The University of Adelaide

# Seiberg-Witten Monopoles on Three-Manifolds

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# Statement

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# Abstract

Seiberg-Witten monopoles were first discovered by Seiberg and Witten in the low energy computations of the Donaldson invariants from $N = 2$ supersymmetric Yang-Mills theories in dimension 4. They are defined on any compact, connected, oriented 4-dimensional manifold $X$ with a smooth Riemannian metric $g$ and a Spin$^c$ structure. The 3-dimensional Seiberg-Witten monopoles on any compact, connected, oriented 3-manifold $Y$ with a Spin$^c$ structure $\mathfrak{s}$ and a Riemannian metric $g$ are the static (translation invariant) Seiberg-Witten monopoles on the cylinder $Y \times \mathbb{R}$. In this thesis, the properties of these 3-dimensional Seiberg-Witten monopoles, including the associated invariants (the Seiberg-Witten invariants and the Floer monopole invariants) were established. Their applications to 4-dimensional Seiberg-Witten monopoles were also discussed.

In construction of the Floer monopole invariants, we gave a detailed account of the infinite dimensional Morse theory for the Chern-Simons-Dirac functional, including the Lojaszewiz-type inequalities, transversality and compactness for the downward gradient flow lines. For $b_1(Y) > 0$ and $c_1(det(\mathfrak{t})) \neq 0$, these monopole homology invariants are independent of the metrics and the perturbations, while for a homology 3-sphere, we studied the equivariant version of the Seiberg-Witten-Floer homology which is a topological invariance. We also define the Fukaya-type monopole homology groups $HFF_*^{(m)}(Y, \mathfrak{s})$ and a pairing between them. They form the mathematical device to study the gluing formulae for the 4-dimensional Seiberg-Witten invariants.

# Contents

# Chapter 1

# Introduction and Statement of Results

## 1.1 Introduction to Seiberg-Witten monopoles

Seiberg-Witten monopoles were first discovered by Seiberg and Witten in the low energy computations of the Donaldson invariants from $N = 2$ supersymmetric Yang-Mills theories in dimension 4 [39][40]. They are defined on any compact, connected, oriented 4-dimensional manifold $X$ with a smooth Riemannian metric $g$ and a Spin$^c$ structure $\mathfrak{s}$.

For each Riemannian metric $g$, there is a principal $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$ bundle of orthonormal frames: $Fr \longrightarrow X$, with two distinguished associated bundles, the bundle $\Lambda^{2,+}$ of self-dual 2-forms and the bundle $\Lambda^{2,-}$ of anti-self-dual 2-forms such that $\Lambda^2 = \Lambda^2(T^*X) = \Lambda^{2,+} \oplus \Lambda^{2,-}$ corresponds to the isomorphism of Lie algebra decomposition: $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Moreover, $Fr \to X$ has a unique Levi-Civita connection which is compatible with the metric $g$ and torsion free.

A Spin$^c$ **structure** on $(X, g)$ is an equivalence class of lifts of $Fr \to X$ to a principal $\text{Spin}^c(4) = \big(SU(2) \times SU(2) \times U(1)\big)/\mathbb{Z}_2$ bundle $P \to X$. Unlike the spin-structure, where there is an obstruction (the second Stiefel-Whitney class of $Fr$) to a lift of the frame bundle to a $Spin(4)$-bundle, it was proved by Hirzebruch-Hopf that every oriented 4-dimensional Riemannian manifold possesses one Spin$^c$ structure. The set of Spin$^c$ structures on $X$ is an affine space modelled on $H^2(X, \mathbb{Z})$.

For each Spin$^c$ structure $\mathfrak{s}$, there are two $U(2)$-bundles $S^{\pm}$ associated to $P$ defined by using the two obvious homomorphisms $\rho_{\pm}$ of Spin$^c(4)$ to $U(2) = (SU(2) \times U(1))/\mathbb{Z}_2$: write

$$\mathrm{Spin}^c(4) = \left\{ \begin{pmatrix} uA_+ & 0 \\ 0 & uA_- \end{pmatrix} \middle| A_{\pm} \in SU(2), u \in U(1), \right\}$$

then the homomorphisms $\rho_{\pm}$ are given by

$$\rho_{\pm} \begin{pmatrix} uA_+ & 0 \\ 0 & uA_- \end{pmatrix} = uA_{\pm}.$$

We choose $S^{\pm}$ such that the projective bundle $\mathbb{P}(S^+)$ is isomorphic to the unit 2-sphere bundle in $\Lambda^{2,+}$, and similarly, $\mathbb{P}(S^-)$ is isomorphic to the unit 2-sphere bundle in $\Lambda^{2,-}$.

As in the spin geometry, the complex endomorphism bundle of the total spinor bundle $S = S^+ \oplus S^-$ is isomorphic to the complexified Clifford bundle of $(T^*X, g)$. This allows us to represent the Clifford multiplication by

$$\Lambda^1 \otimes \mathbb{C} \cong Hom(S^+, S^-) \cong Hom(S^-, S^+), \Lambda^{2,+} \otimes \mathbb{C} \cong End_0(S^+),$$

where $End_0$ denotes the traceless endomorphism of $S^+$. Since Spin$^c(4)$ is a central extension of $SO(4)$ by $U(1)$, a Spin$^c$ connection on $P \to X$ is determined by the Levi-Civita connection on $Fr \to X$ and a $U(1)$-connection, $A$, on the determinant line bundle $\mathcal{L}$ of the Spin$^c$-bundle $P$, where $\mathcal{L} = P \times_{\pi} \mathbb{C}$ and $\pi : \mathrm{Spin}^c(4) \to U(1)$ is given by

$$\pi \begin{pmatrix} uA+ & 0 \\ 0 & uA_- \end{pmatrix} = u^2.$$

The Dirac operator $\not{D}_A$, which is a first order differential operator mapping sections of $S^+$ to sections of $S^-$, is the composition of the Clifford multiplication with the covariant derivative on $S^+$.

The Seiberg-Witten equations are equations for a pair $(A, \psi)$, where $A$ is a $U(1)$-connection on $det(S^+)$ and $\psi$ is a section of $S^+$, they read:

$$\begin{cases} F^+ = q(\psi \otimes \psi^*) + \eta, \\ \not{D}_A \psi = 0, \end{cases} \tag{1.1}$$

where $F^+$ is the self-dual part of the curvature $F_A$, $q$ is the adjoint of the Clifford multiplication: $\Lambda^{2,+} \otimes \mathbb{C} \xrightarrow{\cong} End_0(S^+)$, $\eta$ in the curvature equation is a fixed, imaginary valued, self-dual 2-form on $X$ serving as a perturbation to achieve the smooth structure of the moduli space $\mathcal{M}(X, \mathfrak{s})$ (the solution space to (1.1) modulo the gauge group $C^\infty(X, U(1))$) with the topology inherited from the orbit space. For a 4-manifold $X$ with $b_2^+ \geq 1$, there exist the Seiberg-Witten invariants coming from the moduli space to the Seiberg-Witten equations (1.1). The programme coming from the Seiberg-Witten equations is now called Seiberg-Witten gauge theory.

One advantage of the Seiberg-Witten gauge theory over the instanton (Yang-Mills) gauge theory comes from the fact that the moduli space $\mathcal{M}(X, \mathfrak{s})$ is always compact. Moreover, for $b_2^+ \geq 1$, for generic perturbation $\eta$ in a Baire set of imaginary valued, self-dual 2-forms, the corresponding moduli space $\mathcal{M}$ is a smooth, orientible manifold with dimension given by

$$2m = \frac{1}{4}\Big(c_1(detS^+)^2 - (2\chi + 3\sigma)\Big) = c_2(S^+) \tag{1.2}$$

where $\chi$ and $\sigma$ are the Euler characteristic and the signature of $X$ respectively, and $c_1(detS^+)^2$ (or $c_2(S^+)$) has been evaluated with the fundamental class $[X] \in H_4(X)$. The orientation on $\mathcal{M}$ is determined by assigning an orientation on

$$H^0(X) \otimes det(H^1(X)) \otimes det(H^{2,+}(X)).$$

We adopt here the definition of the Seiberg-Witten invariants from Taubes's work in [45]. Fix a point $x \in X$, let $\mathcal{G}_x$ be the based gauge group whose gauge transformations take value $1 \in U(1)$ at $x$, then we have a based $U(1)$-fibration $\mathcal{M}_x$ of $\mathcal{M}$, which is the solutions to the Seiberg-Witten equations (Cf. (3.15)) modulo the based gauge group $\mathcal{G}_x$. By varying $x$, we obtain a principal $U(1)$-bundle

$$\mathcal{M} = \cup_{x \in X} \mathcal{M}_x$$

over $X \times \mathcal{M}$. Let $c_1(\mathcal{M})$ be the first Chern class of $\mathcal{M}$, then the slant product (integration along the fiber) defines the $\mu$-maps:

$$\mu : H_1(X, \mathbb{Z}) \longrightarrow H^1(\mathcal{M}, Z), \qquad \mu : H_0(X, \mathbb{Z}) \longrightarrow H^2(\mathcal{M}, \mathbb{Z}).$$

For any $\gamma_1 \wedge \cdots \wedge \gamma_p \in \Lambda^p(H_1(X,\mathbb{Z}))$ with $2m-p$ non-negative and even, we can assign

$$\int_{\mathcal{M}} \mu(\gamma_1) \wedge \cdots \mu(\gamma_p) \wedge \Big(\mu([pt])\Big)^{\frac{2m-p}{2}} \in \mathbb{Z}.$$

This defines the Seiberg-Witten invariant $SW_X(\mathfrak{s}) \in \Lambda^*(H^1(X,Z))$, which is a differential invariant (independent of the metric $g$ and the perturbation $\eta$) for $b_2^+ > 1$. In particular, when $X$ is simply-connected, $SW_X(\mathfrak{s})$ is a $\mathbb{Z}$-valued invariant, which can be rewritten as:

$$SW_X(\mathfrak{s}) = \begin{cases} 0 & \text{if } 2m < 0 \text{ or } 2m \text{ is odd,} \\ [\mathcal{M}] & if\ 2m = 0, \\ \int_{\mathcal{M}} \mu([pt]^m) & \text{if } 2m \text{ is positive and even.} \end{cases}$$

Immediately after the inception of the Seiberg-Witten equations, Kronheimer and Mrowka [26] applied these monopole equations to confirm an outstanding conjecture, the Thom conjecture, which asserts that any smooth Riemann surface in $\mathbb{CP}_2$ of degree $d > 0$ must have genus at least $(d-1)(d-2)/2$. A symplectic version of this conjecture was also verified by Morgan-Szabo-Taubes [38]. The most remarkable result coming out of the Seiberg-Witten monopoles is Taubes' theorem "Seiberg-Witten invariants = Gromov-Witten invariants" for a compact, closed symplectic 4-manifold, where the Gromov-Witten invariant is defined by careful counting of certain pseudo-holomorphic curves in the symplectic 4-manifold.

In [26], the Seiberg-Witten monopole equations on a 3-manifold $(Y, \mathfrak{t})$ with a Spin$^c$ structure $\mathfrak{t}$ are derived from the static monopoles on the cylinder $Y \times \mathbb{R}$. In fact, they introduced the Chern-Simons-Dirac functional on the monopole configuration space:

$$C(A,\psi) = -\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \int_Y \langle \psi, \partial\!\!\!/_A \psi \rangle dvol_Y.$$

where $A_0$ is a fixed $U(1)$-connection, $A$ is a connection, $\psi$ is a spinor. The 3-d Seiberg-Witten monopoles appear to be the critical points of the Chern-Simons-Dirac functional. Moreover, the downward gradient flow of this function is the Seiberg-Witten monopole equations on $Y \times \mathbb{R}$ under the temporal gauge (where the $\mathbb{R}$-component of the $U(1)$ connection vanishes).

Since then, there has been steady progress by adapting Floer's idea on his instanton homology to the monopole case [22] [38][27]. There are also intensive investigations of the Seiberg-Witten monopoles on the 3-manifold $Y$, among these, the work of Auckly on the Thurston norm [4] [5], Lim and Chen's work on surgery formulae [29] [14], with the remarkable announcement of Meng-Taubes on their "3d SW = Milnor torsion" [34], which sparked a number of new discoveries about 4-manifolds [19].

The construction of Floer monopole homology seems to be straight forward, but there are many subtleties and analytic difficulties which need care. For example, for a 3-manifold $Y$ with $b_1 > 0$ and with trivial Spin$^c$ structure $\mathfrak{s}$ (the first Chern class is zero), there is no nice way to formulate the corresponding monopole homology. Notice that the Chern-Simons-Dirac functional is an $\mathbb{R}$-valued function on the configuration space (pairs of $U(1)$ connections and spinors modulo the gauge transformations) as long as the perturbation term represents a zero de Rham cohomology class. Unfortunately, the critical points admit a reducible critical set which is a torus $T^{b_1}$ with possible degenerate critical points even for the based configuration space (where the gauge group is the restricted group whose element value $1 \in U(1)$ at a fixed point in $Y$) and in the sense of Bott-Morse. If one tries to perturb the Chern-Simons-Dirac functional by a co-closed one form representing a non-trivial cohomology class, one can get finitely many irreducible critical points which formally generate the Floer chain complexes. The problem arises out of the behaviour of the time independent gradient flow line from one critical point to another. There is no control over the energy along the trajectories due to the ambiguity of the Chern-Simons-Dirac functional, which is not $\mathbb{R}$-valued anymore. So one can't achieve a uniform energy bound for the component of the trajectory moduli space with fixed dimension (Cf. Lemma 3.1.17), hence one can't obtain the compactness or a compactification by adding broken trajectories. Even if one uses Novikov's idea of introducing a power series to distinguish the energy for the component of the trajectory space with fixed dimension, there are some problems relating the dependence of the Novikov ring on the perturbations. This problem is still unclear though the study of the Seiberg-Witten monopoles in this case (eg. on $T^3$) brought new understandings of four-manifolds ([37] [41]).

In this thesis, we settle some of the key analytical pionts of the construction of the Seiberg-Witten-Floer homology theory, and include a brief discussion of their applications.

## 1.2 Statements of the results

Most of Chapter 2 is devoted to giving a self-contained treatment of the Seiberg-Witten monopoles on any closed, oriented 3-manifold $Y$. Some of these were also studied in [4][5] [6] [13] [22] [29]. The author's main contributions here are the detailed studies of the stratification structure of the space of metrics and perturbation $\nu$ with non-trivial kernel for the twisted Dirac operator $\partial\!\!\!/^g_\nu$ and the geometric proof of the wall-crossing formulae for the Seiberg-Witten invariant for a homology 3-sphere.

Section 2.1 sets the notations for this thesis, we introduce the Spin$^c$ structure on a 3-manifold $Y$and give some detailed calculations for the Seiberg-Witten equations. In Section 2.2, we establish the main properties of the Seiberg-Witten monopoles on $Y$ used in this thesis, some of these were also obtained by other researchers in [4] [13] [22] [33] [17] [29] [27].

**Theorem 1.2.1.** *Let $(Y, g, \mathfrak{s})$ be a closed, Riemannian 3-manifold with a* Spin$^c$ *structure $\mathfrak{s}$, then the Seiberg-Witten moduli space for (2.9) has the following properties.*

*(a) The monopole moduli space $\mathcal{M}_{\mathfrak{s},\eta}$ is sequentially compact.*

*(b) Let $(A, \psi)$ be a smooth solution to the perturbed Seiberg-Witten equations (2.3), then $|\psi|^2 \leq \max_{y\in Y}\{0, -s(y) + 2\|\eta\|_{C^0}\}$ where $s(y)$ is the scalar curvature for $(Y, g)$.*

*(c) There exists a Baire set of the co-closed, imaginary valued one forms such that for any perturbation $\eta$ in this set, $\mathcal{M}^*_{\mathfrak{s},\eta}$ is a smooth, oriented, 0-dimensional manifold.*

*(d) For a 3-manifold $Y$ with $b_1(Y) > 0$, for generic $\eta$, $\mathcal{M}_{\mathfrak{s},\eta}$ consists only finitely many irreducible monopoles, moreover the perturbation $\eta$ can be chosen to*

*represent a trivial cohomology class if* $c_1(det(\mathfrak{s})) \neq 0$ *in* $H^2(Y, \mathbb{R})$.

(e) *For a rational homology 3-sphere* $Y$ *(that is* $b_1(Y) = 0$*), if the generic* $\eta = *d\nu$ *satisfies* $Ker(\not{\partial}_\nu) = 0$, *then* $\mathcal{M}_{\mathfrak{s},\eta}$ *consists of only finitely many irreducible monopoles with a unique, isolated, reducible solution* $[\nu, 0]$. *The condition* $Ker\not{\partial}^g_\nu \neq 0$ *determines a subset in the space of metrics and perturbations with real codimension one.*

(f) *There is an invariant associated with the monopole moduli space* $\mathcal{M}_{\mathfrak{s},\eta}(Y, g)$, *which is a topological invariant when* $b_1 > 1$ *and, after a small generic perturbation* $\eta$, *when* $b_1 = 1$.

(g) *The metric and perturbation dependence of the Seiberg-Witten invariant for a* Spin$^c$ *structure* $\mathfrak{s}$ *on a homology 3-sphere* $Y$ *is given by the following wall-crossing formula:*

$$SW_Y(g_1, \nu_1) = SW_Y(g_0, \nu_0) + SF(\not{\partial}^{g_t}_{\nu_t})$$

*where* $SF(\not{\partial}^{g_t}_{\nu_t})$ *is the spectral flow of the twisted Dirac operator along a family of metrics and perturbations which connect* $(g_0, \nu_0)$ *and* $(g_1, \nu_1)$ *in the space of metrics and perturbations.*

It is worthwhile to point out that when $b_1(Y) > 0$, and $c_1(det(\mathfrak{s})) \neq 0$, the perturbation can be chosen to represent a trivial cohomology class. This is vital for the construction of the Seiberg-Witten-Floer homology.

Chapter 3 is the main analytical part of this thesis, where we give a detailed account of the infinite dimensional Morse theory for the Chern-Simons-Dirac functional. We also construct the Seiberg-Witten-Floer homology groups including the equivariant version of the Seiberg-Witten-Floer homology in the case of homology 3-spheres.

The Chern-Simons-Dirac functional $C$ on the infinite dimensional configuration space $\mathcal{B}$ (pairs of $U(1)$-connections and spinors modulo the full gauge group), can be perturbed to achieve finitely many critical points which correspond to the Seiberg-Witten monopoles we discussed in Chapter 2, and whose downward gradient flow line is the Seiberg-Witten monopoles on the cylinder $Y \times \mathbb{R}$. The spectral flow of the Hessian operator is employed to define the relative indices among the critical points. For $b_1(Y) > 1$ and $c_1(det(\mathfrak{s})) \neq 0$, this index is only $\mathbb{Z}_{d(\mathfrak{s})}$ valued, where $d(\mathfrak{s})$ is the divisibility of $c_1(det(\mathfrak{s}))$. It can be lifted to be $\mathbb{Z}$ valued by restricting the gauge group to those gauge transformation $g : Y \to U(1)$ such that:

$$\int_Y c_1(det(\mathfrak{s})) \wedge [g^{-1}dg] = 0.$$

In this lifting, the critical points consist of $\mathbb{Z}$-families of the original critical points $\mathcal{M}_{\mathfrak{s},\eta}$. This index can also be lifted to be $\mathbb{Z}$-valued by using the universal cover of $\mathcal{B}$, where the gauge group is the identity component of the full gauge group. The critical points consist of $H^1(Y, \mathbb{Z})$ families of the critical points $\mathcal{M}_{\mathfrak{s},\eta}$.

Due to the Palais-Smale condition for the Chern-Simons-Dirac function and the $L^2$-distance estimate near the critical points on the $H^1(Y, \mathbb{Z})$ covering, we can confirm that the Lojaszewiz-type inequality holds with the best exponent for the Chern-Simons-Dirac function, which enables us to establish the exponential decay for the downward gradient flow lines approaching the critical points. The decay rates are governed by the least of the absolute eigenvalues of the Hessian operators at the critical points. The Lojaszewiz-type inequality is one of the key technical devices in this thesis.

The exponential decay makes it appropriate to apply the weighted Sobolev space to study the moduli space of the gradient flow line between the critical points. After carefully perturbing the gradient flow equation of the Chern-Simons-Dirac functional, we can achieve the transversality property for the moduli space of the gradient flow lines (See Proposition 3.1.15 and Corollary 3.1.16). In summary, if $c_1(det(\mathfrak{s})) \neq 0$, the moduli space of the gradient flow lines between two critical points $\alpha$ and $\beta$, denoted by $\mathcal{M}(\alpha, \beta)$, consists of infinitely many smooth components with dimension differing by multiples of $d(\mathfrak{s})$, each component is orientable and admits a $\mathbb{R}$-action which comes from the time translation on the gradient flow equations, the quotient $\hat{\mathcal{M}}(\alpha, \beta) = \mathcal{M}(\alpha, \beta)/\mathbb{R}$ is called the moduli space of the time independent trajectories; if $c_1(\mathfrak{s}) = 0$, the trajectories moduli space $\mathcal{M}(\alpha, \beta)$ is also a smooth, orientable manifold of dimension given by the relative index.

The next task is to study the compactness of the time independent trajectory moduli space, where we have to exclude the case of $b_1(Y) > 0$ with $c_1 = 0$. We use the perturbation with trivial cohomology class from this point on. To establish the compactness of the moduli space of time independent trajectories, we need the uniform energy condition on the component with fixed dimension, this requires that the perturbation of the 3D Seiberg-Witten monopoles represents a trivial cohomology class. Then we compactify the moduli space of trajectories with fixed dimension by adding the "broken" trajectories. In particular, the 0-dimensional component of any moduli space $\hat{\mathcal{M}}(\alpha,\beta)$ is compact, and any 1-dimensional component of moduli space $\hat{\mathcal{M}}(\alpha,\gamma)$ is a compact manifold with boundary points given by the "broken" trajectories of simple type (ie., breaking only at one critical point):

$$\cup_\beta \hat{\mathcal{M}}(\alpha,\beta)^0 \times \hat{\mathcal{M}}(\beta,\gamma)^0$$

where the superscript 0 indicates the 0-dimensional component. In general, the component of $\mathcal{M}(\alpha,\beta)$ with fixed dimension is a manifold with corners (Cf. Proposition 3.1.18). All these are obtained in Section 3.1.3, which is concluded by the study of the gluing map of the "broken" trajectories. The glueing theorem and the corresponding gluing maps are the essential ingredient in the gauge theory, and play an important role in this thesis. We only give the detailed proof for one kind of gluing maps, and its variants can be confirmed almost word for word. Our gluing map is Proposition 3.1.19, the proof is rather long and technical, but the idea is simply to apply the exponential decay property of the trajectories to deform the pre-glued solution (an approximate Seiberg-Witten monopole) to an actual Seiberg-Witten monopole.

Using the 0-dimensional and 1-dimensional components of the trajectory moduli spaces, we now can define the (non-equivariant) Seiberg-Witten-Floer homology $HF_*^{SW}(Y,\mathfrak{s})$ for a homology 3-sphere, or $Y$ with $b_1(Y) > 0$ and $c_1(det(\mathfrak{s})) \neq 0$. For the case of $b_1(Y) > 0$ with $c_1(det(\mathfrak{s})) \neq 0$, the corresponding Seiberg-Witten-Floer homology is a topological invariant which means that it is independent of the metric and the perturbation chosen to define them. While $HF_*^{SW}(Y,\mathfrak{s})$ for a homology 3-sphere, after removing the reducible monopole, does depend on the metric and perturbation, note that we use the metric $g$ and perturbation $*d\nu$ with $Ker(\partial\!\!\!/_\nu^g) = 0$. To understand the dependence of the Seiberg-Witten-Floer homology on the metric and perturbation, we have to resort to the equivariant version of the Seiberg-Witten-Floer homology. Section 3.2 is devoted to constructing such equivariant Seiberg-Witten-Floer homology.

Since for a homology 3-sphere $Y$ with a fixed Spin$^c$ structure $\mathfrak{s}$, there is unique reducible critical point $\theta = [\nu, 0]$ for the perturbation $*d\nu$, the stabiliser of this reducible monopole is $U(1)$. It is relatively easy to work out the $U(1)$-equivariant Seiberg-Witten-Floer homology by considering the Chern-Simons-Dirac functional on the based configuration space. The requirement of the Bott-Smale condition on the critical orbits forces the metric and perturbation to satisfy the condition $Ker(\partial\!\!\!/^g_\nu) = 0$.

We construct this $U(1)$-equivariant Seiberg-Witten-Floer homology by looking at the trajectory moduli space between the critical orbits $O_\alpha$ and $O_\beta$, where $O_\alpha$ is the $U(1)$-orbit of $\alpha$ on the based configuration space, which is isomorphic to $U(1)$ if $\alpha$ is irreducible, while $O_\theta = \theta$. We then show that this equivariant Seiberg-Witten-Floer (co) homology is a topological invariant up to the index shifting which is given by the real spectral flow of the twisted Dirac operator $\partial\!\!\!/^{g_t}_{\nu_t}$.

**Theorem 1.2.2.** *Let $(Y, \mathfrak{s})$ be a $\mathbb{Q}$-homology 3-sphere with a* Spin$^c$ *structure $\mathfrak{s}$, let $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_0, \nu_0)$ and $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_1, \nu_1)$ be the equivariant Seiberg-Witten-Floer homology defined for two generic pairs $(g_0, \nu_0)$ and $(g_1, \nu_1)$. Suppose that the spectral flow of the Dirac operators $\partial\!\!\!/^{g_t}_{\nu_t}$ along a path $(g_t, \nu_t)$ is $2m$. Then there exists an isomorphism between $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_0, \nu_0)$ and $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_1, \nu_1)$ with index shifted by $2m$. The analogous claims hold for the equivariant Seiberg-Witten-Floer cohomology.*

In the proof of this topological invariance, we have to deal with the singular time dependent trajectories. Suppose that $(g_t, \nu_t)$ $(t \in [0,1])$ is a family of metrics and perturbations which cross the codimensional one subset (where $Ker(\partial\!\!\!/^g_\nu) \neq 0$) once with the real spectral flow $SF_{\mathbb{R}}(\partial\!\!\!/^{g_t}_{\nu_t}) = 2$, we need to study the time dependent trajectories, the 4D Seiberg-Witten monopoles on $(Y \times \mathbb{R}, g_t + dt^2)$. Let $\theta_0$ and $\theta_1$ be the corresponding reducible critical points of $(g_0, \nu_0)$ and $(g_1, \nu_1)$. The only trouble here is the trajectory moduli space $\mathcal{M}(\theta_1, \theta_0)$ under based gauge group, whose virtual dimension is $-2$, but it cannot be perturbed to be empty by any consistent perturbations. This is due to the fact that $b_2^+ = 0$ for $(Y \times \mathbb{R}, g_t + dt^2)$, there is always a unique reducible 4D monopole in $\mathcal{M}(\theta_1, \theta_0)$ with 2-dimensional $H^2$-cohomology group in the 4D deformation complex. We must take care of this singular monopole in the construction of the chain maps and chain homotopy between the chain complexes for $(g_0, \nu_0)$ and $(g_1, \nu_1)$.

After checking with the obstruction bundle for the gluing map, we eliminate such trouble by showing that this singular monopole doesn't appear in various chain maps (see Proposition 3.3.9), so that the chain maps and chain homotopy are well-defined. Hence the topological invariance of the equivariant Seiberg-Witten-Floer homology is confirmed. We also study the relationship between the non-equivariant Seiberg-Witten-Floer homology and the equivariant Seiberg-Witten-Floer homology by spectral sequences. As an application of our equivariant Seiberg-Witten-Floer homology, we apply the topological invariant property of the equivariant Seiberg-Witten-Floer homology to give a new (algebraic and simpler) proof of the wall-crossing formulae for the Seiberg-Witten invariants on any homology 3-sphere.

Chapter 3 is concluded by a brief account of Fukaya-type monopole homology groups $HFF_*^{(m)}(Y, \mathfrak{s})$ and of a pairing between them. They form the mathematical device needed to address the gluing formulae for the 4-dimensional Seiberg-Witten invariant for a closed 4-manifold $X$ splitting along a 3-dimensional manifold $Y$. Here we consider all the components in the trajectories moduli space instead of only the zero dimensional components (which are naturally compact). The idea comes from the work of Fukaya and Braam-Donaldson's account of Fukaya's construction though the method is quite different. The problem here is that one has to define certain invariants from the higher dimensional components which are non-compact or form manifolds with corners after compactification by adding the broken trajectories.

In the case $b_1(Y) > 0$ and $c_1(det(\mathfrak{s})) \neq 0$, there are only finitely many critical points (the Seiberg-Witten monopoles on $Y$) for the perturbation with trivial cohomology class. Let $\alpha, \beta$ be two such critical points, the time independent trajectory moduli space $\hat{\mathcal{M}}(\alpha, \beta)$ (the Seiberg-Witten monopoles on $(Y, g + dt^2, \mathfrak{s})$ whose asymptotic limits are $\alpha$ and $\beta$ as $t \to -\infty, +\infty$ respectively) has many non-empty components whose dimensions differ by multiples of the divisibility $d(\mathfrak{s})$ of the first Chern class $c_1(det(\mathfrak{s}))$ in $H^2(Y, \mathbb{Z})/Torsion$ ($d(\mathfrak{s})$ is always even). The local dimensions of $\hat{\mathcal{M}}(\alpha,\beta)$ we can assume to be $2k$ or $2k+1$ with $k \geq 0$. If $\hat{\mathcal{M}}(\alpha,\beta)^{2k}$ is one component of $\hat{\mathcal{M}}(\alpha,\beta)$ with fixed dimension $2k$ then there is a natural $U(1)$ fibration, $\hat{\mathcal{M}}^O(\alpha,\beta)^{2k}$, (given by the based gauge transformations) whose associated rank $k$ complex vector bundle over $\hat{\mathcal{M}}(\alpha,\beta)^{2k}$ is given by the natural multiplication of $U(1)$ on $\mathbb{C}^k$, denoted by $\mathcal{E}_{\alpha,\beta,k}$,

$$\mathcal{E}_{\alpha,\beta,k} = \hat{\mathcal{M}}^O(\alpha,\beta)^{2k} \times_{U(1)} \mathbb{C}^k \longrightarrow \hat{\mathcal{M}}(\alpha,\beta)^{2k}.$$

Choose a generic section $s_{\alpha,\beta,k} = (s^1_{\alpha,\beta,k}, s^2_{\alpha,\beta,k}, \cdots, s^k_{\alpha,\beta,k})$ of $\mathcal{E}_{\alpha,\beta,k}$ which is transversal to the zero section. Then $(\mathcal{E}_{\alpha,\beta,k}, s_{\alpha,\beta,k})$ is compatible with the corresponding pairs over the boundary strata of $\hat{\mathcal{M}}(\alpha,\beta)^{2k}$, from which we know that the section $s_{\alpha,\beta,k}$ has no zeros on the boundary by dimension counting. Thus $s^{-1}_{\alpha,\beta,k}(0)$ consists of finitely many oriented points in $\hat{\mathcal{M}}(\alpha,\beta)^{2k}$. Then there is an intrinsic integer associated with $\hat{\mathcal{M}}(\alpha,\beta)^{2k}$, which is the homology class $[s^{-1}_{\alpha,\beta,k}(0)]$, we formally denote this "Euler" number by

$$\int_{\hat{\mathcal{M}}(\alpha,\beta)^{2k}} \Theta^k = \#\left(s^{-1}_{\alpha,\beta,k}(0)\right) \tag{1.3}$$

where $\Theta$ is the first Chern class of the basepoint fibration $\hat{\mathcal{M}}^O(\alpha,\beta)^{2k}$ over $\hat{\mathcal{M}}(\alpha,\beta)^{2k}$. This formal notation proves to be very convenient in the formulation of the calculations of the right hand side of (1.3), even though the left hand side itself is not well-defined since the vector bundle $\mathcal{E}_{\alpha,\beta,k}$ can't be trivialized over the boundary. For example, if $\hat{\mathcal{M}}(\alpha,\gamma)$ has a component with dimension $2k+1$, say $\hat{\mathcal{M}}(\alpha,\gamma)^{2k+1}$, we can associate with it a rank $k$ complex vector bundle,

$$\mathcal{E}_{\alpha,\gamma,k} = \hat{\mathcal{M}}^O(\alpha,\gamma)^{2k+1} \times_{U(1)} \mathbb{C}^k \longrightarrow \hat{\mathcal{M}}(\alpha,\gamma)^{2k+1}$$

and a transversal section $s_{\alpha,\gamma,k}$. Then $s^{-1}_{\alpha,\gamma,k}(0)$ is a one dimensional, compact moduli space whose boundary consists of the zeros of $s_{\alpha,\gamma,k}$ over the codimension one boundary of $\hat{\mathcal{M}}(\alpha,\gamma)^{2k+1}$. From the gluing theorem (cf. Theorem 3.1.19), we know that the codimension one boundary of $\hat{\mathcal{M}}(\alpha,\gamma)^{2k+1}$ consists of

$$\cup_{\beta,l}\hat{\mathcal{M}}(\alpha,\beta)^l \times \hat{\mathcal{M}}(\beta,\gamma)^{2k-l}.$$

We will show that the counting the boundary points of $s^{-1}_{\alpha,\gamma,k}(0)$ with orientation is equivalent to the following formal calculation:

$$\int_{\hat{\mathcal{M}}(\alpha,\gamma)^{2k+1}} d(\Theta^k) = 0.$$

We will give an explicit calculation of the formal expression $\int_{\hat{\mathcal{M}}(\alpha,\gamma)^{2k+1}} d(\Theta^k)$ in terms of the right hand side of (1.3). We will give more details in Section 3.4, where $s_{\alpha,\beta,k}$ will be the main ingredient of our construction of the Fukaya-type homology homology groups $HFF_*^{SW}(Y,\mathfrak{s})$.

Chapter 4 gives an application to the problem of associating invariants to a 4-manifold with boundary and gluing formulae for 4-dimensional monopole invariants. We summarise our main results as follows.

For a 4-manifold $(X_1,\mathfrak{s}_1)$ with cylindrical end modelled on $(Y,\mathfrak{t})$ (in addition, we assume $c_1(det\mathfrak{t}) \neq 0$ if $b_1(Y) > 0$), the Seiberg-Witten equations on $X_1$ define a corresponding $L^2$ moduli space with finite variations of the perturbed Chern-Simons-Dirac functional on the end. We associate these moduli spaces with certain invariants which are used to define the relative Seiberg-Witten invariant of $X_1$. These relative Seiberg-Witten invariants take values in the Fukaya-type homology groups $HFF_*^{SW}(Y,\mathfrak{t})$. We denote this invariant of $X_1$ by $SW_{X_1}$. The definition of the relative Seiberg-Witten invariant involves the canonical basepoint fibrations and its transversal sections. We summarise this definition in the case $b_1(Y) > 0$, let $\mathcal{M}(X_1,\alpha)$ be the $L^2$ moduli space of the Seiberg-Witten monopoles with boundary limit $\alpha$ ($\alpha$ is a Seiberg-Witten monopole on $(Y,\mathfrak{t})$). Then $\mathcal{M}(X_1,\alpha)$ has many components with dimension differing by the multiples of $d(\mathfrak{t})$ (the divisibility of $c_1(\mathfrak{t})$).

There are similar gluing maps which give rise to the boundary term consisting of the factorization of the $L^2$ monopoles and the time independent trajectories on $Y\times\mathbb{R}$. Suppose that $\mathcal{M}(X_1,\alpha)^{2k}$ is one non-empty component with dimension $2k \geq 0$, the basepoint fibration is the $U(1)$-fiber bundle $\mathcal{M}^0(X_1,\alpha)^{2k}$ whose associated rank $k$ bundle is

$$\mathcal{E}_{X_1,\alpha,k} = \mathcal{M}^0(X_1,\alpha)^{2k} \times_{U(1)} \mathbb{C}^k \longrightarrow \mathcal{M}(X_1,\alpha)^{2k}.$$

Choose a transversal section $s_{X_1,\alpha,k} = (s^1_{X_1,\alpha,k}, s^2_{X_1,\alpha,k}, \cdots, s^k_{X_1,\alpha,k})$ which is compatible with the corresponding constructions over the boundary. The zero points of $s_{X_1,\alpha,k}$ define the "Euler" number of the bundle $\mathcal{E}_{X_1,\alpha,k}$, which we formally denote by

$$\int_{\mathcal{M}(X_1,\alpha)^{2k}} \Theta^k = \#\left(s_{X_1,\alpha,k}^{-1}(0)\right). \tag{1.4}$$

In a manner similar to the construction of the Fukaya-type monopole homology, these pairs $(\mathcal{E}_{X_1,\alpha,k}, s_{X_1,\alpha,k})$ will be used to define a relative Seiberg-Witten monopole invariant taking values in the Fukaya-type monopole homology groups.

Our main application is to consider gluing formulae for the Seiberg-Witten invariant on a simply-connected, closed 4-manifold $(X, \mathfrak{s})$ with $b_2^+ > 1$, where $X$ has a decomposition $X_1 \cup_Y X_2$ along a closed 3-manifold $Y$ for which $c_1(\mathfrak{s}|_Y) \neq 0$ if $b_1(Y) > 0$. For $b_1(Y) > 0$, we have a gluing formula which states that the monopole invariant $SW_X(\mathfrak{s})$ for $(X, \mathfrak{s})$ can be obtained through the pairing on the Fukaya-type monopole homology over the two relative invariants $SW_{X_1}$ and $SW_{X_2}$. For $Y$ a homology sphere, though the Seiberg-Witten-Floer homology $HF_*^{SW}(Y)$ depends on the metric and perturbation, as long as the 4-manifold $X$ has $b_2^+(X)$ bigger than one, we still have a gluing formula along $Y$. The fact that the relative invariants for $X_i$ take values in the metric-dependent homology group does not effect this conclusion. There may still be some contribution to the Seiberg-Witten monopole $SW_X(\mathfrak{s})$ from the unique reducible Seiberg-Witten monopole $\theta$ on $Y$. Hence after we study the two versions of Fukaya-type homology groups $HFF_*^{(m)}(Y)$ and $\widetilde{HFF}_*^{(m)}(Y)$ for $Y$, we give two gluing formulae (4.8) and (4.9) for $SW_X(\mathfrak{s})$ ($X = X_1 \cup_Y X_2$) in the two cases: $b_2^+(X_1) \geq 1$, $b_2^+(X_2) \geq 1$ (in which there is no contribution from the reducible critical point $\theta$) and $b_2^+(X_1) > 1$, $b_2^+(X_2) = 0$ (in which there is a contribution from $\theta$) respectively.

Chapter 4 is ended with a similar construction of the Seiberg-Witten invariant for certain contact structures on a closed 3-manifold.

# Chapter 2

# Seiberg-Witten monopoles on 3-manifolds

## 2.1 Spin$^c$ structures and Seiberg-Witten equations

Let $Y$ be an oriented, closed 3-manifold equipped with a Riemannian metric $g$. The tangent space of $Y$ at each point $y$, denoted by $V = T_yY$, is a 3-dimensional real vector space with a positive definite inner product. Choose an orthonormal basis $\{e_1, e_2, e_3\}$ for $V$ which trivializes the tangent bundle $TY$, the real Clifford algebra $Cl(V)$ is generated by $\{e_1, e_2, e_3\}$ with relations:

$$(e_i)^2 = -1, \quad e_ie_j = -e_je_i \quad \text{for } i \neq j.$$

it is of dimension 8 over $\mathbb{R}$, isomorphic to $\mathbb{H} \oplus \mathbb{H}$ as an algebra, where $\mathbb{H}$ is the quaternion algebra:

$$\{\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k | i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}$$

The isomorphism between $\mathbb{H}$ and the first factor is given by sending $1, i, j, k$ to

$$\frac{1 + e_1e_2e_3}{2}, \frac{e_1e_2 - e_3}{2}, \frac{e_2e_3 - e_1}{2}, \frac{e_3e_1 - e_2}{2};$$

and the isomorphism between $\mathbb{H}$ and the second factor is given by sending $1, i, j, k$ to

$$\frac{1 - e_1e_2e_3}{2}, \frac{e_1e_2 + e_3}{2}, \frac{e_2e_3 + e_1}{2}, \frac{e_3e_1 + e_2}{2}.$$

There is a natural $Z_2$-grading on $Cl(V)$ corresponding to the even-odd decomposition of the algebra:

$$Cl(V) = Cl^0(V) \oplus Cl^1(V)$$

which are the $\pm 1$ eigenspaces of the automorphism on $Cl(V)$ by sending the generator $e_i$ to $-e_i$. $Cl^0(V)$ is a subalgebra which is generated by $1, e_1e_2, e_2e_3, e_3e_1$ over $\mathbb{R}$, it can be identified with $\mathbb{H}$ as the diagonal copy in $Cl(V) \cong \mathbb{H} \oplus \mathbb{H}$.

Note that if $v \in V$ is non-zero element, then $v$ is invertible in the Clifford algebra $Cl(V)$, actually, $v^{-1} = -v/|v|^2$.

This motivates the definition of the Pin group $Pin(V)$ and the Spin group $Spin(V)$ as in [28]. $Cl^+(V)$ is defined to be the invertible element in $Cl(V)$ with respect to the multiplication structure, clearly, $Cl^+(V)$ contains all non-zero elements in $V$. Let $Pin(V)$ be the subgroup of $Cl^+(V)$ generated by $v \in V$ with $|v|^2 = 1$. The Spin group $Spin(V)$ is defined by

$$Spin(V) = Pin(V) \cap Cl^0(V)$$

**Lemma 2.1.1.** *$Spin(V) \cong SU(2)$ for $V \cong R^3$, where $SU(2)$ is the special unitary transformation on $\mathbb{C}^2$.*

**Proof.** Note that $SU(2)$ can be identified with the group of unit quaternion in $\mathbb{H}$,

$$SU(2) \ni \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \longleftrightarrow z_1 + jz_2 \in \mathbb{H}.$$

$Spin(V)$ is the subgroup of $Pin(V)$ which can be written as a product of even number of unit-length elements in $V$. Under the identification $Cl(V) \cong \mathbb{H} \oplus \mathbb{H}$,

$$\begin{aligned} V &\cong \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \\ &\cong \{(a, -a) \in \mathbb{H} \oplus \mathbb{H} | a \text{ is a purely imaginary quaternion.}\} \\ &\cong \text{ purely imaginary quaternion} \end{aligned}$$

Therefore,

$$Spin(V) = \{ab | a, b \text{ are purely imaginary unit quaternion}\}$$

Let $a = a_1 i + a_2 j + a_3 k, b = b_1 i + b_2 j + b_3 k$ with $\sum_i a_i^2 = \sum_i b_i^2 = 1$, then

$$ab = -(a_1b_1 + a_2b_2 + a_3b_3) + (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

which is a unit quaternion by direct computation of its norm. This proves that $Spin(V) \hookrightarrow SU(2)$, then the Lemma follows from that $SU(2)$ is connected and has the same dimension as $Spin(V)$. □

As we have seen that $Spin(V) \cong SU(2)$, the spin representation is the standard representation of $SU(2)$ on $\mathbb{C}^2$, whose infinitesimal representation (note that $LieSU(2) = \mathfrak{su}(2) \cong V \subset Cl(V)$, the Lie bracket on $V$ is given by $[v_1, v_2] = v_1v_2 - v_2v_1$.) is as follows:

$$e_1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.1}$$

By the definition, the Spin$^c$ group Spin$^c(V)$ is the subgroup of the multiplicative group of units of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by Spin$^c(V)$ and the unit circle in $\mathbb{C}$, in our case, we see that Spin$^c(\mathbb{R}^3) \cong SU(2) \times_{\mathbb{Z}_2} U(1) \cong U(2)$ where $U(2)$ is the rank two unitary group. The spin representation of $Spin(\mathbb{R}^3)$ has a natural extension to Spin$^c(\mathbb{R}^3)$.

**Definition 2.1.2.** *A* Spin$^c$ *structure* $\mathfrak{s}$ *on an oriented, closed 3-manifold* $(Y, g)$ *is a pair* $(W, \rho)$ *of* $U(2)$ *bundle* $W$ *and a map* $\rho : T^*(Y) \to End(W)$ *satisfying the Clifford relation:*

$$\rho(v_1)\rho(v_2) + \rho(v_2)\rho(v_1) = -2g(v_1, v_2).$$

Due to the natural representation of $U(2)$ on $\mathbb{C}^2$ and our trivialization of $TY$, there is a natural Spin$^c$ structure $W_0 = Y \times \mathbb{C}^2$ with the Clifford multiplication generated by (2.1), we see that this Clifford multiplication gives a natural isomorphism $\rho$:

$$\rho : \qquad \Omega^1(Y, \mathbb{R}) \to \mathfrak{su}(W_0).$$

Locally, let $\{e^1, e^2, e^3\}$ be an oriented local orthonormal basis for $T^*Y$, under the identification of $T^*Y \cong TY$, then

$$\rho(\sum_{i=1}^{3} a_i e^i) = \begin{pmatrix} a_1 i & -a_2 + ia_3 \\ a_2 + ia_3 & -a_1 i \end{pmatrix}$$

rite the spinor (the section of $W_0$) as $\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1, z_2$ are complex-valued function over $Y$, then $i(\psi \otimes \bar{\psi}^t)_0$ (where the subscript 0 denotes the trace-free part) is an element in $\mathfrak{su}(W_0)$, define

$$\sigma(\psi, \psi) = i\rho^{-1}(i(\psi \otimes \bar{\psi}^t)_0) \in \Omega^1(Y, i\mathbb{R}).$$

**Lemma 2.1.3.** *Define the natural Hermitian metric on $W_0$ by*

$$\langle \psi, \phi \rangle = z_1 \bar{z}_1' + z_2 \bar{z}_2'$$

*for $\psi = (z_1, z_2), \phi = (z_1', z_2')$. Then $\langle a.\psi, \psi \rangle = 2g(a, \sigma(\psi, \psi))$ for $a \in \Omega(Y, i\mathbb{R})$, where $g$ is complex bilinear form on $T^*Y \otimes_{\mathbb{R}} \mathbb{C}$ defined by the Riemannian metric, we use . to denote the Clifford multiplication. Therefore, we have*

$$\sigma(\psi, \psi) = \frac{1}{2} \sum_{i=1}^{3} \langle e_i.\psi, \psi \rangle e^i.$$

*$\sigma(\psi, \psi).\psi = -\frac{1}{2}|\psi|^2\psi$, and $|\sigma(\psi, \psi)|^2 = \frac{1}{4}|\psi|^4$.*

**Proof.** Direct calculation gives rise to the following formulae:

(a) $\langle e^1.\psi, \psi \rangle = i(|z_1|^2 - |z_2|^2)$,

(b) $\langle e^2.\psi, \psi \rangle = 2iIm(z_1\bar{z}_2)$,

(c) $\langle e^3.\psi, \psi \rangle = 2iRe(z_1\bar{z}_2)$,

(d) $\sigma(\psi, \psi) = i(\frac{1}{2}(|z_1|^2 - |z_2|^2)e^1 + Im(z_1\bar{z}_2)e^2 + Re(z_1\bar{z}_2)e^3)$.

which complete the proof. □

This definition of $\sigma(\psi, \psi)$ can be extended to a symmetric $\mathbb{R}$-bilinear pairing:

$$\begin{aligned} \sigma(\psi, \phi) &= \tfrac{1}{2}(\sigma(\psi + \phi, \psi + \phi) - \sigma(\psi, \psi) - \sigma(\phi, \phi)) \\ &= \tfrac{i}{2} Im(\langle e_i.\psi, \phi \rangle) e^i \in \Omega^1(Y, i\mathbb{R}). \end{aligned} \tag{2.2}$$

**Lemma 2.1.4.** *There is a quaternionic structure on $\mathbb{C}^2$, under this identification $\mathbb{C}^2 \cong \mathbb{H}$, there is $J$ action defined by the right multiplication of $j$ which commutes with the Clifford multiplication (2.1).*

**Proof.** Define the standard quaternionic structure on $\mathbb{C}^2$ by sending

$$(z_1, z_2) \to z_1 + jz_2$$

then the Clifford multiplications $\rho(e_1), \rho(e_2), \rho(e_3)$ are the left multiplication by $i, j, k$. Hence, these Clifford multiplications commute with the right multiplication by $j$, on $\mathbb{C}^2$, $J$ acts by sending

$$(z_1, z_2) \mapsto (-\bar{z}_2, \bar{z}_1).$$

□

Let Spin$^c(Y)$ denote the set of isomorphism classes of Spin$^c$ structures on $Y$, there is a preferred Spin$^c$ structure $W_0$ as discussed so far. Any other Spin$^c$ structure $\mathfrak{s} = (W, \rho)$ can be constructed from $W_0$ by tensoring with a $U(1)$ line bundle $L^{\frac{1}{2}}$ for an even line bundle $L$ (that is, $c_1(L) = 0(mod\ 2)$), since for 3-manifold $Y$, the short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \to 0$$

gives a long exact sequence in cohomology, from which we see that

$$2H^2(Y, \mathbb{Z}) \oplus H^1(Y, \mathbb{Z}_2) \cong H^2(Y, \mathbb{Z}).$$

$H^1(Y, \mathbb{Z}_2)$ classifies the isomorphism classes of the square root bundle for an even line bundle. Therefore the Spin$^c$ structures on $Y$ are parametrized by $H^2(Y, \mathbb{Z})$. Any Spin$^c$ structure $(W, \rho)$ is given by $W = W_0 \otimes L$ for some line bundle $L$, all the above discussion of Clifford multiplication structure can be developed on $(W, \rho)$ where $\bar{\psi}$ is a section of $\bar{W}_0 \otimes L^*$, therefore the Hermitian metric on $W$ is given by the pairing on $W_0 \otimes L$ and $\bar{W}_0 \otimes L^*$. The claims in Lemma 2.1.3 also hold for these general Spin$^c$ structures.

