$



gradient < 1
$0° < \theta < 45°$

gr = 1
$\theta = 45°$

gradient >
$45° < \theta < 90°$



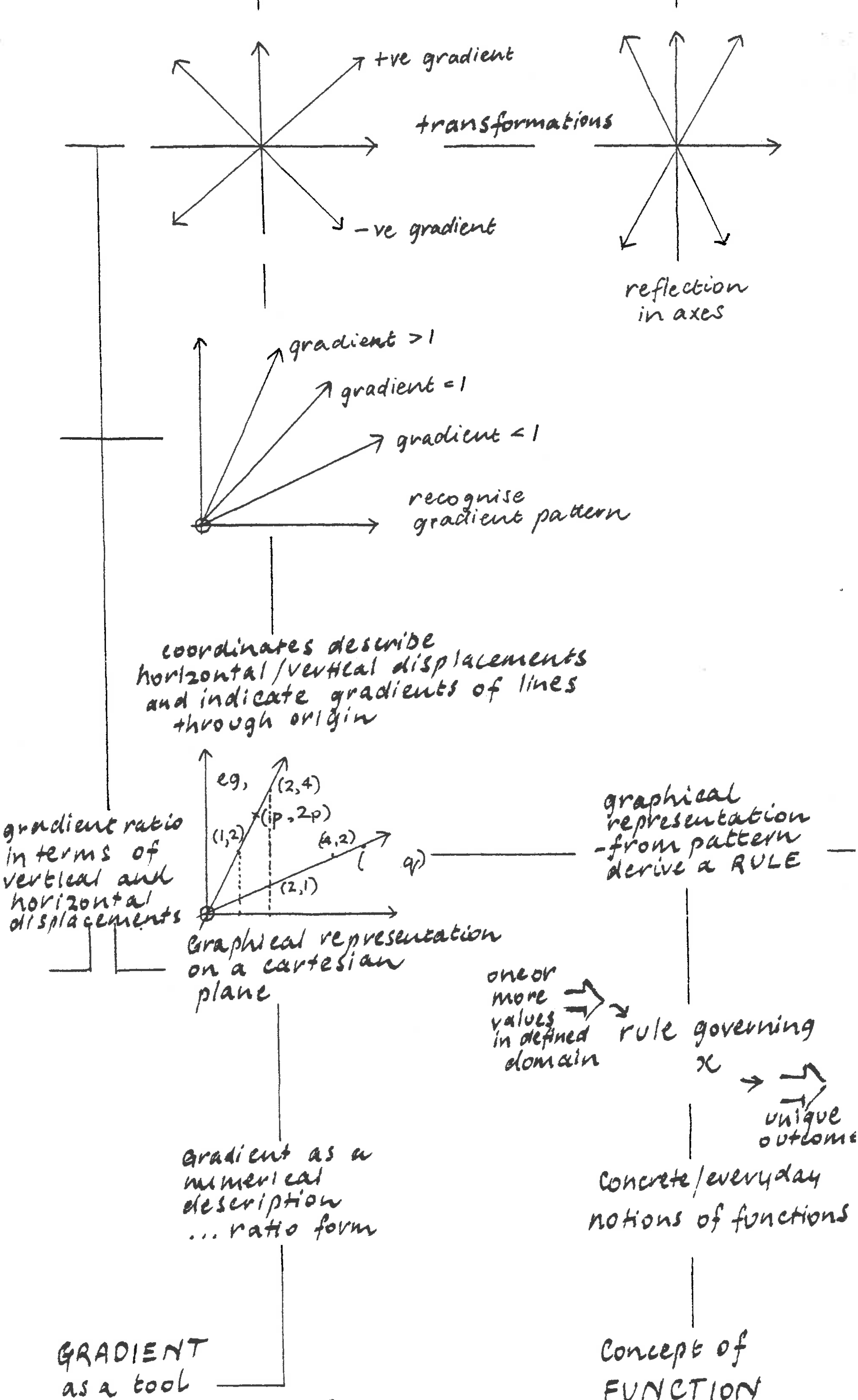
trigonometric
illustration –
steep/gentle gradients

Verbal description
rise compared to run

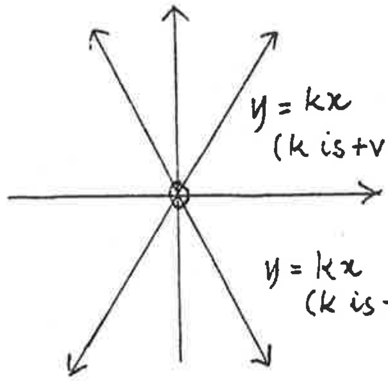
EVERYDAY USE of GRADIENT
Ladder, kite-flying, ramps,
High-rise car-parks, jet taking of
Roofs, ski-slopes, water-ski jump
Animal burrows, steep roads
Leaning tower of Pisa

GRADIEN
as a too

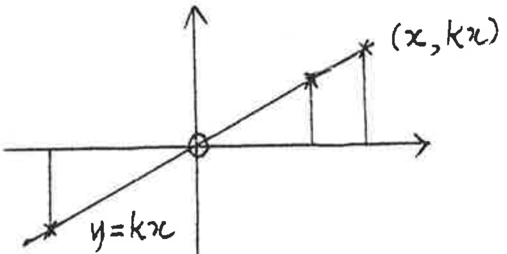
→ to P222


+ve gradient
transformations
-ve gradient
reflection in axes
gradient > 1
gradient = 1
gradient < 1
recognise gradient pattern
coordinates describe horizontal/vertical displacements and indicate gradients of lines through origin
eg,
(2,4)
(p, 2p)
(1,2)
(4,2)
(q)
(2,1)
gradient ratio in terms of vertical and horizontal displacements
graphical representation on a cartesian plane
graphical representation - from pattern derive a RULE
one or more values in defined domain
rule governing x
unique outcome
gradient as a numerical description ... ratio form
Concrete/everyday notions of functions
GRADIENT as a tool
Concept of FUNCTION

→ to P223



reflection in axes (pattern shows in equation)



$\frac{\text{y value}}{\text{x value}}$ yields the gradient

recognising pattern of straight lines passing through origin

Applications: Conversion graphs

Pattern (rule) relates x and y function rule $y = kx$

function as
· mapping
· machine
· process
· transformation

Analysis of ordered pairs within a linear function: points satisfying the function are on the line

Concept of FUNCTION

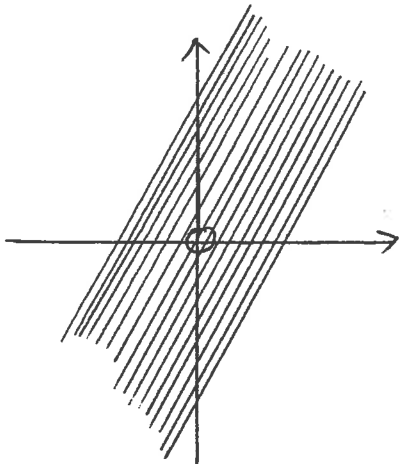
concept of VARIABLE

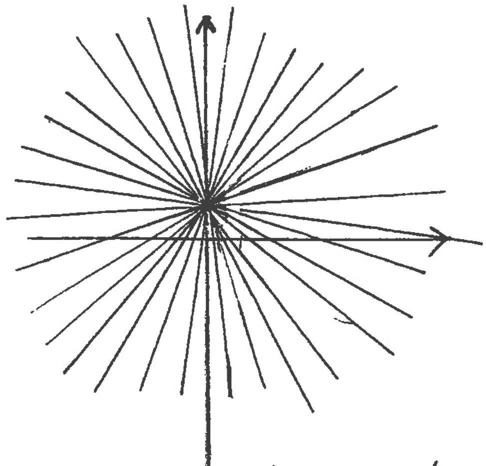
graphical representation

understanding and skill: ordered pairs

A region:
a plane area.
A linear inequation
defines a region

linear
programming





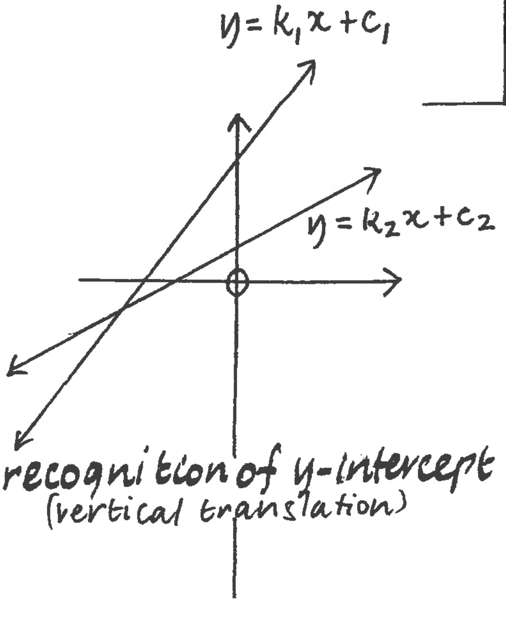
same gradient

same y-intercept

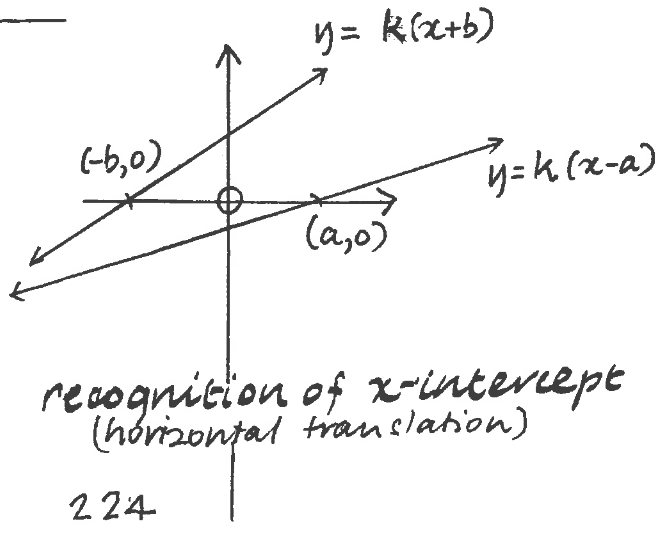
Families of lines
recognised
through patterns in rules

simultaneous
equations

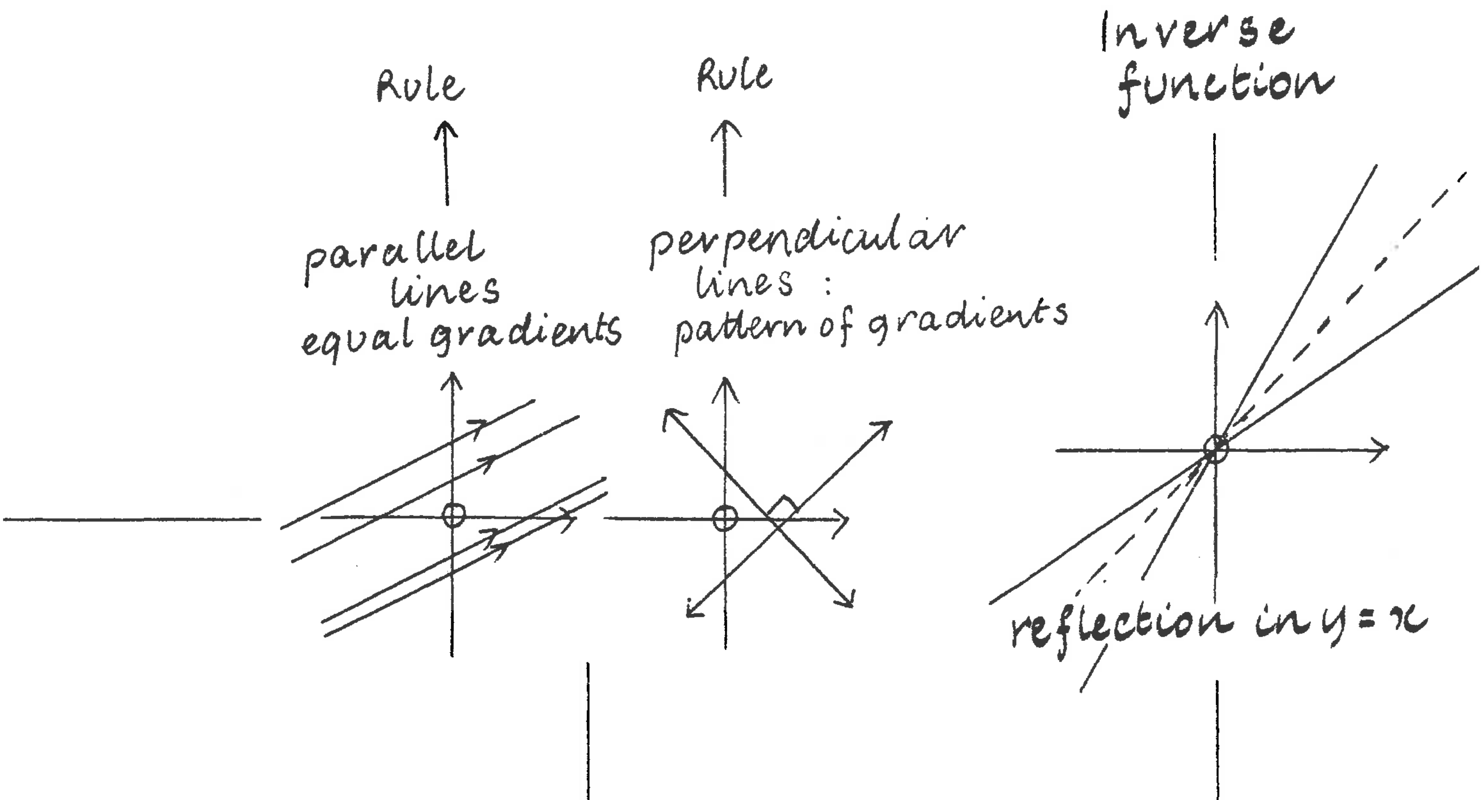
pattern recognition
- prediction of rules -
using
gradient, y-intercept as reference point



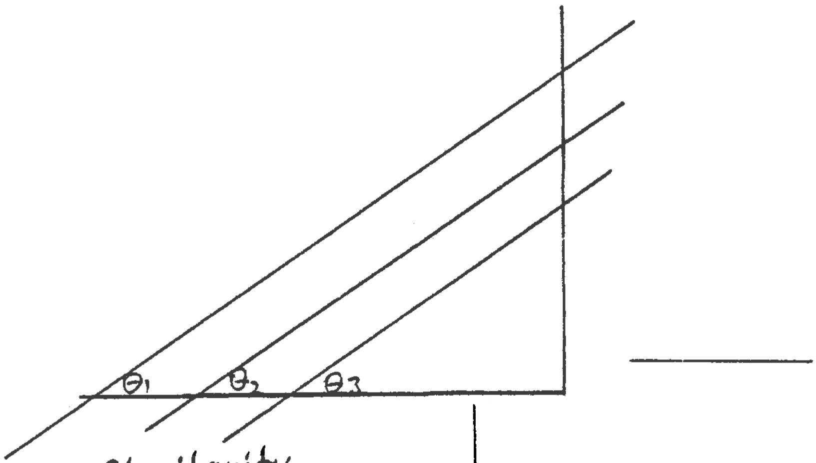
recognition of y-intercept
(vertical translation)



recognition of x-intercept
(horizontal translation)

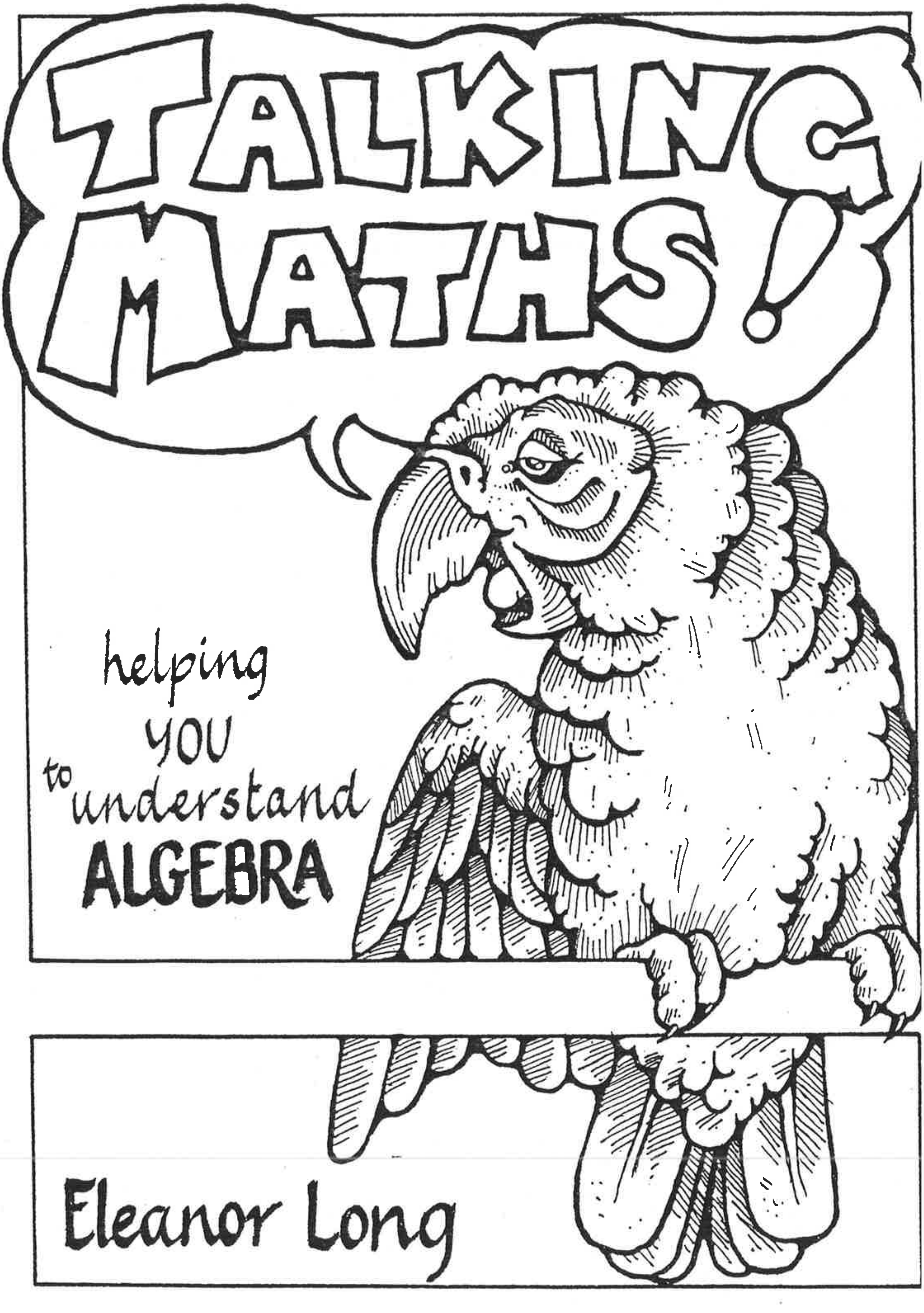

Rule
parallel lines
equal gradients
Rule
perpendicular lines :
pattern of gradients
Inverse function
reflection in $y = x$

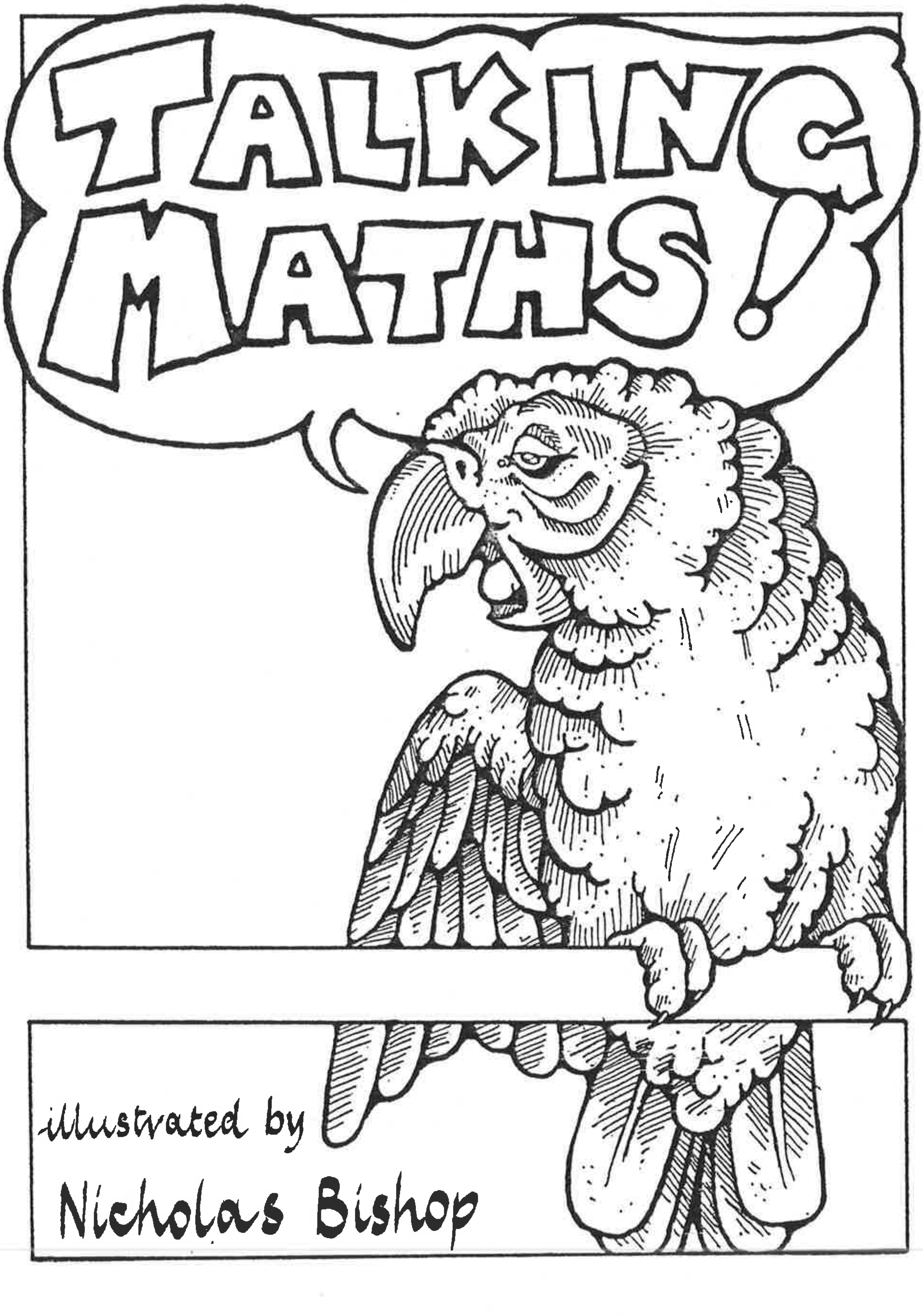
→ to P224



Similarity,
Proportion

Appendix 3


TALKING MATHS!
helping
YOU
to understand
ALGEBRA
Eleanor Long


TALKING MATHS!
illustrated by
Nicholas Bishop

Thank you

- to Nicholas Bishop
- a Year 12 student -
  for giving so generously of his artistic talent

- to 9B Maths classes, Concordia College
  for talking - and enjoying - maths with me.

Eleanor Long

- University of Adelaide -
- 1987 -*

*Note: 1987 edition - draft
1989 edition - post-testing

Thank you

to

Year 9 maths classes and their teachers
Mitcham Girls' High School ...
for their goodwill in using this book
in their classrooms

Nicholas Bishop ...
for giving so generously of his artistic talent

9B maths classes, Concordia College ...
for talking (and enjoying) maths with me.

Eleanor Long

- University of Adelaide -
1989

To teachers and students using this book:

Thank you for using **Talking Maths** as part of your algebra program. **Talking Maths** aims to help students find algebra a meaningful and relevant study. The teaching methods suggested and the work pages included in the book have been used by a number of classes who are working at Year 9 or Year 10 level. Their responses (students and teachers) have been noted in an attempt to present this classroom book as an effective and useful means towards knowing, understanding, and problem solving in school algebra.

Classroom practice guided by theory

The ideas and methods suggested here are developed from a theoretical perspective of classroom learning and problem solving. This framework coordinates a general concept of learning with research findings highlighting specific characteristics of learners. In sum, a theory is provided to interpret classroom experience, and to support and guide practice that aims at more effective learning and problem solving.

The theoretical framework that directs this classroom methodology merges

> Piaget's concept of equilibration, as a fundamental process that governs the acquisition and growth of knowledge in an individual*

with evidence drawn from

> cognitive science, a discipline in which computer systems are developed to simulate human problem-solving behaviour. This evidence has formed the basis for an information-processing approach to cognition that helps us to analyse the learning process.

While Piaget's equilibration seeks to interpret how learning development takes place, the information-processing approach directs more explicitly how to encourage desired learning outcomes. Characteristics of learning and learners, as demonstrated by the theoretical framework that's been developed, form the basis for lesson ideas suggested in this book.

---

* Piaget's post-1970 development of the equilibration concept is not well-known. In educational terms, however, it is of greater significance than Piaget's earlier theorising on 'stages'.

A brief overview of the theoretical framework for learning and problem solving suggests that classroom practice is guided by these fundamental premises :

1. **Learners <u>construct</u> understanding**

This notion of learners building up knowledge towards more adequate understanding has several important implications for algebra learning and teaching :

- Effective learning is rooted in (but then must extend beyond) familiar experience. So what a learner already knows provides a good starting point for the development of new ideas.

- Learners need to find meaning in what they're doing.

- Understanding is a progressive process, so it is quite natural for meaning to develop gradually, bit by bit, and probably with a few hiccups along the way.

- Learners cannot be seen as passive recipients to be 'filled up' or 'rubber stamped'.

2. **To understand something is to know relationships**

This need for learners to recognise connections within and between ideas directs classroom algebra towards practice where

- Teaching highlights and makes specific the relationships present in algebra

- Algebra learning is presented in terms of recognising and creating patterns of knowledge, in both concepts and procedures

- Learners are shown how

  (a) to link new knowledge to known ideas

  (b) to put algebraic knowledge into an overall, coherent context

School algebra is a meaningful study of relationships between numbers. Its function cannot be seen in terms of working through exercises from discrete chapters in a text-book.

It has often been assumed that students who see relationships and make connections for themselves are those who can 'do algebra', while those students who don't make the connections unaided 'cannot do algebra'. The theoretical framework developed for this book demonstrates that students who are directed explicitly to the relationships and contextual links in algebra

- are more likely to understand algebra
- will have their understanding enhanced.

3. **To learn something is to compensate for a gap in knowledge**

When algebra learning is seen as a process in which

> a cognitive disturbance or imbalance is recognised (you become aware that you don't know something that you need to know),
>
> leading to a shift in thinking (in order to compensate for the knowledge gap),
>
> resulting in a growth in knowledge (as a result of coming to terms with the new evidence that's being presented),

then classroom learning will focus on

- shared activities that show students their need for new knowledge
- experiences and information that help learners to broaden and enrich their knowledge
- ways that help learners to be aware of how they are learning
- ways that help learners to monitor their progress in learning.

This indicates that **formative assessment** of student progress should feature significantly in day-to-day lessons. Instead of assessment being seen mainly in terms of summative assessment (typically an end-of-topic test, graded and recorded as a single number or letter), it will become an ongoing process that is an integral and constructive part of the teaching process. As as alternative to students being asked to work alone on text-book exercises, many problems can be solved interactively, with the support of immediate feedback and correction.

4. **Sometimes learning involves a change in conception (seeing things with 'new eyes')**

Sometimes learners' constructions are based on limited thinking or misconceived ideas. The problems that result are unlikely to just disappear, so they must be investigated directly. This will require assertive teacher intervention. Note that this error correction can be a non-threatening, constructive process that

(a) attempts to confront potential (likely) errors before they take root

(b) uses errors as the starting points to working towards a more adequate understanding.

Evidence about students' 'naive' constructions (misconceptions) show that learners cannot be seen as 'experts' who are in a position to decide on their own learning directions.

There is, however, good reason for students to develop a sense of responsibility about their own learning. The title **Talking Maths** indicates the importance that this theoretical perspective attaches to group discussion and active cooperation in learning. Maths is a language - a form of communication - and that includes talking! Cooperation means co-**operation**: students and teacher 'arguing' a case, and making decisions in relationship to the thinking and ideas of others. It includes looking at your own ideas critically, seeing possibilities in the suggestions of others, and being persuded to alter some thinking that is shown to be false or inadequate.

5. Problem solving involves thinking mathematically

If problem solving involves

- constructing a representation of the situation or problem, by means of symbols, graphs, drawings and/or materials that you can handle

and - searching for patterns/relationships/connections, and considering possibilities

before rules and quantification are applied, then classroom practice must be directed towards

- a systematic understanding of algebraic concepts

and - learning how and why to apply relevant procedures through discussion and inquiry.

The theoretical framework developed for this book recognises the importance of essential content to be learned, with vital concepts and procedures to be connected in productive ways. Group workers who must decide on which problem procedures may be appropriate (or which are inappropriate) need to be knowledgeable enough to make <u>informed</u> decisions. From this perpective it is considered that

- systematic investigation, analysis and review of fundamental algebraic knowledge

- supported by well-informed guidance, intervention and monitoring from the teacher

will assist learners' understanding and problem-solving competence in algebra.

To teachers using this book

The course is designed for students who already have some experience in algebra. It assumes that students will be familiar with letter symbols in some contexts, such as

- assigning the letter a numerical value (evaluating)
- recognising the letter as a specific unknown (solving an equation)
- recognising that the letter may take more than one value (for example, plotting points for a graph).

Part 1. VARIABLES (pages 10-37)

In this section students are introduced to the more advanced concept of a letter symbol used as a variable. The ideas of

-a letter representing a range of unspecified values

and -a systematic relationship existing between two such sets of values

are fundamental to school algebra. It is essential that students find meaning in the letter symbols that are used, and are aware of the ideas that are being communicated through them.

This section includes some practice in transformations of algebraic expressions. A transformation involves the presentation of an idea in a different form. This gives us the chance to see the idea from a new perspective. Transformations form an important part of thinking in terms of functions (see page 66).

Part 2. RELATIONS (pages 38-46)

This book gives some examples of how the symbolic language of algebra can communicate relations between numbers. This section is seen as an introductory glimpse into some of the ways that information (including numbers) can be related, and represented.

Relations integrate concepts. They form an integrative function in algebra, in problem solving, and in understanding how we learn mathematics.

Part 3. DIRECTED NUMBERS (pages 47-63)
While students usually find it easy to recall information about the product of directed numbers, they are often less sure about when it is appropriate to use this information. Even more confused, generally, is the distinction between the product of directed numbers and the process of adding directed numbers. Some students do not realise that they represent different processes, nor are they able to recognise when to use one or the other. This section suggests ways to make explicit the distinction between the two processes.

Part 4. FUNCTIONS (pages 64-81)
While the conciseness of mathematics makes it an ideal medium for problem solving, it can also cause confusion for students who do not recognise subtle differences within a single word or symbol.

This part highlights essential ideas inherent in the word function. Several analogies have been chosen in an effort to root this very important mathematical concept in students' familiar experience.

Part 5. LINEAR GRAPHS AND THEIR EQUATIONS (pages 82-126)
Graphs provide an effective, visual means for representing, organising and analysing the behaviour of a function.

It is assumed that the users of this book will have experience in graphing linear functions through calculating linearly-related ordered pairs, and plotting points to represent them. These procedures provide an 'experiential' base to working with straight lines and their equations.

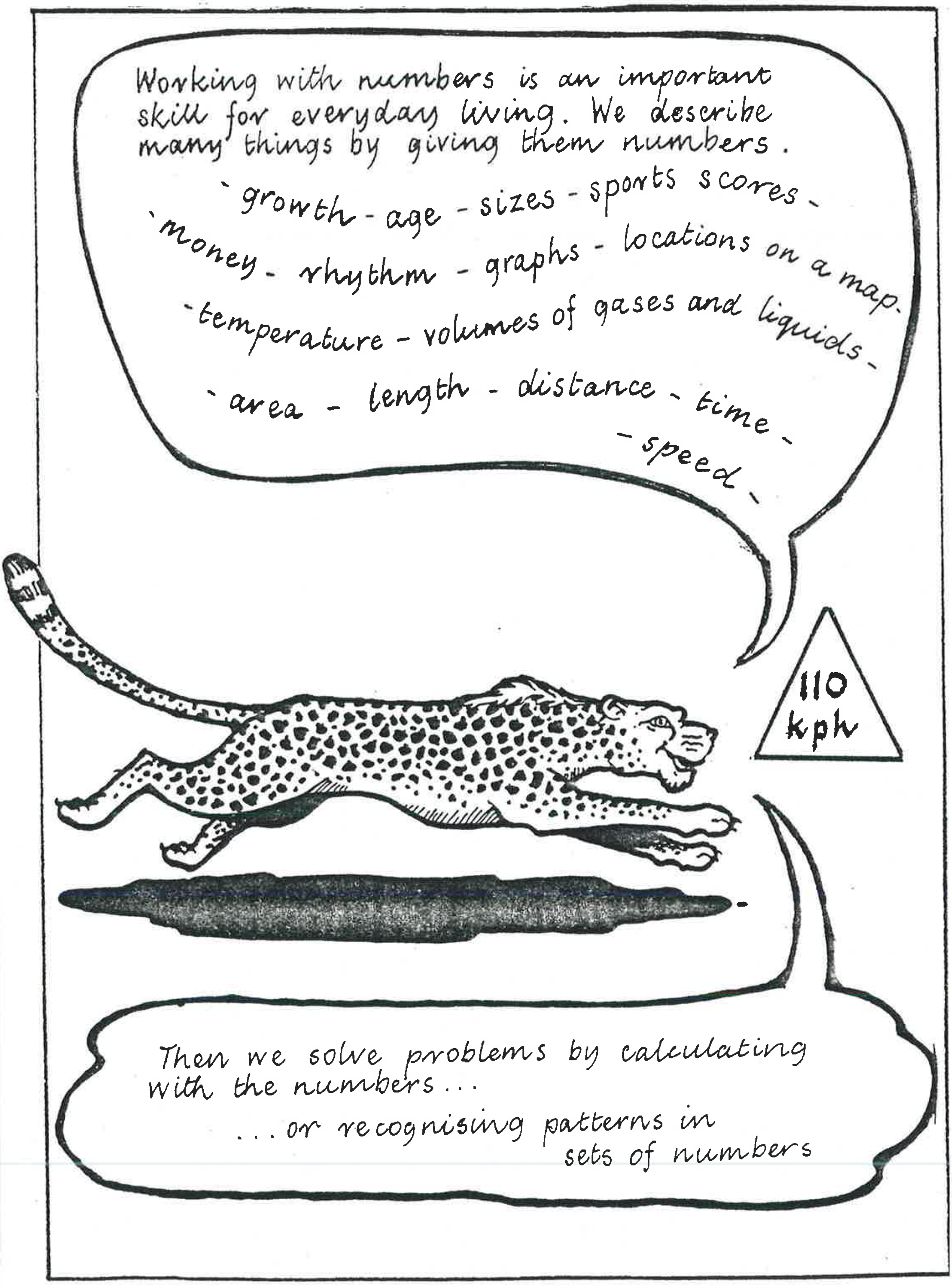
Proficient problem solving involves reasoning that builds on experiential and ground-work skills. Note the processes by which this reasoning might develop in a real life situation:

evidence (information or data) is collected, sorted and organised: this is essential ground work in representing the problem, but it doesn't solve it

the evidence is now investigated in an effort to find patterns/connections/relationships within the data: this search aims to reveal insights about the problem that may be helpful in finding a solution

an interpretation of the evidence from this organised and informed perspective is now directed towards a solution to the problem.

In this section the work sheets aim to develop problem-solving skills through improving competency in investigation, organisation, and analysis of linear graphs and their equations.


Working with numbers is an important skill for everyday living. We describe many things by giving them numbers.
- growth - age - sizes - sports scores - money - rhythm - graphs - locations on a map - temperature - volumes of gases and liquids - area - length - distance - time - speed -
110 kph
Then we solve problems by calculating with the numbers...
...or recognising patterns in sets of numbers

In arithmetic we use SPECIFIC numbers

$36 \quad 27{\cdot}5 \quad 0{\cdot}007 \quad 10{\cdot}99 \quad 0{\cdot}00000321$

$10{,}000{,}000 \quad 5^3 \quad \frac{1}{40}$

$\pi \quad \frac{3}{7} \quad 3{\cdot}3333\ldots \quad 5\frac{2}{3} \quad 16^{\frac{1}{2}}$

In arithmetic problems must be solved separately. Information that uses specific numbers will fit only a single situation.

ALGEBRA
enables us to cover
a **WHOLE SET** of
SITUATIONS
– **AT <u>ONE</u> GO!** –

This gives us extra **POWER** for thinking and solving problems

In algebra we work with VARIABLES

A variable represents an
EMPTY SPACE
that can be replaced by
A NAME
from a chosen set of names.

Think about the set of mountains

In this empty space "   "
you can substitute the name of <u>any</u>
mountain, from a given set of
mountains.

The space can be replaced
by the name of a specific
mountain

say, Mt Everest,
then Mt Kosciusko,
then Mt Fuji,
then Mt Cook,
then Piz Gloria
(and so on)
from that set

Try-to-visualize-an
empty space
that can be replaced
by a name

Think about the set of birds

The empty space can be replaced by the names of specific birds ...

albatross,
pelican,
flamingo,
duck,
gull,
penguin etc.

Note:
The names of these birds could be written more consisely by choosing to use symbols that are more compact than words.

Talking Maths ...
... in small groups

For each person in turn

- State the set that you have chosen.
  Ask others in the group to give a specific name from that set.
  Go around the group at least twice.
- Your task is to show this act of replacement on the board, or on paper if you have a rubber.
  write the name, then rub out,
  write the next name, then rub out,
  then another name ...

VARIABLES are useful in two ways:

1. They make it easy to state LAWS (or rules)

for example,

WHEN YOU PASS 'GO', COLLECT $200

The variable YOU covers all the names of people playing Monopoly.

It's a concise, efficient way of covering any possibility of names from the set of players of the game Monopoly.

Talking Maths in groups

Decide on a time limit ... then

Write down laws or rules that govern

* Students at your school
* Road users
* How much you could earn in a job
* Finding the areas of triangular shapes
* Finding the circumferences of circles
* Making up 1 litre of raspberry cordial

... identifying the variables each time

2. If the solution to a problem is written in terms of variables, then the solution holds true for many individual cases. No new calculations are needed; you just substitute information that fits the specific case.

PROBLEM

How to present a T.V. commercial that will increase fast food sales.

SOLUTION

(1) Show the fast food outlet

(2) Show mouth-watering fast food

(3) Show a smiling staff member

(4) Show a satisfied customer

The variables in the solution have the names fast food outlet, fast food, staff member, customer

What SPECIFIC names could you substitute for them?

In groups

Write solutions in terms of variables for

* more TV commercials
* video recording a TV program
* cooking food in a microwave oven

ALGEBRA is a LANGUAGE

in which the names
and the relationship between these names
are written in symbolic form.

The symbols that we write down represent words
that can be spoken.

In the algebra in this book
the NAMES represent NUMBERS

The language of algebra contains
algebraic expressions.

Expressions that belong to the language of algebra are made up of

* symbols for specific numbers
* symbols for variable numbers
* symbols that link the numbers

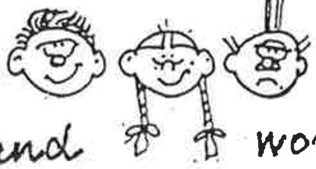
Expressions that belong to the language of arithmetic combine

* symbols for specific numbers

and * symbols that link the numbers

for example,

* $21-10+7$ * $41 \div 5$ * $-\frac{3}{2}$ * $51 \cdot 62 \times 3$ * $(14-2) \div 3$

Most people  feel comfortable speaking and working with these symbols.

In algebra we need symbols that stand for variable numbers.

We must also be able to distinguish between each variable number that is used in an expression.

So it seems necessary to have available a variety of symbols for use in algebraic expressions.

Early mathematicians stopped their search for available, easy-to-write symbols for variable numbers ... when ... they thought of the

alphabet!*

If an algebraic expression involves more than one variable number, we use a different symbol for each variable.

Mathematicians tend to be orderly souls, so if the first variable used in an expression is given the symbol x, the next variable will probably be given the symbol y, with the next variable being given the symbol z.

*François Viète (1540-1603) is credited with first using letter-symbols in algebra. Many years before, the Greeks had used letters

Information about the algebraic symbols used in this book.

Some examples –

A symbol x represents an empty space that can be replaced by **any number** from a set of numbers.

$5x$ refers to the sum of five identical numbers from this set.

$$x + x + x + x + x = 5 \text{ lots of } x$$
$$= \mathbf{5x}$$

$5x$ can also be seen as a product $5 \times x$

$\frac{1}{4}y$ refers to $\frac{1}{4}$ of any number from a set of numbers.

$\frac{1}{4}y$ can also be written $\frac{y}{4}$

$\frac{1}{4}y$ can also be seen as a number divided by 4 : $y \div 4$ or $\frac{y}{4}$

pq refers to a product of two numbers that are **different** (unless it is stated that $p = q$)

Note
- for **pq**, think p times q
- pq is equal to qp

– Talking Maths –

In small groups, decide on how you could rewrite the following arithmetic and algebraic expressions – in a way that is as consise as possible.

*1. 3 lots of 15
$\frac{1}{3}$ of 15
$\frac{2}{3}$ of 15
$2\frac{1}{2}$ lots of 15
$\frac{4}{5}$ of 15

what is the result of
- sharing 15 equally between 5
- sharing 15 equally between 20?

*2. 3 lots of x
$\frac{1}{3}$ of x
$\frac{2}{3}$ of x
$2\frac{1}{2}$ lots of x
$\frac{4}{5}$ of x

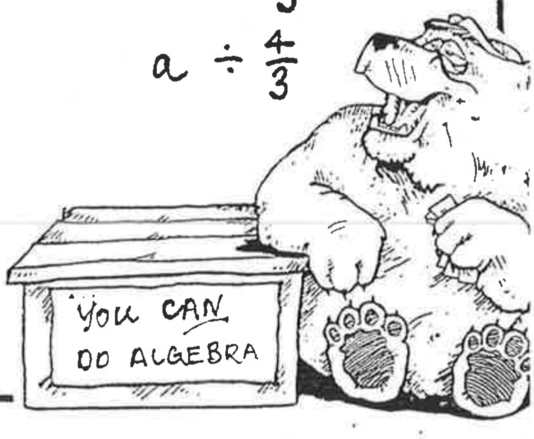
what is the result of
- sharing x equally between 5
- sharing x equally between 20?

*3. 7 lots of y
one-half of z
one-third of t

what is the result of
- sharing a equally between 4
- sharing 4 equally between k
- sharing 4 lots of q. between 3

*4. $3 \times a$
$2 \times 3 \times a$
$a \times 3 \times 2$
$2 \times a \times 3$
$q \times \frac{1}{5}$
$q \times \frac{1}{5} \times 2$
$q \times 3 \times \frac{2}{3}$

*5. $d \div 5$
$5 \div d$
$y \div 3$
$3 \div y$
$t \div 5$
$t \div \frac{1}{5}$
$a \div 3$
$a \div \frac{1}{3}$
$a \div \frac{2}{3}$
$a \div \frac{4}{3}$



UNDERSTANDING IS A MULTI-STEP PROCESS

It is hoped that the use of analogies from everyday life (some suggestions are on pages 12-15) will assist in the understanding of variables. It is important to develop the idea of variables to the point where working with them has meaning. The notion of letter used as a variable is an extended concept requiring thinking that is both sophisticated and flexible. Do not underestimate the nature and extent of the adjustment needed to come to terms with this idea.

However, don't feel that everyone must have a thorough understanding of the concept of variable before you proceed with any further work - or you may never go past this page! An effective (and efficient) approach is to view understanding as a multi-step process, achieved progressively. Sometimes new work requires a shift in thinking on the part of learners - and this may tay take time.

RETURNING TO REVIEW THE CONCEPT: a classroom approach that recognises that algebra learning may not be a straightforward process.

After initial teaching of the concept these steps might be forwarded:

(1) **Go ahead** with basic problems involving the concept that is being learned.

(2) **Return** to further re-investigation and discussion about the concept and its relevant problem-solving procedures. (Although people in the class will be at different stages of understanding, everyone now will have developed an added awareness that will benefit their understanding further.)

(3) **Take** this elaborated knowledge into more demanding problems.

(4) **Support** the development of this knowledge with immediate feedback that focusses on
- awareness of essential features of the problem
- error correction, showing clearly why the incorrect solutions given were not acceptable.

RETURNING TO REVIEW partially understood concepts makes good sense. Each 'return' benefits from the added insight of previous experience, developing a firmer foundation on which to base further understanding. This gradual progress towards understanding - integrated with on-going formative assessment - is likely to facilitate independence, making it time well spent, and possibly even a real time-saver in the end.

TALKING MATHS IN CLASS

The pages so far have introduced a **talking** approach to learning mathematics. Talking over ideas together forms an important part of lessons in other subjects, so why not maths?

There are many ways to take a **talking maths** approach, and the pages to come will introduce further talking approaches. What is really best is to develop ways that suit and benefit the learning needs of the people in the class.

The theoretical framework indicates that these features of classroom talk should be considered as important :

(1) 'Talking Maths' is a means for developing proficiency in questioning, explaining, investigating, reasoning, and solving problems.

(2) Co-operative talk involves striving for achievement in positive but challenging ways.

(3) Talk should be monitored to ensure that it is potentially productive. The 'chairperson' who monitors discussion will often be the teacher - who is recognised as having greater knowledge and expertise in algebra, as well as a more adequate understanding of how learning takes place.

(4) Talk will be thoughtful and searching only if those in the classroom respect others' efforts to improve their knowledge.

(5) Algebraic talk recognises the importance of content knowledge, representing in sorting, organising, and relating ideas.

(6) Sometimes learning will require 'seeing things with new eyes'. Talking things over with others may help to bring about necessary changes in thinking.

(7) 'Talking Maths' with others assists the process of reflection (learning to discuss ideas within your own mind).

ALGEBRAIC EXPRESSIONS

A CLASSROOM IDEA THAT FEATURES MATHS-TALK : A SEQUENCE OF QUIZZES

Note that the quizzes on pages 23 and 24 look at different aspects of transformations of an algebraic expression. These quizzes develop important ideas through a step by step approach, working towards increasing difficulty.

A suggested teaching sequence that integrates formative assessment (constructive feedback) into the learning process:

(1) Write quiz 1 on the board. Allow students a time limit for working through the problems. Faster workers may finish them all in the same time as some students will complete, say, 4 problems.

(2) Students or teacher write solutions on the board, followed by discussion that focuses on how and why. It's important that everyone has a chance to indicate any worries about the solutions for the first 4 problems in the quiz so that the teacher can keep a mental check-list of them.

(3) The same procedure is followed for quiz 2. Everyone starts together and then contributes to the follow-up/feed-back discussion.

A SUGGESTION FOR COPING WITH DIFFERENT LEVELS OF UNDERSTANDING WITHIN THE CLASS.

Some students will see these transformation procedures as self-evident processes; they will be able to simplify and reorganise expressions successfully almost as soon as the ideas are introduced.

Other students will not see the procedures involved as self-evident - and they may even consider them to be contradictory. These students are usually caught up with the arithmetical notion that numbers must be specific. This thinking limits their ability to find meaning in the algebraic notion of variable.

Students who are finding these transformations very easy to manage may begin to work together in small groups (although it is important that they do not rush ahead at the expense of developing understanding and awareness of the significant aspects of the problem). Other students will need extra support, and can continue to work with the teacher as long as necessary. Those who are not at all confident about the work may benefit from a re-teaching of the ideas involved in transformations of algebraic expressions.

