Aspects of Topological Field Theories

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Thesis Submitted for the Degree of
Doctor of Philosophy
in the
Department of Physics and Mathematical Physics,
University of Adelaide

May, 1998

Contents

Abstract

Since their introduction in 1988, topological field theories have attracted a great deal of interest from both mathematicians and physicists. Mathematically they provide alternative formulations for certain topological invariants such as Donaldson invariants. Physically, topological field theories are important as they may be used to test different characteristics of their corresponding physical theory. In this thesis twisted $N = 2$ and $N = 4$ SYM theories in four and six dimensions are studied. We first provide the general background for topological field theories which can be obtained by twisting. A supersymmetric Yang-Mills theory is then constructed on a Calabi-Yau 3-fold by dimensional reduction. It is shown that this theory is a cohomological field theory and the corresponding path integral, in the weak coupling limit, localizes on the moduli space of Donaldson-Uhlenbeck-Yau equations. We also construct a partially twisted theory on a product six-manifold $X \times Y$. When Y is supersymmetrically embedded in a Calabi-Yau manifold M, it is argued how the moduli space on which the path integral localizes can be related to the mirror manifold of M. We also study the twisted $N = 4$ SYM theory on the product four-manifold $\Sigma \times S^2$. We derive the effective theory in the limit where S^2 shrinks. The correlators of the cohomology classes of the BRST operator are then computed in the mass deformed effective theory.

Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

Ali Imaanpur   Date: 25, May, 1998

Acknowledgements

I would like to express my deep gratitude to my advisor, Jim McCarthy. He was a unique teacher and a friend, I appreciate his supervision and his support throughout my study in Adelaide.

I would also like to thank Bryan Wang and Nicholas Buchdahl for useful discussions.

My thanks go to my parents for their constant encouragement in all stages of my study. My father was always with me with his feelings. I am thankful to my wife for her companion and patience.

The support of the Ministry of Culture and Higher Education of Iran is also appreciated.

This thesis is dedicated to my parents, my wife, Maryam, and my son, Sepehr.

Chapter 1

Introduction

Topological field theories (TFT) [1] have proven to be a useful tool in the investigation of the nonperturbative characteristics of supersymmetric gauge theories such as $N = 2$ and $N = 4$ supersymmetric theories. There is an interplay between certain supersymmetric gauge theories and their corresponding topological versions: one can use topological results on smooth manifolds to learn about the underlying physical theory; conversely, one may use physical arguments to gain new insight into the topological structure of the manifold on which the fields are defined.

As an example of the first, in [2] Witten has shown that $N = 2$ supersymmetric Yang-Mills (SYM) theory cannot have a mass gap using some known facts about Donaldson invariants of Kähler manifolds. Then, as an example of the second, he shows how one can use certain nonperturbative facts about $N = 1$ SYM theory to compute the Donaldson invariants of Kähler manifolds. In this way, all Donaldson invariants of Kähler four-manifolds[1] are beautifully determined.

Another example of the use of TFT to learn about a physical theory is found in $N = 4$ SYM theory. This theory has been conjectured to have an exact $SL(2, \mathbf{Z})$ duality [3] providing a correspondence between the weak and strong coupling limits of the same theory. Since this relation involves strong coupling, to test the conjecture one needs quantities such as the partition function to be computed nonperturbatively. This is a formidable task and one actually does not know how to proceed in this direction. This is where topological field theory comes to provide an alternative approach to the

[1] However, there is a topological restriction on these Kähler manifolds that we will discuss later.

problem. Instead of the physical theory, one considers the corresponding topological field theory obtained by a procedure called twisting (discussed in detail later). The basic characteristics of the theory, such as $SL(2,\mathbf{Z})$ invariance, remain intact under twisting, so one hopes to see the realization of this symmetry in the twisted model. In [4] it has been shown how, using known facts about the structure of the moduli space of instantons and the associated Euler characteristic, the partition function of $N=4$ twisted theory on some specific manifolds can be computed. So, in this way, it has become possible to make some exact and nonperturbative statements about the theory and its self-duality properties.

From the mathematical point of view, TFT has provided a new formulation for expressing some typical topological invariants of manifolds in terms of the observables of the theory. In the case of Donaldson theory, this reformulation has been very fruitful since, as we mentioned above, one can use physical arguments in determining the invariants. More importantly, an effective field theory description of Donaldson theory has been discovered [5]. This effective theory has its own invariants, however, they encode all the subtle informations about the Donaldson invariants. Further, as might be expected physically, in the effective theory calculations of the new invariants are much easier.

Although it might seem that TFT have a rather ad hoc appearance in physics, they naturally arise as the effective field theory of some solitonic states in string theory [6]. These solitonic states (D-branes) are generically extended curved objects which appear upon compactifying the string theory on Calabi-Yau manifolds. Since these objects have a curved worldvolume, and since they preserve part of the space-time supersymmetry, the effective field theory living on the worldvolume which describes the low energy excitations of the D-brane is forced to be a topological field theory. Therefore, from this perspective, topological field theories are quite physical.

Let us discuss briefly the contents of this thesis. The introduction in chapter 2 contains a review of relevant literature. The remaining chapters contain our original work. More details on these chapters follows, but let us just state first what the main new results are.

- A cohomological field theory on Calabi-Yau threefolds is constructed.
- It is shown that this theory is indeed a balanced topological field theory.

- We construct a partially twisted theory on product six-manifolds. In a particular limit we determine the moduli space on which the path integral localizes.
- We derive the effective field theory of twisted $N = 4$ SYM theory on $\Sigma \times S^2$ in the limit where S^2 shrinks.
- This theory is then perturbed by a mass term for the hypermultiplet preserving part of the supersymmetry.
- We compute a set of correlation functions in this effective theory.

Finally then, let us return to an outline of the thesis. The second chapter is devoted to a review of topological field theory. The basic construction of TFT from supersymmetric theories, the twisted $N = 4$ SYM, and the higher dimensional analogues of Donaldson-Witten theory are the main topics which are reviewed in this chapter. Among these different topological field theories, twisted $N = 2$ SYM theory has the most basic features of a topological field theory and plays a central role in understanding the others. Therefore, in this chapter, we concentrate on this theory.

In the third chapter we study the analogue of Donaldson-Witten theory in six dimensions. In trying to twist the six-dimensional theory, we face a limitation; the nonanomalous part of the global symmetry is not large enough to allow us to twist the theory on an arbitrary six-manifold. Thus we are limited to consider manifolds with restricted holonomies such as Kähler or product manifolds. This is in sharp contrast with the Donaldson-Witten theory in four dimensions where the theory can be defined on an arbitrary four-manifold, resulting in a set of genuine topological invariants. We construct a cohomological field theory on Calabi-Yau threefolds. This theory, in some respects, parallels the Donaldson theory in four-dimension; it is a theory independent of metric and coupling constant and its correlation functions are topological invariants. However, unlike Donaldson theory, we are here limited to those metric deformations which preserve the holonomy structure of the manifold. In analogy with Donaldson theory, we also write down the cohomology classes of the BRST operator which have topological correlators. Furthermore, we show that there is a balanced formulation of the theory in the sense of [7]. As noted earlier, the cohomological field theories built on Calabi-Yau manifolds are important as they naturally arise in the low energy description of D-branes. In the present case, they describe the low energy physics of euclidean D5-branes wrapping around the whole threefold. Therefore, in studying the various properties of such branes our cohomological theory is valuable.

A partially twisted theory on product six-manifolds is also constructed in chapter three. This theory is useful in studying the D-branes wrapping around a special Lagrangian submanifold of the Calabi-Yau manifold. We study a particular limit of the theory and determine the moduli space on which the path integral localizes.

Motivated by the work of Vafa and Witten [4] to examine the duality properties of $N = 4$ theory using topological methods, in chapter four we study the twisted $N = 4$ theory on the manifold $\Sigma \times S^2$. We consider the limit where S^2 shrinks to zero size. The effective theory in this limit is derived. Following [4], we will see that the partition function of the reduced theory in fact computes the Euler characteristic of the moduli space of flat connections over Σ. Perturbing by a mass term allows us to compute a set of correlation functions in this effective theory. Although perturbing by mass introduces some new fixed points to the original moduli space, it is possible to isolate their contributions to the path integral. We analyze the contribution of the points where a component of the hypermultiplet becomes massless and in particular discuss that these points do not contribute in the case of a nontrivial $SO(3)$ bundle. Using this fact we are able to write down an explicit result for the correlation functions.

In the course of the present investigations many fascinating problems have arisen. Some of them have been incorporated in this thesis. Some remain, and seem to be interesting. We briefly summarize the latter at the end of the chapters.

Chapter 2

Twisted Supersymmetric Yang-Mills theories

2.1 Introduction

There are basically two different ways to construct a topological field theory. One is a mathematical construction starting from the moduli space of some interesting equations, the other starts from a supersymmetric physical theory in flat space and tries to extend it to a general manifold preserving part of the supersymmetry. In this thesis we are interested in the latter, and this chapter will review the status of the field. However, for completeness, let us briefly outline the basics of the first approach, known as the Mathai-Quillen approach [8, 9, 10] which constructs invariants of a vector bundle V over some manifold M. We will not give a mathematical discussion, but rather a physical motivation [4].

Introduce coordinates u^i on M and a basis of sections s of V. The interesting equations above are incorporated as $s = 0$. The idea is to construct quantities which are invariant under small changes in the data u^i and s. A small change in u^i, δu^i, can be identified with a one-form on M. So, as is familiar from supersymmetric quantum mechanics, we write

$$\begin{aligned}\delta u^i &= i\epsilon\psi^i \\ \delta\psi^i &= 0\,,\end{aligned}$$

where ϵ is an anticommuting parameter. As $\delta^2 = 0$, this immediately reminds us of the BRST implementation of gauge fixing, where the ψ^i's are identified with the ghosts. Indeed, let us follow this line to the resulting invariants.

Introduce antighosts χ^a, which are also sections of V, in order to have a nondegenerate action for the ghosts. The auxiliary fields, H^a, then naturally enter the formalism to close the BRST algebra, with the basic structure $\delta\chi^a = \epsilon H^a$, $\delta H^a = 0$. Since we have introduced explicit coordinates and a basis of sections, we need a curvature A_i on V to covariantize the differential structure δ. These transformations are then modified to

$$\begin{aligned}\delta\chi^a &= \epsilon H^a - i\epsilon\psi^i A_i{}^a{}_b \chi^b \\ \delta H^a &= i\epsilon\psi^i A_i{}^a{}_b H^b - i\frac{\epsilon}{2}\psi^i\psi^j F_{ij}{}^a{}_b\chi^b ,\end{aligned}$$

where F_{ij} is the curvature of A_i. Let us initially assume that the equations $s^a = 0$ have only isolated zeros. The invariants we are interested in have the basic form

$$\int \delta\left(s^a\right) \det\left(\frac{\partial s^a}{\partial u^i}\right) ,$$

essentially a sign-weighted counting of the number of solutions to the equations of interest. This, however, can be cast into the path integral representation using the fields introduced above; i.e.,

$$\int \mathcal{D}[H^a, \chi_a, \psi^i]\, e^{-\frac{1}{2\lambda}(-2iH^a s_a + 2\chi_a \frac{\partial s^a}{\partial u^i}\psi^i)} , \tag{2.1}$$

where λ is an arbitrary real parameter. In fact, the "action" in (2.1) can be written as a BRST commutator,

$$S = \delta\Psi \ = \delta\,(\frac{i}{\lambda}\chi^a s_a) ,$$

ensuring, as we will see presently, that nothing depends on the data entering into Ψ. For generalizing to the cases where the space of solutions form a manifold we need to smooth out the above action to a nondegenerate one. This requires that we introduce a metric on the fibers of V to raise or lower the indices. Thus, a more suitable choice for Ψ is

$$\Psi = \frac{1}{2\lambda}(\chi^a H_a + 2i\chi^a s_a) ,$$

so that the action becomes

$$S = \frac{1}{2\lambda}\int\left[H^2 - 2iH^a s_a + 2\chi_a\frac{\partial s^a}{\partial u^i}\psi^i - F_{ijab}\psi^i\psi^j\chi^a\chi^b\right] .$$

This smooths the delta-function to a gaussian weight.

It is now easy to see that the theory defined by the above action is topological, in the sense that it is formally independent of λ, s and the metric. This is so because the variation of the partition function with respect to any of these data is a BRST commutator. For instance, if we vary the partition function Z with respect to λ we have

$$\frac{\delta Z}{\delta \lambda} \sim \langle \delta \Psi \rangle .$$

However, since the measure and the action are BRST invariant, this is zero by

$$0 = \int \mathcal{D}[X]\, e^{-S(X)} \Psi(X) - \int \mathcal{D}[X']\, e^{-S(X')} \Psi(X') = \int \mathcal{D}[X]\, e^{-S(X)} \delta \Psi .$$

Here the fields are represented collectively by X.

Since the theory is formally independent of s and λ, we may compute the partition function using different and arbitrary values for these data. If we set $s = 0$, the Lagrangian simplifies and what we get for the partition function is indeed the usual integral representation for the Euler characteristic of the bundle V

$$\chi(V) = \frac{1}{(2\pi)^{d/2} \cdot d!} \int \mathrm{Pf}(F \wedge F \wedge \ldots \wedge F) ,$$

where d is the rank of the bundle V and the Pfaffian of a skew-symmetric matrix B, $\mathrm{Pf}(B)$, is defined by

$$\det B = \mathrm{Pf}(B)^2 .$$

On the other hand, upon integrating out the auxiliary fields H^a and taking the limit $\lambda \to 0$, the path integral localizes on the moduli space of solutions to the equations $s^a = 0$. Let us for simplicity assume that these equations have only isolated zeroes. Therefore in computing the partition function, we can expand the action around the solutions of these equations keeping only the quadratic terms. Performing the gaussian path integral then results in identical determinants for bosons and fermions (ghosts and antighosts) which up to a sign cancel each other. The independence of the theory from the parameters s and λ thus provides a "proof" of the well-known fact that we may compute, e.g., the Euler class of the bundle V by counting the zeroes of a section s weighted by the appropriate signs.

As we saw above, one of the basic constituents of a topological field theory is the existence of a suitable fermionic symmetry. Since fermionic symmetries arise naturally in supersymmetric theories, we may ask if a topological field theory can be realized in that context. Clearly there is no obstruction to extending the Lagrangian of a flat space supersymmetric theory to an arbitrary spin four-manifold [1]. One just replaces the ordinary derivatives by the covariant ones. However, except on spin manifolds which admit covariantly constant spinors, the Lagrangian will not be supersymmetric. This can be seen as follows. Let α denote the supersymmetry parameter of the supersymmetry transformations, which have the general form

$$\delta\Phi = \bar{\alpha}\Psi$$
$$\delta\Psi = \nabla\alpha\,\Phi\,.$$

On euclidean space, since the action is supersymmetric, the variation of the action under the supersymmetry transformations is by the Noether construction proportional to

$$\int J^M \partial_M \bar{\alpha}\,, \tag{2.2}$$

where J^M is the supersymmetry current. Upon extending the Lagrangian to a curved spin manifold by minimal coupling, the action varies to

$$\int J^M \nabla_M \bar{\alpha}\,, \tag{2.3}$$

which is an obvious generalization of (2.2) (∇_M is the covariant derivative on the manifold), up to terms which are proportional to the Riemann tensor of the manifold. The Riemann tensor may appear, because one usually needs to commute the covariant derivatives. This term does not include the covariant derivatives of α (the fermionic kinetic term is first order in derivative), and hence it cannot cancel the term in (2.3). Note that indeed if they could cancel each other then one would not recover the result for flat space by setting the Christoffel symbols to zero. Thus the necessary condition for the action to be supersymmetric is α to be covariantly constant. Although this argument is more heuristic, the result is very general and we will see an explicit example of this in section 3.4. The above constraint enforces us to consider only those spin manifolds which admit at least one globally defined spinor which is covariantly constant. Being covariantly constant, the spinor is in fact a scalar under the holonomy group of the manifold.

[1]As long as one can define the SYM theory in an arbitrary dimension all these constructions go through for that particular dimension.

A more general procedure which allows part of the fermionic symmetry to survive, is called *twisting* [1]. Twisting basically consists of choosing a new embedding of the holonomy group inside the whole global symmetry (space-time symmetry and the $\mathcal{R}$-symmetry) of the model such that at least one component of the supercharge, Q, transforms as a scalar. As it is a scalar, its global existence on an arbitrary manifold is guaranteed, and obviously a constant scalar is covariantly constant. We note that Q^2 is a scalar bosonic operator. Therefore, if there is no any scalar operator in the supersymmetry algebra, Q^2 must vanish (on-shell and up to a gauge transformation). Hence, on such manifolds, Q is in fact a BRST-like symmetry as expected. Moreover, the newly defined action turns out to be exact with respect to this scalar supercharge; i.e., $S = \{Q, V\}$ for some gauge invariant V. Thus the energy-momentum tensor of the twisted theory, which is the generator of the newly defined space-time symmetry group, is also BRST exact

$$T_{\mu\nu} = \{Q, \delta V/\delta g^{\mu\nu}\}\,.$$

Note, in particular, that, since the action is Q-exact, the supersymmetry follows immediately as Q squares to zero on gauge invariant quantities.

At this point we may formally make two basic observations. Firstly, if we vary the partition function

$$Z = \int \mathcal{D}[X]\, \exp\left(-\frac{1}{e^2}S(X)\right)$$

with respect to coupling constant e, we obtain

$$\frac{\delta Z}{\delta e} \sim \langle S \rangle = \langle \{Q, \ldots\} \rangle\,.$$

Assuming that supersymmetry is not spontaneously broken – i.e., there is at least one vacuum annihilated by Q – the above equation implies $\delta Z/\delta e = 0$. The second observation is that the argument can be repeated to show that Z is also independent of the metric. These are the key properties of the model which will allow us to consider a convenient limit of the coupling or metric in which calculations (mainly perturbative ones around the critical points of the action) become easy or possible.

The organization of this chapter is as follows. We start our discussion with $N = 2$ SYM theory on flat space. We study a variety of properties of the theory such as the global symmetries and the mass gap and compare them to those of $N = 1$ SYM theory. Next we define the twisting of the theory. The importance of zero modes of different fields present in the theory is then argued, and the conditions for their absence is presented. The discussion of $N = 2$ twisted theory ends with an overview of the low energy description of Donaldson theory.

$N = 4$ SYM theory is the next subject that we review. As before, the physical theory and its global symmetries are first discussed. The different possible twistings are then presented. We discuss the relevant equations which appear in the weak coupling limit for a particular twisting of the theory. The analogue of Donaldson theory in higher dimensions is the last subject that we review.

2.2 $N = 2$ SYM theory and its twisting

In [1] Witten introduced a reformulation of Donaldson theory in terms of twisted $N = 2$ SYM theory. He showed how different Donaldson invariants can be identified with some observables of the twisted theory. In this section we begin with a study of the physical theory and its symmetries. The twisted theory and the topological observables are then discussed. At the end we concentrate on the Kähler case.

2.2.1 Physical $N = 2$ SYM theory

In terms of $N = 1$ supersymmetric multiplets, the $N = 2$ SYM theory consists of a gauge multiplet $A = (A_\mu, \lambda)$, for A_μ an $SU(2)$ gauge field and λ a chiral spinor, and a chiral multiplet $\Phi = (\psi, \phi)$ where ψ is again a chiral spinor and ϕ a complex scalar. A and Φ are both in the adjoint representation of the gauge group. The minimal action for the pure – i.e., without matter multiplet (or hypermultiplet) – $N = 2$ SYM theory is [12]

$$\begin{aligned} S = \frac{1}{e^2}\int d^4x \,\mathrm{tr}\, \Big[& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}^{\dot{\alpha}}_{\;i}\sigma^{\mu}_{\alpha\dot{\alpha}}D_\mu\lambda^{\alpha i} - D_\mu\bar{\phi}D^\mu\phi \\ & - \frac{1}{2}[\bar{\phi},\phi]^2 - \frac{i}{\sqrt{2}}\bar{\phi}\epsilon_{ij}[\lambda^{\alpha i},\lambda_{\alpha}^{\;j}] + \frac{i}{\sqrt{2}}\phi\epsilon^{ij}[\bar{\lambda}_{\dot{\alpha}i},\bar{\lambda}^{\dot{\alpha}}_{\;j}]\Big] , \end{aligned} \tag{2.4}$$

where we have grouped λ_α and ψ_α into $\lambda_{\alpha}^{\;i}$, for $i = 1, 2$. The action is invariant under $N = 2$ supersymmetric transformations which read [12]

$$\begin{aligned} \delta A_\mu &= -i\bar{\lambda}^{\dot{\alpha}}_{\;i}(\sigma_\mu)_{\alpha\dot{\alpha}}\xi^{\alpha i} + i\bar{\xi}^{\dot{\alpha}}_{\;i}(\sigma_\mu)_{\alpha\dot{\alpha}}\lambda^{\alpha i} \\ \delta\lambda_{\alpha}^{\;i} &= (\sigma^{\mu\nu})_{\alpha}^{\;\beta}\,\xi_{\beta}^{\;i}F_{\mu\nu} + i\xi_{\alpha}^{\;i}[\phi,\bar{\phi}] + i\sqrt{2}(\sigma^\mu)_{\alpha\dot{\alpha}}D_\mu\phi\,\epsilon^{ij}\bar{\xi}^{\dot{\alpha}}_{\;j} \end{aligned}$$

$$\delta\bar{\lambda}_{\dot{\alpha}i} = (\sigma^{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}\bar{\xi}_{\dot{\beta}i}F_{\mu\nu} - i\bar{\xi}_{\dot{\alpha}i}[\phi,\bar{\phi}] - i\sqrt{2}(\sigma^{\mu})_{\alpha\dot{\alpha}}D_{\mu}\bar{\phi}\,\epsilon_{ij}\xi^{\alpha j}$$
$$\delta\phi = \sqrt{2}\xi^{\alpha i}\lambda_{\alpha i}$$
$$\delta\bar{\phi} = \sqrt{2}\bar{\xi}^{\dot{\alpha}}_{\ i}\bar{\lambda}_{\dot{\alpha}}^{\ i}\,.$$

The corresponding supercharges to $\xi_{\alpha i}$ and $\bar{\xi}_{\dot{\alpha}}^{\ i}$ will be denoted by $Q_{\alpha i}$ and $\bar{Q}_{\dot{\alpha}}^{\ i}$ respectively. It is clear from the Lagrangian that the model has a global symmetry $SU(2)_I \times U(1)_{\mathcal{R}}$. $\lambda_{\alpha i}$ ($Q_{\alpha i}$) transform as a doublet of $SU(2)_I$, and under $U(1)_{\mathcal{R}}$ the fields transform as

$$\lambda_{\alpha i} \longrightarrow e^{i\gamma}\lambda_{\alpha i}$$
$$\phi \longrightarrow e^{2i\gamma}\phi$$

with A_μ being invariant.

The $U(1)_{\mathcal{R}}$ symmetry is a chiral symmetry and thus is quantum mechanically violated by the familiar Adler-Bell-Jackiw anomaly. From the instanton calculation point of view we can easily see how this symmetry is broken. Recall that in the k-instanton field the Dirac equation has $4k$ chiral zero modes with no anti-chiral solutions, so under a $U(1)_{\mathcal{R}}$ rotation the measure in the path integral transforms as (the measure for non-zero modes remains invariant as the spectrum is symmetric for those modes)

$$\mathcal{D}[\lambda_{\alpha i}] \longrightarrow e^{-i8k\gamma}\mathcal{D}[\lambda_{\alpha i}]\,.$$

Notice that as $\lambda_{\alpha i}$ is fermionic, the measure transforms with the inverse Jacobian. Moreover, the result is invariant under Z_{8k} (with group elements $e^{i\beta_n}$, with $\beta_n = 2n\pi/8k$, $n = 1, 2, \ldots, 8k$). Thus instantons explicitly break $U(1)_{\mathcal{R}}$ to Z_8, the smallest symmetry left. Notice that $SU(2)_I$ rotates λ and ψ into each other, and since they have the same zero mode spectrum the measure remains invariant under this global symmetry. In fact, the $SU(2)_I$ part of the global symmetry is believed to be an exact symmetry of the theory [13]. Hence the nonanomalous part of the global symmetry, G, on flat euclidean space [14, 15] is a direct product of the space-time symmetry and the internal one,

$$G = SU(2)_L \times SU(2)_R \times SU(2)_I\,. \tag{2.5}$$

2.2.2 The twisted theory

The two supercharges $Q_{\alpha i}$ and $\bar{Q}_{\dot{\alpha}}^{\ i}$ transform under the global symmetry group G as $(2,1,2)$ and $(1,2,2)$ respectively, where 2 and 1 refer to the dimension of the representations. Twisting is basically defined as follows. Instead of a trivial embedding of the space-time symmetry group K inside the whole global symmetry, one may choose another embedding: take $SU(2)'_R$ to be a diagonal subgroup of $SU(2)_R$ and $SU(2)_I$ and declare $K' = SU(2)_L \times SU(2)'_R$ to be the space-time symmetry. Under K', the supercharges now transform as

$$
\begin{aligned}
Q_{\alpha i} &\sim (2,1,2) \rightarrow (2,2) \\
\bar{Q}_{\dot{\alpha} i} &\sim (1,2,2) \rightarrow (1,3) \oplus (1,1) \,.
\end{aligned}
$$

Thus we do obtain a fermionic supercharge, Q, which transforms as a scalar under the new space-time symmetry. Note that, as argued in the introduction, Q^2, on-shell and up to a gauge transformation, vanishes. Therefore, Q is in fact a BRST-like scalar supercharge. Being a scalar, Q is well defined on an arbitrary smooth four-manifold. Therefore twisting will leave us with just one BRST-like supercharge and a Lagrangian which is invariant under it.

Moreover, it is possible to write the action, up to a topological term, in a form which is BRST exact. This latter property is the most important characteristic of a topological field theory, and as we saw earlier all basic properties of a TFT – such as metric independence – follow from it. As K' is now the new space-time symmetry of the theory, we should decompose the field space into irreducible representations of K' and rewrite the Lagrangian in terms of these new fields. Let us call U the corresponding $U(1)_{\mathcal{R}}$ (ghost symmetry) charge in the twisted theory. The field content of the twisted theory consists of a gauge field A_μ, a (grassmann) odd field (ghost) ψ_μ, to which gauge field transforms, with $U = 1$ and a bosonic field ϕ with $U = 2$. Besides these, there are also the Grassmann odd fields (antighosts) $\chi_{\mu\nu}$ and η with $U = -1$ and a bosonic field $\bar{\phi}$ with $U = -2$. This gives the twisted action of $N = 2$ SYM theory [1]

$$
\begin{aligned}
S \;=\; & \int d^4x \sqrt{g}\, \mathrm{tr}\, \Big[\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} \phi D_\mu D^\mu \bar{\phi} - i \eta D_\mu \psi^\mu + i D_\mu \psi_\nu \, \chi^{\mu\nu} \\
& - \tfrac{i}{8} \phi [\chi_{\mu\nu}, \chi^{\mu\nu}] - \tfrac{i}{2} \bar{\phi} [\psi_\mu, \psi^\mu] - \tfrac{i}{2} \phi [\eta, \eta] - \tfrac{1}{8} [\phi, \bar{\phi}]^2 \Big] \,.
\end{aligned}
\tag{2.6}
$$

The part of supersymmetry transformations generated by the scalar supercharge Q ($Q = \epsilon^{\dot{\alpha} i} \bar{Q}_{\dot{\alpha} i}$) reads

$$
\begin{aligned}
&\delta A_\mu = i \epsilon \psi_\mu \quad && \delta \phi = 0 \quad && \delta \bar{\phi} = 2 i \epsilon \eta \\
&\delta \eta = \tfrac{1}{2} \epsilon [\phi, \bar{\phi}] \quad && \delta \psi_\mu = - \epsilon D_\mu \phi \quad && \delta \chi_{\mu\nu} = \epsilon F^+_{\mu\nu} ,
\end{aligned}
\tag{2.7}
$$

where $F^+ = \frac{1}{2}(F + *F)$ and $*$ is the Hodge duality operation. As noted earlier, although Q-invariant, $\mathcal{L}$ cannot yet be written as $\{Q, ...\}$. To express $\mathcal{L}$ as a Q-trivial object we need to add a topological term

$$k = \frac{1}{2}\int F \wedge F = \frac{1}{4}\int d^4x F_{\mu\nu}(*F)^{\mu\nu}$$

to the action. Note $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. k is the instanton number of the $SU(2)$-bundle E on M, which characterizes the topological type of the bundle. Being a topological invariant and indeed a number in a class, k is invariant under Q and its addition to the action does not disturb the equations of motion.

Since the theory is independent of the coupling we would like to consider the limit in which e goes to zero (notice that since the physical theory is asymptotically free this limit corresponds to high energies, or probing short distances). In this limit, as is clear from the bosonic part of the action, the path integral localizes around the solutions of the following equations

$$\begin{aligned} F^+_{\mu\nu} &= 0 \\ D_\mu\phi &= 0 \\ [\phi, \bar{\phi}] &= 0\,. \end{aligned} \tag{2.8}$$

The first equation is the instanton equation and will be our main concern in the remainder of this section. The fact that the main contribution to the path integral comes from the zeros of the equations in (2.8) can also be explained by an argument due to Witten [16] which we briefly recall in appendix A.

2.2.3 The topological observables of the twisted theory

Let $\mathcal{M}$ be the moduli space of solutions to the instanton equation, i.e.

$$\mathcal{M} = \{A_\mu \in \mathcal{A} \mid F^+_{\mu\nu} = 0\}/\mathcal{G}\,,$$

where $\mathcal{G}$ is the group of gauge transformations and $\mathcal{A}$ the space of all gauge connections. Suppose A is an instanton and $\mathcal{M}$ has the structure of a manifold. The tangent space to $\mathcal{M}$ at A is spanned by the infinitesimal variations δA such that $A + \delta A$ is an instanton, modulo pure gauge transformations. A first variation of the instanton equation gives the equation for δA,

$$0 = D(\delta A) + *D(\delta A)\,, \tag{2.9}$$

where $D = d + [A, \ ]$. In components this reads

$$0 = D_\mu \delta A_\nu - D_\nu \delta A_\mu + \epsilon_{\mu\nu\rho\sigma} D^\rho \delta A^\sigma . \tag{2.10}$$

Demanding that δA is orthogonal to an infinitesimal gauge transformation of A, gives

$$0 = \langle D\alpha, \delta A\rangle \Rightarrow 0 = D_\mu \delta A^\mu . \tag{2.11}$$

We note that the equations (2.10) and (2.11) are exactly the equations for the zero modes of ψ_μ. Thus ψ_μ zero modes are in fact tangent to the moduli space of $\mathcal{M}$ at say A. The real dimension of the moduli space $\mathcal{M}$ is hence generically equal to the number of nontrivial ψ zero modes.

As is well known the number of ψ zero modes can be related to the index of an elliptic complex. Consider the following elliptic complex

$$0 \xrightarrow{i} \Omega^0(\mathrm{g}) \xrightarrow{D} \Omega^1(\mathrm{g}) \xrightarrow{p^+D} \Omega^{2,+}(\mathrm{g}) \xrightarrow{\varphi} 0 , \tag{2.12}$$

where as above $D = d + [A, \ ]$, p^+ is the projector to the self-dual part and $\Omega^n(\mathrm{g})$ is the space of all Lie-algebra valued n-forms. i is the inclusion map and obviously $\ker \varphi = \Omega^{2,+}(\mathrm{g})$. The index of this elliptic complex is defined by

$$\mathrm{ind}\, D = \sum_{i=0}^{3} (-1)^i \dim H^i(\Omega, D) ,$$

where

$$H^i(\Omega, D) \equiv \ker D_i / \mathrm{im} D_{i-1} .$$

In particular, note that the ψ zero modes belong to the first cohomology group of this complex $H^1(\Omega, D)$. To define $d(\mathcal{M})$, the *virtual* dimension of $\mathcal{M}$, it is more convenient to consider the elliptic complex to which the linearized equations (2.10) and (2.11) fit

$$D^\dagger \oplus p^+ D : \Omega^1(\mathrm{g}) \longrightarrow \Omega^0(\mathrm{g}) \oplus \Omega^{2,+}(\mathrm{g}) .$$

Now $d(\mathcal{M})$ is defined to be the index of this complex [17, 18],

$$\begin{aligned} d(\mathcal{M}) &= \dim \{\ker (D^\dagger \oplus p^+ D) - \mathrm{coker}\, (D^\dagger \oplus p^+ D)\} \\ &= \dim \ker (D^\dagger \oplus p^+ D)|_{\Omega^1} - \dim \ker D|_{\Omega^0} - \dim \ker p^+ D^\dagger|_{\Omega^{2,+}} \end{aligned}$$

as coker$D^\dagger$ = ker$D$. This is indeed minus the index of the elliptic complex in (2.12). Therefore the virtual dimension of $\mathcal{M}$ is equal to the number of ψ zero modes minus the number of $\chi_{\mu\nu}$ and η zero modes. Specializing to the $SU(2)$ case, this becomes [17]

$$d(\mathcal{M}) = 8k - \frac{3}{2}(\chi + \sigma)\,. \tag{2.13}$$

Here k is the instanton number, χ and σ are the Euler characteristic and signature of the manifold X. In the case that there are no η and $\chi_{\mu\nu}$ zero modes (an example of which will be discussed below for $SU(2)$ bundles), the real dimension of $\mathcal{M}$ is given by $d(\mathcal{M})$ in (2.13).

We notice that the measure is invariant under $U(1)_{\mathcal{R}}$ if there are no fermionic zero modes. However, in the presence of fermionic zero modes, since under $U(1)_{\mathcal{R}}$ ψ_μ transforms with an opposite sign with respect to η and $\chi_{\mu\nu}$, the measure for the zero modes transforms with a weight, ΔU, equal to the number of ψ zero modes minus the number of $\chi_{\mu\nu}$ and η zero modes. Thus, from the above, ΔU, the net violation of $U(1)_{\mathcal{R}}$ symmetry by the instantons at the quantum level, is indeed equal to the virtual dimension of $\mathcal{M}$.

Let us assume that there are no η or $\chi_{\mu\nu}$ zero modes such that the dimension of moduli space is given by $d(\mathcal{M})$ and see what sort of topological observables can be computed. If $d(\mathcal{M})$ is zero, meaning that basically there are no ψ_μ ghost zero modes, the moduli space is either empty (in which case there are no nontrivial invariants) or consists of a set of discrete solutions. The measure of the path integral transforms invariantly under $U(1)_{\mathcal{R}}$ and we may compute the partition function Z. Later in this section we will argue that in this case it is sufficient, in expanding around one isolated instanton, to keep only the quadratic terms in the action. Since the theory is supersymmetric there is a balance between the bosonic and fermionic degrees of freedom. Therefore, in performing the gaussian integral over the bosonic and fermionic fields we get two identical determinants which up to a sign exactly cancel each other [1]. Having assigned an arbitrary sign for one isolated solution, all other signs associated to the determinants of the remaining instantons can be consistently fixed [1]. Hence the partition function can be written as

$$Z = \sum_i (-1)^{n_i}$$

where i runs over the number of instantons and n_i indicates the associated sign to the

ith instanton.

In the case $d(\mathcal{M}) > 0$, because of the fermionic zero modes, the partition function vanishes. Therefore, we need to insert some gauge invariant operators to soak up the zero modes. However, if we are looking for quantities that are metric independent and therefore topological, then those gauge invariant operators have to be metric independent and BRST closed. For then

$$\frac{\delta}{\delta g^{\mu\nu}}\langle\mathcal{O}\rangle = \langle\mathcal{O}\frac{\delta S}{\delta g^{\mu\nu}}\rangle = \langle\mathcal{O}\{Q,\frac{\delta V}{\delta g^{\mu\nu}}\}\rangle = \langle\{Q,\mathcal{O}\frac{\delta V}{\delta g^{\mu\nu}}\}\rangle = 0\,.$$

On the other hand, since the vacuum is annihilated by Q by assumption, insertion of trivial operators (BRST exact) is trivial. Hence we are led to look for cohomology classes of the operator Q. To construct these classes, it suffices just to look at the following transformations

$$\begin{aligned}\delta A_\mu &= i\epsilon\psi_\mu\\ \delta\psi_\mu &= -\epsilon D_\mu\phi\\ \delta\phi &= 0\,.\end{aligned}\tag{2.14}$$

Since ϕ is closed and no field directly transforms to it, we infer that a gauge invariant operator like $\mathrm{tr}\,\phi(x)^2$ is actually a nontrivial cohomology class. Next let us differentiate it with respect to x (with $\delta = -i\{Q,\cdots\}$)

$$\frac{\partial}{\partial x^\mu}\mathrm{tr}\,\phi(x)^2 = 2i\{Q,\mathrm{tr}\,(\phi\psi_\mu)\}\,.$$

This implies,first that the correlation function $\langle\mathrm{tr}\,\phi(x)^2\rangle$ is x-independent. If $\{\gamma_i\}$ is a basis for the first-homology class of the manifold, then clearly

$$2i\{Q,\oint_{\gamma_i}\mathrm{tr}\,(\phi\psi)\} = \oint_{\gamma_i}d(\mathrm{tr}\,\phi(x)^2) = 0\,.$$

Thus just by differentiating we have constructed another class. Differentiating once more leads to

$$d\mathrm{tr}\,(\phi\psi) = \{Q,\mathrm{tr}\,(\frac{i}{2}\psi\wedge\psi - \phi F)\}\,.$$

This equation implies that the quantity

$$\langle\oint_{\gamma_i}\mathrm{tr}\,(\phi\psi)\rangle$$

depends only on the homology class of γ_i. For example if γ_i is a boundary; $\gamma_i = \partial\Sigma_i$ then

$$\langle \oint_{\partial\Sigma_i} \mathrm{tr}\,(\phi\psi)\rangle = \langle\{Q, \int_{\Sigma_i} \mathrm{tr}\,(\frac{i}{2}\psi\wedge\psi - \phi F)\}\rangle = 0\,.$$

This procedure can be repeated to build up the cohomology classes of Q.

A natural question which arises is whether these are the only cohomology classes of Q that one can construct. This question, to the best of our knowledge, has not been clearly addressed in the literature. Thus, in the following we briefly sketch the related existing views. First, to close the BRST transformations (2.7) off-shell, let us introduce a self-dual auxiliary field $H^{\mu\nu}$ into the transformation laws via

$$\delta\chi_{\mu\nu} = \epsilon H_{\mu\nu}\,, \quad \delta H_{\mu\nu} = 0\,.$$

Q^2 is now zero up to a gauge transformation. Namely, the cohomology of Q only makes sense on the subspace of gauge invariant polynomials. This is a difficult constraint to implement.

To relax this condition, we may introduce the secondary ghost c into the BRST transformations. Then c (and the corresponding anti-ghost $\bar{c}$) enter the Lagrangian in the usual Faddeev-Popov gauge fixing component to completely fix the gauge [19]. We can then define the cohomology of the new BRST charge Q' on the space of arbitrary polynomials of fields [19, 20]. However, by a field redefinition, it is easy to see that this cohomology is trivial. On the other hand, it was further argued [19] that this triviality cannot happen globally. Due to the Gribov problem there are points where the Faddeev-Popov determinant vanishes and thus the secondary ghosts c are not well defined. At most we can define the Faddeev-Popov ghosts locally. Globally, therefore, one does get nontrivial cohomology classes. These classes are then derived by descent equations from the second (for the gauge group $SU(2)$) Chern class of a generalized bundle with the curvature $\mathcal{F} = F + \psi + \phi$. After deriving the cohomology of Q', we must project this cohomology to the cohomology of Q by restricting to the gauge invariant polynomials which do not include c. This, however, does not seem to be a complete answer to the question. Perhaps, to answer the question, one should phrase the problem algebraically and apply the techniques of homological algebra.

