



A THREE-DIMENSIONAL TIDAL MODEL FOR SHALLOW WATERS USING TRANSFORMATIONS AND VARIABLY SPACED GRIDS

by

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GLOSSARY OF SYMBOLS

Symbol	Meaning	Reference Page
a	horizontal grid ratio. (for difference equations)	55
a	amplitude of incident wave.	67
a_0	vertical grid ratio.	59
a_K	vertical grid ratio.	60
a_1	a constant, part of N_η .	19
a_2	a constant, part of N_η .	19
b	subscript indicating sea floor.	8
$b(x)$	curve in x-y plane defining the boundary.	12
b_0	vertical grid ratio.	59
b_K	vertical grid ratio.	60
c	admittance coefficient for radiation boundary condition.	27
c	horizontal grid ratio. (for difference equations)	55
c_0	vertical grid ratio.	59
c_K	vertical grid ratio.	60
c_q	coefficients in eigenvector expansion.	107
$\mathcal{C}$	vector of coefficients.	82
d	a parameter.	74
d_0	vertical grid ratio.	150
e	dummy argument.	32
f	Coriolis parameter.	6
f_a	constant representative of the average value of f .	66
g	gravitational acceleration.	6
h	depth of water below mean sea level.	9
h_a	constant representative of the average value of h .	66

i	square root of -1.	67
i	subscript indicating position in horizontal plain.	32
$i1$	subscript indicating $-\Delta\chi$ from i .	32
$i2$	subscript indicating $\Delta\chi$ from i .	32
j	subscript indicating the column in the horizontal plain.	33
k	subscript indicating depth.	41
m	subscript indicating the row in the horizontal plain.	32
m_χ	wave number component in χ direction.	68
m_λ	wave number component in λ direction.	68
n	superscript indicating time level.	29
n_i	$i=1..4$, parameters.	74
p	grid parameter for variable p -grid.	44
r_k	grid spacing ratio in vertical direction.	43
r_x	grid spacing ratio in horizontal direction.	55
r_m	grid spacing ratio in horizontal direction.	55
r_y	grid spacing ratio in horizontal direction.	211
r_j	grid spacing ratio in horizontal direction.	211
s	subscript indicating sea surface.	8
s_k	integration coefficient for variable vertical grid.	52
t	time.	14
u	Reynolds averaged velocity in x-direction.	6
$\bar{u}$	depth integrated u velocity.	9
v	Reynolds averaged velocity in y-direction.	6
$\bar{v}$	depth integrated v velocity.	9
ϑ	eigen vector of matrix equation.	101
ϑ^n	numerical solution of difference equation.	106

w	Reynolds averaged velocity in z-direction.	6
x	Cartesian coordinate in x-direction.	6
y	Cartesian coordinate in y-direction.	6
z	Cartesian coordinate in z-direction.	6

$A_{\chi,k}, A_{\lambda,k}, B_{\chi}, B_{\lambda}, C_{\chi}, C_{\lambda}, G_{\chi,k}, G_{\lambda,k}, H_{\chi,k}, H_{\lambda,k}, N'_{\chi}, N'_{\lambda}, \overline{N}_a, P, P_b, Q_k, S_{\eta}, f', \beta_{\chi}, \beta_{\lambda}, \mu', \nu', \rho, \Gamma_k^-, \Gamma_k$ and Γ_k^+ are all parameters used in the wave propagation analysis of Chapter 6 for defining the matrices **A** and **B**. Their definitions are given on page 102.

A	matrix.	101
A_{μ}	amplitude response for μ .	121
B	matrix.	101
$B(x)$	curve defining the breadth of the gulf.	12
C	matrix.	121
C_a	a coefficient.	68
C_b	bottom drag coefficient.	25
C_s	surface drag coefficient.	26
$D_{i,0}$	surface derivative boundary value.	59
$D_{i,K}$	sea bed derivative boundary value. (if “slip”)	60
E_a	combined velocity error measure.	120
E_{μ}	μ velocity error measure.	120
E_{ν}	ν velocity error measure.	120
E_{ζ}	surface elevation error measure.	163
$F(\eta)$	the part of N_{η} which varies with depth.	19
F_{χ}	depth variation of analytic solution for μ	67
$\overline{F}_{\chi}$	depth integrated value of F_{χ} .	135

F_λ	depth variation of analytic solution for ν	67
$\overline{F}_\lambda$	depth integrated value of F_λ .	135
$\underline{F}$	vector of F_x and F_λ .	68
$G(\eta)$	inverse function of depth transformation.	21
H	total depth of water ($h + \zeta$).	9
K	the number of vertical grid intervals.	41
K_b	bottom linear friction coefficient.	174
L	length of gulf in the x-direction.	12
M	matrix.	82
M	largest velocity coefficient.	120
N_b	sea bed value of N_η .	72
N_m	maximum value of N_η over η .	75
N_s	surface value of N_η .	71
N_x	constant eddy viscosity coefficient in x-direction.	8
N_y	constant eddy viscosity coefficient in y-direction.	8
N_z	variable eddy viscosity coefficient in z-direction.	8
N_x	equal to N_x .	18
N_λ	equal to N_y .	18
N_η	$N_\eta(\eta) = N(\eta) = N_z(z)$.	19
N_L	number of grid spacings per wave length.	105
P	pressure.	6
P_0	atmospheric pressure.	8
P_μ	phase shift error for μ .	121
$P_{i,k}$	temporary variable. (for difference equations)	58
Q	parameter.	137
$Q_{i,k}$	temporary variable. (for difference equations)	58

$R_{i,k}$	temporary variable. (for difference equations)	58
S	matrix.	68
$S_{i,k}$	temporary variable. (for difference equations)	58
$S_i(\eta)$	$i=1..4$, linearly independent solutions.	69
T	time.	6
U	velocity vector (u, v) .	25
U_n	normal component of depth integrated velocity.	27
W	wind speed.	26
W_{10}	wind speed 10 m above surface.	26
α	a parameter.	73
α_i	$i=1..4$, horizontal grid ratios.	53
β	a parameter.	74
β	a function of χ , λ , $b(\chi)$ and $B(\chi)$.	14
β_i	$i=1..4$, horizontal grid ratios.	53
γ	a function of χ , λ , η , t , ζ and H .	15
γ	parameter.	160
ϵ	tolerance.	86
η	transformed z coordinate.	14
κ	parameter for variable <i>kappa</i> -grid.	41
λ	transformed y coordinate.	14
λ_i	$i=1..2$, parameters.	72
ϕ	dummy argument.	49
ϕ	parameter.	160
$\hat{\phi}$	negative of real part of $\hat{\sigma}$.	114
ψ	imaginary part of σ .	114
$\hat{\psi}$	imaginary part of $\hat{\sigma}$.	114

ϕ_i	$i=1..4$, parameters.	83
θ_1	ratio of horizontal wave number components.	85
θ_2	ratio of $\Delta\lambda$ to $\Delta\chi$.	101
μ	χ -component of the transformed velocity.	17
$\bar{\mu}$	depth integrated μ velocity.	17
μ_a	constant representative of the average value of μ .	66
μ^*	constant coefficient.	67
$\mu_k^\otimes$	finite difference depth coefficient.	101
ν	λ -component of the transformed velocity.	17
$\bar{\nu}$	depth integrated ν velocity.	17
ν_a	constant representative of the average value of ν .	66
ν^*	constant coefficient.	67
$\nu_k^\otimes$	finite difference depth coefficient.	101
ξ	vertical transformation of η .	73
ξ	eigenvalue in wave propagation analysis.	101
ω	η -component of the transformed velocity.	17
χ	transformed x coordinate.	14
ρ	constant density of sea water.	6
σ	circular frequency of numerical surface oscillation.	101
$\hat{\sigma}$	circular frequency of analytic surface oscillation.	67
τ	Reynolds stress term.	6
ζ	elevation of the sea surface above mean sea level.	8
ζ_{max}	maximum value of ζ over a tidal cycle.	21
ζ^*	first approximation for ζ at new time level.	51
$\hat{\zeta}$	specified tidal height at the open boundary.	27
Δ	stepped increments along a coordinate direction.	28

∂	indicates partial derivative.	6
$\sim$	indicates use of second order averaging in χ direction.	49
$\rightarrow$	indicates use of second order averaging in λ direction.	49
$\bar{\cdot}$	indicates averaging in χ direction.	50
$+$	superscript indicates averaging in λ direction.	50
—	over a variable, indicates depth integration.	9

SUMMARY

The aim of this thesis has been to develop an accurate three-dimensional finite difference model for the prediction of tidal heights and velocities in a shallow sea which is forced by tidal motion from any adjoining sea, and also influenced by the surface drag due to any wind blowing across the surface.

The model was designed to be accurate without requiring excessive amounts of computer time and to be general in its applicability to any region where a three-dimensional model is needed to supply detailed information about tidal heights and current flows.

In developing this model a need was seen for a method to assess the stability and accuracy of a three-dimensional model. Various analytic solutions for simplified governing equations and/or simple geometries exist and these are a useful guide for checking the numerical scheme and also for “debugging” the code in the computer program which is involved in modelling the terms in the analytic solutions. However, one problem with these analytic solutions is that land-sea boundary conditions limit the complexity of the solutions, which are therefore not quite as useful as one would like them to be. Another problem with this approach is that many numerical test runs need to be performed with the model and comparisons made to the appropriate analytic solution, to enable model characteristics and behaviour properties to be established. And this applies only for the simplified equations and/or geometry.

If the land-sea boundaries are removed from the problem by considering an infinite horizontal domain, a more complete analytic solution exists for the linearised three-dimensional governing equations. This analytic solution has been used to develop a new wave propagation / stability analysis for any particular three-dimensional finite difference tidal model. The analysis compares the exact solution of the finite difference

equation to the analytic solution when both are given the initial conditions of the analytic solution, which is an infinite travelling wave. Although wave propagation analysis has been used previously to estimate the accuracy of finite difference solutions of the two-dimensional advection-diffusion equations and for the linearised depth averaged tidal equations, neither of these is of any use for analysing a three-dimensional scheme. This new analysis, which is a three-dimensional extension of the depth averaged wave propagation analysis, finds all the numerical modes in the vertical direction which can propagate through the numerical model. The analysis then indicates whether the numerical method is stable for a given Courant number. It also shows if the finite difference equations are consistent with the partial differential equations and can measure the size, as well as locate the position, of the errors introduced by the numerical scheme in comparison to the analytic solution. This procedure finds the relative speed, amplitude response, and errors in the velocity profiles for the numerical scheme.

The analytic solutions, which are used by the wave propagation analysis, exist for various functional forms of the vertical eddy viscosity parameter. One major advantage of this analysis is that realistic forms for the vertical eddy viscosity parameter and realistic values for other parameters, can be used to determine optimal values of these parameters as well as optimise choice of the finite difference approximations.

This analysis has been used in the design of finite difference approximations for the governing equations of the developed numerical tidal model, although it must be remembered that the optimal design for the finite difference equations could be dependent on the magnitude and form of the parameters used in the model.

The final part of this thesis consists of applying the numerical model to the upper parts of Spencer Gulf, South Australia, which is of particular interest due to the heated water discharge from a power station in the area. An extension of this work would be to model the flow of heat by incorporating the transport equation and changing the

currently used equations to incorporate variable density.

SIGNED STATEMENT

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, it contains no material previously published or written by any other person, except where due reference is made in the text of this thesis.

If this thesis is accepted for the award of the degree, then I give my consent that it be made available for photocopying and loan if applicable.

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Malcolm William Stevens

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Chapter 1

INTRODUCTION

The work for this thesis consists of several parts. Firstly a three-dimensional finite difference tidal propagation model is constructed which uses transformations to simplify the application of the finite difference approximations and uses variable grid spacings in all directions to provide increased accuracy where it is required. This is significant in the vertical direction, especially near the sea floor. The second and major part of this thesis develops a technique for analysing, testing and assessing numerical finite difference models. This analysis as well as other standard testing techniques are used to assess the accuracy and stability of the developed numerical model. This testing and assessing has been done in conjunction with the development and construction of the model, where results from the tests and the above mentioned analytic technique feed back into the design of the model to make it more accurate and robust. An application of the model forms the third and final part of this thesis.

This thesis concentrates on the mathematical problem of how to accurately solve the three-dimensional tidal motion equations. The accuracy of the numerical model in solving the given prescribed physical equations determines the quality of the model. If the numerical model accurately solves the equations but does not make accurate predictions for the physical application of the model, then the physical equations need

to be adjusted and the numerical model changed to accurately solve the new equations.

Chapter 2 details the governing hydrodynamic equations which represent the motion of water in the area of application. The solution to these equations determines the velocities and tidal heights of the water under the influence of the tidal effects due to external forcing from adjacent waters and from the effect of surface drag caused by a wind blowing across the surface of the body of water. A transformation which maps an irregular body of water onto a unit cube is employed to facilitate application of finite difference techniques. Details of the transformation and the transformed governing equations are given as well as the boundary conditions which apply at the sea and land boundaries of the application area. Consideration is given to regions of a shape which invalidate the assumptions implicit in the transformations. A method is described to correct this by appropriately restructuring parts of the model boundaries so that the transformation assumptions are not violated.

Chapter 3 describes the structure of the numerical model and gives detail of the organisational and indexing methods employed. The grid structure, which has variable grid spacings in both horizontal directions as well as in the vertical direction, is defined for the model. Although this gives great flexibility, extreme care must be taken in choosing a grid so that the finite difference equations remain accurate. Details of how to retain the accuracy are provided and the validity of this is shown in Chapter 7. The basic outline of the computational procedure is discussed and details of the order of evaluation of the calculations for the finite difference equations are provided.

The model developed in this thesis is deliberately quite general, with the inclusion of as many terms as possible in the physical equations to achieve this aim. The transformation technique also enables the model to be applied to almost any area of interest, and variable vertical and horizontal grids are provided so that resolution and accuracy can be enhanced in areas which are of particular concern or interest.

Explicit details of the finite difference equations are provided in Chapter 4 for the transformed partial difference equations on the grid system defined in Chapter 3. The finite difference approximations are for a variable grid and are second order accurate in each of the spatial directions. The time stepping method has a predictor–corrector approach which increases the accuracy of the model to almost second order in time and also greatly improves the stability. Most of Chapters 2 to 4 and parts of Chapter 7 are included in Noye and Stevens (1987).

The analytic solutions given in Chapter 5 provide an important basis for the work described in Chapter 6, and additionally provide a direct comparison for verification of the tidal model for simplified situations, which forms part of Chapter 7. Analytic solutions exist for various simplifications of either the governing equations and/or the boundary shape. For the analytic solutions which incorporate depth dependence, a solution exists only for certain functional forms of the vertical eddy viscosity coefficient N_η . Some of these forms have been known and used since early this century (for example Sverdrup (1926) and Fjeldstat (1929)), but it has only been since the advent of computers that some of the more complicated analytic solutions have become of practical use. A summary of the functional forms for the vertical eddy viscosity coefficients, for which analytic solutions exist and which previous authors have described and used, is given in Chapter 5. A new form of N_η , for which an analytic solution exists and has not previously been documented, has been found and details of this solution are provided. All the solutions of previous authors have been extended by the inclusion of horizontal diffusion and linearised horizontal advection. Any of the given functional forms for the vertical eddy viscosity coefficient can be combined into a piecewise continuous function, which offers extreme flexibility for the profile of the function. Chapter 5 provides details of both a two piece and a three piece continuous form for N_η .

As mentioned previously, one of the main aims of this thesis is to develop an analytic method for analysing any three-dimensional finite difference tidal model. Previous wave propagation analysis methods have been for the two-dimensional advection-diffusion equations and the linearised depth averaged tidal equations. The new analysis, presented in Chapter 6, extends the previous analyses to the set of linearised equations which govern three-dimensional tidal motion. Parts of Chapter 6 are included in Stevens and Noye (1990).

This analysis can be applied to any finite difference scheme for the linearised equations, either implicit or explicit and for either two or three levels in time. Whereas a fully multi-dimensional implicit numerical finite difference scheme can be analysed by this technique, with current technology it would not be possible to implement this type of scheme for a three-dimensional model, as the scheme would require the inversion of matrices of the order of the number of depth levels multiplied by the number of horizontal elements in the model. The numerical model developed in Chapters 2 to 4 is used for demonstrating the wave propagation analysis. It is shown that suitable choice of the size of spacings in the finite difference grid minimises the error in the numerical solution of the governing equations in the absence of horizontal boundaries.

In Chapter 7 the numerical scheme is analysed using various techniques. The wave propagation analysis of Chapter 6 is used to investigate and improve the numerical scheme. Consistency, stability and convergence are discussed and the time stepping nature of the scheme is analytically shown to be practically second order accurate. Various analytic solutions for tidal flows in channels and bays are used to verify accuracy and check the computer code. A steady state analytic solution for wind driven flows in a long closed channel is used to verify the model's ability to predict the effects of wind stress acting on the water surface. The results highlight the advantages of using an appropriate variable spaced vertical grid to improve model accuracy.

Chapter 8 is an application of the numerical model to the upper reaches of Spencer Gulf in South Australia. This area is of interest due to a power station (in use and with possible extensions) about 5 km south of Port Augusta. The application area is small in size, being only 30 km in length and having a horizontal grid spacing at the finest part of the grid of only 285 metres. This small size is almost at the limit of feasibility for this or any other three-dimensional model, however the developed model is successfully applied to the area by utilising the model's variable grid and boundary edge transformations.

Chapter 9 is the conclusion of this thesis, which is followed by four appendices and the bibliography.

Chapter 2

THE THREE-DIMENSIONAL TIDAL EQUATIONS

2.1 The Basic Equations

The Reynolds averaged forms of the equations governing tidal motion in well-mixed seas are well known (see, for example Nihoul (1975)), and consist of the equation of mass conservation for an incompressible fluid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \quad (2.1)$$

and the three component equations of momentum conservation,

$$\frac{Du}{DT} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{yx}}{\partial y} + \frac{1}{\rho} \frac{\partial \tau_{zx}}{\partial z} + f_v , \quad (2.2)$$

$$\frac{Dv}{DT} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{yy}}{\partial y} + \frac{1}{\rho} \frac{\partial \tau_{zy}}{\partial z} - f_u , \quad (2.3)$$

$$\frac{Dw}{DT} = -g - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{1}{\rho} \frac{\partial \tau_{xz}}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{yz}}{\partial y} + \frac{1}{\rho} \frac{\partial \tau_{zz}}{\partial z} . \quad (2.4)$$

The symbols used in these equations are defined as follows:

x, y, z are rectangular Cartesian coordinates, with the x and y axes lying in the plane of the undisturbed sea surface and the z axis directed vertically upwards,

u, v, w are the corresponding Reynolds averaged velocity components,

T is the time,

P is the pressure,

ρ is the density (assumed constant),

g is the acceleration due to gravity,

f is the Coriolis parameter, which has the form $2\Omega \sin \theta$, where Ω is the angular velocity of the earth, and θ is the latitude, positive when north of the equator,

$\frac{D}{DT}$ is the operator of differentiation following the motion, namely,

$$\frac{D}{DT} \equiv \frac{\partial}{\partial T} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \text{ and}$$

τ is the Reynolds stress, the subscripts indicating the direction of the stress and the orientation of the plane on which it acts: for example, τ_{xy} represents the Reynolds shear stress acting on the plane $x = \text{constant}$ in the y -direction.

The Reynolds stresses in Equations (2.2) and (2.3) can be replaced by their Boussinesq or eddy viscosity forms, namely,

$$\tau_{zx} = \rho N_{zx} \frac{\partial u}{\partial z}, \quad \tau_{zy} = \rho N_{zy} \frac{\partial v}{\partial z}, \quad (2.5)$$

etc., where N_{zx} , N_{zy} , etc., are the appropriate eddy viscosity coefficients, which may be space and time dependent.

In tidal motions the vertical acceleration $\frac{Dw}{DT}$ and the gradients of the Reynolds stresses are small compared with gravitational acceleration g , therefore Equation (2.4) can be replaced by the hydrostatic approximation

$$\frac{\partial P}{\partial z} + \rho g = 0, \quad (2.6)$$

which governs the vertical motions. On integration over depth from the level z to the sea surface ζ , this equation yields

$$P(x, y, z, T) = P_0 + \rho g(\zeta - z) , \quad (2.7)$$

where $\zeta(x, y, T)$ is the elevation of the sea surface above mean sea level and P_0 is the atmospheric pressure, which is considered to be constant over the area of interest. Using Equation (2.7) to eliminate P from Equations (2.2) and (2.3) and substituting the Boussinesq forms of the Reynolds stress terms the following equations are obtained:

$$\begin{aligned} \frac{\partial u}{\partial T} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = & -g \frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial x} \left(N_x \frac{\partial u}{\partial x} \right) \\ & + \frac{\partial}{\partial y} \left(N_y \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(N_z \frac{\partial u}{\partial z} \right) + fv , \end{aligned} \quad (2.8)$$

$$\begin{aligned} \frac{\partial v}{\partial T} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = & -g \frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial x} \left(N_x \frac{\partial v}{\partial x} \right) \\ & + \frac{\partial}{\partial y} \left(N_y \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(N_z \frac{\partial v}{\partial z} \right) - fu , \end{aligned} \quad (2.9)$$

in which $N_{zx} = N_{zy} = N_z$, $N_{xy} = N_{xz} = N_x$ and $N_{yx} = N_{yz} = N_y$.

If the eddy viscosity coefficients N_x , N_y and N_z are known, this gives three Equations (2.1), (2.8) and (2.9) containing four unknowns u , v , w and ζ . The fourth equation required to solve this system is derived from the free surface boundary condition and the sea floor boundary condition, which are

$$w_s = \frac{D\zeta}{DT} = \frac{\partial \zeta}{\partial T} + u_s \frac{\partial \zeta}{\partial x} + v_s \frac{\partial \zeta}{\partial y} , \quad (2.10)$$

and

$$w_b = \frac{D(-h)}{DT} = -u_b \frac{\partial h}{\partial x} - v_b \frac{\partial h}{\partial y} , \quad (2.11)$$

where the subscripts s and b indicate evaluation of the dependent variable at the surface and at the sea floor respectively. A “slip” condition at the sea floor boundary means that there is only flow parallel to the sea floor. That is the component of velocity normal to the sea floor is zero. This means that w_b , u_b and v_b may be all non zero. A “no-slip” condition means that all velocity components are zero and therefore Equation (2.11) is trivially reduced.

When Equation (2.1) is integrated, with respect to z , from the sea floor $z = -h(x, y)$ to the sea surface $z = \zeta(x, y, T)$, giving

$$\int_{-h}^{\zeta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = - \int_{-h}^{\zeta} \frac{\partial w}{\partial z} dz = w_b - w_s . \quad (2.12)$$

The term on the left of this equation can be rewritten using Leibniz’s integration formula, which gives

$$\int_{-h}^{\zeta} \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \left\{ \int_{-h}^{\zeta} u dz \right\} - u_s \frac{\partial \zeta}{\partial x} + u_b \frac{\partial(-h)}{\partial x} , \quad (2.13)$$

$$\int_{-h}^{\zeta} \frac{\partial v}{\partial y} dz = \frac{\partial}{\partial y} \left\{ \int_{-h}^{\zeta} v dz \right\} - v_s \frac{\partial \zeta}{\partial y} + v_b \frac{\partial(-h)}{\partial y} . \quad (2.14)$$

Substitution of these two equations and Equations (2.10) and (2.11) into Equation (2.12) gives

$$\frac{\partial \zeta}{\partial T} + \frac{\partial H \bar{u}}{\partial x} + \frac{\partial H \bar{v}}{\partial y} = 0 , \quad (2.15)$$

where $\bar{u}$ and $\bar{v}$ are the depth averaged values of the horizontal velocities u and v , namely

$$\bar{u} = \frac{1}{H} \int_{-h}^{\zeta} u dz , \quad \bar{v} = \frac{1}{H} \int_{-h}^{\zeta} v dz , \quad (2.16)$$

and H is the total depth of water, so

$$H(x, y, T) = h(x, y) + \zeta(x, y, T) . \quad (2.17)$$

Thus the four equations to solve for the four unknowns ζ , u , v and w are Equations (2.1), (2.8), (2.9) and (2.15).

The system is closed by specifying the eddy viscosity coefficients N_x , N_y and N_z . The results obtained from the solution of this system are not sensitive to the forms of the horizontal coefficients N_x and N_y , which may be given constant values. However, a variable form is required for the vertical eddy viscosity coefficient N_z . Based on a series of numerical experiments using a turbulent kinetic energy (K- ϵ) closure scheme in typical tidal situations, Noye and Arnold (1990) have shown that a suitable form for N_z is one which depends upon the volume of water transported per unit width and varies quadratically with depth. This form (see Equation (2.39)), which is similar to that used by Cooper and Pearce (1977) and Nihoul (1978) is based on the physical structure of tidal currents.

A K- ϵ closure scheme is a possible and natural extension to this model but such extensions have been left for further research work. There are a number of difficulties with the use of a K- ϵ closure scheme which have influenced this decision. To begin with, the computer time and storage required for a three-dimensional model increases markedly in order to solve the extra equations involved with the transport of the quantities K and ϵ . Also, K (turbulent kinetic energy) and ϵ (dissipation rate of energy) are both necessarily positive, and numerical methods of solving the transport equations involving K and ϵ usually produce some negative values at times when the values of K and ϵ are very small, and this can make the procedure unstable. Alternatives can result in the loss of accuracy in the terms being transported or further additional computations. Accurate computation of the positive definite quantities can be achieved with a scheme like the Flux Corrected Transport (FCT) method by Boris and Book (1973) but this is computationally very expensive.

Vager and Kagan (1969a, 1969b, 1971) produced some of the earlier work in this area using the transportation of K in association with ϵ being calculated by an explicit equation in terms of K and the mixing length. Instead of using an explicit equation

for ϵ , Jones and Launder (1972) and Launder and Spaulding (1974) used an additional transportation equation in solving the equations for channel flow. Mellor and Yamada (1974) examine the turbulent equations in detail and present several models which compare favourably with one another. Since then numerous authors have successfully used K- ϵ models for both tidal and wind induced flows.

2.2 The Transformation

Usually the body of water to be modelled is a shape which is not congenial to the use of finite difference methods and therefore requires the use of complicated and sometimes inaccurate boundary approximations in order to fit the difference equations to the irregular shape of the actual boundary. This often requires smaller constant grid spacings to accurately model the shape of the boundaries, which is not desirable because this requires a significant increase in computer time for the model. To avoid this problem a transformation has been chosen so that the transformed region is one on which finite difference approximations for the governing Equations (2.1), (2.8), (2.9) and (2.15) and boundary conditions can be easily applied. In transforming the irregularly shaped body of water onto a shape with boundaries parallel to the new axes, more freedom is available to choose a grid spacing which is appropriate to the model. The size of the grid spacings are determined once the trade-off between the desired accuracy and model efficiency has been established. Finite difference techniques are easily applied to rectangular regions without special boundary approximations being required. Therefore, after scaling, the transformed shape was chosen to be the unit cube. In order to accommodate for bodies of water which invalidate the assumptions of the transformation, and also to incorporate islands and peninsulas, the body of water does not necessarily have to be transformed onto the entire cube. This is illustrated with an example in Section 3.1.

There are still inaccuracies due to small irregularities in the physical boundaries and approximations due to a few dropped terms in the horizontal diffusive terms (see Section 2.3), but if the parameters and orientation of the transformation are chosen wisely, then these introduced errors are minimal. Before transforming the irregularly shaped body of water on to the unit cube, certain aspects of the geometry of the general three-dimensional region must be considered.

The plan view of a typical body of water is shown in Figure 2.1. The number and location of either the open or solid boundaries is immaterial. The x and y axes have been taken to be in the plane of mean sea level and aligned so that two of the boundaries are defined by $x = 0$, $x = L$, where L is the length of the body of water. The other two boundaries are defined by $y = b(x)$ and $y = b(x) + B(x)$, where $b(x)$ is the distance from the closer boundary to an arbitrary x axis and $B(x)$ is the breadth of the area being modelled across a section parallel to the y axis. The boundaries $b(x)$ and $B(x)$ are permitted to vary in a smooth manner such that their slopes with respect to x remain small. In conjunction with the arbitrary orientation of the x axis (the y axis is always at a right angle to the x axis), the variability of $b(x)$ and $B(x)$ permits greater flexibility to minimise the number of “saw-tooth” boundaries. Pederson (1986) shows how “saw-tooth” boundaries, which are characteristic for finite difference models, introduce errors of first order magnitude and suggests careful choice and orientation of the coordinate system to avoid this type of boundary.

A small straight boundary along the line $x = 0$ is used to avoid a singularity being produced by the transformation as the transformation divides by the distance across the body of water. In the example shown in Figure 2.1 the boundary defined by $x = L$ is an open boundary, so there is possible movement of water across it, while the other three boundaries are solid or closed boundaries.

A vertical cross section of the typical body of water is shown in Figure 2.2 with the

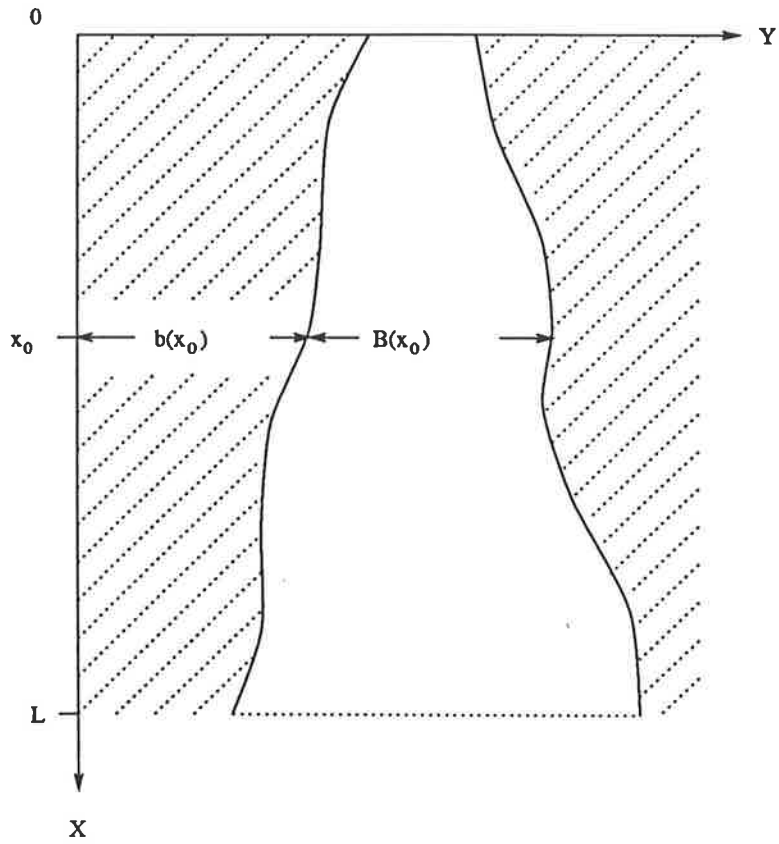


Figure 2.1: Plan view of a typical body of water ($z = 0$ plane).

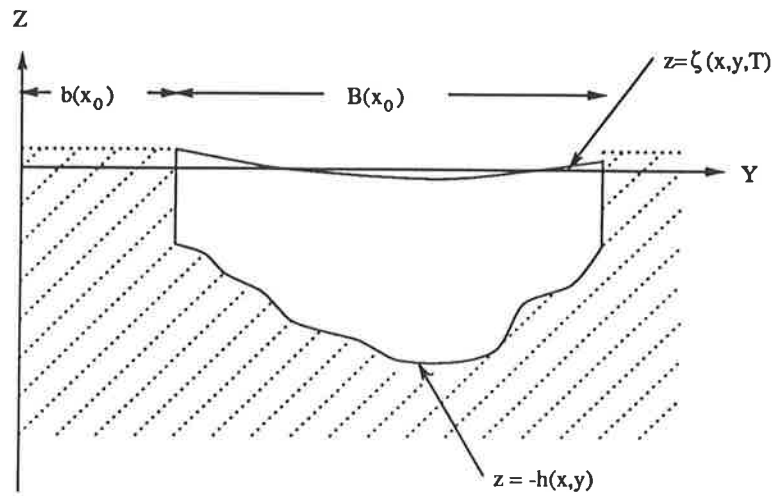


Figure 2.2: The cross sectional view (with the vertical scale exaggerated) at position x_0 along the x axis.

positive z direction upwards. The only notable feature is that all the solid boundaries in this example require short vertical sides to avoid singularities in the depth transformation (see Equation (2.20)). This restriction does not introduce any appreciable errors since the model boundaries can be taken sufficiently close to the actual shoreline so that the mass transport through them is negligible. This is similar to that used in most depth-integrated models presently in use.

The transformation is defined as follows:

$$\begin{array}{ccc} (x, y, z, T) & \longrightarrow & (\chi, \lambda, \eta, t) \\ \text{original system} & & \text{transformed system} \end{array}$$

in which the relationship between the two coordinate systems is defined by

$$\chi(x) = \frac{x}{L}, \quad (2.18)$$

$$\lambda(x, y) = \frac{y - b(x)}{B(x)}, \quad (2.19)$$

$$\eta(x, y, z, T) = \frac{\zeta(x, y, T) - z}{H(x, y, T)}, \quad (2.20)$$

$$t(T) = T, \quad (2.21)$$

where as previously described, L is the length of the body of water being modelled in the x direction, b and B are defined as in Figure 2.1, and depend only on x , h is the depth of the sea floor below mean sea level, and H is the total depth of water, i.e. $H = h + \zeta$.

The transformation defined by Equation (2.20), which maps the total depth of water H on to the interval $0 \leq \eta \leq 1$, is a function of time as well as position. It was first used by Freeman (1970) and has subsequently been used by other workers such as Davies (1980), Owen (1980) and Noye et al. (1982). Care must be taken with this mapping, for example Teubner (1976) has shown that earlier workers overlooked some of the terms in the transformed equations which arise from this mapping. One

advantage of this transformation is that the sea surface and the sea floor correspond to fixed points on the η axis.

The transformations defined by Equations (2.18) and (2.19) were originally used by Boericke and Hall (1974) in an application of depth averaged tidal equations to an estuary. Reid et al. (1977) and Jelesnianski (1976) have also used this type of transformation for the depth averaged tidal equations. More recently, they have been applied by Johns et al. (1981) to the depth averaged equations to calculate heights of storm surges in the Bay of Bengal. The plan view and the cross section of the gulf shown in Figures 2.3 and 2.4 show lines of constant coordinates, namely $\chi = \text{constant}$, $\lambda = \text{constant}$ and $\eta = \text{constant}$.

Using the chain rule and Equations (2.18) to (2.21) the differential operators can be expressed in terms of the new coordinates.

$$\frac{\partial}{\partial x} \equiv \frac{1}{L} \frac{\partial}{\partial \chi} - \frac{1}{LB} \frac{\partial \beta}{\partial \chi} \frac{\partial}{\partial \lambda} + \frac{1}{HL} \left(\frac{\partial \gamma}{\partial \chi} - \frac{1}{B} \frac{\partial \beta}{\partial \chi} \frac{\partial \gamma}{\partial \lambda} \right) \frac{\partial}{\partial \eta}, \quad (2.22)$$

$$\frac{\partial}{\partial y} \equiv \frac{1}{B} \frac{\partial}{\partial \lambda} + \frac{1}{HB} \frac{\partial \gamma}{\partial \lambda} \frac{\partial}{\partial \eta}, \quad (2.23)$$

$$\frac{\partial}{\partial z} \equiv -\frac{1}{H} \frac{\partial}{\partial \eta}, \quad (2.24)$$

$$\frac{\partial}{\partial T} \equiv \frac{(1-\eta)}{H} \frac{\partial \zeta}{\partial t} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial t}, \quad (2.25)$$

where

$$\beta(\chi, \lambda) = b(\chi) + \lambda B(\chi), \quad (2.26)$$

and

$$\gamma(\chi, \lambda, \eta, t) = \zeta(\chi, \lambda, t) - \eta H(\chi, \lambda, t). \quad (2.27)$$

Associated with the new coordinates χ , λ and η , the transformed velocity components $\mu = \frac{D\chi}{DT}$, $\nu = \frac{D\lambda}{DT}$ and $\omega = \frac{D\eta}{DT}$ are defined, where as before $\frac{D}{DT}$ is the derivative following the motion. Expanding the operator $\frac{D}{DT}$ and using Equations (2.18) to (2.25)

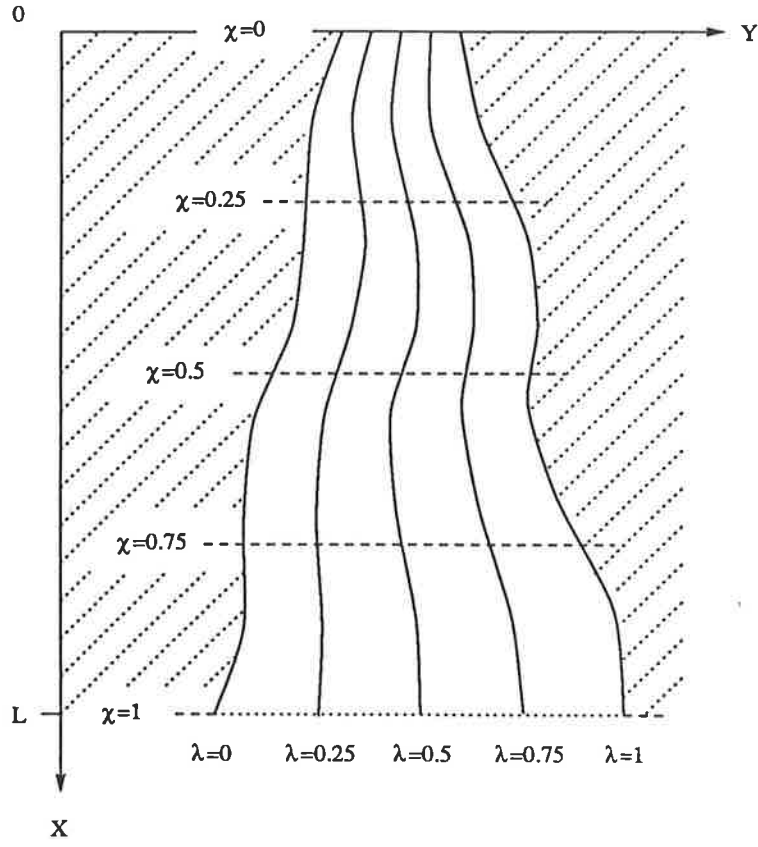


Figure 2.3: The χ and λ coordinates in the $z = 0$ plane for the body of water in the previous figures.

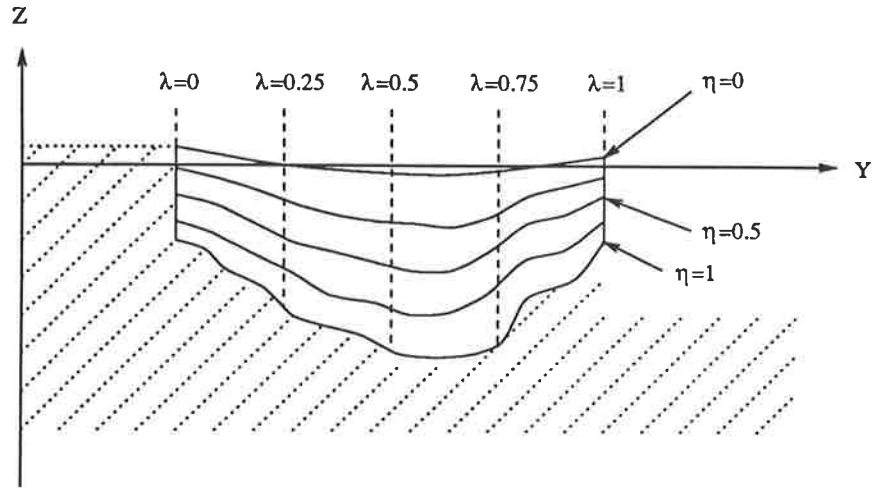


Figure 2.4: The λ and η coordinates in the $\chi = \frac{1}{2}$ plane.

the original velocity components can be expressed in terms of the new components, namely,

$$u = L\mu , \quad (2.28)$$

$$v = B\nu + \mu \frac{\partial \beta}{\partial \chi} , \quad (2.29)$$

$$w = -H\omega + (1 - \eta) \frac{\partial \zeta}{\partial t} + \mu \frac{\partial \gamma}{\partial \chi} + \nu \frac{\partial \gamma}{\partial \lambda} . \quad (2.30)$$

These forms for the velocity components u , v and w are used in the next section to transform the governing tidal equations so that they are expressed in terms of the new coordinate variables χ , λ and η .

Also the depth averaged velocity values $\bar{u}$ and $\bar{v}$ are related by

$$\bar{u} = \frac{1}{H} \int_{-h}^{\zeta} u dz = \frac{1}{H} \int_1^0 L\mu(-(\zeta + h)) d\eta = L \int_0^1 \mu d\eta = L\bar{\mu} , \quad (2.31)$$

and

$$\bar{v} = \frac{1}{H} \int_{-h}^{\zeta} v dz = B\bar{\nu} + \bar{\mu} \frac{\partial \beta}{\partial \chi} , \quad (2.32)$$

where

$$\bar{\mu} = \int_0^1 \mu d\eta \quad \text{and} \quad \bar{\nu} = \int_0^1 \nu d\eta .$$

2.3 The Transformed Equations

Using the differential operators of Equations (2.22) to (2.25) and the velocities given in Equations (2.28) and (2.29), Equation (2.15) is transformed to

$$\frac{\partial \zeta}{\partial t} + \frac{1}{B} \frac{\partial (BH\bar{\mu})}{\partial \chi} + \frac{\partial (H\bar{\nu})}{\partial \lambda} = 0 . \quad (2.33)$$

It can also be shown that

$$\frac{\partial}{\partial T} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \chi} + \nu \frac{\partial}{\partial \lambda} + \omega \frac{\partial}{\partial \eta} , \quad (2.34)$$

and that Equation (2.1) is transformed to

$$\frac{\partial \zeta}{\partial t} + \frac{1}{B} \frac{\partial(BH\mu)}{\partial \chi} + \frac{\partial(H\nu)}{\partial \lambda} + H \frac{\partial \omega}{\partial \eta} = 0. \quad (2.35)$$

In a similar way to Freeman (1970), Spaulding (1973) and Teubner (1976) the horizontal eddy viscosity (or diffusive) terms from Equation (2.8) and (2.9) can be approximately transformed by using

$$\begin{aligned} \frac{\partial}{\partial x} \left(N_x \frac{\partial}{\partial x} \right) &= \left\{ \frac{1}{L} \frac{\partial}{\partial \chi} - \frac{1}{LB} \frac{\partial \beta}{\partial \chi} \frac{\partial}{\partial \lambda} + \frac{1}{HL} \left(\frac{\partial \gamma}{\partial \chi} - \frac{1}{B} \frac{\partial \beta}{\partial \chi} \frac{\partial \gamma}{\partial \lambda} \right) \frac{\partial}{\partial \eta} \right\} \\ &\quad N_x \left\{ \frac{1}{L} \frac{\partial}{\partial \chi} - \frac{1}{LB} \frac{\partial \beta}{\partial \chi} \frac{\partial}{\partial \lambda} + \frac{1}{HL} \left(\frac{\partial \gamma}{\partial \chi} - \frac{1}{B} \frac{\partial \beta}{\partial \chi} \frac{\partial \gamma}{\partial \lambda} \right) \frac{\partial}{\partial \eta} \right\} \\ &\approx \frac{N_x}{L^2} \frac{\partial^2}{\partial \chi^2}, \end{aligned} \quad (2.36)$$

and

$$\begin{aligned} \frac{\partial}{\partial y} \left(N_y \frac{\partial}{\partial y} \right) &= \left\{ \frac{1}{B} \frac{\partial}{\partial \lambda} + \frac{1}{HB} \frac{\partial \gamma}{\partial \lambda} \frac{\partial}{\partial \eta} \right\} N_\lambda \left\{ \frac{1}{B} \frac{\partial}{\partial \lambda} + \frac{1}{HB} \frac{\partial \gamma}{\partial \lambda} \frac{\partial}{\partial \eta} \right\} \\ &\approx \frac{N_\lambda}{B^2} \frac{\partial^2}{\partial \lambda^2}, \end{aligned} \quad (2.37)$$

in which $N_x = N_\chi$ and $N_y = N_\lambda$ are assumed to be constant.

Teubner (1976) compared the two cases of the inclusion and exclusion of the dropped terms and found that there was virtually no difference between the results obtained for either the elevation or the velocity. However this result is only valid under the assumptions that the terms themselves are small and that the coefficients $\left(\frac{\partial \beta}{\partial \chi}\right)$, $\left(\frac{\partial \gamma}{\partial \chi}\right)$ and $\left(\frac{\partial \gamma}{\partial \lambda}\right)$ are also small. The two of these terms which involve γ (see Equation (2.27)) will be small if the depth of water in the model does not change drastically, that is a continental shelf would require special consideration for it to be modelled accurately. In many areas of application the model boundaries represented by $\lambda = B(\chi)$ and $\lambda = b(\chi)$ can follow the land-sea boundary (as in Figure 2.1) without violating the above assumptions. This effectively means that $B(\chi)$ and $b(\chi)$ have

slopes with respect to χ that are bounded by a small number. Some application areas would invalidate these assumptions, that is either $\left(\frac{\partial b}{\partial \chi}\right)$ or $\left(\frac{\partial B}{\partial \chi}\right)$ would be too large for some values of χ . For these cases the model boundaries $B(\chi)$ and $b(\chi)$ can be straightened to satisfy the assumptions that their slopes are small and the land inside the model area is then treated as a peninsula. With careful choice of the coordinate directions and the boundary functions $B(\chi)$ and $b(\chi)$ any plan view of a shallow sea may be represented and modelled accurately without violating the above assumptions. As an example of how this is performed consider the gulf as shown in Figure 2.5. If $B(\chi)$ and $b(\chi)$ followed the land-sea boundary $\left(\frac{\partial \beta}{\partial \chi}\right)$ would be large at the narrowing of the gulf and invalidate the above assumptions. In this case the lines $b(\chi)$ and $B(\chi)$ as shown in Figure 2.5 should be used, even though they do not follow the real land-sea boundaries. When this area is mapped onto the plan view of the unit cube the region corresponding to the water area consists of the plan view of the unit cube, with a rectangular “chunk” missing (see Figure 2.6), which is simply treated as a peninsula or land mass by the numerical model. The finite difference equations are still easily applied to the transformed shape with the “chunks” missing.

The vertical eddy viscosity term, which is far more significant than the horizontal terms, is transformed exactly by means of the relation

$$\frac{\partial}{\partial z} \left(N_z \frac{\partial}{\partial z} \right) = \frac{1}{H^2} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial}{\partial \eta} \right) , \quad (2.38)$$

where N_η can be chosen to be any functional form and is permitted to vary with depth and with time. An example of a form which varies quadratically with depth and depends on the volume of water transported per unit width, could be defined by

$$N_\eta = N(\eta) = a_1 + a_2 H F(\eta) \sqrt{(L\bar{\mu})^2 + \bar{v}^2} , \quad (2.39)$$

with

$$F(\eta) = \left(\frac{1}{2} + \eta\right)(1 - \eta) , \quad (2.40)$$

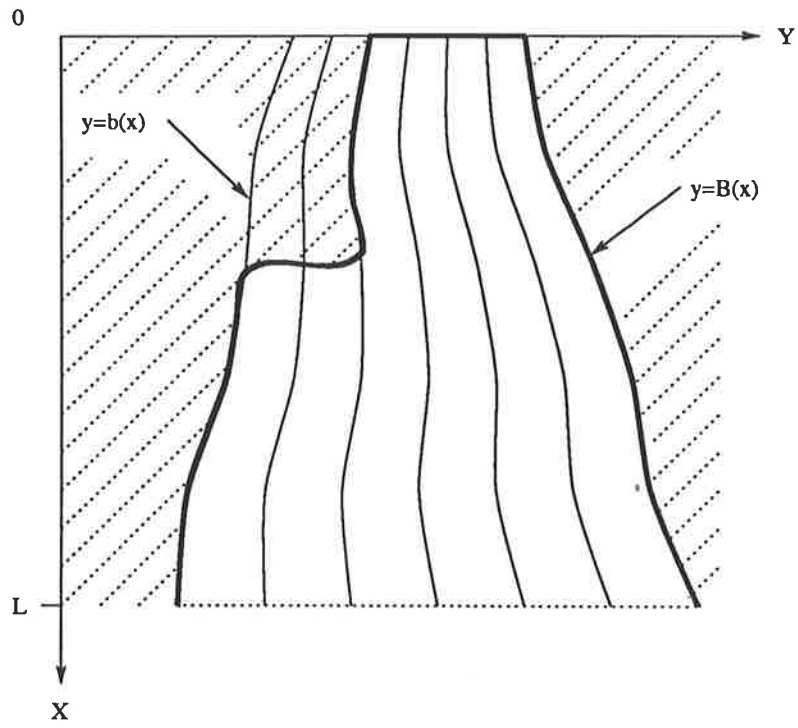


Figure 2.5: The plan view of a body of water which has $b(x)$ and $B(x)$ redefined so that $(\frac{\partial \beta}{\partial x})^2$ is negligible.

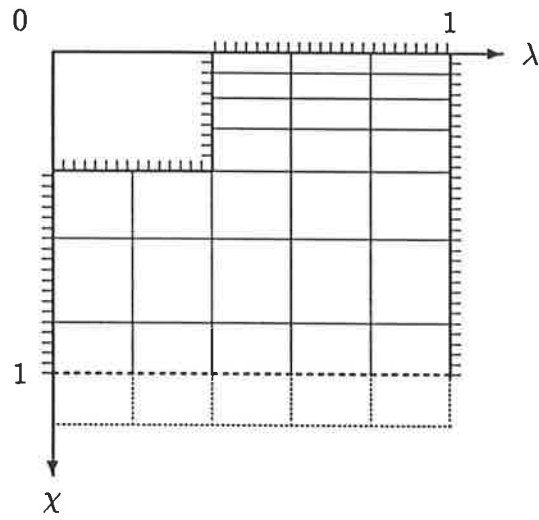


Figure 2.6: The shape of the transformed region of Figure 2.5.

and a_1 and a_2 are constants. The values of the constants a_1 , a_2 , N_χ and N_λ may be determined by an analysis of field data obtained from the area being modelled.

As the transformation maps the total depth of water onto the line segment between 0 and 1 (see Equation (2.20)) and the total depth of water is time dependent, a constant value of η does not correspond to a constant distance above the sea bed or a constant distance below the sea surface. It may be desirable to have a depth variation of the vertical eddy coefficient, N_η , which is constant with respect to the distance above the sea floor. For example it maybe desired that N_η be linear in the bottom two metres of water and constant above that. It is possible to obtain a dependence on physical depth by replacing $F(\eta)$, in Equation (2.39) with $F(G(\eta))$ where

$$G(\eta) = \frac{\eta H - \zeta}{h} = \frac{\eta(\zeta + h) - \zeta}{h} . \quad (2.41)$$

This function is the inverse of Equation (2.20) scaled by a factor of $-1/h$, so that $G(1) = 1$, i.e. the sea floor is still represented by $\eta = 1$. The only difficulty with this is that careful consideration needs to be given to the function $F(\eta)$ near $\eta = 0$, as $G(0)$ varies approximately from $-\zeta_{max}/h$ to ζ_{max}/h , where ζ_{max} is the maximum amplitude of the surface displacement ζ . This formulation will therefore only be applicable when there is no surface stress acting upon the body of water, that is no wind, and therefore $N(\eta)$ would be relatively constant near the sea surface.

If it is desired that N_η be dependent on the physical distance from the sea surface as well as the sea floor, as in the case when there is a wind stress acting on the sea surface, then there would need to be a split definition for $F(\eta)$, incorporating the idea of Equation (2.41), and an “elastic” definition for $F(\eta)$ where the two parts meet. The “elastic” join would ideally need to have continuity of the coefficient N_η and also zero slope with respect to η .

The idea presented above, requires an estimate for ζ_{max} , so it is suggested that a

form such as in Equation (2.40) be used in the initial stages of modelling and these later refinements be made after the initial runs give an estimate for ζ_{max} .

Using Equations (2.34), (2.36), (2.37) and (2.38), Equation (2.8) transforms to

$$\begin{aligned} \frac{\partial \mu}{\partial t} + \mu \frac{\partial \mu}{\partial \chi} + \nu \frac{\partial \mu}{\partial \lambda} + \omega \frac{\partial \mu}{\partial \eta} = & -\frac{g}{L^2} \left(\frac{\partial \zeta}{\partial \chi} - \frac{1}{B} \frac{\partial \zeta}{\partial \lambda} \frac{\partial \beta}{\partial \chi} \right) \\ & + \frac{N_\chi}{L^2} \frac{\partial^2 \mu}{\partial \chi^2} + \frac{N_\lambda}{B^2} \frac{\partial^2 \mu}{\partial \lambda^2} + \frac{1}{H^2} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial \mu}{\partial \eta} \right) + \frac{fv}{L} . \end{aligned} \quad (2.42)$$

Similarly, the transformed version of Equation (2.9) is

$$\begin{aligned} \frac{\partial v}{\partial t} + \mu \frac{\partial v}{\partial \chi} + \nu \frac{\partial v}{\partial \lambda} + \omega \frac{\partial v}{\partial \eta} = & -\frac{g}{B} \frac{\partial \zeta}{\partial \lambda} \\ & + \frac{N_\chi}{L^2} \frac{\partial^2 v}{\partial \chi^2} + \frac{N_\lambda}{B^2} \frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{H^2} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial v}{\partial \eta} \right) - Lf\mu , \end{aligned} \quad (2.43)$$

where this equation retains most of the untransformed v velocity values, as it is more convenient for the computer program to store the v values instead of the values of the transformed velocity component ν .

Subtracting Equation (2.33) from Equation (2.35) and dividing by H gives

$$\frac{\partial \omega}{\partial \eta} - \frac{1}{HB} \frac{\partial [BH(\bar{\mu} - \mu)]}{\partial \chi} - \frac{1}{H} \frac{\partial [H(\bar{\nu} - \nu)]}{\partial \lambda} = 0 . \quad (2.44)$$

This equation can then be integrated from the sea surface, where $\eta = 0$ and $\omega = 0$, to a general depth η below the transformed surface to obtain

$$\omega = \frac{1}{H} \left\{ \frac{1}{B} \frac{\partial}{\partial \chi} \left(BH(\bar{\mu}\eta - \int_0^\eta \mu d\eta) \right) + \frac{\partial}{\partial \lambda} \left(H(\bar{\nu}\eta - \int_0^\eta \nu d\eta) \right) \right\} , \quad (2.45)$$

which automatically satisfies the required boundary condition of $\omega = 0$ both at the surface and at the sea floor. This approach was used by Johns (1978) and has the advantage that ω is calculating directly.

Equation (2.42) may be written in a more suitable manner, in which the term $\mu \frac{\partial \mu}{\partial \chi}$ is replaced by $\frac{1}{2} \frac{\partial \mu^2}{\partial \chi}$. Numerical tests of various finite difference schemes on the non-linear

Burger's Equation, for which an analytic solution is available for comparisons (see Stevens (1981)) have shown that this "conservative" form is more suitable for accurate differencing. These terms are minor terms in the equations which govern the tidal motions and therefore this distinction is not of major importance. The new equation, listed below, is obtained by adding μ times Equation (2.35) to H times Equation (2.42).

$$\begin{aligned} \frac{\partial(H\mu)}{\partial t} + \frac{1}{B} \frac{\partial(BH\mu^2)}{\partial \chi} + \frac{\partial(H\mu\nu)}{\partial \lambda} + H \frac{\partial(\mu\omega)}{\partial \eta} = -\frac{gH}{L^2} \left(\frac{\partial \zeta}{\partial \chi} - \frac{1}{B} \frac{\partial \zeta}{\partial \lambda} \frac{\partial \beta}{\partial \chi} \right) \\ + \frac{HN_x}{L^2} \frac{\partial^2 \mu}{\partial \chi^2} + \frac{HN_\lambda}{B^2} \frac{\partial^2 \mu}{\partial \lambda^2} + \frac{1}{H} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial \mu}{\partial \eta} \right) + \frac{Hfv}{L} . \end{aligned} \quad (2.46)$$

Equation (4.10) may be changed in a similar manner to get

$$\begin{aligned} \frac{\partial(Hv)}{\partial t} + \frac{1}{B} \frac{\partial(BH\mu v)}{\partial \chi} + \frac{\partial(H\nu v)}{\partial \lambda} + H \frac{\partial(v\omega)}{\partial \eta} \\ = -\frac{gH}{B} \frac{\partial \zeta}{\partial \lambda} + \frac{HN_x}{L^2} \frac{\partial^2 v}{\partial \chi^2} + \frac{HN_\lambda}{B^2} \frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{H} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial v}{\partial \eta} \right) - LHf\mu . \end{aligned} \quad (2.47)$$

The four equations to be solved are therefore Equations (2.33), (2.45), (2.46) and (2.47). Since there are five unknowns in these equations, namely ζ , μ , ν , v and ω an additional equation is required to "close" the system. This may be obtained by rearranging Equation (2.29) to give

$$\nu = \frac{1}{B} \left(v - \mu \frac{\partial \beta}{\partial \chi} \right) . \quad (2.48)$$

This is used to obtain ν from μ , v , B and β , thus reducing the unknowns in the above system to ζ , μ , v and ω . As mentioned earlier it is more convenient to use the dependent variable v instead of ν in the equations to be solved. When required, ν is calculated using Equation (2.48) in preference to storing both the ν and v values, which would increase the storage requirements for the program by about twenty five percent. If the body of water being modelled is small or the computer being used is not restricted by variable storage restraints (even virtual memory space can be exhausted

by a large three-dimensional model), then extra storage can be sacrificed in order to increase the computational efficiency of the computer program. The computer program has been designed in such a way that this can be achieved by a simple adjustment.

The above equations are now to be solved in the unit cube defined by, $0 \leq \chi \leq 1$, $0 \leq \lambda \leq 1$ and $0 \leq \eta \leq 1$.

2.4 The Boundary Conditions

The boundary conditions which apply to the transformed equations are listed below

1. At any solid boundary, that is, a land-water interface, the velocity component normal to the boundary is zero, that is, there is no flow across the boundary, but the water is allowed to “slip” parallel to the boundary. Therefore

$$\mu = 0 \quad \text{at any solid boundary where } \chi = \text{constant} , \quad (2.49)$$

$$\nu = 0 \quad \text{at any solid boundary where } \lambda = \text{constant} . \quad (2.50)$$

2. At the sea floor, where $\eta = 1$, either a “no-slip” condition or a “slip” condition can be applied. A “no-slip” condition implies that

$$\mu = \nu = \omega = 0 \quad \text{at } \eta = 1 , \quad (2.51)$$

and therefore from Equation (2.48), $v = 0$ at $\eta = 1$. A “slip” condition, which is an approximation that could be used when there are large derivatives of the horizontal velocities with respect to depth near the sea bed, implies that $\omega = 0$ and the μ and ν values can be respectively determined from the boundary conditions

$$\tau_{bx} = - \frac{\rho L N_{\eta}}{H} \frac{\partial \mu}{\partial \eta} \bigg|_{\eta=1} , \quad (2.52)$$

$$\tau_{b\lambda} = - \frac{\rho N_\eta}{H} \frac{\partial v}{\partial \eta} \bigg|_{\eta=1}, \quad (2.53)$$

where $\tau_{b\chi}$ and $\tau_{b\lambda}$ are the components of the bottom shear stress which need to be determined. The level $\eta = 1$ is assumed to be on the boundary between turbulent and laminar flows, which is assumed to be a small distance above the sea floor. One way of estimating the bottom shear stress is by using the quadratic friction law (see, for example Nihoul (1977)) given by

$$\tau_b = \rho C_b U |U|, \quad (2.54)$$

where $U = (L\mu, v)$ and C_b is the bottom drag coefficient, which can be difficult to estimate. Bowden et al. (1959) suggest $C_b = 3.5 \times 10^{-3}$ while Johns (1978) found that the values for C_b varied greatly within a tidal cycle from 1.24×10^{-3} to 1.39×10^{-1} . Sternberg (1968) experimentally observed this great variation, which suggests that Equation (2.54) will never completely represent the bottom shear stress over a complete tidal cycle. If there are no field measurements available for estimation of C_b for the body of water being modelled then the only means to find the best value for C_b is by calibrating the model by comparing model predictions for surface elevations with observed results. Comparing velocities can also help, if observed data is available.

In the numerical model the far easier option is to use a “no-slip” condition on the sea floor (see Koutitas (1987)), as this avoids the requirement of an “a-priori” knowledge of the bottom shear stress τ_b and the associated determination of parameters. The choice between the “slip” and the “no-slip” condition should be decided by the properties of the flows in the area of application although specific detail of velocity profiles is often very difficult to obtain, especially near the sea floor, which is the most sensitive region.

The “no-slip” boundary condition is by far the more stringent test on a numerical model, as it requires the numerical model to have greater stability near

the sea floor where the condition introduces larger derivatives of the horizontal velocities with respect to depth. If a three-dimensional numerical model works for a “no-slip” condition on the sea floor, it is sure to work with a “slip” condition, so the following work concentrates on using the “no-slip” condition to guarantee robustness of the model.

3. At the sea surface, due to the fact that $\eta = 0$ follows the surface (or is the surface), ω is by definition zero. The vertical boundary condition for the horizontal velocity components u and v are in terms of the shear stresses acting upon the surface due to wind movement over the body of water and are given by:

$$\tau_{sx} = - \frac{\rho L N_\eta}{H} \frac{\partial u}{\partial \eta} \Big|_{\eta=0} , \quad (2.55)$$

$$\tau_{sy} = - \frac{\rho L N_\eta}{H} \frac{\partial v}{\partial \eta} \Big|_{\eta=0} , \quad (2.56)$$

where τ_{sx} and τ_{sy} are the components of the surface shear stress which again may be related to the wind speed and direction by the quadratic friction law, which gives

$$\tau_s = \rho_a C_s W |W| , \quad (2.57)$$

where ρ_a is the density of the atmosphere, C_s is the surface drag coefficient and W is the wind velocity at some reference height above the water surface. Again the coefficient C_s must be experimentally determined. There are many empirical formulations suggested for C_s (see Wilson (1960) or Welander (1961)). Arnold (1985) suggests using the formula of Wu (1982), given by

$$C_s = (0.8 + 0.065 W_{10}) \times 10^{-3} , \quad (2.58)$$

where W_{10} is the wind speed measured at a height 10m above the surface, because this formula appears to be applicable for all wind speeds.

4. The simplest condition to apply at open boundaries, where the modelled region joins another region of water, is specification of the height of the surface elevation, ζ . Unfortunately this simple approach often leads to a boundary which reflects outgoing waves back into the region of interest and can trap initial starting transients or even lead to instability in the numerical model (see Arnold (1987)). Bennett (1976) has formulated several exact open boundary conditions but most are too difficult to be incorporated into numerical models. For tidal flows, Arnold (1987) developed a condition based on an idea proposed by Blumberg and Kantha (1985), who utilised a modified Sommerfeld open boundary condition in a circulation model. Arnold (1987) shows that his condition is an improvement over the Reid and Bodine (1968) condition of

$$U_n = c(\zeta - \hat{\zeta}) , \quad (2.59)$$

where U_n is the normal component of depth integrated velocity, c is an appropriate admittance coefficient (taken as $\sqrt{gH}$), ζ is the calculated tidal height and $\hat{\zeta}$ is the specified tidal height.

Arnold's condition is

$$U_n = c \left(\zeta - \hat{\zeta} + \int_0^t \frac{(\zeta - \hat{\zeta})}{t^*} dt \right) , \quad (2.60)$$

where t^* is a parameter with the dimensions of time and the extra term on the right hand side is a damping term which forces the predicted displacement at the open boundary to closely approximate the specified displacement. Some trial runs need to be performed to find a good value of t^* . As $t^* \rightarrow 0$ Equation (2.60) becomes the height specified condition and when $t^* \rightarrow \infty$ the equation becomes the usual Reid and Bodine condition of Equation (2.59).

Chapman (1985) studied eleven different radiation schemes and recommends using the Orlanski-Sommerfeld condition (see Orlanski (1976)), which is

$$\frac{\partial \zeta}{\partial t} + c \frac{\partial \zeta}{\partial n} = 0 , \quad (2.61)$$

where n is the normal direction to the boundary and where $c = -\frac{\partial \zeta}{\partial t} / \frac{\partial \zeta}{\partial n}$ is modified by the following rules,

$$c = \begin{cases} \Delta n / \Delta t & \text{if } c \geq \Delta n / \Delta t \\ c & \text{if } 0 < c < \Delta n / \Delta t \\ 0 & \text{if } c \leq 0 \end{cases} \quad (2.62)$$

This separates the flow into ingoing and outgoing parts and limits the speed of the outgoing wave to a maximum Courant number of one.