In a busy classroom, teacher time and support must be used as efficiently and as effectively as possible. The use of group work is far more productive than the practice of helping students individually - which is wasteful in terms of teacher time and energy.

Students can be encouraged to discuss and review their progress, checking out difficulties as they proceed. Students with similar problems can then sort them out together in a group, guided by teacher direction and support.

Quiz 1

Simplify:

1. $3x + 5x$
2. $4a + 7a$
3. $9t - 2t$
4. $8s - 3s$
5. $5y + 20y - 18y$

Quiz 2

Simplify:

1. $2xy + 3xy$
2. $5pq + 2qp$
3. $7ab - 6ab$
4. $8rs - sr$
5. $9yz - 6yz - yz$

Quiz 3

Simplify:

1. $3x + 5x + 2$
2. $3x + 5 + 2$
3. $3 + 5x + 2$
4. $3 + 5 + 2x$
5. $3x + 5x + 2x$

Quiz 4

Simplify:

1. $9t - 2t + 3$
2. $9 - 2 + 3t$
3. $9t + 3 - 2$
4. $9t - 2t + 3t$
5. $9 - 2t + 3$

Quiz 5

Simplify:

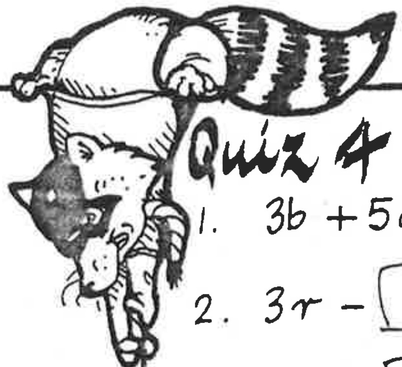
1. $2xy + 3xy + 5y$
2. $2xy + 3xy + 4x$
3. $2xy + 3xy + 4x + 5y$
4. $2xy + 3x + 4x + 5y$
5. $2xy + 3y + 4x + 5y$

Quiz 6

Simplify:

1. $7ab - 2ab + 6ab$
2. $7ab - 2ab + 6a$
3. $7ab - 2ab + 6b$
4. $7ab + 6a - 2a$
5. $7ab + 6a - 2a - 5ab$

Check up on page 127



Quiz 1

1. $5x = 7x - \square$
2. $5x = 2x + x + \square$
3. $5x = 4x + 6x - \square$
4. $5x = 3x + \square - 4x$
5. $5x = \square - 3x$

EXTRA $5x = \square - 3x + x - 2x$

Quiz 2

1. $3y = \square + 2y$
2. $3y = y + 6y - \square$
3. $3y = \square - 4y$
4. $3y = 8y - \square$
5. $3y = 5y + 3y - \square$

EXTRA $3y = 3y + 2y - \square$

Quiz 3

Simplify:

1. $2x + 3x + y + 2y =$
2. $5x + 7y - 4y =$
3. $8x + 2y - 3x + y =$
4. $5y + 10x - 2y - 5x =$
5. $4y + 3x + 2x - y =$

EXTRA $-3y - x + 4y + 3x =$

Quiz 4

1. $3b + 5a - \square = 3b$
2. $3r - \square + 4s - \square = r + s$
3. $2a + \square + 5 + \square = 8a + 7$
4. $e + f - \square + \square = 3f$
5. $3x + \square + \square = 5x + 8$

EXTRA $-2 - d + \square + \square = 5$

Quiz 5

1. $3t + 8 - \square = 8$
2. $4 + 6p - \square = 2 + 6p$
3. $2x + 2y - \square - \square = x + y$
4. $5k + 2 + \square = 11k + \square$
5. $8t + 5 - \square + \square = 12t + 2$

EXTRA $6a - b - 4a + \square = b + \square$

Quiz 6

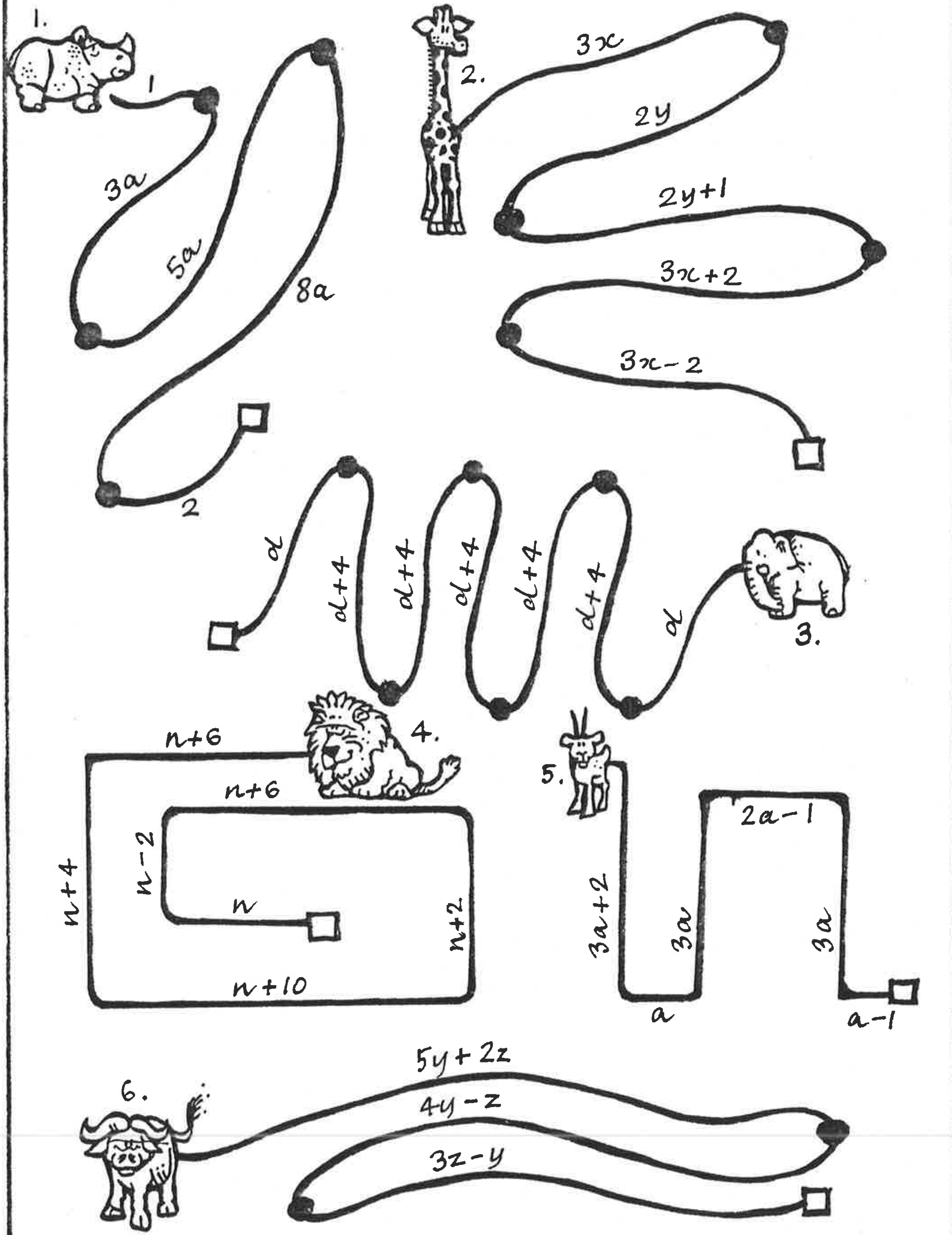
Simplify:

1. $7e + 3f + 5f =$
2. $6 + 2a + 10a - 4 =$
3. $12m - 2m + 3n =$
4. $7 - 4a + 2 =$
5. $8t + 5s - 3t - 2s + 2st =$

EXTRA $7fg - 5g - 2gf - 3f + 7g =$

see p. 127

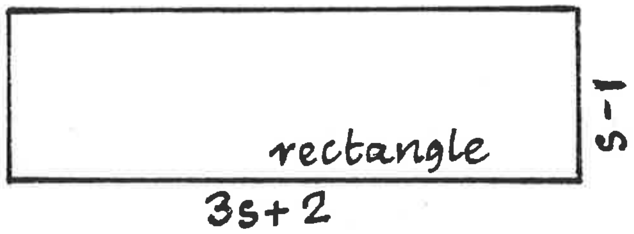
Find the distances of these journeys, from start to finish.



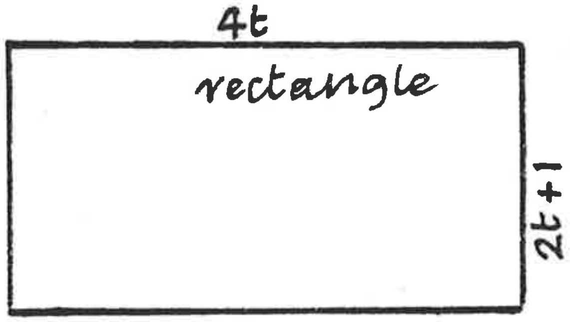
distances are on P 127

Find the perimeters of these shapes:

1.



2.



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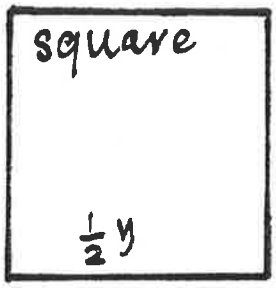

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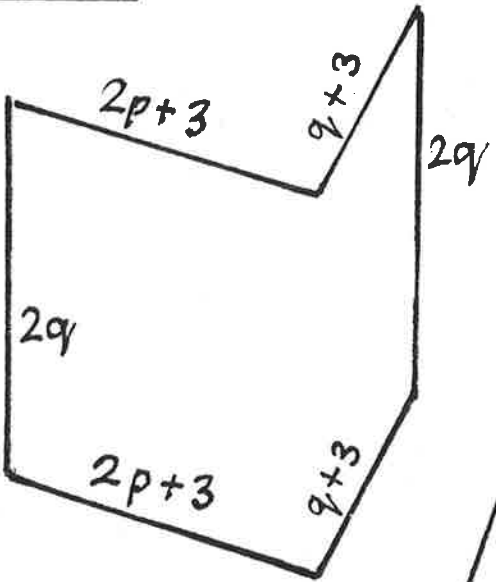

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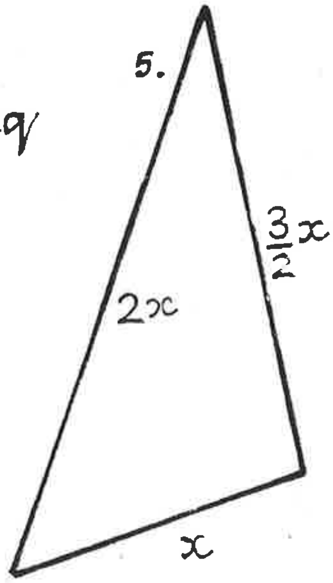

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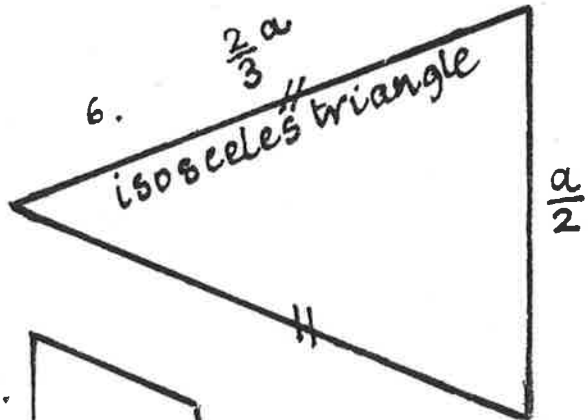

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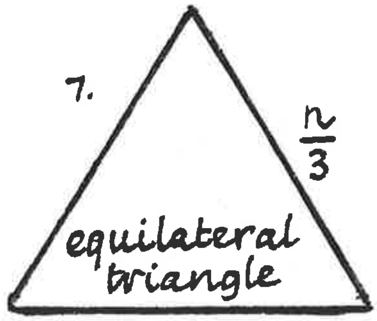

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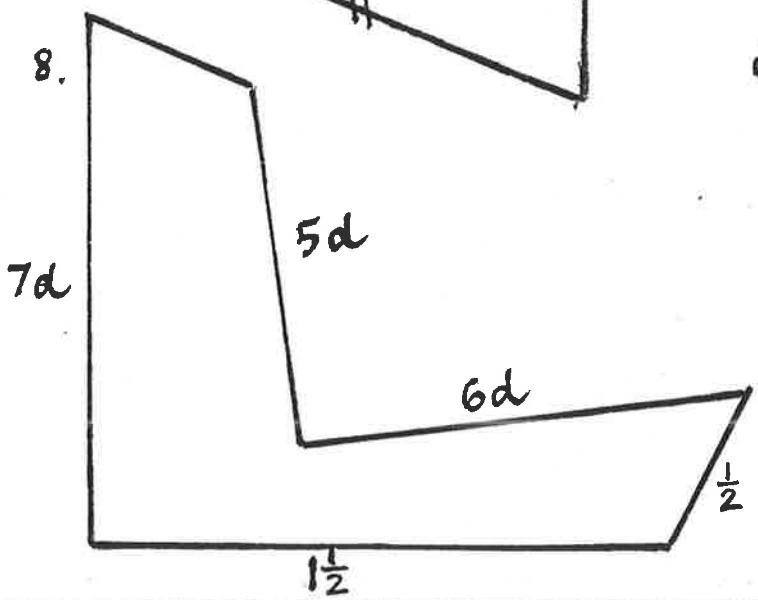
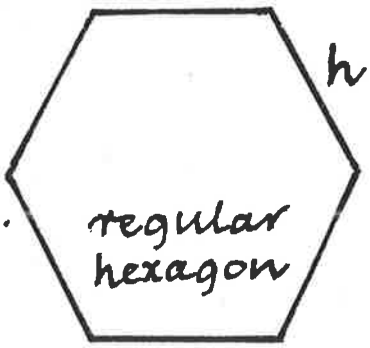


9.



perimeters are on P127

Your turn to sketch ...

Anywhere on this page sketch these shapes:

1. A square with perimeter $8y$
2. A square with perimeter $2y$

3. An equilateral triangle with perimeter $9n$
4. An equilateral triangle with perimeter n

5. A rectangle with perimeter $(6a+4)$
6. Another (different) rectangle with perimeter $(6a+4)$

7. Two different isosceles triangles, each with perimeter $(5t+2)$

Extension: A rectangle with <u>area</u> $(6a+4)$

Share your findings with others

Discuss these please :

Write the next 3 terms in each of these sequences :

1. x, $3x$, $5x$, $7x$, ___, ___, ___

   What term are you adding each time? ___

2. $5a$, $4a$, $3a$, $2a$, ___, ___, ___

   What term are you adding each time? ___

3. 1, $r+2$, $2r+3$, $3r+4$, ___, ___, ___

4. $6t+4$, $5t+3$, $4t+2$, $3t+1$, ___, ___, ___

5. $\frac{n}{4}$, $\frac{n}{2}$, $\frac{3n}{4}$, n, ___, ___, ___

6. $15a$, $12a-b$, $9a-2b$, $6a-3b$, ___, ___, ___

See P127

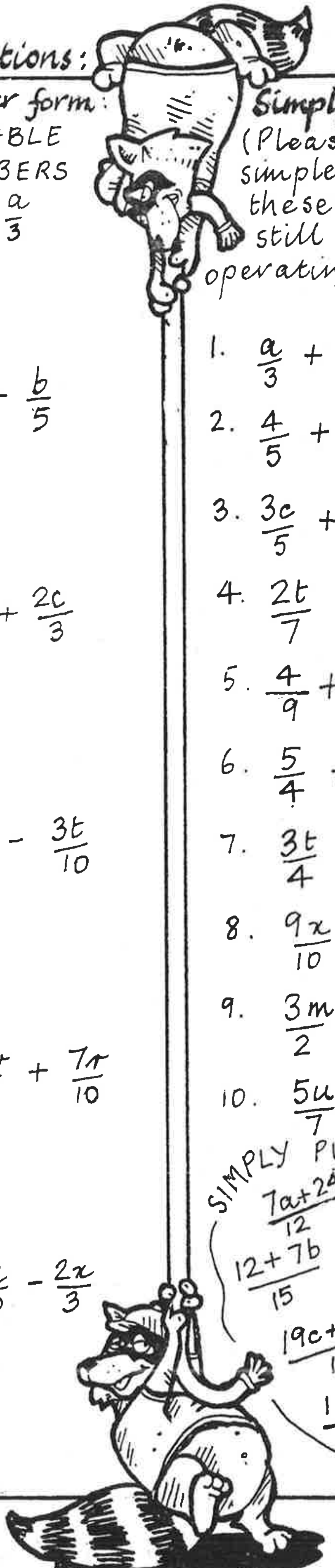
Complete these equivalent fractions :

Specific numbers	Variable numbers
$\frac{1^{\times 3}}{4_{\times 3}} = \frac{}{12}$	$\frac{a}{4} = \frac{}{12}$
$\frac{1}{3} = \frac{}{12}$	$\frac{a}{3} = \frac{}{12}$
$\frac{2}{3} = \frac{}{15}$	$\frac{2b}{3} = \frac{}{15}$
$\frac{1}{5} = \frac{}{15}$	$\frac{b}{5} = \frac{}{15}$
$\frac{3}{5} = \frac{}{15}$	$\frac{3c}{5} = \frac{}{15}$
$\frac{3}{5} = \frac{}{10}$	$\frac{3r}{5} = \frac{}{10}$
$\frac{3}{4} = \frac{}{20}$	$\frac{3t}{4} = \frac{}{20}$
$\frac{3}{10} = \frac{}{20}$	$\frac{3t}{10} = \frac{}{20}$

You'll find the missing numbers here

9 6 4 3 6 3 10 15

Working with fractions: More fractions



Write these in a simpler form:

SPECIFIC NUMBERS	VARIABLE NUMBERS
1. $\frac{1}{4}+\frac{1}{3}$ = =	1. $\frac{a}{4}+\frac{a}{3}$
2. $\frac{2}{3}-\frac{1}{5}$	2. $\frac{2b}{3}-\frac{b}{5}$
3. $\frac{3}{5}+\frac{2}{3}$	3. $\frac{3c}{5}+\frac{2c}{3}$
4. $\frac{3}{4}-\frac{3}{10}$	4. $\frac{3t}{4}-\frac{3t}{10}$
5. $\frac{3}{8}+\frac{7}{10}$	5. $\frac{3r}{8}+\frac{7r}{10}$
6. $\frac{9}{10}-\frac{2}{3}$	6. $\frac{9x}{10}-\frac{2x}{3}$

Simplify these expressions:
(Please note that the simplest way of writing these expressions will still contain an operating symbol: + or − WHY?)

1. $\frac{a}{3}+2+\frac{a}{4}$
2. $\frac{4}{5}+\frac{2b}{3}-\frac{b}{5}$
3. $\frac{3c}{5}+\frac{1}{3}+\frac{2c}{3}$
4. $\frac{2t}{7}-3+\frac{t}{2}$
5. $\frac{4}{9}+\frac{p}{3}+\frac{2}{3}$
6. $\frac{5}{4}+\frac{2q}{5}-\frac{1}{5}+\frac{3q}{4}$
7. $\frac{3t}{4}-\frac{3t}{10}+\frac{3r}{8}+\frac{7r}{10}$
8. $\frac{9x}{10}-\frac{2x}{3}+\frac{4y}{5}-\frac{y}{3}$
9. $\frac{3m}{2}+\frac{4n}{5}-\frac{m}{2}-\frac{n}{2}$
10. $\frac{5u}{7}+\frac{v}{5}-\frac{v}{7}-\frac{3u}{5}$

SIMPLY PUT

$\frac{7a+24}{12}$ $\frac{10+3p}{9}$ $\frac{10m+3n}{10}$ $\frac{12+7b}{15}$ $\frac{21+23q}{20}$ $\frac{4u+2v}{35}$ $\frac{19c+5}{15}$ $\frac{18t+43r}{40}$ $\frac{11t-42}{14}$ $\frac{7x+14y}{30}$

Working with rational numbers

Write these in the simplest form:

Specific numbers	Variable numbers
1. $\frac{1}{4} + \frac{1}{3}$ = =	7. $\frac{a}{4} + \frac{a}{3}$
2. $\frac{2}{3} - \frac{1}{5}$ = =	8. $\frac{2b}{3} - \frac{b}{5}$
3. $\frac{3}{5} + \frac{2}{3}$ = =	9. $\frac{3c}{5} + \frac{2c}{5}$
4. $\frac{3}{5} + \frac{7}{10}$ = =	10. $\frac{3r}{5} + \frac{7r}{10}$
5. $\frac{3}{4} - \frac{3}{10}$ = =	11. $\frac{3t}{4} - \frac{3t}{10}$
6. $\frac{9}{10} - \frac{2}{25}$ = =	12. $\frac{9x}{10} - \frac{2x}{25}$

Check up: (first column only)

$\frac{18}{40}$ $\frac{13}{10}$ $\frac{41}{50}$ $\frac{205}{250}$ $\frac{19}{15}$ $\frac{9}{20}$ $\frac{7}{12}$ $\frac{7}{15}$ $\frac{65}{50}$

note that there are more solutions here than you need.

Tell the others in your group which solutions are not needed ... and why!

STUDENTS' INTUITIVE IDEAS

When students are introduced to a new concept or procedure it cannot be assumed that they are constructing knowledge in ways exactly as the teacher intends. New ideas are always learned in a context of past experience, beliefs and understandings. These prior views interact with the new learning experiences provided by the teacher, influencing how the learner acquires the knowledge. Some of these prior ideas may in effect confuse the learning process; the learner may hold views that are inadequate or confused - or plain incorrect. Often these limited notions or misconceptions are strongly held, and resist change and correction.

In the classroom, thinking that is based on limited and/or confused meanings of algebraic ideas needs to be confronted directly, or it will generate further problems. The situation must be handled sensitively, though, or students are likely to be reticent, fearing loss of self-esteem. A good approach is to value and respect ideas suggested by students (which may include errors) as being **potentially productive** in learning.

The important components of this 'constructive' approach involve classroom experiences

- that use learners' intuitive ideas as a basis for new constructions
- that assist learners to recognise that there is a gap in their knowledge (see pages 4 and 5)
- that give the evidence (preferably in the form of materials that can be handled) of a more adequate conception and/or procedure
- that support learners as they come to terms with this thinking, especially if their intuitive ideas are in conflict with the evidence provided.

This approach requires talking, genuine involvement, and skilled teacher guidance.

USING STUDENTS' INTUITIVE IDEAS IN THE CLASSROOM

Students will be encouraged to use their intuitive thinking productively if they feel confident and free to share their ideas and thinking openly with others in the classroom.

An 'intuitive ideas' approach might proceed in this way:

1. Find out what the students' intuitive ideas about a concept are. These steps might be followed:
   - (a) Write on the board problem examples of the idea that is to be introduced.
   - (b) Ask the students to write down on paper 'reasonable' suggestions/strategies/solutions that are based on their spontaneous, intuitive thinking.
   - (c) Teacher or students can write on the board the suggested ideas for everyone to see.
   - (d) Ask students to defend (give reasons for) their suggestions and perhaps discuss the pros and cons of each suggestion.

   (At this stage, the students' reasonable attempts could be collected for safekeeping, then returned for any required self-correcting <u>after</u> the students have been confronted with evidence that enables an adequate conception of the idea).

2. Use a teaching strategy (such as 'investigation' as oulined on pages 60,61) to provide students with evidence that indicates acceptable thinking and sound procedures. Organise the accepted thinking, procedures and presentation into a form that is manageable, highlighting the key ideas of the concept.

3. Consolidate the ideas through practice, providing <u>immediate, supportive feedback</u> while ideas are fragile and developing.

4. Recognise that coming to terms with new ideas is a **process** that may not proceed easily and in a linear fashion. 'Returning to review' concepts that are still shaky should be seen as a means of student support where and when it is needed.

WORKING WITH BRACKETS : USING STUDENTS' INTUITIVE IDEAS

In the language of algebra, brackets are used for very definite purposes: they are punctuation marks that must be interpreted with specific meaning. Mainly they ensure that we are left in no doubt about which numbers are grouped together for processing.

Students who have used brackets for a variety of purposes in their own written language may not be aware of their explicit use in mathematics. Intuitively, they understand and use brackets in rather haphazard ways without recognising that the use of brackets regulates the interpretation of algebraic expressions in very precise ways. In these cases, skilled teacher intervention performs a vital role in the learning process.

USING STUDENTS' INTUITIVE IDEAS

Working with brackets: Expansion

Working in a whole-class group ...

1. FIND OUT STUDENTS' INTUITIVE IDEAS

(a) Ask students to write in a simpler form:

3 (7 + 3)
3 (3 + 7)
3 (7 − 3) well then, what about 3 (7a + 3a)
3 (3a + 7a)
3 (7a − 3a) ?

<u>REMINDER</u> Use of brackets specifies ... a PRODUCT

(b) Ask students to suggest how these products may be re-written without brackets (a transformation procedure)

3 (7 + a)   3 (a + 7)
3 (7a + 3)   3 (3a + 7)
3 (a + b)
3 (a + 3b)

(Students are given the opportunity to record their intuitive solutions on paper)

(c) Students offer suggested solutions.
<u>ALL</u> suggestions are accepted and written on the board.

(d) Teacher and students discuss pros and cons of the suggestions.

* Note: Some students will intuitively expand the expressions in the accepted way; for others the procedure will not be so clear cut.

2. PRESENT ACCEPTABLE THINKING/PROCEDURES

'Argue' for acceptable approaches by linking new ideas to WHAT THE STUDENTS ALREADY KNOW

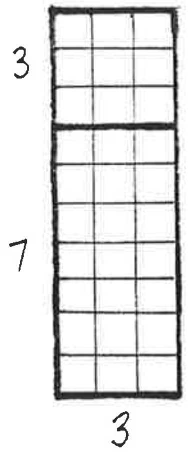
for example,

(a) checking arithmetically

3 (7+3)	3 (7+3)
= 3 × 10	= 3.7 + 3.3
= 30	= 21 + 9
	= 30

(b) illustrating the product of two factors as an area

TRY THIS STORY –

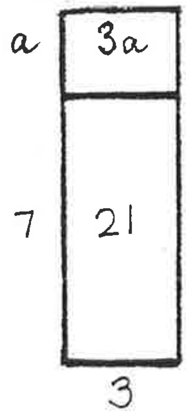


facade of city building:
penthouse
main building

how many windows altogether?

(students are familiar with the procedure of
- finding areas of rectangles
- adding to find a total)

now generalise (extend) idea to variable numbers:



large rectangle: $3(a+7)$

sum of small rectangles: $3a+21$

* Note that the arithmetical check, and the rectangle drawings give the students **evidence from their own experience** that the thinking/procedure presented gives an acceptable solution.

3. DISCUSS APPROPRIATE WAYS OF PRESENTING THE DISTRIBUTIVE LAW:

(a)

X	3
7	21
a	3a

...OR.. $3(7+a)$ ('distributes' fairly)

$\rightarrow$ $21+3a$

...Quiz time now →

EXPANSIONS:

Class-group work

Quiz 1:

Rewrite these expressions as a single number:

$3(2+5)$

$12(5-4)$

$7(6-2)$

$\frac{1}{2}(8+4)$

$\frac{1}{3}(9-6)$

Quiz 2:

Rewrite these expressions without the brackets:

$3(x+5)$

$12(5-b)$

$7(6p-2)$

$\frac{1}{2}(8+4a)$

$\frac{1}{3}(9y-6)$

Quiz 3:

Rewrite these expressions without the brackets then collect like terms:

$2x+3(x+5)$

$7+3(4+y)$

$4+3(2a+1)$

$4(3t+4)-6$

$5(3+y)-2y$

Quiz 4:

Rewrite these expressions without the brackets, then collect like terms:

$3(4p-2)-6p$

$8(2-a)-10$

$4(3-2b)-6$

$a(b+2)-a$

$p(2q+5)+pq$

Quiz 5:

Rewrite without brackets
Collect like terms

$3(x+4)+4(2+3x)$

$5(2y+1)+3(1-y)$

$4(3t+5)+2(5t-8)$

$7(2a+3)+(-4a-9)$

$2(7+b)+(-2b+1)$

Quiz 6:

Rewrite without brackets
Collect like terms

$a(b+7)+2a(3b-1)$

$2p(q+4)+p(-q+1)$

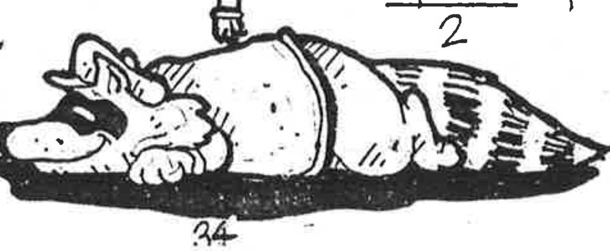
$\frac{1}{3}(6a+3)+\frac{1}{4}(4a+12)$

$\frac{6a+3}{3}+\frac{4a+12}{4}$

$\frac{1}{2}(8p+2)+\frac{1}{4}(4p+8)$

$\frac{8p+2}{2}+\frac{4p+8}{4}$

Are you checking over solutions together?



Talking Maths in small groups -

Activity 1

In everyday life an efficient way to sort and organise things is to find something that is common to all of them.

★ - Take turns to suggest <u>one</u> characteristic that is common to each of these sets.

(1) board surfing
water skiing
wind surfing

(2) Ferrari
Lotus
M.G.

(3) Adelaide
Sydney
Perth

(4) albatross
pelican
flamingo

(5) peach
apricot
nectarine

(6) Dr. No
Live and let die
The living daylights

Activity 2

Even more sophisticated organisation is possible if you can recognise more than one thing that is common to all.

★ - Look again at each set above, and see how many things you can find that are common to each set.

Activity 3

The organising technique that we use here is to find all the factors that are common to each of these sets of numbers or algebraic expressions, and then to select the factor that is the greatest.

For each of these sets, select the greatest factor that is common to all ..... then .....

(1) 14 35 56

(2) 13 7 9

(3) $3a$ 15 $6b$

(4) 48 $16t$ $32s$

(5) $6ab$ $12bc$ $18b$

(6) $6xy$ $4xyz$ $8xz$

(7) $9ax$ $12ay$ $6axy$

(8) $2\pi r$ πr^2 $2\pi rh$

DISCUSS why!

USING STUDENTS' INTUITIVE IDEAS.
– FACTORISATION –

1. Start with the students' prior knowledge.

Quick quiz: For each of these sets, select the greatest common factor.

(1) $8 \quad 12x \quad 20y$
(2) $9xy \quad 3 \quad 3y$
(3) $8t \quad 16rst \quad 4rt$
(4) $6ab \quad 2a \quad abc$
(5) $2pq \quad 8p \quad 12pqr$

Talk about how the greatest common factor is selected.

2. Find out students' intuitive ideas about factorisation:

Quiz a. Write these algebraic expressions as a product of two factors:

(1) $8 + 12x + 20y = 4(\quad + \quad + \quad)$
(2) $9xy + 3 - 3y = 3(\qquad\qquad)$
(3) $8t - 16rst - 4rt = 4t(\qquad\qquad)$
(4) $6ab - 2a + abc =$
(5) $2pq + 8p + 12pqr =$

b. Discuss the pros and cons of suggested ideas.

3. 'Argue' for acceptable thinking, procedures, presentation.
Do expansion procedures help in supporting your argument?

4. Discuss why and when you might need to write an algebraic expression as a product of factors.

5. Factorise fully: (immediate teacher feedback should be provided at the start)

(1) $16ab - 12a + 4abc$
(2) $12pq + 18p - 42pqr$
(3) $3y + 9$
(4) $16 + 12x$
(5) $5a - 25$
(6) $7\pi - \pi r$
(7) $16t + 24st$
(8) $10y + 25yz$
(9) $14a - 21 - 7ab$
(10) $6pq + 9qr$
(11) $12t + 20rt - 24ts$
(12) $2ab + 8a$
(13) $36 - 18t$
(14) $45xz - 30xy$
(15) $\pi r^2 - 2\pi r$

EXPANSION and FACTORISATION

Note that **expansion** and **factorisation** are **reverse** procedures.

The **distributive law** that governs expansion and factorisation can be used

from **product to sum**: $3(x+5) = 3x+15$ EXPANSION

or from **sum to product**: $3x+15 = 3(x+5)$ FACTORISATION

Expansion and **Factorisation** are **transformation** procedures: changing the same expression from one form to another.

Sometimes the **expanded** (sum) form will help to solve a problem.

Sometimes the **factorised** (product) form is most useful.

Please expand:

1. $4(a+3)$
2. $3(t-9)$
3. $3(2x+1)$
4. $5(m-3)$
5. $4a(b+2)$
6. $x(5+2y)$
7. $2m(3+4n)$
8. $5(2p-3q+4)$
9. $5(x+1)+2(x-2)$
10. $3(4y+1)+2(y-1)$
11. $5(3a+2)+3(2a-1)$
12. $5(4t+2)+3(3t+1)$
13. $3g(3h+2)+5g(h+1)$
14. $b(a+9)+b(a+1)$

Please factorise fully:

1. $ax+ay$
2. $3a-6b$
3. $3t+3$
4. $2p-8q$
5. $4ab+8a$
6. $14+7cd$
7. $a\sqrt{2}+b\sqrt{2}$
8. $25-5ab$
9. $2xy-10x$
10. $26\times5-26\times4$
11. $172\times8+172\times2$
12. $156\times13-156\times3$
13. $5lm-25l$
14. $4\sqrt{3}-x\sqrt{3}$

see P127, 128

RELATIONS . . .

The language we speak contains many words (or groups of words) that connect the things or events or people or ideas that we are talking about.

These connections link things together, making our thinking and communicating more organised — and unified.

MATHS TALK in small groups

Create sentences that use these relations:

. . . belongs to . . .

. . . is the sum of . . .

. . . is faster than . . .

. . . are the parents of . . .

. . . is the product of . . .

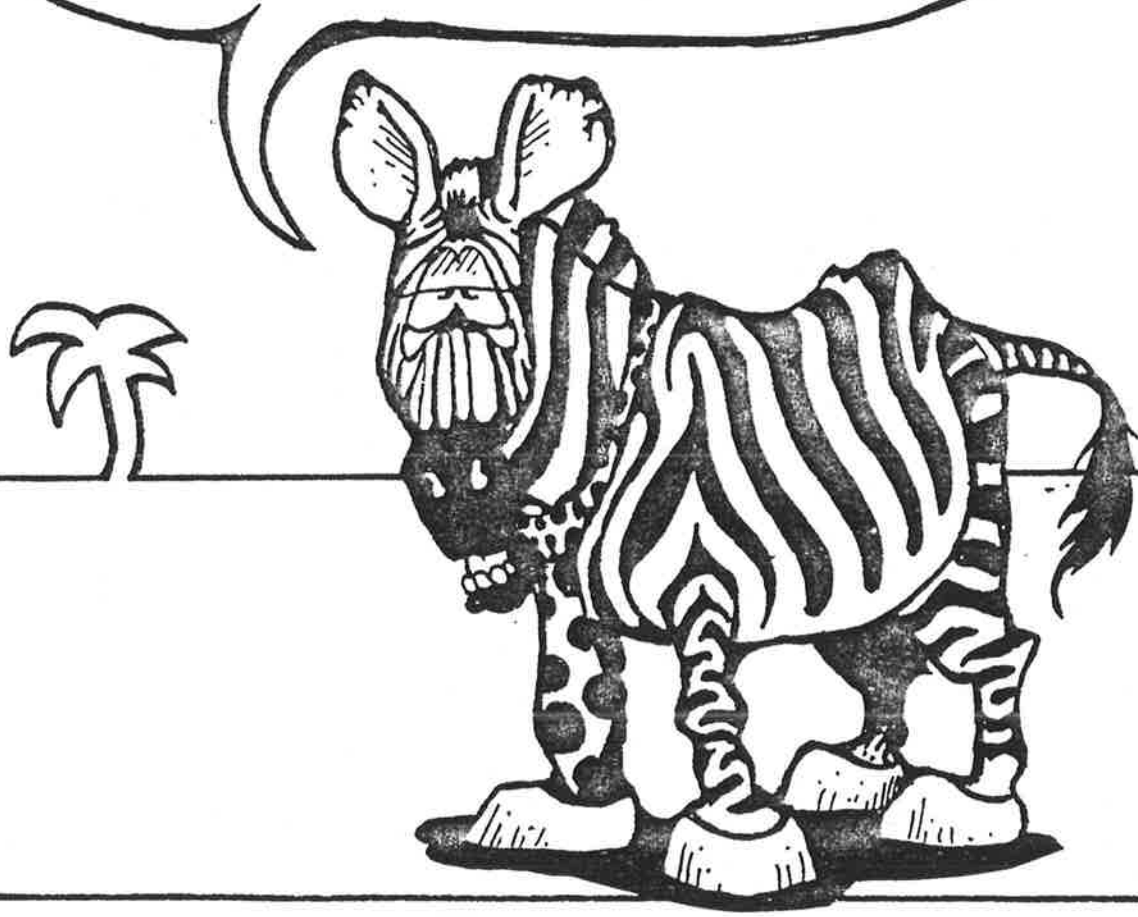
. . . are on . . .

. . . are factors of . . .

. . . are the stars of . . .

. . . plays . . .

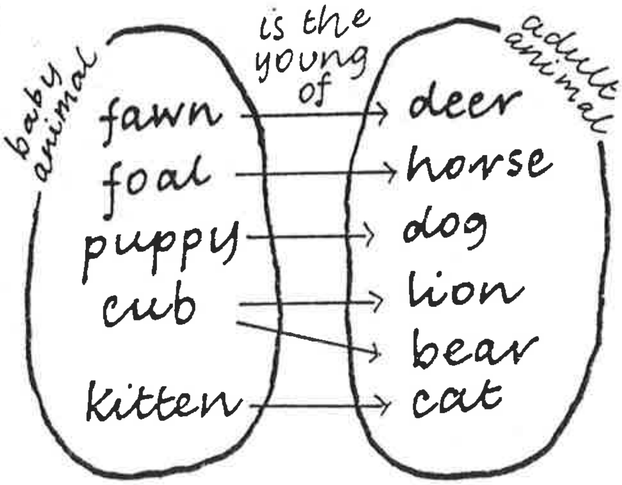
. . . lives in the same city as . . .



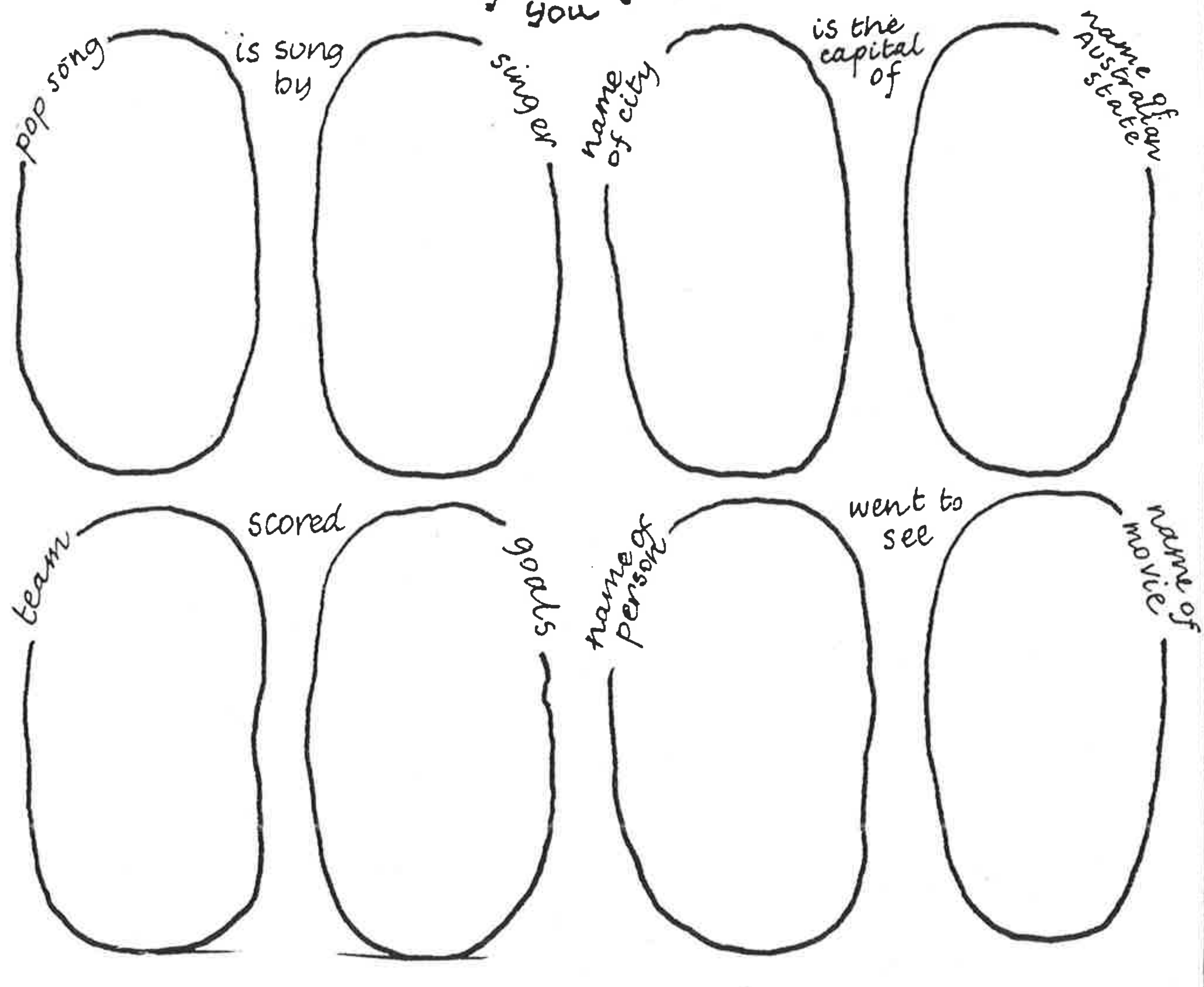
RELATIONS refer to CONNECTIONS between things, or events, or people, or ideas.

One way to organise and work with relations is to put things into sets ...

example:



Your turn now...
Work in small groups if you like



. . . another way is to write their names in a table

tennis netball squash	rounders		
	is played on		
		an oval	a pitch

7	48		72
	is greater than		
		9	

	has a birthday on		

Please note!
Ask a friend to check that the sentences you have constructed are true. For some sentences you might insert many names on either side of the connecting words.

25		64		4
	is the square of			
	3		7	

Notice that relations can involve NUMBERS

In the language of algebra we need to make sense of :

- symbols for the numbers
- symbols that show how the numbers are related.

Some of the relations we find in this book have a symbol to represent them. It is important to read them precisely as they are meant.

symbol	relation
$=$	is equal to
$<$	is less than
$>$	is greater than
$\leqslant$	is less than or equal to
$\geqslant$	is greater than or equal to
$\parallel$	is parallel to
$\perp$	is perpendicular to
$\neq$	is not equal to

Some relations do not have symbols to represent them. Examples are:

... is the square of ...
... is the negative of ...
... is the reciprocal of ...
... is a factor of ...
... is a multiple of ...
... is the sum of ... and ...

a first quantity is greater than a second quantity	$u > v$ (quick?!)
in any circle, the length of the circumference divided by the circle's diameter will give a constant value that is denoted by pi, a letter in the Greek alphabet	$\frac{c}{d} = \pi$ (concise?!)
three times the result of subtracting a certain number from five gives the same result as dividing the number by three.	$3(5 - x) = \frac{x}{3}$ (time saver?!)
two quantities are related so that one is four more than twice the other	$x - 4 = 2y$ OK? or $x =$ ← complete

What could the two numbers be?

(1987 edition)

Sometimes we can write algebraic statements in a very concise form.

These concise symbols help us to work quickly and efficiently.

Here are some relations that are written in English ...
and then rewritten in algebraic symbols.

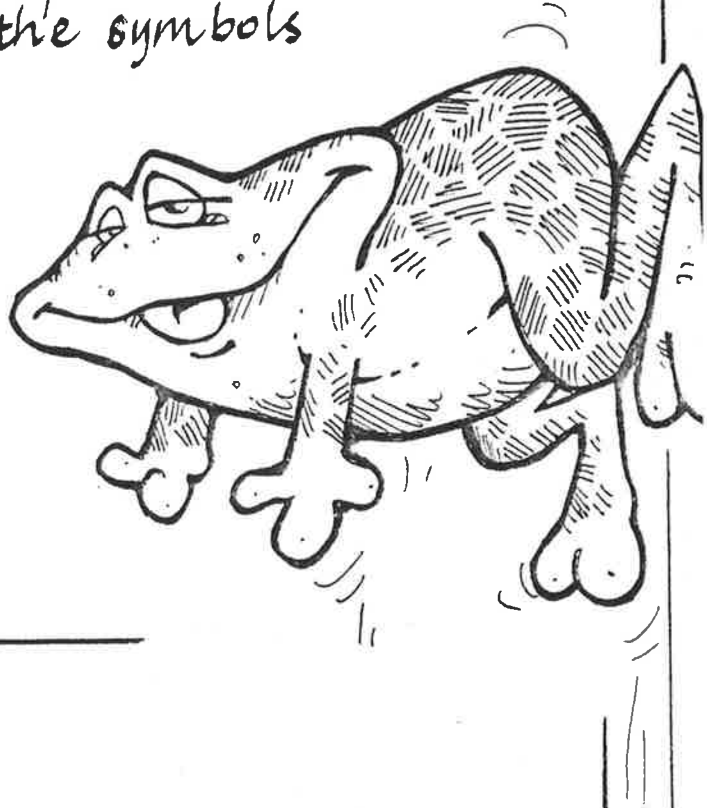
School finishes early when the temperature is thirty-six degrees or more	$t \geqslant 36$
The students using this book are aged between thirteen and sixteen	$13 \leq s \leq 16$
The teachers using this book are over twenty-one	$T > 21$
On average, the students in year 10 are a year older than the students in year 9	$S_{10} = S_9 + 1$
If tickets are \$7 each, the number of dollars raised will be seven times the number of tickets sold	$D = 7t$
... but if the tickets are 50¢ each ...	$D = \frac{1}{2}t$ or $D = \frac{t}{2}$
Drivers who are fined for speeding are charged a levy of \$5, plus eight dollars for every kph over the speed limit.	$C = 5 + 8s$

(What would you be fined for driving at 76 kph in a 60 kph zone?)

(1989 edition)

Reading algebra ...

- Form a small group.
- Read these algebraic statements to others in the group.
- What do you think the symbols could mean?
- Can the others in your group make sense of what you are reading?
- Check your reading with some-one who is experienced in 'talking algebra'.