Having determined the cohomology classes of Q, we can now start looking at their vacuum expectation values. We saw that the dimension of the moduli space of instantons

$\mathcal{M}$ equals the net violation of $U(1)_{\mathcal{R}}$ symmetry by instantons. Also note that the cohomology classes of Q (except $\int F \wedge F$ which does not concern us since it is just a number) all carry a positive $U(1)_{\mathcal{R}}$ charge. Therefore to saturate the fermionic zero modes we should insert an operator $\mathcal{O}$ with the $U(1)_{\mathcal{R}}$ charge equal to $d(\mathcal{M})$.

2.2.4 Reducible connections

So far, we have formally shown that the partition function and the correlators of the cohomology classes of the operator Q are independent of the metric. However, to show that this is really the case, two conditions on the space $\mathcal{M}$ must hold. First of all, the space $\mathcal{M}$ has to be compact [1]. Although this is not always the case, $\mathcal{M}$ can be compactified if some favourable conditions hold [21]. Second and more important is the problem of reducible connections. When there are reducible connections the space $\mathcal{M}$ is not a smooth manifold and has some singularities. As we will see later, all the statements about the topological invariance of the observables finally come down to the intersection theory on $\mathcal{M}$, therefore, for having a topological theory it is crucial to have a smooth manifold $\mathcal{M}$. Let us first discuss the reducible connections and then the conditions under which they do not appear.

A connection A is called *reducible* if there exists an element g of the gauge group which leaves A invariant, i.e.

$$A \to g^{-1}Ag + g^{-1}dg = A\,.$$

Under an infinitesimal gauge transformation by a parameter ϕ, A_μ transforms as

$$A_\mu \to A_\mu + D_\mu \phi\,.$$

Therefore the connection A_μ remains invariant depending upon whether the equation

$$D_\mu \phi = 0 \tag{2.15}$$

has nontrivial solutions. For $SU(2)$ gauge group we can see that reducible connections correspond to the abelian instantons. If ϕ is not identically zero then, being covariantly constant, it never vanishes and, in particular, can be diagonalized globally such that the bundle E splits as a sum of line bundles [18]. Now equation (2.15) reduces to

$$d\phi_3 = 0, \quad [A_\mu, \phi] = 0$$

the second equation implies that A_μ is in the same sub-algebra as ϕ, in particular, $F_{\mu\nu}$ looks like

$$F^{\mu\nu} = \frac{1}{2} F_3^{\mu\nu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $F_3^{\mu\nu}$ is the $U(1)$ curvature. Now the instanton equation turns into an equation for $U(1)$ instantons

$$F_3^+ = 0. \tag{2.16}$$

Note that $dF_3 = 0$ by the Bianchi identity. Since $F_3 = -(*F_3)$ by the above equation, it follows that $d^\dagger F_3 = 0$. Thus F_3 is the representative of a class in the second de Rham cohomology group, $F_3 \in H^2(X, \mathbf{R})$. However, the cocycle condition on transition functions puts an extra condition on the curvature of the bundle (the Dirac quantization law)

$$\int_\Sigma F_3 = 2\pi n$$

for some integer n. Σ is any nontrivial two-cycle in X. Thus in fact $F_3 \in H^2(X, \mathbf{Z})$. On the other hand, as the self-dual part of F_3 is zero by eq. (2.16), we have also $F_3 \in H^2_-(X, \mathbf{R})$. Altogether we conclude that F_3 lies in the intersection of the two cohomology groups:

$$F_3 \in H^2(X, \mathbf{Z}) \cap H^2_-(X, \mathbf{R}).$$

Let b_2^+ denote the dimension of the space of self-dual harmonic two-forms on X. If $b_2^+ = 0$, then $H^2(X, \mathbf{R}) = H^2_-(X, \mathbf{R})$. So any harmonic two-form which satisfies the Dirac quantization law is an instanton (a solution to (2.16)). On the other hand, one can argue that on four manifolds with $b_2^+ = 1$, there is a wall in the space of one-parameter metrics on which abelian instantons appear [21, 22].

Therefore to get rid of the reducible connections, in the case that the gauge group is $SU(2)$ [2], we must restrict our attention to those manifolds with $b_2^+ > 1$. Having this condition guarantees that there are no abelian instantons (or equivalently there are no reducible connections). On Kähler manifolds, however, this condition has more implications as we will see in the next part.

[2] For other gauge groups like $SU(3)$ one needs further restrictions on the bundle E for not having reducible connections.

2.2.5 Existence of zero modes on Kähler manifolds

In this part we will discuss the conditions for not having $\chi_{\mu\nu}$ zero modes. We will see that on Kähler manifolds with $b_2^+ > 1$, in the field of instantons, there are neither η nor $\chi_{\mu\nu}$ zero modes. This, in particular, implies that the formal dimension of $\mathcal{M}$ is equal to its real dimension. Then we discuss the Feynman diagrams which survive in the scaling limit of $e \to 0$.

Let us first discuss the zero modes of $\chi_{\mu\nu}$ field in this setting. The equations are

$$\begin{aligned} F^+_{\mu\nu} &= 0 \\ D^\mu \chi_{\mu\nu} &= 0 \,. \end{aligned} \tag{2.17}$$

On a complex four-manifold, any two-form can be decomposed into components which are a complex $(2,0)$-form, a $(0,2)$-form and a $(1,1)$ form. The $*$ operator maps an (r,s)-form to an $(2-s, 2-r)$-form. In fact, we can see that any $(2,0)$-form ($(0,2)$-form) gets mapped to itself under the $*$ operation. Therefore, we can decompose a self-dual two-form χ as

$$\chi = \chi^{(2,0)} + \chi^{(0,2)} + \chi^{(1,1)+} \,,$$

where $+$ indicates the self-dual part. On a Kähler manifold, however, the $(1,1)$ part of a self-dual two-form is proportional to the Kähler form k. This is easy to see. Let us take the local complex coordinate system of $z^\alpha, z^{\bar\alpha}$. In this local coordinate, the Kähler form simply takes the form $k_{\alpha\bar\beta} = i g_{\alpha\bar\beta}$ and we can write

$$\epsilon_{\alpha\bar\beta\gamma\bar\rho} = \sqrt{g}(k_{\alpha\bar\beta}k_{\gamma\bar\rho} - k_{\gamma\bar\beta}k_{\alpha\bar\rho}) \,,$$

therefore, $\chi^{(1,1)+}$ can be written as

$$\chi_{\alpha\bar\beta} = \frac{1}{\sqrt{g}}\epsilon_{\alpha\bar\beta\gamma\bar\rho}\chi^{\gamma\bar\rho} = (k_{\alpha\bar\beta}k_{\gamma\bar\rho} - k_{\gamma\bar\beta}k_{\alpha\bar\rho})\chi^{\gamma\bar\rho} = k_{\alpha\bar\beta}(k_{\gamma\bar\rho}\chi^{\gamma\bar\rho}) + \chi_{\bar\beta\alpha} \,. \tag{2.18}$$

Thus, on a Kähler manifold, as k nowhere vanishes globally we can write

$$\chi^{(1,1)+} = k\tilde\chi \,,$$

where $\tilde\chi \equiv \frac{1}{2}k_{\gamma\bar\rho}\chi^{\gamma\bar\rho}$ is a scalar of ghost number -1. Further, we learn that if $\chi^{(1,1)+}$ is closed, $\tilde\chi$ must be a constant as the Kähler form is closed. This implies that the only nontrivial cohomology class of $H^{(1,1)+}(X,\mathbf{R})$ is the Kähler form. Hence, on a Kähler manifold we have

$$b^{(1,1)+} = 1\,, \tag{2.19}$$

where $b^{(1,1)+}$ is the dimension of the group $H^{(1,1)+}(X,\mathbf{R})$.

The self-dual part of F can be decomposed similarly. Upon employing the complex coordinate system $\mu = (\alpha, \bar\alpha)$, the eqs. (2.17) read

$$\begin{aligned} &F_{\alpha\beta} = F_{\bar\alpha\bar\beta} = 0\ , \quad k^{\alpha\bar\beta}F_{\alpha\bar\beta} = 0 \\ &D^{\alpha}\chi_{\alpha\beta} + k_{\bar\alpha\beta}D^{\bar\alpha}\tilde\chi = 0\,. \end{aligned}$$

Squaring the last equation we get ($k_{\alpha\bar\beta} = ig_{\alpha\bar\beta}$)

$$0 = \int \mathrm{tr}\left\{|D^{\alpha}\chi_{\alpha\beta}|^2 + |k_{\bar\alpha\beta}D^{\bar\alpha}\tilde\chi|^2 - iD_{\alpha}\chi^{\alpha\beta}D_{\beta}\tilde\chi + iD_{\bar\alpha}\chi^{\bar\alpha\bar\beta}D_{\bar\beta}\tilde\chi\right\}\,. \tag{2.20}$$

However, using the fact that $F_{\alpha\beta} = 0$, we see that

$$\int \mathrm{tr}\left\{D_{\alpha}\chi^{\alpha\beta}D_{\beta}\tilde\chi\right\} = -\frac{1}{2}\int \mathrm{tr}\left\{\chi^{\alpha\beta}[D_{\alpha}, D_{\beta}]\tilde\chi\right\} = 0\,,$$

and thus eq. (2.20) implies

$$D^{\alpha}\chi_{\alpha\beta} = 0 \tag{2.21}$$

$$D_{\alpha}\tilde\chi = D_{\bar\alpha}\tilde\chi = 0\,. \tag{2.22}$$

Since $b_2^+ > 1$ there are no abelian instantons, so eq. (2.22) above implies $\tilde\chi$ must be zero. The equation (2.21) and the equation $F^+ = 0$ are invariant under a $U(1)$ phase transformation

$$\chi_{\alpha\beta} \to e^{i\theta}\chi_{\alpha\beta} \tag{2.23}$$

$$A_{\mu} \to A_{\mu}\,. \tag{2.24}$$

Thus we have to only consider those solutions which are invariant under (2.23), up to a gauge transformation; i.e.,

$$\begin{aligned} &e^{i\theta}\chi_{\alpha\beta} = g(\chi_{\alpha\beta})g^{-1} \\ &A = gAg^{-1} + igdg^{-1}\,. \end{aligned} \tag{2.25}$$

However, the second equation implies that the gauge connection (which in turn is an instanton by the first equation in (2.17)) is reducible – contradicting the fact that on manifolds with $b_2^+ > 1$ there are no abelian instantons. Therefore $\chi_{\alpha\beta}$ must be zero. We conclude that, at least on Kähler manifolds with $b_2^+ > 1$, there are no $\chi_{\mu\nu}$ zero modes. The only fermionic zero modes are those of ψ_μ which are tangent to $\mathcal{M}$.

The case $d(\mathcal{M}) = 0$

As was mentioned the moduli space in this case (if not empty) consists of a discrete set of instanton solutions. The only topological invariant that can be computed is the partition function. In the following we will see that, in the expansion around a single instanton, we need only to keep the quadratic terms in the Lagrangian. The other interaction terms (not present in the kinetic terms) can be ignored, since their contributions die as $e \longrightarrow 0$. For instance, we can pull down the interactions $\phi[\chi_{\mu\nu}, \chi^{\mu\nu}]$ together with $\bar{\phi}[\psi_\mu, \psi^\mu]$ and contract the different fields by replacing them by their corresponding propagators (notice that there are no zero modes). Each propagator gives a factor of e^2 and since we get $1/e^2$ for each vertex, the result scales as e^2 – contradicting the fact that theory is coupling constant independent – thus it must vanish. Therefore all one has to consider are the quadratic terms (expanded around an instanton), which – upon performing the path integral – cancel each other up to a sign. The correlation functions of any other BRST cohomology class vanish simply because there is no ghost number anomaly in this case. So if $d(\mathcal{M}) = 0$, the only topological invariant is the partition function which can be computed by expanding the Lagrangian around instantons and keeping only the quadratic terms.

The case $d(\mathcal{M}) > 0$

In this case we have fermionic, ψ_μ, zero modes. Since these cannot be saturated by the interaction terms already present in the action, the partition function vanishes. Therefore, as mentioned earlier, to saturate the fermionic zero modes we need to insert an operator $\mathcal{O}$ with the $U(1)_\mathcal{R}$ charge equal to $d(\mathcal{M})$.

As an example, let us assume that the dimension of the moduli space is two and try to soak up the zero modes by the interaction terms in the Lagrangian. If we pull down the term $\bar{\phi}[\psi_\mu, \psi^\mu]$ to soak up the two zero modes, to get a nonzero value, we need also to pull down either $\phi[\chi_{\mu\nu}, \chi^{\mu\nu}]$ term or $\phi[\eta, \eta]$ to replace $\phi\bar{\phi}$ by its propagator. But, there is no propagator for the $\eta\eta$ (or $\chi_{\mu\nu}\chi^{\mu\nu}$) system, nor do these have zero modes to be soaked up. Thus the zero modes cannot be absorbed by the interaction terms and the partition function vanishes. On the other hand, we do get a nonvanishing quantity if we instead absorb the two zero modes by inserting the operator ϕ^a. Pulling down the term $\bar{\phi}[\psi_\mu, \psi^\mu]$, and replacing the $\bar{\phi}\phi$ by its propagator, we can see that this term is both coupling constant independent and invariant under metric scaling. Moreover the two zero modes are absorbed by the two fermions.

2.2.6 Integrating out nonzero modes and the integral over $\mathcal{M}$

Putting the condition $b_2^+ > 1$ on the manifold guarantees that there are no reducible connections and thus the kinetic term for the ϕ field is nondegenerate. Therefore, in expanding around instantons and keeping just the quadratic terms, the path integral over ϕ can be done. In this part, we explain this idea by giving an example. Later more general cases are studied.

To begin with, we proceed to calculate the vacuum expectation value (vev) of the operator ϕ^a (index a is the Lie-algebra index). Further, as before, we assume that the dimension of moduli space is 2; i.e., there are only two fermionic zero modes.

First introduce external sources for every fermionic and bosonic field. Take a solution to the fixed point equations (2.8) and expand the Lagrangian around this solution up to quadratic order. Quadratic order would be sufficient since we note that theory is coupling independent and we may go to an arbitrary weak coupling limit. After this, the standard techniques of perturbative quantum field theory can be applied; we pull out the interaction terms from the path integral by simply replacing fields by the derivatives with respect to the corresponding source and then do the gaussian path integral over nonzero modes. This leaves us with an expression which looks like

$$\langle \phi^a(x) \rangle = \frac{\mathrm{Pf}\, D_F}{\sqrt{\det \Delta_B}} \frac{\delta}{\delta J^a(x)} \exp\{V(\delta/\delta J, \delta/\delta\eta)\} \int d\mu\, e^{-\int \left(J\Delta_B^{-1}J + \eta D_F^{-1}\eta + \psi_0\eta\right)}, \quad (2.26)$$

where J and η are sources for the bosonic and fermionic fields respectively and the subscript 0 stands for zeromodes. $d\mu$ is the measure for zero modes

$$d\mu = da_1 da_2 d\psi_1 d\psi_2\,,$$

where a_1, a_2 are the bosonic instanton moduli parameters, while ψ_1 and ψ_2 are the fermionic ones. Now because of supersymmetry, the two operators Δ_B and D_F have the same nonzero spectrum. Therefore the determinants cancel against each other up to a sign. Clearly, since there are no ϕ zero modes, the zero order in this expansion (i.e., without pulling down any interaction term) gives zero and we need to go to the next order. To get a nonzero value we have to pull down the interaction term (in which fields are replaced by functional derivatives) $\bar{\phi}[\psi_\mu, \psi^\mu]$ from the Lagrangian. In this way, $\phi\bar{\phi}$ gets replaced by the propagator Δ_B^{-1}. Fermionic zeromodes are then soaked up by the term $[\psi_\mu, \psi^\mu]$. After all we will have

$$\langle \phi^a(x) \rangle = -i \int d\mu \, d^4y \sqrt{g} \; G^{ab}(x,y)[\psi_\mu, \psi^\mu]^b \tag{2.27}$$

where factors of e have canceled out indicating the coupling independence of this correlator. $G^{ab}(x,y)$ is the inverse operator of Δ_B. After integrating out nonzero modes of ϕ the resulting expression is an integral over the moduli space $\mathcal{M}$.

So in general we start with a nontrivial cohomology class of Q,

$$\mathcal{O}_k = \int_{\gamma_k} W_k \, ,$$

where γ_k is an k-cycle on M and W_k is an operator with ghost charge $U = 4 - k$. In computing the vacuum expectation of $\mathcal{O}_k$, we first integrate over the nonzero modes of ϕ as above. This leaves us with a U-form operator of the general form

$$\mathcal{O}' = \Phi_{i_1 \ldots i_n}(a^k) \, \psi^{i_1} \cdots \psi^{i_n}$$

where ψ^i are fermion zero mode coordinates and Φ is an n-form on $\mathcal{M}$. Performing the Grassmann integrals over fermionic zero modes, we are left with the integral over bosonic zero modes

$$\langle \mathcal{O}_k \rangle = \int_{\mathcal{M}} \Phi_k \, .$$

Therefore, by integrating nonzero modes out in the weak coupling limit, the path integral reduces to a finite dimensional integral over the moduli space $\mathcal{M}$. However, the tangent vectors to the space $\mathcal{M}$ are ψ zero modes which are in turn the variation of an instanton under Q. Therefore, we can conclude that in the weak coupling limit Q acts on the moduli space $\mathcal{M}$ as an exterior derivative. The fact that $\mathcal{O}_k$ are BRST closed, after integration over nonzero modes in the weak coupling limit, translates to the fact that $\mathcal{O}'_k$ are closed forms on $\mathcal{M}$. In the same manner we can see that the BRST exactness of $\mathcal{O}_k$ translates to the triviality of $\mathcal{O}'_k$ on $\mathcal{M}$. We conclude that the integration over nonzero modes indeed maps us from the homology of M to the cohomology of $\mathcal{M}$, i.e.

$$H_k(M) \to H^{4-k}(\mathcal{M}).$$

This procedure can be easily extended to compute the correlation function of n such operators [1]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_n$$

where Φ_i is an $(4-i)$-form on $\mathcal{M}$ such that $\sum_i (4-i) = d(\mathcal{M})$. This formula establishes the relation between twisted $N = 2$ supersymmetric Yang-Mills theory on one hand and the Donaldson theory on the other hand; probing the topology of M by studying the intersection theory on the space $\mathcal{M}$. Using standard techniques in quantum field theory as above, the integration over nonzero modes can be done to reduce the path integral to a finite dimensional integral over $\mathcal{M}$. However, the real task in the calculation of Donaldson invariants is the determination of zeromodes and the integration over them. This turns out to be extremely difficult and any attempt in this direction seems hopeless. However, the fact that these finite integrals which probe the topology of $\mathcal{M}$ are simply particular correlation functions in twisted SYM theory opens a new window for attacking the problem using some well known facts about SYM theories. To see how this comes about we briefly discuss those features of $N = 2$ and $N = 1$ SYM theories which play a key role in the determination of Donaldson invariants.

2.2.7 Mass gap and the twisted theory on Kähler manifolds

So far we have not seriously tried to compute the Donaldson invariants. Mathematically this is a formidable task and except on some specific manifolds Donaldson invariants have not been calculated directly. However, in [2] Witten showed how, in a theory with mass gap, one can compute the Donaldson invariants using physical arguments. In a theory with mass gap, since there are no massless states quantum mechanically, it is ensured that a local expression for the physical observables emerges if one uses the mass of the lightest state to expand perturbatively. In this section first we discuss the absence of a mass gap in $N = 2$ SYM theory. Then we look at the perturbation to $N = 1$ SYM theory and the corresponding twisted theory on Kähler manifolds. The role of the mass gap in the computation of Donaldson theory, and the simplifications that occur in this case, are also discussed.

The fact that $N = 2$ SYM theory does not have a mass gap follows from the work of Seiberg and Witten [13] who showed that the effective low energy theory is indeed a $N = 2$ SYM theory with the $U(1)$ gauge group. The existence of an unbroken $U(1)$ gauge symmetry at low energies shows that there are massless states in the spectrum and the theory does not have a mass gap. In contrast to $N = 2$ SYM theory, $N = 1$ SYM theory is believed to have a mass gap [13] mainly because there are no exact global symmetries preventing fermions from developing a mass.

Since the existence of mass gap is essential in computations via physical arguments (see below), it is important to ask whether one can twist the $N = 1$ SYM theories, or alternatively, whether $N = 2$ twisted theory can be perturbed to have a mass gap without destroying the topological character of the theory.

Regarding the first possibility, we note that, in contrast to $N = 2$ theory, the $N = 1$ SYM theory cannot be defined on a general four-manifold for two reasons. Firstly, pure $N = 1$ SYM theory has a global $U(1)_{\mathcal{R}}$ symmetry which is anomalous and thus cannot be used for twisting. Secondly, even if we try to make this $U(1)_{\mathcal{R}}$ symmetry nonanomalous (by adding more $N = 1$ matter multiplets) it is not large enough for twisting on a general manifold. However, there exists a large class of manifolds on which we can define these $N = 1$ SYM theories by twisting. These are Kähler manifolds, for which the holonomy group is

$$SU(2)_L \times U(1)_R\,,$$

where $U(1)_R$ is a subgroup of $SU(2)_R$ in (2.5). The $U(1)_R$ part of the holonomy group can now be twisted with the global $U(1)_{\mathcal{R}}$ symmetry to produce scalar supercharges [23].

Let us now return to the problem of perturbing the $N = 2$ SYM theory such that it has a mass gap and is still topological. First let us see how twisting works in the case of a Kähler manifold. To twist the theory, we choose a $U(1)_J$ subgroup of $SU(2)_I$ under which spinors transform as

$$\lambda \rightarrow e^{i\beta}\lambda$$
$$\psi \rightarrow e^{-i\beta}\psi$$

and define the new symmetry group to be

$$SU(2)_L \times U(1)'_R\,,$$

where $U(1)'_R$ is obtained by adding the charges of $U(1)_J$ and $U(1)_R$ global symmetries. Therefore, on Kähler manifolds twisting has the following effect

$$\begin{array}{lcllcl}
\lambda_{\alpha 1} & \sim & (2,0,\frac{1}{2}) & \longrightarrow & (2,\frac{1}{2}) \\
\lambda_{\alpha 2} & \sim & (2,0,-\frac{1}{2}) & \longrightarrow & (2,-\frac{1}{2}) \\
\bar{\lambda}_{\dot{1}1} & \sim & (0,\frac{1}{2},\frac{1}{2}) & \longrightarrow & (0,1) \\
\bar{\lambda}_{\dot{2}1} & \sim & (0,-\frac{1}{2},\frac{1}{2}) & \longrightarrow & (0,0) \\
\bar{\lambda}_{\dot{1}2} & \sim & (0,\frac{1}{2},-\frac{1}{2}) & \longrightarrow & (0,0) \\
\bar{\lambda}_{\dot{2}2} & \sim & (0,-\frac{1}{2},-\frac{1}{2}) & \longrightarrow & (0,-1)\,.
\end{array} \tag{2.28}$$

Hence on a Kähler manifold we indeed get two scalar supercharges by twisting.

Let us now perturb the theory by giving a mass term to the Φ multiplet through

$$m\int d^2z d^2\bar{z} d^2\theta \ \mathrm{tr}\,\Phi^2 + h.c. \tag{2.29}$$

This reduces the number of supersymmetries to one (leaving one scalar supercharge unbroken) and leaves a pure $N=1$ supersymmetric effective theory (with the fields A_μ and λ) at energies which are small compared to the mass of Φ. Since we know that $N=1$ theory has mass gap, it follows that $N=2$ theory perturbed by a mass term for the Φ multiplet also has a mass gap.

The procedure of mass perturbation on a general Kähler manifold, however, is not so simple as it may look like. Unlike the action, which contains $d^4xd^4\theta$ as its measure and thus can be defined naturally on a general manifold (i.e., without recourse to a specific coordinate system), the measure for the mass term , $d^4xd^2\theta$, transforms nontrivially under the holonomy group (note according to (2.28), though invariant before twisting, $d^2\theta$ carries one unit of $U(1)_R$ charge after twisting), if we try to generalize the above term on a curved manifold. The remedy for this [2] is to consider that $d^2\bar{z}d^2\theta$ has charge zero under the holonomy group and so is defined naturally. The factor md^2z then can be interpreted as a $(2,0)$-form. Hence, to define the mass term on a Kähler manifold we pick a holomorphic two-form ω as a generalization of md^2z on flat $\mathbf{R}^4$ and declare

$$\int \omega \wedge d^2\bar{z} d^2\theta \ \mathrm{tr}\,\Phi^2 + h.c.$$

to be the mass term as a generalization of (2.29). Therefore, requiring that the mass term be defined on a Kähler manifold implies $H^{(2,0)}(X,\mathbf{R}) \neq 0$. On a Kähler manifold, we have $b^{2,0} = b^{0,2}$, with $b^{2,0}$ being the dimension of $H^{(2,0)}$ and $b^{(1,1)+} = 1$ (see eq. (2.19)). Hence, since the $*$ operator maps $\Omega^{(2,0)}(X,\mathbf{R})$ $(\Omega^{(0,2)}(X,\mathbf{R}))$ to itself, we will have $b_2^+ = 1 + 2b^{2,0}$, which implies that, on a Kähler manifold, $b_2^+ > 1$ – a condition which we have seen before.

Having defined the mass term on a Kähler manifold, let us now see why the mass deformation and the consequent existence of a mass gap, are so important in computations of Donaldson theory via physical arguments.

Consider, for example, the partition function of a theory which has a mass gap. In such a theory, in principle, it is possible to write an effective action for the background gravitational field by expanding around a flat metric. Since there is no massless state in the spectrum of the theory, it is guaranteed that a local expression for the effective action emerges.

Following we consider the one parameter family of metrics, $g_{\mu\nu} \rightarrow t^2 g_{\mu\nu}$ and take t to be arbitrarily large. As the theory is asymptotically free, this limit corresponds to low energies and we may use the mass of the lightest state as a perturbative expansion parameter. Therefore. the general expression for the effective action which emerges is an expansion in terms of successively decreasing powers of t (or decreasing powers of energy) [2]

$$Z = \exp(-L_{\text{eff}}),$$

where

$$L_{\text{eff}} = \int d^4x\sqrt{g}(u + vR + wR^2 + \cdots). \tag{2.30}$$

Here R is the scalar curvature. Note that $\sqrt{g}$ scales as t^4, while the Riemann tensor, $R^\rho{}_{\mu\nu\sigma}$, and thus the Ricci tensor do not scale. Thus $R = g^{\mu\nu}R_{\mu\nu}$ scales as t^{-2}. Therefore, terms shown in (2.30) scale as t^4, t^2 and t^0 respectively. However, the topological invariance of the partition function means that the only local operators that may appear are dimension four. On a four manifold the only topological invariants which can be written as an integral of local operators are the Euler characteristic χ and the signature σ

$$\chi = \int R \wedge R$$

$$\sigma = \int R \wedge *R.$$

Thus

$$Z = \exp(a\chi + b\sigma),$$

for some universal constants a and b. Hence, all remains is to work out the universal coeficients a and b say by comparing to some known results.

2.2.8 Effective low energy description of Donaldson invariants

Although the effective low energy description of the theories which we are going to study does not enter in this thesis, we briefly discuss the low energy description of the Donaldson theory as it is of current interest and has played an important role in determining the basic structure of the Donaldson invariants via introducing some more fundamental invariants.

Shortly after the work of Seiberg and Witten [13], Witten [5] provided an alternative approach to the computation of Donaldson invariants using the low energy effective description of $N = 2$ SYM theory. In [13] it was shown that the effective low energy theory of minimal $N = 2$ SYM theory with gauge group $SU(2)$ is an $N = 2$ SYM theory with the gauge group $U(1)$. Since the microscopic theory is asymptotically free, this weakly coupled effective theory which emerges in the infrared corresponds to the strongly coupled region of field space. There are two important characteristics about this effective description. Firstly, there is no unique effective Lagrangian describing the low energy physics in the whole region of the vacuum manifold – rather, different Lagrangian descriptions are related to one another by $SL(2, \mathbf{Z})$ transformations. Secondly, in the moduli space of vacua, there exist two singular points where a monopole (or dyon) becomes massless and thus the effective low energy description breaks down at these points.

Using this picture to determine the Donaldson invariants, one sums over all contributions coming from the different parts of the moduli space of vacua [2, 24]. As mentioned earlier, this moduli space consists of the whole complex plane with two singularities at say 1 and -1 and it is called sometimes the u-plane. In [24] it has been argued that, upon considering manifolds with $b_2^+ > 1$, there is no contribution to the path integral from any region of the u-plane bounded away from the singularities. This happens mainly because in this case ($b_2^+ > 1$) there are too many fermionic zero modes that cannot be lifted. Thus for manifolds with $b_2^+ > 1$ the only contributions are coming from those two singular points. Manifolds for which the contribution of the u-plane away from the singularities vanishes are called *simple type*. So the form of the invariants for this class of manifolds is now clear; there is a contribution coming from integrating out the heavy fields as well as the high energy modes, there is no subtle topological information in this part and it can simply be worked out by comparing with some known results. The remaining part is the contribution of the massless modes at the singular points. As discussed above, these are a dual $U(1)$ gauge field, a monopole, and of course their supersymmetric partners. In terms of $N = 2$ multiplets, these are just a gauge multiplet

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array} , \tag{2.31}$$

and a hypermultiplet in the fundamental representation of the gauge group describing the monopole

$$\begin{array}{ccc} & \psi_M & \\ M & & \tilde{M} \\ & \tilde{\psi}_M & \end{array} . \tag{2.32}$$

This is the physical field content. In order to define the theory on a general manifold, we still need to twist the above theory. After twisting [25], we obtain from the gauge multiplet – similar to the microscopic description of Donaldson theory – a set of ghost and antighost fields

$$\psi_\mu \ , \ \chi_{\mu\nu} \ , \ \eta \, .$$

However, since the scalars in the hypermultiplet transform as $(1,1,2)$ under $SU(2)_L \times SU(2)_R \times SU(2)_I$ (note that ψ_M and $\tilde{\psi}_M$ are invariant under $SU(2)_I$), after twisting these turn into

$$\begin{aligned} M &\sim (1,1,2) \to (1,2) \\ \tilde{M} &\sim (1,1,2) \to (1,2) \, . \end{aligned}$$

In summary, first we take the $U(1)$ $N=2$ SYM theory with an additional matter multiplet on $\mathbf{R}^4$. Then we redefine the fields by the above twisting prescription. The twisted Lagrangian obtained this way (albeit with adding a multiple of $\int F \wedge F$ to make it BRST trivial) could, in principle, be defined on an arbitrary manifold preserving the BRST symmetry. However, following the above steps, we find out that the twisted theory is not supersymmetric (and indeed cannot be written as a BRST commutator) on a general curved manifold unless we add an extra term proportional to

$$\int d^4x \sqrt{g} R |M|^2$$

to the action. Doing this, it can be seen that in the weak coupling limit the path integral localizes to the moduli space of solutions of the Seiberg-Witten (SW) equations [5]

$$\begin{aligned} F^+_{\mu\nu} &= -\frac{i}{2}\bar{M}\Gamma_{\mu\nu}M \\ \not{D}M &= 0\,. \end{aligned} \tag{2.33}$$

Therefore, the effective low energy description of the microscopic theory has given a new perspective to the computation of Donaldson invariants. Instead of considering the nonabelian equations of instantons to obtain the Donaldson invariants, we can consider the SW equations and derive the same invariants. Indeed, it has been shown [5] that all the subtle topological information about Donaldson invariants are encoded into the SW invariants.

Moreover, the SW equations have much nicer properties than the self-dual instanton equations [5]. Firstly, they have a $U(1)$ gauge invariance which is easier to deal with than the $SU(2)$ invariance of the instanton equations. Secondly, the moduli space of solutions to the SW equations is compact in contrast to that of instantons where compactness fails and one has to compactify the moduli space to ensure a genuine topological behavior. However, as in Donaldson theory, one needs to restrict to manifolds with $b_2^+ > 1$ to have a free action of the gauge group on the moduli space of SW equations.

2.3 $N = 4$ SYM theory and its twisting

We have already discussed in the introduction those properties of $N=4$ theory which can be examined through topological field theory via twisting. In this section we concentrate on a particular twisting of the theory which will be studied more thoroughly in chapter four for a specific product manifold $\Sigma \times S^2$. We start with a brief study of the physical theory. The global symmetry of the model is so large that admits different embeddings of the space-time symmetry and thus resulting to different twistings of the model. Here we look at one of these possible twistings and the related equations.

2.3.1 The physical theory

The fields of $N = 4$ SYM theory can be arranged in terms of $N = 2$ multiplets. Like $N = 2$ with an additional hypermultiplet, the field content of $N = 4$ SYM theory consists of a gauge multiplet as in (2.31) and a hypermultiplet as in (2.32) with the difference that now the hypermultiplet is in the adjoint representation of the gauge group. Hence, we expect to have an increased supersymmetry rotating the gauge and hypermultiplet into each other. Moreover, the internal $\mathcal{R}$-symmetry is also increased from $U(2)$ to $SU(4)$. Spinors, $\lambda_{\alpha i}$, sit in the fundamental representation **4** of $SU(4)$ and scalars, ϕ_{ij}, in the 6-dimensional representation. i and j are $SU(4)$ indices and ϕ_{ij} are components of a real self-dual 2-form.

The $N = 4$ supersymmetric Lagrangian on flat $\mathbf{R}^4$ is [26]

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{e^2}\mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda^{\alpha}{}_{i}(\sigma_\mu)_{\alpha\dot\alpha}D^{\mu}\bar\lambda^{\dot\alpha i} - \frac{1}{4}D_\mu\phi_{ij}D^{\mu}\phi^{ij}\right.\\ & \left. - \frac{i}{\sqrt{2}}\lambda^{\alpha}{}_{i}[\lambda_{\alpha j},\phi^{ij}] + \frac{i}{\sqrt{2}}\bar\lambda_{\dot\alpha}{}^{i}[\bar\lambda^{\dot\alpha j},\phi_{ij}] + \frac{1}{16}[\phi_{ij},\phi_{kl}]^2\right] .\end{aligned}$$

The action is invariant under $N = 4$ supersymmetric transformations:

$$\begin{aligned}\delta A_\mu &= -i\bar\xi_{\dot\alpha}{}^{i}(\sigma_\mu)^{\alpha\dot\alpha}\lambda_{\alpha i} + i\bar\lambda_{\dot\alpha}{}^{i}(\sigma_\mu)^{\alpha\dot\alpha}\xi_{\alpha i}\\ \delta\lambda_{\alpha i} &= -i(\sigma^{\mu\nu})_{\alpha}{}^{\beta}F_{\mu\nu}\,\xi_{\beta i} + i\sqrt{2}\bar\xi^{\dot\alpha j}(\sigma^{\mu})_{\alpha\dot\alpha}D_\mu\phi_{ji} - i\xi_{\alpha k}[\phi_{ij},\phi^{jk}]\\ \delta\phi_{ij} &= \sqrt{2}\left(\xi^{\alpha}{}_{i}\lambda_{\alpha j} - \xi^{\alpha}{}_{j}\lambda_{\alpha i} + \varepsilon_{ijkl}\bar\xi_{\dot\alpha}{}^{k}\bar\lambda^{\dot\alpha l}\right) .\end{aligned}$$

The global symmetry of the action is

$$SU(2)_L \times SU(2)_R \times SU(4) .$$

2.3.2 The twisted model

From what we saw in the $N = 2$ case, it is now clear that to twist the theory, first we should choose a $SU(2)$ subgroup of $SU(4)$ and replace $SU(2)_R \times SU(2)$ by its diagonal subgroup just as we did in the $N=2$ case. Depending on how one chooses this subgroup, one gets different topological field theories. There are three different embeddings [4] of $SU(2) \times SU(2)$ into $SU(4)$ which give rise to singlet supercharges under the twisting. Under these embeddings of $SU(2) \times SU(2)$ subgroup, spinors decompose as

$$
\begin{aligned}
i)\quad & \mathbf{4} \rightarrow (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \\
ii)\quad & \mathbf{4} \rightarrow (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2}) \\
iii)\quad & \mathbf{4} \rightarrow (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \,.
\end{aligned} \tag{2.34}
$$

Let us concentrate on the second case for which scalars trnsform as

$$\mathbf{6} \rightarrow (\mathbf{1},\mathbf{3}) \oplus 3(\mathbf{1},\mathbf{1}) \,.$$

Replacing $SU(2)_R$ by a diagonal subgroup of $SU(2)_R \times SU(2)$, spinors transform as two scalars, two self-dual 2-forms and two vectors under the newly defined space-time symmetry group

$$
\begin{aligned}
(\mathbf{1},\mathbf{2},((\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2}))) &\longrightarrow (\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3},\mathbf{2}) \\
(\mathbf{2},\mathbf{1},((\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2}))) &\longrightarrow (\mathbf{2},\mathbf{2},\mathbf{2})
\end{aligned}
$$

just two copies of the fermionic (ghost and anti-ghost) field content in the twisted $N=2$ theory. For scalars we have

$$(\mathbf{1},\mathbf{1},((\mathbf{1},\mathbf{3}) \oplus 3(\mathbf{1},\mathbf{1}))) \longrightarrow (\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \tag{2.35}$$

transforming as three scalars, say, $\phi, \bar{\phi}$ and C and a self-dual 2-form $B^+_{\mu\nu}$. Knowing how the new fields transform under the new space-time symmetry, we can write the Lagrangian in terms of these fields on flat $\mathbf{R}^4$. Upon covariantizing the derivatives we may extend the Lagrangian on a curved manifold. However, just as in the monopole case, this Lagrangian cannot be written as a BRST commutator unless a curvature term which reads [4]

$$\frac{1}{8e^2}\int d^4x\sqrt{g}\mathrm{tr}\left(B^{\mu\nu}\left(\frac{1}{6}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})R+W^+_{\mu\nu\rho\sigma}\right)B^{\rho\sigma}\right),$$

is added to the Lagrangian. Here W^+ is the self-dual part of the Weyl tensor,

$$W^+_{\mu\nu\rho\sigma}=\frac{1}{2}\left(W_{\mu\nu\rho\sigma}+\frac{1}{2}\varepsilon_{\mu\nu}{}^{\gamma\delta}W_{\gamma\delta\rho\sigma}\right),$$

where

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}\left(R_{\mu\rho}g_{\nu\sigma} - R_{\nu\rho}g_{\mu\sigma} + R_{\nu\sigma}g_{\mu\rho} - R_{\mu\sigma}g_{\nu\rho}\right) + \frac{R}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right).$$

By adding the above curvature term to the Lagrangian on flat $\mathbf{R}^4$, one obtains the twisted Lagrangian of the model [4, 26],

$$\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{e^2}\mathrm{tr}\left\{-D_\mu\lambda D^\mu\phi + \tfrac{1}{2}\tilde{H}^\mu(\tilde{H}_\mu - 2\sqrt{2}D_\mu C + 4\sqrt{2}D^\nu B_{\nu\mu})\right.\\
+\;& \tfrac{1}{2}H^{\mu\nu}(H_{\mu\nu} - 2F^+_{\mu\nu} - 4i[B_{\mu\rho}, B^\rho{}_\nu] - 4i[B_{\mu\nu}, C])\\
+\;& 4\psi_\mu D_\nu\chi^{\mu\nu} + 4\tilde{\chi}_\mu D_\nu\tilde{\psi}^{\mu\nu} + \tilde{\chi}_\mu D^\mu\zeta - \psi_\mu D^\mu\eta\\
+\;& i\sqrt{2}\tilde{\psi}^{\mu\nu}[\tilde{\psi}_{\mu\nu}, \lambda] - i\sqrt{2}\chi^{\mu\nu}[\chi_{\mu\nu}, \phi] + i2\sqrt{2}\tilde{\psi}^{\mu\nu}[\chi_{\mu\nu}, C] + i4\sqrt{2}\tilde{\psi}^{\mu\nu}[\chi_{\mu\rho}, B_\nu{}^\rho]\\
-\;& i\sqrt{2}\chi_{\mu\nu}[\zeta, B^{\mu\nu}] - i\sqrt{2}\tilde{\psi}_{\mu\nu}[\eta, B^{\mu\nu}] + i4\sqrt{2}\psi_\mu[\tilde{\chi}_\nu, B^{\mu\nu}] - i\sqrt{2}\tilde{\chi}_\nu[\tilde{\chi}^\nu, \phi]\\
+\;& i\sqrt{2}\psi_\mu[\psi^\mu, \lambda] - i2\sqrt{2}\psi_\mu[\tilde{\chi}^\mu, C] + \frac{i}{2\sqrt{2}}\zeta[\zeta, \lambda] - \frac{i}{2\sqrt{2}}\eta[\eta, \phi]\\
-\;& \left.\frac{i}{\sqrt{2}}\zeta[\eta, C] + 2[\phi, B^{\mu\nu}][\lambda, B_{\mu\nu}] + 2[\phi, C][\lambda, C] - \tfrac{1}{2}[\phi, \lambda]^2\right\}.
\end{aligned} \tag{2.36}$$

Upon integrating out the auxiliary fields, $\tilde{H}_\mu$ and $H_{\mu\nu}$, we can see that the resulting topological field theory, in the weak coupling limit, localizes on the moduli space of solutions to the following equations (Vafa-Witten equations):

$$\begin{aligned}
&F^+_{\mu\nu} + \frac{1}{2}[C, B^+_{\mu\nu}] + \frac{1}{4}[B^+_{\mu\rho}, B^+_{\nu\sigma}]g^{\rho\sigma} = 0\\
&D^\mu B^+_{\mu\nu} + D_\nu C = 0\,.
\end{aligned}$$

In [4], Vafa and Witten showed that, on Kähler manifolds with $R \geq 0$ and gauge group locally a product of $SU(2)$'s, a suitable vanishing theorem holds such that the solutions to the above equations all have $B^+_{\mu\nu} = C = 0$. In this case they argued that the partition function indeed computes the Euler characteristic of the moduli space of instantons. Further, by using the mathematical results on the structure of the moduli space of instantons, they computed the related Euler characteristic and showed that the partition function is in fact a modular form under S-duality, extending the previous conjectures about the weak-strong coupling duality of $N = 4$ SYM theory.