Chapman does not include tidal forcing in his tests but Bills and Noye (1987) shows how this can be incorporated and also recommends using the Orlanski-Summerfeld radiation condition.

Another approach is to apply a “sponge” layer along the open boundary, where a very large viscosity value is applied to damp out unwanted reflections from the boundary. Schemes of this type can also be applied in conjunction with a radiation condition.

Chapter 3

THE NUMERICAL MODEL

There are two main restrictions placed on the numerical model, which are based on limitations of the computational technique and/or computational power and storage procedures. These restrictions need to be considered before attempting to design a computational procedure and the subsequent computer program.

The first restriction is that the method cannot be implicit in the two horizontal directions, as multi-dimensional implicit equations would require both the use of very large matrix inversions, and more than double the required amount of computer storage space. This is because large amounts of working space would be required for the matrix solving routines and also each velocity variable would require two storage spaces, for the n^{th} and $(n+1)^{th}$ time levels. The model developed here only needs one storage space for each velocity variable because new values overwrite old ones after they have been through a small buffer storage. This means that models with more grid points can be run on computers which are limited in memory size. The inversion of very large matrices is very time consuming (even for sparse matrices and using inversion methods which can be vectorised) and except for very large powerful computers, is beyond the capabilities of normal computers when applied to a full three-dimensional model of moderate size for reasonable amounts of computer time. While super computers are

more common nowadays, they are still difficult to access and their optimal use is design dependent, so writing a model specifically for a particular super computer limits the usefulness of a model. The extra computer space required for the implicit methods is not as important as the time aspect, due to the introduction of virtual memory computers, but none the less, may restrict the maximum size of a model that can be run, even on a virtual memory machine, if small grid spacings are required in the vertical and in the horizontal directions.

The computer program for this model was developed before the author had access to a virtual memory computer and therefore many space saving measures were introduced into the program at the expense of a little extra computational time. However, the computer code and calculations were designed so that the code can be optimised easily either by compiler options or by writing some of the frequently used simpler routines in more time-efficient lower level languages. Also the computer program has been designed and written in such a way that it is very simple to modify the program to be more time-efficient at the cost of using more space. If the computer on which the model is being run is not restricted by the size of its memory, for the given size of program to be run, then it is simple to introduce a few more arrays to store numbers so that they do not have to be calculated at several different places in the program code.

Implicit equations in the η direction of the model are easily accommodated because, as long as only three-point formulae for derivative terms are used, the finite difference equations form the special case of a tri-diagonal set of equations, which are efficiently and easily solved (see Section 4.2). The implicit equations over depth give greater stability to the model in the η direction, where it is most sensitive to instability.

The second restriction is that because the plan view of the area to be modelled can in general be very irregular in shape, even after the transformations of Section 2.2

are used, the model needs to be able to accommodate this. Any number and any orientation of islands and peninsulas should be permissible in the body of water being modelled. To enable this with consideration given to the space saving objective, it was decided not to use the simple element indexing system which references elements by two indices - one for each of the horizontal coordinate directions, but to use a pointer array which indexes a one-dimensional array in conjunction with a system of information, where each element “knows” the locality of any surrounding land or sea boundaries. Each element only “knows” (that is the program has an information system that tells it which are the neighbours of any particular element) its eight neighbours, which is sufficient for up to three point horizontal differencing schemes. As implied earlier, the two horizontal coordinate directions are less sensitive to stability problems, so three point finite difference approximations are sufficiently accurate for any differencing in these directions. The indexing and element information schemes developed, in the next section, are specifically designed for three point differencing schemes, although they could be expanded, quite simply, to cope with any five point differencing scheme, if desired. The element information or identification scheme, developed by Stevens (1981) and Stevens and Noye (1983), has since been adopted and used successfully by Arnold (1985).

3.1 The Horizontal Grid-Scheme

The grid-scheme is the basis of any finite difference model. The plan view of the transformed region, which is a unit square, is divided into rectangular blocks or elements of length $2\Delta\chi$ and breadth $2\Delta\lambda$. Within each of these elements, at each time step, a value for the surface displacement ζ is calculated as well as values for the transformed velocity components μ and ω and the v velocity component, at each depth level.

Elements are labelled with an index number for location purposes (see Figure 3.1), so that the surface displacement and the water velocity from any element in the body of water can be obtained as long as the index number of that element is known. These numbers are sequentially allocated from the corner $\chi = \lambda = 0$, incrementing in the λ direction within the inner loop of the process. The finite difference scheme and the individual finite difference approximations have been designed, so that information is needed only from the adjacent elements when calculating the approximation at the next time level for any of the variables. Each element then only needs information about each of its surrounding elements. Therefore the convention of labelling as shown in Figure 3.2 has been adopted, where for element i the relative positions of the variables are indicated by the subscripts in the following way:

$i1$ indicates the element immediately above (in the plan view) the i^{th} element, so that corresponding variables are located a distance of $2\Delta\chi$ in the negative χ -direction from i ;

$i2$ in a similar way to $i1$, indicates the element immediately below the i^{th} element, so that corresponding variables are located a distance of $2\Delta\chi$ in the positive χ -direction from i ;

$e \pm 1$ indicates the next element to the right (left) of the element e , so that corresponding variables are located a distance of $2\Delta\lambda$ in the positive (negative) λ -direction from the element e , where e may be either i , $i1$ or $i2$;

m is the subscript used on the b and B variables which only depend on χ and thus are the same for each element in a particular row. In Figure 3.2 the m^{th} row is the row which contains the elements indexed by $i - 1$, i and $i + 1$.

Therefore, if the element being considered is indexed by the number 17 in Figure 3.1, then the value of $i1$ is 12, the value of $i2$ is 22 with the values of $i1 \pm 1$, $i \pm 1$ and

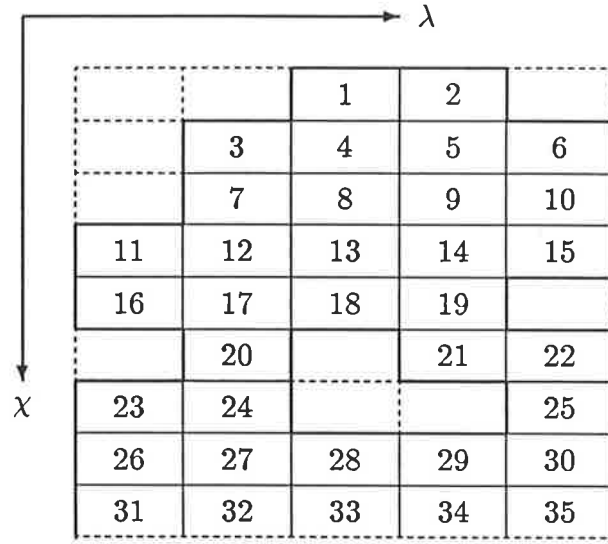


Figure 3.1: The plan view of the index numbers of the elements for a transformed body of water.

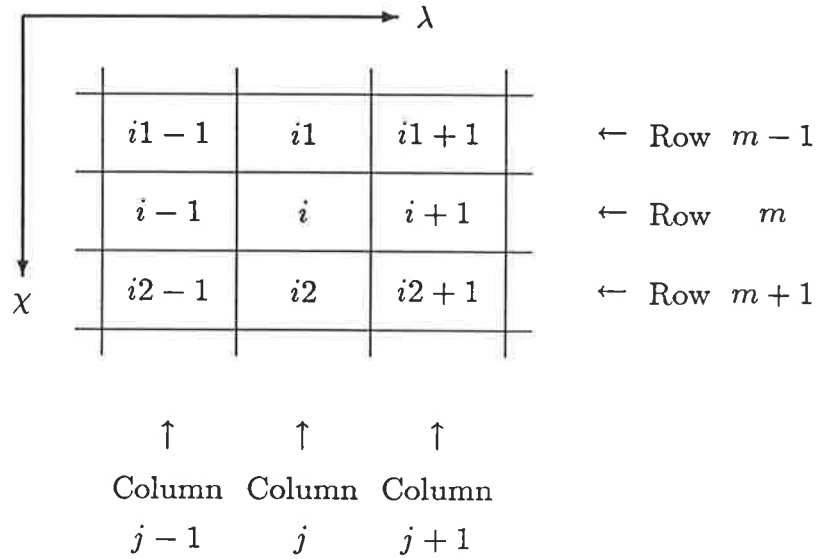


Figure 3.2: The general method of indexing elements which surround the element i .

$i2 \pm 1$ being obvious. The notation $i1$ and $i2$ is used since there may not be a constant number of elements in a each row, such as when there is an island in the area being modelled. The program “knows” the values for $i1$ and $i2$ for each particular element i , by means of an identification number (explained later this section) and a pointer array which, given the row and column numbers, gives the index number for each element. Non-water elements have an index of zero.

The grid spacings in both horizontal coordinate directions, χ and λ , are permitted to vary so that regions of interest where more detail is desired can be covered by more grid points. This variation can be chosen in any manner, but extreme care must be exercised as the accuracy of the finite difference approximations, which are used on this variable grid, depend significantly on the nature of the grid spacing used. The variable grid spacing for the two horizontal directions are chosen by the same method as the variable vertical grid, which is explained in more detail in Section 3.2. Two cases where variable grid spacing in the χ and λ directions give more accurate representation of the coast configurations are shown in Figure 3.3 and 3.4. The grid spacing in the χ direction is defined by setting the values of $\Delta\chi_m$ for each row, where $\Delta\chi_m$ is the width of the m^{th} row of elements. Similarly the grid spacing in the λ direction is defined by setting the values of $\Delta\lambda_j$ for each column.

The three velocity components and the surface displacement are calculated at the positions within each element as indicated in Figure 3.5. At each of the velocity points in an element, a number of depth levels are defined and at each of these levels the three velocity components are calculated. The depth levels are determined by the vertical grid spacing which can be arbitrarily chosen to suit the situation. If there are K depth levels, the vertical grid is defined by assigning values to η_k ranging from $\eta_0 = 0$ at the surface to $\eta_K = 1$ at the sea floor. Thus the subscript k on the velocity components indicates that these velocities are measured a distance of η_k from the surface in the

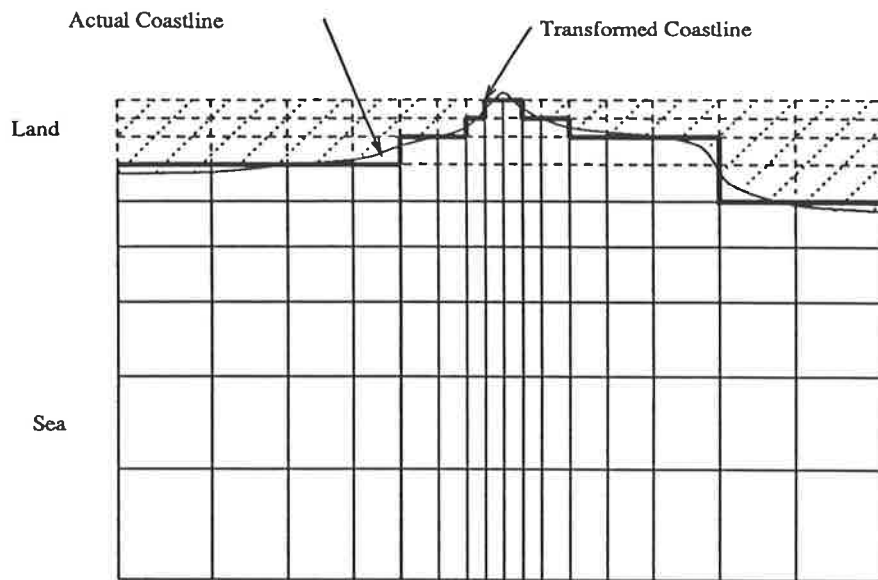


Figure 3.3: The transformed coastline, represented by straight line segments of the suitably defined horizontal grid spacing: Example 1.

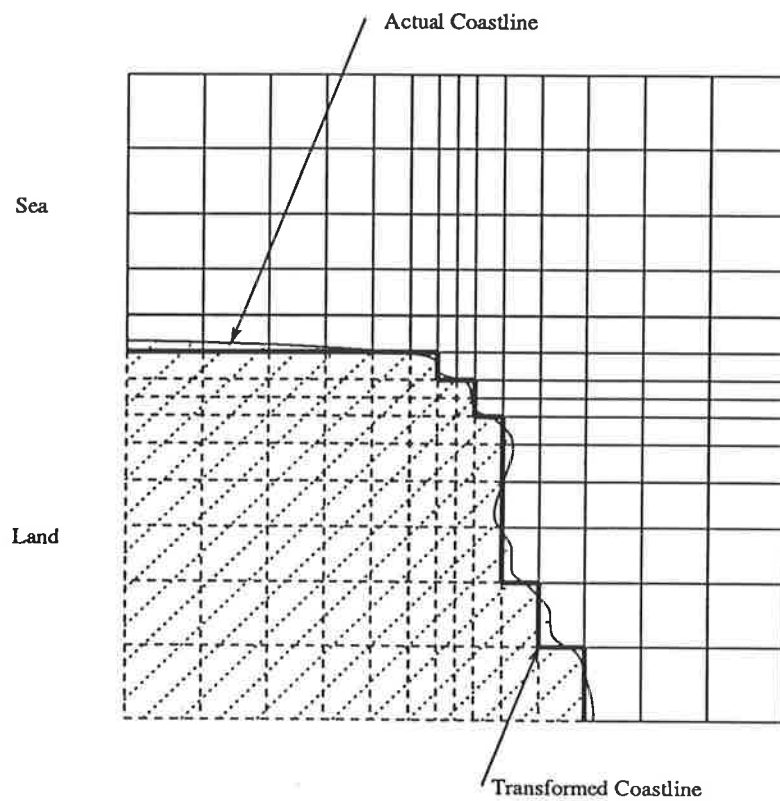


Figure 3.4: The transformed coastline, represented by straight line segments of the suitably defined horizontal grid spacing: Example 2.

transformed system.

In the plan view of an element, such as the one enclosed by the dashed line of Figure 3.5, at each level the vertical velocity component ω is calculated at the same point as the surface displacement ζ is calculated. The two velocity components μ and ν are calculated at points in the middle of the edges of each element, which are in the positive χ and λ directions respectively, from the ζ calculation point.

The closed boundaries of the body of water are aligned so that the horizontal velocity components of each element are perpendicular to the closed boundaries in order to satisfy the boundary conditions (2.49) and (2.50).

On an open boundary, for example along the line $\chi = 1$ in Figure 2.1, which corresponds to $x = L$, the boundary must pass through the points where the surface displacements are computed, since on the open boundary the tidal height is specified as input data to the system.

Element Identification Numbers

The computer program has been designed around a system where each grid point, at which calculations are to be made, “knows” what its surroundings looks like, that is, information about the location of any land or sea boundaries adjacent to this point is available. This system eliminates the need to have numerous different types of computational elements, where calculations are performed depending on the type of the element. The required information, for each calculation point, can be stored in a single integer variable which is considered as a string of boolean flags. This representation is exactly the same as using numbers which are powers of two, where for example the number 8 ($= 2^{4-1}$) is equivalent to the case when the fourth bit from the right in the boolean string is set. The required information can be obtained quickly and efficiently using bit masking procedures, which if not standard can be written in higher level

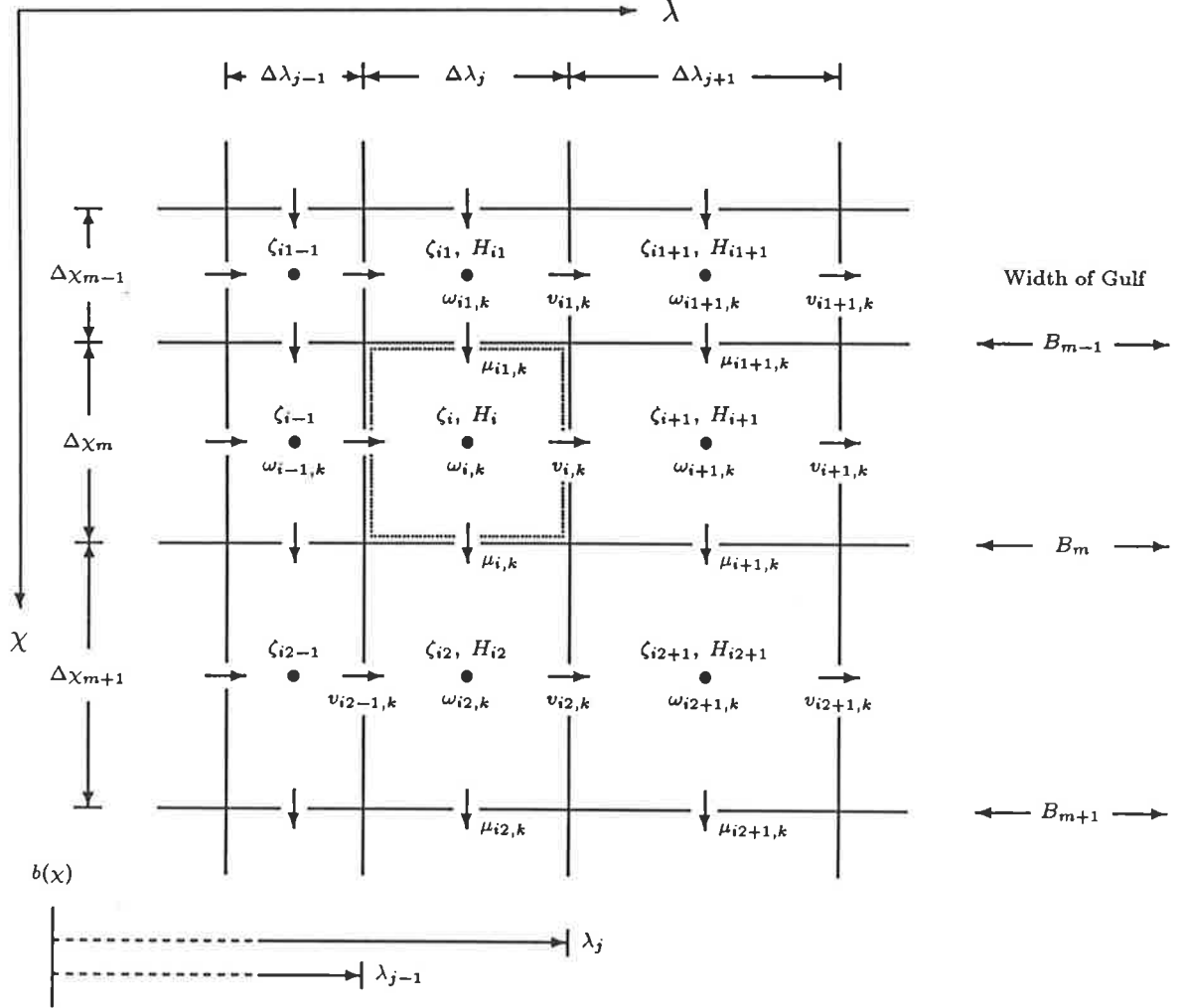


Figure 3.5: The location within the i^{th} and adjacent elements of the position at which the surface displacement and the velocity components, at the k^{th} depth level, are computed. Note that not all variables are labelled, with the names of the unlabelled variables being obvious.

languages if time optimisation is important.

In the computer program an element is considered “active” if the point where the surface displacement ζ is calculated (that is the black dots in the following Figures 3.6 and 3.8) lies within the physical boundaries which define the gulf. Thus in Figure 3.6 there are seven active elements. Calculations are only carried out at active elements as the non-active elements are either land or outside the area being modelled. Open boundary elements are considered to be active elements, even though the ζ values are prescribed and not calculated, as velocity values are still required to be calculated.

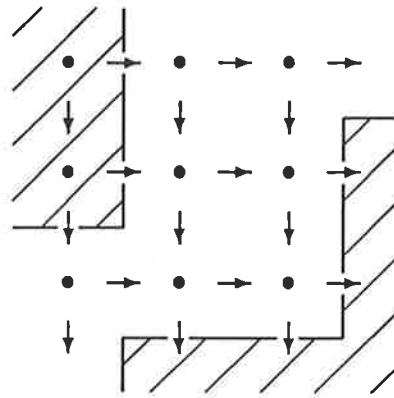


Figure 3.6: Active and non-active elements.

bit 1 $\equiv$ 1	bit 2 $\equiv$ 2	bit 3 $\equiv$ 4
bit 4 $\equiv$ 8		bit 5 $\equiv$ 16
bit 6 $\equiv$ 32	bit 7 $\equiv$ 64	bit 8 $\equiv$ 128

Figure 3.7: The weights given to each of the surrounding elements when determining the identification number for the central element.

Each element has eight neighbours and they are numbered 1 to 8 as shown in

3.2 The Vertical Grid Spacing

Increased resolution required in certain regions, such as near the sea floor because of the great vertical shear in the boundary layer, may be achieved by use of a variably spaced finite difference grid or a further transformation as in Johns et al. (1983). The latter is equivalent to compression of the vertical finite difference grid near the sea floor. For viscous or diffusion dominated flow problems, numerical experiments carried out by Noye (1984a) indicate that differencing on a properly chosen variable grid gives more accurate results than the use of transformations. This is mainly because the latter introduces additional convection-like terms which are much more difficult to discretise accurately than diffusion-like terms. Consequently increased resolution, where necessary, is achieved in this model by use of a variable grid.

When using a variably spaced finite difference grid, it is important that suitable spacing with appropriate differencing is used so that the finite difference approximations used for the first and second order spatial derivatives remain second order accurate. Otherwise any advantage over an equally spaced grid, which produces second order accurate approximations with central differencing, are lost. Most choices of variable grid spacing and differencing used previously (for example, Webber (1980)) result in numerical schemes which are only first order accurate in space. This produces large errors when the relatively large terms such as those involving the vertical eddy viscosity are differenced on such grids.

The variable vertical grid spacing gives flexibility when choosing the distribution of grid points, where the choice may depend on the situation being modelled. For example, if no wind is being modelled a grid can be used which has smaller spacings near the sea floor where velocity gradients with respect to depth are large, and the spacings gradually increase in size to larger spacings at the surface. Or if a wind

stress on the surface is being modelled, a grid which is fine at the top and bottom and increasing in size towards the middle can be used. A fine equally spaced grid over the entire depth is not practical because the increase in storage requirement and the additional number of computations makes the method impractical for all but super computers. The variable grid spacing is achieved by placing the transformed velocity components at the $K + 1$ depth levels defined by

$$\eta = \eta_k , \quad k = 0, 1, \dots, K , \quad (3.1)$$

where the sea surface is represented by $k = 0$ and sea floor by $k = K$ (see Figure 3.9). The values of η_k satisfy $0 = \eta_0 < \eta_1 < \eta_2 \dots < \eta_K = 1$ with the grid spacing defined by

$$\Delta\eta_k = \eta_{k+1} - \eta_k , \quad k = 0, 1, \dots, K - 1 . \quad (3.2)$$

An equally spaced grid can be produced by setting $\eta_k = k/K$. The choice of the vertical grid spacing is most important as many errors may be introduced through the finite difference approximations to the equations of motion, if the grid spacing is not chosen wisely. By trying to minimise these errors the variable grid spacing defined by the κ -method described in Noye (1984a) was developed, namely

$$\Delta\eta_k = \Delta\eta_{k-1}(1 - \kappa\Delta\eta_{k-1}) , \quad k = 0, 1, \dots, K - 1 , \quad (3.3)$$

where κ is a constant of order one. The value of $\kappa = 0$ produces an equally spaced grid and as κ increases the grid becomes finer near the sea floor and coarser near the sea surface (see Figure 3.10). For a fine-coarse-fine grid the bottom half can be mirror imaged onto the top, or calculated with a smaller value of κ if the velocity gradients due to wind stress are not as large as those near the sea floor. Once the number of grid points in the vertical and the type of grid (coarse-fine or fine-coarse-fine) has been chosen, fixing a value of κ uniquely defines the grid.

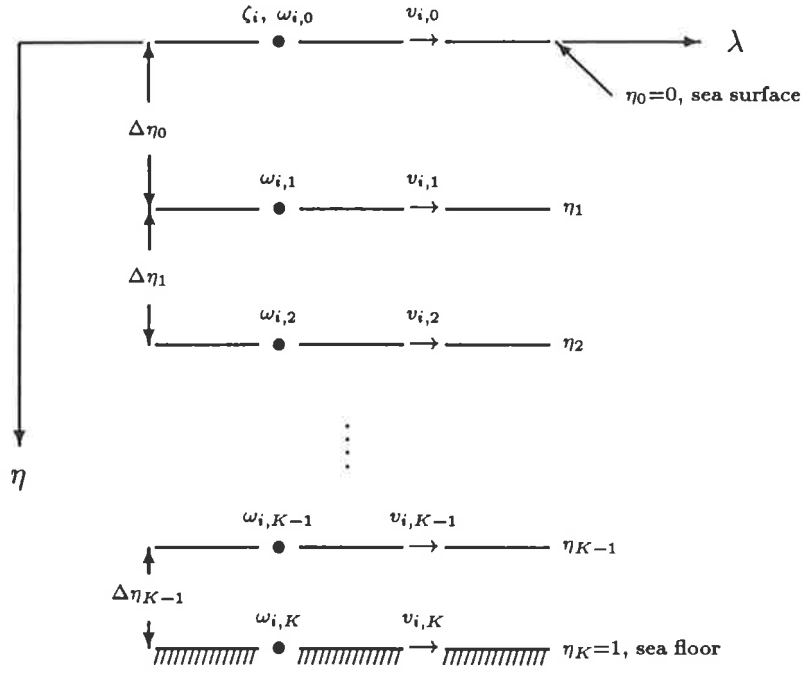


Figure 3.9: The variable vertical grid

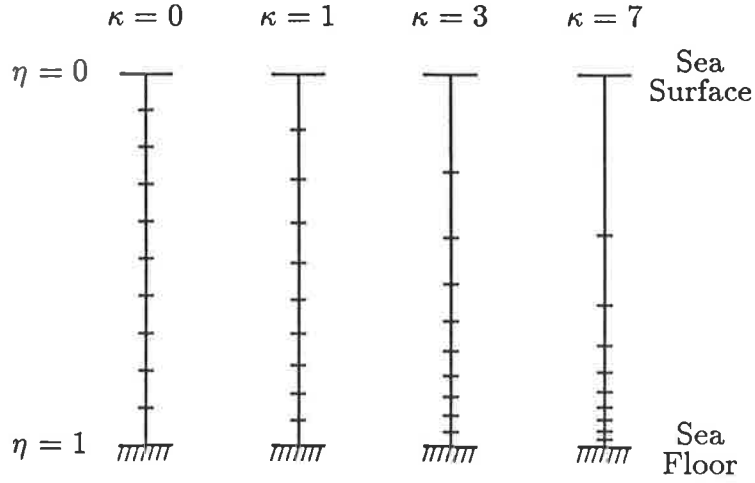


Figure 3.10: Variable vertical grid spacings for eleven depth levels ($K=10$), with different kappa values, namely $\kappa = 0$ (even grid spacing) and $\kappa = 1, 3$ and 7 .

With reference to Figure 3.11, the commonly used approximation

$$\left. \frac{\partial \mu}{\partial \eta} \right|_k \approx \frac{\mu_{k+1} - \mu_{k-1}}{\Delta \eta_k + \Delta \eta_{k-1}}, \quad (3.4)$$

is only first order accurate. It has been shown (see Noye (1984a)) that use of the first term on the right side of the relation given in Equation (3.5), which is obtained by using Taylor series expansions about the point η_k ,

$$\left. \frac{\partial \mu}{\partial \eta} \right|_k = \frac{\mu_{k+1} + (r_k^2 - 1)\mu_k - r_k^2\mu_{k-1}}{(r_k + 1)\Delta \eta_k} - \frac{\Delta \eta_{k-1}\Delta \eta_k}{6} \left. \frac{\partial^3 \mu}{\partial \eta^3} \right|_k + \dots, \quad (3.5)$$

where $r_k = \Delta \eta_k / \Delta \eta_{k-1}$, is a second order accurate approximation to the first derivative and this is clearly better to use than the approximation (3.4). In a similar way the second derivative can be shown to be

$$\begin{aligned} \left. \frac{\partial^2 \mu}{\partial \eta^2} \right|_k &= \frac{\mu_{k+1} - (r_k + 1)\mu_k + r_k\mu_{k-1}}{\frac{1}{2}r_k(r_k + 1)(\Delta \eta_{k-1})^2} - \frac{(r_k - 1)\Delta \eta_{k-1}}{3} \left. \frac{\partial^3 \mu}{\partial \eta^3} \right|_k \\ &\quad - \frac{(r_k^2 - r_k + 1)(\Delta \eta_{k-1})^2}{12} \left. \frac{\partial^4 \mu}{\partial \eta^4} \right|_k + \dots, \end{aligned} \quad (3.6)$$

which yields the approximation

$$\left. \frac{\partial^2 \mu}{\partial \eta^2} \right|_k \approx \frac{\mu_{k+1} - (r_k + 1)\mu_k + r_k\mu_{k-1}}{\frac{1}{2}r_k(r_k + 1)(\Delta \eta_{k-1})^2}. \quad (3.7)$$

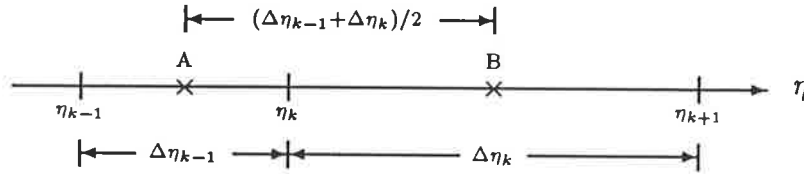


Figure 3.11: A typical non-uniform grid on the η -axis. A is midway between the coordinates η_{k-1} and η_k , and B is midway between η_k and η_{k+1} .

The κ grid spacing is chosen so that the coefficient of the leading term from Equation (3.6), that is $(r_k - 1)\Delta \eta_{k-1}$, is of the order $(\Delta \eta_{k-1})^2$. This specification leads to

Equation (3.3). On a uniform grid where $r_k = 1$, Equation (3.7) is a second order accurate approximation. However, at an abrupt change in grid spacing, such as $r_k = 2$ then the error for the approximation of the second derivative is only first order accurate. A second order accurate difference approximation can be obtained in this case when the spacings are uniform on either side of an abrupt change by using μ_{k-2} instead of μ_{k-1} in the finite difference formulae for a uniform grid spacing (This idea can also be used for $r_k = 3, 4, \dots$). However, this requires the special differencing to be built into the model at a specified point in the grid. This then does not allow a grid to be changed to suit different situations. The usefulness and improved accuracy obtained from varying the vertical grid to give optimal wave speed and amplitude response for the model is shown in Chapters 6 and 7.

Another method for determining the distribution of grid points was tried, and the wave propagation analysis in Chapter 6 has shown that this can give more accurate results than the κ method for certain situations. This method calculates the grid spacings from the relationship

$$\Delta\eta_k = (1 + p)\Delta\eta_{k-1} , \quad (3.8)$$

where p is a parameter which determines the rate of increase or decrease in successive grid spacings.

This type of variable grid shall be referred to as a p grid. The leading error term in Equation (3.6), for a p grid, is of the order $p\Delta\eta_{k-1}$, which suggests that p should be kept as small as possible. A zero value for p produces a uniform grid spacing.

Results obtained in Chapter 7 shows that an optimal value for p can be found and this value, which is non zero, can produce smaller numerical errors in the three-dimensional model than when using the optimal κ grid. An alternative formulation for Equation (3.8) is

$$\Delta\eta_k = (1 + p)^k \Delta\eta_0 . \quad (3.9)$$

3.3 The Computational Procedure

Equations (2.33) and (2.45) to (2.48) are the partial differential equations which govern the behaviour of the tidal model. The finite difference approximations to these equations are detailed in the Chapter 4. The approximations have been designed so that a variable grid spacing can be used in both horizontal directions as well as in the vertical. The model can be used with any variable grid spacing, but it must be remembered that the accuracy of the results obtained depends on the nature of the grid spacing used.

Initially all the surface displacement and velocity variables for the entire gulf are set to zero unless otherwise known. The numerical scheme is then used to calculate the surface displacements and water velocities over the entire gulf Δt later, using the finite difference method together with the boundary conditions and the input surface displacement from the open boundaries. Once the surface displacements and water velocities are all known at the new time level the process is repeated continuously to find the values of all the variables at some particular future time. At each time step the method consists of four separate parts, the first being the calculation of all the ζ variables at the new time level using an explicit form based on a forward time approximation combined with second order spatial differencing. The second part consists of calculating the μ and ν values at the new time level, this being done implicitly over depth using values of ζ from both the new and old time levels. The ν values are calculated before the μ values. All the terms in these two finite difference equations are differenced using second order spatial approximations and with the major terms completely centred about the $(n + \frac{1}{2})$ time level and with the other smaller terms almost centred about this level, thus making the equations practically second order accurate in time. The calculated μ and ν values are then integrated to get $\bar{\mu}$ and $\bar{\nu}$. The third part consists of recalculating the ζ values obtained in the first part of the step, in which

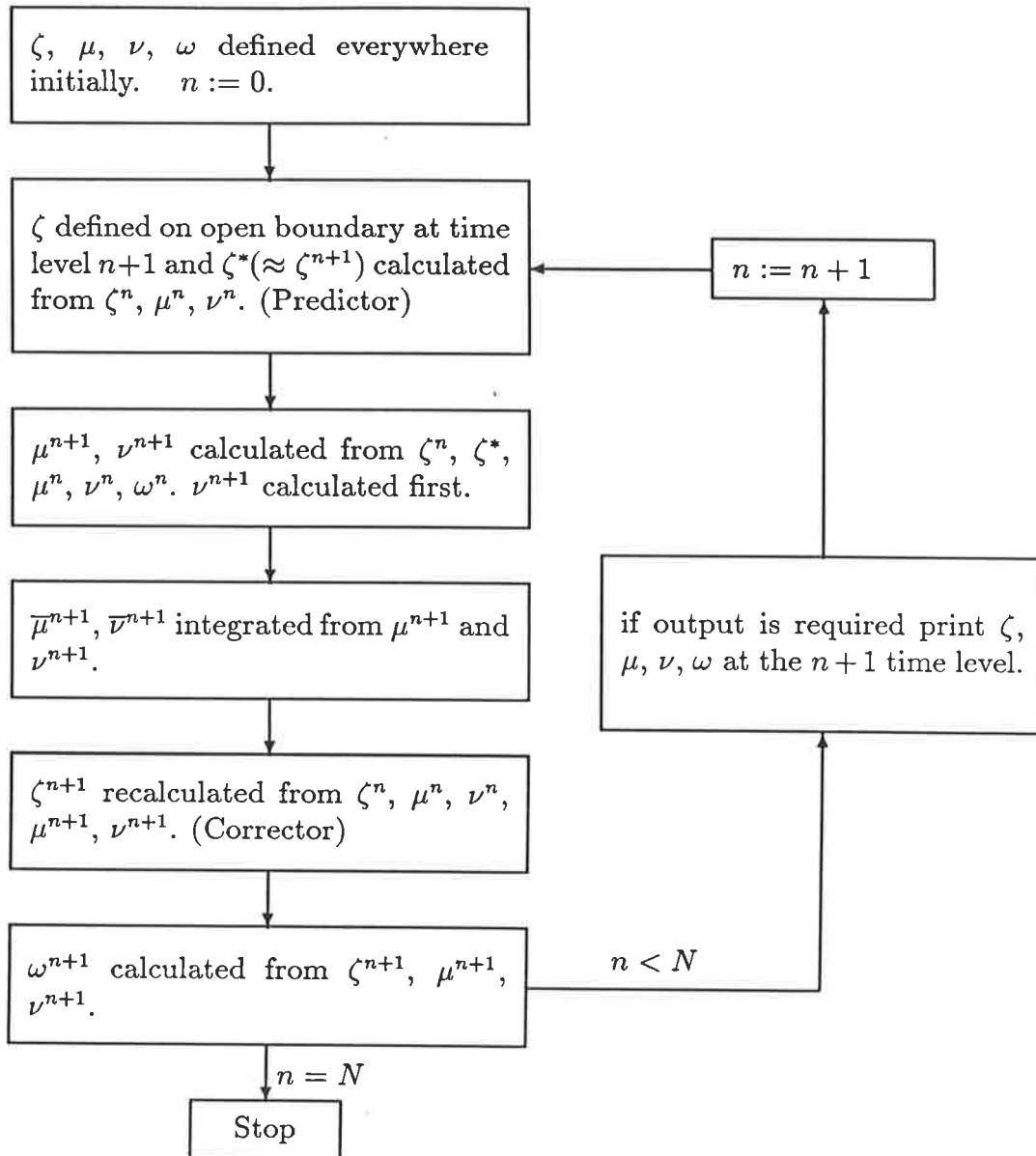


Figure 3.12: A flow chart showing the order of the calculations in the numerical model.

ζ at the new time level was evaluated using a forward time approximation. The final part of the process consists of calculating the ω values at the new time level. These are calculated explicitly from the μ , v and ζ values at the new time level, again using a second order accurate finite difference approximation.

When the scheme was in the process of being developed it initially did not have the third part of the scheme where ζ is recalculated, thus making the scheme essentially forward time central space (FTCS) but this was found to be unstable except for unreasonably small time steps (see Section 7.2). One way of overcoming this instability would be to increase the horizontal eddy viscosity parameters N_x and N_λ to artificially high levels, but this approach was dismissed because, although reasonable model results can be achieved, it was considered that this was a deviation from the original objective of accurately modelling the given equations from a mathematical point of view. Since this is the main objective of this thesis, an alternative way of stabilising the model was sought. The recalculation of ζ is based on the idea present in the Improved Euler Method for first order differential equations where the normal Euler method is analogous to the FTCS approach (both are first order accurate schemes). As the Improved Euler Method increases the accuracy to second order accurate (by re-estimation of the slope of the function) so the recalculation of ζ improves the accuracy of this finite difference method to virtually second order accuracy in time and increases the stability of the method. In this way the first calculation of ζ can be viewed as a “predictor” step and the second calculation as a “corrector” step. An analysis of this time stepping procedure is performed in Sections 7.1 and 7.2, which shows that the finite difference method is exactly second order accurate for the linear case if the horizontal diffusive terms are excluded. As a consequence of the recalculation of ζ the stability of the method was found to increase significantly, to such an extent that the model is stable in the absence of the horizontal diffusive terms, thus enabling the appropriate coefficient for these terms to be chosen, one which reflects the true situation representative

of the model area, not one which is artificially high just to keep the model stable.

The recalculation of ζ uses μ and v values from both the new and old time levels in Equation (2.33), which is centrally differenced in time, thus yielding a finite difference equation which is now second order accurate. In the computer program there is actually only one variable for each velocity point, but all the required information from the old time level is stored in the first approximation for ζ (see Chapter 4 for more detail). This extra recalculation performed for each step is computationally very efficient, as it requires few additional calculations, with most of the computer time being used in the evaluation of μ , v and ω .

Chapter 4

THE FINITE DIFFERENCE EQUATIONS

The indexing system described in Section 3.1 forms the basis of the following difference scheme. The plan view in Figure 3.5 should be used to gain a better understanding of the following symbols and notation, which defined averages in the horizontal plane as follows: a tilde ($\sim$) above a variable indicates an averaging between adjacent elements in the m^{th} row and the $(m+1)^{th}$ row but in the same column and an arrow ($\vec{}$) above a variable indicates an averaging between two adjacent elements in the same row. These averagings are second order accurate finite difference approximations, on an irregular grid for the values of variables, which the computer program does not store at the required position in the plan view of the grid. Thus

$$\begin{aligned}\tilde{H}_{i+\phi}^n &= \frac{(\Delta\chi_m H_{i2+\phi}^n + \Delta\chi_{m+1} H_{i+\phi}^n)}{\Delta\chi_m + \Delta\chi_{m+1}}, \\ \tilde{H}_{i1+\phi}^n &= \frac{(\Delta\chi_{m-1} H_{i+\phi}^n + \Delta\chi_m H_{i1+\phi}^n)}{\Delta\chi_{m-1} + \Delta\chi_m},\end{aligned}$$

where $\phi = -1, 0$ or 1 . Also

$$\begin{aligned}\vec{H}_e^n &= \frac{(\Delta\lambda_j H_{e+1}^n + \Delta\lambda_{j+1} H_e^n)}{\Delta\lambda_j + \Delta\lambda_{j+1}}, \\ \vec{H}_{e-1}^n &= \frac{(\Delta\lambda_{j-1} H_e^n + \Delta\lambda_j H_{e-1}^n)}{\Delta\lambda_{j-1} + \Delta\lambda_j},\end{aligned}$$

where $e = i1, i$ or $i2$ and similarly

$$\begin{aligned}\tilde{f}_i &= \frac{(\Delta\chi_m f_{i2} + \Delta\chi_{m+1} f_{i+})}{\Delta\chi_m + \Delta\chi_{m+1}}, \\ \vec{f}_i &= \frac{(\Delta\lambda_j f_{i+1} + \Delta\lambda_{j+1} f_i)}{\Delta\lambda_j + \Delta\lambda_{j+1}},\end{aligned}$$

where f_i is the value of the Coriolis parameter for the i^{th} element.

The variables $\tilde{\zeta}_i^n$, $\tilde{\omega}_{i,k}^n$ and $\tilde{v}_{i,k}^n$ are defined in the same way as $\tilde{H}_i^n$, and $\vec{\zeta}_i^n$, $\vec{\omega}_{i,k}^n$ and $\vec{\mu}_{i,k}^n$ are defined in the same way as $\vec{H}_i^n$. The variables $\vec{\mu}$ and $\vec{v}$ are the horizontally averaged values of the depth integrated velocities $\bar{\mu}$ and $\bar{v}$. Also $\vec{H}_i^n$ indicates the averaging of the H in both the χ and λ directions.

The numerical values of the widths of the body of water, which are stored as B_m , are measured along the lines through the μ velocity points and therefore the variable β is also measured along those lines and is defined as

$$\beta_i = b_m + \lambda_i B_m,$$

where

$$\lambda_i = \sum_{k=1}^j \Delta\lambda_k.$$

The following symbols are now defined:

$$\begin{aligned}\hat{B}_m &= \frac{(B_{m-1} + B_m)}{2}, \\ \beta_i^+ &= \frac{(\beta_{i-1} + \beta_i)}{2} = b_m + (\lambda_i + \frac{1}{2}\Delta\lambda_j)B_m, \\ \hat{\beta}_i^+ &= \hat{b}_m + (\lambda_i + \frac{1}{2}\Delta\lambda_j)\hat{B}_m,\end{aligned}$$

where the caret (^) above a variable indicates that the variable has been averaged at the centre of the element in the χ direction and the superscript of an addition sign (+) indicates that the variable has been averaged at the centre of the element in the λ direction.

The following finite difference equations are listed for the case where water completely surrounds the central element in which the new values of the variables are being calculated. When there are land boundaries instead of water adjacent to this element, careful consideration needs to be given to the subsequent effect on each finite difference equation. This is explained in Section 4.4.

The differencing used in the following sections is based on information obtained using the wave propagation analysis in Chapter 6. Consideration of the restrictions placed on the numerical scheme discussed at the start of Chapter 3 is also taken into account so the scheme is implicit in the vertical direction but not in either of the two horizontal directions, where the finite difference approximation must be based on three point formulae.

4.1 The Transformed Depth Integrated Continuity Equation

Equation (2.33) evaluated at the n^{th} time level and at the ζ point in the i^{th} element in the body of water is

$$\left. \frac{\partial \zeta}{\partial t} \right|_i^n + \frac{1}{B} \left. \frac{\partial (BH\bar{\mu})}{\partial \chi} \right|_i^n + \left. \frac{\partial (H\bar{\nu})}{\partial \lambda} \right|_i^n = 0 , \quad (4.1)$$

where

$$\bar{\mu} = \int_0^1 \mu d\eta \quad \text{and} \quad \bar{\nu} = \int_0^1 \nu d\eta .$$

The discretised form for this, using forward differencing in time and central differencing in space, becomes after rearrangement

$$\begin{aligned} \zeta_i^* \approx & \zeta_i^n - \frac{\Delta t}{\hat{B}_m \Delta \chi_m} \left(B_m \tilde{H}_i^n \bar{\mu}_i^n - B_{m-1} \tilde{H}_{i1}^n \bar{\mu}_{i1}^n \right) \\ & - \frac{\Delta t}{\Delta \lambda_j} \left(\tilde{H}_i^n \bar{\nu}_i^n - \tilde{H}_{i-1}^n \bar{\nu}_{i-1}^n \right) , \end{aligned} \quad (4.2)$$

where ν is calculated using Equation (2.48) and $\bar{\nu}$ is the depth averaged value of ν . The finite difference approximation for $\bar{\nu}$ obtained using Equation (2.48) is

$$\bar{\nu}_i^n = \frac{1}{\hat{B}_m} \left\{ \bar{\nu}_i^n - \frac{1}{2}(\bar{\nu}_i^n + \bar{\nu}_{i1}^n) \left(\frac{\beta_i - \beta_{i1}}{\Delta\chi_m} \right) \right\} , \quad (4.3)$$

where the values for $\bar{\mu}$ and $\bar{\nu}$ are obtained by using the following rule for approximating an integral using non-uniformly distributed discrete values,

$$\bar{\mu}_i^n = \sum_{k=0}^K s_k \mu_{i,k}^n , \quad \bar{\nu}_i^n = \sum_{k=0}^K s_k \nu_{i,k}^n , \quad (4.4)$$

where K must be an even number and the integration coefficients s_k for a variable grid, are defined by

$$\begin{aligned} s_0 &= \Delta\eta_0(r_1 + 1)(2 - r_1)/6 , \\ s_k &= \begin{cases} \Delta\eta_{k-1}(r_k + 1)^3/6r_k & k \text{ odd} , \\ \Delta\eta_k(r_{k+1} + 1)(2 - r_{k+1})/6 \\ \quad + \Delta\eta_{k-2}(r_{k-1} + 1)(2r_{k-1} - 1)/6r_{k-1} & k \text{ even} , \end{cases} \\ s_K &= \Delta\eta_{K-2}(r_{K-1} + 1)(2r_{K-1} - 1)/6r_{K-1} , \end{aligned}$$

and $r_k = \Delta\eta_k/\Delta\eta_{k-1}$. This integration procedure is second order accurate and on a uniform grid reduces to Simpson's Method.

Equation (4.2) is used to obtain a value for ζ^* , which is the first approximation for ζ at the time level of $n + 1$. This value is then used to calculate μ , ν and ω at the new time level (see Section 3.3). Once the new values for these three velocity components have been found a more accurate value for ζ at the new time level can be obtained

using

$$\begin{aligned}
\zeta_i^{n+1} \approx & \zeta_i^n - \frac{\Delta t}{2\hat{B}_m\Delta\chi_m} (B_m\tilde{H}_i^{n+1}\bar{\mu}_i^{n+1} - B_{m-1}\tilde{H}_{i1}^{n+1}\bar{\mu}_{i1}^{n+1}) \\
& - \frac{\Delta t}{2\Delta\lambda_j} (\vec{H}_i^{n+1}\vec{v}_i^{n+1} - \vec{H}_{i-1}^{n+1}\vec{v}_{i-1}^{n+1}) \\
& - \frac{\Delta t}{2\hat{B}_m\Delta\chi_m} (B_m\tilde{H}_i^n\bar{\mu}_i^n - B_{m-1}\tilde{H}_{i1}^n\bar{\mu}_{i1}^n) \\
& - \frac{\Delta t}{2\Delta\lambda_j} (\vec{H}_i^n\vec{v}_i^n - \vec{H}_{i-1}^n\vec{v}_{i-1}^n) .
\end{aligned} \tag{4.5}$$

This equation is centrally differenced in time about the $(n+\frac{1}{2})^{th}$ time level and is therefore second order accurate in time as well as the space coordinates, whereas Equation (4.2) is only first order accurate in time. The velocity variables in the computer program are only stored for one time level and once they have been calculated at the new time level the values at the old time level have been overwritten, but by making use of the difference between the values of ζ_i^* and ζ_i^n , Equation (4.5) is equivalent to

$$\begin{aligned}
\zeta_i^{n+1} \approx & \zeta_i^n - \frac{\Delta t}{2\hat{B}_m\Delta\chi_m} (B_m\tilde{H}_i^{n+1}\bar{\mu}_i^{n+1} - B_{m-1}\tilde{H}_{i1}^{n+1}\bar{\mu}_{i1}^{n+1}) \\
& - \frac{\Delta t}{2\Delta\lambda_j} (\vec{H}_i^{n+1}\vec{v}_i^{n+1} - \vec{H}_{i-1}^{n+1}\vec{v}_{i-1}^{n+1}) + \frac{\zeta_i^* - \zeta_i^n}{2} .
\end{aligned} \tag{4.6}$$

Since $\zeta_i = H_i - h_i$ and h_i is time independent, h_i can be added to both sides of Equation (4.6) by replacing ζ_i^{n+1} by H_i^{n+1} and ζ_i^n by H_i^n . For evaluation it is necessary to replace $\tilde{H}$ and $\vec{H}$ by the following formulae,

$$\begin{aligned}
\tilde{H}_i^{n+1} &= \frac{(\Delta\chi_m H_{i2}^{n+1} + \Delta\chi_{m+1} H_i^{n+1})}{\Delta\chi_m + \Delta\chi_{m+1}} = \alpha_1 H_{i2}^{n+1} + \beta_1 H_i^{n+1} , \\
\tilde{H}_{i1}^{n+1} &= \frac{(\Delta\chi_{m-1} H_i^{n+1} + \Delta\chi_m H_{i1}^{n+1})}{\Delta\chi_{m-1} + \Delta\chi_m} = \alpha_2 H_i^{n+1} + \beta_2 H_{i1}^{n+1} , \\
\vec{H}_i^{n+1} &= \frac{(\Delta\lambda_j H_{i+1}^{n+1} + \Delta\lambda_{j+1} H_i^{n+1})}{\Delta\lambda_j + \Delta\lambda_{j+1}} = \alpha_3 H_{i+1}^{n+1} + \beta_3 H_i^{n+1} , \\
\vec{H}_{i-1}^{n+1} &= \frac{(\Delta\lambda_{j-1} H_i^{n+1} + \Delta\lambda_j H_{i-1}^{n+1})}{\Delta\lambda_{j-1} + \Delta\lambda_j} = \alpha_4 H_i^{n+1} + \beta_4 H_{i-1}^{n+1} ,
\end{aligned}$$

which give rise to

$$\begin{aligned}
H_i^{n+1} \approx & H_i^n - \frac{\Delta t}{2\hat{B}_m\Delta\chi_m} \left(B_m\bar{\mu}_i^{n+1}(\alpha_1 H_{i2}^{n+1} + \beta_1 H_i^{n+1}) - B_{m-1}\bar{\mu}_{i1}^{n+1}(\alpha_2 H_i^{n+1} + \beta_2 H_{i1}^{n+1}) \right) \\
& - \frac{\Delta t}{2\Delta\lambda_j} \left(\bar{\nu}_i^{n+1}(\alpha_3 H_{i+1}^{n+1} + \beta_3 H_i^{n+1}) - \bar{\nu}_{i-1}^{n+1}(\alpha_4 H_i^{n+1} + \beta_4 H_{i-1}^{n+1}) \right) \\
& + \frac{\zeta_i^* - \zeta_i^n}{2}.
\end{aligned} \tag{4.7}$$

This leads to the following explicit approximation for H_i^{n+1} ,

$$H_i^{n+1} \approx \frac{\left(H_i^n - \frac{\Delta t}{2\hat{B}_m\Delta\chi_m} \left(B_m\bar{\mu}_i^{n+1}\alpha_1 H_{i2}^{n+1} - B_{m-1}\bar{\mu}_{i1}^{n+1}\beta_2 H_{i1}^{n+1} \right) - \frac{\Delta t}{2\Delta\lambda_j} \left(\bar{\nu}_i^{n+1}\alpha_3 H_{i+1}^{n+1} - \bar{\nu}_{i-1}^{n+1}\beta_4 H_{i-1}^{n+1} \right) + (\zeta_i^* - \zeta_i^n)/2 \right)}{1 + \frac{\Delta t}{2\hat{B}_m\Delta\chi_m} \left(B_m\bar{\mu}_i^{n+1}\beta_1 - B_{m-1}\bar{\mu}_{i1}^{n+1}\alpha_2 \right) - \frac{\Delta t}{2\Delta\lambda_j} \left(\bar{\nu}_i^{n+1}\beta_3 - \bar{\nu}_{i-1}^{n+1}\alpha_4 \right)}, \tag{4.8}$$

where due to the order of evaluation H_{i2}^{n+1} and H_{i+1}^{n+1} will be evaluated from $\zeta_{i2}^* + h_{i2}$ and $\zeta_{i+1}^* + h_{i+1}$ respectively. The other two values of H_{i-1}^{n+1} and H_{i1}^{n+1} have both been calculated at the stage Equation (4.8) is used to calculate H_i^{n+1} . This is more accurate and gives greater stability than using Equation (4.6) and evaluating all the H^{n+1} values with the predictor values of ζ^* .

4.2 The Transformed x -Momentum Equation

Equation (2.46) evaluated at the $(n+\frac{1}{2})^{th}$ time level and at the μ point of the i^{th} element in the model, for the variables at the k^{th} depth level below the surface, is

$$\begin{aligned}
& \frac{\partial(H\mu)}{\partial t} \Big|_{i,k}^{n+\frac{1}{2}} + \frac{1}{B} \frac{\partial(BH\mu^2)}{\partial\chi} \Big|_{i,k}^{n+\frac{1}{2}} + \frac{\partial(H\mu\nu)}{\partial\lambda} \Big|_{i,k}^{n+\frac{1}{2}} + H \frac{\partial(\mu\omega)}{\partial\eta} \Big|_{i,k}^{n+\frac{1}{2}} \\
& = - \frac{gH}{L^2} \left(\frac{\partial\zeta}{\partial\chi} - \frac{1}{B} \frac{\partial\zeta}{\partial\lambda} \frac{\partial\beta}{\partial\chi} \right) \Big|_{i,k}^{n+\frac{1}{2}} + \frac{HN_\chi}{L^2} \frac{\partial^2\mu}{\partial\chi^2} \Big|_{i,k}^{n+\frac{1}{2}} \\
& \quad + \frac{HN_\lambda}{B^2} \frac{\partial^2\mu}{\partial\lambda^2} \Big|_{i,k}^{n+\frac{1}{2}} + \frac{1}{H} \frac{\partial}{\partial\eta} \left(N_\eta \frac{\partial\mu}{\partial\eta} \right) \Big|_{i,k}^{n+\frac{1}{2}} + \frac{Hfv}{L} \Big|_{i,k}^{n+\frac{1}{2}}.
\end{aligned} \tag{4.9}$$

In the following approximations, the values of ζ^* , which in the order of calculations have just been calculated using Equation (4.2), are used for the values of ζ^{n+1} , which in

turn are used in calculating H^{n+1} . The second order accurate (in space) finite difference approximations on a variably spaced grid for each term of Equation (4.9) are listed below.

(a)

$$\left. \frac{\partial(H\mu)}{\partial t} \right|_{i,k}^{n+\frac{1}{2}} \approx \left(\frac{\tilde{H}_i^{n+1} \mu_{i,k}^{n+1} - \tilde{H}_i^n \mu_{i,k}^n}{\Delta t} \right) = HU1 \cdot \mu_{i,k}^{n+1} + HU2 , \quad (4.10)$$

where

$$HU1 = \frac{\tilde{H}_i^{n+1}}{\Delta t} \quad \text{and} \quad HU2 = -\frac{\tilde{H}_i^n \mu_{i,k}^n}{\Delta t} .$$

(b)

$$\begin{aligned} \left. \frac{1}{B} \frac{\partial(BH\mu^2)}{\partial \chi} \right|_{i,k}^{n+\frac{1}{2}} &= \mu^2 \left. \frac{\partial H}{\partial \chi} \right|_{i,k}^{n+\frac{1}{2}} + \frac{H}{B} \left. \frac{\partial(B\mu^2)}{\partial \chi} \right|_{i,k}^{n+\frac{1}{2}} \\ &\approx (\mu_{i,k}^n)^2 (HDX^n + HDX^{n+1})/2 \\ &\quad + \frac{\tilde{H}_i^n + \tilde{H}_i^{n+1}}{2B_m \Delta \chi_{m+1} (1 + r_m)} (B_{m+1} (\mu_{i2,k}^n)^2 \\ &\quad + B_m (r_m^2 - 1) (\mu_{i,k}^n)^2 - B_{m-1} r_m^2 (\mu_{i1,k}^{n+1})^2) \\ &= BHUU , \end{aligned} \quad (4.11)$$

where

$$\begin{aligned} HDX^n &= \frac{(\Delta \chi_m + a) H_{i2}^n + c \cdot H_i^n + r_x (\Delta \chi_m - a \cdot r_x) H_{i1}^n}{a^2 r_x (r_x + 1)} , \\ a &= \frac{1}{2} (\Delta \chi_{m-1} + \Delta \chi_m) , \\ c &= (r_x + 1) (a(r_x - 1) - \Delta \chi_m) , \\ r_x &= \frac{\Delta \chi_m + \Delta \chi_{m+1}}{\Delta \chi_{m-1} + \Delta \chi_m} , \\ r_m &= \frac{\Delta \chi_{m+1}}{\Delta \chi_m} . \end{aligned}$$

(c)

$$\begin{aligned}
\left. \frac{\partial(H\mu\nu)}{\partial\lambda} \right|_{i,k}^{n+\frac{1}{2}} &\approx \frac{\vec{H}_i^n + \vec{H}_i^{n+1}}{2B_m\Delta\lambda_j} \left(\vec{\mu}_{i,k}^n \tilde{v}_{i,k}^n - (\vec{\mu}_{i,k}^n)^2 \left[\frac{\hat{\beta}_{i2} - \hat{\beta}_i}{\frac{1}{2}(\Delta\chi_m + \Delta\chi_{m+1})} \right] \right) \\
&\quad - \frac{\vec{H}_{i-1}^n + \vec{H}_{i-1}^{n+1}}{2B_m\Delta\lambda_j} \left(\vec{\mu}_{i-1,k}^n \tilde{v}_{i-1,k}^n - (\vec{\mu}_{i-1,k}^n)^2 \left[\frac{\hat{\beta}_{i2-1} - \hat{\beta}_{i-1}}{\frac{1}{2}(\Delta\chi_m + \Delta\chi_{m+1})} \right] \right) \\
&= HUV ,
\end{aligned} \tag{4.12}$$

where the evaluation of $\vec{\mu}_{i-1,k}^n$, $\tilde{v}_{i,k}^n$ and $\tilde{v}_{i-1,k}^n$ is split between the n^{th} and $(n+1)^{th}$ time level.

(d)

$$H \left. \frac{\partial(\mu\omega)}{\partial\eta} \right|_{i,k}^{n+\frac{1}{2}} \approx HUW1.\mu_{i,k-1}^{n+1} + HUW2.\mu_{i,k}^{n+1} + HUW3.\mu_{i,k+1}^{n+1} + HUW4 , \tag{4.13}$$

where

$$\begin{aligned}
HUW1 &= - \left(\frac{\tilde{H}_i^n + \tilde{H}_i^{n+1}}{4\Delta\eta_k(r_k + 1)} \right) r_k^2 \tilde{\omega}_{i,k-1}^n , \\
HUW2 &= \left(\frac{\tilde{H}_i^n + \tilde{H}_i^{n+1}}{4\Delta\eta_k(r_k + 1)} \right) (r_k^2 - 1) \tilde{\omega}_{i,k}^n , \\
HUW3 &= \left(\frac{\tilde{H}_i^n + \tilde{H}_i^{n+1}}{4\Delta\eta_k(r_k + 1)} \right) \tilde{\omega}_{i,k+1}^n , \\
HUW4 &= (HUW1.\mu_{i,k-1}^n + HUW2.\mu_{i,k}^n + HUW3.\mu_{i,k+1}^n) , \\
r_k &= \frac{\Delta\eta_k}{\Delta\eta_{k-1}} .
\end{aligned}$$

(e)

$$- \frac{gH}{L^2} \left. \frac{\partial\zeta}{\partial\chi} \right|_{i,k}^{n+\frac{1}{2}} \approx - \frac{g}{2L^2} (\tilde{H}_i^n.ZDX^n + \tilde{H}_i^{n+1}.ZDX^{n+1}) = DZDX , \tag{4.14}$$

where

$$ZDX^n = \frac{(\Delta\chi_m + a)\zeta_{i2}^n + c.\zeta_i^n + r_x(\Delta\chi_m - a.r_x)\zeta_{i1}^n}{a^2r_x(r_x + 1)} .$$

(f)

$$\begin{aligned}
\frac{gH}{L^2 B} \frac{\partial \zeta}{\partial \lambda} \frac{\partial \beta}{\partial \chi} \Big|_{i,k}^{n+\frac{1}{2}} &\approx \frac{g}{2L^2 B_m} \left(\frac{\hat{\beta}_{i2}^+ - \hat{\beta}_i^+}{\frac{1}{2}(\Delta\chi_m + \Delta\chi_{m+1})} \right) (\tilde{H}_i^n . ZDY^n + \tilde{H}_i^{n+1} . ZDY^{n+1}) \\
&= DZDY ,
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
ZDY^n &= \frac{\tilde{\zeta}_{i2}^n + (r_y^2 - 1)\tilde{\zeta}_i^n - r_y^2 \tilde{\zeta}_{i1}^n}{\frac{1}{2}(\Delta\lambda_j - \Delta\lambda_{j+1})(r_y + 1)} , \\
r_y &= \frac{\Delta\lambda_j + \Delta\lambda_{j+1}}{\Delta\lambda_{j-1} + \Delta\lambda_j} .
\end{aligned}$$

(g)

$$\begin{aligned}
\frac{HN_\chi}{L^2} \frac{\partial^2 \mu}{\partial \chi^2} \Big|_{i,k}^{n+\frac{1}{2}} &\approx \frac{N_\chi(\tilde{H}_i^n + \tilde{H}_i^{n+1})}{r_m(r_m + 1)L^2 \Delta\chi_m^2} (\mu_{i2,k}^n - (r_m + 1)\mu_{i,k}^n + r_m \mu_{i1,k}^{n+1}) \\
&= HNX .
\end{aligned} \tag{4.16}$$

(h)

$$\begin{aligned}
\frac{HN_\lambda}{B^2} \frac{\partial^2 \mu}{\partial \lambda^2} \Big|_{i,k}^{n+\frac{1}{2}} &\approx \frac{4N_\lambda(\tilde{H}_i^n + \tilde{H}_i^{n+1}) (\mu_{i+1,k}^n - (r_y + 1)\mu_{i,k}^n + r_y \mu_{i-1,k}^{n+1})}{r_y(r_y + 1)B_m^2(\Delta\lambda_{j-1} + \Delta\lambda_j)^2} \\
&= HNY .
\end{aligned} \tag{4.17}$$

(i)

$$\frac{1}{H} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial \mu}{\partial \eta} \right) \Big|_{i,k}^{n+\frac{1}{2}} \approx HNN1 . \mu_{i,k-1}^{n+1} + HNN2 . \mu_{i,k}^{n+1} + HNN3 . \mu_{i,k+1}^{n+1} + HNN4 , \tag{4.18}$$

where

$$\begin{aligned}
HNN1 &= \frac{r_k^2}{DENOM} ((2 - r_k)\tilde{N}_{i,k}^n + r_k \tilde{N}_{i,k-1}^n) , \\
HNN2 &= \frac{-1}{DENOM} (\tilde{N}_{i,k+1}^n - (r_k + 1)(r_k^2 - 3r_k + 1)\tilde{N}_{i,k}^n + r_k^3 \tilde{N}_{i,k-1}^n) , \\
HNN3 &= \frac{1}{DENOM} (\tilde{N}_{i,k+1}^n - (2r_k - 1)\tilde{N}_{i,k}^n) ,
\end{aligned}$$

$$HNN4 = (HNN1.\mu_{i,k-1}^n + HNN2.\mu_{i,k}^n + HNN3.\mu_{i,k+1}^n) ,$$

$$DENOM = (\Delta\eta_k)^2(r_k + 1)(\tilde{H}_i^{n+1} + \tilde{H}_i^n) ,$$

and the vertical eddy viscosity parameter $N(\eta) = N_\eta$ can be defined in any desired form. In the example given by Equations (2.39) and (2.40), this would be evaluated by

$$\tilde{N}_{i,k}^n = a_1 + a_2 \tilde{H}_i^n F(\eta_k) \sqrt{(L\bar{\mu}_i^n) + \left(\frac{\tilde{v}_i^n + \tilde{v}_{i-1}^n}{2}\right)} ,$$

$$\text{and } F(\eta_k) = \left(\frac{1}{2} + \eta_k\right)(1 - \eta_k) .$$

(j)

$$\begin{aligned} \left. \frac{Hfv}{L} \right|_{i,k}^{n+\frac{1}{2}} &\approx \frac{(\tilde{H}_i^{n+1} + \tilde{H}_i^n)\tilde{f}_i^n}{4L} \left(\frac{\Delta\chi_m(v_{i2,k}^n + v_{i2-1,k}^n) + \Delta\chi_{m+1}(v_{i,k}^{n+1} + v_{i-1,k}^{n+1})}{\Delta\chi_m + \Delta\chi_{m+1}} \right) \\ &= HFV . \end{aligned} \tag{4.19}$$

Substitution of Equations (4.10) to (4.19) into Equation (4.9), yields a set of equations for any particular element of the body of water in the form

$$P_{i,k}^{n+1}\mu_{i,k-1}^{n+1} + Q_{i,k}^{n+1}\mu_{i,k}^{n+1} + R_{i,k}^{n+1}\mu_{i,k+1}^{n+1} = S_{i,k}^n , \tag{4.20}$$

where

$$P_{i,k}^{n+1} = HUW1 - HNN1 ,$$

$$Q_{i,k}^{n+1} = HU1 + HUW2 - HNN2 ,$$

$$R_{i,k}^{n+1} = HUW3 - HNN3 ,$$

$$\begin{aligned} S_{i,k}^n &= -HU2 - BHUU - HUV - HUW4 + DZDX + DZDY \\ &\quad + HNX + HNY + HNN4 + HFV . \end{aligned}$$

This set of equations holds for $k = 1, 2, \dots, K - 1$, where $K + 1$ is the number of grid points along the η -axis ($k = K$ at the sea floor), so the equations for the solution of μ at the next time level are implicit in the η -direction.