$x = y$

$a = 2b$

$s = 4t$

$t = \frac{1}{4}s$

$t = \frac{s}{4}$

$C = 2\pi r$

$I = \frac{PRT}{100}$

$A = \frac{b \times h}{2}$

$P = 2(l+b)$

$V = l^3$

$a < b$

$p \geq q$

if $a > 0, b > 0,$
$a + b \neq a - b$

$3 < n < 7$

$V > 18$

$16 < d$

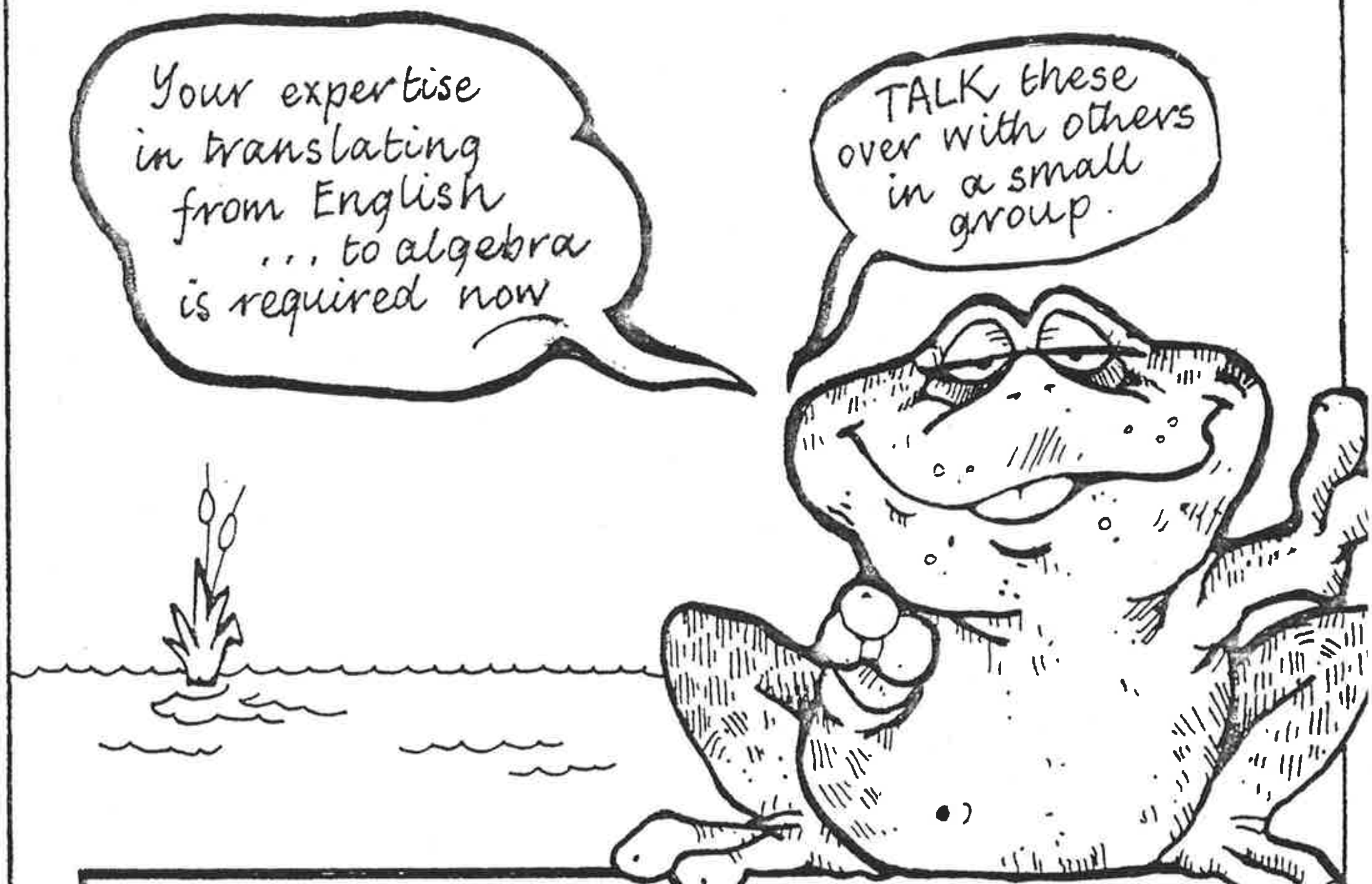
$g = b + 2$

$h = b - 3$

$(n-1) + n + (n+1) = 12$

$n + (n+1) + (n+2) + (n+3) = 30$

$C = 25 + 3n$



1. Two quantities are related so that one is six times as big as the other	1.
2. Two quantities are related so that one is three less than the other	2.
3. The product of two non-zero numbers cannot be zero	3.
4. The sum of two numbers is less than four	4.
5. The difference of two numbers is half the sum of those numbers	5.
6. The area of a square is the square of the length of the side	6.
7. The sum of three consecutive integers is twenty seven	7.

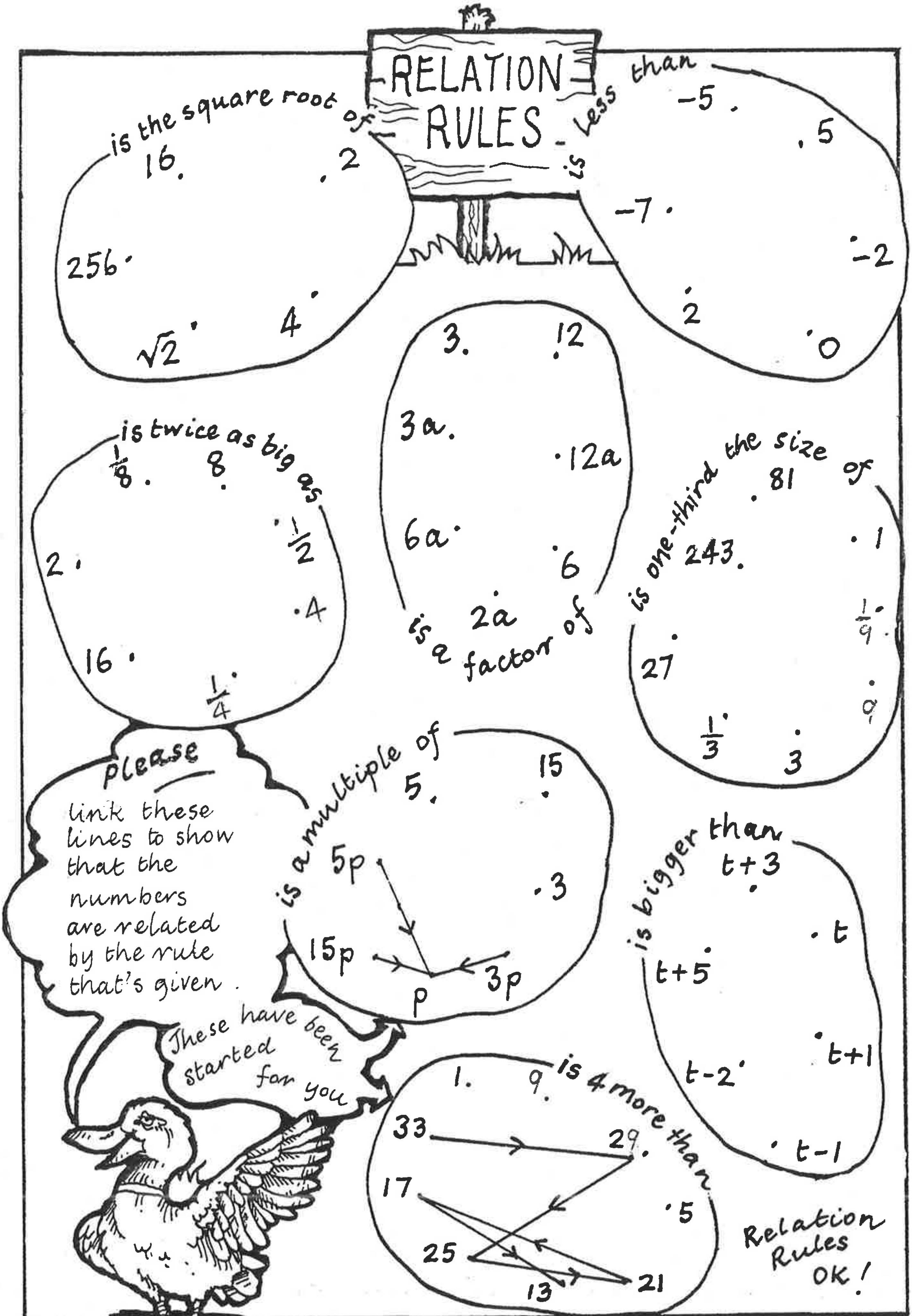
 (1987 edition)

Talking and Writing ALGEBRA ...

Your expertise in translating from English to algebra is required now.

Please work in a small group.

1. In a forest, the number of trees can be found by multiplying the number of rows by the number of trees in each row.	1.
2. If records cost \$16 each, the amount you spend on records will be sixteen times the number of records that you buy.	2.
3. A cricketer's bowling average is found by sharing the number of wickets taken equally amongst the number of balls bowled.	3.
4. The height of a wanted criminal is given as "between 170 and 180 cms".	4.
5. People cannot vote until they are eighteen years old.	5.
6. The temperature range for the day is calculated by subtracting the minimum temperature from the maximum.	6.
7. The sisters are two years apart in age	7.


RELATION RULES
is the square root of
16. .2 256. 4. √2.
is less than
-5. .5 -7. .-2 2 .0
is twice as big as
1/8. 8. .1/2 2. .4 16. 1/4
is a factor of
3. 12 3a. .12a 6a. 6 2a
is one-third the size of
.81 .1 243. 1/9. 27 9 1/3 3
please
link these lines to show that the numbers are related by the rule that's given.
is a multiple of
5. 15 5p .3 15p 3p p
is bigger than
t+3 .t t+5 t+1 t-2 t-1
These have been started for you
is 4 more than
1. 9. 33 29. 17 .5 25 13 21
Relation Rules OK!

Writing relationships in different ways**:

* Extension

1. If $a+b=9$, what can you say about $a+b+6$?

2. If $c+d=7$, what can you say about $c+d+2$?

3. If $p+q=13$, what can you say about $p+q-9$?

4. If $e+f=5$, what can you say about $e+f+g$?

5. If $x+y=8$, what can you say about $x+y-z$?

6. If $a=b+c$ and $a+b+c=10$ what can you say about a?

7. If $p=q-r$ and $q-r+p=12$ what can you say about p?

8. If $r=3s$ and $r+s=12$ what can you say about s?

9. If $u+5=w$ and $3w=27$ what can you say about u?

10. If $a+b=10$ and $b=4a$ what can you say about a? what can you say about b?

11. If $a>3$ what can you say about $5a$?

12. If $b<20$ what can you say about $b/4$?

13. If $a>b$ and $b=3$ what can you say about $a+b$?

14. If $p+q=8$ and $p<q$ what can you say about p?

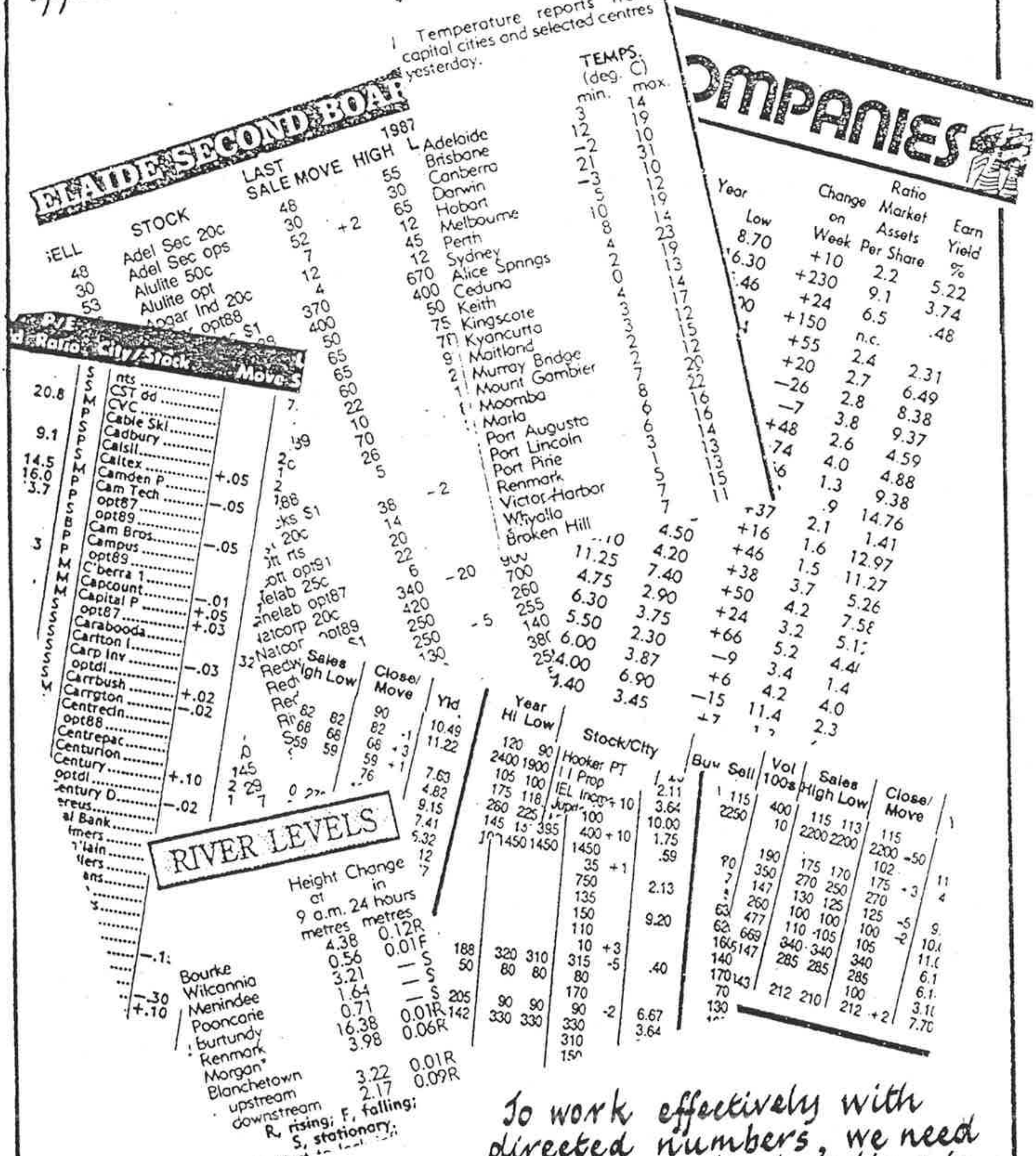
15. If $r+s=12$ and $r>s$ what can you say about r?

16. Is $a+b+c=a+b+d$ always sometimes or never?

** Hart (1981)

CLASS-ROOM TALK

DIRECTED NUMBERS : Numerical variables may take both positive and negative values. These everyday uses of directed numbers show quantities that take opposite directions from a reference point, ZERO.



To work effectively with directed numbers, we need to update our understanding in

(1) ADDITION (2) FINDING THE PRODUCT of directed numbers.

Directed Numbers

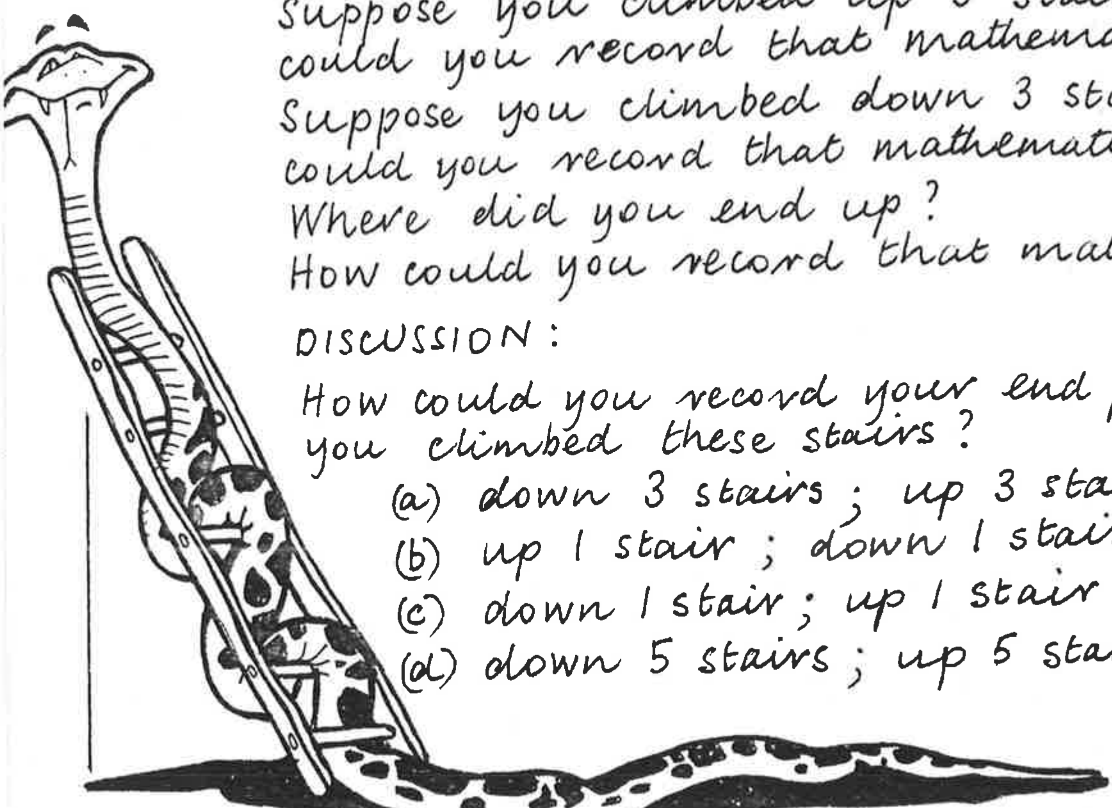
Using students' intuitive ideas
(Working in a whole-class group)

1. **Asking for reasonable suggestions**
   (1) Ask students for solution suggestions to
   $+5 - 2 = ?$ (or $^{+}5 + {}^{-}2$) EASY!
   (2) Ask students for solution suggestions to
   $-5 + 2 = ?$ (or $^{-}5 + {}^{+}2$)
   Record all suggestions on the board for everyone to see. Leave them on display.
   (3) Ask students who offered a solution to say why it seemed reasonable.
2. **Investigation sequence : intervention by teacher**
   - suggest an approach, and provide concrete evidence to show that the approach is adequate.

   (1) Refer to students' familiar knowledge

   Suppose you are standing on one stair in a flight of stairs (or on one rung on a long ladder). Call that stair (or rung) the starting point.
   Suppose you climbed up 3 stairs. How could you record that mathematically?
   Suppose you climbed down 3 stairs. How could you record that mathematically?
   Where did you end up?
   How could you record that mathematically?

   DISCUSSION:
   How could you record your end point if you climbed these stairs?
   (a) down 3 stairs ; up 3 stairs
   (b) up 1 stair ; down 1 stair
   (c) down 1 stair ; up 1 stair
   (d) down 5 stairs ; up 5 stairs



* Talk about how to record mathematically your climbing up and down (or down and up)
* Talk about the reason for each solution
* What rule can you suggest to apply to each of these operations?

(2) Extension to new idea

Quiz : Record mathematically the process of climbing and the end point reached (using signed numbers)

(a) up 3 stairs ; down 2
(b) up 3 ; down 7
(c) down 7 ; up 3
(d) down 4 ; up 6
(e) up 2
(f) down 5
(g) down 5 ; up 7
(h) down 5 ; up 2

DISCUSSION

Decide on appropriate
* ways of recording
* solutions

(3) Return to students' intuitive suggestions

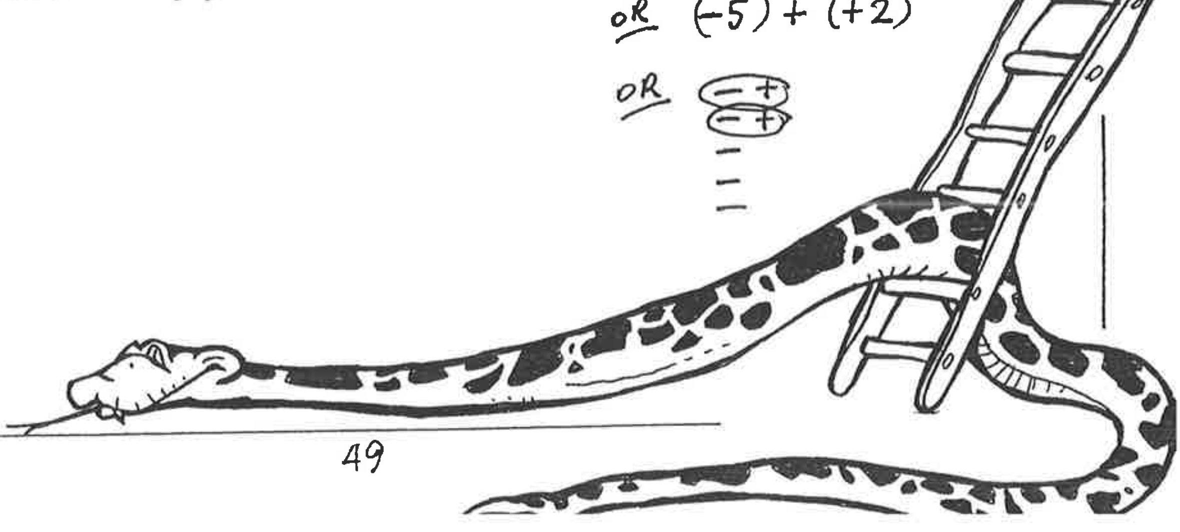
Discuss pros and cons of these suggestions.

$-5 + {}^{+}2$

OR $-5+2$

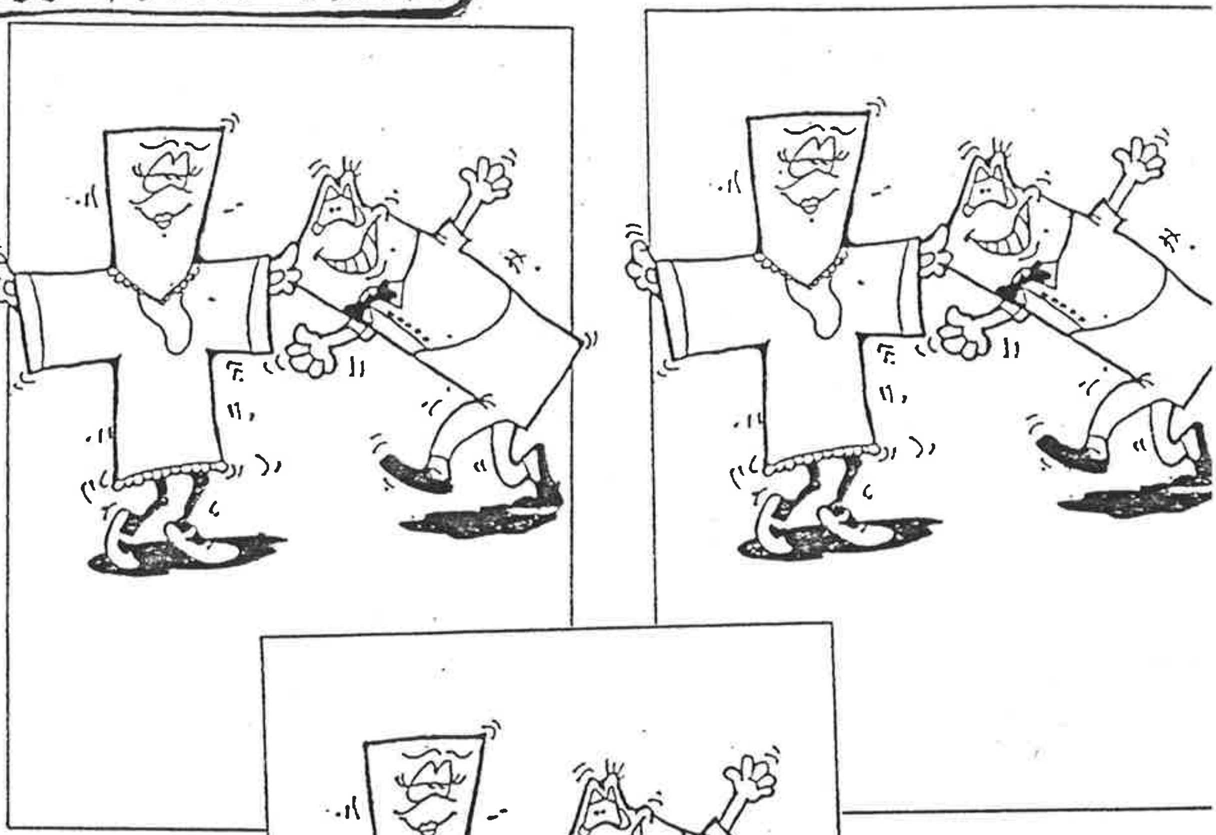
OR $(-5)+(+2)$

OR − + / − + / − / − / −



Positive and Negative numbers have equal status
The addition of $^{+}1$ and $^{-}1$ yields the number
ZERO

Class-room Talk



How do these pictures represent

$\boxed{{}^{+}3 + {}^{-}5}$

or

$\boxed{{}^{-}5 + {}^{+}3}$?

Class-room Talk ...

\+ –
\+ –
\+

These pictures represent:

or

Adding directed numbers: Class-room Talk

1. Add these.

$^{+}5$ and $^{-}5$
$^{-}5$ and $^{+}5$
$^{+}3$ and $^{-}3$
$^{-}3$ and $^{+}3$
$^{+}10$ and $^{-}10$
$^{-}10$ and $^{+}10$

✱ Talk about the outcomes
Why are the outcomes the same?
Can you decide on a rule that holds true in all these cases?

2. Add these

$^{-}7$ and $^{+}4$
$^{+}2$ and $^{-}2$
$^{+}5$ and $^{-}3$
$^{-}3$ and $^{+}2$
$^{-}6$ and $^{-}1$
$^{+}5$ and $^{+}2$

✱ Talk about these values as if they were showing you:
1. temperature changes (°c)
2. a bank balance (in \$1000)

Talk about the outcomes for each of these.

3. Add these

$^{-}7$ and $^{+}4$ and $^{-}2$
$^{+}2$ and $^{-}2$ and $^{-}3$
$^{+}5$ and $^{-}3$ and $^{+}2$
$^{-}3$ and $^{+}2$ and $^{-}1$
$^{-}6$ and $^{-}1$ and $^{-}2$
$^{+}5$ and $^{+}2$ and $^{+}4$
$^{-}10$ and $^{-}6$ and $^{+}14$
$^{+}24$ and $^{-}30$ and $^{-}5$

✱ Talk about the outcomes

Please note

Write the addition of directed numbers as $^{+}5 + {}^{-}5$ or $(^{+}5) + (^{-}5)$ . . . as long as you show clearly that $+$ is the operation 'add'

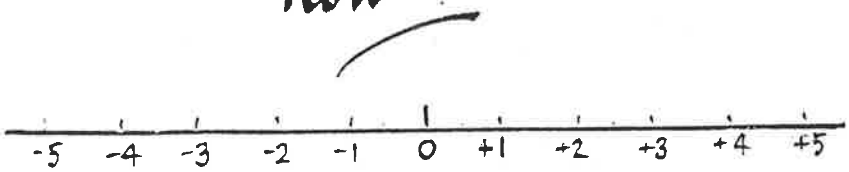
$^{+}5$, $^{-}5$ show the direction signs.

4. Add these

$^{+}5 + {}^{+}2$
$^{+}5 + {}^{-}2$
$^{-}5 + {}^{+}2$
$^{-}5 + {}^{-}2$
$(^{-}7) + (^{+}6)$
$(^{+}7) + (^{-}6)$
$(^{-}7) + (^{-}6)$
$(^{+}7) + (^{+}6)$

✱ Talk about similarities and differences in outcomes

Adding directed numbers: You're on your own now



✱ 1.
$-6 + {}^{-}4 =$
${}^{+}3 + {}^{-}2 =$
${}^{+}6 + {}^{-}3 =$
${}^{-}2 + {}^{+}2 =$
${}^{-}5 + {}^{-}1 =$
${}^{+}6 + {}^{+}2 =$

✱ 2.
$(-7) + (+4) + (-3) =$
$(+2) + (-2) + (-4) =$
$(+5) + (-3) + (+1) =$
$(-3) + (+2) + (-2) =$
$({}^{-}6) + ({}^{-}1) + (-3) =$
$({}^{+}5) + ({}^{+}2) + (+3) =$

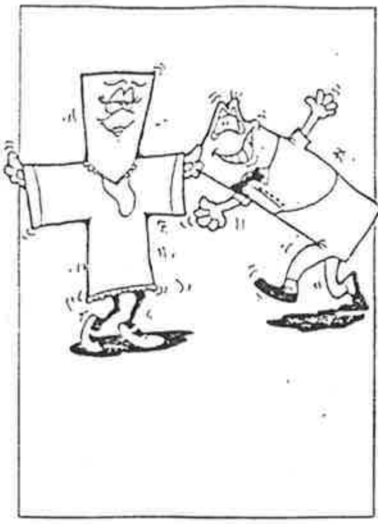
✱ 3.
$({}^{-}6a) + ({}^{-}4a) =$
$({}^{+}3b) + ({}^{-}2b) =$
$({}^{+}6c) + ({}^{-}3c) =$
$(-2d) + ({}^{+}2d) =$
$(-5e) + ({}^{-}e) =$
$({}^{+}6f) + ({}^{+}2f) =$
$({}^{-}7g) + ({}^{+}4g) + ({}^{-}3g) =$
$({}^{+}2h) + ({}^{-}2h) + ({}^{-}4h) =$
$({}^{+}5k) + ({}^{-}3k) + ({}^{+}k) =$
$({}^{+}3p) + ({}^{-}2p) + ({}^{-}5q) + ({}^{-}2q) =$

✱ 4.
$2x - 3x + 4x =$
$5y - 2y - 7y =$
$-3t + 3t - 2t =$
$-4a + 5a + 2a =$
$-n - 2n - 5n =$
$-9z + 2z - 2z =$

✱ 5.
$-2a + 3 - 4a - 4 =$
$2y - 6 - 3 + y =$
$-3t - 1 - 2t - 4 =$
$-6 + 5n - 2n + 1 =$
$-9b - 2c - 3b + c =$
$-5x + 5y - 7y =$

✱ 6. Find the sum of:
$-4, 5, -4, -3$
$-3, -2, -8, 4$
$-2, -8, 1, 7$
$-7, 6, -2, -1$
$-4, 5, -5, 0$
$-3, 14, -12, -16$
$71, -126, -132, 8$

(answers P128)



✱ 3. Complete these tables:

x	-5	-4	-2	0	1	3
$x+6$						

x	-7	-5	-4	-1	0	3
$x+5$						

✱ 1. (1) $5x = 7x + \square$

(2) $5x = 2x + x + \square$

(3) $5x = 4x + 6x + \square$

(4) $5x = 3x + \square - 4x$

(5) $5x = \square - 3x$

(6) $5x = \square - 3x + x - 2x$

What connections do you see between these problems and those in Quiz 1, page 24?

✱ 2. (1) $3t + 8 + \square = 8$

(2) $4 + 6p + \square = 2 + 6p$

(3) $2x + 2y + \square + \square = x + y$

(4) $5k + 2 + \square = 11k + \square$

(5) $8t + 5 + \square + \square = 12t + 2$

(6) $6a - b - 4a + \square = b + \square$

What are the connections between these and Quiz 5, p24?

✱ The missing quantities for ✱2. are all here

$2a$ $2b$ $-y$ 2

$-3t$ -2

$-x$ $6k$ -3 $4t$

✱ 4. Find the value of

$x - 3$

if $x = -3$

$x = -2$

$x = 0$

$x = 1$

$x = 4$

✱ 5. Complete this mapping:

$x \rightarrow x - 2$

$-7 \rightarrow$

$-3 \rightarrow$

$0 \rightarrow$

$2 \rightarrow$

$1 \rightarrow$

✱ 6. Insert $>$, $<$ or $=$

$-4-6 \;\ldots\; -7-3$

$-1+6 \;\ldots\; -3-7$

$6-4 \;\ldots\; -7+3$

$4-6 \;\ldots\; 3-7$

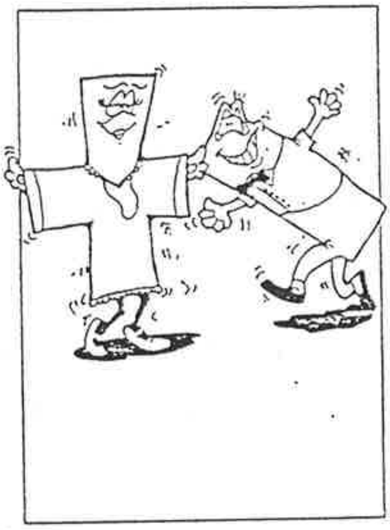
$-6+4 \;\ldots\; -3+7$

$-4+6 \;\ldots\; -7+3$

$4+6 \;\ldots\; 3+7$

✱ 7. Is the average of $-7, -4, 10, -5, -19$

-5 or -25 or 5?



Statistics is a part of mathematics. In statistics, people use samples of data (information that's been selected) to: find totals
find relationships
and predict outcomes
with the information.

Adding directed numbers

is a method that can save time and energy in statistics. It is particularly useful when your data consists of a cluster of numbers.

How you might use directed numbers in statistics.

Suppose you are monitoring your performance on a video game. Your scores for 5 successive games are: 137, 125, 138, 129, 128. Now all these scores cluster around 130, so 130 is a good 'measuring stick' for comparing performances. Some scores are above 130, some are below. So your scores could be organised like this:

which could also be set out like this

Scores	Scores using 130 as a starting point		+	−
137	+7		7	
125	−5			5
138	+8		8	
129	−1			1
128	−2			2
don't you worry about adding this lot!	+7	total	+15	−8 = +7

1. To calculate your **total** video-game score you would:
start with 5 lots of 130 = 650 **(why?)**
then add on +7 7
total score is 657

2. To calculate your **average** score you would:
average (share) 7 between 5 : $\frac{7}{5} = 1{\cdot}4$
Add 1·4 to 130 **(why?)** : average score is 131·4

Statistics: collecting directed numbers

Use these samples of data to find
(1) totals
and/or (2) averages (usually called the **mean** in statistics)
by adding directed numbers.

1. Points scored by your basketball team during a season: 67
72
59
84
73
75
81
54
88
70

* .... If some people choose 70 as the starting point, and some people choose 80, will your outcomes be the same or different?

2. Times for your white rat to negotiate a maze (to reach food!): 3 mins 45 secs
3 mins 57 secs
4 mins 02 secs
4 mins 10 secs
3 mins 43 secs
4 mins 13 secs
4 mins
3 mins 52 secs
3 mins 48 secs
3 mins 38 secs

3. Sales of a top recording in a big record store – week by week:
$966 (rounded to the nearest $)
980
980
1008
1050
1022
1008
994
994
980

4. Collect data on the ages (years and months) of people in the class.

Check your results with a calculator

<u>INVESTIGATING DEFINITIONS</u>: THE PRODUCT OF DIRECTED NUMBERS

The product of directed numbers might be defined in symbols as:

$(+)(+) \rightarrow +$
$(-)(-) \rightarrow +$
$(+)(-) \rightarrow -$
$(-)(+) \rightarrow -$

Students need to recognise the boundaries of the definitions and to work with the symbols quickly and confidently.

A good way to introduce students to definitions is through **investigation**. This approach enables students to build up the appropriate knowledge through an understanding of

<u>how</u> the definitions are derived
and <u>why</u> they are used.

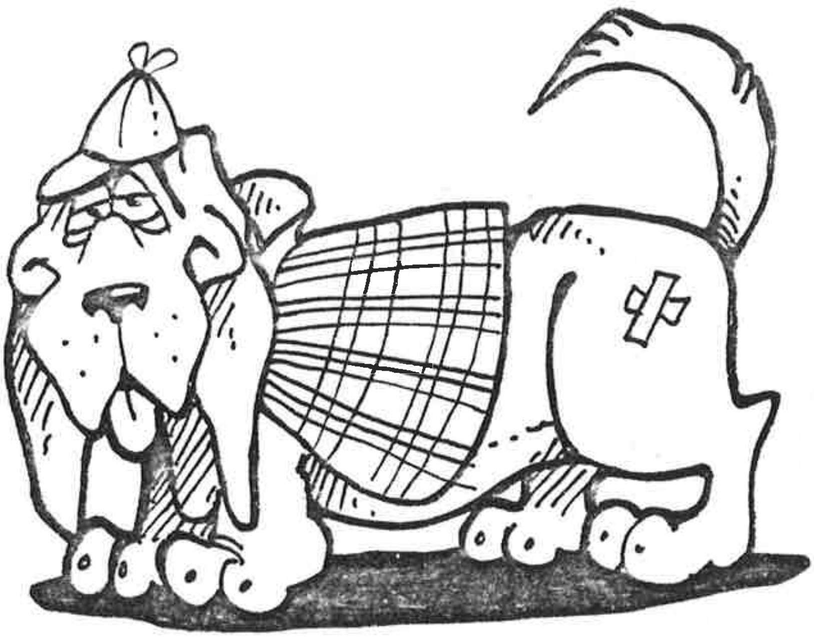
An investigatory approach might follow these steps:

1. **Start with the students' prior knowledge**, exploring procedures and ideas that provide a familiar and meaningful introduction to work that's new.

2. **Extend the familiar** by highlighting the emerging definitions through gradual progress towards new or more advanced procedures or ideas.

3. **Provide immediate feedback** to the students' early attempts to put the definition to use.

4. **Encourage discussion**. It's important that students learn to articulate their thinking, listen to the ideas of others, and follow discussion in a discriminating way. Chances are that this time will be well spent in clarifying thinking, developing partially-formed notions, eliminating misconceptions, and alerting teachers to students' particular learning needs.



<u>QUICK QUIZ</u>: An 'Investigation' Method

A sequence of quick quizzes provides an effective way of investigating how mathematical definitions are derived and why they're used.

Teacher and students work **interactively** (in a whole-class group), focusing on relevant and significant processes, rather than concentrating on a 'finished' presentation of work. You might liken it to a period of 'intensive care' - a high degree of involvement, intense learning, and strong teacher support.

Progress at this stage will be enhanced if

(a) the teacher monitors the students' learning, responding in ways that aim at improved understanding

(b) the students monitor their own learning.

A sequence of quick quizzes designed to promote students' investigations and understanding of the product of directed numbers follows this page.

INVESTIGATING DEFINITIONS: DIRECTED NUMBERS

Ideas for an interactive teaching sequence:

– teacher (assisted by students) directs the work from the board: students participate actively in the thinking/development process.

1. While the teacher writes this number sequence on the board, students complete the sequence on paper. ('Answers only' will do)

$5 - 2 = {}^{+}3$
$5 - 3 = {}^{+}2$
$5 - 4 = {}^{+}1$
$5 - 5 = 0$

establish the pattern through operations that are well known

$5 - 6 =$ ← What happens now? (link to previous work on adding directed numbers')

$5 - 7 =$
$5 - 8 =$
$5 - 9 =$
$5 - 10 =$ ← decreasing sequence

note: increasing sequence

CLASSROOM TALK!

2. Another sequence of familiar operations:

$4 - 0 =$
$4 - 2 =$
$4 - 4 =$
$4 - 6 =$
$4 - 8 =$
$4 - 10 =$
$4 - 12 =$
$4 - 14 =$

So what about
$-8 + 4$
$-10 + 4$
$-12 + 4$
$-14 + 4$??

3. Introducing the product of two negative numbers:

$5 - 3 = 2$
$5 - 2 = 3$
$5 - 1 = 4$
$5 - 0 = 5$
$5 - \quad = 6$
$5 - \quad = 7$
$5 - \quad = 8$
$5 - \quad = 9$
$5 - \quad = 10$ ← increasing sequence

decreasing sequence

OR

$5 - 3 = 2$
$5 - 2 = 3$
$5 - 1 = 4$
$5 - 0 = 5$
$5 - ({}^{-}1) =$
$5 - ({}^{-}2) =$
$5 - ({}^{-}3) =$
$5 - ({}^{-}4) =$
$5 - ({}^{-}5) =$

So, $5 - ({}^{-}1) = 6$
But $5 + 1 = 6$
What follows?

INVESTIGATING DEFINITIONS: DIRECTED NUMBERS

4. Introducing the product of two numbers
...with different signs

The previous sequence in reverse...

$5-(^-3) =$
$5-(^-2) =$
$5-(^-1) =$
$5-0 =$
$5-(^+1) =$
$5-(^+2) =$
$5-(^+3) =$
$5-(^+4) =$

increasing decreasing

...then

$5+3 =$
$5+2 =$
$5+1 =$
$5+0 =$
$5+(^-1) =$
$5+(^-2) =$
$5+(^-3) =$
$5+(^-4) =$

decreasing decreasing

If $5-(^+2) = 3$
and $5-2 = 3$
What follows?

$5+(^-2) = 3$
$5-2 = 3$
What follows?

... and what about $5-(^+2)$ and $5+(^-2)$
How are they related?

5. The product of two positive numbers:

$5+(^-1) =$
$5+0 =$
$5+(^+1) =$
$5+(^+2) =$
$5+(^+3) =$

increasing increasing

6. Now the students (individually or in groups) should define the product of directed numbers
... in words
... in symbols
in a maths work book, and show it to someone else for checking.

7. A final quiz to establish sound procedures...

① find the product	$3-(^-7)$	$3+(^-7)$	$3-(^+7)$	$3+(^+7)$
② collect the numbers	$=3+7$	$=3-7$	$=3-7$	$=3+7$
	$=10$	$=-4$	$=-4$	$=10$

Directed Numbers

– Improving skills –

Quiz 1: products of specific numbers

$(-3)(-2)$

$(-4)(+6)$

$(+3)(-4)$

$(-4)(+3)$

$(-1)(-4)$

$(+4)(-8)$

$(-3)(-3)$

$(-7)(+3)$

$5(-4)$

$(-9)(7)$

how can you tell that you are being asked to <u>multiply</u> here?

Quiz 2: products of a different sort

$-(-3)$

$-(+10)$

$+(-5)$

$+(+7)$

$-(+2)$

$-(-8)$

$+(+12)$

$+(-9)$

What are you being asked to do here?

Quiz 3:

find the product then add – like this

$-1-3(-2)$

$=-1+6$

$=\underline{5}$

1. $7-(-4)$
2. $4-4(+6)$
3. $6+3(-4)$
4. $8-4(+3)$
5. $5-1(-4)$
6. $16-7(-2)$
7. $15+4(-3)$
8. $2-10(2)$
9. $-3(-2)+7$
10. $\frac{5}{2}(-4)+3$
11. $-3(+8)-(-20)$
12. $-5(-5)+5(-5)$
13. $4(-3)-6(2)$
14. $-3-(-5)+(-2)$
15. $-9+(-3)-(-3)$

Match these answers with Quiz 3.

11   −20   13   30

−7   −9   −24   −4   0

−18   −6   −4   3   0   9

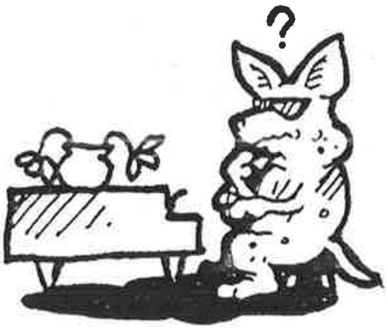
Directed Numbers:

1. If $y = 3x + 2$,
   what can you say
   about y, i
   (1) $x = -5$
   (2) $x = -2$
   (3) $x = 0$
   (4) $x = 4$

2. If $y = -2x + 1$,
   what can you say
   about y, if
   (1) $x = -4$
   (2) $x = -2$
   (3) $x = 1$
   (4) $x = 5$

3. If $y = \frac{5}{2}x - 3$,
   what can you say
   about y, if
   (1) $x = -6$
   (2) $x = -2$
   (3) $x = 0$
   (4) $x = 4$

4. If $2y = 3 - 4x$,
   what can you say
   about y, if
   (1) $x = -5$
   (2) $x = -2$
   (3) $x = 0$
   (4) $x = 1$



5. If $x = -2$,
   what can you say
   about y, if
   (1) $y = 5x - 3$
   (2) $y = -2x + 4$
   (3) $y = 3 - \frac{1}{2}x$
   (4) $2y = 6 - x$
   (5) $3y = 3 - 4x$
   (6) $y = 1 - \frac{4}{3}x$
   (7) $4y = 5x + 4$
   (8) $2x + y = 3$
   (9) $5x + 3 = 2y$
   (10) $2x - y = 5$

In these problems you were asked to LINK a second (output) number to a first (input) number. In maths, two numbers that are linked together by a rule can be written conveniently as an ordered pair. The solutions for these problems are written as ordered pairs.

(-5,-13)	(-6,-18)	(-2,-13)	(-2,-7/2)	
(-2,-4)	(-2,-8)	(-2,8)	(-2,-9)	
(0,2)	(0,-3)	(-2,4)		} why are
(4,14)	(4,7)	(-2,4)		these
(-4,9)	(-5,23/2)	(-2,11/3)		} the
(-2,5)	(-2,11/2)	(-2,11/3)		same?
(1,-1)	(0,3/2)	(-2,-3/2)		
(5,-9)	(1,-1/2)	(-2,7)		

FUNCTIONS

In the 18th century **Leonhard Euler** (1707 - 1783) explained a function as a special kind of connection

– between variables that are distinguished as

independent - varying freely -

and **dependent** - varying under constraint

This explanation provides a useful aspect of the work that we'll be doing in functions in this book.

In the 19th century **Lejeune Dirichlet** (1805 - 1859) clarified thinking about functions by stating that

– for any given x the associated y should be unique.

Recent developments in mathematics have created an even more precise understanding of functions. The language of set theory (founded by **Georg Cantor** - 1845-1918) uses the word **correspondence** to show that a function certain objects are **assigned** certain other objects.

A function involves three things:

- a set A, called the **domain**
- a set B, called the **range**

and a **rule** (using the symbols $f: A \rightarrow B$) that tells us how we must assign each element of set A (as it is defined) - a unique element of set B

(only one)

CORRESPONDENCES exist in everyday life. Think about these examples:

- the students in your maths class are assigned a teacher
- several suburbs or districts are assigned a particular postcode
- members of your family are assigned a family name
- many different 3-sided closed figures are assigned the name 'triangle'.

These assign many-to-one

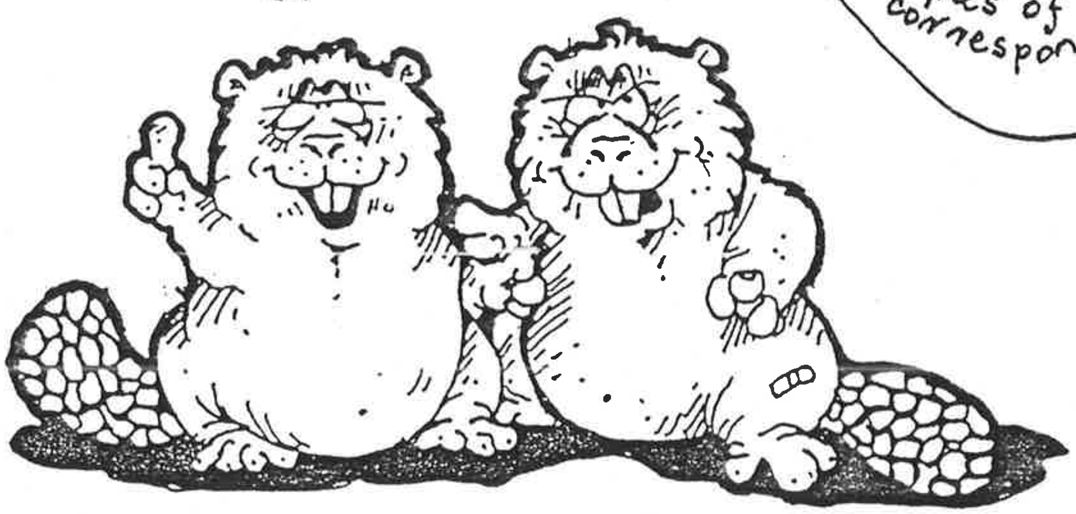
These correspondences might be called 'one-to-one':

- you are assigned a seat in your classroom
- the ticket you buy assigns you a seat in a theatre or stadium
- each car parked in the street or a carpark is assigned a marked space
- each page of a book is assigned a page number
- each key you press on a word processor (input) is assigned a symbol on your screen or page (output)

* In this book, our work with functions involves one-to-one correspondences.