**Remark 2.1.5.** *In case $L$ is trivial, in particular if $Y$ is a oriented integral, homology 3-sphere, the* Spin$^c$ *bundle also has a quaternionic structure as in Lemma 2.1.4.*

By a basic fact of Riemannian geometry, there is a unique torsion free orthogonal connection $\nabla$ on the tangent bundle, it can be viewed as a covariant derivative:

$$\nabla : \qquad \Omega^0(Y, TY) \to \Omega^1(Y, TY)$$

which satisfies:

(a) $d\langle v_1, v_2\rangle = \langle \nabla(v_1), v_2\rangle + \langle v_1, \nabla(v_2)\rangle$;

(b) $\nabla_{v_1} v_2 - \nabla_{v_2} v_1 = [v_1, v_2]$.

for any pair of vector fields $v_1, v_2$ on $Y$. This connection is called the Levi-Civita connection of the Riemannian metric $g$ on $Y$. It can be lifted to a $Spin$-connection on the $Spin$ bundle $W_0$. We need to choose a $U(1)$ connection, $A$, on the determinant bundle $det(W)$ for the lift of $Spin$-connection $\nabla_A$ to the Spin$^c$-bundle $W$. The Dirac operator for the Spin$^c$ structure $(W, \rho)$ is defined by

$$\partial\!\!\!/_A : \qquad \Gamma(W) \xrightarrow{\nabla_A} \Gamma(W \otimes T^*Y) \xrightarrow{\rho} \Gamma(W).$$

In the local frame $\{e^1, e^2, e^3\}$, $\partial\!\!\!/_A(\psi) = \sum_{i=1}^3 \rho(e^i)\nabla_{A_i}(\psi)$.

In our case, we can choose a $U(1)$ connection $A$ on $L$ for the Spin$^c$-lift of the Levi-Civita connection for $det(W) = L^2$.

We review some of basic facts about this Dirac operator:

**Lemma 2.1.6.** *Let $Y$ be a oriented, closed 3-manifold $(Y, g)$ with a* Spin$^c$ *structure $\mathfrak{s} = (W, \rho)$ where $W = W_0 \otimes L$, let $A$ be a $U(1)$-connection on $L$, then*

*(a) $\partial\!\!\!/_A : \Gamma(W) \to \Gamma(W)$ is formally self-adjoint in the sense that*

$$(\partial\!\!\!/_A(\psi_1), \psi_2)_{L^2} = (\psi_1, \partial\!\!\!/_A(\psi_2))_{L^2}$$

*where $( \ , \ )_{L^2}$ is the $L^2$-inner product on the spinor space,*

$$(\psi_1, \psi_2)_{L^2} = \int_Y \langle \psi_1, \psi_2\rangle dvol_Y.$$

*(b) The set of* Spin$^c$*-connections $\nabla_A$ is an affine space of imaginary valued 1-forms $\Omega^1(Y, i\mathbb{R})$, the corresponding Dirac operator is given by*

$$\partial\!\!\!/_{A+a} = \partial\!\!\!/_A + \rho(a)$$

*for $a \in \Omega^1(Y, i\mathbb{R})$. For convenience, if instead, we choose a $U(1)$ connection $A$ on $det(W) = L^2$, then*

$$\partial\!\!\!/_{A+a} = \partial\!\!\!/_A + \frac{1}{2}\rho(a).$$

(c) *In the case that $W$ has trivial determinant line bundle, then there is a quaternionic structure on $\Gamma(W)$ as in Lemma 2.1.4 for which $J$-action commutes with the Dirac operator $\partial\!\!\!/$, and*

$$J\partial\!\!\!/_A = \partial\!\!\!/_{A^*}J$$

*where $A^* = d - \alpha$ if $A = d + \alpha$.*

(d) *The symbol of $\partial\!\!\!/_A$ at a cotangent vector $\xi_y \in T^*_y Y$ is the Clifford multiplication by $\xi_y$, that is*

$$Sym(\partial\!\!\!/_A)_{(y,\xi_y)} = \rho(\xi_y).$$

*Hence, $\partial\!\!\!/_A, \partial\!\!\!/$ are elliptic.*

(e) (**Weitzenböck formula**)

$$\partial\!\!\!/_A\partial\!\!\!/_A(\psi) = \nabla^*_A\nabla_A\psi - \frac{1}{2}(*F_{\hat{A}}).\psi + \frac{s}{4}\psi.$$

*where $\hat{A}$ is the corresponding connection on $det(W) = L^2$ and $s$ is the scalar curvature of $(Y, g)$.*

(f) *$\partial\!\!\!/_A : L^2_k(W) \to L^2_{k-1}(W)$ is a self-adjoint, Fredholm operator. Moreover, any $L^2$-section $\psi \in Ker\partial\!\!\!/_A$ is a smooth section and any $L^2$-section orthogonal to $Im\partial\!\!\!/_A$ under the $L^2$-metric is also smooth.*

**Proof.** Some simple manipulations as in [28] [35] shows that at any $y \in Y$

$$\langle \partial\!\!\!/_A(\psi_1), \psi_2\rangle_y = \langle \psi_1, \partial\!\!\!/_A(\psi_2)\rangle_y - d^*(\sum_{i=1}^{3}\langle e_i.\psi_1, \psi_2\rangle_y e^i).$$

By integrating over $Y$, we obtain the first result. Claim (b) and Claim (c) are obvious. To prove Claim (d), locally at $y$, choose a function $f$ such that $f(y) = 0$, $(df)_y = \xi_y$ (Poincare's Lemma), then by the definition of symbol,

$$\begin{aligned} Sym(\partial\!\!\!/_A)_{(y,\xi_Y)}(\psi)(x) &= \lim_{x\to y}\partial\!\!\!/_A((f(x) - f(y)\psi) \\ &= \rho(df_y)(\psi) = \rho(\xi_y)\psi. \end{aligned}$$

The Weitzenböck formula was proved in Theorem D.12 [28]. Claim (f) follows directly from elementary elliptic theory. Ch. III.5 in [28] gives the proof by using parametrix. Here we use the fundamental inequality (Garding-type inequality) for elliptic operators on Sobolev space:

$$\|\psi\|^2_{L^2_{k+1}} \leq C_k(\|\partial\!\!\!/_A\psi\|^2_{L^2_k} + \|\psi\|^2_{L^2_k})$$

for $\psi \in L^2_{k+1}(W)$ (The proof of this inequality uses the Weitzenböck formula). We also resort to Rellich's theorem which says that the inclusion $L^2_{k+1} \hookrightarrow L^2_k$ is compact (compactness implies that the image of each bounded sequence has a convergent subsequences). Suppose that $Ker\partial\!\!\!/_A$ has infinite dimension, then we can choose a basis $\{x_k\}_{k=1}^{\infty}$ with unit $L^2_k$ norm, then by Garding-type inequality and Rellich's theorem we know that these exists an subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ which converges in $L^2_k$, this leads to the contradiction with the fact $\{x_k\}_{k=1}^{\infty}$ is a basis. Therefore $dimKer\partial\!\!\!/_A < \infty$. This completes the proof of Claim (f) since $\bigcap_{k=0}^{\infty} L^2_k \subset C^{\infty}$. □

**Remark 2.1.7.** $iIm\langle\partial\!\!\!/_A(\psi_1), \psi_2\rangle = iIm\langle\psi_1, \partial\!\!\!/_A(\psi_2)\rangle - 2d^*\sigma(\psi_1, \psi_2)$.

Now we can introduce the Seiberg-Witten equations on $(Y, g)$ with a Spin$^c$-structure $\mathfrak{s} = (W, \rho)$, for a pair of $(A, \psi)$ consisting of a $U(1)$ connection on the determinant bundle of Spin$^c$ structure $\mathfrak{s}$ and a spinor $\psi$ (a section of $W$). They are the following two equations:

$$\begin{cases} \partial\!\!\!/_A\psi = 0, \\ *F_A = \sigma(\psi, \psi). \end{cases} \tag{2.3}$$

Denote by $\mathcal{A}(Y, \mathfrak{s})$ the Seiberg-Witten configuration space on $(Y, \mathfrak{s})$ consisting of pairs $(A, \psi)$ where $A$ is a U(1) connection on $det(W)$ and $\psi \in \Gamma(W)$. $\mathcal{A}_{\mathfrak{s}}$ is an affine space modelled on $\Omega(X, i\mathbb{R}) \times \Gamma(W)$. For analytic reasons, we shall use the $L^2_1$-integrable configurations, denoted by $\mathcal{A}_{L^2_1}$.

The automorphism group of $det(W)$ is the gauge group $\mathcal{G}_Y = \mathcal{G}_{L^2_2}$ of maps of $Y$ to $U(1)$ locally modelled on the Lie algebra $\Omega^0_{L^2_2}(Y, i\mathbb{R})$. The gauge group $\mathcal{G}_{L^2_2}$ acts on $\mathcal{A}_{L^2_1}$ as follows (by gauge transformations):

$$(A, \psi) \xrightarrow{u} (A - 2u^{-1}du, u\psi) \tag{2.4}$$

It is easy to see that $\mathcal{G}_{L^2_2}$ acts freely on $\{(A,\psi)|\psi \neq 0\}$, those elements are said to be irreducible, while $(A,\psi)$ with $\psi = 0$ is said to be reducible. Denote by $\mathcal{A}^*_{L^2_1}$ the open subset of irreducible configurations, denote by $\mathcal{B}^* = \mathcal{A}^*_{L^2_1}/\mathcal{G}_{L^2_2}$ the quotient space of $\mathcal{A}^*_{L^2_1}$ modulo the gauge group $\mathcal{G}_{L^2_2}$, then $\mathcal{B}^*$ is a Hausdorff space.

Simple computation using Lemma 2.1.6, shows that the above Seiberg-Witten monopole equations for $(Y,\mathfrak{s})$ are invariant under the gauge transformations $u \in \mathcal{G}_{L^2_2}$.

**Definition 2.1.8.** *The moduli space $\mathcal{M}_{\mathfrak{s}}$ of Seiberg-Witten monopoles on $(Y,\mathfrak{s})$ is the solution space of the Seiberg-Witten equations (2.3) modulo the gauge transformation group $\mathcal{G}_{L^2_2}$. The moduli space $\mathcal{M}^*_{\mathfrak{s}}$ is the irreducible part of $\mathcal{M}_{\mathfrak{s}}$.*

Without any difficulty, we know that $\mathcal{M}^*_{\mathfrak{s}}$ can be viewed as zero-point set of a section for an infinite dimensional bundle over $\mathcal{B}^*$ which leads to the topological quantum field theoretical interpretation of the Seiberg-Witten invariants on $Y$ [11]. The bundle over $\mathcal{B}^*$ is the associated bundle of $\mathcal{A}^* \to \mathcal{B}^*$ via $\mathcal{G}_{L^2_2}$-action on $\Omega^1_{L^2_1} \times L^2_1(W)$ by $(a,\phi) \xrightarrow{u} (a, u\phi)$. The corresponding section $s$ can be described by the following diagram:

$$\begin{array}{ccc} \mathcal{A}^* \times_{\mathcal{G}} (\Omega^1_{L^2}(Y, i\mathbb{R}) \times L^2(W)) & \ni & [A, \psi, *(F_A - \sigma(\psi,\psi)), \not\partial_A(\psi)] \\ \downarrow & & s \nearrow \\ B^* & \ni & [A,\psi] \end{array} \tag{2.5}$$

Actually, we will see that $s$ defines a section on the $L^2$-tangent bundle on $B^*$. Now, $\mathcal{M}^* = s^{-1}(0)$. $\mathcal{M} - \mathcal{M}^*$ are the reducible solutions to (2.3), it consists of

$$\{(A,0)|\ F_A = 0\}/\mathcal{G}_{L^2_2}.$$

The structures of $\mathcal{M}$, $\mathcal{M}^*$ will be discussed in the following section. The following proposition is the direct consequence of the Weizenböck formula for Dirac operator in Lemma 2.1.6.

**Lemma 2.1.9.** *Suppose that $\mathcal{M}_{\mathfrak{s}}$ is non-empty for a* Spin$^c$ *structure $\mathfrak{s}$ on $(Y,g)$, then the $L^2$-norm of the curvature $F_A$ is bounded by the $L^2$-norm of the scalar curvature $s$ as follows:*

$$||F_A|| \leq \frac{||s||}{2}.$$

**Proof.** Let $(A,\psi)$ be a solution to the Seiberg-Witten equations (2.3), from the curvature equation, we know that $|\psi|^4 = 4|F_A|^2$ (Cf. Lemma 2.1.3), pairing the Weizenböck formula with $\psi$ and applying the integration over $Y$, we have

$$\int_Y (|\nabla_A\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{4}|\psi|^4) dvol_Y = 0,$$

which implies the inequality in the Lemma. □

**Corollary 2.1.10.** *There are only finitely many* $\mathrm{Spin}^c$ *structures on* $(Y,g)$ *with non-empty Seiberg-Witten monopoles to the Seiberg-Witten equations.*

## 2.2 Moduli space for Seiberg-Witten monopoles

Let $(A_0,\psi_0) \in \mathcal{A}^*_{L^2_1}$ be an irreducible solution to the Seiberg-Witten equations (2.3), there is a natural associated elliptic complex which incorporates the linearization of the gauge action (2.4) and the linearization of equations (2.3) as follows. The tangent space $\mathcal{A}_{L^2_1}$, can be identified with $\Omega^1_{L^2_1}(Y, i\mathbb{R}) \oplus L^2_1(W)$.

$$0 \to \Omega^0_{L^2_2}(Y, i\mathbb{R}) \xrightarrow{G} \Omega^1_{L^2_1}(Y, i\mathbb{R}) \oplus L^2_1(W) \xrightarrow{L} \Omega^1_{L^2}(Y, i\mathbb{R}) \oplus L^2(W) \tag{2.6}$$

where $G$ is the infinitesimal gauge transformation:

$$G: \qquad f \longmapsto (-2df, f\psi_0),$$

and $L$ is the linearization of the Seiberg-Witten equations (2.3) at $(A_0,\psi_0)$:

$$(a,\phi) \mapsto (*da - 2\sigma(\psi_0,\phi), \not{\partial}_{A_0}\phi + \frac{1}{2}a.\psi_0),$$

where $\sigma(\psi_0,\phi)$ is the $\mathbb{R}$-bilinear symmetric pairing as defined (2.2).

**Lemma 2.2.1.** $L \circ G = 0$.

**Proof.** Since $\sigma(\psi,\phi)$ is $\mathbb{R}$-bilinear and symmetric, and from the definition (2.2), we know that $\sigma(\psi,\phi)$ is complex linear with respect to the first factor, complex anti-linear with respect to the second factor. This implies that $\sigma(f\psi,\psi) = 0$ for $f \in \Omega^0(Y, i\mathbb{R}), \psi \in \Gamma(W)$. Hence

$$L \circ G(f) = L(-2df, f\psi_0) = \Big(-2\sigma(f\psi_0,\psi_0), \not{\partial}_{A_0}(f\psi) + \frac{1}{2}(-2df.\psi_0)\Big) = 0.$$

□

From the deformation complex (2.6), we can read off the virtual dimension and Zariski tangent space to $\mathcal{M}$ at $[A_0, \psi_0]$. We need to put the deformation complex in the following form:

$$\Omega^0_{L^2_2}(Y, i\mathbb{R}) \oplus \Omega^1_{L^2_1}(Y, i\mathbb{R}) \oplus L^2_1(W) \xrightarrow{T_{(A_0,\psi_0)}} \Omega^1_{L^2}(Y, i\mathbb{R}) \oplus L^2(W) \oplus \Omega^0_{L^2}(Y, i\mathbb{R}) \quad (2.7)$$

where $T_{(A_0,\psi_0)} = G \oplus L \oplus G^*$ in which $G^*$ is the dual operator of $G$ under the $L^2$-inner product on $\Omega^1_{L^2_1}(Y, i\mathbb{R}) \oplus L^2_1(W)$:

$$< (a, \phi), (b, \psi) >= \int_Y (-a \wedge *b + 2Re\langle \phi, \psi \rangle dvol_Y), \quad (2.8)$$

**Lemma 2.2.2.** $G^*(a, \phi) = -2(d^*a + iIm\langle \psi_0, \phi \rangle)$.

**Proof.**

$$\begin{aligned} < (-2df, f\psi_0), (a, \phi) > &= (f, -2d^*a) + 2Re\langle f\psi_0, \phi \rangle \\ &= (f, -2d^*a) + 2ifIm\langle \psi_0, \phi \rangle \\ &= (f, -2d^*a - 2iIm\langle \psi_0, \phi \rangle) \\ &= (f, G^*(a, \phi)) \end{aligned}$$

□

Since $\mathcal{G}$ acts on $\mathcal{B}^*$ freely and the tangent space of the gauge orbit $\mathcal{G}.(A, \psi)$ is

$$\{(-2df, f\psi) | f \in \Omega^0_{L^2_2}(Y, i\mathbb{R})\}$$

the $L^2$-orthogonal complement to this tangent space defines the $L^2$-tangent space of $\mathcal{B}^*$ at $[A, \psi]$, which can be identified with

$$\{(a, \phi) \in \Omega^1_{L^2}(Y, i\mathbb{R}) \oplus L^2(W) \mid d^*a + iIm\langle \psi, \phi \rangle = 0\}.$$

Actually, $T_{(A_0,\psi_0)}$ is a compact perturbation of the sum of the signature operator and the twisted Dirac operator:

$$\begin{pmatrix} *d & -2d \\ -2d^* & 0 \end{pmatrix} \oplus \partial\!\!\!/_{A_0}$$

which has index 0. We have applied the following key $C^0$-estimate for the Seiberg-Witten monopoles to deduce the above compact perturbation. First note that the

$C^\infty$ pairs $(A, \psi)$ are dense in the configuration space $\mathcal{A}$, every $(A, \psi)$ is gauge equivalent to a pair $(A', \psi')$ which lies in the slice of gauge group through a $C^\infty$ object. Restrict the Seiberg-Witten equations (2.3) to the slice through a $C^\infty$ object, we see that as a solution to an elliptic equation with $C^\infty$ coefficients, every solution to (2.3) is gauge equivalent to $C^\infty$ solution.

**Lemma 2.2.3.** *Let $(A_0, \psi_0)$ be a smooth solution to (2.3). Then*

$$|\psi_0|^2 \leq \max_{y\in Y}\{0, -s(y)\}$$

*where $s(y)$ is the scalar curvature on $(Y, g)$.*

**Proof.** Apply the Weitzenböck formula in Lemma 2.1.6 at the maxima of $|\psi_0|^2$, we obtain:

$$\begin{aligned} 0 &\geq \Delta|\psi_0|^2 = 2|\nabla_{A_0}\psi_0|^2 - 2\langle\nabla^*_{A_0}\nabla_{A_0}\psi_0, \psi_0\rangle \\ &\geq -2\langle\nabla^*_{A_0}\nabla_{A_0}\psi_0, \psi_0\rangle \\ &= \frac{s}{2}|\psi_0|^2 - \langle *F_A.\psi_0, \psi_0\rangle && \text{(by the Weitzenböck formula )} \\ &= \frac{s}{2}|\psi_0|^2 - \langle\sigma(\psi_0, \psi_0).\psi_0, \psi_0\rangle && \text{(by equation (2.3))} \\ &\geq \frac{s}{2}|\psi_0|^2 + \frac{1}{2}|\psi_0|^4. && \text{(by Lemma 2.1.3)} \end{aligned}$$

This completes the proof of Lemma. □

**Remark 2.2.4.** *This $C^0$-bound in Lemma 2.2.3 for any Seiberg-Witten monopole implies that any 3-manifold with a positive scalar curvature metric has only reducible solutions.*

**Proposition 2.2.5.** *Given a 3-manifold $Y$ with a Riemannian metric, there are only finitely many* Spin$^c$ *structures (up to isomorphism) such that the Seiberg-Witten equations (2.3) admit solutions. For each* Spin$^c$ *structure $\mathfrak{s}$, any solution $(A, \psi)$ to the equations (2.3) enjoys the following estimates:*

$$|\psi|^2 \leq \max_{y\in Y}\{0, -s(y)\} \triangleq s_g$$

$$||\nabla_A(\psi)||^2_{L^2} \leq \frac{s_g^2}{4}vol(Y)$$

$$|F_A| = \frac{1}{2}|\psi|^2 \leq \frac{s_g}{2}$$

$$||F_A||^2_{L^2_1} \leq C \qquad \textit{for some constant } C \textit{ depending only on } Y.$$

**Proof.** The first and third inequalities in the proposition follow directly from Lemma 2.2.3 and Lemma 2.1.3. From the Weitzenböck formulae and pairing with $\psi$, we know that

$$\|\nabla_A(\psi)\|_{L^2}^2 + \frac{1}{2}\|\psi\|_{L^4}^4 + \frac{1}{4}\langle s\psi, \psi\rangle_{L^2} = 0$$

Therefore, $\|\nabla_A(\psi)\|_{L^2}^2 \leq -\frac{1}{4}\langle s\psi, \psi\rangle_{L^2} \leq \frac{s_g^2}{4} vol(Y)$. The bound of the curvature $F_A$ means that $\frac{iF_A}{2\pi}$ lies in a compact subset inside $H^2(Y, \mathbb{R})$, being an integral class in $H^2(Y, \mathbb{Z})$ implies that there are only finitely many Spin$^c$ structures (parametrized by $H^2(Y, \mathbb{Z})$) which admit the Seiberg-Witten monopoles. The fourth inequality follows from the third by integrating over $Y$ and the following estimate:

$$\begin{aligned} \|dF_A\|_{L^2}^2 &= \|d(*\sigma(\psi, \psi))\|_{L^2}^2 \\ &\leq C_0\|\nabla_A(\psi)\|_{L^2}^2\|\psi\|_{C^0} \end{aligned}$$

for some constant $C_0$. □

Using the bootstrapping technique and the Seiberg-Witten equations (2.3), we can improve the $L_2^2$-bound on $A$ and $C^0$-bound on $\psi$ to $C^\infty$-bound on both $A$ and $\psi$ which gives rise to the compactness of the moduli space $\mathcal{M}$. Note that we can obtain the $C^\infty$-bound for each $\psi$ immediately from Garding-type inequality Lemma 2.1.6, but now we must obtain a uniform bound for a sequence of solutions. The first lemma we need is the gauge fixing condition and the estimate of $L_k^2$-bound on connections via an $L_{k-1}^2$ bound on curvature. In Yang-Mills theory, this is called Uhlenbeck's lemma, for 4-dimensional Seiberg-Witten theory, a similar lemma is obtained by Morgan in [35] where the $L_k^2$-norm of the gauged connection is bounded by the $L_{k-1}^2$-norm of the self-dual part of the curvature. Our lemma concerns the 3-dimensional case.

**Lemma 2.2.6.** *Let $L$ be a complex linear bundle over $Y$ with a Hermitian metric. Fix a $U(1)$-connection $A_0$ on $L$, then for any $k \geq 0$, these exist constants $C_1, C_2 > 0$ depending only on $Y, A_0, k$ such that the following hold: for any $L_k^2$ $U(1)$-connection*

*A on L, there is an $L^2_{k+1}$ gauge transformation u such that $u^*(A) = A_0 + \alpha$ where $\alpha \in \Omega^1_{L^2_k}(Y, i\mathbb{R})$ satisfies*

$$d^*\alpha = 0$$
$$\|\alpha\|^2_{L^2_k} \leq C_1 \|F_A\|^2_{L^2_{k-1}} + C_2.$$

**Proof.** Write $A = A_0 + \alpha_0$ for $\alpha \in \Omega^1_{L^2_k}(Y, i\mathbb{R})$. On the $L^2$-orthogonal space of constant functions in $\Omega^0_{L^2_{k-1}}(Y, i\mathbb{R})$, the Laplacian $\Delta$ has an inverse $\Delta^{-1}$. Let

$$f_0 = -\Delta^{-1}(d^*(\alpha_0)) \in \Omega^0_{L^2_{k+1}},$$

let $u_0 = exp(-f_0)$ which is $L^2_{k+1}$ integrable, then

$$u_0(A) = A_0 + \alpha_0 + df_0 \triangleq A_0 + \alpha_1$$

where $\alpha_1 = \alpha_0 + df_0$ is $d^*$-closed and $L^2_k$-integrable. By the Hodge theorem, we can write $\alpha_1 = h + \alpha_2$ where $h$ is a harmonic 1-form and $\alpha_2 \in H^\perp$ the subspace $L^2$-orthogonal to the harmonic 1-forms. On $H^\perp$, $(d^* + d)$ is a bounded operator, therefore, there is a constant $C$, depending only on $Y$ and $k$ such that

$$\begin{aligned} \|\alpha_2\|^2_{L^2_k} &\leq C\|(d^*\alpha_2 + d\alpha_2)\|^2_{L^2_{k-1}} \qquad (d^*\alpha_2 = 0) \\ &= C\|F_A - F_{A_0}\|^2_{L^2_{k-1}} \\ &\leq C\|F_A\|^2_{L^2_{k-1}} + C, \end{aligned}$$

after rescaling the constant $C$, depending on $Y, A_0$ and $k$. We need only to estimate $h$ (the harmonic part of $\alpha_1$) by further transformation but without affecting $\alpha_2$. Notice that $u_0$ belongs to the identity component of $\mathcal{G}$ and $h \in iH^1(Y, \mathbb{R})$. Note also $\pi_0(\mathcal{G}) = H^1(Y, \mathbb{Z})$ and $iH^1(Y, \mathbb{R})/iH^1(Y, \mathbb{Z})$ is a compact torus, we can choose a gauge transformation $f : Y \to U(1)$ in an appropriate component such that

$$f(u_0(A)) = A_0 + h_1 + \alpha_2.$$

where $h_1$ is $L^2_k$-bounded. In detail, we have $\pi_0(\mathcal{G}) \cong iH^1(Y, \mathbb{Z})$, those harmonic 1-forms with period in $i\mathbb{Z}$, i.e, those harmonic 1-forms $h_0 = f^{-1}df$ where $f : Y \to S^1$ is harmonic. Write $h = h_0 + h_1$ where $h_0 = f^{-1}df$ represents an element in $iH^1(Y, \mathbb{Z})$ and $h_1$ is $L^2_k$-bounded. Apply the gauge transformation $f : Y \to S^1$, we obtain that

$$\begin{aligned} f(u_0(A_0 + \alpha_0)) &= f(A_0 + h + \alpha_2) \\ &= A_0 + h - f^{-1}df + \alpha_2 \\ &= A_0 + h_1 + \alpha_2. \end{aligned}$$

Setting $\alpha = h_1 + \alpha_2$, we have $d^*\alpha = 0$ and

$$\|\alpha\|^2_{L^2_k} \leq \|h_1\|^2_{L^2_k} + \|\alpha\|^2_{L^2_k} \leq C_1\|F_A\|^2_{L^2_{k-1}} + C_2,$$

for some constants $C_1, C_2$ depending only on $Y, A_0$ and $k$. This completes the proof of the Lemma. □

**Corollary 2.2.7.** *Fix a smooth, $U(1)$-connection $A_0$ on $det(\mathfrak{s})$ for a* Spin$^c$ *structure $\mathfrak{s}$. These exists a constant $C$ depending only on $Y$ and $A_0$ such that for any solution $(A, \psi)$ to the Seiberg-Witten equations, we have a gauge equivalent connection $A' = A_0 + \alpha$ with the properties:*

$$\begin{cases} d^*\alpha = 0 \\ \|\alpha\|^2 \leq C. \end{cases}$$

**Proof.** This is the direct consequence of Proposition 2.2.5 and Lemma 2.2.6. □

Now we are ready to begin our bootstrapping argument to obtain a uniform bound for solutions to the Seiberg-Witten equations (2.3).

**Theorem 2.2.8.** *Fix a smooth, U(1)-connection $A_0$ on $det(\mathfrak{s})$, suppose that $(A, \psi)$ is a solution to the Seiberg-Witten equations (2.3) such that $A = A_0 + \alpha$ with $d^*\alpha = 0$ and the harmonic projection of $\alpha$ has a bound only depending on $Y, A_0$. For every $k \geq 2$, there exists a constant $C(k)$, depending only on $Y, A_0$ and $k$ such that*

$$\|\alpha\|^2_{L^2_k} + \|\psi\|^2_{L^2_k} \leq C(k),$$

*where the $L^2_k$-norm of the spinor is taken with respect to $\nabla_{A_0}$.*

**Proof.** We will repeatedly apply the Sobolev embedding theorem (see [3]) and the Sobolev multiplication theorem, so we summarize them as follows.

- **Sobolev embedding theorem**: the inclusion $L^p_k \hookrightarrow L^q_l$ is a bounded map if $k - \frac{3}{p} \geq l - \frac{3}{q}$ (here 3 is the dimension of our 3-manifold, in general, it should be the dimension of the underlying manifold, $w(k,p) = k - \frac{n}{p}$ is called the "scaling weight" for $L^p_k$).

- **Sobolev multiplication theorem**: $L^2_k \times L^p_l \hookrightarrow L^p_l$ is a bounded map if $k \geq l$ and $k \geq 2$( using the Leibnitz's rule for derivatives and the fact: for $k \geq 2$, $L^2_k \hookrightarrow C^0$ is bounded in dimension 3). There is a general Sobolev multiplication theorem, the above Sobolev multiplications are enough for us.

**Step 1:** We already have $\psi$ is $C^0$-bounded, $\nabla_A\psi$ is $L^2$-bounded and $\alpha$ is $L^2_2$-bounded, therefore, we have

$$\begin{aligned} ||\psi||^2_{L^2_1} &= ||\psi||^2_{L^2} + ||\nabla_{A_0}\psi||^2_{L^2} \\ &\leq ||\psi||^2_{L^2} + ||\nabla_A\psi||^2_{L^2} + ||\alpha||^2_{L^2}||\psi||_{C^0}. \end{aligned}$$

That means $\psi$ is $L^2_1$-bounded.

**Step 2:** By the Dirac equation, we know that

$$\partial\!\!\!/_{A_0}\psi = -\frac{1}{2}\alpha.\psi$$

Since $\alpha$ is $L^2_2$-bounded and the Sobolev embedding $L^2_2 \hookrightarrow L^4$ is a bounded map, we see that $\partial\!\!\!/_{A_0}\psi$ is $L^4$-bounded (note here we only use the $C^0$-bound on $\psi$).
$L^2$-decomposition: write $\psi = \psi_0 + \phi$ with $\partial\!\!\!/_{A_0}\phi = 0$ and $\psi_0 \perp Ker\partial\!\!\!/_{A_0}$ under the $L^2$-inner product. First, since $\psi$ is $L^2$-bounded and $(\psi_0, \phi)_{L^2} = 0$, $\phi$ is also $L^2$-bounded, together with $\partial\!\!\!/_{A_0}\phi = 0$ and $\partial\!\!\!/_{A_0}$ is elliptic, we know that $\phi$ is $C^0$-bounded (elliptic regularity). Secondly, $\psi_0$ is orthogonal to $Ker\partial\!\!\!/_{A_0}$ and $\partial\!\!\!/_{A_0}$ is elliptic, from the injectivity of the symbol map ($\partial\!\!\!/_{A_0}$ is elliptic), we see that there exists a constant $C_3$ depending only on $Y$ and $A_0$ such that

$$\begin{aligned} ||\psi_0||_{L^4_1} &\leq C_3||\partial\!\!\!/_{A_0}\psi_0||_{L^4} \\ &= C_3||\partial\!\!\!/_{A_0}\psi||_{L^4} \leq C_4 \qquad \text{(from Step 1).} \end{aligned}$$

Hence, $\psi_0$ is $L^4_1$-bound, then $\psi = \psi_0 + \phi$ is also $L^4_1$-bound.

Using the Sobolev multiplication and Sobolev embeddings:

$$L^2_2 \times L^4_1 \hookrightarrow L^4_1 \hookrightarrow L^3_1,$$

$\partial\!\!\!/_{A_0}\psi = -\frac{1}{2}\alpha.\psi$ is $L^3_1$-bounded. Repeat the $L^2$-decomposition procedure as above, we know that $\psi$ is $L^3_2$-bounded. Using the Sobolev multiplication and Sobolev embeddings:

$$L^2_2 \times L^3_2 \hookrightarrow L^3_2 \hookrightarrow L^2_2,$$

we know that $\partial\!\!\!/_{A_0}\psi = -\frac{1}{2}\alpha.\psi$ is $L^2_2$-bounded, repeat the $L^2$-decomposition procedure again, we know that $\partial\!\!\!/_{A_0}\psi$ is $L^2_2$-bounded, which yields that $\psi$ is $L^2_3$-bounded.

From the curvature equation in the Seiberg-Witten equations (2.3) and the Sobolev multiplication $L^2_3 \times L^2_3 \hookrightarrow L^2_3$, we obtain that $F_A$ is $L^2_3$-bounded, then by Lemma 2.2.6, $\alpha$ is $L^2_4$-bounded.

**Step 3:** Suppose by induction that for some $k \geq 3$, we have $L^2_k$-bounds of $\alpha$ and $\psi$. Then from

$$\partial\!\!\!/_{A_0}\psi = -\frac{1}{2}\alpha.\psi$$

and the Sobolev multiplication: $L^2_k \times L^2_k \hookrightarrow L^2_k$, we know that $\partial\!\!\!/_{A_0}\psi$ is $L^2_k$-bounded, then the $L^2$-decomposition procedure yields that $\psi$ is $L^2_{k+1}$-bounded. From the curvature equation and the Sobolev multiplication: $L^2_{k+1} \times L^2_{k+1} \hookrightarrow L^2_{k+1}$, $F_A$ is $L^2_{k+1}$-bounded, hence $\alpha$ is $L^2_{k+2}$-bounded. By induction, we complete the proof of the Theorem. □

**Corollary 2.2.9.** *Let $\{(A_n, \psi_n)\}$ be any sequence of solutions to the Seiberg-Witten equations. Then after passing to a subsequence and applying $L^2_2$ gauge transformations $\{h_n\}$, $\{h_n(A_n, \psi_n)\}$ are $C^\infty$ and converge in $C^\infty$-topology to a limit which is also a solution to the Seiberg-Witten equations. Therefore, the moduli space $\mathcal{M}_{\mathfrak{s}}$ of Seiberg-Witten equations (2.3) is sequentially compact.*

From the index of the complex (2.7) and the above discussion, we know that $\mathcal{M}$ consists of only finitely many points if $\mathcal{M}$ is smooth. In general, we need to perturb the Seiberg-Witten equations (2.3) to achieve the smoothness of $\mathcal{M}$, that is, for generic perturbation, $T_{(A,\psi)}$ in (2.7) will be surjective. There are two kinds of perturbations: one is to perturb the curvature equation by adding an imaginary, co-closed 1-form $\eta$ on $*F_A - \sigma(\psi, \psi) = 0$. $\eta$ must be co-closed, since $\sigma(\psi, \psi)$ is co-closed if $\psi$ satisfies the Dirac equation and $*F_A$ is also co-closed (Bianchi identity).

The other perturbation is to perturb the Dirac equation by Clifford multiplication of any imaginary 1-form. Either of them is sufficient to obtain the smoothness of $\mathcal{M}$. Here we only consider the first kind of perturbation by adding a co-closed 1-form to the curvature equation, though transversality of the second kind of perturbation is simpler to prove.

Let $\eta \in \Omega^1_{L^2_2}(Y, i\mathbb{R})$ be a co-closed 1-form. The perturbed Seiberg-Witten equations can be written as follows:

$$\begin{cases} \not{\partial}_A \psi = 0, \\ *F_A = \sigma(\psi, \psi) + \eta. \end{cases} \tag{2.9}$$

For a small perturbation, the solutions to (2.9) enjoy all the properties of the solutions to the unperturbed Seiberg-Witten equations (2.3) almost word by word, we won't repeat them in this perturbed situation but we just restate the $C^0$-estimate Lemma 2.2.3 to indicate the small modification due to the perturbation, the proof is the same.

**Remark 2.2.10.** *Let $(A, \psi)$ be a smooth solution to the perturbed Seiberg-Witten equations (2.9),*

$$|\psi|^2 \leq \max_{y \in Y}\{0, -s(y) + 2\|\eta\|_{C^0}\}$$

*where $s(y)$ is the scalar curvature for $(Y, g)$.*

The corresponding moduli space is denoted by $\mathcal{M}_{\mathfrak{s},\eta}$, $\mathcal{M}^*_{\mathfrak{s},\eta}$ its irreducible part. Denote by $Z^1(Y, i\mathbb{R})$ the space of co-closed, $L^2_2$-integrable, imaginary 1-forms on $Y$.

The perturbed Seiberg-Witten equations define an $\mathcal{G}$-equivariant map:

$$\mathcal{A}_{L^2_1} \times Z^1(Y, i\mathbb{R}) \xrightarrow{\tilde{s}} \Omega^1_{L^2}(Y, i\mathbb{R}) \times L^2(W),$$

$$(A, \psi, \eta) \mapsto (*F_A - \sigma(\psi, \psi) - \eta, \not{\partial}_A \psi).$$

Over $\mathcal{B}$, $\tilde{s}$ defines a section whose zero set is the parametrized moduli space $\mathfrak{M}$:

$$\mathfrak{M} = \bigcup_{\eta \in Z^1(Y, i\mathbb{R})} \mathcal{M}_{\mathfrak{s},\eta}.$$

The fiber of the natural projection $\mathfrak{M} \to Z^1(Y, i\mathbb{R})$ at $\eta \in Z^1(Y, i\mathbb{R})$ is the perturbed moduli space $\mathcal{M}_{\mathfrak{s},\eta}$. Denote by $\mathfrak{M}^*$ the irreducible part of $\mathfrak{M}$.

At any solution $(A_0, \psi_0, \eta_0)$, the linearization of the perturbed Seiberg-Witten equations and the linearization of gauge transformations define a corresponding extended Hessian operator (cf. 2.7). Here $T_{(A_0,\psi_0,\eta_0)}$:

$$\Omega^0_{L^2_2}(Y, i\mathbb{R}) \oplus \Omega^1_{L^2_1}(Y, i\mathbb{R}) \oplus L^2_1(W) \oplus Z^1(Y, i\mathbb{R}) \to \Omega^1_{L^2}(Y, i\mathbb{R}) \oplus L^2(W) \oplus \Omega^0_{L^2}(Y, i\mathbb{R})$$

is defined by assigning to $(f, a, \phi, \epsilon)$ the element:

$$(*da - 2\sigma(\psi_0, \phi) - 2df + \epsilon, \partial\!\!\!/_{A_0}\phi + \frac{1}{2}a.\psi_0 + f\psi_0, -2d^*a - 2iIm\langle\psi_0, \phi\rangle). \quad (2.10)$$

**Lemma 2.2.11.** *At any solution $(A_0, \psi_0, \eta_0)$ to the perturbed equations (2.9) with $\psi_0 \neq 0$, $T_{(A_0,\psi_0,\eta_0)}$ is surjective. In particular, the parametrized irreducible moduli space $\mathfrak{M}^*$ is smooth.*

**Proof.** Let $(a_1, \phi_1, f_1)$ be $L^2$-orthogonal to the image of $T_{(A_0,\psi_0,\eta_0)}$ under the $L^2$-inner product. By varying $\epsilon \in Z^1(Y, i\mathbb{R})$ alone, we see that $a_1$ must be an exact 1-form. Then from the direct calculations, we know that $(a_1, \phi_1, f_1)$ satisfies the following equations (note that the $L^2$-metric on spinors is twice of the real part of the Hermitian metric $\langle , \rangle$):

$$\begin{cases} (1). \quad d^*a_1 + iIm\langle\psi_0, \phi_1\rangle = 0, \\ (2). \quad df_1 + \sigma(\psi_0, \phi_1) = 0, \qquad \text{see (2.2)} \\ (3). \quad \partial\!\!\!/_{A_0}\phi_1 + \frac{1}{2}a_1.\psi_0 + f_1\psi_0 = 0. \end{cases} \quad (2.11)$$

Apply $d^*$ to the second equation (2), we see that

$$\begin{aligned} d^*df_1 &= -d^*\sigma(\psi_0, \phi_1) \\ &= -\frac{i}{2}Im\langle\psi_0, \partial\!\!\!/_{A_0}\phi_1\rangle && \text{(by Remark 2.1.7 )} \\ &= \frac{i}{2}Im(\langle\psi_0, \frac{1}{2}a_1.\psi_0 + f_1\psi_0\rangle) && \text{(by the the equation (3))} \\ &= \frac{i}{2}Im\langle\psi_0, f_1\psi_0\rangle && \text{(since } \langle\psi_0, a_1.\psi_0\rangle \text{ is real)} \\ &= \frac{1}{2}\langle\psi_0, f_1\psi_0\rangle = -f_1\langle\psi_0, \psi_0\rangle \end{aligned}$$

Therefore, we have $d^*df_1 + \frac{|\psi_0|^2}{2}f_1 = 0$, from which on $Y - \psi_0^{-1}(0)$, we have $f_1 = 0$. On $\psi_0^{-1}(0)$, we see that $\sigma(\psi_0, \phi_1) = 0$, hence,

$$\sigma(\psi_0, \phi_1) \equiv 0.$$

This implies that $df_1 \equiv 0$, that is $f_1$ is a constant function on $Y$, as a solution to the Dirac equation, $\psi_0 \neq 0$ can not vanish on an open set, then $f_1$ must be zero. Locally from the Spin$^c$ representation (2.1), we get

$$\phi_1 = i\xi\psi_0, \tag{2.12}$$

on the complement of $\psi_0^{-1}(0)$ for some smooth function $\xi : Y - \psi_0^{-1}(0) \to \mathbb{R}$. By the unique continuation theorem for the Dirac equation, $\xi$ can be extended to all of $Y$ and $\phi_1 = i\xi\psi_0$ on the whole of $Y$. Insert $\phi_1 = i\xi\psi_0$ into the third equation (3) (note that $f_1 = 0$ already), we know that

$$id\xi.\psi_0 + \frac{1}{2}a_1.\psi_0 = 0. \tag{2.13}$$

Since the Spin$^c$ representation is the standard representation of $U(2)$ on $\mathbb{C}^2$, we have $a_1 = -2id\xi$ on $Y - \psi_0^{-1}(0)$. Insert it in the first equation (1), we see that, on $Y - \psi_0^{-1}(0)$,

$$d^*d\xi + \xi|\psi_0|^2 = 0.$$

From the maximum principle, $\xi$ must be zero, hence, $\phi_1 = 0$, and $d^*a_1 = 0$, which leads to $a_1 = 0$ since $a_1$ is exact. This completes the proof of the surjectivity of $T_{(A_0,\psi_0,\eta_0)}$.

The smoothness of $\mathfrak{M}^*$ is the conclusion of the implicit function theorem together with the fact that the gauge group acts on $\mathfrak{M}^*$ freely with the slice at $[A_0, \psi_0]$ given by the Coulomb gauge condition:

$$KerG^* = \{(\alpha, \phi) \in \Omega^1_{L^2}(Y, i\mathbb{R}) \oplus L^2(W) | d^*\alpha + iIm\langle\psi_0, \phi\rangle = 0\}.$$

□

Apply the Sard-Smale theorem, we see that there exists a Baire set $\Sigma$ in $Z^1(Y, i\mathbb{R})$, such that for $\eta \in \Sigma$, $\mathcal{M}^*_{\mathfrak{s},\eta}$ is a smooth, 0-dimensional manifold. If we can choose a perturbation such that the perturbed Seiberg-Witten equations have no reducible solutions, then $\mathcal{M}_{\mathfrak{s},\eta} = \mathcal{M}^*_{\mathfrak{s},\eta}$ is a compact, smooth, 0-dimensional manifold, hence consists of finitely many points in $\mathcal{B}^*$. This is indeed the case except when the first Betti number $b_1(Y) = 0$ ($Y$ is a homology 3-sphere). In the following

Lemma, we will discuss the appearance of the reducible solutions to the perturbed Seiberg-Witten equations.

**Lemma 2.2.12.** *For any 3-manifold $Y$ with non-trivial first Betti number, there exists an open dense set of perturbations in $Z^1(Y, i\mathbb{R})$ such that the perturbed Seiberg-Witten equations have no reducible solutions. If $Y$ is a homology 3-sphere, there is always a unique reducible solution (up to gauge transformation) which cannot be perturbed away.*

**Proof.** Suppose $(A, 0)$ is a reducible solution to the perturbed Seiberg-Witten equations (2.9), then

$$*F_A = \eta.$$

Since $\eta \in Z^1(Y, i\mathbb{R})$ is co-closed, write $\eta = \eta_0 + *d\nu$ where $\eta_0$ is an imaginary-valued, harmonic 1-form and $\nu \in \Omega^0(Y, i\mathbb{R})$, therefore,

$$F_A = *\eta_0 + d\nu. \tag{2.14}$$

Certainly, if the first Betti number $b_1(Y) \geq 1$, the equation (2.14) cannot have a solution for an open dense set (those $\eta$ whose harmonic part is not $[*F_A]$ since $[*F_A]$ is only a point in the lattice $2\pi i H^1(Y, \mathbb{Z}) \in H^1(Y, i\mathbb{R})$. The perturbation with the harmonic part representing $[*F_A]$ is a co-dimension $b_1(Y)$ subspace of $Z^1(Y, i\mathbb{R})$. If Y is a homology 3-sphere, then $[iF_A/(2\pi)]$ is a torsion element in $H^2(Y, \mathbb{Z})$ which classifies the line bundle, and $H^2(Y, i\mathbb{R}) = 0$ implies that $\eta = *d\nu$. For each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, $*F_A = d\eta$ always has a unique solution $A$ whose connection 1-form is given by $\nu$ (up to gauge transformation). □

**Corollary 2.2.13.** *For a 3-manifold $Y$ with $b_1(Y) > 0$, there exists a Baire set of perturbations in $Z^1(Y, i\mathbb{R})$ such that for any perturbation $\eta$ in this open dense set, the perturbed Seiberg-Witten moduli space $\mathcal{M}_{\mathfrak{s},\eta}$ consists of finitely many smooth points in $\mathcal{B}^*$. Moreover, the perturbation $\eta$ can be chosen to represent a trivial cohomology class if $c_1(det\mathfrak{s}) \neq 0$ in $H^2(Y, \mathbb{R})$.*

**Proof.** We only need to prove that we can choose $[\eta] = 0$ as a de Rham cohomology class. Checking through the proof of Lemma 2.2.11 again, we can choose $\eta$ to be a co-closed 1-form $*d\mu$, where $\mu$ is an imaginary 1-form on $Y$. Then we need to verify surjectivity of $T_{(A_0,\psi_0,\eta_0)}$ in Lemma 2.2.11. Now the equations in (2.11) don't require that $a_1$ be exact (cf. the proof of Lemma 2.2.11), but the exactness of $a_1$ can be obtained from equation (2.13) and the unique continuation property of the Dirac equation. The rest of the proof goes in the same way as the proof of Lemma 2.2.11, after noting that we need the first Chern class of the Spin$^c$ structure to be non-trivial in $H^2(Y,\mathbb{R})$ so there is no reducible solution in $\mathcal{M}_{\mathfrak{s},\eta}$ for a $*d$-exact one-form $\eta$. □

For a homology 3-sphere $Y$ we can perturb the Seiberg-Witten equations further such that the reducible solution is isolated in $\mathcal{M}_{\mathfrak{s},\eta}$. Then $\mathcal{M}^*_{\mathfrak{s},\eta}$ also consists of finitely many smooth points in $\mathcal{B}^*$.

Notice that a homology 3-sphere $Y$ has only finitely many Spin$^c$ structures, for each Spin$^c$ structure $\mathfrak{s}$ there is only one reducible Seiberg-Witten monopole (up to gauge transformation) for the perturbation $\eta = *d\nu$. Without confusion, denote by $\nu$ the corresponding connection on $det(\mathfrak{s})$.