At the sea surface the second order accurate finite difference approximation for an irregular grid is

$$\left. \frac{\partial \mu}{\partial \eta} \right|_{\eta=0} = a_0 \mu_{i,0} + b_0 \mu_{i,1} + c_0 \mu_{i,2} = D_{i,0} , \quad (4.21)$$

where

$$a_0 = \frac{-r_1 - 2}{\Delta \eta_0 (r_1 + 1)} , \quad b_0 = \frac{r_1 + 1}{\Delta \eta_0 r_1} \quad \text{and} \quad c_0 = \frac{-1}{\Delta \eta_0 r_1 (r_1 + 1)} . \quad (4.22)$$

The surface boundary condition of Equation (2.55) is used to obtain a value for $D_{i,0}$, giving

$$D_{i,0} = \frac{-\tilde{H}_i \tau_{s\chi}}{\rho L N_\eta(0)} .$$

Rearranging Equation (4.21) in terms of $\mu_{i,2}$ at the $(n+1)^{th}$ time level and substituting this into Equation (4.20) with $k = 1$ gives

$$(P_{i,1}^{n+1} - \frac{a_0 R_{i,1}^{n+1}}{c_0}) \mu_{i,0}^{n+1} + (Q_{i,1}^{n+1} - \frac{b_0 R_{i,1}^{n+1}}{c_0}) \mu_{i,1}^{n+1} = S_{i,1}^n - \frac{D_{i,0}^{n+1} R_{i,1}^{n+1}}{c_0} , \quad (4.23)$$

which is in the same form as Equation (4.20) if P , Q , R and S are defined by

$$\begin{aligned} P_{i,0}^{n+1} &= 0 , \\ Q_{i,0}^{n+1} &= P_{i,1}^{n+1} - r_1(r_1 + 2)R_{i,1}^{n+1} , \\ R_{i,0}^{n+1} &= Q_{i,1}^{n+1} + (r_1 + 1)^2 R_{i,1}^{n+1} , \\ S_{i,0}^n &= S_{i,1}^n + \Delta \eta_0 r_1 (r_1 + 1) D_{i,0}^{n+1} R_{i,1}^{n+1} . \end{aligned}$$

If a “no-slip” condition applies on the sea floor, then Equation (4.20) with $k = K - 1$, simplifies to

$$P_{i,K-1}^{n+1} \mu_{i,K-2}^{n+1} + Q_{i,K-1}^{n+1} \mu_{i,K-1}^{n+1} = S_{i,K-1}^n . \quad (4.24)$$

Thus Equations (4.20) for $k = 1, 2, \dots, K - 2$, Equation (4.23) and Equation (4.24) form a tri-diagonal set of equations (K equations and K unknowns as $\mu_{i,K} = 0$) which can be implicitly solved to give all the μ velocity values over depth for the grid element indexed by i , at the new time level.

If a “slip” condition is required at the sea floor then the second order accurate finite difference approximation for an irregular grid is

$$\left. \frac{\partial \mu}{\partial \eta} \right|_{\eta=1} = a_K \mu_{i,K} + b_K \mu_{i,K-1} + c_K \mu_{i,K-2} = D_{i,K} , \quad (4.25)$$

where

$$a_K = \frac{-2r_{K-1} - 1}{\Delta \eta_{K-1}(r_{K-1} + 1)} , \quad b_K = \frac{r_{K-1} + 1}{\Delta \eta_{K-1}} \quad \text{and} \quad c_K = -\frac{r_{K-1}^2}{\Delta \eta_{K-1}(r_{K-1} + 1)} , \quad (4.26)$$

$D_{i,K}$ is evaluated using Equation (2.52), giving

$$D_{i,K} = \frac{-\tilde{H}_i \tau_{bx}}{\rho L N_\eta(1)} ,$$

and where $\eta = 1$ (or $k = K$) represents the boundary between the turbulent and laminar flows, some small distance above the physical sea floor.

Rearranging Equation (4.25) in terms of $\mu_{i,K-2}$ and substituting this into Equation (4.20) with $k = K - 1$ gives

$$(Q_{i,K-1}^{n+1} - \frac{b_K P_{i,K-1}^{n+1}}{c_K}) \mu_{i,K-1}^{n+1} + (R_{i,K-1}^{n+1} - \frac{a_K P_{i,K-1}^{n+1}}{c_K}) \mu_{i,K}^{n+1} = S_{i,K-1}^n - \frac{D_{i,K}^{n+1} P_{i,K-1}^{n+1}}{c_K} , \quad (4.27)$$

which is in the same form as Equation (4.20) if P , Q , R and S are defined by

$$\begin{aligned} P_{i,K}^{n+1} &= Q_{i,K-1}^{n+1} + \frac{(r_{K-1} + 1)^2}{r_{K-1}^2} P_{i,K-1}^{n+1} , \\ Q_{i,K}^{n+1} &= R_{i,K-1}^{n+1} - \left(\frac{2r_{K-1} + 1}{r_{K-1}^2} \right) P_{i,K-1}^{n+1} , \\ R_{i,K}^{n+1} &= 0 , \\ S_{i,K}^n &= S_{i,K-1}^n + \left(\frac{\Delta \eta_{K-1}(r_{K-1} + 1)}{r_{K-1}^2} \right) D_{i,K}^{n+1} P_{i,K-1}^{n+1} . \end{aligned}$$

Thus with a “slip” condition Equations (4.20) for $k = 1, 2, \dots, K-1$, Equation (4.23) and Equation (4.27) form the set of equations to be solved for the values of μ at the new time level and at each depth level, for the grid element indexed by i . This is one more unknown, $\mu_{i,K}$ and one more equation in the tri-diagonal set of equations than for the “no-slip” condition and also introduces the problem of representing the bottom shear stress τ_b correctly.

The tri-diagonal set of equations, with either the “slip” or the “no-slip” condition on the sea floor, can easily and efficiently be solved using the algorithm developed by Thomas (1949).

The y -momentum Equation (2.47), which is evaluated at the $(n+\frac{1}{2})^{th}$ time level and at the ν point of the i^{th} element in the model, is differenced and solved in a similar way to the x -momentum equation (see Appendix A).

Note that the computational procedure requires the v variables to be evaluated using the y -momentum equation before the μ variables are calculated using the above difference formula. This is so that the Coriolis terms in each of the momentum equations will be balanced – each with two values at the new and at the old time levels.

4.3 The Transformed Continuity Equation used to Compute ω

At the stage when the ω variables need to be calculated at the new time level, all variables except ω are known at this level. Therefore Equation (2.45) is evaluated at the new time level, at the ζ point of the i^{th} element in the body of water and at the k^{th} depth level, giving

$$\omega_{i,k}^{n+1} = \frac{1}{H} \left\{ \frac{1}{B} \frac{\partial}{\partial \chi} \left(BH(\eta\bar{\mu} - \int_0^\eta \mu d\eta) \right) + \frac{\partial}{\partial \lambda} \left(H(\eta\bar{\nu} - \int_0^\eta \nu d\eta) \right) \right\} \Bigg|_{i,k}^{n+1}. \quad (4.28)$$

The finite difference form of Equation (4.28), obtained using centred spatial difference approximations, is

$$\begin{aligned} \omega_{i,k}^{n+1} = & \frac{1}{H_i^{n+1}} \left\{ \frac{1}{\tilde{B}_m \Delta \chi_m} \left[B_m \tilde{H}_i^{n+1} (\eta_k \bar{\mu}_i^{n+1} - \int_0^{\eta_k} \mu_i^{n+1} d\eta) \right. \right. \\ & \left. \left. - B_{m-1} \tilde{H}_{i1}^{n+1} (\eta_k \bar{\mu}_{i1}^{n+1} - \int_0^{\eta_k} \mu_{i1}^{n+1} d\eta) \right] \right. \\ & + \frac{1}{\Delta \lambda_j} \left[\tilde{H}_i^{n+1} (\eta_k \bar{\nu}_i^{n+1} - \int_0^{\eta_k} \nu_i^{n+1} d\eta) \right. \\ & \left. \left. - \tilde{H}_{i-1}^{n+1} (\eta_k \bar{\nu}_{i-1}^{n+1} - \int_0^{\eta_k} \nu_{i-1}^{n+1} d\eta) \right] \right\} , \quad (4.29) \end{aligned}$$

which is evaluated for $k = 0, 1, \dots, K$ for each “active” element in the body of water being modelled. If a “no-slip” condition exists on the sea floor then this equation does not need to be evaluated for $k = K$, as the result is zero. The integrations from the sea surface to a depth η_k are evaluated using the same formula as the integrations in $\bar{\mu}$ and $\bar{\nu}$, when k is an even number (see Section 4.1). A simple method to use when k is odd, is to use the trapezoidal approximation for the integral between η_{k-1} and η_k and add this to the integral from η_0 to η_{k-1} , which because $k - 1$ is an even number, is calculated with the usual approximation. However, the trapezoidal approximation is not as accurate as the rest of the approximation and it is better to use the second order accurate formula given by

$$\int_{\eta_{k-1}}^{\eta_k} \mu_k d\eta = \frac{\Delta \eta_{k-1}}{6r_k(1+r_k)} \left\{ -\mu_{k+1} + (1+3r_k)(1+r_k)\mu_k + (2r_k+3r_k^2)\mu_{k-1} \right\} , \quad (4.30)$$

which is used with k odd.

Equation (4.29) calculates the new values of ω directly from the new level values of all the other variables, without the need for a time derivative approximation.

4.4 The Horizontal Boundary Conditions

The finite difference equations listed in Sections 4.1 to 4.3 are those used for calculating ζ , μ , ν and ω for an element i which is completely surrounded by water (i.e.

with identification number of 0). When one or more of the surrounding elements are land elements, some of the finite difference expressions for the terms in Equations (4.1), (4.9) and (4.28) have to be modified or omitted depending on which of the surrounding elements are land. The third term of Equation (4.9) will be used to illustrate an example of how this works. Equation (4.9) is used to calculate $\mu_{i,k}^{n+1}$, for all k . The finite difference form for the third term is

$$\begin{aligned} \frac{\partial(H\mu\nu)}{\partial\lambda}\bigg|_{i,k}^{n+\frac{1}{2}} &\approx \frac{\vec{H}_i^n}{B_m\Delta\lambda_j} \left(\vec{\mu}_{i,k}^n \tilde{\nu}_{i,k}^n - (\vec{\mu}_{i,k}^n)^2 \left[\frac{\hat{\beta}_{i2} - \hat{\beta}_i}{\frac{1}{2}(\Delta\chi_m + \Delta\chi_{m+1})} \right] \right) \\ &\quad - \frac{\vec{H}_{i-1}^n}{B_m\Delta\lambda_j} \left(\vec{\mu}_{i-1,k}^n \tilde{\nu}_{i-1,k}^n - (\vec{\mu}_{i-1,k}^n)^2 \left[\frac{\hat{\beta}_{i2-1} - \hat{\beta}_{i-1}}{\frac{1}{2}(\Delta\chi_m + \Delta\chi_{m+1})} \right] \right) . \end{aligned} \quad (4.31)$$

In this example the four elements represented by $i \pm 1$ and $i2 \pm 1$ (see Figures 3.2 and 3.5) are of importance, because information from these positions as well as the element $i2$ is needed in the calculations, as given in Equation (4.31), for this term. The element represented by $i2$ will always represent water, otherwise from the boundary condition of Equation (2.49), $\mu_{i,k}^{n+1} = 0$, for all k , and evaluation of Equation (4.9) is not necessary. For this term it is therefore not important if any of the elements represented by $i1$ or $i1 \pm 1$ are land or sea elements.

If one or more of the elements represented by $i \pm 1$ and $i2 \pm 1$ are land elements then the finite difference approximation given in Equation (4.31) has to be modified. Consider the cases in Figure 4.1 where the elements $i + 1$ and $i2 + 1$ are land elements. When this occurs both $\nu_{i,k}$ and $\nu_{i2,k}$ are zero due to the boundary condition in Equation (2.50) and therefore $\tilde{\nu}_{i,k}$ is zero. This means that $\tilde{\nu}_{i,k} - \vec{\mu}_{i,k}(\partial\beta/\partial\chi)$ is zero and Equation (4.31) reduces to

$$\frac{\partial(H\mu\nu)}{\partial\lambda}\bigg|_{i,k}^{n+\frac{1}{2}} \approx -\frac{\vec{H}_{i-1}^n}{B_m\Delta\lambda_j} \left(\vec{\mu}_{i-1,k}^n \tilde{\nu}_{i-1,k}^n - (\vec{\mu}_{i-1,k}^n)^2 \left[\frac{\hat{\beta}_{i2-1} - \hat{\beta}_{i-1}}{\frac{1}{2}(\Delta\chi_m + \Delta\chi_{m+1})} \right] \right) . \quad (4.32)$$

This process is carried out for every term in Equations (4.1), (4.9) and (4.28) and also for the y momentum equation and for each different combination of surrounding

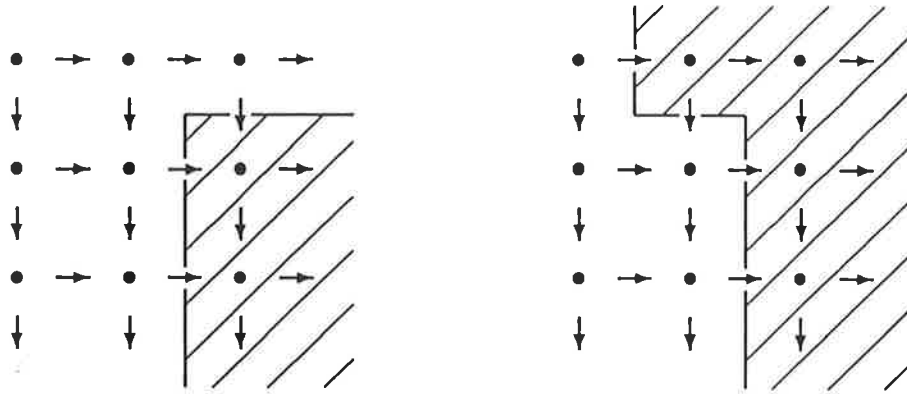


Figure 4.1: Two situations where, with respect to the central element i , the two elements $i + 1$ and $i2 + 1$, which are used in the calculation of the term in Equation (4.31), are both land elements.

land elements. For many of these cases no modification of the finite difference approximation is necessary, but some of the terms require careful consideration to keep the approximation accurate and stable. Programming of the approximations for each term of each equation is quite straight forward once all the possible cases of surrounding land have been considered.

Chapter 5

ANALYTIC SOLUTIONS TO THE SIMPLIFIED TIDAL EQUATIONS

Analytic solutions to the governing tidal motion equations of Equations (2.1), (2.8), (2.9) and (2.15) may never be found because the equations are non-linear and very complicated, especially when the complex boundary conditions of the sea geometry are taken into consideration. In this chapter the analytic solutions used by various authors have been extended to include the horizontal diffusion terms and the quasi-linearised horizontal advection terms, a more complete repertoire of available vertical eddy viscosity profiles is given and a new profile not previously documented before is presented. Even though full analytical solutions may never be found, analytic solutions to the governing tidal motion equations, which may have been simplified or where the sea geometry has been simplified, are very useful for two reasons. Firstly they can be used for model verification, where the model is run on the same simplified situation and the results from the model are compared to the analytic answers. This gives an indication of the accuracy of the model, and also more importantly indicates that the computer program has been coded correctly for the terms that are present in the simplified situation. The more terms and spatial dimensions that can be included in

the tidal motion equations that can be solved analytically the more useful that solution is for model verification. Model verification using the analytical solutions developed in this chapter is discussed in Chapter 7. The second reason for looking at analytic solutions is that some of them enable a wave propagation analysis to be performed, where the differencing of each term in the tidal equations can be analytically studied to find the best method of discretisation. A new three-dimensional wave propagation analysis, which also gives an indication of the stability and accuracy of each possible discretisation method, is developed in Chapter 6.

A new analytic solution to Equations (5.1) to (5.4) for a functional form for N_η , that has not been used or documented previously in this field of study, is given in part (g) of Section 5.2.

The locally linearised forms of the governing tidal Equations (2.1), (2.8) and (2.9) for constant b and B are

$$\frac{\partial \zeta}{\partial t} + h_a \frac{\partial \bar{\mu}}{\partial \chi} + h_a \frac{\partial \bar{\nu}}{\partial \lambda} = 0 , \quad (5.1)$$

$$\frac{\partial \mu}{\partial t} + \mu_a \frac{\partial \mu}{\partial \chi} + \nu_a \frac{\partial \mu}{\partial \lambda} = -\frac{g}{L^2} \frac{\partial \zeta}{\partial \chi} + \frac{N_\chi}{L^2} \frac{\partial^2 \mu}{\partial \chi^2} + \frac{N_\lambda}{B^2} \frac{\partial^2 \mu}{\partial \lambda^2} + \frac{1}{h_a^2} \frac{\partial}{\partial \eta} (N_\eta \frac{\partial \mu}{\partial \eta}) + \frac{f_a B \nu}{L} , \quad (5.2)$$

$$\frac{\partial \nu}{\partial t} + \mu_a \frac{\partial \nu}{\partial \chi} + \nu_a \frac{\partial \nu}{\partial \lambda} = -\frac{g}{B^2} \frac{\partial \zeta}{\partial \lambda} + \frac{N_\chi}{L^2} \frac{\partial^2 \nu}{\partial \chi^2} + \frac{N_\lambda}{B^2} \frac{\partial^2 \nu}{\partial \lambda^2} + \frac{1}{h_a^2} \frac{\partial}{\partial \eta} (N_\eta \frac{\partial \nu}{\partial \eta}) - \frac{f_a L \mu}{B} , \quad (5.3)$$

where

$$\bar{\mu} = \int_0^1 \mu d\eta , \quad \bar{\nu} = \int_0^1 \nu d\eta , \quad (5.4)$$

and μ_a , ν_a , h_a and f_a are constants representative of μ , ν , H and f over a tidal cycle in the neighbourhood of (χ, λ, η) . The horizontal eddy viscosity coefficients N_χ and N_λ are taken as constant and the vertical eddy viscosity coefficient N_η is depth dependent. In linearising the tidal equations, it is assumed that the surface displacement ζ is small compared to the depth of water. Because of this, the vertical advective terms and

the vertical component of velocity may be considered negligible and can therefore be omitted. Absence of wind stress implies that the boundary conditions at the sea surface are

$$\left. \frac{\partial \mu}{\partial \eta} \right|_{\eta=0} = \left. \frac{\partial \nu}{\partial \eta} \right|_{\eta=0} = 0 , \quad (5.5)$$

and a “no-slip” condition implies that the boundary conditions at $\eta = 1$ are

$$\mu|_{\eta=1} = \nu|_{\eta=1} = 0 . \quad (5.6)$$

In the absence of horizontal boundaries, an analytic solution to these locally linearised governing equations can be obtained by seeking a solution in the form

$$\zeta(\chi, \lambda, t) = a \exp\{i(\hat{\sigma}t + m_\chi \chi + m_\lambda \lambda)\} , \quad (5.7)$$

$$\mu(\chi, \lambda, \eta, t) = \mu^* F_\chi(\eta) \exp\{i(\hat{\sigma}t + m_\chi \chi + m_\lambda \lambda)\} , \quad (5.8)$$

$$\nu(\chi, \lambda, \eta, t) = \nu^* F_\lambda(\eta) \exp\{i(\hat{\sigma}t + m_\chi \chi + m_\lambda \lambda)\} , \quad (5.9)$$

where the real parts of the right hand sides of these equations give the desired solution, and where

i is $\sqrt{-1}$,

a is a real number which is the amplitude of the surface oscillation,

$\hat{\sigma}$ is the circular frequency of the surface oscillation,

m_χ and m_λ are the wave number components in the χ and λ directions respectively,

$F_\chi(\eta)$ and $F_\lambda(\eta)$ are the parts of the velocity solutions, which vary with depth. These functions are dependent on the choice of the vertical eddy viscosity, N_η .

5.1 The Differential Equations

If the constants μ^* and ν^* are chosen to be

$$-\frac{gm_\chi m_\lambda a}{L^2 B \hat{\sigma}^2} \quad \text{and} \quad -\frac{gm_\chi m_\lambda a}{LB^2 \hat{\sigma}^2}, \quad (5.10)$$

respectively, then the substitution of Equations (5.7) to (5.9) into Equations (5.2) and (5.3) leads to a pair of linked second order differential equations, given by

$$\frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial \tilde{F}}{\partial \eta} \right) + h_a^2 \mathbf{S} \tilde{F} = -i\hat{\sigma}^2 h_a^2 \begin{bmatrix} \frac{B}{m_\lambda} \\ \frac{L}{m_\chi} \end{bmatrix}; \quad (5.11)$$

where

$$\tilde{F} = \begin{bmatrix} F_\chi(\eta) \\ F_\lambda(\eta) \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} -i\hat{\sigma} - C_a & f_a \\ -f_a & -i\hat{\sigma} - C_a \end{bmatrix}, \quad (5.12)$$

and

$$C_a = \frac{N_\chi m_\chi^2}{L^2} + \frac{N_\lambda m_\lambda^2}{B^2} + im_\chi \mu_a + im_\lambda \nu_a.$$

A particular solution to Equation (5.11) is

$$\tilde{F}_p = \begin{bmatrix} F_{p\chi} \\ F_{p\lambda} \end{bmatrix} = -i\hat{\sigma}^2 \mathbf{S}^{-1} \begin{bmatrix} \frac{B}{m_\lambda} \\ \frac{L}{m_\chi} \end{bmatrix}, \quad (5.13)$$

which gives

$$F_{p\chi} = \frac{i\hat{\sigma}[(i\hat{\sigma} + C_a)B/m_\lambda + f_a L/m_\chi]}{[(i\hat{\sigma} + C_a)^2 + f_a^2]},$$

and

$$F_{p\lambda} = \frac{i\hat{\sigma}[(i\hat{\sigma} + C_a)L/m_\chi - f_a B/m_\lambda]}{[(i\hat{\sigma} + C_a)^2 + f_a^2]}.$$

The solution to the homogeneous equation corresponding to Equation (5.10) can be found by considering

$$\tilde{F} = \begin{bmatrix} F_1(\eta) + F_2(\eta) \\ -i(F_1(\eta) + F_2(\eta)) \end{bmatrix}, \quad (5.14)$$

which leads to two independent second order differential equations for solving $F_1(\eta)$ and $F_2(\eta)$, namely

$$\frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial F_1(\eta)}{\partial \eta} \right) - h_a^2 (C_a + i(\hat{\sigma} - f_a)) F_1(\eta) = 0, \quad (5.15)$$

$$\frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial F_2(\eta)}{\partial \eta} \right) - h_a^2 (C_a + i(\hat{\sigma} + f_a)) F_2(\eta) = 0 . \quad (5.16)$$

Equations (5.15) and (5.16) can only be solved analytically for various forms of N_η . If it is assumed that the two linearly independent solutions to Equation (5.15) are $S_1(\eta)$ and $S_2(\eta)$ and the two solutions to Equation (5.16) are $S_3(\eta)$ and $S_4(\eta)$, then the solution to Equation (5.11) is

$$\begin{aligned} F_x(\eta) &= c_1 S_1(\eta) + c_2 S_2(\eta) + c_3 S_3(\eta) + c_4 S_4(\eta) + F_{px} , \\ F_\lambda(\eta) &= i(c_1 S_1(\eta) + c_2 S_2(\eta) - c_3 S_3(\eta) - c_4 S_4(\eta)) + F_{p\lambda} . \end{aligned} \quad (5.17)$$

The boundary conditions of Equations (5.5) and (5.6) imply that

$$F'_x(0) = F'_\lambda(0) = F_x(1) = F_\lambda(1) = 0 , \quad (5.18)$$

which lead to the following equations for the constants c_1 to c_4 .

$$\begin{aligned} c_1 &= \frac{-S'_2(0)}{2(S_1(1)S'_2(0) - S'_1(0)S_2(1))} (F_{px} - iF_{p\lambda}) , \\ c_2 &= \frac{S'_1(0)}{2(S_1(1)S'_2(0) - S'_1(0)S_2(1))} (F_{px} - iF_{p\lambda}) , \\ c_3 &= \frac{-S'_4(0)}{2(S_3(1)S'_4(0) - S'_3(0)S_4(1))} (F_{px} + iF_{p\lambda}) , \\ c_4 &= \frac{S'_3(0)}{2(S_3(1)S'_4(0) - S'_3(0)S_4(1))} (F_{px} + iF_{p\lambda}) . \end{aligned} \quad (5.19)$$

5.2 The Vertical Eddy Viscosity Profiles

There are a number of forms of the vertical eddy viscosity coefficient, N_η , for which the previously listed second order differential equations can be solved analytically. A number of authors have previously solved and used most of the profiles which lead to the analytic solutions of Equations (5.1) to (5.4), but without the inclusion of the horizontal advection or horizontal diffusion terms (see Table 5.1). The analytic solutions are extended in this work by the inclusion of the horizontal advection and diffusion terms

2D $z-x$	3D $z-x-y$	Horiz. Advection	Horiz. Diffusion	Vertical Eddy Viscosity Type	Author
	•	No	No	Constant	Sverdrup (1926)
	•	No	No	Exponential	Fjeldstat (1929)
	•	No	No	Constant	Bowden et al. (1959)
	•	No	No	Two piece linear & constant	Kagan (1966)
	•	No	No	Quadratic	Johns (1966, 67, 68, 69b)
	•	No	No	Parabolic – solved numerically	Johns (1969a)
	•	Yes	No	Two piece constant & quadratic	Johns (1970)
	•	No	No	Two piece constant & quadratic	Johns & Dyke (1971, 72)
	•	No	No	Constant	Durance (1976)
	•	No	No	Parabolic	Nihoul (1977)
	•	Yes	No	Constant, Parabolic, Two piece constant & parabolic	Ianniello (1977)
	•	No	No	Constant	Kundu et al. (1981)
	•	No	No	Constant, Linear	Prandle (1982)
	•	No	No	Constant, Parabolic – solved numerically, Two piece constant & quadratic, Two piece parabolic & quadratic – solved numerically, Exponential, Two piece constant & linear	Tee (1979, 87)

Table 5.1: Table summarising the eddy viscosity profiles used by various authors

in the differential equations. The coefficients of these terms effect the solution through the parameter C_a , which in essence changes the circular frequency of the tidal analytic solution. A table summarising the types of solutions used by various authors is given in Table (5.1). In Equations (5.2) and (5.3) the horizontal advection terms have been linearised, whereas Johns (1970) and Ianniello (1977) considered the non-linear term to second order accuracy in their two-dimensional work by using perturbation analysis.

The two-dimensional case (one horizontal and one vertical direction) can be examined by considering Equations (5.1) and (5.2) with $\nu = \bar{\nu} = \nu_a = f_a = N_\lambda = 0$. This case is useful because the analytic solution can have a solid boundary at $\chi = 0$, and so can be used for verifying a numerical model, when applied to this channel flow situation. This verification is performed in Chapter 7.

The analytic solution for Equations (5.1) to (5.4) requires the derivative of each of the solutions $S_i(\eta)$, $i = 1 \dots 4$, for each of the cases discussed in the following pages. The solutions need to be differentiated so that the boundary condition of Equation (5.5) can be applied. Also the integral, from the sea floor to surface, of each solution is required, so that $\bar{\mu}$ and $\bar{\nu}$ can be calculated for use in Equation (5.1). The significance of Equation (5.1) is discussed in Section 5.3, where it leads to Equation (5.42). The first seven vertical eddy viscosity profiles are graphed in Figure 5.1. The $\sin * \sin$ (or $\sin^2$) profile is the new profile developed in this work.

(a) Constant Eddy Viscosity

This solution has been used by numerous authors dating back to Sverdrup (1926).

For

$$N_\eta = N_s = \text{constant} ,$$

Equations (5.15) and (5.16) are two constant coefficient differential equations with the

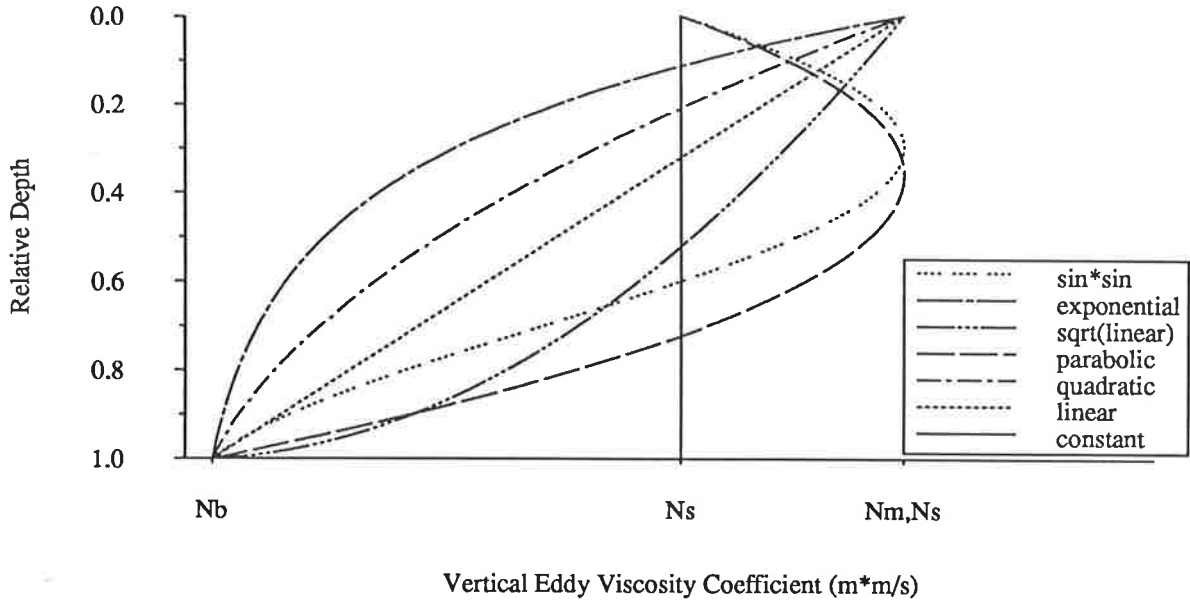


Figure 5.1: Seven different vertical eddy viscosity profiles, which lead to analytic solutions for Equations (5.1) to (5.4)

simple independent solutions of

$$\begin{aligned} S_1(\eta) &= e^{\lambda_1 \eta}, & S_2(\eta) &= e^{-\lambda_1 \eta}, \\ S_3(\eta) &= e^{\lambda_2 \eta}, & S_4(\eta) &= e^{-\lambda_2 \eta}, \end{aligned} \quad (5.20)$$

where $\lambda_1 = h_a \sqrt{\frac{-C_a - i(\hat{\sigma} - f_a)}{N_s}}$, $\lambda_2 = h_a \sqrt{\frac{-C_a - i(\hat{\sigma} + f_a)}{N_s}}$ and the derivatives and integrals of these solutions are trivial.

(b) Linear Eddy Viscosity

Kagan (1966) and Prandle (1982) have used this solution in their work, Kagan using it in conjunction with the constant solution of part (a) giving a piecewise continuous form for N_η (see part (h) for more detail).

For

$$N_\eta = N_s + (N_b - N_s)\eta,$$

where N_s is the surface value and N_b is the sea bed value for the eddy viscosity coeffi-

cient, the transformation given by

$$\xi = \sqrt{N_s + (N_b - N_s)\eta} , \quad (5.21)$$

transforms Equations (5.15) and (5.16) into two Bessel differential equation of order zero, with solutions of

$$\begin{aligned} S_1(\eta) &= J_0(\lambda_1 \xi) , \quad S_2(\eta) = Y_0(\lambda_1 \xi) , \\ S_3(\eta) &= J_0(\lambda_2 \xi) , \quad S_4(\eta) = Y_0(\lambda_2 \xi) , \end{aligned} \quad (5.22)$$

where $\lambda_1^2 = \frac{-4h_a^2(C_a + i(\hat{\sigma} - f_a))}{(N_b - N_s)^2}$ and $\lambda_2^2 = \frac{-4h_a^2(C_a + i(\hat{\sigma} + f_a))}{(N_b - N_s)^2}$. These functions require evaluation for the complex arguments $\lambda_1 \xi$ and $\lambda_2 \xi$ and if these functions are not available as library routines, can be evaluated using the appropriate power series expansions, using complex arithmetic in the computer program. The derivatives of these solutions can be found using formulae in Abramowitz and Stegun (1972), namely $J_0'(\lambda_i \xi) = -\lambda_i J_1(\lambda_i \xi)$ and $Y_0'(\lambda_i \xi) = -\lambda_i Y_1(\lambda_i \xi)$. The required integrals with respect to ξ turn out to be of the form $\int \xi J_0(\lambda_i \xi) d\xi$, which can be evaluated from formulae in Erdélyi et al. Vol. 2 (1953), namely

$$\int \xi J_0(\lambda_i \xi) d\xi = \frac{\xi}{\lambda_i} J_1(\lambda_i \xi) ,$$

and replacing J_0, J_1 by Y_0, Y_1 respectively gives the other required formulae.

(c) Quadratic Eddy Viscosity

This form of the eddy viscosity was introduced by Johns (1966) for a channel flow problem in two-dimensions and later in a the three-dimensional solution for an off-shore tidal stream in some of his subsequent papers. N_η is defined by

$$N_\eta = N_b(1 + \alpha(1 - \eta))^2 ,$$

where $\alpha = -1 + \sqrt{\frac{N_s}{N_b}}$, N_s is the surface value and N_b is the sea bed value for the eddy viscosity. With this form for N_η Equations (5.15) and (5.16) are Euler-Cauchy

differential equations with the obvious solutions of

$$\begin{aligned} S_1(\eta) &= (1 + \alpha(1 - \eta))^{n_1} , \quad S_2(\eta) = (1 + \alpha(1 - \eta))^{n_2} , \\ S_3(\eta) &= (1 + \alpha(1 - \eta))^{n_3} , \quad S_4(\eta) = (1 + \alpha(1 - \eta))^{n_4} , \end{aligned} \quad (5.23)$$

where $n_1, n_2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{C_a + i(\hat{\sigma} - f_a)}{\alpha^2 N_b}}$ and $n_3, n_4 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{C_a + i(\hat{\sigma} + f_a)}{\alpha^2 N_b}}$. The derivatives and integrals of these solutions are easily evaluated.

(d) Parabolic Eddy Viscosity

The parabolic form is of particular interest as a number of authors have used either a time varying or time constant vertical eddy viscosity of this form, for their numerical models. In studying wind driven flows Nihoul (1977) and Engelund (1978) used parabolic forms, which varied with time through a velocity component and Koutitas (1978, 1987) used a time invariant form (the same as that given below with $\alpha = 0$, $\beta = 1$). A similar parabolic form of the eddy viscosity was also used by Ellison and Turner (1954) in a simplified model investigating the effect of a density difference on the velocity profile of a tidal flow. Noye et al. (1982) used a time varying parabolic profile for their numerical model of flow in Spencer Gulf, South Australia.

The parabolic form is far more realistic in shape than many of the other forms for which analytic solutions exist. McGregor (1972) described the analytic solution for this form, but linearised it straight away and never used the full solution. Johns (1969a) and Tee (1979) use this form of the eddy viscosity profile but solve the differential equations with finite difference methods. Ianniello (1977) also uses this analytic solution for the two-dimensional case, presumably by evaluating the Confluent Hypergeometric function with self written complex number functions. The solution can be stated easily in terms of the Legendre functions of order zero with complex degree or in terms of the Confluent Hypergeometric function as shown by McGregor (1972). The approach here uses the Legendre functions. For

$$N_\eta = d(\eta - \alpha)(\eta - \beta) ,$$

where

$$d = N_s + N_b - 2N_m - 2\sqrt{(N_m - N_s)(N_m - N_b)} ,$$

$$\alpha, \beta = \frac{1}{2} - \frac{N_b - N_s}{2d} \pm \sqrt{\left(\frac{1}{2} - \frac{N_b - N_s}{2d}\right)^2 - \frac{N_s}{d}} ,$$

and where N_s is the surface value, N_m is the maximum value and N_b is the sea bed value for the eddy viscosity, the transformation given by

$$\xi = \frac{2\eta - \alpha - \beta}{\beta - \alpha} , \quad (5.24)$$

transforms Equations (5.15) and (5.16) into two Legendre differential equations with solutions of

$$S_1(\eta) = P_{\nu_1}(\xi) , \quad S_2(\eta) = Q_{\nu_1}(\xi) ,$$

$$S_3(\eta) = P_{\nu_2}(\xi) , \quad S_4(\eta) = Q_{\nu_2}(\xi) , \quad (5.25)$$

where $P_\nu(\xi) = P_\nu^0(\xi)$ and $Q_\nu(\xi) = Q_\nu^0(\xi)$ are the Legendre functions of the first and second kinds respectively, and where

$$\nu_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{C_a + i(\hat{\sigma} - f_a)}{d}} \quad \text{and} \quad \nu_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{C_a + i(\hat{\sigma} + f_a)}{d}} .$$

These functions can be evaluated using formulae and transformations given by Erdélyi et al. Vol. 1 (1953). $P_\nu(\xi)$ can be evaluated using the transformation formula of

$$P_\nu(\xi) = F(-\nu, \nu + 1; 1; \frac{1 - \xi}{2}) , \quad (5.26)$$

and $Q_\nu(\xi)$ can be evaluated using

$$Q_\nu(\xi) = \frac{\pi}{2 \sin \nu \pi} (\cos \nu \pi P_\nu(\xi) - P_\nu(-\xi)) , \quad (5.27)$$

where $F(a, b; c; \xi)$ is the Hypergeometric function given in Abramowitz and Stegun (1972). Special functions on the computer needed to be written to evaluate the Hypergeometric function for complex parameters a , b and c , and also for the Gamma function $\Gamma(z)$, which is also required to be evaluated for a complex z . The Gamma function which is

used in the complex power series evaluation of $F(a, b; c; \xi)$, can be calculated using a power series expansion of its reciprocal, where the first 26 coefficients of the expansion are given to 16 decimal places. This is given in Abramowitz and Stegun (1972). The computer functions for the Legendre, Hypergeometric and Gamma functions have been written in Fortran as complex arithmetic is easily handled by this language. The code for these functions is listed in Appendix C and was written because no appropriate library functions were available.

The derivatives of these solutions can be calculated using

$$(\xi^2 - 1) \frac{\partial P_\nu(\xi)}{\partial \xi} = (\nu + 1) [P_{\nu+1}(\xi) - \xi P_\nu(\xi)] , \quad (5.28)$$

which also applies for $Q_\nu(\xi)$ in stead of $P_\nu(\xi)$ or by the differentiation formulae for the Hypergeometric function given in Abramowitz and Stegun (1972).

No relevant integration formulae are available for $P_\nu(\xi)$ and $Q_\nu(\xi)$, however the definite integrals of these functions can be found by integrating term by term the power series representation of the Hypergeometric function and can thus be expressed in terms of Hypergeometric functions with different parameters, if the first terms are ignored. That is

$$\int_{\xi_1}^{\xi_2} F(-\nu, \nu + 1; 1; \frac{1 - \xi}{2}) d\xi = \frac{2}{\nu(\nu + 1)} F(-\nu - 1, \nu; 0; \frac{1 - \xi}{2}) \Big|_{\xi_1}^{\xi_2} , \quad (5.29)$$

as long as the first term of each evaluation of the Hypergeometric function on the right hand side of Equation (5.29) is neglected.

(e) Squareroot(Linear) Eddy Viscosity

This form for N_η has not been documented as being used by any of the authors working in this field. Arnold (1987) used this form of vertical eddy viscosity in his work with wind induced currents in lakes. Due to its convex shape this form of profile seems to be more realistic than the concave nature of the quadratic profile first used

by Johns (1966) and the exponential profile used by Fjeldstat (1929) and Tee (1987) (see Figure 5.1), and is closer to the parabolic shape which is widely used. Another advantage of this form is that the derivative near the sea floor is the largest of all of the forms graphed in Figure 5.1. This gives rise to velocity profile which increases more rapidly from the sea floor. This profile has further potential when considered as part of a two or three piece composite profile explained in parts (h) and (i).

For

$$N_\eta = \sqrt{N_s^2 + (N_b^2 - N_s^2)\eta} ,$$

where N_s is the surface value and N_b is the sea bed value for the eddy viscosity, the transformation given by

$$\xi = (N_s^2 + (N_b^2 - N_s^2)\eta)^{\frac{3}{4}} , \quad (5.30)$$

transforms Equations (5.15) and (5.16) into a pair of associated Bessel differential equations, which have solutions expressed in terms of Bessel functions of order one third, namely

$$\begin{aligned} S_1(\eta) &= \sqrt[3]{\xi} J_{\frac{1}{3}}(\frac{2}{3}\sqrt{\lambda_1}\xi) , & S_2(\eta) &= \sqrt[3]{\xi} J_{-\frac{1}{3}}(\frac{2}{3}\sqrt{\lambda_1}\xi) , \\ S_3(\eta) &= \sqrt[3]{\xi} J_{\frac{1}{3}}(\frac{2}{3}\sqrt{\lambda_2}\xi) , & S_4(\eta) &= \sqrt[3]{\xi} J_{-\frac{1}{3}}(\frac{2}{3}\sqrt{\lambda_2}\xi) , \end{aligned} \quad (5.31)$$

where $\lambda_1 = \frac{-4h_a^2(C_a + i(\hat{\sigma} - f_a))}{(N_s^2 - N_b^2)^2}$ and $\lambda_2 = \frac{-4h_a^2(C_a + i(\hat{\sigma} + f_a))}{(N_s^2 - N_b^2)^2}$. Complex Fortran functions were required to be written for evaluation of these functions as no library functions that take complex arguments were available. The code for these functions is listed in Appendix C. The evaluation can be achieved by summing the appropriate power series to any desired accuracy and using recurrence relations in conjunction with constants obtained from the complex Gamma function mentioned earlier. Alternatively the solutions may be expressed in terms of the Airy functions. The derivatives of these solutions can be found using formulae given in Abramowitz and Stegun (1972), namely

$$\frac{\partial}{\partial \xi} (\xi^\phi J_\phi(\xi)) = \xi^\phi J_{\phi-1}(\xi) , \quad (5.32)$$

$$\frac{\partial}{\partial \xi} \left(\xi^{-\phi} J_{\phi}(\xi) \right) = -\xi^{-\phi} J_{\phi+1}(\xi) . \quad (5.33)$$

Both sides of the equations in these two formulae can be integrated to give formulae which are the relevant integrals required to find $\bar{\mu}$ and $\bar{\nu}$, from the four linearly independent solutions S_1 to S_4 .

(f) Exponential

This form of N_{η} has been used by Fjeldstat (1929) where he developed his own series solution for the problem and by Tee (1987) who stated that the solution was calculated analytically. The solution here is expressed in terms of Bessel functions of order one which have complex argument.

For

$$N_{\eta} = N_s \exp \left(\eta \ln \left(\frac{N_b}{N_s} \right) \right) ,$$

where N_s is the surface value and N_b is the sea bed value for the eddy viscosity, the transformation given by

$$\xi = N_s \exp \left(-\frac{1}{2} \eta \ln \left(\frac{N_b}{N_s} \right) \right) , \quad (5.34)$$

transforms Equations (5.15) and (5.16) into a pair of associated Bessel differential equations, which have solutions expressed in terms of Bessel functions of order one, namely

$$\begin{aligned} S_1(\eta) &= \xi J_1(\sqrt{\lambda_1} \xi) , \quad S_2(\eta) = \xi Y_1(\sqrt{\lambda_1} \xi) , \\ S_3(\eta) &= \xi J_1(\sqrt{\lambda_2} \xi) , \quad S_4(\eta) = \xi Y_1(\sqrt{\lambda_2} \xi) , \end{aligned} \quad (5.35)$$

where $\lambda_1 = \frac{-4h_a^2(C_a + i(\bar{\sigma} - f_a))}{N_s(\ln(\frac{N_b}{N_s}))^2}$ and $\lambda_2 = \frac{-4h_a^2(C_a + i(\bar{\sigma} + f_a))}{N_s(\ln(\frac{N_b}{N_s}))^2}$. The derivatives of these solutions can be found using formulae given in Equations (5.32) and (5.33). It seems that these solutions cannot be analytically integrated. However, for this case numerical integration can be performed to yield results to any desirable level of accuracy.

(g) Sin Squared

This form for N_η leads to an analytic solution which has not been documented or used in any previously known publication in this field of study. As can be seen from Figure 5.1 this profile has many of the features of the parabolic profile which is generally accepted as being quite realistic and as can be seen in Section 5.4 can produce realistic velocity profiles.

For

$$N_\eta = a \sin^2 \left(\frac{\pi(\eta - d)}{b} \right) ,$$

where

$$a = N_m ,$$

$$b = \frac{\pi}{\pi - \sin^{-1} \left(\sqrt{\frac{N_s}{N_m}} \right) - \sin^{-1} \left(\sqrt{\frac{N_b}{N_m}} \right)} ,$$

$$d = \frac{-b}{\pi} \sin^{-1} \left(\sqrt{\frac{N_s}{N_m}} \right) ,$$

the transformation given by

$$\xi = \frac{1}{2} + \frac{i}{2} \cot \left(\frac{\pi(\eta - d)}{b} \right) , \quad (5.36)$$

transforms Equations (5.15) and (5.16) into a pair of equations which have Hypergeometric Functions as solutions (see Abramowitz and Stegun (1972)), namely

$$\begin{aligned} S_1(\eta) &= \xi F(n_1 + 1, n_2 + 1; 2; \xi) , & S_2(\eta) &= (1 - \xi) F(-n_1, -n_2; 2; 1 - \xi) , \\ S_3(\eta) &= \xi F(n_3 + 1, n_4 + 1; 2; \xi) , & S_4(\eta) &= (1 - \xi) F(-n_3, -n_4; 2; 1 - \xi) , \end{aligned} \quad (5.37)$$

where $n_1, n_2 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{C_a + i(\hat{\sigma} - f_a)b^2}{a\pi^2}}$ and $n_3, n_4 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{C_a + i(\hat{\sigma} + f_a)b^2}{a\pi^2}}$. The evaluation of these functions proves to be very difficult due mainly to the fact that the Hypergeometric function does not converge for some of the required values of ξ and $\xi - 1$ for reasonable values of a , b and d . Fortunately there are many transformation

formulae and analytic continuations given in both Abramowitz and Stegun (1972) and Erdélyi et al. Vol. 1 (1953), which enable the evaluation of the solutions over the entire range of values of ξ . These are given in Appendix B.

The derivatives of these solutions can be obtained by differentiating each term in the power series representation of each solution. This is quite tedious as all of the analytic continuations need to be differentiated term by term to obtain the full function. The presence of the “cot” term in the transformation of Equation (5.36) means that there is no analytic function for the integral of these solutions. As with the previous profile, numerical integration is a viable alternative, for this case.

(h) Two Piece Composite

Any of the above seven vertical eddy viscosity formulations can be combined so that one form for N_η applies from $\eta = 1$ to some depth, $\eta = \eta_1$ and another form applies from $\eta = \eta_1$ to $\eta = 0$. Numerous authors (Ianniello (1977), Johns (1970), Kagan (1966) and Tee (1987)) have used this approach (see Table 5.1).

Figure 5.2 shows the two piece and three piece composite profiles for N_η used as examples in this section. Figure 5.3 shows the two piece composite functions used by the authors listed in Table 5.1. These authors have used this approach to achieve a more rapid increase in N_η near the sea floor. This means that N_η can increase quickly from molecular viscosity at the sea floor, through a thin laminar sublayer and then vary at a slower rate suitable for turbulent flow above the laminar layer.

The analytic solutions can be found by applying the normal boundary conditions at $\eta = 0$ and $\eta = 1$ and by matching the two solutions at $\eta = \eta_1$ by making the solution and its first derivative continuous at this depth level.

The choice of two piece functions is large, but they all follow the same pattern once the above seven analytic solutions have been found. The example used here details the

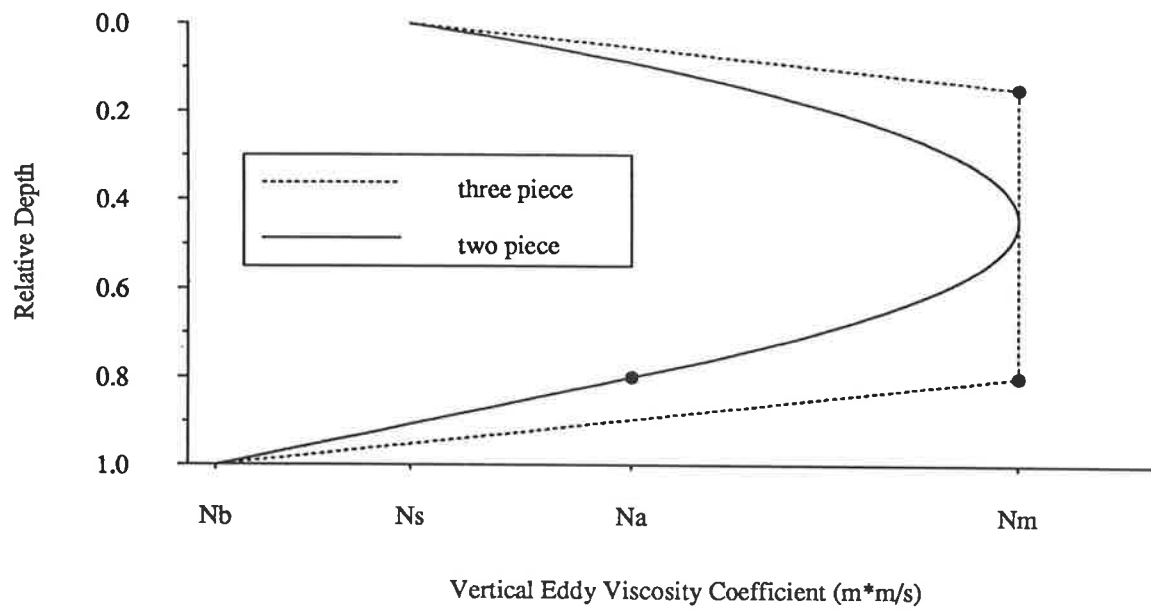


Figure 5.2: The two and three piece composite profiles used for an example in part (h) and part (i)

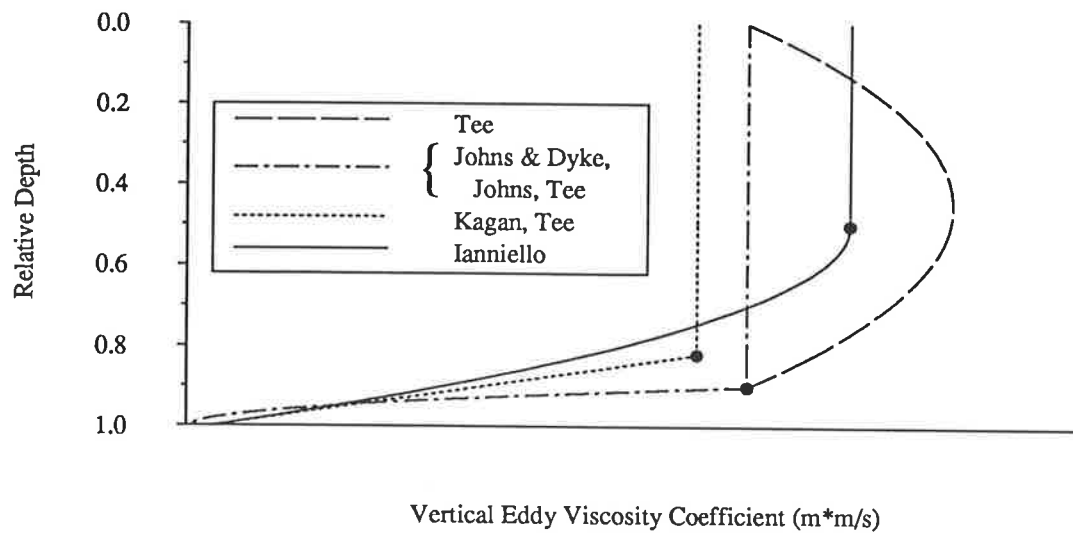


Figure 5.3: The two piece composite functions used by the authors in Table 5.1

solution for a vertical eddy viscosity composed of a linear profile from $\eta = 1$ to $\eta = \eta_1$ and a parabolic profile from $\eta = \eta_1$ to $\eta = 0$ (see Figure 5.2).

When N_η is defined by

$$N_\eta = \begin{cases} d(\eta - \alpha)(\eta - \beta), & \text{for } 0 \leq \eta \leq \eta_1, \\ \left(\frac{N_a - N_b}{\eta_1 - 1}\right)\eta + \frac{\eta_1 N_b - N_a}{\eta_1 - 1}, & \text{for } \eta_1 < \eta \leq 1, \end{cases}$$

where

$$d = \frac{(N_s + N_a - 2N_m - 2\sqrt{(N_m - N_s)(N_m - N_a)})}{\eta_1^2},$$

$$\alpha, \beta = \frac{\eta_1}{2} - \frac{N_a - N_s}{2d\eta_1} \pm \sqrt{\left(\frac{\eta_1}{2} - \frac{N_a - N_s}{2d\eta_1}\right)^2 - \frac{N_s}{d}},$$

and where N_s is the surface value, N_m is the maximum value, N_a is the value at $\eta = \eta_1$ and N_b is the sea bed value for the eddy viscosity, the transformation given by

$$\xi = \begin{cases} \frac{2\eta - \alpha - \beta}{\beta - \alpha}, & \text{for } 0 \leq \eta \leq \eta_1, \\ \sqrt{\left(\frac{N_a - N_b}{\eta_1 - 1}\right)\eta + \frac{\eta_1 N_b - N_a}{\eta_1 - 1}}, & \text{for } \eta_1 < \eta \leq 1, \end{cases} \quad (5.38)$$

leads to the solution represented by

$$\begin{aligned} F_x(\eta) &= c_1 P_{\nu_1}(\xi) + c_2 Q_{\nu_1}(\xi) + c_3 P_{\nu_2}(\xi) + c_4 Q_{\nu_2}(\xi) + F_{px}, \\ F_\lambda(\eta) &= i(c_1 P_{\nu_1}(\xi) + c_2 Q_{\nu_1}(\xi) - c_3 P_{\nu_2}(\xi) - c_4 Q_{\nu_2}(\xi)) + F_{p\lambda}, \end{aligned} \quad (5.39)$$

for $0 \leq \eta \leq \eta_1$ and

$$\begin{aligned} F_x(\eta) &= c_5 J_0(\lambda_1 \xi) + c_6 Y_0(\lambda_1 \xi) + c_7 J_0(\lambda_2 \xi) + c_8 Y_0(\lambda_2 \xi) + F_{px}, \\ F_\lambda(\eta) &= i(c_5 J_0(\lambda_1 \xi) + c_6 Y_0(\lambda_1 \xi) - c_7 J_0(\lambda_2 \xi) - c_8 Y_0(\lambda_2 \xi)) + F_{p\lambda}, \end{aligned} \quad (5.40)$$

for $\eta_1 < \eta \leq 1$. The constants are defined by $\nu_1, \nu_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{C_a + i(\hat{\sigma} \mp f_a)}{d}}$ and $\lambda_1^2, \lambda_2^2 = \frac{-4k_a^2(1-\eta_1)^2(C_a + i(\hat{\sigma} \mp f_a))}{(N_b - N_s)^2}$.

The boundary and matching conditions of $F_x(\eta_1^-) = F_x(\eta_1^+)$, $F'_x(\eta_1^-) = F'_x(\eta_1^+)$, $F_\lambda(\eta_1^-) = F_\lambda(\eta_1^+)$, $F'_\lambda(\eta_1^-) = F'_\lambda(\eta_1^+)$, give eight equations to solve for the unknown

coefficients, c_j , $j = 1..8$, which can be found by solving the 8×8 complex coefficient matrix equation given by

$$\mathbf{M}\boldsymbol{\xi} = \mathbf{0} , \quad (5.41)$$

where

$$\mathbf{M} = \begin{bmatrix} P'_{\nu_1}(\xi_0) & Q'_{\nu_1}(\xi_0) & P'_{\nu_2}(\xi_0) & Q'_{\nu_2}(\xi_0) & 0 & 0 & 0 & 0 \\ P'_{\nu_1}(\xi_0) & Q'_{\nu_1}(\xi_0) & -P'_{\nu_2}(\xi_0) & -Q'_{\nu_2}(\xi_0) & 0 & 0 & 0 & 0 \\ P_{\nu_1}(\xi_1) & Q_{\nu_1}(\xi_1) & P_{\nu_2}(\xi_1) & Q_{\nu_2}(\xi_1) & -J_0(\phi_1) & -Y_0(\phi_1) & -J_0(\phi_2) & -Y_0(\phi_2) \\ P_{\nu_1}(\xi_1) & Q_{\nu_1}(\xi_1) & -P_{\nu_2}(\xi_1) & -Q_{\nu_2}(\xi_1) & -J_0(\phi_1) & -Y_0(\phi_1) & J_0(\phi_2) & Y_0(\phi_2) \\ P'_{\nu_1}(\xi_1) & Q'_{\nu_1}(\xi_1) & P'_{\nu_2}(\xi_1) & Q'_{\nu_2}(\xi_1) & -J'_0(\phi_1) & -Y'_0(\phi_1) & -J'_0(\phi_2) & -Y'_0(\phi_2) \\ P'_{\nu_1}(\xi_1) & Q'_{\nu_1}(\xi_1) & -P'_{\nu_2}(\xi_1) & -Q'_{\nu_2}(\xi_1) & -J'_0(\phi_1) & -Y'_0(\phi_1) & J'_0(\phi_2) & Y'_0(\phi_2) \\ 0 & 0 & 0 & 0 & -J_0(\phi_3) & -Y_0(\phi_3) & -J_0(\phi_4) & -Y_0(\phi_4) \\ 0 & 0 & 0 & 0 & -J_0(\phi_3) & -Y_0(\phi_3) & J_0(\phi_4) & Y_0(\phi_4) \end{bmatrix} ,$$

$$\boldsymbol{\xi} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ,$$

and where

$$\xi_0 = \xi|_{\eta=0} , \quad \phi_1 = \lambda_1 \lim_{\eta \rightarrow \eta_1^+} \xi , \quad \phi_3 = \lambda_1 \xi|_{\eta=1} ,$$

$$\xi_1 = \xi|_{\eta=\eta_1} , \quad \phi_2 = \lambda_2 \lim_{\eta \rightarrow \eta_1^+} \xi , \quad \phi_4 = \lambda_2 \xi|_{\eta=1} .$$

Note that the definition of the vertical eddy viscosity coefficient N_η for this form ensures that N_η is continuous. The solution will always be continuous and have a continuous first derivative due to the matching conditions at the join, and by careful selection of the functions in the viscosity profile and the values of N_s , N_m , N_a and N_b , the total composite eddy viscosity coefficient form can be constructed so that at the join its first derivative is also continuous. That is, N_η can be chosen with a smooth profile if that is desirable.

The integration and differentiation of the solutions can be carried out by piecewise integration and differentiation of each profile.

(j) Three Piece Composite

A three or more piece composite profile for the vertical eddy viscosity is possible by generalising the two piece composite profile. This gives more flexibility in choosing the profile which would suit the situation being modelled, especially if there are wind stresses on the surface, which require the values of the coefficient to approach zero as η approaches the surface.

The example used to illustrate a three piece composite profile for a vertical eddy viscosity is for a linear profile from $\eta = 1$ to $\eta = \eta_1$ with a constant profile from $\eta = \eta_1$ to $\eta = \eta_2$ and a linear profile from $\eta = \eta_2$ to $\eta = 0$ (see Figure 5.3). However, any other combination can be chosen.

The working for this is completely analogous to the two piece composite profile in part (h), with extra matching conditions at the other join. This gives rise to a 12×12 complex coefficient matrix used for the solution of the unknown coefficients c_j , $j = 1..12$.

5.3 The Relationship between $\hat{\sigma}$, m_χ and m_λ

Substitution of Equations (5.7) to (5.9) into Equation (5.1) gives the following relationship between $\hat{\sigma}$, m_χ and m_λ :

$$\hat{\sigma}^3 = \frac{gh_a m_\chi m_\lambda}{LB} \left(\frac{m_\chi}{L} \int_0^1 F_\chi(\eta) d\eta + \frac{m_\lambda}{B} \int_0^1 F_\lambda(\eta) d\eta \right) . \quad (5.42)$$

Two types of analytic solutions are produced depending on whether $\hat{\sigma}$ is chosen to be a real value or alternatively whether the values of m_χ and m_λ are chosen to be real.

The two depth dependence functions $F_\chi(\eta)$ and $F_\lambda(\eta)$ (see the previous section), are each made up of a constant, with respect to depth, which depends on $\hat{\sigma}$, m_χ and m_λ added to a function of depth which depends on $\hat{\sigma}$, but not on m_χ and m_λ . These two functions are the solution to the pair of linked second order differential equations of Equation (5.11), which are solved once a congenial form for the vertical eddy viscosity coefficient, N_η , has been chosen. These two functions along with $\hat{\sigma}$, m_χ and m_λ must satisfy Equation (5.42).

Assuming that m_χ and m_λ are in a given ratio, say $m_\lambda = \theta_1 m_\chi$, then there are four unknowns for the three equations, represented by Equations (5.11) and (5.42). This gives one degree of freedom and thus one of these unknowns can be chosen. By choosing $\hat{\sigma}$ to be a real constant, say $1.405 \times 10^{-4} s^{-1}$ (semidiurnal tide), this determines the value of $F_\chi(\eta)$, $F_\lambda(\eta)$ and m_χ (and therefore m_λ), or choosing a real constant for m_χ will determine the value of $F_\chi(\eta)$, $F_\lambda(\eta)$ and $\hat{\sigma}$.

The simplest solution occurs when $\hat{\sigma}$ is chosen to be real and therefore m_χ and m_λ become complex. This solution is used by Johns (1966 to 1970), Kagan (1966), Tee (1979, 1987) and others. Having chosen a value for $\hat{\sigma}$ and the constant ratio θ_1 , Equation (5.42) can be rearranged to find m_χ explicitly in terms of $\hat{\sigma}$, $F_\chi(\eta)$, $F_\lambda(\eta)$ and θ_1 . The dependence of the constant parts of $F_\chi(\eta)$ and $F_\lambda(\eta)$ on m_χ reduces to dependence on θ_1 when used in Equation (5.42).

A real value for $\hat{\sigma}$ means that the analytic solution is a wave which is periodic in time. That is, after one or more periods in time of the wave, the solution is exactly the same for all points in space as the initial distribution, and because m_χ and m_λ will be complex, at any given time the wave at a point one wavelength away will be lagging or leading and attenuated or amplified by some amount depending on the magnitudes and signs of the real and imaginary parts of m_χ and m_λ .