GROUP WORK

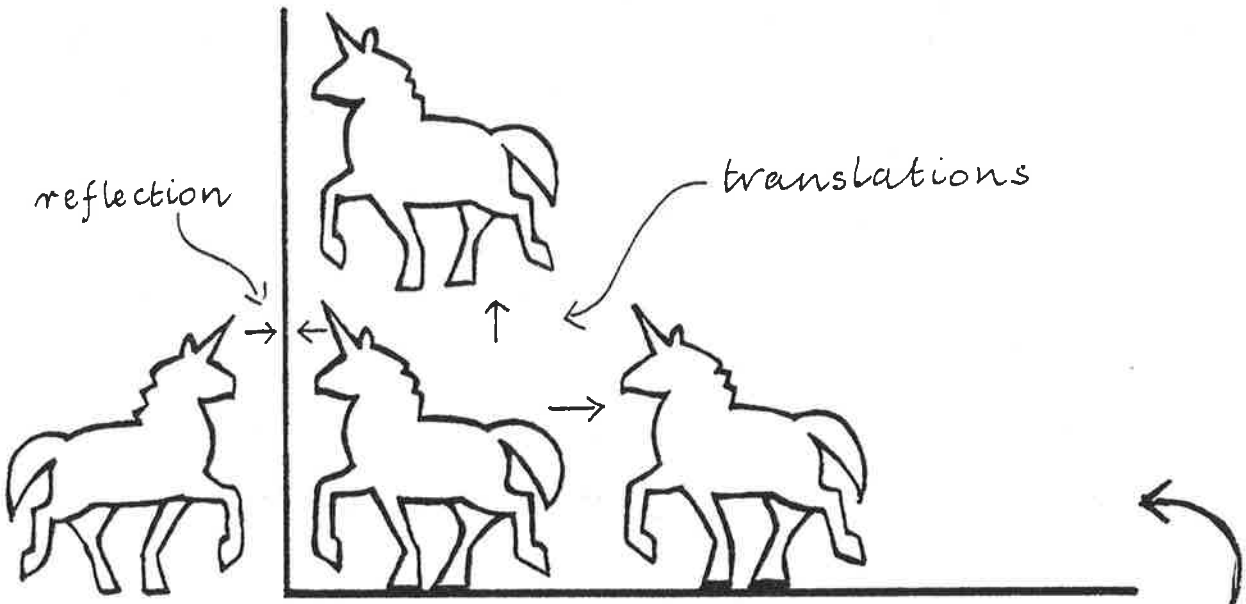
think of other examples of correspondences



Because maths is a concise language,
one word (or a single symbol)
can often have several meanings
– all similar, but with subtle differences.
Knowing these subtle differences is important.

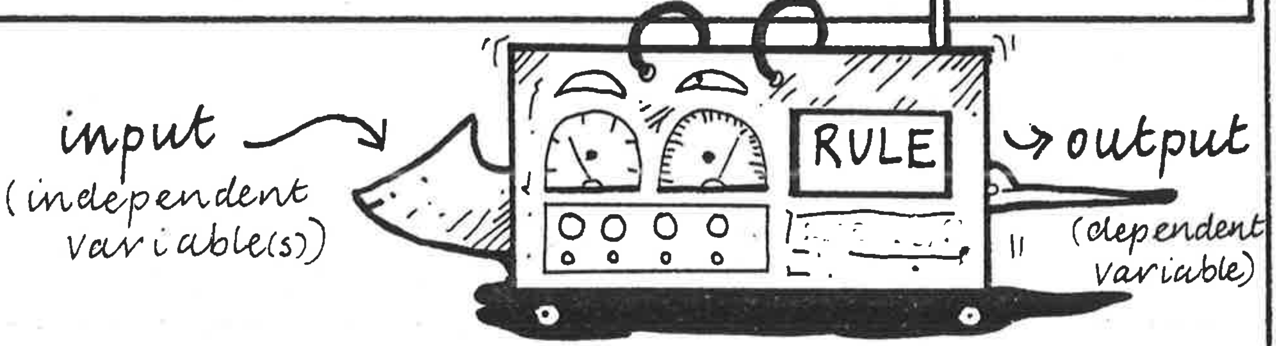
The word FUNCTION may refer to a number of similar mathematical acts.

Here are some:



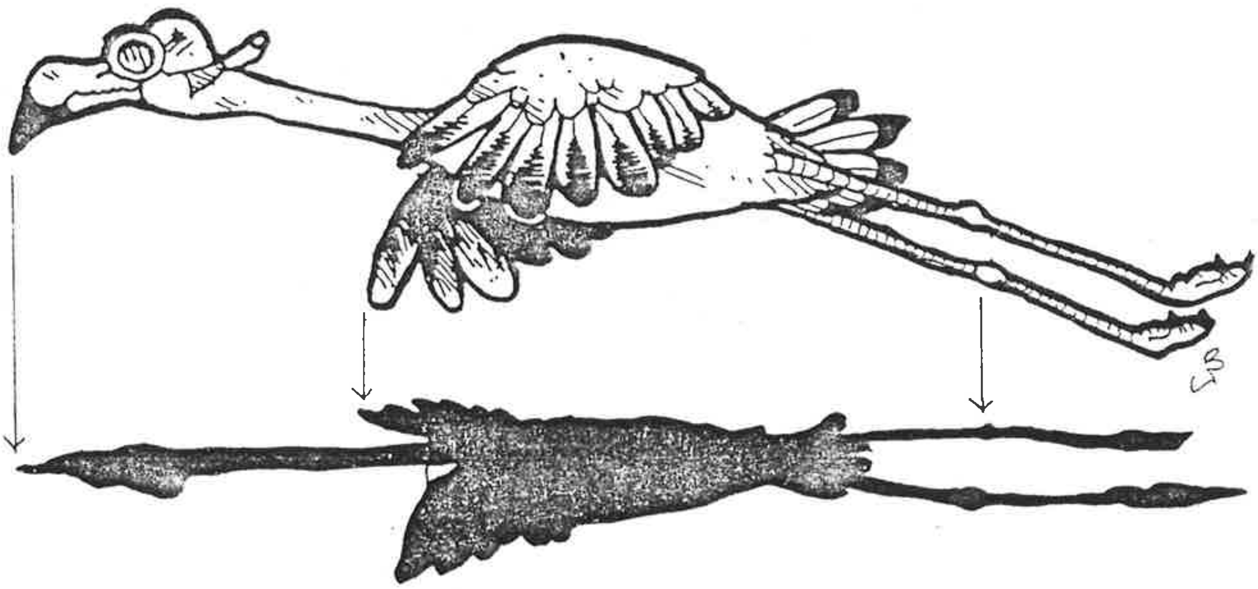
FUNCTIONS act as TRANSFORMATIONS
– one point is an <u>image</u> of another: where the image is depends on the nature of the tranformation

FUNCTIONS
act like
MACHINES



FUNCTIONS are MAPPINGS

– one point maps directly onto another — look for mappings in atlases, street directories, photographs, shadows, video images, scaled models, etc.



FUNCTIONS refer to a PROCESS of CHANGE...

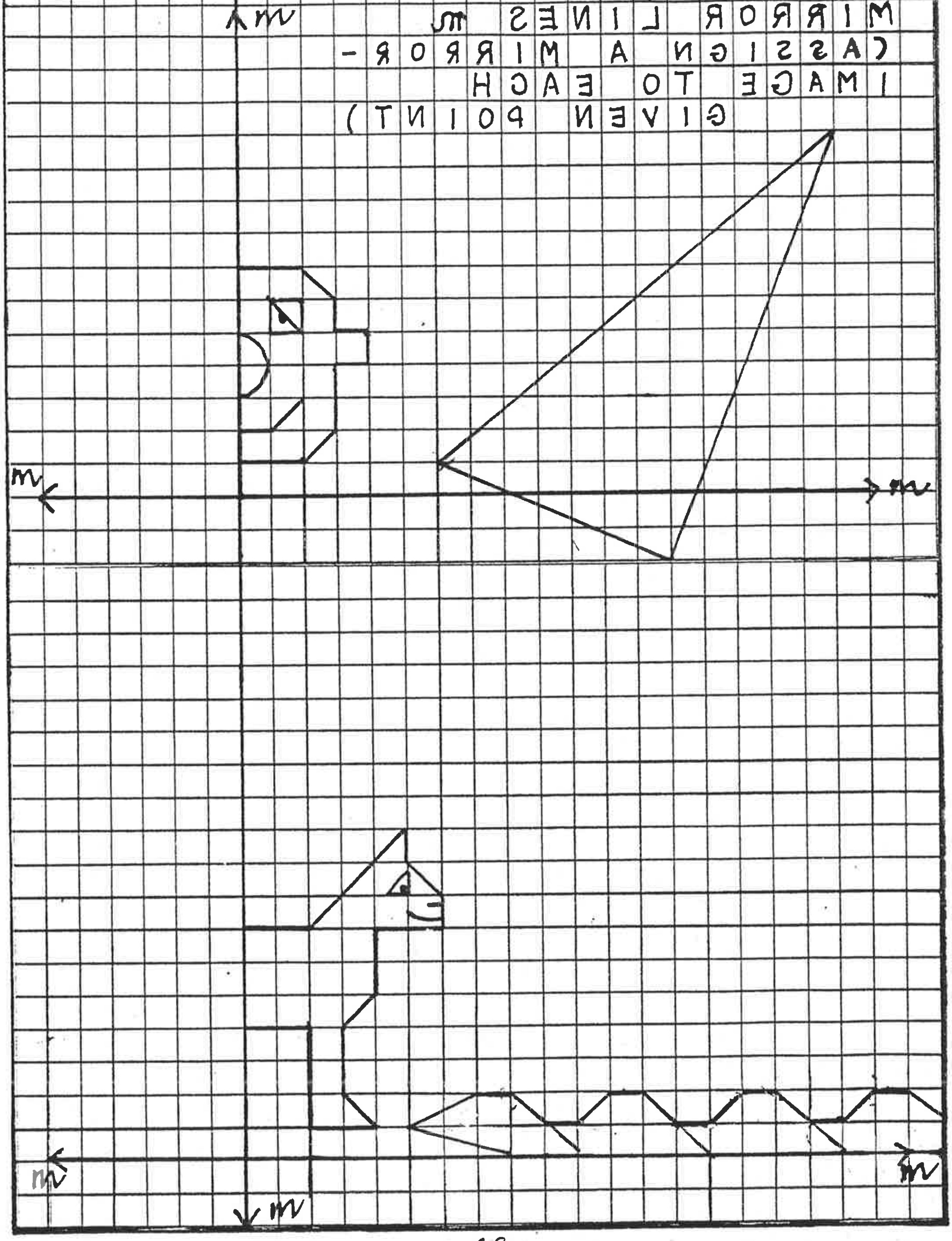
In this book we work with changes that are dependent on one simple rule.

In everyday life, changes are complex and depend on many factors. The outcomes are not always simple to predict. Mathematics models simple, well defined changes.

FUNCTIONS as TRANSFORMATIONS:

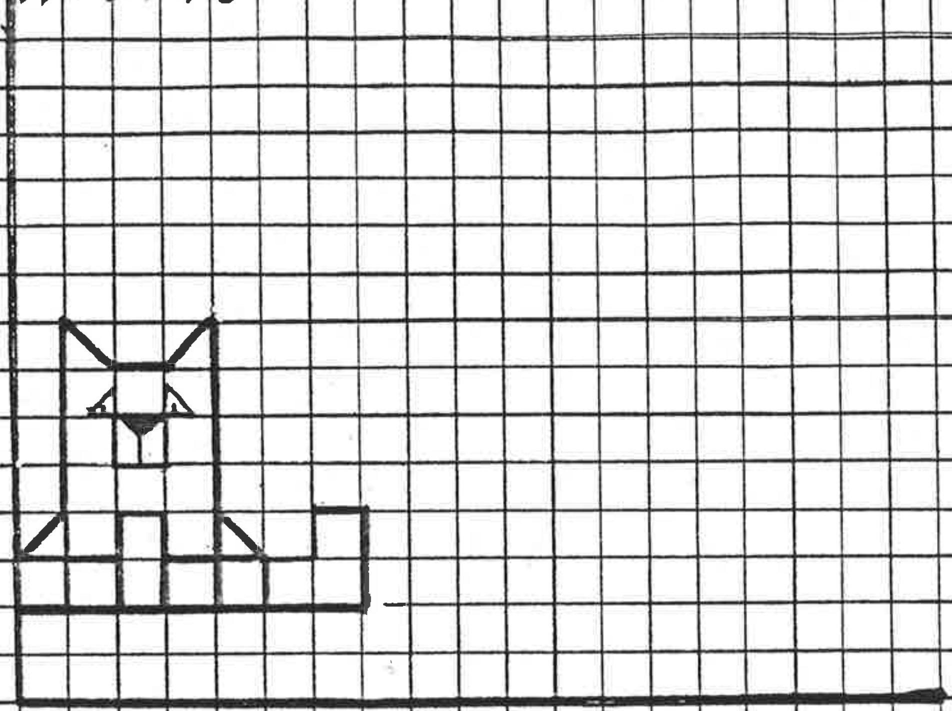
REFLECTIONS

REFLECT THESE SHAPES IN THE MIRROR LINES m (ASSIGN A MIRROR-IMAGE TO EACH GIVEN POINT)

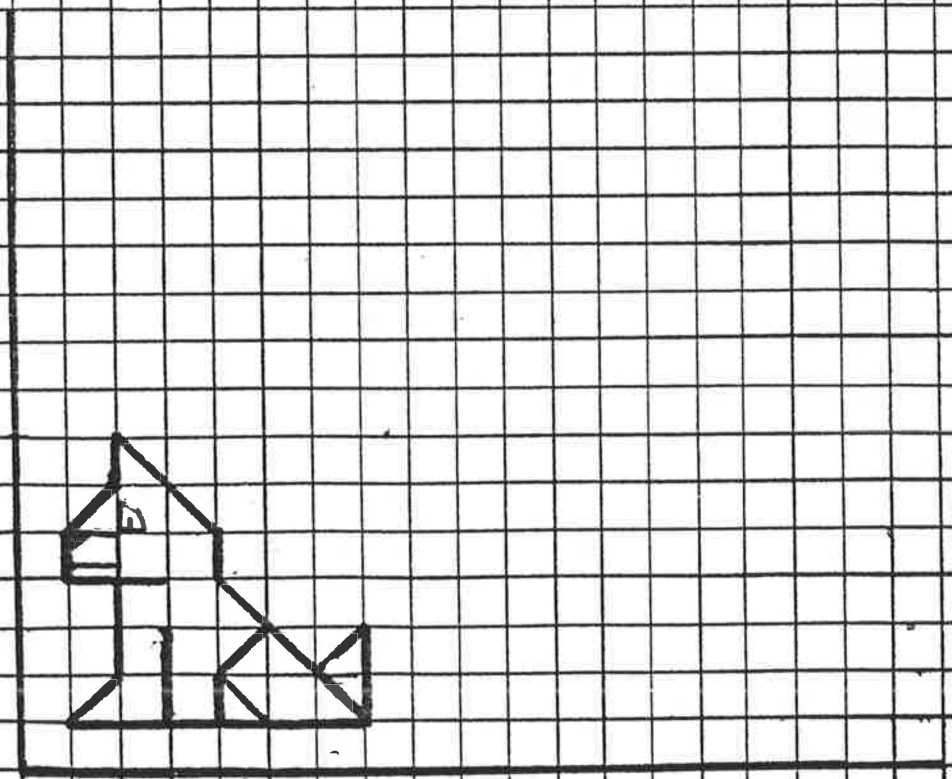


FUNCTIONS as TRANSFORMATIONS
TRANSLATIONS

Translate this shape [10,-1]
10 squares to the right, 1 square down



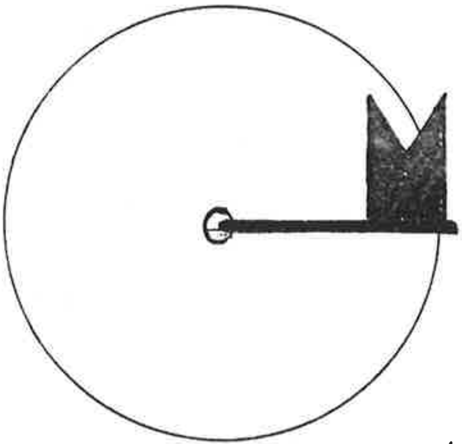
Translate [9,7]



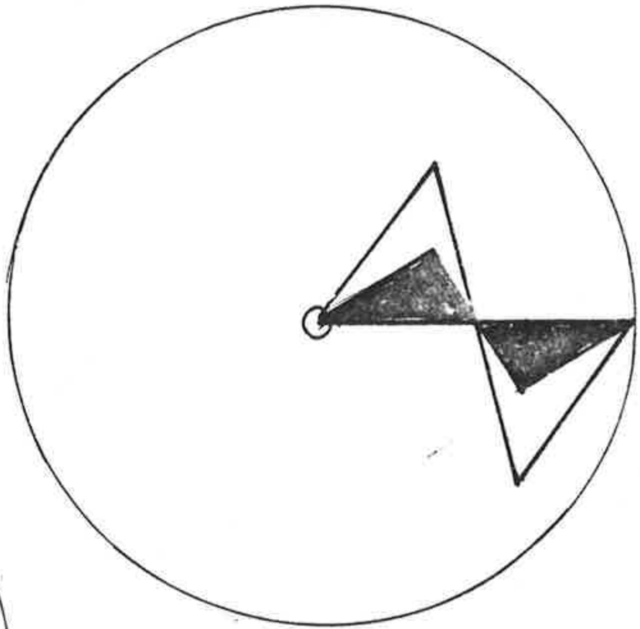
FUNCTIONS as TRANSFORMATIONS
ROTATIONS

ROTATIONS

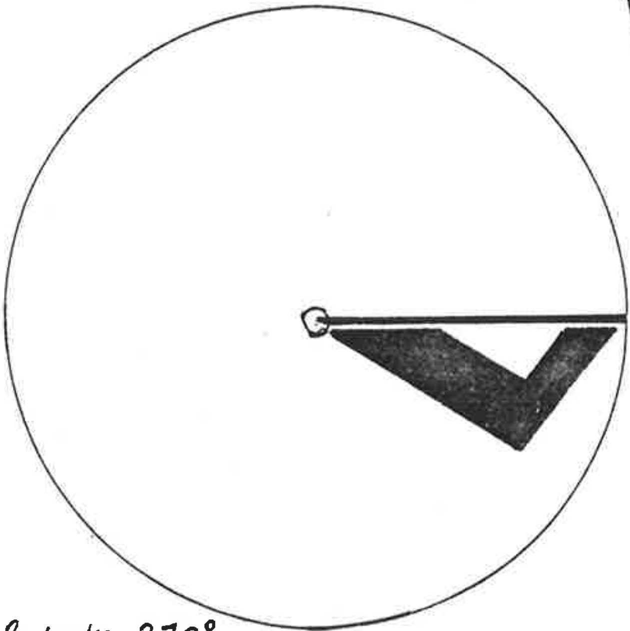
When you rotate anti-clockwise ↺ you rotate positively...



Rotate positively a quarter-turn about O. (This is +90°)



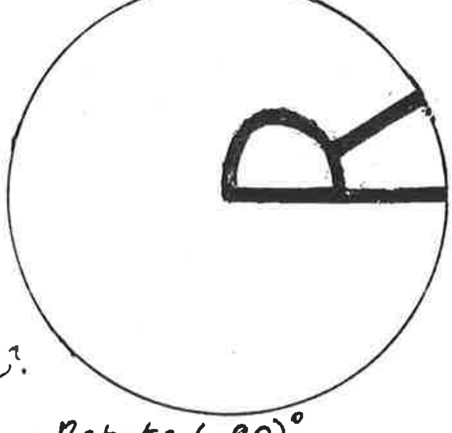
Rotate positively a half-turn about O (+180°)



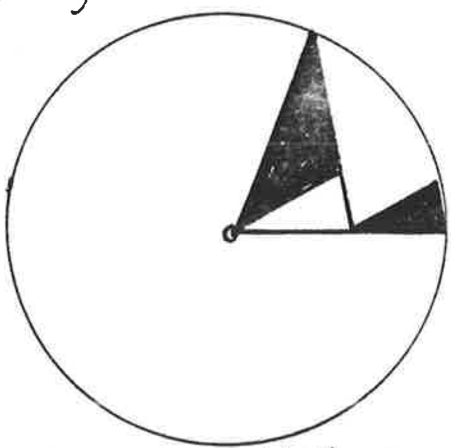
Rotate 270°

When you rotate clockwise ↻ you rotate negatively.

Why not find some tracing paper and a pin?



Rotate (−90)°



Rotate negatively a half-turn about O. (−180°)

Functions act like MACHINES . . . In these activities you will

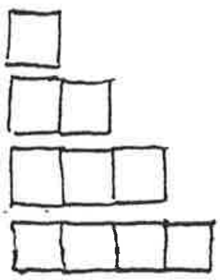
select a value,
OPERATE on it (according to a rule)
and assign it an outcome. . .

- For these activities you will need some square shapes cut from cardboard
- What to do:
  1. Place the square shapes in a sequence, as shown.
  2. Write in the number sequences
     ... for the independent variable
     ... for the dependent variable
  3. Look for a pattern in the number sequences.
  4. When you have worked out the rule, write it in.

Activity 1

(this activity has been completed for you)

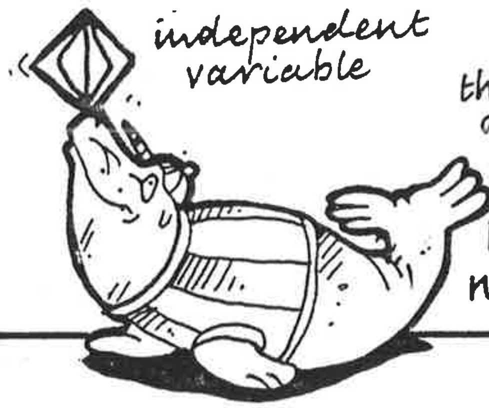
Place the squares edge to edge.



Number of squares	Number of vertical edges
1	2
2	3
3	4
4	5
⋮	⋮
n	$n+1$

independent variable

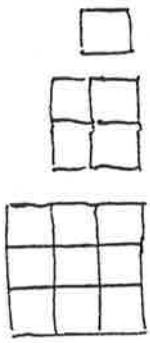
the dependent variable



RULE:
$n \rightarrow n+1$

Activity 2

Place the small squares to form a large square



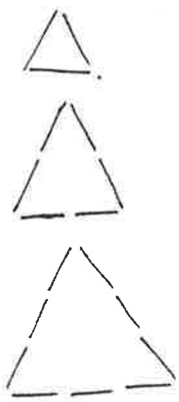
Length of side of large square	Number of small squares
1	1^2 (= 1)
2	2^2 (=)
3	
⋮	
n	

note: the small 2 says 'squared'. Three squared (3^2) means how many small squares there are in a big square with side 3.

Functions as machines.

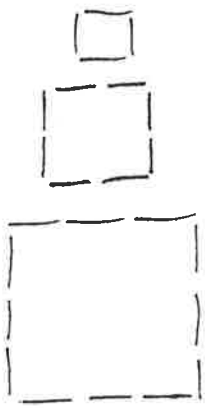
- You will need toothpicks for these activities.
- What to do:
  1. Place the toothpicks in sequence, as shown.
  2. Write in the number sequences
  3. Look for a pattern in the number sequences
  4. Write in the rule

Activity 3



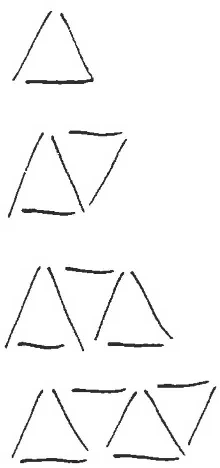
Length of side of triangle	Number of toothpicks

Activity 4



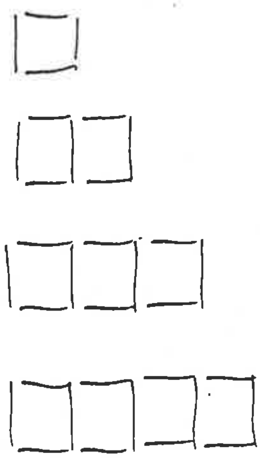
Length of side of square	Number of toothpicks

Activity 5



Number of triangles	Number of toothpicks

Activity 6



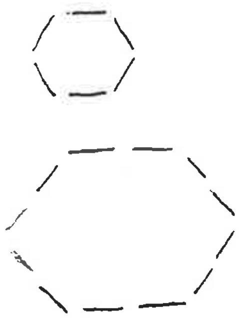
Number of squares	Number of toothpicks

Functions as machines

- More toothpick activities.

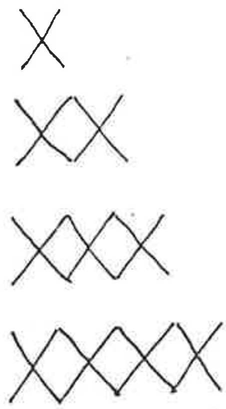
- What to do:
  1. Place the toothpicks in sequence, as shown
  2. Write in the number sequences
  3. Look for a pattern in the number sequences
  4. Write in the rule

Activity 7



Length of side of hexagon	Number of toothpicks

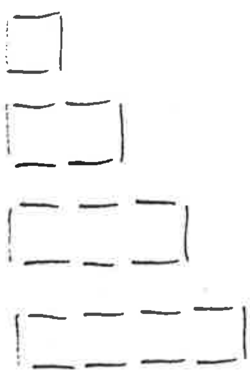
Activity 8



How many?

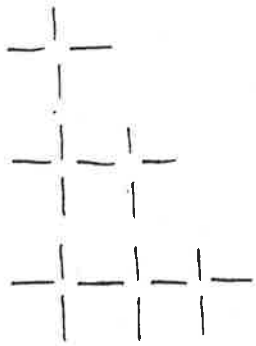
diamonds	crosses	toothpicks

Activity 9



How wide is the rectangle?	How many toothpicks?

Activity 10



How many crosses?	How many toothpicks?

Functions act like machines

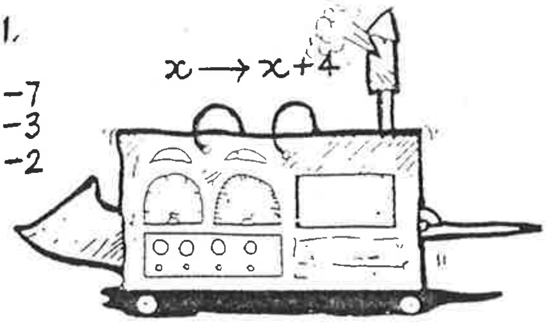
Process these numbers according to the rules given

<u>Idea</u>! Work in groups of three, and check results with each other!

1.

-7
-3
-2

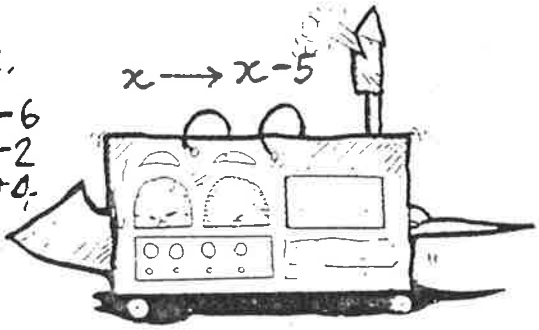
$x \longrightarrow x+4$



2.

-6
-2
$+0$

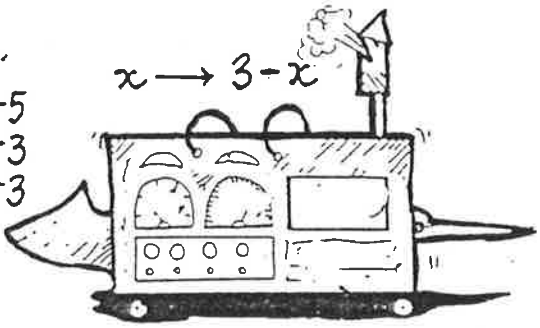
$x \longrightarrow x-5$



3.

-5
-3
$+3$

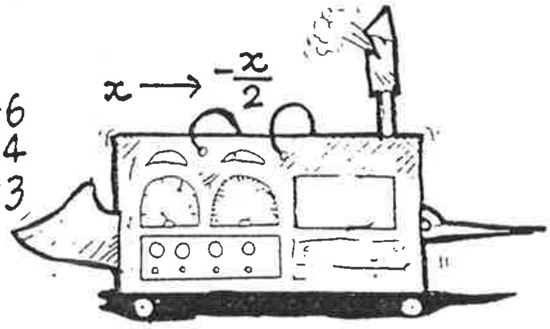
$x \longrightarrow 3-x$



4.

-6
-4
$+3$

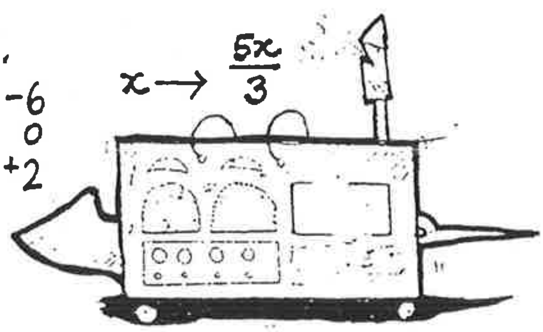
$x \longrightarrow -\dfrac{x}{2}$



5.

-6
0
$+2$

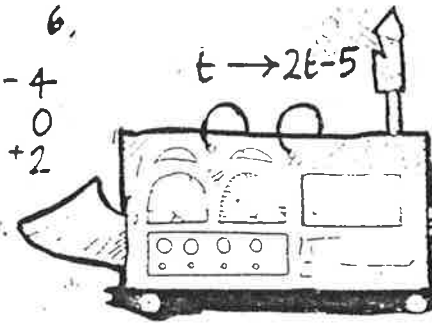
$x \longrightarrow \dfrac{5x}{3}$



6.

-4
0
$+2$

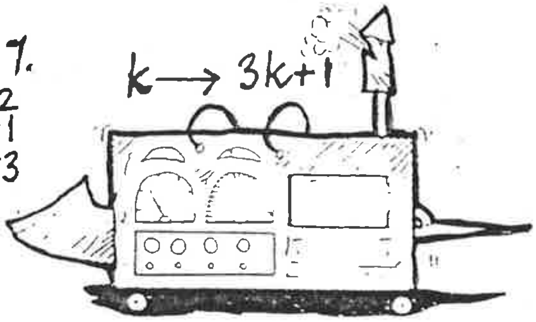
$t \longrightarrow 2t-5$



7.

-2
-1
$+3$

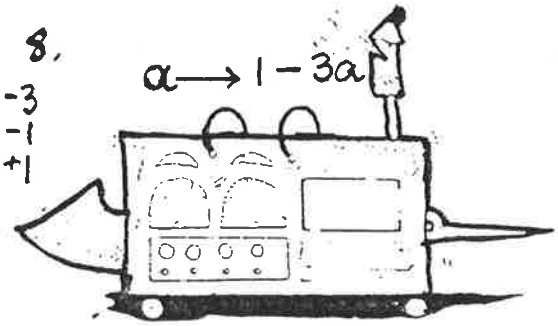
$k \longrightarrow 3k+1$



8.

-3
-1
$+1$

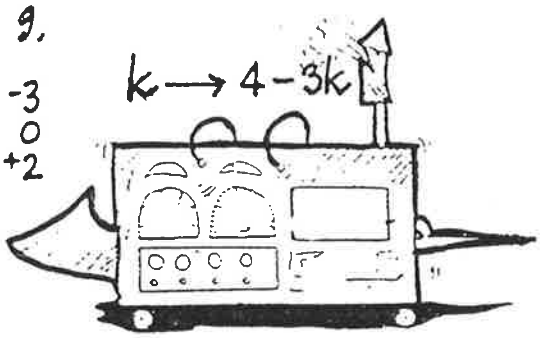
$a \longrightarrow 1-3a$



9.

-3
0
$+2$

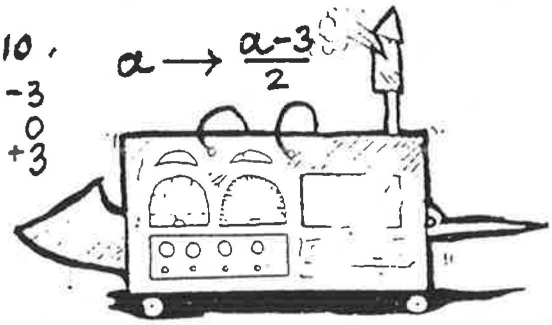
$k \longrightarrow 4-3k$



10.

-3
0
$+3$

$a \longrightarrow \dfrac{a-3}{2}$



Check up on P 128

Functions can be likened to a PROCESS that brings about change.

COMPLETE each table according to the rule given

1. $x \to 5x$

7	11	6	9	5	0
35	55				

2. $x \to \frac{1}{4}x$

8	32	40	24	6	9
		10			

3. $x \to x + 8$

7	11	6	9	5	0

4. $x \to x - 6$

15	25	35	12	32	42

5. $x \to \frac{x}{4}$

8	32	40	16	6	9

6. $x \to \frac{1}{2}x$

10	12	6	5	15

7. $x \to x \div \frac{1}{2}$

10	12	6	5	15

8. $x \to x + 3{\cdot}7$

4	6	6·3	16·3	2·4

9. $x \to x - 0{\cdot}5$

7	5	5·5	5·3	5·1

10. $x \to 3x + 4$ [X3, then +4]

7	2	10	0	5
25				

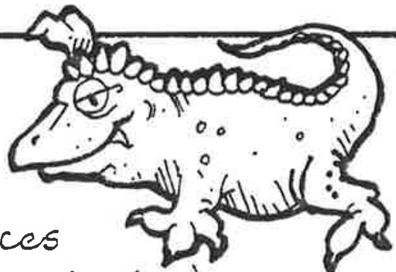
11. $x \to 5x - 2$ [X5, then −2]

7	11	6	9	5

PLEASE discuss your findings with others!

see P128

Functions as a PROCESS



1. Complete the sequences of numbers.
2. Look for a pattern in the number sequences
3. When you have worked out the rule, write it in.

1.

a ↓	1	2	3	
$a+\boxed{}$	5	6	7	

6.

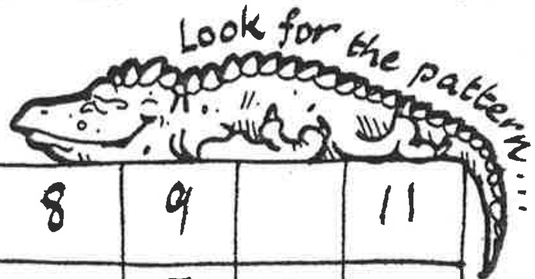
p ↓	1	2	3	4
	3	6	9	

2.

n ↓	5	6	7	
	8	9	10	

7.

p ↓	1	2	3	4
	2	5	8	



3.

k ↓	8	9		11
	6	7	8	

8.

t ↓	1	2	3	4
	5	10	15	

4.

x ↓	1	2	3	4
	4	8	12	

9.

t ↓	1	2	3	4
	2	7	12	

5.

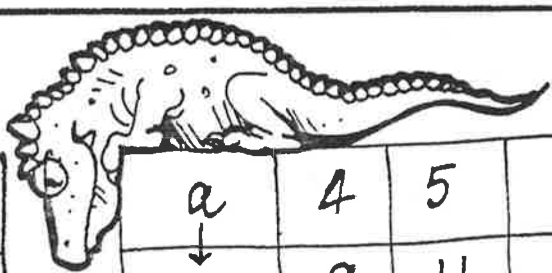
x ↓	1	2	3	4
	5	9	13	

10.

y ↓	1	2	3	4
	1	3	5	

See P128

Functions as a PROCESS



1.

n	1	2	3	4
↓ □+4	7	10	13	

6.

a	4	5	6	7
↓	9	11		15

Complete each table and find the rule.

2.

p	5	6	7	8
↓	10	12	14	

7.

x	5	6	7	8
↓	17		23	26

3.

k	1	2	3	4
↓ 2k-□	1	3	5	

8.

y	1	2	3	4
↓	−3	−1		3

4.

t	3	4	5	6
↓ 3t-□	8	11		17

9.

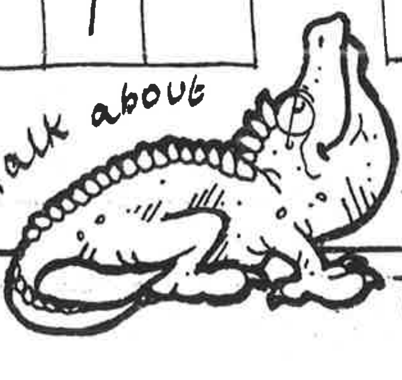
d	1	2	3	4
↓	1		9	

5.

m	1	2	3	4
↓	−1	0	1	

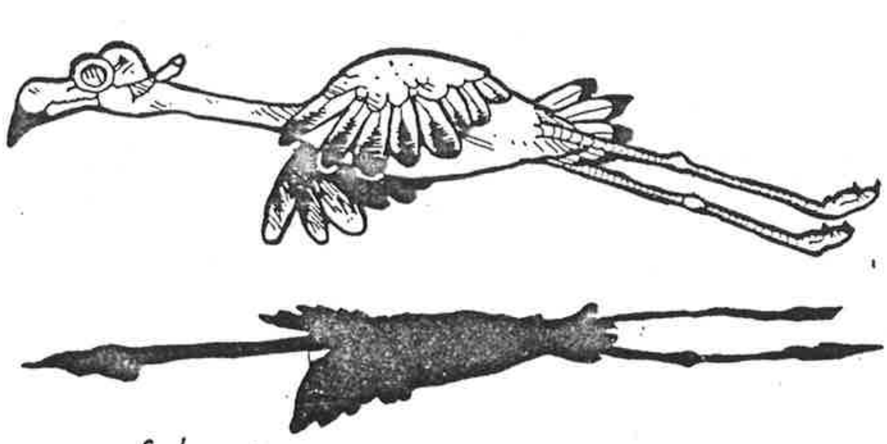
10.

u	−3	−2	−1	
↓		−5	−2	1

Talk about  how you saw the pattern.

See P128

Functions are MAPPINGS

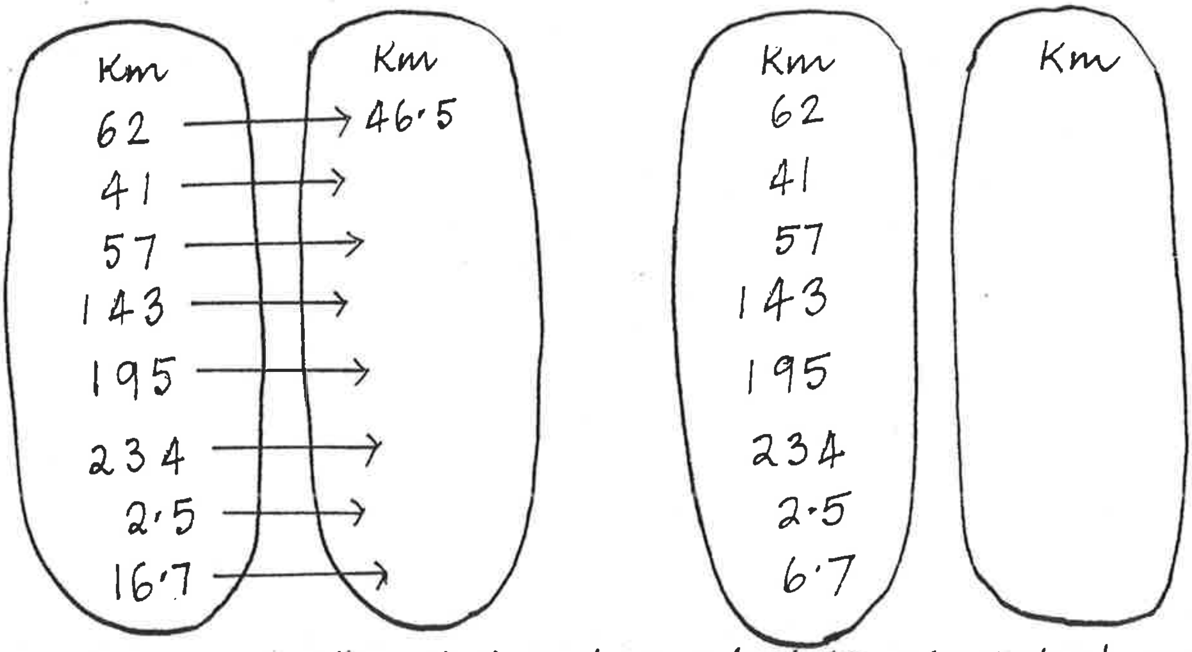


Mappings
give us a good picture of how
one element (or member of a set)
is assigned → another element (or member of a set)

Look at the maps of Yorke Peninsula.
The **reduction** is a mapping that shows a particular length of coastline **corresponding to** an <u>image</u> that is $\frac{3}{4}$ of the <u>original</u> length.
The **enlargement** is a mapping that shows a particular length of coastline **corresponding** to an image that is $\frac{5}{4}$ the original length (that is, $1\frac{1}{4}$ times as long).

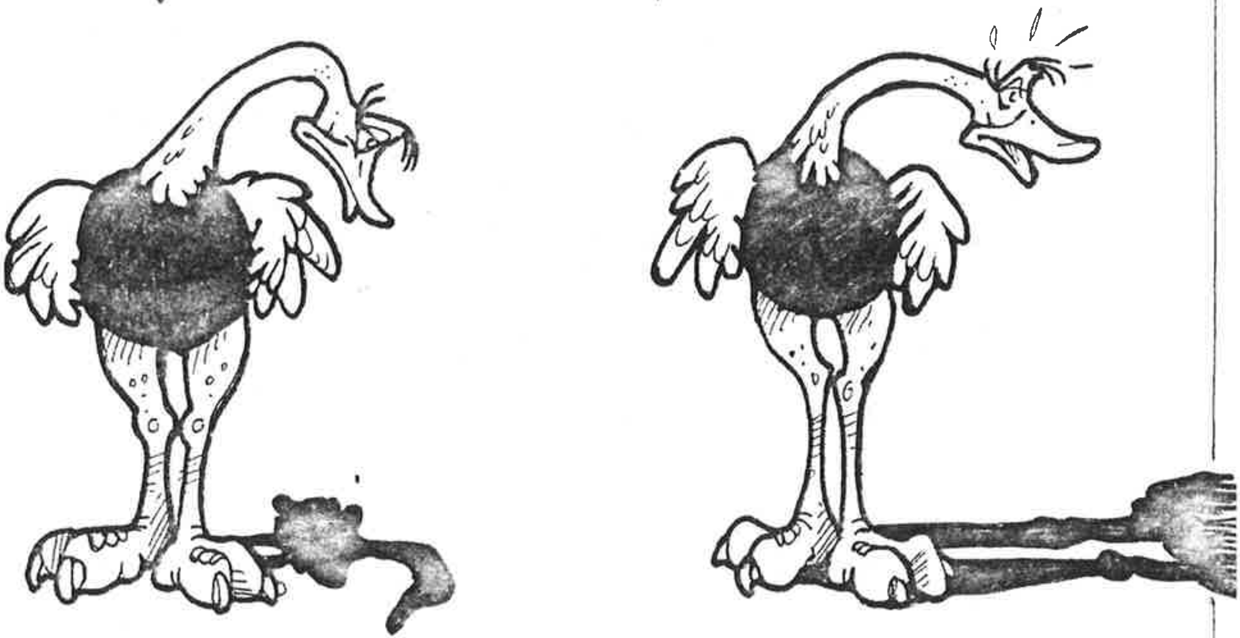
In these **Set Graphs**, calculate the image lengths that will correspond to the original lengths. (Use a calculator if you wish)

Reduction | **Enlargement**



Why not share these calculations (and check answers) with others in a small group?

Form small groups please ...



Functions are MAPPINGS ...

Mappings give us a good picture of how one element (or member of a set) is assigned another element (or member of a set).

** THE MAPPING MACHINE **

... you need a calculator for this game ...

How many 4s in $\boxed{1317}$?

Guess	Result
300	1200
325	1300
330	1320
329·5	1318
329·25	1317 Spot on!

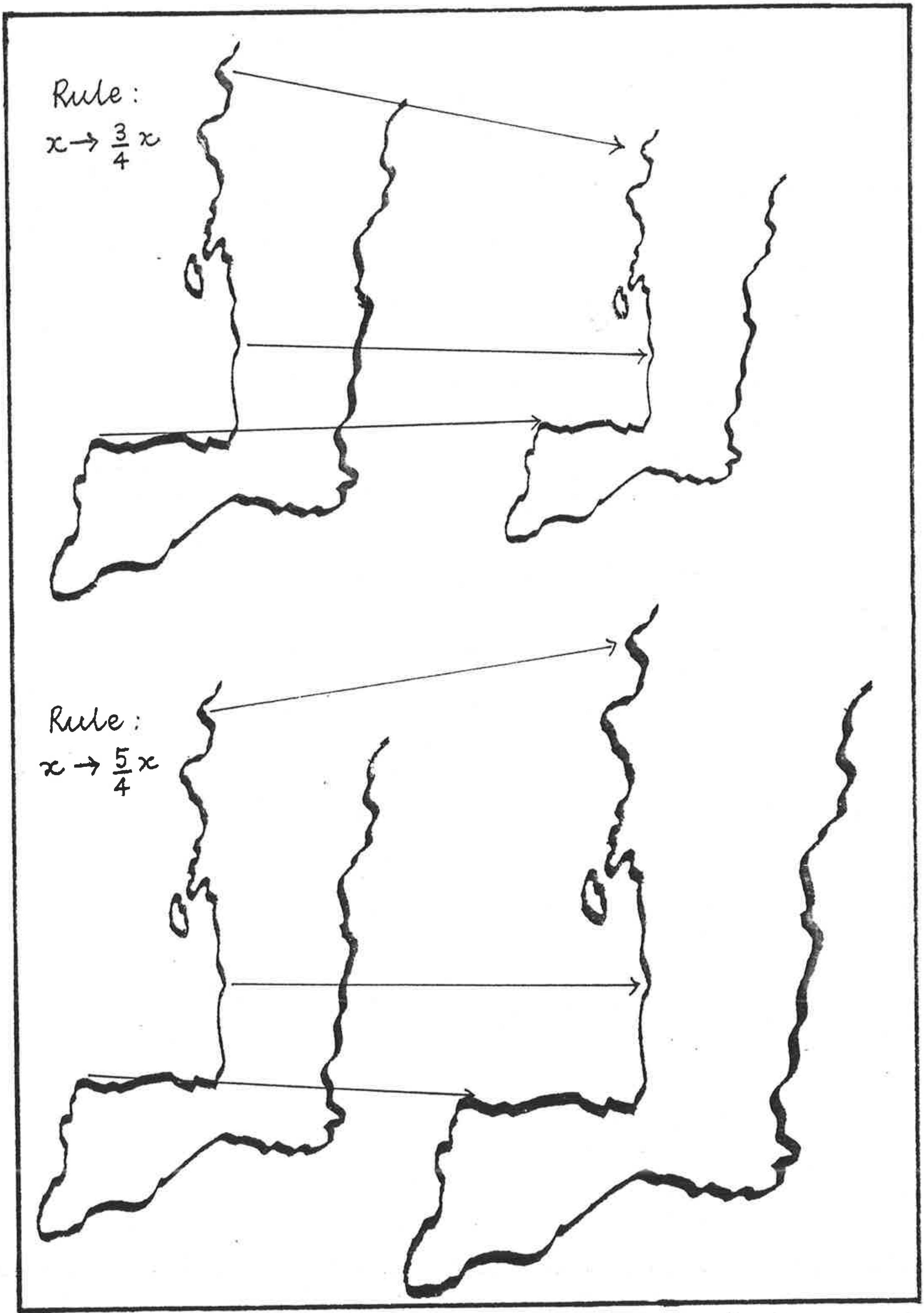
Rule: $x \rightarrow 4x$

How many 8s in $\boxed{7791}$?

Your turn now

Guess	Result
1000	8000

Rule: $x \rightarrow \boxed{}$



7.9 (1987 edition)

THE MAPPING MACHINE

How many 15s in $\boxed{1455}$?