**Lemma 2.2.14.** *Let $(Y, g, \mathfrak{s})$ be a homology 3-sphere with a* Spin$^c$ *structure $\mathfrak{s}$, denote by $[\nu, 0]$ the unique reducible point in the perturbed Seiberg-Witten moduli space $\mathcal{M}_{\mathfrak{s},*d\nu}$. If the twisted Dirac operator $\partial\!\!\!/_\nu$ has trivial kernel, then $[\nu, 0]$ is isolated in $\mathcal{M}_{\mathfrak{s},*d\nu}$ in the $C^\infty$-topology.*

**Proof.** We use perturbation theory by expanding the solution $[A,\psi]$ to the perturbed equations (2.9) near $[\nu, 0]$. Write $(A,\psi) = (\nu + a, \psi)$, then $(a, \psi)$ satisfies the following equations:

$$\begin{cases} (i) & d^*a = 0 \\ (ii) & *da = \sigma\psi,\psi) \\ (iii) & \partial\!\!\!/_\nu(\psi) + \frac{1}{2}a.\psi = 0 \end{cases}$$

Expand $(a, \psi)$ as

$$a = \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \cdots$$
$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \cdots .$$

Then from the Dirac equation (iii) and $Ker\partial\!\!\!/_\nu = 0$, we know that $\psi_1 = 0$. Using this fact and the curvature equation, we know that $a_i$ $(i = 1, 2, 3)$ are closed and co-closed for a homology 3-sphere, this leads to $a_i (i = 1, 2, 3)$ must be zero. Repeating the above procedure, we get $[a, \psi]$ is zero. This proves the Lemma. □

The following Proposition proves that the condition $Ker \partial\!\!\!/^g_\nu = 0$ holds for generic metrics $g$ and perturbations $\nu$.

**Proposition 2.2.15.** *Denote by $Met$ the space of all Riemannian metrics on a homology 3-sphere $Y$, consider the twisted Dirac operator $\partial\!\!\!/^g_\nu$ associated with the chosen metric $g$ and the connection $\nu$. The condition $Ker\partial\!\!\!/^g_\nu \neq 0$ determines a subset in the space of $Met \times Z^1(Y, i\mathbb{R})$ with real codimension one.*

**Proof.** Let $g_0$ be a metric on $Y$ such that $Ker\partial\!\!\!/^{g_0}_{\nu_0} \neq 0$ for the connection $\nu_0$ on $det(\mathfrak{s})$. We can decompose the spinor space as $\mathcal{H} \oplus \mathcal{H}^\perp$, under the $L^2$ inner product, where $\mathcal{H} = Ker\partial\!\!\!/^{g_0}_{\nu_0}$, equipped with a Hermitian metric from the Spin$^c$ structure. Consider the Dirac operator $\partial\!\!\!/^g_\nu$ for $(g, \nu)$ sufficiently close to $(g_0, \nu_0)$ under the $C^\infty$-topology. Under the isometry identification of the spinor spaces for $g_0$ and $g$, the Dirac operator for $g$ can be conjugated to act on the spinor space of $g_0$, still denoted by $\partial\!\!\!/^g_\nu$.

<u>Claim 1 :</u> $\partial\!\!\!/^g_\nu$ acting on the spinor space of $g_0$ is self-adjoint if and only if they define the same volume element (up to a scalar).

Suppose $dvol_g = f\ dvol_{g_0}$ for a positive function $f$ on $Y$, then from the direct calculation, we have

$$< \partial\!\!\!/^g_\nu \psi, \phi >_{g_0} = < \psi, f\partial\!\!\!/^g_\nu (f^{-1}\phi) >_{g_0}$$

Then the Claim 1 is obvious. Denote by $Met^0$ the space of metrics which have the same volume element as $g_0$ up to a scalar.

<u>Claim 2 :</u> If two metrics $g_1$ and $g_2$ are conformal, that is, $g_1 = e^{2u} g_2$ for a real function $u$ on $Y$, then the multiplication by $e^{-u}$ defines an isomorphism between $Ker\partial\!\!\!/^{g_1}_\nu$ and $Ker\partial\!\!\!/^{g_2}_\nu$.

This is the consequence of the following relations: under the isometry identification of the the spinor spaces for $g_1$ and $g_2$, we have (see Proposition 1.3 in [24] or Theorem 5.24 in [28])

$$\partial\!\!\!/^{g_1}_\nu = e^{-u} \partial\!\!\!/^{g_2}_\nu e^u.$$

Therefore, we only need to prove that the condition $Ker\partial\!\!\!/^g_\nu \neq 0$ determines a real codimension one subset in the space of $Met^0 \times Z^1(Y, i\mathbb{R})$.

We want to reduce the problem of the existence of solutions of the equation $\partial\!\!\!/^g_\nu\psi = 0$ to a finite dimensional problem on $\mathcal{H}$.

As a map from $Met^0 \times Z^1(Y, i\mathbb{R}) \times \Gamma(W)$ to $\Gamma(W)$, the linearization of the equation $\partial\!\!\!/^g_\nu\psi = 0$ at $(g_0, \nu_0, 0)$ is invertible when restricted to $\mathcal{H}^\perp$. Thus, the implicit function theorem provides a unique map $q : \mathcal{U} \to \mathcal{H}^\perp$ defined on a neighbourhood $\mathcal{U}$ of $(g_0, \nu_0, 0)$ in $Met^0 \times Z^1(Y, i\mathbb{R}) \times \mathcal{H}$, such that

$$(1 - \Pi)\partial\!\!\!/^g_\nu(\phi + q(g, \nu, \phi)) = 0$$

for all $(g, \nu, \phi) \in \mathcal{U}$, where $\Pi$ is the projection onto $\mathcal{H}$.

Therefore, the operator $\partial\!\!\!/^g_\nu$ has non-trivial kernel if and only if the equation

$$\Pi\partial\!\!\!/^g_\nu(\phi + q(g, \nu, \phi)) = 0$$

admits solutions in $\mathcal{H}$. This is a finite dimensional problem. Define a map

$$L : (Met^0 \times Z^1(Y, i\mathbb{R})) \cap \mathcal{U} \longrightarrow U(\mathcal{H})$$

by sending

$$L(g, \nu)(\phi) = \Pi\partial\!\!\!/^g_\nu(\phi + q(g, \nu, \phi)).$$

Direct calculation implies that $L(g, \nu)$ is a Hermitian transformation of the space $\mathcal{H}$, that means $L(g, \nu) \in U(\mathcal{H})$, where $U(\mathcal{H})$ is the space of Hermitian transformations on $\mathcal{H}$. The kernel of $\partial\!\!\!/^g_\nu$ is non-trivial if and only if the kernel of $L(g, \nu)$ is non-trivial. There is a real-valued function on $U(\mathcal{H})$, which is the determinant function. Therefore, we have a real-valued function $f(g, \nu) = det(L(g, \nu))$ on the neighborhood of $(g_0, \nu_0)$ in $Met^0 \times Z^1(Y, i\mathbb{R})$. Those $(g, \nu)$ with non-trivial kernel have value 0 for this function.

Now we only need to check that the derivative of $f(g, \nu)$ at $(g_0, \nu_0)$ is surjective, then the Lemma follows from the Morse theory. It can be checked by differentiating $f(g, \nu)$ at $(g_0, \nu_0)$ along $(0, \alpha)$-direction, for $\alpha \in \Omega^1(Y, i\mathbb{R})$. Since

$$Df_{(g_0, \nu_0)}(0, \alpha) = Tr(\phi \mapsto \Pi(\frac{1}{2}\alpha.\phi)),$$

which is non-zero for suitable choice of $\alpha$. □

In summary, for the homology 3-spheres case, from this Proposition, we see that the complement of the codimension one subspace in the open dense subset of $Mat \times Z^1(Y, i\mathbb{R})$ has many chambers, for a pair $(g, \nu)$ in each chamber, the moduli space $\mathcal{M}_{\mathfrak{s},\nu}$ has only one reducible point $[\nu, 0]$ which is isolated, therefore,

$$\mathcal{M}^*_{\mathfrak{s},\nu} = \mathcal{M}_{\mathfrak{s},\nu} - \{[\nu, 0]\}$$

consists of finitely many smooth points. From the proof of Corollary 2.2.13, we know that for a 3-manifold $Y$ with $b_1 > 1$, the open dense set, In which the perturbed Seiberg-Witten moduli space $\mathcal{M}_{\mathfrak{s},\eta}$ consists of finitely many smooth points, is path-connected; while for a 3-manifold $Y$ with $b_1 = 1$, this open dense set is not path-connection since the perturbation with the harmonic part representing $[*F_A]$ (when the perturbed equations (2.9) have reducible solutions) is a co-dimension $b_1(Y) = 1$ subspace of $Z^1(Y, i\mathbb{R})$.

## 2.3 Seiberg-Witten invariants on a 3-manifold

In the last section, we analysed the structure of the perturbed Seiberg-Witten moduli space for a closed, oriented 3-manifold $(Y, g)$ with a Spin$^c$ structure $\mathfrak{s}$. Now we can define the Seiberg-Witten invariant for $(Y, g, \mathfrak{s})$ if the smooth point is oriented, that is $\mathcal{M}^*_{\mathfrak{s},\nu}$ is a finite set of points with sign.

To see that $\mathcal{M}^*_{\mathfrak{s},\nu}$ is orientable, we only need to show that the determinant bundle of the deformation complex (2.7)

$$\mathfrak{L} = \wedge^{\max}(KerT) \otimes \wedge^{\max}(CokerT)$$

is a trivial real line bundle over $\mathcal{M}^*_{\mathfrak{s},\nu}$. At each point $[A_0, \psi_0]$, $T_{(A_0,\psi_0)}$ is homotopic to the sum of the signature operator and the twisted Dirac operator:

$$\begin{pmatrix} *d & -2d \\ -2d^* & 0 \end{pmatrix} \oplus \not{\partial}_{A_0}$$

Since the Dirac operator $\not{\partial}_{A_0}$ is a complex operator, therefore the determinant bundle $\wedge^{\max}(CokerT)$ is naturally trivial over $\mathcal{B}^*$ and oriented by the complex structure. The kernel and the cokernel of the signature operator are both $H^0(Y, i\mathbb{R}) \oplus H^1(Y, i\mathbb{R})$, which are independent of the base point, therefore, its determinant bundle is trivial over $\mathcal{B}^*$. A choice of orientation of $H^0(Y, i\mathbb{R}) \oplus H^1(Y, i\mathbb{R})$ determines an orientation of $\mathcal{M}_{\mathfrak{s},\eta}$ at any smooth, irreducible points. The same argument can be employed for the parametrized moduli space if the parameter space is oriented.

To determine the sign for the point $[A, \psi] \in \mathcal{M}^*_{\mathfrak{s},\eta}$ (in the case where $b_1(Y) > 0$, $\mathcal{M}^*_{\mathfrak{s},\eta} = \mathcal{M}_{\mathfrak{s},\eta}$), we need to compare the orientation of $\mathfrak{L}_{[A,\psi]}$ with the natural orientation of a trivial real bundle $\underline{\mathbb{R}}$ over $\mathcal{M}^*_{\mathfrak{s},\eta}$. Notice that the orientation changes if and only if the orientation of $\mathfrak{L}$ changes along a path connecting two points in $\mathcal{M}^*_{\mathfrak{s},\eta}$, this happens only where $T$ (2.7) is non-invertible ($T$ is invertible at each point in $\mathcal{M}^*_{\mathfrak{s},\eta}$). Therefore, we can use the spectral flow of the extended operator $T$ in (2.7) to determine the Seiberg-Witten invariants (up to an overall sign).

For a homology 3-sphere, we have the a priori solution (the reducible) $[\nu, 0]$ from which we recover the overall sign, the sign at each point $[A, \psi]$ is determined by

$$(-1)^{SF^{[A,\psi]}_{[\nu,0]}(T)}$$

where $SF^{[A,\psi]}_{[\nu,0]}(T)$ is the spectral flow of $T$ (2.7) along a path from a representation of $[\nu, 0]$ to that of $[A, \psi]$ in $\mathcal{A}$. This spectral flow is independent of the choices of the path and the lifts of $[\nu, 0]$ and $[A, \psi]$ in $\mathcal{A}$, here we need to choose a small positive $\epsilon$ and use the $(-\epsilon)$-spectral flow to define $SF^{(A,\psi)}_{(\nu,0)}(T)$ (see [2]) since $T_{[\nu,0]}$ has one dimensional kernel. Or one could choose an irreducible pair $(A_1, \psi_1)$ sufficiently close to $(\nu, 0)$ with trivial kernel, then apply the ordinary spectral flow (counting the eigenvalues crossing 0), one needs to add a term $\delta_{(A_1,\psi_1)}$ to eliminate the dependence of the spectral flow on the choice of $(A_1, \psi_1)$. One way to realise this is to choose a path connecting $(\nu, 0)$ and $(A_1, \psi_1)$ such that $T_s$ on this path has no kernel except $s = 0$. Let $\lambda(s)$ ($\lambda(0) = 0$) be the eigenvalue of $T_s$ near 0, then

$$\delta_{(A_1,\psi_1)} = \begin{cases} 0 & \text{if } \lambda(s) \geq 0, \\ -1 & \text{if } \lambda(s) \leq 0. \end{cases}$$

One can check that these two definitions of the spectral flow are the same by using the additivity property of the spectral flow. In this thesis, we use the convention of $(-\epsilon)$-spectral flow if the operator has non-trivial kernel at the end-points.

**Remark 2.3.1.** *For a family of self-adjoint, Fredholm operators $D_s$ $(s \in [0,1])$ with only zero eigenvalue in $[-\epsilon, \epsilon]$ for $D_0, D_1$, the $\epsilon$-spectral flow and the $(-\epsilon)$-spectral flow of $D_s$ can be related by the following formula:*

$$(\epsilon - SF)(D_s) = ((-\epsilon) - SF)(D_s) + dimKer(D_0) - dimKer(D_1).$$

We will reiterate this spectral flow in the next section where we define the relative indices for solutions in $\mathcal{M}_{\mathfrak{s},\eta}$. So we won't spend time here to justify the arguments.

Now we can define the Seiberg-Witten invariant for any closed, oriented 3-manifold $(Y, g)$ with a Spin$^c$ structure $\mathfrak{s}$.

**Definition 2.3.2.** *Let $Y$ be an oriented, closed 3-manifold $(Y, g)$ with a* Spin$^c$ *structure $\mathfrak{s}$, $\mathcal{M}^*_{\mathfrak{s},\eta}$ $(\mathcal{M}^*_{\mathfrak{s},\eta} = \mathcal{M}_{\mathfrak{s},\eta}$ for $b_1(Y) > 0)$ is the perturbed Seiberg-Witten moduli space which consists of a finite set of signed, smooth points. Counting the points in $\mathcal{M}^*_{\mathfrak{s},\eta}$ defines the Seiberg-Witten invariant:*

$$\lambda_{SW}(Y, g, \eta, \mathfrak{s}) = \sum_{x \in \mathcal{M}^*_{\mathfrak{s},\eta}} sign(x).$$

**Lemma 2.3.3.** *If $b_1(Y) > 1$, then $\lambda_{SW}(Y, \mathfrak{s})$ is a topological invariant for each* Spin$^c$ *structure $\mathfrak{s}$, that is $\lambda_{SW}(Y, g, \eta, \mathfrak{s})$ is independent of the metric and perturbation.*

**Proof.** This is the standard oriented cobordism argument for the perturbed Seiberg-Witten moduli space. Note that the appearance of the reducible solutions to the perturbed equations can only happen on a codimension $b_1(Y) > 1$ subset in $Met \times Z^1(Y, i\mathbb{R})$ (which is path-connected). Suppose $(g_0, \eta_0)$, $(g_1, \eta_1)$ are two pairs with the smooth, finitely many solutions (after modulo gauge transformations) $\mathcal{M}_{\mathfrak{s},\eta_0}(Y, g_0)$ and $\mathcal{M}_{\mathfrak{s},\eta_1}(Y, g_1)$ respectively. We can choose a path $(g_t, \eta_t)$ for $t \in [0,1]$ connecting $(g_0, \eta_0)$ and $(g_1, \eta_1)$ such that each moduli space $\mathcal{M}_{\mathfrak{s},\eta_0}(Y, g_t)$ at $t$ consists of finitely many, smooth points. Then the parametrized moduli spaces

$$\bigcup_{t \in [0,1]} \mathcal{M}_{\mathfrak{s},\eta_t}(Y, g_t)$$

defines an oriented cobordism between $\mathcal{M}_{\mathfrak{s},\eta_0}(Y, g_0)$ and $\mathcal{M}_{\mathfrak{s},\eta_1}(Y, g_1)$. By the definition of the Seiberg-Witten invariant, we see that

$$\lambda_{SW}(Y, g_0, \eta_0, \mathfrak{s}) = \lambda_{SW}(Y, g_1, \eta_2, \mathfrak{s}).$$

□

Therefore, the Seiberg-Witten invariant for $Y$ with $b_1(Y) > 1$ is a $\mathbb{Z}$-valued function over the Spin$^c$-structures:

$$\lambda_{SW}(Y): \qquad \mathrm{Spin}^c(Y) \cong H^2(Y,\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

### 2.3.1 Special Case I: $b_1(Y) = 0$

From our earlier discussion, we know that, for a homology 3-sphere $Y$, the Seiberg-Witten invariant is only defined for $(g,\eta)$ whose Dirac operator $\partial\!\!\!/^g_\nu$ has trivial kernel (here $\eta = *d\nu$). In this case, the unique reducible point in $\mathcal{M}_{\mathfrak{s},\nu}$ is isolated, hence, $\mathcal{M}^*_{\mathfrak{s},\nu}$ consists of finitely many smooth points with sign. Proposition 2.2.15 tells us that

$$\mathcal{W} = \{(g, *d\nu) |\ Ker\partial\!\!\!/^g_\nu \neq 0\} \tag{2.15}$$

is a codimension one subset in $Met \times Z^1(Y, i\mathbb{R})$. There are many connected components in $Met\times Z^1(Y,i\mathbb{R}) - \mathcal{W}$. Therefore, the Seiberg-Witten invariant $\lambda_{SW}(Y,\mathfrak{s},g,\nu)$ depends on the metric and perturbation. As a family of $(g_t, *d\nu_t) \in Met \times Z^1(Y, i\mathbb{R})$ for $t \in [-1,1]$ crosses $\mathcal{W}$ transversally, we know that there is a family of reducible solutions $[\nu_t, 0]$, then the Seiberg-Witten invariants for $(g_0, *d\nu_0)$ and $(g_1, *d\nu_1)$ are related by the following "wall-crossing" formula:

$$\lambda_{SW}(Y,\mathfrak{s},g_1,\eta_1) = \lambda_{SW}(Y,\mathfrak{s},g_0,\eta_0) + SF_{[\nu_t,0]}(\partial\!\!\!/^{g_t}_{\nu_t}) \tag{2.16}$$

where $SF_{[\nu_t,0]}(\partial\!\!\!/^{g_t}_{\nu_t})$ is the spectral flow of the Dirac operator $\partial\!\!\!/^{g_t}_{\nu_t}$ for $t \in [0,1]$. There exist two approaches to verify this formula, one is to study the geometric structure of the parametrized moduli space at the reducible whose twisted Dirac operator has non-trivial kernel. The other method (simpler) is to use the algebraic structure of the equivariant Seiberg-Witten-Floer homology. Both of these approaches were applied by Marcolli and myself in [33]. In this subsection, we give a geometric interpretation of the wall-crossing formulae for $b_1 = 0$.

From Proposition 2.2.15, we know that $\mathcal{W}$ is a codimension one subset in

$$Met \times Z^1(Y, i\mathbb{R}).$$

We need to know the dimension of the kernel of $\partial\!\!\!/_\nu^g$ for a generic pair $(g,\nu)$ in $\mathcal{W}$, this is the issue of the following Lemma, it is Chen's work [13] that motivates us to study the structure of $\mathcal{W}$.

**Lemma 2.3.4.** *$\mathcal{W}$ is a stratified space with the highest stratum consisting of those $(g,\nu)$ with $Ker\partial\!\!\!/_\nu^g \cong \mathbb{C}$, in general, the set of pairs $(g,\nu)$ with $Ker\partial\!\!\!/_\nu^g \cong \mathbb{C}^n$ is a codimension $2n-1$ subset in $Met \times Z^1(Y, i\mathbb{R})$.*

**Proof.** As in the proof of Proposition 2.2.15, we only need to prove the Lemma for $(g,\nu)$ in $Met^0 \times Z^1(Y,i\mathbb{R})$ (see Claim 1 and Claim 2 in the proof of Proposition 2.2.15). Consider a real Hilbert bundle $\mathcal{L}$ over

$$Met^0 \times Z^1(Y,i\mathbb{R}) \times (L_1^2(W) - \{0\})$$

whose fiber over $(g,\nu,\psi)$ is

$$\mathcal{L}_{(g,\nu,\psi)} = \{\phi \in L_1^2(W) | Re\langle\phi, i\psi\rangle_g = 0\}$$

Define a section $\zeta$ of $\mathcal{L}$ by assigning to $(g,\nu,\psi)$ the element $\partial\!\!\!/_\nu^g\psi$.

<u>Claim :</u> $\zeta$ is transverse to the zero section of $\mathcal{L}$.

We need to prove the differential map of $\zeta$ is surjective at a zero of $\zeta$. Suppose that $(g_0,\nu_0,\psi_0)$ $(\psi \neq 0)$ satisfies $\partial\!\!\!/_{\nu_0}^{g_0}\psi_0 = 0$. Differentiate $\zeta$ with respect to the directions tangent to $Z^1(Y,i\mathbb{R}) \times (L_1^2(W)/\{0\})$ only, then we see that the differential map is

$$\begin{aligned} \mathcal{D}\zeta: \quad \Omega^1_{L_2^2}(Y,i\mathbb{R}) \oplus L_1^2(W) &\longrightarrow \{\phi \in L_1^2(W) | Re\langle\phi, i\psi_0\rangle_g = 0\} \\ (\nu_1,\psi_1) &\mapsto \partial\!\!\!/_{\nu_0}^{g_0}\psi_1 + \frac{1}{2}\nu_1.\psi_0. \end{aligned}$$

If $\phi \in \mathcal{L}_{(g_0,\nu_0,\psi_0)}$ is orthogonal to the image of $\mathcal{D}\zeta$, then $\phi$ satisfies:

$$\begin{cases} (1).\ Re\langle\phi, i\psi_0\rangle_{g_0} = 0, \\ (2).\ Re\langle\phi, \nu_1.\psi_0\rangle_{g_0} = 0, \qquad \text{for any } \nu_1 \in \Omega^1(Y,i\mathbb{R}). \\ (3).\ \partial\!\!\!/_{\nu_0}^{g_0}\phi = 0. \end{cases}$$

From the second equation,we see that there exists a function $f: Y \to \mathbb{R}$ such that $\phi = if\psi_0$. Plugging into the equation (3), using $\partial\!\!\!/_{\nu_0}^{g_0}\psi_0 = 0$, we obtain $df = 0$ which implies $f = C$ is a constant function. Then from (1), we can see that

$$Re\langle iC\psi_0, i\psi_0\rangle_{g_0} = C|\psi_0|^2 = 0$$

and we get $C = 0$. Therefore, $\phi = 0$, which means, $\mathcal{D}\zeta$ is surjective at $(g_0, \nu_0, \psi_0)$. It is easy to see that the index of $\mathcal{D}\zeta$ is the index of $\partial\!\!\!/^{g_0}_{\nu_o}$, which is 1, since $i\psi_0$ is orthogonal to the image of $\partial\!\!\!/^{g_0}_{\nu_o}$.

From the Claim 1, $\zeta^{-1}(0)$ is a Banach manifold and the projection

$$\Pi : \zeta^{-1}(0) \to Met^0 \times \Omega^1(Y, i\mathbb{R})$$

is Fredholm with index 1. Note that for any $(g, \nu) \in Met^0 \times \Omega^1(Y, i\mathbb{R})$, we have $\Pi^{-1}(g, \nu) = Ker\partial\!\!\!/^{g}_{\nu}$.

Moreover, at $(g_0, \nu_0, \psi_0)$, we have

$$\begin{cases} Ker(\Pi_*) = \{\phi \in \Gamma(W) | \partial\!\!\!/^{g_0}_{\nu_0}\phi = 0\} \\ dimKer(\Pi_*) - dimCoker(\Pi_*) = 1. \end{cases}$$

at $(g_0, \nu_0)$. Therefore, $dimCoker(\Pi_*) = dimKer(\Pi_*) - 1$, then the Lemma follows with the highest stratum of codimension one (from Proposition 2.2.15). □

Now we can prove the wall-crossing formula (2.16) by studying the local structure of the parametrized moduli space near the degenerate reducible point. Note that a family of metrics and perturbations $(g_t, \nu_t)$, which don't cross the "wall" $\mathcal{W}$ (2.15), yield the same Seiberg-Witten invariants. From Lemma 2.3.4, we can choose a path of metrics and perturbations $(g_t, \nu_t)$ which cross $\mathcal{W}$ at the highest stratum (a codimension one subset). Suppose that $(g_t, \nu_t)$ ($t \in [-1, 1]$) cross $\mathcal{W}$ only once at $t = 0$ with $Ker\partial\!\!\!/^{g_0}_{\nu_0} \cong \mathbb{C}$. For $|t|$ sufficiently small, write the eigenvalue near 0 for $\partial\!\!\!/^{g_t}_{\nu_t}$ as $\lambda(t)$ with $\lambda(0) = 0$ and $\lambda'(0) \neq 0$. Then we know that

$$SF^{t>0}_{t<0}(\partial\!\!\!/^{g_t}_{\nu_t}) = \begin{cases} 1 & \text{when } \lambda'(0) > 0, \\ -1 & \text{when } \lambda'(0) < 0. \end{cases}$$

Denote by $\mathcal{M}(g, \nu)$ the moduli space of the Seiebrg-Witten equations on $(Y, \mathfrak{s})$ with metric $g$ and perturbation $\nu$. The unique reducible point $[\nu, 0]$ is isolated if

$Ker\partial\!\!\!/^g_\nu = 0$. We want to analyze the local structure of the parametrized moduli space

$$\bar{\mathcal{M}} = \{\mathcal{M}(g_t, \nu_t) \times \{t\} | t \in [-1, 1]\}$$

at the degenerate reducible point $\vartheta_0 = (\theta_0, 0)$, where $\theta_0 = [\nu_0, 0]$ is the class of the reducible solution of the Seiberg-Witten equation on $(Y, \mathfrak{s})$ with the metric and perturbation $(g_0, \nu_0)$. There is a family of reducibles $\vartheta_t$ in $\bar{\mathcal{M}}$. Let $\bar{\mathcal{M}}^*$ be the irreducible set in $\bar{\mathcal{M}}$, $\mathcal{U}$ be a sufficiently small neighbourhood of $\theta_0$ in $\bar{\mathcal{M}}$, and $\mathcal{U}^*$ be the irreducible part of $\mathcal{U}$.

We construct a bundle over neighbourhood of $\vartheta_0$ in $\mathcal{A} \times [-1, 1]$, together with a section $\varsigma$ such that

$$\mathcal{U}^* = (\varsigma^{-1}(0) - \{(\vartheta_t, t)\})/\mathcal{G}.$$

**Lemma 2.3.5.** *The based gauge group $G_0 = \mathcal{G}/U(1)$ acting on the configuration space $\mathcal{A}$ has the following slice models:*

*(a) The slice of the $\mathcal{G}/U(1)$-action at a point $(A_0, 0)$ is $V_{(A_0,0)} = Ker(d^*) \times \Gamma_{L^2}(S)$,*

*(b) The slice of the $\mathcal{G}/U(1)$-action at a point $(A, \psi)$ is*

$$V_{(A,\psi)} = \{(\alpha, \phi) | d^*(\alpha) + iIm\langle \phi, \psi \rangle \text{ is a constant function on } Y.\}$$

*(c) For $(A, \psi)$ close to $(A_0, 0)$ there is an isomorphism*

$$\lambda_{(A,\psi)} : \qquad V_{(A,\psi)} \to V_{(A_0,0)}$$

**Proof.** Properties (a) and (b) follow by direct computation. For (c), choose $(\alpha, \phi)$ in $V_{(A,\psi)}$ and define $\lambda_{(A,\psi)}(\alpha, \phi)$ to be

$$(\alpha - 2d\xi_{(\alpha,\phi)}, \xi_{(\alpha,\phi)}\psi + \phi)$$

where $\xi_{(\alpha,\phi)}$ is the unique solution of the following equations:

$$\begin{cases} 2d^*d\xi_{(\alpha,\phi)} = d^*\alpha \\ \int_Y \xi_{(\alpha,\phi)} dv = 0 \end{cases}$$

Direct computation shows that $\lambda_{(A,\psi)}$ is an isomorphism. □

The above Lemma shows that $V$ is a locally trivial vector bundle over the space of connections and spinors $\mathcal{A}$ endowed with a $U(1)$-action.

Define the section $\varsigma$

$$\varsigma : \mathcal{A} \times [-1,1] \to V$$

to be

$$\varsigma(A,\psi,t) = \lambda_{(A,\psi)}(*_{g_t}(F_A - d\nu_t) - \sigma(\psi,\psi), \partial\!\!\!/_A \psi).$$

Near $\vartheta_0$, we know that $\mathcal{U} = \varsigma^{-1}(0)/\mathcal{G}$. Therefore, the local structure of $\mathcal{U}^*$ at $\vartheta_0$ is given by the Kuranishi model of $\varsigma^{-1}(0)/\mathcal{G}$ at $\vartheta_0$.

Suppose $(A_t, \psi_t)$ is an element in $\mathcal{U}^*$. Consider a formal expansion at $\vartheta_0$ of the form

$$A_t = \nu_t + t\alpha_1 + t^2\alpha_2 + \cdots, \psi_t = t\psi_1 + t^2\psi_2 + \cdots .$$

Near the degenerate point $\vartheta_0$, the section $\varsigma$ is approximated by the following pair:

$$\Big(*_{g_0} d(\nu_t + t\alpha_1 + t^2\alpha_2 + \cdots) - *_{g_0} d\nu_t - \sigma(t\psi_1 + t^2\psi_2 + \cdots, t\psi_1 + t^2\psi_2 + \cdots),$$
$$\partial\!\!\!/^{g_0}_{\nu_t}(t\psi_1 + t^2\psi_2 + \cdots) + (t\alpha_1 + t^2\alpha_2 + \cdots)\cdot(t\psi_1 + t^2\psi_2 + \cdots)\Big),$$

where we are perturbing in a neighbourhood of the wall $W$ just by changing the perturbation and fixing the metric $g_0$.

The zero set of the section therefore determines the conditions $*d\alpha_1 = 0$ and $d^*\alpha_1 = 0$, which imply $\alpha_1 = 0$ on a homology sphere. Moreover, we have $d^*\alpha_2 = 0$ and $*d\alpha_2 = \sigma(\psi_1, \psi_1)$. On the kernel of $d^*$ the operator $*d$ is invertible, hence we have $\alpha_2 = (*d)^{-1}\sigma(\psi_1, \psi_1)$.

The Kuranishi model near $\vartheta_0$ is given by a $U(1)$-equivariant map

$$\mathcal{S} : \mathbb{R} \times Ker(\partial\!\!\!/^{g_0}_{\nu_0}) \to Coker(\partial\!\!\!/^{g_0}_{\nu_0}),$$

where $U(1)$ acts on $Ker(\partial\!\!\!/^{g_0}_{\nu_0}) \cong Coker(\partial\!\!\!/^{g_0}_{\nu_0}) \cong \mathbb{C}$ by the natural $U(1)$ multiplication on $\mathbb{C}$.

We know that there exists a sufficiently small $\delta > 0$ such that, for $t \in [-\delta, \delta]$, we have that $\partial\!\!\!/^{g_0}_{\nu_t}$ has exactly one small eigenvalue $\lambda(t)$ with eigenvector $\phi_t$ ($\lambda(0) = 0$ and $\lambda'(0) \neq 0$), that is $\partial\!\!\!/^{g_0}_{\nu_t}\phi_t = \lambda(t)\phi_t$.

The map $\mathcal{S}$ is given by

$$\mathcal{S} : \mathbb{R} \times Ker(\partial\!\!\!/_{\nu_0}^{g_0}) \to Ker(\partial\!\!\!/_{\nu_0}^{g_0}),$$

$$\mathcal{S}(t, w\phi) = \Pi_{Ker(\partial\!\!\!/_{\nu_0}^{g_0})}(\partial\!\!\!/_{A_t} w\phi).$$

Here we assume that $\phi$ is a spinor in $Ker(\partial\!\!\!/_{\nu_0}^{g_0})$ with $||\phi|| = 1$, so that $Ker(\partial\!\!\!/_{\nu_0}^{g_0}) \cong \mathbb{C}\phi$, $\Pi_{Ker(\partial\!\!\!/_{\nu_0}^{g_0})}$ is the $L^2$ projection onto $Ker(\partial\!\!\!/_{\nu_0}^{g_0})$.

**Lemma 2.3.6.** *Consider the expression $\langle \partial_{\nu_t}^{g_0}\phi, \phi\rangle = z(t)$, then we have*

$$z(0) = 0, \qquad z'(0) = \lambda'(0).$$

**Proof.** We write formally $\lambda(t) \sim t\lambda'(0)$, $\phi_t \sim \phi + t\phi_1$ and the Dirac operator $\partial_{\nu_t}^{g_0} \sim \partial_{\nu_0}^{g_0} + tC$, where $\nu_t \sim \nu_0 + t\nu_1$ and $C$ acts as Clifford multiplication by $\nu_1$. We can write the first order term in the relation $\partial_{\nu_t}^{g_0}\phi_t = \lambda(t)\phi_t$ as

$$\langle \partial_{\nu_0}^{g_0}\phi_1, \phi\rangle + t\langle C\phi, \phi\rangle = t\lambda'(0) + \lambda(0).$$

Here the term $\langle \partial_{\nu_0}^{g_0}\phi_1, \phi\rangle = \langle \phi_1, \partial_{\nu_0}^{g_0}\phi\rangle$ vanishes, and also $\lambda(0) = 0$. Thus, we have the relation $\langle C\phi, \phi\rangle = \lambda'(0)$. On the other hand, we have $\langle C\phi, \phi\rangle = z'(0)$ from the expansion of $\partial_{\nu_t}^{g_0}\phi = z(t)\phi$. □

Assume $\psi_1 = re^{i\theta}\phi$, then $\sigma(\psi_1, \psi_1) = r^2\sigma(\phi, \phi)$, the map $\mathcal{S}$ can be rewritten as

$$\begin{aligned} \mathcal{S}(t, w\phi) &= z(t)w\phi + t^2\langle \alpha_2\phi, \phi\rangle w\phi + O(t^3) \\ &= \Big(z(t) + t^2r^2\langle (*d)^{-1}\sigma(\phi, \phi), \sigma(\phi, \phi)\rangle\Big)w\phi + O(t^3). \end{aligned}$$

whose zero set can be approximated by

$$z(t) + t^2r^2\langle (*d)^{-1}\sigma(\phi, \phi), \sigma(\phi, \phi)\rangle = 0. \tag{2.17}$$

The term $\tau(Y, g_0, \nu_0) = \langle (*d)^{-1}\sigma(\phi, \phi), \sigma(\phi, \phi)\rangle$ is a constant that only depends on the manifold $(Y, \mathfrak{s}, g_0, \nu_0)$. Generically, we can assume that $\tau(Y, g_0, \nu_0)$ is non-zero.

Notice that we have $(\mathbb{R} \times Ker(\partial\!\!\!/_{\nu_0}^{g_0}) - \{0\})/U(1) = \mathbb{R} \times \mathbb{R}^+$. The irreducible part of $\varsigma^{-1}(0)/\mathcal{G}$ is tangent to $\{0\} \times \mathbb{R}^+$ as $t$ approaches 0.

The difference between the Seiberg-Witten invariants at $t = \pm\delta$ can be evaluated by counting the number (with sign) of oriented lines in $\varsigma^{-1}(0)/\mathcal{G}$, with

$t \in [-\delta, \delta]$, that are tangent to $\{0\} \times \mathbb{R}^+ \times \{0\}$. Here we identify $\mathcal{U}^*$ with the set $(\mathcal{S}^{-1}(0) - \{w = 0\})/U(1)$. From the approximated zero set (2.17), we see that the zero set $(\mathcal{S}^{-1}(0) - \{w = 0\})/U(1)$ is given by the condition

$$\lambda'(0) + tr^2\tau(Y, g_0, \nu_0) = 0.$$

Thus, we have only one line in $\mathcal{U}^*$ given by

$$t = -\frac{\lambda'(0)}{r^2\tau(Y, g_0, \nu_0)},$$

whose contribution to $\lambda_{SW}(Y, \mathfrak{s}, g_t, \nu_t)$ $(t \in [-\delta, \delta])$ is given by the $(mod\ 2)$ spectral flow.

Without loss of generality, suppose $\lambda'(0) > 0$, that is $SF_{[\nu_t, 0]}(\partial\!\!\!/^{g_t}_{\nu_t}) = 1$, by our convention of the spectral flow, we see that

$$\begin{cases} SF^{\vartheta_0}_{\vartheta_{-1}}(\partial\!\!\!/^{g_t}_{\nu_t}) = 1, \\ SF^{\vartheta_1}_{\vartheta_0}(\partial\!\!\!/^{g_t}_{\nu_t}) = 0. \end{cases}$$

When $\tau(Y, g_0, \nu_0) > 0$, there are only one irreducible solutions approaching the degenerate reducible point $\vartheta_0$ which happens as $t < 0$ with $(-1)$ contribution to the Seiberg-Witten invariant, hence,

$$\lambda_{SW}(Y, \mathfrak{s}, g_1, \nu_1) = \lambda_{SW}(Y, \mathfrak{s}, g_{-1}, \nu_{-1}) + 1.$$

When $\tau(Y, g_0, \nu_0) < 0$, we see that there are only one irreducible solutions approaching the degenerate reducible point $\vartheta_0$ which happens as $t > 0$ with $(+1)$ contribution to the Seiberg-Witten invariant, hence,

$$\lambda_{SW}(Y, \mathfrak{s}, g_1, \nu_1) = \lambda_{SW}(Y, \mathfrak{s}, g_{-1}, \nu_{-1}) + 1.$$

This completes the geometric proof the wall-crossing formula (2.16). After introducing the equivariant Seiberg-Witten-Floer homology theory, we will give a more general proof of the the wall-crossing formula (2.16).

We end this subsection by two remarks: For the trivial Spin$^c$ structure on $Y$, we know that there is a quaternionic structure on $\Gamma(W)$ as in Lemma 2.1.4 for which $J$-action commutes with the pure Dirac operator $\partial\!\!\!/$ (see Lemma 2.1.6), one can perturb the Seiberg-Witten equations on $Y$ such that the moduli space admits this

$Z_2$-action, this has been worked out by Chen [13] to prove that the Seiberg-Witten invariant is an integer lifting of the Rohlin $Z_2$ $\mu$-invariant for an integral homology 3-sphere.

### 2.3.2 Special Case II: $b_1(Y) = 1$

For $Y$ with $b_1(Y) = 1$, we know that reducible solutions happen at a codimension one subspace

$$\mathfrak{R} = \left\{ (g,\eta) \left| \begin{array}{l} (1).\quad \eta = \eta^0 + *_g d\nu, \text{ where } \eta^0 \text{ is harmonic, ,} \\ (2).\quad \eta^0 = \frac{2\pi}{i} *_g [c_1(det(W)] \text{ under the Hodge decomposition.} \end{array} \right. \right\}$$

There is a projection map $\pi : Met \times Z^1(Y, i\mathbb{R}) \to H^1(Y, i\mathbb{R})$ defined by sending

$$(g,\eta) \to \eta^0$$

the harmonic part of $\eta$ for the metric $g$. By fixing the metric $g_0$, we see that $\mathfrak{R}_{g=g_0}$ separates $Z^1(Y, i\mathbb{R})$ into two chambers. By varying the metric, therefore, $\mathfrak{R}$ divides $Met \times Z^1(Y, i\mathbb{R})$ into two chambers, at each chamber, apply the same proof as above, $\lambda_{SW}(Y)$ is constant. As a family of $(g_t, \eta_t)(t \in [-1,1]$ crosses $\mathfrak{R}$ once at $t = 0$, the parametrized moduli space

$$\mathcal{M}^{[-1,1]} = \bigcup_{t\in[-1,1]} \mathcal{M}_{\mathfrak{s},\eta_t}(Y, g_t)$$

has reducible solutions at $t = 0$. As in the proof of Lemma 2.2.6, the reducible part of $\mathcal{M}_{\mathfrak{s},\eta_0}(Y, g_0)$, denoted $\mathcal{M}_0^{red}$,

$$\mathcal{M}_0^{red} = \{(A,0)| * F_A = \eta_0 = \eta_0^0 + *d\nu_0\}/\mathcal{G}$$

can be identified as a circle. Let $\omega_0$ be the connection on $det(W)$ with curvature $F_{\omega_0} = *\eta_0^0$, then any connection $A$ with $*F_A = \eta_0^0 + *d\nu_0$ can be written as:

$$A = \omega_0 + \nu_0 + a$$

where $a$ is an imaginary 1-form with $da = 0$, using the gauge transformation in the identity component, we can require $d^*a = 0$, hence, $a \in H^1(Y, i\mathbb{R})$. Applying a further gauge transformation in an appropriate component, we see that $a \in H^1(Y, i\mathbb{R})/H^1(Y, i\mathbb{Z}) \cong S^1$ since $\pi_0(\mathcal{G}) \cong H^1(Y, i\mathbb{Z})$.

In $\mathcal{M}_{\mathfrak{s},\eta_{-1}}(Y,g_{-1}), \mathcal{M}_{\mathfrak{s},\eta_1}(Y,g_1)$, the contributions to the Seiberg-Witten invariants, from those points which are not path-connected to $\mathcal{M}_0^{red}$ in the parametrized moduli space $\mathcal{M}^{[-1,1]}$, are the same for $(g_{-1},\eta_{-1})$ and $(g_1,\eta_1)$. Therefore we need only consider the structure of the neighbourhood of $\mathcal{M}_0^{red}$ in $\mathcal{M}^{[-1,1]}$.

Similarly to Lemma 2.2.14, the following Lemma clarifies when the irreducible solutions approach a point in $\mathcal{M}_0^{red}$.

**Lemma 2.3.7.** *Suppose that $\mathcal{M}_0^{red}$ is indexed by $a \in H^1(Y,i\mathbb{R})/H^1(Y,i\mathbb{Z})$ with the corresponding connection $A_a = \omega_0+\nu_0+a$, then if $Ker\not{\partial}^{g_0}_{A_a} = 0$, there is no irreducible solution connecting $A_a$ in $\mathcal{M}^{[-1,1]}$.*

**Proof.** Let a $(A_t,\psi_t)$ be a family of solutions to the perturbed equations (2.9) with respect to $(g_t,\eta_t)$. Suppose $(A_t,\psi_t)_{t=0} = (A_a,0)$, then differentiating the Dirac equation $\not{\partial}^{g_t}_{A_t}\psi_t = 0$ with respect to $t$, one obtains the proof of the Lemma. □

Therefore, if we can perturb the Seiberg-Witten equations further so as to achieve the result that $\mathcal{M}_0^{red}$ is isolated from the irreducible solutions in $\mathcal{M}^{[-1,1]}$, then we can claim that the Seiberg-Witten invariant $\lambda_{SW}(Y,\mathfrak{s})$ is also a topological invariant as long as the invariant is defined away from $\mathfrak{R}$. Can this situation really happen for a 3-manifold $Y$ with $b_1(Y) = 1$? Unfortunately, the answer is "**No**". We will discuss this as follows.

Notice that we can choose $\nu \in \Omega^1(Y,i\mathbb{R})$ for the perturbation $\eta = \eta^0 + *d\nu$ whose harmonic projection under the metric $g_0$ is $\eta^0 = *F_{\omega_0}$. Denote by $Z_{\eta^0}$, the $\nu$-space (those $\eta$ with harmonic part $\eta^0$). Using the arguments similar to Lemma 2.2.15, we know that

$$\mathfrak{S} = \{(*d\nu, a_\theta) | Ker\not{\partial}^{g_0}_{\omega_0+a_\theta+\nu} \neq 0\}$$

is a co-dimension one subset in $Z_{\eta^0} \times S^1$. Therefore, there is an open dense set $Z^0_{\eta^0}$ in $Z_{\eta^0}$ such that any $*d\nu \in Z^0_{\eta^0}$, $\{*d\nu\} \times S^1$ is transversal to the highest stratum in that co-dimension one subset $\mathfrak{S}$. Then there are only finitely many $\theta \in S^1$ such that $Ker\not{\partial}^{g_0}_{\omega_0+a_\theta}$ is non-trivial.

From Lemma 2.3.7, generically, there are only finitely many points on $\mathcal{M}_0^{red}$ which can meet the irreducible solutions in $\mathcal{M}^{[-1,1]}$. We call these points the singular points. The local structures at singular points determine the wall-crossing formula for the Seiberg-Witten invariants defined for $(g,\eta)$ in each chamber separated by $\mathfrak{R}$.

**Proposition 2.3.8.** *Suppose $c_1(det(W)) = 2n * \Theta$ where $\Theta$ is the generator of $H^1(Y, i\mathbb{Z})$ and defines the orientation on $H^1(Y, i\mathbb{R})$, let $(g_{-1}, \eta_{-1})$ and $(g_1, \eta_1)$ belong to the two different chambers separated by $\mathfrak{R}$. Then*

$$\lambda_{SW}(Y, \mathfrak{s}, g_1, \eta_1) = \lambda_{SW}(Y, \mathfrak{s}, g_{-1}, \eta_{-1}) + n.$$

**Proof.** First choose a family of metrics and perturbations $(g_t, \eta_t)$ $(t \in [-1,1])$ such that it crosses $\mathfrak{R}$ once at $t=0$, with only finite singular points $A(\theta_i)$ on

$$\mathcal{M}_0^{red} = \{A(\theta) = \omega_0 + \nu_0 + \theta\Theta | \theta \in [0, 2\pi]\}$$

Using the Kuranishi model at the singular point at $A(\theta_i)$, one can see that the contribution to $\lambda_{SW}(Y, \mathfrak{s}, g_1, \eta_1) - \lambda_{SW}(Y, \mathfrak{s}, g_{-1}, \eta_{-1})$ from the singular point is the same as the contribution of the spectral flow of the twisted Dirac operator along $\mathcal{M}_0^{red}$ from this singular point (the proof of this claim is essentially the same as the geometric proof of the "wall-crossing" formula for $b_1 = 0$ in next subsection). Therefore,

$$\lambda_{SW}(Y, \mathfrak{s}, g_1, \eta_1) - \lambda_{SW}(Y, \mathfrak{s}, g_{-1}, \eta_{-1}) = SF(\partial\!\!\!/^{g_0}_{A(\theta)})$$

Note that $A(\theta)$ defines a connection on $det(W)$ over $Y \times S^1$, hence, the spectral flow is the index of the corresponding Dirac operator on $Y \times S^1$, which is

$$\frac{1}{4}(c_1^2 + sign(Y \times S^1)) = n,$$

where $c_1 = 2n * \Theta + \dfrac{\Theta}{2\pi} d\theta$. □

One of the most striking results about the Seiberg-Witten invariants for $b_1(Y) > 0$ was announced by Meng and Taubes [34], where they considered the 3-manifold $Y$ with $b_1 > 0$ and zero Euler characteristic (possibly with boundaries consisting of disjoint tori), then the Seiberg-Witten invariants for all the Spin$^c$ structures define a unique element $\underline{SW}$ in $\mathbb{Z}[H]/H$ for $b_1 > 1$, and $\mathbb{Z}[[H]]/H$ where $H = H^2_{compact}(Y, \mathbb{Z})/Torsion$. They claimed that this version of Seiberg-Witten invariant equals the Milnor torsion constructed from the combinatorial topology.

# Chapter 3

# Seiberg-Witten-Floer homology

The Seiberg-Witten monopoles contain more information than we discussed in the last section. They can define a homology group for each 3-manifold with a $\mathrm{Spin}^c$ structure. This chapter begins with a framework for the infinite dimensional Morse theory for the Chern-Simons-Dirac functional whose critical points are defined by the Seiberg-Witten equations (2.3). The irreducible critical points generate the Seiberg-Witten-Floer (non-equivariant) complex with the boundary defined by counting the gradient flows connecting the critical points with relative index one. We also construct the equivariant Seiberg-Witten-Floer homology which takes account of reducibles.

## 3.1 Morse theory for Chern-Simons-Dirac functional

Let $Y$ be a closed, oriented 3-manifold with a Riemannian metric, $g$, and a $\mathrm{Spin}^c$ structure, $\mathfrak{s} = (W, \rho)$, as in last section. $\mathcal{A}_{\mathfrak{s}}$ is the $L^2_1$-configuration space for the Seiberg-Witten equations, the tangent space is the space of $L^2_1$-sections: $\Omega^1_{L^2_1}(Y, i\mathbb{R}) \oplus L^2_1(W)$ with the $L^2$-inner product $< , >$ (see (2.8)):

$$< (a, \phi), (b, \psi) >= \int_Y (-a \wedge *b + 2Re\langle \phi, \psi \rangle dvol_Y),$$

$\mathcal{G}$ is the $L^2_2$ gauge transformation group on $Y$, its tangent space is $\Omega^0_{L^2_2}(Y, i\mathbb{R})$.