2.3.3 Twisted theory on $\Sigma \times C$

Twisted $N = 2$ and $N = 4$ SYM theories on product manifolds $\Sigma \times C$, where Σ and C are both Riemann surfaces, have been studied in [27]. There it has been shown that, upon shrinking C to zero size, the effective theory which generically emerges is a two-dimensional sigma model describing maps from Σ to $\mathcal{M}$. In the $N=2$ case, $\mathcal{M}$ is the moduli space of flat connections on C. For $N=4$ theory, it turns out that $\mathcal{M}$ is the moduli space of solutions to the Hitchin's equations [27]. In the following we overview the basic points. In chapter 4 we study the case where C is a Riemann sphere.

Let us consider the $N=2$ case in the limit of shrinking C. We denote the indices on Σ by $i,j,\cdots$ and those on C by $a,b,\cdots$. Upon scaling the metric on C by $g_{ab}\to\epsilon g_{ab}$, the bosonic part of the action (2.6) becomes

$$S=\tfrac{1}{4}\int d^4x\sqrt{g}\,\mathrm{tr}\Big[\epsilon F_{ij}F^{ij}+F_{ia}F^{ia}+\frac{1}{\epsilon}F_{ab}F^{ab}+2\epsilon\phi D_iD^i\bar{\phi}+2\phi D_aD^a\bar{\phi}-\frac{\epsilon}{2}[\phi,\bar{\phi}]^2\Big]\,.$$

This now shows that in the limit of $\epsilon\to 0$, the path integral localizes on the flat connections over C. At the level of equations, this can also be seen as follows. Consider the instanton equation on such a manifold. This manifold has the holonomy of $U(1)\times U(1)$ and thus is Kähler. So, as before, the self-dual two-form F^+, can be decomposed as

$$F^+=F^{(2,0)}+F^{(0,2)}+F^{(1,1)+}\,,$$

where $F^{(1,1)+}=\frac{1}{2}k(k_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}})$. Thus the instanton equation, $F^+=0$, on Kähler manifolds reduces to

$$\begin{aligned}&F^{(2,0)}=F^{(0,2)}=0\\&k_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}}=0\,,\end{aligned}$$

as $k_{\alpha\bar{\beta}}=ig_{\alpha\bar{\beta}}$, the second equation is

$$g^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}}=0\,. \tag{2.37}$$

Let $z,\bar{z}$ and $w,\bar{w}$ indicate the complex coordinate on Σ and C respectively. Then eq. (2.37) becomes

$$g^{z\bar{z}}F_{z\bar{z}}+g^{w\bar{w}}F_{w\bar{w}}=0\,.$$

Let us now shrink C by scaling its metric, $g_{w\bar{w}}\to\epsilon g_{w\bar{w}}$, and taking the limit $\epsilon\to 0$. In this limit, the above equation reduces to the equation of flat connections over C,

$$F_{w\bar{w}}=0\,. \tag{2.38}$$

To get rid of the reducible flat connections, consider those bundles which restrict nontrivially over C. The solutions to (2.38) can be parametrized by the moduli parameters which in general depend on the coordinates of Σ. Therefore, a flat connection on C can be written as

$$A_C(w,\bar{w};z,\bar{z}) = A_C(w,\bar{w};X(z,\bar{z}))\,,$$

where X^I are coordinates on the moduli space of flat connections $\mathcal{M}$. The tangent space to $\mathcal{M}$ can be found by varying A_C to a nearby flat connection

$$0 = F_C(A_C+\delta A_C) = F_C(A_C) + d(\delta A_C) + A_C\wedge\delta A_C + \delta A_C\wedge A_C = D_C(\delta A_C)\,. \quad (2.39)$$

As usual, we are interested in those variations of A_C which cannot be obtained by a gauge transformation, i.e., $\delta A_C \neq D_C\beta$. On the other hand, $F_C = 0$ implies that $D_C^2 = 0$, therefore $\delta A_C \in T_{A_C}\mathcal{M}$ belongs to the first cohomology group of the operator D_C

$$\delta A_C \in H^1(E, D_C) \equiv \ker D_C/\mathrm{im}\, D_C\,.$$

As before, this is related to the virtual dimension of moduli space of flat connections. Since $D_C^2 = 0$, we can define the following elliptic complex

$$0 \xrightarrow{i} \Omega^0(\mathbf{g}) \xrightarrow{D_C} \Omega^1(\mathbf{g}) \xrightarrow{D_C} \Omega^2(\mathbf{g}) \xrightarrow{\varphi} 0\,. \quad (2.40)$$

The linearized equation (2.39) fits in the two-term elliptic complex of

$$D\oplus D^\dagger : \Omega^1(\mathbf{g}) \longrightarrow \Omega^0(\mathbf{g})\oplus\Omega^2(\mathbf{g})\,.$$

The virtual dimension of $\mathcal{M}$ is defined by the index of this complex

$$\begin{aligned} d(\mathcal{M}) &= \dim\{\ker(D\oplus D^\dagger)|_{\Omega^1} - \mathrm{coker}\,(D\oplus D^\dagger)|_{\Omega^0\oplus\Omega^2}\} \\ &= \dim\ker(D\oplus D^\dagger)|_{\Omega^1} - \dim\ker D|_{\Omega^0} - \dim\ker D^\dagger|_{\Omega^2}\,. \end{aligned}$$

We have chosen the bundle, E, such that there are no reducible flat connections, thus $\dim\ker D|_{\Omega^0} = 0$. However, in two dimensions, this further implies that $\dim\ker D^\dagger|_{\Omega^2} = 0$. To see this, let $B_{\mu\nu}$ be a two-form on C. If B is in the kernel of $D^\dagger$, then we have

$$D^\mu B_{\mu\nu} = 0\,.$$

In complex coordinate this becomes

$$D^w B_{w\bar{w}} = 0\,, \quad D^{\bar{w}} B_{\bar{w}w} = 0\,.$$

Note that in two dimension we can write $B_{w\bar{w}} = \epsilon_{w\bar{w}} b$, for some scalar b. Therefore the above equations reduce to

$$D_{\bar{w}} b = 0 \,, \quad D_w b = 0 \,.$$

Since there are no reducible flat connections, we conclude that $b = 0$, or $B_{\mu\nu} = 0$. Therefore, for such bundles the dimension of moduli space of flat connections is given by the virtual dimension of $\mathcal{M}$, which is given by an index theorem to be

$$d(\mathcal{M}) = \mathrm{dimG}\,(2g - 2) \,,$$

where g is the genus of the two-dimensional surface and G is the gauge group.

Let α_I's denote the bases for the first cohomology group, $H^1(E, D_C)$, of the elliptic complex in (2.40). Now a tangent vector to the space of flat connections can in general be written as

$$\frac{\delta A_C}{\delta X^I} = \alpha_I + D_C E_I \,,$$

where $D_C E_I$ is a gauge transformation. Therefore, upon fixing the gauge, the tangent space to $\mathcal{M}$ can be represented by α_I. Notice that, since the path integral has localized on $\mathcal{M}$ and since there are no reducible flat connections, we can mod out the gauge group completely by working on $\mathcal{M}$ where there are no gauge degrees of freedom left. Furthermore, ignoring the terms which are order of ϵ, A_Σ does not depend on the coordinates of Σ and can be integrated out by its equation of motion. The only degrees of freedom which are left are thus the moduli parameters X^I. Therefore the problem reduces to a path integral over X and it can be shown [27] that the effective theory describes the maps from Σ to $\mathcal{M}$. For $N = 4$ theory a similar effective theory emerges.

When C is a Riemann sphere, we get a different effective description in the limit where C shrinks. This case is the subject of chapter 4 where we discuss it in detail. Here we point out the two main differences which arise in this case:

Firstly, if, as above, we consider bundles which restrict nontrivially over C, the moduli space $\mathcal{M}$ (either in $N = 2$ or $N = 4$ case) is empty. This can be seen as follows.

Let us cover S^2 with two patches S_+ and S_-. Since $F = 0$, the gauge connection on S_+ can be written as $g_+^{-1} dg_+$. In the same way, let us write the gauge connection on S_- as $g_-^{-1} dg_-$. We may perform a gauge transformation to set $A_- = 0$ and $A_+ = g^{-1} dg$. On $S_+ \cap S_-$, these two connections are related by

$$g^{-1} dg = t^{-1} dt \,, \tag{2.41}$$

where t is the transition function. From this we see that g and t have to be in the same topological class such that t can be smoothly extended all over S_+ to g. However, g is a map from S_+ to $SO(3)$ and thus is topologically trivial and so the transition function t, also must be trivial. We conclude that a flat $SO(3)$ bundle over a sphere must be trivial.

Since in this case the moduli space $\mathcal{M}$ is empty, the effective theory is trivial and the partition function, for instance, vanishes. On the other hand, if we consider bundles which restrict trivially over C, there are some gauge degrees of freedom left after shrinking C which are effectively described by a two-dimensional SYM theory.

Secondly and more importantly, the dimension of the space of self-dual harmonic 2-forms, b_2^+, is one in this case. Hence there exist metrics for which the connection is reducible. It follows then that the path integral may get a contribution from the u-plane [24, 28].

2.4 Higher dimensional analogues of Donaldson-Witten theory

Extended supersymmetric Yang-Mills theories can, in principle, be derived from supersymmetric theories in higher dimensions by dimensional reduction [29]. For instance, $N = 2$ and $N = 4$ SYM theories in four dimensions can simply be obtained by dimensional reduction of $N = 1$ SYM theories in six and ten dimensions respectively. The highest dimension in which we can define a pure SYM theory is ten with a minkowskian signature $(-1, 1, \cdots, 1)$. As this ten-dimensional theory has a basic role in the forthcoming discussions, let us study it in more detail.

2.4.1 $N = 1$ SYM theory in ten dimensions

The Lagrangian of the super Yang-Mills theory in ten dimensions is

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^{MN} + \frac{i}{2}\overline{\Psi}\Gamma^M D_M \Psi \,. \tag{2.42}$$

To balance the degrees of freedom between the bosons and fermions we put the following constraints on spinors consistent with the space-time symmetry (see appendix B for more detail).

$$\overline{\Psi} = \Psi^t C$$

$$\Gamma_{11}\Psi = -\Psi\,,$$

Ψ is then called a Majorana-Weyl spinor. The supersymmetry transformations are

$$\begin{aligned} \delta A_M &= i\bar{\alpha}\Gamma_M\Psi \\ \delta\Psi &= \tfrac{1}{2}F_{MN}\Gamma^{MN}\alpha \\ \delta\overline{\Psi} &= -\tfrac{1}{2}\bar{\alpha}F_{MN}\Gamma^{MN}\,, \end{aligned} \tag{2.43}$$

where α is a constant anticommuting Majorana-Weyl spinor with the same chirality as Ψ.

2.4.2 Supersymmetry of $\mathcal{L}$ and supercurrent

Let us quickly demonstrate the supersymmetry of the action. First note that (we allow $\alpha = \alpha(x)$ to derive the supercurrent for completeness)

$$\begin{aligned} \delta F_{MN} &= i\partial_M\bar{\alpha}\Gamma_N\Psi - i\partial_N\bar{\alpha}\Gamma_M\Psi + i\bar{\alpha}\Gamma_N D_M\Psi - i\bar{\alpha}\Gamma_M D_N\Psi \\ \delta(\overline{\Psi}\Gamma^M D_M\Psi) &= -\frac{1}{2}\bar{\alpha}\Gamma_{MN}F^{MN}\Gamma^L D_L\Psi + \frac{1}{2}\overline{\Psi}\Gamma^L\Gamma^{MN}D_L(\alpha F_{MN}) + i\overline{\Psi}\Gamma^M[\bar{\alpha}\Gamma_M\Psi,\Psi]\,, \end{aligned}$$

so

$$\begin{aligned} \delta(-\frac{1}{4}\int d^{10}x\, F_{MN}F^{MN}) &= -i\int d^{10}x\,\left[F^{MN}\partial_M\bar{\alpha}\Gamma_N\Psi + \bar{\alpha}F^{MN}\Gamma_N D_M\Psi\right] \\ &= i\int d^{10}x\,(D_M F^{MN})\bar{\alpha}\Gamma_N\Psi\,, \end{aligned}$$

where in the last equality we have dropped the total divergence term. Since α and Ψ are both anticommuting Majorana-Weyl spinors and using eq. (B.5) we have

$$D_L\overline{\Psi}\Gamma^L\Gamma_{MN}\alpha = \bar{\alpha}\Gamma_{MN}\Gamma^L D_L\Psi\,.$$

Thus

$$\begin{aligned} \delta\int d^{10}x\,(\overline{\Psi}\Gamma^M D_M\Psi) &= \int d^{10}x\,\Big[-\frac{1}{2}\bar{\alpha}\Gamma_{MN}F^{MN}\Gamma^L D_L\Psi \\ &\qquad + \frac{1}{2}D_L(\bar{\alpha}F^{MN})\Gamma^{MN}\Gamma^L\Psi + i\overline{\Psi}\Gamma^M[\bar{\alpha}\Gamma_M\Psi,\Psi]\Big] \\ &= \int d^{10}x\,\Big[\partial_L\bar{\alpha}\Gamma_{MN}F^{MN}\Gamma^L\Psi + \bar{\alpha}D_L F^{MN}\Gamma_{MN}\Gamma^L\Psi \\ &\qquad + i\overline{\Psi}\Gamma^M[\bar{\alpha}\Gamma_M\Psi,\Psi]\Big]\,. \end{aligned}$$

Therefore up to a total divergence term, we can write

$$\delta S = -i\int d^{10}x \left[D_N F^{MN}\bar{\alpha}\Gamma_M\Psi - \frac{1}{2}D_L F^{MN}\bar{\alpha}\Gamma_{MN}\Gamma^L\Psi - \frac{1}{2}F^{MN}\partial_L\bar{\alpha}\Gamma_{MN}\Gamma^L\Psi \right.$$
$$\left. -\frac{i}{2}\bar{\Psi}\Gamma^M[\bar{\alpha}\Gamma_M\Psi, \Psi]\right]. \tag{2.44}$$

The first and the second term in the integrand combine to

$$\begin{aligned} \frac{i}{2}D_L F_{MN}\bar{\alpha}(\Gamma^{MN}\Gamma^L + 2\eta^{LN}\Gamma^M)\Psi &= \frac{i}{2}D_L F_{MN}\bar{\alpha}(\Gamma^{MN}\Gamma^L + \eta^{LN}\Gamma^M - \eta^{LM}\Gamma^N)\Psi \\ &= \frac{i}{2}D_L F_{MN}\bar{\alpha}\Gamma^{MNL}\Psi = 0\,, \end{aligned}$$

the last equality follows by using the Bianchi identity. If the trilinear term is zero then one could conclude that the action is supersymmetric for constant α's.

In appendix B we show that the trilinear term is indeed vanishing which proves the supersymmetry of the action in 10 dimensions. Note that we can now also read the supersymmetry current from (2.44)

$$J^K = \frac{i}{2}F^{MN}\Gamma_{MN}\Gamma^K\Psi\,. \tag{2.45}$$

2.4.3 Reduction to lower dimensions

We may derive the whole bunch of extended SYM theories by the dimensional reduction. Let us briefly outline a few examples.

The ten-dimensional SYM theory can be reduced to four dimensions with euclidean signature simply by demanding that fields do not depend on the extra six coordinates. This breaks the space-time symmetry as follows

$$SO(9,1) \to SO(5,1) \times SO(4)\,.$$

Thus $SO(4)$ is the remaining space-time symmetry and $SO(5,1)$ is now the global symmetry of the theory left from the original ten-dimensional Lorentz symmetry. This symmetry is in fact the $\mathcal{R}$-symmetry group of reduced $N = 4$ theory. In the case of $N = 2$, we start with $N = 1$ in six dimensions. Putting the constraint that fields only depend on four coordinates breaks the Lorentz symmetry as

$$SO(5,1) \to SO(1,1) \times SO(4)$$

leaving a $SO(1,1)$ $\mathcal{R}$-symmetry group which is the same group (after complexification) that we called $U(1)_{\mathcal{R}}$ earlier. In these derivations, scalars appear as the components of the gauge fields in the directions normal to the reduced space-time. In this sense, it is easy to see how scalars appear and how they transform under the $\mathcal{R}$-symmetry group.

As we noticed earlier, the essential ingredient in the twisting of SYM theories is the existence of a suitable global $\mathcal{R}$-symmetry which of course must be anomaly free. Let us see what sort of supersymmetric Yang-Mills theories can be constructed by reduction from ten to lower dimensions, and what sort of global $\mathcal{R}$-symmetry in this way are produced. First consider the dimensional reduction to eight dimensions. The space-time symmetry breaks as

$$SO(9,1) \to SO(1,1) \times SO(8).$$

The $\mathcal{R}$-symmetry $SO(1,1)$ (or equivalently $U(1)$) for this theory turns out to be anomalous, therefore, it cannot be used to twist the theory (at least on eight-dimensional Kähler manifolds which have $SU(4) \times U(1)$ as their holonomy group). Hence, the only manifolds on which one may hope to construct a topological field theory are those with a reduced holonomy such that they admit globally defined covariant spinors. For instance, Calabi-Yau four folds with the $SU(4)$ holonomy are in this category. In fact, topological field theories of this type have been constructed in [30, 31, 32] where it is shown that the corresponding theory is invariant under the metric deformations which preserve the holonomy structure of the manifold (note that the holonomy is uniquely characterized by the metric).

The above cohomological field theories, obtained from the dimensional reduction of $N = 1$, $d = 10$ SYM theory, also arise in the effective description of D-branes.

D-branes are intrinsically nonperturbative objects in string theory. They have played an important role in unraveling the nonperturbative behavior of string theory in recent years (for an introductory review of D-branes see [33]). Roughly speaking, D-branes are extended geometrical objects on which open strings can end. The dynamics of a D-brane is thus inherited from that of the open strings attached to it. There exist different types of D-branes depending on their dimension, orientation, topology and so on. Let us denote a p-dimensional D-brane by Dp-brane. To describe a D-brane, first one defines the coordinates which are embeddings of the D-brane into the 10-dimensional target space-time. Moreover, a D-brane carries gauge degrees of freedom associated with the open strings attached to it. A D-brane can be described by the Born-Infeld action which is obtained at the tree level of string theory by requiring the anomaly cancelations. However, if we restrict ourselves to low energies, we may expand the Born-Infeld action to derive an effetive theory for low energy excitations of a D-brane. Thus the effective theory is nothing but a SYM theory living on the worldvolume of the D-brane. For flat Dp-branes, this SYM theory can be obtained by dimensional reduction of $d = 10$ SYM theory down to $p+1$ dimensions [34]. One may also study curved D-branes which naturally arise in the compactification of string theory on curved manifolds. In [6] it has been argued that the effective theory of such curved branes is a topological field theory which lives on the worldvolume of the brane.

The eight-dimensional cohomological field theory that we just mentioned above, for example, can be thought of as the effective field theory of an euclidean D7-brane. The D-brane in this case is wrapping around the whole eight-manifold M. The two scalars left from the reduction of ten-dimensional SYM theory simply specify the location of the D-brane in the ambient ten-dimensional space. There are some examples where the D-brane is wrapping around a supersymmetric submanifold [35] Y of M. In these cases, the effective field theory living on the worldvolume of the brane can again be obtained by the dimensional reduction of the ten-dimensional SYM theory. However, since the ambient space of the brane is now curved, scalars turn out to be sections of the normal bundle of Y [6]. This could be understood if we recall how the scalars appear in the reduced theory. They are indeed components of the gauge field in the normal directions. Take p to be a point on Y, then the tangent bundle at p decomposes as

$$T_pM = T_pY \oplus N_pY\,.$$

This equation then tells us that scalars must transform as sections of $N(Y)$.

Next we move to consider the dimensional reduction to six dimensions [29, 36]. The Lorentz symmetry group in this case breaks as

$$SO(9,1) \rightarrow SO(3,1) \times SO(6)\,.$$

The global $\mathcal{R}$-symmetry, $SO(3,1)$, is again anomalous. In fact there is a subgroup, $SU(2)_V$, of this $SO(3,1)$ which is anomaly free and thus can be used in twisting. For example, on Kähler manifolds we may choose a $U(1)$ subgroup of $SU(2)_V$ to twist with the $U(1)$ part of the holonomy group (Kähler manifolds have a $SU(3) \times U(1)$ holonomy in six dimensions). Another type of manifolds on which twisting is possible are product manifolds of $X \times Y$, where X and Y are both three manifolds. However, since the holonomy group in this case is $SU(2)_X \times SU(2)_Y$, twisting can be done on just one of the two manifolds. The latter example is specially interesting when we are considering D5-branes wrapping around say Y supersymmetrically embedded in a Calabi-Yau manifold. We will discuss this particular case in detail in chapter 3.

As in eight dimensions, we may also construct cohomological field theories on manifolds with a reduced holonomy without a need for twisting. These theories turn out to describe the effective theory of the euclidean D5-branes wrapping around the whole manifold.

Chapter 3

Supersymmetric Gauge Theory on Calabi-Yau 3-folds

3.1 Introduction

Extended supersymmetric Yang-Mills theories in various dimensions have been intensively studied by both physicists and mathematicians. From the mathematical point of view, these theories are interesting since they can give rise to a physical formulation of topological invariants of manifolds in different dimensions. As in four dimensions, where one reformulates the Donaldson theory in terms of twisted $N = 2$ SYM theory, one may hope that a similar construction exists in higher dimensions, by which topological invariants are expressed in terms of physical observables of a supersymmetric Yang-Mills theory. Of course, this reformulation crucially depends on the existence of a suitable global symmetry to be able to twist the theory. For instance, as we discussed in the second chapter, SYM theory in eight dimensions does not have a nonanomalous global symmetry and thus its existence is limited to those manifolds which admit globally defined covariant spinors. In this chapter we will see that a similar restriction arises for twisting the SYM theory on a general six-manifold. However, there is a class of manifolds (Kähler six-manifolds) for which twisting is possible. Partial twisting [27] is another option which can be considered on product six-manifolds. We will discuss this case in detail in section five.

From the physical point of view, extended supersymmetric Yang-Mills theories –

which arise from the reduction of SYM theory in ten dimensions – have become a major field of research as they describe the low energy effective theory of D-branes [34]. As mentioned in the second chapter, the low energy effective worldvolume theory of a Dp-brane (where p indicates the spatial dimension of the brane) is described by the SYM theory obtained from the dimensional reduction of SYM theory in ten dimensions down to $p+1$ dimensions.

We start this chapter by reducing the ten-dimensional SYM theory down to six dimensions and deriving the Lagrangian on flat $\mathbf{R}^6$. Since the nonanomalous part of the global symmetry is not large enough, we are led to consider those manifolds with reduced holonomy. In section four, we define the theory on a Calabi-Yau 3-fold [37]. The Lagrangian is derived and it is shown that the resultant theory is cohomological in the sense that it is invariant under the metric deformations which preserve the holonomy structure of the manifold. This cohomological field theory in fact describes the low energy degrees of freedom of euclidean D5-branes wrapping around the whole manifold. A balanced formulation [7] of the theory is also presented in this section. In section five, we consider the product six-manifold $X \times Y$, where X and Y are both three-manifolds. We partially twist the theory on one of the three-manifolds and study the limit where that particular manifold shrinks to zero size.

3.2 The reduced 6-dimensional theory

Constructing lower dimensional theories through dimensional reduction of theories in higher dimensions goes back to the idea of Kaluza [38] and Klein [39] of unifying general relativity and electromagnetism (see [40] for further reference). To explain the idea, let us start from a five-dimensional theory of gravity defined on the space-time $X \times S^1$, where X is a Lorentz four-manifold. Let $M, N = 0,1,2,3,5$ indicate the coordinate indices on the whole manifold, and $\mu, \nu = 0,1,2,3$ indicate the indices on X. The metric now decomposes as

$$g_{\mu\nu}\,, \quad g_{\mu 5} \equiv A_\mu\,, \quad g_{55} \equiv \phi\,.$$

Assuming the periodicity in the fifth direction, we can Fourier expand the fields. Then it is easy to see that the nonzero modes in this expansion have a mass proportional to $1/r$, where r is the radius of S^1. Therefore, in the limit of very small r, nonzero modes decouple from the zero modes. Keeping only the zero modes, the five-dimensional Einstein-Hilbert action reduces to the corresponding Einstein-Hilbert action in four dimensions plus the Maxwell action for the A_μ. Moreover, in this setting, the gauge invariance can be recognized as part of the general coordinate transformations invariance of the five-dimensional theory.

Although the Kaluza-Klein idea was not successful in achieving its goal of unifying gravity with electromagnetism, it emphasized the role of higher dimensional theories in the understanding of the physical theories in four dimensions. In this section we will derive a six-dimensional theory by the dimensional reduction of the SYM theory in ten dimensions. Along the way, we will see how the different fields, transforming differently under the reduced Lorentz symmetry, appear and how the extra global symmetries, left from the original space-time symmetry, emerge.

3.2.1 Field decomposition

The reduction can be achieved conveniently by splitting the coordinates as $M = (I, \mu + 3)$, where $I = 0, \ldots, 3$ and $\mu = 1, \ldots, 6$. Assuming fields to be independent of x^I coordinates breaks the part of the Lorentz symmetry which rotates x^μ and x^I into each other and thus reduces it to

$$SO(9,1) \to SO(3,1) \times SO(6)\,,$$

where $SO(6)$ acts on x^μ coordinate. Since fields do not depend on x^I coordinates but transform as irreducible representations of $SO(3,1)$, this subgroup plays the role of an internal global symmetry. A tensor field like A_M now decomposes to the scalars A_I transforming as a vector under $SO(3,1)$ and the gauge fields A_μ which are scalar under $SO(3,1)$. Since A_I components of the gauge field are scalar under the space-time symmetry $SO(6)$, it is more convenient to introduce the notation $\phi_I \equiv A_I$. Thus A_M decomposes as

$$A_M = (A_\mu, \phi_I)\,.$$

To see how spinors decompose, let us choose the representation of the Γ_M's to be

$$\begin{aligned} \Gamma^I &= \tilde{\gamma}^I \otimes \gamma_7 \\ \Gamma^{\mu+3} &= 1_4 \otimes \gamma^\mu\,, \end{aligned} \tag{3.1}$$

where $\tilde{\gamma}^I$ generate a $[3+1]$ Clifford algebra and γ^μ a $[6+0]$ one. Using the definitions of $\tilde{\gamma}_5$ and γ_7;

$$\gamma_7 = i\gamma_0 \cdots \gamma_5 \ , \quad \tilde{\gamma}_5 = i\tilde{\gamma}_0 \cdots \tilde{\gamma}_3 \,,$$

we find that

$$\Gamma_{11} = \tilde{\gamma}_5 \otimes \gamma_7 \,.$$

Further, if we take

$$\begin{aligned} \sigma_d^{(6)} &= -\sigma_t^{(6)} = -1 \\ \sigma_d^{(4)} &= -\sigma_t^{(4)} = 1 \,, \end{aligned}$$

then we can take

$$\mathsf{C} = \tilde{C} \otimes C \ ,$$

where $\tilde{C}$ and C are the unitary charge conjugation matrices in 4 and 6 dimensions respectively. For the sake of explicitness, we choose the chiral representation for the 4d γ's. In terms of Pauli matrices, these are

$$\begin{aligned} &\tilde{\gamma}^0 = 1_2 \otimes (i\sigma^2) \\ &\tilde{\gamma}^i = \sigma^i \otimes \sigma^1 \quad i = 1,2,3 \,, \end{aligned}$$

or

$$\tilde{\gamma}^I = \begin{pmatrix} 0 & \sigma^I \\ \bar{\sigma}^I & 0 \end{pmatrix}$$

with

$$\begin{aligned} \tilde{\gamma}_5 &= 1_2 \otimes \sigma^3 \\ \tilde{C} &= i\sigma^2 \otimes \sigma^3 \,. \end{aligned}$$

Let e_a, $a = 1,2$ denote the eigen-basis of σ^3. Then a 10d spinor in this representation can be written most generally as

$$\Psi = e_a \otimes e_b \otimes \psi^{ab} \,,$$

where ψ^{ab} is a 6d spinor. Imposing the Weyl condition on Ψ yields

$$\begin{aligned} \Gamma_{11}\Psi &= (\tilde{\gamma}_5 \otimes \gamma_7)(e_a \otimes e_b \otimes \psi^{ab}) = 1_2 e_a \otimes \sigma^3 e_b \otimes \gamma_7 \psi^{ab} \\ &= e_a \otimes e_1 \otimes \gamma_7 \psi^{a1} - e_a \otimes e_2 \otimes \gamma_7 \psi^{a2} \\ &= -\Psi = -(e_a \otimes e_1 \otimes \psi^{a1} + e_a \otimes e_2 \otimes \psi^{a2}) \,, \end{aligned} \tag{3.2}$$

therefore,

$$\gamma_7\psi^{a1} = -\psi^{a1}, \ \gamma_7\psi^{a2} = \psi^{a2}.$$

Hence $\psi^{a1} \equiv \psi_L^a$ and $\psi^{a2} \equiv \psi_R^a$ are left and right handed 6d spinors respectively and Ψ can be written as

$$\Psi = e^a \otimes e_1 \otimes \psi_{La} + e_{\dot a} \otimes e_2 \otimes \psi_R^{\dot a}. \tag{3.3}$$

Note that

$$\begin{aligned} \overline{\Psi} = \Psi^\dagger \Gamma_0 &= -(e^{\dot a} \otimes e_1 \otimes (\psi_{La})^\dagger + e_a \otimes e_2 \otimes (\psi_R^{\dot a})^\dagger)(1_2 \otimes i\sigma^2 \otimes \gamma_7) \\ &= (e^{\dot a} \otimes e_2 \otimes (\psi_{La})^\dagger + e_a \otimes e_1 \otimes (\psi_R^{\dot a})^\dagger), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \Psi^c = \Psi^t \mathsf{C} &= (e^a \otimes e_1 \otimes \psi_{La}^t + e_{\dot a} \otimes e_2 \otimes \psi_R^{t\dot a})(i\sigma^2 \otimes \sigma^3 \otimes C) \\ &= e^a (i\sigma^2) \otimes e_1 \sigma^3 \otimes \psi_{La}^t C + e_{\dot a}(i\sigma^2) \otimes e_2 \sigma^3 \otimes \psi_R^{t\dot a} C \\ &= \epsilon^{ab} e_b \otimes e_1 \otimes \psi_{La}^t C - \epsilon_{\dot a \dot b} e^{\dot b} \otimes e_2 \otimes \psi_R^{t\dot a} C. \end{aligned}$$

So, if we define $(\psi_a)^\dagger \equiv \bar\psi_{\dot a}$ and $(\psi^{\dot a})^\dagger \equiv \bar\psi^a$, the 10d Majorana condition implies

$$\begin{aligned} (\psi_{Lb})^\dagger &= \bar\psi_{L\dot b} = -\psi_R^{t\dot a} C \epsilon_{\dot a \dot b} \\ (\psi_R^{\dot b})^\dagger &= \bar\psi_R^b = \psi_{La}^t C \epsilon^{ab}. \end{aligned} \tag{3.5}$$

3.2.2 Lagrangian and the supersymmetry transformations

In the last part we deduced how the 10d spinors can be decomposed in terms of spinor representation of $SO(6)$, i.e., ψ_{Lb} and $\psi_R^{\dot b}$. In this subsection we show that the resulting 6d Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm F}, \tag{3.6}$$

where

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi_I D^\mu \phi^I - \frac{1}{4}([\phi_I, \phi_J])^2,$$

and

$$\mathcal{L}_{\rm F} = \frac{i}{2} \bar\psi_{L\dot a} \gamma^\mu D_\mu \psi_R^{\dot a} + \frac{i}{2} \bar\psi_R^a \gamma^\mu D_\mu \psi_{La} + i \bar\psi_R^a [\phi_I \sigma_{a\dot b}^I, \psi_R^{\dot b}].$$

We also show that the action is invariant under the supersymmetry transformations, which read

$$\begin{aligned}
\delta A_\mu &= i\bar{\alpha}^a_R\gamma_\mu\psi_{La} + i\bar{\alpha}_{L\dot{a}}\gamma_\mu\psi^{\dot{a}}_R \\
\delta\phi_I &= -i\bar{\alpha}_{L\dot{a}}\bar{\sigma}^{\dot{a}b}_I\psi_{Lb} + i\bar{\alpha}^a_R(\sigma_I)_{a\dot{b}}\psi^{\dot{b}}_R \\
\delta\psi_{La} &= \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\alpha_{La} + \gamma^\mu D_\mu\phi_I\sigma^I_{a\dot{b}}\alpha^{\dot{b}}_R + [\phi_I,\phi_J](\sigma^{IJ})_a{}^b\alpha_{Lb} \\
\delta\psi^{\dot{a}}_R &= \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\alpha^{\dot{a}}_R - \gamma^\mu D_\mu\phi_I\bar{\sigma}^{I\dot{a}b}\alpha_{Lb} + [\phi_I,\phi_J](\bar{\sigma}^{IJ})^{\dot{a}}{}_{\dot{b}}\alpha^{\dot{b}}_R\,.
\end{aligned} \tag{3.7}$$

Further we derive the global $SO(3,1)$ symmetry of the reduced theory. These are

$$\begin{aligned}
&\delta\psi_{La} = i(v_i + ia_i)(\frac{\sigma^i}{2})_a{}^b\psi_{Lb} \\
&\delta\psi^{\dot{a}}_R = i(v_i - ia_i)(\frac{\sigma^i}{2})^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}_R \\
&\delta\phi_0 = -a_i\phi^i \\
&\delta\phi_i = -a_i\phi_0 + \epsilon_{ijk}v_k\phi_j\,.
\end{aligned} \tag{3.8}$$

The derivations of these results is straightforward. The bosonic part of the action is easily derived, if we just note that

$$\begin{aligned}
&F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\
&F_{\mu I} = \partial_\mu\phi_I + [A_\mu,\phi_I] = D_\mu\phi_I \\
&F_{IJ} = [\phi_I,\phi_J]\,.
\end{aligned} \tag{3.9}$$

The fermionic part of the action can also simply be obtained by replacing the decomposed spinors in the 10d action. Substituting (3.4), (3.1) and (3.3) into $\mathcal{L}_{\rm F}$ we obtain

$$\begin{aligned}
\mathcal{L}_{\rm F} &= \frac{i}{2}\bar{\Psi}\Gamma^M D_M\Psi \\
&= \frac{i}{2}(e^{\dot{a}}\otimes e_2\otimes\bar{\psi}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\psi}^a_R)(1_4\otimes\gamma^\mu)D_\mu\,(e^b\otimes e_1\otimes\psi_{Lb} + e_{\dot{b}}\otimes e_2\otimes\psi^{\dot{b}}_R) \\
&+ \frac{i}{2}(e^{\dot{a}}\otimes e_2\otimes\bar{\psi}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\psi}^a_R)(\tilde{\gamma}^I\otimes\gamma_7)\left[\phi_I\,,(e^b\otimes e_1\otimes\psi_{Lb} + e_{\dot{b}}\otimes e_2\otimes\psi^{\dot{b}}_R))\right] \\
&= \frac{i}{2}\bar{\psi}_{L\dot{a}}\gamma^\mu D_\mu\psi^{\dot{a}}_R + \frac{i}{2}\bar{\psi}^a_R\gamma^\mu D_\mu\psi_{La} \\
&+ \frac{i}{2}(e^{\dot{a}}\otimes e_2\otimes\bar{\psi}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\psi}^a_R)\left[\phi_0\,,(e^b\otimes i\sigma_2 e_1\otimes\gamma_7\psi_{Lb} + e_{\dot{b}}\otimes i\sigma_2 e_2\otimes\gamma_7\psi^{\dot{b}}_R)\right] \\
&+ \frac{i}{2}(e^{\dot{a}}\otimes e_2\otimes\bar{\psi}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\psi}^a_R)\left[\phi_i\,,(\sigma^i e^b\otimes\sigma^1 e_1\otimes\gamma_7\psi_{Lb} + \sigma^i e_{\dot{b}}\otimes\sigma^1 e_2\otimes\gamma_7\psi^{\dot{b}}_R)\right] \\
&= \frac{i}{2}\bar{\psi}_{L\dot{a}}\gamma^\mu D_\mu\psi^{\dot{a}}_R + \frac{i}{2}\bar{\psi}^a_R\gamma^\mu D_\mu\psi_{La} + \frac{i}{2}\bar{\psi}_{L\dot{a}}[\phi_0(\sigma^0)^{\dot{a}b},\psi_{Lb}] + \frac{i}{2}\bar{\psi}^a_R[\phi_0(\sigma^0)_{a\dot{b}},\psi^{\dot{b}}_R]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{i}{2}\bar{\psi}_{L\dot{a}}[\phi_i(\sigma^i)^{\dot{a}b}\ ,\psi_{Lb}] + \frac{i}{2}\bar{\psi}^a_R[\phi_i(\sigma^i)_{a\dot{b}}\ ,\psi^{\dot{b}}_R] \\
&= \frac{i}{2}\bar{\psi}_{L\dot{a}}\gamma^\mu D_\mu\psi^{\dot{a}}_R + \frac{i}{2}\bar{\psi}^a_R\gamma^\mu D_\mu\psi_{La} + i\bar{\psi}^a_R[\phi_I\sigma^I_{a\dot{b}}\ ,\psi^{\dot{b}}_R]\,,
\end{aligned}
$$

The last equality follows by noticing that

$$
\begin{aligned}
f^{ABC}\bar{\psi}^A_{L\dot{a}}(\sigma^0)^{\dot{a}b}\psi^B_{Lb} &= -f^{ABC}\psi^{t\dot{c}A}_R C\epsilon_{\dot{c}\dot{a}}(\sigma^0)^{\dot{a}b}\psi^B_{Lb} = f^{ABC}\psi^{tB}_{Lb}C^t\epsilon^{ab}(\sigma^0)_{a\dot{c}}\psi^{\dot{c}A}_R \\
&= -f^{ABC}\bar{\psi}^{Ba}_R(\sigma^0)_{a\dot{c}}\psi^{\dot{c}A}_R = f^{ABC}\bar{\psi}^{aA}_R(\sigma^0)_{a\dot{c}}\psi^{\dot{c}B}_R\,,
\end{aligned}
$$

as

$$
\epsilon_{\dot{c}\dot{a}}(\sigma^0)^{\dot{a}b} = \epsilon^{ab}(\sigma^0)_{a\dot{c}}
$$

and

$$
\begin{aligned}
-f^{ABC}\bar{\psi}^A_{L\dot{a}}(\sigma^i)^{\dot{a}b}\psi^B_{Lb} &= -f^{ABC}\psi^{t\dot{c}A}_R C\epsilon_{\dot{c}\dot{a}}(\sigma^i)^{\dot{a}b}\psi^B_{Lb} = f^{ABC}\psi^{tB}_{Lb}C^t\psi^{\dot{c}A}_R\epsilon_{\dot{c}\dot{a}}(\sigma^i)^{\dot{a}b} \\
&= f^{ABC}\psi^{tB}_{Lb}C^t\psi^{\dot{c}A}_R\epsilon^{bc}\sigma^i_{c\dot{c}} = -f^{ABC}\bar{\psi}^{cB}_R\psi^{\dot{c}A}_R\sigma^i_{c\dot{c}} = f^{ABC}\bar{\psi}^{cA}_R\psi^{\dot{c}B}_R\sigma^i_{c\dot{c}}\,,
\end{aligned}
$$

where we used the reality condition (3.5) and $\epsilon_{\dot{c}\dot{a}}(\sigma^i)^{\dot{a}b} = -\epsilon_{\dot{c}\dot{a}}\epsilon^{bc}\epsilon^{\dot{a}\dot{b}}\sigma^i_{c\dot{b}} = \epsilon^{bc}\sigma^i_{c\dot{c}}$.