Guess	Result

How many 16s in $\boxed{2388}$?

Guess	Result

IDEA! Play THE MAPPING MACHINE a few more times with numbers you have thought up.

Challenge a friend to reach 'Spot on' in as few guesses as possible, provided

(1) a guess is given within 2 seconds

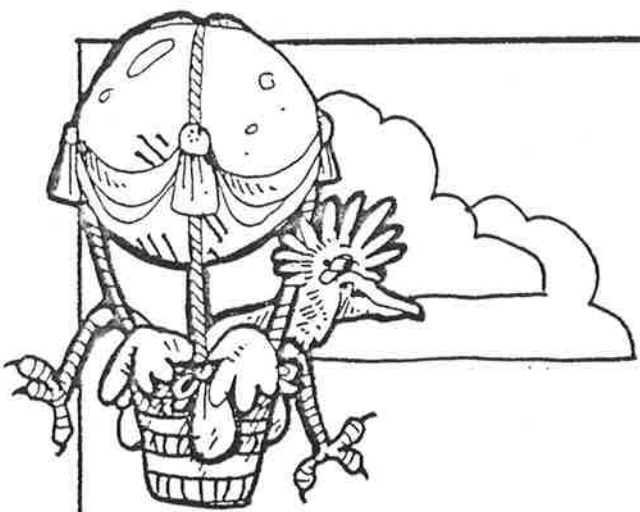
(2) calculators are not used for guesses

YORKE PENINSULA

Scale: 1 cm. represents 12 Km.

Complete this chart:

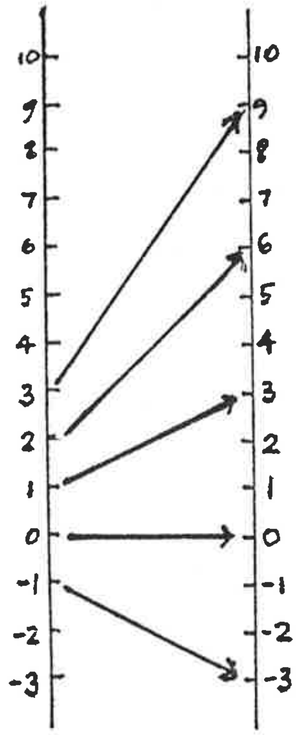
Journey taken	Distance on map (in cms)	Actual distance (in Km)
Maitland to Yorketown		
Wardang Is to Pt. Victoria		
Yorketown to Corny Point		
Yorketown to Corny Point via Stenhouse Bay		
Maitland to Moonta		



Functions are Mappings...

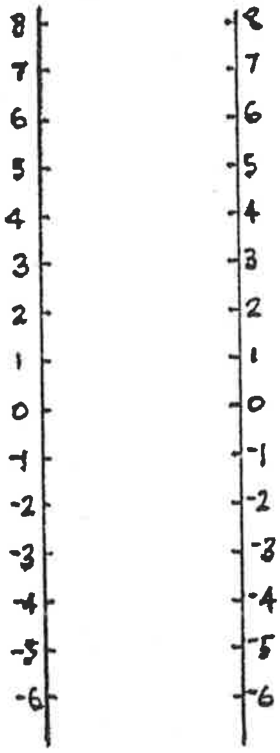
Mappings are sometimes represented on two parallel number lines

The mapping $x \to 3x$ might look like this:

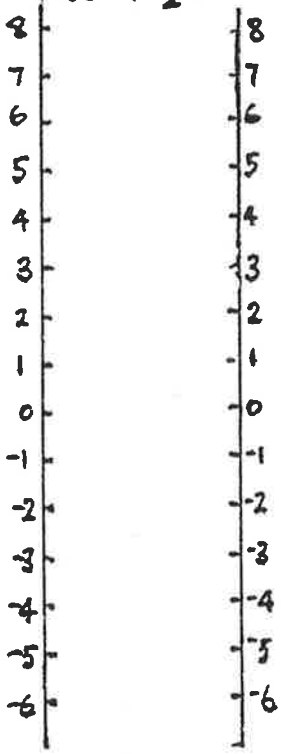


On these number lines please represent these mappings:

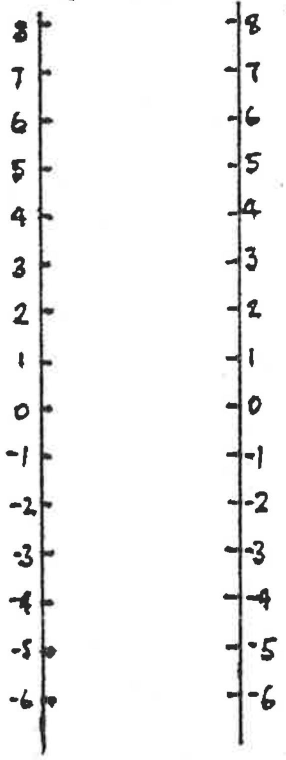
$x \to 2x$

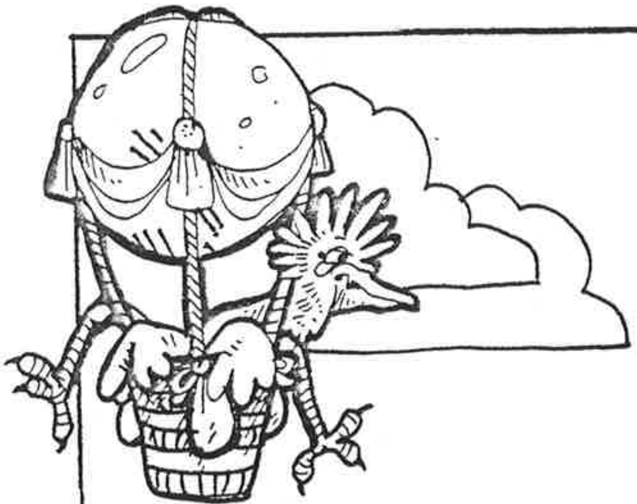


$x \to \frac{1}{2}x$



$x \to x+2$





Investigating Mappings

... in small groups

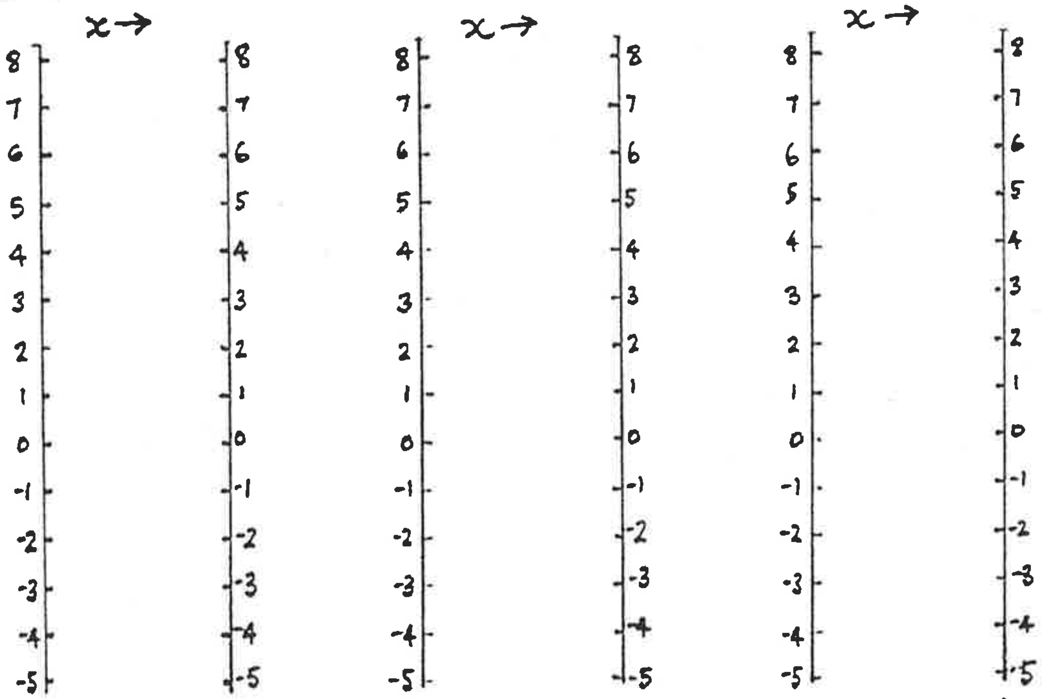
Share these mappings amongst yourselves:

$x \rightarrow 1\frac{1}{2}x$ $x \rightarrow x+1$ $x \rightarrow \frac{6}{x}$

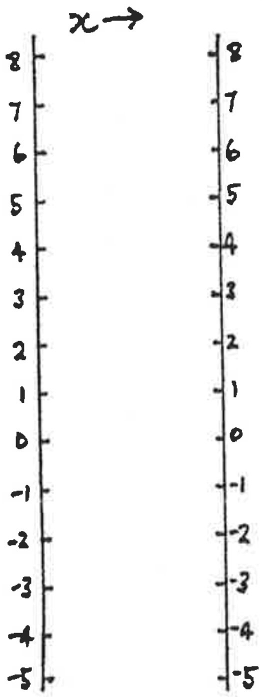
$x \rightarrow \frac{1}{4}x$ $x \rightarrow x-1$ $x \rightarrow \frac{1}{x}$

$x \rightarrow -x$ $x \rightarrow x-2$ $x \rightarrow 1-x$

$x \rightarrow -2x$



Look at the mappings drawn by people in your group. Discuss answers to these questions in a class group



1. What is the connection between $x \rightarrow 2x$ and $x \rightarrow \frac{1}{2}x$?
2. What is the connection between $x \rightarrow x+1$ and $x \rightarrow x-1$ (or between $x \rightarrow x+2$ and $x \rightarrow x-2$)?
3. What is the difference between $x \rightarrow x-1$ and $x \rightarrow 1-x$?
4. What is the difference between $x \rightarrow 2x$ and $x \rightarrow x+2$?
5. How can you tell whether a mapping is $x \rightarrow 2x$ or $x \rightarrow 3x$ or $x \rightarrow 4x$?
6. Which arrows are:
   a. straight across
   b. parallel
   c. not parallel ?

GRAPHS

A graph is a very good means for displaying numerical data. A graph has more visual impact than tables, charts and lists of numbers.

A graph also provides a clear and powerful way of showing how numbers are related. A function can be represented by means of a graph. To do this you need to construct a number plane by drawing horizontal and vertical reference lines. These reference lines (axes) are numbered so that points can be located on a grid. The position of each point is located by using ordered pairs. Ordered pairs that belong to a particular function will locate points that display the function as a graph.

The development of Analytic Geometry

Analytic geometry fuses together

	-	the representation of geometry
with	-	the calculations of algebra
and	-	the thinking of functions.

It provides an important means towards our understanding of the world.

The idea of putting a coordinate grid on a geometric surface in order to measure and locate is a very old one. In the second century A.D., Claudius Ptolemy constructed a map of the world, representing points on a globe by pairs of angles that measured the number of degrees north and east of a fixed reference point. Late in the fifteenth century mathematical map making developed into a very sophisticated technique, to meet the demands of seafarers navigating in the open ocean for long periods of time.

In the seventeenth century, French mathematicians Rene Descartes and Pierre Fermat combined the notation and problem solving capacity of algebra with the constructions of geometry. This was done through developing a system of coordinates, named cartesian coordinates after Descartes.

The notation and presentation used nowadays in analytic geometry is largely the work of Leonhard Euler (1707-1783). The value of analytical geometry in predicting and problem solving dates back to that time.

Talking Maths in a small group

Match	Score
1	5
2	4
3	7
4	12
5	2
6	15
7	9

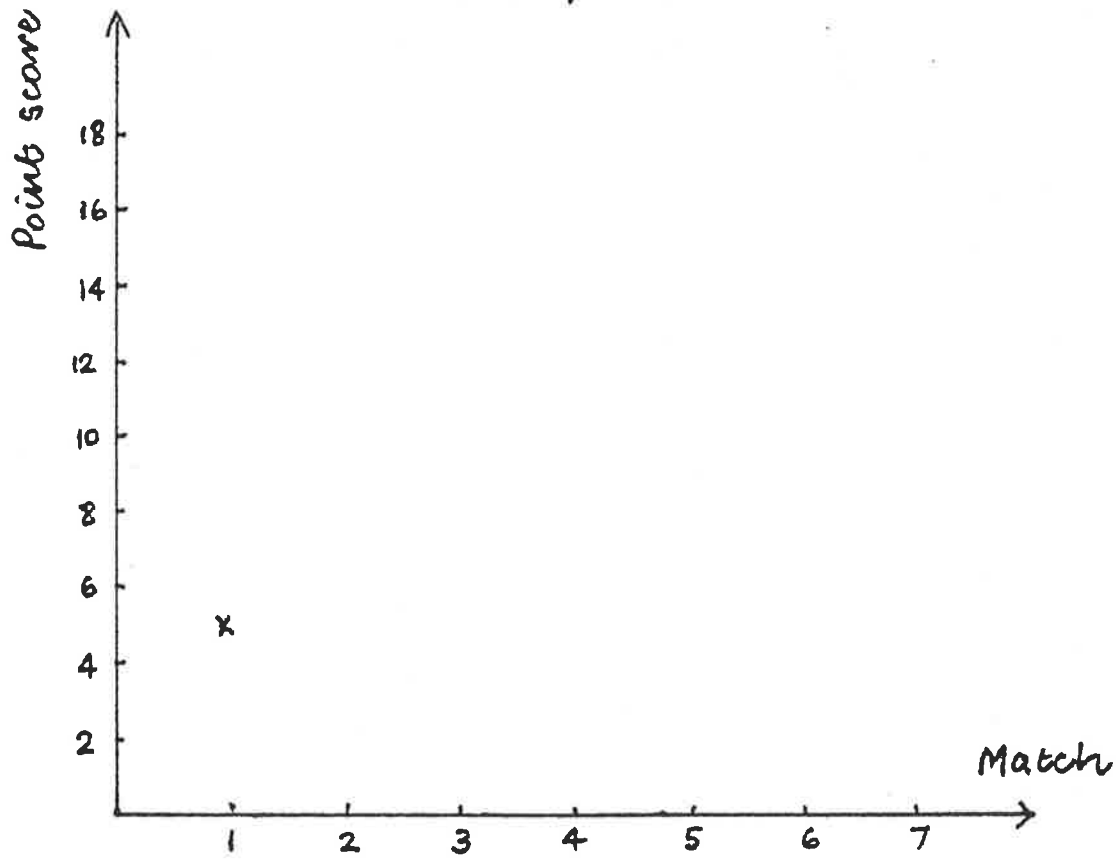
This table shows your point-scoring progress in the first round of a basketball season

This information can expressed as ordered pairs:

(1,5) (4,12) (7,9)
(2,4) (5,2)
(3,7) (6,15)

Please plot them on the number plane below.

- Scatter-graph showing -
Point scoring progress

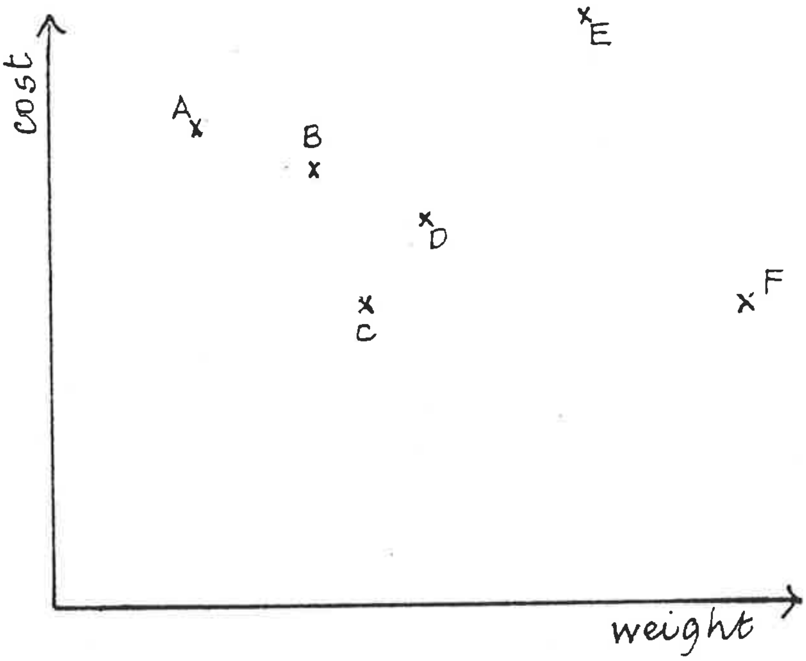


Talk

- Ask questions about your progressive performance.
- Are you able to predict your score for the 8th match?

Shopping for Cereals

The points on this scattergraph represent approximately the link between weight and cost of each cereal.



(a) Match a packet of cereal to each point.

(b) Which cereal provides the best value for money? How can you tell?

(c) Which is better value: Muesli Munch or Sugar Frosties? How can you tell?

(d) There are 3 cereals that give equal value for money. What are they? How can you tell?

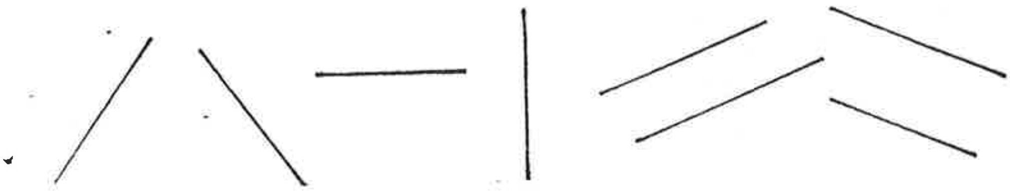
— Talk these over with other people —

Investigating Linear Graphs . . .

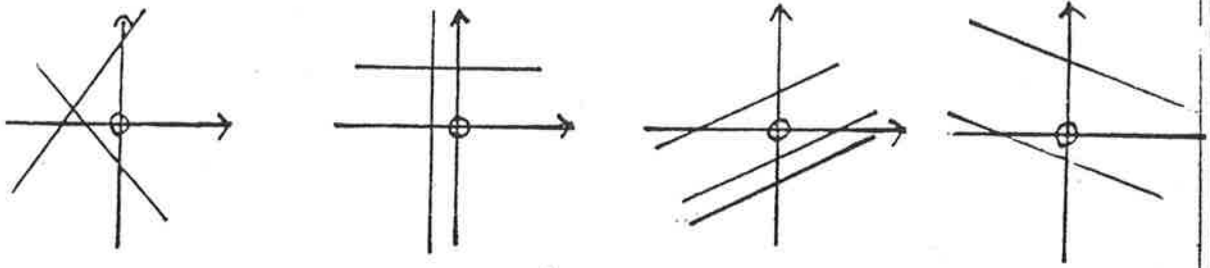
Linear Graphs are easy to recognise because they're all straight lines!

The way to tell one from the other is by looking at . . .

its **Gradient**



and its **Position** with reference to the coordinate axes.



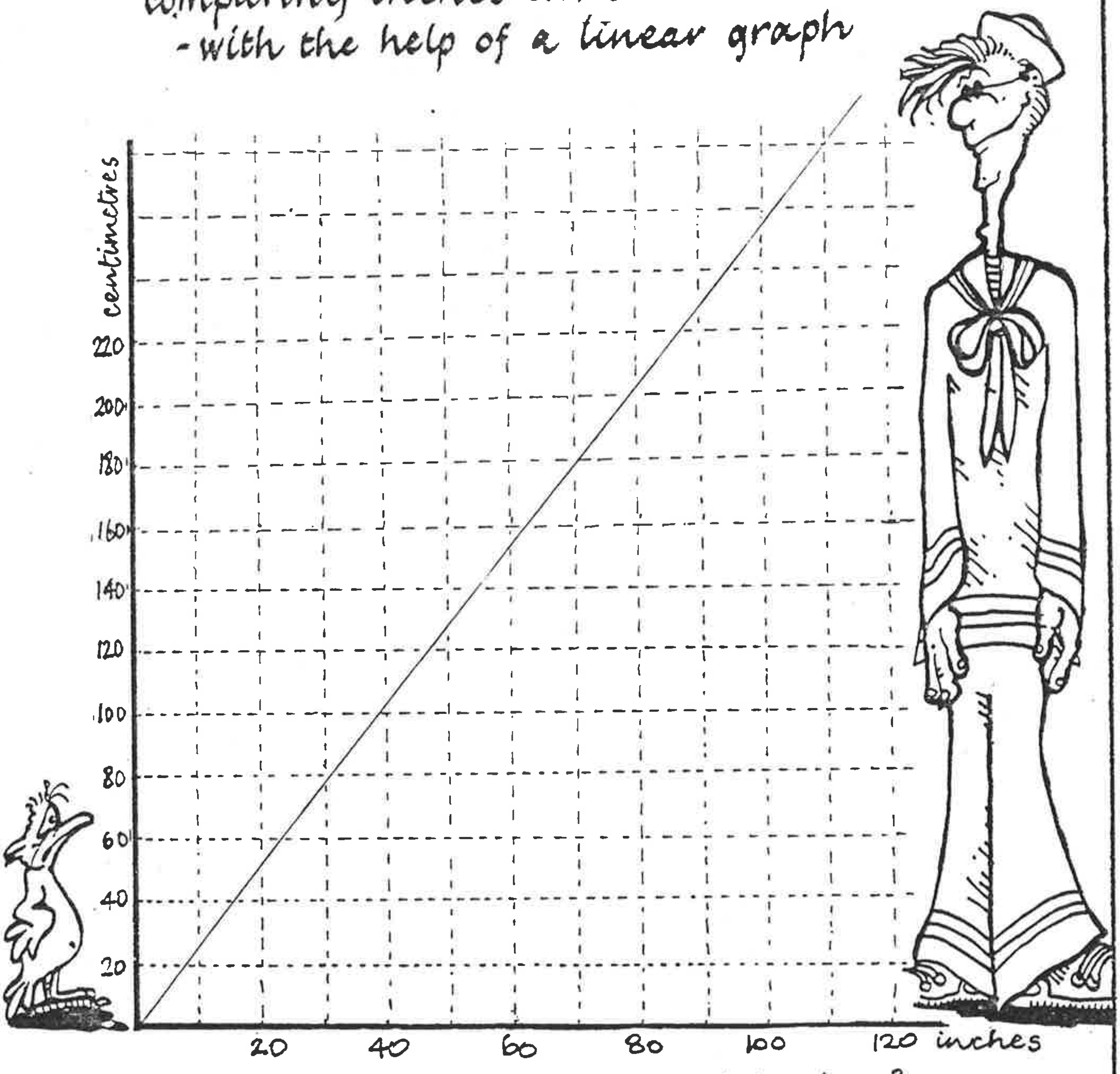
On a cartesian plane the position of each line can be located with reference to the X- and Y-axes. In this book we will be looking at the usefulness of the Y-axis as a reference line.

. . . . we begin our investigation of straight line graphs with a look at GRADIENTS, – and where we might find them and use them . . . .

Small groups please . . .

CONVERSION GRAPH

comparing inches and centimetres - with the help of a linear graph

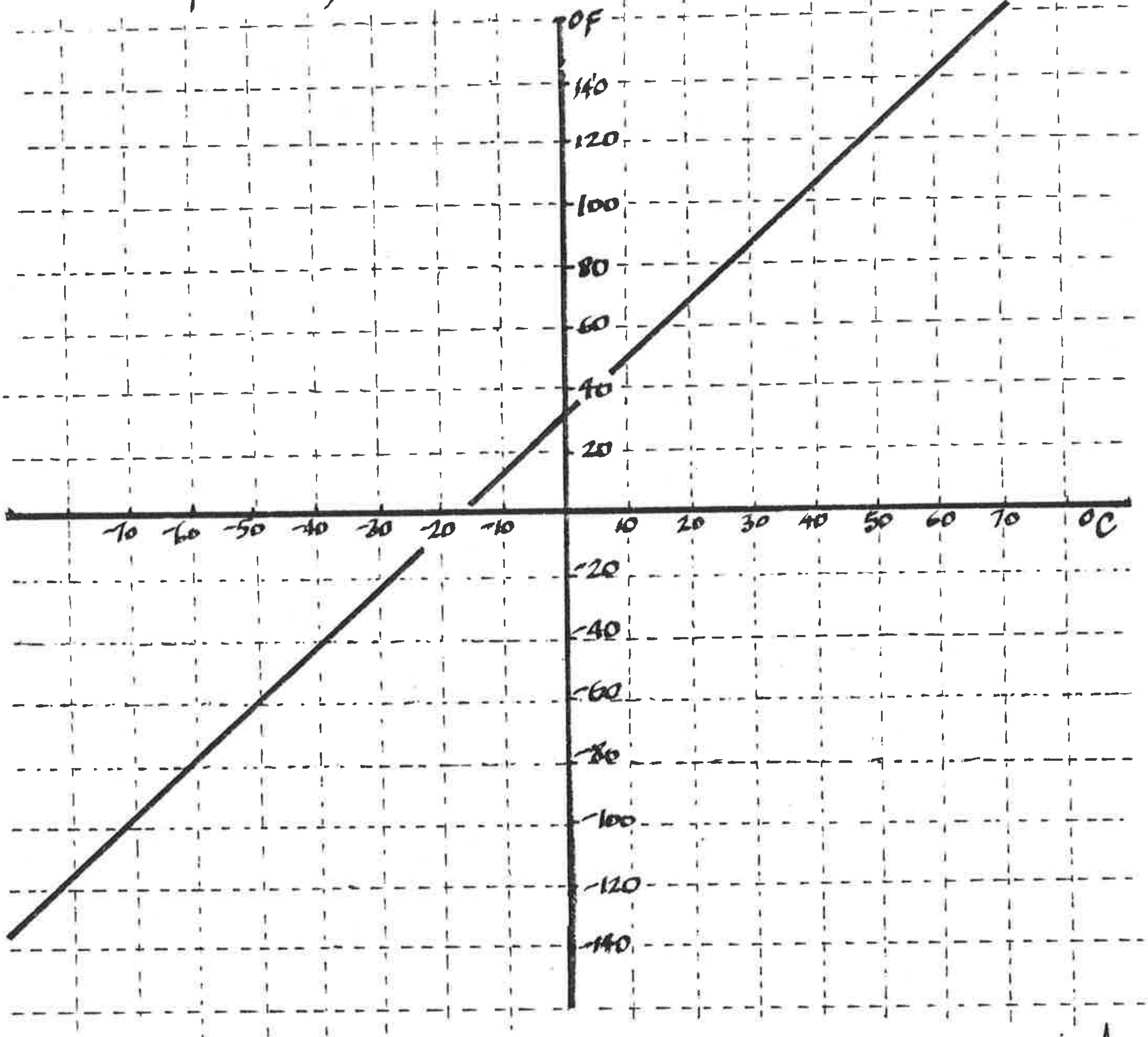


1. What is your height in feet and inches? Convert this height to centimetres.
2. What is the height in centimetres of a person (a) 5 feet tall (b) 6 feet tall
3. In professional basketball the player at centre usually ranges in height from $6\frac{1}{2}$ – $7\frac{1}{2}$ feet. What is this range in centimetres?
4. In 1905 an Irishman called Giant Machnow appeared at the London Hippodrome as an entertainment attraction. He was (so they say) 280 cms tall. What was his height in feet?

Small groups please...

CONVERSION GRAPH

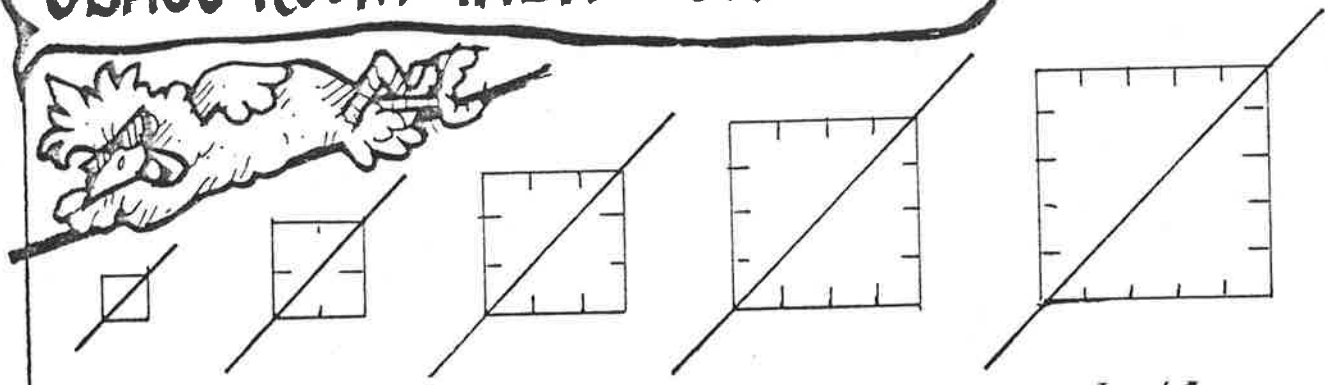
...comparing °Celsius and °Fahrenheit



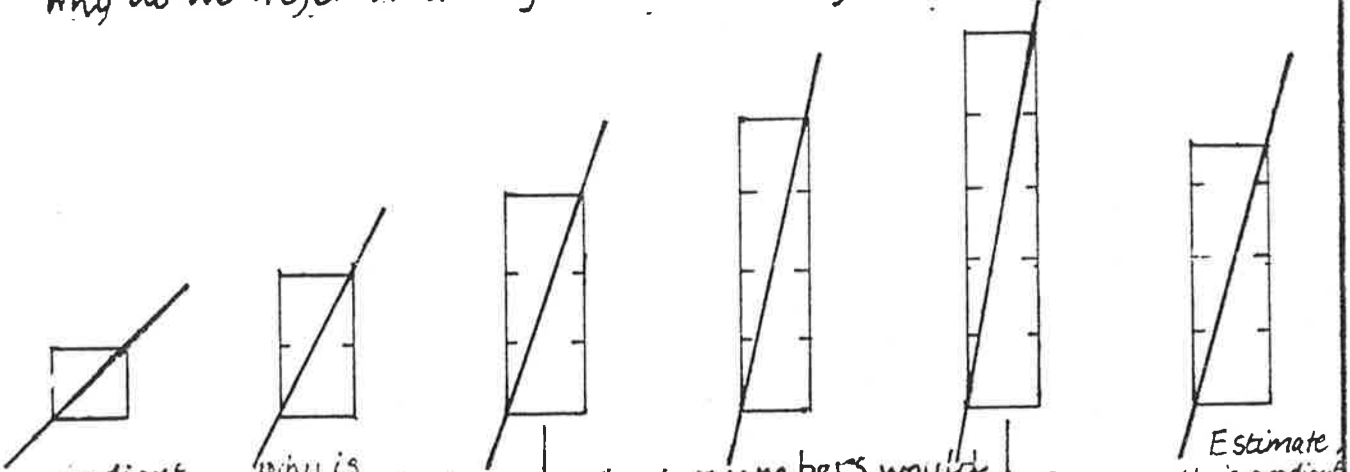
Read off from the conversion graph an approximate temperature for each of these.

1. The hottest temperature ever recorded on earth (at Al'Azizyah, Libya) is 58°C. This is ☐ °F.
2. The coldest temperature ever recorded on earth (at Vostok, Antarctica) is ⁻88°C. This is ☐ °F.
3. Australia's hottest recorded temperature (at Cloncurry, Qld) is 128°F. This is ☐ °C.
4. The coldest temperature recorded outside of Antarctica (at Oymyakon, Siberia) is ⁻90°F. This is ☐ °C
5. Note that the graph does not go through (0,0). This is because 0°C = ☐ °F.

CLASS-ROOM TALK : GRADIENTS



Do you notice that these lines have the same gradient?
Why do we refer to this gradient as a gradient of 1?

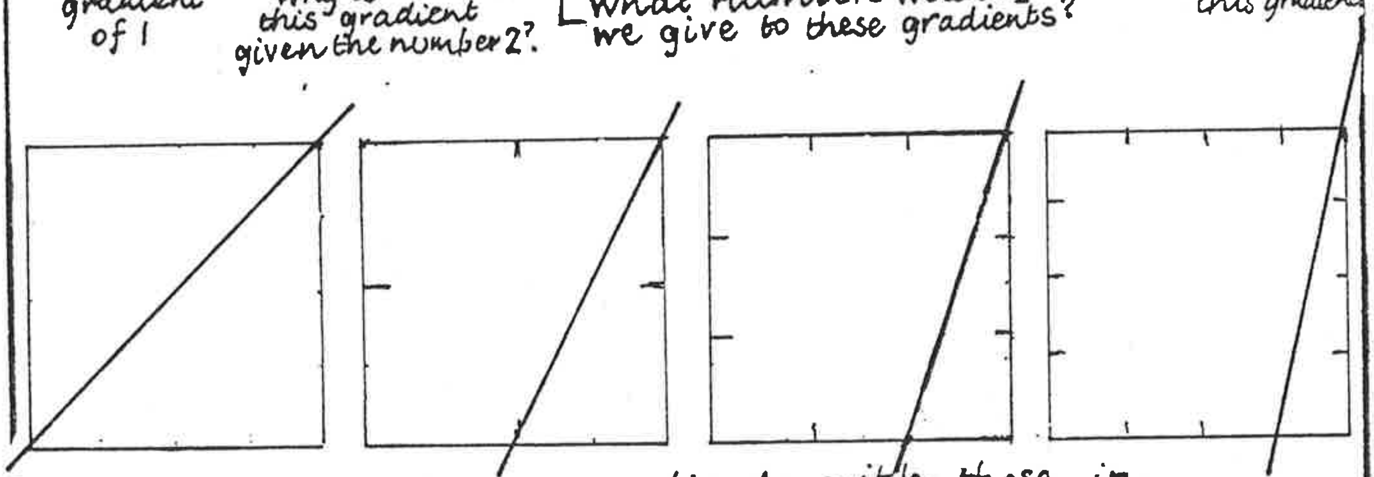


gradient of 1

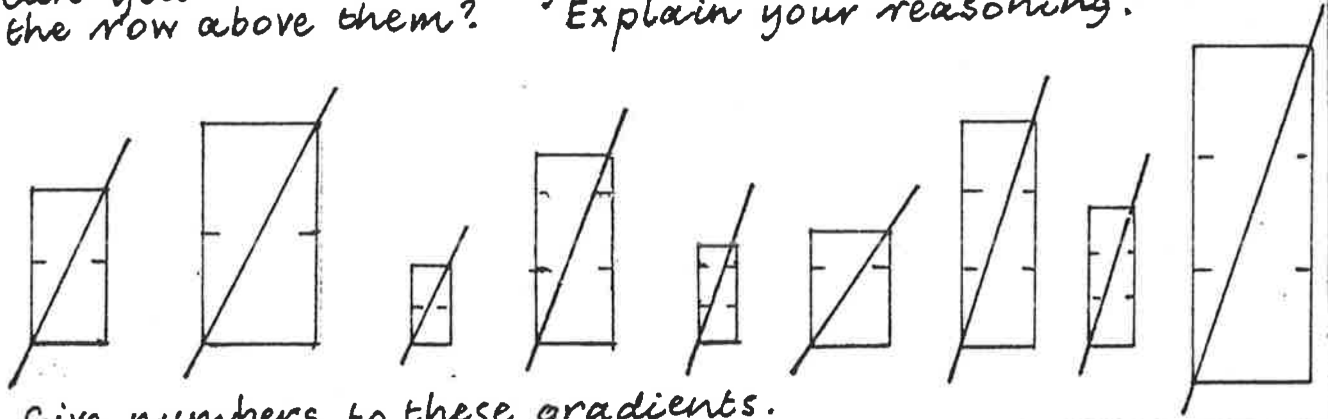
Why is this gradient given the number 2?

What numbers would we give to these gradients?

Estimate this gradient



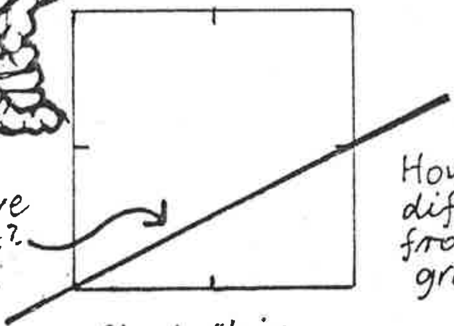
Can you match these gradients with those in the row above them? Explain your reasoning.



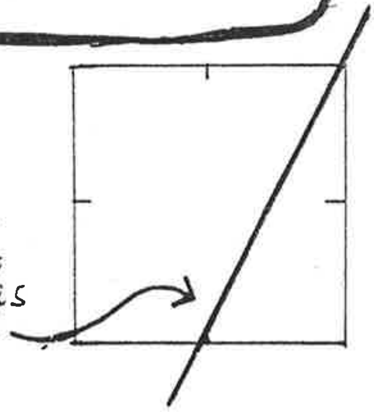
Give numbers to these gradients.

CLASS-ROOM TALK : GRADIENTS

What number would you give to this gradient?



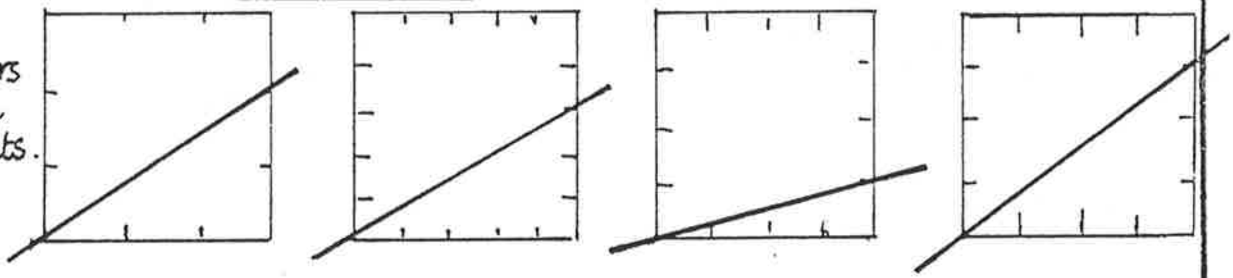
How is it different from this gradient?



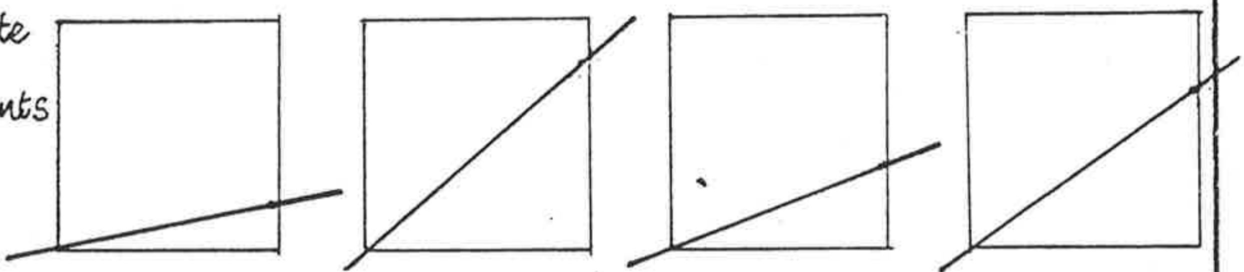
Where would you find this gradient in real life?
In naturally occurring things?
In things made by people?

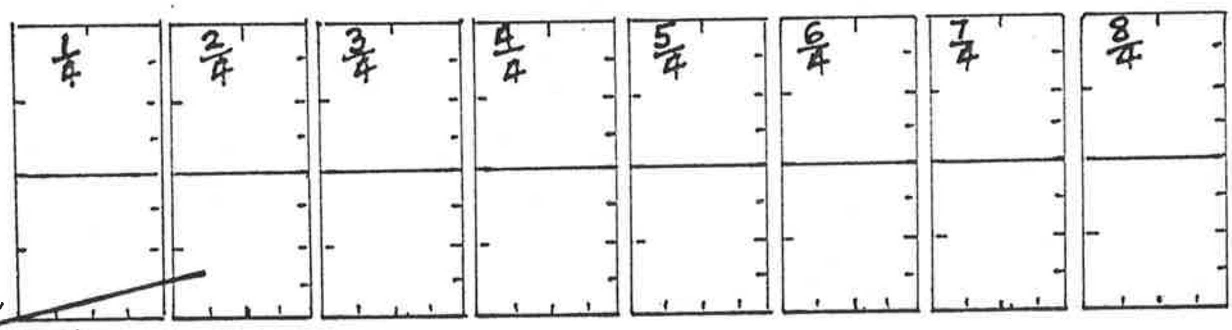
Where would you be likely to find this gradient?

Give numbers to these gradients.



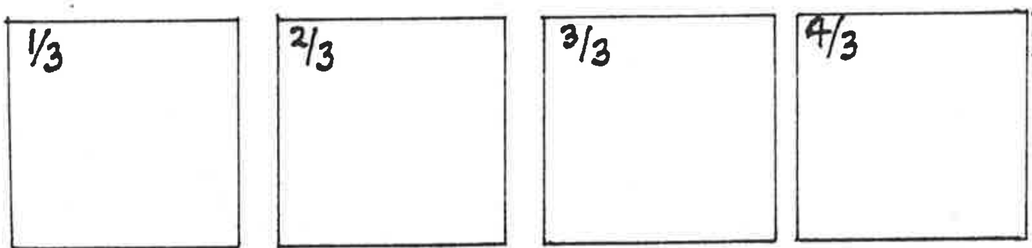
Estimate these gradients



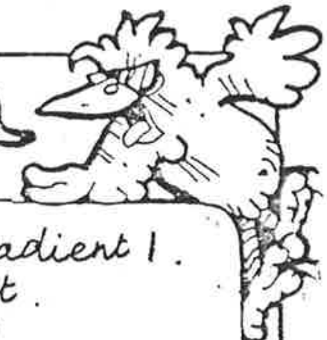


Draw these gradients

Sketch these gradients



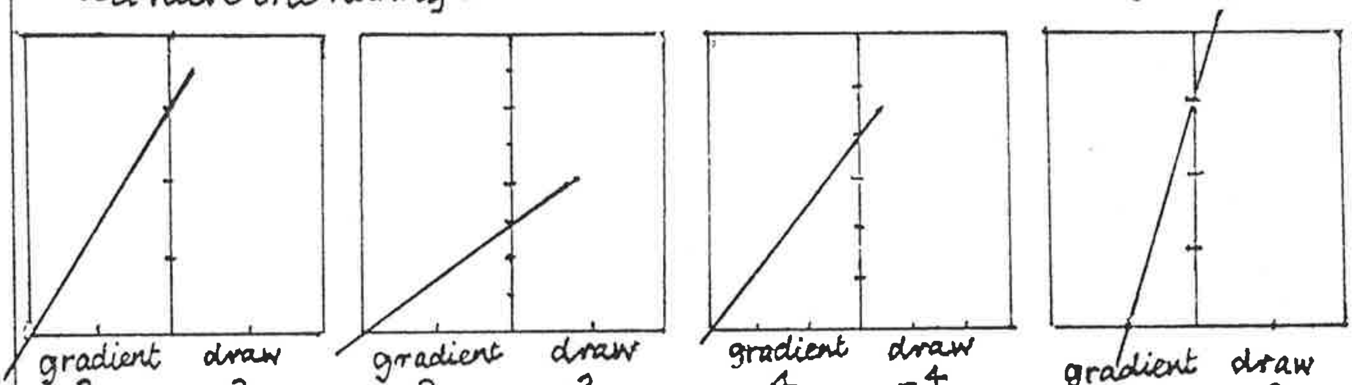
CLASS-ROOM TALK: GRADIENTS



ACTIVITY | YTIVITJA

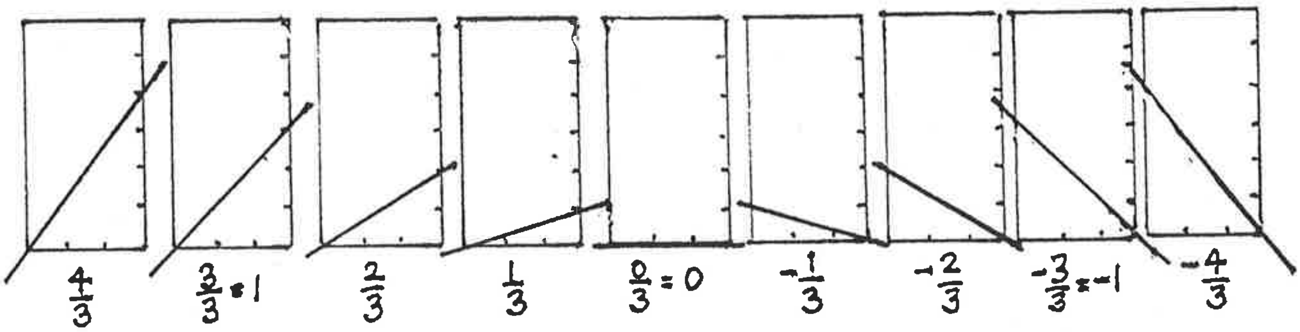
On a spare sheet of paper rule a line of gradient 1.
Turn the sheet over, and hold it to the light.
You see a REFLECTION of the line you ruled.
What number would you give to the gradient of this reflected line?
Rule another line: this time with gradient 2.
What is the gradient of the reflection of this line?
Rule a line with gradient −2. What is the gradient of its reflection?

(or use a mirror if you have one handy)



gradient $\frac{3}{2}$	draw $-\frac{3}{2}$	gradient $\frac{3}{4}$	draw $-\frac{3}{4}$	gradient $\frac{4}{3}$	draw $-\frac{4}{3}$	gradient 3	draw −3

Discussion:
Tell someone else, as clearly as you can, why gradients that are directed ↗ are given positive numbers, while gradients directed ↖ are given negative numbers.
(Note: your thinking on directed numbers needs to be up-to-date!)



$\frac{4}{3}$ $\quad \frac{3}{3}=1$ $\quad \frac{2}{3}$ $\quad \frac{1}{3}$ $\quad \frac{0}{3}=0$ $\quad -\frac{1}{3}$ $\quad -\frac{2}{3}$ $\quad -\frac{3}{3}=-1$ $\quad -\frac{4}{3}$

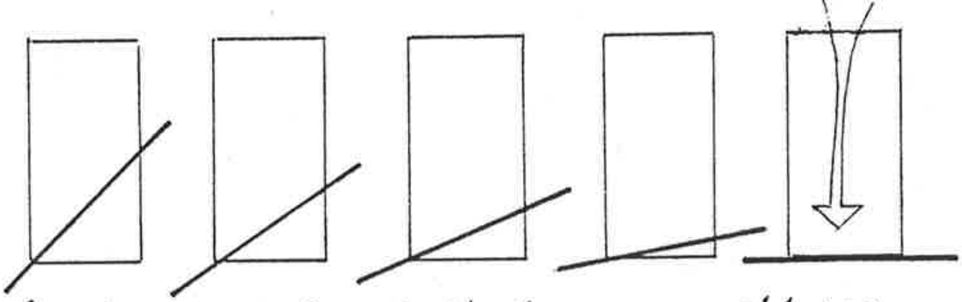
Discussion: Tell someone else why
(1) the gradients change in steepness
(2) the gradients change in direction

USING STUDENTS' INTUITIVE IDEAS
... a special gradient (1)

1. Find out students' intuitive ideas

These gradients are decreasing.

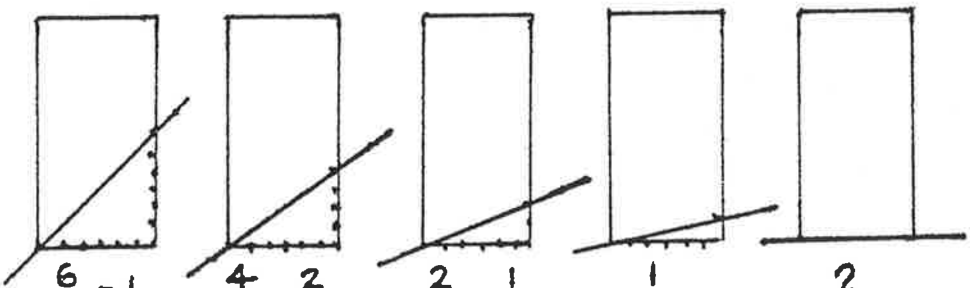
What number would you give to this gradient?