Fix a $C^\infty$ connection $A_0$ on $det(W)$, then there is a functional, called the Chern-

Simons-Dirac function on $\mathcal{A}_{\mathfrak{s}}$:

$$\mathcal{C}(A,\psi) = -\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \int_Y \langle \psi, \partial\!\!\!/_A \psi \rangle dvol_Y. \tag{3.1}$$

The $L^2$-gradient vector field of $\mathcal{C}$ at $(A,\psi)$, denoted by $\nabla\mathcal{C}_{(A,\psi)}$, is the $L^2$-tangent vector for which the following equation holds for any $L_1^2$-tangent vector $(a,\phi) \in \Omega^1_{L_1^2}(Y, i\mathbb{R}) \oplus L_1^2(W)$ at $(A,\psi)$:

$$< \nabla\mathcal{C}_{(A,\psi)}, (a,\phi) >= \partial_t(\mathcal{C}(A+ta, \psi+t\phi))|_{t=0}$$

**Lemma 3.1.1.** $\nabla\mathcal{C}_{(A,\psi)} = (*F_A - \sigma(\psi,\psi), \partial\!\!\!/_A\psi)$.

**Proof.** This follows from the infinite dimensional variation with respect to the tangent vector

$$(a,\phi) \in \Omega^1_{L_1^2}(Y, i\mathbb{R}) \oplus L_1^2(W)$$

$$\begin{aligned}
& \partial_t(\mathcal{C}(A+ta, \psi+t\phi)|_{t=0} \\
= \ & -\frac{1}{2}\int_Y (a \wedge (F_A + F_{A_0}) + (A - A_0) \wedge da - \langle \psi, a.\psi \rangle dvol_Y \\
& + \int_Y (\langle \phi, \partial\!\!\!/_A\psi \rangle + \langle \psi, \partial\!\!\!/_A\phi \rangle) dvol_Y \\
= \ & \int_Y a \wedge (-F_A + *\sigma(\psi,\psi)) + \int_Y 2Re\langle \partial\!\!\!/_A\psi, \phi \rangle dvol_Y \\
= \ & < (*F_A - \sigma(\psi,\psi), \partial\!\!\!/_A\psi), (a,\phi) > .
\end{aligned}$$

□

There is a surjective homomorphism: $\mathcal{G} \to H^1(Y, Z)$ which sends each $L_2^2$ map $g : Y \to U(1)$ to a cohomology class $[g] = [-2\pi i g^{-1} dg]$ on $U(1)$. The kernel of this map is the identity component of $\mathcal{G}$. For $g \in \mathcal{G}$, we have

$$\mathcal{C}(g.(A,\psi)) - \mathcal{C}(A,\psi) = \int_Y c_1(det(W)) \wedge [g]. \tag{3.2}$$

Therefore, $\mathcal{C}$ descends to a map $\mathcal{B} \to \mathbb{R}/d\mathbb{Z} \cong U(1)$ where $d$ is the divisibility of $c_1(\mathfrak{s}) = c_1(det(W))$ in $H^2(Y,Z)/Torsion$. Obviously, the critical points of $\mathcal{C}$ on $\mathcal{B}$ form the moduli space $\mathcal{M}_{\mathfrak{s}}$ of the Seiberg-Witten equations (2.3).

In the case of $c_1(det(W)) = 0$, we see from (3.2) that $\mathcal{C}$ descends to a map $\mathcal{B} \to \mathbb{R}$. If $c_1(det(W)) \neq 0$, we know that $\mathcal{C}$ descends to a map $\mathcal{B} \to S^1$. In order to obtain a $\mathbb{R}$-valued function over certain quotient space, we need a $H^1(Y, Z)$-cover of $\mathcal{B}$, which is the quotient of $\mathcal{A}$ by the identity component $\mathcal{G}^0$ of $\mathcal{G}$. Denote by $\tilde{\mathcal{B}}$ the resulting quotient. Then the critical points of $\mathcal{C}$ on $\tilde{\mathcal{B}}$ consist of $H^1(Y, Z)$-copies of $\mathcal{M}_{\mathfrak{s}}$, denoted by $\tilde{\mathcal{M}_{\mathfrak{s}}}$ (see the following diagram).

$$\begin{array}{ccccc} & & \tilde{\mathcal{B}} & \hookleftarrow & \tilde{\mathcal{M}_{\mathfrak{s}}} \\ & \nearrow^{\mathcal{G}^0} & \downarrow{\scriptstyle H^1(Y,\mathbb{Z})} & & \downarrow{\scriptstyle H^1(Y,\mathbb{Z})} \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{B} & \hookleftarrow & \mathcal{M}_{\mathfrak{s}} \end{array} \tag{3.3}$$

Another nice lifting of $\mathcal{C}$ (in the case of $c_1(det(W)) \neq 0$) on $\mathcal{B}$ was discovered by R. Wang [47] where he defined a subgroup of $\mathcal{G}$:

$$\mathcal{G}_1 = \Big\{ g \in \mathcal{G} \Big| \int_Y c_1(det(W)) \wedge [g] = 0 \Big\}. \tag{3.4}$$

Then there is a exact sequence

$$0 \to \mathcal{G}_1 \to \mathcal{G} \to H^1(Y, \mathbb{Z})/Ker(\xi) \to 0,$$

where $\xi$ is the map on $H^1(Y, \mathbb{Z})$ given by

$$\xi(g) = \int_Y c_1(det(W)) \wedge [g],$$

with $H^1(Y, \mathbb{Z})/Ker(\xi) \cong \mathbb{Z}$ for $c_1(det(W)) \neq 0$. The corresponding quotient space $\mathcal{B}_1 = \mathcal{A}_{\mathfrak{s}}/\mathcal{G}_1$ is a $\mathbb{Z}$-cover of $\mathcal{B}$. This cover is universal in the sense that

$$\mathcal{B}_1 = \mathcal{B} \times_{S^1} \mathbb{R},$$

and any cover of $\mathcal{B}$ such that $\mathcal{C}$ descends to an $\mathbb{R}$-valued function is a cover of $\mathcal{B}_1$. The critical point set $\underline{\mathcal{M}_{\mathfrak{s}}}$ on $\mathcal{B}_1$ is a $\mathbb{Z}$-cover of $\mathcal{M}_{\mathfrak{s}}$.

$$\underline{\mathcal{M}_{\mathfrak{s}}} \xrightarrow{\mathbb{Z}} \mathcal{M}_{\mathfrak{s}}. \tag{3.5}$$

**Lemma 3.1.2.** *The Chern-Simons-Dirac functional can be perturbed by a co-closed, imaginary, $L^2_2$ 1-form $\eta \in Z^1(Y, i\mathbb{R})$,*

$$\mathcal{C}_\eta(A, \psi) = -\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0} - *2\eta) + \langle \psi, \partial\!\!\!/_A \psi \rangle dvol_Y,$$

*with the $L^2$-gradient flow given by $(*F_A - \sigma(\psi, \psi) - \eta, \partial\!\!\!/_A \psi)$, whose critical points on $\mathcal{B}$ give the perturbed Seiberg-Witten moduli space $\mathcal{M}_{\mathfrak{s},\eta}$ for (2.9). These critical points are non-degenerate in the sense that the Hessian operator is invertible at the critical point.*

**Proof.** Applying the variation as in Lemma 3.1.1, we see that the critical points are the gauge equivalence classes of the solutions to the equations

$$\begin{cases} *F_A = \sigma(\psi, \psi) - \eta, \\ \partial\!\!\!/_A \psi = 0. \end{cases}$$

The rest of the claims are the consequences of Corollary 2.2.13, Lemma 2.2.14 and Proposition 2.2.15, since the Hessian operator $Q_{[A_0, \psi_0]}$ acts on the tangent space of $\mathcal{B}^*$ at $[A_0, \psi_0] \in \mathcal{M}_\eta$ with domain

$$\{(a, \phi) \in \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(W) | d^*a + iIm\langle \psi_0, \phi \rangle = 0\}.$$

and $Q$ sends $(a, \phi)$ to

$$(*da - 2\sigma(\psi_0, \phi), \partial\!\!\!/_{A_0} \phi + \frac{1}{2} a.\psi_0).$$

This Hessian operator has $\mathcal{G}$ equivariant extension to $T\mathcal{A}^* \oplus T\mathcal{G}$, which is the operator $T$ defined in (2.10). Both $Q$ and $T$ are essentially self-adjoint, their kernels have the same dimensions at the irreducible critical points. □

For $c_1(det(W)) \neq 0$, we will take the perturbation $*\eta$ to be an exact 2-form (Cf. Corollary 2.2.13), then under the gauge transformation the perturbed Chern-Simons-Dirac functional $\mathcal{C}_\eta$ behaves in the same ways as (3.2).

On $\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}^0$, $\mathcal{C}_\eta$ is an $\mathbb{R}$-valued function, the corresponding critical points consist of $H^1(Y, \mathbb{Z})$-copies of $\mathcal{M}_{\mathfrak{s},\eta}$, denoted by $\tilde{\mathcal{M}}_{\mathfrak{s},\eta}$. We introduce this lifting in order to establish the Lojaszewiz-type inequality for $\mathcal{C}_\eta$.

For generic perturbation $\eta$, the functional $\mathcal{C}_\eta$, which has only finitely many critical points on $\mathcal{B}$, has the following nice property, since it satisfies the Palais-Smale condition.

**Lemma 3.1.3.** *For any $\epsilon > 0$, there is $\lambda > 0$ such that if $[A, \psi] \in \mathcal{B}$ has the $L^2_1$-distance at least $\epsilon$ from all the critical points in $\mathcal{M}_{\mathfrak{s},\eta}$ for sufficiently small perturbation $\eta$, then*

$$||\nabla \mathcal{C}_\eta([A, \psi])||_{L^2} > \lambda.$$

**Proof.** Suppose there is a sequence $(A_i, \psi_i)$ in $\mathcal{B}$ whose $L_1^2$-distance is at least $\epsilon$ from all the critical points in $\mathcal{M}_\eta$ for which

$$\|\nabla \mathcal{C}_\eta([A_i, \psi_i])\|_{L^2} \to 0 \qquad \text{as } i \to \infty.$$

Then as $i \to \infty$, we have

$$\| *F_{A_i} - \sigma(\psi_i, \psi_i) - \eta\|_{L^2} \to 0$$
$$\|\partial\!\!\!/_{A_i}(\psi_i)\|_{L^2} \to 0$$

where $\eta$ is sufficiently small. This means that there is a constant $C > 0$ such that

$$\int_Y | *F_{A_i} - \sigma(\psi_i, \psi_i) - \eta|^2 + |\partial\!\!\!/_{A_i}(\psi_i)|^2 < C.$$

Resorting to the Weitzenbock formula for $\partial\!\!\!/_{A_i}^2$ and for $\eta$ sufficiently small, the above inequality reads as

$$\int_Y |F_{A_i}|^2 + |\sigma(\psi_i, \psi_i)|^2 + \frac{s}{2}|\psi_i|^2 + 2|\nabla_{A_i}\psi_i|^2 < 2C.$$

It follows that $\|\psi_i\|_{L^4}, \|F_{A_i}\|_{L^2}, \|\nabla_{A_i}\psi_i\|_{L^2}$ are all bounded independent of $i$. Now using the standard elliptic argument as in the proof of compactness, we know that there is a subsequence converging in $L_1^2$-topology to a solution of (2.9). This contradicts the assumption that the $(A_i, \psi_i)$ in $\mathcal{B}$ have $L_1^2$-distance at least $\epsilon$ from all the critical points in $\mathcal{M}_\eta$. □

Near the non-degenerate critical points of $\mathcal{C}_\eta$, we have the following $L^2$-distance estimate.

**Lemma 3.1.4.** *Suppose $\alpha$ is a non-degenerate critical point of $\mathcal{C}_\eta$, there exists a constant $C_\alpha$ such that if the $L^2$-distance from $[A, \psi]$ to $\alpha$ is sufficiently small, then the $L^2$-distance from $[A, \psi]$ to $\alpha$ (denoted by $dist_{L^2}([A, \psi], \alpha)$ is bounded by*

$$C_\alpha \|\nabla \mathcal{C}_\eta(A, \psi)\|_{L^2(Y)}.$$

*The claims are also true for the $\mathcal{G}^0$-quotient cases with critical point set $\tilde{\mathcal{M}}_{s\eta}$ on the quotient space $\tilde{\mathcal{B}}$.*

**Proof.** By Lemma 3.1.5,

$$\lambda_\alpha = max\{\frac{1}{|\lambda_i|} | \ \lambda_i \text{ is a eigenvalue of the Hessian operator } Q \text{ at } \alpha\}$$

exists and is bounded. We know that $[A,\psi] \mapsto \nabla\mathcal{C}_\eta(A,\psi)$ defines a $L^2$-tangent section, which is smooth and transverse to zero at $\alpha$. Hence, we may choose a small neighbourhood of $\alpha$ (denoted by $U_\alpha$) which may be identified with a small neighbourhood of 0 in the $L^2$-tangent space of $\mathcal{B}^*$ at $\alpha$. We then have

$$\nabla\mathcal{C}_\eta([A,\psi]) \approx Q_\alpha([A,\psi]),$$

which implies that

$$\begin{aligned} & dist_{L^2}([A,\psi],\alpha) \\ \leq\ & \frac{3}{2}\|Q_\alpha^{-1}(\nabla\mathcal{C}_\eta([A,\psi]))\|_{L^2(Y)} \\ \leq\ & \frac{3}{2}\lambda_\alpha\|\nabla\mathcal{C}_\eta([A,\psi])\|_{L^2(Y)}. \end{aligned}$$

Choose the constant $C_\alpha$ with

$$C_\alpha > \frac{3}{2}\lambda_\alpha,$$

then the Lemma follows. □

### 3.1.1 Relative indices and spectral flow

Note that the tangent space of $\mathcal{B}^*$ at $[A,\psi]$ is the $L^2_1$-completion of

$$KerG^*_{[A,\psi]} = \{(a,\phi) \in \Omega^1(Y, i\mathbb{R}) \oplus \Gamma(W) | d^*a + iIm\langle\psi,\phi\rangle = 0\}.$$

The Hessian operator of $\mathcal{C}$ at $[A,\psi] \in \mathcal{B}^*$ is given by

$$Q_{[A,\psi]} : (a,\phi) \mapsto (*da - 2\sigma(\psi,\phi) - 2df, \not{\partial}_A\phi + \frac{1}{2}a.\psi + f\psi)$$

where $f$ is the unique solution in $\Omega^0_{L^2_2}(Y, i\mathbb{R})$ of the equation

$$(d^*d + \frac{1}{2}|\psi|^2)f = iIm\langle\not{\partial}_A\psi,\phi\rangle.$$

Notice that if $[A,\psi]$ is a critical point of $\mathcal{C}$, then $f = 0$ by the maximum principle.

**Lemma 3.1.5.** *$Q_{[A,\psi]}$ defines a closed, unbounded, essentially self-adjoint, Fredholm operator on the $L^2$-completion of $KerG^*_{[A,\psi]}$, and its eigenvectors form an $L^2$-complete orthonormal basis for $KerG^*_{[A,\psi]}$. The domain of $Q_{[A,\psi]}$ is the $L^2_1$-tangent space of $\mathcal{B}^*$. The eigenvalues form a discrete subset of the real line which has no accumulation points, and which is unbounded in both directions. Each eigenvalue has finite multiplicity.*

**Proof.** We follow the arguments in [42]. Since $Q_{[A,\psi]}$ defines a bounded, essentially self-adjoint, Fredholm operator on the $L^2_1$-completion of $KerG^*_{[A,\psi]}$ to the $L^2$-completion of $KerG^*_{[A,\psi]}$, the forgetful map from $L^2_1$ to $L^2$ is compact, and the resolvent of $Q_{[A,\psi]}$ is compact, then $Q_{[A,\psi]}$ has no essential spectrum (accumulation points or isolated eigenvalue with infinite multiplicity). Therefore $Q_{[A,\psi]}$ has discrete spectrum without accumulation points and each eigenvalue has finite multiplicity. The remaining statements are standard in the elliptic theory on compact manifolds.

□

**Remark 3.1.6.** *From the arguments in the last section, we see that after a generic perturbation by a co-closed 1-form $\eta$, $\mathcal{M}^*_{\mathfrak{s},\eta}$ consists of finitely many non-degenerate critical points of $C_\eta$, in the sense that the Hessian operator $Q_\eta$ has no kernel at these critical points. Then for any two critical points $[A_1, \psi_1]$ and $[A_2, \psi_2]$, the spectral flow of the Hessian operator, $Q_\eta$, for the generic perturbation, $\eta$, defines a locally constant function on the space of continuous paths between $[A_0, \psi_0]$ and $[A_1, \psi_1]$. This function depends only on the homotopy class of the path between $[A_0, \psi_0]$ and $[A_1, \psi_1]$, but its mod $(d(\mathfrak{s}))$ reduction doesn't depend on the homotopy class of the path, where $d(\mathfrak{s})$ is the divisibility of the first Chern class $c_1(det(W))$ in $H^2(Y, Z)/Torsion$.*

**Proof.** We only need to prove the claims regarding the spectral flow of $Q_\eta$. Let $[A(t), \psi(t)]$ $(t \in [0, 1])$ be a path in $\mathcal{B}^*$ which connects $[A_0, \psi_0]$ and $[A_1, \psi_1]$ as $t$ varies. The eigenvalues near 0 vary and if the zero crossings are not transverse, we can perturb $Q_\eta$ with the two endpoints fixed, by adding a smooth family of self-adjoint, $Q_\eta$-relatively compact operators $p(t)$, such that the perturbed family has its eigenvalues crossing zero transversely. Then the spectral flow of the Hessian operator $Q_\eta$ is the number of eigenvalues which cross zero with positive slope minus the number of eigenvalues which cross zero with negative slope. It is finite and independent of a sufficiently small perturbation $p(t)$. This spectral flow depends only on the homotopy class of the path between $[A_0, \psi_0]$ and $[A_1, \psi_1]$. Since $\mathcal{A}^*$ is simply connected, $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ is not simply connected,

$$\pi_1(\mathcal{B}^*) = \pi_0(\mathcal{G}) \cong H^1(Y, Z).$$

Any two paths in $\mathcal{B}^*$, connecting $[A_0, \psi_0]$ and $[A_1, \psi_1]$, define a closed loop in $\mathcal{B}^*$ based at $[A_0, \psi_0]$, which belongs to the component $[g] \in H^1(Y, \mathbb{Z})$ for $g \in \mathcal{G}$. The difference of the spectral flows along two paths equals the spectral flow along the closed loop. By [2], it is the index of operator

$$\partial_t - Q_\eta.$$

This index equals the index of the deformation complex for the Seiberg-Witten equations on $Y \times S^1$ with the pull-back Spin$^c$ structure $\pi^*(\mathfrak{s})$ ($\pi : Y \times S^1 \to Y$ is the projection). The positive spinor bundle for $Y \times S^1$ can be identified with the negative spinor bundle via the Clifford multiplication of $dt$. Therefore, the difference of the spectral flows along the two paths equals

$$\begin{aligned} & \frac{1}{4}(c_1(\pi^*(\mathfrak{s}))^2 - 2\chi(Y \times S^1) - 3\sigma(Y \times S^1)) \\ = & \int_Y c_1(det(W)) \wedge [g] \\ = & 0 (mod\ d(\mathfrak{s})), \end{aligned}$$

where $d(\mathfrak{s})$ is the divisibility of $c_1(det(W))$. Note that $d(\mathfrak{s})$ is always even. □

We know that $Q_{[A,\psi]}$ has a gauge equivariant extension to $T\mathcal{A}^* \oplus T\mathcal{G}$, which is the operator $T$ as similarly defined in (2.10) where $T$ was only defined for the critical points. Denote by $T_{(A,\psi)}$ the extended Hessian operator at $(A, \psi)$, on $\mathcal{A}^*$, let $(A_0, \psi_0)$ and $(A_1, \psi_1)$ be two lifts of $[A_0, \psi_0]$ and $[A_1, \psi_1]$ respectively. By the construction, the kernel of $T_{(A_i,\psi_i)}$ is trivial, then the spectral flow of $T_{(A,\psi)}$ from $(A_0, \psi_0)$ to $(A_1, \psi_1)$ is a well defined $\mathbb{Z}$-function and

$$SF_{[A_0,\psi_0]}^{[A_1,\psi_1]} Q_\eta = SF_{(A_0,\psi_0)}^{(A_1,\psi_1)} T (mod\ d(\mathfrak{s})). \tag{3.6}$$

Recall that the definition of the Seiberg-Witten invariant for $(Y, g, \mathfrak{s})$ (See Definition 2.3.2), where we use the spectral flow of $T_{(A,\psi)}$, up to a sign, is:

$$\lambda_{SW}(Y)(\mathfrak{s}) = \sum_{[A,\psi]} (-1)^{SF_{[A_0,\psi_0]}^{[A,\psi]} Q_\eta} \tag{3.7}$$

here $[A_0, \psi_0]$ is a fixed critical point in $\mathcal{M}^*_{\mathfrak{s},\eta}$. Note that for a 3-manifold with $b_1(Y) > 0$, we can choose the perturbation $\eta$ so that $\mathcal{M}_{\mathfrak{s},\eta} = \mathcal{M}^*_{\mathfrak{s},\eta}$. For a homology 3-sphere $Y$, we need to choose the perturbation $\eta = *d\nu$ so that the unique reducible critical point $[\nu, 0]$ is isolated, that is, $(g, \eta)$ is away from the codimension one set $\mathcal{W}$ (2.15). In the latter case, we have a priori choice of the critical point, the reducible one $[\nu, 0]$, then we can replace $[A_0, \psi_0]$ by $[\nu, 0]$ in (3.7). $Q$ is invertible at $[\nu, 0]$, but $T$ has one dimensional kernel at $[\nu, 0]$, therefore the spectral flow of $T$ from $[\nu, 0]$ to an irreducible critical point is defined to be the intersection number of the eigenvalues with the line $\lambda = -\epsilon$ where $\epsilon$ is any sufficiently small positive number (see [2]).

Therefore, using the spectral flow of the Hessian operator $Q_\eta$, we have defined relative indices for the critical set $\mathcal{M}_{\mathfrak{s},\eta}$:

$$\mathfrak{i}: \quad \mathcal{M}_{\mathfrak{s},\eta} \longrightarrow \mathbb{Z}_{d(\mathfrak{s})} \tag{3.8}$$

by sending $[A, \psi]$ to $SF_{[A_0,\psi_0]}^{[A,\psi]} Q_\eta$. In particular, the relative index between the two critical points $\alpha, \beta$ is a well-defined $\mathbb{Z}_{d(\mathfrak{s})}$-valued function:

$$\mathfrak{i}(\alpha, \beta) = \mathfrak{i}(\alpha) - \mathfrak{i}(\beta) \tag{3.9}$$

which is the spectral flow of the Hessian operator along any path connecting $\alpha, \beta$ on $\mathcal{B}$ $(mod\ d(\mathfrak{s}))$. When $c_1(\mathfrak{s})$ is a torsion class, which happens, if $Y$ is a homology 3-sphere, then $\mathfrak{i}$ is an integer-valued function. When $c_1(\mathfrak{s}) \neq 0$, the $\mathbb{Z}_{d(\mathfrak{s})}$-valued relative indices can be lifted to the $\mathbb{Z}$-cover of $\mathcal{M}_{\mathfrak{s}}$ (see (3.5)) to define a $\mathbb{Z}$-valued indices

$$\mathfrak{i}: \quad \underline{\mathcal{M}_{\mathfrak{s},\eta}} \longrightarrow \mathbb{Z}. \tag{3.10}$$

### 3.1.2 Moduli space for the gradient flows

The downward gradient flow equation for $C_\eta$ on $\mathcal{A}$ is given by the path of pairs $(A(t), \psi(t))$ that satisfies the equations

$$\begin{cases} \dfrac{dA(t)}{dt} = - * F_{A(t)} + \sigma(\psi, \psi) + \eta, \\ \dfrac{d\psi(t)}{dt} = -\partial\!\!\!/_{A(t)} \psi(t). \end{cases} \tag{3.11}$$

These are the Seiberg-Witten equations on $(Y \times \mathbb{R}, g + dt^2)$ for the pull-back Spin$^c$ structure with respect to the temporal gauge (the $dt$ component of $A$ on $det(W) \to Y \times \mathbb{R}$ vanishes identically) [26]. The two solutions to the equations

(3.11) are said to be equivalent if they (as paths in $\mathcal{A}$) are gauge equivalent under the action of $\mathcal{G}$.

Under the projection $\pi : Y \times \mathbb{R} \to Y$, we identify

$$\Omega^{2,+}(Y \times \mathbb{R}, i\mathbb{R}) \cong \pi^*(\Omega^1(Y, i\mathbb{R})) \tag{3.12}$$

by sending $\rho(t)$ to $*\rho(t) + \rho(t) \wedge dt$, and identify

$$\Omega^1(Y \times \mathbb{R}, i\mathbb{R}) \cong \pi^*(\Omega^0(Y, i\mathbb{R}) \oplus \Omega^1(Y, i\mathbb{R})) \tag{3.13}$$

by sending $(f(t), \rho(t))$ to $\rho(t) + f(t)dt)$. Let the Spin$^c$ structure on $Y \times \mathbb{R}$ be the pull-back of $(\mathfrak{s}, W)$. Using the Clifford multiplication of $dt$, we can identify the positive and negative spinor bundles $W^{\pm}$ both as $\pi^*(W)$ as follows:

$$\begin{array}{ccc} & \pi^*(W) & \\ \swarrow^{\rho^+} & & \searrow^{\rho^-} \\ W^+ & \xrightarrow{\;dt.\;} & W^- \end{array} \tag{3.14}$$

such that for 1-form $a \in \Omega^1(Y, i\mathbb{R})$, $\psi \in W$, we have

$$\rho^+(a.\psi) = a.dt.\rho^+(\psi) = a.\rho^-(\psi).$$

**Lemma 3.1.7.** *(Identification of Dirac operators) Let $\psi(t)$ be a section of $\pi^*(W)$ and $A(t)$ be a family of connections on $det(W)$. Then we can view $A(t)$ as a connection on $det(\psi^*W)$, and the twisted Dirac operator $\not{D}_A$ for $(Y \times \mathbb{R}, \pi^*(W))$ can be expressed as $\partial_t + \not{\partial}_{A(t)}$ in the sense of*

$$\not{D}_A(\rho^+\psi) = \rho^-\Big((\partial_t + \not{\partial}_{A(t)})\psi(t)\Big).$$

**Proof.** Let $\{e_i\}_{i=1}^3$ be an orthonormal basis for $TY$, then by the definition of Dirac operator on $(Y \times \mathbb{R}, \pi^*(W))$, we know

$$\begin{aligned}
& \not{D}_A(\rho^+\psi) \\
= & \textstyle\sum_{i=1}^3 e^i.\nabla_{A_i}(\rho^+\psi) + dt.\partial_t(\rho^+\psi) \\
= & dt.\textstyle\sum_{i=1}^3 e^i.dt.\rho^+(\nabla_{A_i}\psi) + dt.\partial_t(\rho^+\psi) \\
= & dt.\rho^+(\textstyle\sum_{i=1}^3 e^i.\nabla_{A_i}\psi + \partial_t\psi) \\
= & dt.\rho^+((\partial_t + \partial\!\!\!/_{A(t)})\psi(t)) \\
= & \rho^-((\partial_t + \partial\!\!\!/_{A(t)})\psi(t)).
\end{aligned}$$

□

Then the downward gradient flow equation (3.11) becomes the Seiberg-Witten equations on $(Y \times \mathbb{R}, \pi^*(W))$ in the temporal gauge [26] [49]:

$$\begin{cases} F_A^+ = q(\psi, \psi) + \eta^+ \\ \not{D}_A\psi = 0 \end{cases} \tag{3.15}$$

where $F_A^+$ is the self-dual part of the curvature $F_A$, $\eta^+ = *_3\eta + \eta \wedge dt$, and $q(\psi, \psi)$ is the self-dual 2-form given in local orthonormal frame $\{e_1, e_2, e_3, e_4 = \partial_t\}$ by

$$\frac{1}{4}\langle e_i.e_j.\psi, \psi\rangle e^i \wedge e^j.$$

We say that two solutions to the Seiberg-Witten equations (3.15) are gauge equivalent under $Map(Y \times \mathbb{R}, U(1))$ if and only if the paths in $\mathcal{A}$ they determine in temporal gauge are gauge equivalent under gauge group the $\mathcal{G}_Y = Map(Y, U(1))$.

For any solution $S(t) = (A(t), \psi(t))$ of the downward gradient flow equations (3.11), we have

$$\begin{aligned}
& \mathcal{C}_\eta(S(t_1)) - \mathcal{C}_\eta(S(t_2)) \\
= & \int_{t_2}^{t_1} \frac{\partial \mathcal{C}_\eta(S(t))}{\partial t} dt \\
= & -\int_{t_2}^{t_1} \|\nabla \mathcal{C}_\eta(S(t))\|_{L^2}^2 dt \\
= & \int_{t_1}^{t_2} \|\nabla \mathcal{C}_\eta(S(t))\|_{L^2}^2 dt
\end{aligned}$$

for any $t_1 < t_2$. Hence, we will say that any gradient flow line $S(t) = (A(t), \psi(t))$ on $\mathcal{A}$ has " a finite variation of $\mathcal{C}_\eta$" if $S(t)$ satisfies the "finite energy" condition:

$$\int_{-\infty}^{\infty} \|\nabla \mathcal{C}_\eta(S(t))\|_{L^2}^2 dt < \infty.$$

**Lemma 3.1.8.** *Suppose $S(t) = (A(t), \psi(t))$ to be any solution to the equations (3.11) with finite variation of $\mathcal{C}_\eta$, and $\eta$ to be sufficiently small, then there exists a constant $s$ such that $|\psi(t)| < s$ and*

*(a)* $\frac{\partial \mathcal{C}_\eta}{\partial t} < 0$ *for all $t$ or $[S(t)] = \alpha$,*

*(b)* $\lim_{t\to\pm\infty} S(t)$ *exist and represent some critical points in $\mathcal{M}_{\mathfrak{s},\eta}$,*

*(c) There exist $T_0 >> 0$ and a positive constant $C_0$ such that for $t > T_0$ the following inequalities hold:*

$$|\mathcal{C}_\eta(S(t)) - \mathcal{C}_\eta(S(+\infty))|^{1/2} \leq C_0 \|\nabla \mathcal{C}_\eta(S(t))\|_{L^2(Y)}$$

$$|\mathcal{C}_\eta(S(-t)) - \mathcal{C}_\eta(S(-\infty))|^{1/2} \leq C_0 \|\nabla \mathcal{C}_\eta(S(-t))\|_{L^2(Y)}$$

**Proof.** The $C^0$-bound of $\psi$ is a standard application of the Weitzenböck formula. The first claim comes from the following equality:

$$\begin{aligned} & \frac{\partial}{\partial t}\mathcal{C}_\eta(S(t)) \\ = \ & \langle \nabla \mathcal{C}_\eta(S(t)), \frac{\partial S(t)}{\partial t} \rangle \\ = \ & -\|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2} \qquad \text{(by the downward gradient flow)} \end{aligned}$$

Notice that

$$\begin{aligned} & \int_{t_1}^{t_2} \|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2(Y)} dt \\ = \ & -\int_{t_1}^{t_2} \|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2} \\ = \ & \mathcal{C}_\eta(S(t_1)) - \mathcal{C}_\eta(S(t_2)) < \infty \end{aligned}$$

for any $t_1, t_2$, hence the finite variation of $\mathcal{C}_\eta(S(t))$ and the claim (a) imply that $\lim_{t\to\pm\infty} \nabla \mathcal{C}_\eta(S(t)) = 0$. As a solution to an elliptic equation, $S(t)$ is gauge equivalent to a smooth solution, therefore, $\lim_{t\to\pm\infty} S(t)$ exist and represent two critical points of $\mathcal{C}_\eta$.

The inequalities in claim (c) follow from the following lemma which is the infinite dimensional version of the Lojaszewiz inequalities with best exponent proposed by L. Simon for nonlinear geometrical equations.

**Lemma 3.1.9.** *For each critical point $\alpha \in \tilde{\mathcal{M}}_{\mathfrak{s},\eta}$ (see Diagram (3.3)), there is a neighborhood $U_\alpha \subset \tilde{\mathcal{B}}$ of $\alpha$ and a positive constant $C_\alpha$ such that for every $S \in U_\alpha$:*

$$|\mathcal{C}_\eta(S) - \mathcal{C}_\eta(\alpha)| < C_\alpha \|\nabla \mathcal{C}_\eta(S)\|^2_{L^2(Y)} \tag{3.16}$$

Proof of Lemma 3.1.9 This is an application of Lemma 3.1.4. Let $U_\alpha \subset \tilde{\mathcal{B}}$ be as determined by Lemma 3.1.4. Choose any path $S(t)$ ($t \in [t_1, t_2]$) in $U_\alpha$ which connects $S$ and $\alpha$ with the least $L^2$-distance, $\mathcal{C}_\eta$ is $\mathbb{R}$-valued on , then

$$\begin{aligned} & |\mathcal{C}_\eta(S) - \mathcal{C}_\eta(\alpha)| \\ \leq \ & \int_{t_1}^{t_2} |\frac{\partial \mathcal{C}_\eta(S(t))}{\partial t}| dt \\ \leq \ & \|\nabla \mathcal{C}_\eta(S)\|_{L^2} \int_{t_1}^{t_2} |\frac{\partial S(t)}{\partial t}| dt \\ \leq \ & C_\alpha \|\nabla \mathcal{C}_\eta(S)\|^2_{L^2(Y)} \end{aligned}$$

where $C_\alpha$ is the constant in Lemma 3.1.4. □

Let $\alpha, \beta$ be two critical points in $\mathcal{M}_{\mathfrak{s},\eta}$. Denote by $\mathcal{M}(\alpha, \beta)$ the moduli space of the gradient flow equations (3.11) with the asymptotic limits $\alpha, \beta$ as $t \to -\infty, +\infty$. In the following we will prove that, generically, this moduli space is a smooth manifold with dimension prescribed by the index theorem. This property depends on an accurate choice of perturbation. First, we need to put $\mathcal{M}(\alpha, \beta)$ in suitable function spaces. We choose the weighted Sobolev spaces by studying the decay rates of the gradient flows in $\mathcal{M}(\alpha, \beta)$ as t $\to \pm\infty$.

We remind the reader that we always identify a path $(A(t), \psi(t))$ in $\mathcal{A}$ with a pair consisting of connection and spinor for $(Y \times \mathbb{R}, \pi^*(W))$ under temporal gauge.

We need to perturb the gradient flow equations in the following form:

$$\begin{cases} \frac{\partial A}{\partial t} = - * F_A + \sigma(\psi, \psi) + \eta + E(A, \psi) \\ \frac{\partial \psi}{\partial t} = -\partial\!\!\!/_A(\psi) \end{cases} \tag{3.17}$$

where $E$ is a $\Omega^1(Y, i\mathbb{R})$-valued, $\mathcal{G}_Y$-invariant function on $\mathcal{A}$ satisfying

$$\|E(A, \psi)\|_{L^2} < \frac{1}{4} \|\nabla \mathcal{C}_\eta(A, \psi)\|_{L^2}. \tag{3.18}$$

**Remark 3.1.10.** *This perturbation is to preserve the translation invariance of the gradient flow equation under the action of $\mathbb{R}$. We call the solution to (3.17) the perturbed gradient flow line of $\mathcal{C}_\eta$, or simply, the gradient flow line. Denote again by $\mathcal{M}(\alpha, \beta)$ the moduli space of the perturbed gradient flow lines on $\mathcal{B}$ which connect $\alpha$ and $\beta$. For this kind of perturbation, Lemma 3.1.8 and the Lojasiewicz inequality (3.16) still hold.*

The corresponding perturbation of the Seiberg-Witten equations on $Y \times \mathbb{R}$ is given by

$$\begin{cases} F_A^+ = q(\psi, \psi) + \eta^+ + E(A_1, \psi)^+ \\ \not{D}(\psi) = 0 \end{cases} \quad (3.19)$$

where $A_1$ is the $Y$-component of $A$ and $E(A_0, \psi)^+ = *_3 E(A_1, \psi) + E(A_1, \psi) \wedge dt$.

Another kind of perturbation is given by Froyshov [22] using the involution of $\mathcal{C}_\eta$ with certain cut-off functions. For the perturbation as in (3.17) satisfying (3.18), we have the following lemma, which comes from a direct calculation.

**Lemma 3.1.11.** *Let $\alpha, \beta \in \mathcal{M}_\eta$, suppose that $S(t) = (A(t), \psi(t))$ is a solution of (3.17) with finite variation of $\mathcal{C}_\eta$, representing a perturbed gradient flow line on $\mathcal{B}$ connecting $\alpha$ and $\beta$, then for any $t_1, t_2 \in \mathbb{R}$,*

$$\frac{1}{2}\int_{t_1}^{t_2} \|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2(Y)} < \mathcal{C}_\eta(S(t_1)) - \mathcal{C}_\eta(S(t_2)) < 2\int_{t_1}^{t_2} \|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2(Y)}.$$

Applying the Lojasiewicz inequality (3.16), we have the following exponential decay estimates for the perturbed gradient flow connecting $\alpha$ and $\beta$.

**Proposition 3.1.12.** *Let $S(t) = [A(t), \psi(t)]$ be a perturbed gradient flow connecting $\alpha$ and $\beta$, satisfying the "finite energy" condition:*

$$\int_{-\infty}^{\infty} \|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2(Y)} dt < \infty$$

*then there exist gauge representatives $(A(t), \psi(t))$ for $S(t)$, $(A_\alpha, \psi_\alpha)$ for $\alpha$ and $(A_\beta, \psi_\beta)$ for $\beta$ such that*

*(a) $A - A_\alpha$ and $\psi - \psi_\alpha$ decay exponentially along with their first derivatives as $t \to -\infty$,*

*(b) $A - A_\beta$ and $\psi - \psi_\beta$ decay exponentially along with their first derivatives as $t \to \infty$.*

**Proof.** We will only establish the claims in (b), the others are similar. Choose $T >> 1$, let $t > T$, define

$$\mathcal{E}(t) = \int_t^\infty \|\nabla \mathcal{C}_\eta(S(t_1))\|^2_{L^2(Y)} dt_1.$$

The same arguments as in (Lemma 6.14, [38]) can be employed to prove

$$dist_{L^2_1}(S(t), \beta) \leq C_1 \int_{t-1}^\infty \|\nabla \mathcal{C}_\eta(S(t_1))\|^2_{L^2(Y)} dt_1 = C_1 \mathcal{E}(t-1)$$

for some constant $C_1$. Lift $S(t)$ to a path on $\mathcal{A}$, then it is easy to see this lifted path has a finite variation of $\mathcal{C}_\eta$. Continue to denote the lifted path by $S(t)$. Then by Lemma 3.1.8, we know that $\lim_{t\to\pm\infty} S(t) = S(\pm\infty)$ exist and represent $\alpha$ and $\beta$. We will prove that $\mathcal{E}(t)(t > T)$ decays exponentially as $t \to \infty$. Note that

$$\begin{aligned} & -\frac{\partial}{\partial t}|\mathcal{C}_\eta(S(t)) - \mathcal{C}_\eta(S(\infty)| \\ = \;& \langle \nabla \mathcal{C}_\eta(S(t)), -\partial_t S(t)\rangle_{L^2(Y)} \\ \geq \;& \frac{3}{4}\|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2(Y)}. \end{aligned}$$

Applying integration over $[t, \infty)$, we get

$$\begin{aligned} \frac{3}{4}\mathcal{E}(t) &\leq |\mathcal{C}_\eta(S(t)) - \mathcal{C}_\eta(S(\infty))| \\ &\leq C_\beta \|\nabla \mathcal{C}_\eta(S(t))\|^2_{L^2(Y)} \qquad (\text{ by Lojaszewiz inequality (3.16)}) \\ &= -C_\beta \frac{\partial \mathcal{E}(t)}{\partial t}. \end{aligned}$$

This implies

$$\frac{\partial \mathcal{E}(t)}{\partial t} \leq -\frac{3}{4C_\beta}\mathcal{E}(t).$$

Therefore, there exists a positive constant $C_1$ such that

$$\mathcal{E}(t) \leq C_1 \exp(-\frac{3}{4C_\beta} t)$$

for $t > T$. □

**Remark 3.1.13.** *By our choice of* $C_\beta$ ($C_\beta > \frac{3}{2\lambda_{min}(\beta)}$, *see the proof of Lemma 3.1.4), we know the decay rate is* $\delta_\beta = \frac{3}{4C_\beta}$, *which is less than* $\frac{\lambda_{min}(\beta)}{2}$, *where* $\lambda_{min}(\beta)$ *is the least absolute eigenvalue of the Hessian operator at the non-degenerate critical point* $\beta$. *From Proposition 3.1.12, we see that the perturbed gradient flow lines always have exponential decay with rate* $\delta_\beta$ *depending the critical points which the gradient flow lines approach as* $t \to \pm\infty$.

Since we work on the non-compact manifold $Y \times \mathbb{R}$, the ordinary Fredholm theory won't be enough because the Sobolev embedding theorems and the Sobolev multiplications fail. By Proposition 3.1.12, we know that we can take the weighted Sobolev space for the purpose of transversality and index calculations. See [30] for the relevant background of the weighted Sobolev spaces, which satisfy the Sobolev embedding theorems and the Sobolev multiplications.

### 3.1.3 Transversality and gluing procedure

We now study the geometric structure of the moduli space $\mathcal{M}(\alpha, \beta)$ for the gradient flow lines connecting two different critical points $\alpha, \beta$. The space $\mathcal{M}(\alpha, \beta)$ is also the moduli space of solutions to the perturbed Seiberg-Witten equations on $(Y \times \mathbb{R}, g + dt^2, \pi^*(\mathfrak{s}))$, which are asymptotic to $\alpha$ at $-\infty$ and $\beta$ at $\infty$. Let $\delta$ be the decay rate of the gradient flow line in $\mathcal{M}(\alpha, \beta)$ (see Proposition 3.1.12 and Remark 3.1.13).

Consider the space of pairs of connections and spinor sections of

$$\Big(Y \times \mathbb{R}, g + dt^2, \pi^*(\mathfrak{s})\Big),$$

topologized with the weighted Sobolev norms of weight $\delta$ as in [30]. Here the weight function is $e_\delta(t) = e^{\tilde{\delta}t}$, where $\tilde{\delta}$ is a smooth function with bounded derivatives, $\tilde{\delta} : \mathbb{R} \to [-\delta, \delta]$ such that $\tilde{\delta}(t) \equiv -\delta$ for $t \le -1$ and $\tilde{\delta}(t) \equiv \delta$ for $t \ge 1$. The $L^2_{k,\delta}$ norm is defined as

$$\|f\|_{L^2_{k,\delta}} = \Big(\int_{Y\times\mathbb{R}} e_\delta(t)(|f|^2 + |\nabla f|^2 + |\nabla^2 f|^2 + \cdots + |\nabla^k f|^2) dt dvol_Y\Big)^{\frac{1}{2}}$$

The weight $e_\delta$ imposes an exponential decay as an asymptotic condition along the cylinder.

We summarize some of the Sobolev embedding theorems and the Sobolev multiplications for the weighted Sobolev spaces (see [30]).

**Proposition 3.1.14.** *Let $Y$ be a compact oriented three-manifold endowed with a fixed Riemannian metric $g$. Consider the cylinder $Y \times \mathbb{R}$ with the metric $g + dt^2$. The weighted Sobolev spaces $L^2_{k,\delta}$ on the manifold $Y \times \mathbb{R}$ satisfy the following Sobolev properties.*

*(a) The embedding $L^2_{k,\delta} \hookrightarrow L^2_{k-1,\delta}$ is compact for all $k \geq 1$.*

*(b) If $k > m + 2$ we have a continuous embedding $L^2_{k,\delta} \hookrightarrow C^m$.*

*(c) If $k > m + 3$ the embedding $L^2_{k,\delta} \hookrightarrow C^m$ is compact.*

*(d) If $2 < k'$ and $k \leq k'$ the multiplication map $L^2_{k',\delta} \otimes L^2_{k,\delta} \stackrel{m}{\to} L^2_{k,2\delta}$ is continuous.*

*(e) The operator $\frac{\partial}{\partial t} + T(t)$ is a Fredholm operator on the weighted Sobolev spaces if $\frac{\delta}{2}$ is not an eigenvalue of the operators $\lim_{t\to\pm\infty} T(t)$, moreover, the index of $\frac{\partial}{\partial t} + T(t)$ is the spectral flow of the operator $T(t)$.*

*Consider a metric $g_t + dt^2$ on the cylinder $Y \times \mathbb{R}$ such that for a fixed $T$ we have $g_t \equiv g_0$ for $t \geq T$ and $g_t \equiv g_1$ for $t \leq -T$ and $g_t$ varies smoothly when $t \in [-1, 1]$. The same Sobolev embedding theorems hold for the $L^2_{k,\delta}$ spaces on $Y \times \mathbb{R}$.*

An element $\gamma_0 \in \mathcal{M}(\alpha, \beta)$ determines a path in $\mathcal{A}/\mathcal{G}$, also denoted by $\gamma_0$. Let $\tilde{\gamma}_0 = (A_0(t), \psi_0(t))$ be a lift of $\gamma_0$ to $\mathcal{A}(Y, \mathfrak{s})$. There is a spectral flow of $T(t)$ along the path $\tilde{\gamma}_0$ (see (3.6) for the relation with the spectral flow of the Hessian operator $Q$), denoted by $\mathfrak{i}(\gamma_0)$. We see that

$$\mathfrak{i}(\gamma_0) = SF_{\gamma_0}(Q) = \mathfrak{i}(\alpha, \beta)(mod\ d(\mathfrak{s}))$$

where $\mathfrak{i}(\alpha, \beta)$ is the relative index between $\alpha$ and $\beta$, defined in (3.8) and (3.9), note that $\mathfrak{i}(\gamma_0)$ is $\mathbb{Z}$-valued.

For $k \geq 2$, let $\mathcal{A}_{k,\delta}(\alpha, \beta)$ be the space of pairs of connections and spinor sections $(A, \psi)$ on $Y \times \mathbb{R}$ satisfying

$$(A, \psi) \in (A_0(t), \psi_0(t)) + (\Omega^1_{L^2_{k,\delta}}(Y \times \mathbb{R}, i\mathbb{R}) \oplus L^2_{k,\delta}(W^+)).$$

The gauge transformation group $\mathcal{G}_{k+1,\delta}(\alpha,\beta)$ is locally modelled on

$$L^2_{k+1,\delta}(\Omega^0(Y\times\mathbb{R}, i\mathbb{R}))$$

and approaches elements in the stabilizers $G_\alpha$ and $G_\beta$ as $t\to\pm\infty$. This gauge group acts on $\mathcal{A}_{k,\delta}(\alpha,\beta)$ freely. We can form the quotient $\mathcal{B}_{k,\delta}(\alpha,\beta)$, a smooth Banach manifold with tangent space at $[A,\psi]$

$$\{(a,\phi)\in\Omega^1_{L^2_{k,\delta}}\oplus L^2_{k,\delta}(W^+)\ |(e_{-\delta}d^*e_\delta)a+iIm\langle\psi,\phi\rangle=0\}.$$

From Proposition 3.1.12, we know that the component of $\mathcal{M}(\alpha,\beta)$ containing the given gradient flow line $\gamma_0$, denoted by $\mathcal{M}(\alpha,\beta)_{\gamma_0}$, can be identified with

$$\left\{(A,\Psi)\in\mathcal{A}_{k,\delta}(\alpha,\beta)\ \middle|\ \begin{array}{l}(A,\psi)\text{ satisfies the}\\ \text{monopole equations (3.19).}\end{array}\right\}/\mathcal{G}_{k+1,\delta}(\alpha,\beta).$$

Note that for a homology 3-sphere, each component of the moduli space $\mathcal{M}(\alpha,\beta)$ with the relative index $\mathfrak{i}(\alpha,\beta)$ has the constant dimension $\mathfrak{i}(\alpha,\beta)$.

The perturbation parameter space $\mathcal{P}$ is the space of $\Omega^1_{L^2_{k+1}}(Y,i\mathbb{R})$-valued, $\mathcal{G}_Y$-invariant functions on $\mathcal{A}$ with the $L^2(Y)$-norm less than the $L^2(Y)$-norm of $\frac{1}{4}\nabla\mathcal{C}_\eta$ at each point $(A,\psi)$ (see equations (3.17), (3.19 and (3.18)). Therefore, each class in $\mathcal{P}$ defines a compact perturbation $E(A_0,\psi)^+$ with respect to the $L^2_{k,\delta}$-norm.