It is also possible, although much more difficult, to obtain a solution where m_χ and

m_λ are real and $\hat{\sigma}$ is complex, thus giving a solution where at a point in space one or more wavelengths away the solution is exactly the same. That is the solution is periodic in space but not in time and the wave will either decay or amplify in height after one period in time and will not reach its maximum height at the same relative time each oscillation. This type of solution is required for the wave propagation analysis discussed in Chapter 6.

The difficulty in calculating a value of $\hat{\sigma}$ for a specific value of m_x and m_λ , using Equation (5.42), is that the variant parts of the functions $F_x(\eta)$ and $F_\lambda(\eta)$ depend on $\hat{\sigma}$. This requires $\hat{\sigma}$ to be found by an iterative method, where $F_x(\eta)$ and $F_\lambda(\eta)$ are recalculated in each iteration, until the value of $\hat{\sigma}$ converges to a value which satisfies Equation (5.42), to any required accuracy.

A scheme which converges is

$$\hat{\sigma}_0 = \frac{m_x}{L}, \quad (5.43)$$

$$\hat{\sigma}_{n+1} = \sqrt{\frac{gh_a m_x^3 \theta_1}{LB \hat{\sigma}_n} \left(\frac{1}{L} \int_0^1 F_x(\eta, \hat{\sigma}_n) d\eta + \frac{\theta_1}{B} \int_0^1 F_\lambda(\eta, \hat{\sigma}_n) d\eta \right)}, \quad n > 0, \quad (5.44)$$

where the value of $\hat{\sigma}_{n+1}$ is taken with a negative real part. The functions $F_x(\eta)$ and $F_\lambda(\eta)$ are calculated for each $\hat{\sigma}_n$ and the stopping criteria used is

$$\left| \frac{m_x}{L} - \sqrt[3]{\frac{\hat{\sigma}_{n+1}^3 B}{gh_a \theta_1 \left(\frac{1}{L} \int_0^1 F_x(\eta, \hat{\sigma}_n) d\eta + \frac{\theta_1}{B} \int_0^1 F_\lambda(\eta, \hat{\sigma}_n) d\eta \right)}} \right| < \epsilon, \quad (5.45)$$

where $\epsilon > 0$ is a specified tolerance, which basically requires m_x to be practically real.

There would be schemes which converge at a faster rate than this one, but since the calculations only need to be performed once, finding schemes which converge faster is not of importance. The above scheme converges in about 20 iterations to an accuracy of about 14 decimal places of $\frac{m_x}{L}$.

5.4 The Velocity Profiles

The magnitude of eddy viscosity parameters can vary the characteristics of the velocity profile markedly. The most sensitive part of the eddy viscosity profile is the sea bed, where small variations in the sea bed value for the viscosity dramatically affect the shape and characteristics of the velocity profile, especially when the sea bed value is small. If a two or three piece composite eddy viscosity profile is used, a small variation in the depth at which the join occurs has a marked effect on the resulting velocity profiles when the join is close to the sea bed. By choosing different profile shapes for the eddy viscosity and/or varying the join height, velocity profiles can be produced which vary from profiles which resemble a “slip” condition near the sea floor (even though there is still a “no-slip” boundary condition at the sea floor) to the typical logarithmically shaped profile which occurs with “no-slip” conditions. Figure 5.4 shows the velocity profiles at maximum flow for a parabolic and a three piece linear eddy viscosity profile, which are substantially different near the sea bed. The parabolic case ($N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.001 \text{ m}^2/\text{s}$) could be used to test a numerical model with a “no-slip” condition, either in the wave propagation analysis of Chapter 6 or by directly running the model with the same parameters. The three piece linear case ($N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.004 \text{ m}^2/\text{s}$, $N_b = 0.0000001 \text{ m}^2/\text{s}$, depth changes at $\eta = 0.2, 0.97$) could be used when testing the model with a “slip” condition at the sea bed.

There are very many views about the ideal shape of the eddy viscosity profile and whether there should be a “no-slip” boundary condition at the sea floor or whether the approximation of the “slip” condition should be used to accommodate for the inability of a numerical model to handle the large gradients near the sea bed for some flow situations. The wave propagation analysis developed in Chapter 6, which gives a detailed error analysis over depth of the numerical errors produced by the model

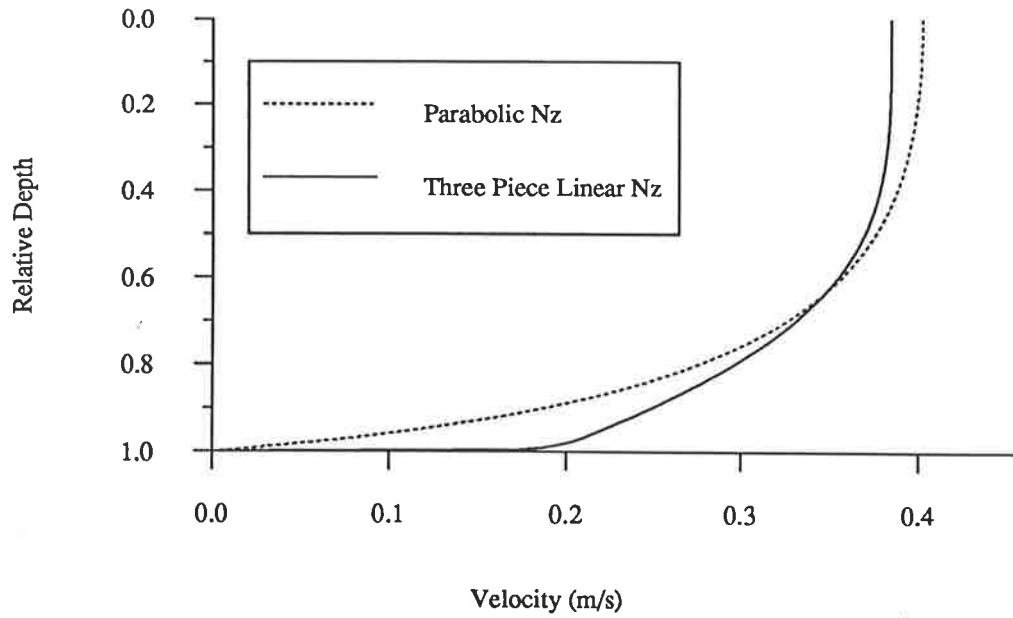


Figure 5.4: By varying the Eddy viscosity profile the analytic solutions for the velocity profile can be made to look that where a bottom “slip” condition may be used or one for which there would be no problems in the numerical model when using a “no-slip” condition.

for a given eddy viscosity profile and for a given vertical grid spacing, can be used to carefully select a vertically variable grid so that the the large errors produced near the sea bed can be minimised. If these errors are still too large near the sea bed for the eddy viscosity profile which should be used in the modelling, then it would be better to use a bottom “slip” condition.

A major factor influencing these different views is the fact that there are many different physical conditions in the areas being modelled by different workers. For some areas being modelled the choice of “no-slip” or “slip” bottom boundary condition is not important. This is true when the surface elevation is more important and detailed structure of the velocity flow is not required, in which case it may be more appropriate to use a two-dimensional depth integrated model. For other situations and applications it may be very important to model the bottom few metres accurately and for these cases some areas would be best modelled by “slip” conditions while others could demand a

“no-slip” condition. It all depends on the particular application for which the model is being used and where it is being used.

In making a choice for the vertical eddy viscosity parameter it is important to have an idea of the conditions which apply for the area being modelled. If available, velocity profiles over depth, from instantaneous measurements averaged over a suitable time length at various times through the tidal cycle, would be very useful especially if there is detail near the sea bed. Unfortunately there are many problems with obtaining detailed profiles near the sea bed, ranging from mechanical positioning without affecting the flow to fouling by drifting seaweed and obtaining accurate information can be a time consuming and very costly exercise. New technology, which uses radar to measure the velocity at a certain depth and can scan the entire depth almost instantly is an exciting prospect, which could provide much information for oceanographers and numerical modellers. The data that this type of measurement could provide would be very useful for setting up and calibrating a three-dimensional numerical model.

If these velocity profiles are available, then by altering both the parameters to the analytic solutions and the type of eddy viscosity profile (either single or composite N_η) for which the analytic solution exists, similar velocity profiles can be produced from the analytic solution. The eddy viscosity profile, which produced this similar velocity profile, should be used in the numerical model and also in the wave propagation analysis described in Chapter 6. The analytic solution cannot simulate a wind stress on the surface and so if the model application is run with a non-zero wind stress parameter then the upper part of the viscosity profile would need to be adjusted accordingly.

In the following analysis all the analytic solutions are used in their two-dimensional form as the velocity profile consideration relies more on the general characteristics of the flow which are similar for both velocity components in the three-dimensional case. For each eddy viscosity formulation the graphs of the velocity profiles and the associated

Reynolds shear stress terms can be calculated as functions of depth. Figures 5.5 and 5.6 show typical results. The general shape of the shear stresses are in agreement with stresses calculated from measurements by Bowden et al. (1959). Figures 5.7 shows the velocity profiles for the seven different non-composite vertical eddy viscosity profiles, at maximum flow, with parameters $L = 200,000$ m, Period = 12 hrs, $h_a = 20$ m, $\mu_a = 0$, $N_x = 0$, $N_s = 0.002$ m²/s, $N_b = 0.001$ m²/s, and $N_m = 0.005$ m²/s. The shape of the velocity profiles vary considerably for the different vertical eddy viscosity forms and even greater variation can be achieved by changing the top, bottom and maximum values for N_η .

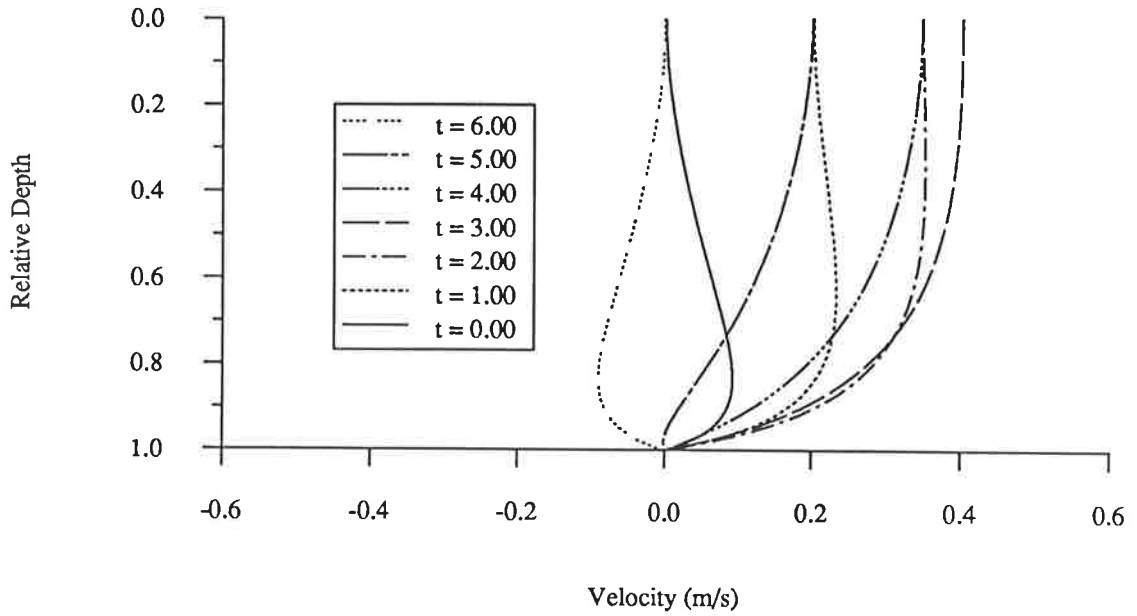


Figure 5.5: A typical velocity profile from the analytical solution, for a parabolic N_η with $N_s = 0.003$ m²/s, $N_m = 0.005$ m²/s, $N_b = 0.001$ m²/s.

Figures 5.8 shows two examples of composite forms for N_η . The two piece form is linear from the sea floor to some point above which the form is parabolic to the surface. The three piece form is linear, constant and then linear again although any of the seven single forms could be used for either the two or three piece form of N_η .

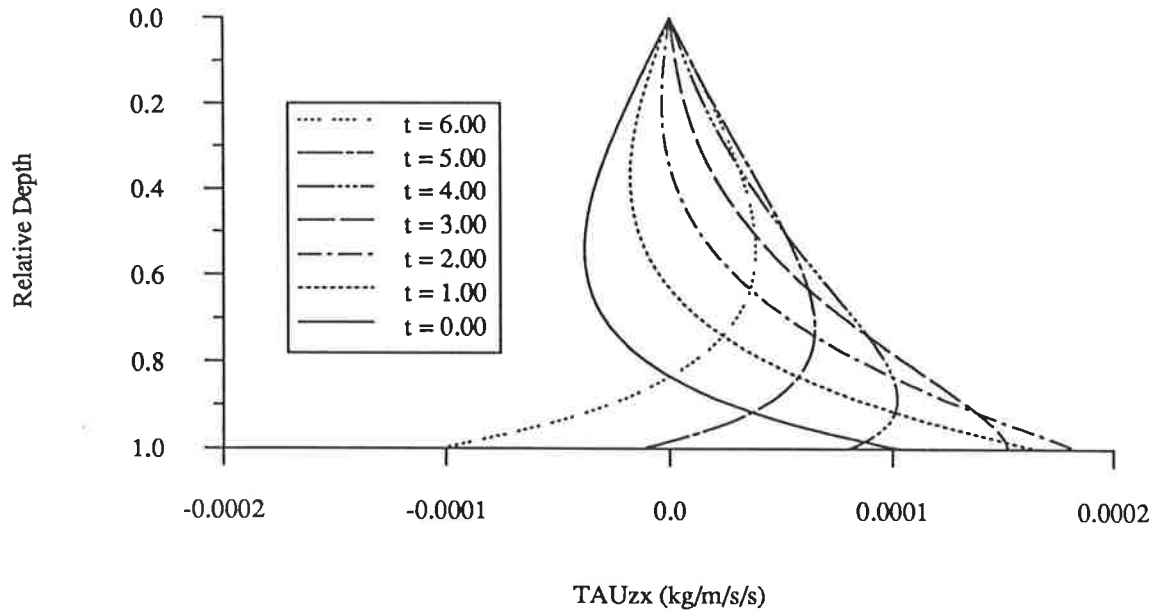


Figure 5.6: The Reynolds shear stress term τ_{zx} for the velocity profiles in Figure 5.5.

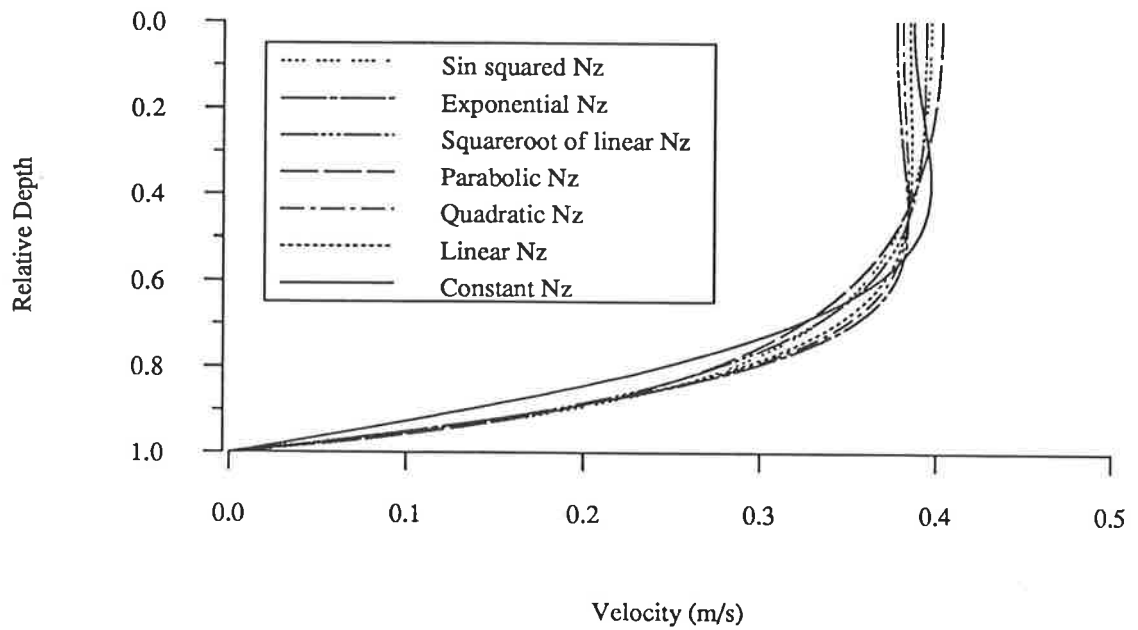


Figure 5.7: Maximum flow velocity profiles for the seven different vertical eddy viscosity forms.

The maximum velocity profile for each of these two forms is graphed for two different values of the bottom eddy viscosity value, $N_b = 0.001 \text{ m}^2/\text{s}$ and $N_b = 0.000001 \text{ m}^2/\text{s}$, with the other values being $N_s = 0.003 \text{ m}^2/\text{s}$ and $N_m = 0.005 \text{ m}^2/\text{s}$. The smaller value of N_b produces velocity profiles which have very large gradients near the sea floor.

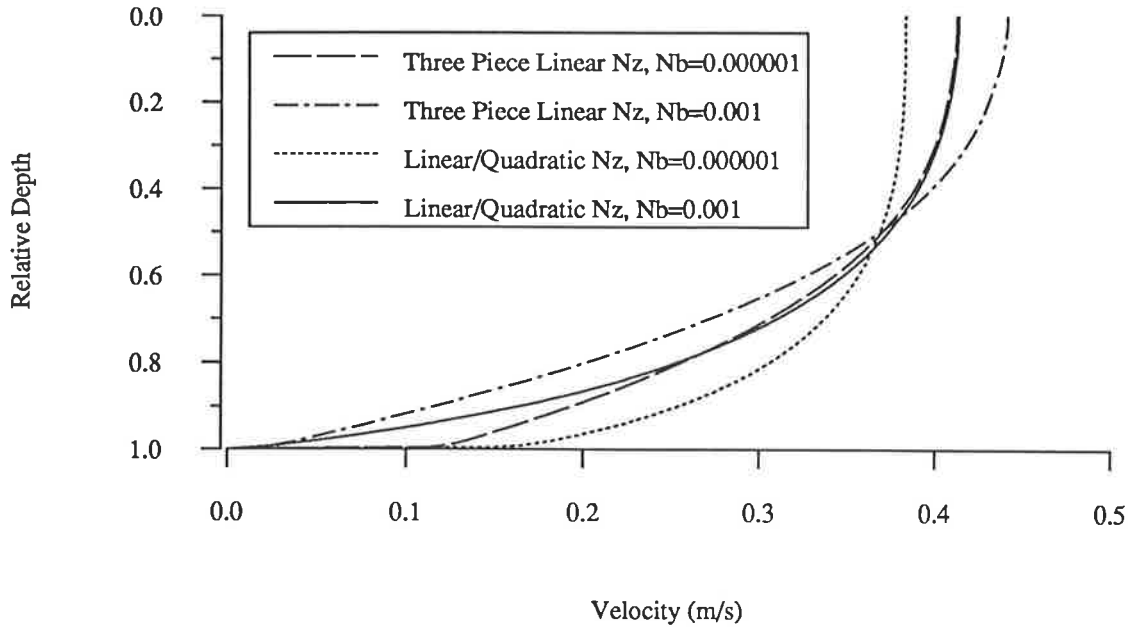


Figure 5.8: Maximum flow velocity profiles for the two composite forms for the vertical eddy viscosity coefficient, each for two different values of the bottom value of the coefficient.

There is a large amount of flexibility in the choice of profile for the vertical eddy viscosity parameter N_η and in the size of the surface, maximum and bottom values of the parameter. It is not suggested that any one particular form or size is the “correct one” to use, although some are clearly more realistic than others, because it depends on the situation being modelled. Whether for direct testing of a model on a channel or for use in the wave propagation analysis (see Chapter 6) it is important to find a vertical eddy viscosity of a shape and size which produces velocity profiles and shear stresses which have similar characteristics to those in the area of application for the numerical model.

Chapter 6

WAVE PROPAGATION ANALYSIS

The accuracy of a finite difference method for solving the governing partial differential equations for tidal flow may be estimated in several ways, the most common being comparison with analytic solutions to a corresponding simplified set of governing equations and/or simplification of the sea geometry. This approach is used in Chapter 7 in testing the three-dimensional model developed in Chapters 3 and 4. An alternative approach is to determine, for an infinite initial wave, some characteristics of the wave which propagates according to the numerical technique relative to the wave which is propagated by the partial differential equation. In order to achieve this, the same finite difference technique used in the model is applied to a linearised forms of the governing three-dimensional partial differential equations. Then the exact solution to the resulting finite difference equations is compared to the analytic solution of the partial differential equations with an infinitely long wave train as the initial condition. This procedure also gives an indication of the stability of the finite difference equations.

Fourier analysis, upon which this method is based, has been widely used in the past to examine the accuracy of various discretisation schemes applied to relatively simple model problems, such as pure advection, by Morton (1971). It has also been used in

a wave propagation analysis of the two-dimensional advection-diffusion equations by Leendertse (1967), Abbott (1979) and Tong (1980) and the linearised depth averaged tidal equations by Sobey (1970) and Noye (1984b). In the following, the analysis is extended to the set of linearised equations which govern three-dimensional tidal motion. Noye and Stevens (1987) first incorporated the depth dimension into this procedure but did not account for all the computational modes which exist in a three-dimensional model. The method presented here is a refinement on that and, as a consequence, reveals a great deal of information about the vertical structure of the errors produced by a finite difference scheme for a three-dimensional tidal model. Some of the material in this chapter is described in Stevens and Noye(1990).

6.1 Defining the Wave Propagation Analysis

The ultimate aim of this work is to determine the stability and accuracy of any fully three-dimensional non-linear finite difference model for tidal motion in well mixed seas. The present analysis, which extends previous two-dimensional analyses to three-dimensions, is a partial step towards the goal. Since a stability analysis for the three-dimensional non-linear model is practically impossible to achieve, the stability of the numerical method for the corresponding linearised equations is investigated by a Fourier analysis to give an indication of the Courant stability for the non-linear equations. This approach also permits estimates to be made of wave propagation characteristics, such as amplitude response, relative wave speed and other error measures. All evidence points towards the linearised equations being a reasonable first approximation for the problem of coastal tides. A number of authors have successfully applied tidal models, which omit a number of the non-linear terms, and still give good agreement with the available data for their applications (see, for example, Leendertse (1967) and Heaps (1969)). This suggests that any conclusions obtained by this method of analysis

of a numerical scheme applied to the linearised equations should be applicable to the non-linear equations (see Morton (1971)).

The linearised versions of the transformed tidal equations are given in Chapter 5 by Equations (5.1) to (5.4) for constant b and B . In order to find an analytic solution for Equations (5.1) to (5.4) all horizontal boundary conditions have been removed and the motion considered as an infinitely long travelling wave. For the given wave number components m_χ , m_λ an analytic solution can then be obtained in the form

$$\zeta(\chi, \lambda, t) = a \exp\{i(\hat{\sigma}t + m_\chi\chi + m_\lambda\lambda)\} , \quad (6.1)$$

$$\mu(\chi, \lambda, \eta, t) = \left(\frac{-gm_\chi m_\lambda a}{L^2 B \hat{\sigma}^2}\right) F_\chi(\eta) \exp\{i(\hat{\sigma}t + m_\chi\chi + m_\lambda\lambda)\} , \quad (6.2)$$

$$\nu(\chi, \lambda, \eta, t) = \left(\frac{-gm_\chi m_\lambda a}{LB^2 \hat{\sigma}^2}\right) F_\lambda(\eta) \exp\{i(\hat{\sigma}t + m_\chi\chi + m_\lambda\lambda)\} . \quad (6.3)$$

See Chapter 5 for more detail. The forms of $F_\chi(\eta)$ and $F_\lambda(\eta)$ depend on the choice of the vertical eddy viscosity coefficient N_η , with analytic solutions existing only for certain functional forms of N_η . The form of N_η used for illustration in this analysis is the quadratic function given by

$$N_\eta = N(\eta) = d(\eta - \alpha)(\eta - \beta) , \quad (6.4)$$

where

$$d = N_s + N_b - 2N_m - 2\sqrt{(N_m - N_s)(N_m - N_b)} , \quad (6.5)$$

$$\alpha, \beta = \frac{1}{2} - \frac{N_b - N_s}{2d} \pm \sqrt{\left(\frac{1}{2} - \frac{N_b - N_s}{2d}\right)^2 - \frac{N_s}{d}} , \quad (6.6)$$

in which N_s is the surface value, N_m is the maximum value and N_b is the sea bed value for the eddy viscosity. The above definition of d , α and β is only valid if N_m occurs in the range $0 \leq \eta \leq 1$. This form for N_η leads to solutions for $F_\chi(\eta)$ and $F_\lambda(\eta)$ expressed in terms of Legendre functions of complex degree (see Section 5.2). Also note that Equation (5.42) must be satisfied by the analytic solution.

The wave propagation analysis used in this article is based on changes over time to the amplitude and wave speed of an infinitely long travelling wave and the resulting

wave response parameters expressed as functions of the number of grid spacings in a wavelength. This requires that the analytic solution to the linearised equations has real values for m_χ and m_λ , thus requiring $\hat{\sigma}$ to be complex. As F_χ and F_λ both depend on m_χ , m_λ and $\hat{\sigma}$, finding this solution is not trivial. However it can be achieved by using an iterative scheme to find the appropriate value of $\hat{\sigma}$ for given values of m_χ and m_λ . Section 5.3 gives details of a method which does this.

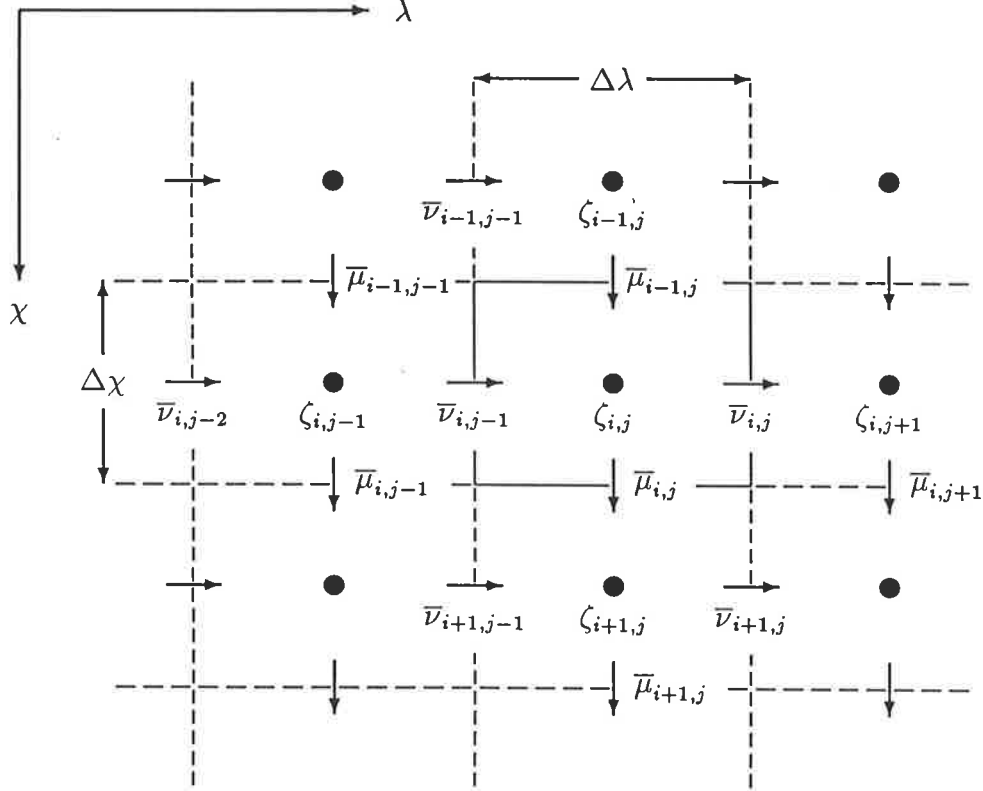


Figure 6.1: Position of variables ζ (grid points $\bullet$), μ (grid points $\downarrow$) and ν (grid points $\rightarrow$) in the two-dimensional array.

The same method of differencing that is used to discretise the original non-linear governing equations of tidal motion described in Chapter 4 should be used to discretise Equations (5.1) to (5.4). The discretisation scheme used for illustration is that described in Chapter 4, with the variable vertical grid, but with uniform grid spacings $\Delta\chi$ and $\Delta\lambda$ in the χ and λ directions and with the index of the horizontal spatial

subscript changed to the double indexed form shown in Figure 6.1. Using a constant time step Δt , the discretisation gives, for Equation (5.1),

$$\frac{1}{\Delta t}(\zeta_{i,j}^* - \zeta_{i,j}^n) + \frac{h_a}{\Delta \chi}(\bar{\mu}_{i,j}^n - \bar{\mu}_{i-1,j}^n) + \frac{h_a}{\Delta \lambda}(\bar{\nu}_{i,j}^n - \bar{\nu}_{i,j-1}^n) = 0 , \quad (6.7)$$

where the depth integrated values of $\bar{\mu}_{i,j}^n$ and $\bar{\nu}_{i,j}^n$ are approximated by the formulae

$$\bar{\mu}_{i,j}^n = \sum_{k=0}^K s_k \mu_{i,j,k}^n , \quad \bar{\nu}_{i,j}^n = \sum_{k=0}^K s_k \nu_{i,j,k}^n , \quad (6.8)$$

and where the integration coefficients s_k for a variable grid are defined as in Section 4.2. At the sea surface the variables are subscripted by $k = 0$ and at the sea floor by $k = K$. On a uniform grid this integration procedure reduces to Simpson's method.

With rearrangement of Equation (6.7) and substitution of Equation (6.8), the first approximation for $\zeta_{i,j}^{n+1}$, $\zeta_{i,j}^*$ may be computed explicitly as follows:

$$\zeta_{i,j}^* = \zeta_{i,j}^n - \frac{h_a \Delta t}{\Delta \chi} \sum_{k=0}^K s_k (\mu_{i,j,k}^n - \mu_{i-1,j,k}^n) - \frac{h_a \Delta t}{\Delta \lambda} \sum_{k=0}^K s_k (\nu_{i,j,k}^n - \nu_{i,j-1,k}^n) . \quad (6.9)$$

The corresponding ‘‘corrector’’ equation is used to recalculate ζ at the new time level, and is based on discretising Equation (5.1) using time centred differencing for all terms, giving

$$\begin{aligned} \frac{1}{\Delta t}(\zeta_{i,j}^{n+1} - \zeta_{i,j}^n) + \frac{h_a}{2\Delta \chi} \sum_{k=0}^K s_k (\mu_{i,j,k}^{n+1} - \mu_{i-1,j,k}^{n+1} + \mu_{i,j,k}^n - \mu_{i-1,j,k}^n) \\ + \frac{h_a}{2\Delta \lambda} \sum_{k=0}^K s_k (\nu_{i,j,k}^{n+1} - \nu_{i,j-1,k}^{n+1} + \nu_{i,j,k}^n - \nu_{i,j-1,k}^n) = 0 . \end{aligned} \quad (6.10)$$

This equation is used after the velocities at the new time level have been calculated. It corrects instabilities introduced by the forward time approximation of Equation (6.9) and results in a more accurate approximation for $\zeta_{i,j}^{n+1}$.

The discretisation of the μ velocity component Equation (5.2), is

$$\begin{aligned}
& \frac{1}{\Delta t}(\mu_{i,j,k}^{n+1} - \mu_{i,j,k}^n) + \frac{\mu_a}{2\Delta\chi}(\mu_{i+1,j,k}^n - \mu_{i-1,j,k}^n) + \frac{\nu_a}{2\Delta\lambda}(\mu_{i,j+1,k}^n - \mu_{i,j-1,k}^n) \\
& = -\frac{g}{2L^2\Delta\chi}(\zeta_{i+1,j}^* - \zeta_{i,j}^* + \zeta_{i+1,j}^n - \zeta_{i,j}^n) \\
& + \frac{N_\chi}{L^2(\Delta\chi)^2}(\mu_{i+1,j,k}^n - 2\mu_{i,j,k}^n + \mu_{i-1,j,k}^n) + \frac{N_\lambda}{B^2(\Delta\lambda)^2}(\mu_{i,j+1,k}^n - 2\mu_{i,j,k}^n + \mu_{i,j-1,k}^n) \\
& + \frac{\left(\begin{aligned} & [N_{k+1} + (2r_k - 1)N_k](\mu_{i,j,k+1}^{n+1} + \mu_{i,j,k+1}^n) \\ & - [N_{k+1} - (r_k + 1)(r_k^2 - 3r_k + 1)N_k + r_k^3N_{k-1}](\mu_{i,j,k}^{n+1} + \mu_{i,j,k}^n) \\ & + r_k^2[(2 - r_k)N_k + r_kN_{k-1}](\mu_{i,j,k-1}^{n+1} + \mu_{i,j,k-1}^n) \end{aligned} \right)}{2h_a^2(\Delta\eta_k)^2(r_k + 1)} \\
& + \frac{f_a B}{4L}(\nu_{i,j-1,k}^{n+1} + \nu_{i,j,k}^{n+1} + \nu_{i+1,j-1,k}^n + \nu_{i+1,j,k}^n), \tag{6.11}
\end{aligned}$$

where $r_k = \Delta\eta_k/\Delta\eta_{k-1}$ and $N_k = N_\eta(\eta_k)$. A similar finite difference equation for the ν velocity component is obtained by discretising Equation (5.3), giving

$$\begin{aligned}
& \frac{1}{\Delta t}(\nu_{i,j,k}^{n+1} - \nu_{i,j,k}^n) + \frac{\mu_a}{2\Delta\chi}(\nu_{i+1,j,k}^n - \nu_{i-1,j,k}^n) + \frac{\nu_a}{2\Delta\lambda}(\nu_{i,j+1,k}^n - \nu_{i,j-1,k}^n) \\
& = -\frac{g}{2B^2\Delta\lambda}(\zeta_{i,j+1}^* - \zeta_{i,j}^* + \zeta_{i,j+1}^n - \zeta_{i,j}^n) \\
& + \frac{N_\chi}{L^2(\Delta\chi)^2}(\nu_{i+1,j,k}^n - 2\nu_{i,j,k}^n + \nu_{i-1,j,k}^n) + \frac{N_\lambda}{B^2(\Delta\lambda)^2}(\nu_{i,j+1,k}^n - 2\nu_{i,j,k}^n + \nu_{i,j-1,k}^n) \\
& + \frac{\left(\begin{aligned} & [N_{k+1} + (2r_k - 1)N_k](\nu_{i,j,k+1}^{n+1} + \nu_{i,j,k+1}^n) \\ & - [N_{k+1} - (r_k + 1)(r_k^2 - 3r_k + 1)N_k + r_k^3N_{k-1}](\nu_{i,j,k}^{n+1} + \nu_{i,j,k}^n) \\ & + r_k^2[(2 - r_k)N_k + r_kN_{k-1}](\nu_{i,j,k-1}^{n+1} + \nu_{i,j,k-1}^n) \end{aligned} \right)}{2h_a^2(\Delta\eta_k)^2(r_k + 1)} \\
& - \frac{f_a L}{4B}(\mu_{i-1,j,k}^{n+1} + \mu_{i,j,k}^{n+1} + \mu_{i-1,j+1,k}^n + \mu_{i,j+1,k}^n). \tag{6.12}
\end{aligned}$$

Equations (6.11) and (6.12) may be expressed entirely in terms of the variables ζ^n ,

μ^n and μ^{n+1} ν^n and ν^{n+1} by substitution of Equation (6.9) giving

$$\begin{aligned}
& \frac{1}{\Delta t}(\mu_{i,j,k}^{n+1} - \mu_{i,j,k}^n) + \frac{\mu_a}{2\Delta\chi}(\mu_{i+1,j,k}^n - \mu_{i-1,j,k}^n) + \frac{\nu_a}{2\Delta\lambda}(\mu_{i,j+1,k}^n - \mu_{i,j-1,k}^n) \\
& = + \frac{gh_a\Delta t}{2L^2(\Delta\chi)^2} \sum_{k=0}^K s_k(\mu_{i+1,j,k}^n - 2\mu_{i,j,k}^n + \mu_{i-1,j,k}^n) \\
& + \frac{gh_a\Delta t}{2L^2\Delta\chi\Delta\lambda} \sum_{k=0}^K s_k(\nu_{i+1,j,k}^n - \nu_{i+1,j-1,k}^n - \nu_{i,j,k}^n + \nu_{i,j-1,k}^n) - \frac{g}{L^2\Delta\chi}(\zeta_{i+1,j}^n - \zeta_{i,j}^n) \\
& + \frac{N_\chi}{L^2(\Delta\chi)^2}(\mu_{i+1,j,k}^n - 2\mu_{i,j,k}^n + \mu_{i-1,j,k}^n) + \frac{N_\lambda}{B^2(\Delta\lambda)^2}(\mu_{i,j+1,k}^n - 2\mu_{i,j,k}^n + \mu_{i,j-1,k}^n) \\
& + \frac{\left([N_{k+1} + (2r_k - 1)N_k](\mu_{i,j,k+1}^{n+1} + \mu_{i,j,k+1}^n) \right.}{2h_a^2(\Delta\eta_k)^2(r_k + 1)} \\
& \quad \left. - [N_{k+1} - (r_k + 1)(r_k^2 - 3r_k + 1)N_k + r_k^3N_{k-1}](\mu_{i,j,k}^{n+1} + \mu_{i,j,k}^n) \right. \\
& \quad \left. + r_k^2[(2 - r_k)N_k + r_kN_{k-1}](\mu_{i,j,k-1}^{n+1} + \mu_{i,j,k-1}^n) \right) \\
& + \frac{f_aB}{4L}(\nu_{i,j-1,k}^{n+1} + \nu_{i,j,k}^{n+1} + \nu_{i+1,j-1,k}^n + \nu_{i+1,j,k}^n), \quad k = 1, 2, \dots, K-1, \quad (6.13)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\Delta t}(\nu_{i,j,k}^{n+1} - \nu_{i,j,k}^n) + \frac{\mu_a}{2\Delta\chi}(\nu_{i+1,j,k}^n - \nu_{i-1,j,k}^n) + \frac{\nu_a}{2\Delta\lambda}(\nu_{i,j+1,k}^n - \nu_{i,j-1,k}^n) \\
& = + \frac{gh_a\Delta t}{2B^2\Delta\chi\Delta\lambda} \sum_{k=0}^K s_k(\mu_{i,j+1,k}^n - \mu_{i-1,j+1,k}^n - \mu_{i,j,k}^n + \mu_{i-1,j,k}^n) \\
& + \frac{gh_a\Delta t}{2B^2(\Delta\lambda)^2} \sum_{k=0}^K s_k(\nu_{i,j+1,k}^n - 2\nu_{i,j,k}^n + \nu_{i,j-1,k}^n) - \frac{g}{B^2\Delta\lambda}(\zeta_{i,j+1}^n - \zeta_{i,j}^n) \\
& + \frac{N_\chi}{L^2(\Delta\chi)^2}(\nu_{i+1,j,k}^n - 2\nu_{i,j,k}^n + \nu_{i-1,j,k}^n) + \frac{N_\lambda}{B^2(\Delta\lambda)^2}(\nu_{i,j+1,k}^n - 2\nu_{i,j,k}^n + \nu_{i,j-1,k}^n) \\
& + \frac{\left([N_{k+1} + (2r_k - 1)N_k](\nu_{i,j,k+1}^{n+1} + \nu_{i,j,k+1}^n) \right.}{2h_a^2(\Delta\eta_k)^2(r_k + 1)} \\
& \quad \left. - [N_{k+1} - (r_k + 1)(r_k^2 - 3r_k + 1)N_k + r_k^3N_{k-1}](\nu_{i,j,k}^{n+1} + \nu_{i,j,k}^n) \right. \\
& \quad \left. + r_k^2[(2 - r_k)N_k + r_kN_{k-1}](\nu_{i,j,k-1}^{n+1} + \nu_{i,j,k-1}^n) \right) \\
& - \frac{f_aL}{4B}(\mu_{i-1,j,k}^{n+1} + \mu_{i,j,k}^n + \mu_{i-1,j+1,k}^{n+1} + \mu_{i,j+1,k}^n), \quad k = 1, 2, \dots, K-1. \quad (6.14)
\end{aligned}$$

The “no-slip” bottom boundary equation implies that

$$\mu_{i,j,K}^n = \nu_{i,j,K}^n = 0, \quad (6.15)$$

and using the same discretisation as described in Section 4.2, the surface boundary equation with no wind stress implies that

$$a_0\mu_{i,j,0}^n + b_0\mu_{i,j,1}^n + c_0\mu_{i,j,2}^n = 0 \quad \text{and} \quad a_0\nu_{i,j,0}^n + b_0\nu_{i,j,1}^n + c_0\nu_{i,j,2}^n = 0, \quad (6.16)$$

where a_0 , b_0 and c_0 are defined by Equations (4.22) and are dependent on the vertical grid spacing chosen. These equations give a second order accurate boundary approximation at the surface.

We now consider the behaviour of the system of finite difference Equations (6.10) and (6.13) to (6.16) in the large, that is with the computational element assumed far away from the disturbing influence of any boundary. Therefore a solution is sought to this system of finite difference equations with the initial conditions (real part intended) on $-\infty < \chi, \lambda < \infty$

$$\zeta^0(\chi, \lambda) = \sum_p \zeta_p^* \exp\{i(m_{\chi,p}\chi + m_{\lambda,p}\lambda)\} , \quad (6.17)$$

$$\mu^0(\chi, \lambda, \eta) = \sum_p \mu_p^* F_{\chi,p}(\eta) \exp\{i(m_{\chi,p}\chi + m_{\lambda,p}\lambda)\} , \quad 0 \leq \eta \leq 1 , \quad (6.18)$$

$$\nu^0(\chi, \lambda, \eta) = \sum_p \nu_p^* F_{\lambda,p}(\eta) \exp\{i(m_{\chi,p}\chi + m_{\lambda,p}\lambda)\} , \quad 0 \leq \eta \leq 1 , \quad (6.19)$$

in which

$m_{\chi,p}$ is the wave number component of the p^{th} Fourier mode in the χ direction,

$m_{\lambda,p}$ is the wave number component of the p^{th} Fourier mode in the λ direction,

$F_{\chi,p}(\eta)$ is the depth variation of the p^{th} mode for μ ,

$F_{\lambda,p}(\eta)$ is the depth variation of the p^{th} mode for ν ,

i is the value of $\sqrt{-1}$,

$\left. \begin{array}{l} \zeta^n(\chi, \lambda) \\ \mu^n(\chi, \lambda, \eta) \\ \nu^n(\chi, \lambda, \eta) \end{array} \right\}$ is the solution of Equations (6.10), (6.13) and (6.14) at time $n\Delta t$,

$\left. \begin{array}{l} a_p \\ \mu_p^* \\ \nu_p^* \end{array} \right\}$ are the amplitudes of the Fourier components of ζ^0 , μ^0 and ν^0 .



Since the system of Equations (6.10), (6.13) and (6.14) is linear, superposition of solutions is valid and only one component of the Fourier series of the initial conditions on $-\infty < \chi, \lambda < \infty$ need be considered, namely the real parts of

$$\zeta^0(\chi, \lambda) = a \exp\{i(m_\chi \chi + m_\lambda \lambda)\} , \quad (6.20)$$

$$\mu^0(\chi, \lambda, \eta) = \mu^* F_\chi(\eta) \exp\{i(m_\chi \chi + m_\lambda \lambda)\} , \quad 0 \leq \eta \leq 1 , \quad (6.21)$$

$$\nu^0(\chi, \lambda, \eta) = \nu^* F_\lambda(\eta) \exp\{i(m_\chi \chi + m_\lambda \lambda)\} , \quad 0 \leq \eta \leq 1 . \quad (6.22)$$

A solution to the set of finite difference equations is sought as the real part of

$$\zeta_{i,j}^n = a \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} , \quad (6.23)$$

$$\mu_{i,j,k}^n = \mu_k^\otimes \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} , \quad k = 0 \dots K , \quad (6.24)$$

$$\nu_{i,j,k}^n = \nu_k^\otimes \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} , \quad k = 0 \dots K , \quad (6.25)$$

where $\mu_k^\otimes$ and $\nu_k^\otimes$ are the finite difference solutions for $\mu^* F_\chi(\eta)$ and $\nu^* F_\lambda(\eta)$ respectively at the depth level $\eta = \eta_k$.

The sea floor boundary condition (6.15) implies that $\mu_K^\otimes = \nu_K^\otimes = 0$. Substituting Equations (6.23) to (6.25) into the finite difference Equations (6.10), (6.13) and (6.14), for $k = 1, 2, \dots, K-1$, and using Equation (6.16) with $\mu_k^\otimes$ and $\nu_k^\otimes$ replacing $\mu_{i,j,k}^n$ and $\nu_{i,j,k}^n$, provides $2K+1$ equations for the $2K+1$ unknowns $\zeta^\otimes$ and $\mu_k^\otimes, \nu_k^\otimes, k = 0, 1, \dots, K-1$.

Assuming that $m_\lambda = \theta_1 m_\chi, \Delta \lambda = \theta_2 \Delta \chi, L = B$ and allowing for the fact that $\mu_{i,j,k}$ is evaluated at $((i + \frac{1}{2})\Delta \chi, j\Delta \lambda, \eta_k)$ and $\nu_{i,j,k}$ is evaluated at $(i\Delta \chi, (j + \frac{1}{2})\Delta \lambda, \eta_k)$ as shown in Figure 6.1, the above substitution yields the matrix equation

$$\mathbf{A} \underline{v} = \xi \mathbf{B} \underline{v} , \quad (6.26)$$

where $\underline{v} = [\zeta^\otimes \mu_0^\otimes \mu_1^\otimes \dots \mu_{K-1}^\otimes \nu_0^\otimes \nu_1^\otimes \dots \nu_{K-1}^\otimes]^T$ contains the $2K+1$ unknowns, the matrix $\mathbf{A}$ contains the coefficients of the unknowns at the n^{th} time level

and $\xi \mathbf{B}$ are the coefficients at the $(n+1)^{th}$ time level, $\xi = \exp\{i\sigma\Delta t\}$ being the multiplier which maps $\zeta_{i,j}^n$ onto $\zeta_{i,j}^{n+1}$. The matrices $\mathbf{A}$ and $\mathbf{B}$ have the forms given on the following two pages; their components are defined by

$$\begin{aligned}
A_{\chi,k} &= i\rho h_a s_k \sin(\beta_\chi) , & A_{\lambda,k} &= \frac{i\rho h_a s_k \sin(\beta_\lambda)}{\theta_2} , & \rho &= \frac{\Delta t}{\Delta \chi} , \\
\beta_\chi &= \frac{1}{2} m_\chi \Delta \chi , & \beta_\lambda &= \frac{1}{2} \theta_1 \theta_2 m_\chi \Delta \chi , & S_\eta &= \frac{\bar{N}_a \Delta t}{2(h_a \Delta \eta_k)^2} , \\
B_\chi &= \frac{2iC_\chi^2}{\rho h_a} \sin(\beta_\chi) , & B_\lambda &= \frac{2iC_\lambda^2}{\rho h_a} \sin(\beta_\lambda) , & \text{Period} &= \frac{L\Delta \chi N_L}{\sqrt{g h_a}} , \\
C_\chi &= \frac{\Delta t \sqrt{g h_a}}{L\Delta \chi} , & C_\lambda &= \frac{C_\chi}{\theta_2} , & \bar{N}_a &= \int_0^1 N(\eta) d\eta , \\
H_{\chi,k} &= -1 + i\mu' \sin(2\beta_\chi) + i\nu' \sin(2\beta_\lambda) + 4N'_\chi \sin^2(\beta_\chi) + 4N'_\lambda \sin^2(\beta_\lambda) + G_{\chi,k} , \\
H_{\lambda,k} &= -1 + i\mu' \sin(2\beta_\chi) + i\nu' \sin(2\beta_\lambda) + 4N'_\chi \sin^2(\beta_\chi) + 4N'_\lambda \sin^2(\beta_\lambda) + G_{\lambda,k} , \\
\Gamma_k^- &= \frac{S_\eta}{\bar{N}_a(r_k + 1)} \{r_k^2(2 - r_k)N_k + r_k^3 N_{k-1}\} , \\
\Gamma_k &= \frac{S_\eta}{\bar{N}_a(r_k + 1)} \{-N_{k+1} + (r_k + 1)(r_k^2 - 3r_k + 1)N_k - r_k^3 N_{k-1}\} , \\
\Gamma_k^+ &= \frac{S_\eta}{\bar{N}_a(r_k + 1)} \{N_{k+1} + (2r_k - 1)N_k\} , & f' &= f_a \Delta t , \\
G_{\chi,k} &= 2C_\chi^2 \sin^2(\beta_\chi) s_k , & P &= f' \cos(\beta_\lambda) \exp(i\beta_\chi)/2 , \\
G_{\lambda,k} &= 2C_\lambda^2 \sin^2(\beta_\lambda) s_k , & P_b &= f' \cos(\beta_\lambda) \exp(-i\beta_\chi)/2 , \\
Q_k &= 2C_\chi C_\lambda \sin(\beta_\chi) \sin(\beta_\lambda) s_k , & \mu' &= \frac{\mu_a \Delta t}{\Delta \chi} , \\
N'_\chi &= \frac{N_\chi \Delta t}{L^2 \Delta \chi^2} , & N'_\lambda &= \frac{N_\lambda \Delta t}{\theta_2^2 L^2 \Delta \chi^2} , & \nu' &= \frac{\nu_a \Delta t}{\theta_2 \Delta \chi} .
\end{aligned}$$

In the matrices $\mathbf{A}$ and $\mathbf{B}$ row number 1 corresponds to Equation (6.10), row 2 corresponds to the surface boundary condition for μ , rows 3 to $K+1$ correspond to Equations (6.13), row $K+2$ corresponds to the surface boundary condition for ν , and rows $K+3$ to $2K+1$ correspond to Equations (6.14). The column numbers correspond to the unknown variables in the vector ψ . The Q and G terms are introduced by the “corrector” scheme, used in this example, where integrated velocity values, which are approximated by summation, originate from the first approximation ζ^* used in the

														<i>Row no.</i>
	1	$-A_{\chi,0}$	$-A_{\chi,1}$	$-A_{\chi,2}$	$-A_{\chi,3}$	...	$-A_{\chi,K-1}$	$-A_{\lambda,0}$	$-A_{\lambda,1}$	$-A_{\lambda,2}$	$-A_{\lambda,3}$	...	$-A_{\lambda,K-1}$	1
	0	$-a_0$	$-b_0$	$-c_0$	0	...	0	0	0	0	0	...	0	2
	$-B_\chi$	$\Gamma_1^- - G_{\chi,0}$	$\Gamma_1 - H_{\chi,1}$	$\Gamma_1^+ - G_{\chi,2}$	$-G_{\chi,3}$	...	$-G_{\chi,K-1}$	$-Q_0$	$P - Q_1$	$-Q_2$	$-Q_3$	...	$-Q_{K-1}$	3
	$-B_\chi$	$-G_{\chi,0}$	$\Gamma_2^- - G_{\chi,1}$	$\Gamma_2 - H_{\chi,2}$	$\Gamma_2^+ - G_{\chi,3}$	...	$-G_{\chi,K-1}$	$-Q_0$	$-Q_1$	$P - Q_2$	$-Q_3$	...	$-Q_{K-1}$	4
	$-B_\chi$	$-G_{\chi,0}$	$-G_{\chi,1}$	$\Gamma_3^- - G_{\chi,2}$	$\Gamma_3 - H_{\chi,3}$	...	$-G_{\chi,K-1}$	$-Q_0$	$-Q_1$	$-Q_2$	$P - Q_3$	...	$-Q_{K-1}$	5
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
A =	$-B_\chi$	$-G_{\chi,0}$	$-G_{\chi,1}$	$-G_{\chi,2}$	$-G_{\chi,3}$	...	$\Gamma_{K-1} - H_{\chi,K-1}$	$-Q_0$	$-Q_1$	$-Q_2$	$-Q_3$	...	$P - Q_{K-1}$	$K + 1$
	0	0	0	0	0	...	0	$-a_0$	$-b_0$	$-c_0$	0	...	0	$K + 2$
	$-B_\lambda$	$-Q_0$	$-P - Q_1$	$-Q_2$	$-Q_3$	...	$-Q_{K-1}$	$\Gamma_1^- - G_{\lambda,0}$	$\Gamma_1 - H_{\lambda,1}$	$\Gamma_1^+ - G_{\lambda,2}$	$-G_{\lambda,3}$	...	$-G_{\lambda,K-1}$	$K + 3$
	$-B_\lambda$	$-Q_0$	$-Q_1$	$-P - Q_2$	$-Q_3$	...	$-Q_{K-1}$	$-G_{\lambda,0}$	$\Gamma_2^- - G_{\lambda,1}$	$\Gamma_2 - H_{\lambda,2}$	$\Gamma_2^+ - G_{\lambda,3}$	...	$-G_{\lambda,K-1}$	$K + 4$
	$-B_\lambda$	$-Q_0$	$-Q_1$	$-Q_2$	$-P - Q_3$	...	$-Q_{K-1}$	$-G_{\lambda,0}$	$-G_{\lambda,1}$	$\Gamma_3^- - G_{\lambda,2}$	$\Gamma_3 - H_{\lambda,3}$	...	$-G_{\lambda,K-1}$	$K + 5$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	$-B_\lambda$	$-Q_0$	$-Q_1$	$-Q_2$	$-Q_3$	...	$-P - Q_{K-1}$	$-G_{\lambda,0}$	$-G_{\lambda,1}$	$-G_{\lambda,2}$	$-G_{\lambda,3}$	...	$\Gamma_{K-1} - H_{\lambda,K-1}$	$2K + 1$
<i>Col.no.</i>	1	2	3	4	5	...	$K + 1$	$K + 2$	$K + 3$	$K + 4$	$K + 5$	...	$2K + 1$	

$$\mathbf{B} = \begin{bmatrix}
 1 & A_{X,0} & A_{X,1} & A_{X,2} & A_{X,3} & \dots & A_{X,K-1} & A_{\lambda,0} & A_{\lambda,1} & A_{\lambda,2} & A_{\lambda,3} & \dots & A_{\lambda,K-1} \\
 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 0 & -\Gamma_1^- & 1-\Gamma_1 & -\Gamma_1^+ & 0 & \dots & 0 & 0 & -P_b & 0 & 0 & \dots & 0 \\
 0 & 0 & -\Gamma_2^- & 1-\Gamma_2 & -\Gamma_2^+ & \dots & 0 & 0 & 0 & -P_b & 0 & \dots & 0 \\
 0 & 0 & 0 & -\Gamma_3^- & 1-\Gamma_3 & \dots & 0 & 0 & 0 & 0 & -P_b & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & 0 & 0 & \dots & 1-\Gamma_{K-1} & 0 & 0 & 0 & 0 & \dots & -P_b \\
 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & P_b & 0 & 0 & \dots & 0 & -\Gamma_1^- & 1-\Gamma_1 & -\Gamma_1^+ & 0 & \dots & 0 \\
 0 & 0 & 0 & P_b & 0 & \dots & 0 & 0 & -\Gamma_2^- & 1-\Gamma_2 & -\Gamma_2^+ & \dots & 0 \\
 0 & 0 & 0 & 0 & P_b & \dots & 0 & 0 & 0 & -\Gamma_3^- & 1-\Gamma_3 & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & 0 & 0 & \dots & P_b & 0 & 0 & 0 & 0 & \dots & 1-\Gamma_{K-1}
 \end{bmatrix}$$

Row
no.

1

2

3

4

5

$\vdots$

$K+1$

$K+2$

$K+3$

$K+4$

$K+5$

$\vdots$

$2K+1$

Col. no. 1 2 3 4 5 ... $K+1$ $K+2$ $K+3$ $K+4$ $K+5$... $2K+1$

velocity equations. In matrices **A** and **B**, elements in rows 2 to $K+1$ and columns $K+2$ to $2K+1$ and in rows $K+2$ to $2K+1$ and columns 2 to $K+1$ link the two velocity component equations, hence the P terms, which are the Coriolis terms, run diagonally through those submatrices. The dominant terms running almost tri-diagonally through both **A** and **B** are the Γ terms which arise from the vertical eddy viscosity term in each of Equations (6.13) and (6.14).

A non-trivial solution of the matrix Equation (6.26) exists if and only if ξ is an eigenvalue of this extended eigen-equation – the normal eigen problem is obtained by premultiplication of Equation (6.26) by the inverse of the matrix **B**. The $2K+1$ eigenvalues ξ can be found using a computer package (for example the NAG routine F02GJF or the IMSL routine GVCCG).

By defining N_L as the number of $\Delta\chi$ grid spacings in a wave length of the χ component of the horizontal wave motion, namely

$$N_L = 2\pi/(m_\chi \Delta\chi) , \quad (6.27)$$

the parameters β_χ and β_λ may be expressed in terms of N_L by means of the relations

$$\beta_\chi = \frac{\pi}{N_L} , \quad \beta_\lambda = \frac{\theta_1 \theta_2 \pi}{N_L} . \quad (6.28)$$

Therefore the eigenvalues ξ obtained by solving Equation (6.26) are functions of the non-dimensional parameters N_L , C_χ , S_η , θ_1 , θ_2 , μ' , ν' , N'_χ , N'_λ and f' .

Clearly, the finite difference Equations (6.10) and (6.13) to (6.16) will be stable if

$$|\xi| \leq 1 , \quad (6.29)$$

for each N_L for a particular Courant number C_χ . Since the flow being modelled is oscillatory, this criterion for stability has 1 on the right hand side; for exponential growth models it is taken to be $1 + O(\Delta t)$. Courant instability occurs when the

Courant number exceeds some characteristic value above which some of the values of $|\xi|$ are larger than the value one for at least one of the different values of N_L .

Stability of a numerical scheme does not imply that the results obtained are accurate. Provided the finite difference equations are consistent with the original partial differential equations, stability is a necessary and sufficient condition for convergence in the limit as Δt , $\Delta \chi$, $\Delta \lambda$ and $\Delta \eta_k$ all $\rightarrow 0$ (see Lax and Richtmyer (1956)). However, as a numerical model is run for finite values of the grid spacings, it is of practical use to determine the accuracy of a numerical scheme when the grid spacings are finite. In such a case, with the initial conditions (6.20) to (6.22), the solution of the numerical scheme given by Equations (6.10), (6.13) and (6.14) may be compared to the analytic solution of Equations (5.1) to (5.4), which is the true wave solution obtained in Chapter 5 and given by Equation (6.1) to (6.3), to determine the accuracy of the numerical scheme.

The numerical solution, which approximates the true wave solution, will be a linear combination of all the different numerical modes propagating in the model. These numerical modes will be the eigenvectors of Equation (6.26). By defining

$$\begin{aligned} \underline{v}^n &= \underline{v} \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} \\ &= \left[\zeta_{i,j}^n \quad \mu_{i,j,0}^n \quad \mu_{i,j,1}^n \quad \cdots \quad \mu_{i,j,K-1}^n \quad \nu_{i,j,0}^n \quad \nu_{i,j,1}^n \quad \cdots \quad \nu_{i,j,K-1}^n \right]^T, \end{aligned} \quad (6.30)$$

where $\underline{v}^n$ is the numerical solution at the n^{th} time level, then

$$\begin{aligned} \xi \underline{v}^n &= \exp\{i \sigma \Delta t\} \underline{v} \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} \\ &= \underline{v} \exp\{i(\sigma(n+1) \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} \\ &= \underline{v}^{n+1} \end{aligned} \quad (6.31)$$

and the matrix Equation (6.26) can be written in the form

$$\underline{v}^{n+1} = \mathbf{B}^{-1} \mathbf{A} \underline{v}^n. \quad (6.32)$$

Therefore

$$\begin{aligned}
v^n &= B^{-1} A v^{n-1} \\
&= (B^{-1} A)^2 v^{n-2} \\
&= \vdots \\
&= (B^{-1} A)^n v^0,
\end{aligned} \tag{6.33}$$

or by using eigenvector theory

$$v^n = \sum_{q=1}^{2K+1} c_q (\xi_q)^n v_q, \tag{6.34}$$

where ξ_q is the q^{th} eigenvalue of Equation (6.26), v_q the corresponding eigenvector and c_q a coefficient determined by the initial condition v^0 and the eigenvectors, that is the c_q are the coefficients of the linear combination of the eigenvectors which make up v^0 .

$$v^0 = \sum_{q=1}^{2K+1} c_q v_q. \tag{6.35}$$

For the numerical method considered as an illustration for defining this analysis, the value of K is 16, therefore there are 33 eigenvalues and 33 corresponding eigenvectors. (All other details of this case are given later.) The μ component of each eigenvector (that is elements 2 to $K+1$) can be considered as a vertical mode propagating in the model and have been graphed in Figures 6.2 to 6.5. As can be seen from Equation (6.34) the μ component of total numerical motion is a linear combination of all these 33 modes. For this particular case c_{23} is (0.99983, 0.00181) and the largest of other coefficients is c_{24} with (−0.00064, −0.00328). Other coefficients slightly smaller than c_{24} are c_{25} to c_{29} , c_{32} and c_{33} . The modulus of c_{23} is 300 times larger than the modulus of c_{24} , therefore the dominant mode is that corresponding to the 23rd eigenvalue, which is also the value of σ closest to the true wave solution. For the example used here, Table 6.1 lists all the eigenvalues, the modulus of the coefficients c_q and the modulus of the coefficients of the eigenvectors after one period, that is $c_q(\xi_q)^{\text{Period}/\Delta t}$ and from this it can be seen

that most of the modes in the numerical model contribute nothing to the solution after one period of time.

The errors in the numerical model, either from any initial conditions, or from round off errors, also propagate as a linear combination of the $2K+1$ different modes graphed in Figures 6.2 to 6.5. All of the eigenvalues in this example are less than one (if at least one was greater than one then the method is unstable for the given parameters) and most are significantly less than one, therefore any numerical errors propagating in these modes will diminish rapidly. In the numerical model the mode which is closest to the true wave solution, that is the mode whose ξ_q minimises $|-i \log(\xi_q)/\Delta t - \hat{\sigma}|$, is called the true numerical mode and its value of ξ_q will be denoted ξ . The errors which will be propagated without diminishing in the numerical model are those whose modes correspond to the eigenvalues which have magnitudes close to or greater than the value of ξ , and less than or equal to the value one. For the example used here, the eigenvalues ξ_{24} and ξ_{25} have a modulus very close to that of ξ_{23} , and the eigenvalues ξ_{32} and ξ_{33} have a modulus of exactly one. This means that the part of any numerical error in the modes corresponding to ξ_{32} and ξ_{33} , will propagate indefinitely and never be lost from the system. This is not drastic as the scheme is still stable and any errors introduced for one step in the numerical scheme are just as likely to be negated in the next step.

Two parameters which indicate discrepancies between the true wave propagating in the $\chi-\lambda$ plane and the corresponding approximate motion produced by the numerical model, are the ratios of the amplitudes and the speeds of the propagating waves. These can be derived from the values of ξ for the numerical wave and $\hat{\sigma}$ for the true wave, in the following manner.