2. Ask students to defend their suggestions

Classroom talk: discuss the pros and cons of these suggestions

3. Look at the gradients again, using numbers

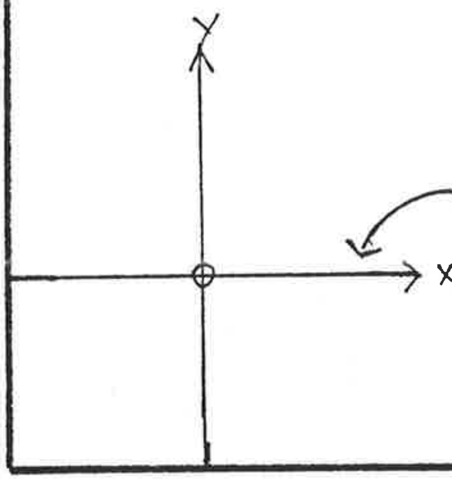


$\frac{6}{6} = 1$ $\quad$ $\frac{4}{6} = \frac{2}{3}$ $\quad$ $\frac{2}{6} = \frac{1}{3}$ $\quad$ $\frac{1}{6}$ $\quad$?

Can this gradient be written as a fraction $\frac{?}{6}$.

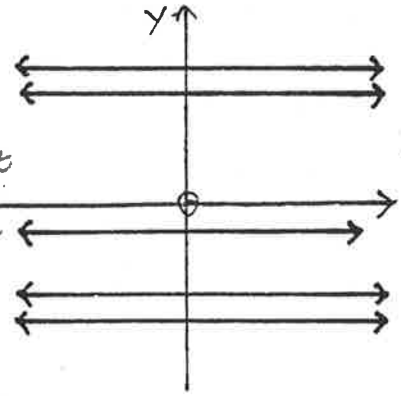
What is this fraction simplified?

4. Investigate further



What number can we give to the gradient of the X-axis

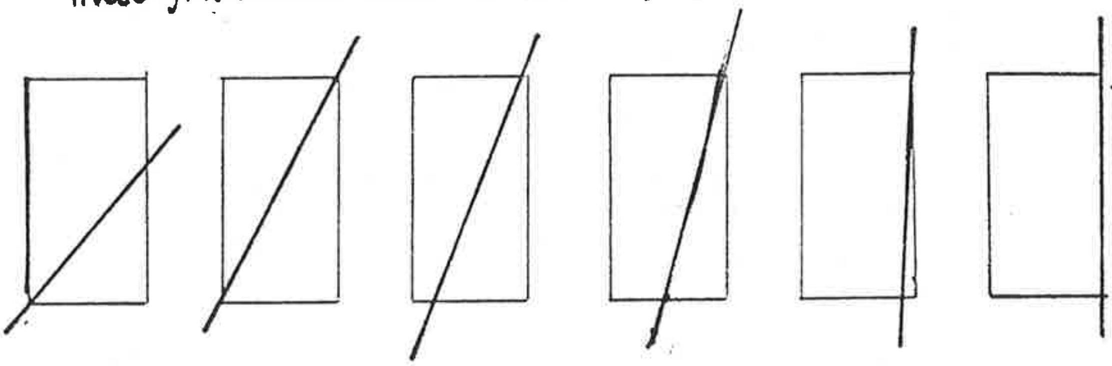
What is the gradient of lines parallel to the X-axis?



USING STUDENTS' INTUITIVE IDEAS
...a special gradient (2)

1. Find out students' intuitive ideas

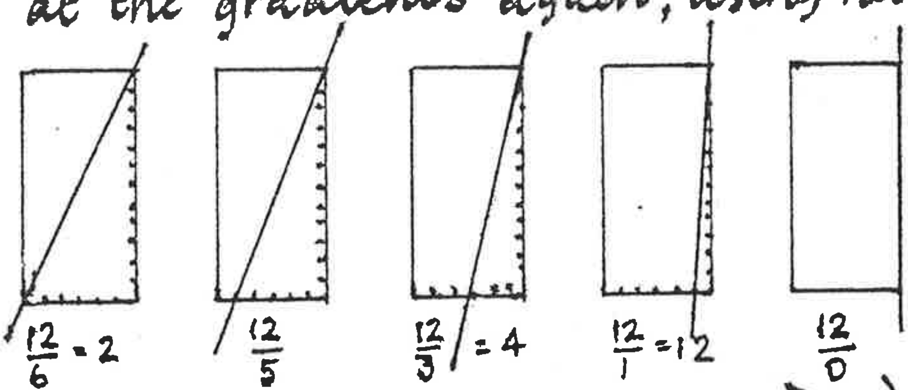
These gradients are increasing...



What would you say about this gradient?

2. Discuss the suggestions

3. Look at the gradients again, using numbers



$\frac{12}{6} = 2$ $\quad \frac{12}{5}$ $\quad \frac{12}{3} = 4$ $\quad \frac{12}{1} = 12$ $\quad \frac{12}{0}$

...but

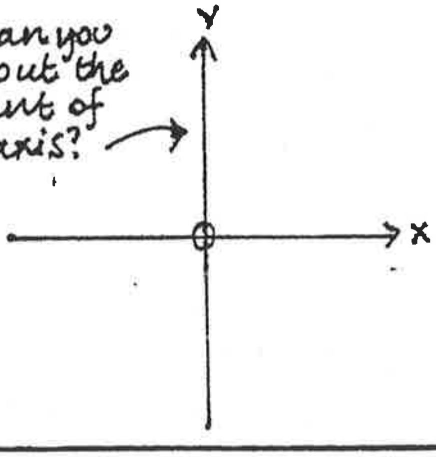
DIVISION BY ZERO CANNOT BE PERFORMED

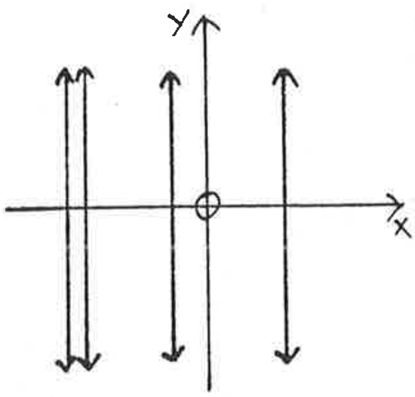
– so a vertical gradient is UNDEFINED

☆ This fundamental theorem of mathematics is not always easy to accept – or remember

4. Investigate further...

What can you say about the gradient of the Y axis?





What can you say about the gradient of lines parallel to the y-axis?

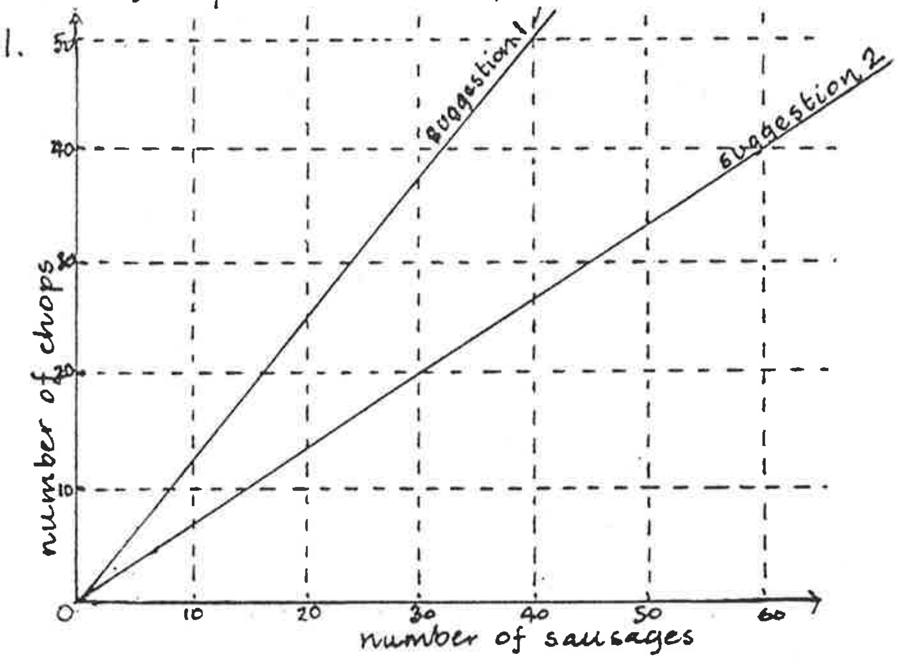
Catering for a barbecue

Some teenagers were planning the meat order for a barbecue. In discussion, there were two suggestions of how the quantity of chops and sausages could be divided.

Suggestion 1 – the proportion chops : sausages should be 5:4

Suggestion 2 – the proportion chops : sausages should be 2:3

The graphs below represent these suggestions.

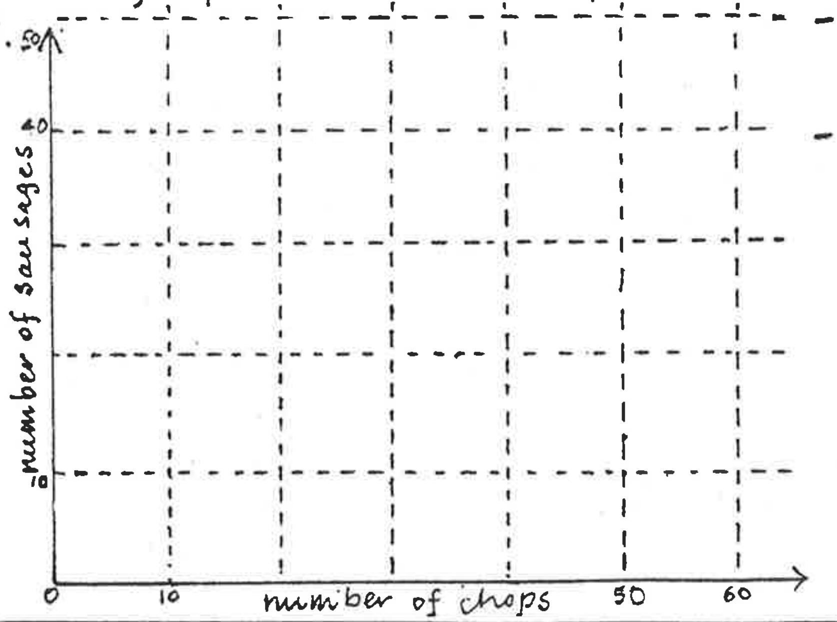
1. 

(1) What is the gradient of the line that represents
(a) Suggestion 1 ?
(b) Suggestion 2 ?

(2) Rule lines for each of these suggestions
(a) one chop for every two sausages.
(b) no chops, only sausages (what is the gradient of this line?)
(c) no sausages, only chops.

The graphs above are represented so that
- the number of sausages is the independent variable
- the number of chops is the dependent variable.

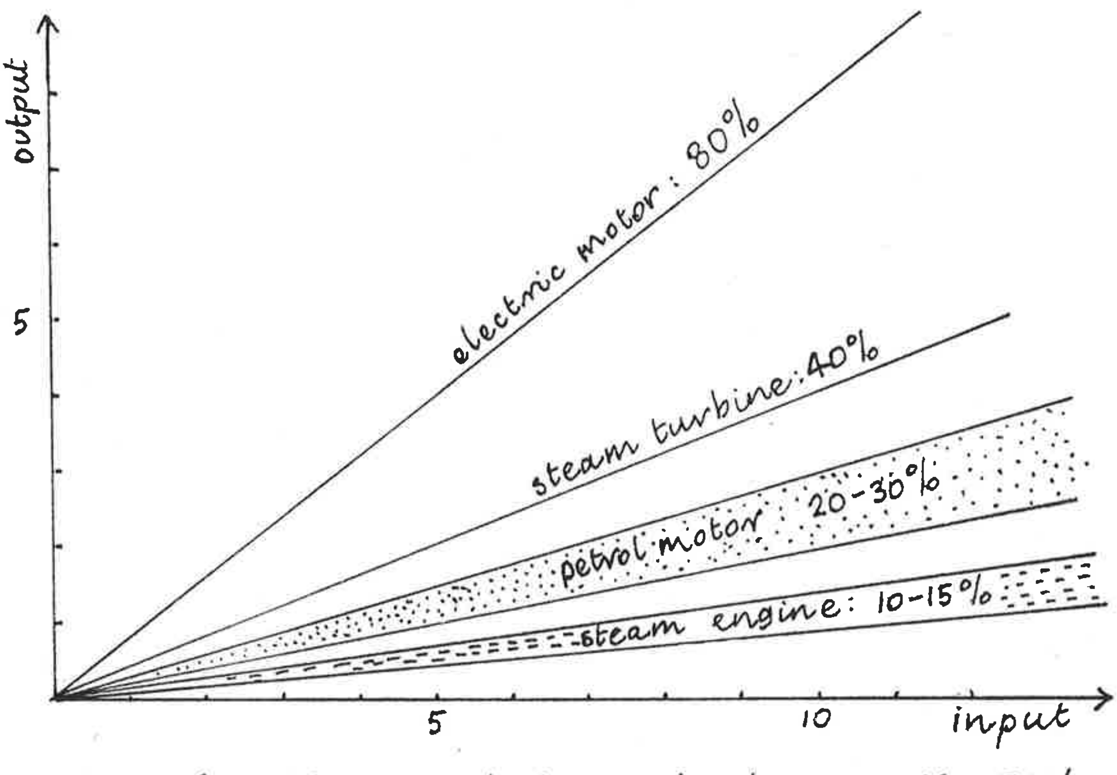
Represent the same information on this plane with the independent and dependent variables reversed.

2. 

MACHINES

For a machine to be useful, it needs to work efficiently.
To measure the efficiency of a machine, COMPARE the work it does (OUTPUT) with the fuel it needs (INPUT).
This comparison OUTPUT : INPUT is a RATIO.

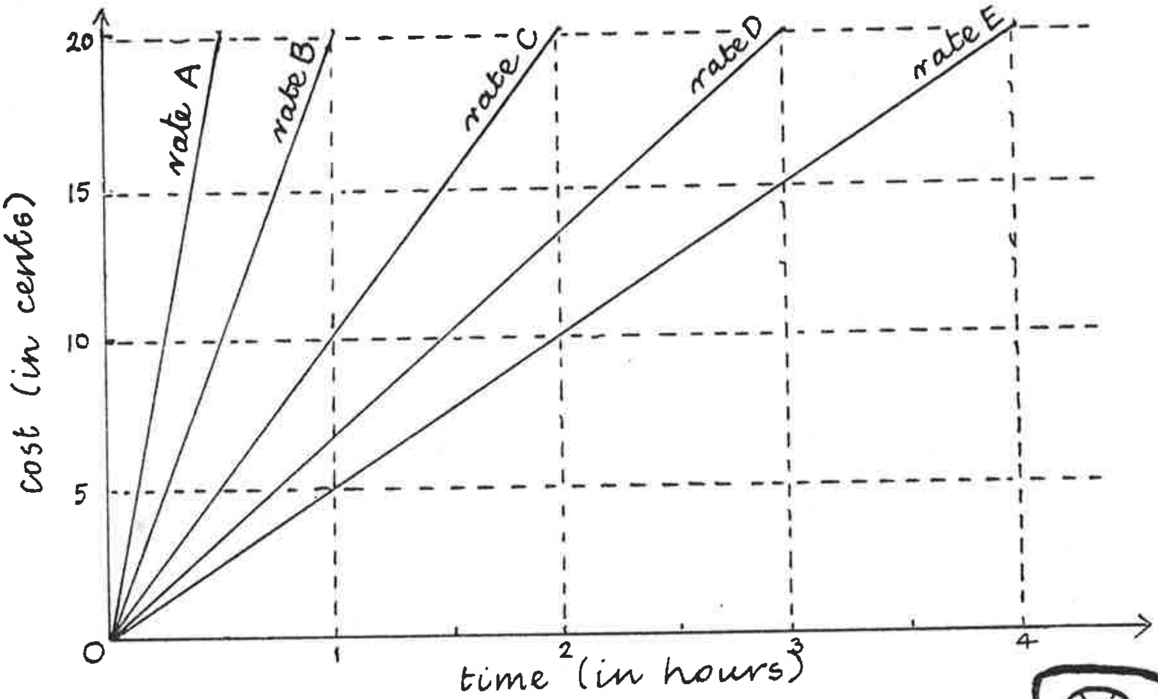
The human body can be seen as a machine. Its efficiency can be measured by comparing the work it can do with the fuel (food) it takes in.
These graphs show the usual efficiency of some machines.



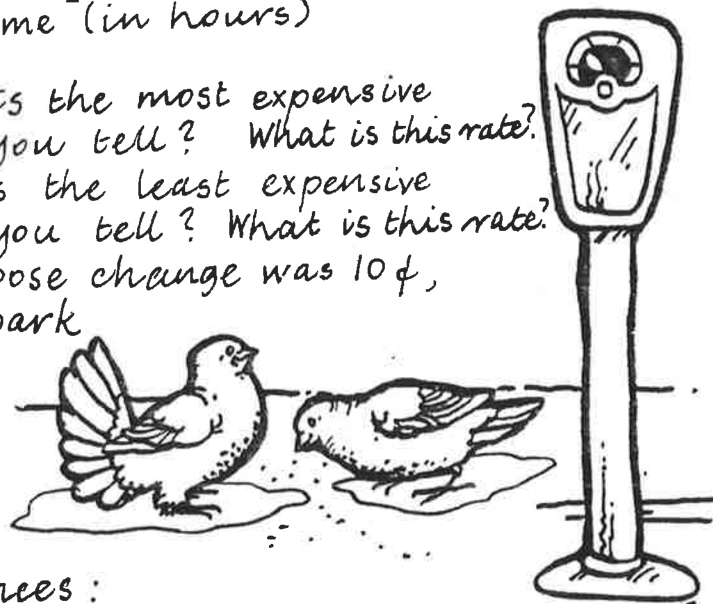
1. The efficiency of the human body varies between 16 - 56%, depending on a person's fitness and food intake.
   On the diagram above, rule straight line graphs to show the region bounded by the efficiency ratios 16% and 56%.
   Label this region : human body : 16-56%
2. The gradient (slope) ratios are given as percentages.
   Write the gradients in simplest (fraction) terms for each of these:
   a. electric motor
   b. steam turbine
   c. the range for the steam engine
   d. the range for the human body

Comparing PARKING METER rates. group work

Parking meter rates vary across the city.
The straight line graphs below show comparisons between the different rates.
(Note that the rate is a RATIO. The rate is the measure of COST per unit TIME. These graphs show the ratio cost : time. For example, rate A is 20¢ per $\frac{1}{2}$ hour, or 40¢ per hour)



1. Which rate represents the most expensive parking? How can you tell? What is this rate?
2. Which rate represents the least expensive parking? How can you tell? What is this rate?
3. If all you had in loose change was 10¢, how long could you park
   on A?
   B?
   C?
   D?
   E?



4. Complete these sentences:
   a. The number given to the gradient A is twice that of ______
   b. The number given to the gradient B is twice that of ______
   c. The number given to the gradient C is twice that of ______

Extension...

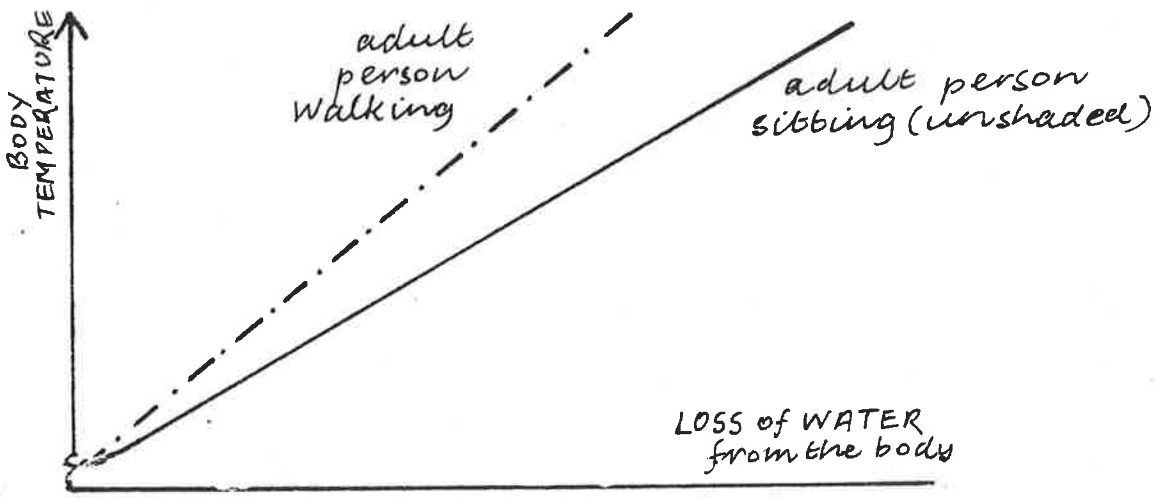
5. Look carefully at rate E. Every point on that line satisfies the rule $C = 5t$. (c = cost in cents, t = time in hours)
   Write the rules for (a) rate C: ________ (b) rate D: ________

CLASS-ROOM TALK.

THE DESERT Humans are not well adapted to living in hot, desert regions. To keep the body temperature at the right level, people have sweat glands that release water, allowing the body to cool through evaporation.

If there is no drinking water available to replace the fluids lost by sweating, we are unable to sweat. In hot conditions the blood quickly becomes over-heated, and heat-stroke results.

These graphs show how a person's body temperature rises as water is lost from the body through evaporation.



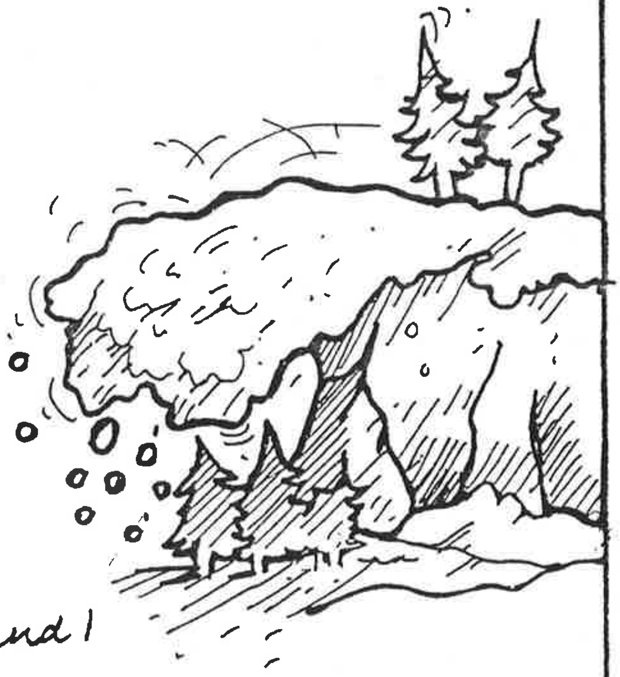
Talk about possible solutions to these questions. Record (write down) the solutions that seem best.

1. From the information given you in the graphs above:
   (1) What variable factors determine the start of heat stroke in an adult person?
   (2) Would you be likely to suffer heat stroke more quickly by walking or sitting?
   How do the graphs tell you that?
2. Draw in <u>your estimations</u> of straight line graphs that show how likely a person is to get heat-stroke when
   (1) jogging
   (2) sitting in the shade.
   — Explain your reasons carefully.
3. The vertical axis shows variations in body temperature. What is likely to be the lowest point that you would mark on the vertical axis above?

CLASS-ROOM TALK:

GRADIENTS of STRAIGHT LINES

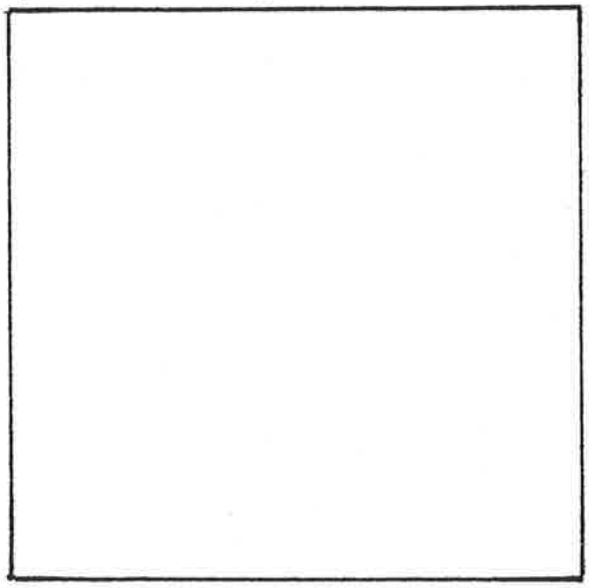
AVALANCHES



The most dangerous gradients for avalanches are those with gradients between $\frac{3}{5}$ and 1

In the space below rule lines to show gradients $\frac{3}{5}$ and 1

Shade in the area that represents gradients that are most dangerous for avalanches.



While you're doing this, think about why this range of gradients should be the most dangerous.

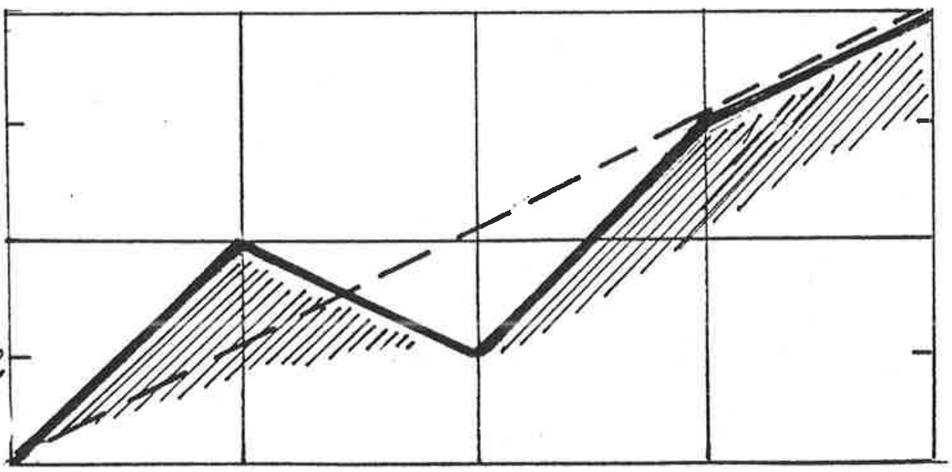
Alternatively, think about why the gradients greater than 1 / less than $\frac{3}{5}$ should be less dangerous

Share your reasoning with others.

MOUNTAIN CHALLENGE -

Find the average of the gradients of this mountain

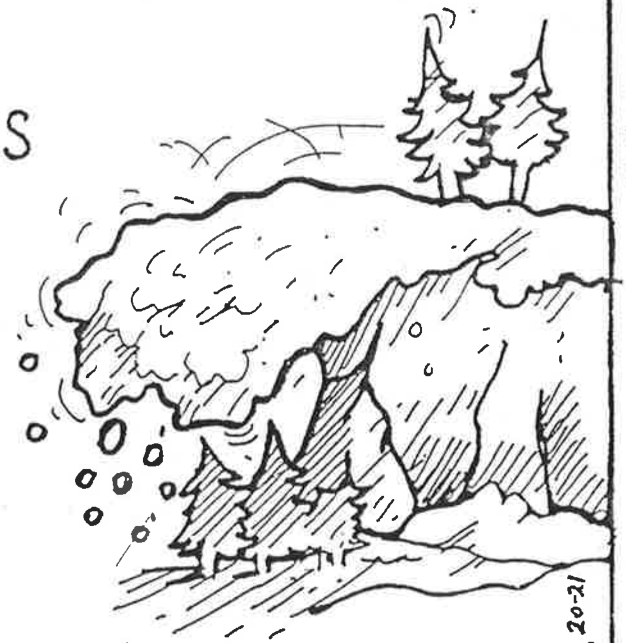
Now compare it with the overall (bottom-to-top) gradient



CLASS-ROOM TALK:

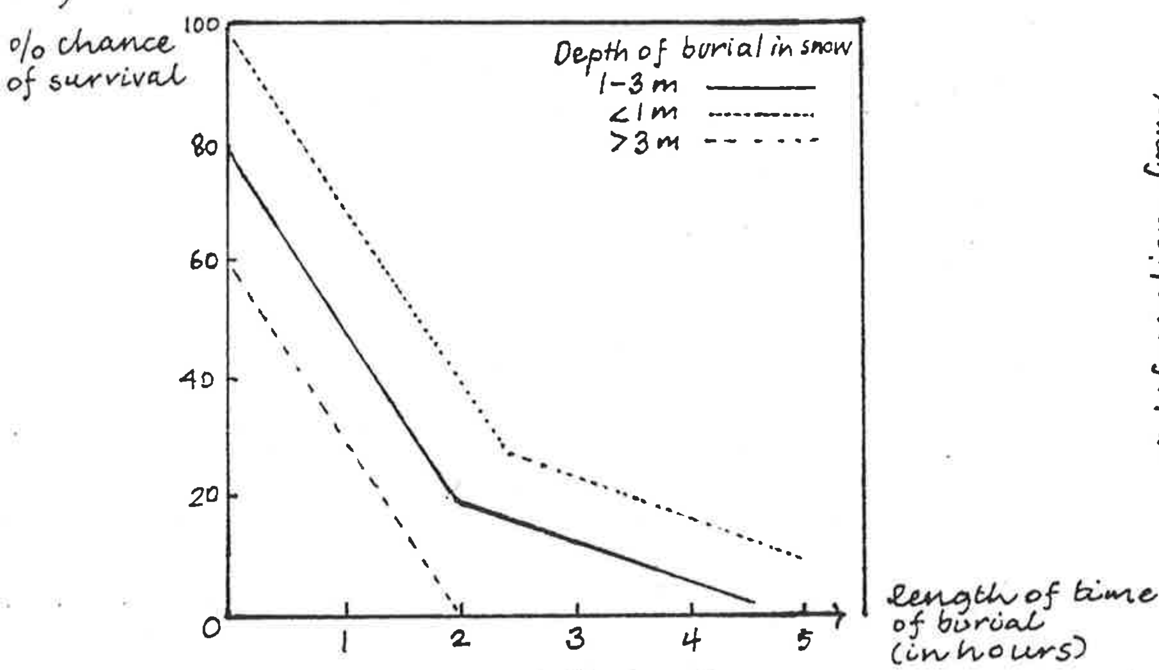
GRADIENTS of STRAIGHT LINES

AVALANCHES



The sketch* straight-line graphs below indicate roughly your chances of surviving after being buried by an avalanche.

(Notice that your chance of survival (the outcome) is dependent on the length of time that you remain buried under the snow.)



* Information from *Mathematics in School*, 1987, 16:2, 20-21

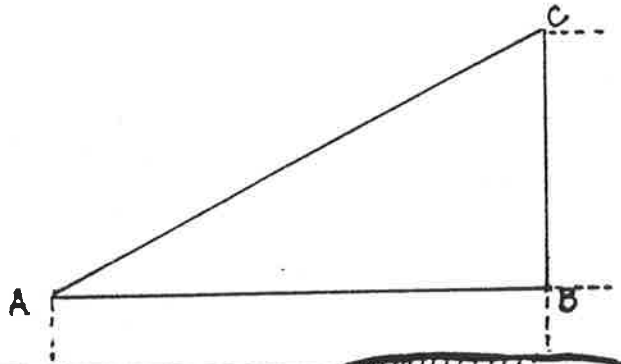
1. (1) These graphs have negative gradients. What do these negative slopes tell you about your chances of survival as time goes on?
   (2) Two of the graphs show changes in gradient. What do these changes tell you? What about the graph that has no change in gradient? Explain your reasoning carefully.

2. Notice that these graphs show some parallel lines. What do you observe about the gradients of lines that are parallel? What information do parallel lines tell you about your chances of survival?

3. Additional comparative information about survival chances is given by drawing three graphs. So survival must be dependent on another variable factor — besides length of time of burial. What is the other variable factor?

DEFINING A GRADIENT

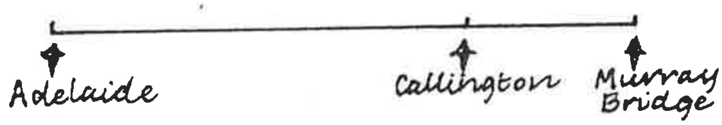
This roadside sign indicating a steep hill refers to the GRADIENT of the hill



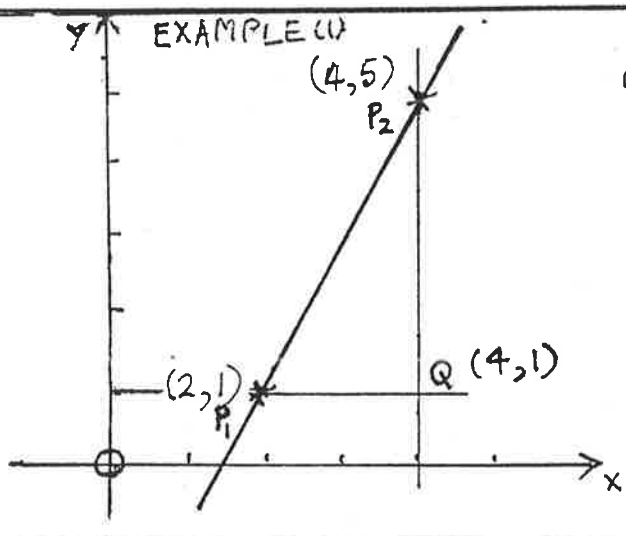
In mathematics, GRADIENT is defined as the ratio of the difference in height BC to the horizontal distance AB

CALCULATING GRADIENTS

Suppose you travelled from Adelaide to Murray Bridge via Callington. A diagram of your journey might look like this:



Now if you know that the distance between Adelaide and Murray Bridge is 76km and the distance between Adelaide and Callington is 55km then the distance between Callington and Murray Bridge is (76 − 55) km



P_1 is (2,1)
and P_2 is (4,5)
Q has the same x-coordinate as P_2
and the same y-coordinate as P_1
So the coordinates of Q are (4,1)
Using a similar approach to the one above, we can say that P_2Q is $5-1$
and P_1Q is $4-2$

Gradient P_1P_2 is $\frac{5-1}{4-2}$
$= 2$

continued

CALCULATING GRADIENTS (continued)

EXAMPLE (2)

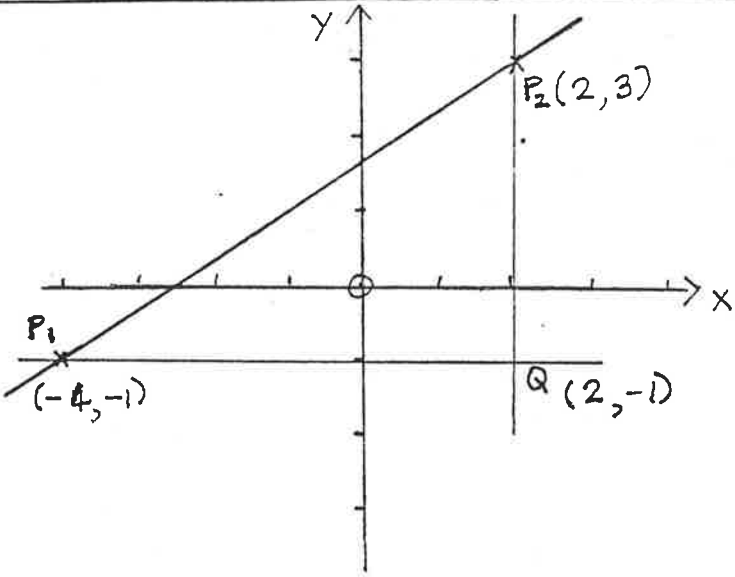
In this figure,
P_1 is $(-4,-1)$ P_2 is $(2,3)$
Q is $(2,-1)$

P_2Q is $3-(-1)$
and P_1Q is $2-(-4)$

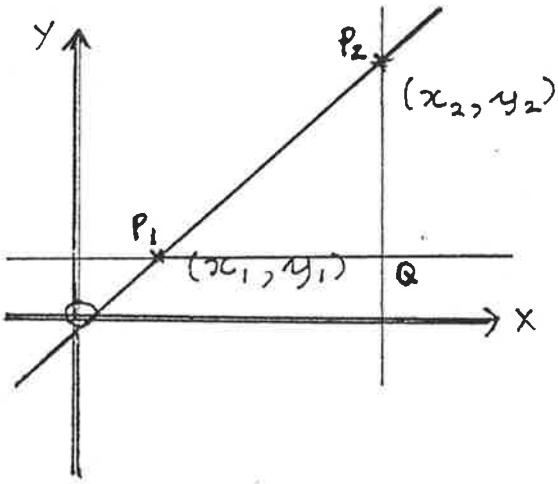
Gradient P_1P_2 is $\frac{3-(-1)}{2-(-4)}$

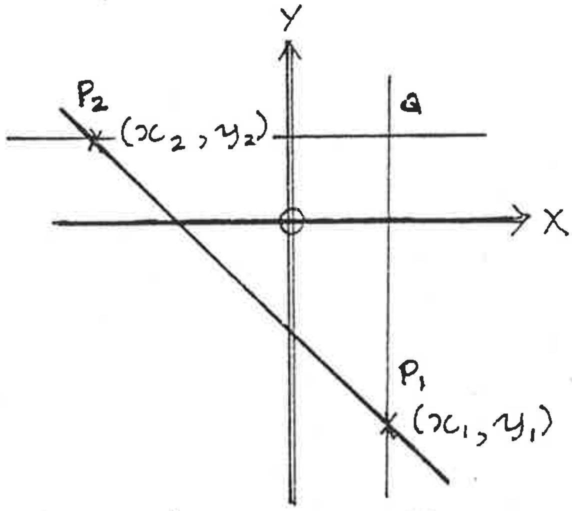
$= \frac{3+1}{2+4}$

$= \frac{2}{3}$



More generally, gradients may be shown like this:





GRADIENT P_1P_2 is $\frac{y_2 - y_1}{x_2 - x_1}$

What happens if the numbers are negative?

Note that the symbol 'm' is often used for 'gradient'

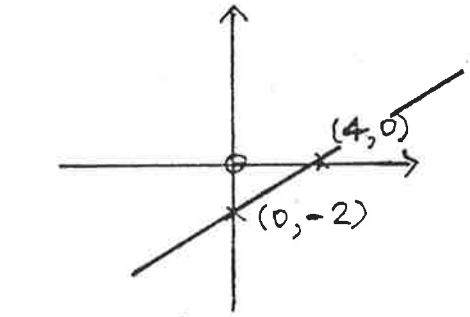
Discuss with others why

m is $\frac{y_2 - y_1}{x_2 - x_1}$ OR $\frac{y_1 - y_2}{x_1 - x_2}$

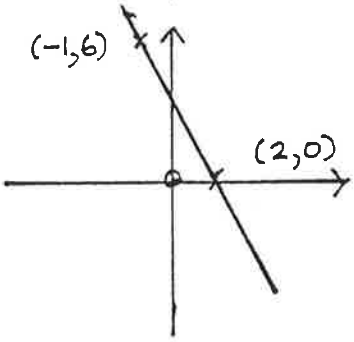
Calculating GRADIENTS

Calculate the gradients of these lines:

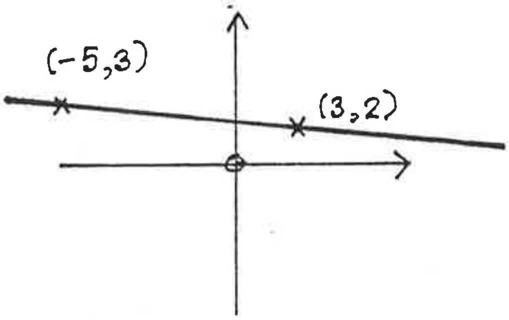
1.



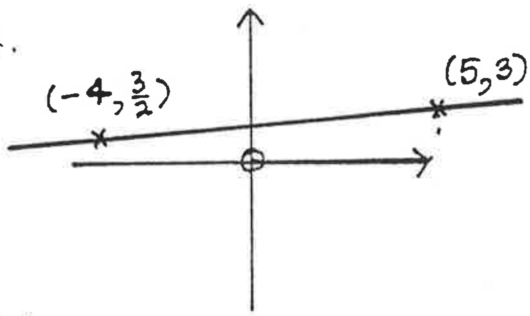
2.



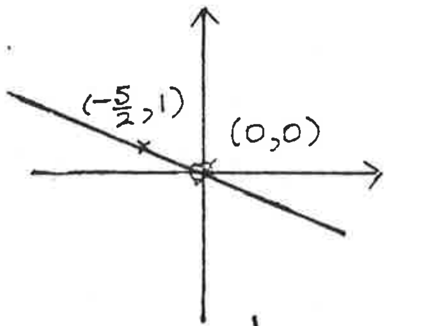
3.



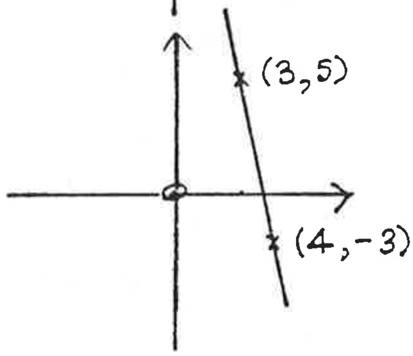
4.



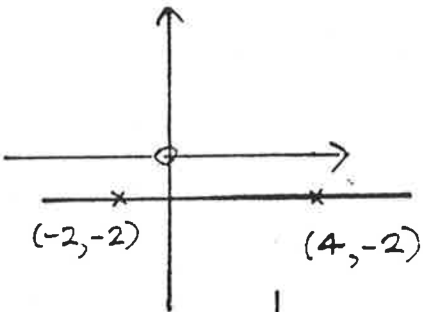
5.



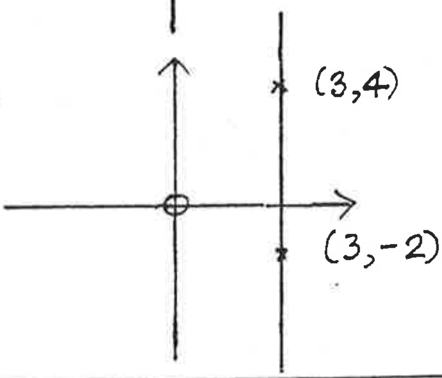
6.



7.



8.



Assorted gradients: $\frac{1}{6}$ $\frac{1}{2}$ $-\frac{2}{5}$ -2 -8 $-\frac{1}{8}$ 0 undefined

Calculating GRADIENTS

If 3 or more points lie on the same line we say they are COLLINEAR

Quiz 1

For each, find the gradient of a line that passes through points with these coordinates

1. $(-3,0)$ $(0,-4)$
2. $(0,0)$ $(3,4)$
3. $(3,5)$ $(-3,-5)$
4. $(2,-4)$ $(5,-4)$
5. $(-2,5)$ $(4,-3)$
6. $(6,-2)$ $(6,2)$
7. $(3,\frac{3}{2})$ $(\frac{5}{2},2)$

$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	0	$-\frac{4}{3}$	undefined	-1

Quiz 2

Given a point O (0,0) locate a point A so that the line segment OA has the gradient:

1. $\frac{2}{5}$
2. $-\frac{3}{4}$
3. $\frac{5}{3}$
4. -2
5. 1

Note: (Many points A could give these gradients. Talk about possible locations of A with other students)

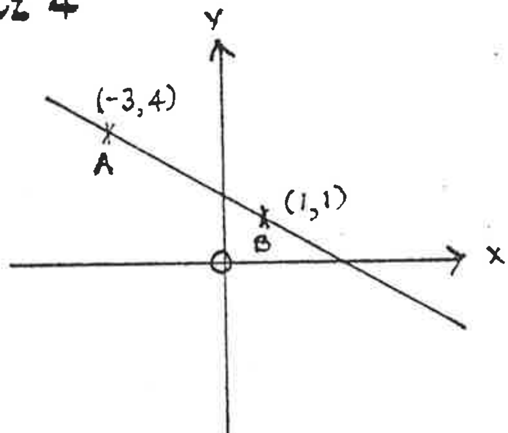
Quiz 3

For each, determine whether the points with these coordinates are collinear.

(Check your solutions with others in the class)

1. P $(7,0)$ Q $(1,3)$ R $(3,2)$
2. P $(2,3)$ Q $(-4,3)$ R $(3,0)$
3. P $(-3,-10)$ Q $(5,-2)$ R $(-2,-9)$
4. P $(3,-2)$ Q $(-1,-2)$ R $(-5,-2)$

Quiz 4



Plot (and give the coordinates of) any point collinear with A, B

CLASS-ROOM TALK

Lines that have the same gradient but are distinct from each other are parallel.

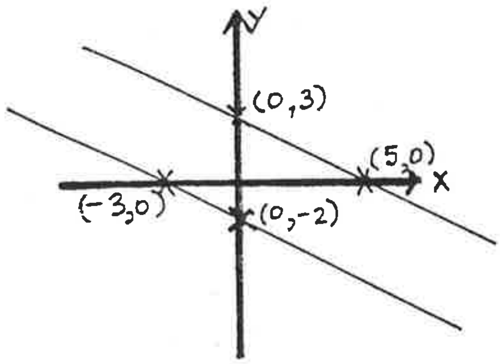
Discussion:

- How can you tell whether two lines (passing through points with given coordinates) are parallel?
- How many points must you be given for each line?