**Proposition 3.1.15.** *For a generic perturbation $E\in\mathcal{P}$, $\mathcal{M}(\alpha,\beta)_{\gamma_0}$, the component of $\mathcal{M}(\alpha,\beta)$ containing $\gamma_0$, is an oriented, smooth manifold of dimension $\mathfrak{i}(\gamma_0)-d_\alpha$ (if $\mathfrak{i}(\gamma_0)-d_\alpha>0$), where $d_\alpha=1$ if $\alpha$ is reducible and $d_\alpha=0$ otherwise. The space $\mathcal{M}(\alpha,\beta)_{\gamma_0}$ is empty if $\mathfrak{i}(\gamma_0)-d_\alpha\le 0$.*

**Proof.** Suppose $(A(t),\psi(t),E)$ is a solution of the perturbed gradient flow equation (3.17) or (3.19). The linearization of (3.19), together with the gauge fixing condition $G^*$ at $(A(t),\psi(t),E)$ is the operator

$$\mathcal{D}_{A,\psi,E}:\quad \mathcal{P}\times L^2_{k,\delta}(i\Omega^1\oplus W^+)\to L^2_{k-1,\delta}(i\Omega^0\oplus i\Omega^{2,+}\oplus W^-)$$

which sends $(\kappa,b,\phi)$ to

$$(e_{-\delta}G^*e_\delta(b,\phi),d^+b+\kappa(A,\psi)^++K(b,\phi)+((Dq_\psi)\phi)^+,\not{\partial}_A(\phi)+\frac{1}{2}b.\psi),$$

where $K(b,\phi)$ is a bounded operator which is the linearization of $E(A,\psi)^+$ at $(A,\psi)$, $Dq_\psi$ is the linearization of $q(\psi,\psi)$ at $\psi$. Under the identifications of (3.12), (3.13), (3.14) and Lemma 3.1.7, for a sufficiently small $E(A,\psi)$ (fixed), up to a compact perturbation,

$$\mathcal{D}_{A,\psi,E} \approx \partial_t + T(t)$$

where $T(t)$ is the extended Hessian operator of $\mathcal{C}_\eta$ at $(A(t),\psi(t))$. By the choice of our weight $\delta$ and Proposition 3.1.14, we know that for a fixed perturbation $E(A,\psi)$, $\mathcal{D}_{A,\psi,E}$ is a Fredholm operator of index $\mathrm{i}(\gamma_0) - d_\alpha$. The appearance of $d_\alpha$ is due to the fact that $T(-\infty)$ has a one dimensional kernel at the reducible point.

. Now we need to verify that $\mathcal{D}_{A,\psi,E}$ is surjective whenever $(A,\psi,E)$ is a solution to the equation (3.19) in $\mathcal{A}_{k,\delta}(\alpha,\beta)$.

Let $(\xi,a,\phi) \in L^2_{-\delta}(i\Omega^0 \oplus i\Omega^{2,+} \oplus W^-)$, the dual space of $L^2_\delta(i\Omega^0 \oplus i\Omega^{2,+} \oplus W^-)$, such that $(\xi,a,\phi)$ is $L^2$-orthogonal to the range of $\mathcal{D}_{A,\psi,E}$. Varying $\kappa(A,\psi)$ alone, we see that $a=0$. Then $(\xi,\phi)$ satisfies the following equations:

$$\begin{cases} (a). & -2e_\delta d e_{-\delta}\xi = \sigma(\psi,\phi), \\ (b). & \not{D}_A\phi + \xi\psi = 0. \end{cases}$$

from which we can obtain the following equation for $\xi$:

$$-4d^*\Big(e_\delta d e_{-\delta}\xi\Big) = 2d^*\sigma(\psi,\phi) = iIm\langle\psi, \not{D}_A\phi\rangle = |\psi|^2\xi,$$

that is equivalent to

$$4\Big(e_{-\frac{\delta}{2}}d^* e_{\frac{\delta}{2}}\Big)\Big(e_{\frac{\delta}{2}} d e_{-\frac{\delta}{2}}\Big)\Big(e_{-\frac{\delta}{2}}\xi\Big) + |\psi|^2\Big(e_{-\frac{\delta}{2}}\xi\Big) = 0.$$

Note that $e_{-\frac{\delta}{2}}\xi$ is $L^2$-bounded, hence we know that $\xi = 0$ by the maximum principle, since $\psi \neq 0$. Then $\phi$ must satisfy the following equations (see the equations (a) and (b)):

$$\begin{cases} (c). & \not{D}_A\phi = 0, \\ (d). & \sigma(\psi,\phi) = 0, \\ (e). & \sigma(\psi, b.\phi) = 0 \qquad \text{for any } b \in \pi^*(i\Omega^1_Y). \end{cases}$$

where the equation (e) comes from the fact that $(0,0,\phi)$ is $L^2$-orthogonal to the range of $\mathcal{D}_{A,\psi,E}$. From (e), we know that $\phi = if\psi$ for a real function $f$. Plug $\phi = if\psi$ into the equation (d), to get $f = 0$, hence $\phi = 0$.

Therefore, the moduli space $\mathcal{M}^{\mathcal{P}}$ of triples $(A,\psi,E)$ in $\mathcal{A}_{k,\delta}(\alpha,\beta)\times\mathcal{P}$ to the equation (3.19) is a smooth, infinite dimensional manifold.

The projection $\Pi : \mathcal{M}^{\mathcal{P}} \to \mathcal{P}$ linearizes to a surjective Fredholm operator with index $\mathfrak{i}(\gamma_0) - d_\alpha$. Then the Sard-Smale theorem implies that the moduli space $\mathcal{M}(\alpha,\beta)_{\gamma_0}$, for generic perturbation $E \in \mathcal{P}$, is the inverse image under the projection map $\Pi$ of a regular value. Thus $\mathcal{M}(\alpha,\beta)_{\gamma_0}$, the component of $\mathcal{M}(\alpha,\beta)$ containing $\gamma_0$, is a smooth manifold of dimension $\mathfrak{i}(\gamma_0) - d_\alpha$ if $\mathfrak{i}(\gamma_0) - d_\alpha > 0$, and is empty otherwise.

The orientation of $\mathcal{M}(\alpha,\beta)_{\gamma_0}$ is determined by the determinant line bundle of $\mathcal{D}_{A,\psi,E}$ over $\mathcal{A}_{k,\delta}(\alpha,\beta)$ for a fixed perturbation $E$. Notice that we consider the configuration space of $\delta$-exponential decay connections and spinor sections. Then the arguments in [49] and V.6 of [35] can be employed to get the orientation of $\mathcal{M}(\alpha,\beta)$ by choosing an orientation of

$$\wedge^{top} H^1_\delta(Y\times\mathbb{R}, i\mathbb{R}) \otimes \wedge^{top} H^{2,+}_\delta(Y\times\mathbb{R}, i\mathbb{R}) \otimes H^0_\delta(Y\times\mathbb{R}, i\mathbb{R}).$$

Note that there are no exponentially decaying harmonic forms on $Y\times\mathbb{R}$. Therefore,

$$H^1_\delta(Y\times\mathbb{R}, i\mathbb{R}) = H^{2,+}_\delta(Y\times\mathbb{R}, i\mathbb{R}) = H^0_\delta(Y\times\mathbb{R}, i\mathbb{R}) = 0.$$

This implies that $\mathcal{M}(\alpha,\beta)$ has a canonical orientation from the complex structure of the spinor sections. □

Proposition 3.1.15 has an immediate corollary.

**Corollary 3.1.16.** *Let $(Y, g, \mathfrak{s})$ be a closed, oriented 3-manifold with Riemannian metric $g$ and the* Spin$^c$ *structure $\mathfrak{s} = (W = W_0 \otimes L, \rho)$, $\mathcal{M}_{\mathfrak{s},\eta}$ be the set of non-degenerate critical points of $\mathcal{C}_\eta$ as in Lemma 3.1.2, let $\mathcal{M}(\alpha,\beta)$ be the moduli space of gradient flow lines (for a generic perturbation) of $\mathcal{C}_\eta$ which connect two different critical points $\alpha$ and $\beta$. Then,*

*(a) If $c_1(\mathfrak{s})$ is a torsion element, in particular, in the case of $Y$ a homology 3-sphere, $\mathcal{M}(\alpha,\beta)$ is empty if $\mathfrak{i}(\alpha) - \mathfrak{i}(\beta) - d_\alpha \le 0$, otherwise, $\mathcal{M}(\alpha,\beta)$ is an oriented, smooth manifold of dimension $\mathfrak{i}(\alpha) - \mathfrak{i}(\beta) - d_\alpha$.*

(b) *If $c_1(\mathfrak{s})$ is non-zero in $H^2(Y,\mathbb{R})$, $\mathcal{M}(\alpha,\beta)$ has many components, each of them is an oriented, smooth manifold with dimension given by $\mathfrak{i}(\gamma) - d_\alpha > 0$, where $\gamma$ is a chosen element in that component, the dimensions of two non-empty components are different by a multiple of $d(\mathfrak{s})$, where $d(\mathfrak{s})$ is the divisibility of $c_1(\mathfrak{s})$ in $H^2(Y,Z)/Torsion$.*

Fix a critical point $\alpha_0$ in $\mathcal{M}_\eta$ for the case of a 3-manifold $Y$ with $b_1(Y) > 0$, we define

$$\mathfrak{i}(\alpha) = \mathfrak{i}(\alpha,\alpha_0) \in \{0,1,2,\cdots,d(\mathfrak{s})-1\} \tag{3.20}$$

then the relative index $\mathfrak{i}(\alpha,\beta) = \mathfrak{i}(\alpha) - \mathfrak{i}(\beta)$.

The primary components of concern in Corollary 3.1.16(b) are the non-empty component with the least dimension. We will mainly study the components of $\mathcal{M}(\alpha,\beta)$ with dimension between one and $d(\mathfrak{s})-1$. Denoted by $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ the least dimensional components in $\mathcal{M}(\alpha,\beta)$ for $\mathfrak{i}(\alpha) > \mathfrak{i}(\beta)$ where $\mathfrak{i}(\alpha)$ and $\mathfrak{i}(\beta)$ are defined by (3.20). There are similar properties for the higher dimensional components of $\mathcal{M}(\alpha,\beta)$.

**Lemma 3.1.17.** *For a homology 3-sphere $(Y,\mathfrak{s})$ or $(Y,\mathfrak{s})$ with $b_1(Y) > 0$ and non-trivial first Chern class $c_1(\mathfrak{s})$ in $H^2(Y,\mathbb{R})$, the perturbation $\eta$ is chosen to be $*d$-exact as in Corollary 2.2.13, then there exists a uniform finite energy $E$:*

$$\int_{-\infty}^{\infty} \|\nabla \mathcal{C}_\eta(S(t))\|_{L^2} dt < E,$$

*for any $S(t)$ in a component of $\mathcal{M}(\alpha,\beta)$ with fixed dimension, in particular, on $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$.*

**Proof.** For a homology 3-sphere $(Y,\mathfrak{s})$, the claim is obvious, since the perturbed Chern-Simons-Dirac function on $\mathcal{B}$ is $\mathbb{R}$-valued. For $(Y,\mathfrak{s})$ with $b_1(Y) > 0$ and non-trivial first Chern class $c_1(\mathfrak{s})$ in $H^2(Y,\mathbb{R})$, the perturbation $*\eta$ can be chosen to be an exact 1-form by Corollary 2.2.13. We can lift $\mathcal{C}_\eta$ to be an $\mathbb{R}$-valued function on $\mathcal{B}_0$ where $\mathcal{B}_0$ is the intermediate covering space of $\mathcal{B}$ determined by $c_1(\mathfrak{s})$. To be specific, denote by $Ker(c_1(\mathfrak{s}))$ the kernel of the map $H^1(Y,\mathbb{Z}) \to \mathbb{Z}$ defined by sending $[u]$ to

$$\Big(c_1(\mathfrak{s}) \wedge [u]\Big)([Y]).$$

Then $\tilde{\mathcal{B}}$ (cf. Diagram 3.3) is a covering space of $\mathcal{B}_0$ with fiber $Ker(c_1(\mathfrak{s}))$:

$$\tilde{\mathcal{B}} \xrightarrow{Ker(c_1(\mathfrak{s}))} \mathcal{B}_0 \xrightarrow{H^1(Y,\mathbb{Z})/Ker(c_1(\mathfrak{s}))} \mathcal{B}.$$

Fix a point $\alpha^{(0)}$ in $\mathcal{B}_0$ over $\alpha \in \mathcal{B}$. By the dimension formula and the definition of energy, any lifting of $S(t) \in \mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ to a path on $\mathcal{B}_0$ starting from $\alpha^{(0)}$ (as $t \to -\infty$) will approach a fixed critical point in $\mathcal{B}_0$ over $\beta$. Therefore, we have a uniform finite energy for the gradient flow lines in $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$. The same arguments can be applied to prove this Lemma for a component in $\mathcal{M}(\alpha, \beta)$ with fixed dimension. □

There is a natural $\mathbb{R}$-action on $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$, since the perturbed gradient flow equation (3.17) is $\mathbb{R}$-translation invariant. Denote by $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ the quotient of $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ by $\mathbb{R}$. For $\alpha, \beta$ with $\mathfrak{i}(\alpha) > \mathfrak{i}(\beta)$, $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha,\beta)}$ is an oriented, smooth manifold of dimension $\mathfrak{i}(\alpha) - \mathfrak{i}(\beta) - d_\alpha - 1$.

Due to the $\mathbb{R}$-action on $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$, the moduli space $\mathcal{M}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ is not compact, unlike the Seiberg-Witten moduli space on a compact manifold where the $C^0$-bound on the spinor can lead to the $L^2_k$-bounds. Though we have the $C^0$-bound on the spinor, the non-compactness of $Y \times R$ blows up the $L^2_k$-bounds. Notice that our solutions to (3.17) or (3.19) with a small perturbation satisfying (3.18), after the quotient by $\mathbb{R}$, $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ is sequentially compact.

For any two gradient flow lines $S_1(t) \in \mathcal{M}(\alpha, \beta)$, $S_2(t) \in \mathcal{M}(\beta, \gamma)$, after reparametrization, we can identify $(S_1(t), S_2(t))$ as a "broken trajectory" of the gradient flows in $\mathcal{M}(\beta, \gamma)$. With this convention, we show that $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ is a manifold with corners, the following Proposition claims that it is also compact. If we already know the fixed quantity $\mathfrak{i}(\alpha) - \mathfrak{i}(\beta) = n$, we would rather use the notation $\hat{\mathcal{M}}^n(\alpha, \beta)$ for $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$.

**Proposition 3.1.18.** *Suppose that $Y$ is a homology 3-sphere, or $(Y, \mathfrak{s})$ is a 3-manifold with $b_1(Y) > 0$ and $c_1(det(\mathfrak{s})) \neq 0$ in $H^2(Y, \mathbb{R})$, choose the perturbation $\eta$ on $Y$ as in Corollary 2.2.13, then $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ is sequentially compact, namely, any sequence in $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ has a subsequence which either converges to a trajectory in $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ or converges to a broken trajectory. The component of $\hat{\mathcal{M}}(\alpha, \beta)$ with fixed dimension for $b_1(Y) > 0$ is also sequentially compact.*

**Proof.** We choose a lift of $\hat{\mathcal{M}}^{i(\alpha)-i(\beta)}$ in $\mathcal{M}^{i(\alpha)-i(\beta)}$ by requiring that the gradient flow line $[A(t), \psi(t)]$ have equal energy on $(-\infty, 0]$ and $[0, \infty)$:

$$\int_{-\infty}^{0} \|\nabla \mathcal{C}_\eta([A(t), \psi(t)])\|_{L^2} dt = \int_{0}^{\infty} \|\nabla \mathcal{C}_\eta([A(t), \psi(t)])\|_{L^2} dt. \tag{3.21}$$

Note that as a solution to the elliptic equation, $\nabla \mathcal{C}_\eta([A(t), \psi(t)])$ is smooth. This lift is unique since in the family of gradient flow lines $\mathbb{R}.[A(t), \psi(t)]$ in $\mathcal{M}^{i(\alpha)-i(\beta)}$, there exists a unique gradient flow line satisfying the equal energy condition (3.21).

Let $[A_i(t), \psi_i(t)]$ be a sequence of gradient flow lines on $\mathcal{B}$ with the equal energy condition. By Lemma 3.1.17, we know that $[A_i(t), \psi_i(t)]$ have uniformly finite energy $E$. By the Palais-Smale condition (Lemma 3.1.3), we can find $T >> 1$ (choose $T > \frac{E}{\lambda}$ where $\lambda$ is the constant appearing in Lemma 3.1.3), such that for $|t| > T$, $[A_i(t), \psi_i(t)]$ lie in a very small $\epsilon$-neighbourhood of $\alpha$ or $\beta$.

On $Y \times [-T-1, T+1]$ (a compact 4-manifold), we can apply the compactness result for the Seiberg-Witten moduli space [26], then we follow the arguments in [27] to prove that $(A_i(t), \psi_i(t))$ represents a subsequence of $[A_i(t), \psi_i(t)]$ that converges to a solution to (3.19) on $Y \times [-T-1, T+1]$ after a gauge transformation.

Now we assume that there is a subsequence with the uniform exponential decay, then we can find a subsequence converging strongly on $Y \times ((-\infty, -T] \cup [T, \infty))$. On $Y \times [-T-1, T+1]$, after passing to a further subsequence, there exist gauge transformations $u_i \in L^2_{k+1}(Y \times [-T-1, T+1])$ such that the transformed solutions converge strongly on $Y \times [-T-1, T+1]$. We need to merge $\{u_i\}$ over the overlap $K = Y \times ([-T-1, -T] \cup [T, T+1])$. This can be done by choosing a cut-off function $c$ equal to 1 on $Y \times [-T, T]$ and 0 on $Y \times ((-\infty, -T-1] \cup [T+1, \infty))$. Over $K$, there exists a subsequence of $\{u_i\}$ converging strongly to a gauge transformation $u$, such that for a sufficiently large $N$ and for all $i > N$, the $C^0$-bound $|u_i - u_N| < \frac{1}{2}$ is satisfied. Then for all $i > N$, $u_i = u_N exp(2\pi i \theta_i)$ for a unique function $\theta_i$ on $K$ satisfying $|\theta_i| < \frac{1}{2}$. Now define gauge transformations $\{v_i\}$ on $Y \times \mathbb{R}$ by

$$v_i = \begin{cases} u_i & \text{on } Y \times [-T, T], \\ u_N exp(2\pi i c\theta_i) & \text{on } Y \times ([-T-1, -T] \cup [T, T+1]), \\ u_N & \text{on } Y \times ((-\infty, -T-1] \cup [T+1, \infty)). \end{cases}$$

Then the sequences $[v_i(A_i, \psi_i)]$ converges strongly on $Y \times \mathbb{R}$ with limit representing a trajectories from $\alpha$ to $\beta$.

For the other situation such that the sequence is not uniformly exponential decay on $Y \times ((-\infty, -T] \cup [T, \infty))$, after reparametrization, let $[A_i(t \pm T_i), \psi_i(t \pm T_i)]$ be the resultant sequence with uniform decay on $Y \times (-\infty, -T]$ or $Y \times [T, \infty)$, then the above argument implies that there exists a subsequence converging in $C^\infty_{loc}$-topology to a broken gradient flow line from $\alpha$ to $\gamma$ and then from $\gamma$ to $\beta$. Repeat these procedures, there exists a subsequence converging to a broken trajectory. See [33] for the another proof of the appearance of the broken trajectories. This completes the proof of the Proposition. □

We will not distinguish $\hat{\mathcal{M}}(\alpha, \beta)$ from its compactification by adding the 'corners'. After we understand the following gluing model in Proposition 3.1.19, we will see that the boundary of $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$ consists of "broken trajectories". Now we construct the gluing map of two gradient flows. This enables us to analyse the boundary of the moduli space $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$.

**Proposition 3.1.19.** *Given $\alpha, \beta$ and $\gamma$ in $\mathcal{M}_{\mathfrak{s},\eta}(Y)$ with $\mathfrak{i}(\alpha) > \mathfrak{i}(\beta) > \mathfrak{i}(\gamma)$, introduce the spaces $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)}$, $\hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\gamma)}$ and $\hat{\mathcal{M}}^{\mathfrak{i}(\beta)-\mathfrak{i}(\gamma)}$ as before. We further assume that*

$$max\{\mathfrak{i}(\alpha) - \mathfrak{i}(\beta), \mathfrak{i}(\beta) - \mathfrak{i}(\gamma)\} < \frac{d(\mathfrak{s})}{2}.$$

*(a) If $\beta$ is irreducible, then for large enough $T$ there is an orientation preserving, local diffeomorphism*

$$g: \qquad \hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)} \times [T, \infty) \times \hat{\mathcal{M}}^{\mathfrak{i}(\beta)-\mathfrak{i}(\gamma)} \hookrightarrow \hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\gamma)}.$$

*(b) If $\beta$ is reducible, then for large enough $T$ there is a orientation preserving, local diffeomorphism*

$$\hat{g}: \qquad \hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\beta)} \times \hat{\mathcal{M}}^{\mathfrak{i}(\beta)-\mathfrak{i}(\gamma)} \times U(1) \times [T, \infty) \hookrightarrow \hat{\mathcal{M}}^{\mathfrak{i}(\alpha)-\mathfrak{i}(\gamma)}.$$

**Remark 3.1.20.** *There is a general gluing map on $\mathcal{M}(\alpha, \beta) \times \mathcal{M}(\beta, \gamma)$ without the assumption that $max\{\mathfrak{i}(\alpha) - \mathfrak{i}(\beta), \mathfrak{i}(\beta) - \mathfrak{i}(\gamma)\} < \frac{d(\mathfrak{s})}{2}$. Our assumption is to ensure* *that the image of the gluing map stays in the component of the least dimension. This assumption is always satisfied in the construction of the Seiberg-Witten-Floer homology.*

**Proof.** (a). Suppose that the perturbations for $\mathcal{M}(\alpha,\beta)$ and $\mathcal{M}(\beta,\gamma)$ are $E_1$ and $E_2$. Let $[A_1(t),\psi_1(t)]$ and $[A_2(t),\psi_2(t)]$ be two elements in the moduli spaces $\hat{\mathcal{M}}^{\mathrm{i}(\alpha)-\mathrm{i}(\beta)}$ and $\hat{\mathcal{M}}^{\mathrm{i}(\beta)-\mathrm{i}(\gamma)}$ respectively.

We lift $[A_i(t),\psi_i(t)]$ to $\mathcal{B}$ with equal energy as in (3.21). Then there exists a sufficiently large $T$, such that $[A_1(t),\psi_1(t)]|_{[T,\infty)}$ and $[A_2(t),\psi_2(t)]|_{(-\infty,-T]}$ decay exponentially to $\beta$ as $t\to\pm\infty$ respectively.

We want to construct a path in the $\epsilon$-neighbourhood of $\beta$ from $[A_1(t),\psi_1(t)]|_{[T,\infty)}$ to $[A_2(t),\psi_2(t)]|_{(-\infty,-T]}$. We identify the $\epsilon$-neighbourhood of $\beta$ with the $\epsilon$-ball in $T_\beta\mathcal{B}^*$ centered at 0. Write

$$[A_1(t),\psi_1(t)]|_{[T,\infty)}=\beta+(a_1(t),\phi_1(t)),$$
$$[A_2(t),\psi_2(t)]|_{(-\infty,-T]}=\beta+(a_2(t),\phi_2(t)),$$

where $x_i(t)=(a_i(t),\phi_i(t))$ decay exponentially to 0 and satisfy the perturbed gradient flow equation on the $\epsilon$-ball in $T_\beta\mathcal{B}^*$ centered at 0:

$$(\partial_t+Q)(a_i(t),\phi_i(t))=N(a_i(t),\phi_i(t)). \tag{3.22}$$

Here $Q$ is the Hessian operator at $\beta$, $N(a_i(t),\phi_i(t))$ is the non-linear term of $\sigma(\phi_i,\phi_i)$ and the term from the perturbation.

Choose a smooth function $\rho(t)$ on $[-1,1]$, which varies only in $[-\frac{1}{2},\frac{1}{2}]$, is 1 near $-1$ and 0 near 1. Define

$$x(t)=(a_1,\psi_1)\sharp_T^0(a_2,\psi_2)=\rho(t)x_1(t+2T)+(1-\rho(t))x_2(t-2T)$$

for $-1\le t\le 1$. Then define the pre-gluing map:

$$[A_1(t),\psi_1(t)]\sharp_T^0[A_2(t),\psi_2(t)]=\begin{cases}[A_1(t+2T),\psi_1(t+2T)] & \text{for} \quad t\le -1\\ \tilde{x}(t)=\beta+x(t) & \text{for} \quad -1\le t\le 1\\ [A_2(t-2T),\psi_2(t-2T)] & \text{for} \quad t\ge 1\end{cases}$$

Using the exponential decay of $(a_i(t), \phi_i(t))$, we see that

$$[A_1(t), \psi_1(t)]\sharp^0_T[A_2(t), \psi_2(t)]$$

is an approximate solution to (3.22) for a sufficiently large $T >> 1$. Then one can find an actual monopole on $Y \times \mathbb{R}$ in a neighbourhood of $[A_1(t), \psi_1(t)]\sharp^0_T[A_2(t), \psi_2(t)]$ which connects the critical points $\alpha$ and $\gamma$. This is a standard application of the implicit function theorem.

Given a compact set $K$ in $\mathcal{M}(\alpha, \beta) \times \mathcal{M}(\beta, \gamma)$ and a sufficiently large $T_0$, consider the Hilbert bundles, $\mathcal{T}_1$ and $\mathcal{T}_0$, that are defined respectively as pullbacks via the map $\#^0_T$ of the $L^2_{1,\delta}$ and of the $L^2_{0,\delta}$ tangent bundles of $\mathcal{B}(\alpha, \gamma)$ on the base space $K \times [T_0, \infty)$.

The perturbed gradient flow defines the fibre restriction of a bundle map from $\mathcal{T}_1$ to $\mathcal{T}_2$ defined on a neighbourhood of the zero section in $\mathcal{T}_1$. The linearization $\mathcal{L}$ at the approximate solution

$$[A_1(t), \psi_1(t)]\sharp^0_T[A_2(t), \psi_2(t)]$$

is the fibre derivative of the above bundle map. Since the linearizations $\mathcal{L}_{[A_1(t),\psi_1(t)]}$ and $\mathcal{L}_{[A_2(t),\psi_2(t)]}$ are surjective, then

$$\mathcal{K} = \bigcup_{K \times [T_0, \infty)} Ker(\mathcal{L}_{[A_1(t),\psi_1(t)]}) \times Ker(\mathcal{L}_{[A_2(t),\psi_2(t)]})$$

is a subbundle of $\mathcal{T}_1$ over $K \times [T_0, \infty)$. Thus we can define a space $\mathcal{T}_\chi^\perp$ for $\chi \in K \times [T_0, \infty)$ given by all elements of $\mathcal{T}_1$ that are orthogonal to the image of $Ker(\mathcal{L}_{[A_1(t),\psi_1(t)]}) \times Ker(\mathcal{L}_{[A_2(t),\psi_2(t)]})$ under the linearization $L_\#$ of the pre-gluing $\#^0_T$.

**Lemma 3.1.21.** *There exist a bound $T(K) \geq T_0$ such that, for all $T \geq T(K)$ and for all broken trajectories*

$$((A_1(t), \psi_1(t)), (A_2(t), \psi_2(t))) \in K,$$

*the Fredholm operator $\mathcal{L}_\chi$*

$$\mathcal{L}_\chi : \mathcal{T}_{1\chi} \to \mathcal{T}_{0\chi}$$

*is surjective, where $\chi$ is the approximate solution under the pre-gluing map. Moreover, composition of the pre-gluing map $\#_T^0$ with the orthogonal projection on $Ker\mathcal{L}_\chi$ gives an isomorphism*

$$Ker(\mathcal{L}_{[A_1(t),\psi_1(t)]}) \times Ker(\mathcal{L}_{[A_2(t),\psi_2(t)]}) \xrightarrow{\cong} Ker\mathcal{L}_\chi.$$

Proof of Lemma 3.1.21: We know that $\mathcal{L}_\chi$ is Fredholm of index $\mathrm{i}(\alpha) - \mathrm{i}(\gamma)$ and $dimKer(\mathcal{L}_{[A_1(t),\psi_1(t)]}) = \mathrm{i}(\alpha) - \mathrm{i}(\beta)$, $dimKer(\mathcal{L}_{[A_2(t),\psi_2(t)]}) = \mathrm{i}(\beta) - \mathrm{i}(\gamma)$.

We need to show that for any pair $([A_1(t),\psi_1(t)],[A_2(t),\psi_2(t)])$ there is a bound $T_0 = T(x_1,x_2)$ such that $\mathcal{L}_\chi$ is surjective for $T \geq T_0$. The compactness of $K$ will ensure that there is a uniform such bound $T(K)$.

It is therefore enough to prove the following crucial step.

**Lemma 3.1.22.** *There exist $T_0$ and a constant $C > 0$ such that*

$$||\mathcal{L}_\chi \xi||_{L^2_{0,\delta}} \geq C||\xi||_{L^2_{1,\delta}}.$$

*for all $T \geq T_0$ and $\xi$ in $\mathcal{T}_\chi^\perp$.*

Proof of Lemma 3.1.22: Suppose there are sequences $T_k \to \infty$ and $\xi_k \in \mathcal{T}_\chi^\perp$ such that $||\xi_k|| = 1$ and $||\mathcal{L}_\chi \xi_k|| \to 0$.

We first show that the supports of the $\xi_k$ become more and more concentrated at the asymptotic ends as $k \to \infty$. We consider the operator $\mathcal{L}_\beta = \frac{\partial}{\partial t} + Q_\beta$. If $\zeta : \mathbb{R} \to [0,1]$ is a smooth function which is equal to 1 on $[-1/2,1/2]$ and equal to zero outside $(-1,1)$, let $\zeta_k(t) = \zeta(\frac{t}{2T_k})$. Then we have

$$\begin{aligned} ||\mathcal{L}_\beta\ \zeta_k\xi_k|| &\leq ||\zeta_k'\xi_k|| + ||\zeta_k\mathcal{L}_b\xi_k|| \\ &\leq \tfrac{1}{2T_k}\max|\zeta'| + ||(\mathcal{L}_\chi - \mathcal{L}_\beta)\xi_k|| + ||\mathcal{L}_\chi\xi_k|| \\ &\leq \tfrac{1}{T_k}\max|\zeta'| + \sup\nolimits_{t\in[-T_k,T_k]}||Q_{\chi(t)} - Q_\beta||\ \cdot ||\xi_k|| + ||\mathcal{L}_\chi\xi_k||. \end{aligned}$$

Thus, $||\mathcal{L}_\beta\ \zeta_k\xi_k|| \to 0$ as $k \to \infty$. In fact, the term $\sup_{t\in[-2T_k,2T_k]}||Q_{\chi(t)} - Q_\beta||$ is bounded by

$$\sup_{t\in[-1,1]}||Q_{\chi(t)} - Q_\beta|| + \sup_{t\in[-2T_k,-1]}||Q_{x_1(t+2T_k)} - Q_\beta|| + \sup_{t\in[1,2T_k]}||Q_{x_2(t-2T_k)} - \beta||.$$

where $x_i(t) = [A_i(t), \psi_i(t)]$. All these terms tend to zero because of the exponential decay to the critical point $\beta$ of the trajectories $(A_1(t), \psi_1(t))$ and $(A_2(t), \psi_2(t))$. The operator $\mathcal{L}_\beta$ is an isomorphism between the spaces $L^2_{1,\delta}(\mathbb{R}, T(\mathcal{B}))$ and $L^2_{0,\delta}(\mathbb{R}, T(\mathcal{B}))$, hence we have $\xi_k \to 0$ in the $L^2_1$ norm over $Y \times [-2T_k, 2T_k]$.

This result allows us to rephrase the convergence condition $||\mathcal{L}_X \xi_k|| \to 0$ in terms of the Fredholm operators $\mathcal{L}_{x_1}$ and $\mathcal{L}_{x_2}$:

$$||\mathcal{L}_{x_1}(\rho^-_{1-T_k}\xi_k^{-T_k})|| \leq ||\rho'^- \xi_k|| + ||\rho^- \mathcal{L}_X \xi_k|| \leq$$

$$\leq C||\xi_k||_{Y\times[-1,1]} + ||\mathcal{L}_X \xi_k|| \to 0,$$

where $\rho^-_{1-T_k}(t) = \rho(t+1-T_k)$ and $\xi_k^{-T_k}(t) = \xi_k(t-T_k)$. This implies $\rho^-_{1-T_k}\xi_k^{-T_k} \to v$ where $v \in Ker(\mathcal{L}_{x_1})$, since $\mathcal{L}_{x_1}$ is a Fredholm operator. Thus, $||\rho^-_1 \xi_k - v^{T_k}|| \to 0$. Similarly we obtain an element $u$ in $Ker(\mathcal{L}_{x_2})$ such that $||\rho^+_{-1}\xi_k - u^{-T_k}|| \to 0$.

We now use these estimates to derive a contradiction with the assumption that $||\xi_k|| = 1$ and $\xi_k \in \mathcal{T}_X^\perp$. We have

$$1 = \lim_k ||\xi_k|| = \lim_k \langle \rho^-_1 \xi_k, \xi_k \rangle + \langle \rho^+_{-1}\xi_k, \xi_k \rangle,$$

since the remaining term satisfies

$$\langle (1 - \rho^-_1 - \rho^+_{-1})\xi_k, \xi_k \rangle = 0$$

for large $k$ because $(1 - \rho^-_1 - \rho^+_{-1})$ is supported in $[-2, 2]$. Thus the equality can be rewritten as

$$1 = \lim_k \langle \rho^- v, \xi_k \rangle + \lim_k \langle \rho^+ u, \xi_k \rangle = \lim_k \langle L_\#(u, v), \xi_k \rangle = 0.$$

The last equality holds since, by construction, $\xi_k \in \mathcal{T}^\perp$ is orthogonal to the image of the linearization $L_\#$ of the pre-gluing map.

This completes the proof of Lemma 3.1.22. The proof of Lemma 3.1.21 follows from dimension counting.

Now we want to define the actual gluing map # that provides a solution of the flow equations in $\mathcal{M}(O_a, O_c)$. This means that we want to obtain a section $e$ of $\mathcal{T}_1$ such that the image under the bundle homomorphism given by the flow equation is zero in $\mathcal{T}_0$. Moreover, we want this element $e(x_1, x_2, T)$ to converge to zero sufficiently rapidly as $T \to \infty$, so that the glued solution will converge to the broken trajectory $(x_1, x_2)$ in the limit $T \to \infty$.

The result is obtained as a fixed point theorem in Banach spaces. Consider the right inverse map of $\mathcal{L}$ restricted to $\mathcal{T}^\perp$,

$$G : \mathcal{T}_0 \to \mathcal{T}^\perp.$$

There is a $T(K)$ and a constant $C > 0$ such that

$$||G_\chi \xi|| \le C||\xi||$$

for $\chi \in K \times [T(K), \infty)$. The proof of uniform $C$ for $\chi \in K \times [T(K), \infty)$ is similar to the proof Lemma 3.1.22.

We aim at using the contraction principle. Namely, suppose we are given a smooth map $f : E \to F$ between Banach spaces of the form

$$f(x) = f(0) + Df(0)x + N(x),$$

with $Ker(Df(0))$ finite dimensional, with a right inverse $Df(0) \circ G = Id_F$, and with the nonlinear part $N(x)$ satisfying the estimate

$$||GN(x) - GN(y)|| \le C(||x|| + ||y||)||x - y|| \tag{3.23}$$

for some constant $C > 0$ and $x$ and $y$ in a small neighbourhood $B_{\epsilon(C)}(0)$. Then, with the initial condition $||G(f(0))|| \le \epsilon/2$, there is a unique zero $x_0$ of the map $f$ in $B_\epsilon(0) \cap G(F)$. This satisfies $||x_0|| \le \epsilon$.

The map $f$ is given in our case by the flow equation, viewed as a bundle homomorphism $\mathcal{T}_1 \mapsto \mathcal{T}_0$. We write $f$ as a sum of a linear and a non-linear term, where the linear term is $\mathcal{L}$ and the nonlinear term is $N_{(A(t),\psi(t))}(a, \phi)$. It is easy to see that the estimate (3.23) holds for our equations.

We have to verify the initial condition. This is provided by the exponential decay. In fact, we have

$$||f(A_1 \#_T^0 A_2, \psi_1 \#_T^0 \psi_2)|| \le C(||(\alpha_1, \phi_1)^{2T}||_{Y \times [-1,0]} + ||(\alpha_2, \phi_2)^{-2T}||_{Y \times [0,1]}).$$

The exponential decay of $(A_1(t), \psi_1(t))$ and $(A_2(t), \psi_2(t))$ to the critical point $\beta$ implies a decay

$$||f(A_1 \#_T^0 A_2, \psi_1 \#_T^0 \psi_2)|| \le Ce^{-\delta T} \tag{3.24}$$

for all $T \geq T_0$. The constant $C$ and the lower bound $T_0$ can be chosen uniformly due to the compactness of $K$.

This provides the existence of a unique correction term

$$e(x_1, x_2, T) \in B_\epsilon(0) \cap \mathcal{T}^\perp$$

satisfying $f(e) = 0$. The implicit function theorem ensures that $e$ is smooth. The exponential decay (3.24) ensures an analogous decay for $e$, hence the glued trajectory approaches the broken trajectory when $T$ is very large. The gluing map is given by

$$(A_1 \#_T A_2, \psi_1 \#_T \psi_2) = (A_1 \#_T^0 A_2, \psi_1 \#_T^0 \psi_2) + e([A_1, \psi_1], [A_2, \psi_2], T).$$

By dimension counting, we see that $g$ maps

$$\hat{\mathcal{M}}^{i(\alpha)-i(\beta)} \times \hat{\mathcal{M}}^{i(\beta)-i(\gamma)} \times [T, \infty) \to \hat{\mathcal{M}}^{i(\alpha)-i(\gamma)}$$

and is a local diffeomorphism which is orientation preserving.

The proof of (b) is analogous except that we need to work with the framed configuration space near the reducible critical point $\beta$, we omit the details. □

Notice that Lemma 3.1.22 and the fixed point argument that completes the proof of Proposition 3.1.19 construct an $L^2_{1,\delta}$ solution. However, on the manifold $Y \times \mathbb{R}$ this is not enough to guarantee that the solution is continuous. Since we are interested in the moduli space of gauge orbits, we have to improve the regularity of the estimate of Lemma 3.1.22. A similar problem is analysed in [15], 7.2.3. We can consider an equivalent model of the moduli space by taking $L^p_{k,\delta}$ connections and sections acted upon by $L^p_{k+1,\delta}$ gauge transformations. We have the following version of the Sobolev embeddings of Proposition 3.1.14.

**Proposition 3.1.23.** *Let $Y$ be a compact oriented three-manifold endowed with a fixed Riemannian metric $g_0$. Consider the cylinder $Y \times \mathbb{R}$ with the metric $g_0 + dt^2$. The weighted Sobolev spaces $L^p_{k,\delta}$ on the manifold $Y \times \mathbb{R}$ satisfy the following Sobolev embeddings.*

*(a) The embedding $L^q_{k,\delta} \hookrightarrow L^p_{l,\delta}$ is continuous whenever $\frac{1}{p} \leq \frac{1}{q} - \frac{k-l}{4}$.*

*(b) If $1 \geq \frac{1}{p} > \frac{1}{q} - \frac{k}{4} > 0$, then we have a compact embedding $L^q_{k,\delta} \hookrightarrow L^p_{0,\delta}$.*

(c) *If* $k > \alpha + \frac{4}{p}$, *with* $0 \leq \alpha < 1$, *we have a compact embedding* $L^p_{k,\delta} \hookrightarrow C^\alpha$.

(d) *If* $2 \geq k$ *and* $1 + \frac{4}{p} \geq \frac{4}{q} + k$, *then the multiplication map* $L^p_{k,\delta} \otimes L^4_{k,\delta} \xrightarrow{m} L^q_{k,2\delta}$ *is continuous.*

In particular this means that we can rephrase the estimate of Lemma 3.1.22 as

$$||\mathcal{L}_{x\#^0_T y}\xi||_{L^p_{0,2\delta}} \geq C_0||\xi||_{L^p_{1,2\delta}},$$

for all $T \geq T_0$ and

$$\xi \in (L_\#(Ker(\mathcal{L}_x) \times Ker(\mathcal{L}_y)))^\perp.$$

Now, if we choose $p > 4$ the solution in $L^p_{1,\delta}$ is continuous. We can bootstrap from the relation

$$f(x\#^0_T y) + \mathcal{L}_{x\#^0_T y}\xi + \mathcal{N}(\xi) = 0$$

at the fixed point $\xi$. We obtain the estimates

$$||\xi||_{L^p_{k,2\delta}} \leq C_1(||\mathcal{N}(\xi)||_{L^q_{k-1,2\delta}} + ||f(x\#^0_T y)||_{L^q_{k-1,2\delta}}.$$

Proposition 3.1.23 provides the bound

$$||\mathcal{N}(\xi)||_{L^q_{k-1,2\delta}} \leq C_2||\xi||_{L^p_{k-1,\delta}}||\xi||_{L^4_{k-1,\delta}} \leq C_3||\xi||^2_{L^p_{k-1,\delta}}.$$

Thus, we can bootstrap to bound higher Sobolev norms and obtain an equivalent moduli space of smooth solutions.

Proposition 3.1.19 gives an explicit picture for the boundary of various moduli spaces. We give here just one example of these arguments. We will meet several other interesting boundaries of the various moduli space. Proposition 3.1.19 also tells us how a trajectory connecting two critical points $\alpha, \gamma$ ($\mathrm{i}(\alpha) > \mathrm{i}(\gamma)$) breaks into two pieces, which become two trajectories satisfying (3.17) and by breaking at a "middle" critical point $\beta$ with $\mathrm{i}(\alpha) > \mathrm{i}(\beta) > \mathrm{i}(\gamma)$.

**Corollary 3.1.24.** *Suppose* $\alpha, \gamma$ *are two irreducible critical points in* $\mathcal{M}_{\mathfrak{s},\eta}$ *with the relative index* $\mathrm{i}(\alpha) - \mathrm{i}(\gamma) = 2$, *Then the boundary of* $\hat{\mathcal{M}}^2(\alpha, \gamma)$ *(an oriented, compact 1-manifold) consists of the union*

$$\bigcup_{\beta \in \mathcal{M}^*_{\mathfrak{s},\eta}} \hat{\mathcal{M}}^1(\alpha, \beta) \times \hat{\mathcal{M}}^1(\beta, \gamma)$$

*where* $\beta$ *only runs over those critical points with* $\mathrm{i}(\alpha) - \mathrm{i}(\beta) = 1$.

## 3.2 Non-equivariant Floer monopole homology

Let $(Y, g, \mathfrak{s})$ be an oriented, closed 3-manifold with a Riemannian metric $g$ and a Spin$^c$ structure $\mathfrak{s}$, if $b_1(Y) > 0$, we require that $c_1(\mathfrak{s})$ is non-trivial in $H^2(Y, \mathbb{R})$. For generic perturbations $\eta$ with zero cohomology class (see Corollary 2.2.13), the critical points of $\mathcal{C}_\eta$ consist of finitely many, non-degenerate points $\mathcal{M}_{\mathfrak{s},\eta}(Y)$. If $b_1(Y) > 0$, then every critical point is irreducible, that is,

$$\mathcal{M}_{\mathfrak{s},\eta}(Y) = \mathcal{M}^*_{\mathfrak{s},\eta}(Y).$$

If $b_1(Y) = 0$, then there is a unique, non-degenerate reducible critical point $\vartheta = [\nu, 0]$ for the generic perturbation $\eta = *d\nu$ with $Ker \partial\!\!\!/^g_\nu = 0$.

The Floer complex $C_*(Y, \mathfrak{s})$ is generated freely by the critical points in $\mathcal{M}^*_{\mathfrak{s},\eta}(Y)$ with grading given by the relative indices (3.8) or (3.9):

$$C_k(Y, \mathfrak{s}) = \oplus_{\alpha: i(\alpha,\alpha_0)=k} \mathbb{Z}. < \alpha >,$$

where $\alpha_0$ is a fixed critical point in $\mathcal{M}_{\mathfrak{s},\eta}$.

The boundary operator $\partial$ on $C_*(Y, \mathfrak{s})$ is defined by

$$\partial : \qquad C_k(Y) \longrightarrow C_{k-1}(Y)$$

$$\partial(< \alpha >) = \sum_{\beta \in \mathcal{M}^*_{\mathfrak{s},\eta}} n_{\alpha\beta} < \beta >,$$

where $n_{\alpha\beta}$ is given by counting the points in $\hat{\mathcal{M}}^1(\alpha, \beta)$ (an oriented, smooth, compact 0-manifold) with sign.

If $Y$ is a homology sphere ($b_1(Y) = 0$), then by choosing $\alpha_0 = \vartheta$ the unique reducible point in $\mathcal{M}_{\mathfrak{s},\eta}$, we see that $C_*(Y, \mathfrak{s})$ is $\mathbb{Z}$-graded, which is a finitely generated complex. If $c_1(det(\mathfrak{s}))$ is a torsion class for $b_1(Y) > 0$, then $C_*(Y, \mathfrak{s})$ is also $\mathbb{Z}$-graded, though we don't have a priori $\alpha_0$.

If $c_1(det(\mathfrak{s}))$ is non-zero in $H^2(Y, \mathbb{R})$, then $C_*(Y, \mathfrak{s})$ is a $\mathbb{Z}_{d(\mathfrak{s})}$-graded complex as follows (where $d(\mathfrak{s})$ is the divisibility of $c_1(det(W))$).

$$\begin{array}{ccccccc}
C_{d(\mathfrak{s})-1} & \xrightarrow{\partial} & C_{d(\mathfrak{s})-2} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_4 \\
\uparrow \partial & & & & & & \downarrow \partial \\
C_0 & \xleftarrow{\partial} & C_1 & \xleftarrow{\partial} & C_2 & \xleftarrow{\partial} & C_3
\end{array} \tag{3.25}$$

**Lemma 3.2.1.** $\partial \circ \partial = 0$.

**Proof.** By definition,

$$\begin{aligned} \partial^2(<\alpha>) &= \sum_{\beta \in \mathcal{M}^*_{\mathfrak{s}}} n_{\alpha\beta}\partial(<\beta>) \\ &= \sum_{\beta \in \mathcal{M}^*_{\mathfrak{s}}} \sum_{\gamma \in \mathcal{M}^*_{\mathfrak{s}}} n_{\alpha\beta} n_{\beta\gamma} <\gamma> . \end{aligned}$$

where $\beta$ runs over the critical points with the relative index $\mathfrak{i}(\alpha,\beta) = 1$, $\gamma$ runs over the critical points with the relative index $\mathfrak{i}(\alpha,\gamma) = 2$. We want

$$\sum_{\beta \in \mathcal{M}^*_{\mathfrak{s}} \ \mathfrak{i}(b)=\mathfrak{i}(\alpha)-1} n_{\alpha\beta} n_{\beta\gamma} = 0$$

for any $\gamma \in \mathcal{M}^*_{\mathfrak{s},\eta}$ with $\mathfrak{i}(\gamma) = \mathfrak{i}(\alpha) - 2$. We know that the number

$$\sum_{\beta \in \mathcal{M}^*_{\mathfrak{s}}} n_{\alpha\beta} n_{\beta\gamma}$$

is the number of oriented boundary points of $\hat{\mathcal{M}}^2(\alpha,\gamma)$ (Corollary 3.1.24), hence is zero. □

Now we can define the non-equivariant Seiberg-Witten-Floer homology as the homology groups for the Floer complex $(C_*(Y,\mathfrak{s}), \partial)$.

**Definition 3.2.2.** *Let $(Y, g, \mathfrak{s})$ be an oriented, closed 3-manifold with metric $g$ and* Spin$^c$ *structure (if $b_1(Y) > 0$, $c_1(det(\mathfrak{s})) \neq 0$), let $\mathcal{M}^*_{\mathfrak{s},\eta}$ be the non-degenerate, irreducible critical points of the Chern-Simons-Dirac functional $C_\eta$ with a generic perturbation $\eta$ ( a co-closed, $i\mathbb{R}$-valued 1-form on $Y$ representing a trivial de Rham cohomology class (Cf. Corollary 2.2.13))). Let $(C_*(Y,\mathfrak{s}), \partial)$ be the Floer complex generated freely by the points in $\mathcal{M}^*_{\mathfrak{s},\eta}$ with the boundary operators given by counting the 1-dimensional, oriented, gradient flow moduli space against the natural $\mathbb{R}$-translation action. Then the Seiberg-Witten-Floer homology is defined to be*

$$HF^{SW}_k(Y,\mathfrak{s}) = Ker\partial_k / Im\partial_{k+1} \tag{3.26}$$

In the definition, we have used the metric and perturbation, the dependence of the Seiberg-Witten-Floer homology is summarized as follows: for a 3-manifold $Y$

with $b_1(Y) > 0$, the Seiberg-Witten-Floer homology is a topological invariant, which doesn't depend on the metrics or the perturbations. This is the standard proof as in Floer's paper on instanton homology [20]. This is the subject of the next subsection.

For a homology 3-sphere $Y$, there are only finitely many Spin$^c$ structures, for each Spin$^c$ structure, there is a unique reducible critical point $\vartheta = [\nu, 0]$ in $\mathcal{M}_{\mathfrak{s},\eta}$, which is also non-degenerate. But the non-degenerate condition is a codimension-one condition in the space of metrics and perturbations $*d\nu$. To understand how $HF_*^{SW}(Y, \mathfrak{s})$ depends on the metric and the perturbation, we need to consider the equivariant Floer monopole homology groups, which were constructed in a joint paper with M . Marcolli. In the following section, we will discuss the construction of the equivariant Seiberg-Witten-Floer homology theory for a homology 3-sphere.