The supersymmetry transformations of the reduced 6-dimensional theory can be obtained similarly from the field decomposition. First we note that, in this decomposition, the supersymmetry transformations (2.43) read

$$
\begin{aligned}
\delta A_\mu &= i(e^{\dot{a}}\otimes e_2\otimes\bar{\alpha}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\alpha}^a_R)(1_4\otimes\gamma_\mu)(e^b\otimes e_1\otimes\psi_{Lb} + e_{\dot{b}}\otimes e_2\otimes\psi^{\dot{b}}_R) \\
&= i\bar{\alpha}^a_R\gamma_\mu\psi_{La} + i\bar{\alpha}_{L\dot{a}}\gamma_\mu\psi^{\dot{a}}_R \\
\delta\phi_i &= i(e^{\dot{a}}\otimes e_2\otimes\bar{\alpha}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\alpha}^a_R)(\sigma_i\otimes\sigma^1\otimes\gamma_7)(e^b\otimes e_1\otimes\psi_{Lb} + e_{\dot{b}}\otimes e_2\otimes\psi^{\dot{b}}_R) \\
&= -i\bar{\alpha}_{L\dot{a}}\bar{\sigma}^{\dot{a}b}_i\psi_{Lb} + i\bar{\alpha}^a_R(\sigma_i)_{a\dot{b}}\psi^{\dot{b}}_R \\
\delta\phi_0 &= -i(e^{\dot{a}}\otimes e_2\otimes\bar{\alpha}_{L\dot{a}} + e_a\otimes e_1\otimes\bar{\alpha}^a_R)(1_2\otimes i\sigma^2\otimes\gamma_7)(e^b\otimes e_1\otimes\psi_{Lb} + e_{\dot{b}}\otimes e_2\otimes\psi^{\dot{b}}_R) \\
&= -i\bar{\alpha}_{L\dot{a}}\bar{\sigma}^{\dot{a}b}_0\psi_{Lb} + i\bar{\alpha}^a_R(\sigma_0)_{a\dot{b}}\psi^{\dot{b}}_R\,, \qquad (3.10)
\end{aligned}
$$

or

$$
\delta\phi_I = -i\bar{\alpha}_{L\dot{a}}\bar{\sigma}^{\dot{a}b}_I\psi_{Lb} + i\bar{\alpha}^a_R(\sigma_I)_{a\dot{b}}\psi^{\dot{b}}_R\,.
$$

For the 10d spinor transformation, using the definition of Γ_{IJ} and (3.1), it is easy to see

$$
\begin{aligned}
\Gamma^{\mu+3,\nu+3} &= 1_4\otimes\gamma^{\mu\nu} \\
\Gamma^{I,\mu+3} &= \tilde{\gamma}^I\otimes\gamma_7\gamma^\mu \\
\Gamma^{IJ} &= \tilde{\gamma}^{IJ}\otimes 1_6\,,
\end{aligned}
$$

where

$$\tilde{\gamma}^{IJ} = 2\begin{pmatrix} \sigma^{IJ} & 0 \\ 0 & \bar{\sigma}^{IJ} \end{pmatrix}.$$

So, the spinor transformation reads

$$\begin{aligned} \delta(e^a\otimes e_1\otimes \psi_{La} &+ e_{\dot{a}}\otimes e_2\otimes \psi_R^{\dot{a}}) \\ &= \frac{1}{2}\left\{F_{\mu\nu}(1_4\otimes \gamma^{\mu\nu}) + 2D_\mu\phi_I(\tilde{\gamma}^I\otimes \gamma^\mu\gamma_7) + [\phi_I,\phi_J](\tilde{\gamma}^{IJ}\otimes 1_6)\right\} \\ &\times (e^b\otimes e_1\otimes \alpha_{Lb} + e_{\dot{b}}\otimes e_2\otimes \alpha_R^{\dot{b}}). \end{aligned} \tag{3.11}$$

Therefore, equating the similar terms in this equation we get the spinor transformations in (3.7).

In verifying the supersymmetry of the action one may proceed directly, using the above supersymmetry transformations laws. However, this can also be proved using the fact that the higher 10d action is supersymmetric. In reducing to 6 dimensions we assumed that fields do not depend on the (compactified) coordinates x^I. Hence one can see that the variation of the 6-dimensional Lagrangian under the supersymmetry transformations is exactly the equation (2.44) where now fields depend only on the coordinates x^μ, $\mu = 1,\ldots,6$. One uses the same proof to show that the fermionic trilinear term in (2.44) is zero. However, the term $D_LF_{MN}(\bar{\alpha}\Gamma^{MNL}\Psi)$, which is the sum of the first two terms, now splits to

$$\begin{aligned} D_LF_{MN}(\bar{\alpha}\Gamma^{MNL}\Psi) &= D_\mu F_{\nu\rho}(\bar{\alpha}\Gamma^{\mu\nu\rho}\Psi) + [\phi_K,[\phi_I,\phi_J]](\bar{\alpha}\Gamma^{IJK}\Psi) \\ &+ D_\mu[\phi_I,\phi_J](\bar{\alpha}\Gamma^{IJ\mu}\Psi) + 2[\phi_J,D_\mu\phi_I](\bar{\alpha}\Gamma^{IJ\mu}\Psi) \\ &+ 2D_\mu D_\nu\phi_I(\bar{\alpha}\Gamma^{\mu\nu I}\Psi) + [\phi_I,F_{\mu\nu}](\bar{\alpha}\Gamma^{\mu\nu I}\Psi). \end{aligned}$$

The first and the second term vanish by Bianchi and Jacobi identities respectively. The third and the fourth term combine to

$$([\phi_J,D_\mu\phi_I] + [\phi_I,D_\mu\phi_J])(\bar{\alpha}\Gamma^{IJ\mu}\Psi)$$

which is zero since $\Gamma^{IJ\mu}$ is antisymmetric in I and J. And finally it is obvious that the two last terms cancel each other. This proves the supersymmetry of the reduced 6-dimensional action on flat $\mathbf{R}^6$.

As a comment, note that the supercurrent of the 6-dimensional theory can be read from (2.45), the corresponding expression for the supersymmetric current in 10 dimensions, just by replacing the decomposed field expressions (3.3) and (3.9) into (2.45),

$$J^\rho = \Big(\frac{i}{2}F_{\mu\nu}(1_4\otimes\gamma^{\mu\nu}) - iD_\mu\phi_I(\tilde{\gamma}^I\otimes\gamma_7\gamma^\mu) + \frac{i}{2}[\phi_I,\phi_J](\tilde{\gamma}^{IJ}\otimes 1_6)\Big)(1_4\otimes\gamma^\rho)\Psi\,. \quad (3.12)$$

In components this becomes

$$\begin{aligned}\begin{pmatrix} J^\rho_{La} \\ J^{\rho\dot{a}}_R \end{pmatrix} &= \left(\frac{i}{2}F_{\mu\nu}\gamma^{\mu\nu}\gamma^\rho\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i[\phi_I,\phi_J]\gamma^\rho\begin{pmatrix}\sigma^{IJ} & 0\\ 0 & \bar{\sigma}^{IJ}\end{pmatrix}\right)\begin{pmatrix}\psi_{La}\\ \psi^{\dot{a}}_R\end{pmatrix}\\ &- i(D_\mu\phi_I)\gamma^\mu\gamma^\rho\begin{pmatrix}0 & \sigma^I\\ \bar{\sigma}^I & 0\end{pmatrix}\begin{pmatrix}-\psi_{La}\\ \psi^{\dot{a}}_R\end{pmatrix}.\end{aligned}$$

Thus the expression for the supercurrent in 6 dimensions reads

$$\begin{aligned} J^\rho_{La} &= \frac{i}{2}F_{\mu\nu}\gamma^{\mu\nu}\gamma^\rho\psi_{La} + i[\phi_I,\phi_J]\gamma^\rho(\sigma^{IJ})_a{}^b\psi_{Lb} - i(D_\mu\phi_I)\gamma^\mu\gamma^\rho\sigma^I_{a\dot{a}}\psi^{\dot{a}}_R \\ J^{\rho\dot{a}}_R &= \frac{i}{2}F_{\mu\nu}\gamma^{\mu\nu}\gamma^\rho\psi^{\dot{a}}_R + i[\phi_I,\phi_J]\gamma^\rho(\bar{\sigma}^{IJ})^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}_R + i(D_\mu\phi_I)\gamma^\mu\gamma^\rho(\bar{\sigma}^I)^{\dot{a}a}\psi_{La}\,. \quad (3.13)\end{aligned}$$

Finally, let us see how different fields in the reduced theory transform under the $SO(3,1)$ global symmetry. This subgroup is generated by the Γ_{IJ} matrices. Thus, under this subgroup, spinors transform as

$$\delta\Psi = \frac{1}{2}\omega_{IJ}\Gamma^{IJ}\Psi\,.$$

Writing this in terms of spinor representation of $SO(6)$ we have

$$\begin{aligned}\delta\psi_{La} &= \omega_{IJ}(\sigma^{IJ})_a{}^b\psi_{Lb}\\ \delta\psi^{\dot{a}}_R &= \omega_{IJ}(\bar{\sigma}^{IJ})^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}_R\,.\end{aligned}$$

ϕ_I is a vector under $SO(3,1)$, so it transforms as

$$\delta\phi_I = 2\eta_{IJ}\omega^{JK}\phi_K\,.$$

If we define $a_i \equiv 2\omega_{i0}, v_i \equiv \epsilon_{ijk}\omega_{jk}$ and write all this in components, we obtain the transformations in (3.8). Upon taking the complex conjugate of the transformations (3.8), we get

$$\begin{aligned}\delta(\psi_{La})^\dagger = \delta\bar{\psi}_{L\dot{a}} &= -i(v_i - a_i)\bar{\psi}_{L\dot{b}}(\frac{\sigma^i}{2})^{\dot{b}}{}_{\dot{a}}\\ \delta(\psi^{\dot{a}}_R)^\dagger = \delta\bar{\psi}^a_R &= -i(v_i + ia_i)\bar{\psi}^b_R(\frac{\sigma^i}{2})_b{}^a\end{aligned}$$

one can clearly see the invariance of the action under $SO(3,1)$. One may also check the consistency of the $SO(3,1)$ symmetry with the reality conditions (3.5):

$$
\begin{aligned}
\delta(\psi_R^{t\dot a}\epsilon_{\dot b\dot a}) &= i(v_i - ia_i)\psi_R^{t\dot c}(\frac{\sigma^{it}}{2})_{\dot c}{}^{\dot a}\epsilon_{\dot b\dot a} \\
&= i(v_i - ia_i)(\psi_R^{t\dot c}\epsilon_{\dot c\dot a})(\frac{\sigma^i}{2})^{\dot a}{}_{\dot b} \\
&= -i(v_i - ia_i)\bar\psi_{L\dot a}(\frac{\sigma^i}{2})^{\dot a}{}_{\dot b} = \delta\bar\psi_{L\dot b}
\end{aligned}
$$

using $(\sigma^{it})_{\dot c}{}^{\dot a}\epsilon_{\dot a\dot b} = -\epsilon_{\dot c\dot a}(\sigma^i)^{\dot a}{}_{\dot b}$. Taking a_i and v_i to be x-dependent, the associated Noether currents are found to be

$$
\begin{aligned}
J^i_{\mu V} &= -\tfrac{1}{2}\bar\psi_{L\dot a}\gamma_\mu(\sigma^i)^{\dot a}{}_{\dot b}\psi_R^{\dot b} - \tfrac{1}{2}\bar\psi_R^a\gamma_\mu(\sigma^i)_a{}^b\psi_{Lb} \\
J^i_{\mu A} &= \tfrac{i}{2}\bar\psi_{L\dot a}\gamma_\mu(\sigma^i)^{\dot a}{}_{\dot b}\psi_R^{\dot b} - \tfrac{i}{2}\bar\psi_R^a\gamma_\mu(\sigma^i)_a{}^b\psi_{Lb}\,.
\end{aligned}
\tag{3.14}
$$

Or, more concisely,

$$
\begin{aligned}
J^i_{\mu V} &= -\overline\Psi\gamma_\mu\Sigma^i\Psi \\
J^i_{\mu A} &= i\overline\Psi\gamma_\mu\gamma_7\Sigma^i\Psi\,,
\end{aligned}
$$

where

$$
\Sigma^i \equiv \tfrac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.
$$

Later on, we will exploit the $SO(3)_V$ subgroup for partial twisting on 6-dimensional product manifolds.

3.3 Reduction to manifolds with $SU(3)$ holonomy

Having derived the 6-dimensional supersymmetric theory on euclidean space, the natural question which arises is whether the theory can be defined on an arbitrary six-manifold. Unfortunately, in contrast with $N = 2$ SYM in four dimensions, the nonanomalous part of the $\mathcal{R}$-symmetry group ($SU(2)_V$ above) is not large enough to allow us to twist the theory on a general six-manifold. However, there are some special manifolds with reduced holonomy for which twisting is possible or even trivial. The case that is considered here is a Calabi-Yau manifold with the $SU(3)$ holonomy. We also comment on twisting on the Kähler manifolds.

First of all, Let us point out the main characteristcs of a Calabi-Yau manifold [41, 42] (see [43, 44] for application to physics). Let M be an even dimensional manifold. Let p be an arbitrary point on M. Define the tensor J, *the almost complex structure*, by

$$J_p \cdot \partial/\partial x^\mu = \partial/\partial y^\mu \quad J_p \cdot \partial/\partial y^\mu = -\partial/\partial x^\mu .$$

We can now decompose the complexified tangent space on p into the holomorphic and antiholomorphic parts by diagonalizing the tensor J. In this basis, the complexified tangent space splits as

$$T_pM^C = T_pM^+ \oplus T_pM^- ,$$

where T_pM^+ and T_pM^- are spanned by $\partial/\partial z^\mu$ and $\partial/\partial \bar{z}^\mu$ respectively, and J becomes

$$J^\alpha{}_\beta = i\delta^\alpha{}_\beta , \quad J^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^{\bar{\alpha}}{}_{\bar{\beta}} . \tag{3.15}$$

Therefore, at the level of tangent space, we can always diagonalize the tensor J and decompose the tangent space accordingly. However, to patch J across charts and define it globally as in the form of (3.15), the Nijenhuis tensor

$$N^\rho{}_{\mu\nu} = J^\sigma{}_\mu(\partial_\sigma J^\rho{}_\nu - \partial_\nu J^\rho{}_\sigma) - J^\sigma{}_\nu(\partial_\sigma J^\rho{}_\mu - \partial_\mu J^\rho{}_\sigma)$$

must vanish [45]. This is the criterion for M to be a complex manifold. If, in addition, J happens to be covariantly constant, M is said to be Kähler. A Kähler n-fold has a reduced holonomy group of $U(n) \subset SO(2n)$. On the other hand, Calabi-Yau manifolds have an even more restricted holonomy group. These are the Kähler manifolds which admit a covariantly constant holomorphic n-form. This, in particular, implies that the canonical line bundle of the manifold is trivial and the holonomy group is thus contained in $SU(n) \subset U(n)$.

In the next subsection we will see how all these structures follow from the existence of a covariantly constant spinor. Next we show how fields in the different representations of $SO(6)$ decompose under the $SU(3)$, the structure group of the manifold. This is similar to what we did in reducing from ten to six dimensions; i.e., determining the $SO(6)$ irreducible representations embedded in $SO(9,1)$.

Fermions transform in the fundamental representations of $SU(4)$, the universal covering group of $SO(6)$. Under $SU(3) \times U(1)$ these branch as follows

$$\begin{aligned} \mathbf{4} &= \mathbf{1}^{-3} + \mathbf{3}^{1} \\ \bar{\mathbf{4}} &= \mathbf{1}^{3} + \bar{\mathbf{3}}^{-1} , \end{aligned} \tag{3.16}$$

where the superscripts indicate the $U(1)$ charges. Introduce the commuting c-number spinor θ, which is the left-handed $SU(3)$ singlet, normalized by

$$\theta^\dagger\theta = 1\,.$$

In six dimensions we can choose a representation of the Clifford algebra for which the generators are all antisymmetric. In this case we may take $C = 1$, so $\gamma_7^t = -\gamma_7$, and θ^* is a right handed singlet spinor

$$\gamma_7^\dagger\theta = -\theta \Rightarrow \gamma_7^t\theta^* = -\theta^* \Rightarrow \gamma_7\theta^* = \theta^*\,.$$

3.3.1 A Fierz identity and covariantly constant tensors

The projector $\theta\theta^\dagger$ can be expanded using the complete set of bases

$$\mathbf{1}\ ,\ \gamma_\mu\ ,\ \gamma_7\gamma_\mu\ ,\ \gamma_{\mu\nu}\ ,\ \gamma_7\gamma_{\mu\nu}\ ,\ \gamma_{\mu\nu\lambda}\ ,$$

for the 8-dimensional matrices. Since θ and $\theta^\dagger$ are both left handed, the only matrices that can appear in this expansion are $(1-\gamma_7)$ and $(1-\gamma_7)\gamma_{\mu\nu}$

$$\theta\theta^\dagger = a(1-\gamma_7) + b_{\mu\nu}(1-\gamma_7)\gamma^{\mu\nu}\,.$$

Using the orthogonality of the bases with respect to trace, and the fact that

$$\mathrm{tr}(\gamma_\mu\gamma_\nu) = 8\eta_{\mu\nu} \tag{3.17}$$

$$\mathrm{tr}(\gamma^{\mu\nu}\gamma^{\rho\sigma}) = 8(\eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma})\,, \tag{3.18}$$

we see that

$$\theta\theta^\dagger = \frac{1}{8}(1-\gamma_7) - \frac{1}{16}(\theta^\dagger\gamma_{\mu\nu}\theta)(1-\gamma_7)\gamma^{\mu\nu}\,. \tag{3.19}$$

Multiplying this identity by $\theta^\dagger$ from left and θ from right we learn that

$$(\theta^\dagger\gamma^{\mu\nu}\theta)(\theta^\dagger\gamma_{\mu\nu}\theta) = -(\theta^\dagger\gamma^{\mu\nu}\theta)(\theta^t\gamma_{\mu\nu}\theta^*) = -6\,. \tag{3.20}$$

Also using the relations

$$\gamma^\lambda\gamma_\mu\gamma_\lambda = -4\gamma_\mu \quad , \quad \gamma^\lambda\gamma_{\mu\nu}\gamma_\lambda = 2\gamma_{\mu\nu}\,,$$

where the last one follows from

$$[\gamma_{\mu\nu},\gamma_\lambda] = 2(\eta_{\nu\lambda}\gamma_\mu - \eta_{\mu\lambda}\gamma_\nu),$$

we see that

$$\gamma^\lambda \theta\theta^\dagger \gamma_\lambda = \frac{3}{4}(1+\gamma_7) - \frac{1}{8}(\theta^\dagger \gamma_{\mu\nu}\theta)(1+\gamma_7)\gamma^{\mu\nu}\,.$$

Upon taking the complex conjugate we get

$$\gamma^\lambda \theta^* \theta^t \gamma_\lambda = \frac{3}{4}(1-\gamma_7) - \frac{1}{8}(\theta^t \gamma_{\mu\nu}\theta^*)(1-\gamma_7)\gamma^{\mu\nu}\,,$$

where we used that $\gamma_\mu^* = -\gamma_\mu$. Therefore, these two equations and (3.19) imply

$$\begin{aligned}\theta\theta^\dagger + \frac{1}{2}\gamma^\lambda \theta^*\theta^t \gamma_\lambda &= \frac{1}{2}(1-\gamma_7)\\ \theta^*\theta^t + \frac{1}{2}\gamma^\lambda \theta\theta^\dagger \gamma_\lambda &= \frac{1}{2}(1+\gamma_7)\,.\end{aligned} \tag{3.21}$$

This shows the decomposition of the projectors $\frac{1}{2}(1 \pm \gamma_7)$ into the pieces projecting to the singlet and the triplet of $SU(3)$ respectively.

Since θ is an $SU(3)$ singlet it is invariant under parallel transport around a loop, and it follows that θ is in fact covariantly constant. Using θ, we may define further covariantly constant tensors; e.g., introduce

$$k_{\mu\nu} \equiv i\theta^\dagger \gamma_{\mu\nu}\theta$$

and

$$J_\mu{}^\lambda = g^{\lambda\nu} k_{\mu\nu}\,.$$

First note that

$$k_{\mu\nu}k^{\mu\nu} = 6\,,$$

and that

$$\begin{aligned} J_\mu{}^\lambda J_\lambda{}^\nu &= g^{\lambda\rho} g^{\nu\sigma} k_{\mu\rho} k_{\lambda\sigma} = -g^{\lambda\rho} g^{\nu\sigma} (\theta^\dagger \gamma_{\mu\rho}\theta)(\theta^\dagger \gamma_{\lambda\sigma}\theta)\\ &= -g^{\lambda\rho} g^{\nu\sigma} (\theta^\dagger \gamma_{\mu\rho}) \left(\frac{1}{8}(1-\gamma_7) - \frac{1}{16}(\theta^\dagger \gamma_{\delta\eta}\theta)(1-\gamma_7)\gamma^{\delta\eta}\right)(\gamma_{\lambda\sigma}\theta)\\ &= -\frac{1}{4}(\theta^\dagger \gamma_{\mu\rho}\gamma^{\rho\nu}\theta) + \frac{1}{8}(\theta^\dagger \gamma^{\delta\eta}\theta)(\theta^\dagger \gamma_{\mu\rho}\gamma_{\delta\eta}\gamma^{\rho\nu}\theta)\\ &= -\frac{1}{4}\delta_\mu{}^\nu - \theta^\dagger \gamma_\mu \gamma^\nu \theta - \frac{3}{4}\delta_\mu{}^\nu - \frac{1}{2}(\theta^\dagger \gamma^{\nu\eta}\theta)(\theta^\dagger \gamma_\mu \gamma_\eta \theta) + \frac{1}{2}(\theta^\dagger \gamma_\mu{}^\eta \theta)(\theta^\dagger \gamma_\eta \gamma^\nu \theta)\\ &= -2\delta_\mu{}^\nu - J_\mu{}^\eta J_\eta{}^\nu\,. \end{aligned}$$

So,

$$J_\mu{}^\lambda J_\lambda{}^\nu = -\delta_\mu{}^\nu\,, \tag{3.22}$$

showing that $J_\mu{}^\lambda$ in fact defines the almost complex structure. Notice that $k_{\mu\nu}$, the Kähler form, (or equivalently $J_\mu{}^\lambda$) is covariantly constant by construction. As noted earlier, this is the necessary and sufficient condition for a complex manifold to be Kähler (or having a $U(3)$ holonomy).

We can still define one more covariantly constant tensor; i.e., the 3-form

$$\Omega_{\mu\nu\lambda} \equiv \theta^\dagger \gamma_{\mu\nu\lambda} \theta^* \,.$$

Being a 3-form, $\Omega_{\mu\nu\lambda}$ is certainly invariant under $SU(3)$, however, the fact that it is covariantly constant implies that it nowhere vanishes and thus the canonical line bundle is trivial, the specific characteristic of Calabi-Yau manifolds which distinguishes them from Kähler manifolds. There are no more covariant tensors that one can construct. Consider for example the tensor $\theta^\dagger \gamma_\lambda \theta^*$. This vanishes by

$$\theta^\dagger \gamma_\lambda \theta^* = (\theta^\dagger \gamma_\lambda \theta^*)^t = -\theta^\dagger \gamma_\lambda \theta^* \,, \tag{3.23}$$

where we used that the generators are chosen to be antisymmetric ($\gamma_\mu^t = -\gamma_\mu, \gamma_{\mu\nu}^t = -\gamma_{\mu\nu}$ and $\gamma_{\mu\nu\lambda}^t = \gamma_{\mu\nu\lambda}$).

At this point it is useful to make a special choice of θ which corresponds to the standard choice of complex coordinates. This reduces the problem to the standard construction of the spinor representation of $SO(6)$ via linear combinations of the Clifford algebra generators which obey the algebra of fermionic oscillators. First introduce the combinations (taking $\mu = (\alpha, \bar{\alpha})$ in flat (local frame) coordinates)

$$\begin{aligned} \hat{\gamma}^\alpha &= \frac{1}{\sqrt{2}}(\gamma^\alpha + i\gamma^{\alpha+3}) \\ \hat{\gamma}^{\bar{\alpha}} &= \frac{1}{\sqrt{2}}(\gamma^\alpha - i\gamma^{\alpha+3}) \,. \end{aligned} \tag{3.24}$$

Then $\hat{\gamma}^\alpha$ satisfy the algebra of fermionic harmonic oscillators; i.e.,

$$\{\hat{\gamma}^\alpha, \hat{\gamma}^{\bar{\beta}}\} = 2\delta^{\alpha\bar{\beta}} \,,$$

with all other anticommutators being zero. The generators of $SO(6)$ in this representation are

$$\begin{aligned} \hat{\Sigma}_{\alpha\beta} &= \tfrac{1}{2}[\hat{\gamma}_\alpha, \hat{\gamma}_\beta] = \hat{\gamma}_\alpha \hat{\gamma}_\beta \,, \\ \hat{\Sigma}_{\bar{\alpha}\beta} &= \tfrac{1}{2}[\hat{\gamma}_{\bar{\alpha}}, \hat{\gamma}_\beta] = \hat{\gamma}_{\bar{\alpha}} \hat{\gamma}_\beta \,, \end{aligned}$$

and their complex conjugate. Among these, $\hat{\Sigma}_{\bar{\alpha}\beta}$ form the generators of $U(3)$ subgroup of $SO(6)$. Thus subtracting the trace part, $SU(3)$ generators are

$$\hat{\Sigma}^{\alpha}{}_{\beta} = \hat{\gamma}^{\alpha}\hat{\gamma}_{\beta} - \frac{1}{3}\delta^{\alpha}_{\beta}\,\hat{\gamma}^{\gamma}\hat{\gamma}_{\gamma}\,.$$

To simplify the notation from now on we drop the hat sign and remember that we are in the representation defined by (3.24). Requiring that θ is an $SU(3)$ singlet (with appropriate $U(1)$ charge -3) then fixes

$$\gamma_{\alpha}\theta = 0\,, \tag{3.25}$$

so that

$$k_{\alpha\beta} = k_{\bar{\alpha}\bar{\beta}} = 0$$
$$k_{\alpha\bar{\beta}} = \frac{i}{2}\theta^{\dagger}\gamma_{\alpha}\gamma_{\bar{\beta}}\theta = i\theta^{\dagger}(-\frac{1}{2}\gamma_{\bar{\beta}}\gamma_{\alpha} + \delta_{\alpha\bar{\beta}})\theta = i\delta_{\alpha\bar{\beta}}\,.$$

Moreover, the only nonvanishing components of $\Omega_{\mu\nu\lambda}$ are

$$\Omega_{\alpha\beta\gamma} = \theta^{\dagger}\gamma_{\alpha\beta\gamma}\theta^{*}\,.$$

3.3.2 Field content and the Lagrangian

In this part the Lagrangian and the supersymmetry transformations on a Calabi-Yau 3-fold are derived. As implied by equations (3.21) θ and θ^{*} can be used to project out the spinor representations of $SU(4)$ to the irreducible representations (singlet and triplet) of $SU(3)$. In terms of these irreducible spinor representations, defined in (3.27) and (3.28), the result is

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta} - \tfrac{1}{2}F_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} - D_{\alpha}\phi_{I}D^{\alpha}\phi^{I} \\ &- \tfrac{1}{4}([\phi_{I},\phi_{J}])^{2} + i\epsilon^{ab}\psi_{a}D_{\alpha}\psi_{b}{}^{\alpha} + i\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{a}}D^{\alpha}\bar{\psi}_{\dot{b}\alpha} \\ &- \tfrac{i}{8}\Omega_{\alpha\beta\gamma}\epsilon^{ab}\psi_{a}{}^{\alpha}D^{\beta}\psi_{b}{}^{\gamma} - \tfrac{i}{8}\Omega^{\alpha\beta\gamma}\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{a}\alpha}D_{\beta}\bar{\psi}_{\dot{b}\gamma} \\ &- i\bar{\sigma}^{I\dot{a}b}\bar{\psi}_{\dot{a}}[\phi_{I},\psi_{b}] - \tfrac{i}{2}\bar{\sigma}^{I\dot{a}b}\bar{\psi}_{\dot{a}\alpha}[\phi_{I},\psi_{b}{}^{\alpha}]\,, \end{aligned} \tag{3.26}$$

which is invariant, upon using the equations of motion, under the following supersymmetry transformations

$$\delta A_{\alpha} = i\bar{\epsilon}_{\dot{a}}\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}\alpha}$$
$$\delta A_{\bar{\alpha}} = i\epsilon_{a}\epsilon^{ab}\psi_{b\bar{\alpha}}$$

$$
\begin{aligned}
\delta\phi_I &= -i\bar\epsilon_{\dot a}\bar\sigma_I^{\dot a b}\psi_b - i\epsilon_b\bar\sigma_I^{\dot a b}\bar\psi_{\dot a} \\
\delta\psi_a &= \epsilon_a\delta_{\alpha\bar\beta}F^{\alpha\bar\beta} + [\phi_I,\phi_J](\sigma^{IJ})_a{}^b\epsilon_b \\
\delta\psi_{a\bar\alpha} &= -\tfrac{1}{2}\Omega_{\bar\alpha\bar\beta\bar\gamma}F^{\bar\beta\bar\gamma}\epsilon_a - 2\bar\epsilon_{\dot a}\epsilon_{ba}\bar\sigma^{I\dot a b}D_{\bar\alpha}\phi_I \\
\delta\bar\psi_{\dot a} &= -\bar\epsilon_{\dot a}\delta_{\alpha\bar\beta}F^{\alpha\bar\beta} - [\phi_I,\phi_J](\bar\sigma^{IJ})^{\dot b}{}_{\dot a}\bar\epsilon_{\dot b} \\
\delta\bar\psi_{\dot a\alpha} &= -\tfrac{1}{2}\Omega_{\alpha\beta\gamma}F^{\beta\gamma}\bar\epsilon_{\dot a} - 2\epsilon_b\epsilon^{ab}\sigma^I_{a\dot a}D_\alpha\phi_I\,.
\end{aligned}
$$

To begin with, first we show how spinors decompose under $SU(3)\times U(1)$. Equation (3.21) implies

$$\psi_{La} = \frac{1}{2}(1-\gamma_7)\psi_{La} = \theta(\theta^\dagger\psi_{La}) + \frac{1}{2}\gamma^{\bar\alpha}\theta^*(\theta^t\gamma_{\bar\alpha}\psi_{La})\,,$$

so we recognize the components $\theta^\dagger\psi_{La}$ and $\theta^t\gamma_\lambda\psi_{La}$ as a singlet and triplet under $SU(3)$ respectively. Similarly $\psi_R^{\dot a}$ decomposes as

$$\psi_R^{\dot a} = \frac{1}{2}(1+\gamma_7)\psi_R^{\dot a} = \theta^*(\theta^t\psi_R^{\dot a}) + \frac{1}{2}\gamma^\alpha\theta(\theta^\dagger\gamma_\alpha\psi_R^{\dot a})\,.$$

However, note that the reality condition (3.5) reads (with $C=1$)

$$\psi_R^{\dot a} = \epsilon^{\dot b\dot a}\bar\psi_{L\dot b}^t\,.$$

Therefore, defining $\psi_a \equiv \theta^\dagger\psi_{La}$ or $\bar\psi_{\dot a}\equiv(\psi_a)^\dagger = \bar\psi_{L\dot b}\theta = \theta^t\bar\psi_{L\dot b}^t$, we see that

$$\theta^t\psi_R^{\dot a} = \epsilon^{\dot b\dot a}\theta^t\bar\psi_{L\dot b}^t = \epsilon^{\dot b\dot a}\bar\psi_{\dot b}\,.$$

Likewise, defining $\psi_{a\bar\alpha} = \theta^t\gamma_{\bar\alpha}\psi_{La}$ or $\bar\psi_{\dot a\alpha}\equiv(\psi_{a\bar\alpha})^\dagger = \bar\psi_{L\dot a}\gamma_\alpha\theta^* = -\theta^\dagger\gamma_\alpha\bar\psi_{L\dot a}^t$, we have

$$\theta^\dagger\gamma_\alpha\psi_R^{\dot a} = \theta^\dagger\gamma_\alpha\epsilon^{\dot b\dot a}\bar\psi_{L\dot b}^t = \epsilon^{\dot a\dot b}\bar\psi_{\dot b\alpha}\,.$$

Hence, by imposing the reality conditions we can write

$$\psi_{La} = \theta\psi_a + \tfrac{1}{2}\gamma^{\bar\alpha}\theta^*\psi_{a\bar\alpha} \tag{3.27}$$

and

$$\psi_R^{\dot a} = -\theta^*\epsilon^{\dot a\dot b}\bar\psi_{\dot b} + \tfrac{1}{2}\gamma^\alpha\theta\epsilon^{\dot a\dot b}\bar\psi_{\dot b\alpha}\,. \tag{3.28}$$

Upon replacing the above decomposed fields in (3.6) and using $\epsilon^{ab}\epsilon^{\dot a\dot b}\sigma^I_{a\dot a} = -\bar\sigma^{I\dot b b}$, we arrive at the Lagrangian in (3.26). As an example, let us look at the following term in detail

$$
\begin{aligned}
\tfrac{i}{2}\bar\psi_{L\dot a}\gamma^\mu D_\mu\psi_R^{\dot a} &= \tfrac{i}{2}(\bar\psi_{\dot a}\theta^\dagger + \tfrac{1}{2}\bar\psi_{\dot a\alpha}\theta^t\gamma^\alpha)(\gamma^\beta D_\beta + \gamma^{\bar\beta}D_{\bar\beta})(-\theta^*\epsilon^{\dot a\dot b}\bar\psi_{\dot b} + \tfrac{1}{2}\gamma^\alpha\theta\epsilon^{\dot a\dot b}\bar\psi_{\dot b\alpha}) \\
&= \tfrac{i}{2}(\bar\psi_{\dot a}\theta^\dagger\gamma^{\bar\beta}D_{\bar\beta} + \bar\psi_{\dot a\beta}\theta^t D^\beta + \tfrac{1}{2}\bar\psi_{\dot a\alpha}\theta^t\gamma^\alpha\gamma^\beta D_\beta)(-\theta^*\epsilon^{\dot a\dot b}\bar\psi_{\dot b} + \tfrac{1}{2}\gamma^\alpha\theta\epsilon^{\dot a\dot b}\bar\psi_{\dot b\alpha}) \\
&= i\epsilon^{\dot a\dot b}\bar\psi_{\dot a}D^\alpha\bar\psi_{\dot b\alpha} - \tfrac{i}{8}\Omega^{\alpha\beta\gamma}\epsilon^{\dot a\dot b}\bar\psi_{\dot a\alpha}D_\beta\bar\psi_{\dot b\gamma}\,,
\end{aligned} \tag{3.29}
$$

where we repeatedly used (3.23) and (3.25). Also note that since γ_μ are hermitian and antisymmetric then $\gamma^*_{\alpha} = -\gamma_{\bar{\alpha}}$ and therefore $\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = -\theta^t\gamma_{\bar{\alpha}}\gamma_{\bar{\beta}}\gamma_{\bar{\gamma}}\theta$.

Choosing the singlet supersymmetry parameters as

$$\alpha_{La} = \theta\epsilon_a \ , \quad \alpha^{\dot{a}}_R = -\theta^*\epsilon^{\dot{a}\dot{b}}\bar{\epsilon}_{\dot{b}} \, ,$$

let us look at the supersymmetry transformations in (3.7). For the gauge field A_α we will have

$$\begin{aligned} \delta A_\alpha &= i(\epsilon_b\epsilon^{ba}\theta^t)\gamma_\alpha(\theta\psi_a + \tfrac{1}{2}\gamma_\beta\theta^*\psi^{\beta}_a) + (\bar{\epsilon}_{\dot{a}}\theta^\dagger)\gamma_\alpha(-\theta^*\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}} + \tfrac{1}{2}\gamma^\beta\theta\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}\beta}) \\ &= \tfrac{i}{2}\bar{\epsilon}_{\dot{a}}(\theta^\dagger\gamma_\alpha\gamma_{\bar{\alpha}}\theta)\epsilon^{\dot{a}\dot{b}}\bar{\psi}^{\bar{\alpha}}_{\dot{b}} = i\bar{\epsilon}_{\dot{a}}\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}\alpha} \, . \end{aligned} \tag{3.30}$$

Similarly

$$\begin{aligned} \delta A_{\bar{\alpha}} &= i\epsilon_a\epsilon^{ab}\psi_{b\bar{\alpha}} \\ \delta\phi_I &= -i\bar{\epsilon}_{\dot{a}}\bar{\sigma}^{\dot{a}b}_I\psi_b - i\epsilon_b\bar{\sigma}^{\dot{a}b}_I\bar{\psi}_{\dot{a}} \, . \end{aligned} \tag{3.31}$$

Next consider the fermions. From

$$\delta(\theta\psi_a + \tfrac{1}{2}\gamma_\beta\theta^*\psi^\beta_a) = (\tfrac{1}{2}\gamma_{\mu\nu}F^{\mu\nu})\theta\epsilon_a + \gamma_\mu D^\mu\phi_I\sigma^I_{a\dot{b}}(-\theta^*\epsilon^{\dot{b}\dot{a}}\bar{\epsilon}_{\dot{a}}) + [\phi_I,\phi_J](\sigma^{IJ})_a{}^b\theta(\epsilon_b) \, , \tag{3.32}$$

we have

$$\begin{aligned} \delta\psi_a &= \epsilon_a\delta_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} + [\phi_I,\phi_J](\sigma^{IJ})_a{}^b\epsilon_b \\ \delta\psi_{a\bar{\alpha}} &= -\tfrac{1}{2}\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}}F^{\bar{\beta}\bar{\gamma}}\epsilon_a - 2\bar{\epsilon}_{\dot{a}}\epsilon^{\dot{b}\dot{a}}\sigma^I_{a\dot{b}}D_{\bar{\alpha}}\phi_I = -\tfrac{1}{2}\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}}F^{\bar{\beta}\bar{\gamma}}\epsilon_a - 2\bar{\epsilon}_{\dot{a}}\epsilon_{ba}\bar{\sigma}^{I\dot{a}b}D_{\bar{\alpha}}\phi_I \, , \end{aligned}$$

and so, since $\epsilon_{\dot{a}\dot{d}}\epsilon^{\dot{b}\dot{c}}(\bar{\sigma}^{IJ})^{\dot{a}}{}_{\dot{b}} = -(\bar{\sigma}^{IJ})^{\dot{c}}{}_{\dot{d}}$,

$$\begin{aligned} \delta\bar{\psi}_{\dot{a}} &= -\bar{\epsilon}_{\dot{a}}\delta_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} - [\phi_I,\phi_J](\bar{\sigma}^{IJ})^{\dot{b}}{}_{\dot{a}}\bar{\epsilon}_{\dot{b}} \\ \delta\bar{\psi}_{\dot{a}\alpha} &= -\tfrac{1}{2}\Omega_{\alpha\beta\gamma}F^{\beta\gamma}\bar{\epsilon}_{\dot{a}} - 2\epsilon_b\epsilon^{ab}\sigma^I_{a\dot{a}}D_\alpha\phi_I \, . \end{aligned}$$

To close the supersymmetry algebra off-shell – i.e., without using the equations of motion – we introduce the following auxiliary fields into the supersymmetry transformations,

$$H = i\delta_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} \quad \text{and} \quad H_\alpha = \tfrac{i}{2}\Omega_{\alpha\beta\gamma}F^{\beta\gamma} \, .$$

As we will see, these definitions are consistent with the field equations for H and H_α in (3.36). The transformations of these auxiliary fields are obtained by demanding the off-shell closure of the supersymmetry transformations up to a gauge transformation.