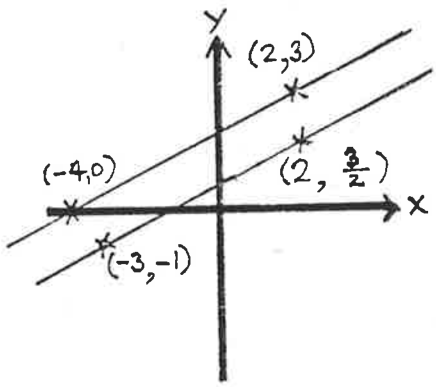
---

1. Are these lines parallel?

(1)



(2)

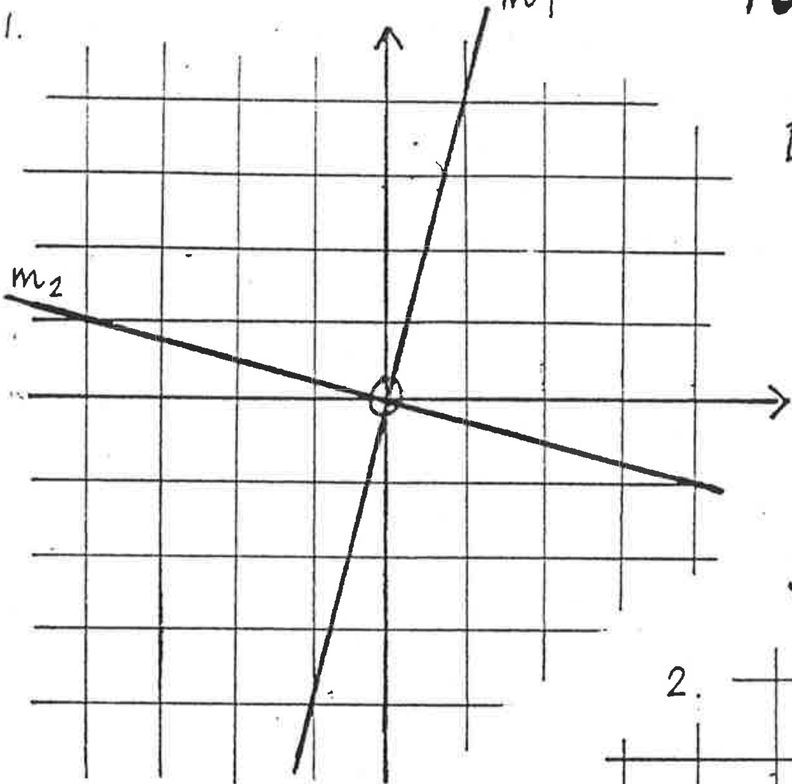


2. Two points have coordinates: A (-2, 3) B (4, 1)
   Are the points C, D collinear/parallel/neither with A, B
   if (1) C is (0,0) D is (6, -2)
   (2) C is (1, 2) D is (7, 0)
   (3) C is (1, 2) D is (2, -1)

3. Four points have coordinates: A (2, -1) B (3, 5) C (0, 6) D (-1, 0)
   Is ABCD a parallelogram?
   How can you tell?

Investigating a Definition: Gradients of Perpendicular Lines

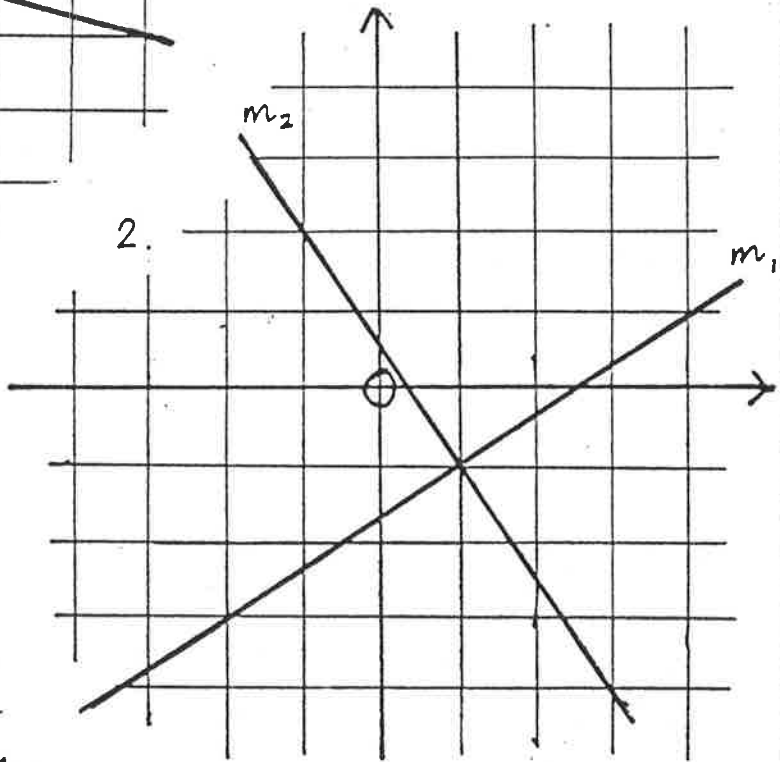
1.



In each of these, the lines are perpendicular to each other.

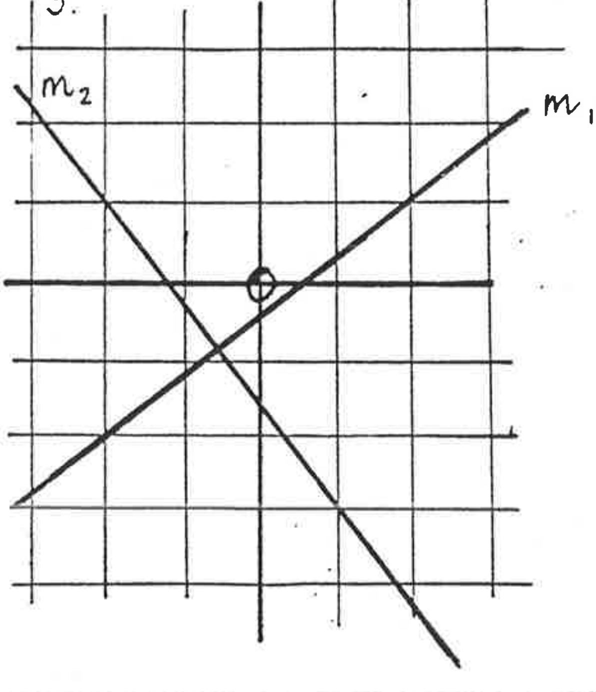
<u>Note</u> m_1 is the symbol for first gradient
m_2 for second gradient

2.



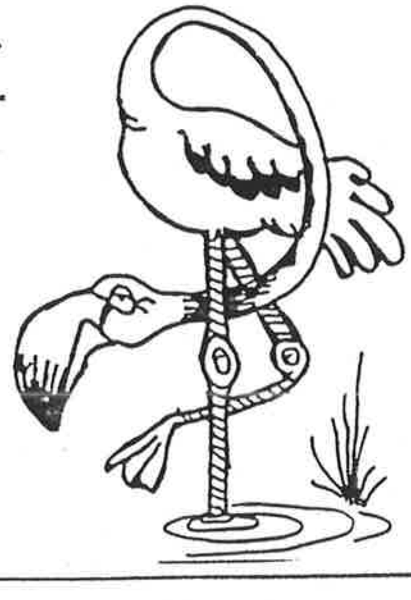
1. a. What is the number for m_1?
   b. What is the number for m_2?
   c. What is the product $m_1 m_2$?

3.

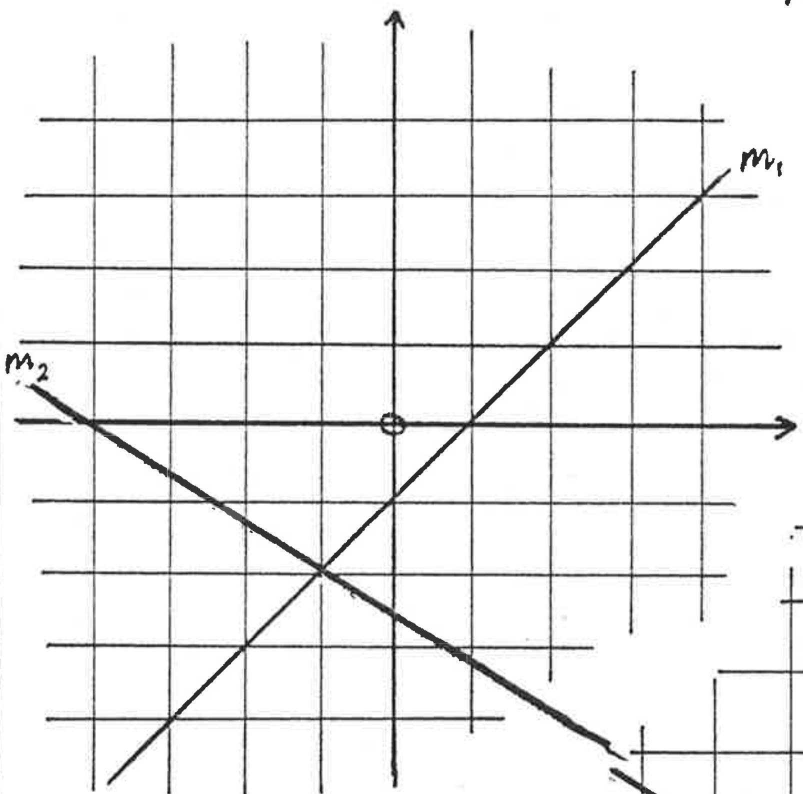


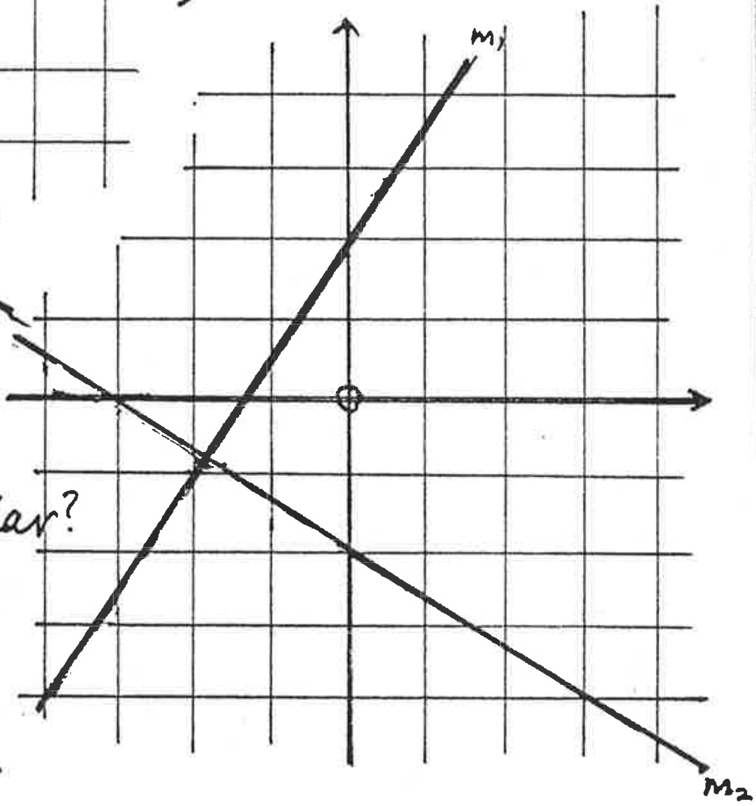
2. m_1 is ___
   m_2 is ___
   $m_1 . m_2$ is ___

3. m_1 is ___
   m_2 is ___
   $m_1 m_2$ is ___



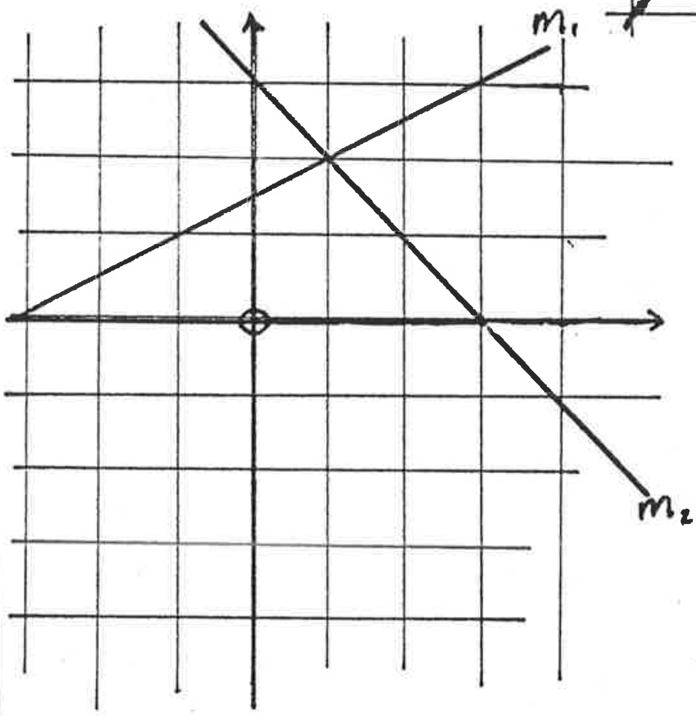
Investigating a Definition: Gradients of Perpendicular Lines





Are these pairs of lines perpendicular?

How can you tell?



Complete this definition:
If two lines are perpendicular to each other, the product of their gradients $m_1 m_2$ is ___

PERPENDICULAR LINES

Use the definition that refers to the gradients of perpendicular lines to answer these questions.

1. The gradients of some lines are given below. What is the gradient of any line that is perpendicular to each of these?

(1) m_1 is $\frac{-4}{3}$

(2) m_1 is $\frac{3}{2}$

(3) m_2 is $-\frac{1}{4}$

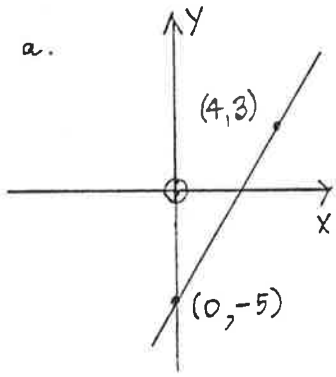
(4) m_2 is 5

(5) m_1 is undefined

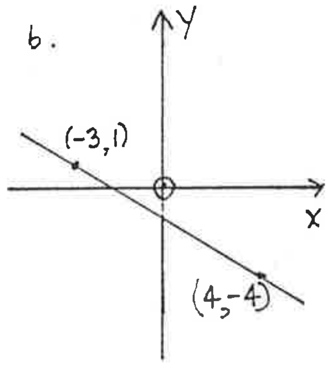
(6) m_1 is $4\frac{1}{2}$

2 (1) What are the gradients of lines that are perpendicular to the lines drawn here?

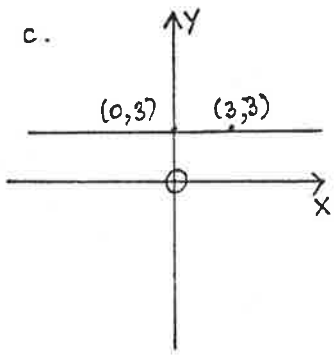
a.



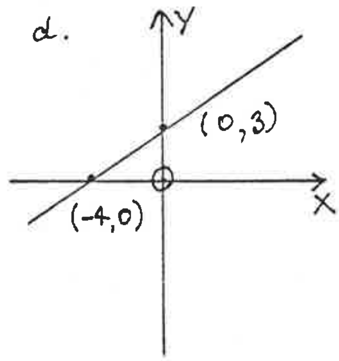
b.



c.



d.



(2) On each of the above cartesian planes rule MANY LINES that are perpendicular to the given line

3. Are the following pairs of lines (a) perpendicular
(b) parallel
(c) neither of these?

(1) AB and CD, where A is (7,-2) B is (4,-1) C is (6,-1) D is (7,2)

(2) PQ and RS, where P is (-2,4) Q is (6,-3) R is (-3,-2) S is (4,6)

(3) WX and YZ, where W is (4,5) X is (3,-2) Y is (6,1) Z is (-2,5)

(4) KL and MN, where K is (5,-10) L is (2,-8) M is (9,-7) N is (11,-4)

(5) EF and GH, where E is (3,b) F is (4,a) G is (5,3b) H is (8,3a)

(6) AB and XY, where A is (-1,2k) B is (-2,3k) X is (k,5) Y is (1,6)

Note: your working must show clearly why you have selected (a), (b) or (c)

CHECK UP if you need to

1. $\frac{3}{4}$ $-\frac{2}{3}$ 4 $-\frac{1}{5}$ 0 $-\frac{2}{9}$

2. $-\frac{1}{2}$ $\frac{7}{5}$ undefined $-\frac{4}{3}$

3. (a) → (a) → (c) → (b) → (b) → (a)

(5) $\frac{3(a-b)}{3} = a-b$

(6) $(k-1) \times \frac{1}{(1-k)} = -1$

- GRADIENTS -

1. A line has a gradient of 2.
   If the line passes through
   (a) $(2, k)$ and $(1, 2)$, find k
   (b) $(2a, 3)$ and $(3, -3)$, find a
   (c) $(3, y)$ and $(-2, 3y)$, find y

2. Three points $A(5, -5)$ $B(3, -2)$ and $C(k, -1)$ are collinear. Find k.

3. A line passes through the points $X(-5, 2)$ and $Y(-2, 4)$
   (a) If XY is parallel to UV where U is $(4, k)$ and V is $(1, 3)$ find k
   (b) If XY is perpendicular to RS where R is $(3t, 7)$ and S is $(5, 4)$ find t.

4. A line passes through the points $A(4, -3)$ and $B(9, -4)$
   (a) If AB is parallel to CD where C is $(-5, 2)$ and D is $(0, 3a)$ find a
   (b) If AB is perpendicular to EF, where E is $(4, 3)$ and F is $(b, -2)$ find b

5. $A(2, -1)$ $B(-8, -3)$ $C(-5, 1)$ $D(5, 3)$ are four points on a cartesian plane. Is the figure $ABCD$ a parallelogram? How can you tell without sketching the figure? Show your reasoning.

6. $A(-2, -3)$ $B(4, 6)$ $C(-5, 1)$ are three points on a cartesian plane. Is the triangle ABC right angled? How can you tell? Show your reasoning.

7. $A(-3, 5)$ $B(0, 7)$ $C(4, 1)$ $D(1, -1)$ are four points on a cartesian plane. Is the figure $ABCD$ a rectangle? How can you tell? Show your reasoning.

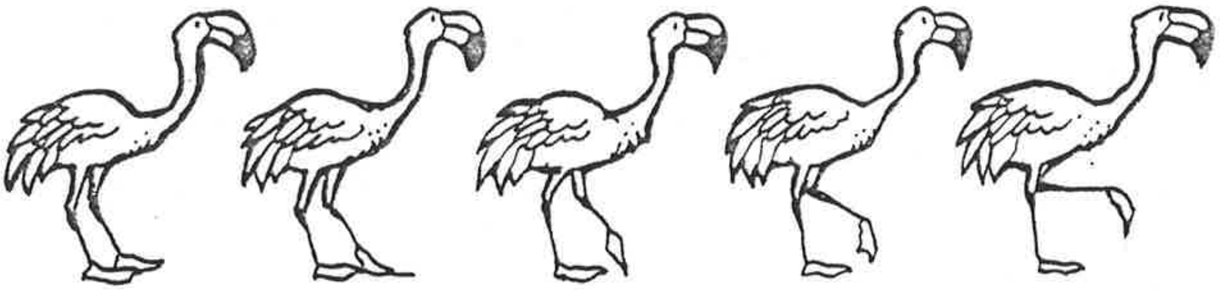
8. $A(-5, -2)$ $B(3, 4)$ are the end-points of a diameter AB of a circle. C is a point $(-4, 5)$. Does C lie on the circumference of a circle? How can you tell? Show your reasoning

Specific Unknowns

Here are the values

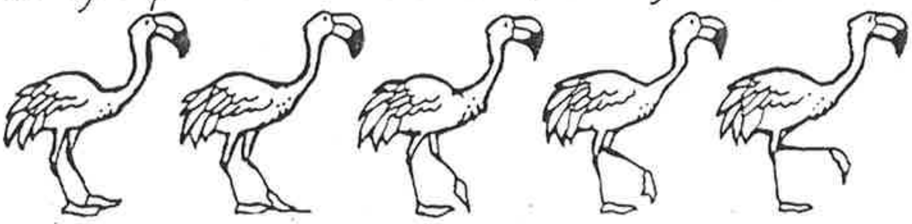
1. $k = 4$	3. $k = 5$
$a = 3$	$t = 1$
$y = -5$	4. $a = 1/3$
2. $k = 7/3$	$b = 5$

Investigating Linear Graphs



Cartoonists are able to create an illusion of movement by creating many drawings that, in effect, 'run' together.

Linear graphs can be viewed in a similar way. The plotting of many points close together can be seen as representing a continuous line (or curve). If the ordered pairs satisfy a linear pattern (or rule) the graph will be a straight line.

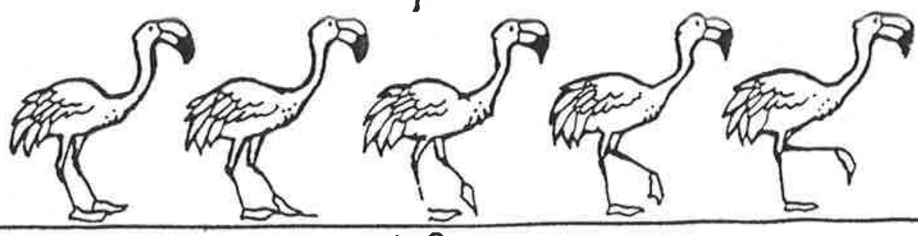


Problem solving involves . . .

- LOOKING FOR A PATTERN
- LOOKING FOR SIGNIFICANT FEATURES IN THE PATTERN

Two significant features of LINEAR PATTERNS that we will consider are:

the **gradient**
and the **position** with reference to the coordinate axes on the plane



A function can be represented as a formula

In this formula,

- variables are introduced for the elements of the domain of definition - and - of the range.

  We frequently use x as a symbol for the numbers in the domain (independent variable)

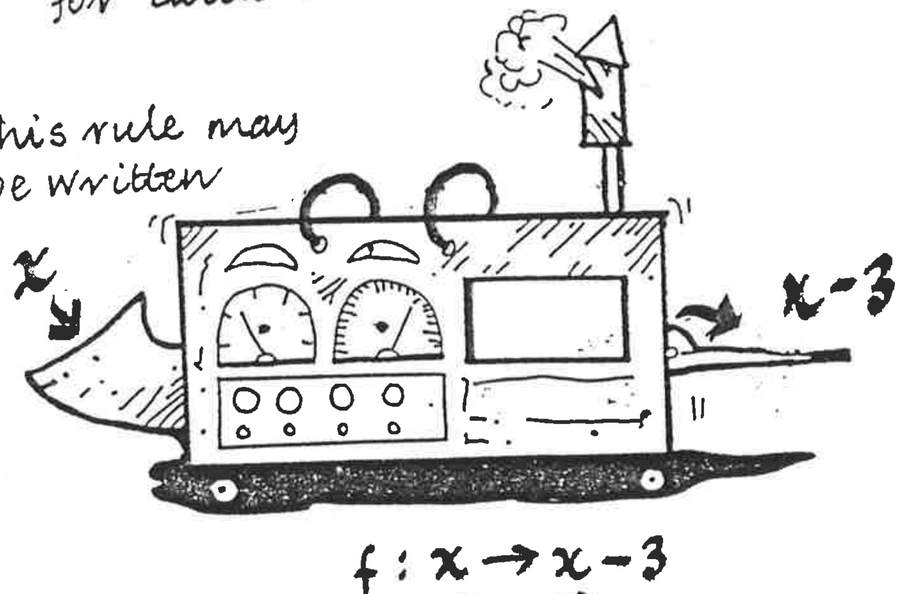
  and y as a symbol for the numbers in the range (dependent variable)

  - but other letter symbols may be used.

- the rule for the correspondence is defined with the help of the variables by means of an equation.

  This equation gives us a rule for calculation

This rule may be written



$$f: x \rightarrow x-3$$

OR $y = (x-3)$ called the equation of the function

The following pages consider equations that can be represented as LINEAR GRAPHS

Writing rules for LINEAR GRAPHS

Form a small group

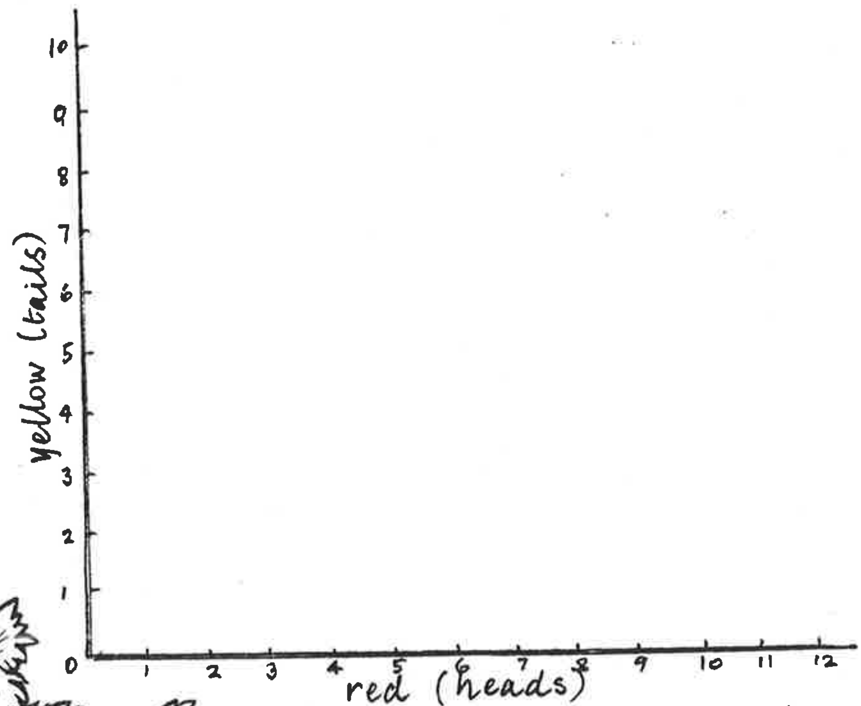
For this investigation your group will need 10 flip blocks (squares that are red on one side and yellow on the other) or 10 coins.

What to do

- Take turns to toss the flip blocks or coins.
- Record each toss in an organised way, for example:

red (heads)	yellow (tails)
6	4
10	0
3	7
4	6
8	2
⋮	⋮

- Rewrite each toss (not including repeats) as an ordered pair:
  (6, 4)
  (10, 0) and so on
- Record these ordered pairs on the cartesian plane below:



TASK: Decide on a rule that you think is adequate to describe every toss.

- Take your suggestion to a class-group discussion.

Class-group discussion

1. As you recorded each toss, a pattern emerged in the numbers. What was it?
2. Suggest the rule that your small group decided was adequate to describe every toss.
3. Did you notice a pattern emerging on the graph, too? What is it? Does it have a name?

Follow-up small-group activity

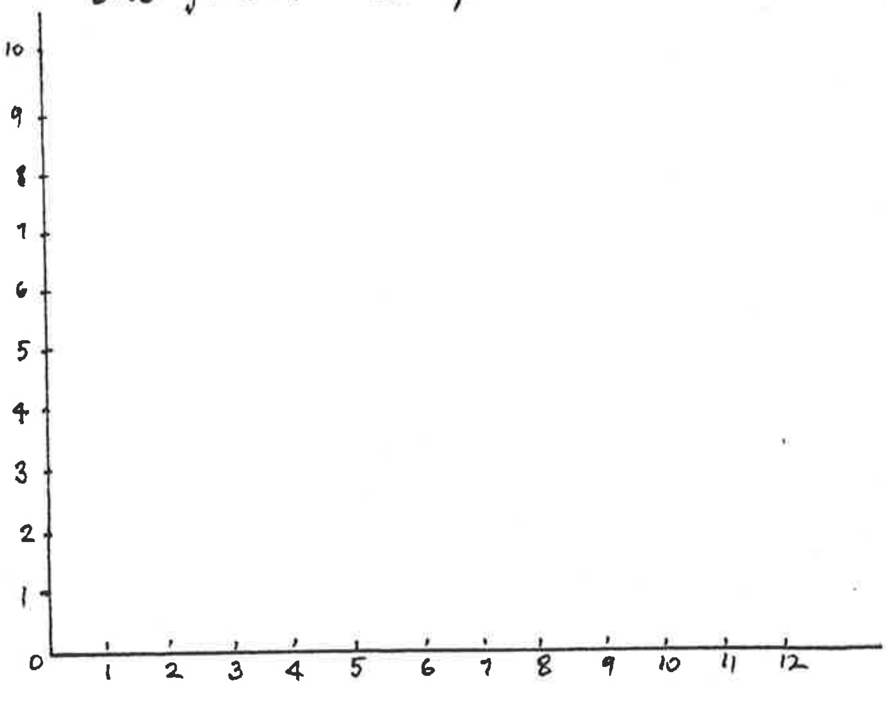
Follow the same 'toss-record-graph' procedure as before, this time using a <u>different</u> number of flip-blocks or coins.

- Use the same cartesian plane for each graph.
- an interesting pattern will emerge.
- You should talk about the important features of this pattern in a whole-class group.

FOLLOW-UP ACTIVITY (using positive numbers only)

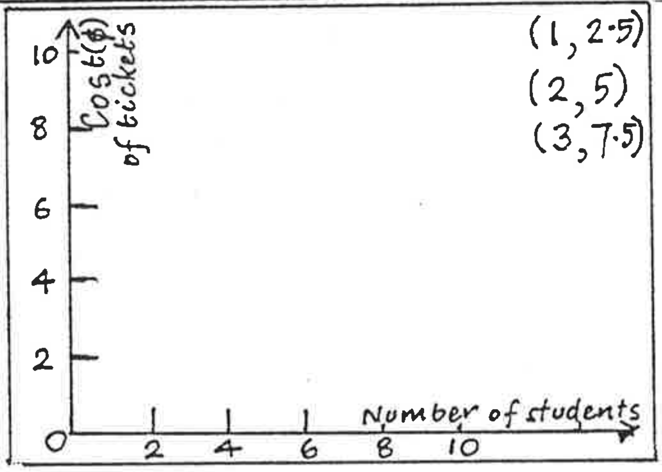
1. Write down a list of ordered pairs that yield a difference of 2 when the first number is subtracted from the second, eg, (3,5). Graph them.
2. Now list the ordered pairs that yield a difference of 2 when the second number is subtracted from the first. Graph them.

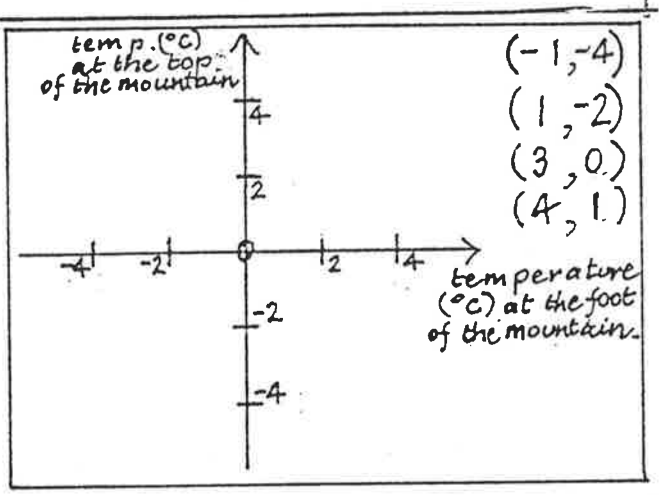
What do you notice?

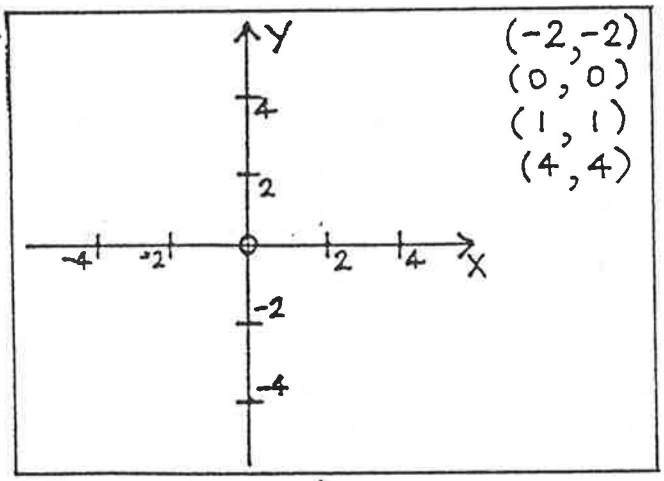


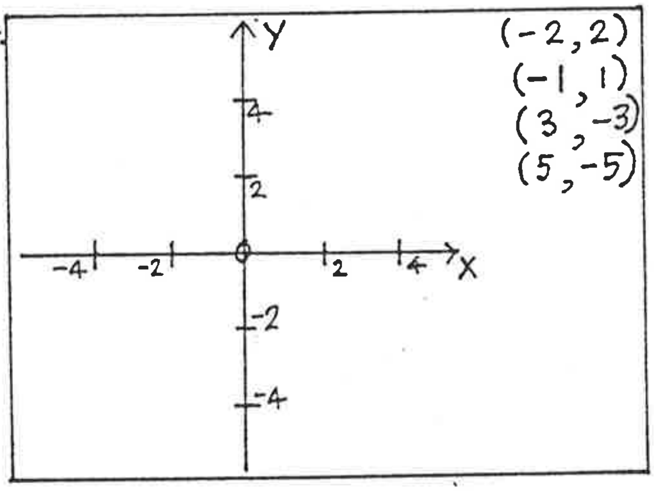
Writing RULES for LINEAR (straight line) GRAPHS

The sets of axes sketched on this page form CARTESIAN PLANES.
On each plane, plot the points represented by the ordered pairs that are listed alongside.
Join the points, using a ruler. Extend the graph, both ends.

1.


2.


3.


4.


For each graph that you have drawn:

(1) List two extra ordered pairs that fit the same pattern as the ordered pairs already given.
Can these two extra ordered pairs be located on the straight line you've drawn? Why is this important?

(2) Write a rule for each linear graph.

1. $y = 2{\cdot}5x$ or $\frac{y}{x} = 2{\cdot}5$ 2. $y = x - 3$ or [illegible] 3. $y = x$ 4. $y = -x$ also written $x + y = 0$

Writing rules for Linear graphs –

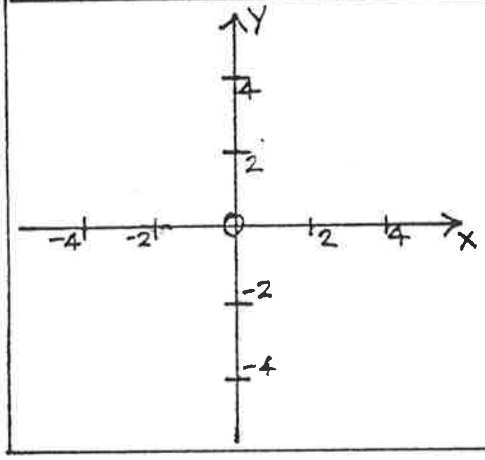
For each:

(1) Plot and join the points represented by the ordered pairs. Extend the graph.

(2) Write a RULE.

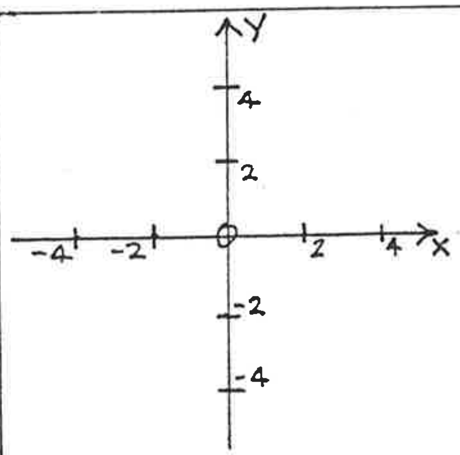
(3) Add another ordered pair that satisfies the rule.

1.



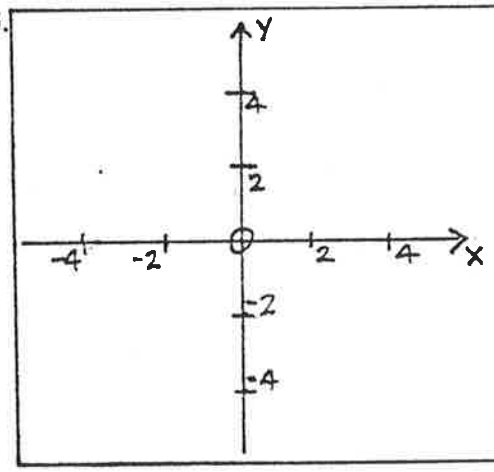
$(-4, -2)$
$(-3, -\frac{3}{2})$
$(-1, -\frac{1}{2})$
$(2, 1)$

2.



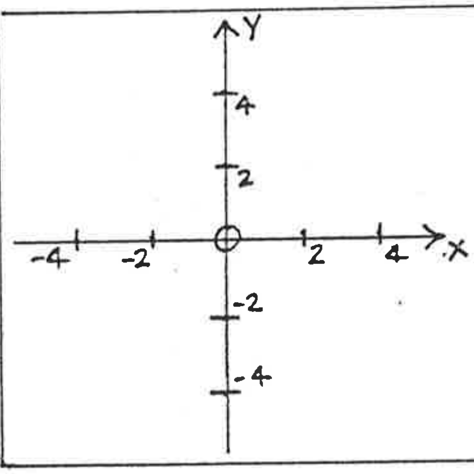
$(-4, 2)$
$(-3, \frac{3}{2})$
$(-1, \frac{1}{2})$
$(2, -1)$

3.



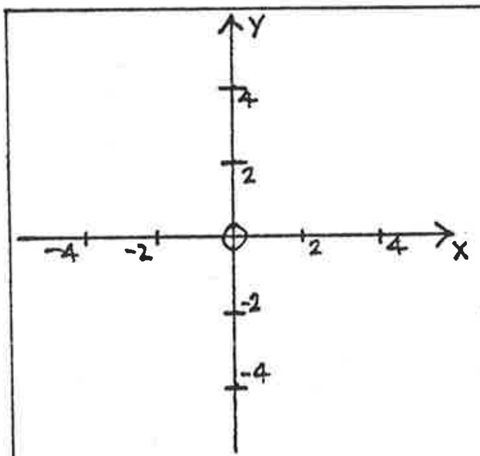
$(-4, -3)$
$(-3, -2)$
$(-1, 0)$
$(1, 2)$

4.



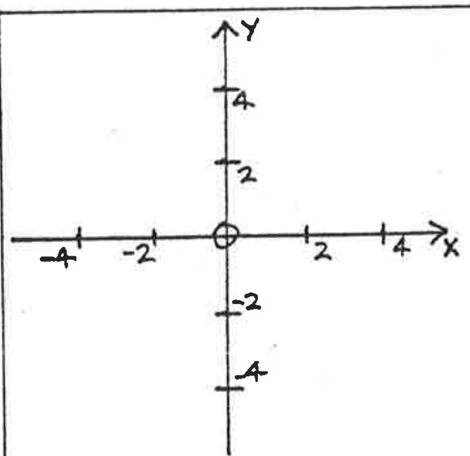
$(2, -1)$
$(2, 0)$
$(2, 2)$
$(2, 3)$

5.



$(-2, 3)$
$(-1, 2)$
$(0, 1)$
$(1, 0)$

6.



$(-4, -3)$
$(-1, -3)$
$(0, -3)$
$(2, -3)$

SCRAMBLED RULES . . .

$x + y = 1$ (or $y = 1 - x$) $\quad y = -\frac{1}{2}x \quad y = -3$

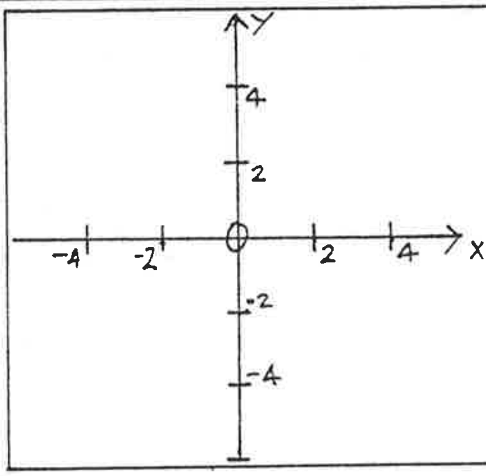
$y = \frac{1}{2}x$ (or $y = \frac{x}{2}$) $\quad y = x + 1 \quad x = 2$

Writing rules for Linear graphs:

Why 3?

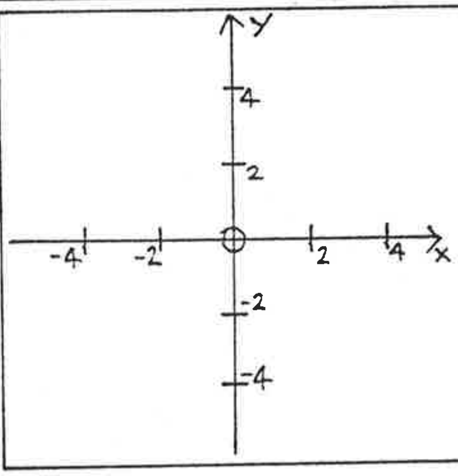
On this side of the sheet:	On this side of the sheet:
(1) Draw the graph	(1) Plot 3 points that fit the rule
(2) What is the gradient of the graph?	(2) Draw the graph
(3) Write the RULE for the graph	(3) What is the gradient of the graph?
(4) Now draw the REFLECTION of the graph. (Just sketch it: don't plot points)	(4) Sketch the REFLECTION of the graph.
(5) What is the gradient of the new graph?	(5) What is the gradient of the new graph?
(6) Write the RULE for the new graph.	(6) Write the RULE for the new graph.

1.



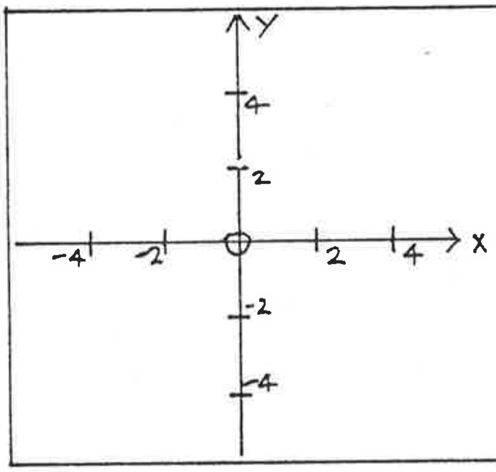
$(-4, -6)$
$(-2, -3)$
$(1, \frac{3}{2})$

4.



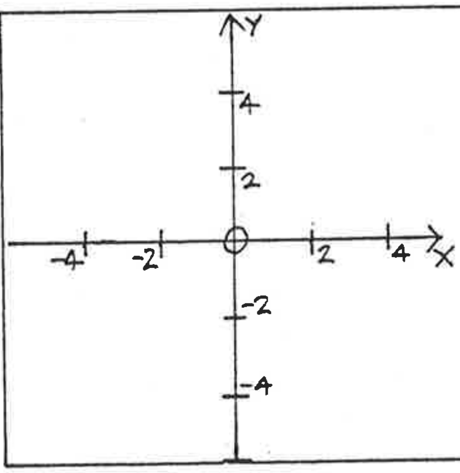
Rule:
$y = 2x$
(in each ordered pair, the second number must be twice the quantity of the first)

2.



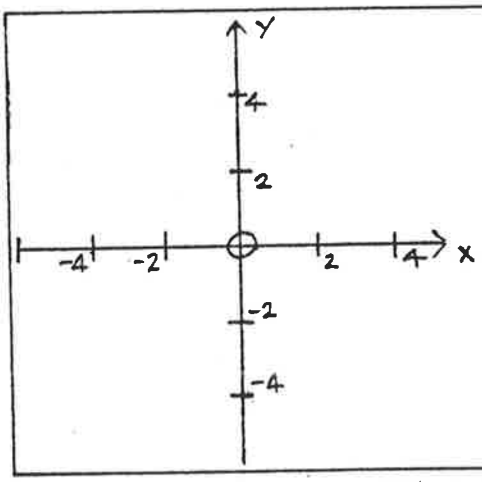
$(-4, -3)$
$(2, \frac{3}{2})$
$(4, 3)$

5.



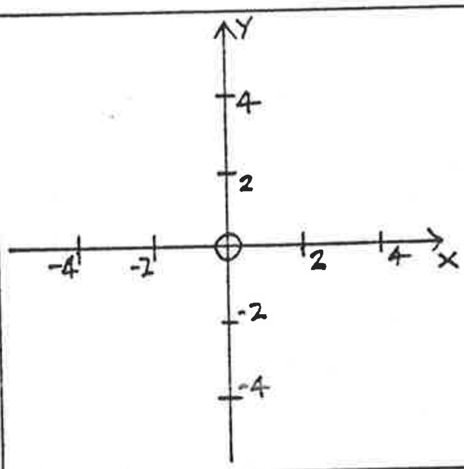
Rule:
$y = 3x$

3.



$(-6, -4)$
$(-3, -2)$
$(3, 2)$

6.



Rule:
$y = \frac{5x}{2}$

Do you notice a connection between the gradient and the rule??

Do you notice connections between a graph and its reflection?

check P12

— EQUATIONS of STRAIGHT LINES —

– looking at gradients . . .

Working from students' prior knowledge

1. Find out students' understanding of the relationship between gradient and equation.

   a. Ask students to sketch on graph paper (probably on the same axes) their understanding of the graphs

      $y = x$
      $y = 2x$
      $y = 3x$
      $y = 4x$

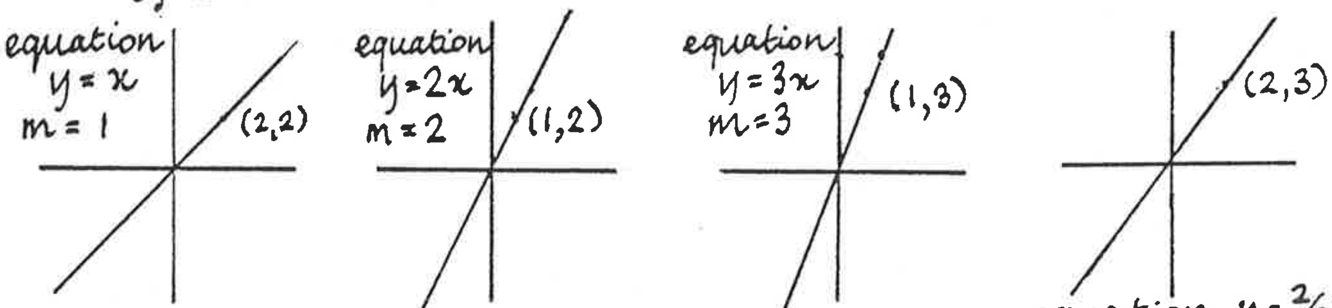
   b. Ask several students to sketch their representations on the board

   c. Teacher and students discuss the pros and cons of the possibilities suggested.

   d. If necessary, an accurate plotting of ordered pairs that satisfy the given equations will show conclusively the correct representations.

2. Make explicit the relationship between gradient and equation.

   Investigate and talk about the way that a linear equation can reveal what the gradient of the line is.



equation $y = \frac{2}{3}x$
m is ?

3. Quiz: to provide immediate, supportive feedback.

(1)

Equation	Gradient
$y = 5x$	
$y = \frac{1}{2}x$	
$y = -3x$	
$y = \frac{x}{4}$	
$y = -\frac{5x}{2}$	

(2)

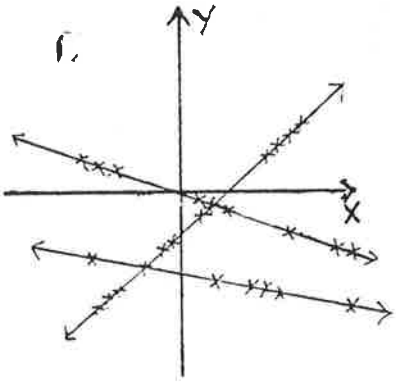
Equation	Gradient
$y = -\frac{x}{3}$	
$2y = x$	
$x + y = 0$	
$y - 4x = 0$	
$-x = y$	

(3)

Equation	Gradient
$3x + y = 0$	
$x - 4y = 0$	
$2x + 5y = 0$	
$3x - 2y = 0$	
$5y - 2x = 0$	

Classroom Talk:
Investigating equations of straight lines
Ideas for an interactive teaching sequence

1\.

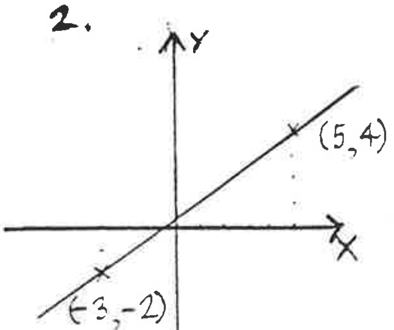


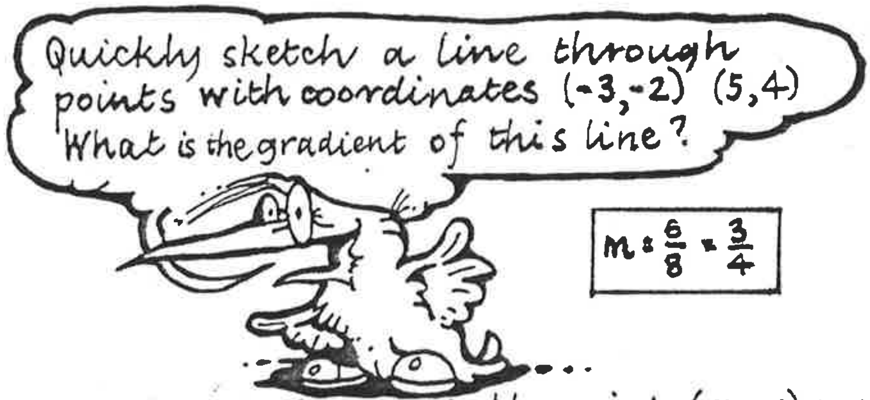
If you draw a straight line on a cartesian plane, then the coordinates of every point on that line will satisfy a particular rule. This rule is called the equation of the line.