**Remark 3.2.3.** *The Seiberg-Witten invariant for* $(Y, \mathfrak{s})$ *(defined in Definition 2.3.2, see also the equation (3.7) is the Euler characteristic of the Seiberg-Witten-Floer homology* $HF_*^{SW}(Y, \mathfrak{s})$*:*

$$\begin{aligned} \chi(HF_*^{SW}(Y,\mathfrak{s})) &= \textstyle\sum_k (-1)^k \dim HF_k^{SW}(Y, g) \\ &= \textstyle\sum_k (-1)^k \dim C_k(Y) \\ &= \textstyle\sum_{\alpha \in \mathcal{M}_{\mathfrak{s}}^*} (-1)^{i(\alpha,\alpha_0)} \\ &= \lambda_{SW}(Y)(\mathfrak{s}) \end{aligned} \tag{3.27}$$

*Therefore, the Seiberg-Witten-Floer homology is a refinement of the Seiberg-Witten invariant on a 3-manifold* $(Y, \mathfrak{s})$ *for* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

### 3.2.1 Topological invariance for $Y$ with $b_1(Y) > 0$

For the 3-manifold $Y$ with $b_1(Y) > 0$ and non-trivial Spin$^c$ structure $\mathfrak{s}$ (i.e, $c_1(\mathfrak{s}) \neq 0$), we can prove that the Seiberg-Witten Floer monopole homology groups are in fact independent of the metric and the perturbation. The idea comes from Floer's paper [20], we only sketch the initial procedure and leave the details to the reader.

Given two pairs of metric and perturbation $(g_0, \eta_0)$, $(g_1, \eta_1)$ for which $HF_*^{SW}(Y, \mathfrak{s}, g_0, \eta_0)$ and $HF_*^{SW}(Y, \mathfrak{s}, g_1, \eta_1)$ are well-defined. We need to understand the time dependent trajectories on $(Y \times \mathbb{R}, g_t + dt^2)$ with the product metric outside $Y \times [0, 1]$.

The time dependent trajectory is the solution to the perturbed Seiberg-Witten monopole equations on $(Y \times \mathbb{R}, g_t + dt^2)$:

$$\begin{cases} \frac{dA(t)}{dt} = - *_t F_{A(t)} + \sigma_t(\psi, \psi) + \eta_t, \\ \frac{d\psi(t)}{dt} = -\partial\!\!\!/^{g_t}_{A(t)}\psi(t). \end{cases} \tag{3.28}$$

where $*_t$, $\sigma_t$ and $\partial\!\!\!/^{g_t}_{A(t)}$ are the corresponding operators with respect the time dependent metric $g_t$ on $Y$. Consider the perturbed Seiberg-Witten equations on $Y \times \mathbb{R}$, then away from a compact set, these equations are equivalent to the perturbed gradient flow equations (3.17) on the two ends with respect to $(g_0, \eta_0)$ and $(g_1, \eta_1)$. Any solution to the Seiberg-Witten equations (3.28) decays exponentially as $t \to \pm\infty$ to the critical orbits of $\mathcal{C}_\eta$ for $(g_0, \eta_0)$ and $(g_1, \eta_1)$ respectively. Let $\mathcal{M}(\alpha, \alpha')$ be the moduli space of the perturbed Seiberg-Witten equations (3.28) on $(Y \times \mathbb{R}, g_t + dt^2)$ with asymptotic limits $\alpha$ and $\alpha'$ as $t \to \pm\infty$.

The local virtual dimension of $\mathcal{M}(\alpha, \alpha')$ is given by the index of linearization of equations (3.28), which is the spectral flow of the time dependent extended Hessian operator $T(t)$ on the weighted Sobolev space. Choose an element $\gamma$ in $\mathcal{M}(\alpha, \alpha')$, the local structure of $\mathcal{M}(\alpha, \alpha')$ at $\gamma$ can be studied as in the proof of Proposition 3.1.15 (see also Proposition 3.3.2), we know that generically (that means, after a compactly supported, sufficiently small perturbation) is a smooth, oriented manifold of dimension

$$SF_\gamma(T_t) \geq 0.$$

For each $\alpha$ (the Seiberg-Witten monopole for $(g_0, \eta_0)$) with fixed grading, we can assign a grading for the Seiberg-Witten monopoles on $(Y, g_1, \eta_1)$ by the spectral flow of the time dependent extended Hessian operator $T(t)$. Then we also have the compactness and gluing property for the time dependent trajectories, namely, the component of $\mathcal{M}(\alpha, \alpha')$ can be compactified by adding the lower dimensional "broken" trajectories, in particular, the 0-dimensional component of $\mathcal{M}(\alpha, \alpha')$ is always compact, this provides a degree zero chain map $I$ from the chain complex $C_*(Y, g_0)$ to the chain complex $C_*(Y, g_1)$ by counting the solutions in the 0-dimensional component of $\mathcal{M}(\alpha, \alpha')$ with sign given by the orientation. This chain map gives an isomorphism between the homology groups $HF_*^{SW}(Y, g_0)$ and $HF_*^{SW}(Y, g_1)$ as in the original proof of Floer's instanton homology (CF. [20]).

## 3.3 Equivariant Floer monopole homology

In this section, we only concentrate on 3-manifolds $Y$ which have the same homology groups as $S^3$ over the rational coefficient $\mathbb{Q}$, sometimes, $Y$ is called a $\mathbb{Q}$-homology sphere. A special $\mathbb{Q}$-homology sphere is the integral homology 3-sphere, for which the homology groups $H_0(Y, \mathbb{Z})$, $H_1(Y, \mathbb{Z})$, $H_2(Y, \mathbb{Z})$ and $H_3(Y, \mathbb{Z})$ are $\mathbb{Z}, 0, 0, \mathbb{Z}$ respectively. For a $\mathbb{Z}$-homology 3-sphere, there is only one Spin$^c$ structure $W = W_0$. Let $\vartheta = [\nu, 0]$ be the unique reducible critical point for a fix Spin$^c$ structure $\mathfrak{s}$.

In order to take account of the reducible critical point, we need to consider the framed configuration space on $Y$. It is the quotient of $\mathcal{A}(Y, \mathfrak{s})$ by the based gauge group

$$\mathcal{G}_0 = \{u \in \mathcal{G} | u(x_0) = 1 \text{ for a fixed point } x_0 \in Y.\} \tag{3.29}$$

Actually, $\mathcal{G}_0 = \mathcal{G}/U(1)$, therefore, $\mathcal{G}_0$ acts on $\mathcal{A}(Y, \mathfrak{s})$ freely, the quotient being denoted by $\mathcal{B}_0$.

The Chern-Simons-Dirac $\mathcal{C}_\eta$ ( where $\eta = *d\nu$ is the generic perturbation with $Ker \not{\partial}_\nu^g = 0$) can descend to $\mathcal{B}_0$ as an $\mathbb{R}$-valued function since we consider only homology 3-spheres. Also we can write $\eta = *d\nu$ for $\nu \in \Omega^1(Y, i\mathbb{R})$. The critical point set $\mathcal{M}_{\mathfrak{s},\nu}^0$ on $\mathcal{B}_0$ is a $U(1)$-fibration over $\mathcal{M}_{\mathfrak{s},\nu}$. The $U(1)$-action on $\mathcal{M}_{\mathfrak{s},\nu}^0$ is free except at one point $\vartheta$ (the reducible point), therefore,

$$\mathcal{M}_{\mathfrak{s},\nu}^0 = \bigcup_{\alpha \in \mathcal{M}_{\mathfrak{s},\nu}^*} O_\alpha \cup \{\vartheta\} \to \mathcal{M}_{\mathfrak{s},\nu},$$

where $O_\alpha$ is the critical$U(1)$-orbit over $\alpha$. Note that $\mathcal{C}_\eta$ is a Morse-Bott function over $\mathcal{B}_0$ in the sense that the Hessian is non-degenerate on the normal bundle to the critical $U(1)$-orbit (denote by $O_\vartheta = \vartheta$).

The $\mathbb{Z}$-valued index $\mathfrak{i}$ on $\mathcal{M}_{\mathfrak{s},\nu}$ defines the corresponding relative Morse index on $\mathcal{M}_{\mathfrak{s},\nu}^0$, denoted that $\mathfrak{i}(\vartheta)$, and

$$\mathfrak{i}(O_\alpha) = ind(\alpha, \vartheta).$$

On the framed configuration space $\mathcal{B}_0$, all the claims in Lemma 3.1.8, Lemma 3.1.11, Proposition 3.1.12 regarding the (perturbed) gradient flow lines hold except that the critical point $\alpha, \beta$ is replaced by the critical $U(1)$-orbits $O_\alpha, O_\beta$ and so on. So we summarize the main feature of the gradient flow lines connecting two critical $U(1)$-orbits without the details of the proofs (see [33]) if one wants to see the details). One way to understand these is to lift the gradient flows on $B$ equivariantly to its $U(1)$-fibration $\mathcal{B}_0$.

**Theorem 3.3.1.** *Let $(Y, \mathfrak{s})$ be a rational homology 3-sphere with a* Spin$^c$ *structure $\mathfrak{s}$, the critical points set $\mathcal{M}^0_{\mathfrak{s},\nu}$ of the Chern-Simons-Diary functional $\mathcal{C}_\nu$ on the framed configuration space $\mathcal{B}_0$ is the $U(1)$-fibration over $\mathcal{M}_{\mathfrak{s},\nu}$ with only one singular fiber over the unique reducible point $\vartheta$. The gradient flow line on $\mathcal{B}_0$ is the solution to the equation (3.11) modulo the based gauge group $\mathcal{G}_0$ (3.29).*

*(a) Suppose $[A(t), \psi(t)]$ on $\mathcal{B}_0$ to be the gradient flow line with finite energy*

$$E([A(t), \psi(t)]) = \int_{-\infty}^{\infty} \|\nabla \mathcal{C}_\nu([A(t), \psi(t)])\|^2_{L^2(Y)} dt < \infty$$

*then $lim_{t\to\pm\infty}[A(t), \psi(t)]$ belongs to two critical $U(1)$-orbits in $\mathcal{M}^0_{\mathfrak{s},\nu}$.*

*(b) Let $[A(t), \psi(t)]$ be a gradient flow line of $\mathcal{C}_\nu$ on $\mathcal{B}_0$ with asymptotic limits in $O_\alpha, O_\beta$ respectively, then $[A(t), \psi(t)]$ approaches the critical orbits $O_\alpha, O_\beta$ exponentially fast with the decay rate $\delta$, where $\delta$ is the same decay rate for $\alpha, \beta$ as in Proposition 3.1.12, which is less than $\frac{1}{2}$ of the absolute eigenvalues of the Hessian operators at $\alpha, \beta$ (i.e., the Hessian operator restricted to the orthogonal directions to the critical orbits in $\mathcal{B}_0$).*

*These claims are also true for the perturbed gradient flow lines to the equation (3.17) with the property (3.18).*

As in the unframed case, we need the weighted Sobolev spaces to study the moduli space of the gradient flow lines between the critical orbits $O_\alpha, O_\beta$. Denote by $\mathcal{M}(O_\alpha, O_\beta)$ the corresponding moduli space (we use this notation the perturbed gradient flow lines between $O_\alpha, O_\beta$).

For a gradient flow $[A_0(t), \psi_0(t)]$ on $\mathcal{B}_0$ which connects the critical orbits $O_\alpha, O_\beta$, lift it to be a path in $\mathcal{A}(Y, \mathfrak{s})$, denoted by $(A_0(t), \psi_0(t))$.

For $k \geq 2$, let $\mathcal{A}_{k,\delta}(O_\alpha, O_\beta)$ be the space of pairs of connections and spinor sections $(A, \psi)$ on $Y \times \mathbb{R}$ satisfying

$$(A, \psi) \in (A_0(t), \psi_0(t)) + (\Omega^1_{L^2_{k,\delta}}(Y \times \mathbb{R}, i\mathbb{R}) \oplus L^2_{k,\delta}(W^+))$$

The gauge transformation group $\mathcal{G}_{k+1,\delta}(O_\alpha, O_\beta)$ is the based gauge group (identity over a fixed point $(x_0, t_0)$), elements of which are locally $L^2_{k+1,\delta}$ and converge to elements of the stabilizers $G_\alpha$ and $G_\beta$ as $t \to \pm\infty$. Then $\mathcal{M}(O_\alpha, O_\beta)$ is the moduli space of solutions to the equations (3.19) modulo the gauge group $\mathcal{G}_{k+1,\delta}(O_\alpha, O_\beta)$.

**Proposition 3.3.2.** *For the Chern-Simons-Dirac functional over $\mathcal{B}_0$ with the critical orbits $\mathcal{M}_0$, the moduli spaces of the gradient flow lines between the critical orbits satisfy the following transversality and gluing properties:*

*(a) For a generic perturbation $E \in \mathcal{P}$, $\mathcal{M}(O_\alpha, O_\beta)$ is an oriented, smooth $U(1)\times\mathbb{R}$-manifold of dimension*

$$i(O_\alpha) - i(O_\beta) + dim(O_\alpha) \geq 2.$$

*Moreover, there are smooth, $U(1) \times \mathbb{R}$-equivariant endpoint maps:*

$$e^+_\alpha : \quad \mathcal{M}(O_\alpha, O_\beta) \to O_\alpha, \qquad e^-_\beta : \quad \mathcal{M}(O_\alpha, O_\beta) \to O_\beta.$$

*If $i(O_\alpha) - i(O_\beta) + dim(O_\alpha) < 2$, then $\mathcal{M}(O_\alpha, O_\beta)$ is empty for a generic perturbation.*

*(b) Denote by $\hat{\mathcal{M}}(O_\alpha, O_\beta)$ the quotient of $\mathcal{M}(O_\alpha, O_\beta)$ by the $\mathbb{R}$-translation action. Then we have the following gluing model:*

$$g: \qquad \hat{\mathcal{M}}(O_\alpha, O_\beta) \times_{O_\beta} \hat{\mathcal{M}}(O_\beta, O_\gamma) \times [T, \infty) \hookrightarrow \hat{\mathcal{M}}(O_\alpha, O_\gamma)$$

*in the case where all the involved moduli spaces are non-empty and $g$ is a local diffeomorphism near the boundary of $\hat{\mathcal{M}}(O_\alpha, O_\gamma)$. The endpoint maps $e^+_\alpha$ and $e^-_\beta$ descend to $\hat{\mathcal{M}}(O_\alpha, O_\beta)$ $U(1)$-equivariantly as the endpoint maps of $\hat{\mathcal{M}}(O_\alpha, O_\beta)$ go to $O_\alpha$ and $O_\beta$ respectively.*

**Proof.** The proofs of the transversality and the gluing model are in the same vein as the proof of Proposition 3.1.15 and Proposition 3.1.19. See [33] for details. □

As an immediate corollary, we have the boundary information for $\hat{\mathcal{M}}(O_\alpha, O_\beta)$. The existence of the gluing map $g$ implies that the broken trajectories appear in the boundary of $\hat{\mathcal{M}}(O_\alpha, O_\beta)$.

**Corollary 3.3.3.** *For a generic perturbation as in Proposition 3.3.2, $\hat{\mathcal{M}}(O_\alpha, O_\beta)$ has a compactification obtaining by adding boundary strata of the factorizations of gradient flow lines breaking through intermediate critical orbits.*

$$\bigcup_{\gamma_1,\gamma_2,\cdots,\gamma_k} \hat{\mathcal{M}}(O_\alpha, O_{\gamma_1}) \times_{O_{\gamma_1}} \hat{\mathcal{M}}(O_{\gamma_1}, O_{\gamma_2}) \times_{O_{\gamma_2}} \cdots \times_{O_{\gamma_k}} \hat{\mathcal{M}}(O_{\gamma_k}, O_\beta).$$

*Here the union is taken over all (possibly empty) sequences of critical orbits with decreasing indices strictly bigger than $\mathrm{i}(O_\beta)$ in order that all the moduli spaces involved are non-empty. The number of breaking points gives the codimension of the stratum. Moreover, the endpoint maps $e_\alpha^+$ and $e_\beta^-$ extend smoothly over the lower strata of the boundary and on the boundary they coincide with $e_\alpha^+$ and $e_\beta^-$ on $\hat{\mathcal{M}}(O_\alpha, O_{\gamma_1})$ and $\hat{\mathcal{M}}(O_{\gamma_k}, O_\beta)$ respectively. This compactification is compatible with all the orientations.*

**Proof.** We only need to prove that for any sequence in $\hat{\mathcal{M}}(O_\alpha, O_\beta)$, there exists a subsequence whose limit is either an interior point of $\hat{\mathcal{M}}(O_\alpha, O_\beta)$ or the broken trajectory as in the gluing model. As in the proof of Proposition 3.1.18, we know that there exists a convergent subsequence. If all the trajectories are away from the critical orbits except $\alpha, O_\beta$, then the limit is an interior point following from the fact that $\mathcal{M}(O_\alpha, O_\beta)$ is cut out transversely by the equation (3.19). On the other case, there is a subsequence approaching to a critical orbit (this orbit must be an intermediate critical orbit), then by the gluing map, passing to further subsequence, they approach a broken trajectory at that intermediate critical orbit. This completes the proof of the Lemma. □

Now we are ready to construct the equivariant Seiberg-Witten-Floer complex and its (co)homology. Before we do that, let us briefly review some background of the $G$-equivariant (co)homology for a $G$-manifold for a compact Lie group $G$, our situation is the simplest case of $G = U(1)$, the basic reference is [9].

### 3.3.1 Brief review of equivariant (co)homology

Let $X$ be a finite dimensional compact $G$-manifold, the equivariant cohomology $H^*_G(X,\mathbb{R})$ is defined to be the ordinary cohomology of the homotopy quotient $EG\times_G X$, where $EG \to BG$ is the classifying space for the group $G$. One model to calculate these equivariant cohomology group is to use "Cartan" model: the complex of equivariant differential forms $(\Omega^*_G(X), d_G)$, where

$$\Omega^k_G(X) = \bigoplus_{2i+j=k} \left(S^i(\mathfrak{g}^*)\otimes\Omega^j(X)\right)^G \tag{3.30}$$

where $S^*(\mathfrak{g}^*)$ is the symmetric algebra of polynomial on $\mathfrak{g}$, $\Omega^*(X)$ is the de Rham complex of $X$, the Lie algebra of $G$ (linear functions have degree 2), the differential $d_G = 1\otimes d - C$ where $C$ is the multiplication on $S^*(\mathfrak{g}^*)$ tensored with contraction on $\Omega^*(X)$ with the universal element $V\in\mathfrak{g}^*\otimes Vect(M)$ ( $V(\xi)(x)=\xi_x$ for $x\in X$ and $\xi\in\mathfrak{g}$, here $\xi_x$ is the infinitesimal vector field generated by $\xi$).

Recall that for a free G-action on $X$,

$$H^*_G(X,\mathbb{R}) \cong H^*(X/G,\mathbb{R}),$$

and $H^*_G(pt,\mathbb{R}) = (S^*(\mathfrak{g}^*))^G$, the $G$-invariant symmetric polynomial on $\mathfrak{g}$. The equivariant homology is defined to be $H_{G,*}(X,\mathbb{R}) = H_*(EG\times_G X,\mathbb{R})$ which is isomorphic to the homology of the following complex:

$$\left(\Omega_{G,k}(X),\partial_G\right) = \Big(\bigoplus_{2i+j=k}\left(S^i(\mathfrak{g})\otimes\Omega^{dim(X)-j}(X)\right)^G, 1\otimes d - c\Big) \tag{3.31}$$

where $c$ is the contraction by the universal element $V\in\mathfrak{g}^*\otimes Vect(M)$. This boundary operator has degree $-1$ and square 0.

If $f: X\to\mathbb{R}$ is a $G$-invariant Morse-Bott function in the sense that the critical orbits (denoted by $G_a$, $G_b$,...,) are isolated and the Hessian of $f$ is non-degenerate in the normal directions, whose ranks of the negative normal directions define the indices of the critical orbits (denoted by $\mathfrak{i}(G_a)$). Take the moduli space of the gradient flow lines between two critical orbits, then the quotient by the $\mathbb{R}$-translation action, denoted by $\hat{\mathcal{M}}(G_a,G_b)$, is an oriented, smooth manifold with dimension

$$\mathfrak{i}(G_a) - \mathfrak{i}(G_b) + dimG_a - 1.$$

There are also the endpoint maps:

$$e_a^+ : \quad \hat{\mathcal{M}}(G_a, G_b) \to G_a, \qquad\qquad e_b^- : \quad \hat{\mathcal{M}}(G_a, G_b) \to G_b,$$

which can be extended smoothly to the (codimensional one) boundary where the trajectory breaks once at an intermediate critical orbit. Then Austin and Braam [8] constructed a Morse-Bott (co)-complex which computes the equivariant (co)homology of $X$:

$$\Big( \bigoplus_{\mathfrak{i}(G_a)+j=k} \Omega_G^j(G_a), \qquad D \Big)$$

with differential $D$ given by a matrix

$$D_{a,b}\omega = \begin{cases} d_G\omega & \text{if } G_a = G_b \\ (-1)^{r(\omega)}(e_a^+)_*(e_b^-)^*\omega & \text{if } \mathfrak{i}(G_a) > \mathfrak{i}(G_b) \\ 0 & \text{otherwise} \end{cases}$$

where $\omega \in \Omega_G^j(G_b)$, $r(\omega)$ is the de Rham degree of the equivariant form $\omega$, $(e_a^+)_*$ is the "slant product" (the integration along the fiber of $e_a^+ : \hat{\mathcal{M}}(G_a, G_b) \to G_a$). This complex is a generalization of the Morse-Witten complex, and computes the equivariant cohomology of $X$. There is also an analogous complex by using

$$\Big( \bigoplus_{\mathfrak{i}(G_b)+j=k} \Omega_{G,j}(G_b), \qquad \delta \Big)$$

which computes the equivariant homology of $X$. See [8] for details.

### 3.3.2 Equivariant Seiberg-Witten-Floer (co)homology

We follow Austin and Braam's idea to construct the Morse-Bott complex for the Chern-Simons-Dirac functional $\mathbb{C}_\eta$ on the framed configuration space $\mathcal{B}_0$.

Using the notation as in the brief review, we can consider the complexes of the equivariant forms on $O_\alpha$: $\Omega_G^*(O_\alpha)$ and $\Omega_{G,*}(O_\alpha)$. Since $O_\alpha$ is a $U(1)$-orbit, at the regular orbit,

$$\Omega_G^*(O_\alpha) = \mathbb{R}[u] \otimes 1_\alpha \oplus \mathbb{R}[u] \otimes \eta_\alpha, \qquad \Omega_{G,*}(O_\alpha) = \mathbb{R}[t] \otimes 1_\alpha \oplus \mathbb{R}[t] \otimes \eta_\alpha$$

where $t, u$ are the generators of $Lie U(1) = \mathfrak{u}(1)$ and its dual $\mathfrak{u}(1)^*$, their degrees are 2 and the contraction between them are $< u, t >= 1$. $1_\alpha$ is the constant function 1 on $O_\alpha$, $\eta_\alpha$ is the canonical 1-form on $\Omega_\alpha \cong U(1)$. Notice their degrees are different in $\Omega_G^*(O_\alpha)$ and $\Omega_{G,*}(O_\alpha)$ according to (3.30) and (3.31). At the singular orbit $O_\vartheta = \vartheta$, $\Omega_G^*(O_\vartheta)$ and $\Omega_{G,*}(O_\vartheta)$ become $\mathbb{R}[u]$ and $\mathbb{R}[t]$ respectively.

The equivariant Seiberg-Witten-Floer complex to compute the equivariant Floer monopole cohomology $HF_G^{SW,*}(Y,\mathfrak{s})$ is defined by the following bigraded complex:

$$\Big(C^k_{U(1)}(Y,\mathfrak{s}) = \bigoplus_{\mathfrak{i}(O_\beta)+j=k} \Omega^j_{U(1)}(O_\beta), \qquad D\Big) \tag{3.32}$$

with the differential $D$ given by a matrix

$$D_{\beta\alpha}\omega = \begin{cases} d_{U(1)}\omega & \text{if } O_\alpha = O_\beta \\ (-1)^{r(\omega)}(e^+_\alpha)_*(e^-_\beta)^*\omega & \text{if } \mathfrak{i}(G_\alpha) > \mathfrak{i}(G_\beta) \\ 0 & \text{otherwise} \end{cases}$$

here $\omega \in \Omega^j_{U(1)}(O_\beta)$ with maximum de Rham degree $r(\omega)$, $(e^+_\alpha)_*$ is the integration along the fiber of $e^+_\alpha$, $(e^-_\beta)^*$ is pull-back map. Note that $d_{U(1)} = 1\otimes d - u\otimes i_T$ where $u$ acts by the usual multiplication on $S^*(\mathfrak{u}(1)^*)$ and $i_T$ is the contraction by the vector field generated by the infinitesimal $U(1)$-action.

The analogous complex that computes the equivariant monopole homology is given by

$$\Big(C_{U(1),k}(Y,\mathfrak{s}) = \bigoplus_{\mathfrak{i}(O_\alpha)+j=k} \Omega_{U(1),j}(O_\alpha), \qquad \delta\Big) \tag{3.33}$$

with the boundary operator $\delta$ given by a matrix

$$\delta_{\alpha,\beta}\omega = \begin{cases} (1\otimes d - c)\omega & \text{if } O_\alpha = O_\beta \\ (-1)^{r(\omega)-1}(e^-_\beta)_*(e^+_\alpha)^*\omega & \text{if } \mathfrak{i}(G_\alpha) > \mathfrak{i}(G_\beta) \\ 0 & \text{otherwise} \end{cases}$$

here $\omega \in \Omega^j_{U(1)}(O_\alpha)$ with maximum de Rham degree $r(\omega)$. Note that $c$ acts on $\Omega_{U(1),j}(O_\alpha)$ by the contraction of $u\otimes i_T$.

By the dimension argument, $D_{\beta,\alpha}$ and $\delta_{\alpha,\beta}$ are zero if $\mathfrak{i}(O_\alpha) \geq \mathfrak{i}(O_\beta) + 3$.

**Lemma 3.3.4.** *The differential operators $D$ and the boundary operators $\delta$ have square zero, this means that*

$$D^2_{\gamma,\alpha} = \textstyle\sum_\beta D_{\gamma,\beta}\circ D_{\beta,\alpha} = 0$$

$$\delta_{\alpha,\gamma} = \textstyle\sum_\beta \delta_{\alpha,\beta}\circ \delta_{\beta,\gamma} = 0,$$

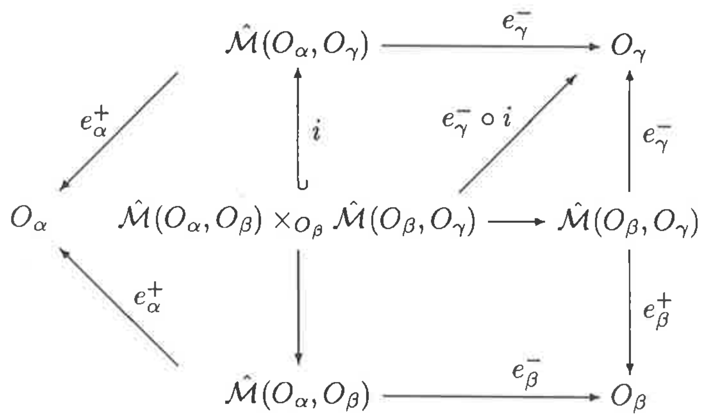
*where $O_\beta$ runs over the critical orbits with $\mathfrak{i}(O_\alpha) \geq \mathfrak{i}(O_\beta) \geq \mathfrak{i}(O_\gamma)$. Moreover, there is a pairing on $C^*_{U(1)}(Y, \mathfrak{s}) \times C_{U(1),*}(Y, \mathfrak{s})$ given by tensoring the pairing on $S^*(\mathfrak{u}(1)^*) \times S^*(\mathfrak{u}(1))$ with the wedge product and integrating, this pairing $< \ , \ >$ satisfies*

$$< D\omega_1, \omega_2 > = < \omega_1, \delta\omega_2 > .$$

**Proof.** We only prove the case for the boundary operators $\delta$, note that $(1 \otimes d - c)^2 = -L_T$ ( where $L$ is the Lie derivative) which is zero on $\Omega_{U(1),*}(O_\alpha)$, therefore the statement is true for $O_\alpha = O_\gamma$. For critical orbits $O_\alpha$, $O_\gamma$ with $\mathfrak{i}(O_\alpha) > \mathfrak{i}(O_\beta)$, we have the following expression for $\delta^2_{\alpha,\gamma}\omega$ for $\omega \in \Omega_{U(1),*}(O_\alpha)$:

$$(-1)^{r(\omega)}\Big((e^-_\gamma)_*\partial_{U(1)}\varpi - \partial_{U(1)}(e^-_\gamma)_* + \sum_\beta (-1)^{r((e^-_\beta)_*\varpi)}(e^-_\gamma)_*(e^+_\beta)^*(e^-_\beta)_*\varpi\Big)$$

where $\varpi = (e^+_\alpha)^*\omega \in \Omega_{U(1),*}\hat{\mathcal{M}}(O_\alpha, O_\gamma)$ and $r((e^-_\beta)_*\varpi) = r(\omega) - \mathfrak{i}(\alpha, \beta) - 1$. To prove that the above expression is zero, we need to analyse the structure of various moduli spaces by applying Proposition 3.3.2 and Corollary 3.3.3. We illustrate them as in the following diagram:




where $i$ is the embedding as the codimension one boundary of $\hat{\mathcal{M}}(O_\alpha, O_\beta)$, $e^-_\gamma$ is the fibration with boundary fibration $e^-_\gamma \circ i$. Apply the Stokes' theorem for the fibration $e^-_\gamma : \hat{\mathcal{M}}(O_\alpha, O_\gamma) \to O_\gamma$ to $(e^-_\gamma)_*(d\varpi)$, then we have

$$(e^-_\gamma)_*(d\varpi) = d((e^-_\gamma)_*\varpi) + (-1)^{r(\varpi)+\mathfrak{i}(\alpha,\gamma)}(e^-_\gamma \circ i)_*(\varpi|_{Im(i)}),$$

See [8] for the proof of the Stokes theorem for general fibration with boundary, where the local expression of $\varpi$ was used for the fiber bundle with boundary. By integration by parts, we see that

$$\begin{aligned}
&(e_\gamma^- \circ i)_*(\varpi|_{Im(i)}) \\
= \; &\textstyle\sum_\beta (-1)^{\mathfrak{i}(\beta,\gamma)} (e_\gamma^-)_*(e_\beta^+)^*(e_\beta^-)_*(\varpi|_{Im(i)}) \\
= \; &\textstyle\sum_\beta (-1)^{\mathfrak{i}(\beta,\gamma)} (e_\gamma^-)_*(e_\beta^+)^*(e_\beta^-)_*(e_\alpha^+)^*\omega
\end{aligned}$$

where the sign comes from the difference of the boundary orientation and the orientation on $\hat{\mathcal{M}}(O_\alpha, O_\beta) \times_{O_\beta} \hat{\mathcal{M}}(O_\beta, O_\gamma)$ (see Proposition 3.1.19 and Proposition 3.3.2). Putting these calculations together, we know that $\delta^2_{\alpha,\gamma} = 0$ for any $\mathfrak{i}(O_\alpha) > \mathfrak{i}(O_\gamma)$ since the differentials commute with the action of $S^*(\mathfrak{u}(1)^*)$.

For the statement about the pairing, it is sufficient to verify that

$$< D(\omega_1)_\alpha, \omega_2 >=< \omega_1, \delta(\omega_2)_\beta >$$

for the equivariant forms $\omega_1 \in \Omega^*_{U(1)}(O_\beta)$ and $\omega_2 \in \Omega_{U(1),*}(O_\alpha)$. If $O_\alpha = O_\beta$, this is a simple consequence of integration by parts $< C\omega_1, \omega_2 >=< \omega_1, c\omega_2 >$. If $\mathfrak{i}(O_\alpha) > \mathfrak{i}(O_\beta)$, it follows from the identity

$$\int_{O_\alpha} (e_\alpha^+)_*(e_\beta^-)^*\omega_1 \wedge \omega_2 = \int_{\hat{\mathcal{M}}(O_\alpha,O_\beta)} (e_\beta^-)^*\omega_1 \wedge (e_\alpha^+)^*\omega_2 = \int_{O_\beta} \omega_1 \wedge (e_\beta^-)_*(e_\alpha^+)^*\omega_2.$$

□

**Remark 3.3.5.** *(Stokes theorem: [8] or Lemma 3.2 in [7]) Let $\pi : E \to B$ be a fiber bundle with fiber $F$, a compact manifold with boundary. Denote by $\pi \circ i$ the fiber bundle restricted to the boundary of $F$. Then for $\omega \in \Omega^k(E)$,*

$$\pi_*(d\omega)) - d\pi_*(\omega) = (-1)^{deg(\omega)-dim(F)+1}(\pi \circ i)_*(\omega).$$

**Definition 3.3.6.** *We define the equivariant Seiberg-Witten-Floer cohomology and homology for $(Y, \mathfrak{s}, \nu)$ with real coefficients to be*

$$HF^{SW,*}_{U(1)}(Y, \mathfrak{s}) = H^*(C^*_{U(1)}(Y, \mathfrak{s}), D)$$

*and*

$$HF^{SW}_{U(1),*}(Y, \mathfrak{s}) = H^*(C_{U(1),*}(Y, \mathfrak{s}), \delta).$$

For a given regular critical orbit $O_\alpha$, the complex of the equivariant forms in $C_{U(1),*}(Y,\mathfrak{s})$ is given by

$$\Omega_{U(1),j}(O_\alpha) = \bigoplus_{2k+l=j} \mathbb{R}t^k \otimes \Omega_0^{1-l}(O_\alpha)$$

with the total grading $j + \mathfrak{i}(O_\alpha)$, where $\Omega_0^{1-l}(O_\alpha)$ denotes the $U(1)$-invariant forms. We see that

$$\Omega_0^0(O_\alpha) = \mathbb{R}1_\alpha, \qquad \Omega_0^1(O_\alpha) = \mathbb{R}\eta_\alpha,$$

where $1_\alpha$ is the constant function 1 and $\eta_\alpha$ is the canonical 1-form on $O_\alpha$. We normalize $\eta$ such that $\int_{O_\alpha} \eta_\alpha = 1$. Note that $\Omega_{U(1),*}(\Omega_\vartheta) = \mathbb{R}[t] \otimes 1_\vartheta$.

Therefore, we can write the complex for the equivariant Seiberg-Witten-Floer homology in more detail:

$$\begin{aligned} C_{U(1),2k}(Y,\mathfrak{s}) &= \left(\textstyle\bigoplus_{\alpha_0} \mathbb{R}1 \otimes \eta_{\alpha_0}\right) \oplus \left(\textstyle\bigoplus_{\alpha_1} \mathbb{R}1 \otimes 1_{\alpha_1}\right) \oplus \left(\textstyle\bigoplus_{\alpha_2} \mathbb{R}t \otimes \eta_{\alpha_2}\right) \oplus \cdots \\ &\quad \oplus \left(\textstyle\bigoplus_{\alpha_{2k-1}} \mathbb{R}t^{k-1} \otimes 1_{\alpha_{2k-1}}\right) \oplus \mathbf{R}\mathbf{t^k} \otimes \mathbf{1}_\vartheta \oplus \left(\textstyle\bigoplus_{\alpha_{2k}} \mathbb{R}t^k \otimes \eta_{\alpha_{2k}}\right) \\ &\quad \oplus \left(\textstyle\bigoplus_{\beta_1} \mathbb{R}t^k \otimes 1_{\beta_1}\right) \oplus \left(\textstyle\bigoplus_{\beta_2} \mathbb{R}t^{k+1} \otimes \eta_{\beta_2}\right) \oplus \cdots \end{aligned}$$

and

$$\begin{aligned} C_{U(1),2k-1}(Y,\mathfrak{s}) &= \left(\textstyle\bigoplus_{\alpha_1} \mathbb{R}1 \otimes \eta_{\alpha_1}\right) \oplus \left(\textstyle\bigoplus_{\alpha_2} \mathbb{R}1 \otimes 1_{\alpha_2}\right) \oplus \left(\textstyle\bigoplus_{\alpha_3} \mathbb{R}t \otimes \eta_{\alpha_3}\right) \oplus \cdots \\ &\quad \oplus \left(\textstyle\bigoplus_{\alpha_{2k}} \mathbb{R}t^{k-1} \otimes 1_{\alpha_{2k}}\right) \oplus \left(\textstyle\bigoplus_{\beta_1} \mathbb{R}t^k \otimes \eta_{\beta_1}\right) \\ &\quad \oplus \left(\textstyle\bigoplus_{\beta_2} \mathbb{R}t^k \otimes 1_{\beta_2}\right) \oplus \left(\textstyle\bigoplus_{\beta_3} \mathbb{R}t^{k+1} \otimes \eta_{\beta_3}\right) \oplus \cdots \end{aligned}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_{2k-1}, \alpha_{2k}$ range over the regular critical orbits with indices

$$\mathfrak{i}(\alpha_0) = 2k,\ \mathfrak{i}(\alpha_1) = 2k-1\ , ...,\ \mathfrak{i}(\alpha_{2k-1}) = 1,\ \mathfrak{i}(\alpha_{2k}) = 0,$$

and where $\beta_1.\beta_2, \cdots$, range over the regular critical orbits with indices $\mathfrak{i}(\beta_1) = -1$, $\mathfrak{i}(\beta_2) = -2$, $\mathfrak{i}(\beta_3) = -3$ and so on. Note that $\mathbf{t^k} \otimes \mathbf{1}_\vartheta$ only appears in the complex with even (here it is $2k$) degree.

On these components, the equivariant boundary $\delta$ can be expressed as follows:

$$\delta: \quad \begin{aligned} t^k \otimes 1_\vartheta &\mapsto -n_{\vartheta\alpha} t^k \otimes 1_\alpha \\ t^k \otimes 1_\alpha &\mapsto -n_{\alpha\beta} t^k \otimes 1_\beta \\ t^k \otimes \eta_\alpha &\mapsto n_{\alpha\beta} t^k \otimes \eta_\beta + m_{\alpha\gamma} t^{k-1} \otimes 1_\gamma + t^{k-1} \otimes 1_\alpha. \end{aligned} \tag{3.34}$$

Here we assume that $\eta_\vartheta = 0$ whenever it appears in the expressions, and

$$n_{\alpha\beta} = \#(\hat{\mathcal{M}}(\alpha, \beta))$$

for $\mathrm{i}(\alpha, \beta) = 1$. Notice that this $n_{\alpha\beta}$ is exactly the same invariant that appeared in the definition of the non-equivariant Seiberg-Witten-Floer homology in the last subsection. If $\mathrm{i}(O_\alpha) = -2$, then $\mathcal{M}(\vartheta, \alpha)$ is also a 0-dimensional, compact, oriented manifold, which defines $n_{\vartheta\alpha}$. If $\mathrm{i}(\alpha, \gamma) = 2$, then we define the following useful invariant

$$m_{\alpha\gamma} = (e_\gamma^-)_*(e_\alpha^+)^*\eta_\alpha$$

for the endpoint maps:

$$\begin{array}{ccc} & \hat{\mathcal{M}}(O_\alpha, O_\gamma) & \\ {\scriptstyle e_\alpha^+}\swarrow & & \searrow{\scriptstyle e_\gamma^-} \\ O_\alpha & & O_\gamma \end{array} \tag{3.35}$$

**Lemma 3.3.7.** *Let $\alpha, \gamma$ be irreducible critical points on the unframed configuration space $\mathcal{B}$. Suppose the relative indices $\mathrm{i}(\alpha, \gamma) = 2$ and we have defined $m_{\alpha\gamma}$ to be $(e_\gamma^-)_*(e_\alpha^+)^*\eta_\alpha$ as above, then $\{m_{\alpha\gamma} | \mathrm{i}(\alpha, \gamma) = 2\}$ are integers which satisfy the following property:*

$$\sum_{\alpha_2} n_{\alpha_1\alpha_2} m_{\alpha_2\alpha_4} = \sum_{\alpha_3} m_{\alpha_1\alpha_3} n_{\alpha_3\alpha_4}$$

*for any pair $\alpha_1, \alpha_4$ with $\mathrm{i}(\alpha_1, \alpha_4) = 3$, $\alpha_2$ run over the critical orbits with $\mathrm{i}(\alpha_1, \alpha_2) = 1$ and $\alpha_3$ run over the critical orbits with $\mathrm{i}(\alpha_1, \alpha_3 = 2$. This Lemma is true for any closed 3-manifold, see Lemma 2.7 and Lemma 2.8 in [12].*

**Proof.** From the end point maps (see Diagram (3.35), we know that $e_\alpha^+, e_\gamma^-$ induce homomorphisms on their cohomology groups:

$$H^1(O_\alpha, \mathbb{R}) \xrightarrow{(e_\alpha^+)^*} H^1(\hat{\mathcal{M}}(O_\alpha, O_\gamma), \mathbb{R}) \xrightarrow{(e_\gamma^-)_*} H^1(O_\gamma, \mathbb{R}).$$

Note that $\eta_\alpha$ is an integral class in $H^1(O_\alpha, \mathbb{R})$, therefore, $m_{\alpha\gamma}$ is an integer.

The identity follows from the fact that the codimension-one boundary of $\hat{\mathcal{M}}(\alpha_1, \alpha_4)$ is given by

$$\left(\hat{\mathcal{M}}(O_{\alpha_1}, O_{\alpha_2}) \times \hat{\mathcal{M}}(O_{\alpha_2}, O_{\alpha_4})\right) \bigcup \left(\hat{\mathcal{M}}(O_{\alpha_1}, O_{\alpha_3}) \times \hat{\mathcal{M}}(O_{\alpha_3}, O_{\alpha_4})\right)$$

and applying the Stokes theorem to

$$(e^-_{\alpha_4})_*(d(e^-_{\alpha_1})^*\eta_{\alpha_1})$$

on the fibration $\hat{\mathcal{M}}(O_{\alpha_1}, O_{\alpha_4}) \to O_{\alpha_4}$. □

Using this Lemma, one can check directly that $\delta$ (in (3.34)) is a square zero operator. Also from Lemma and the expressions of $\delta$ in (3.34) we can define the equivariant Seiberg-Witten-Floer (co)homology with integral coefficients from which we can recover the torsion elements.

### 3.3.3 Topological invariance of the equivariant monopole homology

For each metric and perturbation $(g, \nu)$ with $Ker\partial\!\!\!/^g_\nu = 0$ on a (rational) homology 3-sphere, we have constructed the equivariant Seiberg-Witten (co)homology. In this subsection, we will discuss the dependence on the metric and perturbations.

For two pairs of metrics and perturbations $(g_0, \nu_0)$ and $(g_1, \nu_1)$, by Proposition 2.2.15 and Lemma 2.3.4, we can choose a generic path $(g_t, \nu_t)$ ($t \in [0,1]$) which connects $(g_0, \nu_0)$ and $(g_1, \nu_1)$ such that $\partial\!\!\!/^{g_t}_{\nu_t}$ is invertible for all $t \in [0,1]$ except at finitely many points $\{t_1, t_2, \cdots, t_n\}$ with $Ker\partial\!\!\!/^{g_{t_i}}_{\nu_{t_i}} \cong \mathbb{C}$. The path can be chosen to be transverse to the codimension one subset $\mathcal{W}$.

Without loss of generality, we suppose that there is only one $t$ with $Ker\partial\!\!\!/^{g_t}_{\nu_t} \cong \mathbb{C}$. Assume that the spectral flow of the Dirac operators $\partial\!\!\!/^{g_t}_{\nu_t}$ is one. Then we will show that the equivariant Seiberg-Witten-Floer (co)homology for $(g_0, \nu_0)$ and $(g_1, \nu_1)$ are isomorphic up to an index shift by 2 (decrease by 2).

**Theorem 3.3.8.** *Let $(Y, \mathfrak{s})$ be a $\mathbb{Q}$-homology 3-sphere with a* Spin$^c$ *structure $\mathfrak{s}$, let $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_0, \nu_0)$ and $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_1, \nu_1)$ be the equivariant Seiberg-Witten-Floer homology defined for two generic pairs $(g_0, \nu_0)$ and $(g_1, \nu_1)$. Suppose that the spectral flow of the Dirac operators $\partial\!\!\!/^{g_t}_{\nu_t}$ along a path $(g_t, \nu_1)$ is 2. Then there exists an isomorphism between $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_0, \nu_0)$ and $HF^{SW}_{U(1),*}(Y, \mathfrak{s}, g_1, \nu_1)$ with index* *shifted by 2. The analogous claims hold for the equivariant Seiberg-Witten-Floer cohomology.*

**Proof.** Denote $\mathcal{M}_0(g_0) = \{\vartheta, O_\alpha, O_\beta, \cdots,\}$ the critical orbits for the metric $g_0$ and $\mathcal{M}_0(g_1) = \{\vartheta', O_{\alpha'}, O_{\beta'}, \cdots,\}$ the critical orbits for the metric $g_1$ with the indices

$$\mathfrak{i}: \quad \mathcal{M}_0(g_0) \to \mathbb{Z}, \qquad \mathfrak{i}': \quad \mathcal{M}_0(g_1) \to \mathbb{Z}.$$

Give the manifold $Y \times \mathbb{R}$ the metric $g_t + dt^2$ on $Y \times [0,1]$ extended as the product metric outside $Y \times [0,1]$. From the assumption of the spectral flow of Dirac operators $\partial\!\!\!/^{g_t}_{\nu_t}$, we know that $\mathfrak{i}(\vartheta) - \mathfrak{i}(\vartheta') = 2$, therefore, we can shift the indices on $\mathcal{M}_0(g_1)$ by 2 such that $\mathfrak{i} = \mathfrak{i}' + 2$ on $\mathcal{M}_0(g_1)$.

Consider the perturbed Seiberg-Witten equations on $Y \times \mathbb{R}$, then away from a compact set, these equations are equivalent to the perturbed gradient flow equations (3.17) on the two ends with respect to $(g_0, \nu_0)$ and $(g_1, \nu_1)$. Any solution to the Seiberg-Witten equations on $Y \times \mathbb{R}$ decays exponentially as $t \to \pm\infty$ to the critical orbits of $\mathcal{C}_\nu$ for $(g_0, \nu_0)$ and $(g_1, \nu_1)$ respectively. Let $\mathcal{M}(O_\alpha, O'_\alpha)$ be the moduli space of the perturbed Seiberg-Witten equations on $Y \times \mathbb{R}$ with asymptotic limits in $O_\alpha, O'_\alpha$ respectively. As in the proof of Proposition 3.1.15 (see also Proposition 3.3.2), we know that generically (that means, after a compactly supported, sufficiently small perturbation), suppose that at least one of $\alpha, \alpha'$ is irreducible, then $\mathcal{M}(O_\alpha, O'_\alpha)$ is a smooth, oriented $U(1)$-manifold of dimension

$$\mathfrak{i}(O_\alpha) - \mathfrak{i}(O'_\alpha) + \dim O_\alpha$$

which is empty if $\mathfrak{i}(O_\alpha) - \mathfrak{i}(O'_\alpha) + \dim O_\alpha \leq 1$. There are also similar gluing models as in Proposition 3.3.2) which give the structure of the codimension one boundary in $\mathcal{M}(O_\alpha, O'_\alpha)$. Note that $\mathcal{M}(\theta, \theta')$ is 2-dimensional manifold which is smooth at irreducible point, it admits one reducible monopole for $b_2^+ = 0$ for any perturbations.

Recall that we have shifted the indices of $\mathcal{M}_0(g_1)$ by 2. Define a degree zero homomorphism of the Floer complexes

$$I: \qquad C_{U(1),k}(Y, \mathfrak{s}, g_0, \nu_0) \to C_{U(1),k}(Y, \mathfrak{s}, g_1, \nu_1)$$

by sending $\omega$ to $(e^-_{\alpha'})_*(e^+_\alpha)^*\omega$ where $\omega \in \Omega_{U(1),\ k-\mathrm{i}(O_\alpha)}(O_\alpha)$, $e^-_{\alpha'}$ and $e^+_\alpha$ are the two endpoint maps:

$$O_\alpha \xleftarrow{e^+_\alpha} \mathcal{M}(O_\alpha, O'_\alpha) \xrightarrow{e^-_{\alpha'}} O_{\alpha'}.$$

<u>**Claim 1:**</u> The map $I$ is well-defined and a chain homomorphism.

To show that $I$ is well-defined, we need to ensure that the fiber of $e^-_{\alpha'}$ is compact so the push forward is well-defined. It is easy to see that $\mathcal{M}(\theta, \theta')$ doesn't involve in the definition of $I$, then follows the compactness of the fiber of $e^-_{\alpha'}$.

We need also to prove that $I \circ D - D \circ I = 0$. Since $I$ commutes with $c = u \otimes i_T$, the expression of $I \circ D - D \circ I$ at $O_\alpha \in \mathcal{M}_0(g_0)$ and $\beta' \in \mathcal{M}_0(g_1)$ is given by

$$(e^-_{\beta'})_*(e^+_\alpha)^*d\omega + (-1)^{r(\omega)}\textstyle\sum_\beta (e^-_{\beta'})_*(e^+_\beta)^*(e^-_\beta)_*(e^+_\alpha)^*\omega$$
$$-d(e^-_{\beta'})_*(e^+_\alpha)^*\omega - (-1)^{r(\omega)-\mathrm{i}(\alpha,\alpha')-1}\textstyle\sum_{\alpha'}(e^-_{\beta'})_*(e^+_{\alpha'})^*(e^-_{\alpha'})_*(e^+_\alpha)^*\omega$$

where $\beta$ ranges over the critical orbits in $\mathcal{M}_0(g_0)$ with $\mathrm{i}(\alpha,\beta) > 0$ and $\alpha'$ ranges over the critical orbits in $\mathcal{M}_0(g_1)$ with $\mathrm{i}(\alpha',\beta') > 0$.