For the true wave solution, Equation (5.42) and the iterative scheme described in Section 5.3, can be used to find the the complex value of $\hat{\sigma}$, which may be written in

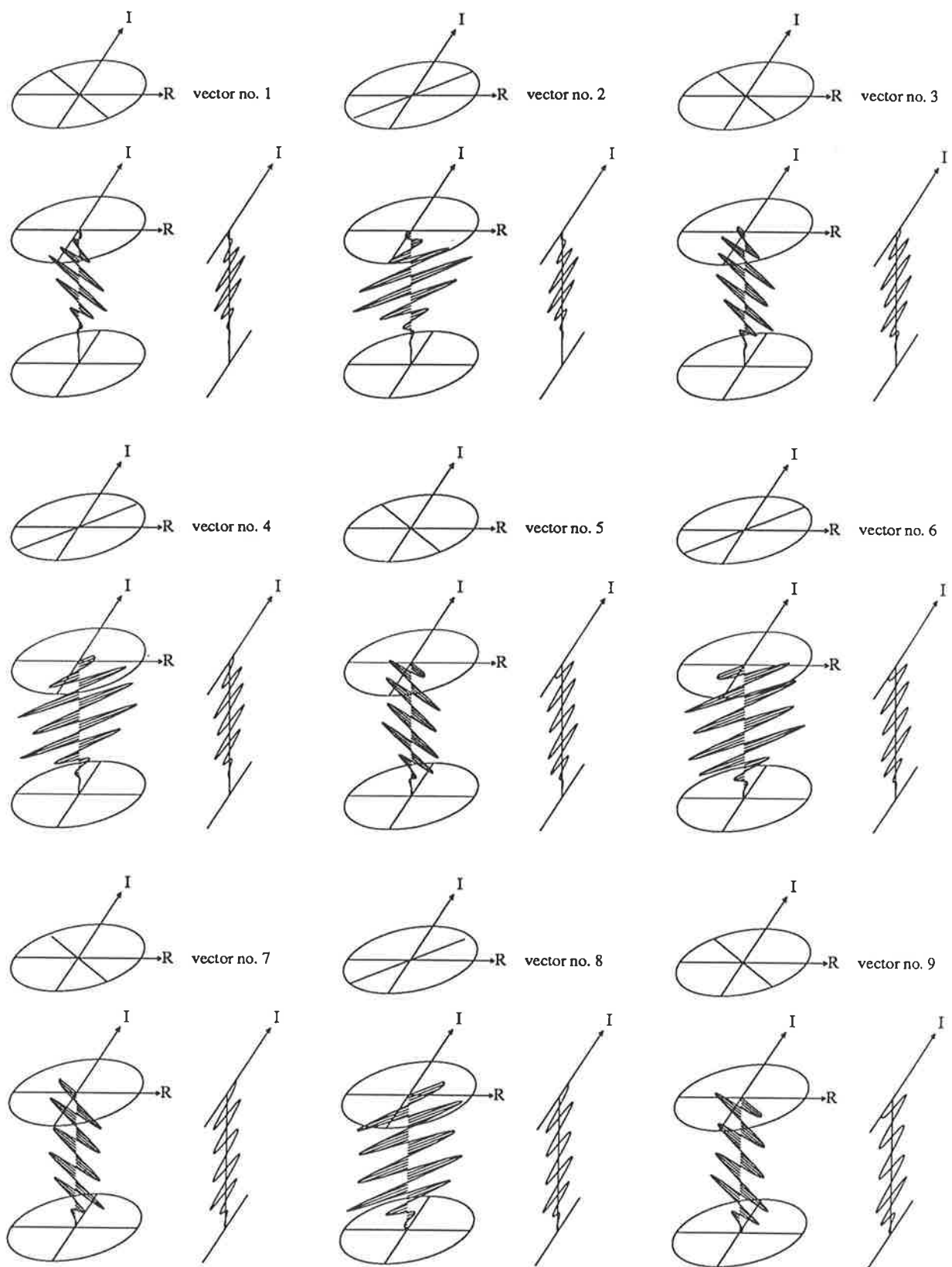


Figure 6.2: The μ components of eigenvectors 1 to 9 from Equation (6.26).

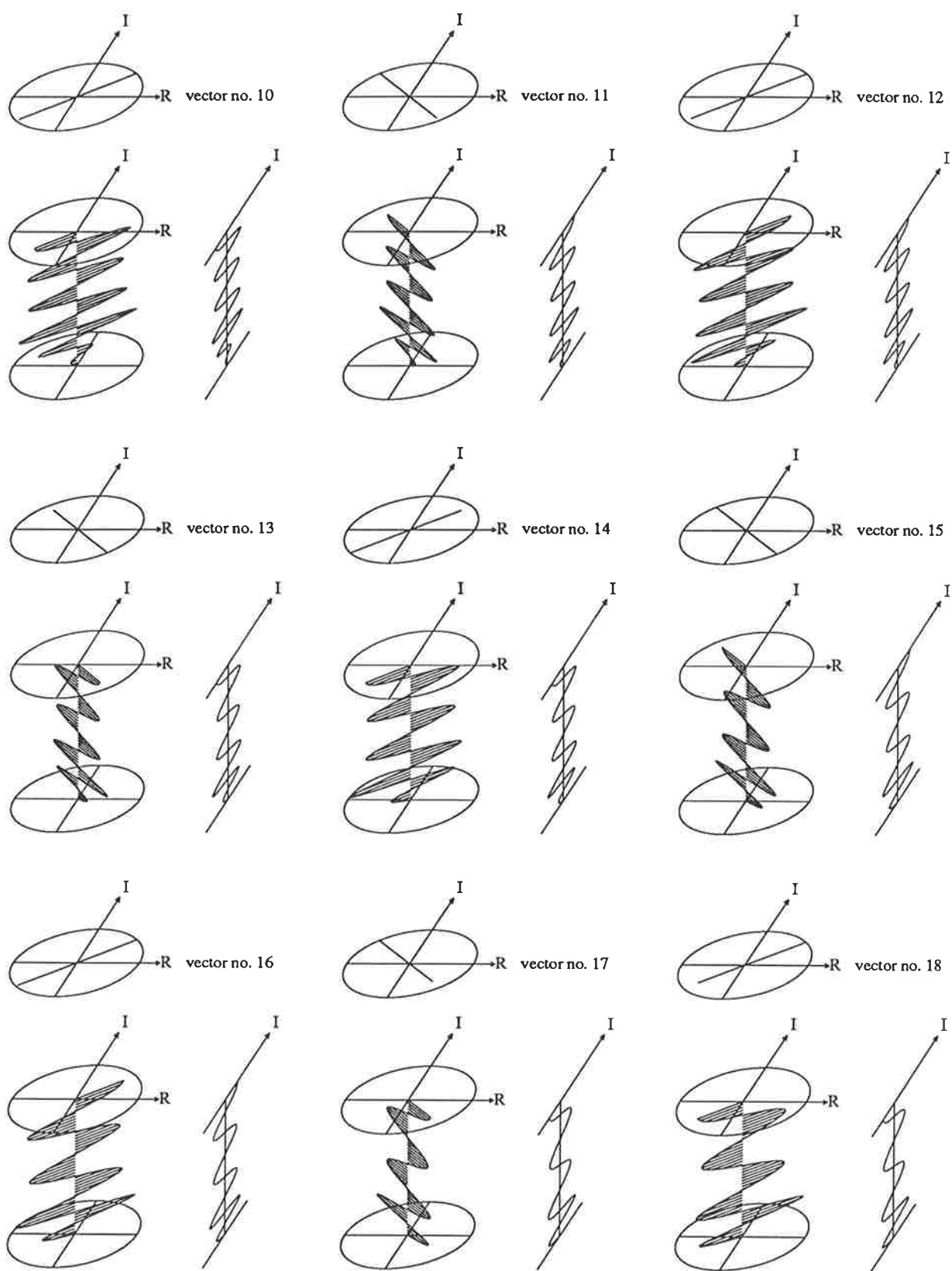


Figure 6.3: The μ components of eigenvectors 10 to 18 from Equation (6.26).

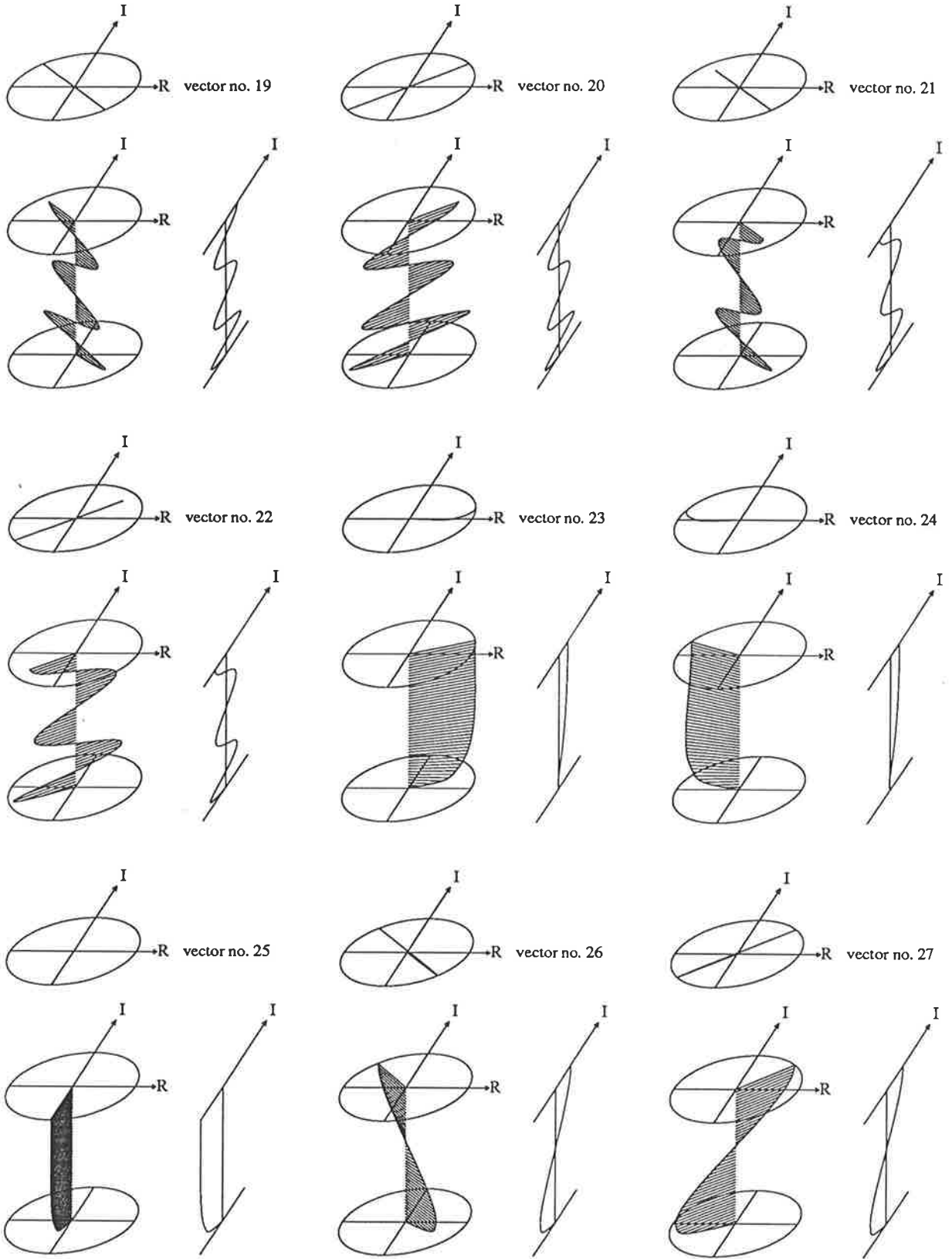


Figure 6.4: The μ components of eigenvectors 19 to 27 from Equation (6.26).

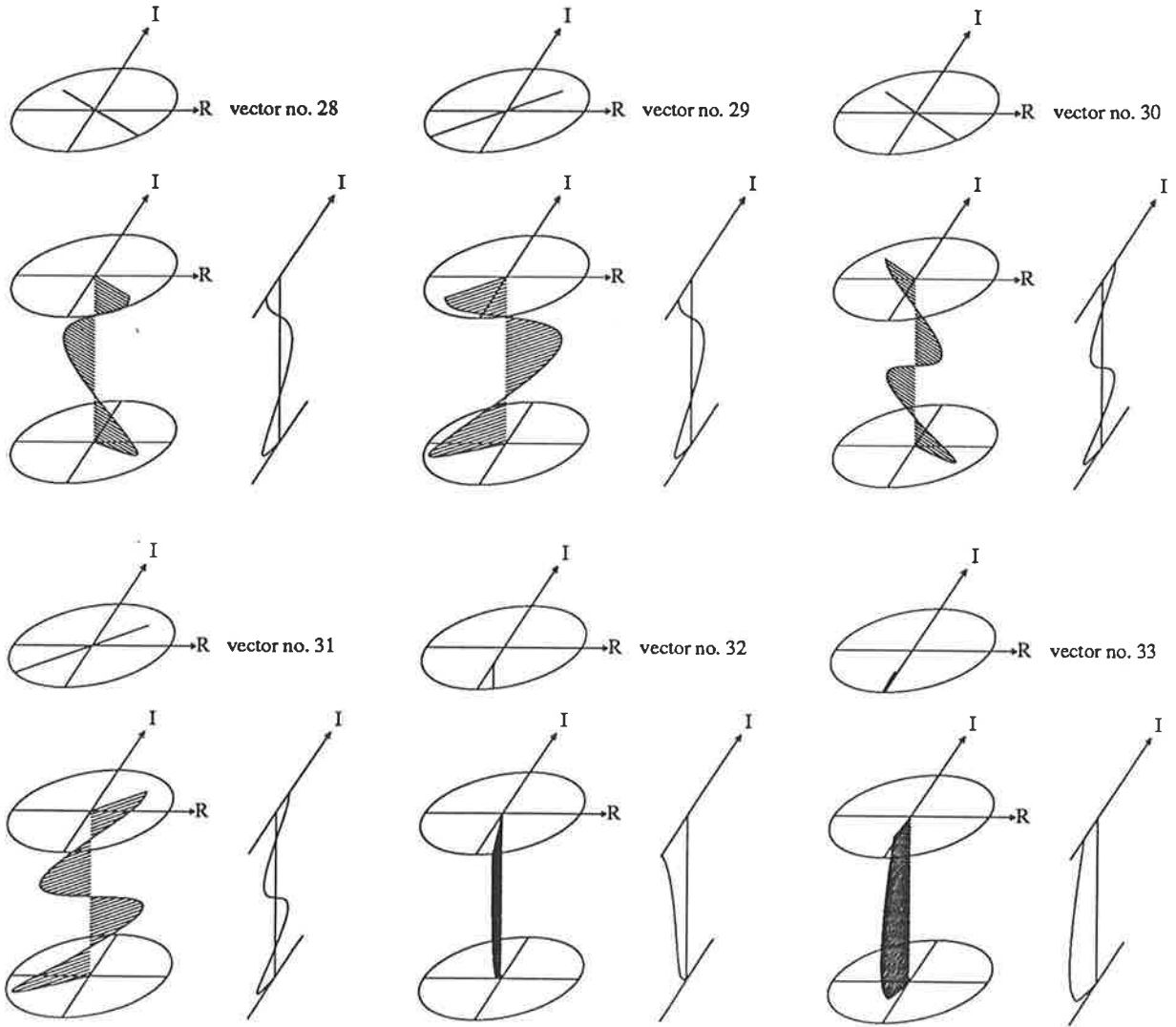


Figure 6.5: The μ components of eigenvectors 28 to 33 from Equation (6.26).

p	ξ_q	$ c_q $	$ c_q(\xi_q)^{\text{Period}/\Delta t} $
1	(0.85435, -0.00086)	0.00000	0.00000
2	(0.85435, 0.00076)	0.00000	0.00000
3	(0.86830, -0.00087)	0.00000	0.00000
4	(0.86830, 0.00077)	0.00000	0.00000
5	(0.88225, -0.00089)	0.00000	0.00000
6	(0.88225, 0.00078)	0.00000	0.00000
7	(0.89641, -0.00090)	0.00000	0.00000
8	(0.89641, 0.00080)	0.00000	0.00000
9	(0.91059, -0.00091)	0.00000	0.00000
10	(0.91059, 0.00081)	0.00000	0.00000
11	(0.92448, -0.00092)	0.00000	0.00000
12	(0.92448, 0.00082)	0.00000	0.00000
13	(0.93776, -0.00094)	0.00000	0.00000
14	(0.93776, 0.00083)	0.00000	0.00000
15	(0.95017, -0.00095)	0.00000	0.00000
16	(0.95016, 0.00084)	0.00000	0.00000
17	(0.96151, -0.00096)	0.00000	0.00000
18	(0.96151, 0.00085)	0.00000	0.00000
19	(0.97168, -0.00097)	0.00001	0.00000
20	(0.97168, 0.00086)	0.00000	0.00000
21	(0.98055, -0.00098)	0.00003	0.00000
22	(0.98055, 0.00087)	0.00002	0.00000
23	(0.99988, -0.00279)	0.99983	0.76946
24	(0.99988, 0.00273)	0.00334	0.00258
25	(0.99984, -0.00005)	0.00125	0.00088
26	(0.99784, -0.00099)	0.00146	0.00001
27	(0.99784, 0.00089)	0.00062	0.00000
28	(0.99382, -0.00099)	0.00037	0.00000
29	(0.99382, 0.00088)	0.00022	0.00000
30	(0.98800, -0.00098)	0.00007	0.00000
31	(0.98800, 0.00088)	0.00004	0.00000
32	(1.00000, 0.00000)	0.00053	0.00053
33	(1.00000, 0.00000)	0.00053	0.00053

Table 6.1: Table of eigenvalues ξ_q , the modulus of the coefficients c_q and the modulus of the coefficients of the eigenvectors after one period.

the form

$$\hat{\sigma} = -\hat{\phi} + i\hat{\psi} , \quad (6.36)$$

where $\hat{\phi} = -\text{Re}\{\hat{\sigma}\}$, $\hat{\psi} = \text{Im}\{\hat{\sigma}\}$. Then Equation (6.1) becomes

$$\zeta(\chi, \lambda, t) = a \exp\{-\hat{\psi}t\} \cdot \exp\{i(m_\chi\chi + m_\lambda\lambda - \hat{\phi}t)\} , \quad (6.37)$$

and

$$\text{Re}\{\zeta(\chi, \lambda, t)\} = a \exp\{-\hat{\psi}t\} \cdot \cos(m_\chi\chi + m_\lambda\lambda - \hat{\phi}t) . \quad (6.38)$$

For the true wave, after a time of $2\pi/|\hat{\phi}|$ (the time between two successive maximum tidal heights) the amplitude of the wave is multiplied by the factor

$$\alpha_T = \exp\left\{\frac{-2\pi\hat{\psi}}{|\hat{\phi}|}\right\} = \exp\left\{\frac{-2\pi\text{Im}\{\hat{\sigma}\}}{|\text{Re}\{\hat{\sigma}\}|}\right\} . \quad (6.39)$$

A similar expression holds for α_N for the numerical approximation, with ψ replacing $\hat{\psi}$, where

$$\psi = \text{Im}\{\sigma\} \quad \text{and} \quad \sigma = \frac{-i \log(\xi)}{\Delta t} , \quad (6.40)$$

and ξ is the eigenvalue for which $|\sigma - \hat{\sigma}|$ is closest to zero. Therefore the amplitude response, which is the ratio of the amplitude of the numerical wave to the true wave, is

$$A_r = \frac{\alpha_N}{\alpha_T} = \exp\left\{\frac{2\pi}{\text{Re}\{\hat{\sigma}\}}(\text{Im}\{\sigma\} - \text{Im}\{\hat{\sigma}\})\right\} , \quad \text{Re}\{\hat{\sigma}\} < 0 . \quad (6.41)$$

The relative wave speed which is the ratio of the numerical wave speed to that of the true wave is

$$S_r = \frac{\text{Re}\{\sigma\}}{\text{Re}\{\hat{\sigma}\}} . \quad (6.42)$$

In the expected range of values of θ_1 , θ_2 , μ' , ν' , N'_χ , N'_λ , f' , and S_η for the tidal motions being modelled, the values of A_r and S_r can be computed as functions of N_L and the Courant number C_χ . Ideally, both A_r and S_r should take the value one for all values of N_L and C_χ , whatever the values of S_η .

Of the $2K+1$ eigenvalues for each different value of N_L and C_x , one will correspond to the true wave solution and all the others will correspond to computational modes. The chosen eigenvalue is the one which corresponds to a value of the complex circular frequency σ which is closest to the value of $\hat{\sigma}$ from the true wave solution (that is $|\sigma - \hat{\sigma}|$ is minimised). As shown earlier the numerical solution is a linear combination of all the numerical modes but due to the initial conditions the mode corresponding to the chosen value of ξ has 300 times the amplitude of the next largest component, and the analysis is therefore restricted to looking at this one largest component. This chosen eigenvalue ξ will be the number from which σ , A_r and S_r are calculated. Also, for this eigenvalue a corresponding eigenvector $\mathbf{v}$ can be calculated, which can be normalised so that $\zeta^\otimes = a$. This eigenvector can be used to analyse the distribution over depth of the errors introduced by the numerical scheme. By comparing $\mu_k^\otimes$ and $\nu_k^\otimes$ with the analytic solutions of $\mu^* F_\chi(\eta_k)$ and $\nu^* F_\lambda(\eta_k)$ at the different depth levels it can be seen how the numerical scheme differs from the analytic solution and an error distribution over depth can be graphed.

A typical set of results obtained by this analysis, using $\theta_1 = 1$, $\theta_2 = 1$, $K = 16$, $\kappa = 0$, $h_a = 20$ m, $g = 9.81$ m²/s, $\mu_a = 0.2$ m/s, $\nu_a = 0.2$ m/s, $N_\chi = 100$ m²/s, $N_\lambda = 100$ m²/s, $N_s = 0.003$ m²/s, $N_m = 0.005$ m²/s and $N_b = 0.0002$ m²/s, is graphed in Figures 6.6 to 6.12, where κ is a grid parameter (see Section 3.2), which gives a uniform grid spacing when set to zero. The period (time taken to travel one wavelength at speed $\sqrt{gh_a}$) is taken to be 12 hours, therefore $m_\chi/L = 1.03836 \times 10^{-4}$ m⁻¹ which leads to a value of $\hat{\sigma} = (-2.16627 \times 10^{-4}, 8.28604 \times 10^{-6})$ s⁻¹. Figures 6.6 and 6.7 show the amplitude response and relative wave speed plotted against the number of grid spacings per wavelength N_L for various Courant numbers C_x .

The graphs in Figures 6.6 and 6.7 do not tend to the number one exactly because as $N_L \rightarrow \infty$ for a fixed Courant number, only Δt , $\Delta\chi$ and $\Delta\lambda \rightarrow 0$, while the vertical

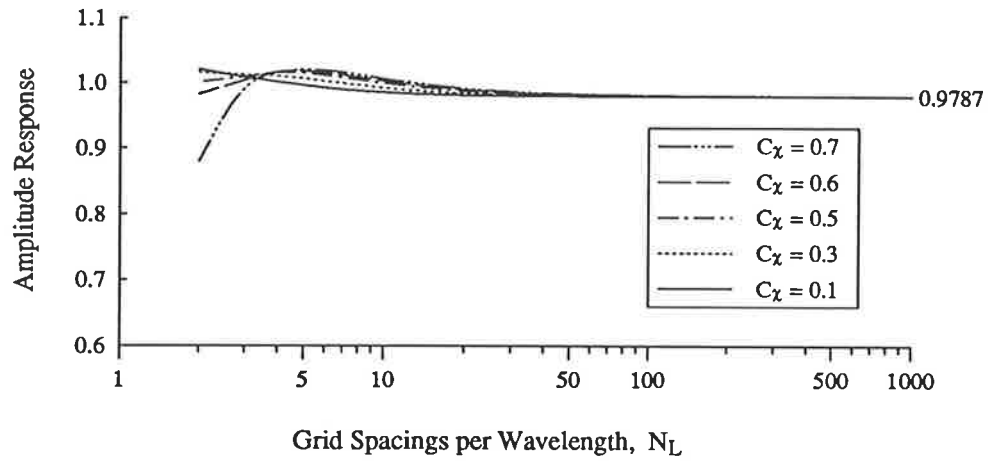


Figure 6.6: Amplitude response for the numerical scheme under consideration.

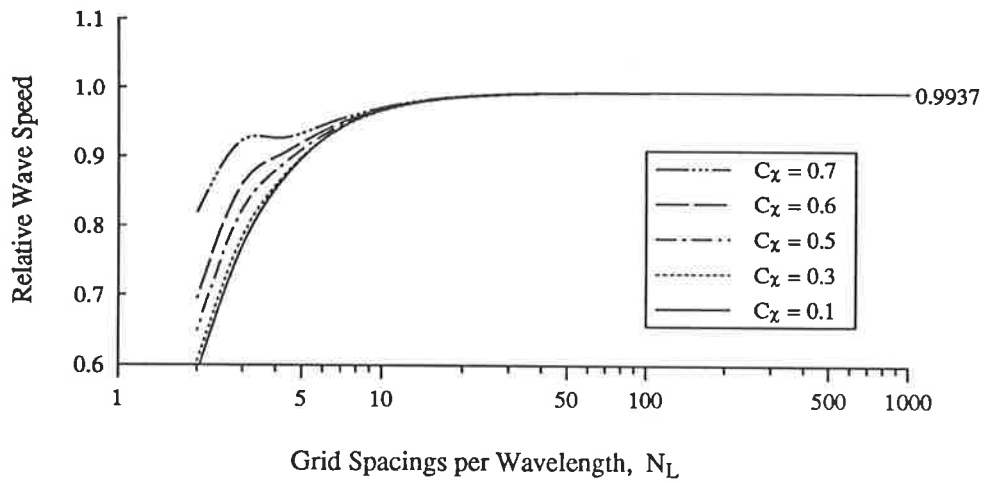


Figure 6.7: Relative wave speed for the numerical scheme under consideration.

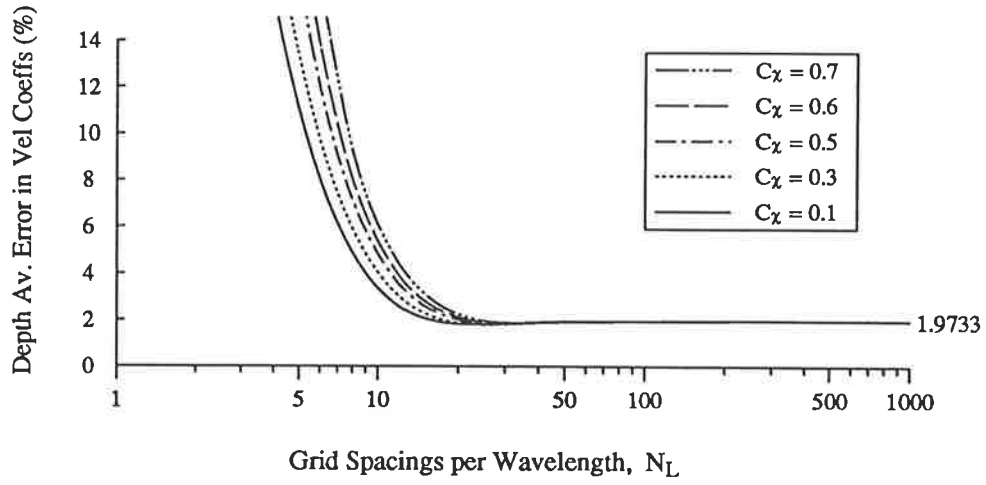


Figure 6.8: Error estimate for the numerically approximated values of $\mu_k^\otimes$ and $\nu_k^\otimes$.

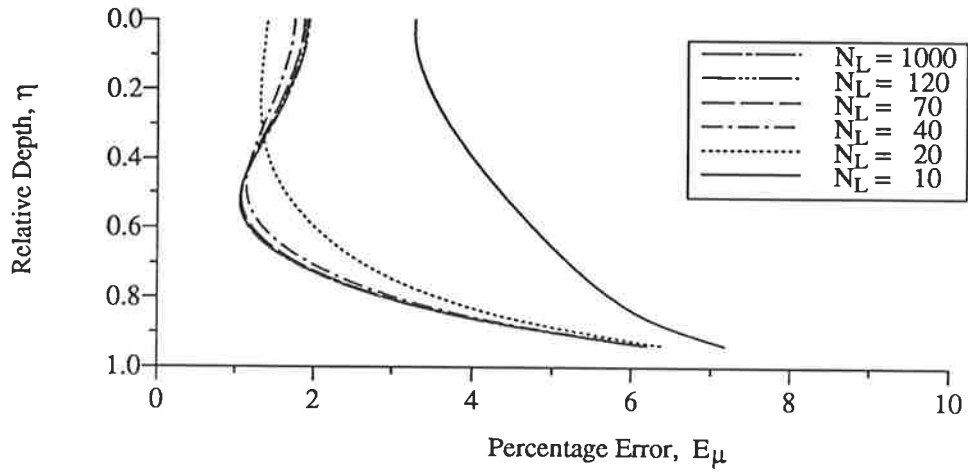


Figure 6.9: Error profiles for $\mu_k^\otimes$ for $C_\chi = 0.3$.

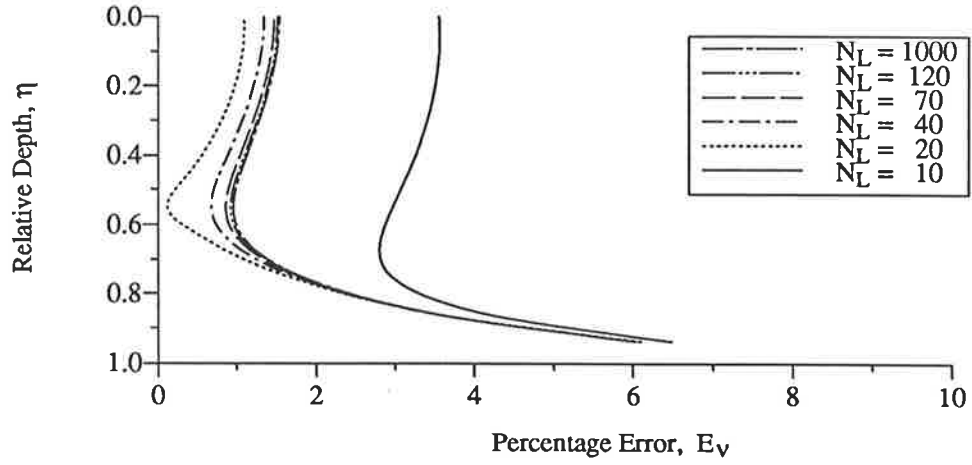


Figure 6.10: Error profiles for $\nu_k^\otimes$ for $C_x = 0.3$.

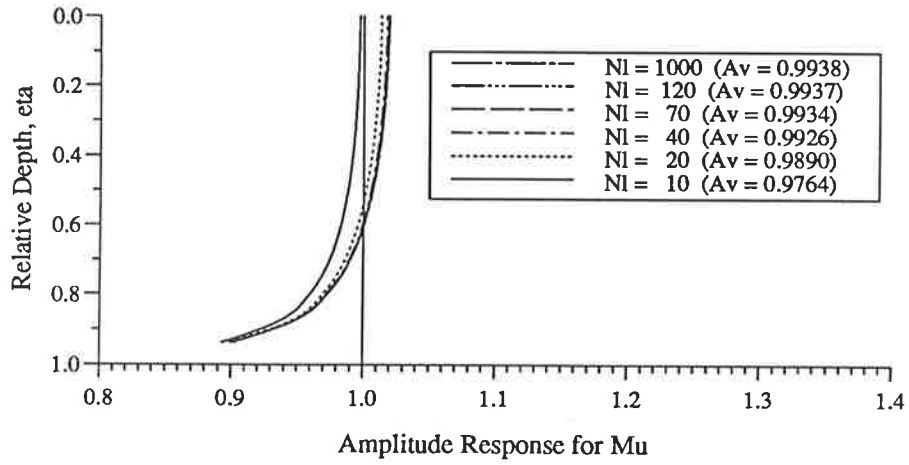


Figure 6.11: Variation over depth of amplitude response for $\mu_k^\otimes$, for $C_x = 0.3$.

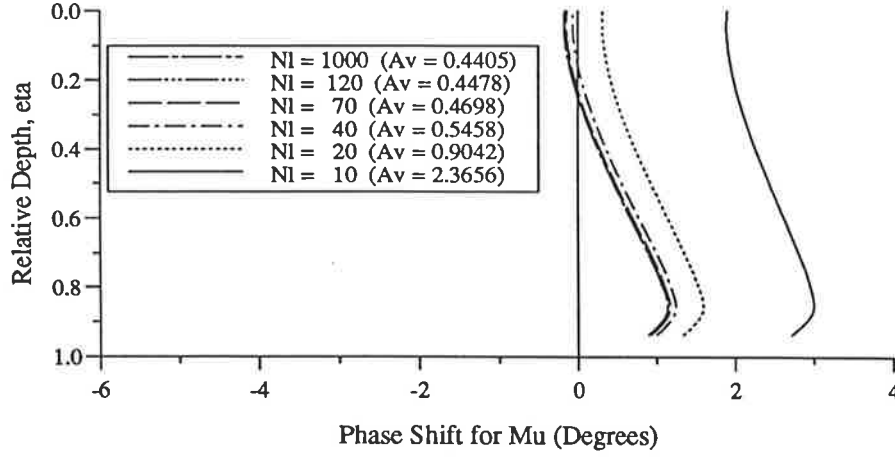


Figure 6.12: Variation over depth of phase shift error for $\mu_k^\otimes$, for $C_\chi = 0.3$.

grid spacings $\Delta\eta_k$ are fixed. Therefore the limiting values as $N_L \rightarrow \infty$ incorporate an error associated with the differencing on the vertical grid. These limits tend to one exactly as K tends to infinity, that is as $\Delta\eta_k$ tend to zero, but since all numerical models have to run for a finite value of K (the smaller the value of K the more cost effective the method) this analysis provides measures for the numerical model for any particular grid spacing. Also if the number of vertical grid spacings is kept constant and the distribution is changed, then there may either be an increase or decrease in the accuracy of the model (reflected in the limiting values of A_r and S_r), depending on whether the redistribution gives a better approximation to the velocity profile in the vertical or not. Thus for a given number of depth levels for the numerical model this analysis offers the possibility of finding an optimal grid distribution in the vertical for the specified parameters which should be chosen to match the conditions in the area that is to be modelled by the numerical scheme.

The limiting values of the amplitude response, relative wave speed and velocity error measure E_a , are the values for which $\Delta\chi$ and $\Delta\lambda$ tend to zero for a given vertical

grid distribution, but as can be seen from Figures 6.6 to 6.8 these are the same values as the values for $N_L = 40$, which means that the numerical model does not need to use a finer horizontal grid spacing. This also shows that the vertical direction is the one which dominates the error limits and is the one which needs to be modelled or represented accurately in a three-dimensional tidal model.

Figure 6.8 shows an averaged error estimate for the errors between the $\mu_k^\otimes$, $\nu_k^\otimes$ and their analytic values. This depth averaged percentage error in velocity coefficients is denoted and defined by

$$E_a = \frac{1}{2KM} \left(\sum_{k=0}^{K-1} L \left| \mu_k^\otimes - \mu^* F_\chi(\eta_k) \right| + B \left| \nu_k^\otimes - \nu^* F_\lambda(\eta_k) \right| \right), \quad (6.43)$$

$$M = \max_{\eta} \left(L \left| \mu^* F_\chi(\eta) \right|, B \left| \nu^* F_\lambda(\eta) \right| \right) / 100. \quad (6.44)$$

That is E_a is the average complex distance between analytical and numerical velocity coefficients expressed as a percentage relative to the largest velocity coefficient.

As $N_L \rightarrow \infty$ the limiting value of the error, as with the amplitude response and relative wave speed, incorporates an error associated with the differencing on the vertical grid. As $\Delta\eta_k \rightarrow 0$ the error E_a tends to zero.

Figures 6.9 and 6.10 are graphed for the specific value of the Courant number $C_\chi = 0.3$ and show the distribution over depth of the errors graphed in Figure 6.8 for various values of N_L . The difference between the two graphs is due to the effect of Coriolis. The error measures used at different depth levels in Figures 6.9 and 6.10 are

$$E_\mu(\eta_k) = \frac{L}{M} \left| \mu_k^\otimes - \mu^* F_\chi(\eta_k) \right|, \quad E_\nu(\eta_k) = \frac{B}{M} \left| \nu_k^\otimes - \nu^* F_\lambda(\eta_k) \right|. \quad (6.45)$$

As $\mu^* F_\chi(\eta_k)$ is a complex number it can be considered as an amplitude and phase change to the exponential term in the analytic solution. Therefore by comparing $\mu_k^\otimes$

and $\mu^* F_\chi(\eta_k)$ an amplitude response over depth for μ can be calculated as

$$A_\mu = \frac{|\mu_k^\otimes|}{|\mu^* F_\chi(\eta_k)|} . \quad (6.46)$$

Also a phase shift error over depth for μ can be expressed as

$$P_\mu = \frac{\text{phase}(\mu_k^\otimes) - \text{phase}(\mu^* F_\chi(\eta_k))}{\pi/180} , \quad (6.47)$$

where the phase error is in degrees. Similar expressions can be deduced from $\nu_k^\otimes$ and $\nu^* F_\lambda(\eta_k)$. The graphs in Figures 6.11 and 6.12 show A_μ and P_μ for several values of N_L and for $C_\chi = 0.3$.

Little is gained by comparing these results with those from corresponding two-dimensional analyses as the numerical models are completely different due to the inclusion of discretisation in the vertical direction in this three-dimensional case. The only point worthy of note is that previous analyses of two-dimensional numerical schemes have only provided information similar to Figures 6.6 and 6.7.

6.2 Analysis for a Three Level Numerical Scheme

The wave propagation analysis method described in Section 6.1 is illustrated with the two level numerical scheme described in this thesis. This analysis can also be applied to three (or more) level numerical schemes. The procedure is just the same except after substituting Equations (6.23) to (6.25) into the finite difference equations, the matrix equation obtained is of the form

$$\mathbf{A}v = \xi \mathbf{B}v + \xi^2 \mathbf{C}v , \quad (6.48)$$

with the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ being similar to those given in Section 6.1, depending on the numerical scheme. The eigenvalues and eigenvectors of this equation can be found by making the substitution

$$\xi v = u , \quad (6.49)$$

which gives for Equation (6.48)

$$\mathbf{A}\mathbf{v} - \mathbf{B}\mathbf{u} = \xi\mathbf{C}\mathbf{u} . \quad (6.50)$$

Combining Equations (6.49) and (6.50) into a matrix equation give

$$\begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{A} & -\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \xi \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} , \quad (6.51)$$

where matrices $\mathbf{O}$ and $\mathbf{I}$ are the zero and identity matrices which are the same size as the other matrices, that is $(2K+1) \times (2K+1)$. The eigenvalues and the vector $\mathbf{v}$ from each corresponding eigenvector of Equation (6.51) are the eigenvalues and eigenvectors of Equation (6.48) and therefore the ones required for the wave propagation analysis. Similar principles can be applied for higher level methods.

6.3 Application of the Analysis

There are many applications for this analysis. It can be used to compare different numerical schemes, to determine the best finite difference form for each term in the governing differential equations or the best surface boundary approximation, or it can be used to optimise some of the parameters associated with numerical schemes, such as the number and location of the non-uniformly spaced vertical grid points. The analysis can also be used to estimate the magnitude and the distribution of the errors produced by a numerical method. Such results have previously not been available for a three-dimensional tidal model. The analysis can also be used to find the range of values of the non-dimensional parameters for which the difference equations used to solve the linearised governing equations are stable.

Before using this analysis appropriate values for the parameters for the numerical model need to be determined. The depth h_a , length L , breadth B , and the gravitational and Coriolis parameters are easily determined. The other parameters μ_a , ν_a , N_χ , N_λ

and N_η can be determined by measurements in the area being modelled or can be estimated from results obtained from previous model applications and/or experience. A major advantage of this wave propagation analysis is that it may be used with realistic forms for N_η which produce velocity profiles which match those of the region to be modelled. This particular set of N_η and other parameters used in the wave propagation analysis indicates the type of errors introduced by the numerical method for the model application. The form of the parameter N_η , which is variable over depth, effects the characteristics of the flow, so it is important to use the form which yields properties similar to those which occur in the area for which the numerical model will be applied. The predicted tidal elevations are fairly insensitive to this vertical eddy viscosity parameter, but the vertical velocity profile varies considerably when N_η is changed in either its distribution over depth or in its magnitude.

Even though the analytic solutions used in this analysis all have a “no-slip” condition at the sea floor, certain alterations to the form of N_η can vary the velocity profile from one which resembles a “slip” condition approximation near the sea floor, to one which resembles the typically logarithmically shaped profile which occurs with “no-slip” conditions. Figure 6.13 shows these two conditions: the parabolic case has $N_\eta = 0.003, 0.005, 0.001 \text{ m}^2/\text{s}$ at $\eta = 0, 0.414, 1$ respectively and the three piece linear case has $N_\eta = 0.003, 0.004, 0.004, 0.0000001 \text{ m}^2/\text{s}$ at $\eta = 0, 0.2, 0.97, 1$ respectively. Analytic solutions for these and other forms of N_η exist (see, for example Sverdrup (1926), Fjeldstat (1929), Tee (1987) and Section 5.2) and so can be used in this wave propagation analysis.

An example of the way in which this method can expose inaccurate differencing of a term in the tidal motion equations is seen by considering the Coriolis term. In the differencing given in Equations (6.13) and (6.14) the Coriolis terms are centred in time about the $(n+\frac{1}{2})$ time level. If the terms are evaluated with all the velocity

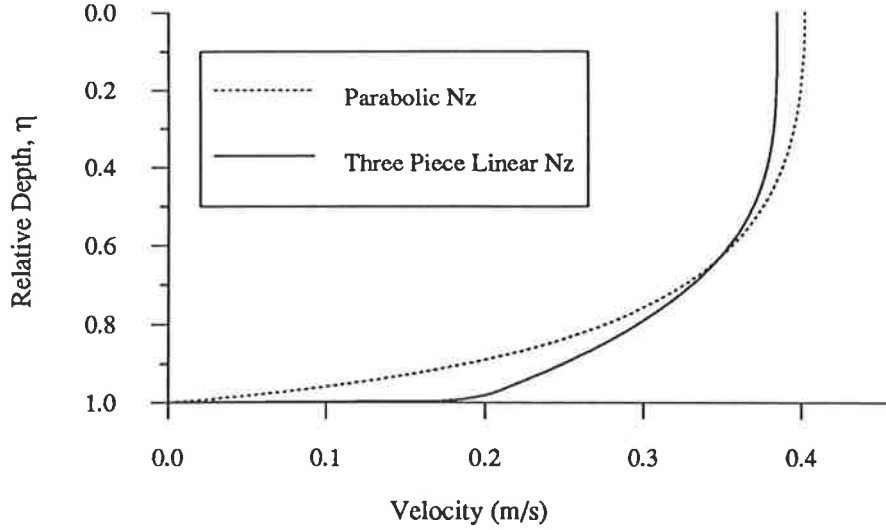


Figure 6.13: Two velocity profiles, which give analytic solutions to Equations (5.1) to (5.4) from the many profiles available by choosing particular forms for N_η . The profiles are at maximum flow.

values at the n^{th} time level, the graph of the amplitude response (see Figure 6.14) for the resulting numerical method has a large peak centred near 6 grid spacings per wave length. This means that any waves in the initial conditions or in the round-off error distribution, propagating with a wavelength about 6 times the horizontal grid spacing of the model and with a Courant number of 0.5 or larger, will have an amplitude which increases by more than 13% over each true wave period.

This analysis also enables an optimal grid spacing in the vertical to be determined. Clearly, this would be specific to the conditions for which the model would be run. This emphasises the need to choose a vertical eddy viscosity form which provides an analytic solution which closely matches the conditions that are to be modelled. By varying the number and distribution of vertical grid spacings and analysing the resultant numerical errors, a balance between computational effort and accuracy can be achieved. The optimal distribution of vertical grid spacings depends on the form of N_η and the resulting velocity profile, especially near the sea floor. If the grid spacing is not

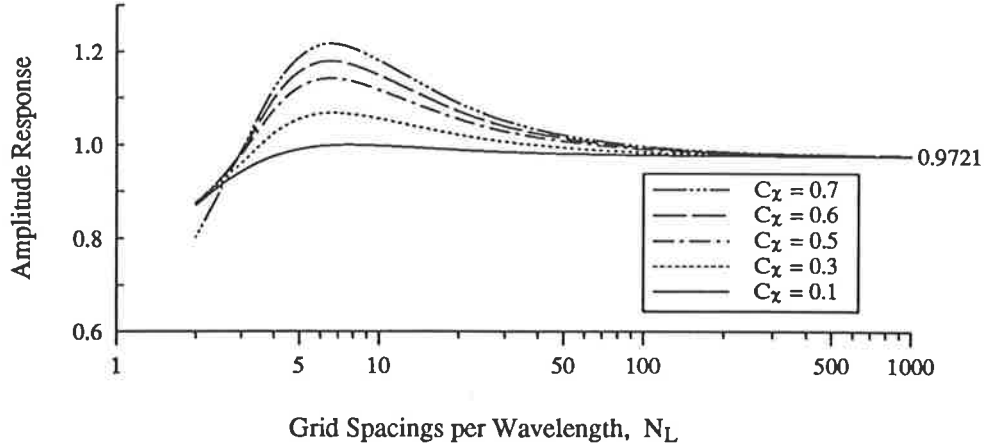


Figure 6.14: When the Coriolis term is not differenced accurately, the amplitude response indicates that something is wrong.

chosen wisely large errors are introduced through the finite difference approximations. In order to minimise these errors the variable grid spacing defined by the “ κ method” was developed by Noye (1984a), see Section 3.2. This method ensures that the vertical spatial differencing remains second order accurate and minimises errors. The value of $\kappa = 0$ produces an equally spaced grid, and as κ increases the grid becomes finer near $\eta = 1$ and coarser near $\eta = 0$ as illustrated in Figure 3.10.

The optimal value for κ is highly dependent on the form and magnitude of N_η , and as long as a realistic form for N_η is used in the wave propagation analysis, the optimal value of κ obtained will be a good choice to use in the numerical model. An advantage of using this wave propagation analysis is that the error estimates are calculated by analytic means and there is no need to run many time consuming numerical tests with the model for different values of κ in order to determine its optimal value.

Before optimising the value of κ the objectives for minimisation need to be established. There are several ways of measuring performance, and their relative importance needs to be established for the particular case for which the model is to be run. The aim

may be to minimise the amplitude response errors, minimise the relative wave speed errors, minimise the total error in the velocity components, or it may be important to minimise the errors in velocity near the sea bed. Once the relative importance of these objectives has been established, the analysis may be used to produce a set of graphs similar to Figures 6.6 to 6.10 for several different values of κ in order to determine the best value to use. Figure 6.15 shows the percentage error for different κ values for the first three of the above mentioned criteria. Notice that the performance of the numerical model with a non-uniform grid can be greatly improved over the corresponding uniform grid ($\kappa = 0$) for the same number of depth levels.

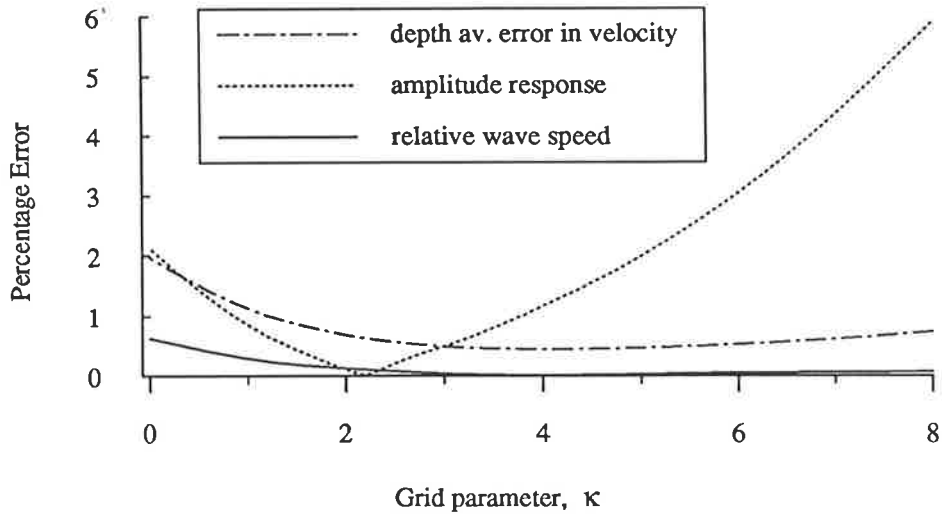


Figure 6.15: The percentage errors for various criteria that may be optimised by appropriate selection of the vertical grid parameter κ .

Figures 6.16 to 6.22 are the corresponding graphs for Figures 6.6 to 6.12 for a vertical grid with a value of $\kappa = 3$. As can be seen from these graphs this gives errors for the numerical model significantly smaller than for the uniform grid spacing ($\kappa = 0$) graphed in Figures 6.6 to 6.12.

A feature of this analysis is its ability to obtain error distributions over depth for

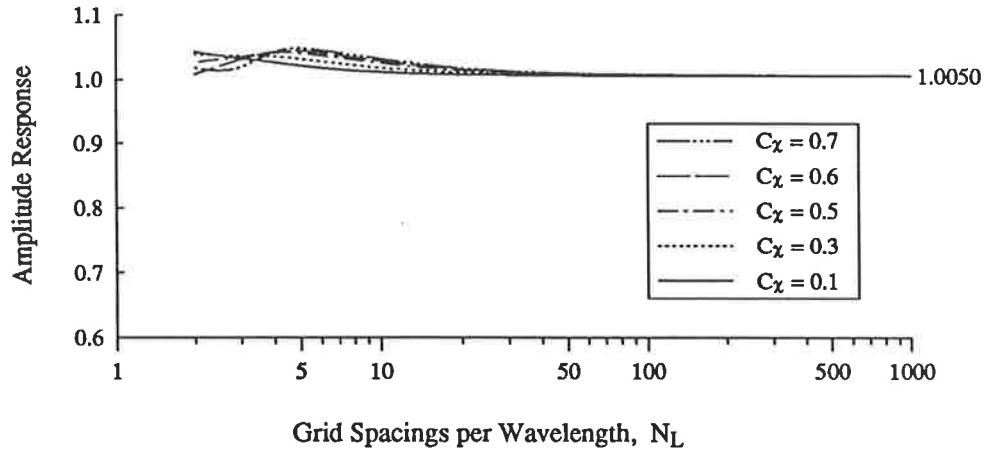


Figure 6.16: Amplitude response for the numerical scheme under consideration with $\kappa = 3$.

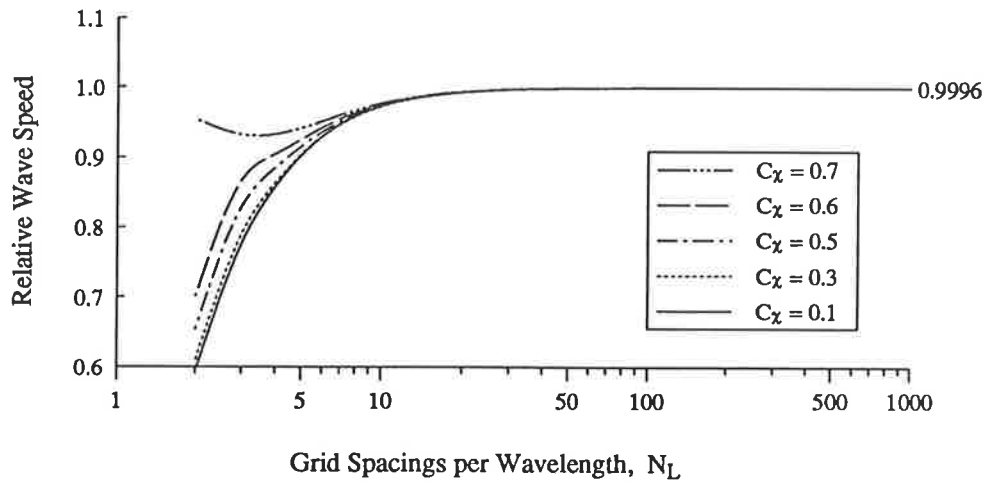


Figure 6.17: Relative wave speed for the numerical scheme under consideration with $\kappa = 3$.

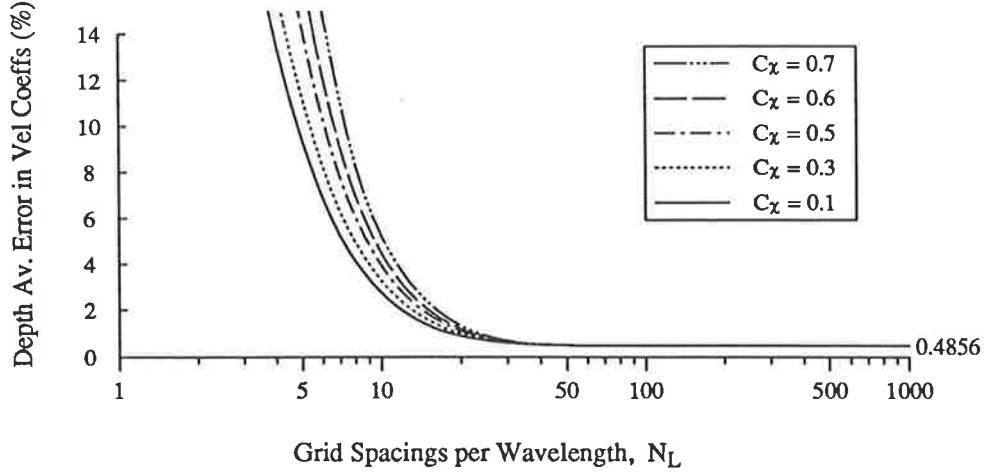


Figure 6.18: Error estimate for the numerically approximated values of $\mu_k^\otimes$ and $\nu_k^\otimes$ for $\kappa = 3$.

the velocity components, as shown in Figures 6.9 to 6.12. This analysis can be used to detect the errors near the sea floor, where steep velocity gradients can cause larger errors in the finite difference approximations. Compared to the uniform grid case, for the $\kappa = 3$ case, the amplitude response in the limit as $N_L \rightarrow \infty$, improves from 0.9787 to 1.0050, the relative wave speed improves from 0.9937 to 0.9996, and the depth averaged error in the velocity coefficients E_a improves from 1.97% to 0.49%. The reason for this improvement is evident in the comparison of Figures 6.9 to 6.12 with Figures 6.19 and 6.22. This comparison clearly shows the improvement gained by using a variable grid in the vertical direction to obtain increased resolution near the sea floor. Due to this improvement in the representation of the velocity profile in the vertical, the limiting values of the other three graphs are closer to the true value of one than those obtained when using a uniform grid spacing.

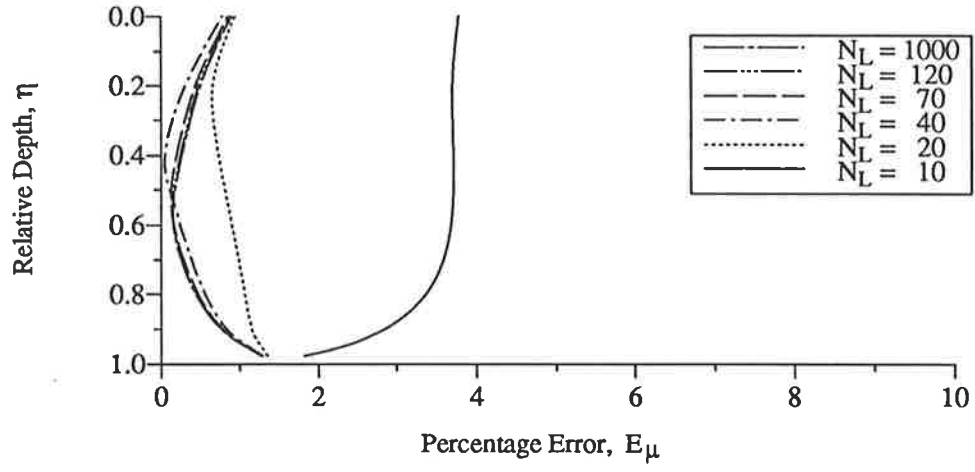


Figure 6.19: Error profiles for $\mu_k^{\otimes}$ for $C_x = 0.3$ and $\kappa = 3$.

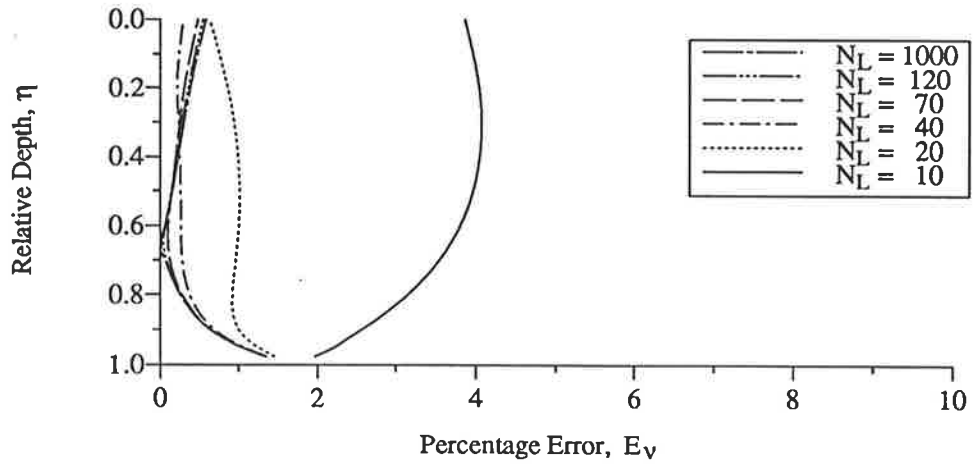


Figure 6.20: Error profiles for $\nu_k^{\otimes}$ for $C_x = 0.3$ and $\kappa = 3$.

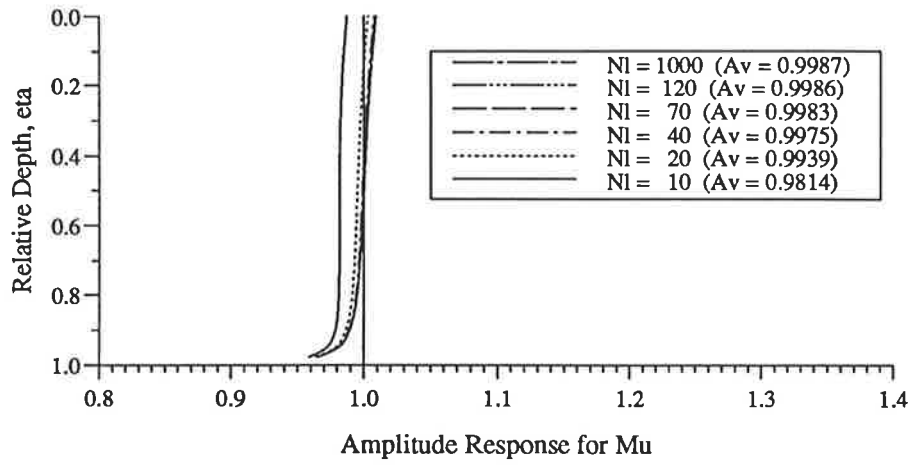


Figure 6.21: Variation over depth of amplitude response for $\mu_k^{\otimes}$, for $C_x = 0.3$ and $\kappa = 3$.

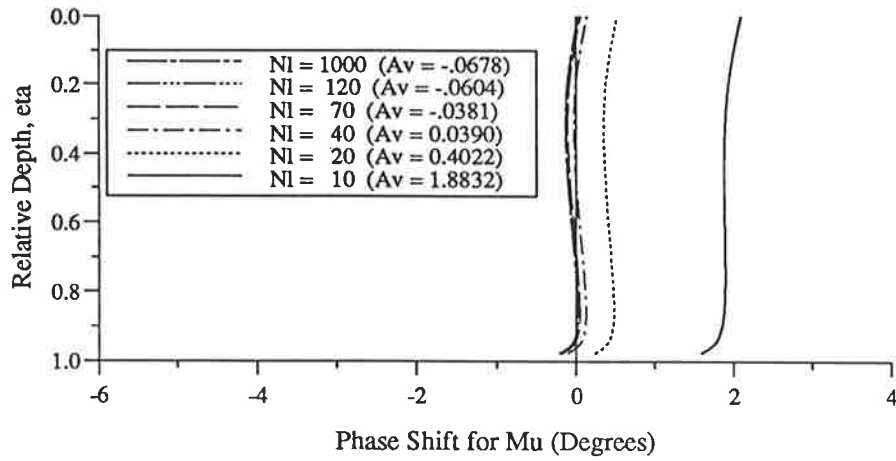


Figure 6.22: Variation over depth of phase shift error for $\mu_k^{\otimes}$, for $C_x = 0.3$ and $\kappa = 3$.

6.4 Conclusion of the Analysis

A method for wave propagation analysis for three-dimensional hydrodynamic numerical tidal models has been developed. For conditions which can be selected to match those in the region being modelled, the analysis can be used to optimise or detect deficiencies in the finite difference schemes, as illustrated with the non-centred differencing in time of the Coriolis term in Section 6.2, and/or to optimise the grid spacing parameters. The analysis uses analytic solutions and therefore offers the possibility of saving large amounts of computer time, which in the past has been consumed in performing the many numerical test runs needed to determine optimal values for parameters such as κ , K , $\Delta\chi$ and $\Delta\lambda$. In deciding the optimum values for these parameters consideration needs to be given to the trade off between desired accuracy and computer time usage. The information provided by this analysis enables a wise choice to be made for these parameters, allowing a specified accuracy for predictions to be achieved without excessive use of computer time, which could be caused by using many grid points when a smaller number produces the same accuracy. That is, this analysis can provide a limit beyond which further refinement of a grid is unnecessary. For the numerical model and parameters used as an example in this chapter the results show that there is no improvement in accuracy of the model if the horizontal grid used contains more than 40 grid points per wavelength.

The analysis uses a functional form of the vertical eddy viscosity parameter N_η for which there exists an analytic solution to the linearised tidal equations. The choice for N_η must be one for which an analytic solution is presently available. A wide selection of these exist, see, for example, Sverdrup (1926), Fjeldstat (1929), Tee (1987) and Chapter 5.

It seems possible that a similar two-dimensional analysis involving one horizontal

and one vertical direction, as opposed to previous two-dimensional analyses involving two horizontal directions, could be developed. In this case two Fourier modes travelling in opposite directions may be combined to form a standing wave with a horizontal boundary coinciding with a node. Such an analysis could be used to test stability and accuracy of the numerical scheme near a land boundary as distinct from the present analysis which assumes absence of such boundaries.

Chapter 7

ANALYSIS OF THE NUMERICAL MODEL

This chapter tests and analyses the numerical scheme in several ways. Section 7.1 investigates the time stepping nature of the numerical scheme and the wave propagation analysis, which was developed in Chapter 6, is used in Section 7.2 to analyse the numerical scheme. Several examples of how the numerical scheme was improved are included. Section 7.3 discusses consistency, stability and convergence of the numerical scheme and Section 7.4 compares the results from numerical model to the analytic solution for flow in a channel of constant depth and width. The flow is variable over depth and has analytic solutions for certain functional forms of the vertical eddy viscosity parameter N_η . Section 7.5 compares the numerical model to a two-dimensional analytic solution for depth integrated flow in a bay, which is open on one side and under the influence of the Coriolis force. The numerical model is compared to a second order accurate solution for one-dimensional non-linear flow in Section 7.6. The ability of the model to predict wind generated currents through a wind stress acting on the surface is tested in Section 7.7. The results from the numerical model are compared to a two-dimensional steady analytic solution.

7.1 The Order of Accuracy of the Time Step

As mentioned in Section 3.3 the predictor–corrector type recalculation of ζ^{n+1} makes the numerical model almost second order accurate. It is exactly second order accurate in time if the horizontal diffusive terms are ignored for the linear model. As with the wave propagation analysis, this can be shown by examining the linearised governing Equations (5.1) to (5.4) and applying the same finite difference techniques and numerical scheme to these equations as is used on the full governing equations. By letting $\Delta\chi$, $\Delta\lambda$ and $\Delta\eta$ approach zero the limit of the finite difference approximations will be equivalent to the partial derivatives in these three spatial dimensions. The time step can then be expanded as a Taylor series to determine the accuracy of the time stepping procedure.