In the previous pages on straight line graphs you were given lists of ordered pairs that showed fairly clearly the linear rule (that is, what the equation is)

What you need now is a method of finding the equation of a line that doesn't require a list of ordered pairs – a method that is more general (one that covers a whole range of situations.)

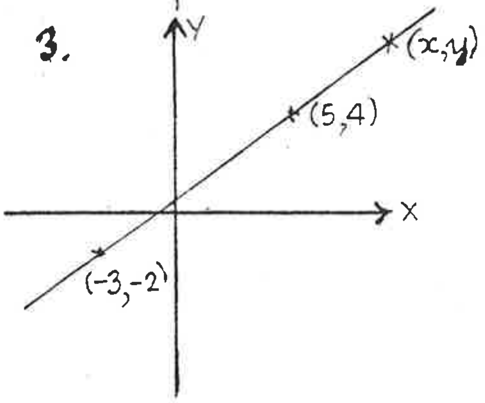
2\.





$\boxed{m = \frac{6}{8} = \frac{3}{4}}$

3\.



Now mark a variable point (x, y) anywhere on the line. (Note that the 3 points given are collinear)

To find the gradient of a line you only need two points, so now you are able to write the gradient in 3 different ways:

① $\dfrac{4+2}{5+3} = \dfrac{3}{4}$ ② $\dfrac{y-4}{x-5}$ ③ $\dfrac{y+2}{x+3}$

specific number

using variables

Because they all refer to the gradient of the same line, they must be equal.

So, construct an equation using the specific number, and selecting either of the variable forms.

$$\frac{y-4}{x-5} = \frac{3}{4}$$

...continued...

Investigating equations of straight lines (continued)

3. (continued)

$\Rightarrow 4(y-4) = 3(x-5)$

distributive law $\quad 4y-16 = 3x-15$

re-form into an accepted way of writing an equation $\quad 3x-4y+1=0$ is the equation of the line passing through $(-3,-2)$ and $(5,4)$.

* see below

PLEASE NOTE:

(1) When you mark (x,y) on a straight line it means that as x and y vary they must be related according to the rule (equation) of that line.

(2) If you choose (x,y) to be the coordinates of the variable point, then the equation of the line will be in terms of x and y. If you choose a point with coordinates (p,q), then the equation will be given in terms of p and q.

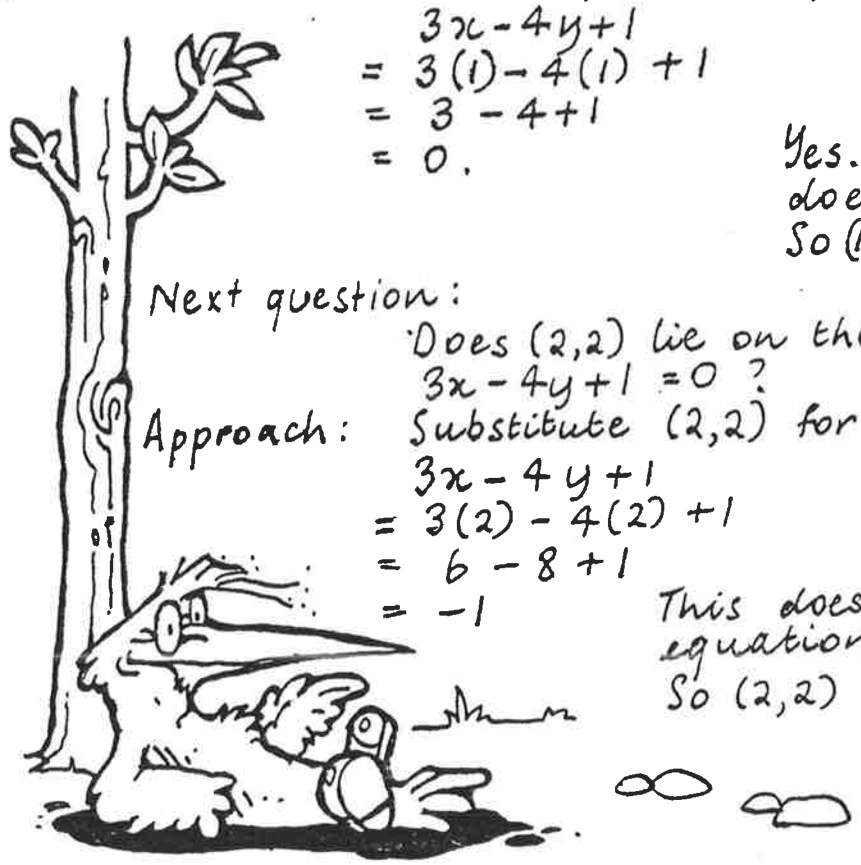
4. The straight line with equation $3x-4y+1=0$ passes through $(-3,-2)$ and $(5,4)$.

QUESTION: Does the point $(1,1)$ lie on this line?

Approach: Substitute $(1,1)$ for the variable point (x,y) – that is, let $x=1$, $y=1$

$$\begin{aligned} &3x-4y+1 \\ &= 3(1)-4(1)+1 \\ &= 3-4+1 \\ &= 0. \end{aligned}$$

Yes. The ordered pair $(1,1)$ does satisfy the equation. So $(1,1)$ does lie on the line



Next question:

Does $(2,2)$ lie on the line with equation $3x-4y+1=0$?

Approach: Substitute $(2,2)$ for (x,y)

$$\begin{aligned} &3x-4y+1 \\ &= 3(2)-4(2)+1 \\ &= 6-8+1 \\ &= -1 \end{aligned}$$

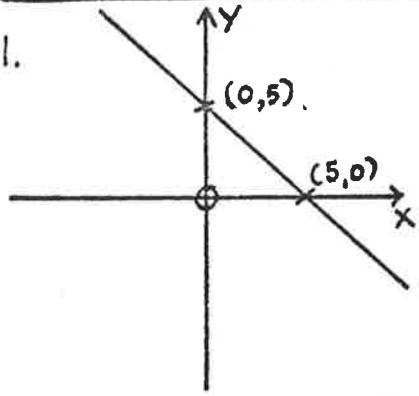
This does not match the equation $3x-4y+1=0$. So $(2,2)$ does not lie on the line.

* $3x-4y+1=0$ is called the GENERAL FORM of a linear equation. (x value, then y value, then constant:

EQUATIONS : Work Sheet

Find the equation of each of these lines.
(Hint: first plot a variable point (x,y) anywhere on the line)

1.



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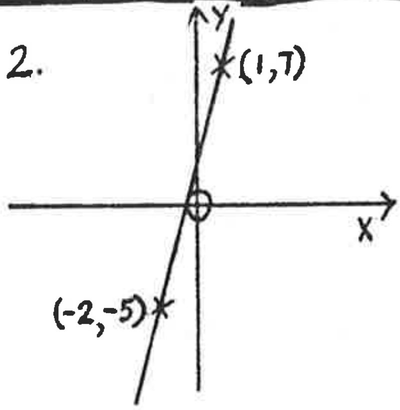

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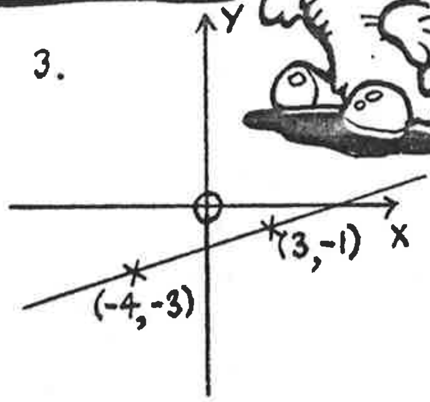

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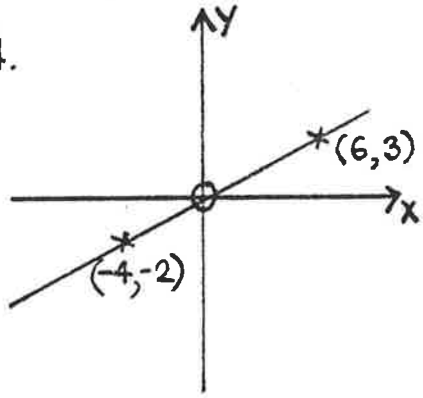

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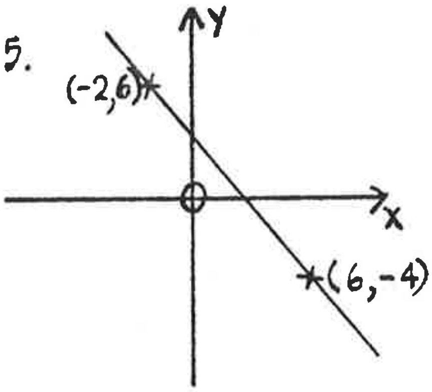

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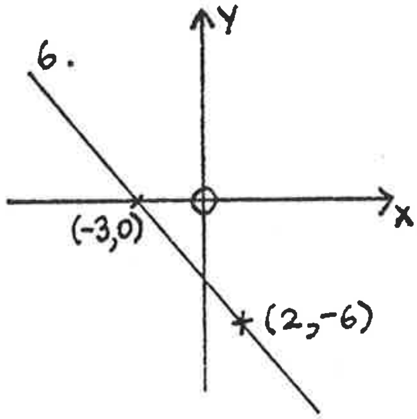

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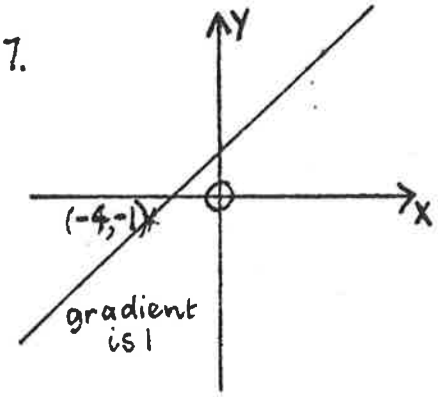

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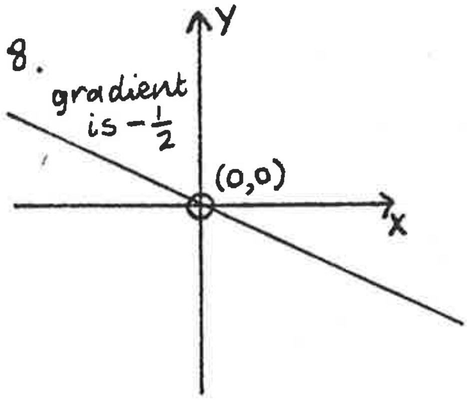

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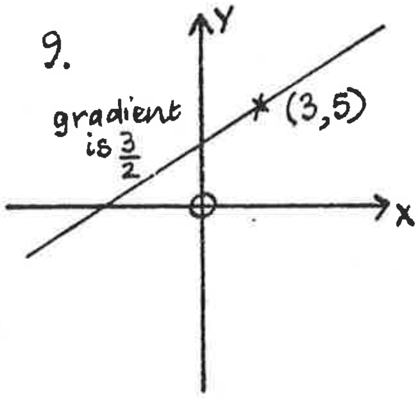
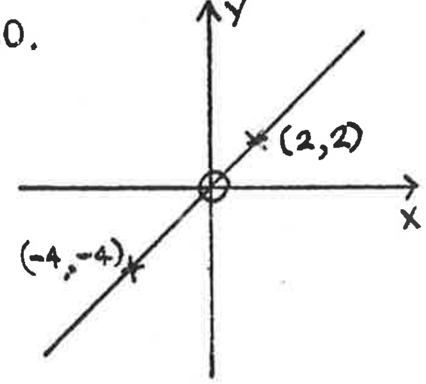


The variable point shouldn't be necessary in these...

10.



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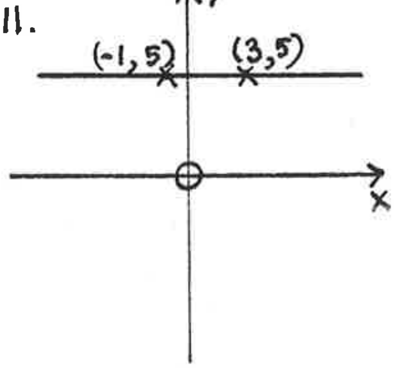

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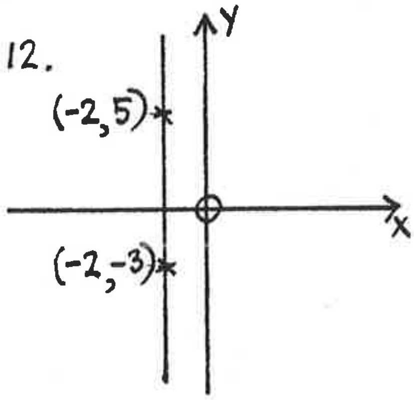


... go to page 129

EQUATIONS of straight lines : points on a line

Quiz 1

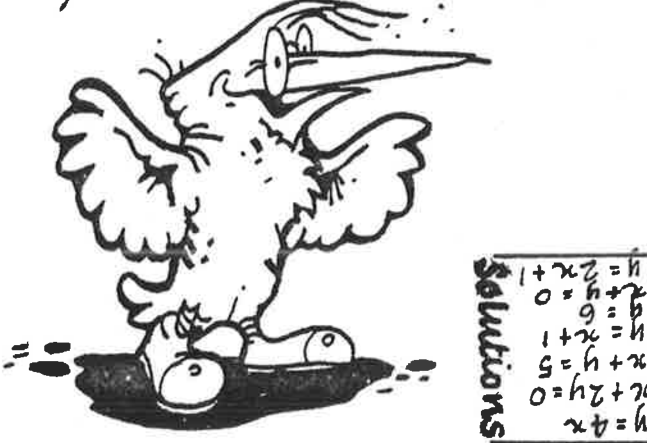
In this quiz some equations are given. For each, find ordered pairs that satisfy the equation.

1. $2x - 3y + 4 = 0$
   a. $(0, \ )$ c. $(\ , 2)$
   b. $(\ , 0)$ d. $(4, \ )$
2. $y = 4$
   a. $(0, \ )$ c. $(-3, \ )$
   b. $(-4, \ )$ d. $(\ , 4)$
3. $-3x + 2y = 0$
   a. $(0, \ )$ c. $(\frac{4}{3}, \ )$
   b. $(-4, \ )$ d. $(\ , \frac{9}{2})$

Quiz 2

For each, find whether the points listed lie on a line with the equation given

1. $4x - 2y = 6$ $(1,2)$ $(2,1)$
2. $x - y = 2$ $(5,-3)$ $(5,3)$
3. $5x + 2y - 4 = 0$ $(-2,-3)$ $(0,2)$
4. $y = -3$ $(5,-3)$ $(-3,4)$
5. $x = 5$ $(5,7)$ $(-3,5)$
6. $y = x$ $(3,3)$ $(-2,-2)$



Quiz 3

1. If $(1,2)$ lies on $2x + y = d$, find d.
2. If $(3,-4)$ lies on $x + 3y - c = 0$, find c.
3. If $(-2,1)$ lies on $y = 3x + b$ find b
4. If $(-1,-2)$ lies on $y = mx + 5$ find m
5. If $(5,3)$ lies on $ax - 4y = 8$ find a

Quiz 4

If these points lie on the same straight line, what is the equation of the line?

1. $(1,4)$ $(2,8)$ $(3,12)$ $(4,16)$
2. $(-4,2)$ $(-2,1)$ $(2,-1)$ $(4,-2)$
3. $(0,5)$ $(1,4)$ $(2,3)$ $(3,2)$
4. $(-5,-4)$ $(-3,-2)$ $(1,2)$ $(2,3)$
5. $(-3,6)$ $(-2,6)$ $(0,6)$ $(2,6)$
6. $(-3,3)$ $(0,0)$ $(2,-2)$ $(4,-4)$
7. $(-3,-5)$ $(-2,-3)$ $(0,1)$ $(1,3)$ $(4,9)$

Quiz 5 (a fast quiz)

Decide whether each ordered pair is a solution of the equation written next to it.

1. $(1,0)$: $x + y = 1$	5. $(4,3)$: $x - y = 2$
2. $(-1,2)$: $x + y = 1$	6. $(1,-1)$: $x - y = 2$
3. $(\frac{1}{2}, \frac{1}{2})$: $x + y = 1$	7. $(1,-3)$: $y = -3$
4. $(1,2)$: $x + y = 1$	8. $(-2,2)$: $x + y = 0$

Solutions

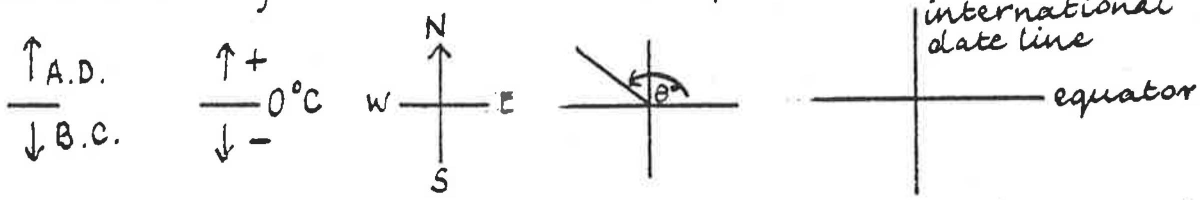
1. $(0,-\frac{4}{3})$ $(-2,0)$ $(1,2)$ $(4,4)$
   $(0,4)$ $(-4,4)$ $(-3,4)$ (varying value, 4)
   $(0,0)$ $(-4,-6)$ $(\frac{4}{3},2)$ $(3,\frac{9}{2})$
2. $(2,1)$ $(5,3)$ $(0,2)$ $(5,-3)$ $(5,7)$ both
3. $d=4$ $c=9$ $b=7$ $m=1$ $a=4$
4. $y = 4x$
   $x + 2y = 0$
   $x + y = 5$
   $y = x + 1$
   $y = 6$
   $x + y = 0$
   $y = 2x + 1$

EQUATIONS of STRAIGHT LINES using students' intuitive ideas

LOOKING at y-intercepts

1. **Find out students' understanding of the use of a reference line (or reference point) in maths.**

Reference lines and points are commonly used to indicate in a scale a starting point or different direction.
Well known reference lines and points are :

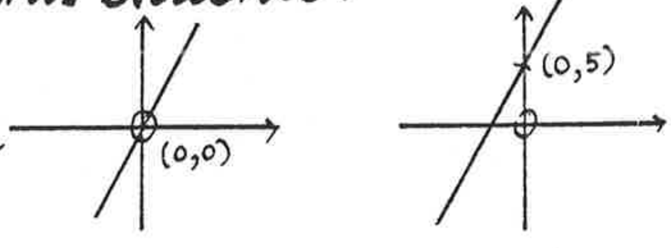


On a cartesian plane the reference lines are the x-axis and the y-axis, and the reference point is the origin

2. **Find out students' intuitive understanding of linear representation, with reference to the y-axis.**
   a. Ask students to sketch $y = 2x$
   b. Ask students to imagine how $y = 2x + 3$ would be represented. Would it have the same gradient? Would it go through the origin?
   c. Ask students to <u>sketch</u> (on graph paper) their <u>reasonable</u> <u>representation</u> of $y = 2x + 3$.
   d. Some students can sketch their reasonable view of what $y = 2x + 3$ might look like on the board. All suggestions must be considered valuable; they are all reasonable attempts.

3. **Refer intuitive ideas to accurate evidence**
   a. Students plot points (from a table, if necessary) to produce an accurate representation of $y = 2x + 3$
   b. Note explicitly : **gradient** is the same as $y = 2x$
   graph passes through the y-axis not at origin but at $y = 3$.

4. **Generalise from this evidence.**
   a. Sketch $y = 2x + 5$
   b. In effect the same line ... but positioned differently with reference to y-axis

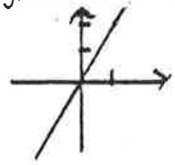


EQUATIONS of STRAIGHT LINES
– looking at y-intercepts

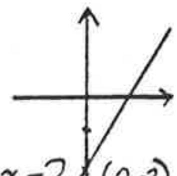
4. c. Ask students to sketch $y = 2x + 1$
$y = 2x - 2$
$y = 2x - 5$

The significance of * gradient
* y-intercept
should be made explicit

NOTE: Some students find it difficult to sketch a line with a given gradient
Try this: sketch $y = 2x$



now translate the line.



$y = 2x - 2$ $(0, -2)$

5. Generalise to lines with other gradients. Quiz method (with immediate feedback)
Sketch and/or talk about

① $y = -x$
$y = -x + 1$
$y = 1 - x$
$y = -x + 3$
$y = -x - 5$
$x + y + 2 = 0$

② $y = \frac{1}{2}x$
$y = \frac{1}{2}x + 3$
$y = \frac{x}{2} - 4$
$2y = x + 4$
$2y = x - 5$

③ $y = -4x$
$y = -4x + 5$
$y = -4x - 1$
$4x + y = 2$
$4x + y + 3 = 0$

④ $y = -\frac{1}{3}x$
$3y = -x$
$3y = -x + 6$
$3y = -x - 9$
$x + 3y + 3 = 0$

⑤ $y = \frac{4}{5}x$
$y = \frac{4}{5}x + 3$
$y = \frac{4}{5}x - 1$
$5y = 4 + 20$
$5y = 4x - 15$

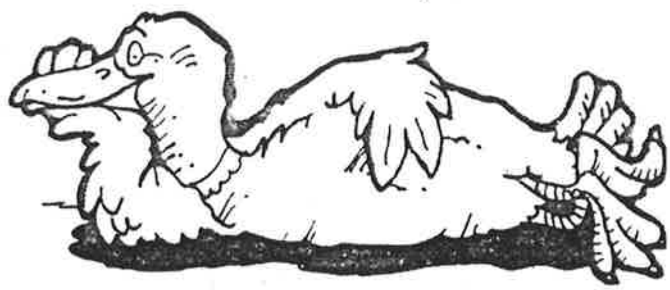
⑥ $y = \frac{7}{3}x$
$y = \frac{7}{3}x + 2$
$y = \frac{7}{3}x - 4$
$3y = 7x + 15$
$3y = 7x - 6$

* Some students may prefer to work on ④, ⑤, ⑥ alone; others may continue to work with the teacher.

~ EQUATIONS of STRAIGHT LINES ~

WORK SHEET : gradient; y-intercept (where the graph cuts the y-axis)

1. Sketch these straight line graphs.
   (1) gradient 2, y intercept 3
   (2) gradient $\frac{1}{2}$, y intercept -4
   (3) gradient $\frac{5}{2}$, y intercept -1
   (4) gradient 5, y intercept -3
   (5) gradient $\frac{5}{3}$, y intercept 2



2. For each of these linear equations write
   a. the gradient
   b. the y-intercept

   (1) $y = -\frac{1}{2}x + 3$
   (2) $y = \frac{x}{3} - \frac{1}{2}$
   (3) $y = 3 - 2x$
   (4) $2y = 3 - 2x$
   (5) $2y = x - 4$
   (6) $3y = 8 - 3x$
   (7) $x + y = 2$
   (8) $x - y = 2$
   (9) $x + y + 3 = 0$
   (10) $4x + 2y + 1 = 0$

3. Which of these linear equations have the same gradients.
   (Make a prediction just by looking at the equations; check by analysing)
   (1) $2x - 3y = 3$
   (2) $3x - 2y = 5$
   (3) $3y + 4 = 2x$
   (4) $y = 8 + \frac{2}{3}x$
   (5) $\frac{2x+5}{3} = y$
   (6) $\frac{1}{3}(y+3) = 2x$
   (7) $\frac{1}{3}(2x - 9) = y$

4. For each of these linear equations write
   a. the gradient of the line
   b. the gradient of the perpendicular to the line.

   (1) $3y = x + 1$
   (2) $2x + y = 5$
   (3) $2x + y + 1 = 0$
   (4) $3x - y = 6$
   (5) $\frac{5+x}{3} = y$

5. Say whether or not these pairs of lines are perpendicular.
   (1) $y = -3x + 1$ $\quad y = \frac{1}{3}x + 1$
   (2) $y = -2x - 1$ $\quad y = \frac{x}{2} + 4$
   (3) $2x - y = 3$ $\quad x + 2y = 5$
   (4) $3x + 2y = 2$ $\quad 2x - 3y = 7$

2. $-\frac{1}{2}, 3$ $\quad \frac{1}{3}, -\frac{1}{2}$ $\quad -2, 3$ $\quad -1, \frac{3}{2}$ $\quad \frac{1}{2}, -2$
   $-1, \frac{8}{3}$ $\quad -1, 2$ $\quad 1, -2$ $\quad -1, -3$ $\quad -2, -\frac{1}{2}$
3. gradients are: $\frac{2}{3}$ $\frac{3}{2}$ $\frac{2}{3}$ $\frac{2}{3}$ $\frac{2}{3}$ $6\frac{2}{3}$
4. $\frac{1}{3}, -3$ $\quad -2, \frac{1}{2}$ $\quad -2, \frac{1}{2}$ $\quad 3, -\frac{1}{3}$ $\quad \frac{1}{3}, -3$
5. gradients are
   $-3, \frac{1}{3}$ $\quad -2, \frac{1}{2}$ $\quad 2, -\frac{1}{2}$ $\quad -\frac{3}{2}, \frac{2}{3}$ all perpendicular

EQUATIONS of STRAIGHT LINES ~

Work Sheet : gradient / y-intercept (continued)

6. A line has an equation
   $2x - 3y + 4 = 0$
   Say whether the following lines are parallel to / perpendicular to / or the same line as the given line.
   (1) $y = 2x + 3$
   (2) $\frac{y-1}{x} = \frac{2}{3}$
   (3) $y = \frac{4-3x}{2}$
   (4) $4x - 6y + 8 = 0$
   (5) $2x = 3y + 4$
   (6) $y = \frac{2x+4}{3}$
   (7) $3x + 2y = 5$

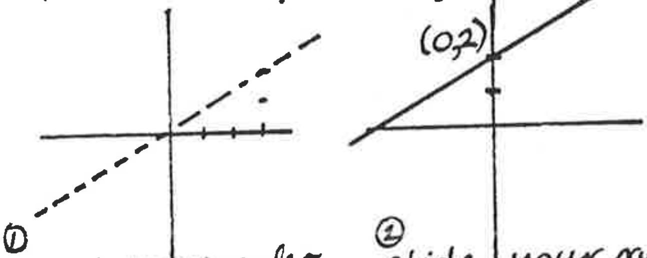
7. If 2 parallel lines have equations
   (1) $y = 3x + 2$
   $y = (k+1)x - 3$
   what must be the value of k?
   (2) $2x + y = 5$
   $4 - y = mx$
   what must be the value of m?
   (3) $3x + 2y + 5 = 0$
   $(t+1)x + y + 2 = 0$
   what must be the value of t?

6. gradients are $\boxed{2/3}$ 2 2/3 −3/2 2/3 2/3 2/3 −3/2
7. k = 2 m = −2 t = ½
8. k = 6 m = −4/3 t = 7/3

8. If 2 perpendicular lines have equations
   (1) $y = 5x - 1$
   $y = (k-1)x + 3$
   what must be the value of k?
   (2) $2x - y = 3$
   $5 - mx = 2y$
   what must be the value of m?
   (3) $4x + 3y + 2 = 0$
   $(t-1)x + y + 3 = 0$
   what must be the value of t?

9. Sketch these graphs
   using only
   ... the gradient
   ... the y-intercept
   (1) $y = \frac{2}{3}x + 2$ (see sketch below)
   (2) $y = 3x - 1$
   (3) $y = -3x + 5$
   (4) $3y = x + 3$
   (5) $4y = 3x + 12$
   (6) $x + y = 4$
   (7) $x + 2y + 3 = 0$
   (8) $5x - 4y = 2$
   (9) $6x - 3y = 1$
   (10) $4x + 2y + 3 = 0$

Reminder : for easy sketching



① hold your ruler at $y = \frac{2}{3}x$

② Slide your ruler (keeping it parallel so that the line passes through the y-axis at (0, 2) : $y = \frac{2}{3}x + $

EQUATIONS of STRAIGHT LINES : Work Sheet

1. For the following list of linear equations
   (a) give the gradient of each
   (b) say which lines are
   parallel
   perpendicular
   the same line

(1) $y = \frac{4}{5}x$

(2) $3y = 5x$

(3) $y = 3$

(4) $x = 5$

(5) $4x - 5y = 0$

(6) $x + 4 = 0$

(7) $-y = x$

(8) $3x + 5y = 0$

(9) $x - y = 0$

(10) $3y + x = 0$

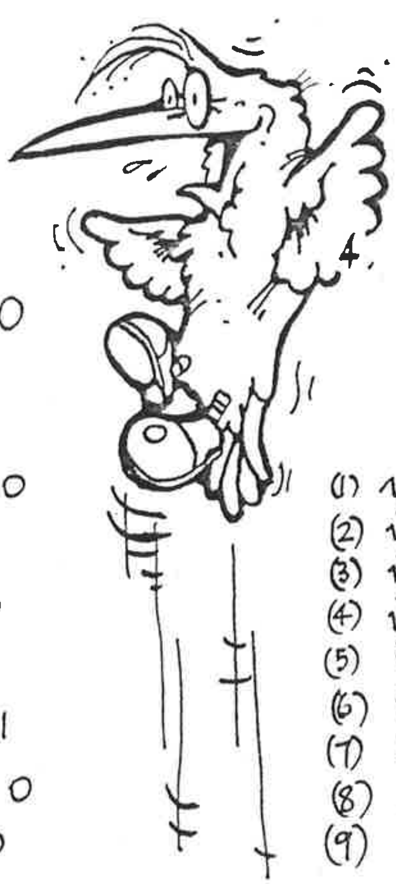
(11) $y = 3x$

(12) $y = 3x + 1$

(13) $3x - 5y = 0$

(14) $2y + 7 = 0$

(15) $-\frac{3}{5}x = y$



2. The gradients of some pairs of lines are given. If the lines are parallel to each other, find the value of t in each of these:

(1) $t+2$, 5

(2) $\frac{t}{4}$, 8

(3) $\frac{t-3}{2}$, 4

(4) 5, $\frac{2t+1}{3}$

(5) $\frac{2}{3}$, $\frac{5t-1}{6}$

3. The gradients of some pairs of lines are given. If the lines are perpendicular to each other, find the value of k in each of these.

(1) $4k$, $\frac{1}{4}$

(2) $\frac{k}{3}$, 3

(3) $k+2$, $\frac{1}{2}$

(4) -3, $2k$

(5) $\frac{k-3}{3}$, 1

(6) $\frac{k+4}{2}$, $\frac{-2}{3}$

4. Write equations for these reflected lines. Draw sketches to help you if you wish.
   What is the equation if

(1) $y=x$ is reflected in the Y axis

(2) $y=x$ is reflected in the X axis

(3) $y=\frac{1}{2}x$ is reflected in the X axis

(4) $y=3x$ is reflected in the X axis

(5) $y=2x$ is reflected in the Y axis

(6) $x+y=0$ is reflected in the Y axis

(7) $3y=2x$ is reflected in the Y axis

(8) $2y=3x$ is reflected in the X axis

(9) $y=3x$ is reflected in the line $y=x$

(10) $y=2x$ is reflected in the line $y=x$.

SOLUTIONS

1. $4/5$ $5/3$ 0 undefined $4/5$ undefined -1 $-3/5$ 1 $-1/3$ 3 3 $3/5$ 0 $-3/5$

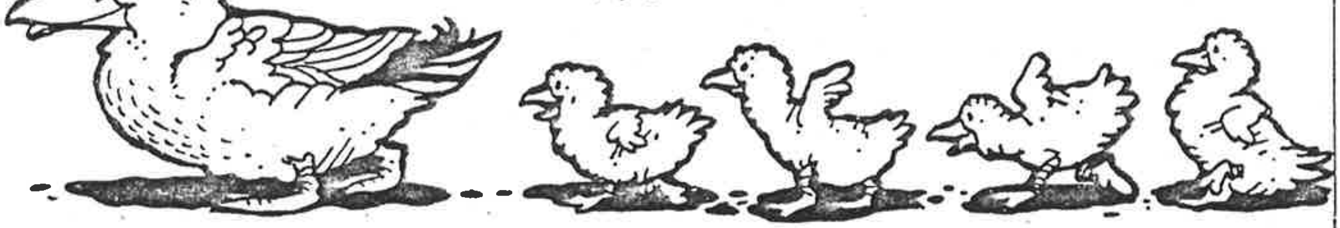
2. $t = 3, 32, 11, 7, 1$

3. $k = -1, 1, -4, 1/6, 0, -1$

4. $y=-x$ $y=-x$ $y=-\frac{1}{2}x$ $y=-3x$ $y=-2x$ $y=x$ $3y=-2x$ $2y=-3x$ $x=3y$ $(y=\frac{1}{3}x)$ $x=2y$ $(y=\frac{1}{2}x)$

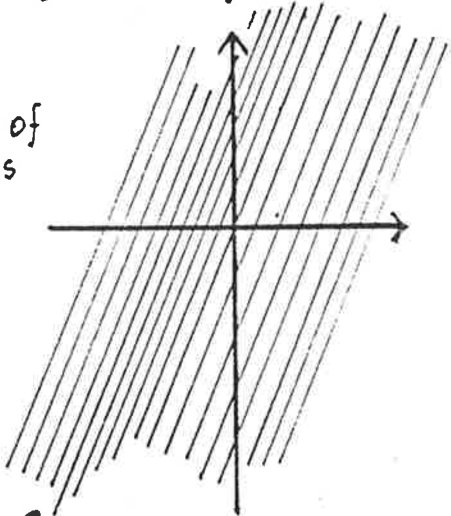
Families of Straight Lines

...Gradients



This family has a gradient of 3.

Every one of these lines has an equation of the form



$$y = 3x + c$$

The gradient must be 3
The y-intercept c varies, depending on the displacement of the line.
(c may be zero, or it may take any positive or negative value)

Group Quiz:

1. In the family of lines with slope 3 $(y = 3x + c)$
   (1) Find the member with y-intercept of 3.
   (Locate it on the graph above, then write the equation)
   (2) Find the equation of the line that passes through the point $(-1,-1)$
   (3) Find the equation of the line that passes through the x-axis at $(-2,0)$.
   (it has an x-intercept 2)

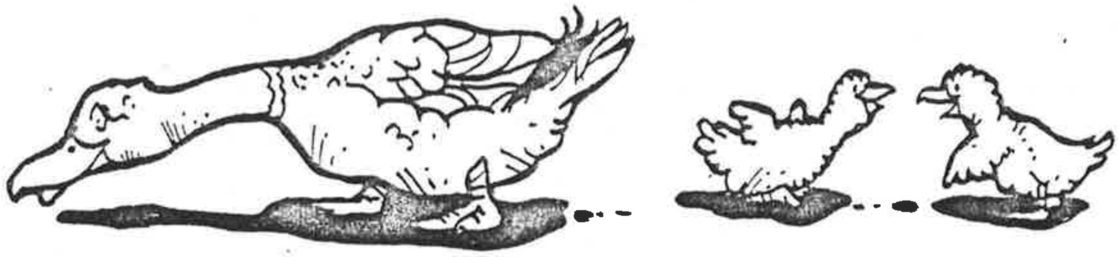
2. Write the equation for the family of lines that has a gradient -2.
   (1) Find the member that has a y-intercept of 5.
   (2) Find the member that has an x-intercept -3
   (3) Find the member that passes through $(-3,7)$

3. Write the equation for the family of lines that has a gradient $\frac{2}{3}$
   (1) Find the equation for the line that has a y-intercept of 3.
   (2) Find the equation for the line with x-intercept of 2
   (3) Find the equation for the line passing through $(3,-2)$

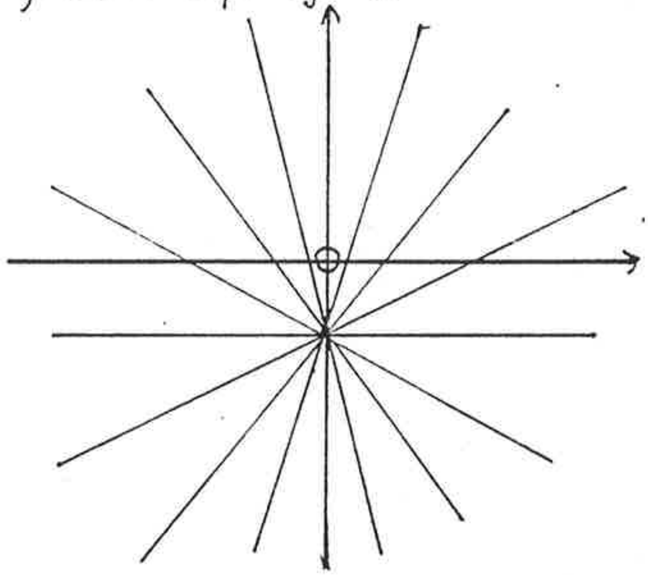
SOLUTIONS:

1. $y = 3x+3$
   $y = 3x+2$
   $y = 3x+6$
2. $y = -2x+c$
   $y = -2x+5$
   $2x+y+6=0$
   $2x+y=1$
3. $y = \frac{2}{3}x + c$
   $y = \frac{2}{3}x - 3$
   or $3y = 2x - 9$
   $3y = 2x - 4$
   $3y = 2x - 12$

Families of Straight Lines
... y-intercepts



This family of lines has a y-intercept of -1.



Every one of these lines has an equation of the form

$$y = mx - 1$$

The line must pass through $(0, -1)$. The gradient varies.

Group Quiz:

1. In the family of lines with y-intercept -1 $(y = mx - 1)$
   (1) Find the member with gradient $\frac{1}{2}$
   (2) Find the equation of the line that passes through the point $(1, 3)$
   (3) Find the equation of the line that passes through $(2, 0)$ (x-intercept 2)

2. Write the equation for the family of lines with y-intercept $\frac{5}{2}$
   (1) Find the member of the family that has gradient -3
   (2) Find the member that has x-intercept 2
   (3) Find the member that passes through $(-2, 5)$

3. Write the equation for the family of lines with y-intercept -2
   (1) Find the equation for the line with gradient $\frac{4}{3}$
   (2) Find the equation for the line with x-intercept -4
   (3) Find the equation for the line passing through $(5, 3)$

SOLUTIONS:

1. $2y = x - 2$
   $y = 4x - 1$
   $2y = x - 2$
2. $y = mx + \frac{5}{2}$
   $6x + 2y = 5$
   $5x + 4y = 10$
   $5x + 4y = 10$
3. $y = mx - 2$
   $4x - 3y = 6$
   $x + 2y + 2 = 0$
   $y = x - 2$

Solutions

P23

1. $8x, 11a, 7t, 5s, 7y$
2. $5xy, 7pq, ab, 7rs, 2yz$
3. $8x+2,\ 3x+7,\ 5+5x, 8+2x, 10x$
4. $7t+3, 7+3t, 9t+1, 10t, 12-2t$
5. $5xy+5y,\ 5xy+4x,\ 5xy+4x+5y$
   $2xy+7x+5y,\ 2xy+4x+8y$
6. $11ab,\ 5ab+6a,\ 5ab+6b$
   $7ab+4a, 2ab+4a$

P24

1. $2x,\ 2x,\ 5x,\ 6x,\ 8x,\ 9x$
2. $y,\ 4y,\ 7y,\ 5y,\ 5y,\ 2y$
3. $5x+3y,\ 5x+3y,\ 5x+3y$
   $5x+3y,\ 5x+3y,\ 2x+y$
4. $5a, 2r/3s,\ 6a/2,\ e/2f, 2x/8, 7/d$
5. $3t,\ 2,\ x/y,\ 6k/2,\ 3/4t, 2b/2a$
6. $7e+8f,\ 2+12a, 10m+3n,\ 9-4a,$
   $5t+3s+2st,\ 5fg+2g-3f$

P25.
1. $16a+3$
2. $9x+4y+1$
3. $7d+20$
4. $7n+26$
5. $13a$
6. $8y+4z$

P26
1. $8s+2$
2. $12t+2$
3. $2y$
4. $4p+6q+12$
5. $4\frac{1}{2}x$ or $\frac{9x}{2}$
6. $\frac{11a}{6}$
7. n
8. $18d+2$
9. $6h$

P28 (top section)

1. $9x, 11x,\ 13x \qquad 2x$
2. $a,\ 0,\ -a \qquad -a$
3. $4r+5,\ 5r+6,\ 6r+7$
4. $2t,\ t-1,\ -2$
5. $\frac{5n}{4},\ \frac{3n}{2},\ \frac{7n}{4}$
6. $3a-4b,\ -5b,\ -3a-6b$

P34

1. $21, 12, 28, 6, 1$
2. $3x+15,\ 60-12b, 42p-14,$
   $4+2a,\ 3y-2.$
3. $5x+15,\ 19+3y, 6a+7,$
   $12t+10,\ 15+3y$
4. $6p-6,\ 6-8a,\ 6-8b,$
   $ab+a,\ 3pq+5p.$
5. $15x+20, 7y+8, 22t+4,$
   $10a+12,\ 15$
6. $7ab+5a,\ pq+9p,$
   $3a+4,\ 3a+4,$
   $5p+3,\ 5p+3$

P37 LHS.

1. $4a+12$
2. $3t-27$
3. $6x+3$
4. $5m-15$
5. $4ab+8a$
6. $5x+2xy$
7. $6m+8mn$
8. $10p-15q+20$
9. $7x+1$
10. $14y+1$
11. $21a+7$
12. $29t+13$
13. $14gh+11g$
14. $2ab+10b$

Solutions

P37 (continued)

1. $a(x+y)$
2. $3(a-2b)$
3. $3(t+1)$
4. $2(p-4q)$
5. $4a(b+2)$
6. $7(2+cd)$
7. $\sqrt{2}(a+b)$

R.H.S.

8. $5(5-ab)$
9. $2x(y-5)$
10. $26 \times 1 = 26$
11. $172 \times 10 = 1720$
12. $156 \times 10 = 1560$
13. $5l(m-5)$
14. $\sqrt{3}(4-x)$

P46

1. 15
2. 9
3. 4
4. $5+g$
5. $8-z$
6. $2a=10,\ a=5$
7. $2p=12,\ p=6$
8. $4s=12,\ s=3$
9. $w=9,\ u=4$
10. $a=2,\ b=8$
11. $5a>15$
12. $\frac{b}{4}<5$
13. $a>3,\ a+b>6$
14. if $p=q$,
    $p=q=4$
    but $p<q$
    so $p<4$
15. $r>6$
16. sometimes, when $c=d$

P54

1. -10, 1, 3, 0, -6, 8
2. -6, -4, 3, -3, -10, 10
3. $-10a$, b, $3c$, 0, $-6e$, $8f$
4. $3x$, $-4y$, $-2t$, $3a$, $-8n$, $-9z$
5. $-6a-1$, $3y-9$, $-5t-5$, $-5+3n$, $-12b-c$, $-5x-2y$
6. -6, -9, -2, -4, -4, -17, -179

P57

1. 72.3
2. 3 mins 54.8 secs
3. \$994.20

P74

1.	-3	1	2
2.	-11	-7	-1
3.	8	6	0
4.	3	2	$-\frac{3}{2}$
5.	-10	0	$\frac{10}{3}$
6.	-13	-5	-1
7.	-5	-2	10
8.	10	4	-2
9.	13	4	-2
10.	-3	$-\frac{3}{2}$	0

P75

1.	35	55	30	45	25	0
2.	2	8	10	6	$\frac{3}{2}$	$\frac{9}{4}$
3.	15	19	14	17	13	8
4.	9	19	29	6	26	36
5.	2	8	10	4	$\frac{3}{2}$	$\frac{9}{4}$
6.	5	6	3	$\frac{5}{2}$	$\frac{15}{2}$	
7.	20	24	12	10	30	
8.	7.7	9.7	11	21	61	
9.	6.5	4.5	5.0	4.8	4.6	
10.	25	10	34	4	19	
11.	33	53	28	43	23	

P76

1. $a+4$
2. $n+3$
3. $k-2$
4. $4x$
5. $4x+1$
6. $3p$
7. $3p-1$
8. $5t$
9. $5t-3$
10. $2y-1$

P77

1. $n+4$
2. $2p$
3. $2k-1$
4. $3t-1$
5. $m-2$
6. $2a+1$
7. $3x+2$
8. $2y-5$
9. d^2
10. $3u+1$

Solutions

P114

1. $y = \frac{3}{2}x$ gradient $\frac{3}{2}$
2. $y = \frac{3}{4}x$ gradient $\frac{3}{4}$
3. $y = \frac{2}{3}x$ gradient $\frac{2}{3}$
4. gradient $2, -2$
5. gradient $3, -3$
6. gradient $\frac{5}{2}, -\frac{5}{2}$

P118

1. $x + y = 5$
2. $y = 4x + 3$
3. $2x - 7y = 13$
4. $x - 2y = 13$
5. $5x + 4y = 14$
6. $6x + 5y + 18 = 0$
7. $y = x + 3$
8. $x + 2y = 0$
9. $3x - 2y + 1 = 0$
10. $y = x$
11. $y = 5$
12. $x = -2$

Appendix 4

THE COURSE: METHODOLOGICAL APPROACHES (Implementation)

What do you see as the aims of the 'Talking Maths' course?

To what extent, do you feel, are they being achieved in your classroom?

In what ways does the 'Talking Maths' book *help you
*fail to help you
maintain and sustain these directions?

Has any other professional support helped you to achieve these aims? If so, please indicate the nature of the support and how it has helped you.

Have you been using the teaching ideas that are suggested in 'Talking Maths'?
If so, which ones have you used, and how have you used them?

To what extent do you feel that the teaching ideas make, or do not make, good sense in your classroom?

To what extent do you personally find them useful/helpful/effective? Or not so?

In what ways (if any) are these approaches different from the ones you normally use?

THE COURSE: THEORETICAL BASIS

Have you read/understood the theoretical ideas presented in the typewritten sections of 'Talking Maths'?

To what extent do you see the theory as relevant/useful in your classroom?

Do you see any links between this theoretical basis and the teaching methods suggested? If so, please say how you think the theory and practice are linked.

To what extent do you see the theory (as outlined) as a support in making your algebra teaching more effective?

PROFESSIONAL ASPECTS

To what extent/In what ways
is the course providing a means of professional development for you?

Does participation in the course place additional pressures on you? In what ways?

Has participation in the course afforded you any benefits professionally?
In what ways?

Has participation in the course caused you to make any changes in your ways of teaching?

If so, in what ways?

Do you feel that participation in the course has increased your awareness or understanding of

algebraic knowledge? Please give particulars.

approaches and methods that may improve the effectiveness of your teaching? Please give particulars.

specific teaching techniques that assist teachers to enhance student learning? Please give particulars.

Do you think the effort involved in participating in the course has been worthwhile?

<u>THE COURSE: ALGEBRA FOR YEAR 9</u>

(Teachers' personal perceptions)

Do you see any differences in the way algebra is presented in 'Talking Maths' compared with the approaches of other textbooks?

In what ways is 'Talking Maths' different?

To what extent is 'Talking Maths' a useful link with

prior algebra (e.g., Year 8 algebra)?

future maths (e.g., algebra/analytic geometry/calculus in the next two or three years)?

In what ways?

'<u>TALKING MATHS: The book and its presentation</u>

In what ways do you like

do you not like (or feel uneasy about)

the format and presentation of the book?

Please comment on the algebra in 'Talking Maths' with reference to

the variety of content and presentation

the usefulness of the algebra chosen

your enjoyment of the work

the level of challenge provided

attention to different ability levels

clarity of the instructions given

THE STUDENTS: TEACHERS' PERCEPTIONS

Could you please comment on your perceptions of student responses to 'Talking Maths', with reference to

enjoyment of the course

attitude to algebra

confidence in doing algebra

interest in the work

making meaning of algebra

talking about algebraic ideas

usefulness of the work

parts that were not understood

Do you feel that the students consider 'Talking Maths' to be different from other algebra text books?

If so, in what ways?

To what extent do you feel that students have benefited
have not benefited
from using 'Talking Maths'?

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