The two successive supersymmetry transformations δ_1 and δ_2, act on A_α as

$$(\delta_1\delta_2 - \delta_2\delta_1)A_\alpha = 2i(\bar{\epsilon}^1_{\dot{a}}\epsilon^2_b - \bar{\epsilon}^2_{\dot{a}}\epsilon^1_b)D_\alpha\phi_I\,,$$

which is an infinitesimal pure gauge transformation. However, an infinitesimal gauge transformation of the gauge field, $A_\mu \to A_\mu + i\alpha D_\mu\phi$, induces the gauge transformation of $\psi \to \psi - i\alpha[\phi, \psi]$ on the fields. Therefore, the above operator should act on spinors as (for the two successive transformations give a gauge transformation)

$$\begin{aligned}
(\delta_1\delta_2 - \delta_2\delta_1)\psi_a &= -2i(\bar{\epsilon}^1_{\dot{b}}\epsilon^2_b - \bar{\epsilon}^2_{\dot{b}}\epsilon^1_b)\bar{\sigma}^{I\dot{b}b}[\phi_I, \psi_a] \\
(\delta_1\delta_2 - \delta_2\delta_1)\bar{\psi}_{\dot{a}\alpha} &= -2i(\bar{\epsilon}^1_{\dot{b}}\epsilon^2_b - \bar{\epsilon}^2_{\dot{b}}\epsilon^1_b)\bar{\sigma}^{I\dot{b}b}[\phi_I, \bar{\psi}_{\dot{a}\alpha}]\,.
\end{aligned}$$

Consequently the supersymmetry transformations of the auxiliary fields are worked out to be

$$\begin{aligned}
\delta H &= \epsilon_a\bar{\sigma}^{I\dot{a}a}[\phi_I, \bar{\psi}_{\dot{a}}] - \bar{\epsilon}_{\dot{a}}\bar{\sigma}^{I\dot{a}a}[\phi_I, \psi_a] \\
\delta H_\alpha &= -4\epsilon_b\epsilon^{ab}D_\alpha\psi_a + 2\epsilon_b\bar{\sigma}^{I\dot{a}b}[\phi_I, \bar{\psi}_{\dot{a}\alpha}]\,.
\end{aligned}$$

Introduce the generators, Q and $\bar{Q}$, by

$$\delta = i\bar{\epsilon}_{\dot{a}}\bar{Q}^{\dot{a}} - i\epsilon_a Q^a\,. \tag{3.33}$$

Then Q and $\bar{Q}$ act by commutator (anticommutator) on bosonic (fermionic) fields; i.e.,

$$\delta A = i\bar{\epsilon}_{\dot{a}}\{\bar{Q}^{\dot{a}}, A\} - i\epsilon_a\{Q^a, A\}\,.$$

The sign is such that δ is hermitian. This enables us to rewrite all the above supersymmetry transformations more succinctly as:

Field	$\bar{Q}^{\dot{a}}$	Q^{a}
ψ_b	0	$H\delta^a_b + i\sigma^{IJa}{}_b[\phi_I, \phi_J]$
$\bar{\psi}_{\dot{b}}$	$H\delta^{\dot{a}}_{\dot{b}} + i\bar{\sigma}^{IJ\dot{a}}{}_{\dot{b}}[\phi_I, \phi_J]$	0
$\psi_{b\bar{\alpha}}$	$2iD_{\bar{\alpha}}\phi_I\bar{\sigma}^{I\dot{a}a}\epsilon_{ab}$	$H_{\bar{\alpha}}\delta^a_b$
$\bar{\psi}_{\dot{b}\alpha}$	$H_\alpha\delta^{\dot{a}}_{\dot{b}}$	$2iD_\alpha\phi_I\epsilon^{ab}\sigma^I_{b\dot{b}}$
A_α	$\epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}\alpha}$	0
$A_{\bar{\alpha}}$	0	$-\epsilon^{ab}\psi_{b\bar{\alpha}}$
ϕ_I	$-\bar{\sigma}_I^{\dot{a}b}\psi_b$	$\bar{\sigma}_I^{\dot{b}a}\bar{\psi}_{\dot{b}}$
H	$i\bar{\sigma}^{I\dot{a}b}[\phi_I, \psi_b]$	$i\bar{\sigma}^{I\dot{b}a}[\phi_I, \bar{\psi}_{\dot{b}}]$
H_α	0	$4i\epsilon^{ab}D_\alpha\psi_b + 2i\bar{\sigma}^{I\dot{b}a}[\phi_I, \bar{\psi}_{\dot{b}\alpha}]$
$H_{\bar{\alpha}}$	$4i\epsilon^{\dot{a}\dot{b}}D_{\bar{\alpha}}\bar{\psi}_{\dot{b}} + 2i\bar{\sigma}^{I\dot{a}b}[\phi_I, \psi_{b\bar{\alpha}}]$	0

3.3.3 The cohomology classes of the BRST operator

In this subsection, first we show that the theory defined by the Lagrangian (3.26) is in fact a cohomological theory. This is done by showing that the action, up to a topological term, can be written in a BRST exact way. The BRST cohomology classes are then obtained.

The following definitions will break the explicit $SO(3,1)$ covariance; however, they are useful in constructing the cohomology classes of the BRST operator. Introduce

$$\begin{aligned} \eta &= -\bar{\psi}_{\dot{2}}\,,\ \zeta = \bar{\psi}_{\dot{1}}\,,\ \psi_\alpha = \bar{\psi}^{\dot{1}}_\alpha\,,\ \chi_\alpha = \bar{\psi}^{\dot{2}}_\alpha \\ \varphi &= \phi_0 - \phi_3\,,\ \varphi' = \phi_0 + \phi_3\,,\ \phi_\pm = \phi_1 \pm i\phi_2\,. \end{aligned} \tag{3.34}$$

Further, define

$$Q \equiv i\bar{Q}^{\dot{1}} - iQ^1\,,\quad \tilde{Q} \equiv i\bar{Q}^{\dot{2}} - iQ^2\,.$$

Let

$$\begin{aligned} V &= \frac{1}{8}\left\{\chi_\alpha(-iH^\alpha - \Omega^{\alpha\beta\gamma}F_{\beta\gamma}) + 2\chi^\alpha D_\alpha\phi_+ - 2\varphi' D_\alpha\psi^\alpha - 2\varphi'[\phi_+, \eta]\right\} \\ &- \frac{1}{4}\zeta\left\{iH + 2ik^{\bar{\beta}\alpha}F_{\alpha\beta} + \frac{1}{2}[\phi_-, \phi_+] - \frac{1}{2}[\varphi, \varphi']\right\}. \end{aligned} \tag{3.35}$$

then, noticing $\{Q^1, V\} = 0$, we have that

$$
\begin{aligned}
\{Q, V\} &= \{(i\bar{Q}^{\bar{1}} - iQ^1), V\} = \{i\bar{Q}^{\bar{1}}, V\} \\
&= \frac{1}{8}\left\{-iH_\alpha(-iH^\alpha - \Omega^{\alpha\beta\gamma}F_{\beta\gamma}) - \chi_\alpha(-2[\varphi, \chi_\alpha] + 4D^\alpha\eta + 2[\psi^\alpha, \phi_-] - \Omega^{\alpha\beta\gamma}D_{[\beta}\psi_{\gamma]})\right\} \\
&- \frac{1}{2}D^\alpha\phi_- D_\alpha\phi_+ - \frac{i}{2}\chi^\alpha D_\alpha\bar{\eta} - \frac{i}{4}\chi_\alpha[\psi_\alpha, \phi_+] \\
&+ \frac{i}{2}\bar{\zeta}D_\alpha\psi^\alpha - \frac{i}{4}\varphi'[\psi_\alpha, \psi^\alpha] - \frac{1}{2}\varphi' D_\alpha D^\alpha\varphi + \frac{i}{2}\bar{\zeta}[\phi_+, \eta] - \frac{i}{2}\varphi'[\bar{\eta}, \eta] + \frac{1}{4}\varphi'[\phi_+, [\varphi, \phi_-]] \\
&- \frac{1}{4}(iH - \frac{1}{2}[\phi_-, \phi_+] + \frac{1}{2}[\varphi, \varphi'])(iH + 2ik^{\bar{\beta}\alpha}F_{\alpha\beta} + \frac{1}{2}[\phi_-, \phi_+] - \frac{1}{2}[\varphi, \varphi']) \\
&+ \frac{i}{2}\zeta\left([\varphi, \bar{\zeta}] + [\phi_-, \bar{\eta}] - ik^{\bar{\beta}\alpha}D_{\bar{\beta}}\psi_\alpha\right) .
\end{aligned}
$$

Adding the complex conjugate part, we define

$$\bar{\mathcal{L}} \equiv \{Q, (V + \bar{V})\} .$$

H_α and H have no dynamics and can be solved algebraically

$$H_\alpha = \tfrac{i}{2}\Omega_{\alpha\beta\gamma}F^{\beta\gamma} \quad , \quad H = k_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} . \tag{3.36}$$

Substituting these solutions back to $\bar{\mathcal{L}}$ we obtain

$$\bar{\mathcal{L}} \;=\; \mathcal{L} - \tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + \tfrac{1}{2}F_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} - \tfrac{1}{2}(k^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}})^2 , \tag{3.37}$$

where use has been made of

$$\Omega_{\alpha\beta\gamma}\Omega^{\alpha}{}_{\bar{\beta}\bar{\gamma}} = 8(k_{\gamma\bar{\beta}}k_{\beta\bar{\gamma}} - k_{\gamma\bar{\gamma}}k_{\beta\bar{\beta}}) .$$

Interestingly, the integral of the extra terms in (3.37) turns out to be a topological invariant [46] [1]

$$-\tfrac{1}{2}\int k \wedge \mathrm{tr}\,(F \wedge F) = \tfrac{1}{2}\int \mathrm{tr}\,(F_{\alpha\beta}F^{\alpha\beta} - F_{\alpha\bar{\beta}}F^{\alpha\bar{\beta}} + (k^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}})^2)$$

which only depends on the topological class of the vector bundle E and the Kähler class[2] of the metric. If we change the metric in its class, k at most changes by an exact form

[1]There is a typographical mistake in the equation (16.7.14) page 539 of the above reference.

[2]Given a Kähler metric g on the manifold X, the corresponding Kähler form, since it is a closed 2-form, defines an element of $H^2(X, \mathbf{R})$. This element is called the Kähler class of the Kähler metric g [46].

and thus the integral remains invariant. It is straightforward to show the above identity. Let us write k and F in complex coordinates

$$k = \tfrac{1}{2}k_{\mu\nu}\ dx^\mu \wedge dx^\nu = k_{\alpha\bar\beta}\ dx^\alpha \wedge dx^{\bar\beta}$$
$$F = \tfrac{1}{2}F_{\mu\nu}\ dx^\mu \wedge dx^\nu = F_{\alpha\bar\beta}\ dx^\alpha \wedge dx^{\bar\beta} + \tfrac{1}{2}F_{\alpha\beta}\ dx^\alpha \wedge dx^\beta + \tfrac{1}{2}F_{\bar\alpha\bar\beta}\ dx^{\bar\alpha} \wedge dx^{\bar\beta}\,.$$

Hence

$$F\wedge F = F_{\alpha\bar\beta}F_{\rho\bar\sigma}dx^\alpha\wedge dx^{\bar\beta}\wedge dx^\rho\wedge dx^{\bar\sigma} + \tfrac{1}{2}F_{\alpha\beta}F_{\bar\rho\bar\sigma}dx^\alpha\wedge dx^\beta\wedge dx^{\bar\rho}\wedge dx^{\bar\sigma}\,.$$

The integrand now can be written as

$$\begin{aligned}
k\wedge F\wedge F &= (k_{\eta\bar\zeta}F_{\alpha\bar\beta}F_{\rho\bar\sigma})\ dx^\eta\wedge dx^{\bar\zeta}\wedge dx^\alpha\wedge dx^{\bar\beta}\wedge dx^\rho\wedge dx^{\bar\sigma} \\
&+ \tfrac{1}{2}(k_{\eta\bar\zeta}F_{\alpha\beta}F_{\bar\rho\bar\sigma})\ dx^\eta\wedge dx^{\bar\zeta}\wedge dx^\alpha\wedge dx^\beta\wedge dx^{\bar\rho}\wedge dx^{\bar\sigma} \\
&= \left\{\varepsilon^{\eta\bar\zeta\alpha\bar\beta\rho\bar\sigma}k_{\eta\bar\zeta}F_{\alpha\bar\beta}F_{\rho\bar\sigma} + \tfrac{1}{2}\varepsilon^{\eta\bar\zeta\alpha\beta\bar\rho\bar\sigma}k_{\eta\bar\zeta}F_{\alpha\beta}F_{\bar\rho\bar\sigma}\right\} d^6x \\
&= \left\{i\varepsilon^{\eta\alpha\rho}\varepsilon^{\bar\beta\bar\zeta\bar\sigma}(ig_{\eta\bar\zeta})F_{\alpha\bar\beta}F_{\rho\bar\sigma} + \tfrac{i}{2}\varepsilon^{\eta\alpha\beta}\varepsilon^{\bar\zeta\bar\rho\bar\sigma}(ig_{\eta\bar\zeta})F_{\alpha\beta}F_{\bar\rho\bar\sigma}\right\} d^6x \\
&= \left\{(g^{\alpha\bar\beta}g^{\rho\bar\sigma} - g^{\alpha\bar\sigma}g^{\rho\bar\beta})F_{\alpha\bar\beta}F_{\rho\bar\sigma} - \tfrac{1}{2}(g^{\alpha\bar\rho}g^{\beta\bar\sigma} - g^{\alpha\bar\sigma}g^{\beta\bar\rho})F_{\alpha\beta}F_{\bar\rho\bar\sigma}\right\}\sqrt{g}d^6x \\
&= \left\{(F_{\alpha\bar\beta}F^{\alpha\bar\beta}) - (F_{\alpha\beta}F^{\alpha\beta}) - (k^{\alpha\bar\beta}F_{\alpha\bar\beta})^2\right\}\sqrt{g}d^6x\,,
\end{aligned} \tag{3.38}$$

using the shorthand notation d^6x for $dx^1\wedge dx^2\wedge dx^3\wedge dx^4\wedge dx^5\wedge dx^6$. This proves the identity of (3.37), and shows that on a compact Calabi-Yau manifold the Lagrangian, up to a topological term, can be written in a BRST exact form

$$\mathcal{L} + \tfrac{1}{2}k\wedge \mathrm{tr}\,(F\wedge F) = \{Q, (V+\bar V)\}\,. \tag{3.39}$$

Using the supercharges Q^2 and $\bar Q^{\dot 2}$, we now show that V itself can also be written as a BRST commutator. In fact we can write

$$V = \{(i\bar Q^{\dot 2} - iQ^2), W\} = \{\tilde Q, W\}\,,$$

where

$$W = \frac{1}{8}\left\{\chi_\alpha\psi^\alpha - i\phi_+(H + 2ik^{\bar\beta\alpha}F_{\alpha\bar\beta} - \tfrac{1}{2}[\varphi,\varphi']) + \tfrac{i}{2}\Omega^{\alpha\beta\gamma}\mathsf{CS}(A)_{\alpha\beta\gamma}\right\}\,,$$

with

$$\mathsf{CS}(A)_{\alpha\beta\gamma} = A_\alpha F_{\beta\gamma} - \tfrac{1}{3}A_\alpha[A_\beta, A_\gamma]\,. \tag{3.40}$$

Similarly,

$$\bar V = -\{\tilde Q, \bar W\}\,.$$

By this, we have been able to write the action quadratic in BRST charges. Although this is not still in a balanced form (see the next part for the definition), it is interesting to note that there are different ways of writing the action quadratic in Q's.

Suppose the gauge group has rank r. Since $\{Q,\varphi\}=0$, we can write the first obvious BRST and gauge invariant operator which is just $\mathrm{tr}\,\varphi^r$. In the following, we use the language of forms for clarity and compactness.
First note that

$$2D\varphi = 2(D_\alpha\varphi dx^\alpha + D_{\bar\alpha}\varphi dx^{\bar\alpha}) = -i\{Q^1,\psi_\alpha\}dx^\alpha + i\{\bar{Q}^1,\psi_{\bar\alpha}dx^{\bar\alpha}\} = \{Q,\psi\}\,.$$

Likewise, since

$$\begin{aligned}
\{\bar{Q}^1, F_{\alpha\beta}\} &= D_{[\alpha}\psi_{\beta]} \\
\{Q^1, F_{\bar\alpha\bar\beta}\} &= -D_{[\bar\alpha}\psi_{\bar\beta]} \\
\{Q^1, F_{\alpha\bar\beta}\} &= -D_\alpha\psi_{\bar\beta} \\
\{\bar{Q}^1, F_{\alpha\bar\beta}\} &= -D_{\bar\beta}\psi_\alpha\,,
\end{aligned}$$

we find that

$$\begin{aligned}
\{Q,F\} &= \frac{i}{2}D_{[\alpha}\psi_{\beta]}dx^\alpha\wedge dx^\beta + \frac{i}{2}D_{[\bar\alpha}\psi_{\bar\beta]}dx^{\bar\alpha}\wedge dx^{\bar\beta} \\
&+ (iD_\alpha\psi_{\bar\beta} - iD_{\bar\beta}\psi_\alpha)dx^\alpha\wedge dx^{\bar\beta} \\
&= \frac{i}{2}D_{[\mu}\psi_{\nu]}dx^\mu\wedge dx^\nu = iD\psi\,.
\end{aligned}$$

For the example of $SU(2)$ gauge group we have the familiar BRST cohomology classes. We start with the BRST and gauge invariant operator $\mathrm{tr}\,\varphi^2(x)$. Differentiating this operator with respect to x we obtain

$$d(\mathrm{tr}\,\varphi^2(x)) = \{Q,\varphi\psi\}\,.$$

Repeating this procedure results in a set of BRST invariant operators,

$$\mathcal{O}^{(\gamma)} = \int_\gamma W_{k_\gamma}\,,$$

where γ is a k_γ dimensional homology cycle on the six-manifold, and W_{k_γ} are differential forms of degree k_γ defined by

$$W_0 = \tfrac{1}{2}\mathrm{tr}\,\varphi^2$$

$$
\begin{aligned}
W_1 &= \mathrm{tr}\,(\varphi\psi)\\
W_2 &= \mathrm{tr}\,(\tfrac{1}{4}\psi^2 - i\varphi F)\\
W_3 &= \mathrm{tr}\,(\psi F)\\
W_4 &= -\tfrac{1}{2}\mathrm{tr}\,F^2\,.
\end{aligned}
$$

Here we have omitted the wedge product sign, so for example F^2 stands for $F\wedge F$. As in the second chapter, it is easy to see that $\mathcal{O}^k$'s are indeed BRST invariant

$$
\{Q,\mathcal{O}^k\} = \int_{\gamma_k}\{Q,W_k\} = \int_{\gamma_k} dW_{k-1} = 0\,.
$$

For gauge groups of higher rank one can construct more BRST invariant operators. For instance, for $SU(3)$ there is an additional gauge and BRST invariant operator, that is $\frac{1}{3!}\mathrm{tr}\,\varphi^3(x)$. One can construct these operators just by successive differentiating of this operator:

$$
\begin{aligned}
&\tfrac{1}{3!}d\mathrm{tr}\,\varphi^3(x) = \tfrac{1}{4}\{Q,\mathrm{tr}\,(\varphi^2\psi)\}\\
&d\mathrm{tr}\,(\varphi^2\psi) = \{Q,\tfrac{1}{2}\mathrm{tr}\,(\varphi\psi^2 - i\varphi^2 F)\}\\
&d\mathrm{tr}\,(\tfrac{1}{2}\varphi\psi^2 - i\varphi^2 F) = \tfrac{1}{2}\{Q,\mathrm{tr}\,(\tfrac{1}{3!}\psi^3 - i\varphi[F,\psi])\}\\
&d\mathrm{tr}\,(\tfrac{1}{3!}\psi^3 - i\varphi[F,\psi]) = -\{Q,\mathrm{tr}\,(\tfrac{i}{2}\psi^2 F + \varphi F^2)\}\\
&d\mathrm{tr}\,(\tfrac{i}{2}\psi^2 F + \varphi F^2) = \tfrac{1}{2}\{Q,\mathrm{tr}\,(\psi F^2)\}\\
&d\mathrm{tr}\,(\psi F) = -\tfrac{i}{3}\{Q,\mathrm{tr}\,(F^3)\}\,,
\end{aligned}
$$

note that $[F,\psi] = F\wedge\psi + \psi\wedge F$. Thus for $SU(3)$ gauge group we have additional BRST invariant operators

$$
\mathcal{O}'^{(\gamma)} = \int_\gamma W'_{k_\gamma}
$$

with

$$
\begin{aligned}
W'_0 &= \tfrac{1}{3!}\mathrm{tr}\,\varphi^3(x)\\
W'_1 &= \mathrm{tr}\,(\varphi^2\psi)\\
W'_2 &= \mathrm{tr}\,(\varphi\psi^2 - i\varphi^2 F)\\
W'_3 &= \mathrm{tr}\,(\tfrac{1}{3!}\psi^3 - i\varphi[F,\psi])\\
W'_4 &= \mathrm{tr}\,(\tfrac{i}{2}\psi^2 F + \varphi F^2)\\
W'_5 &= \mathrm{tr}\,(\psi F^2)\\
W'_6 &= -\tfrac{i}{3!}\mathrm{tr}\,(F^3)\,.
\end{aligned}
$$

Clearly W_6' is just the third Chern class of the $SU(3)$ bundle.

3.3.4 A balanced cohomological field theory

A balanced topological field theory [7] has the properties of being invariant under the two topological symmetries as well as having a global sl_2 symmetry. The two BRST supercharges, d^1 and d^2, transform as a doublet under this global symmetry. The important fact about these topological theories is that the action can be obtained from an action potential W

$$S = d^1 d^2 W\,,$$

such that the critical points of this potential are identical to the fixed points of the BRST transformations.

In this subsection we will show that our theory is indeed a balanced topological field theory. To write the action in balanced form, we first choose the supercharges such that they transform as doublets under the $SO(2,1)$ subgroup of $SO(3,1)$. Under $SO(3,1)$, the spinors transform as

$$\delta\psi_a = \omega_{IJ}(\sigma^{IJ})_a{}^b\psi_b\,,$$

where ω_{IJ} are the rotation parameters. Under the $SO(2,1)$ subgroup generated by $\sigma^{01} = \frac{1}{2}\sigma^1, \sigma^{02} = \frac{1}{2}\sigma^2$ and $\sigma^{12} = \frac{i}{2}\sigma^3$,

$$\delta\psi_a = \tfrac{1}{2}(\omega_{01}\sigma^1 + \omega_{02}\sigma^2 + i\omega_{12}\sigma^3)_a{}^b\psi_b\,.$$

Or, introducing $\omega_\pm = \frac{1}{2}(\omega_{01} \pm i\omega_{02})$ and $\omega_0 = \frac{1}{2}\omega_{12}$,

$$\begin{aligned}\delta\psi_1 &= \omega_-\psi_2 + i\omega_0\psi_1\\ \delta\psi_2 &= \omega_+\psi_1 - i\omega_0\psi_2\,.\end{aligned}$$

Upon conjugation we get

$$\begin{aligned}\delta\bar\psi_{\dot 1} &= \omega_+\bar\psi_{\dot 2} - i\omega_0\bar\psi_{\dot 1}\\ \delta\bar\psi_{\dot 2} &= \omega_-\bar\psi_{\dot 1} + i\omega_0\bar\psi_{\dot 2}\,.\end{aligned}$$

Hence, under $SO(2,1)$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \sim \begin{pmatrix}\bar\psi_{\dot 2}\\ \bar\psi_{\dot 1}\end{pmatrix}.$$

Using $\psi^a = \epsilon^{ab}\psi_b$ and working similarly, we get

$$\begin{pmatrix} -\psi^2 \\ \psi^1 \end{pmatrix} \sim \begin{pmatrix} \bar{\psi}^{\dot{1}} \\ -\bar{\psi}^{\dot{2}} \end{pmatrix} .$$

Therefore, the spinorial charge

$$d^A = i \begin{pmatrix} (\bar{Q}^{\dot{1}} - Q^2) \\ (\bar{Q}^{\dot{2}} - Q^1) \end{pmatrix}$$

also transforms as a doublet of $SO(2,1)$. In the following we show that the action can be written in an $SO(2,1)$ invariant way,

$$S + \tfrac{1}{4} \int k \wedge F \wedge F = \tfrac{1}{2}\epsilon_{AB} d^A d^B W ,$$

where

$$\begin{aligned} W &= \frac{1}{2}\phi_3\, k^{\bar{\beta}\alpha} F_{\alpha\bar{\beta}} + \frac{1}{4}(\eta\zeta + \bar{\zeta}\bar{\eta}) - \frac{1}{8}\left(\psi^\alpha \psi_\alpha + \chi_\alpha \chi^\alpha\right) \\ &+ \frac{i}{16}\Omega^{\alpha\beta\gamma}(A_\alpha F_{\beta\gamma} - \frac{1}{3}A_\alpha[A_\beta, A_\gamma]) - \frac{i}{16}\Omega_{\alpha\beta\gamma}(A^\alpha F^{\beta\gamma} - \frac{1}{3}A^\alpha[A^\beta, A^\gamma]) . \end{aligned}$$

such that the critical points of W are identical to the fixed points of the BRST action.

First note that, apart from those topological terms, action can be written as

$$\bar{S} = d^1 V' = i(\bar{Q}^{\dot{1}} - Q^2)V' ,$$

where V' is

$$\begin{aligned} V' &= \frac{1}{8}\left\{\chi_\alpha(iH^\alpha - \Omega^{\alpha\beta\gamma}F_{\beta\gamma}) + \psi^\alpha(iH_\alpha + \Omega_{\alpha\beta\gamma}F^{\beta\gamma}) + 2\chi^\alpha D_\alpha \phi_+ + 2\psi_\alpha D^\alpha \phi_+\right\} \\ &+ \frac{1}{4}\left\{-\phi' D_\alpha \psi^\alpha - \phi D^\alpha \chi_\alpha - \phi'[\phi_+, \eta] + \phi[\phi_+, \bar{\zeta}]\right\} \\ &- \frac{1}{4}\zeta\left(iH + 2iF_{\alpha\bar{\beta}}k^{\bar{\beta}\alpha} + \frac{1}{2}[\phi_-, \phi_+] - \frac{1}{2}[\phi, \phi']\right) \\ &- \frac{1}{4}\bar{\eta}\left(iH + 2iF_{\alpha\bar{\beta}}k^{\bar{\beta}\alpha} - \frac{1}{2}[\phi_-, \phi_+] - \frac{1}{2}[\phi, \phi']\right) . \end{aligned}$$

Now V' in turn can be written as a BRST exact term

$$V' = d^2 W = i(\bar{Q}^{\dot{2}} - Q^1)W .$$

In the variation of the Chern-Simons term we note that

$$\{Q^1, \Omega^{\alpha\beta\gamma}(A_\alpha F_{\beta\gamma} - \frac{1}{3}A_\alpha[A_\beta, A_\gamma])\} = 0 ,$$

and

$$
\begin{aligned}
&\{i\bar{Q}^{\dot{2}}\,,\,\Omega^{\alpha\beta\gamma}(A_\alpha F_{\beta\gamma} - \frac{1}{3}A_\alpha[A_\beta, A_\gamma])\} \\
&= \Omega^{\alpha\beta\gamma}\left(i\chi_\alpha F_{\beta\gamma} - i\chi_\gamma D_\beta A_\alpha + i\chi_\beta D_\gamma A_\alpha - i\chi_\alpha[A_\beta, A_\gamma]\right) \\
&= \Omega^{\alpha\beta\gamma}\left(i\chi_\alpha F_{\beta\gamma} + i\chi_\alpha\left\{D_\beta A_\gamma - D_\gamma A_\beta - [A_\beta, A_\gamma]\right\}\right) \\
&= 2i\Omega^{\alpha\beta\gamma}\chi_\alpha F_{\beta\gamma}\,. \qquad (3.41)
\end{aligned}
$$

It now can be seen that the critical points of W are the same as the fixed points of the supersymmetry transformations. Differentiating W with respect to ϕ_3 and A_α respectively yields

$$
\begin{aligned}
&k^{\bar{\beta}\alpha}F_{\alpha\bar{\beta}} = 0 \\
&\frac{1}{4}\Omega^{\alpha\beta\gamma}F_{\beta\gamma} + D^\alpha\phi_3 = 0\,.
\end{aligned}
$$

For a compact Calabi-Yau manifold, the second equation, after squaring, reduces to $F_{\alpha\beta} = 0$. Together with the complex conjugate equations we have the Kähler-Yang-Mills (Donaldson-Uhlenbeck-Yau) equations:

$$
\begin{aligned}
&k^{\bar{\beta}\alpha}F_{\alpha\bar{\beta}} = 0 \\
&F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0 \qquad (3.42)
\end{aligned}
$$

of course with $D_\alpha\phi_3 = D_{\bar{\alpha}}\phi_3 = 0$.

3.3.5 $N = 2$ reduction

The theory defined by the Lagrangian (3.26) admits a truncation consistent with some of the supersymmetry. If we want just to keep the Q^1 supersymmetry then we may set

$$
\eta = \bar{\eta} = \phi_+ = \phi_- = 0
$$

which is consistent with Q^1 symmetry but destroys Q^2 symmetry. From equation (3.39) it is clear that $N = 2$ reduced Lagrangian still can be written in a BRST exact form

$$
\mathcal{L} + \tfrac{1}{2}k \wedge \mathrm{tr}\,(F \wedge F) = \{Q, (V + \bar{V})\}
$$

where V is now

$$
V = \frac{1}{8}\left\{\chi_\alpha(-iH^\alpha - \Omega^{\alpha\beta\gamma}F_{\beta\gamma}) - 2\varphi' D_\alpha\psi^\alpha\right\} - \frac{1}{4}\zeta\left\{iH + 2ik^{\bar{\beta}\alpha}F_{\alpha\bar{\beta}} - \frac{1}{2}[\varphi, \varphi']\right\}\,. \qquad (3.43)
$$

Therefore, the truncated $N = 2$ theory is still a cohomological one; i.e. it is independent of the metric deformations which preserves the holonomy structure of the manifold. To see the fixed point equations of this cohomological theory, we write out the supersymmetry transformations in the table below.

Field	$\bar{Q}^{\mathrm{i}}$	Q^1
ζ	$H - \frac{i}{2}[\varphi, \varphi']$	0
$\bar{\zeta}$	0	$H + \frac{i}{2}[\varphi, \varphi']$
χ_α	$-H_\alpha$	0
$\chi_{\bar{\alpha}}$	0	$-H_{\bar{\alpha}}$
ψ_α	0	$2iD_\alpha\varphi$
$\psi_{\bar{\alpha}}$	$-2iD_{\bar{\alpha}}\varphi$	0
A_α	ψ_α	0
$A_{\bar{\alpha}}$	0	$-\psi_{\bar{\alpha}}$
φ	0	0
φ'	$-2\bar{\zeta}$	2ζ
H	$-i[\varphi, \bar{\zeta}]$	$-i[\varphi, \zeta]$
H_α	0	$2i[\varphi, \chi_\alpha]$
$H_{\bar{\alpha}}$	$2i[\varphi, \chi_{\bar{\alpha}}]$	0

The fixed points of the action of $\bar{Q}^{\mathrm{i}}$ on fermionic fields are obtained by setting

$$\{\bar{Q}^{\mathrm{i}}, \zeta\} = \{\bar{Q}^{\mathrm{i}}, \chi_\alpha\} = \{\bar{Q}^{\mathrm{i}}, \psi_\alpha\} = 0\,,$$

together with the complex conjugate variations, and upon using the equations of motion for H and H_α the fixed point equations are found to be

$$\begin{aligned} F_{\alpha\beta} &= F_{\bar{\alpha}\bar{\beta}} = 0 \\ k^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} &= \tfrac{i}{2}[\varphi, \varphi'] \\ D_\alpha\varphi &= D_{\bar{\alpha}}\varphi = 0\,. \end{aligned}$$

Therefore we conclude that the truncated $N = 2$ theory, like the unreduced $N = 4$ theory, localizes on the moduli space of solutions to Kähler-Yang-Mills (Donaldson-Uhlenbeck-Yau) equations.

3.3.6 The case of Kähler 3-fold

Before ending this section let us briefly discuss the twisting procedure on Kähler manifolds. Twisting on Kähler 3-folds has also been outlined in [47] from a group theoretical point of view without explicit derivation of the Lagrangian. A Kähler 3-fold has the holonomy of $SU(3) \times U(1)$ under which spinors transform as in (3.16). Recall that our six-dimensional theory has a global nonanomalous $SU(2)_V$ symmetry. Let us choose a $U(1)_V$ subgroup of this global symmetry with the transformations derived from (3.8) by setting $a_i = 0, v_1 = v_2 = 0$. With a normalization of the charges, the spinors in (3.27) transform under $U(1)_V$ as

$$(\psi_1, \psi_{1\bar{\alpha}}) \sim \mathbf{1}^{-3}, \quad (\psi_2, \psi_{2\bar{\alpha}}) \sim \mathbf{1}^{+3}.$$

Similarly the two complex scalars $\phi_+ \equiv \phi_1 + i\phi_2$ and $\phi_- \equiv \phi_1 - i\phi_2$ transform as

$$\phi_+ \sim \mathbf{1}^{+6}, \quad \phi_- \sim \mathbf{1}^{-6},$$

with A_α, ϕ_0 and ϕ_3 being invariant. Now we twist the $U(1)$ part of the holonomy group with $U(1)_V$ (simply by adding the $U(1)$ charges). Under the new holonomy group, fields transform as

$$\begin{aligned}
\psi_1 &\sim (\mathbf{1}^{-3}, \mathbf{1}^{-3}) \longrightarrow \mathbf{1}^{-6} \\
\psi_2 &\sim (\mathbf{1}^{-3}, \mathbf{1}^{+3}) \longrightarrow \mathbf{1}^{0} \\
\psi_{1\bar{\alpha}} &\sim (\mathbf{3}^{+1}, \mathbf{1}^{-3}) \longrightarrow \mathbf{3}^{-2} \\
\psi_{2\bar{\alpha}} &\sim (\mathbf{3}^{+1}, \mathbf{1}^{+3}) \longrightarrow \mathbf{3}^{+4} \\
\bar{\psi}_{\dot{1}} &\sim (\mathbf{1}^{+3}, \mathbf{1}^{+3}) \longrightarrow \mathbf{1}^{+6} \\
\bar{\psi}_{\dot{2}} &\sim (\mathbf{1}^{+3}, \mathbf{1}^{-3}) \longrightarrow \mathbf{1}^{0} \\
\bar{\psi}_{\dot{1}\alpha} &\sim (\bar{\mathbf{3}}^{-1}, \mathbf{1}^{+3}) \longrightarrow \bar{\mathbf{3}}^{+2} \\
\bar{\psi}_{\dot{2}\alpha} &\sim (\bar{\mathbf{3}}^{-1}, \mathbf{1}^{-3}) \longrightarrow \bar{\mathbf{3}}^{-4} \\
\phi_+ &\sim (\mathbf{1}^{0}, \mathbf{1}^{+6}) \longrightarrow \mathbf{1}^{+6} \\
\phi_- &\sim (\mathbf{1}^{0}, \mathbf{1}^{-6}) \longrightarrow \mathbf{1}^{-6}
\end{aligned}$$

with all other fields being invariant under twisting. Since we have only changed the $U(1)$ charges of the fields, the Lagrangian is the same as (3.26) with the difference that now the covariant derivatives have the appropriate $U(1)$ connection. Notice that, instead of four scalar supercharges in the Calabi-Yau case, we now have two scalar supercharges Q^1 and $\bar{Q}^1$ after twisting. This still allows us to write the Lagrangian as a BRST commutator just as in section 3.3.3,

$$\bar{\mathcal{L}} = \{Q, (V + \bar{V})\} .$$

Therefore, the twisted theory on Kähler manifold is a cohomological one.

3.4 Partial twisting on product six manifolds

Consider the product six manifold $M = X \times Y$, where X and Y are both three manifolds. We choose the metric to be diagonal $g = g_X \oplus g_Y$, thus the holonomy of M is just $SO(3) \times SO(3)$. Spinors sit in the $\mathbf{4}$ of $SU(4)$ the universal covering group of $SO(6)$. Under the $SU(2) \times SU(2)$ subgroup of $SU(4)$ this representation decomposes as

$$\mathbf{4} = \mathbf{2} \oplus \mathbf{2} .$$

Therefore under the nonanomalous part of the global symmetry $SU(2)_X \times SU(2)_Y \times SU(2)_V$, the spinors, ψ_{La} and $\psi_R^{\dot{a}}$, will transform as

$$\begin{aligned} \psi_{La} &\sim (\mathbf{2}, \mathbf{2}, \mathbf{2}) \\ \psi_R^{\dot{a}} &\sim (\mathbf{2}, \mathbf{2}, \mathbf{2}) . \end{aligned}$$

Obviously there is no twisting which leaves a scalar supercharge on the whole manifold. However, one may still twist partially [27] on one of the three manifolds, say Y, to get a scalar supercharge in the Y direction. Let $i, j, \cdots$ and $m, n, \cdots$ indicate the indices on X and Y respectively. Let us choose the following representation for the Gamma matrices

$$\begin{aligned} \gamma^i &= \sigma^i \otimes \mathbf{1} \otimes \sigma^1 \\ \gamma^m &= \mathbf{1} \otimes \sigma^m \otimes \sigma^2 , \end{aligned}$$

so that

$$\gamma_7 = -\mathbf{1} \otimes \mathbf{1} \otimes \sigma^3 , \quad \text{and} \quad C = i\sigma^2 \otimes i\sigma^2 \otimes \sigma^1 .$$

Now a spinor can be decomposed in general as

$$\psi = \psi^{\alpha\dot{\alpha}}\, e_\alpha \otimes e_{\dot{\alpha}} \otimes e_1 + \lambda^{\alpha\dot{\alpha}}\, e_\alpha \otimes e_{\dot{\alpha}} \otimes e_2 ,$$

where α and $\dot{\alpha}$ indicate the two dimensional representations of $SU(2)_X$ and $SU(2)_Y$ respectively. Moreover

$$\gamma_7 \psi = -\psi^{\alpha\dot{\alpha}}\ e_\alpha \otimes\ e_{\dot{\alpha}} \otimes\ e_1 + \lambda^{\alpha\dot{\alpha}}\ e_\alpha \otimes\ e_{\dot{\alpha}} \otimes\ e_2 \,.$$

Thus $\psi^{\alpha\dot{\alpha}}$ is a left-handed spinor while $\lambda^{\alpha\dot{\alpha}}$ is a right-handed one.