As the vertical grid spacing $\Delta\eta$ tends toward zero (and K tends towards infinity) terms with vertical differencing tend to partial derivatives with respect to η and the summations in the vertical tend to integrals and therefore Equation (6.13) tends to

$$\begin{aligned}
& \frac{1}{\Delta t}(\mu_{i,j,k}^{n+1} - \mu_{i,j,k}^n) + \frac{\mu_a}{2\Delta\chi}(\mu_{i+1,j,k}^n - \mu_{i,j,k}^n + \mu_{i,j,k}^{n+1} - \mu_{i-1,j,k}^{n+1}) \\
& + \frac{\nu_a}{2\Delta\lambda}(\mu_{i,j+1,k}^n - \mu_{i,j,k}^n + \mu_{i,j,k}^{n+1} - \mu_{i,j-1,k}^{n+1}) \\
& = \frac{gh_a\Delta t}{2L^2(\Delta\chi)^2}(\bar{\mu}_{i+1,j}^n - 2\bar{\mu}_{i,j}^n + \bar{\mu}_{i-1,j}^n) \\
& + \frac{gh_a\Delta t}{2L^2\Delta\chi\Delta\lambda}(\bar{\nu}_{i+1,j}^n - \bar{\nu}_{i+1,j-1}^n - \bar{\nu}_{i,j}^n + \bar{\nu}_{i,j-1}^n) - \frac{g}{L^2\Delta\chi}(\zeta_{i+1,j}^n - \zeta_{i,j}^n) \\
& + \frac{N_\chi}{L^2(\Delta\chi)^2}(\mu_{i+1,j,k}^n - 2\mu_{i,j,k}^n + \mu_{i-1,j,k}^n) + \frac{N_\lambda}{B^2(\Delta\lambda)^2}(\mu_{i,j+1,k}^n - 2\mu_{i,j,k}^n + \mu_{i,j-1,k}^n) \\
& + \frac{1}{2h_a^2} \frac{\partial}{\partial\eta} \left[N_\eta \frac{\partial}{\partial\eta} (\mu_{i,j,k}^n + \mu_{i,j,k}^{n+1}) \right] \\
& + \frac{f_a B}{4L}(\nu_{i,j-1,k}^{n+1} + \nu_{i,j,k}^{n+1} + \nu_{i+1,j-1,k}^n + \nu_{i+1,j,k}^n), \tag{7.1}
\end{aligned}$$

and similarly Equation (6.14) becomes

$$\begin{aligned}
& \frac{1}{\Delta t}(\nu_{i,j,k}^{n+1} - \nu_{i,j,k}^n) + \frac{\mu_a}{2\Delta\chi}(\nu_{i+1,j,k}^n - \nu_{i,j,k}^n + \nu_{i,j,k}^{n+1} - \nu_{i-1,j,k}^{n+1}) \\
& + \frac{\nu_a}{2\Delta\lambda}(\nu_{i,j+1,k}^n - \nu_{i,j,k}^n + \nu_{i,j,k}^{n+1} - \nu_{i,j-1,k}^{n+1}) \\
& = \frac{gh_a\Delta t}{2B^2\Delta\chi\Delta\lambda}(\bar{\mu}_{i,j+1}^n - \bar{\mu}_{i-1,j+1}^n - \bar{\mu}_{i,j}^n + \bar{\mu}_{i-1,j}^n) \\
& + \frac{gh_a\Delta t}{2B^2(\Delta\lambda)^2}(\bar{\nu}_{i,j+1}^n - 2\bar{\nu}_{i,j}^n + \bar{\nu}_{i,j-1}^n) - \frac{g}{B^2\Delta\lambda}(\zeta_{i,j+1}^n - \zeta_{i,j}^n) \\
& + \frac{N_\chi}{L^2(\Delta\chi)^2}(\nu_{i+1,j,k}^n - 2\nu_{i,j,k}^n + \nu_{i-1,j,k}^n) + \frac{N_\lambda}{B^2(\Delta\lambda)^2}(\nu_{i,j+1,k}^n - 2\nu_{i,j,k}^n + \nu_{i,j-1,k}^n) \\
& + \frac{1}{2h_a^2}\frac{\partial}{\partial\eta}\left[N_\eta\frac{\partial}{\partial\eta}(\nu_{i,j,k}^n + \nu_{i,j,k}^{n+1})\right] \\
& - \frac{f_aL}{4B}(\mu_{i-1,j,k}^{n+1} + \mu_{i,j,k}^n + \mu_{i-1,j+1,k}^{n+1} + \mu_{i,j+1,k}^n) . \tag{7.2}
\end{aligned}$$

A solution to Equations (6.10), (7.1) and (7.2) is sought in the form

$$\zeta(\chi, \lambda) = a \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} , \tag{7.3}$$

$$\mu(\chi, \lambda, \eta) = \mu^* F_\chi(\eta) \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} , \tag{7.4}$$

$$\nu(\chi, \lambda, \eta) = \nu^* F_\lambda(\eta) \exp\{i(\sigma n \Delta t + i \Delta \chi m_\chi + j \Delta \lambda m_\lambda)\} . \tag{7.5}$$

Substitution of Equation (7.3) to (7.5) into Equation (6.10), after simplification, yields

$$(\xi - 1)a + \frac{i\Delta t h_a}{\Delta\chi} \bar{F}_\chi \sin(\beta_\chi)(\xi + 1)\mu^* + \frac{i\Delta t h_a}{\Delta\lambda} \bar{F}_\lambda \sin(\beta_\lambda)(\xi + 1)\nu^* = 0 , \tag{7.6}$$

where $\xi = \exp\{i\sigma\Delta t\}$, $\beta_\chi = m_\chi\Delta\chi/2$, $\beta_\lambda = m_\lambda\Delta\lambda/2$, $\bar{F}_\chi = \int_0^1 F_\chi(\eta) d\eta$ and $\bar{F}_\lambda = \int_0^1 F_\lambda(\eta) d\eta$. Now as $\Delta\chi \rightarrow 0$, $\sin(\beta_\chi) \rightarrow m_\chi/2$ and similarly as $\Delta\lambda \rightarrow 0$, $\sin(\beta_\lambda) \rightarrow m_\lambda/2$, so Equation (7.6) tends towards

$$(\xi - 1)a + i\Delta t h_a \bar{F}_\chi m_\chi \left(\frac{\xi + 1}{2}\right)\mu^* + i\Delta t h_a \bar{F}_\lambda m_\lambda \left(\frac{\xi + 1}{2}\right)\nu^* = 0 . \tag{7.7}$$

Substitution of Equation (7.3) to (7.5) into Equation (7.1), after simplification,

yields

$$\begin{aligned}
& a \left(\frac{2ig\Delta t \sin(\beta_x)}{L^2 \Delta \chi} \right) + \mu^* \left(\frac{2g(\Delta t)^2 h_a \bar{F}_x \sin^2(\beta_x)}{(L\Delta \chi)^2} + \frac{\Delta t(\xi + 1)}{2h_a^2} \frac{\partial}{\partial \eta} \left[N_\eta \frac{\partial F_x}{\partial \eta} \right] \right) \\
& + \mu^* F_x \left(\xi - 1 + \frac{\mu_a \Delta t}{\Delta \chi} (\exp\{2i\beta_x\} - 1 + \xi(1 - \exp\{2i\beta_x\})) \right. \\
& \quad \left. + \frac{\nu_a \Delta t}{\Delta \lambda} (\exp\{2i\beta_\lambda\} - 1 + \xi(1 - \exp\{2i\beta_\lambda\})) \right. \\
& \quad \left. + \frac{4N_x \Delta t \sin^2(\beta_x)}{(L\Delta \chi)^2} + \frac{4N_\lambda \Delta t \sin^2(\beta_\lambda)}{(B\Delta \lambda)^2} \right) \\
& + \nu^* \left(\frac{2g(\Delta t)^2 h_a \bar{F}_\lambda \sin(\beta_x) \sin(\beta_\lambda)}{L^2 \Delta \chi \Delta \lambda} - \frac{2\Delta t f_a B}{L} F_\lambda \cos(\beta_\lambda) (\xi \exp\{-i\beta_x\} + \exp\{i\beta_x\}) \right) \\
& = 0 .
\end{aligned} \tag{7.8}$$

As $\Delta \chi \rightarrow 0$ and $\Delta \lambda \rightarrow 0$ Equation (7.8) tends towards

$$\begin{aligned}
& a \left(\frac{ig\Delta t m_x}{L^2} \right) + \mu^* \left(\frac{g(\Delta t)^2 h_a \bar{F}_x(m_x)^2}{2L^2} - \frac{\Delta t(\xi + 1)}{2h_a^2} \frac{\partial}{\partial \eta} \left[N_\eta \frac{\partial F_x}{\partial \eta} \right] \right) \\
& + \mu^* F_x \left(\xi - 1 + \mu_a i \Delta t m_x \left(\frac{1 + \xi}{2} \right) + \nu_a i \Delta t m_\lambda \left(\frac{1 + \xi}{2} \right) \right. \\
& \quad \left. + \frac{N_x \Delta t (m_x)^2}{L^2} + \frac{N_\lambda \Delta t (m_\lambda)^2}{B^2} \right) \\
& + \nu^* \left(\frac{g(\Delta t)^2 h_a \bar{F}_\lambda m_x m_\lambda}{2L^2} - \frac{\Delta t f_a B}{2L} F_\lambda (\xi + 1) \right) \\
& = 0 .
\end{aligned} \tag{7.9}$$

Now since $F_x(\eta)$ satisfies Equation (5.11) then

$$\frac{1}{h_a^2} \frac{\partial}{\partial \eta} \left[N_\eta \frac{\partial F_x}{\partial \eta} \right] = F_x \left(i\hat{\sigma} + im_x \mu_a + im_\lambda \nu_a + \frac{N_x(m_x)^2}{L^2} + \frac{N_\lambda(m_\lambda)^2}{B^2} \right) - f_a F_\lambda - \frac{i\hat{\sigma}^2 B}{m_\lambda} . \tag{7.10}$$

Equation (7.10) can be substituted into Equation (7.9) and cancellation gives

$$\begin{aligned}
& a \left(\frac{ig\Delta t m_x}{L^2} \right) + \mu^* \left(\frac{g(\Delta t)^2 h_a \bar{F}_x(m_x)^2}{2L^2} + F_x(\xi - 1 + \Delta t Q) \right) \\
& + \mu^* \left(\frac{\Delta t(\xi + 1)}{2} \left[-F_x(i\hat{\sigma} + Q) + f_a F_\lambda + \frac{i\hat{\sigma}^2 B}{m_\lambda} \right] \right) \\
& + \nu^* \left(\frac{g(\Delta t)^2 h_a \bar{F}_\lambda m_x m_\lambda}{2L^2} - \frac{\Delta t f_a B}{2L} F_\lambda (\xi + 1) \right) \\
& = 0 ,
\end{aligned} \tag{7.11}$$

where $Q = \frac{N_x(m_x)^2}{L^2} + \frac{N_\lambda(m_\lambda)^2}{B^2}$. Similarly Equation (7.2) yields

$$\begin{aligned}
a \left(\frac{ig\Delta t m_\lambda}{B^2} \right) &+ \nu^* \left(\frac{g(\Delta t)^2 h_a \bar{F}_\lambda (m_\lambda)^2}{2B^2} + F_\lambda (\xi - 1 + \Delta t Q) \right) \\
&+ \nu^* \left(\frac{\Delta t (\xi + 1)}{2} \left[-F_\lambda (i\hat{\sigma} + Q) - f_a F_x + \frac{i\hat{\sigma}^2 L}{m_x} \right] \right) \\
&+ \mu^* \left(\frac{g(\Delta t)^2 h_a \bar{F}_x m_x m_\lambda}{2B^2} + \frac{\Delta t f_a L}{2B} F_\lambda (\xi + 1) \right) \\
&= 0 .
\end{aligned} \tag{7.12}$$

Equations (7.7), (7.11) and (7.12) form three equations for the solution of a , μ^* and ν^* . The solution of interest here, which is the numerical mode closest to the true wave solution, has $L\mu^* = B\nu^*$, so if V^* is defined as $V^* = L\mu^* = B\nu^*$ these equations can be simplified to

$$(\xi - 1)a + ih_a \Delta t \left(\frac{\xi + 1}{2} \right) \left(\frac{m_x}{L} \bar{F}_x + \frac{m_\lambda}{B} \bar{F}_\lambda \right) V^* = 0 , \tag{7.13}$$

$$\begin{aligned}
&\left(\frac{ig\Delta t m_x m_\lambda}{LB} \right) a \\
&+ \left\{ \frac{gh_a(\Delta t)^2 m_x m_\lambda}{2LB} \left(\frac{m_x}{L} \bar{F}_x + \frac{m_\lambda}{B} \bar{F}_\lambda \right) + \frac{m_x F_x}{B} (\xi - 1 + \Delta t Q) \right. \\
&\quad \left. - \frac{\Delta t (\xi + 1) F_x m_\lambda}{2B} (i\hat{\sigma} + Q) + \frac{\Delta t (\xi + 1) i\hat{\sigma}^2}{2} \right\} V^* = 0 .
\end{aligned} \tag{7.14}$$

The complex circular frequency $\hat{\sigma}$ satisfies the relationship given in Equation (5.42) and with substitution of Equation (7.13) into Equation (7.14), non zero solutions for V^* will exist if

$$\begin{aligned}
&\left(\frac{\xi - 1}{2} \right) (\Delta t)^2 \hat{\sigma}^3 + \frac{m_\lambda F_x}{B} (\xi - 1)^2 + \frac{m_\lambda F_x}{B} \Delta t Q (\xi - 1) \\
&- \frac{F_x m_\lambda}{2B} \Delta t (\xi^2 - 1) (i\hat{\sigma} + Q) + \left(\frac{\xi^2 - 1}{2} \right) \Delta t i \hat{\sigma}^2 + \left(\frac{\xi + 1}{2} \right) (\Delta t)^2 \hat{\sigma}^3 = 0 .
\end{aligned} \tag{7.15}$$

This equation is a quadratic in ξ and can therefore be rearranged in the form of

$$\begin{aligned} & \xi^2 \left(\frac{F_\chi m_\lambda}{2B} (2 - \Delta t i \hat{\sigma} - \Delta t Q) + \frac{\Delta t i \hat{\sigma}^2}{2} \right) \\ & \xi \left(\frac{F_\chi m_\lambda}{2B} (-4 + 2\Delta t Q) + \frac{(\Delta t)^2 \hat{\sigma}^3}{2} \right) \\ & \left(\frac{F_\chi m_\lambda}{2B} (2 + \Delta t i \hat{\sigma} - \Delta t Q) - \frac{\Delta t i \hat{\sigma}^2}{2} \right) = 0. \end{aligned} \quad (7.16)$$

Now the root of Equation (7.16) with the positive sign in front of the square root, corresponds to the true wave solution. If this value of ξ is expressed as a Taylor series in Δt the result is

$$\xi = 1 + i\hat{\sigma}\Delta t - \frac{\hat{\sigma}^2(\Delta t)^2}{2} + \frac{iQ\hat{\sigma}F_\chi m_\lambda(\Delta t)^2}{2(F_\chi m_\lambda + B\hat{\sigma})} + O(\Delta t)^3. \quad (7.17)$$

This is the value of ξ for the numerical scheme obtained in the limit as $\Delta\chi$, $\Delta\lambda$ and $\Delta\eta \rightarrow 0$, and which corresponds to the true wave solution. The value of ξ is formed from $\exp(i\sigma\Delta t)$, where σ is the circular frequency of the numerical wave which is closest to $\hat{\sigma}$. The Taylor series expansion of $\exp(i\sigma\Delta t)$ is

$$\xi = 1 + i\hat{\sigma}\Delta t - \frac{\hat{\sigma}^2(\Delta t)^2}{2} - \frac{i\hat{\sigma}^3(\Delta t)^3}{6} + O(\Delta t)^4. \quad (7.18)$$

By comparing Equation (7.17) with Equation (7.18) it can be seen that if the coefficient Q , which is derived from the horizontal eddy viscosity coefficients, is zero (that is if these terms are considered to be relatively small and are removed from the equations) then Equation (7.17) shows that the numerical method has a value of ξ which is second order accurate in the time step Δt . As the horizontal diffusive terms are relatively small the numerical scheme for the full tidal modelling equations can be considered virtually second order accurate in time.

7.2 Wave Propagation Analysis of the Numerical Model

Once the wave propagation analysis, which was presented in Chapter 6, was developed and applied to the developed numerical scheme, it became obvious that much work could be done investigating the relationship between the magnitude and form of the vertical eddy viscosity coefficient N_v and how variations in these affect the optimal finite difference approximations, the overall finite difference scheme, the optimal vertical grid spacing and the number of grid points to use in both the horizontal and vertical directions. The wave propagation analysis, as a tool for a three-dimensional tidal model, will enable these type of interactions to be studied in more detail than was previously possible. To show the usefulness of the wave propagation analysis four examples are described where the analysis was used to improve the numerical model developed in this thesis.

The first example, which was previewed in Chapter 6, involves the differencing of the Coriolis term. At one stage of the development of the numerical model it was proposed, for simplicity, to have the Coriolis terms in the u and v momentum equations differenced at the n^{th} time level not at the $(n+\frac{1}{2})^{th}$ level as the rest of the terms, as it was thought that it would not affect the performance of the numerical model. After the wave propagation analysis was developed it was quickly discovered that this introduced large errors. The amplitude response graph shown in Figure 6.14 of Section 6.3 clearly indicates that the numerical scheme could be improved and that the method of differencing of the Coriolis terms is important. Fortunately, within the limitations considered in Chapter 3, it was possible to difference the Coriolis terms about the $(n+\frac{1}{2})^{th}$ time level by taking the average of the four surrounding velocities, which are required in the calculation, with two of the values at the $(n+1)^{th}$ time level and the other two at the n^{th} time level. Thus the finite difference approximations for

the Coriolis term in the μ momentum equation was changed from

$$\frac{Hfv}{L} \Big|_{i,k}^{n+\frac{1}{2}} \approx \frac{\tilde{H}_i^n \tilde{f}_i^n}{2L} \left(\frac{\Delta\chi_m(v_{i2,k}^n + v_{i2-1,k}^n) + \Delta\chi_{m+1}(v_{i,k}^n + v_{i-1,k}^n)}{\Delta\chi_m + \Delta\chi_{m+1}} \right), \quad (7.19)$$

to

$$\frac{Hfv}{L} \Big|_{i,k}^{n+\frac{1}{2}} \approx \frac{(\tilde{H}_i^{n+1} + \tilde{H}_i^n) \tilde{f}_i^n}{4L} \left(\frac{\Delta\chi_m(v_{i2,k}^n + v_{i2-1,k}^n) + \Delta\chi_{m+1}(v_{i,k}^{n+1} + v_{i-1,k}^{n+1})}{\Delta\chi_m + \Delta\chi_{m+1}} \right), \quad (7.20)$$

and similarly for the Coriolis term in the v momentum equation. When the new finite difference approximations were used for the Coriolis terms the graphs of Figures 6.6 and 6.16 show the resultant improvement in the amplitude response of the numerical model.

As can be seen from Equation (7.20) the two values of v required at the new time level are $v_{i,k}^{n+1}$ and $v_{i-1,k}^{n+1}$. For the v momentum equation the two values of μ required at the new time level are $\mu_{i1,k}^{n+1}$ and $\mu_{i1+1,k}^{n+1}$. Thus for each grid element i , so long as the new values of $v_{i,k}^{n+1}$ are calculated before the new values of $\mu_{i,k}^{n+1}$, the improved scheme can be implemented.

The second example of model improvement involves the introduction of the recalculation of ζ^{n+1} (or corrector step) using Equation (4.5). As previously mentioned the numerical scheme was first proposed without the corrector step, which is calculated after the velocity values at the new time level. Without this corrector step the wave propagation analysis shows that the numerical method is only stable for Courant numbers $C_x \leq 0.07$ where as with the recalculation of ζ the method is stable for all $C_x \leq 0.7$. The amplitude response, relative wave speed, and error in velocity coefficients graphs for the case without the corrector step are shown in in Figures 7.1 to 7.3. These are calculated with the same parameters as are used in Chapter 6, except for the Courant numbers and with the vertical grid parameter $\kappa = 3$.

For a value of the number of grid points per wavelength, N_L , greater than 30 the graphs of Figures 7.1 to 7.3 are very similar in shape and in their limiting value to the

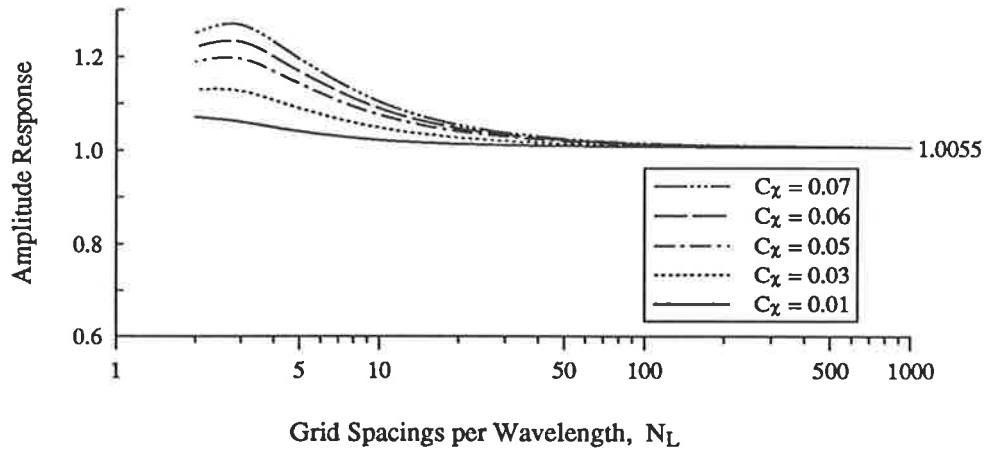


Figure 7.1: Amplitude response for the numerical scheme without the recalculation of ζ^{n+1} and with $\kappa = 3$.

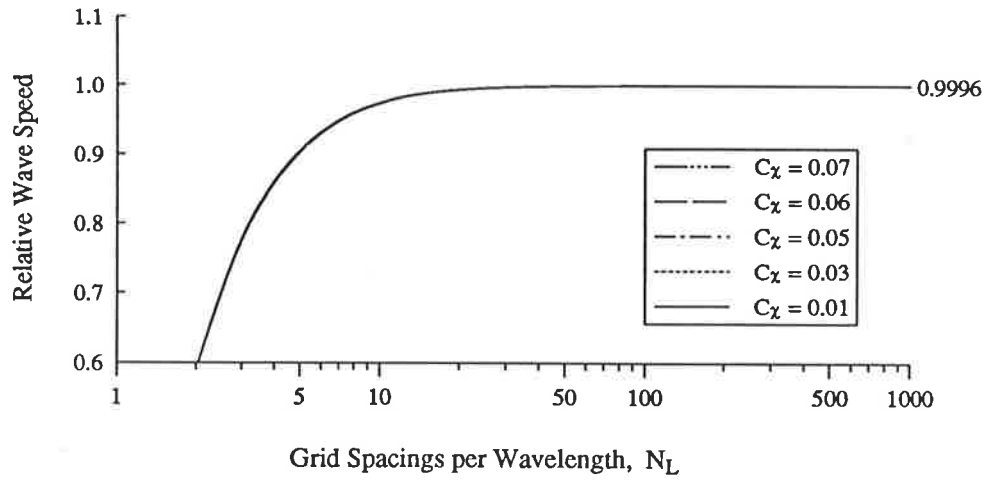


Figure 7.2: Relative wave speed for the numerical scheme without the recalculation of ζ^{n+1} and with $\kappa = 3$.

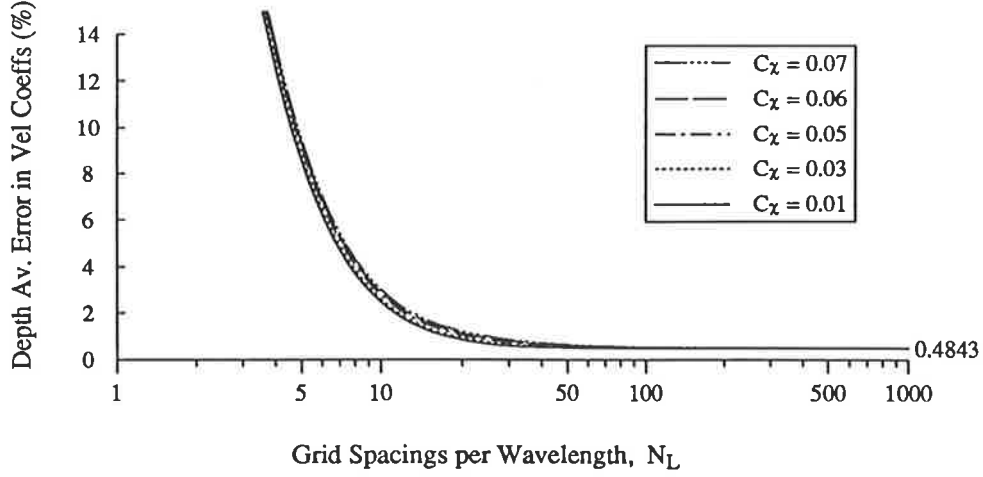


Figure 7.3: Error estimate for the numerically approximated values of $\mu_k^\otimes$ and $\nu_k^\otimes$ for the numerical scheme without the recalculation of ζ^{n+1} and with $\kappa = 3$.

graphs of Figures 6.16 to 6.18. This indicates that the scheme without the corrector step is just as accurate as the scheme including the corrector step, but the problem is that the Courant numbers are $C_x = 0.01, 0.03, 0.05, 0.06$ and 0.07 whereas the graphs of Figures 6.16 to 6.18 are for $C_x = 0.1, 0.3, 0.5, 0.6$ and 0.7 . This means that, with all else held constant, the scheme with the corrector step added can use a time step 10 times larger than that of the time step used by the scheme without the corrector step and also the wave propagation analysis shows through Figures 6.16 and 7.1 that the amplitude response with the corrector step is slightly more accurate. For N_L less than 50 the graphs of Figures 7.1 to 7.3 show a loss of accuracy for the larger Courant numbers for the numerical scheme without the recalculation of ζ^{n+1} .

The recalculation of ζ^{n+1} is computationally very cheap as most of the computer time is used in calculating the new values of the velocity variables at each depth, for each grid point. If, for example, the corrector step takes 10% of the total number of floating point operations for a complete time step (and this is a rather large estimate),

then without the corrector step there is a 10% saving in computer time, but by using the corrector step and the larger time step there are the extra calculations to perform, but they are performed 10 times less often for a fixed length of time. Thus there is a 90% saving of computer time for the scheme using the corrector step, which also has a slightly improved accuracy.

The third possibility investigated using the wave propagation analysis is the comparison between two finite difference approximations for the vertical diffusive terms. The term from the μ velocity equation is

$$\frac{1}{H} \frac{\partial}{\partial \eta} \left(N_{\eta} \frac{\partial \mu}{\partial \eta} \right) \Big|_{i,k}^{n+\frac{1}{2}}, \quad (7.21)$$

and the first differencing form is given in Section 4.2. An alternate differencing can be achieved if the derivative of the vertical eddy viscosity is known, in which case the term of Equation (7.21) can be expressed as

$$\frac{1}{H} \frac{\partial}{\partial \eta} \left(N_{\eta} \frac{\partial \mu}{\partial \eta} \right) \Big|_{i,k}^{n+\frac{1}{2}} = \frac{1}{H} \frac{\partial N_{\eta}}{\partial \eta} \frac{\partial \mu}{\partial \eta} \Big|_{i,k}^{n+\frac{1}{2}} + \frac{1}{H} N_{\eta} \frac{\partial^2 \mu}{\partial^2 \eta} \Big|_{i,k}^{n+\frac{1}{2}}. \quad (7.22)$$

After using the second order accurate finite difference approximations for an irregular grid (given in Chapter 3) and with some rearranging the finite difference form of this equation can be expressed as

$$\begin{aligned} \frac{1}{H} \frac{\partial}{\partial \eta} \left(N_{\eta} \frac{\partial \mu}{\partial \eta} \right) \Big|_{i,k}^{n+\frac{1}{2}} &= \frac{\mu_{i,k-1}^{n+1} + \mu_{i,k-1}^n}{(\tilde{H}_i^{n+1} + \tilde{H}_i^n)(r_k + 1)\Delta\eta_k} \left(\frac{2r_k N_{\eta}}{\Delta\eta_{k-1}} - r_k^2 N'_{\eta} \right) \\ &\quad - \frac{\mu_{i,k-1}^{n+1} + \mu_{i,k-1}^n}{(\tilde{H}_i^{n+1} + \tilde{H}_i^n)(r_k + 1)\Delta\eta_k} \left(\frac{-2(r_k + 1)N_{\eta}}{\Delta\eta_{k-1}} + (r_k^2 + 1)N'_{\eta} \right) \\ &\quad + \frac{\mu_{i,k-1}^{n+1} + \mu_{i,k-1}^n}{(\tilde{H}_i^{n+1} + \tilde{H}_i^n)(r_k + 1)\Delta\eta_k} \left(\frac{2N_{\eta}}{\Delta\eta_{k-1}} + N'_{\eta} \right), \end{aligned} \quad (7.23)$$

where N'_{η} is the derivative of N_{η} with respect to η .

This form of finite differencing tested by the wave propagation analysis gave the results graphed in Figures 7.4 to 7.6. These should be compared with those of Figures 6.16 to 6.18.

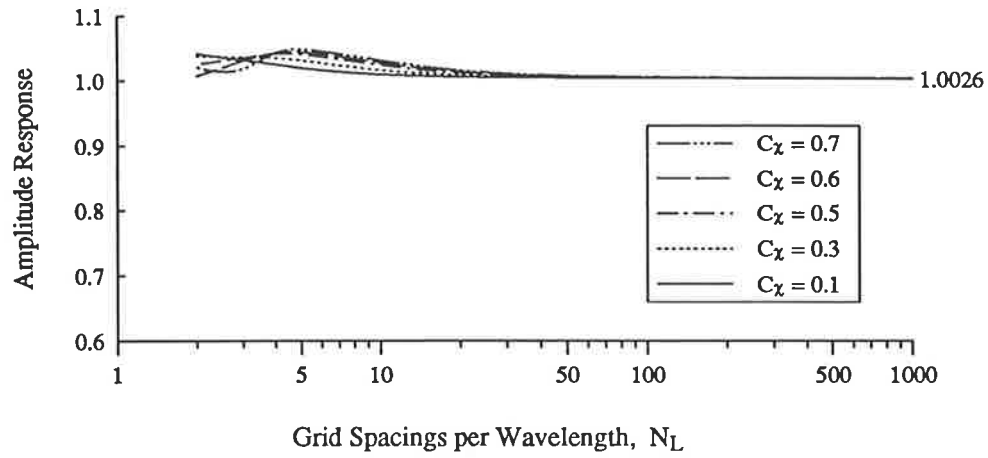


Figure 7.4: Amplitude response for the numerical scheme with the alternate differencing of the vertical eddy viscosity term and with $\kappa = 3$.

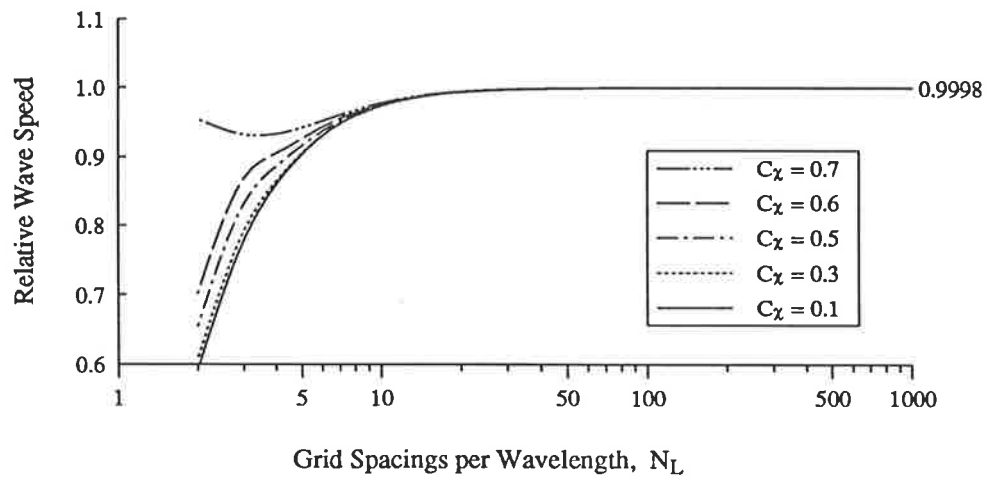


Figure 7.5: Relative wave speed for the numerical scheme with the alternate differencing of the vertical eddy viscosity term and with $\kappa = 3$.

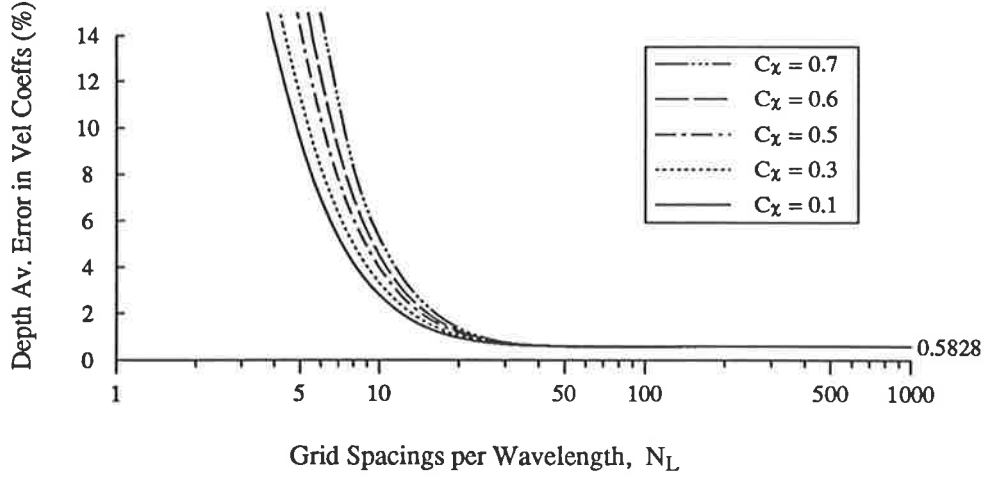


Figure 7.6: Error estimate for the numerically approximated values of $\mu_k^\otimes$ and $\nu_k^\otimes$ for the numerical scheme with the alternate differencing of the vertical eddy viscosity term and with $\kappa = 3$.

Using the alternate finite difference form given by Equation (7.23) the limiting value of the amplitude response improves from 1.0050 to 1.0026, that of the relative wave speed improves from 0.9996 to 0.9998 but the error measure for the velocity coefficients deteriorates from 0.4856% to 0.5828%. These results are not conclusive as to which differencing is better for this particular situation. To make a more definite statement further research work needs to be carried out with a more detailed investigation involving finding the optimum vertical grid spacing for each of these two possibilities, for a wide range of different vertical eddy viscosity forms, N_η , and with a variety of different magnitudes for N_η and other model parameters. It is suspected that the choice of differencing for the numerical model will depend on the form and magnitude of N_η as well as the criteria for deciding which vertical grid gives the optimal results.

The last example provided is that of the surface boundary condition of Equation (4.21), which is a second order accurate approximation for a variable grid. If there is no wind stress at the surface and a first order accurate approximation is chosen in

the form

$$\left. \frac{\partial \mu}{\partial \eta} \right|_{\eta=0} = \left(\frac{\mu_{i,1} - \mu_{i,0}}{\Delta \eta_0} \right) = 0, \quad (7.24)$$

then this implies that $\mu_{i,0} = \mu_{i,1}$. Similarly $v_{i,0} = v_{i,1}$ is used as the corresponding approximation for the v velocity equation.

The wave propagation graphs for a numerical scheme with the same parameters as the scheme with wave propagation graphs shown in Figures 6.16 to 6.22 but using the boundary condition of Equation (7.24) are graphed in Figures 7.7 to 7.13.

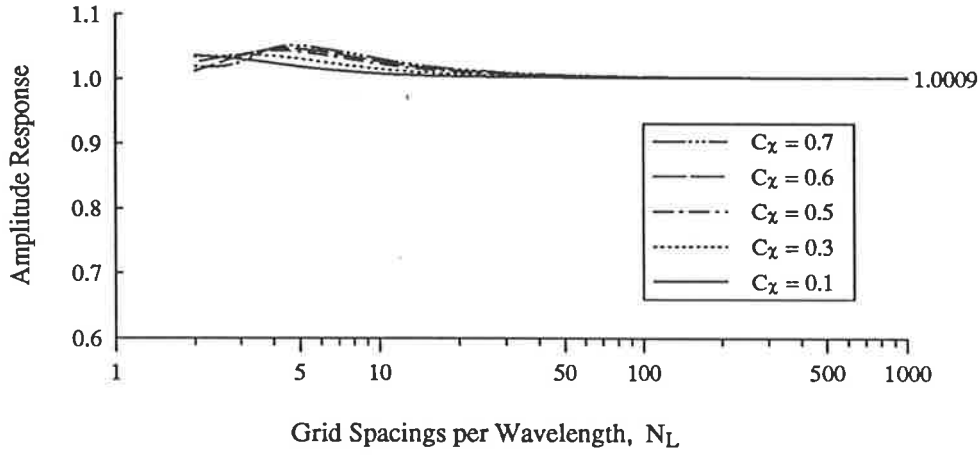


Figure 7.7: Amplitude response for the numerical scheme with a first order approximation for the surface boundary condition and with $\kappa = 3$.

Figure 7.7 shows an improvement in the amplitude response of the numerical model given the prescribed conditions, where the limiting value improves from 1.0050 to 1.0009. However the relative wave speed (Figure 7.8) and velocity error measure (Figure 7.9) both are significantly worse, going from 0.9996 and 0.4856% respectively to 1.0018 and 0.8256%. Figures 7.10 and 7.11 which show the distribution over depth of the errors in velocity components, clearly indicates that there is a loss of accuracy near the sea surface. Figures 7.12 and 7.13, which show the total errors from Figures 7.10 separated into an amplitude and a phase error, suggest that the main source of inac-

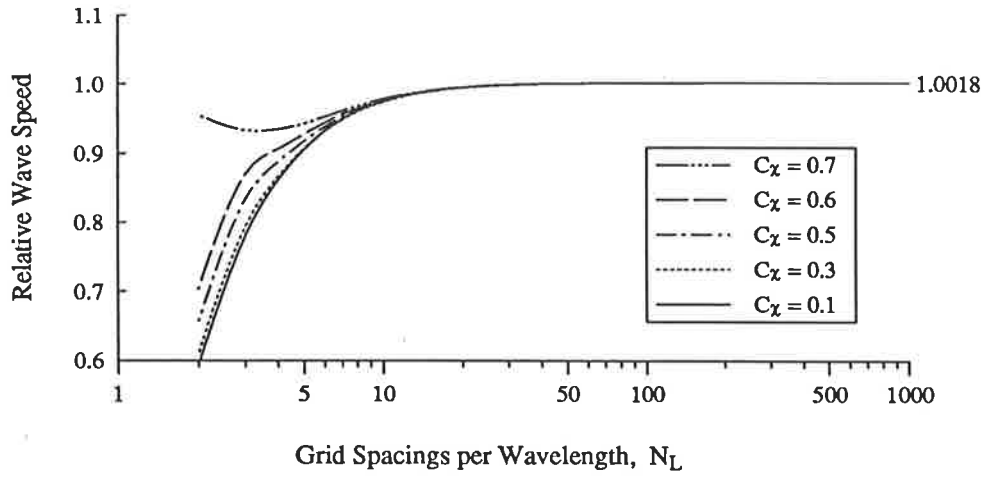


Figure 7.8: Relative wave speed for the numerical scheme with a first order approximation for the surface boundary condition and with $\kappa = 3$.

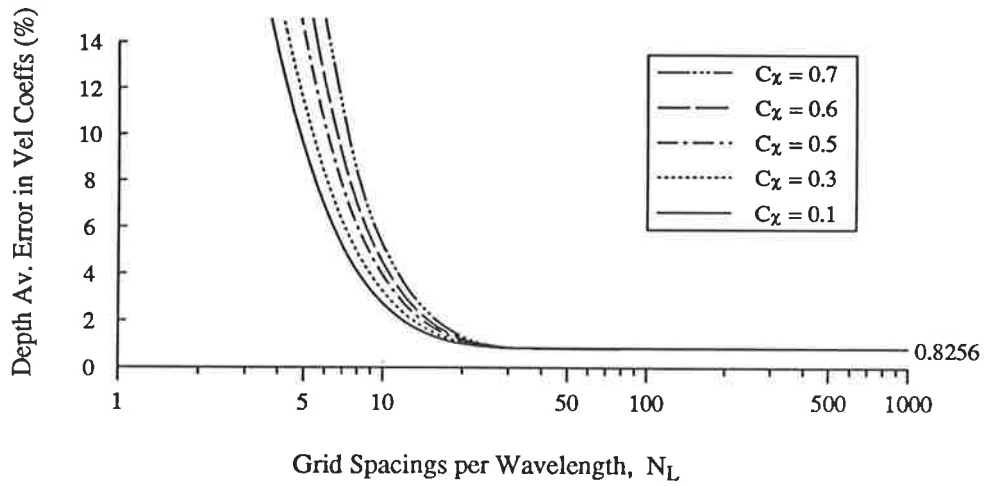


Figure 7.9: Error estimate for the numerically approximated values of $\mu_k^\otimes$ and $\nu_k^\otimes$ for the numerical scheme with a first order approximation for the surface boundary condition and with $\kappa = 3$.

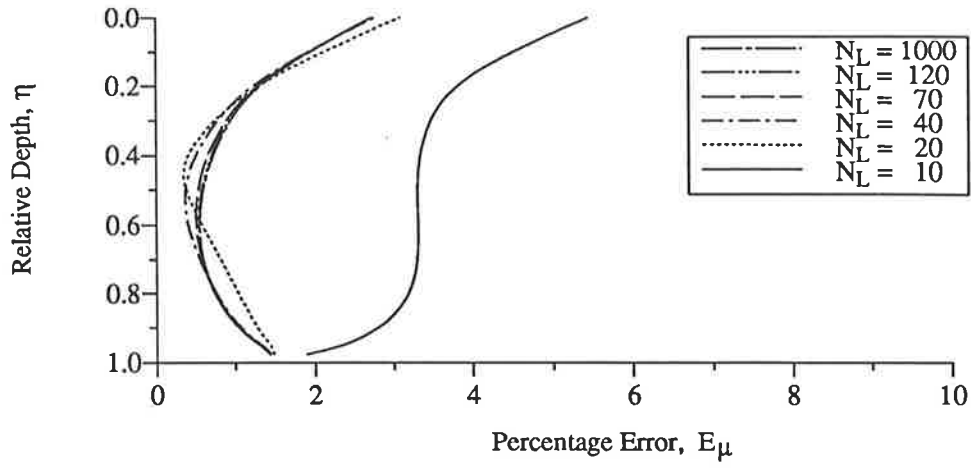


Figure 7.10: Error profiles for $\mu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_x = 0.3$ and $\kappa = 3$.

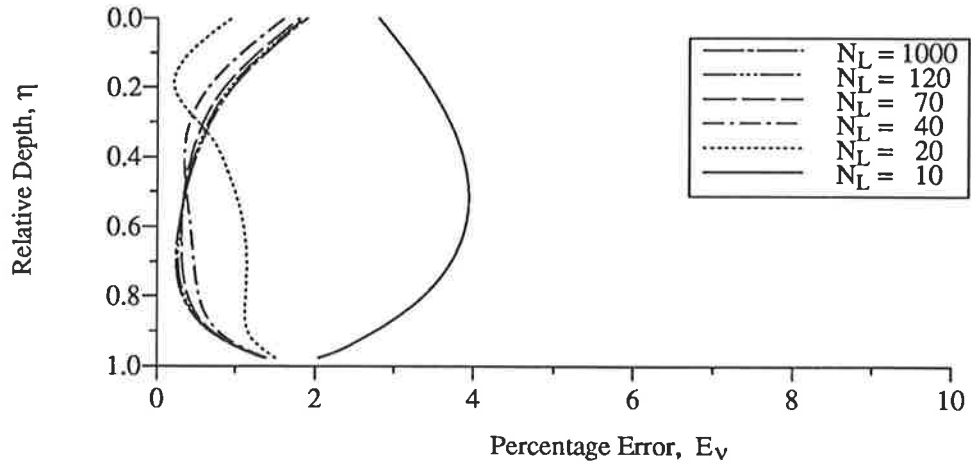


Figure 7.11: Error profiles for $\nu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_x = 0.3$ and $\kappa = 3$.

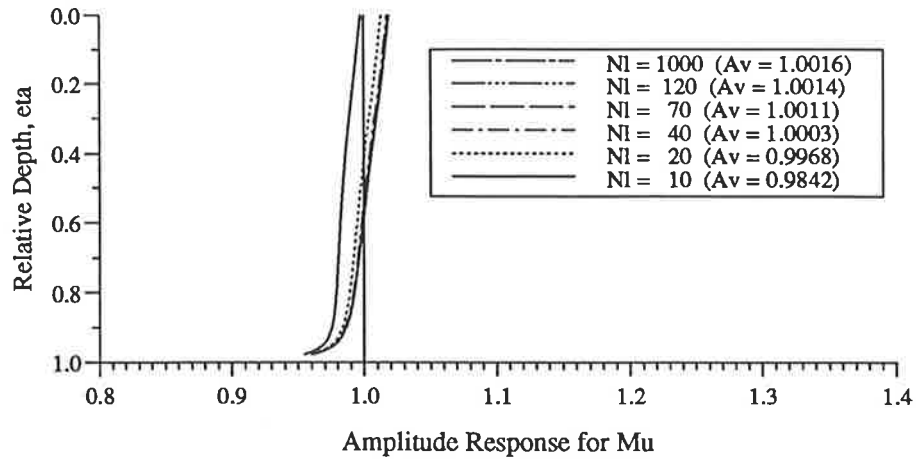


Figure 7.12: Variation over depth of amplitude response for $\mu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_\chi = 0.3$ and $\kappa = 3$.

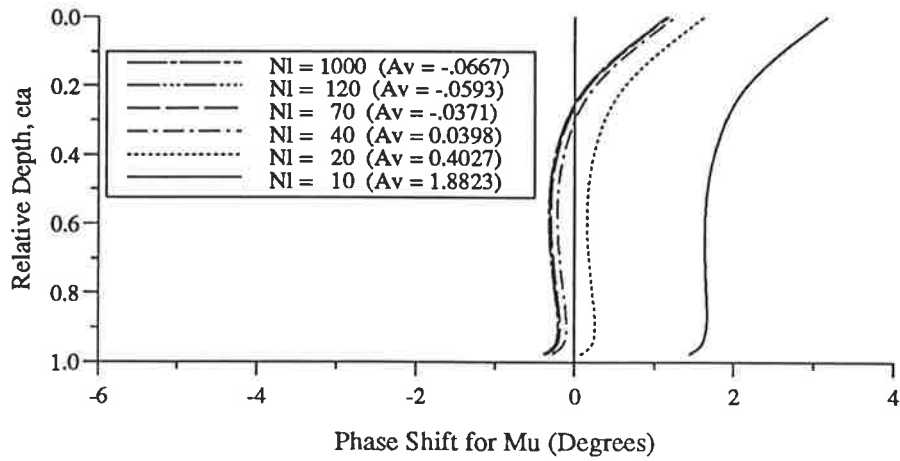


Figure 7.13: Variation over depth of phase shift error for $\mu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_\chi = 0.3$ and $\kappa = 3$.

curacy due to using a first order boundary approximation at the surface, comes from a phase shift of the velocity component near the sea surface. This is the probable cause of the loss of accuracy in the relative wave speed of the dominant mode through the numerical scheme.

A third order accurate surface boundary condition was also tried to see if this would improve the overall accuracy of the numerical model. The surface boundary condition used instead of Equation (4.21), with zero wind stress, is

$$\left. \frac{\partial \mu}{\partial \eta} \right|_{\eta=0} = a_0 \mu_{i,0} + b_0 \mu_{i,1} + c_0 \mu_{i,2} + d_0 \mu_{i,3} = 0 , \quad (7.25)$$

where

$$\begin{aligned} a_0 &= \frac{\Delta \eta_0(3\Delta \eta_0 + 2\Delta \eta_1) + (2\Delta \eta_0 + \Delta \eta_1)(\Delta \eta_1 + \Delta \eta_2)}{\Delta \eta_0(\Delta \eta_0 + \Delta \eta_1)(\Delta \eta_0 + \Delta \eta_1 + \Delta \eta_2)} , \\ b_0 &= \frac{(\Delta \eta_0 + \Delta \eta_1)(\Delta \eta_0 + \Delta \eta_1 + \Delta \eta_2)}{\Delta \eta_0 \Delta \eta_1 (\Delta \eta_1 + \Delta \eta_2)} , \\ c_0 &= \frac{\Delta \eta_0(\Delta \eta_0 + \Delta \eta_1 + \Delta \eta_2)}{\Delta \eta_1 \Delta \eta_2 (\Delta \eta_0 + \Delta \eta_1)} , \\ d_0 &= \frac{\Delta \eta_0(\Delta \eta_0 + \Delta \eta_1)}{\Delta \eta_2(\Delta \eta_1 + \Delta \eta_2)(\Delta \eta_0 + \Delta \eta_1 + \Delta \eta_2)} . \end{aligned}$$

Figures 7.14 to 7.20 show the graphs from the wave propagation analysis for the model with the same parameters as before with the third order accurate surface boundary condition.

Figure 7.14 shows an improvement in the amplitude response of the numerical model given the prescribed conditions, where the limiting value improves from 1.0050 to 1.0013, the relative wave speed (Figure 7.15) has a slight loss of accuracy from a value of 0.9996 to 0.9995, and the velocity error measure (Figure 7.16) has a slight improvement from 0.4856% to 0.4228%. Figures 7.19 and 7.20 which show the total errors from Figures 7.17 separated into an amplitude and a phase error indicate (in comparison to Figures 6.21 and 6.22) that the amplitude error in the velocity component is more

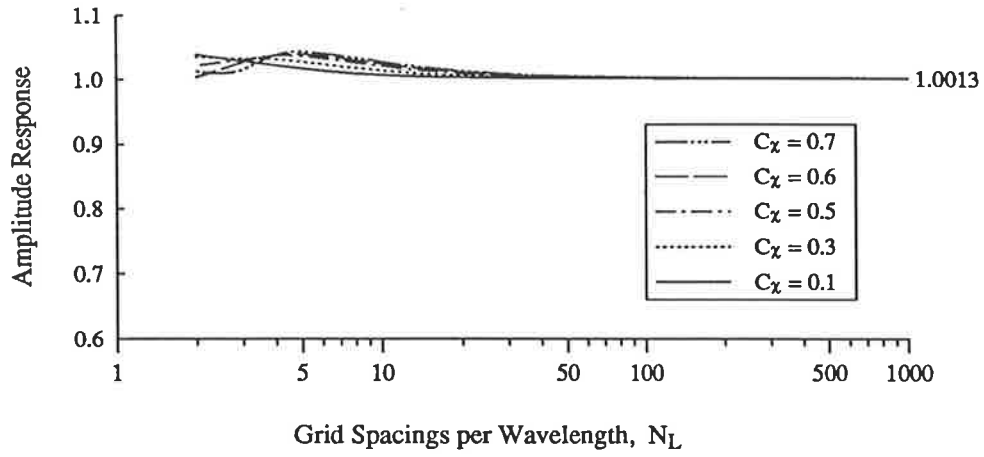


Figure 7.14: Amplitude response for the numerical scheme with a first order approximation for the surface boundary condition and with $\kappa = 3$.

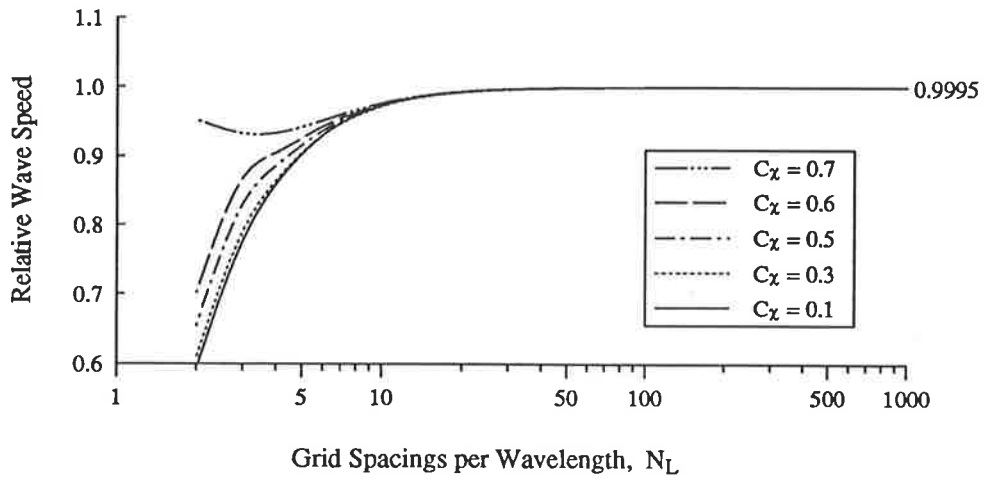


Figure 7.15: Relative wave speed for the numerical scheme with a first order approximation for the surface boundary condition and with $\kappa = 3$.

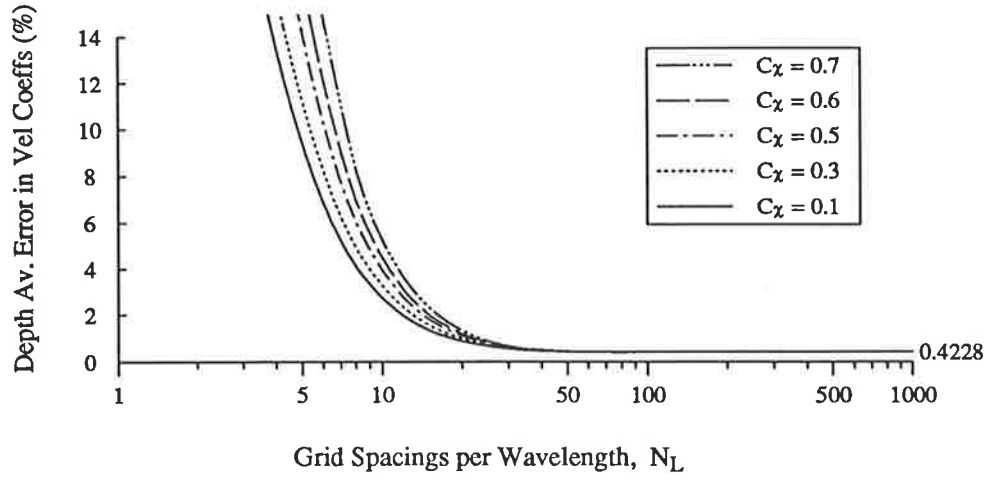


Figure 7.16: Error estimate for the numerically approximated values of $\mu_k^\otimes$ and $\nu_k^\otimes$ for the numerical scheme with a first order approximation for the surface boundary condition and with $\kappa = 3$.

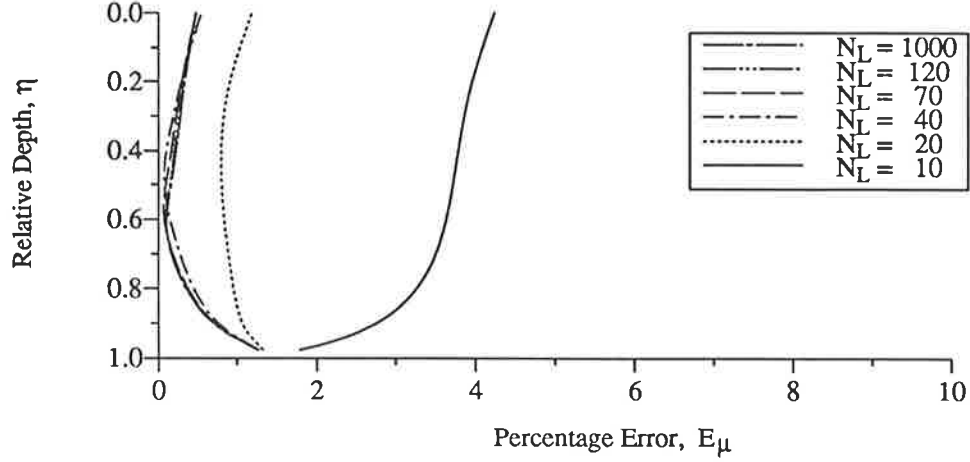


Figure 7.17: Error profiles for $\mu_k^\otimes$ for the scheme with a first order approximation for the surface boundary condition and with $C_x = 0.3$ and $\kappa = 3$.

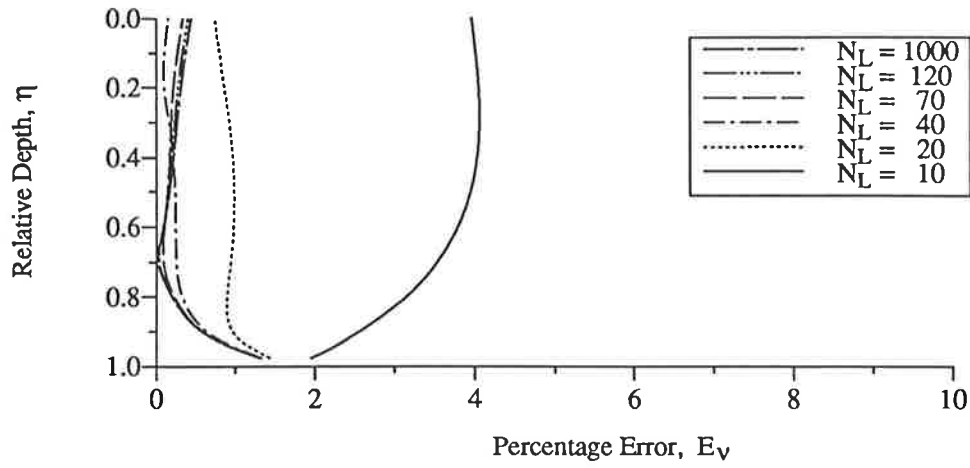


Figure 7.18: Error profiles for $\nu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_\chi = 0.3$ and $\kappa = 3$.

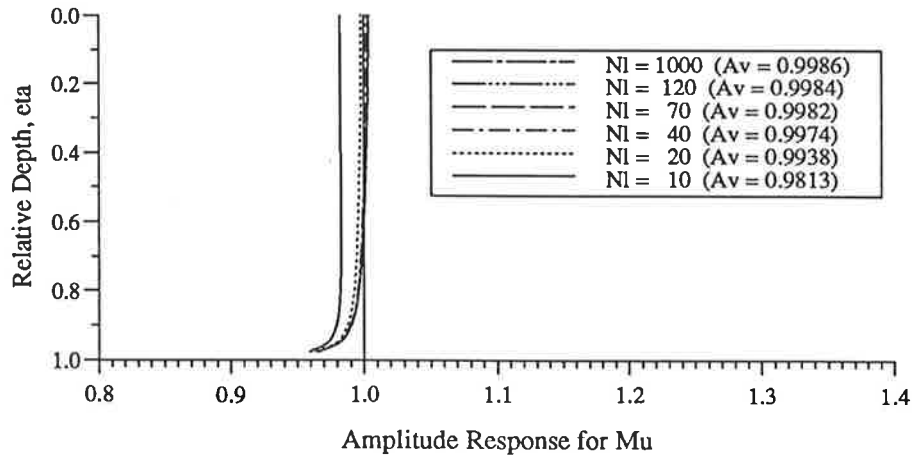


Figure 7.19: Variation over depth of amplitude response for $\mu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_\chi = 0.3$ and $\kappa = 3$.

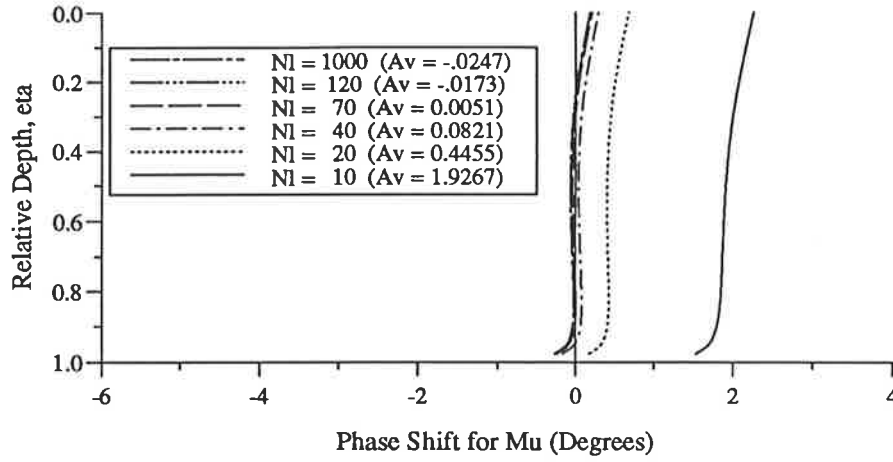


Figure 7.20: Variation over depth of phase shift error for $\mu_k^{\otimes}$ for the scheme with a first order approximation for the surface boundary condition and with $C_x = 0.3$ and $\kappa = 3$.

accurate near the sea surface with the third order accurate surface boundary condition, but the error due to phase shift near the sea surface is larger.

It appears that there is no significant increase in accuracy when using the third order accurate surface boundary condition and there is a practical problem which must be overcome in order to implement this surface approximation. The problem is due to the presence of the velocity values from the 3rd depth level and three simultaneous equations need to be solved in order to convert to tri-diagonal form the set of equations, which are solved to give the velocity components over depth. The effort involved to solve this is not justified by the slight improvement in the numerical results.

The overall characteristics, obtained from the wave propagation analysis, of the numerical model outlined in Chapters 3 and 4 could be summarised as

- The model can be run for Courant numbers up to 0.7.

- The model characteristics will be constant and the accuracy will not be improved once the horizontal grid is refined by more than 40 grid spacings per wave length.
- Once a form for the vertical eddy viscosity coefficient has been found which is characteristically representative of the area to be modelled, the wave propagation analysis can be used to find an optimal vertical grid distribution, so that errors in amplitude response, relative wave speed and velocity distribution over depth can be minimised.
- It is possible to choose the vertical grid distribution so that the amplitude response or relative wave speed is exactly 1.
- With the optimal vertical grid distribution the error in the velocity coefficients over depth compared to the true wave solution, can be made as small as desired by increasing the number of depth levels, and was as low as 0.23% with only 16 depth levels for the example described in this section.

It must be emphasised that these observations are influenced by the vertical grid distribution and the number of vertical grid points, both of which have been held constant for these comparisons, and that any conclusions may only be valid for these conditions. Further research work needs to be done using this procedure before it is possible to state general conclusions. The wave propagation analysis developed in Chapter 6 has created a means to do this, whereas without this tool the differencing and numerical scheme would need to be based on supposition that may not have any justifiable reasoning. Comparing numerical models with any analytic solution for a simplified situation only provides an error measure which is based on differences between numerical and exact solutions. It cannot provide valuable information such as wave speed and amplitude of the dominant numerical wave which propagates through the finite difference equations, both of which are obtained from the wave propagation analysis.

7.3 Consistency, Stability and Convergence of the Numerical Scheme

As discussed in Chapter 6 the limiting values of the graphs for the amplitude response, relative wave speed and the velocity error measure are functions of the vertical grid spacing, because these limiting values are for $\Delta\chi$ and $\Delta\lambda \rightarrow 0$ but with a fixed number and distribution of the vertical grid points. This is well suited for the wave propagation analysis because any three-dimensional numerical model has to be run for a finite number of vertical grid points (the fewer the better) and the analysis provides this information for the given vertical grid spacing, and also enables the optimum distribution of these grid points to be found. If the finite difference solution is convergent then the limiting values of the amplitude response and the relative wave speed should tend exactly to the value one and the velocity error measure should tend to exactly zero as $\Delta\chi$, $\Delta\lambda$ and $\Delta\eta \rightarrow 0$.

The consistency of the finite difference equations given in Chapter 4 can easily be verified by Taylor series expansions of the finite difference approximation for each term in the equations. There is no point in doing this formally as the approximations used were second order accurate in space and Section 7.1 shows that for the linear system the scheme is almost second order accurate in time (exactly if the horizontal diffusive terms are removed). Also, if the finite difference approximations were not consistent with the partial differential equations then the wave propagation analysis would detect this through the non convergence of the numerical solution with the analytic solution.

The stability of the numerical method is tested quite simply by finding which Courant numbers have all the eigenvalues of Equation (6.26) less than or equal to the number one. For the numerical model developed in this thesis this is true for all Courant numbers less than or equal to 0.7.

By Lax's equivalence theorem the stability of the numerical scheme and the consistency of the finite difference equations with the given linear partial differential equations implies that the numerical solution is convergent to the exact solution of the partial differential equations. Local application of this procedure to linearised versions of the non-linear partial differential equations can then be used to indicate convergence of the complete set of governing equations. The wave propagation analysis can be used to demonstrate the convergence of the numerical solution and also to show that the rate of convergence is influenced by the structure of the vertical grid.

The convergence of the finite difference solution of the numerical model can be demonstrated by letting the number of vertical grid spacings, K , tend to infinity. In the wave propagation analysis the eigenvalues and eigenvectors of a $2K+1$ by $2K+1$ complex matrix must be found, so it is not possible to let K become very large. On the Pyramid computer used for this work the largest value tried was $K = 250$, which gives a 501 by 501 complex eigenvalue problem. This took about 8 hours of computing time and used a large portion of the computer's available virtual memory, so this is close to the limit of possibility on the Pyramid. However, this was easily large enough to show that the finite difference solution was converging to the solution of the given partial differential equations.

The limiting values for the amplitude response, relative wave speed and the velocity error measure were calculated for three different vertical distributions. The three different initial distributions of the vertical grid points chosen were the uniform distribution ($\kappa = 0$), a κ grid with parameter $\kappa = 2$ and a p grid (see Section 3.2) with parameter $p = 0.13$ (each grid spacing is 13% larger than the previous one), each with 16 grid spacings. The number of vertical grid spacings were then increased from 16 to 30, 60, 120 and 250. In the case of the p grid, when the number of grid points doubles the parameter p was halved in order to keep a grid with approximately the

same properties, thus the values of p used for the different number of grid spacings were 0.13, 0.063, 0.031, 0.0154 and 0.0071. The graph of the limiting values of the amplitude response, relative wave speed and the velocity error measure are shown in Figures 7.21 to 7.23 respectively.