From the gluing model, we see that the co-dimension one boundary of $\mathcal{M}(O_\alpha, O'_\beta)$ is of the form

$$\bigcup_{\alpha'}\Big(\mathcal{M}(O_\alpha, O_{\alpha'}) \times_{\alpha'} \mathcal{M}(O_{\alpha'}, O_{\beta'})\Big) \bigcup_{\beta}\Big(\hat{\mathcal{M}}(O_\alpha, O_\beta) \times \mathcal{M}(O_\beta, O_{\beta'})\Big).$$

Applying the Stokes theorem to the fibration $\mathcal{M}(O_\alpha, O'_\beta) \xrightarrow{e^-_{\beta'}} O'_\beta$, noting that the dimension of fiber is $\mathrm{i}(\alpha,\alpha') + dimO_\alpha - dimO_{\alpha'}$, we can express

$$(e^-_{\beta'})_*(e^+_\alpha)^*d\omega - d(e^-_{\beta'})_*(e^+_\alpha)^*\omega$$

as

$$(-1)^{r(\omega)-\mathrm{i}(\alpha,\beta')}(e^-_{\beta'} \circ i)_*((e^+_\alpha)^*\omega)$$

where $e^-_{\beta'} \circ i$ the boundary fibration of $e^-_{\beta'}$. From the boundary orientation, we see that $I \circ D - D \circ I = 0$.

Now reverse the path $(g_t, \nu_t)$, called by $\tilde{g}_t$, we can construct the corresponding map $J$:

$$J: \qquad C_{U(1),k}(Y, \mathfrak{s}, g_1, \nu_1 \to C_{U(1),k}(Y, \mathfrak{s}, g_0, \nu_0).$$

<u>**Claim 2:**</u> The map $J$ is well-defined and a chain homomorphism.

In the definition of $J$, we meet a potential trouble involving the singular monopole in $\mathcal{M}(\theta',\theta)$ whose virtually dimension is $-2$, but cannot be perturbed away, we must be careful to study the definition of $J$ when $\mathcal{M}(\theta',\theta)$ could enter. We can perturb the monopole equations such that $\mathcal{M}(\theta',\theta)$ consisting only the unique reducible monopole $x$ on $Y\times\mathbb{R}$. The obstruction bundle must be applied to show that the pre-glued pair from this singular monopole with any trajectories in $\mathcal{M}(\theta,\alpha)$ cannot be deformed to an actual monopole. This leads to following Proposition:

**Proposition 3.3.9.** *Let $x$ be the unique singular monopole in $\mathcal{M}(\theta',\theta)$, suppose that $y$ is a time independent trajectory in $\hat{\mathcal{M}}(\theta,\alpha)$ where $\mathrm{i}(\alpha)\geq -3$, then the pre-glued, approximate monopole $x\#^0_T y$ can be not be deformed to an actual solution for any large $T$. The same claim holds for the pre-gluing map on $\hat{\mathcal{M}}(\alpha',\theta')\times\mathcal{M}(\theta',\theta)$ where $\mathrm{i}(\alpha')\leq 0$.*

We will prove this proposition later. Apply this proposition, we can show that $J$ is well-defined, the chain homomorphism is similar to the proof in case for $I$ except that we have to take Proposition 3.3.9 into account. From now on, we use the nation $\alpha_k$ and $\alpha'_k$ to indicate their indices to be $k$. Then to ensure that the coefficient of $1_\theta$ in $J(\eta_{\alpha'_0})$, we must show that $\mathcal{M}(\alpha'_0,\theta)$ is compact. The compactness argument could fail if there is a glued solution from $\hat{\mathcal{M}}(\alpha'_0,\theta')\times\mathcal{M}(\theta',\alpha)$. This is exactly ruled out by Proposition 3.3.9. The other case is the coefficient of $1_{\alpha_{-3}}$ in $J(1_{\theta'})$, we can show in the same way that $\mathcal{M}(\theta',\alpha_{-3})$ (0-dimensional since $\mathrm{i}(\theta')=-2$) is also compact. Therefore, $J$ is well defined.

The statement of the theorem now follows if we show that there is a chain homotopy $H$ on $C_{U(1),*}(Y,\mathfrak{s},g_0,\nu_0)$ such that

$$id_k-(JI)_k=D_{k+1}H_k+H_{k-1}D_k. \tag{3.36}$$

In order to define $H$ let us consider the manifold $Y\times\mathbb{R}$ endowed with the metric

$\sigma_1$ which is

$$\sigma_1 = \begin{cases} g_0 + dt^2 & t < -2 \\ g_{t+2} + dt^2 & t \in [-2,-1] \\ g_1 + dt^2 & t \in [-1,1] \\ \tilde{g}_{2-t} + dt^2 & t \in [1,2] \\ g_0 + dt^2 & t > 2. \end{cases}$$

Consider a path of metrics $\sigma_s$ with $s \in [0,1]$, that connects $\sigma_0 = g_0 + dt^2$ to $\sigma_1$, such that for all $s$ the metric $\sigma_s$ is the product metric $g_0 + dt^2$ outside a fixed large interval $[-T, T]$.

Let $\mathcal{M}^P(O_\alpha, O_\beta)$ be the parametrized moduli space of $(A(t), \psi(t), \sigma)$, solutions of the perturbed Seiberg-Witten equations with respect to the metric $\gamma_\sigma$, modulo gauge transformation. Consider a family of perturbations such that $\mathcal{M}^P(O_\alpha, O_\beta)$ is a smooth manifold of dimension $\mathrm{i}(\alpha,\beta) + 1 + dim O_\alpha$ which is cut out transversely. Denote by $\tilde{e}_a^+$ and $\tilde{e}_b^-$ the end point maps from $\mathcal{M}^P(O_\alpha, O_\beta)$ to $O_\alpha$ and $O_\beta$ respectively.

Now we can define the degree-one map $H$ to be

$$H : C_{U(1),k}(Y, g_0, \nu_0) \to C_{U(1),k}(Y, g_0, \nu_0)$$

$$H_{\alpha,\beta} : \quad \omega \to (\tilde{e}_\beta^-)_*(\tilde{e}_\alpha^+)^*\omega,$$

with $\omega \in \Omega_{U(1),\ k-\mathrm{i}(O_\alpha)}(O_\alpha)$.

The identity (3.36) which proves that $H$ is a chain homotopy can be rewritten as the following two identities:

$$\begin{aligned} & id_{\alpha,\alpha} - \textstyle\sum_{\alpha'} I_{\alpha,\alpha'} J_{\alpha',\alpha} \\ = \ & \textstyle\sum_{\{\beta:\mathrm{i}(\beta,\alpha)\le 0\}} D_{\alpha,\beta} H_{\beta,\alpha} + \sum_{\{\beta:\mathrm{i}(\beta,\alpha)\ge 0\}} H_{\alpha,\beta} D_{\beta,\alpha} \end{aligned}$$

and, for $\alpha \neq \beta$,

$$\begin{aligned} & -\textstyle\sum_{\alpha'} I_{\alpha,\alpha'} J_{\alpha',\beta} \\ = \ & \textstyle\sum_{\{\gamma:\mathrm{i}(\gamma,\alpha)\le 0\}} D_{\alpha,\gamma} H_{\gamma,\beta} + \sum_{\{\gamma:\mathrm{i}(\gamma,\alpha)\ge 0\}} H_{\alpha,\gamma} D_{\gamma,\beta}. \end{aligned}$$

These identities can be proved by applying Stokes theorem again to their explicit expression, in the way we discussed already, and using the fact that the co-dimension one boundary for $\mathcal{M}^P(O_\alpha, O_\alpha)$ and $\mathcal{M}^P(O_\alpha, O_\beta)$ is of the form (for $\alpha \neq \beta$):

$$\begin{aligned}\partial\mathcal{M}^P(O_\alpha, O_\alpha) = \;& \bigcup_{\alpha'}\Big(\mathcal{M}(O_\alpha, O_{\alpha'}) \times_{O_{\alpha'}} \mathcal{M}(O_{\alpha'}, O_\alpha)\Big) \cup \{-\alpha\} \\ & \bigcup_{\{\beta: i(\beta,\alpha)\geq 0\}}\Big(-\mathcal{M}^P(O_\alpha, O_\beta) \times_{O_\beta} \hat{\mathcal{M}}(O_\beta, O_\alpha)\Big) \\ & \bigcup_{\{\beta: i(\beta,\alpha)\leq 0\}}\Big(\hat{\mathcal{M}}(O_\alpha, O_\beta) \times_{O_\beta} \mathcal{M}^P(O_\beta, O_\alpha)\Big)\end{aligned}$$

$$\begin{aligned}\partial\mathcal{M}^P(O_\alpha, O_\beta) = \;& \bigcup_{\alpha'}\Big(\mathcal{M}(O_\alpha, O_{\alpha'}) \times_{O_{\alpha'}} \mathcal{M}(O_{\alpha'}, O_\beta)\Big) \\ & \bigcup_{\{\gamma: i(\gamma,\alpha)\geq 0\}}\Big(-\mathcal{M}^P(O_\alpha, O_\gamma) \times_{O_\gamma} \hat{\mathcal{M}}(O_\gamma, O_\beta)\Big) \\ & \bigcup_{\{\gamma: i(\gamma,\alpha)\leq 0\}}\Big(\hat{\mathcal{M}}(O_\alpha, O_\gamma) \times_{O_\gamma} \mathcal{M}^P(O_{\gamma}, , O_\alpha)\Big).\end{aligned}$$

We have applied Proposition 3.3.9 in the study of the boundary structures. □

**Proof of Proposition 3.3.9**: For simplicity, we only prove that there is no glue map on $\mathcal{M}(\theta', \theta) \times \hat{\mathcal{M}}(\theta, \alpha_{-3})$, the proof for the other cases are similar. We adopt the notation in the proof of the gluing theorem (Cf. Theorem 3.1.19).

Since at the singular point $x$, the operator $\mathcal{L}_x$ has cokernel which is isomorphic to $\mathbb{C}$ and trivial kernel. We need to study the obstruction bundle as in [16] [43] [44]. The unique feature here is that once the Seiberg-Witten monopole is irreducible, we can always ensure that it is a smooth point which means that the linearization has trivial cokernel.

Follow the work in [33], we write the gradient flow equation as $s(A, \psi) = 0$, if the pre-glued element $x\#^0_T y$ can be deformed to an actual solution, then there exists a small $(\alpha, \phi) \in Ker(\mathcal{L}_{x\#^0_T y})^\perp$ such that we have

$$[x\#^0_T y + (\alpha, \phi)] \in \mathcal{M}(O_a, O_b),$$

that is, the following equation is satisfied

$$s(x\#^0_T y) + \mathcal{L}_{x\#^0_T y}(\alpha, \phi) + \mathcal{N}_{x\#^0_T y}(\alpha, \phi) = 0. \tag{3.37}$$

Here $\mathcal{N}$ denotes the nonlinear term in the equation, as in the proof of the gluing theorem. The presence of a non-trivial kernel (and of small eigenvalues since the problem is non-linear) of $\mathcal{L}^*_{x\#^0_T y}$ can generate obstructions to solving equation (3.37) for $(\alpha, \phi)$. In fact, the hypothesis that $\mathcal{L}_{x\#^0_T y}$ has a trivial cokernel is essential in the proof of the gluing theorem: the same argument cannot be extended to a case with a non-trivial cokernel. We need to show that there is in fact an obstruction for the Seiberg-Witten monopoles.

We follow [44] and introduce open sets $\mathcal{U}(\mu)$ of elements $[x\#^0_T y]$ such that $\mu > 0$ is not an eigenvalue of the unbounded operator $\mathcal{L}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}$ acting on $L^2$ connections and sections. There projection maps $\Pi(\mu, x\#^0_T y)$ onto the span of the eigenvectors of $\mathcal{L}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}$ with eigenvalue $< \mu$. These are smooth maps of $[x\#^0_T y] \in \mathcal{U}(\mu)$.

Since the element $(\alpha, \phi)$ is in $Ker(\mathcal{L}_{x\#^0_T y})^\perp$, we have

$$(\alpha, \phi) = \mathcal{L}^*_{x\#^0_T y}(\beta, \xi).$$

If $[x\#^0_T y + (\alpha, \phi)]$ is a solution, and we assume that $(\beta, \xi)$ can be chosen so that

$$(\beta, \xi) \in Ker(\Pi(\mu, x\#^0_T y)),$$

then $(\beta, \xi)$ solves the equations

$$\mathcal{L}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}(\beta, \xi) + (1 - \Pi(\mu, x\#^0_T y))\left(\mathcal{N}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}(\beta, \xi) + s(x\#^0_T y)\right) = 0, \quad (3.38)$$

$$\Pi(\mu, x\#^0_T y)\left(\mathcal{N}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}(\beta, \xi) + s(x\#^0_T y)\right) = 0. \quad (3.39)$$

The following lemma ensures that it is always possible to find a small solution of equation (3.38), hence the problem of deforming an approximate solution to an actual solution depends on whether equation (3.39) can also be solved. The latter has a geometric interpretation as the section of an obstruction bundle, as described in the following.

**Lemma 3.3.10.** *There is an $\epsilon > 0$ such that, for $\mu \geq \epsilon^{1/2}$, the equation (3.38) has a unique solution*

$$(\beta, \xi) \in Ker(\Pi(\mu, x\#^0_T y))$$

*with $||(\beta, \xi)||_{L^2_2} \leq \epsilon$.*

**Proof.** For $\mu > 0$ and any $(A, \psi)$, the operator $H_{(A,\psi)} = \mathcal{L}_{(A,\psi)}\mathcal{L}^*_{(A,\psi)}$, where the operator $\mathcal{L}^*_{(A,\psi)}$ is the dual operator under the weighted Sobolev space, has a bounded inverse, when restricted to the image of $(1 - \Pi(\mu, (A, \psi))$ in the space of $L^2$ connections and sections.

Thus, the equation (3.38) can be rephrased as a fixed point problem

$$(\beta,\xi) = -H^{-1}_{x\#^0_T y}(1-\Pi(\mu, x\#^0_T y))(\mathcal{N}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}(\beta,\xi) + s(x\#^0_T y)).$$

We need to prove that the right hand side is a contraction. For all $(\beta,\xi)$ and $(\beta',\xi')$, we have an estimate

$$\begin{aligned} & \|H^{-1}_{x\#^0_T y}(1-\Pi(\mu, x\#^0_T y))\,(\mathcal{N}\mathcal{L}^*(\beta,\xi) - \mathcal{N}\mathcal{L}^*(\beta',\xi'))\,\|_{L^2} \\ \leq\ & \tfrac{1}{\mu}\|(1-\Pi)(\mathcal{N}\mathcal{L}^*(\beta,\xi) - \mathcal{N}\mathcal{L}^*(\beta',\xi'))\|_{L^2} \\ \leq\ & \tfrac{1}{\mu}\|\mathcal{N}\mathcal{L}^*(\beta,\xi) - \mathcal{N}\mathcal{L}^*(\beta',\xi')\| \\ \leq\ & \tfrac{\tilde{C}}{\mu}(\|\mathcal{L}^*(\beta,\xi)\| + \|\mathcal{L}^*(\beta',\xi')\|)\|\mathcal{L}^*((\beta,\xi)-(\beta',\xi'))\|_{L^2_1} \\ \leq\ & \tfrac{C}{\mu}(\|(\beta,\xi)\| + \|(\beta',\xi')\|)\|(\beta,\xi)-(\beta',\xi')\|_{L^2_2}. \end{aligned}$$

Here the two last estimates follow as in the proof of the gluing theorem.

Suppose given $\epsilon$ such that $2C\epsilon^{1/2} < 1$, and $\mu$ such that $\mu \geq \epsilon^{1/2}$. Then the map is a contraction on the ball $\|(\beta,\xi)\|_{L^2_2} \leq \epsilon$. □

Using the result of the Lemma, we can construct the obstruction bundle and the canonical section . There is a local bundle over $\mathcal{U}(\mu)$ and a section

$$s_\mu(x\#^0_T y) = \Pi(\mu, x\#^0_T y)\left(\mathcal{N}_{x\#^0_T y}\mathcal{L}^*_{x\#^0_T y}(\beta,\xi) + s(x\#^0_T y)\right) \tag{3.40}$$

such that the following property holds. $x\#^0_T y$ can be deformed to a solution of the flow equations iff $x\#^0_T y \in s_\mu^{-1}(0)$.

This describes completely whether a given point of $\mathcal{U}(\mu)$ can be deformed to a solution or not. After a generic perturbation, Proposition 3.3.9 follows from the dimension counting. See [33] for more details.

From Theorem 3.3.8, we have the following corollary which claims that the equivariant Seiberg-Witten-Floer homology is a topological invariant.

**Corollary 3.3.11.** *Let $(Y,\mathfrak{s})$ be a homology 3-sphere with a* Spin$^c$ *structure. Suppose that $(g_0,\nu_0)$ and $(g_1,\nu_1)$ are two pairs such that their equivariant Seiberg-Witten-Floer homology groups are well-defined, that means, $Ker(\partial\!\!\!/^{g_i}_{\nu_i} = 0$. Let $n$ be the spectral flow of the twisted Dirac operator $\partial\!\!\!/^{g}_{\nu}$ from $(g_0,\nu_0)$ to $(g_1,\nu_1)$. Then there is an isomorphism between the equivariant Seiberg-Witten-Floer homology groups for $(g_0,\nu_0)$ and $(g_1,\nu_1)$ up to an index shift by $2n$:*

$$HF^{SW}_{U(1),k}(Y,\mathfrak{s},g_0,\nu_0) \cong HF^{SW}_{U(1),k-2n}(Y,\mathfrak{s},g_0,\nu_0).$$

### 3.3.4 Comparison with non-equivariant Floer theory

We want to compare the equivariant Floer homology with the ordinary Floer homology in the case of homology 3-spheres. The general relationships are discussed in [33].

From the detailed descriptions of the coboundary (see (3.34), we know that there is an equivariant Floer monopole homology of $\mathbb{Z}$-coefficients. The corresponding complex, still denoted by $C_{U(1),2k}(Y,\mathfrak{s})$, is

$$\begin{aligned} C_{U(1),2k}(Y,\mathfrak{s}) &= \left(\bigoplus_{\alpha_0}\mathbb{Z}1\otimes\eta_{\alpha_0}\right)\oplus\left(\bigoplus_{\alpha_1}\mathbb{Z}1\otimes 1_{\alpha_1}\right)\oplus\left(\bigoplus_{\alpha_2}\mathbb{Z}t\otimes\eta_{\alpha_2}\right)\oplus\cdots\\ &\oplus\left(\bigoplus_{\alpha_{2k-1}}\mathbb{Z}t^{k-1}\otimes 1_{\alpha_{2k-1}}\right)\oplus\mathbb{Z}\mathbf{t}^{\mathbf{k}}\otimes\mathbf{1}_{\vartheta}\oplus\left(\bigoplus_{\alpha_{2k}}\mathbb{Z}t^k\otimes\eta_{\alpha_{2k}}\right)\\ &\oplus\left(\bigoplus_{\beta_1}\mathbb{Z}t^k\otimes 1_{\beta_1}\right)\oplus\left(\bigoplus_{\beta_2}\mathbb{Z}t^{k+1}\otimes\eta_{\beta_2}\right)\oplus\cdots \end{aligned}$$

here $\alpha_0,\alpha_1,\cdots,\alpha_{2k-1},\alpha_{2k}$ range over the regular critical orbits with indices

$$\mathfrak{i}(\alpha_0)=2k,\ \mathfrak{i}(\alpha_1)=2k-1\ ,...,\ \mathfrak{i}(\alpha_{2k-1})=1,\ \mathfrak{i}(\alpha_{2k})=0$$

and $\beta_1.\beta_2,\cdots,$ range over the regular critical orbits with indices $\mathfrak{i}(\beta_1)=-1$, $\mathfrak{i}(\beta_2)=-2$, $\mathfrak{i}(\beta_3)=-3$ and so on. The coboundary operator $\delta$ is given by 3.34.

Now we can define a chain homomorphism that maps the equivariant to the non-equivariant complex.

$$i_k: C_{U(1),k}(Y,\mathfrak{s})\to C_k(Y,\mathfrak{s}),$$

acts on the generators as follows

$$i_k:\bigoplus_{\mathfrak{i}(\alpha)+j=k}\Omega_{U(1),j}(O_\alpha)\to\sum_{\mathfrak{i}(\alpha)=k}\mathbb{Z}<\alpha>,$$

$$i_k(t^n\otimes 1_\alpha)=0,$$

for all values of $n$ and $\mathfrak{i}(O_a)$,

$$i_k(1\otimes\eta_\alpha)=\alpha,$$

if $\mathfrak{i}(O_\alpha)=k$, and in all other cases

$$i_k(t^n\otimes\eta_\alpha)=0.$$

It is easy to see that $\partial_{k-1} \circ i_k = i_{k-1} \circ \delta_k$. Therefore, $i_*$ defines a subcomplex of $C_{U(1),k}(Y, \mathfrak{s})$ given by $Q_* = Ker(i_*)$ with the restriction of the boundary operator $\delta$.

**Theorem 3.3.12.** *Let $(Y, \mathfrak{s})$ be a $\mathbb{Q}$-homology sphere with a* Spin$^c$ *structure $\mathfrak{s}$. Fix a metric $g$ and a perturbation $\nu$ such that $HF^{SW}_*(Y, \mathfrak{s})$ and $HF^{SW}_{U(1),*}(Y, \mathfrak{s})$ are well-defined. Then there is an exact sequence*

$$\cdots \to H_{k,U(1)}(\vartheta) \to HF^{SW}_{U(1),k}(Y) \xrightarrow{i} HF^{SW}_k(Y) \to \cdots$$

*where $H_{2n,U(1)}(\vartheta) \cong \mathbb{Z}t^n \otimes \vartheta$ for $n \geq 0$, 0 otherwise, is the $U(1)$-equivariant homology at the fixed point $\vartheta$. Moreover, for $k < 0$,*

$$HF^{SW}_k(Y, \mathfrak{s}) \cong HF^{SW}_{U(1),k}(Y, \mathfrak{s}).$$

*For $k \geq 0$, we have the following (finitely many) short exact sequences*

$$0 \to HF^{SW}_{U(1),2k+1}(Y, \mathfrak{s}) \xrightarrow{i_{2k+1}} HF^{SW}_{2k+1}(Y, \mathfrak{s}) \xrightarrow{\Delta_k} \mathbb{Z}t^k \to$$

$$\to HF^{SW}_{U(1),2k}(Y, \mathfrak{s}) \xrightarrow{i_{2k}} HF^{SW}_{2k}(Y, \mathfrak{s}) \to 0.$$

**Proof.** The complexes $C_{U(1),*}$, $Q_*$, and $C_*$ all have a filtration by indices of $\mathcal{M}_{\mathfrak{s},\eta}$. For $C_{U(1),k}$, the filtration is given by

$$C_{U(1),k}(n) = \bigoplus_{\mathfrak{i}(O_\alpha)+j=k, \mathfrak{i}(O_\alpha) \leq n} \Omega_{U(1),j}(O_\alpha). \tag{3.41}$$

The filtration on $Q_*$ is induced from (3.41). The filtration on the non-equivariant complex is given by

$$C_k(n) = \bigoplus_{\mathfrak{i}(a)=k \leq n} \mathbb{Z} < \alpha > .$$

Let $E_Q$ be the spectral sequences associated to the filtration of the complexes $Q_*$. We know that the $E^1$-term is precisely the equivariant cohomology of $\vartheta$. Hence, $H_*(Q_*) \cong H_{U(1),*}(\vartheta) = \mathbb{Z}[t]$ with $deg(t) = 2$.

Now the short exact sequences of

$$0 \to Q_* \to C_{U(1),*}(Y, \mathfrak{s}) \to C_*(Y, \mathfrak{s}) \to 0$$

gives the long exact sequence of the homology groups:

$$\cdots \to H_{k,U(1)}(\vartheta) \to HF^{SW}_{U(1),k}(Y,\mathfrak{s}) \xrightarrow{\imath} HF^{SW}_{k}(Y,\mathfrak{s}) \xrightarrow{\Delta_k} H_{k-1,U(1)}(\vartheta) \to \cdots$$

The splitting of this long exact sequence follows from the structure of $H_{k,U(1)}(\vartheta)$. □

The connecting homomorphism $\Delta_{2k}$ in the above long exact sequences is particularly interesting.

**Proposition 3.3.13.** *Suppose we are given a representative* $\sum_\alpha x_\alpha < \alpha >$ *in* $HF^{SW}_{2k+1}(Y)$. *Then the connecting homomorphism in the long exact sequence is*

$$\Delta_k : \qquad HF^{SW}_{2k+1} \to H_{U(1),2k}(\vartheta) = \mathbb{Z}t^k \otimes \vartheta, \tag{3.42}$$

*where*

$$\Delta_k\Big(\sum_\alpha x_\alpha < \alpha >\Big) = \sum x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} \cdots m_{\alpha_{k-1}\alpha_k} n_{\alpha_k\vartheta} t^k \otimes \vartheta. \tag{3.43}$$

*Here the sum is understood over all the repeated indices, that is over all critical points with indices* $\mathrm{i}(\alpha) = 2k+1$, $\mathrm{i}(\alpha_1) = 2k-1$, $\mathrm{i}(\alpha_{k-1}) = 3$, $\mu(\alpha_k) = 1$.

**Proof.** The map is defined by the standard diagram chase and by adding boundary terms in order to find a representative of the form (3.43). For a cycle in $HF^{SW}_{2k+1}$ represented by

$$\sum_{\{\alpha:\mathrm{i}(\alpha)=2k+1\}} x_\alpha < \alpha >$$

with $\sum_{\{\alpha:\mathrm{i}(\alpha)=2k+1\}} x_\alpha n_{\alpha,\beta} = 0$ for any $\beta$ with $\mathrm{i}(\alpha,\beta) = 1$. Since $\imath_{2k+1}$ is surjective (see the definition of $i_{2k+1}$), the element $\sum_{\{\alpha:\mathrm{i}(\alpha)=2k+1\}} x_\alpha < \alpha >$ has a preimage in $C_{U(1),2k+1}(Y,\mathfrak{s})$:

$$\sum_{\{\alpha:\mathrm{i}(\alpha)=2k+1\}} x_\alpha 1 \otimes \eta_\alpha$$

under $i_{2k+1}$. Adding the element (which is in the kernel of $i_{2k+1}$)

$$\sum_{\alpha,\alpha_1} x_\alpha m_{\alpha\alpha_1} t \otimes \eta_{\alpha_1}$$

applying the boundary operator $\delta_{2k+1}$, we have

$$\delta(x_\alpha 1 \otimes \eta_\alpha + x_\alpha m_{\alpha\alpha_1} t \otimes \eta_{\alpha_1}) = \sum_{\alpha,\alpha_1,\alpha_2} x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} 1 \otimes 1_{\alpha_2}$$

which is the kernel of $i_{2k}$. Next adding the element

$$\sum_{\alpha,\alpha_1,\alpha_2} x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} t^2 \otimes \eta_{\alpha_2}$$

which is in the kernel of $i_{2k+1}$, applying the boundary operator $\delta_{2k+1}$ again, we have

$$\begin{aligned} & \delta(x_\alpha 1 \otimes \eta_\alpha + x_\alpha m_{\alpha\alpha_1} t \otimes \eta_{\alpha_1} + x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} t^2 \otimes \eta_{\alpha_2}) \\ = & \textstyle\sum_{\alpha,\alpha_1,\alpha_2} x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} m_{\alpha_2\alpha_3} 1 \otimes 1_{\alpha_3} \end{aligned}$$

which is in the kernel of $i_{2k}$, repeat this procedure by adding the element

$$\begin{gathered} \textstyle\sum_{\alpha,\alpha_1,\cdots,\alpha_3} x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} m_{\alpha_2\alpha_3} t^3 \otimes \eta_{\alpha_3} \\ + \cdots + \\ \textstyle\sum_{\alpha,\alpha_1,\cdots,\alpha_k} x_\alpha m_{\alpha\alpha_1} \cdots m_{\alpha_{k-1}\alpha_k} t^k \otimes \eta_{\alpha_k} + \cdots \end{gathered}$$

and the diagram chasing, we obtain that $\delta_{2k+1}(\sum_\alpha x_\alpha < \alpha >)$ can be represented by

$$\sum x_\alpha m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} \cdots m_{\alpha_{k-1}\alpha_k} n_{\alpha_k\vartheta} t^k \otimes \vartheta.$$

We use the fact that $C_{U(1),2k+1}(Y, \mathfrak{s})$ is finitely generated over $\mathbb{Z}$. □

Let $\alpha$ be a critical point of $\mathcal{C}_\nu$ with index $2k+1$, denote by

$$m(\alpha, \vartheta) = m_{\alpha\alpha_1} m_{\alpha_1\alpha_2} m_{\alpha_2\alpha_3} \cdots m_{\alpha_{k-1}\alpha_k} n_{\alpha_k\vartheta}$$

the number presented in $\Delta_{2k+1}$. Summing over $\mathrm{i}(\alpha_1) = 2k-1$, $\mathrm{i}(\alpha_2) = 2k-3$, $\cdots$, $\mathrm{i}(\alpha_{k-1}) = 3$, $\mathrm{i}(\alpha_k) = 1$ we see that

$$\begin{aligned} Im(i_{2k+1}) &= Ker(\Delta_k) \\ &= \left\{ \sum_\alpha x_\alpha \alpha \;\middle|\; \begin{array}{ll} (1) & \sum_\alpha x_\alpha \alpha \in HF^{SW}_{2k+1}(Y, \mathfrak{s}) \\ (2) & \sum_\alpha x_\alpha m(\alpha, \theta) = 0 \end{array} \right\}. \end{aligned}$$

Therefore, $Ker(\Delta_{2k+1})$ measures the interaction of $HF^{SW}_*(Y, g)$ with the reducible critical points. There is a similar analogue for Seiberg-Witten-Floer cohomology.

As an application of Theorem 3.2.2 and Corollary 3.3.11, we can prove the wall-crossing formulae for the Seiberg-Witten invariant on a homology 3-sphere $(Y, \mathfrak{s})$ with a Spin$^c$ structure $\mathfrak{s}$.

Note that the Seiberg-Witten-Floer homology is only well-defined for a metric $g$ and perturbation $\nu$ with $Ker(\partial\!\!\!/^g_\nu) = 0$. We call $(g, \nu)$ with trivial kernel a good pair. From Proposition 2.2.15 and Proposition 2.3.4,

$$\{(g, \nu) | Ker(\partial\!\!\!/^g_\nu) = 0\}$$

has many components separated by the codimensional one subset $\mathcal{W}$ (see (2.15). The usual cobordism argument can be adopted to prove that the non-equivariant Seiberg-Witten-Floer homology is independent of $(g, \nu)$ in the same component. For two good pairs $(g_0, \nu_0)$ and $(g_1, \nu_1)$, we have (see Corollary 3.3.11)

$$HF^{SW}_{U(1),k}(Y, \mathfrak{s}, g_0, \nu_0) \cong HF^{SW}_{U(1),k-2n}(Y, \mathfrak{s}, g_0, \nu_0). \tag{3.44}$$

where $n$ is the spectral flow of the twisted Dirac operator $\partial\!\!\!/^g_\nu$ from $(g_0, \nu_0)$ and $(g_1, \nu_1)$. Then we claim that the Seiberg-Witten invariants for $(g_0, \nu_0)$ and $(g_1, \nu_1)$ are related by

$$\lambda_{SW}(Y, \mathfrak{s}, g_1, \nu_1) = \lambda_{SW}(Y, \mathfrak{s}, g_0, \nu_0) + n \tag{3.45}$$

as claimed by the wall-crossing formula (2.16).

**Proof of Claim (3.45):** Without loss of generality, we can assume that we are in one of the following two cases for the Floer homology group $HF^{SW}_*(Y, \mathfrak{s}, g_0, \nu_0)$:

Case 1: There exists an integer $N$ such that $HF^{SW}_*(Y, \mathfrak{s}, g_0, \nu_0) = 0$ for all $p \geq 2N$ but $HF^{SW}_*(Y, \mathfrak{s}, g_0, \nu_0) \neq 0$.

Case 2: There exists an integer $N$ such that $HF^{SW}_*(Y, \mathfrak{s}, g_0, \nu_0) = 0$ for all $p \geq 2N+1$ but $HF^{SW}_*(Y, \mathfrak{s}, g_0, \nu_0) \neq 0$.

For case 1, we can assume $N \geq 0$, otherwise, we can change the orientation of $(Y, \mathfrak{s})$. Therefore, from the exact sequences in Theorem 3.3.12, we get

$$\text{(a)} \quad HF^{SW}_{U(1),k}(Y, \mathfrak{s}, g_0, \nu_0) = \begin{cases} 0 & k > 2N, \text{ k is odd} \\ \mathbb{Z}t^m & k = 2m \geq 2N. \end{cases}$$

(b) There are the following identities which come from Theorem 3.3.12:

$$\begin{aligned}
&\dim HF^{SW}_{2N-2}(Y, g_0, \nu_0) - \dim HF^{SW}_{2N-1}(Y, g_0, \nu_0)\\
&\quad = \dim HF^{SW}_{U(1),2N-2}(Y, g_0, \nu_0) - \dim HF^{SW}_{U(1),2N-1}(Y, g_0, \nu_0) - 1;\\
&\dim HF^{SW}_{2N-4}(Y, g_0) - \dim HF^{SW}_{2N-3}(Y, g_0, \nu_0)\\
&\quad = \dim HF^{SW}_{U(1),2N-4}(Y, g_0, \nu_0) - \dim SWH_{U(1),2N-3}(Y, g_0, \nu_0) - 1;\\
&\cdots\cdots\cdots\\
&\dim HF^{SW}_{0}(Y, g_0, \nu_0) - \dim HF^{SW}_{1}(Y, g_0, \nu_0)\\
&\quad = \dim HF^{SW}_{U(1),0}(Y, g_0, \nu_0) - \dim HF^{SW}_{U(1),1}(Y, g_0, \nu_0) - 1.
\end{aligned}$$

(c) For $k < 0$, $HF^{SW}_k(Y, g_0, \nu_0) = HF^{SW}_{U(1),k}(Y, g_0, \nu_0)$.

Given the above information, we can calculate the Seiberg-Witten invariant for the metric and perturbation $(g_0, \nu_0)$ as,

$$\begin{aligned}
&\lambda_{SW}(Y, \mathfrak{s}, g_0, \nu_0)\\
= \ &\textstyle\sum_k (-1)^k dim HF^{SW}_k(Y, \mathfrak{s}, g_0, \nu_0) \qquad (3.46)\\
= \ &\textstyle\sum_{k<N}\Big(dim HF^{SW}_{U(1),2k}(Y, \mathfrak{s}, g_0, \nu_0) - \dim HF^{SW}_{U(1),2k+1}(Y, \mathfrak{s}, g_0, \nu_0)\Big) - N.
\end{aligned}$$

From the isomorphism (3.44), we have

$$HF^{SW}_{U(1),k}(Y, \mathfrak{s}, g_1, \nu_1) \cong \begin{cases} 0 & k > 2N-2, \text{ k is odd}\\ \mathbb{Z}t^m & k = 2m \geq 2N-2n\\ HF^{SW}_{U(1),k}(Y, \mathfrak{s}, g_0, \nu_0) & k \leq 2N-2n. \end{cases}$$

Apply Theorem 3.3.12 once more to $HF^{SW}_{U(1),k}(Y, \mathfrak{s}, g_1, \nu_1)$, we obtain that

$$\begin{aligned}
&\lambda_{SW}(Y, \mathfrak{s}, g_1, \nu_1)\\
= \ &\textstyle\sum_k (-1)^k dim HF^{SW}_k(Y, \mathfrak{s}, g_1, \nu_1)\\
= \ &\textstyle\sum_{k<N-n}\Big(dim HF^{SW}_{U(1),2k}(Y, g_1, \nu_1) - \dim HF^{SW}_{U(1),2k+1}(Y, g_0, \nu_0)\Big) - (N-n)\\
= \ &\textstyle\sum_{k<N}\Big(dim HF^{SW}_{U(1),2k}(Y, g_0, \nu_0) - \dim HF^{SW}_{U(1),2k+1}(Y, g_0, \nu_0)\Big) - N + n\\
= \ &\lambda_{SW}(Y, \mathfrak{s}, g_0, \nu_0) + n \qquad \text{(by equation (3.46))}
\end{aligned}$$

This proves claim (3.45) for case 1.

For case 2, similarly, we have

(a) $$HF^{SW}_{U(1),k}(Y,\mathfrak{s},g_0,\nu_0) = \begin{cases} 0 & k > 2N, \text{ k is odd} \\ \mathbb{Z}t^m & k = 2m \geq 2N+2. \end{cases}$$

(b) There are the following identities which come from Theorem 3.3.12:

$$\dim HF^{SW}_{2N}(Y,g_0,\nu_0) = \dim HF^{SW}_{U(1),2N}(Y,g_0,\nu_0) - 1;$$

$$= \dim HF^{SW}_{2N-2}(Y,g_0,\nu_0) - \dim HF^{SW}_{2N-1}(Y,g_0,\nu_0)$$

$$\dim HF^{SW}_{U(1),2N-2}(Y,g_0,\nu_0) - \dim HF^{SW}_{U(1),2N-1}(Y,g_0,\nu_0) - 1;$$

$$\dim HF^{SW}_{2N-4}(Y,g_0) - \dim HF^{SW}_{2N-3}(Y,g_0,\nu_0)$$

$$= \dim HF^{SW}_{U(1),2N-4}(Y,g_0,\nu_0) - \dim SWH_{U(1),2N-3}(Y,g_0,\nu_0) - 1;$$

..........

$$\dim HF^{SW}_{0}(Y,g_0,\nu_0) - \dim HF^{SW}_{1}(Y,g_0,\nu_0)$$

$$= \dim HF^{SW}_{U(1),0}(Y,g_0,\nu_0) - \dim HF^{SW}_{U(1),1}(Y,g_0,\nu_0) - 1.$$

(c) For $k < 0$, $HF^{SW}_k(Y,g_0,\nu_0) = HF^{SW}_{U(1),k}(Y,g_0,\nu_0)$.

Then the Seiberg-Witten invariant for $(g_o,\nu_0)$ in this case is given by

$$\begin{aligned} &\lambda_{SW}(Y,\mathfrak{s},g_0,\nu_0) \\ = \;& \textstyle\sum_k (-1)^k dim HF^{SW}_k(Y,\mathfrak{s},g_0,\nu_0) \\ = \;& \textstyle\sum_{k<N}\Big(dim HF^{SW}_{U(1),2k}(Y,\mathfrak{s},g_0,\nu_0) - \dim HF^{SW}_{U(1),2k+1}(Y,\mathfrak{s},g_0,\nu_0)\Big) \\ & -N + dim HF^{SW}_{U(1),2N}(Y,\mathfrak{s},g_0,\nu_0) - 1. \end{aligned} \tag{3.47}$$

From the isomorphism $HF^{SW}_{U(1),k}(Y,\mathfrak{s},g_1,\nu_1) = SWH_{U(1),k+2n}(Y,\mathfrak{s},g_0,\nu_0)$, we know that

$$HF^{SW}_{U(1),k}(Y,\mathfrak{s},g_1,\nu_1) = \begin{cases} 0 & k > 2N-2n, \text{ k is odd} \\ \mathbb{Z}t^m & k = 2m \geq 2N+2-2n \\ HF^{SW}_{U(1),k}(Y,Y,\mathfrak{s},g_0,\nu_0) & k < 2N+2-2n. \end{cases}$$

Repeat the procedure in case 1, we get

$$\begin{aligned}
& \lambda_{SW}(Y, \mathfrak{s}, g_1, \nu_1) \\
= & \sum_k (-1)^k dim HF_k^{SW}(Y, \mathfrak{s}, g_1, \nu_1) \\
= & \sum_{k<N-n} \Big( dim HF^{SW}_{U(1),2k}(Y, \mathfrak{s}, g_1, \nu_1) - \dim HF^{SW}_{U(1),2k+1}(Y, \mathfrak{s}, g_0, \nu_0) \Big) \\
& -(N-n) + dim HF^{SW}_{U(1),2N-2n}(Y, \mathfrak{s}, g_1, \nu_1) - 1 \\
= & \sum_{k<N} \Big( dim HF^{SW}_{U(1),2k}(Y, \mathfrak{s}, g_0, \nu_0) - \dim HF^{SW}_{U(1),2k+1}(Y, \mathfrak{s}, g_0, \nu_0) \Big) \\
& -N + n + dim HF^{SW}_{U(1),2N}(Y, \mathfrak{s}, g_0, \nu_0) - 1 \\
= & \lambda_{SW}(Y, \mathfrak{s}, g_0, \nu_0) + n \qquad \text{(by equation (3.47))}
\end{aligned}$$

This proves claim (3.45) in case 2.

## 3.4 Fukaya-Floer monopole homology

As in the instanton case, there is a version of Fukaya-type Floer homology for the monopoles on 3-manifolds. These were constructed in a joint paper with A. Carey in [12].

Note that the Seiberg-Witten-Floer complex for $(Y, \mathfrak{t})$ with non-zero $c_1(\mathfrak{t})$ has a period of $d(\mathfrak{t})$, see Diagram 3.25. We can lift these periodic complex to a $\mathbb{Z}$-graded complex by the repeated patterns $\{C_j\}_{j\in\mathbb{Z}}$. Since there is no fixed $C_0$, we can fix any chain group $C_*$ to be $C_0$. In the specific applications later, there is a natural way to get a fixed chain complex. This $\mathbb{Z}$-graded complex is the Seiberg-Witten-Floer complex for the Chern-Simons-Dirac function on the covering space $\tilde{\mathcal{B}}$ of $\mathcal{B}$

$$\tilde{\mathcal{B}} \xrightarrow{H^1(Y,\mathbb{Z})/Ker(c_1(\mathfrak{t}))} \mathcal{B}$$

where $Ker(c_1(\mathfrak{t}))$ is the kernel of the map: $[u] \mapsto ([u] \cup c_1(\mathfrak{t}))[Y]$.

Suppose that we have a $\mathbb{Z}$-graded complex $\{C_j\}_{j\in\mathbb{Z}}$. Define the chain complex at grading $j$ to be

$$CFF_j^{(m)}(Y, \mathfrak{t}) = \oplus_{0\le n\le m} C_{j-2n}(Y, \mathfrak{t})$$

with the boundary operators $\partial$ given by

$$\partial_{(m)}: \quad CFF_j^{(m)}(Y, \mathfrak{t}) \to CFF_{j-1}^{(m)}(Y, \mathfrak{t})$$

whose non-zero entry is

$$\binom{l}{k} n_{\alpha,\beta}^{(l-k)}$$

only at $\alpha \in C_{j-2k}(Y, \mathfrak{t})$, $\beta \in C_{j-1-2l}(Y, \mathfrak{t})$ with $m \geq l \geq k \geq 0$, which is given by

$$n_{\alpha,\beta}^{(l-k)} = \begin{cases} n_{\alpha,\beta} & \text{as defined in section 3.2} \quad \text{for } l = k, \\ \int_{\hat{\mathcal{M}}^{2(l-k)}(\alpha,\beta)} \Theta^{l-k} & \text{for } l > k. \end{cases} \tag{3.48}$$

Here $\hat{\mathcal{M}}(\alpha, \beta)^{2(l-k)}$ is the non-empty component of $\hat{\mathcal{M}}(\alpha, \beta)$ with dimension $2(l-k)$, and $\Theta$ is the first Chern class of the based moduli space over $\hat{\mathcal{M}}(\alpha, \beta)^{2(l-k)}$, here we use the convention explained in the introduction. We will justify this formal notation since $\hat{\mathcal{M}}(\alpha, \beta)^{2(l-k)}$ is a manifold with boundary.

Using the based gauge group, there is a $U(1)$ fibration over each trajectory moduli space, which is actually the moduli space of the gradient flow line on the configuration space (modulo the based gauge group). On this based configuration space, the critical point set consists of finitely many $U(1)$ orbits where $U(1)$ acts freely. We denote by $O_\alpha$ the $U(1)$ orbit corresponding to the critical point $\alpha$. Then the based fibration over $\hat{\mathcal{M}}(\alpha, \beta)$ is $\hat{\mathcal{M}}(O_\alpha, O_\beta)$ (the time independent trajectory moduli space between $O_\alpha, O_\beta$). We denote this moduli space by $\hat{\mathcal{M}}^O(\alpha, \beta)$. There are two end-point maps:

$$e_\alpha^- : \hat{\mathcal{M}}(O_\alpha, O_\beta) \longrightarrow O_\alpha, \qquad e_\beta^+ : \hat{\mathcal{M}}(O_\alpha, O_\beta) \longrightarrow O_\beta.$$

We can construct a complex vector bundle by using the standard $U(1)$ multiplication on a complex vector space. Suppose that $\hat{\mathcal{M}}(\alpha, \beta)^{2k}$ is one non-empty component of $\hat{\mathcal{M}}(\alpha, \beta)$ with dimension $2k$ (or $2k+1$), then we associate a rank k complex vector bundle $\mathcal{E}_{\alpha,\beta,k}$

$$\mathcal{E}_{\alpha,\beta,k} = \hat{\mathcal{M}}(O_\alpha, O_\beta)^{2k} \times_{U(1)} \mathbb{C}^k$$

with a transversal section $s_{\alpha,\beta,k} = (s^1_{\alpha,\beta,k}, s^2_{\alpha,\beta,k}, \cdots, s^k_{\alpha,\beta,k})$ with the following compatibility conditions on the boundary (corners).

The gluing map in the based moduli space setting, that is, for a sufficiently large

$T$, the gluing maps define the following commutative diagram:

$$\begin{array}{ccc} \hat{\mathcal{M}}(O_\alpha, O_\beta) \times_{O_\beta} \hat{\mathcal{M}}(O_\beta, O_\gamma) & \xrightarrow{g_T} & \hat{\mathcal{M}}(O_\alpha, O_\gamma) \\ \big\downarrow & & \big\downarrow \\ \hat{\mathcal{M}}(\alpha, \beta) \times \hat{\mathcal{M}}(\beta, \gamma) & \xrightarrow{g_T} & \hat{\mathcal{M}}(\alpha, \gamma). \end{array} \tag{3.49}$$

There are natural isomorphisms between the pull-back bundles $g_T^* \mathcal{E}_{\alpha,\gamma,k}$ and the bundles

$$\pi_{\alpha,\beta}^* \mathcal{E}_{\alpha,\beta,k_1} \oplus \pi_{\beta,\gamma}^* \mathcal{E}_{\alpha,\beta,k-k_1}.$$

The transversal sections are also compatible with the gluing maps.

In the definition of the boundary operator (cf. 3.48), we used the following convention:

$$\int_{\hat{\mathcal{M}}(\alpha,\beta)^{2(l-k)}} \Theta^{l-k} = \#\left(s_{\alpha,\beta,l-k}^{-1}(0)\right)$$

where $s_{\alpha,\beta,l-k}$ is the transversal section of $\mathcal{E}_{\alpha,\beta,l-k}$, $s_{\alpha,\beta,l-k}^{-1}$ is zero dimensional, with no zeros from the boundary (by the dimension counting and the compatibility conditions), hence $s_{\alpha,\beta,l-k}^{-1}(0)$ has only finitely many oriented points.

Suppose that $\hat{\mathcal{M}}(\alpha,\gamma)^{2n+1}$ is the non-empty component of $\hat{\mathcal{M}}(\alpha,\gamma)$ with dimension $2n+1$, then $s_{\alpha,\gamma,n}^{-1}(0)$ is a one dimensional, smooth, compact manifold whose boundary points coming from the zero points of the corresponding sections over the co-dimension one boundary (the simply-broken trajectories, simply-broken means that there is one broken point). For example, the contribution from the boundary component $\hat{\mathcal{M}}(\alpha,\beta)^{2k} \times \hat{\mathcal{M}}(\beta,\gamma)^{2n-2k}$ is

$$\cup_{i_1,\cdots,i_k}\left((s_{\alpha,\beta,k}^{i_1}, \cdots, s_{\alpha,\beta,k}^{i_k})^{-1}(0) \times (s_{\beta,\gamma,n-k}^{j_1}, \cdots, s_{\beta,\gamma,n-k}^{j_{n-k}})^{-1}(0)\right)$$

where $\{j_1, \cdots, j_{n-k}\} = \{1, \cdots, n\} - \{i_1, \cdots, i_k\}$. Counting the boundary points with orientation $\pm 1$, we know that the contribution from $\hat{\mathcal{M}}(\alpha,\beta)^{2k} \times \hat{\mathcal{M}}(\beta,\gamma)^{2n-2k}$ is given by

$$\binom{n}{k} \#\left(s_{\alpha,\beta,k}^{-1}(0)\right) \#\left(s_{\beta,\gamma,n-k}^{-1}(0)\right) = \binom{n}{k} \int_{\hat{\mathcal{M}}(\alpha,\beta)^{2k}} \Theta^k \int_{\hat{\mathcal{M}}(\beta,\gamma)^{2n-2k}} \Theta^{n-k}$$

where on the right hand side, we use our formal notation. Then counting the boundary points of $s^{-1}_{\alpha,\gamma,n}(0)$ with orientation, which is zero of course, is equivalent to

$$\sum_{\beta,k} \binom{n}{k} \#\Big((s^{-1}_{\alpha,\beta,k}(0)\Big)\#\Big((s^{-1}_{\beta,\gamma,n-k}(0)\Big) = 0.$$

This identity is a formal Stokes formula calculation applied to $d(\Theta^n)$ on the moduli space $\hat{\mathcal{M}}(\alpha,\beta)^{2n}$ with codimension one boundary.