Our starting point is the Lagrangian (3.6), where now ψ_{La} and $\psi_R^{\dot{a}}$ are both bispinors. In the following we use the reality conditions (3.5) to write down the Lagrangian only in terms of $\psi^{\alpha\dot{\alpha}}$. The fermionic part of the Lagrangian then reads

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} &= \tfrac{i}{2}\epsilon^{ba}\psi^t_{Lb} C \gamma^\mu D_\mu \psi_{La} + h.c. \\ &+ i\epsilon^{ba}\psi^t_{Lb} C \sigma^I_{a\dot{c}}[\phi_I, C^{-1}\bar{\psi}^t_{L\dot{b}}\epsilon^{\dot{b}\dot{c}}] \,. \end{aligned}$$

Twisting on the Y component of the manifold essentially consists of promoting the global $SU(2)_V$ index a to a dot index $\dot{\alpha}$ and decomposing the resultant tensor product representation under $SU(2)'_Y$, the diagonal subgroup of $SU(2)_Y \times SU(2)_V$. Hence, by twisting we simply mean

$$\psi_{\alpha\dot{\alpha}a} \to \psi_{\alpha\dot{\alpha}\dot{\beta}} \,.$$

This in turn decomposes to a singlet and a triplet (in the Y direction) under the new space time symmetry $SU(2)_X \times SU(2)'_Y$,

$$\psi_{\alpha\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi_\alpha + \epsilon_{\dot{\alpha}\dot{\gamma}}\sigma^{m\dot{\gamma}}_{\dot{\beta}}\chi_{m\alpha} \,.$$

Note that $\epsilon_{\dot{\alpha}\dot{\gamma}}\sigma^{m\dot{\gamma}}_{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\gamma}}\sigma^{m\dot{\gamma}}_{\dot{\alpha}}$, thus we have

$$\psi_{\alpha\dot{\alpha}}{}^{\dot{\beta}} = \epsilon^{\dot{\beta}\dot{\gamma}}\psi_{\alpha\dot{\alpha}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\ \dot{\beta}}\psi_\alpha - \sigma^{m\dot{\beta}}_{\dot{\alpha}}\chi_{m\alpha} \,.$$

Let us see how the Lagrangian looks like in terms of these new fields. The fermionic kinetic terms become

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} &= \tfrac{i}{2}\left\{\epsilon^{\dot{\beta}\dot{\eta}}(\epsilon_{\dot{\alpha}\dot{\beta}}\psi_\alpha + \epsilon_{\dot{\alpha}\dot{\gamma}}\sigma^{m\dot{\gamma}}_{\dot{\beta}}\chi_{m\alpha})\epsilon^{\alpha\gamma}\sigma^i_{\gamma\beta}D_i\epsilon^{\dot{\alpha}\dot{\rho}}(\epsilon_{\dot{\rho}\dot{\eta}}\psi^\beta + \epsilon_{\dot{\rho}\dot{\delta}}\sigma^{n\dot{\delta}}_{\dot{\eta}}\chi^\beta_n)\right\} \\ &+ \tfrac{i}{2}\left\{i\epsilon^{\dot{\beta}\dot{\eta}}(\epsilon_{\dot{\alpha}\dot{\beta}}\psi_\alpha + \epsilon_{\dot{\alpha}\dot{\gamma}}\sigma^{m\dot{\gamma}}_{\dot{\beta}}\chi_{m\alpha})\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\rho}}\sigma^{n\dot{\delta}}_{\dot{\rho}}D_n(\epsilon_{\dot{\delta}\dot{\eta}}\psi_\beta + \epsilon_{\dot{\delta}\dot{\sigma}}\sigma^{r\dot{\sigma}}_{\dot{\eta}}\chi_{r\beta})\right\} + h.c. \\ &= -i\psi^\alpha\sigma^i_{\alpha\beta}D_i\psi^\beta + i\chi^\alpha_m\ \sigma^i_{\alpha\beta}D_i\chi^{m\beta} - 2\psi^\alpha D_m\chi^m_\alpha - i\epsilon^{mnr}\chi^\alpha_m D_n\chi_{r\alpha} + h.c.\,. \end{aligned}$$

The potential is

$$V = -i\bar{\psi}_{L\dot{a}}\bar{\sigma}^{I\dot{a}a}[\phi_I, \psi_{La}] = i(\bar{\psi}^\alpha\epsilon^{\dot{\alpha}\dot{\beta}} - \epsilon^{\dot{\alpha}\dot{\gamma}}\sigma^{m\dot{\beta}}_{\dot{\gamma}}\overline{\chi^\alpha_m})[\phi, (\epsilon_{\dot{\alpha}\dot{\beta}}\psi_\alpha + \epsilon_{\dot{\alpha}\dot{\eta}}\sigma^{n\dot{\eta}}_{\dot{\beta}}\chi_{n\alpha})]$$

$$
\begin{aligned}
&- \; i(\bar{\psi}^{\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} - \epsilon^{\dot{\alpha}\dot{\gamma}}\sigma^{n\dot{\beta}}_{\dot{\gamma}}\overline{\chi}^{\alpha}_{n})\sigma^{m\dot{\gamma}}_{\dot{\beta}}[\phi_m, (\epsilon_{\dot{\alpha}\dot{\gamma}}\psi_\alpha + \epsilon_{\dot{\alpha}\dot{\eta}}\sigma^{n\dot{\eta}}_{\dot{\gamma}}\chi_{n\alpha})] \\
&= \; 2i\bar{\psi}^{\alpha}[\phi, \psi_\alpha] - 2i\overline{\chi}^{\alpha}_{m}[\phi, \chi^m_\alpha] - 2i\bar{\psi}^{\alpha}[\phi_m, \chi^m_\alpha] + 2i\overline{\chi}^{\alpha}_{m}[\phi^m, \psi_\alpha] \\
&- \; 2\epsilon^{mnr}\overline{\chi}^{\alpha}_{m}[\phi_n, \chi_{r\alpha}],
\end{aligned}
$$

where we defined $\phi \equiv \phi_0$. The bosonic part of the Lagrangian is of course much easier to write

$$
\begin{aligned}
\mathcal{L}_{\mathrm{B}} \;&=\; -\tfrac{1}{4}\left\{F_{ij}F^{ij} + F_{mn}F^{mn} + 2F_{im}F^{im}\right\} \\
&-\; \tfrac{1}{2}\left\{D_i\phi_m D^i\phi^m + D_n\phi_m D^n\phi^m - D_i\phi D^i\phi - D_m\phi D^m\phi\right. \\
&+\; \left.\tfrac{1}{2}[\phi_m, \phi_n]^2 - [\phi_m, \phi]^2\right\}. \qquad (3.44)
\end{aligned}
$$

Let us now work out the supersymmetry transformations of the decomposed fields. Like $\psi_{L\dot{\beta}}$, the supersymmetry parameter $\alpha_{L\dot{\beta}}$, can be decomposed as

$$
\alpha_{L\dot{\beta}} = \alpha_{\alpha\dot{\alpha}\dot{\beta}}\; e^{\alpha}\otimes e^{\dot{\alpha}}\otimes e^{1}.
$$

Since we are only interested in the singlet part of the supersymmetry, we set $\alpha_{\alpha\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_\alpha$. The supersymmetry transformations (3.10) now look like

$$
\begin{aligned}
\delta A^i &= -2i\epsilon^{\alpha}\sigma^{i\beta}_{\alpha}\psi_\beta + 2i\bar{\psi}^{\alpha}\sigma^{i\beta}_{\alpha}\bar{\epsilon}_\beta \\
\delta A^m &= -4\epsilon^{\alpha}\chi^m_\alpha - 4\bar{\epsilon}^{\alpha}\overline{\chi}^m_\alpha \\
\delta\phi &= 2i\bar{\epsilon}^{\alpha}\psi_\alpha - 2i\epsilon^{\alpha}\bar{\psi}_\alpha \\
\delta\phi_m &= -2i\bar{\epsilon}^{\alpha}\chi_{\alpha m} + 2i\epsilon^{\alpha}\overline{\chi}_{\alpha m}.
\end{aligned}
$$

For spinors, note that by reality condition we have $\bar{\alpha}^a_R = \alpha^t_{Lb}C\epsilon^{ab}$. Thus

$$
\alpha_{R\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\epsilon}_\alpha\; e^{\alpha}\otimes e^{\dot{\alpha}}\otimes e^{2},
$$

and the transformation laws (3.7) become

$$
\begin{aligned}
\delta\psi_{La} \;&\equiv\; \delta\left((\epsilon_{\dot{\alpha}\dot{\beta}}\psi_\alpha + \sigma^{m\dot{\gamma}}_{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\gamma}}\chi_{m\alpha})\;\; e^{\alpha}\otimes e^{\dot{\alpha}}\otimes e^{1}\right) \\
&=\; \left\{\tfrac{i}{2}\epsilon^{ijk}F_{ij}(\sigma_k\otimes \mathbf{1}\otimes \mathbf{1})\delta^{\dot{\gamma}}_{\dot{\beta}} + \tfrac{i}{2}\epsilon^{mnr}F_{mn}(\mathbf{1}\otimes \sigma_r\otimes \mathbf{1})\delta^{\dot{\gamma}}_{\dot{\beta}}\right. \\
&+\; \left.F_{im}(\sigma^i\otimes \sigma^m\otimes i\sigma^3)\delta^{\dot{\gamma}}_{\dot{\beta}} + \tfrac{i}{2}\epsilon_{mnr}[\phi^m, \phi^n]\sigma^{r\dot{\gamma}}_{\dot{\beta}} - [\phi_m, \phi]\sigma^{m\dot{\gamma}}_{\dot{\beta}}\right\}\epsilon_\alpha\epsilon_{\dot{\alpha}\dot{\gamma}}(e^{\alpha}\otimes e^{\dot{\alpha}}\otimes e^{1}) \\
&-\; \left\{(D_i\phi\epsilon_{\dot{\alpha}\dot{\beta}} - D_i\phi_m\sigma^{m\dot{\gamma}}_{\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\alpha}})(\sigma^i\otimes \mathbf{1}\otimes \sigma^1)\right. \\
&+\; \left.(D_m\phi\epsilon_{\dot{\alpha}\dot{\beta}} - D_m\phi_n\sigma^{n\dot{\gamma}}_{\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\alpha}})(\mathbf{1}\otimes \sigma^m\otimes \sigma^2)\right\}\bar{\epsilon}_\alpha\;(e^{\alpha}\otimes e^{\dot{\alpha}}\otimes e^{2}).
\end{aligned}
$$

Multiplying first by $\epsilon^{\dot{\alpha}\dot{\beta}}$ and then by $\epsilon^{\dot{\beta}\dot{\rho}}(\sigma^s)_{\dot{\rho}}^{\ \dot{\alpha}}$ we conclude

$$
\begin{aligned}
\delta\psi_\alpha &= \tfrac{i}{2}\epsilon_{ijk}F^{ij}\sigma_\alpha^{k\beta}\epsilon_\beta - D_i\phi\ \sigma_\alpha^{i\beta}\bar{\epsilon}_\beta - iD_m\phi^m\bar{\epsilon}_\alpha \\
\delta\chi_{m\alpha} &= -\tfrac{i}{2}\epsilon_{mnr}F^{nr}\epsilon_\alpha - iF_{im}\sigma_\alpha^{i\beta}\epsilon_\beta - D_i\phi_m\sigma_\alpha^{i\beta}\bar{\epsilon}_\beta - iD_m\phi\ \bar{\epsilon}_\alpha \\
&- \epsilon_{mnr}D^n\phi^r\bar{\epsilon}_\alpha + (\tfrac{i}{2}\epsilon_{mnr}[\phi^n,\phi^r] - [\phi_m,\phi])\epsilon_\alpha\,.
\end{aligned}
$$

Supersymmetry

It is not clear whether the action is invariant under the supersymmetry transformations we just wrote down. Supersymmetry transformations require the existence of a globally defined spinor ϵ_α on X. On the other hand, to have an invariant action under these transformations, the supersymmetry parameter must be covariantly constant. Thus the holonomy of X must be such to admit a covariantly constant spinor. For instance, one may consider 3-manifolds with $U(1)$ holonomy. For simplicity, let us take X to be flat $\mathbf{R}^3$ so that the above supersymmetry transformations make sense. Even so, in verifying the supersymmetry of the action, one may meet commutators of covariant derivatives and the Riemann tensor of Y might appear. In our case this happens in the variation of the following terms

$$
\begin{aligned}
&\delta(-i\epsilon^{mnr}\chi_m^\alpha D_n\chi_{r\alpha}) = 2i\chi^{m\alpha}(D_nD^n\phi_m - D_nD_m\phi^n)\bar{\epsilon}_\alpha + \cdots \\
&\delta(\tfrac{1}{2}\phi_m D_nD^n\phi^m) = -2i\chi_m^\alpha D_nD^n\phi^m\bar{\epsilon}_\alpha + \cdots \\
&\delta(2\chi^{m\alpha}D_m\psi_\alpha) = 2i\chi^{m\alpha}D_mD_n\phi^n\bar{\epsilon}_\alpha + \cdots
\end{aligned}
$$

thus

$$
\begin{aligned}
\delta S &\sim \int 2i\chi^{m\alpha}[D_m,D_n]\phi^n\ \bar{\epsilon}_\alpha + h.c. \\
&= \int -2(iR_{mn}\chi^{m\alpha}\phi^n - i\chi^{m\alpha}[F_{mn},\phi^n])\bar{\epsilon}_\alpha + h.c.
\end{aligned}
$$

as

$$[D_m,D_n]\phi^m = R_{mn}\phi^m + [F_{mn},\phi^m]\,.$$

But the action is supersymmetric on flat $\mathbf{R}^6$, so to make it supersymmetric on $\mathbf{R}^3\times Y$ we just need to add the term

$$-\frac{1}{2}\int R_{mn}\phi^m\phi^n$$

to the action. Now the bosonic part of the action in the Y direction reads

$$\begin{aligned}
& - \tfrac{1}{4}\int \Big\{F_{mn}F^{mn} + [\phi_m,\phi_n]^2 - 2F_{mn}[\phi^m,\phi^n] \\
& + \; 2D_n\phi_m D^n\phi^m + 2R_{mn}\phi^m\phi^n + 2F_{mn}[\phi^m,\phi^n]\} \\
& = \; -\tfrac{1}{4}\int \Big\{(F_{mn} - [\phi_m,\phi_n])^2 + 2(D_m\phi^m)^2 + (D_{[m}\phi_{n]})^2\Big\} \,.
\end{aligned}$$

Thus if we scale down the metric on Y, the path integral will localize on the moduli space of the following equations

$$\begin{aligned}
& F_{mn} = [\phi_m,\phi_n] \\
& D_{[m}\phi_{n]} = 0 \\
& D_m\phi^m = 0\,.
\end{aligned} \tag{3.45}$$

3.4.1 Discussion

To finish this chapter, we briefly discuss the case of $U(1)$ gauge group, its relation to D-brane physics and a number of related conjectures. We follow the work of [6] in four dimensions and the discussion of Calabi-Yau mirror manifolds in [48].

If the gauge group is $U(1)$ then equations (3.45), in the language of forms, simply reduce to

$$F = 0, \quad d\Phi = 0, \quad d^\dagger\Phi = 0\,, \tag{3.46}$$

where $F = \frac{1}{2}F_{mn}dx^m \wedge dx^n$ and $\Phi = \phi_m dx^m$. That is a flat $U(1)$ connection and a harmonic one-form Φ. The moduli space of flat $U(1)$ bundles can be described as follows. Let us perturb a flat connection A to a nearby connection $A' = A + \delta A$ and ask whether it is flat. Flatness of A' requires

$$dA' = d(A + \delta A) = 0\,.$$

Since A is flat then we get

$$d\delta A = 0\,.$$

Moreover, we demand that δA cannot be derived by a gauge transformation, i.e.

$$(d\alpha, \delta A) = 0\,,$$

for an arbitrary gauge parameter α. This implies $d^\dagger\delta A = 0$. Therefore, for A' to be a flat connection, δA must be a harmonic one-form. Hence the tangent space to the moduli space of flat connections at A is $H^1(Y,\mathbf{R})$. However, there are still some gauge degrees of freedom among these harmonic one-forms δA that must be removed. These are the non-trivial global gauge transformations which map non-trivial one-cycles to $U(1)$. Such a gauge transformations fall in different isomorphism classes characterized by integers. Let us represent a flat connection A gauge invariantly by a Wilson loop $\oint_{C_i} A$ (with C_i a one-cycle in Y), then a non-trivial global gauge transformation acts like

$$\oint_{C_i} A \to \oint_{C_i} A + 2\pi n$$

for n an integer. Therefore, if we mod out this global gauge symmetry, then δA is a harmonic one-form with values in $[0,1]$ and we end up with a torus T^{b_1} (with b_1 the first Betti number of Y) as the moduli space of flat $U(1)$ connections. Thus the moduli space of solutions to the eqs. (3.46) is

$$\mathcal{M}_{fl} \times H^1(Y,\mathbf{R})\,, \tag{3.47}$$

where $\mathcal{M}_{fl}$, as we saw, is parametrized by the torus T^{b_1}. Notice that the moduli parameters are in fact arbitrary functions of the coordinates on $\mathbf{R}^3$, i.e., they are maps from $\mathbf{R}^3$ to $\mathcal{M}$.

Now let us consider a variant of the above problem. Take f to be an embedding of Y into a Calabi-Yau 3-fold M which has a mirror manifold $\widetilde{M}$. The concept of mirror manifolds is not important for us (see [49] for details). What we really need here is that M and $\widetilde{M}$ locally look like $T^3 \times T^3$ [48]. $f(Y)$ is said to be a special Lagrangian submanifold of M (or sometimes we say Y is supersymmetrically embedded in M) [50] if the following conditions on f hold

$$f^*k = 0 \quad \text{and} \quad f^*(\operatorname{Im}\Omega) = 0\,. \tag{3.48}$$

Here k is the Kähler form, and Ω the Calabi-Yau form, on M, and $*$ indicates the pull-back operation. We denote the moduli space of all special Lagrangian submanifolds inside M by $\mathcal{M}_{sl}$. The tangent space of $\mathcal{M}_{sl}$ at Y can be found as follows [51, 48]. Suppose f is a map which satisfies the special Lagrangian condition (3.48). Consider a one-parameter family $f(t)$ of $f = f(0)$ parametrized by t. Under what conditions does $f(t_1)$ preserve the special Lagrangian condition? If t_1 is infinitesimally small then obviously we must have

$$\frac{d}{dt} f^*k = 0 \quad \text{and} \quad \frac{d}{dt} f^*(\operatorname{Im}\Omega) = 0\,.$$

However, in [51] it has been shown that these conditions are equivalent to

$$d\theta = d^{\dagger}\theta = 0\,,$$

where $\theta_m = \dot{f}^{\beta} k_{\alpha\beta} \partial_m f^{\alpha}$ is a one-form on Y. Therefore, $T_Y\mathcal{M}_{sl}$ is isomorphic to $H^1(Y, \mathbf{R})$; any harmonic one-form on Y corresponds to an infinitesimal deformation of $f(Y)$ to a nearby special Lagrangian submanifold. Hence if Y is a special Lagrangian submanifold of M, the moduli space of solutions to $d\Phi = d^{\dagger}\Phi = 0$ is equivalent to $T_Y\mathcal{M}_{sl}$ and the moduli space $\mathcal{M}$ in (3.47) becomes

$$\mathcal{M}_{fl} \times T_Y\mathcal{M}_{sl}\,.$$

The partial twisted theory discussed above in fact arises in the effective low energy description of a D3-brane (or a D5-brane after compactifying two directions in $\mathbf{R}^3$ and using T-duality operations) wrapping around the Y (or $S^1 \times S^1 \times Y$ for a D5-brane) supersymmetrically embedded in M. Consider type II string theory with the target space $M_4 \times M$, where M_4 is the four-dimensional flat Minkowski space-time and M is a Calabi-Yau 3-fold. In string theory the structure of the target space is partly fixed by demanding the vacuum of the theory preserves part of the supersymmetry. In the compactification of type IIB string theory on Calabi-Yau 3-folds, there are BPS states (i.e. solitonic states which preserve part of the supersymmetry) corresponding to D5-branes wrapping around $S^1 \times S^1 \times Y$, where Y is a 3-cycles in M and S^1's are the two compactified directions in M_4. However, the fact that these configurations preserve part of the supersymmetry requires Y to be a special Lagrangian submanifold of M and the $U(1)$ connection on the brane to be a flat one [35]. The bosonic degrees of freedom of a D5-brane consists of a $U(1)$ gauge field and 4 scalars which specify the location of brane in the ambient space. Clearly, scalars have to be in the normal direction to the brane. If we decompose the tangent bundle to $\mathbf{R} \times M$ on Y, we have

$$T(\mathbf{R} \times M) = TY \oplus N_Y\,,$$

where N_Y indicates the normal bundle of Y. Hence, the scalars are sections of N_Y. Out of these 4 scalars, one of them is a section of $\mathbf{R}$ and we may call it the real scalar ϕ. The remaining three 'scalars' are then sections of the normal bundle of Y inside M. In fact, as Y is a Lagrangian submanifold, any normal vector to Y can be converted to a one-form on Y by the Kähler form on M. This is exactly the bosonic spectrum which appears in the partially twisted Lagrangian in (3.44); a gauge field A_m, a one-form ϕ_m and a scalar ϕ. Therefore, following [34, 6], the partial twisted theory that we constructed in the last subsection is conjectured to describe the low energy excitations of D5-branes in the type IIB string theory.

As we saw earlier, this effective field theory, which lives on the worldvolume of a D5-brane, localizes, in the limit of small Y, to $\mathcal{M}_{fl} \times T_Y\mathcal{M}_{sl}$ which is conjectured to be the local description of the mirror manifold $\widetilde{M}$ [48]. As in [6], this could be used to derive the moduli space of wrapped branes or to discuss the existence of the bound states of branes. Furthermore, we may derive, following [27], the effective theory which arises in this limit. There are indications that the effective theory which emerges is a sigma model with the target discussed above. In this way, one should be able to construct a relation between the invariants of the six-dimensional product manifold and the invariants associated to the three-dimensional sigma model.

Chapter 4

$N = 4$ SYM on $\Sigma \times S^2$ and its Topological Reduction

4.1 Introduction

In this chapter, we will study the twisted $N = 4$ SYM theory on a product four-manifold $\Sigma \times S^2$, where Σ is a Riemann surface of genus g [52]. We derive the effective theory in the limit where S^2 shrinks and then perturb the effective theory by a mass term for the hypermultiplet. In principle, one should get the same effective theory perturbed by mass, if one instead first perturbs the $N = 4$ theory by a mass term for the hypermultiplet [1] and then takes the limit where S^2 shrinks.

Although the mass term reduces the number of supersymmetries to two, the massive theory is still believed to be S-dual [11]. S-duality relates the behaviour of the theory, with gauge group G, in the strong coupling region to the behaviour of the same theory, with the dual gauge group $\tilde{G}$, in the weak coupling region. Here we take the gauge group to be $SU(2)$ and hence its dual is $SO(3)$. Therefore, to probe the duality properties of the massive theory, we need to compute quantities like the partition function nonperturbatively – and for both gauge groups $SU(2)$ and $SO(3)$. With this aim in mind, we will compute, in the limit where S^2 shrinks to zero size, the correlation functions of a set of specific operators in the twisted theory and for the gauge group $SO(3)$. In our

[1] The hypermultiplet is the same as the one in (2.32), however, note that for $N = 4$ theory the gauge multiplet and the hypermultiplet are both in the adjoint representation of the gauge group.

case, there are three different types of $SO(3)$ bundles to be considered. To see this, let us first discuss the classification of bundles on $\Sigma \times S^2$.

$SU(2)$ bundles on a four-manifold are simply characterized by the instanton number k. In contrast, $SO(3)$ bundles are classified by two topological invariants: k, the instanton number, and $w_2(E)$, the second Stieffel-Whitney class [45] of the bundle. $w_2(E)$ takes values in $\mathbf{Z}_2$ telling us whether the restriction of the bundle to a specific two-dimensional cycle is trivial or not (notice that $SO(3)$ bundles over a two-dimensional surface are classified by $\pi_1(SO(3)) = \mathbf{Z}_2$). Therefore, for a fixed instanton number k, there exist 2^{b_2} different types of $SO(3)$ bundles. Here we consider the manifold $\Sigma \times S^2$. Since in this case $b_2 = 2$ (see the next section), there are two independent two-cycles and we may take them to be Σ and S^2. Therefore we have four different types of $SO(3)$ bundles depending on how the bundle restricts over Σ or S^2. $SO(3)$ bundles which restrict trivially over both Σ and S^2 are identical with $SU(2)$ bundles.

We consider $SO(3)$ bundles such that the restricted bundle over S^2 is trivial. As was argued in section 2.3.3, bundles which restrict nontrivially over S^2 give zero contribution to the path integral. Therefore, we are left to consider two types of $SO(3)$ bundles which restrict trivially over S^2; bundles which restrict nontrivially over Σ and bundles which are trivial over Σ and hence are identical with $SU(2)$ bundles.

The organization of this chapter is as follows: In section 2 we consider the twisted $N = 4$ Lagrangian on $\Sigma \times S^2$. In the limit where S^2 shrinks it is shown how the four-dimensional theory reduces to an effective two dimensional theory. The fixed point equations imply, in the case of a nontrivial $SO(3)$ bundle over Σ, that the partition function of this reduced theory is in fact the Euler characteristic of the moduli space of flat connections on Σ. A mass perturbation makes the path integral calculation more tractable – particularly for the limiting two-dimensional theory. In section 3, we show how this comes about. Perturbing by the mass allows most fields to be integrated out, and reduces the path integral to a finite dimensional integral which can be easily performed. In section 4 we discuss the result. Although we have not given an explicit check of S-duality, we have isolated the problems involved.

4.2 Twisted $N = 4$ on $\Sigma \times S^2$ and its reduction

As was discussed in the second chapter, the key point in twisting [1] is to redefine the global space-time symmetry such that at least one component of the supercharge becomes scalar under the new defined space-time symmetry. This procedure crucially depends on the existence of a suitable global $\mathcal{R}$-symmetry. $N = 4$ SYM theory in four dimensions has a large global $\mathcal{R}$-symmetry, $SU(4)$, and thus there are different possible nontrivial embedding of space-time symmetry in the global symmetry of the theory. As in [4] we will consider the embedding (ii) in (2.34) where, after twisting, two components of the supercharges turn out to be singlets and therefore square to zero. The scalar fields of the physical theory, which transform under the $\mathbf{6}$ of $SU(4)$, now transform under the new rotation group, $SU(2)_L \times SU(2)'_R$, as $3(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$, three singlets and one self-dual 2-form.

Having determined how the new fields transform under the new symmetry group, what remains is to rewrite the Lagrangian in terms of these new fields on flat $\mathbf{R}^4$. This Lagrangian can then be defined on an arbitrary smooth four-manifold while preserving those two BRST like symmetries.

Let us start our discussion with the twisted $N = 4$ Lagrangian[2] in 4 dimensions [4, 26],

$$\begin{aligned}
\mathcal{L} &= \frac{1}{e^2}\mathrm{tr}\,\Big\{-D_\mu\lambda D^\mu\phi + \tfrac{1}{2}\tilde{H}^\mu(\tilde{H}_\mu - 2\sqrt{2}D_\mu C + 4\sqrt{2}D^\nu B_{\nu\mu}) \\
&+ \tfrac{1}{2}H^{\mu\nu}(H_{\mu\nu} - 2F^+_{\mu\nu} - 4i[B_{\mu\rho}, B^\rho{}_\nu] - 4i[B_{\mu\nu}, C]) \\
&+ 4\psi_\mu D_\nu\chi^{\mu\nu} + 4\tilde{\chi}_\mu D_\nu\tilde{\psi}^{\mu\nu} + \tilde{\chi}_\mu D^\mu\zeta - \psi_\mu D^\mu\eta \\
&+ i\sqrt{2}\tilde{\psi}^{\mu\nu}[\tilde{\psi}_{\mu\nu}, \lambda] - i\sqrt{2}\chi^{\mu\nu}[\chi_{\mu\nu}, \phi] + i2\sqrt{2}\tilde{\psi}^{\mu\nu}[\chi_{\mu\nu}, C] + i4\sqrt{2}\tilde{\psi}^{\mu\nu}[\chi_{\mu\rho}, B_\nu{}^\rho] \\
&- i\sqrt{2}\chi_{\mu\nu}[\zeta, B^{\mu\nu}] - i\sqrt{2}\tilde{\psi}_{\mu\nu}[\eta, B^{\mu\nu}] + i4\sqrt{2}\psi_\mu[\tilde{\chi}_\nu, B^{\mu\nu}] - i\sqrt{2}\tilde{\chi}_\nu[\tilde{\chi}^\nu, \phi] \\
&+ i\sqrt{2}\psi_\mu[\psi^\mu, \lambda] - i2\sqrt{2}\psi_\mu[\tilde{\chi}^\mu, C] + \frac{i}{2\sqrt{2}}\zeta[\zeta, \lambda] \\
&- \frac{i}{\sqrt{2}}\zeta[\eta, C] + 2[\phi, B^{\mu\nu}][\lambda, B_{\mu\nu}] + 2[\phi, C][\lambda, C]\Big\}.
\end{aligned} \tag{4.1}$$

As mentioned, the action is invariant under two BRST transformations. However, for

[2]The Lagrangian that we use is actually different from the one in (2.36) by a BRST exact term $-\frac{i}{4}\delta(\eta[\phi, \lambda])$.

us it is enough to consider one of them, which reads [26]

$$\begin{array}{ll} \delta A_\mu = -2\psi_\mu & \delta\zeta = 4i[C,\phi] \\ \delta\psi_\mu = -\sqrt{2}D_\mu\phi & \delta\lambda = \sqrt{2}\eta \\ \delta\phi = 0 & \delta\eta = 2i[\lambda,\phi] \\ \delta B_{\mu\nu} = \sqrt{2}\tilde{\psi}_{\mu\nu} & \delta\tilde{\chi}_\mu = \tilde{H}_\mu \\ \delta\tilde{\psi}_{\mu\nu} = 2i[B_{\mu\nu},\phi] & \delta\tilde{H}_\mu = 2\sqrt{2}i[\tilde{\chi}_\mu,\phi] \\ \delta C = \frac{1}{\sqrt{2}}\zeta & \delta\chi_{\mu\nu} = H_{\mu\nu} \\ \delta F_{\mu\nu} = -2(D_\mu\psi_\nu - D_\nu\psi_\mu) & \delta H_{\mu\nu} = 2\sqrt{2}i[\chi_{\mu\nu},\phi]. \end{array}$$

Here we choose ϕ and λ to be two independent real scalars. This will render the Lagrangian to be hermitian and allow us to treat ϕ and λ independently. The generators of the $SU(2)$ group are chosen to be hermitian $T^a = \frac{1}{\sqrt{2}}\sigma^a$ with $\mathrm{tr}\,(T^aT^b) = \delta^{ab}$.

The theory enjoys an exact $U(1)$ ghost symmetry under which $\psi_\mu, \tilde{\psi}_{\mu\nu}, \zeta$ have charge 1, $\chi_{\mu\nu}, \eta, \tilde{\chi}_\mu$ charge -1, while ϕ and λ have charges 2 and -2 respectively. All other fields have ghost number zero.

Take the underlying manifold to be $\Sigma \times S^2$. Let us denote the indices on Σ by $i, j, \cdots$ and those on S^2 by $a, b, \cdots$. We define

$$\begin{array}{ll} F_{ij} = \dfrac{1}{\sqrt{g_1}}\epsilon_{ij} f & \chi_{ij} = \dfrac{1}{\sqrt{g_1}}\epsilon_{ij}\chi \\ B_{ij} = \dfrac{1}{2\sqrt{g_1}}\epsilon_{ij} b & \tilde{\psi}_{ij} = \dfrac{1}{2\sqrt{g_1}}\epsilon_{ij}\tilde{\psi}\ , \end{array} \tag{4.2}$$

and the same for indices on S^2

$$B_{ab} = \frac{1}{2\sqrt{g_2}}\epsilon_{ab} b'\ , \quad \chi_{ab} = \frac{1}{\sqrt{g_2}}\epsilon_{ab}\chi'\ , \quad \tilde{\psi}_{ab} = \frac{1}{2\sqrt{g_2}}\epsilon_{ab}\tilde{\psi}'. \tag{4.3}$$

Here g_1 and g_2 denote the determinant of the metric on Σ and S^2 respectively.

The fields $H_{\mu\nu}$, $B_{\mu\nu}$, $\chi_{\mu\nu}$ and $\tilde{\psi}_{\mu\nu}$ are all self-dual. Note that

$$\begin{aligned} B_{ij} = *B_{ij} \Rightarrow \frac{1}{2\sqrt{g_1}}\epsilon_{ij} b &= \frac{1}{2\sqrt{g}}\epsilon_{ij}{}^{ab} B_{ab} = \frac{1}{2\sqrt{g}}\epsilon_{ij}\epsilon^{ab}(\frac{1}{2\sqrt{g_2}}\epsilon_{ab} b') \\ &= \frac{1}{4\sqrt{g g_2}} 2g_2\ \epsilon_{ij} b' = \frac{1}{2\sqrt{g_1}}\epsilon_{ij} b'\ , \end{aligned} \tag{4.4}$$

where we have used

$$\epsilon^{ab}\epsilon_{ab} = \epsilon^{ab}\epsilon^{a'b'} g_{aa'} g_{bb'} = 2g_2$$

and

$$g^{aa'}g^{bb'}\epsilon_{a'b'} = \epsilon^{ab}\,.$$

Also we chose $\epsilon^{12} = 1$ and so $\epsilon_{12} = g_2$; thus, for example, we have $B^{ab} = \frac{1}{2\sqrt{g_2}}\epsilon^{ab}b'$. Hence we conclude that

$$b = b'\;,\quad \chi = \chi'\;,\quad \tilde{\psi} = \tilde{\psi}'.$$

For the manifold $\Sigma \times S^2$, by the Künneth formula we can write

$$H^2(\Sigma \times S^2) = [H^2(\Sigma) \otimes H^0(S^2)] \oplus [H^1(\Sigma) \otimes H^1(S^2)] \oplus [H^0(\Sigma) \otimes H^2(S^2)]\,.$$

Since $H^1(S^2)$ is trivial, $H^2(\Sigma \times S^2)$ is spanned by two generators; ω_1 which generates $H^2(\Sigma)$ and is dual to Σ and ω_2 which generates $H^2(S^2)$ and is dual to S^2. Thus in this case $b_2 = 2$. ω_1 and ω_2 are normalized such that

$$\int \omega_1 \wedge \omega_2 = 1$$

so the intersection form looks like

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Hodge star operator maps ω_1 and ω_2 into each other such that we can define (note $*^2 = 1$)

$$\begin{aligned} *\,\omega_1 &= \alpha\,\omega_2 \\ *\,\omega_2 &= \frac{1}{\alpha}\,\omega_1\,, \end{aligned}$$

where α is some real coeficient depending on the metric. One may choose another basis for $H^2(\Sigma \times S^2)$ by introducing

$$\begin{aligned} \omega^+ &= \frac{1}{\sqrt{2\alpha}}(\omega_1 + \alpha\,\omega_2) \\ \omega^- &= \frac{1}{\sqrt{2\alpha}}(\omega_1 - \alpha\,\omega_2)\,. \end{aligned}$$

In this basis the intersection form is diagonal

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So ω^+ spans the group $H^2_+(\Sigma \times S^2)$ and thus $b_2^+ = 1$.

In [27] it was shown that upon shrinking the metric on Σ, one gets an effective 2-dimensional sigma model governing the maps from S^2 to $\mathcal{M}$, where $\mathcal{M}$ is the moduli space of solutions to the Hitchin's equations. Although the twisted theory is supposed to be metric independent, one may not get the same effective theory if one instead shrinks S^2. In this case we will see that the effective theory which emerges is a 2-dimensional twisted SYM theory. This happens mainly because of the following reason: recall that a topological theory is independent of the metric as long as, in varying the metric, the Lagrangian remains nondegenerate. Here this fails to be the case because the space of self-dual harmonic 2-forms is one-dimensional and thus there are metrics for which abelian instantons exist and the Lagarangian degenerates.