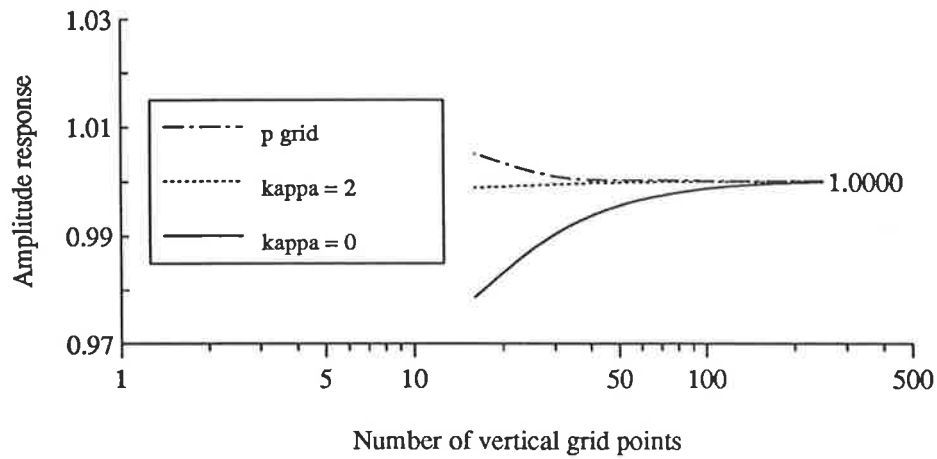


Figure 7.21: The limiting values of the amplitude response compared for different values of the number of vertical grid points.

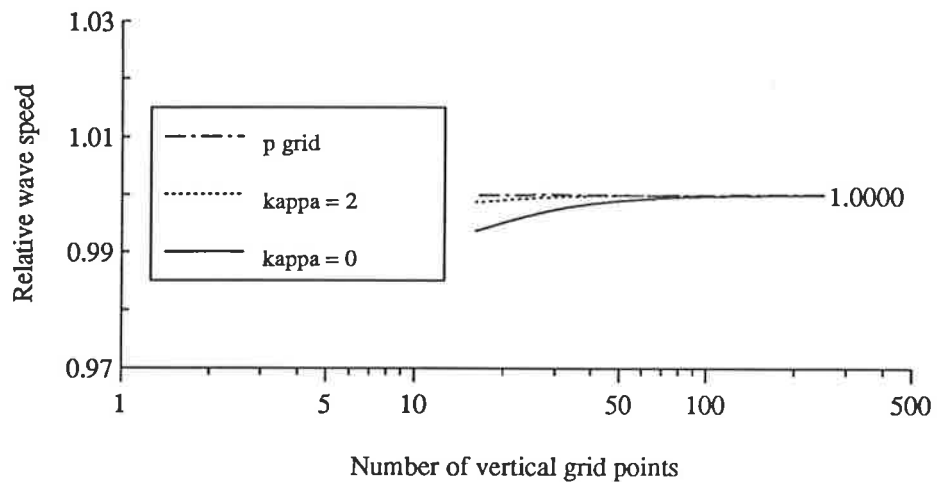


Figure 7.22: The limiting values of the relative wave speed compared for different values of the number of vertical grid points.

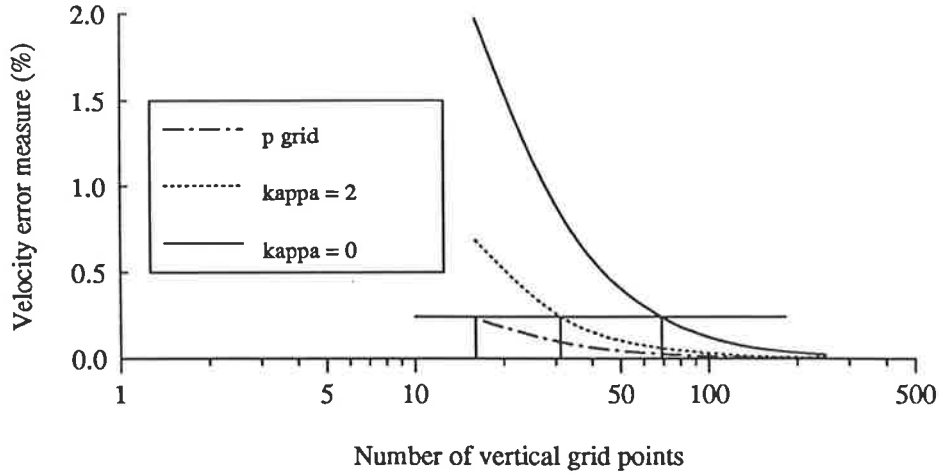


Figure 7.23: The limiting values of the velocity error measure compared for different values of the number of vertical grid points.

From these figures it is seen that the limiting values quickly converge to the desired result thus showing that the finite difference solution converges to the solution of the governing partial differential equations, although the rate of convergence is clearly dependent on the size and nature of their vertical grid distribution.

Using Figure 7.23 it can be seen that to obtain the same level of accuracy (0.25% error) in the velocity coefficients of the numerical model, would require 16 grid points with the p grid ($p = 0.13$ giving a 13% increase per grid spacing from the sea floor), 32 grid points for a vertical κ grid with parameter $\kappa = 2$ and 70 grid points for a uniform grid ($\kappa = 0$). For this particular form and magnitude of the vertical eddy viscosity parameter and other parameters this clearly demonstrates the ability of a variable vertical grid, with second order accurate finite differencing, to increase accuracy without significantly increasing the amount of computer time used for the numerical model. Extra computer time is required to calculate finite difference approximations on a variable grid but for here the alternative of using 70 grid points on an uniform grid is much more computationally expensive.

7.4 Comparison with Channel Flow

The numerical model has been modified to solve the problem first described by Johns (1966). For a rectangular basin or channel of constant depth in the absence of Coriolis force, he developed an exact solution for the predicted surface displacements ζ and the μ velocity component if the amplitude of the tidal input is small compared with the total depth of water. The channel is closed on both sides and one end and open on the remaining end along which the surface displacement is the input.

This analytic solution given by Johns requires the vertical eddy viscosity coefficient to be of the form

$$N_\eta(\eta) = \phi \left(1 + \gamma(1 - \eta) \right)^2, \quad (7.26)$$

where ϕ is the value of N_η at the sea floor and $\phi(1 + \gamma)^2$ is the value at the surface.

The analytic solution can also be obtained by reducing the three-dimensional analytic solution given in Chapter 5 down to two dimensions (χ and η) and neglecting the horizontal advective and diffusive terms as well as the Coriolis terms. That is $\nu = \nu_a = 0$, $\mu_a = 0$, $N_\chi = N_\lambda = 0$ and $f_a = 0$. An identical wave travelling in the opposite direction is then superimposed on the original wave to permit the introduction of a closed boundary at $\chi = 0$, thus giving the solution for tides in a channel closed at one end with tidal height prescribed at the open end. Therefore all the different vertical eddy viscosity profiles from Chapter 5 are available for use, with their corresponding exact solutions. Exact solution (c) in the two-dimensional case, with 2 waves superimposed, is the solution given by Johns (1966). For this test two different profiles for N_η were used. The first is a parabolic vertical eddy viscosity form (exact solution (d) for the two-dimensional case in Chapter 5), that is

$$N_\eta = d(\eta - \alpha)(\eta - \beta), \quad (7.27)$$

and d , α and β are defined in Chapter 5 in terms of the surface, maximum and bottom

values of N_η and the analytic solution is in terms of complex Legendre functions. The second form used for N_η is the form used by Johns (1966, 1967, 1968, 1969b) given by Equation (7.26).

For the parabolic form for N_η , two different values of N_η at the sea floor are used. The first of these produces moderate shear stresses and therefore moderate gradients in the velocity profile near the sea floor ($N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.002 \text{ m}^2/\text{s}$) while the other which gives large shear stresses and velocity gradients near the sea floor ($N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.0002 \text{ m}^2/\text{s}$). The latter case is a more stringent test for a numerical model with a no-slip condition at the sea floor.

The numerical model was forced by prescribing the analytic solution for the elevation at the open boundary, commencing with all elevation and velocity values initialised to zero. The dimensions and other model parameters that were used in the comparison are: $g = 9.81 \text{ m}^2/\text{s}$, $H = 20\text{m}$, $L = 95 \text{ km}$, $a = 0.5 \text{ m}$, $L\Delta\chi = 5 \text{ km}$, $\hat{\sigma} = 1.454 \times 10^{-4} \text{ s}^{-1}$, $K = 16$ and $\Delta t = 300 \text{ s}$ (therefore Courant number ≈ 0.84).

The numerical model was run for 10 periods (120 hrs) from a cold start, so that any starting transients had time to decay, and then the results were compared to the analytic solution. Figure 7.24 shows the velocity predictions and analytic solution 117 hours after starting and at hourly intervals for 3 hours before and after this time for the numerical model with the above parameters and a uniform grid at the point 80 km from the solid boundary at $x = \chi = 0$.

As can be seen in Figure 7.24 the numerical solution is very accurate, for the uniform grid. Two error measures which have been used to assess the accuracy of the

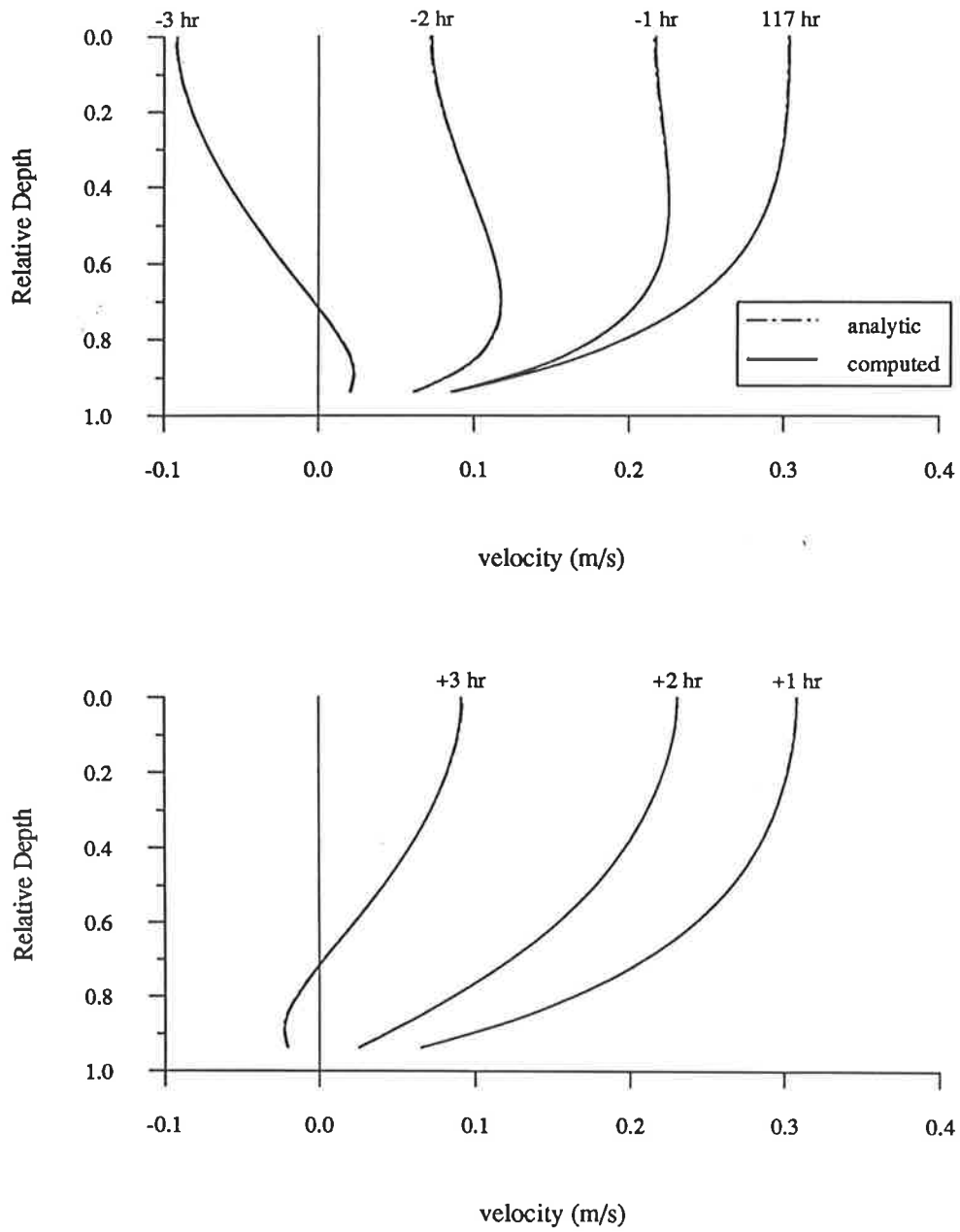


Figure 7.24: Computed and analytic results using a parabolic vertical eddy viscosity parameter which has $N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.002 \text{ m}^2/\text{s}$ and with a uniform grid ($\kappa = 0$).

numerical solution are the surface elevation error

$$E_{\zeta} = \frac{100}{aS} \sum_{1 \text{ period}} \left\{ \sum_{\text{all elements}} \left| \zeta_{\text{analytic}} - \zeta_{\text{numerical}} \right| \right\}, \quad (7.28)$$

where $S = (\text{number of time steps per period}) \times (\text{number of elements in model})$ and the velocity error measure

$$E_{\mu} = \frac{100}{SK\mu_{\max}} \sum_{p=0}^{K-1} \left\{ \sum_{1 \text{ period}} \left\{ \sum_{\text{all elements}} \left| \mu_{\text{analytic}}(\eta_p) - \mu_{\text{numerical}}|_{\eta=\eta_p} \right| \right\} \right\}. \quad (7.29)$$

These error measures are averaged over a tidal cycle, averaged over the entire simulated area and for the velocity error averaged over depth. For the above example with a uniform grid spacing the error measures are $E_{\zeta} = 0.0164\%$ and $E_{\mu} = 0.0803\%$. The performance of the model can be improved slightly by choosing a grid parameter with a value for κ which is between 0.1 and 0.4. The improvement cannot be noticed on a diagram like that in Figure 7.24, but the errors graphed in Figure 7.25 (which are for the same point and at the time 117 hours), show the improvement of the model when a value of $\kappa = 0.4$ is used. This is also reflected in the improvement in the velocity error measure, which for this value of κ , has $E_{\mu} = 0.0648\%$. The tidal height error measure E_{ζ} is smallest for a vertical grid with $\kappa = 0.1$ which gives a value of $E_{\zeta} = 0.0076\%$.

By graphing the values of E_{ζ} and E_{μ} for different values of the grid parameter κ , as in Figure 7.26, an optimal value for the κ grid can be determined in much the same way as in the wave propagation analysis, except these values are obtained by running the model, not by analytical means. The wave propagation analysis can be used on this simplified situation by considering only one horizontal direction and using zero coefficients for the terms not being modelled by this comparison. The error measures obtained from the analysis, using the same parameters as this comparison, are graphed in Figure 7.27.

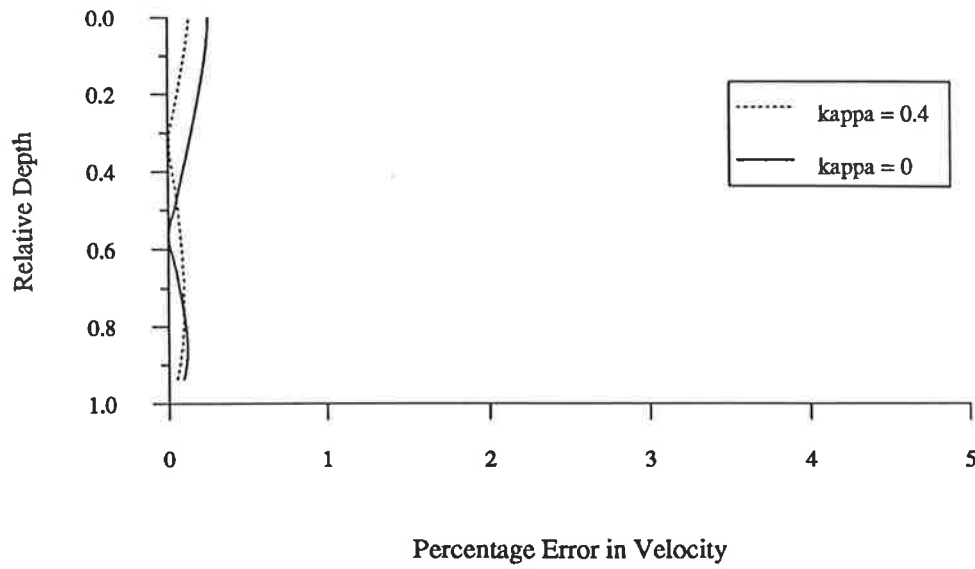


Figure 7.25: The errors between the computed and analytic solutions at 117 hours for the uniform grid ($\kappa = 0$) and the optimal κ grid.

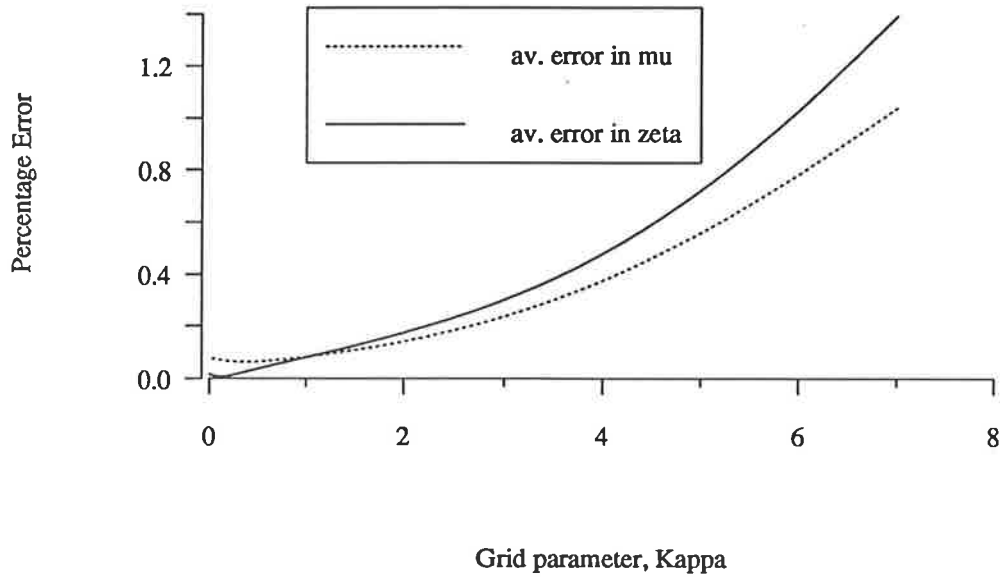


Figure 7.26: The values of E_ζ and E_μ , obtained by running the numerical model, graphed against a changing κ value for the κ grid.

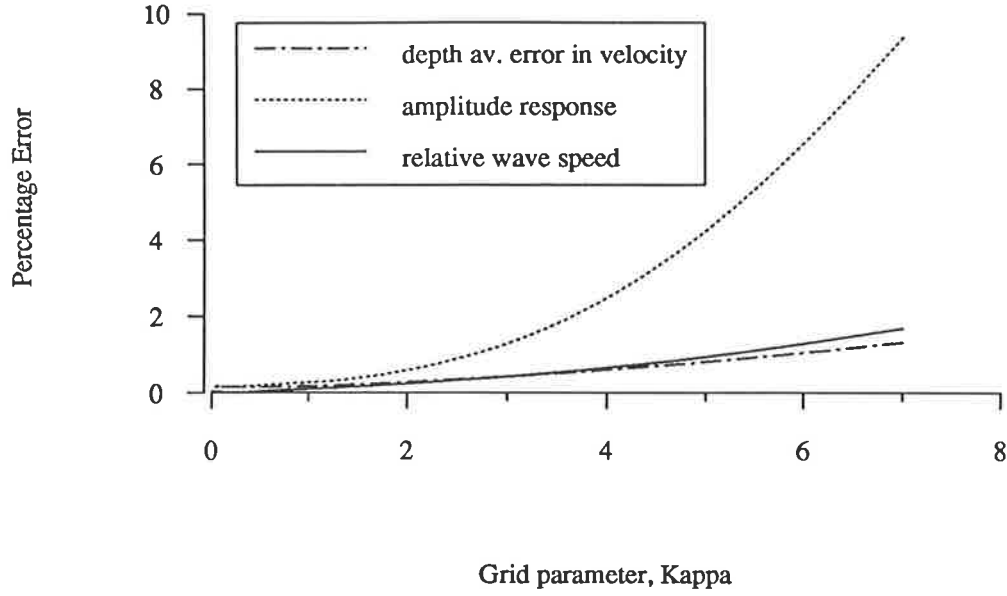


Figure 7.27: The wave propagation error measures graphed against a changing κ value and for the same parameters as Figure 7.27.

The wave propagation analysis is, in a number of ways, different from the numerical comparison in this section. The wave propagation analysis involves a complex value for the circular frequency σ for given real values of the wave number components m_x , m_λ (that is the solution has a decaying amplitude as time increases and a constant amplitude in space at any given moment), whereas this comparison involves a real value of σ making the solution periodic in time (that is the amplitude is constant over time for each position in space, but varies in space). This also causes the velocity profile coefficients $F_x(\eta)$ and $F_\lambda(\eta)$ to have slightly different distributions. The wave propagation analysis also has an infinite travelling wave without any horizontal boundaries, whereas this comparison has two related solutions ($\pm m_x$ as wave numbers) superimposed to form a standing wave with a closed boundary able to be introduced at the node where velocities are zero. By the nature of the error measures used, the same error measures cannot be used in both the wave propagation analysis and the numerical comparison. Error measures in the wave propagation analysis assess a difference of complex numbers and so can give amplitude and phase (or wave speed) errors but

the numerical comparison can only measure real differences between the numerical and analytic solutions.

Although the wave propagation analysis is significantly different from the numerical comparison in this section, there is remarkable agreement between Figures 7.26 and 7.27 with the respective error measures in velocity both predicting an optimal grid parameter of about $\kappa = 0.4$. This verifies both the theory of Chapter 6 and the accuracy of the numerical model. A p grid (see Section 3.2), with the smaller spacings near the sea bed, produces similar results to the κ grid and has a minimum for E_ζ and E_μ for a percentage increase grid with parameter around 1%. Therefore, both types of variable vertical grids have an optimal parameter which is very close to a uniform grid. For this case it may well be more economical to use a fixed uniform vertical grid in the numerical model and use a few more grid points in the vertical direction.

The second case is the parabolic profile for N_η , that is with $N_b = 0.0002 \text{ m}^2/\text{s}$ and therefore steeper velocity gradients near the sea floor. In this case the numerical model does not perform as well with the uniform grid. The analytic and numerical results are graphed in Figure 7.28 at the point 80 km from the solid boundary at $x = \chi = 0$. The error measures for this case are $E_\zeta = 0.5492\%$ and $E_\mu = 1.0156\%$.

Figure 7.29 shows similar results to Figure 7.28 except the numerical model has been run with a grid parameter of $\kappa = 3.2$. The error measures for this case are $E_\zeta = 0.0726\%$ and $E_\mu = 0.2040\%$. As can be seen from Figure 7.29 and from the error measures, the variable κ grid results in a substantial increase in accuracy (over 5 times better for this example) than for the uniform grid.

The difference between velocity profiles of the analytic and numerical solutions for the two cases graphed in Figures 7.28 and 7.29 (at the time 117 hours), is graphed in Figure 7.30. This shows the improvement in the modelling of the velocity profile

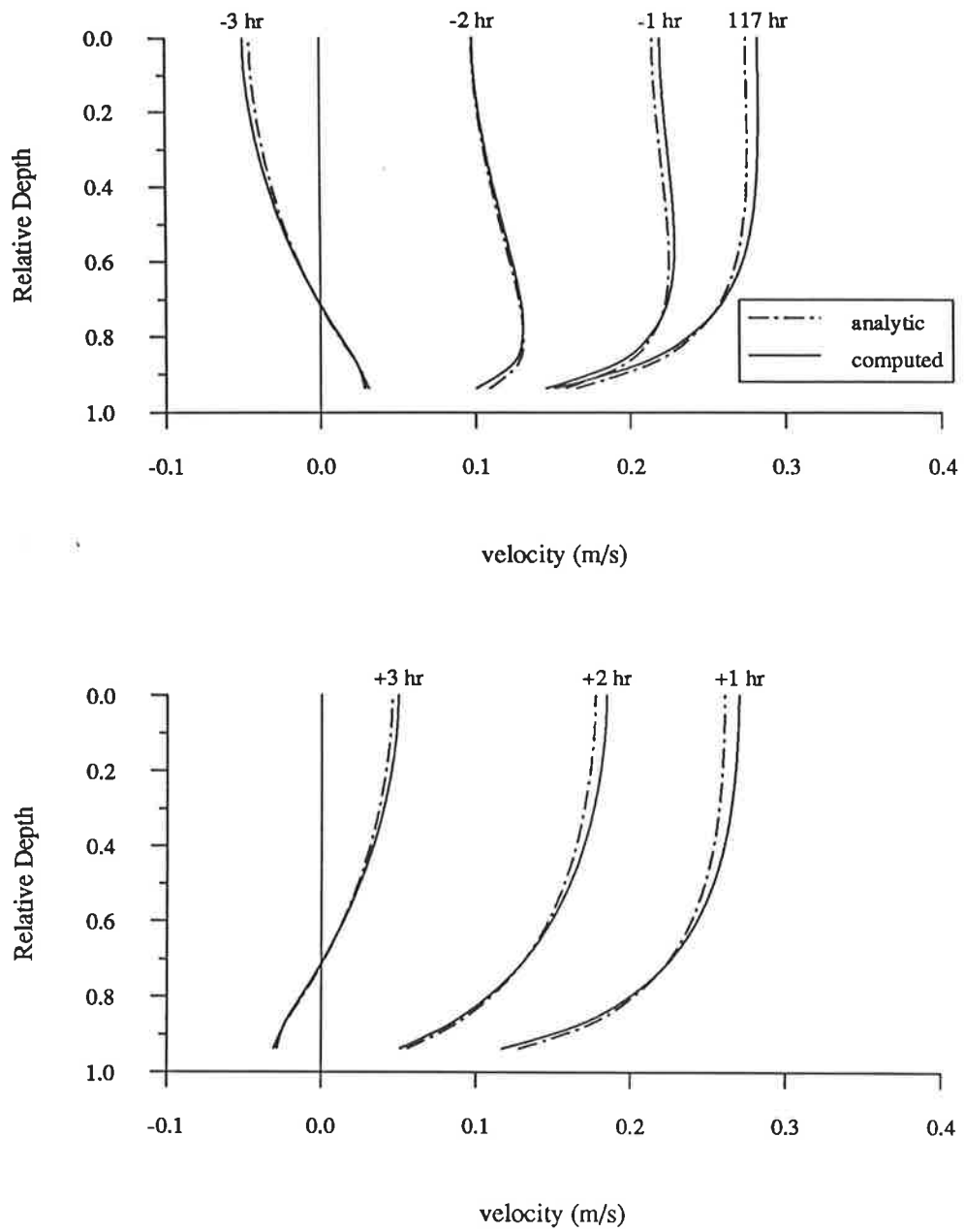


Figure 7.28: Computed and analytic results using a parabolic vertical eddy viscosity parameter which has $N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.0002 \text{ m}^2/\text{s}$ and with a uniform grid ($\kappa = 0$).

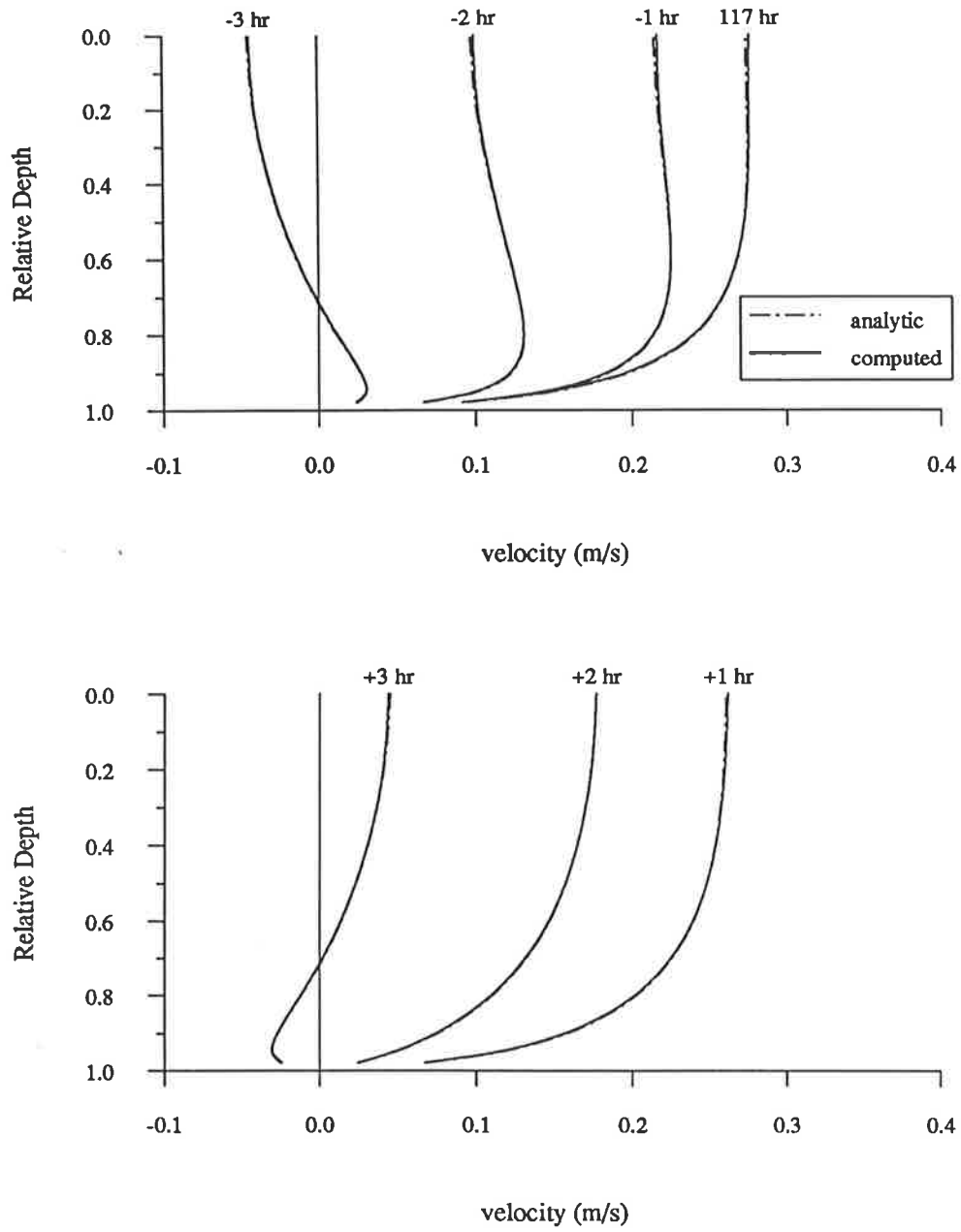


Figure 7.29: Computed and analytic results using a parabolic vertical eddy viscosity parameter which has $N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.0002 \text{ m}^2/\text{s}$ and with a variable vertical grid with $\kappa = 3.2$.

when the variable vertical grid is used. These results agree remarkably well (in general characteristics) with the analytically predicted results for $N_L = 120$ from the wave propagation analysis which have velocity profile errors graphed over depth in Figure 6.9 (uniform grid) and Figure 6.19 ($\kappa = 3.0$).

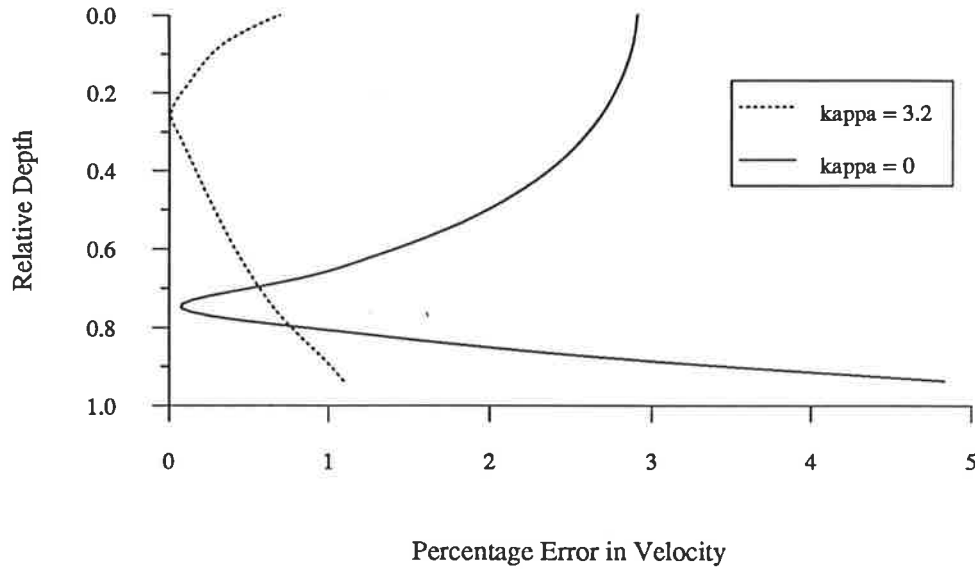


Figure 7.30: Error in the velocity profile for the numerical model with a parabolic vertical eddy viscosity parameter which has $N_s = 0.003 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$ and $N_b = 0.0002 \text{ m}^2/\text{s}$ and with two different vertical grids.

Figures 7.31 and 7.32 show the numerical error estimates and the wave propagation error estimates graphed against a varying vertical grid parameter κ . Again, as with the first case, there is a good agreement for the optimal value of κ between the analytically predicted and the numerically tested results.

For this case the family of variable vertical grids which are defined by the κ method do not provide the best grid distribution for minimising the errors of the numerical model. Figure 7.33 shows the error estimates for the numerical model graphed against a changing value of p for the p grid defined in Section 3.2. The minimum value of E_ζ is 0.0250% for $p = 0.11$ and the minimum value of E_μ is 0.0788% for $p = 0.14$.

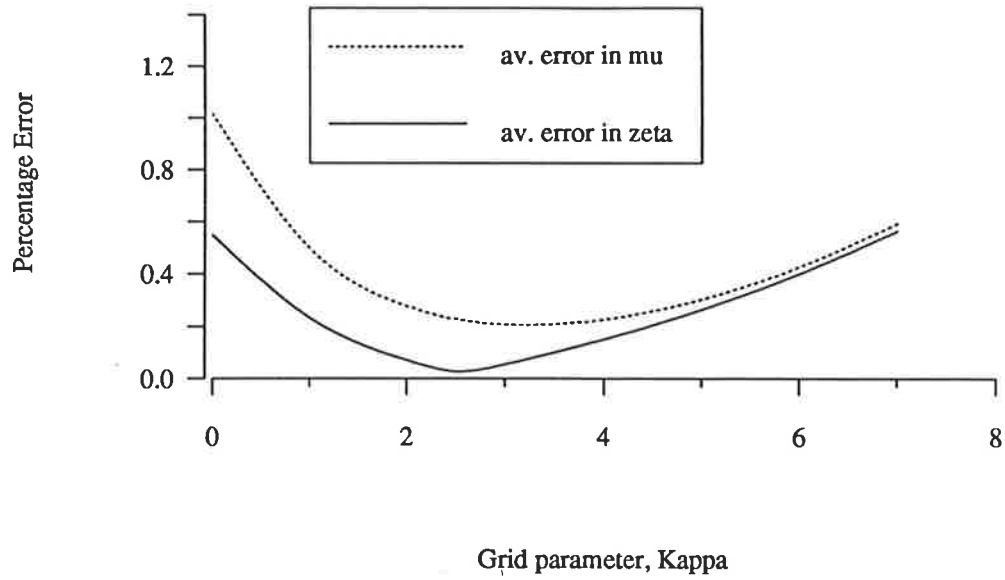


Figure 7.31: The values of E_ζ and E_μ , obtained by running the numerical model, graphed against a changing κ value for the κ grid, for the second parabolic N_η case ($N_b = 0.0002 \text{ m}^2/\text{s}$).

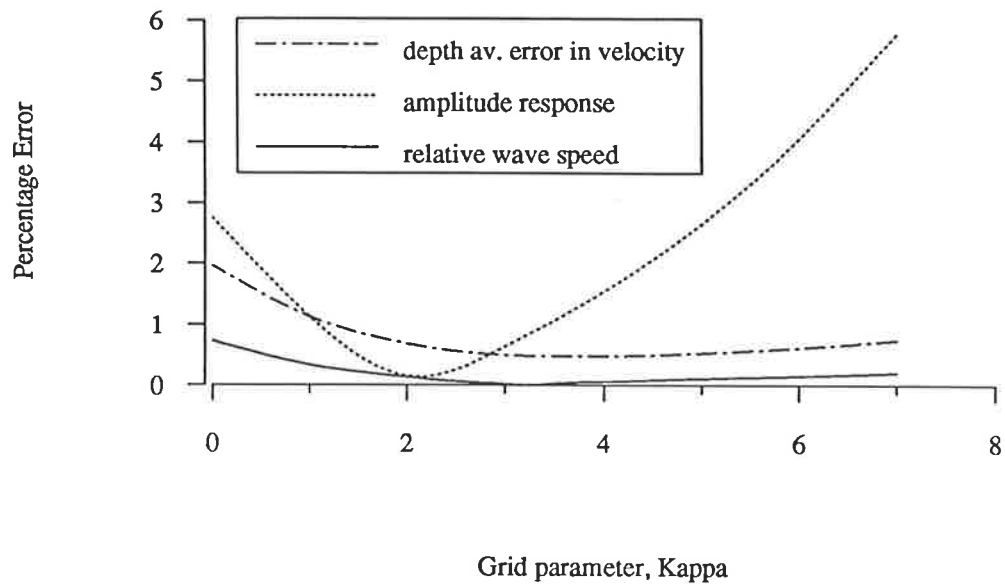


Figure 7.32: The wave propagation error measures graphed against a changing κ value and for the same parameters as Figure 7.31.

The respective minimum from the κ grid family was 0.0278% for $\kappa = 2.6$ and 0.2040% for a $\kappa = 3.2$ grid. The increase in the accuracy of the ζ (tidal height) is not large but there is a substantial increase in the accuracy of the predicted velocity profile for this p grid distribution over the κ grid method. The wave propagation analysis also predicts this behaviour for this type of vertical grid spacing. Both types of variable grid distributions provide a large improvement in the accuracy of the numerical model compared to using a uniform grid.

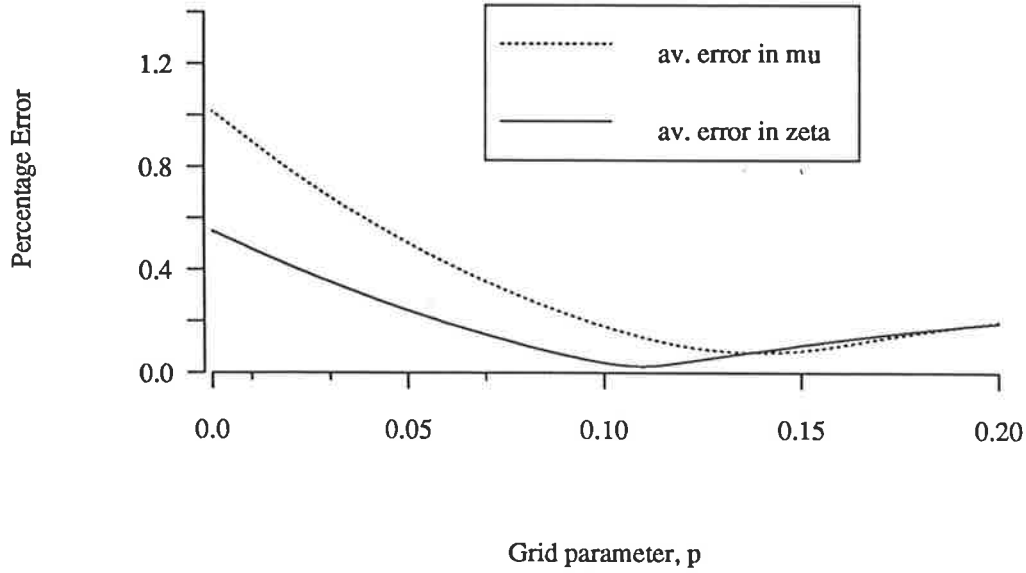


Figure 7.33: The values of E_ζ and E_μ , obtained by running the numerical model, graphed against a changing p value for the p grid, for the second parabolic N_η case ($N_b = 0.0002 \text{ m}^2/\text{s}$).

The third case considered in this comparison is for the quadratic form of N_η which Johns (1966, 1967, 1968, 1969b) has used in some of his work. The surface and sea floor viscosity values were chosen to be the same as the values that he used in the last of the articles cited, that is $N_s = 0.100 \text{ m}^2/\text{s}$, and $N_b = 0.0010 \text{ m}^2/\text{s}$. The characteristics of the flow with this form of the vertical eddy viscosity parameter are quite different from those of the parabolic form, especially at the +3 and -3 hour times of the cycle. The numerical model performs very well in this situation and the value of the optimal grid parameter κ is around a value of 1.1, which gives error estimates of $E_\zeta = 0.1254\%$ and

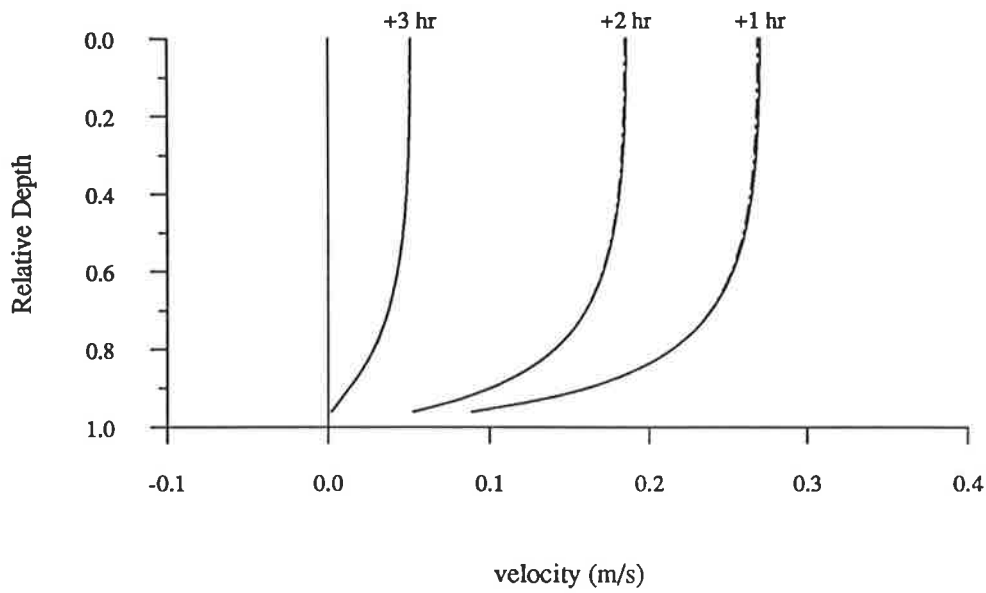
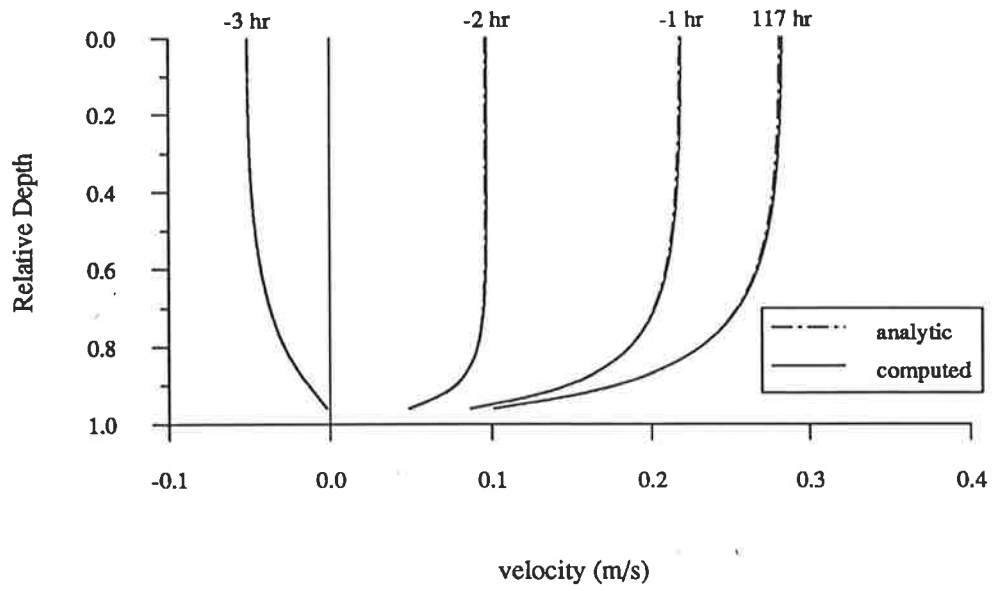


Figure 7.34: Computed and analytic results using a quadratic vertical eddy viscosity parameter which has $N_s = 0.100 \text{ m}^2/\text{s}$, and $N_b = 0.0010 \text{ m}^2/\text{s}$ and with a vertical grid with parameter $\kappa = 1.1$.

$E_\mu = 0.1527\%$. Figure 7.34 shows the analytic and computed velocity profiles at the point 80 km from the solid boundary, when the vertical grid in the model was defined with a grid parameter of $\kappa = 1.1$.

7.5 Comparison with a Truncated Series Solution for a Kelvin Wave

This section describes a comparison between results from the model, adapted to simulate the idealised conditions described by Rienecker and Teubner (1980), and their solution to a depth integrated system of tidal motion equations. Because this is a depth integrated solution, it is not able to test the differencing in the vertical direction and the main use of this test is to check the accuracy of the numerical scheme in the horizontal directions, especially that of the Coriolis terms. The wave propagation analysis encompasses these features as well as including depth variation of the velocities. Thus the development of the wave propagation analysis has diminished the usefulness of this comparison. Nevertheless the comparison is useful for checking and debugging the coding of the computer program, and a several of typographical errors were eliminated from the code in this manner.

The depth integrated equations with constant Coriolis parameter, linearised friction terms, surface displacement considered negligible compared to the sea depth and zero transverse velocity across the open boundary, have been solved for a rectangular bay of constant depth which is open at one end. Analytic solutions for ζ , $\bar{u}$ and $\bar{v}$, given as an infinite series in Rienecker and Teubner (1980), are obtainable to any degree of accuracy by evaluating sufficient terms in the series. The analytic solution was used to supply open boundary surface displacements at each time step of the numerical model, and to give exact solutions in the interior of the bay for comparison with output from the modified numerical tidal model.

The equations modelled are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial H\bar{\mu}}{\partial \chi} + \frac{\partial H\bar{\nu}}{\partial \lambda} = 0, \quad (7.30)$$

$$\frac{\partial \bar{\mu}}{\partial t} - \frac{fB\bar{\nu}}{L} = -\frac{g}{L^2} \frac{\partial \zeta}{\partial \chi} - \frac{K_b\bar{\mu}}{H}, \quad (7.31)$$

$$\frac{\partial \bar{\nu}}{\partial t} + \frac{fL\bar{\mu}}{B} = -\frac{g}{B^2} \frac{\partial \zeta}{\partial \lambda} - \frac{K_b\bar{\nu}}{H}, \quad (7.32)$$

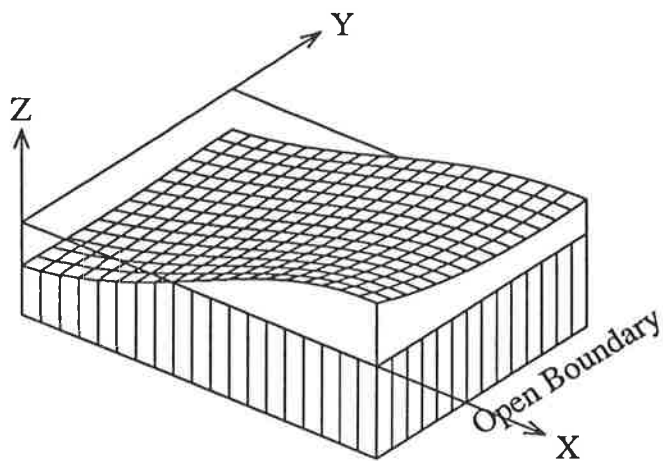
where all variables are as previously defined, K_b is a linear friction coefficient for the term representing shear stress at the sea floor and the surface elevation is considered small relative to the depth of water, h .

The rectangular bay has solid boundaries at $x = \chi = 0$, $y = \lambda = 0$ and $y = B$ or $\lambda = 1$. The usual boundary conditions apply at these boundaries, that is, $\mu = 0$ at $\chi = 0$ and $\nu = 0$ at $\lambda = 0$ and 1 . To solve the problem the boundary condition of $\nu = 0$ also needs to be stipulated at the open boundary, where $x = L$ or $\chi = 1$. This ensures a unidirectional input along the open boundary.

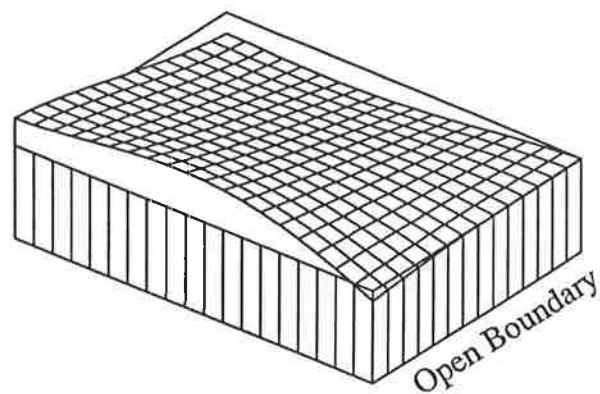
The dimension of the bay and the model parameters used in this comparison are $\Delta t = 300$ s (5 minutes), $K_b = 0.001$ m/s (linear friction coefficient), $L = 290,000$ m (length 290 km), $B = 90,000$ m (width 90 km), $\Delta x = 7436$ m (x space step), $\Delta y = 5000$ m (y space step), $h = 20$ m (constant depth), $f = -0.000073$ (i.e. latitude 30°S) and $a = 1/6$ m (incident amplitude).

The open boundary forcing in this case produced a Kelvin wave with a maximum elevation amplitude of approximately 0.53 m, which rotated in a clockwise direction around the bay. Figure 7.35 shows the solution from the numerical model at four stages through the 12 hour cycle. The analytic solution is exactly the same when the graphs are drawn on that scale.

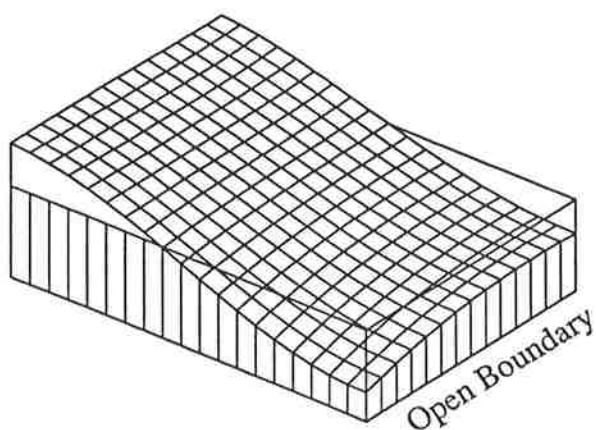
Comparison of the analytic solution with the model's results indicate an average numerical error in surface elevation of 0.0006 m when the maximum elevation was



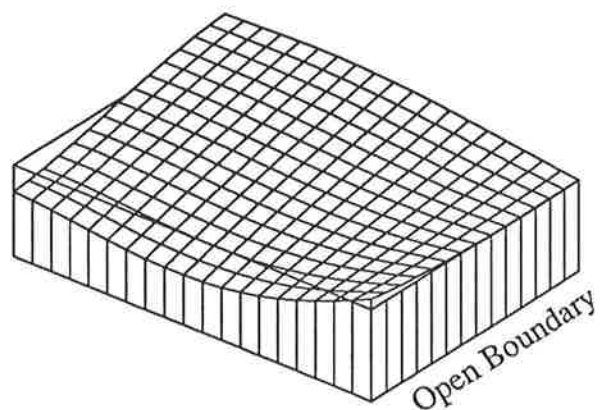
time = 120 hours



time = 123 hours



time = 126 hours



time = 129 hours

Figure 7.35: Four graphs of the Kelvin wave solution at three hour intervals through the 12 hour cycle. The analytic and model results are indistinguishable.

0.53 m (0.1132% error). The maximum error in surface elevation was 0.002 m (0.3774% error). The average numerical error in the depth averaged x-component of velocity was 0.0004 m/s in a maximum value of 0.308 m/s (0.1299% error) and the average error in the depth averaged y-component was 0.0001 m/s in a maximum value of 0.038 m/s (0.2632% error).

7.6 Comparison with a Second Order Solution to Non-Linear Channel Flow

A second order solution to the depth integrated, one-dimensional tidal motion equations has been described by Knight (1973). This solution can be used to test the differencing of some of the non-linear terms in the three-dimensional model, which has been modified to model the same conditions and equations solved by Knight. The analytic solution was used to supply open boundary surface displacements at each time step of the numerical model, and to give values for comparison with output from the modified numerical model.

The equations modelled are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial H\bar{\mu}}{\partial \chi} = 0, \quad (7.33)$$

$$\frac{\partial \bar{\mu}}{\partial t} + \bar{\mu} \frac{\partial \bar{\mu}}{\partial \chi} = -\frac{g}{L^2} \frac{\partial \zeta}{\partial \chi} - \frac{f_b \bar{\mu} |\bar{\mu}|}{2h\alpha}, \quad (7.34)$$

where all the symbols are as previously defined, f_b is the Darcy resistance coefficient (English) and α is the ratio of the hydraulic radius to mean depth ($\alpha = R/h$, where R is the hydraulic radius and b is the breadth of the water).

The value of this comparison is that it models the non-linear advection term, however it is only one-dimensional (depth integrated) and the analytic solution is only available to second order accuracy.

The dimension of the bay and the model parameters used are the same as those used by Knight, namely, $L = 20$ m (length), $B = 0.15$ m (constant width), $h = 0.15$ m (constant depth), $f_b = 0.005$ (Darcy friction coefficient) and $a = 0.02$ m (incident amplitude). The numerical model was run with ten grid points in the horizontal direction, so $\Delta\chi$ was equal to 1.05 m, and with a time step of 1 second.

Comparison of Knight's solution with model results indicate an average error in surface elevation of 0.00094 m relative to a maximum elevation of 0.032 m (2.938% error). The average error in the depth averaged x-component of velocity was 0.00472 m/s in a maximum value of 0.193 m/s (2.44% error).

Knight's analytic solution is only accurate to second order, so it is possible that part of the apparent error is due to the low order of accuracy of his solution. For the same case with an incident amplitude of 0.01 m, when Knight's solution is more accurate than the case described above, the relative error in surface elevation and horizontal velocity was less than half that obtained in the first test case.

7.7 Testing of Wind Stresses on the Surface

The part of the numerical model which simulates a wind blowing across the sea surface and thus creating a horizontal surface stress can be tested by using an analytic solution for a long narrow channel closed on both ends and of constant depth.

The problem to be modelled is two-dimensional, with variation in the vertical and one horizontal direction, and the linear equations to be solved are

$$\frac{\partial\zeta}{\partial t} + \frac{\partial H\bar{\mu}}{\partial\chi} = 0 , \quad (7.35)$$

$$\frac{\partial\mu}{\partial t} = -\frac{g}{L^2} \frac{\partial\zeta}{\partial\chi} + \frac{1}{H^2} \frac{\partial}{\partial\eta} \left(N_\eta \frac{\partial\mu}{\partial\eta} \right) , \quad (7.36)$$

where all the symbols are as previously defined and the surface elevation is considered small relative to the depth of water, h . Therefore $H \approx h$.

The boundary conditions at the closed edges are $\mu = 0$ at $\chi = 0, L$ and $\mu = 0$ at $\eta = 1$ with the boundary condition at the surface supplying the input to the system by means of the surface stress. This boundary condition is that of Equation (2.55), namely

$$\frac{-\rho L N_\eta}{H} \frac{\partial \mu}{\partial \eta} = \tau_s, \quad \text{at } \eta = 0, \quad (7.37)$$

where τ_s is the wind stress at the surface. The wind stress has been estimated from the wind speed, measured at 10 metres above the sea level, using Equations (2.57) and (2.58). For this comparison a wind speed of $W_{10} = 10$ m/s is used. The model was also run with wind speeds of 5 m/s and 20 m/s but as the errors in the numerical model, expressed as percentages of the maximum velocity, were exactly the same, all subsequent testing was only performed for a wind speed of 10 m/s. For this wind speed, Equations (2.57) and (2.58) give a surface stress of 0.187 N/m².

A solution can be found for various functional forms of the vertical eddy viscosity parameter N_η and for a wind stress which is homogeneous in space, varying sinusoidally with time and with amplitude τ_0 , that is

$$\tau_s = \tau_0 \exp\{i\sigma t\}. \quad (7.38)$$

Walsh (1974) provided the solution for a constant N_η and Arnold (1985) has given the solution for a range of N_η forms. In fact all the profiles used in the analytic solution given in Section 5.2 will give rise to analytic solutions for this problem. Arnold has given the solutions for all of these except the $\sin^2$ profile. For each of the cases he also gives the equilibrium solutions for a steady uniform wind by letting $\sigma \rightarrow 0$.

The rest of this section follows the slightly different notation used by Arnold (1985), where $\eta = 1 - \theta$ (θ is used to correspond to Arnold's different definition of η). The

profile for N_η chosen to test the model is the parabolic profile in the form

$$N(\theta) = N(1 - \eta) = d(\theta + \alpha)(\theta + \beta) , \quad (7.39)$$

where

$$\begin{aligned} d &= N_s + N_b - 2N_m - 2\sqrt{(N_m - N_s)(N_m - N_b)} , \\ \alpha, \beta &= \frac{N_t \pm \sqrt{N_t^2 - 4N_b d}}{2d} , \\ N_t &= 2N_m - 2N_b + 2\sqrt{(N_m - N_s)(N_m - N_b)} , \end{aligned}$$

and N_s , N_b and N_m are the sea surface, sea floor and maximum value vertical eddy viscosity values.

Using the depth transformation given by

$$\xi(\theta) = \frac{\theta + \beta}{\beta - \alpha} , \quad (7.40)$$

Arnold gives the equilibrium solution as

$$\begin{aligned} u(x, \eta) &= B'_4 \tau_0 \left\{ \frac{-\log |\xi - 1|}{\log |\xi_b - 1|} + 1 \right. \\ &\quad \left. - R'_4 \left(\frac{\log |\xi - 1|}{\log |\xi_b - 1|} \log \frac{|\xi_b - 1|}{|\xi_b|} - \log \frac{|\xi - 1|}{|\xi|} \right) \right\} , \end{aligned} \quad (7.41)$$

$$\zeta(x) = \frac{-B'_4 d}{gH^2 \log |\xi_b - 1|} \left\{ 1 - R'_4 \log \frac{|\xi_b - 1|}{|\xi_b|} \right\} \left(x - \frac{L}{2} \right) , \quad (7.42)$$

where

$$\begin{aligned} R'_4 &= \left[\xi_s - \xi_b - \frac{(\xi_s - 1)(\log |\xi_s - 1| - 1)}{\log |\xi_b - 1|} + \frac{(\xi_b - 1)(\log |\xi_b - 1| - 1)}{\log |\xi_b - 1|} \right] / , \\ &\quad \left[(\xi_s - 1) \left\{ \frac{(1 - \log |\xi_s - 1|)}{\log |\xi_b - 1|} \log \frac{|\xi_b - 1|}{|\xi_b|} + \log |\xi_s - 1| - 1 \right\} \right. \\ &\quad \left. - (\xi_b - 1) \left\{ \frac{(1 - \log |\xi_b - 1|)}{\log |\xi_b - 1|} \log \frac{|\xi_b - 1|}{|\xi_b|} + \log |\xi_b - 1| - 1 \right\} \right. \\ &\quad \left. - \xi_s(\log \xi_s - 1) + \xi_b(\log \xi_b - 1) \right] \end{aligned} \quad (7.43)$$

$$B'_4 = \frac{H(\rho d(\beta - \alpha)\xi_s)^{-1}}{\left(\frac{-1}{\log|\xi_b - 1|} - R'_4 \left(\frac{1}{\xi_s} - \frac{\log(|\xi_b - 1|/|\xi_b|)}{\log|\xi_b - 1|} \right) \right)}, \quad (7.44)$$

and $\xi_s = (\beta + 1)/(\beta - \alpha)$ and $\xi_b = \beta/(\beta - \alpha)$.

The numerical model was initially at rest with the wind introduced as a step function at the start. The model was then run for 120 hours with the constant wind stress applied to the surface. By this time practically all of the starting transients have decayed and the motion is steady. The model parameters which were used in this comparison are $\Delta t = 300$ s (5 minutes), $L = 100,000$ m (length 100 km), $\Delta x = 5000$ m (x space step), $h = 20$ m (constant depth), $N_s = 0.001$ m²/s, $N_m = 0.005$ m²/s, $N_b = 0.0010$ m²/s, $W_{10} = 10$ m/s (constant wind speed), $\rho = 1,030$ kg/m³ (constant sea density) and $\rho_a = 1.290$ kg/m³ (constant air density).

The velocity profile over depth, with water flowing in the same direction as the wind stress near the surface and with an underflow in the opposite direction, is graphed in Figure 7.36 for the analytic solution and the numerically computed solution. The numerical model for this graph was run with 16 depth levels, and a uniform vertical grid spacing ($\kappa = 0$).

The error measure used to test the accuracy of the model for this comparison is

$$\begin{aligned} E_\mu &= \frac{100}{K\mu_{\max}} \int_{\eta=0}^{\eta=1} \left| \mu_{\text{analytic}}(\eta) - \mu_{\text{numerical}} \right|_\eta d\eta, \\ &\approx \frac{100}{K\mu_{\max}} \sum_{k=0}^K s_k \left| \mu_{\text{analytic}}(\eta) - \mu_{\text{numerical}} \right|_\eta, \end{aligned} \quad (7.45)$$

where the values of s_k are defined in Section 4.1.

As can be seen from Figure 7.36 the numerical model has a significant error, with the error measure $E_\mu = 5.5280\%$ also showing this. The accuracy of the numerical model can be improved by using a variable vertical grid with the grid distribution chosen

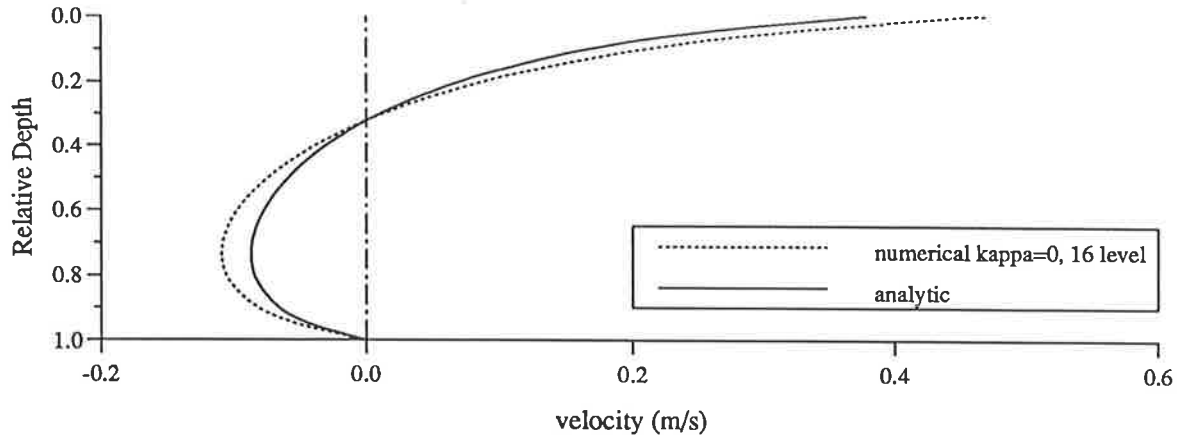


Figure 7.36: Computed and analytic results using a parabolic vertical eddy viscosity parameter which has $N_s = 0.001 \text{ m}^2/\text{s}$, $N_m = 0.005 \text{ m}^2/\text{s}$, and $N_b = 0.0010 \text{ m}^2/\text{s}$, with $W = 10 \text{ m/s}$ and with a uniform vertical grid ($\kappa = 0$) with 16 depth levels.

carefully. Both the κ grid and the percentage increase grid improve the accuracy of the numerical model dramatically. With the wind stress acting on the surface and the "no-slip" condition at the sea floor, the vertical grid has been chosen to be finer at the ends and coarser in the middle. The κ or the percentage grid method is used to calculate a vertical grid distribution for half the grid and the mirror image used for the other half.