For a homology 3-sphere $Y$ with a good metric and perturbation, there is also a version of Fukaya-type homology group defined in the same way. (We ignore the unique reducible critical point, as in the case of $c_1(\mathfrak{t})$ non-zero.) There is another variation of the Fukaya-type homology group where we take account of the reducible critical point, we will discuss this after the following Proposition.

**Proposition 3.4.1.** *(a)* $\partial^2_{(m)} = 0$ *on* $CFF^{(m)}_*(Y,\mathfrak{t})$, *therefore there are homology groups* $HFF^{(m)}_*(Y,\mathfrak{t})$ *which are independent of the metric and small perturbations.*

*(b) Denote by* $(\bar{Y},\bar{\mathfrak{t}})$ *the 3-manifold* $(Y,\mathfrak{t})$ *with the opposite orientation and the induced* Spin$^c$ *structure* $\bar{\mathfrak{t}}$. *Fix the chain group for* $(\bar{Y},\bar{\mathfrak{t}})$ *by requiring that the following pairing makes sense. Then there is a natural pairing* $\sigma_m$*:*

$$HFF^{(m)}_j(Y,\mathfrak{t}) \times HFF^{(m)}_{-j+2m}(\bar{Y},\bar{\mathfrak{t}}) \xrightarrow{\sigma_m} \mathbb{Z}$$

*which is given by*

$$((\psi_0,\psi_1,\cdots,\psi_m),(\phi_0,\phi_1,\cdots,\phi_m)) \mapsto \sum_k \binom{m}{k}\psi_k\phi_{m-k}.$$

*on the corresponding chain groups.*

**Proof.** The proof of Proposition (a) goes in the same way as in the previous subsection by applying the Stokes formula to the manifold with boundary. We briefly indicate why $\partial^2_{(m)} = 0$ and leave the details of the boundary information to the reader. Note that the entry of $d^2_{(m)}$ coming from

$$C_{j-2k} \xrightarrow{\binom{l}{k} n^{(l-k)}_{*,*}} \oplus_{0\leq l\leq m} C_{j-1-2l} \xrightarrow{\binom{n}{l} n^{(n-l)}_{*,*}} C_{j-2-2n}$$

can be written as

$$\begin{aligned}
&\sum_l \sum_\beta \binom{l}{k} n_{\alpha,\beta}^{(l-k)} \binom{n}{l} n_{\beta,\gamma}^{(n-l)} \\
= &\sum_l \sum_\beta \binom{n}{k} \binom{n-k}{l-k} n_{\alpha,\beta}^{(l-k)} n_{\beta,\gamma}^{(n-l)} \\
= &\binom{n}{k} \sum_l \sum_\beta \binom{n-k}{l-k} n_{\alpha,\beta}^{(l-k)} n_{\beta,\gamma}^{(n-l)},
\end{aligned}$$

where $\sum_l \sum_\beta \binom{n-k}{l-k} n_{\alpha,\beta}^{(l-k)} n_{\beta,\gamma}^{(n-l)}$ is the formal integration of $\Theta^{n-k}$ over the co-dimension one boundary of the moduli space $\hat{\mathcal{M}}^{2(n-k)+1}(\alpha,\gamma)$ (the $2(n-k)+1$ dimensional component of $\hat{\mathcal{M}}(\alpha,\gamma)$), which is zero as we explained above. In the case that $Y$ is a homology 3-sphere, we have to show that there is no contribution from $\hat{\mathcal{M}}(\alpha,\theta) \times U(1) \times \hat{\mathcal{M}}(\theta,\beta)$ in $s^{-1}_{\alpha,\beta,2k+1}(0)$, this is due to the fact that there is a $U(1)$-action on the pull-back bundle and the pull-back section is transversal and $U(1)$-equivariant on $\hat{\mathcal{M}}(\alpha,\theta) \times U(1) \times \hat{\mathcal{M}}(\theta,\beta)$. The proof of Proposition (b) is straight forward. □

Here we briefly discuss another variation of the Fukaya-type monopole homology for a homology sphere $(Y, g, \eta)$ whose monopole moduli space $\mathcal{M}_\eta(Y)$ consists of finitely many irreducible, non-degenerate critical points and a unique reducible critical point $\theta$. The $Z$-graded chain complex is the same as the non-equivariant Seiberg-Witten-Floer chain complex except at degree 0:

$$\begin{aligned}
\tilde{C}_j(Y, g, \eta) &= C_j(Y) \qquad\qquad \text{for } j \neq 0, \\
\tilde{C}_0(Y, g, \eta) &= C_0(Y) \oplus \mathbb{Z}.\langle\theta\rangle.
\end{aligned} \tag{3.50}$$

Then as in the previous construction, define the Fukaya-type complex as

$$C\tilde{F}F_j^{(m)}(Y, \mathfrak{t}) = \oplus_{0 \leq n \leq m} \tilde{C}_{j-2n}(Y, g, \eta)$$

with the boundary operator $\partial_{(m)}$ given by (3.48) while $\beta$ can be the reducible solution $\theta$, and any entry $n_{\theta,\beta}^{l-k} = 0$. Then there is a homology group $C\tilde{C}F_*^{(m)}(Y, g, \eta)$ which satisfies Proposition 3.4.1. This version of Fukaya-type monopole homology is the right model to study the gluing formulae for a simply-connected, 4-manifold $X = X_1 \cup_Y X_2$ splitting along a homology 3-sphere for which $b_2^+(X_1) > 1$ and $b_2^+(X_2) = 0$. The Fukaya-type monopole homology (by removing the reducible point

$\theta$) will be applied to the case of both $b_2^+(X_1) \geq 1$ and $b_2^+(X_2) \geq 1$ where the Seiberg-Witten monopole invariant has no contribution from the gluing monopoles along $\theta$. There is also a version of Fukaya-type monopole homology for the equivariant Seiberg-Witten-Floer homology defined in [33], which is used to investigate the 4-manifold with $b_2^+(X) = 1$ splitting along a homology sphere. This last issue is very subtle, we will leave it for future research.

# Chapter 4

# Applications of Monopole Homology Theory

## 4.1 Gluing formulae for 4-d SW invariants

### 4.1.1 Relative SW invariants for 4-manifolds with boundary

Let $X$ be an oriented Riemannian 4-manifold with cylindrical end modelled on $Y \times [0, \infty)$. Let $\mathfrak{s}$ be a Spin$^c$ structure on $X$ whose induced Spin$^c$ structure on $Y$ is $\mathfrak{t}$. $\mathcal{M}_{\mathfrak{t},\eta}$ is the moduli space of the Seiberg-Witten monopoles on $(Y, \mathfrak{t})$, which consists of finitely many irreducible solutions for $b_1(Y) > 0$, with a unique reducible solution $\theta_\eta$ for a homology 3-sphere $Y$. The relative Seiberg-Witten invariant for $X$ is defined by studying the relative Seiberg-Witten moduli space on $X$ with asymptotic limit representing a critical point in $\mathcal{M}_{\mathfrak{t},\eta}$.

We begin with $b_1(Y) > 0$: for each $\alpha \in \mathcal{M}_{\mathfrak{t},\eta}$, define $\mathcal{M}(X, \alpha)$ to be the moduli space of the Seiberg-Witten equations on $X$ with asymptotic limits representing $\alpha$ modulo the gauge group of $X$. Then there is an asymptotic value map:

$$\partial_\infty : \qquad \mathcal{M}_X(\mathfrak{s}) = \bigcup_{\alpha \in \mathcal{M}_{\mathfrak{t},\eta}} \mathcal{M}(X, \alpha) \longrightarrow \tilde{\mathcal{M}}_{\mathfrak{t},\eta}(Y) \tag{4.1}$$

where $\tilde{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$ is the moduli space of the Seiberg-Witten monopoles on $(Y, \mathfrak{t})$ modulo the gauge transformations which can be extended to $X$. Note that $\tilde{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$ is an

$H^1(Y,\mathbb{Z})/Im(i^*)$-covering space of $\mathcal{M}_{\mathfrak{t},\eta}(Y)$:

$$\pi: \qquad \tilde{\mathcal{M}}_{\mathfrak{t},\eta}(Y) \xrightarrow{H^1(Y,\mathbb{Z})/Im(i^*)} \mathcal{M}_{\mathfrak{t},\eta}(Y)$$

with $i^*: H^1(X,\mathbb{Z}) \longrightarrow H^1(Y,\mathbb{Z})$ is the induced map of the embedding $i: Y \to X$.

The functional space for the monopole equations modelled on $\alpha$ are $L^2_1$-Sobolev spaces with '$\delta$-decay' where $\delta > 0$ is determined by the least absolute eigenvalue of the Hessian of $C_\eta$ at $\alpha$ as in Remark 3.1.13. Let $\Gamma_\alpha = (A_\alpha, \Phi_\alpha)$ be a Spin$^c$ connection and a spinor on $X$ which agrees with the pull-back of some gauge representative of $\alpha$ on the cylindrical end. Define the configuration space to be the following $\delta$-weighted Sobolev space ($l > 4$),

$$\mathcal{C}(X,\Gamma_\alpha) = \{(A,\psi) | A - A_\alpha \in L^2_{l,\delta}, \psi - \Phi_\alpha \in L^2_{l,\delta,A_0}\}$$

with gauge group given by

$$\mathcal{G}(X) = \{u : X \to \mathbb{C} | |u| = 1, 1 - u \in L^2_{l+1,\delta}\}.$$

Fix $r > l$, choose $\mu \in \Gamma(isu(W^+)) \cong \Lambda^{2,+}(X)$ such that $\mu - ic\eta \in C^r_\delta$ ($\delta$-decaying $C^r$-forms) where $c : X \to \mathbb{R}^+$ is a cut-off function supported away from a compact set and equaling 1 on the end.

The perturbed monopole equations on $X$ are given by

$$\begin{cases} F^+_A = q(\psi) + \mu \\ \not{D}_A \psi = 0. \end{cases} \tag{4.2}$$

The corresponding $L^2$-moduli space $\mathcal{M}(X,\Gamma_\alpha)$ is the set of $\mathcal{G}(X)$-equivalence classes of pairs $(A,\psi) \in \mathcal{C}(X,\Gamma_\alpha)$ solving equations (4.2) with finite variation on the end for $\mathcal{C}_\eta$. Note that the asymptotic limit of $[A,\psi]$ takes valued in $\tilde{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$. If $\Gamma^{(1)}_\alpha, \Gamma^{(2)}_\alpha$ are two such classes in $\tilde{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$ representing $\alpha$, then there is a gauge transformation $u$ on $Y$ such that $u(\Gamma^{(1)}_\alpha) = \Gamma^{(2)}_\alpha$, where $[u]$ lies in $H^1(Y,\mathbb{Z})/Im(i^*)$, then the virtual dimensions $\mathrm{i}_X(\Gamma^{(1)}_\alpha)$ and $\mathrm{i}_X(\Gamma^{(2)}_\alpha)$ of $\mathcal{M}(X,\Gamma^{(1)}_\alpha)$ and $\mathcal{M}(X,\Gamma^{(2)}_\alpha)$ (the indices of the deformation complexes for the Seiberg-Witten monopoles) has the following relations:

$$\mathrm{i}_X(\Gamma^{(2)}_\alpha) = \mathrm{i}_X(\Gamma^{(1)}_\alpha) + \Big([u] \cup c_1(\mathfrak{t})\Big)[Y]$$

**Proposition 4.1.1.** *If* $i(\Gamma_\alpha) < 0$, *then there exists a Baire set of perturbations such that* $\mathcal{M}_\eta(\Gamma_\alpha)$ *is empty; if* $i(\Gamma_\alpha) \geq 0$, *then there exists a Baire set of perturbations such that* $\mathcal{M}_\eta(\Gamma_\alpha)$ *is a smooth, oriented manifold with dimension* $i(\Gamma_\alpha)$.

**Proof.** This is the application of standard transversality arguments as in subsection 3.1.3. □

Let $\mathcal{M}(\Gamma_\alpha^{(0)})$ be the non-empty component of $\mathcal{M}(X,\alpha)$ with the least dimension $i_X(\Gamma_\alpha^{(0)})$, denote by $\mathcal{M}(\Gamma_\alpha^{(n)})$ be the non-empty component of $\mathcal{M}(X,\alpha)$ with dimension $i_X(\Gamma_\alpha^{(0)}) + n\ d(\mathfrak{t})$. We can view $\{\Gamma_\alpha^{(n)} : n \geq 0\}$ as in the fiber of the following covering space:

$$\hat{\mathcal{M}}_{\mathfrak{t},\eta}(Y) \xrightarrow{H^1(Y,\mathbb{Z})/Ker(c_1(\mathfrak{t}))} \mathcal{M}_{\mathfrak{t},\eta}(Y) \tag{4.3}$$

where $Ker(c_1(\mathfrak{t}))$ is the kernel of the map $[u] \mapsto ([u] \cup c_1(\mathfrak{t}))[Y]$, for $c_1(\mathfrak{t}) \neq 0$,

$$H^1(Y,\mathbb{Z})/Ker(c_1(\mathfrak{t})) \cong \mathbb{Z}$$

hence, $\hat{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$ is a $\mathbb{Z}$-covering space of $\mathcal{M}_{\mathfrak{t},\eta}(Y)$, and the relative $Z_{d(\mathfrak{t})}$-indices can be lifted to $\mathbb{Z}$-valued indices on $\hat{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$, this was first discovered by R. Wang[47].

Then we know that generically $\mathcal{M}(X,\alpha)$ is the union of the non-empty components $\{\mathcal{M}(\Gamma_\alpha^{(n)}) : n \geq 0\}$. We will define the relative Seiberg-Witten invariant $SW_X(\mathfrak{s})$ with values in the Fukaya-Floer monopole homology groups from the compact components in $\{\mathcal{M}(\Gamma_\alpha^{(n)}) : n \geq 0\}$.

We know that on each component $\mathcal{M}(X,\Gamma_\alpha^{(n)})$, there is a canonical $U(1)$ fibration $\tilde{\mathcal{M}}^0(\Gamma_\alpha^{(n)})$ with associated vector bundles.

If $i_X(\Gamma_\alpha^{(0)}) = 0$, we define

$$SW_{X,\alpha^{(0)}} = \#\Big(\mathcal{M}(X,\Gamma_\alpha^{(0)})\Big),$$

and for any $n > 0$, let $\mathcal{M}^0(\Gamma_\alpha^{(n)})$ be the based moduli space over $\mathcal{M}(\Gamma_\alpha^{(n)})$. Denote by $\Theta$ its first Chern class. Then we construct a rank $d$ complex vector bundle (where $2d$ is the dimension of $\mathcal{M}(X,\Gamma_\alpha^{(n)})$)

$$\mathcal{E}_{X,\alpha,d} = \mathcal{M}^0(\Gamma_\alpha^{(n)}) \times_{U(1)} \mathbb{C}^d$$

with a transversal section $s_{X,\alpha,d} = (s^1_{X,\alpha,d}, \cdots, s^d_{X,\alpha,d})$ which is compatible with the pairs $(\mathcal{E}_{\alpha,\beta,*}, s_{\alpha,\beta,*})$ via the gluing map:

$$g_T : \mathcal{M}(X,\alpha) \times \mathcal{M}(\alpha,\beta) \longrightarrow \mathcal{M}(X,\beta).$$

Then we define

$$SW_{X,\alpha^{(n)}} = \#\Big(s^{-1}_{X,\alpha,d}(0)\Big).$$

We formally denote it by $\int_{\mathcal{M}(X,\Gamma^{(n)}_\alpha)} \Theta^d$. Now $\#\Big(s^{-1}_{X,\alpha,d}(0)$ is well-defined by an argument similar to the corresponding construction in the definition of Fukaya-type homology groups.

If $\mathfrak{i}_X(\Gamma^{(0)}_\alpha) > 0$ is an odd number, define $n_{X,\alpha^{(n)}} = 0$. If $\mathfrak{i}_X(\Gamma^{(0)}_\alpha > 0$ is even, we again use the transversal section on the associated vector bundle $\mathcal{E}_{X,\alpha,d}$ (where $2d$ is the dimension of $\mathcal{M}(X,\Gamma^{(n)}_\alpha)$) to define $SW_{X,\alpha^{(n)}}$.

**Definition 4.1.2.** *The relative Seiberg-Witten invariant for $(X,\mathfrak{s})$ is defined to be the following formal series*

$$SW_X(\mathfrak{s}) = \sum_{\alpha\in\mathcal{M}_{\mathfrak{t},\eta}(Y)} \sum_{n\geq 0} SW_{X,\alpha^{(n)}} \langle t^n\alpha\rangle. \tag{4.4}$$

For each 4-manifold with a Spin$^c$ structure $\mathfrak{s}$, there is a natural $\mathbb{Z}$-lifting of the periodic Seiberg-Witten-Floer chain complex which is given by the grading

$$\mathfrak{i}_X(\alpha^{(n)}) = \mathfrak{i}_X(t^n\alpha) = n\ d(\mathfrak{t}) + \mathfrak{i}_X(\Gamma^{(0)}). \tag{4.5}$$

This can be extended to give a grading on $\hat{\mathcal{M}}_{\mathfrak{t},\eta}(Y)$ (cf (4.3)).

**Proposition 4.1.3.** *Let $(X,\mathfrak{s})$ be a Riemannian 4-manifold with cylindrical end modelled on $(Y\times[0,\infty),\mathfrak{t})$ and $Y$ be a closed 3-manifold with $b_1(Y) > 0$ and $c_1(\mathfrak{t}) \neq 0$. Let $\eta$ be a perturbation such that $\mathcal{M}_{\mathfrak{t},\eta}(Y)$ consists of only finitely many irreducible, non-degenerate critical points. Define the Seiberg-Witten invariant for $(X,\mathfrak{s})$ to be (4.4). Then for any $2m > 0$, we can extract a relative Seiberg-Witten invariant $SW^{(m)}_X(\mathfrak{s})$ from $SW_X(\mathfrak{s})$ by only considering those terms with $\mathfrak{i}_X(t^n\alpha) \leq 2m$ such that $SW^{(m)}_X(\mathfrak{s})$ represents a homology class in $HFF^{(m)}_*(Y,\mathfrak{t})$.*

**Proof.** Note that there are only finitely many terms in the expression of $SW^{(m)}_X(\mathfrak{s})$. The cocycle condition can be checked through the boundary information of $\mathcal{M}(\Gamma^{(n)}_\alpha)$

involved in and following from the formal Stokes formulae explained in the Section 3.4. □

For a 4-manifold $(X, \mathfrak{s})$ with boundary a homology 3-sphere $Y$, we can define the relative invariant from the Seiberg-Witten monopole moduli space on the corresponding 4-manifold with cylindrical end. We assume that $b_2^+(X) \geq 1$ and the induced metric $g_Y$ on $Y$ is a 'good' metric in the sense that $\mathcal{M}_\eta(Y, g)$ consists of finitely many irreducible non-degenerate points and a unique reducible point $\theta$. The Seiberg-Witten moduli space $\mathcal{M}_X(\mathfrak{s})$ on $X$ with finite variation of $\mathcal{C}_\eta$ on the cylindrical end can be studied by using the weighted Sobolev space as before.

Any monopole in $\mathcal{M}_X(\mathfrak{s})$ has an asymptotic limit as we approach the end, which decays exponentially to a critical point in $\mathcal{M}_\eta(Y, g)$, therefore,

$$\mathcal{M}_X(\mathfrak{s}) = \cup_{\alpha \in \mathcal{M}_\eta(Y,g)} \mathcal{M}_X(\mathfrak{s}, \alpha)$$

where $\mathcal{M}_X(\mathfrak{s}, \alpha)$ is non-empty only if the virtual dimension (the index of the deformation complex) $i_X(\alpha) \geq 0$. The non-empty component $\mathcal{M}_X(\mathfrak{s}, \alpha)$ is an oriented, compact, smooth manifold of dimension $i_X(\alpha)$ after a standard perturbation argument.

If $i_X(\alpha) = 0$, we can count the points in $\mathcal{M}_X(\mathfrak{s}, \alpha)$ (an oriented, compact 0-manifold) with sign, we get a number $SW_{X,\alpha}$, if $i_X(\alpha) = 2d > 0$, we can define the corresponding monopole invariant $SW_{X,\alpha}$ as in the Definition 4.1.2 (c) by integrating $\Theta^d$ over $\mathcal{M}_X(\mathfrak{s}, \alpha)$, similarly, if $i_X(\alpha)$ is odd, we assign $SW_{X,\alpha}$ to be zero. Note that we take account of the reducible point $\theta \in \mathcal{M}_\eta(Y, g)$. Define

$$SW_X(\mathfrak{s}) = \sum_{\alpha \in \mathcal{M}_\eta(Y,g)} SW_{X,\alpha} \langle \alpha \rangle \in \tilde{C}_*(Y, g, \eta). \tag{4.6}$$

Then for any $2m > 0$, we can retract a relative Seiberg-Witten invariant $SW_X^{(m)}(\mathfrak{s})$ from $SW_X(\mathfrak{s})$ by only considering those terms with $i_X(\alpha) \leq 2m$, then $SW_X^{(m)}(\mathfrak{s})$ represents an element in the Fukaya-type homology groups $\widetilde{HFF}_*^{(m)}(Y, g, \eta)$. Moreover $SW_X^{(m)}(\mathfrak{s})(mod\, \mathbb{Z}.\langle \theta \rangle)$ defines an element in the Fukaya-type homology groups $HFF_*^{(m)}(Y, g, \eta)$.

### 4.1.2 Gluing formulae for 4-d monopole invariants

Our application is mainly to consider a simply-connected, closed 4-manifold $(X, \mathfrak{s})$ with $b_2^+ > 1$ and a Spin$^c$ structure $\mathfrak{s}$. Suppose that $X$ has a decomposition:

$$X_1 \cup_Y X_2$$

along a 3-dimensional submanifold $Y$. Here since the Seiberg-Witten invariant for 4-manifold $X$ with $b_2^+ > 1$ is independent of the chosen metric, we choose a metric $g_T$ on $X$ such that $X$ contains a very long 'neck' $Y \times [T, -T]$.

Let $i_k$ denote the embedding maps of $Y$ into $X_k$ as the boundary. Suppose the restriction of $\mathfrak{s}$ on $Y$ is a Spin$^c$-structure $\mathfrak{t}$ such that $\mathcal{M}_\eta(Y, \mathfrak{t})$ consists of finitely many, non-degenerate, irreducible critical points of $\mathcal{C}_\eta$ for $b_1(Y) > 0$ with a unique reducible point $\theta$ for a homology 3-sphere $Y$.

First we assume that $Y$ is a closed 3-manifold with $b_1(Y) > 0$. Let $\mathfrak{s}_1 = \mathfrak{s}|_{X_1}$, $\mathfrak{s}_2 = \mathfrak{s}|_{X_2}$, then the Spin$^c$-structure on $X$, denoted by Spin$^c(X; \mathfrak{s}_1, \mathfrak{s}_2)$, which agrees with $\mathfrak{s}_i$ over $X_i$ is an affine space over

$$H^1(Y,\ Z)/Imi_1^* \times Imi_2^*$$

where $i_1^* : H^1(X_1, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$, $i_2^* : H^1(X_2, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$.

By the Mayer-Vietoris sequence, we know that

$$\begin{cases} Imi_1^* \cap Imi_2^* = 0 \\ Imi_1^* \times Imi_2^* = H^1(Y, \mathbb{Z}). \end{cases}$$

from which we see that Spin$^c(X; \mathfrak{s}_1, \mathfrak{s}_2)$ consists of only one point, that is $\{\mathfrak{s}\}$.

As $T$ is chosen sufficiently large, each solution to the Seiberg-Witten equations on $X$ is approximated by a pair of solutions on $X_{i,cyl}$ where cylindrical ends are attached.

From Definition (4.1.2), there exists a relative $SW$-invariant denoted by $SW_{X_i}$ which take values in $HF_*^{SW}(Y, \mathfrak{t})$ and $HF_*^{SW}(\bar{Y}, \bar{\mathfrak{t}})$ respectively. Note that $HF_*^{SW}(\bar{Y}, \bar{\mathfrak{t}})$ is constructed from the same complex as $HF_*^{SW}(Y, \mathfrak{t})$ with reversed orientation.

$$SW_{X_i} = \sum_{\alpha \in \mathcal{M}_{\mathfrak{t},\eta}(Y)} \sum_{n \geq 0} SW_{X_i, \alpha^{(n)}} \langle t^n \alpha \rangle,$$

where $SW_{X_i,\alpha^{(n)}}$ is the Seiberg-Witten monopole invariant corresponding to the compact moduli space $\mathcal{M}(X_i,\Gamma_\alpha^{(n)})$ with even dimension $\mathrm{i}_{X_i}(\Gamma_\alpha^{(n)})$.

The Seiberg-Witten monopole theory on a closed 4-manifold ([26]), tells us that the moduli space $\mathcal{M}_X(\mathfrak{s})$ is an oriented, compact, smooth manifold with dimension given by

$$\frac{1}{4}(c_1(\mathfrak{s})^2-(2\chi+3\sigma))=d_X(\mathfrak{s})\geq 0$$

after a generic perturbation.

If $d_X(\mathfrak{s})=0$, counting the points in $\mathcal{M}_X(\mathfrak{s})$ with sign gives rise to the Seiberg-Witten invariant $SW_X(\mathfrak{s})$. If $d_X(\mathfrak{s})$ is odd, we define $SW_X(\mathfrak{s})$ to be 0. If $d_X(\mathfrak{s})=2d>0$, then $SW_X(\mathfrak{s})$ is defined to be

$$\int_{\mathcal{M}_X(\mathfrak{s})}\Theta^d$$

where $\Theta$ is the first Chern class of the based moduli space over $\mathcal{M}_X(\mathfrak{s})$.

When $T$ is sufficiently large, the gluing theorem of the Seiberg-Witten monopoles identifies $\mathcal{M}_X(\mathfrak{s})$ with the following product:

$$\mathcal{M}_X(\mathfrak{s})\cong\cup_{\alpha^{(n)}\in\Pi_X}\mathcal{M}(X_1,\Gamma_\alpha^{(n)})\times\mathcal{M}(X_2,\Gamma_\alpha^{(n)})\tag{4.7}$$

where $\Pi_X=\{\alpha^{(n)}:\mathrm{i}_{X_1}(\alpha^{(n)})+\mathrm{i}_{X_2}(\alpha^{(n)})=d_X(\mathfrak{s})\}$. From (4.7) we see that any non-empty component $\mathcal{M}(X_i,\Gamma_\alpha^{(n)})$ appearing in the gluing model can be compactified by adding lower dimensional "broken" monopoles. Those components with even dimensions will have contributions to the monopole invariant $SW_X(\mathfrak{s})$ for simply connected 4-manifold $X$. We summary our gluing formulae in the following proposition.

**Proposition 4.1.4.** *Suppose that $(X,\mathfrak{s})$ is a closed, simply-connected 4-manifold with $b_2^+>1$, which splits along a closed 3-manifold $(Y,\mathfrak{t})$ with $b_1(Y)>0$. Then the Seiberg-Witten invariant $SW_X(\mathfrak{s})$ for $(X,\mathfrak{s})$ with $d_X(\mathfrak{s})=2d$ can be expressed as follows:*

$$SW_X(\mathfrak{s})=\sum_{\alpha^{(n)}\in\Pi_X}\binom{d}{\frac{\mathrm{i}_{X_1}(\alpha^{(n)})}{2}}SW_{X_1,\alpha^{(n)}}SW_{X_2,\alpha^{(n)}}$$

*where $\Pi_X=\{\alpha^{(n)}:\mathrm{i}_{X_1}(\alpha^{(n)})+\mathrm{i}_{X_2}(\alpha^{(n)})=d_X(\mathfrak{s})\}$. If we grade the Seiberg-Witten-Floer chain complex by the indices defined in (4.5), and denote by $SW_{X_i}^{(d)}$ the relative* *invariants valued in the Fukaya-type homology* $HFF_*^{(d)}(Y, \mathfrak{t})$ *as in Proposition 4.1.3, then*

$$SW_X(\mathfrak{s}) = \sigma_d(SW_{X_1}, SW_{X_2})$$

*where* $\sigma_d$ *is the pairing defined in Proposition 3.4.1.*

**Proof.** We give here only the formal proof by using our notation convention, one can reinterpret each identity in terms of the canonical vector bundle and their transversal and compatible sections as we did in section 3.4. From the gluing model (4.7), we know that the first Chern class $\Theta$ on $\mathcal{M}_X(\mathfrak{s})$ can be decomposed as $\Theta_1 + \Theta_2$ (the sum of the first Chern classes) on any component $\mathcal{M}(X_1, \Gamma_\alpha^{(n)}) \times \mathcal{M}(X_2, \Gamma_\alpha^{(n)})$,from which the contribution to $SW_X(\mathfrak{s})$ can be expressed as

$$\begin{aligned}
&\int_{\mathcal{M}(X_1,\Gamma_\alpha^{(n)})\times\mathcal{M}(X_2,\Gamma_\alpha^{(n)})} (\Theta_1 + \Theta_2)^d \\
= &\binom{d}{\frac{\mathfrak{i}_{X_1}(\alpha^{(n)})}{2}} \int_{\mathcal{M}(X_1,\Gamma_\alpha^{(n)})} \Theta^{\frac{\mathfrak{i}_{X_1}(\alpha^{(n)})}{2}} \int_{\mathcal{M}(X_2,\Gamma_\alpha^{(n)})} \Theta^{\frac{\mathfrak{i}_{X_2}(\alpha^{(n)})}{2}} \\
= &\binom{d}{\frac{\mathfrak{i}_{X_1}(\alpha^{(n)})}{2}} SW_{X_1,\alpha^{(n)}} SW_{X_2,\alpha^{(n)}}.
\end{aligned}$$

Then the gluing formula follows from the definition of the relative monopole invariants and the pairing $\sigma_d$. □

Now we suppose that the simply-connected 4-manifold $X = X_1 \cup_Y X_2$ splits along a homology 3-sphere $Y$. We further assume that $b_2^+(X_i) \geq 1$ so that $b_2^+(X) \geq 2$, then we will discuss the case when $b_2^+(X_1) > 1$ and $b_2^+(X_2) = 0$.

For the first case where $b_2^+(X_i) \geq 1$, we know that the Seiberg-Witten invariant is independent of metric and perturbation, then we can choose a good metric such that there is no reducible solution on $X_1$ and $X_2$, and the metric near $Y$ has a sufficient long 'neck' structure, we can also require that $\mathcal{M}_\eta(Y)$ has only finitely many critical points with only one isolated reducible solution.

The gluing model in this situation gives rise to the following isomorphism, in which $\mathcal{M}_X(\mathfrak{s})$ is the Seiberg-Witten moduli space with even dimension $d(\mathfrak{s}) = 2d$:

$$\mathcal{M}_X(\mathfrak{s}) \cong \cup_{\alpha\in\Pi_X} \Big(\mathcal{M}_{X_1}(\alpha) \times \mathcal{M}_{X_2}(\alpha)\Big) \cup \Big(\mathcal{M}_{X_1}(\theta) \times \mathcal{M}_{X_2}(\theta) \times U(1)\Big)$$

where $\Pi_X = \{\alpha \in \mathcal{M}^*_\eta(Y) : \mathrm{i}_{X_1}(\alpha) + \mathrm{i}_{X_2}(\alpha) = d(\mathfrak{s})\}$, $\mathcal{M}_\eta(Y) = \mathcal{M}^*_\eta(Y) \cup \{\theta\}$.

We can see from the definition, the contribution of $M_{X_1}(\theta) \times \mathcal{M}_{X_2}(\theta) \times U(1)$ to $SW_X(\mathfrak{s})$ vanishes, so we can use the $mod(\mathbb{Z}\,\theta)$ version of the relative invariants (cf (4.6)) for $X_i$, which can be written as

$$SW_{X_i} = \sum_{\alpha \in \mathcal{M}^*_\eta(Y)} SW_{X_i,\alpha}\langle\alpha\rangle.$$

The gluing formula in this case is the pairing $\sigma_d$ of $SW_{X_1}$ and $SW_{X_2}$:

$$\begin{aligned} SW_X(\mathfrak{s}) &= \textstyle\sum_{\alpha\in\Pi_X} \begin{pmatrix} d \\ \frac{i_{X_1}(\alpha)}{2} \end{pmatrix} SW_{X_1,\alpha} SWX_2, \alpha \\ &= \sigma_d(SW_{X_1}, SW_{X_2}) \end{aligned} \tag{4.8}$$

In the case of $b_2^+(X_1) > 1$ and $b_2^+(X_2) = 0$, there is always a unique reducible solution (also denoted by $\theta$) in the moduli space $\mathcal{M}_{X_2}$. We can choose a metric on $X$ such that $\mathcal{M}_{X_1}$ has no reducible solution, and the induced metric on $Y$ is 'good' with respect to the perturbation $\eta$. We assume that there is no cokernel of the twisted Dirac operator with $\theta$ on $X_2$ in the weighted Sobolev space, therefore there is no obstruction in the gluing of the monopoles in $\mathcal{M}_{X_1}(\theta)$ with $\theta$. Then the above gluing model gives the following isomorphism:

$$\begin{aligned} \mathcal{M}_X(\mathfrak{s}) \;\cong\; & \cup_{\alpha\in\Pi_X}\Big(\mathcal{M}_{X_1}(\alpha) \times \mathcal{M}_{X_2}(\alpha)\Big) \\ & \cup\Big(\mathcal{M}_{X_1}(\theta) \times \mathcal{M}^*_{X_2}(\theta) \times U(1)\Big) \cup \Big(\mathcal{M}_{X_1}(\theta) \times \{\theta\}\Big). \end{aligned}$$

From this gluing model, we see that $d(\mathfrak{s}) = \mathrm{i}_{X_1}(\theta) = 2d \geq 0$ and the contribution to $SW_X(\mathfrak{s})$ from $\mathcal{M}_{X_1}(\theta) \times \mathcal{M}^*_{X_2}(\theta) \times U(1)$ is zero since $M^*_{X_2}(\theta)$ (the irreducible part) is empty by dimension counting. The contribution to $SW_X(\mathfrak{s})$ from $\mathcal{M}_{X_1}(\theta) \times \{\theta\}$ is $SW_{X_1}(\theta)$.

The gluing formula in this case is given by

$$\begin{aligned} SW_X(\mathfrak{s}) &= SW_{X_1}(\theta) + \textstyle\sum_{\alpha\in\Pi_X} \begin{pmatrix} d \\ \frac{i_{X_1}(\alpha)}{2} \end{pmatrix} SW_{X_1,\alpha} SWX_2, \alpha \\ &= SW_{X_1}(\theta) + \sigma_d(SW_{X_1}, SW_{X_2}) \end{aligned} \tag{4.9}$$

where $\sigma_d$ is the pairing on $\widetilde{HFF}_*^{(d)}(Y, g, \eta)$.

Even if there is non-trivial cokernel for the twisted Dirac operator with $\theta$ on the weighted Sobolev space over $X_2$, the above gluing formula (4.9) remains to be true by studying the obstruction bundle as in Donaldson theory [16]. We need to understand the obstruction bundle over $\mathcal{M}_{X_1}(\theta)\times\{\theta\}$, suppose that $dimCoker(\not{D}_\theta)\cong\mathbb{C}^k$, then $dim\mathcal{M}_{X_1}(\theta)=2d+2k$. The obstruction bundle is

$$\Pi=\tilde{\mathcal{M}}_{X_1}(\theta)\times_{U(1)}\mathbb{C}^k$$

with a canonical section $\Delta$ whose zero points are the actual monopoles obtained by deforming the pre-gluing monopoles of the form

$$(A,\psi)\#\{\theta\}\in\mathcal{M}_{X_1}(\theta)\times\{\theta\}.$$

It is easy to see that the obstruction bundle $\Theta$ is the associated bundle with the basepoint fibration, let $\mathcal{L}$ be the line bundle associated with the basepoint fibration $\tilde{\mathcal{M}}_{X_1}(\theta)$, then

$$\Pi=\oplus_{k-copies}\mathcal{L}$$

from which we see that the homology class $[\Delta^{-1}(0)]$ is just the Poincare dual of $\Theta^k$ (the first Chern class of the basepoint fibration) in $\mathcal{M}_X(\mathfrak{s})$, hence the contribution to $SW_X(\mathfrak{s})$ is $SW_{X_1}(\theta)$.

**Remark 4.1.5.** *Though the Seiberg-Witten-Floer homology for a homology 3-sphere depends on the metric and perturbation, the gluing formulae to calculate the monopole invariant on $(X,\mathfrak{s})$ makes sense since we always assume $b_2^+(X)>1$. In case of $b_2^+(X)=1$, we know that the monopole invariant on $(X,\mathfrak{s})$ depends on the metric, therefore, one may except that the wall-crossing formula for $SW_X(\mathfrak{s})$ could be worked out by using the gluing formulae in the equivariant Seiberg-Witten-Floer homology setting. We will study this application in thc futurc.*

## 4.2 Monopole invariants for contact structures

In this subsection, we will apply the techniques in [27] [45] to define certain invariants taking values in the Floer homology groups of a 3-manifold $Y$ with contact structure $\xi$. A contact structure $\xi$ on $Y$ is an oriented 2-plane field which is nowhere integrable. This requires that the 1-form $\alpha$ which annihilates $\xi$ satisfies $\alpha\wedge d\alpha>0$.

Let $\Xi$ denote the space of oriented 2-plane fields on $Y$. For each $\xi \in \Xi$, there is a natural Spin$^c$-structure $\mathfrak{t} = (W, \rho)$ associated with $\xi$, where

$$W = \xi \oplus \underline{\mathbb{C}}.$$

This Spin$^c$ structure $\mathfrak{t}$ also admits a unit-length spinor $\Phi \in \underline{\mathbb{C}}$ unique up to gauge transformations $u : Y \to U(1)$. On the other hand, for $\mathfrak{t} = (W, \rho) \in \text{Spin}^c(Y)$ and $\Phi \in \Gamma(W)$ such a unit-length spinor, if we write $W = \mathbb{C}\Phi^{\perp} \oplus \mathbb{C}\Phi$ locally, then

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \Gamma(i\mathfrak{su}(W))$$

defines a unique 1-form (of unit-length) $\alpha$ such that $\rho(\alpha)$ is the above form. This 1-form defines an oriented 2-plane field $\xi = ker(\alpha)$. There is a projective map from $\pi_0(W)$ (the connected component group of $\Xi$) to the isomorphic classes of Spin$^c$ structures on $Y$: $p : \quad \pi_0(\Xi) \to \text{Spin}^c(Y)$ whose fiber over $\mathfrak{t} = (W, \rho)$ can be identified with the set of homology classes of unit-length spinors of $W$ modulo the action of automorphisms of $\mathfrak{t}$. This set can be identified with $\mathbb{Z}/d(\mathfrak{t})\mathbb{Z}$ by using relative Euler classes as in [27]. We briefly recall this identification as follows: let $\Phi_1, \Phi_2$ be two unit-length spinors of $W$, then we can define a difference element $\delta(\Phi_1, \Phi_2) \in \mathbb{Z}$ given by the relative Euler class:

$$\delta(\Phi_1, \Phi_2) = e([0,1] \times W; \Phi_0, \Phi_1)[[0,1] \times Y, \partial].$$

If $\Phi_1, \Phi_2$ are homotopic, then $\delta(\Phi_1, \Phi_2) = 0$. If $\Phi_2 = u(\Phi_1)$ where $u : Y \to U(1)$, then

$$\begin{aligned} \delta(\Phi_1, u(\Phi_1)) &= c_2(W \times S^1, A - u_{-1}du)([Y \times S^1]) \\ &= ([u] \cup c_1(W))([Y]) \in d(\mathfrak{t})\ Z. \end{aligned}$$

where $[u] \in H^1(Y, \mathbb{Z})$ corresponds to the connect component determined by $u$. Fix $[\xi_0] \in p^{-1}(\mathfrak{t})$, then for any $[\xi_1] \in p^{-1}(\mathfrak{t})$, the above difference element defines $\delta(\xi_1, \xi_2) \in \mathbb{Z}/d(\mathfrak{t})\mathbb{Z}$ which identifies the fiber $p^{-1}(\mathfrak{t})$ with $\mathbb{Z}/d(\mathfrak{t})\mathbb{Z}$.

Suppose that $\xi \in p^{-1}(\mathfrak{t})$ is a contact structure on $Y$, then there exists a unique Riemannian metric $g_1$ on $Y$ such that

(a) $|\alpha| = 1$,

(b) $d\alpha = 2 * \alpha$,

(c) The canonical complex structure $J$ on $\xi$ ( $J^2 = -1$, $(e, Je)$ is a positively oriented basis for any vector $e \in \xi$) is an isometry.

For this contact manifold $(Y, \xi)$, there is a symplectisation on $[2, \infty) \times Y$ whose symplectic form $\omega$ is given by

$$\omega = \frac{1}{2} d(t^2 d\alpha) = t dt \wedge \alpha + \frac{1}{2} t^2 d\alpha$$

and the compatible metric on $[2, \infty) \times Y$ is given by $dt^2 + t^2 g_1$. Choose a smooth positive function $f : [-1, 1] \to [0, 1]$ such that

$$f|_{[-1,0]}(t) = t, \qquad f|_{[1/2,1]}(t) = 1, \qquad 0 < f'(t) < 4.$$

Then $(Y \times \mathbb{R})$ with metric $g$

$$g = \begin{cases} dt^2 + t^2 g_1 & t \in (-\infty, -1] \\ dt^2 + f(t) g_1 & t \in [-1, 1] \\ dt^2 + g_1 & t \in [1, \infty) \end{cases} \tag{4.10}$$

has one cylindrical end and one conical end. If $Y$ is a homology 3-sphere, we can further assume that the cylindrical end is modelled on $Y$ with metric away from the codimension one subset $\mathcal{W}$ in Proposition 2.2.15.

On the symplectic part $[2, \infty) \times Y$, there is a canonical Spin$^c$ structure as in [45]:

$$W^+ = \Lambda^{0,0} \oplus \Lambda^{2,0}$$

associated with a unique spin connection $A_0$ such that $\not{D}_{A_0} \Phi_0 = 0$ for $\Phi$ is the constant unit-length spinor in $\Lambda^{0,0}$.

Write a spin connection as $A_0 + a$, where $a$ is a $i\mathbb{R}$-valued 1-form on $[2, \infty) \times Y$ and let

$$\psi = (\rho(\beta_0).\Phi_0, \rho(\beta_1).\Phi_0)$$

be an ordered pair where $(\beta_0, \beta_1) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$. Then the perturbed monopole equations can be written as follows:

$$\begin{aligned} &\bar{\partial}_a\beta_0 + \bar{\partial}_a^*\beta_1 = 0 \\ &F_a^{\omega} = \frac{i}{8}(1 - |\beta_0|^2 + |\beta_1|^2)\omega \\ &F_a^{0,2} = \frac{1}{4}\bar{\beta}_0\beta_1. \end{aligned} \tag{4.11}$$

where $F_a^{\omega} = \frac{1}{2}\langle F_a, \omega\rangle$.

Suppose that $b_1(Y) > 0$ such that $\mathcal{C}_\eta$ has only finite, non-degenerate, irreducible critical point set $\mathcal{M}_{\mathfrak{t},\eta}(Y)$. Let $\mathcal{M}_\eta(Y,\xi)$ be the $L^2$-moduli space of monopole equations on $(Y \times \mathbb{R}, g)$ whose restriction to the cylindrical end gives a finite variation of $\mathcal{C}_\eta$ and whose restriction to the conical end satisfies the above equations (4.11). Then any Seiberg-Witten monopole in $\mathcal{M}_\eta(Y,\xi)$ decays exponentially to a Seiberg-Witten monopole in $\mathcal{M}_{\mathfrak{t},\eta}(Y)$ as $t \to -\infty$ (see Proposition 3.1.12), and decays exponentially to $(A_0, \Phi_0)$ as $t \to \infty$ (see [27]). Denoted by $\mathcal{M}_\eta(Y,\xi,\alpha)$ the corresponding moduli space, let $i(\xi,\alpha)$ be the index of the linearization of the monopole equations on $(Y \times \mathbb{R}, g, \xi)$ for $\mathcal{M}_\eta(Y,\xi,\alpha)$ then

$$i(\alpha,\beta) = i(\xi,\beta) - i(\xi,\alpha)(mod\ d(\mathfrak{t})).$$

The standard transversality and compactness argument in subsection $\mathcal{M}_\eta(Y,\xi,\alpha)$.

**Proposition 4.2.1.** *There is a Baire set of perturbations $\eta$ such that $\mathcal{M}_\eta(Y,\xi,\alpha)$ is a smooth, oriented, compact manifold with dimension $i(\xi,\alpha)$ whenever $i(\xi,\alpha) \geq 0$ and is empty when $i(\xi,\alpha) > 0$.*

Now we come to define the monopole invariants for the contact structure $\xi \in p^{-1}(\mathfrak{t})$ for a Spin$^c$-structure $\mathfrak{t}$ on $Y$.

**Definition 4.2.2.** *Let $\xi$ be a contact structure in $p^{-1}(\mathfrak{t})$, denote by $\mathcal{M}_\eta(Y,\xi,\alpha)$ the moduli space of Seiberg-Witten monopoles on $(Y,g)$ which satisfy the perturbed gradient flow equation of $\mathcal{C}_\eta$ on $Y \times (-\infty,-1]$ with asymptotic limit $\alpha \in \mathcal{M}_{\mathfrak{t},\eta}(Y)$ as $t \to -\infty$ and satisfy the equations (4.11) on $Y \times [2,\infty)$. We also assume that $c_1(\mathfrak{t}) \neq 0$ if $b_1(Y) > 1$.*

*(a) If $i(\xi,\alpha) = 0$, define $n_{\xi,\alpha} = \#(\mathcal{M}_\eta(Y,\xi,\alpha))$,*

(b) *If* $i(\xi, \alpha)$ *is an odd number, define* $n_{\xi,\alpha} = 0$,

(c) *If* $i(\xi, \alpha) = 2m > 0$, *let* $\mathcal{M}^0_\eta(Y, \xi, \alpha)$ *be the based moduli space over* $\mathcal{M}_\eta(Y, \xi, \alpha)$, *it is a* $U(1)$ *bundle. Denote by* $\Theta$ *its first Chern class and then define*

$$n_{\xi,\alpha} = \int_{\mathcal{M}^0_\eta(Y,\xi,\alpha)} \Theta^m.$$

*where we use our formal convention as before,* $n_{\xi,\alpha}$ *is actually the "Euler" number of the complex vector bundle associated with the basepoint fibration.*

*Then the monopole invariant for contact structure* $\xi \in p^{-1}(\mathfrak{t})$ *is defined to be*

$$SW(Y, \xi) = \sum_{\alpha \in \mathcal{M}_{\iota,\eta}(Y)} n_{\alpha,\xi} \langle \alpha \rangle. \tag{4.12}$$

**Proposition 4.2.3.** *For each contact structure* $\xi \in p^{-1}(\mathfrak{t})$, *its Seiberg-Witten invariant* $SW(Y, \xi) \in HF^{SW}_*(Y, \mathfrak{t})$.

**Proof.** Applying the boundary operator $\partial$ to $SW(Y, \xi)$, we see that

$$\partial(SW(Y, \xi)) = \sum_\alpha \sum_{\beta: i(\alpha,\beta)=1} n_{\alpha,\xi} n_{\alpha,\beta} \langle \beta \rangle.$$

We need to check $\sum_\alpha n_{\alpha,\xi} n_{\alpha,\beta} = 0$ for any $\beta$ with $i(\alpha, \beta) = 1$. For those $\alpha$ with $i(\xi, \alpha) = 0$, then contribution comes from the boundary of 1-manifold $\mathcal{M}_\eta(Y, \xi, \beta)$, which is zero. The other terms are also zero by the Stokes formula. □

**Remark 4.2.4.** *We can define the monopole invariant for a contact structure on a homology 3-sphere, where we need only to ignore any contribution from the unique reducible solution in* $\mathcal{M}_\eta(Y)$.

**Remark 4.2.5.** *In [27], the monopole invariant for a 4-manifold with contact boundary is defined through the Seiberg-Witten monopoles on the 4-manifold with conical end. This kind of invariant can be obtained by the gluing formulae between the Seiberg-Witten invariant on a 4-manifold with cylindrical end and the Seiberg-Witten invariant for contact structure. We leave the formulation of this gluing formulae to the reader.*

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# Index