Thereto, we now scale the metric on S^2 by a factor of ϵ. Notice that the definitions (4.2) and (4.3) are consistent with this scaling, since both sides of the self-duality constraints scale with the same power of ϵ.
After integrating out the auxiliary fields, the bosonic part of the Lagrangian reads

$$\mathcal{L}_{\rm B} = \frac{1}{e^2}\mathrm{tr}\left\{-D_\mu\lambda D^\mu\phi - (D_\mu C - 2D^\nu B_{\nu\mu})^2 - \tfrac{1}{2}(F^+_{\mu\nu} + 2i[B_{\mu\rho}, B^\rho{}_\nu] + 2i[B_{\mu\nu}, C])^2\right\}, \tag{4.5}$$

where $F^+ = \frac{1}{2}(F + *F)$ and $*$ is the Hodge duality operation. Thus we can write

$$-\tfrac{1}{2}\int\sqrt{g}F^+_{\mu\nu}F^{\mu\nu +} = -\tfrac{1}{4}\int\sqrt{g}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}\int\sqrt{g}\,(*F)_{\mu\nu}F^{\mu\nu}.$$

The last term is the instanton number and is metric independent. Using this, and the fact that $B_{\mu\nu}$ is self-dual, we write the last term in (4.5) as

$$\begin{aligned}
- \ & \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2iF^{\mu\nu}([B_{\mu\rho}, B^\rho{}_\nu] + [B_{\mu\nu}, C]) + 2([B_{\mu\rho}, B^\rho{}_\nu] + [B_{\mu\nu}, C])^2 - \tfrac{1}{4}(*F)_{\mu\nu}F^{\mu\nu} \\
= \ & -\tfrac{1}{4}\left\{F_{ij}F^{ij} + 8iF^{ij}([B_{ij}, C] + [B_{ia}, B^a{}_j]) - 8([B_{ij}, C] + [B_{ia}, B^a{}_j])^2 + (*F)_{ij}F^{ij}\right. \\
+ \ & \left. F_{ab}F^{ab} + 8iF^{ab}([B_{ab}, C] + [B_{ai}, B^i{}_b]) - 8([B_{ab}, C] + [B_{ai}, B^i{}_b])^2 + (*F)_{ab}F^{ab}\right\} \\
- \ & (F^+_{ai} + 2i[B_{aj}, B^j{}_i] + 2i[B_{ab}, B^b{}_i] + 2i[B_{ai}, C])^2 \\
= \ & -\tfrac{1}{4}(F_{ij} + 4i[B_{ij}, C])^2 - \tfrac{1}{4}(F_{ab} + 4i[B_{ai}, B^i{}_b])^2 - 2iF^{ij}[B_{ia}, B^a{}_j] - 2iF^{ab}[B_{ab}, C] \\
+ \ & 4[B^{ij}, C][B_{ia}, B^a{}_j] + 4[B^{ab}, C][B_{ai}, B^i{}_b] - \tfrac{1}{4}(*F)_{ij}F^{ij} - \tfrac{1}{4}(*F)_{ab}F^{ab}) \\
- \ & (F^+_{ai} + 2i[B_{aj}, B^j{}_i] + 2i[B_{ab}, B^b{}_i] + 2i[B_{ai}, C])^2.
\end{aligned}$$

In the last equality we noted that for a self-dual antisymmetric tensor S, we have

$S_{ij}^2 = S_{ab}^2$. Thus, in particular, we have

$$[B_{ab}, C]^2 = [B_{ij}, C]^2$$
$$\mathrm{tr}\,([B_{ai}, B^i{}_b][B^{aj}, B_j{}^b]) = \mathrm{tr}\,([B_{ia}, B^a{}_j][B^{ib}, B_b{}^j]).$$

The proof of this is as follows. Since S_{ij} is self-dual, we have

$$S_{ij} = *S_{ij} = \frac{1}{2\sqrt{g}}\epsilon_{ij}{}^{ab}S_{ab}\,.$$

Therefore

$$\begin{aligned} S_{ij}S^{ij} &= \frac{1}{2\sqrt{g}}\epsilon_{ij}\epsilon^{ab}S_{ab} \times \frac{1}{2\sqrt{g}}\epsilon^{ij}\epsilon_{cd}S^{cd} \\ &= \frac{1}{2g_2}\epsilon^{ab}\epsilon_{cd}S_{ab}S^{cd} = S_{ab}S^{ab}\,, \end{aligned} \tag{4.6}$$

where we used

$$\epsilon^{ab}\epsilon_{cd} = g_2(\delta^a{}_c\delta^b{}_d - \delta^a{}_d\delta^b{}_c)\,.$$

After scaling the metric, then, the Lagrangian splits to three parts;

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0 + \mathcal{L}_{-1},$$

where $\mathcal{L}_n$ scales as ϵ^n. Specifically,

$$\begin{aligned} \mathcal{L}_1 &= \frac{\epsilon}{e^2}\mathrm{tr}\left\{-D_i\lambda D^i\phi - D_iCD^iC - D_ibD^ib + \frac{2}{\sqrt{g_1}}\epsilon^{ij}D_ibD_jC - \frac{1}{2}(f + 2i[b, C])^2\right. \\ &+ \frac{4}{\sqrt{g_1}}\epsilon^{ij}\psi_iD_j\chi + \frac{2}{\sqrt{g_1}}\epsilon^{ij}\tilde{\chi}_iD_j\tilde{\psi} + \tilde{\chi}_iD^i\zeta - \psi_iD^i\eta \\ &+ i\sqrt{2}\tilde{\psi}[\tilde{\psi}, \lambda] - i4\sqrt{2}\chi[\chi, \phi] + i4\sqrt{2}\tilde{\psi}[\chi, C] - 2i\sqrt{2}\chi[\zeta, b] \\ &- i\sqrt{2}\tilde{\psi}[\eta, b] + i\frac{2\sqrt{2}}{\sqrt{g_1}}\epsilon^{ij}\psi_i[\tilde{\chi}_j, b] - i\sqrt{2}\tilde{\chi}_i[\tilde{\chi}^i, \phi] + i\sqrt{2}\psi_i[\psi^i, \lambda] \\ &- i2\sqrt{2}\psi_i[\tilde{\chi}^i, C] + \frac{i}{2\sqrt{2}}\zeta[\zeta, \lambda] - \frac{i}{\sqrt{2}}\zeta[\eta, C] \\ &+ 2[\phi, b][\lambda, b] + 2[\phi, C][\lambda, C]\}\,, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{e^2}\mathrm{tr}\left\{-D_a\lambda D^a\phi - (D_aC + \frac{1}{\sqrt{g_2}}\epsilon_{ab}D^bb - 2D^iB_{ia})^2 + 4(D^aB_{ai})(D^iC - 2D^jB_j{}^i)\right. \\ &- (F^+_{ai} + 2i[B_{aj}, B^j{}_i] + 2i[B_{ab}, B^b{}_i] + 2i[B_{ai}, C])^2 - \tfrac{1}{4}(*F)_{ij}F^{ij} - \tfrac{1}{4}(*F)_{ab}F^{ab} \\ &- 2i(F_{ij} + 2i[B_{ij}, C])[B^{ia}, B_a{}^j] - 2i(F_{ab} + 2i[B_{ai}, B^i{}_b])[B^{ab}, C] \end{aligned}$$

$$
\begin{aligned}
&+\ 4\psi_{[i}D_{a]}\chi^{ia}+4\tilde{\chi}_{[i}D_{a]}\tilde{\psi}^{ia}+\tilde{\chi}_aD^a\zeta-\psi_aD^a\eta+4\psi_aD_b\chi^{ab}+4\tilde{\chi}_aD_b\tilde{\psi}^{ab}\\
&+\ 2i\sqrt{2}\tilde{\psi}^{ai}[\tilde{\psi}_{ai},\lambda]-2i\sqrt{2}\chi^{ai}[\chi_{ai},\phi]+i4\sqrt{2}\tilde{\psi}^{ai}[\chi_{ai},C]+i4\sqrt{2}\tilde{\psi}^{ab}[\chi_{ai},B_b{}^i]\\
&+\ i4\sqrt{2}\tilde{\psi}^{ia}[\chi_{ib},B_a{}^b]+i4\sqrt{2}\tilde{\psi}^{ai}[\chi_{ab},B_i{}^b]+i4\sqrt{2}\tilde{\psi}^{ai}[\chi_{aj},B_i{}^j]+i4\sqrt{2}\tilde{\psi}^{ia}[\chi_{ij},B_a{}^j]\\
&+\ i4\sqrt{2}\tilde{\psi}^{ij}[\chi_{ia},B_j{}^a]-2i\sqrt{2}\chi_{ai}[\zeta,B^{ai}]-2i\sqrt{2}\tilde{\psi}_{ai}[\eta,B^{ai}]+i4\sqrt{2}\psi_a[\tilde{\chi}_b,B^{ab}]\\
&+\ i4\sqrt{2}\psi_a[\tilde{\chi}_i,B^{ai}]+i4\sqrt{2}\psi_i[\tilde{\chi}_a,B^{ia}]-i\sqrt{2}\tilde{\chi}_a[\tilde{\chi}^a,\phi]+i\sqrt{2}\psi_a[\psi^a,\lambda]\\
&-\ i2\sqrt{2}\psi_a[\tilde{\chi}^a,C]+4[\phi,B^{ai}][\lambda,B_{ai}]\Big\}\,, \qquad (4.8)
\end{aligned}
$$

and

$$
\mathcal{L}_{-1}=\frac{1}{\epsilon e^2}\mathrm{tr}\left\{-4(D^aB_{ai})^2-\frac{1}{4}(F_{ab}+4i[B_{ai},B^i{}_b])^2\right\}. \qquad (4.9)
$$

Now, in sending ϵ to zero path integral localizes around the solutions of the following equations

$$
\begin{aligned}
&F_{ab}+4i[B_{ai},B^i{}_b]=0\\
&D^aB_{ai}=0. \qquad (4.10)
\end{aligned}
$$

In appendix C we show that these equations imply

$$
F_{ab}=B_{ai}=0,
$$

and that from $F_{ab}=0$ it follows that the instanton number vanishes. A flat connection on sphere can be written globally as

$$
A_a=g^{-1}\partial_ag
$$

for some gauge group element g. Therefore, the connection A is

$$
A=A_idx^i+(g^{-1}\partial_ag)dx^a\,.
$$

We gauge transform A such that it lies in Σ direction

$$
A\to gAg^{-1}+gdg^{-1}=g(A_idx^i)g^{-1}+g(\partial_ig^{-1})dx^i=A'_idx^i\,.
$$

Setting $A_a=0$ and $B_{ai}=0$, $\mathcal{L}_0$ greatly simplifies. However, because of the zero modes of the operator ∂_a, one has to still keep the order ϵ terms in $\mathcal{L}_1$. We expand all fields in terms of eigenfunctions of ∂_a and denote the zero modes by a 0 superscript. Effectively we do the following substitution

$$
\Phi(z,\bar{z};w,\bar{w})\to\Phi^0(z,\bar{z})+\Phi(z,\bar{z};w,\bar{w})
$$

where $\Phi(z,\bar{z};w,\bar{w})$ on the RHS stands for the nonzero modes. The kinetic part of $\mathcal{L}_0$ then reads

$$\begin{aligned}\mathcal{L}_{0\ \mathrm{kin}} &= \frac{1}{e^2}\mathrm{tr}\left\{-\partial_a\lambda\partial^a\phi - (\partial_a C + \epsilon_{ab}\partial^b b)^2 - (\partial_a A_i)^2\right.\\ &+ \left. 4\psi_{[i}\nabla_{a]}\chi^{ia} + 4\tilde{\chi}_{[i}\nabla_{a]}\tilde{\psi}^{ia} + \tilde{\chi}_a\nabla^a\zeta - \psi_a\nabla^a\eta + 4\psi_a\nabla_b\chi^{ab} + 4\tilde{\chi}_a\nabla_b\tilde{\psi}^{ab}\right\}.\end{aligned} \tag{4.11}$$

Since χ_{ai} and $\tilde{\psi}_{ai}$ are self-dual and since there are no holomorphic one-forms on the sphere (see appendix C), $\mathcal{L}_{0\ \mathrm{kin}}$ is nondegenerate. Thus in doing the integral over nonzero modes, one may drop the terms which are order of ϵ. Keeping terms of order one, the integral over $\eta,\zeta,\chi,\tilde{\psi},\psi_i$ and $\tilde{\chi}_i$ results in a set of delta functions imposing the following constraints

$$\begin{aligned}&\nabla_a\chi^{ai} = 0,\ \ \nabla_a\tilde{\psi}^{ai} = 0\\ &\nabla_a\psi^a = 0,\ \ \epsilon^{ab}\nabla_a\psi_b = 0\\ &\nabla_a\tilde{\chi}^a = 0,\ \ \epsilon^{ab}\nabla_a\tilde{\chi}_b = 0.\end{aligned} \tag{4.12}$$

As was mentioned, these equations have no nontrivial solutions on sphere. Setting these fields to zero, $\mathcal{L}_0$ reduces to

$$\mathcal{L}_0 = \frac{1}{e^2}\mathrm{tr}\left\{-\partial_a\lambda\partial^a\phi - (\partial_a C)^2 - (\partial_a b)^2 - (\partial_a A_i)^2\right\}$$

where the fields are all nonzero modes. Using the equation of motion for A_i we obtain

$$d^\dagger dA_i + \text{terms proportional to } \epsilon = 0$$

as A_i is a nonzero mode this equation implies that, up to ϵ order, $A_i = 0$. The same happens for ϕ, b and C fields. So in the limit $\epsilon \to 0$ all nonzero modes can be set to zero and one is left with a copy of $\mathcal{L}_1$ in which fields now depend only on coordinates on Σ. From now on we call this reduced Lagrangian $\mathcal{L}$ and drop the 0 superscript on zero modes.

The reduced Lagrangian, $\mathcal{L}$, which now describes a two-dimensional TFT, can be obtained by the BRST variation of V, where

$$\begin{aligned}V &= \frac{1}{e^2}\int_\Sigma \mathrm{tr}\,\{\tfrac{1}{2}\tilde{\chi}^i(\tilde{H}_i - 2\sqrt{2}D_iC + \frac{2\sqrt{2}}{\sqrt{g}_1}\epsilon_{ji}D^jb) + \chi(2H - 2f - 4i[b,C])\\ &- \frac{1}{2\sqrt{2}}\,\lambda(2D_i\psi^i + 2\sqrt{2}i[\tilde{\psi},b] + \sqrt{2}i[\zeta,C])\},\end{aligned} \tag{4.13}$$

and the BRST transformations of the two-dimensional fields are ($\delta \equiv \{Q, \cdots\}$)

$$
\begin{array}{llll}
\delta A_i = -2\psi_i & \delta b = \sqrt{2}\tilde{\psi} & \delta C = \frac{1}{\sqrt{2}}\zeta & \\
\delta \psi_i = -\sqrt{2} i D_i \phi & \delta \tilde{\psi} = -2[b, \phi] & \delta \zeta = -4[C, \phi] & \\
\delta \tilde{\chi}_i = i\tilde{H}_i & \delta \chi = iH & \delta \lambda = \sqrt{2}\eta & \delta \phi = 0 \\
\delta \tilde{H}_i = 2\sqrt{2} i[\tilde{\chi}_i, \phi] & \delta H = 2\sqrt{2} i[\chi, \phi] & \delta \eta = -2[\lambda, \phi]. &
\end{array}
$$

The fixed points around which path integral localizes are those configurations that are BRST invariant. Thus, setting $\delta\chi = H = 0$ and $\delta\tilde{\chi}_i = \tilde{H}_i = 0$ and using the equation of motion for H and $\tilde{H}_i$ we find the fixed point equations

$$
\begin{aligned}
& f + 2i[b, C] = 0 \\
& D_i C + \frac{1}{\sqrt{g_1}} \epsilon_{ij} D^j b = 0.
\end{aligned} \tag{4.14}
$$

After squaring, these imply

$$
\begin{aligned}
& f = 0 \;, \; [b, C] = 0 \\
& D_i C = D_i b = 0.
\end{aligned} \tag{4.15}
$$

Requiring that there are no reducible connections (as is the case for flat non-trivial $SO(3)$ bundles) it follows that the only solutions are $C = b = 0$. Let us see in this case what topological invariant the partition function corresponds to. First recall that, in Witten-type topological field theories, the partition function computes the Euler number of the bundle of antighosts zero modes (here χ and $\tilde{\chi}_i$) over Σ [4]. However, as there are no reducible connections, there are no χ zero modes and we need only to consider the zero modes of $\tilde{\chi}_i$ which are in fact cotangent to the moduli space of flat connections $\mathcal{M}$. Therefore, the bundle of antighost zero modes is really the cotangent bundle of $\mathcal{M}$ which has the same Euler characteristic as the space $\mathcal{M}$. We conclude that, when there are no reducible connections, the partition function is nothing but the Euler characteristic of the moduli space of flat connections over Σ.

4.3 Perturbing by mass term

The theory discussed so far does not have a mass gap [34]. To make the calculations more feasible we perturb the theory such that it has a mass gap. This enables us to integrate out most fields and reduce the path integral to a finite dimensional one.

The reduced 2-dimensional theory has a $U(1)$ ghost number symmetry coming directly from the nonanomalous $U(1)$ symmetry of the underlying 4-dimensional $N = 4$ SYM theory. Because of supersymmetry, the measure for nonzero modes is invariant under the $U(1)$ action. The ghost and the antighost zero modes, on the other hand, obey the same equations of motion such that there are equal number of ghost and antighost zero modes. This renders the measure invariant under the ghost symmetry of the action. Therefore the ghost symmetry is anomaly free. As the measure is invariant under this symmetry, the correlation function of any operator that has a ghost charge is zero. Therefore, this symmetry allows us to perturb the Lagrangian, by adding gauge invariant terms with nonzero ghost number, without changing the partition function.

Thus, for example, since the mass term for the hypermultiplet,

$$-\frac{m}{\sqrt{2}g_1}\epsilon_{ij}\tilde{\chi}^i\tilde{\chi}^j - 2im\lambda[b, C] + \frac{i}{2}\bar{m}\delta_m V'' ,$$

where V'' is given in (4.18), consists of a term with negative ghost number and a term which is BRST exact, one expects that the partition function is invariant under perturbing the Lagrangian by a mass term for the hypermultiplet. In [4] and [53] it has been argued that even an additional mass term for the chiral multiplet Φ (which contains the fields ψ_α and ϕ in (2.31)) still leaves the partition function invariant.

However, in the following we are interested in the correlation functions of a set of BRST cohomology classes of the form

$$I(\varepsilon) = \frac{1}{4\pi^2}\int_\Sigma \mathrm{tr}\left(\frac{i}{\sqrt{2}}\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{32\pi^2}\int_\Sigma \mathrm{tr}\,\phi^2. \tag{4.16}$$

Notice that, since we are not concerned with the partition function, the above mass independency argument does not apply here. This can be seen roughly as follows. The $\psi\wedge\psi$ factor in (4.16) contains part of the mass term for the chiral fields λ_α and ψ_α (in the notation of (2.31)). One can easily complete the mass term for the ψ_α component with an extra BRST exact term, which at the same time gives a mass to the ϕ field [2]. The remaining part of (4.16), which are not contributing to the mass term of the Φ multiplet, will in general have a nonvanishing expectation value in the mass deformed theory. This, in particular, implies that – in contrast to the partition function – the correlation functions of $I(\varepsilon)$ (in the theory perturbed by mass for the hypermultiplet) may depend on the mass parameter.

The next problem is to give a mass to $\chi, \eta, \lambda, \tilde{\psi}$ and ζ. This can be achieved by adding V' and V'' to V, where

$$V' = -\frac{2}{e^2}\int_\Sigma d\mu \ \mathrm{tr}\{\chi\lambda\} \tag{4.17}$$

$$V'' = \frac{1}{e^2}\int_\Sigma d\mu \ \mathrm{tr}\{\tilde{\psi}C - \tfrac{1}{2}\zeta b\}. \tag{4.18}$$

Notice that no field transforms to C, b or $\tilde{\chi}_i$ under δ_m. Therefore, a mass term for these fields cannot be obtained simply by adding the BRST exact terms to the Lagrangian. Alternatively, to produce the mass terms, one may think of changing the BRST transformation rules, keeping the BRST symmetry of the action. In our case, this is possible if we change the BRST transformation rules for $\tilde{H}_i, \tilde{\psi}$ and ζ to the following ones

$$\begin{aligned}
\delta_m \tilde{H}_i &= 2\sqrt{2}i[\tilde{\chi}_i, \phi] + \frac{\sqrt{2}}{\sqrt{g_1}} m\epsilon_{ij}\tilde{\chi}^j \\
\delta_m \tilde{\psi} &= -2[b, \phi] + imC \\
\delta_m \zeta &= -4[C, \phi] - 2imb.
\end{aligned} \tag{4.19}$$

As we will see shortly, this perturbation will give a mass to the fields C, b and $\tilde{\chi}_i$. The coeficients in V'' and those of the above deformed transformations are all fixed by demanding invariance under the new supersymmetry transformations δ_m. Even though the metric is explicitly introduced via the above first BRST transformation rule, note that the extra term is still invariant under metric rescaling ($\epsilon_{ij} \sim g_1$).

Thus, in the following we will consider the theory defined by the deformed action

$$\begin{aligned}
S &= I(\varepsilon) + i\delta_m(V + tV' + \frac{1}{2}\bar{m}V'') \\
&= I(\varepsilon) + \frac{1}{e^2}\int_\Sigma d\mu \ \mathrm{tr}\{D_i\lambda D^i\phi + D_iCD^iC + D_ibD^ib - \frac{2}{\sqrt{g_1}}\epsilon^{ij}D_ibD_jC \\
&+ \tfrac{1}{2}(f + 2i[b, C] + t\lambda)^2 + 2i\sqrt{2}t\chi\eta - \tfrac{1}{2}|m|^2C^2 - \tfrac{1}{2}|m|^2b^2 \\
&+ \frac{i\bar{m}}{\sqrt{2}}\zeta\tilde{\psi} - \frac{m}{\sqrt{2g_1}}\epsilon_{ij}\tilde{\chi}^i\tilde{\chi}^j + 2i\bar{m}\phi[b, C] - 2im\lambda[b, C] \\
&+ \frac{4i}{\sqrt{g_1}}\epsilon^{ij}\psi_iD_j\chi + \frac{2i}{\sqrt{g_1}}\epsilon^{ij}\tilde{\chi}_iD_j\tilde{\psi} + i\tilde{\chi}_iD^i\zeta - i\psi_iD^i\eta \\
&- \sqrt{2}\tilde{\psi}[\tilde{\psi}, \lambda] + 4\sqrt{2}\chi[\chi, \phi] - 4\sqrt{2}\tilde{\psi}[\chi, C] + 2\sqrt{2}\chi[\zeta, b] \\
&+ \sqrt{2}\tilde{\psi}[\eta, b] - \frac{2\sqrt{2}}{\sqrt{g_1}}\epsilon^{ij}\psi_i[\tilde{\chi}_j, b] + \sqrt{2}\tilde{\chi}_i[\tilde{\chi}^i, \phi] - \sqrt{2}\psi_i[\psi^i, \lambda] + 2\sqrt{2}\psi_i[\tilde{\chi}^i, C] \\
&- \frac{1}{2\sqrt{2}}\zeta[\zeta, \lambda] + \frac{1}{\sqrt{2}}\zeta[\eta, C] - 2[\phi, b][\lambda, b] - 2[\phi, C][\lambda, C]\}.
\end{aligned} \tag{4.20}$$

Notice that although the new BRST charge does not square to a gauge transformation (because of those new terms proportional to m), Lagrangian remains BRST invariant. This can be understood if we notice that δ_m^2 acting on fields generates (up to a gauge transformation) a $U(1)$ action. Let $\delta_T \equiv \frac{1}{i\sqrt{2}m}\delta_m^2$ and $\beta \equiv b + iC$, $\psi \equiv \tilde{\psi} + \frac{i}{2}\zeta$, then $U(1)$ group acts as

$$\delta_T \beta = -i\beta \quad , \quad \delta_T \psi = -i\psi$$
$$\delta_T \tilde{\chi}_i = \frac{1}{\sqrt{g}_1}\epsilon_{ij}\tilde{\chi}^j \quad , \quad \delta_T \tilde{H}_i = \frac{1}{\sqrt{g}_1}\epsilon_{ij}\tilde{H}^j$$

thus the fields β, ψ, $\tilde{\chi}_{\bar{z}}$ and $\tilde{H}_{\bar{z}}$ all have charge -1, with their complex conjugate having charge $+1$. All other fields have zero charge under this $U(1)$ group. The fact that S is invariant under δ_m then follows since V, V' and V'' all have zero $U(1)$ charge.

Before continuing the analysis, it is important to understand the relation between the perturbed and unperturbed theories. Since the perturbing terms proportional to t and $\bar{m}$ are BRST exact, one may expect that correlation functions are going to be independent of these two parameters, but actually this is not true in general: adding $\delta_m V'$ and $\delta_m V''$ to the Lagrangian may result in some new set of fixed points flowing in from infinity and deforming the original moduli space of solutions [54] such that the path integral gets contribution from these new fixed points. The theory will be independent of t and $\bar{m}$ if in varying these parameters Lagrangian remains nondegenerate and the perturbation does not introduce new components to the moduli space of fixed points.

We first discuss the situation for $t = 0$ with arbitrary m and $\bar{m}$. The fixed point equations are those of (4.14) together with (setting $\delta_m \tilde{\psi} = \delta_m \zeta = \delta_m \eta = \delta_m \psi_i = 0$)

$$[\beta, \phi] = \tfrac{1}{2} m \beta \ , \quad [\lambda, \phi] = 0 \ , \quad D_i \phi = 0. \tag{4.21}$$

If ϕ is not identically zero then, being covariantly constant, it never vanishes and, in particular, can be diagonalized globally such that the bundle E splits as a sum of line bundles [18]. Moreover, if $\beta \neq 0$, the first equation in (4.21) fixes ϕ (up to a sign)

$$\phi = \frac{m}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.22}$$

with β as

$$\beta = \tilde{\beta}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.23}$$

Now the equations (4.14) become

$$\tilde{f} + 2|\tilde{\beta}|^2 = 0$$
$$\bar{D}\tilde{\beta} = (\delta_{\bar{z}} - iA_{\bar{z}})\tilde{\beta} = 0$$

(notice $f = \frac{1}{2}\tilde{f}\sigma_3$, where $\tilde{f}$ here is the $U(1)$ curvature). Note that $\phi = \frac{1}{4}m\sigma_3$ corresponds to a point, in the classical moduli space of vacua, where a component of the hypermultiplet becomes massless[3]. The relevant fixed points are then determined by the above equations. Clearly one can then argue that the path integral over massless modes computes the Euler characteristic of the moduli space of $U(1)$ flat connections $\mathcal{M}_{fl}$, over Σ. A similar argument to the one in chapter 3 shows that $\mathcal{M}_{fl}$ is parametrized by the torus T^{b_1}. Here $b_1 = 2g$ and T^{b_1} is identical to Σ, thus

$$\chi(T^{2g}) = 2 - 2g\,.$$

To evaluate the contribution of this singular point to the path integral, however, one still has to do the integral over the massive modes.
This is not an easy task, but there is a special case where this point ($\phi = \frac{1}{4}m\sigma_3$) does not make any contribution. This occurs upon restricting to the nontrivial $SO(3)$ bundles. As discussed above, a nonzero ϕ breaks the gauge group down to $U(1)$. In particular, $SO(3)$ bundles split as

$$E = L \oplus \mathcal{O} \oplus L^{-1}, \tag{4.24}$$

where L is the $U(1)$ line bundle and $\mathcal{O}$ is a trivial line bundle. In this case, $w_2(E)$, which measures the nontriviality of the bundle E, turns out to be the mod two reduction of $c_1(L)$, the first Chern class of L [4]. Thus if $f = 0$, as is required by eqs. (4.14), $w_2(E)$ has to be zero – implying that flat nontrivial $SO(3)$ bundles do not admit reducible connections. Therefore, in this case, the point $\phi = \frac{1}{4}m\sigma_3$ does not contribute to the path integral.

Let us now discuss the case that $t \neq 0$. The fixed point equations (4.14) turn into the following equations ($\beta \equiv b + iC$ with $\epsilon_{z\bar{z}} = i\sqrt{g}_1 g_{z\bar{z}}$)

$$f + [\bar{\beta}, \beta] + t\lambda = 0$$
$$\bar{D}\beta = 0 \quad , \quad D\bar{\beta} = 0. \tag{4.25}$$

[3]As eq. (4.21) fixes ϕ up to a sign, there are indeed two such singular points in the classical moduli space of vacua.

The vanishing argument now fails; $f = \beta = 0$ (and $\lambda = 0$) are not the only solutions, there are new fixed points with $f \neq 0$ contributing to the partition function. Since the connection is not bounded to be flat any more, a set of $U(1)$ connections, in all classes of $U(1)$ bundles, appear in the moduli space of solutions. Moreover, the point $\phi = \frac{1}{4}m\sigma_3$ may contribute to the path integral even for nontrivial bundles. In the following we single out this point from our discussion and treat it independently.

4.3.1 Integrating λ, η and χ

Perturbing by V' now allows us to integrate out the fields λ, η and χ. Using the equations of motion for λ and η we get

$$\begin{aligned} t^2\lambda &= D^2\phi - t(f + 2i[b, C]) + 2im[b, C] + \sqrt{2}[\psi_i, \psi^i] \\ &+ \sqrt{2}[\widetilde{\psi}, \widetilde{\psi}] + \frac{1}{2\sqrt{2}}[\zeta, \zeta] + 2[b, [\phi, b]] + 2[C, [\phi, C]] \end{aligned} \tag{4.26}$$

and

$$\chi = \frac{1}{2\sqrt{2}t}\left\{-D_i\psi^i + i\sqrt{2}[b, \widetilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta]\right\}.$$

Putting these back into the Lagrangian yields

$$\begin{aligned} S = I(\varepsilon) &+ \frac{1}{e^2}\int_\Sigma d\mu \ \mathrm{tr}\Big\{D_iCD^iC + D_ibD^ib - \frac{2}{\sqrt{g_1}}\epsilon^{ij}D_ibD_jC + \frac{2i}{\sqrt{g_1}}\epsilon^{ij}\tilde{\chi}_iD_j\widetilde{\psi} + i\tilde{\chi}_iD^i\zeta \\ &- \frac{2\sqrt{2}}{\sqrt{g_1}}\epsilon^{ij}\psi_i[\tilde{\chi}_j, b] + 2\sqrt{2}\psi_i[\tilde{\chi}^i, C] + \sqrt{2}\tilde{\chi}_i[\tilde{\chi}^i, \phi] - \tfrac{1}{2}|m|^2C^2 - \tfrac{1}{2}|m|^2b^2 + \frac{i\bar{m}}{\sqrt{2}}\zeta\widetilde{\psi} \\ &- \frac{m}{\sqrt{2g_1}}\epsilon_{ij}\tilde{\chi}^i\tilde{\chi}^j + 2i\bar{m}\phi[b, C] + \frac{1}{t}\Big\{ \ (f + 2i[b, C]) \\ &\times \left(D^2\phi + 2im[b, C] + \sqrt{2}[\psi_i, \psi^i] + \sqrt{2}[\widetilde{\psi}, \widetilde{\psi}] + \frac{1}{2\sqrt{2}}[\zeta, \zeta] + 2[b, [\phi, b]] + 2[C, [\phi, C]]\right) \\ &+ \frac{i}{2\sqrt{2}}\left(-D_i\psi^i + i\sqrt{2}[b, \widetilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta]\right)\left(\frac{-4}{\sqrt{g_1}}\epsilon^{kl}D_k\psi_l - i4\sqrt{2}[C, \widetilde{\psi}] - 2i\sqrt{2}[\zeta, b]\right)\Big\} \\ &+ \frac{1}{2t^2}\Big\{\left(D^2\phi + 2im[b, C] + \sqrt{2}([\psi_i, \psi^i] + [\widetilde{\psi}, \widetilde{\psi}] + \frac{1}{4}[\zeta, \zeta]) + 2[b, [\phi, b]] + 2[C, [\phi, C]]\right)^2 \\ &+ \sqrt{2}\left(-D_i\psi^i + i\sqrt{2}([b, \widetilde{\psi}] + \frac{1}{2}[C, \zeta])\right)\Big[\left(-D_l\psi^l + i\sqrt{2}([b, \widetilde{\psi}] + \frac{1}{2}[C, \zeta])\right), \phi\Big]\Big\}\Big\}. \end{aligned} \tag{4.27}$$

Using

$$\delta_m(f + 2i[b, C]) = -2\epsilon^{ij}D_i\psi_j + 2i\sqrt{2}[\widetilde{\psi}, C] + \sqrt{2}i[b, \zeta],$$

$$\delta_m\left(-D_i\psi^i + i\sqrt{2}[b, \widetilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta]\right)$$

$$= i\sqrt{2}\left(D^2\phi + 2im[b,C] + \sqrt{2}[\psi_i,\psi^i] + \sqrt{2}[\tilde{\psi},\tilde{\psi}] + \frac{1}{2\sqrt{2}}[\zeta,\zeta] + 2[b,[\phi,b]] + 2[C,[\phi,C]]\right),$$

and

$$\delta_m\left(D^2\phi + 2im[b,C] + \sqrt{2}[\psi_i,\psi^i] + \sqrt{2}[\tilde{\psi},\tilde{\psi}] + \frac{1}{2\sqrt{2}}[\zeta,\zeta] + 2[b,[\phi,b]] + 2[C,[\phi,C]]\right)$$
$$= -2i\left[\left(-D_i\psi^i + i\sqrt{2}[b,\tilde{\psi}] + \frac{i}{\sqrt{2}}[C,\zeta]\right),\phi\right],$$

it is easy to see that terms proportional to $1/t$ are indeed BRST trivial, and can be written

$$\frac{-i}{\sqrt{2}t}\delta_m\left\{(f + 2i[b,C])\left(-D_i\psi^i + i\sqrt{2}[b,\tilde{\psi}] + \frac{i}{\sqrt{2}}[C,\zeta]\right)\right\}.$$

Terms proportional to $1/t^2$ are also combining into

$$\frac{-i}{2\sqrt{2}t^2}\delta_m\left\{\left(-D_i\psi^i + i\sqrt{2}[b,\tilde{\psi}] + \frac{i}{\sqrt{2}}[C,\zeta]\right)\times\right.$$
$$\left.\left(D^2\phi + 2im[b,C] + \sqrt{2}[\psi_l,\psi^l] + \sqrt{2}[\tilde{\psi},\tilde{\psi}] + \frac{1}{2\sqrt{2}}[\zeta,\zeta] + 2[b,[\phi,b]] + 2[C,[\phi,C]]\right)\right\}.$$

In the effective Lagrangian (4.27), the kinetic terms are nondegenerate for all values of t and since those terms proportional to t are still in a BRST exact form, the path integral does not depend on t.

4.3.2 Large t Limit and The Integration over b, C, ζ, $\tilde{\psi}$

As argued above, for nontrivial $SO(3)$ bundles the point $\phi = \frac{1}{4}m\sigma_3$ does not contribute.

For $t \neq 0$, because of the supersymmetry, even after integrating out λ, η and χ the singularity still persists at $\operatorname{tr}\phi^2 = \frac{1}{8}m^2$. As we have chosen ϕ to be a real scalar field, reality of the action requires that m to be a real parameter. However, to regulate the contribution of the points in the neighborhood of $\operatorname{tr}\phi^2 = \frac{1}{8}m^2$, we allow m to have a small imaginary part. If there is going to be any singularity when ϕ approaches m, it has to show up in the final result when we take the limit $\operatorname{Im} m \to 0$. This can be thought of as a kind of regularization by analytic continuation.

Now let us consider the large limit of t. Since the kinetic terms remain nondegenerate we can actually take $t \to \infty$. This amounts to droping a BRST exact term from the Lagrangian and we are arguing that this is allowed since the remaining part of the

Lagrangian is nondegenerate. Using the auxiliary field $\tilde{H}_i$, in this limit we are left with the action

$$\begin{aligned} S &= \frac{1}{e^2}\int_\Sigma d\mu \ \mathrm{tr}\left\{-\tfrac{1}{2}\tilde{H}^i(\tilde{H}_i - 2\sqrt{2}D_iC + \frac{2\sqrt{2}}{\sqrt{g}_1}\epsilon_{ji}D^jb) + \frac{2i}{\sqrt{g}_1}\epsilon^{ij}\tilde{\chi}_iD_j\tilde{\psi} + i\tilde{\chi}_iD^i\zeta\right. \\ &- \tfrac{1}{2}|m|^2C^2 - \tfrac{1}{2}|m|^2b^2 + \frac{i\bar{m}}{\sqrt{2}}\zeta\tilde{\psi} - \frac{m}{\sqrt{2g_1}}\epsilon_{ij}\tilde{\chi}^i\tilde{\chi}^j + 2i\bar{m}\phi[b,C] \\ &- \left.\frac{2\sqrt{2}}{\sqrt{g}_1}\epsilon^{ij}\psi_i[\tilde{\chi}_j,b] + 2\sqrt{2}\psi_i[\tilde{\chi}^i,C] + \sqrt{2}\tilde{\chi}_i[\tilde{\chi}^i,\phi]\right\} + I(\varepsilon). \end{aligned}$$

$\mathcal{L}$ can still be written as a sum of BRST exact term

$$i\delta_m\left\{\frac{1}{e^2}\mathrm{tr}\left\{\tfrac{1}{2}\tilde{\chi}^i(\tilde{H}_i - 2\sqrt{2}D_iC + \frac{2\sqrt{2}}{\sqrt{g}_1}\epsilon_{ji}D^jb) + \frac{1}{2}\bar{m}(\tilde{\psi}C - \tfrac{1}{2}\zeta b)\right\}\right\}$$

and $I(\varepsilon)$. The integral over C gives a factor of $\left(\det\left(\frac{1}{2e^2}|m|^2\right)\right)^{-\frac{3}{2}}$ and leaves

$$\begin{aligned} S &= \frac{1}{e^2}\int_\Sigma \mathrm{tr}\left\{-\tfrac{1}{2}\tilde{H}^i\tilde{H}_i + \sqrt{2}\tilde{\chi}^i[\tilde{\chi}_i,\phi] - \frac{m}{\sqrt{2g_1}}\epsilon_{ij}\tilde{\chi}^i\tilde{\chi}^j + \frac{2i}{\sqrt{g}_1}\epsilon^{ij}\tilde{\chi}_iD_j\tilde{\psi} + i\tilde{\chi}_iD^i\zeta\right. \\ &+ \frac{1}{|m|^2}(D_i\tilde{H}^i - 2[\tilde{\chi}_i,\psi^i])^2 - 2\frac{\bar{m}}{m}[b,\phi]^2 + \frac{2i\sqrt{2}}{m}[b,\phi](D_i\tilde{H}^i - 2[\tilde{\chi}_i,\psi^i]) \\ &- \left.\frac{\sqrt{2}}{\sqrt{g}_1}\epsilon_{ji}\tilde{H}^iD^jb - \tfrac{1}{2}|m|^2b^2 + \frac{i\bar{m}}{\sqrt{2}}\zeta\tilde{\psi} - \frac{2\sqrt{2}}{\sqrt{g}_1}\epsilon^{ij}\psi_i[\tilde{\chi}_j,b]\right\} + I(\varepsilon). \end{aligned}$$

Next we would like to integrate out b,ζ and $\tilde{\psi}$. It is easy to integrate out ζ and $\tilde{\psi}$ using their equations of motion. In the evaluation of determinants, which appear in doing the integral over b and finally over χ_i, we always assume that ϕ is a constant field. This can be justified finally when the integral over the gauge fields constrains ϕ to be constant. The equation of motion for b yields

$$b^A = \frac{\sqrt{2}}{|m|^2}K^{AB}\left(-\frac{1}{\sqrt{g}_1}\epsilon^{ij}D_j\tilde{H}_i + \frac{2}{\sqrt{g}_1}\epsilon^{ij}[\tilde{\chi}_i,\psi_j] - \frac{2i}{m}[(D_i\tilde{H}^i - 2[\tilde{\chi}_i,\psi^i]),\phi]\right)^B, \quad (4.28)$$

where we have defined (A and B are Lie-algebra indices)

$$K^{AB} \equiv (1 - \frac{8}{m^2}\mathrm{tr}\,\phi^2)^{-1}(\delta^{AB} - \frac{8}{m^2}\phi^A\phi^B).$$

Replacing b in the action, we obtain

$$\begin{aligned} S &= I(\varepsilon) + \frac{1}{e^2}\int_\Sigma d\mu \ \mathrm{tr}\left\{-\frac{1}{2}\tilde{H}^i\tilde{H}_i + \sqrt{2}\tilde{\chi}^i[\tilde{\chi}_i,\phi]\right. \\ &- \left.\frac{m}{\sqrt{2g_1}}\epsilon_{ij}(\tilde{\chi}^i\tilde{\chi}^j - \frac{4i}{|m|^2}D_l\tilde{\chi}^lD^i\tilde{\chi}^j) + \frac{1}{|m|^2}(D_i\tilde{H}^i - 2[\tilde{\chi}_i,\psi^i])^2\right\} \end{aligned}$$

$$+\frac{1}{|m|^2}\left(-\frac{1}{\sqrt{g}_1}\epsilon^{ij}D_j\tilde{H}_i+\frac{2}{\sqrt{g}_1}\epsilon^{ij}[\tilde{\chi}_i,\psi_j]-\frac{2i}{m}[(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i]),\phi]\right)^A K^{AB}$$
$$\times\left(-\frac{1}{\sqrt{g}_1}\epsilon^{kl}D_l\tilde{H}_k+\frac{2}{\sqrt{g}_1}\epsilon^{kl}[\tilde{\chi}_k,\psi_l]-\frac{2i}{m}[(D_l\tilde{H}^l-2[\tilde{\chi}_l,\psi^l]),\phi]\right)^B ,$$

and a factor of $(\det(\frac{1}{\sqrt{2e^2}}\bar{m})^3)(\det\frac{1}{2e^2}|m|^2)^{-3}(\det(1-\frac{8}{m^2}\mathrm{tr}\,\phi^2)^2)^{-\frac{1}{2}}$.
The following are easily derived,

$$\begin{aligned}\delta_m(D_i\tilde{\chi}^i) &= i(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i])\\ \delta_m(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i]) &= 2\sqrt{2}i[D_i\tilde{\chi}^i,\phi]+\frac{\sqrt{2}m}{\sqrt{g}_1}\epsilon^{ij}D_i\tilde{\chi}_j\end{aligned}$$

$$\begin{aligned}\delta_m\left\{(D_l\tilde{\chi}^l)(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i])\right\} &= i(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i])^2-2\sqrt{2}i(D_i\tilde{\chi}^i)[D_l\tilde{\chi}^l,\phi]\\ &+ \frac{\sqrt{2}m}{\sqrt{g}_1}\epsilon^{ij}D_i\tilde{\chi}_jD_l\tilde{\chi}^l ,\end{aligned}$$

and

$$\delta_m\left\{\frac{1}{\sqrt{g}_1}\epsilon^{ij}D_j\tilde{\chi}_i+\frac{2i}{m}[D_i\tilde{\chi}^i,\phi]\right\}=\frac{i\epsilon^{ij}}{\sqrt{g}_1}D_j\tilde{H}_i-\frac{2i}{\sqrt{g}_1}\epsilon^{ij}[\tilde{\chi}_i,\psi_j]-\frac{2}{m}[(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i]),\phi]$$
$$\delta_m^2\left\{\frac{1}{\sqrt{g}_1}\epsilon^{ij}D_j\tilde{\chi}_i+\frac{2i}{m}[D_i\tilde{\chi}^i,\phi]\right\}=i\sqrt{2}mD_l\tilde{\chi}^l-\frac{4i\sqrt{2}}{m}\left[[D_l\tilde{\chi}^l,\phi],\phi\right].$$

Using these, the action can be written as

$$\begin{aligned}S &= I(\varepsilon)+\frac{1}{e^2}\int_\Sigma d\mu\ \mathrm{tr}\left(-\tfrac{1}{2}\tilde{H}^i\tilde{H}_i+\sqrt{2}\tilde{\chi}^i[\tilde{\chi}_i,\phi]-\frac{m}{\sqrt{2g_1}}\epsilon_{ij}\tilde{\chi}^i\tilde{\chi}^j\right)\\ &- \frac{i}{e^2|m|^2}\int_\Sigma d\mu\ \ \delta_m\left\{\mathrm{tr}\ \left((D_l\tilde{\chi}^l)(D_i\tilde{H}^i-2[\tilde{\chi}_i,\psi^i])\right)\right.\\ &+ \left(\frac{\epsilon^{ij}}{\sqrt{g}_1}D_j\tilde{\chi}_i+\frac{2i}{m}[D_i\tilde{\chi}^i,\phi]\right)^A K^{AB}\\ &\times \left.\left(\frac{\epsilon^{kl}}{\sqrt{g}_1}(D_l\tilde{H}_k-2[\tilde{\chi}_k,\psi_l])+\frac{2i}{m}[(D_l\tilde{H}^l-2[\tilde{\chi}_l,\psi^l]),\phi]\right)^B\right\}.\end{aligned} \tag{4.29}$$

Note that the integration over b, C, ζ and $\tilde{\psi}$ has not destroyed the manifest BRST exactness of the action, in particular, the variation of S with respect to $\bar{m}$ is still a BRST commutator.

4.3.3 Large $\bar{m}$ Limit and The Final Reduction

We note the partition function is formally independent of $\bar{m}$ (since the variation of the partition function with respect to $\bar{m}$ gives an BRST exact expression) and is really independent of $\bar{m}$ if in varying $\bar{m}$ the Lagrangian remains nondegenerate with a good behaviour at infinity in field space. The mass term for $\tilde{\chi}_i$, the term $\tilde{H}^i\tilde{H}_i$, and the form of the cohomology classes that we have added by hand, guarantee that this is actually the case. Having this freedom in the value of $\bar{m}$, we simply set $\bar{m} = \infty$. This leaves us with the action

$$S = I(\varepsilon) + \frac{1}{e^2}\int_\Sigma d\mu \left\{ -\tfrac{1}{2}\tilde{H}^{iA}\tilde{H}_i{}^A - \tilde{\chi}^{iA}\left(\frac{1}{\sqrt{2}g_1} m\epsilon_{ij}\delta_{AB} - 2if_{ABC}\phi_C g_{ij}\right)\tilde{\chi}^{jB}\right\},$$

and partition function reads

$$Z[\varepsilon, m] = \int \mathcal{D}(A_i, \psi_i, \phi, \tilde{H}_i, \tilde{\chi}_i)\left(\frac{\det(\frac{1}{\sqrt{2}e^2}\bar{m})^3}{(\det \frac{1}{2e^2}|m|^2)^3(\det(1 - \frac{8}{m^2}\mathrm{tr}\,\phi^2)^2)^{\frac{1}{2}}}\right)_{\Omega^0} e^{-S},$$

where Ω^0 indicates the determinant has to be evaluated in the space of zero-forms. The explicit appearance of m on the LHS reminds us that, although independent of $\bar{m}$, Z does depend on m. This is so because m was introduced through the BRST transformation laws. This is reminiscent of holomorphicity of $N = 1$ theories in four dimensions.

Doing the integral over $\tilde{\chi}^i$ gives a similar determinant, but this time over the space of one-forms. Putting all pieces together one gets

$$Z[\varepsilon, m] = \int \mathcal{D}(A_i, \psi_i, \phi)\left(\frac{\left[\det m^3(1 - \frac{8}{m^2}\mathrm{tr}\,\phi^2)\right]^{\frac{1}{2}}_{\Omega^1}}{\left[\det m^3(1 - \frac{8}{m^2}\mathrm{tr}\,\phi^2)\right]_{\Omega^0}}\right) e^{\left(\frac{-1}{4\pi^2}\int_\Sigma \mathrm{tr}\left(\frac{i}{\sqrt{2}}\phi F + \frac{1}{2}\psi\wedge\psi\right) - \frac{\varepsilon}{32\pi^2}\int_\Sigma \mathrm{tr}\,\phi^2\right)}. \tag{4.30}$$

Notice that, as expected, $\bar{m}$ cancels out between the fermionic and bosonic determinants. The integral over ψ_i provides a symplectic measure for the gauge fields A_i [54]. Performing the path integral over ϕ and A_i is now straightforward. Indeed, apart from the determinant factor of the integrand, (4.30) is the path integral of a two-dimentional Yang-Mills theory. In appendix D, using the Faddeev-Popov gauge fixing technique, we show that the integral over the gauge fields constrains ϕ to be constant and hence the path integral calculation reduces to a finite dimensional integral over constant ϕ. Explicitly, for $SO(3)$ gauge group we have

$$Z[\varepsilon, m] = m^{3(g-1)}\sum_{n\in\mathbf{Z}}\int d\phi\ \phi^{2-2g}(1 - \frac{8}{m^2}\phi^2)^{g-1}\exp\left(-i\sqrt{2}\frac{\phi(n+\frac{1}{2})}{4\pi} - \frac{\varepsilon\phi^2}{32\pi^2}\right). \tag{4.31}$$

4.4 Discussion

We have reduced the calculation of the correlation functions in the mass deformed theory to a finite dimensional integral in (4.31). In the case that Σ is a Riemann sphere, the integrand has a singularity when ϕ approaches m as the imaginary part of m is removed. However, for higher genus surfaces the integrand is regular. Using

$$(1+x)^N = \sum_{n=0}^{N} \frac{N!}{(N-n)!\; n!}\, x^{N-n}$$

the eq. (4.31) can be written as

$$Z[\varepsilon, m] = (-8m)^{g-1} \sum_{r=0}^{g-1} \frac{(g-1)!}{(g-1-r)!\; r!} \left(-\frac{m^2}{8}\right)^{g-1-r} Z_r(\varepsilon),$$

where

$$Z_r(\varepsilon) = \sum_{n\in\mathbf{Z}} \int d\phi\; \phi^{2(1+r-g)} \exp\left(-i\sqrt{2}\frac{\phi(n+\frac{1}{2})}{4\pi} - \frac{\varepsilon\phi^2}{32\pi^2}\right). \tag{4.32}$$

Here $Z_r(\varepsilon)$ is actually the partition function of the Yang-Mills theory on a Riemann surface of genus $g-r$ [55]. For the $SO(3)$ gauge group this is known to be [54]

$$Z_r(\varepsilon) = \frac{1}{2(8\pi^2)^{g-1-r}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \exp(-\varepsilon\pi^2 n^2)}{n^{2(g-1-r)}}.$$

So the final expression is

$$Z[\varepsilon, m] = (-8m)^{g-1} \sum_{r=0}^{g-1} \frac{(g-1)!}{2(g-1-r)!\; r!} \left(-\frac{m^2}{64\pi^2}\right)^{g-1-r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \exp(-\varepsilon\pi^2 n^2)}{n^{2(g-1-r)}}. \tag{4.33}$$

To this one still has to add the contribution of the point $\phi = \frac{1}{4}m\sigma_3$. However, note that from the discussion we had in section 3, for nontrivial $SO(3)$ bundles, this point contributes only if we perturb to $t \neq 0$[4]. Thus if we are interested in the limit of $t=0$, we can just ignore the contribution of this point.

Differentiating (4.32) $g-1-r$ times with respect to ε we get

$$\frac{\delta^{g-1-r} Z_r(\varepsilon)}{\delta\varepsilon^{g-1-r}} = \left(\frac{-1}{16\pi^2}\right)^{g-1-r} \sqrt{\frac{32\pi^2}{\varepsilon}} \sum_{n\in\mathbf{Z}} \exp\left(-\frac{(n+\frac{1}{2})^2}{\varepsilon}\right) \tag{4.34}$$

upon integrating up with respect to ε we get a polynomial in ε and terms which are exponentially small.

[4]As is discussed in [54], the contribution of the original moduli space is invariant under perturbing to $t \neq 0$.

Finally let us discuss the origin of these exponential terms. Recall that if the equation $D_i\phi = 0$ has nontrivial solutions, then bundle splits to a sum of line bundles. Moreover, the perturbed eqs. (4.25) show that, for $t \neq 0$ and $\beta = 0$ ($\beta \neq 0$ in this case corresponds to the point $\phi = \frac{1}{4}m\sigma_3$), f is not zero. This implies that the corresponding line bundles are not necessarily trivial. Indeed for a bundle E as in (4.24), $w_2(E)$ is the mod two reduction of the first Chern class of L. Thus the nontrivial part of f is (if we choose ϕ in the 3rd direction)

$$f = 2\pi(n + \tfrac{1}{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.35}$$

where n is an integer. The classical action for such a configuration is

$$-\frac{c_1(L)^2}{\varepsilon} = \frac{-1}{8\pi^2\varepsilon}\int_\Sigma d\mu \ \mathrm{tr}\, f^2 = -\frac{(n+\frac{1}{2})^2}{\varepsilon}\ ,$$

where $c_1(L)$ is the first Chern class of the bundle L. We note that these are the same exponents appearing in (4.34).

We conclude that the perturbation by V' introduces a new component to the moduli space of fixed points where $f \neq 0$ and the gauge field is a $U(1)$ connection. Thus the exponentially small terms in the final result can be recognized as the contribution of this new component of moduli space to the partition function. Apart from these exponentially small terms, there are polynomial terms in ϵ coming from the original moduli space with $t = 0$. In this sense, we have been able to compute (in a chamber where S^2 shrinks) some specific correlation functions for the $N = 4$ SYM theory broken to $N = 2$ by the mass term for the hypermultiplet.

In conclusion we note two observations. Firstly, the result is m-dependent as might be expected from the discussion in section 3. Note in particular that the expression (4.31) has the right behavior when $m \to \infty$; in this limit the kinetic terms of the heavy fields are negligible compared to their mass terms meaning that these fields are so heavy that do not propagate and all interactions between the heavy and light fields can be ignored. Therefore in this limit the heavy fields decouple from the light ones such that we are left with the corresponding correlation functions in the say pure $N = 2$ theory. The remaining factor, $m^{3(g-1)}$, is left from the integration over the heavy fields in that limit. Notice that the power of m is in accord with the dimension of the moduli space of flat connections which is

$$\dim(\mathcal{M}) = 6g - 6.$$

Any two zero modes of χ_i are absorbed by the corresponding mass term in the Lagrangian and gives a power of m.

Secondly, we recall that S-duality relates the strong and weak couplings and swaps the gauge group with its dual group. Thus to provide an explicit check on S-duality, one still needs to do the calculations for the $SU(2)$ case. As noted earlier, the main difficulty which arises in this case is due to the contribution of the singular points where a component of the hypermultiplet becomes massless. Although we recognized the contribution of the massless modes on these points, the integral over the massive modes remains to be done. Turning around the problem, a better understanding of the S-duality action in this particular case will allow us to infer properties of this contribution by *demanding* S-duality.

The problem that we considered in this chapter is also useful in studying the low energy description of D-branes wrapping around spheres which are holomorphically embedded in a Calabi-Yau 2-fold [6]. This configuration arises when one studies the solitonic states (D-branes) upon compactifying the string theory on a Calabi-Yau 2-fold. The low energy physics of such D-branes wrapping around the sphere (holomorphically embedded in the Calabi-Yau 2-fold) is described by the same twisted theory that we studied in chapter four, however, the four-manifold is now $\mathbf{R} \times S^1 \times S^2$ where S^2 is holomorphically embedded in the Calabi-Yau manifold.

Another route for further investigation is to establish the wall crossing formula in this particular case. In [27] it was shown that upon shrinking Σ, instead of S^2, one gets a two-dimensional sigma model governing maps from S^2 to $\mathcal{M}$, where $\mathcal{M}$ is the moduli space of solutions to Hitchin's equations on Σ. Having the results for the two extreme limits, one should, in principle, be able to work out the wall crossing formula.

Appendix A

The Fixed Point Theorem

In this appendix, Witten's fixed point theorem [16] is discussed. This is a general theorem about theories with a fermionic symmetry, so its application is not limited to topological field theories.

Let $\mathcal{E}$ denote the space of fields on which we wish to integrate and F a symmetry group which acts freely on the space of fields. Thus, instead of $\mathcal{E}$, we can consider the fibered space $\mathcal{E}/F$ and perform the integral over it. The integral over the fibers simply gives the volume of F. Upon considering F-invariant observables we have

$$\int_{\mathcal{E}} e^{-S}\mathcal{O} = \mathrm{vol}(F) \int_{\mathcal{E}/F} e^{-S}\mathcal{O}\,. \tag{A.1}$$

Now let F be the BRST symmetry of Q. Since Q has a fermionic character, its volume vanishes

$$\int d\theta \cdot 1 = 0\,.$$

Thus, if Q acts freely then (A.1) implies that the correlation function of any operator $\mathcal{O}$ vanishes. However, Q does not, in general, act freely and there is a subspace $\mathcal{E}_0$ invariant under the action of Q. Take a small tubular neighbourhood of $\mathcal{E}_0$ and call its complement $\mathcal{E}'$. Since Q acts freely on $\mathcal{E}'$, it gives no contribution to the path integral by the above argument. The whole contribution thus comes from $\mathcal{E}_0$ and its close neighbourhood, such that one can expand the action around the fixed points $\mathcal{E}_0$ up to the second order and does the path integral. Note, however, the integral over $\mathcal{E}_0$ must be done exactly. In the light of the fixed point theorem, the origin of eqs. (2.8) now becomes clear. They simply arise by looking for the fixed point action of Q in (2.7). Setting $\delta\chi_{\mu\nu} = \delta\psi_\mu = \delta\eta = 0$, we arrive at the same equations in (2.8).

Appendix B

Majorana-Weyl spinors in ten dimensions

Let the Minkowski signature be $(-1, 1, 1, \ldots, 1)$ in D spacetime dimensions. The Clifford algebra generators are Γ^M such that

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN},$$

for $M = 0, \ldots, 9$. Introduce Γ_{11}

$$\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$$

which anticommutes with all Γ^M and satisfies

$$\Gamma_{11}^\dagger = \Gamma_{11}, \ \Gamma_{11}^2 = 1.$$

The unitary charge conjugation matrix C is defined by

$$\mathsf{C}\Gamma_M\mathsf{C}^{-1} = \sigma_d \Gamma_M^t \tag{B.1}$$

$$\mathsf{C}^t = \sigma_t \mathsf{C}\,. \tag{B.2}$$

where σ_d, $\sigma_t \in \{\pm 1\}$, also note that $\mathsf{C}\Gamma_{11}\mathsf{C}^{-1} = -\Gamma_{11}^t$. Further, define the Dirac conjugate

$$\overline{\Psi} = \Psi^\dagger \Gamma_0,$$

and Majorana conjugate

$$\overline{\Psi}_c = \Psi^t \mathsf{C},$$

Ψ is Majorana if

$$\overline{\Psi} = \overline{\Psi}_c$$

or $\Psi^\dagger = -\Psi^t C\Gamma_0$. This implies $\Psi^t\Gamma_0^* = \Psi^\dagger C^* = -\Psi^t C\Gamma_0 C^*$ and therefore

$$\Gamma_0^\dagger = -\sigma_t C^\dagger \Gamma_0^t C = -\sigma_d\sigma_t\Gamma_0$$

concluding that $\Gamma_0^\dagger\Gamma_0 = \sigma_d\sigma_t$. Thus having Majorana spinors requires that $\sigma_d\sigma_t = 1$. Weyl spinors are defined using the projector Γ_{11}; $\Gamma_{11}\Psi = \pm\Psi$. In 10 dimensions it is consistent to constrain a spinor to be simultaneously Weyl and Majorana

$$\Gamma_{11}\Psi = \Gamma_{11}\Gamma_0 C^{-1}\Psi^* = -\Gamma_0\Gamma_{11}C^{-1}\Psi^* = \Gamma_0 C^{-1}\Gamma_{11}^t\Psi^* = \pm\Gamma_0 C^{-1}\Psi^* = \pm\Psi$$

where Ψ was assumed to be Weyl ($\Gamma_{11}^t\Psi^* = \pm\Psi^*$). A spinor in ten dimensions has 32 independent complex components. The Majorana-Weyl condition reduces this representation to the one with 16 independent real components. As the fermionic equations of motion are first order, there are in fact 8 degrees of freedom on the representation space (on-shell). This is the same number of degrees of freedom of a gauge field (on-shell). Thus in a system of gauge fields and fermions in 10 dimensions, one can balance the number of degrees of freedom between bosons and fermions by simply putting the Majorana-Weyl condition.

B.1 A 10d Fierz identity and the proof of supersymmetry

In the following we show that the last term in (2.44) vanishes. Writing out the Lie-algebra indices of this term explicitly we have

$$-\frac{i}{2}f^{abc}(\bar{\alpha}\Gamma^M\Psi^b)(\overline{\Psi}^a\Gamma_M\Psi^c)\,.$$

We can expand $\Psi^b\overline{\Psi}^a$, a 32×32 matrix, using the complete set of independent matrices

$$\mathbf{1}\ ,\ \Gamma_M\ ,\ \Gamma_{MN}\ ,\ \Gamma_{MNL}\ ,\ \Gamma_{MNLK}\ ,\ \Gamma_{MNLKI}\ ,\ \Gamma_M\Gamma_{11}\ ,$$
$$\Gamma_{MN}\Gamma_{11}\ ,\ \Gamma_{MNL}\Gamma_{11}\ ,\ \Gamma_{MNLK}\Gamma_{11}\ ,\ \Gamma_{11}\,,$$

spanning a 32×32-dimensional vector space. This basis is orthogonal with respect to the inner product defined by trace. Since Ψ is Weyl, $\Gamma_{11}\Psi = -\Psi$, $\overline{\Psi}\Gamma_{11} = \overline{\Psi}$, the most general form of this expansion is

$$\Psi^b\overline{\Psi}^a = b^{ab}_{1M}\Gamma^M(1+\Gamma_{11}) + b^{ab}_{2MNK}\Gamma^{MNK}(1+\Gamma_{11}) + b^{ab}_{3MNKIJ}\Gamma^{MNKIJ}(1+\Gamma_{11}) . \quad \text{(B.3)}$$

Notice that

$$\overline{\Psi}\Gamma^{(M)}\Psi = 0 \,, \qquad \text{for } |M| \text{ even} .$$

Multiplying (B.3) by Γ_M, Γ_{MNL} and Γ_{MNLKI} respectively and using the orthogonality of the bases, one finds out that

$$\begin{aligned}\Psi^b\overline{\Psi}^a &= -\frac{1}{32}\overline{\Psi}^a\Gamma_M\Psi^b\Gamma^M(1+\Gamma_{11}) + \frac{1}{32\cdot 3!}\overline{\Psi}^a\Gamma_{MNK}\Psi^b\Gamma^{MNK}(1+\Gamma_{11}) \\ &- \frac{1}{32\cdot 5!}\overline{\Psi}^a\Gamma_{MNKIJ}\Psi^b\Gamma^{MNKIJ} . \end{aligned} \quad \text{(B.4)}$$

This is the Fierz rearrangement formula in 10 dimensions. Let us choose $\sigma_d^{(10)} = \sigma_t^{(10)} = -1$ in (B.1) and (B.2), then we have

$$\Gamma^t_{MN} = -\mathsf{C}\Gamma_{MN}\mathsf{C}^{-1} . \quad \text{(B.5)}$$

Now, since

$$\Gamma_{MNK} = \Gamma_{MN}\Gamma_K - \eta_{NK}\Gamma_M + \eta_{MK}\Gamma_N \,,$$

(B.1) and (B.5) imply

$$\Gamma^t_{MNK} = \mathsf{C}\Gamma_{MNK}\mathsf{C}^{-1} .$$

Therefore, the second term in (B.4) is symmetric in a and b

$$\overline{\Psi}^a\Gamma_{MNK}\Psi^b = -\Psi^{bt}\Gamma^t_{MNK}\mathsf{C}^t\Psi^a = \overline{\Psi}^b\Gamma_{MNK}\Psi^a .$$

As this term gets contracted by f^{abc} in the trilinear term, it gives no contribution. Finally since $\Gamma^M\Gamma_N\Gamma_M = -8\Gamma_N$ and $\Gamma^L\Gamma^{MNKIJ}\Gamma_L = 0$, we conclude that the trilinear term is indeed zero.

B.2 Conventions

For the spinor representations of SO(3,1) we use notation along the lines of Wess and Bagger [12],

$$\begin{aligned}\sigma^I &= (1, \sigma^i) \\ \bar{\sigma}^I &= (-1, \sigma^i) ,\end{aligned}$$

such that

$$\bar{\sigma}^{I\dot{a}a} = -\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{I}_{b\dot{b}}\,.$$

We choose $\epsilon^{12} = 1$ and $\epsilon^{ab}\epsilon_{bc} = -\delta^{a}_{c}$. The dot distinguishes between the two spinor representations, ψ_a and $\bar{\psi}^{\dot{a}}$, for which the SO(3,1) generators are

$$\begin{aligned} \sigma^{IJ} &\equiv \tfrac{1}{4}(\sigma^{I}\bar{\sigma}^{J} - \sigma^{J}\bar{\sigma}^{I}) \\ \bar{\sigma}^{IJ} &\equiv \tfrac{1}{4}(\bar{\sigma}^{I}\sigma^{J} - \bar{\sigma}^{J}\sigma^{I})\,. \end{aligned}$$

It is straightforward to see that with

$$(\psi_a)^{\dagger} \equiv \bar{\psi}_{\dot{a}}\,, \qquad (\bar{\psi}^{\dot{a}})^{\dagger} \equiv \psi^{a}\,,$$

we have, eg,

$$\bar{\psi}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}}\,.$$

The following identities will be useful

$$\begin{aligned} \sigma_{Ia\dot{b}}\bar{\sigma}^{I\dot{c}d} &= 2\delta^{d}_{a}\delta^{\dot{c}}_{\dot{b}}\,, \qquad \bar{\sigma}^{\dot{a}b}_{I}\bar{\sigma}^{I\dot{c}d} = -2\epsilon^{\dot{a}\dot{c}}\epsilon^{bd}\,, \qquad \sigma_{Ia\dot{b}}\sigma^{I}_{c\dot{d}} = -2\epsilon_{ac}\epsilon_{\dot{b}\dot{d}} \\ (\bar{\sigma}^{I}\sigma^{J})^{\dot{a}}{}_{\dot{b}} &= \eta^{IJ}\delta^{\dot{a}}_{\dot{b}} + 2\bar{\sigma}^{IJ\dot{a}}{}_{\dot{b}}\,, \qquad (\sigma^{I}\bar{\sigma}^{J})_{a}{}^{b} = \eta^{IJ}\delta^{b}_{a} + 2\sigma^{IJb}_{a} \\ \bar{\sigma}^{I}\sigma^{J}\bar{\sigma}^{K} &= -i\epsilon^{IJKL}\bar{\sigma}_{L} - \eta^{IK}\bar{\sigma}^{J} + \eta^{IJ}\bar{\sigma}^{K} + \eta^{JK}\bar{\sigma}^{I} \\ \mathrm{tr}\,(\sigma^{IJ}\sigma^{KL}) &= -\frac{1}{2}(\eta^{IK}\eta^{JL} - \eta^{IL}\eta^{JK}) + \frac{i}{2}\epsilon^{IJKL}\,. \end{aligned} \tag{B.6}$$

Appendix C

The Vanishing Argument

In this appendix we want to discuss the solutions to eqs. (4.10):

$$k = F_{ab} + 4i[B_{ai}, B^{i}{}_{b}] = 0$$
$$s = D^a B_{ai} = 0.$$

Let first analyze the second equation. After squaring we get

$$\begin{aligned}
\int \mathrm{tr}\,(D^a B_{ai})^2 &= -\int \mathrm{tr}\, B^{ai}(D_a D_b B^b{}_i) \\
&= -\int \mathrm{tr}\,\left(B^{ai} D_b D_a B^b{}_i + B^{ai}[D_a, D_b]B^b{}_i\right) \\
&= \int \mathrm{tr}\,\left((D_a B^b{}_i)(D_b B^{ai}) + R_{ab} B^{ai} B^b{}_i - iB^{ai}[F_{ab}, B^b{}_i]\right) \\
&= \int \mathrm{tr}\,\left((D_a B^b{}_i + D^b B_{ai} - D^b B_{ai})(D_b B^{ai}) + \tfrac{1}{2}R\; B^{ai}B_{ai} - iB^{ai}[F_{ab}, B^b{}_i]\right) \\
&= \int \mathrm{tr}\,\left((D_b B_{ai})^2 - \tfrac{1}{2}(D_{[a}B_{b]i})^2 + \tfrac{1}{2}R\; B^{ai}B_{ai} - iB^{ai}[F_{ab}, B^b{}_i]\right) \qquad \text{(C.1)}
\end{aligned}$$

where we used the fact that in two dimensions, Ricci tensor takes a simple form

$$R_{ab} = \tfrac{1}{2} g_{ab} R$$

and

$$\begin{aligned}
&[D_a, D_b]B^{ci} = R^c{}_{dab} B^{di} + i[F_{ab}, B^{ci}] \\
&[D_a, D_b]B^{ai} = R_{ab} B^{ai} + i[F_{ab}, B^{ai}]. \qquad \text{(C.2)}
\end{aligned}$$

Since $B_{\mu\nu}$ is self-dual, we have $B_{w\bar{z}} = B_{\bar{w}z} = 0$, hence

$$\begin{aligned}
(D_{[a}B_{b]i})(D^{[a}B^{b]i}) &= (D_{\bar{w}} B_{wz})(D^{\bar{w}} B^{wz}) + (D_w B_{\bar{w}\bar{z}})(D^w B^{\bar{w}\bar{z}}) \\
&= (D^w B_{wz})(D_w B^{wz}) + (D^{\bar{w}} B_{\bar{w}\bar{z}})(D_{\bar{w}} B^{\bar{w}\bar{z}}) \\
&= (D^a B_{ai})(D_b B^{bi}). \qquad \text{(C.3)}
\end{aligned}$$

Putting this back into (C.1) we get

$$\tfrac{3}{2}\int \mathrm{tr}\,(D^a B_{ai})^2 = \int \mathrm{tr}\,\left((D_b B_{ai})^2 + \tfrac{1}{2}R\; B^{ai}B_{ai} - iB^{ai}[F_{ab}, B^b{}_i]\right). \qquad \text{(C.4)}$$

Upon adding the squares of the sections k and s, we have

$$\begin{aligned}
\int \mathrm{tr}\,(\tfrac{1}{4}k^2 + 3s^2) &= \int \mathrm{tr}\,\left\{\tfrac{1}{4}(F_{ab})^2 - 4[B_{ai}, B^i{}_b]^2 + 2iF_{ab}[B^{ai}, B_i{}^b] + 2(D_b B_{ai})^2\right. \\
&+ \left. R\; B^{ai}B_{ai} - 2iB^{ai}[F_{ab}, B^b{}_i]\right\} \\
&= \int \mathrm{tr}\,\left\{\tfrac{1}{4}(F_{ab})^2 - 4[B_{ai}, B^i{}_b]^2 + 2(D_b B_{ai})^2 + R\; B^{ai}B_{ai}\right\}
\end{aligned}$$

the right hand side vanishes if and only if $k = s = 0$. However, for sphere $(R > 0)$ all terms on the RHS are positive definite so a solution to $k = s = 0$ has necessarily $B^{ai} = 0$. This leaves us with the equation

$$F_{ab} = 0$$

this equation implies that the connection is locally a pure gauge $A_a = u^{-1}d_a u$ for some $SU(2)$ matrix u. However, as the transition functions for $SU(2)$ bundles on sphere are trivial, the connection can be written globally as a pure gauge and be gauged away. Moreover, one can argue that this can be done continuously all over Σ. Thus we can set $A_a = 0$ everywhere.

More rigorously if $\{U_\alpha\}$ is an open covering of Σ by contractible sets and $\{V_i\}$ is an open covering of S^2 by such sets, the sets $U_\alpha \times V_i$ give an open cover of $\Sigma \times S^2$ by contractible sets. On the intersection of two patches, the connection A now satisfies

$$A_{\alpha i} = g^{-1}_{\alpha i\beta j}A_{\beta j}g_{\alpha i\beta j} + g^{-1}_{\alpha i\beta j}dg_{\alpha i\beta j},$$

or

$$dg_{\alpha i\beta j} + A_{\beta j}g_{\alpha i\beta j} - g_{\alpha i\beta j}A_{\alpha i} = 0.$$

Since the S^2 component of the curvature is zero we have that $(A_a)_{\alpha i} = u^{-1}_{\alpha i}d_a u_{\alpha i}$. Putting this in the above equation yields

$$d_a(u_{\alpha i}g_{\alpha i\beta j}u^{-1}_{\beta j}) = 0.$$

Therefore $\bar{g}_{\alpha i\beta j} \equiv u_{\alpha i}g_{\alpha i\beta j}u^{-1}_{\beta j}$ does not depend on the coordinates of S^2. This implies that $\bar{g}_{\alpha i\beta j}$'s are a set of locally constant transition functions equivalent to $g_{\alpha i\beta j}$ and for a fixed point on Σ define a map from S^1 to $SU(2)$. This map is trivial so $\bar{g}_{\alpha i \alpha j}$ belongs to the conjugacy class of identity

$$\bar{g}_{\alpha i \alpha j} = \bar{g}_{\alpha i} \bar{g}_{\alpha j}^{-1} = u_{\alpha i} g_{\alpha i \alpha j} u_{\alpha j}^{-1}$$

or $(\bar{g}_{\alpha i}^{-1} u_{\alpha i}) g_{\alpha i \alpha j} (\bar{g}_{\alpha j}^{-1} u_{\alpha j})^{-1} = 1$. Now consider $(\bar{g}_{\alpha i}^{-1} u_{\alpha i}) g_{\alpha i \beta j} (\bar{g}_{\beta j}^{-1} u_{\beta j})^{-1}$. This is a constant matrix in the S^2 direction. Since $g_{\alpha i \beta j} = g_{\alpha i \beta i} g_{\beta i \beta j}$ it is equal to $(\bar{g}_{\alpha i}^{-1} u_{\alpha i}) g_{\alpha i \beta i} (\bar{g}_{\beta i}^{-1} u_{\beta i})^{-1}$, and since $g_{\alpha i \beta j} = g_{\alpha i \alpha j} g_{\alpha j \beta j}$ it is equal to $(\bar{g}_{\alpha j}^{-1} u_{\alpha j}) g_{\alpha j \beta j} (\bar{g}_{\beta j}^{-1} u_{\beta j})^{-1}$. Thus it is in fact independent of the index i and therefore defines a matrix $\tilde{g}_{\alpha\beta}$ depending only on $x \in U_{\alpha\beta}$ and satisfying the cocycle condition[1].

Since the transition functions are independent of i, therefore $(A_\Sigma)_{\alpha i}$ do not depend on i index and A_a can be gauged away.

It is now easy to see that the flatness condition, $F_{ab} = 0$, necessarily requires the instanton number to be zero. The curvature locally takes the form

$$F = dA + A \wedge A$$

therefore *locally* we can write

$$\mathrm{tr}\,(F \wedge F) = d\ \mathrm{tr}\,(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A),$$

but since $A_a = 0$, instanton number reads

$$k = \frac{1}{8\pi^2} \int_{\Sigma \times S^2} \mathrm{tr}\, F \wedge F = \frac{1}{8\pi^2} \int_{\Sigma \times S^2} d_C\ \mathrm{tr}\,(A_\Sigma \wedge d_C A_\Sigma)$$

where the subindex C indicates differentiating with respect to the coordinates on S^2. Note that the integrand is still a local one. However, we showed that the transition functions are independent of the local coordinates on S^2. Therefore, for a fixed point on Σ, A_Σ is globally defined on S^2. This means that the integral over S^2 is a total divergence and gives zero for the instanton number. In summary, we have learned that if the bundle E admits a flat connection in S^2 direction then it has to be trivial (for those bundles that are classified only by instanton number) and k, the instanton number, is zero.

[1]The proof of this part was provided by Nicholas Buchdahl.

Appendix D

Faddeev-Popov gauge fixing

In this appendix we want to show how the eq. (4.31) is obtained starting from (4.30). To evaluate the path integral over gauge fields and ϕ, following [55], we choose the so called unitary gauge in which one rotates the lie algebra valued field ϕ^a to the Cartan subalgebra by conjugation, i.e. we choose $\phi_\pm = 0$, where

$$\phi = \phi_3 \tau_3 + \phi_+ \tau_+ + \phi_- \tau_- .$$

This gauge can always be achieved at least locally, but there might be some topological obstruction to impose it globally [55]. Implementing this gauge in the path integral requires to introduce the Faddeev-Popov ghosts c and antighosts $\bar{c}$ together with a bosonic auxiliary field b. These fields transform under a BRST operator δ like

$$\begin{aligned} &\delta\phi_\pm = \pm i c_\pm \phi_3, \quad \delta\phi_3 = 0, \quad \delta c_\pm = 0, \\ &\delta\bar{c}_\pm = b_\pm, \quad \delta b_\pm = 0. \end{aligned} \tag{D.1}$$

The Faddeev-Popov prescription consists of adding a BRST-trivial term

$$i\delta(\bar{c}_-\phi_+ + \bar{c}_+\phi_-) = ib_-\phi_+ + ib_+\phi_- + \bar{c}_-\phi_3\, c_+ - \bar{c}_+\phi_3\, c_-$$

to the action in (4.30). It is now clear that the integration over b will impose the gauge condition; $\phi_\pm = 0$. We have

$$\mathrm{tr}\,\phi F = \phi_3 F_3 = \phi_3(dA_3 + (A\wedge A)_3) = \phi_3(dA_3 + i\sqrt{2}A_1 \wedge A_2)\,,$$

therefore, defining $\phi \equiv \phi_3, A \equiv A_3$ and $F \equiv F_3$, the action in (4.30) turns into

$$S = \frac{1}{4\pi^2}\int_\Sigma \left(\frac{i}{\sqrt{2}}\phi\, dA - \phi A_1 \wedge A_2 + \frac{\varepsilon}{8}\phi^2\right) + \int_\Sigma d\mu\,(\bar{c}_-\phi\, c_+ - \bar{c}_+\phi\, c_-)\,.$$

Integration over Faddeev-Popov ghosts gives

$$[\det \phi^2]_{\Omega^0(\Sigma_g)},$$

while over A_1 and A_2 results in

$$\left[\det \phi^2\right]^{-1/2}_{\Omega^1(\Sigma_g)}.$$

Using the Hodge decomposition theorem we can express the product of these two determinants as

$$\frac{[\det \phi^2]_{H^0(\Sigma_g)}}{[\det \phi^2]^{1/2}_{H^1(\Sigma_g)}}.$$

Note that the reduced $U(1)$ bundle is not necessarily trivial (see eq. (4.35)), so we write the curvature as

$$F = 2\pi(n + \tfrac{1}{2})\omega + dA,$$

where ω is the volume form ($\int_\Sigma \omega = 1$) and

$$n + \tfrac{1}{2} = \frac{1}{2\pi}\int_\Sigma F$$

is the first Chern class which characterizes the $U(1)$ bundle. To gauge fix the residual $U(1)$ symmetry

$$A \to A + d\alpha,$$

we again appeal to the Faddeev-Popov prescription. We demand that a selected slice be normal to the gauge orbit,

$$\langle d\alpha, A\rangle = 0,$$

which implies that $d^\dagger A = 0$. Imposing this gauge, the action is

$$\frac{1}{4\pi^2}\int_\Sigma \left(i\sqrt{2}\pi(n + \frac{1}{2})\phi\omega + \frac{\varepsilon}{8}\phi^2 + \frac{i}{\sqrt{2}}(\phi dA + bd * A + \bar{c}d * dc)\right).$$

The kinetic term for A vanishes for A a harmonic one-form, i.e. when $dA = 0$ and $d^\dagger A = 0$. Hence there is still a residual symmetry under

$$A \to A + \gamma$$
$$b \to b + \text{constant}$$
$$c \to c + \text{constant},$$

where γ is a harmonic one-form. Integration over the zero modes of b and c and over the harmonic one-forms gives an unspecified constant factor that can be simply absorbed in the normalization. Therefore we need only be concerned about the nonzero modes. Dropping the harmonic part of A, it can be written globally and uniquely as

$$A = d\alpha + *d\beta,$$

for some zero-forms α and β. The action then looks like

$$\frac{1}{4\pi^2}\int_\Sigma \left(i\sqrt{2}\pi(n+\frac{1}{2})\phi\omega + \frac{\varepsilon}{8}\phi^2 + \frac{i}{\sqrt{2}}(\phi d * d\beta + bd * d\alpha + \bar{c}d * dc) \right),$$

and the measure is

$$\mathcal{D}A = \mathcal{D}\alpha\mathcal{D}\beta \, \det[dd^\dagger]_{\Omega_0}. \tag{D.2}$$

Note that $*^2 = (-1)^p$ when acting on a p-form and $d^\dagger = - * d*$. The integral over b and α results in a determinant, $\det[dd^\dagger]^{-1}_{\Omega_0}$, which cancels the jacobian in (D.2). Also the integral over β gives a delta function

$$\delta(dd^\dagger\phi) = \det[dd^\dagger]^{-1}_{\Omega_0}\, \delta(\phi). \tag{D.3}$$

Notice that since we are integrating over nonzero modes the delta function on the right hand side is a delta function on nonconstant ϕ's. The determinant in eq. (D.3) gets cancelled against the determinant coming from the ghosts. At the end we are left with a finite dimensional integral over constant ϕ fields

$$Z[\varepsilon, m] = \sum_{n\in Z}\int d\phi \left(\frac{\left[\det m^3(1-\frac{8}{m^2}\phi^2)\right]^{\frac{1}{2}}_{H^1}}{\left[\det m^3(1-\frac{8}{m^2}\phi^2)\right]_{H^0}} \right) \frac{[\det\phi^2]_{H^0}}{[\det\phi^2]^{1/2}_{H^1}} \exp\left(-i\sqrt{2}\frac{\phi(n+\frac{1}{2})}{4\pi} - \frac{\varepsilon\phi^2}{32\pi^2}\right).$$

Using the defenition of Euler characteristic of a Riemann surface $\chi(\Sigma_g) = 2b^0 - b^1 = 2-2g$, where b^i is the dimension of the H^i i-th de Rham cohomology group, g is the genus of the Riemann surface Σ, and since ϕ is now a constant, we can write the partition function as

$$Z[\varepsilon, m] = m^{3(g-1)}\sum_{n\in Z}\int d\phi\ \phi^{2-2g}(1-\frac{8}{m^2}\phi^2)^{g-1}\exp\left(-i\sqrt{2}\frac{\phi(n+\frac{1}{2})}{4\pi} - \frac{\varepsilon\phi^2}{32\pi^2}\right) \tag{D.4}$$

which is the equation (4.31).

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