The first diagram in Figure 7.37 shows the computed and analytic solutions for (1) a κ grid with grid parameter $\kappa = 9$ ($K = 16$ depth levels), (2) a κ grid with grid parameter $\kappa = 8$ ($K = 26$ depth levels) and (3) a p grid with grid parameter $p = 0.31$ ($K = 16$ depth levels). All of these are very accurate compared to the results obtained with a uniform grid. The second diagram in Figure 7.37 shows the differences between the computed and analytic solutions for the three different cases. The error measures for these three cases are (1) $E_\mu = 0.3249$, (2) $E_\mu = 0.1631$ and (3) $E_\mu = 0.1697$.

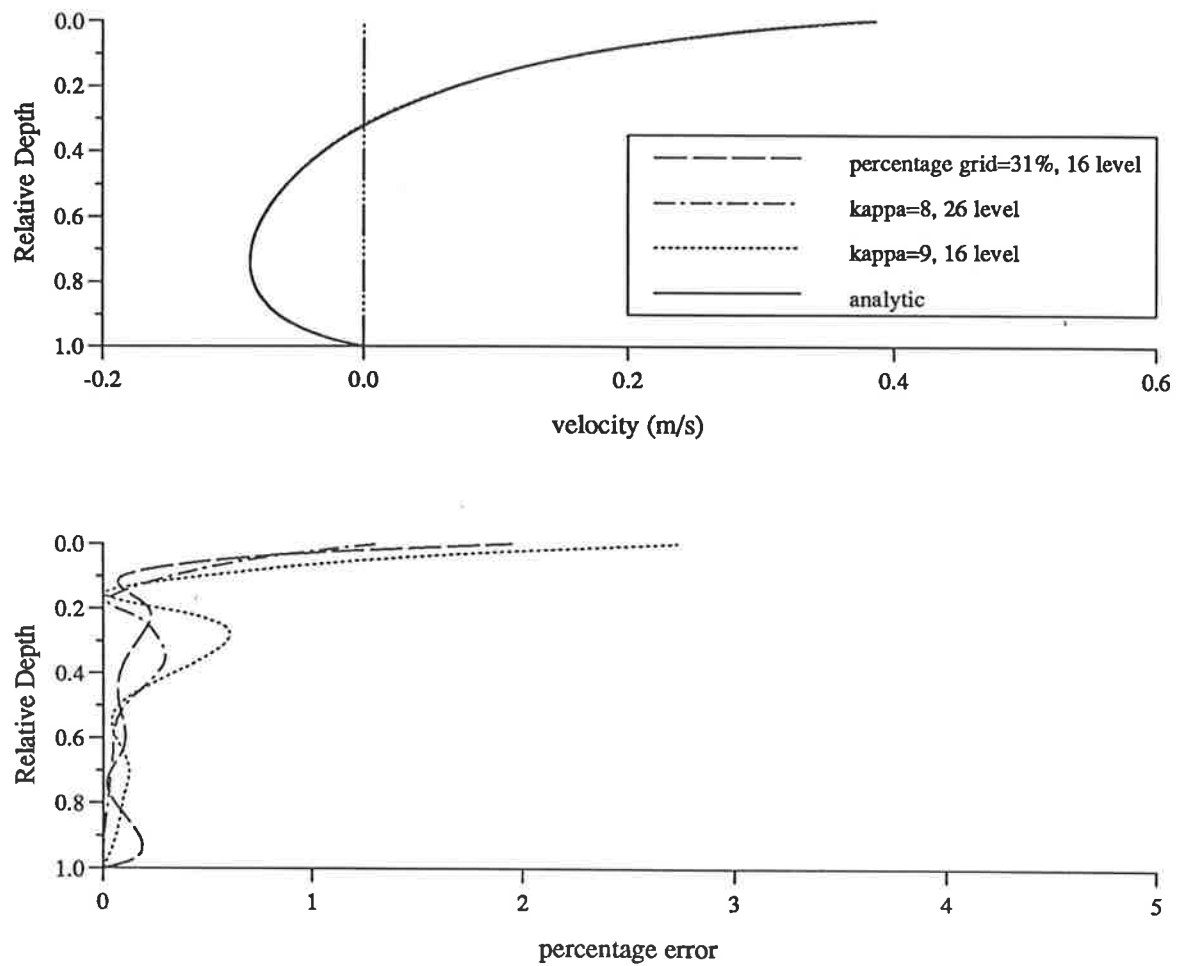


Figure 7.37: Velocity profiles and percentage difference from the analytic solution for three vertical grid distributions.

As expected the main source of the numerical error is near the sea surface where the wind stress is applied and all cases have a much larger error in that zone. For the two cases with 16 depth levels the percentage grid method has a smaller error near the surface, a larger error near the sea floor but overall the error measure is significantly smaller. For the two κ grids the increase in the number of grid spacings from 16 to 26 gives rise to an halving of both the maximum and average errors.

By comparing Figures 7.36 and 7.37, it can be seen clearly that the variable vertical grid is able to greatly increase the accuracy of the numerical scheme compared to a uniform grid for the same number of vertical grid spacings.

Chapter 8

CALIBRATION OF THE NUMERICAL MODEL FOR AN APPLICATION

This chapter describes the setting up and calibration of the numerical model for a specific application. The numerical model is more general than is required for this application, but this application demonstrates the applicability of the numerical model. As with most numerical models, it requires some customising and calibration in order to achieve the required accuracy. Section 8.1 gives some background information about the application area and Section 8.2 describes the setting up of the model. The specific details of the calibration process used for this particular application are given in Section 8.3.

8.1 Introduction

The area chosen for the application of the developed numerical model is the upper reaches of Spencer Gulf, South Australia. This area, shown in Figures 8.1 and 8.2, is of particular interest due to the location of a power station about 5 km south from Port Augusta. The power station is run by the Electricity Trust of South Australia

(ETSA) and is powered by coal transported from Leigh Creek, South Australia, 230 km north of Port Augusta. One of the objectives for setting up and calibrating the numerical model for this region has been to permit ETSA to use it to further the understanding of the area and to help predict the effect and extent of the warm water, which is discharged into the gulf from the power station.

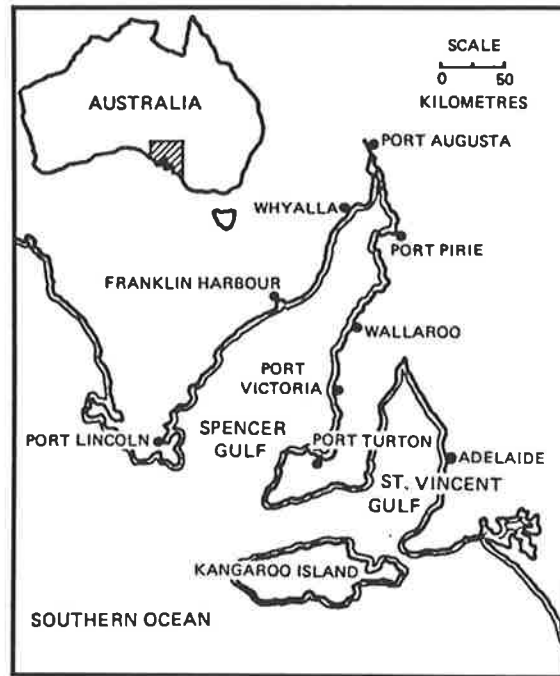


Figure 8.1: Map showing the location of Spencer Gulf. The area modelled is at the top of Spencer Gulf, which is shown in more detail in Figure 8.2 and the model boundary and horizontal grid are shown in Figure 8.3.

The station consists of two sections, the older Playford Power Station and the newer Northern Power Station. The Playford Power Station was built in two phases. The “A” phase with a capacity of 90 Mega Watts was retired in 1987/88 and the “B” phase with a capacity of 240 Mega Watts (4×60) currently operates in a peak load role and

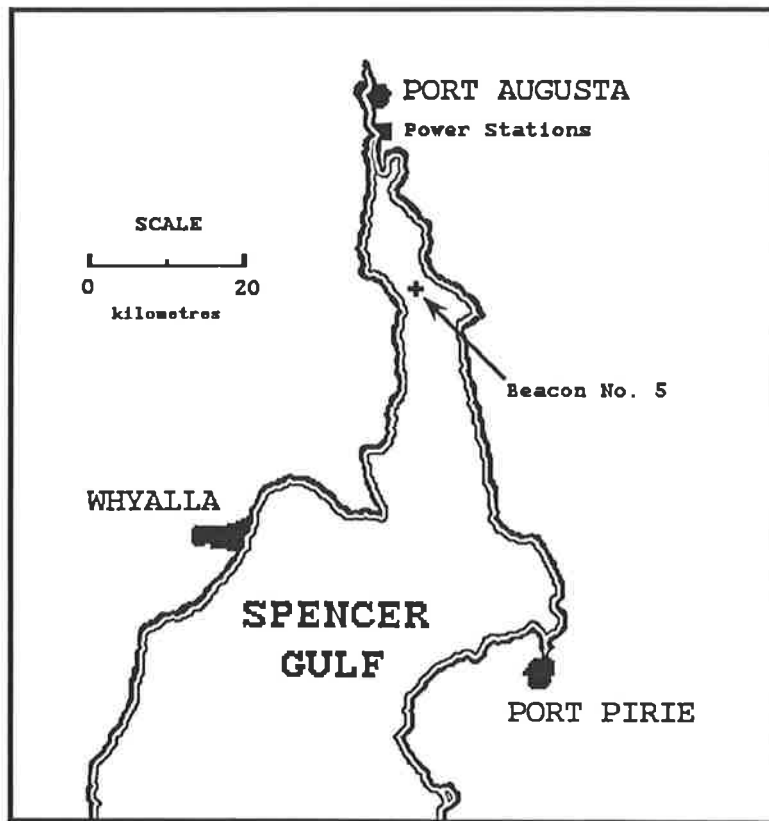


Figure 8.2: Map showing the upper reaches of Spencer Gulf. The model application area is from the top of the gulf to Beacon No. 5.

also provides capacity support when South Australia's other stations are unavailable. The newer Northern Power Station has two 250 Mega Watt generators, which for the 1988/89 financial year provided 38% of South Australia's power supply, namely 3,437 Giga Watt hours (GWh) out of the states total of 8,988 GWh while Playford "B" provided only 1.1% or 105 GWh. This indicates that the Northern power Station was running at an average (with down time of only 6.8% included) of 78% of capacity for the year. The average for Playford "B" was 5% of capacity. Both power stations draw water from the gulf to cool their condensers and discharge the resulting warm

water back into the gulf. The Northern Power Station returns this heated water with a temperature increase of no more than 7°C. Economically the optimal temperature rise in order to minimise generating costs is between 8° and 12°C, but the temperature difference needs to be kept as low as possible to minimise the environmental impact.

The water inlet and outlet for the Northern Station are separated by the use of two adjacent channels. The inlet channel is barricaded at its mouth in such a way that water is drawn from the gulf at a depth of more than 8 metres. As the heated water leaves the outlet channel, it mixes with the cooler water in the gulf. The resultant mixture, at a temperature about 4° or 5°C more than that at the point of intake (8 metres down) tends to form a layer on the surface of the gulf and does not affect the temperature of the intake water.

For the Northern Power station the maximum cooling water flow rate is $4.06 \times 10^6 \text{ m}^3/\text{day}$ ($1.37 \times 10^6 \text{ m}^3/\text{day}$ for Playford “B”) and the maximum heat load is $104 \times 10^6 \text{ MJ}/\text{day}$ ($36 \times 10^6 \text{ MJ}/\text{day}$ for Playford “B”). The average heat loads are about 75% of these figures.

Tide height information is known at Port Augusta and at Beacon No. 5 situated about 26 km south of Port Augusta and in limited form at the power stations (see Figure 8.2). Tidal constants for Port Augusta and Beacon No. 5 have been supplied by the Tidal Laboratory of the Flinders Institute for Atmospheric and Marine Science (F.I.A.M.S) at The Flinders University of South Australia, Copyright reserved. The tidal constants used in this work are given in Tables 8.1 and 8.2 in Section 8.3. ETSA has also been collecting data near the power stations in the form of tide height logs, water temperature surveys and a few velocity measurements.

8.2 Setting up the Model

The area of application extends from the top of the gulf, 5 km north of Port Augusta down to Beacon No. 5. The southern limit was chosen so that the known tidal constants at Beacon No. 5 could be used to specify the tidal height for the open boundary of the model. The model does not incorporate wetting and drying of sand flats so the coastline for the model is taken as the coastline when the water level is at mean sea level. The known tidal constants from Port Augusta can then be used to validate and calibrate the numerical model. The grid used for the numerical model is shown in Figure 8.3.

The tidal constants for Port Augusta computed by F.I.A.M.S. were obtained from an analysis of tide heights from a standard tide well for the period 1/1/83 to 31/12/83 (with nine gaps of 11 to 51 hours). The data was analysed for 112 constituents using a Gauss-Seidel iterative method for solving the set of algebraic equations involved. The data from Beacon No. 5 was measured with a pressure gauge on the sea bed for the period of 104 days starting from 8/4/83. Atmospheric pressure was removed from the data and the sea height calculated. This was then analysed for 40 tidal constituents using a Gauss pivotal condensation method and by stipulating local relationships between some of the constituents, namely between P1, S1 and K1 and between T2, S2 and K2, which have frequencies so close that they are not able to be separated for the given length of data.

ETSA has measured water temperatures and tidal heights in the gulf waters near the power stations and has also taken some measurements of water velocities. They have used the data to predict the extent of the warm water plume in the surrounding gulf waters. Their tidal height measurements only exist as an analogue recording and have not been digitised or analysed to give the tidal constituents. From 5:30 am to 3:00

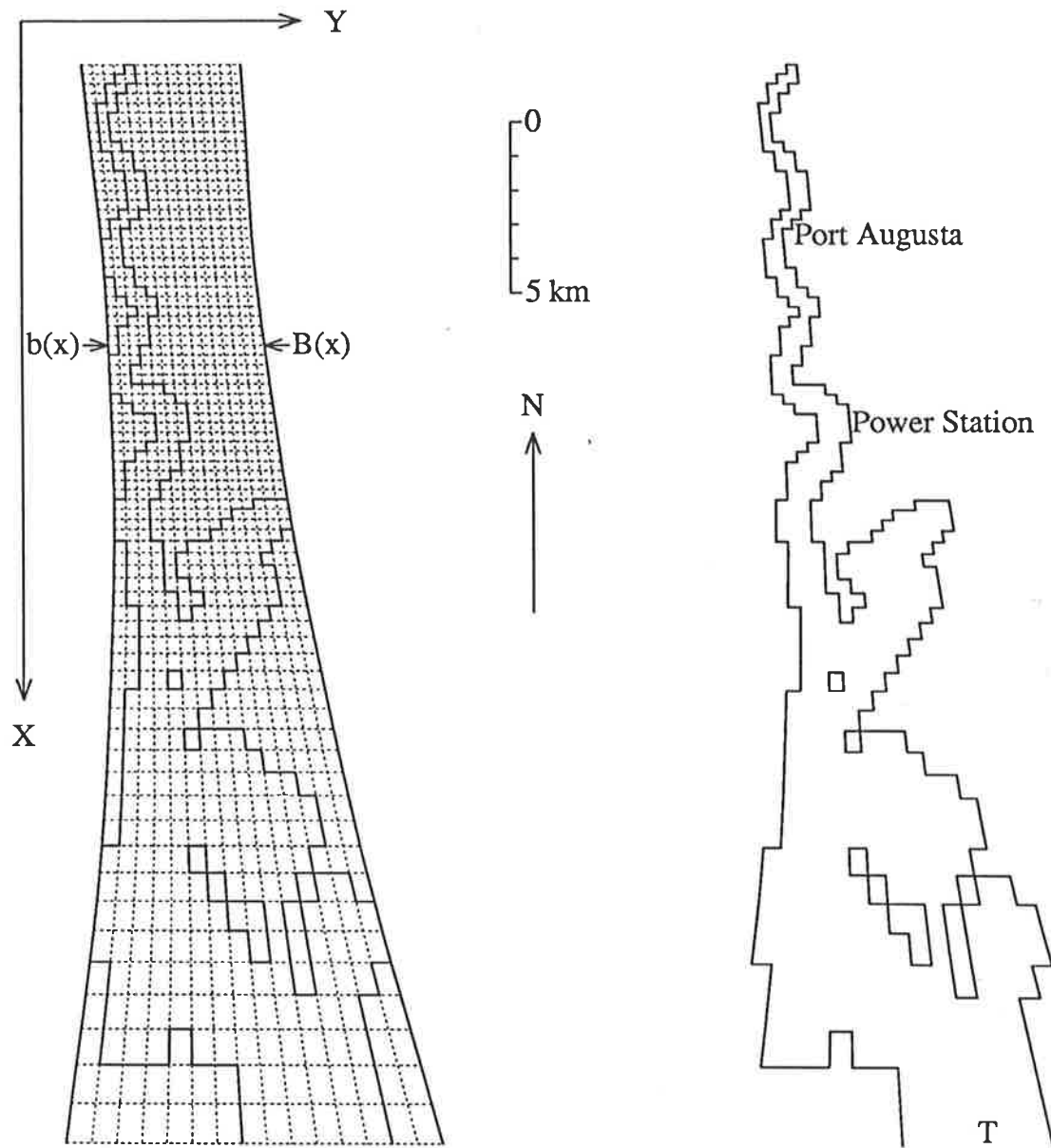


Figure 8.3: Shows the approximate coastline and horizontal grid distribution used for the application of the model. "T" is the location of tidal information from a marker known as Beacon No. 5.

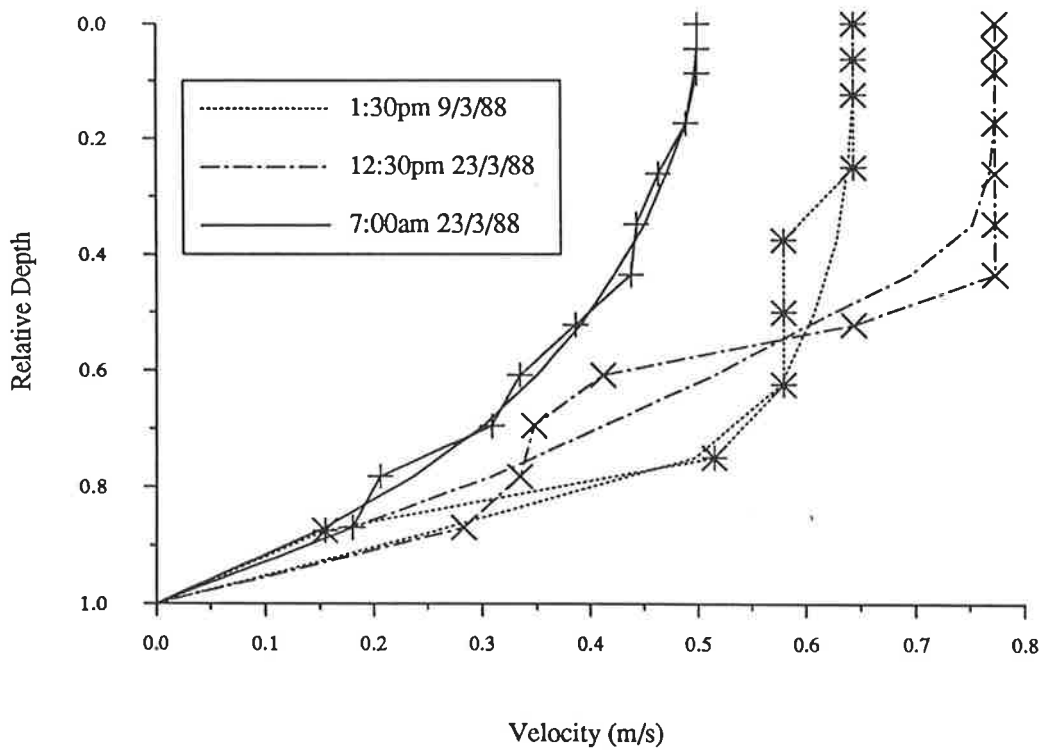


Figure 8.4: Velocity profiles at local maxima in the flow measured in the tidal stream adjacent to the power stations. The symbols on the plots are the actual measurements and the smoother curves are smoothed estimates

pm on the 9th of March 1988 and from 6:00 am to 5:30 pm on the 23rd of March 1988 the velocity of the tidal stream was measured at 15 minute intervals. Figure 8.4 shows the measured velocity at various depths at three different times of maximum flow of the tidal stream adjacent to the power stations. The velocity was measured by lowering to different depths over the side of a 6 metre boat a device for measuring current direction and speed. The boat used was that normally employed for measuring the temperature profile of the water by the lowering of a temperature gauge. Unfortunately difficulty was encountered in keeping the boat still while the velocity gauge was used, and even though two anchors were employed they often dragged and the boat shifted especially when the current or the wind was strong. The velocity profiles obtained

give a reasonable indication of the surface velocity and of the gradient (with respect to depth) of the profile near the sea floor.

An application area of this very small size with the given grid size is almost at the limits of feasibility for three-dimensional numerical modelling, as even when the model is run with the relatively large Courant number of 0.7, due to the size of the horizontal grid spacing, the time step required is 15 seconds. This extremely small time step means that even with only 8 depth levels the numerical model runs at about 28% of real time on the Pyramid computer in the Applied Mathematics Department of The University of Adelaide. That is, a simulation of 100 hours requires 28 hours central processing time¹.

The variably spaced horizontal grid and slowly varying boundary functions $b(x)$ and $B(x)$, see Figure 8.3, have been utilised for this application in order to minimise the total number of computational elements in the model and to increase the accuracy of the numerical results at the head of the gulf. A finer horizontal grid spacing in the numerical model was required both near the power station and at Port Augusta to adequately resolve the bends and narrowness of the channel. If the finer horizontal grid spacing was used over the whole application area a total number of about 1100 grid elements is required, but use of the variably spaced horizontal grid with the coordinate transformations requires only 390 elements. This is a 64.5% reduction in the number of computational elements and therefore a 64.5% reduction in computer time for a numerical run compared with a similar run using a uniformly spaced fine grid.

Besides the difficulties imposed by the small size of the application area, the shallowness and narrowness of the northern region of the area provide a severe test for the model. The depth of the water at mean sea level varies from at most 17 metres near

¹Just prior to submitting this thesis the numerical model was run on a Sun Sparc Station, on which it ran over 6 times faster therefore only requiring 5 hours central processing time for a 100 hour simulation. This improves the feasibility of this modelling application markedly.

Beacon No. 5 to only 4 metres at a number of places in the area. The average depth is between 7 and 8 metres.

8.3 Details of the Calibration Process

The frequency, phase and amplitude of each of the 6 constituents of largest amplitude for Beacon No. 5 are shown in Table 8.1. These were reconstituted into a tidal wave which was specified as the tidal height inputs for the numerical model. The values for these constituents at Port Augusta, where they are still the 6 largest, are given in Table 8.2. These were also reconstructed into a tidal wave and used for comparison with the model predictions. The model predictions are also analysed to get phase and amplitude information for direct comparison with the values in Table 8.2.

	Freq. (°/hr.)	Phase (°)	Amplitude (m)
O1	13.9430356	48.050	0.2501
P1	14.9589314	73.375	0.0988
K1	15.0410686	73.375	0.4704
M2	28.9841042	196.017	0.5793
S2	30.0000000	255.173	0.6197
K2	30.0821373	255.173	0.1778

Table 8.1: The Frequency, Phase and Amplitude data for the 6 major constituents, used as input at the open boundary of the model simulation. Note that when F.I.A.M.S produced these figures they assumed a relationship between the phases and amplitudes for P1 and K1 and also for S2 and K2.

In calibrating the numerical model several test runs were conducted with various values for the model parameters, as well as different forms for the vertical eddy viscosity coefficient and the open boundary condition. The tests performed were not exhaustive but were sufficient to show that the numerical model performs up to expectations. Each test case was initially run for 150 simulated hours and accuracy was determined

from the average absolute difference between the numerical results and the expected results, for the part of the run between 50 and 150 hours. In the following this measure is denoted E_{ζ} . Then most of the test cases were continued further until at least 29 simulated days had been modelled. Appendix D lists an example input, which specifies the initial data for the program.

	Freq. ($^{\circ}$ /hr.)	Phase ($^{\circ}$)	Amplitude (m)
O1	13.9430356	52.799	0.2834
P1	14.9589314	81.901	0.1066
K1	15.0410686	82.743	0.4395
M2	28.9841042	212.247	0.6397
S2	30.0000000	272.324	0.6861
K2	30.0821373	271.897	0.2133

Table 8.2: The Frequency, Phase and Amplitude data for the 6 major constituents, obtained from recorded data at Port Augusta.

All the numerical runs were conducted with a “cold start”, where all tide heights are at the mean sea level and all velocities are zero. For this application the model takes very little time for the starting transients to dissipate because of the small time step used.

(a1) Initial Run

The parameters used for the initial run were : the number of vertical levels was $K = 8$, the time step was $\Delta t = 15$ seconds (so the Courant number was $C_r = 0.7$), the vertical grid used was Fine-Coarse-Fine with $\kappa = 2.0$, the horizontal eddy viscosity parameters were $N_x = N_y = 5 \text{ m}^2/\text{s}$ and the vertical eddy viscosity parameters were $a_1 = 0.0005 \text{ m}^2/\text{s}$ and $a_2 = 0.0100$. The results for the first 140 hours are graphed in Figure 8.5. The large difference between predicted and expected results at the start of the time series shown in Figure 8.5 are due to the initial transients, which dissipated in about 15 hours.

The error estimate for this run is $E_\zeta = 0.1725$, which is 7.3% of the maximum amplitude of 2.37 metres. The wave in the numerical predictions is leading the expected results by a phase difference of around 15 minutes. This was assumed to be due to friction being too low in the numerical model.

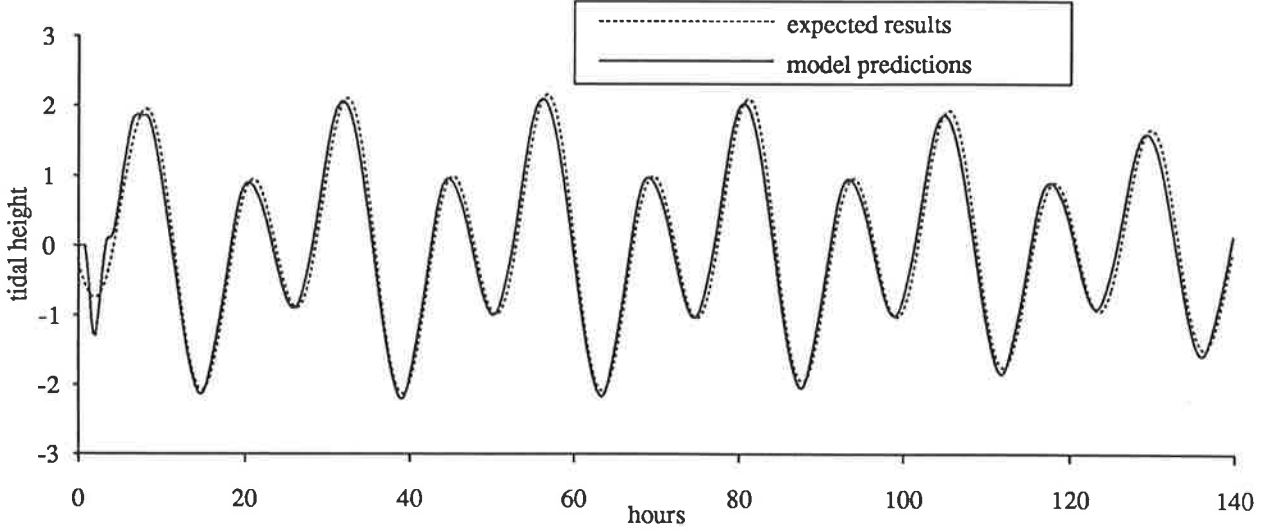


Figure 8.5: Run (a1): initial run.

(a2) Increased Friction

For this run the effect of friction in the numerical model was increased by changing the values for the eddy viscosity parameters. The horizontal values N_x and N_y were increased to $100 \text{ m}^2/\text{s}$ and the vertical eddy viscosity was increased by setting $a_1 = 0.0010 \text{ m}^2/\text{s}$ and $a_2 = 0.0200$. The vertical grid spacing was also changed to Coarse-Fine (that is coarse at the surface to fine at the sea floor) with $\kappa = 2.0$. The results for the first 140 hours are graphed in Figure 8.6. The error estimate for this run is $E_\zeta = 0.0976$ or 4.1%, which is a large improvement over run (a1). The wave in the numerical results no longer leads the expected results at times of low water but still leads it at high water. This may be due to the shallowness of the area causing the vertical eddy viscosity coefficient to significantly differ in shape between low and high waters. The function $F(\eta)$ in the vertical eddy viscosity coefficient has a constant

shape for the relative depth, but when the depth varies from an average 5.7 metres to 10.3 metres (8 ± 2.3), the physical profile that this represents changes significantly.

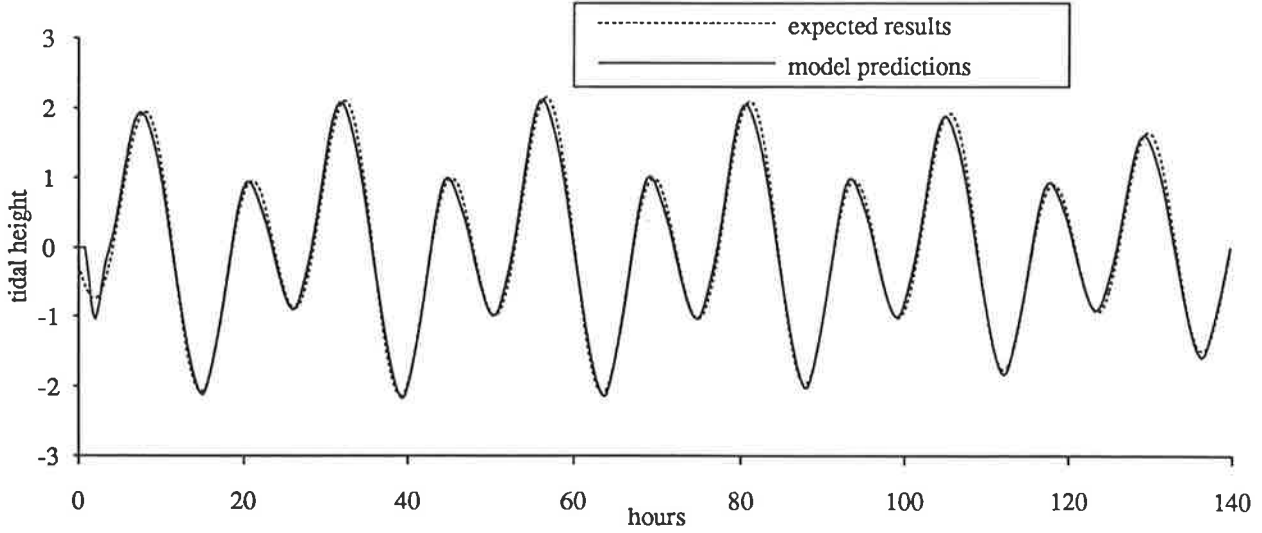


Figure 8.6: Run (a2): increased friction.

Notice that the increase in the effect of friction is evident in the increased damping in the initial transients shown in Figure 8.6 when compared to those from run (a1) shown in Figure 8.5.

(b) Vertical Eddy Viscosity with Dependence on Physical Depth

For this run Equation (2.41), which gives the inverse of the depth transformation, has been used to implement a vertical eddy viscosity profile which is constant with respect to physical depth not relative depth. The form chosen for $F(\eta)$ is linear from the sea floor to a height of 2 metres above, linear from the sea surface to 2 metres below, and constant in between.

The same horizontal eddy viscosity parameters as in run (a2) were used and the vertical grid spacing was Fine-Coarse-Fine, with $\kappa = 3.0$. The results for the first 140 hours are graphed in Figure 8.7. The error estimate for this run is $E_\zeta = 0.0821$ or 3.5%. This error is smaller than the previous two cases, but the numerical wave seems

to flow in too quickly and then flow out too slowly.

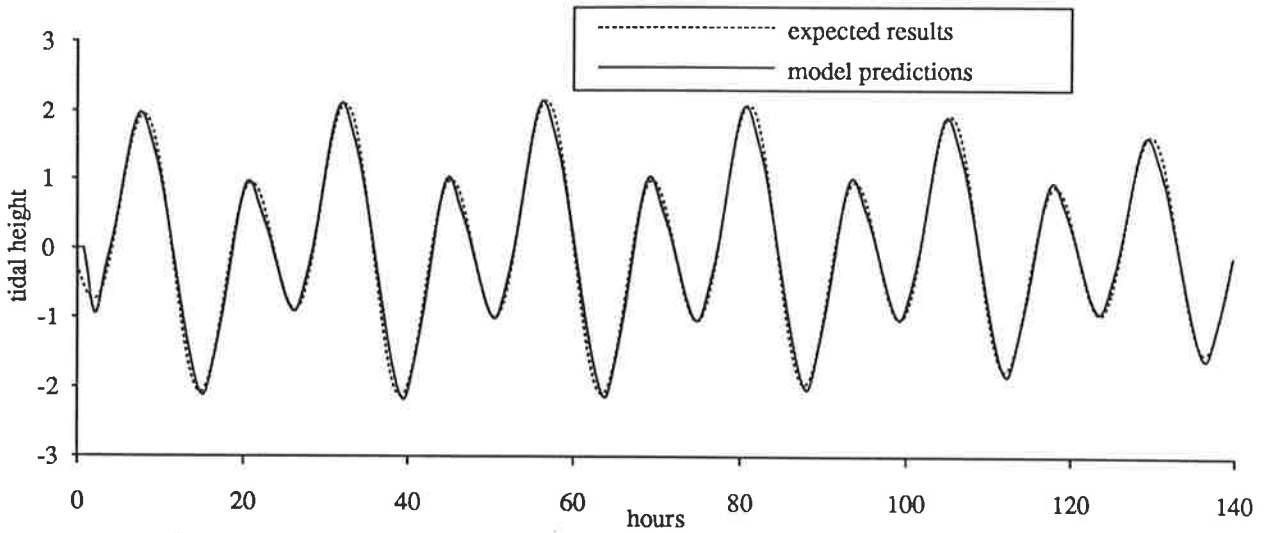


Figure 8.7: Run (b): vertical eddy viscosity with dependence on physical depth.

(c) Radiation Boundary Condition

It was thought that the fast inflow and the slow outflow evident in run (b) may be due to the boundary condition at the open boundary, where the tide height is prescribed. In order to test this an Orlanski-Sommerfeld radiation boundary condition, using Equations (2.61) and (2.62), was tried at the open boundary of the model. The vertical eddy viscosity profile of run (b) and the same parameters and grid spacing as used in that run were used for this run. The results for the first 140 hours are graphed in Figure 8.8. The error estimate obtained was $E_{\zeta} = 0.0820$ or 3.5%. As this is virtually unchanged from run (b) it seems that the fast inflow and the slow outflow are not dependent on the nature of the open boundary condition.

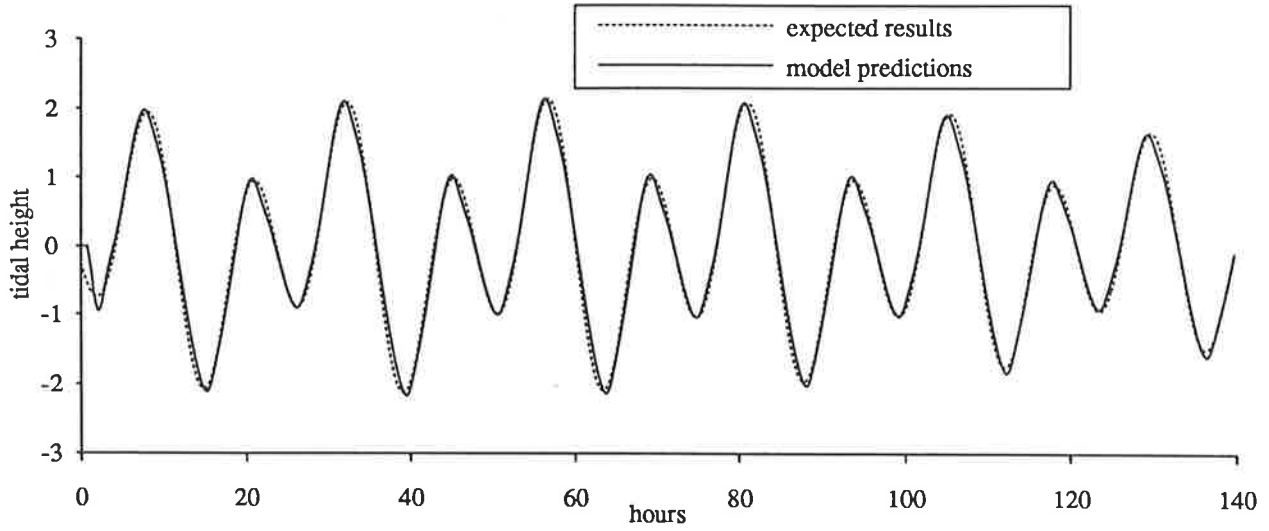


Figure 8.8: Run (c): radiation boundary condition.

(d1) New Eddy Viscosity Formulation

An inspection of Figures 8.7 and 8.8 shows that the predicted wave is too peaked at the top and the bottom of the tides. In looking at the formulation of the vertical eddy coefficient in Equation (2.39) it can be noted that when the combined velocity of $(L\bar{\mu})^2 + \bar{v}^2$ is small the overall coefficient will be small. In order to increase the effect of friction at times of high and low water, Equation (2.39) was modified to

$$N_\eta = N(\eta) = a_1 + a_2 H F(\eta) \max \left(U_{\max}/2, \sqrt{(L\bar{\mu})^2 + \bar{v}^2} \right), \quad (8.1)$$

where

$$U_{\max} = \sqrt{(L\bar{\mu}_{\max})^2 + \bar{v}_{\max}^2}.$$

This in effect stops the vertical eddy viscosity coefficient from becoming too small when the tide turns at high and low water and the velocity term in Equation (2.39) normally is small.

The same eddy viscosity, other parameters and grid spacing as used in the previous two runs were used in this test. The results for the first 140 hours are graphed in

Figure 8.9. The error estimate for this run is $E_\zeta = 0.0819$ or 3.5%. This is virtually unchanged from the previous two runs, but the graph shows that the maximum and minimum tidal height are now correct, which was not the situation for the previous two cases. However the problem of the fast inflow and the slow outflow is still evident.

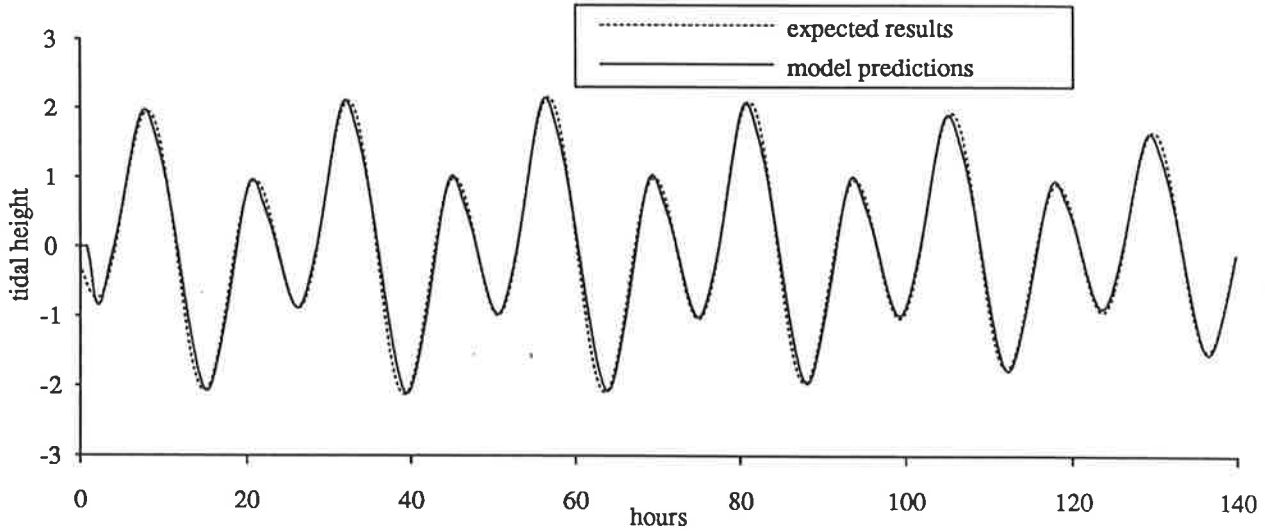


Figure 8.9: Run (d1): new eddy viscosity formulation.

(d2) Different Vertical Grid

This run is the same as (d1) except the vertical grid used was Coarse-Fine with $\kappa = 2$. The results for the first 140 hours are shown in Figure 8.10. The error estimate for this run is $E_\zeta = 0.0923$ or 3.9%. The error is larger for this run therefore the vertical Fine-Coarse-Fine grid with $\kappa = 3$ seems more appropriate for this particular application.

(d3) More Depth Levels

This run is the same as (d1) except that 14 depth levels ($K = 14$) have been used in the numerical model instead of only 8. The results for the first 140 hours are graphed in Figure 8.11. The error estimate for this run is $E_\zeta = 0.0803$ or 3.4%. This is a slight improvement over run (d1) but has the disadvantage of increasing the running time

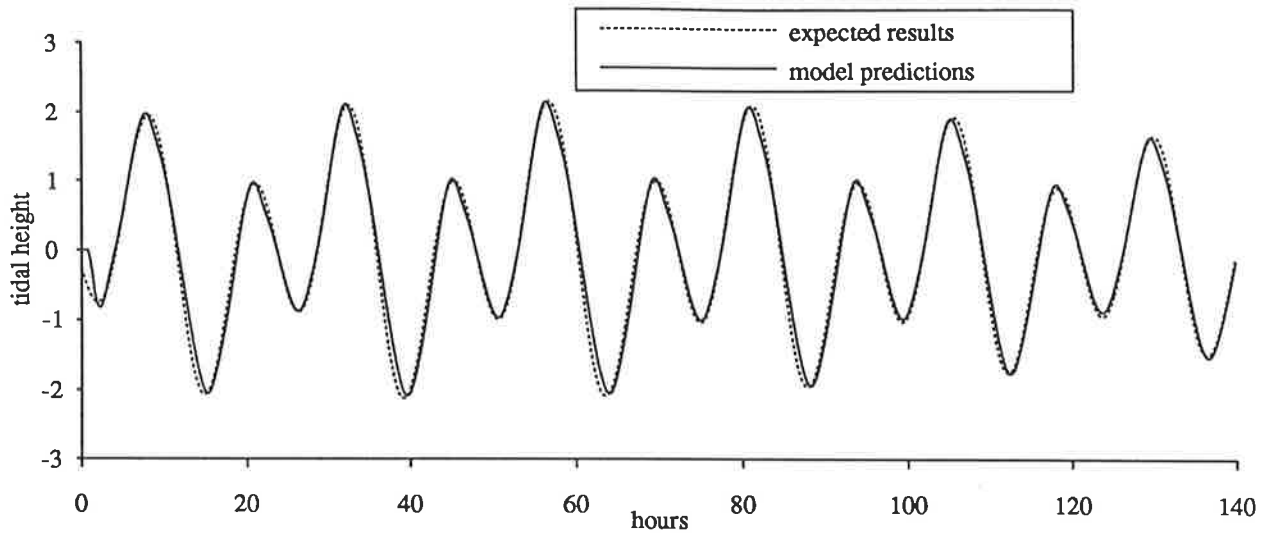


Figure 8.10: Run (d2): different vertical grid.

of the numerical model by 75%. The slight improvement in accuracy does not seem to justify the extra computational time required.

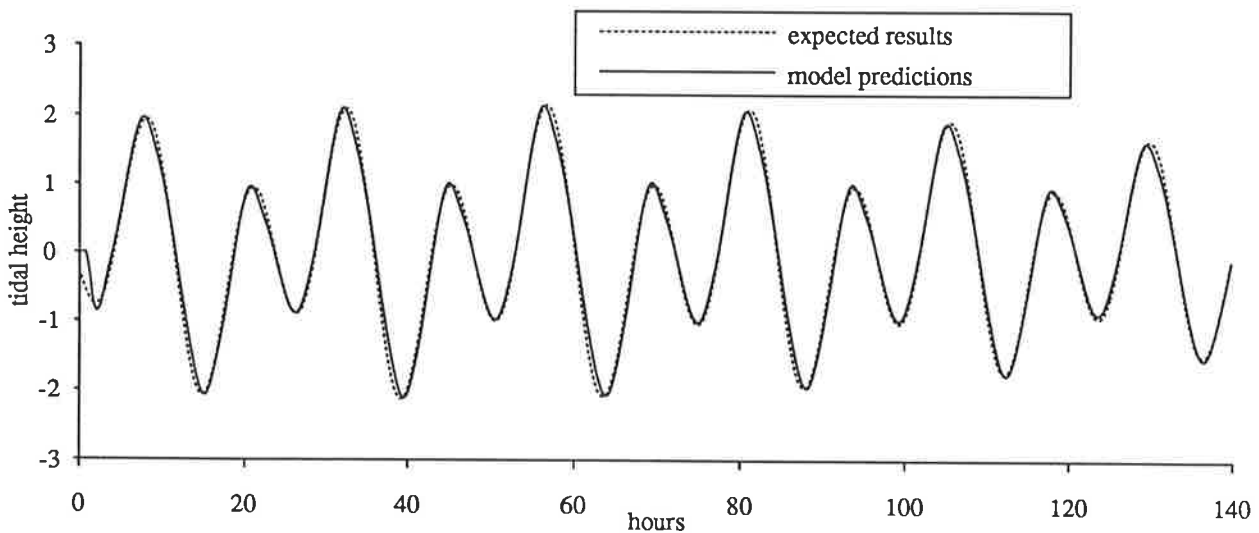


Figure 8.11: Run (d3): more depth levels.

The numerical model for each of the runs (b) to (d3) was then continued up to a duration of 32 simulated days (768 hours) after initialisation. The graph of run (d1) from 0 hours to 700 hours is shown in Figure 8.12. The graphs from 0 hours to 700 hours for all the other cases are not shown as the results are not sufficiently

distinguishable for this scale, but the values of E_ζ for the 32 day run are given in Table 8.3, which summarises the runs (a1) to (d3).

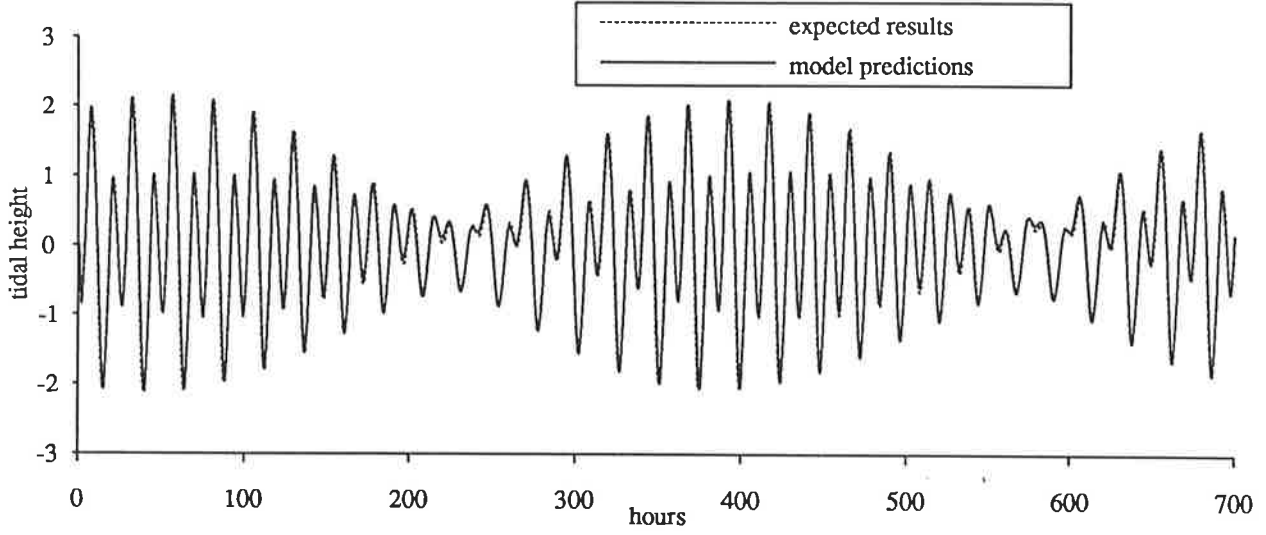


Figure 8.12: run (d1): new eddy viscosity formulation graphed for the first 700 hours of simulation.

run	N_z		$N_x = N_y$	Grid	κ	K	E_ζ	E_ζ
	a_1	a_2					50-150 hrs	50-700 hrs
(a1)	0.0005	0.0100	5.0	F-C-F	2.0	8	0.1725	
(a2)	0.0010	0.0200	100.0	C-F	2.0	8	0.0976	
(b)	0.0010	0.0200	100.0	F-C-F	3.0	8	0.0821	0.0717
(c)	0.0010	0.0200	100.0	F-C-F	3.0	8	0.0820	0.0716
(d1)	0.0010	0.0200	100.0	F-C-F	3.0	8	0.0819	0.0639
(d2)	0.0010	0.0200	100.0	C-F	2.0	8	0.0923	0.0662
(d3)	0.0010	0.0200	100.0	F-C-F	3.0	14	0.0803	0.0690

Table 8.3: Summary of runs (a1) to (d3).

(d4) to (d6) Varying the Number of Tidal Constituents Used

The number of constituents input at the open boundary for the previous runs was 6 and consisted of the O1, P1, K1, M2, S2 and K2 constituents. For run (d4) the largest 26 constituents at Beacon No. 5 were used, for run (d5) the largest 4 (O1, K1, M2 and S2) were used and for run (d6) only M2 was used. Runs (d4) to (d6) are

graphed in Figures 8.13 to 8.15 and their error estimates are summarised in Table 8.4. For the graphs and error estimates the appropriate constituents were reconstituted to obtain the tide height at Port Augusta.

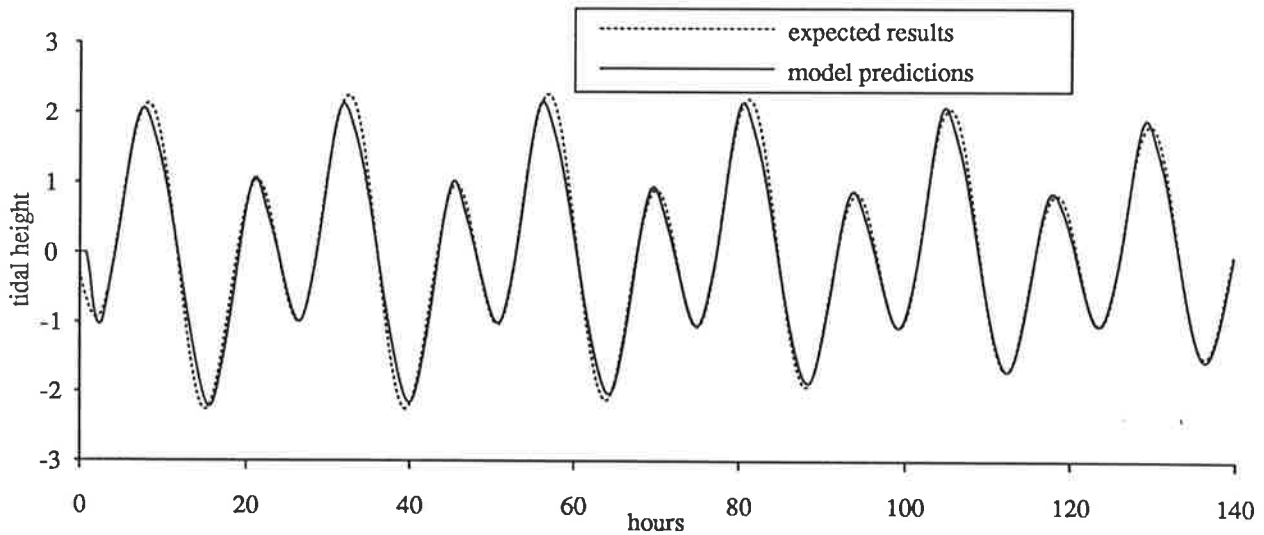


Figure 8.13: Run (d4): 26 constituent inputs at the open boundary.

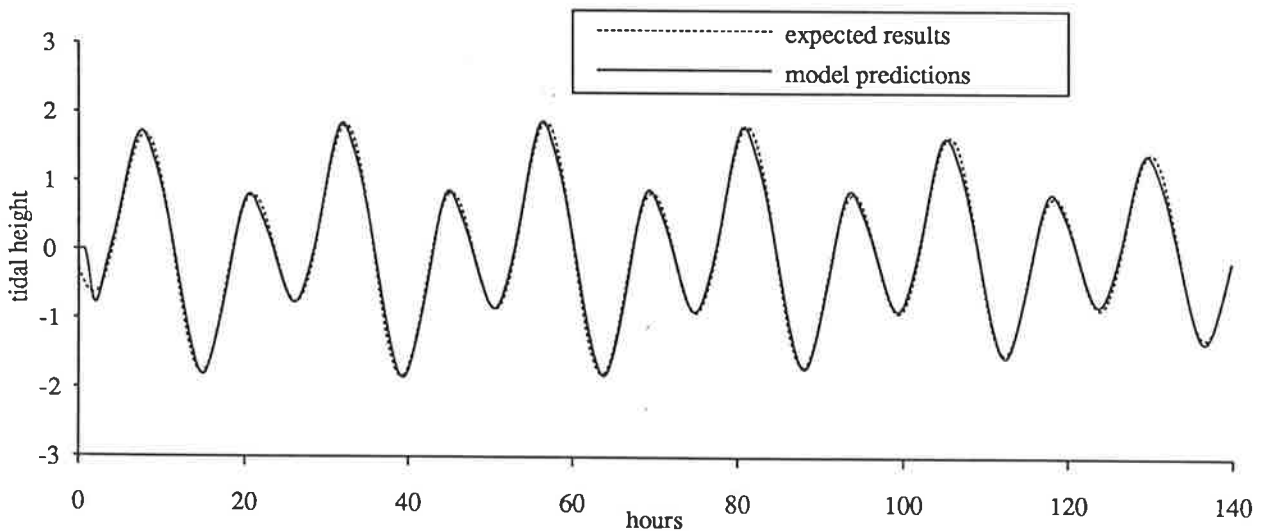


Figure 8.14: Run (d5): 4 constituent inputs at the open boundary.

The error estimates for runs (d3) to (d6) increase with the number of constituents used in the run. Figure 8.13, for the 26 constituents run, shows errors just after high tide and just before low tide, when compared to Figure 8.11, for the 6 constituents run

(d3). As the extra 20 constituents used are all quite small in amplitude this relatively large error indicates a non-linear interaction between some of the constituents in the numerical model. Run (d5), for 4 constituents, is similar to that of run (d3), for 6 constituents. The error for run (d6), which had just M2 as input, can be seen in Figure 8.15 as mainly a phase error, as there is no non-linear interaction with other constituents to produce the effects evident in the other runs.

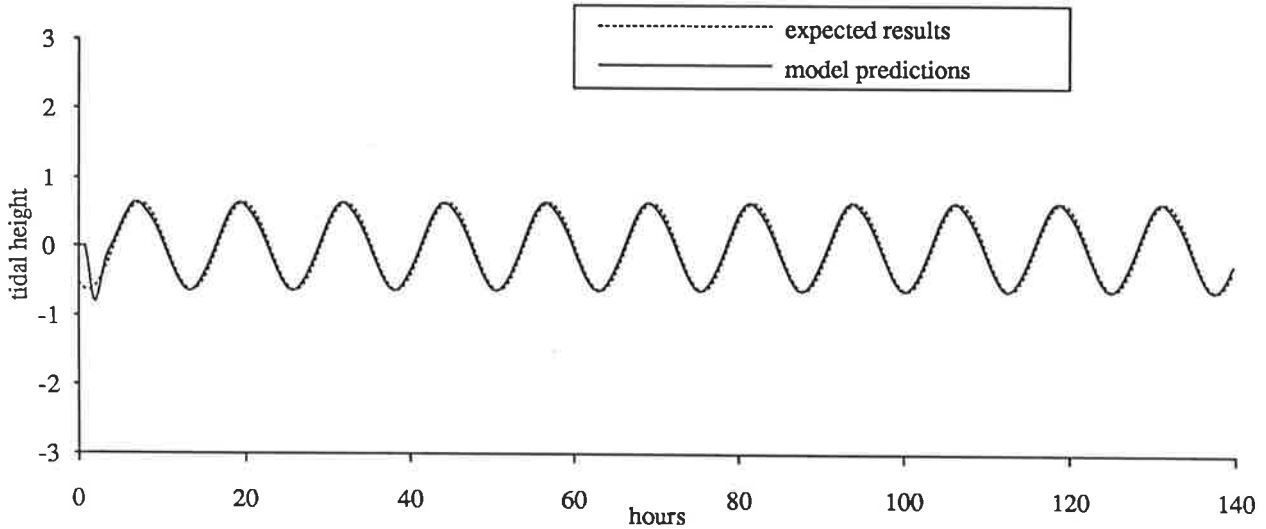


Figure 8.15: Run (d6): M2 constituent as input at the open boundary.

run	Constituents	E_{ζ} 50-150 hrs	E_{ζ} 50-700 hrs
(d3)	O1,P1,K1,M2,S2,K2	0.0803	0.0690
(d4)	largest 26 constituents	0.0867	0.0854
(d5)	O1,K1,M2,S2	0.0631	0.0585
(d6)	M2	0.0547	0.0561

Table 8.4: Summary of runs (d3) to (d6).

(e) The Effect of Reducing the Courant Number

This run was exactly the same as run (d5) except that the time step was halved to $\Delta t = 7.5$ seconds and therefore the Courant number was reduced from 0.7 to 0.35.

The purpose of this run was to see if some of the numerical errors in the previous runs were due to the model being run with a relatively high Courant number. The results of the first 140 hours of this run are graphed in Figure 8.16. The surprising result is that there is no appreciable difference between the the graphs for runs (d5) and (e) and that the error estimate E_ζ , for 50 to 150 hours, increased slightly from 0.0631 to 0.0645, and for 50 to 700 hours increased slightly from 0.0585 to 0.0597 (see Table 8.5). This shows that for this application the model is best run at the higher Courant number which therefore has the advantage of being twice as computationally efficient.

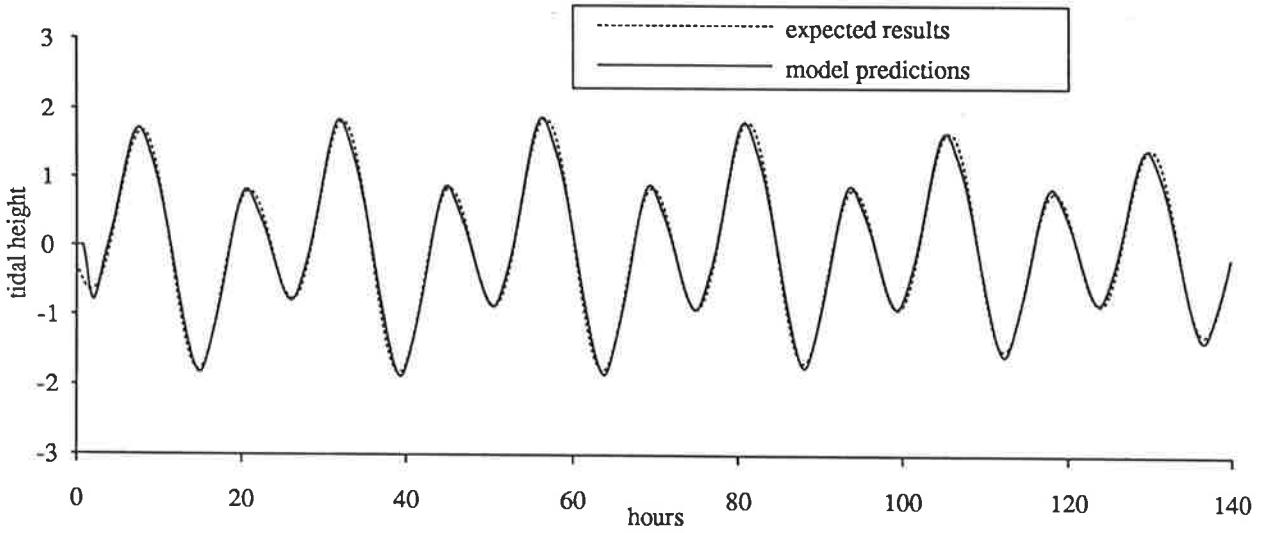


Figure 8.16: Run (e): 4 constituents as in Figure 8.14 with the time step halved to $\Delta t = 7.5$ seconds and therefore the Courant number is $C_r = 0.35$. the open boundary.

run	Constituents	Courant Number	E_ζ 50-150 hrs	E_ζ 50-700 hrs
(d5)	O1,K1,M2,S2	0.70	0.0631	0.0585
(e)	O1,K1,M2,S2	0.35	0.0645	0.0597

Table 8.5: Summary of runs (d5)and (e).

Analysing for Constituent Frequencies

The numerical model results of tidal heights at Port Augusta can be analysed to obtain the amplitude and phase for each of the constituent frequencies which compose the signal. A computer program, originally written by Easton (1977) but modified and improved by Mitchell (1982), has been used for the analysis. The program takes an hourly time series of length at least 29 days and produces the amplitude and phases for 25 constituents of differing frequencies. It uses a least squares error minimisation approach, which has the advantage that the actual tidal frequencies can be specified. Unfortunately this analysis cannot separate either the P1 and K1 or the S2 and K2 constituents with the specified length (29 days) of data as the frequencies are too close together. To separate these constituents requires a time series of at least 6 months (preferably 12 months) and if this amount of data is available then a more complete analysis could be used to obtain these constituents. Therefore only the runs (d5) to (e) can be analysed with this program as these do not contain the P1 and K2 constituents. The results for runs (d5) and (e), compared to the expected values, are displayed in Table 8.6 and results from run (d6) are shown in Table 8.7.

constituent	Amplitude			Phase		
	Predicted		Expected	Predicted		Expected
	run (d5)	run (e)		run (d5)	run (e)	
O1	0.273	0.274	0.283	54.6	55.2	52.8
K1	0.476	0.477	0.440	81.2	81.6	82.7
M2	0.614	0.615	0.640	210.4	210.1	212.2
S2	0.670	0.671	0.686	270.0	269.7	271.9

Table 8.6: Amplitude and Phase data obtained by analysing the numerical results for Port Augusta from runs (d5) and (e).

Both runs (d5) and (d6) have the same input for the M2 constituent as well as the same value for all model parameters. Comparing the M2 constituent in Tables 8.6

constituent	Amplitude		Phase	
	Predicted	Expected	Predicted	Expected
M2	0.631	0.640	204.9	212.2

Table 8.7: Amplitude and Phase data obtained by analysing the numerical results for Port Augusta from run (d6) using TIDANAL.

and 8.7 shows that the numerical results are quite non-linear, as the other constituents in run (d5) interact to give different amplitude and phase values from the analysis. A difficulty with using the constituent data from Beacon No. 5 and comparing the model results with the constituent data at Port Augusta is that the data at Beacon No. 5, which has been supplied by F.I.A.M.S., has been obtained by assuming local relations between P1, S1 and K1 and between T2, S2 and K2. In particular P1 and K1 have the same phase and a fixed distribution of the energy in the band for which they are analysed together and similarly for S2 and K2. This assumption leads to part of the difference between expected and numerical results. It is difficult to tell if this is significant as this could only be resolved by having a longer time series of measured data at Beacon No. 5. Considering this, the results in Tables 8.6 and 8.7 indicate that the model is predicting results quite accurately.

It would be difficult to expect the numerical model to do any better than is shown by these tests as it is likely that the expected amplitudes and phases are 3 figure accurate at best, especially those in Table 8.1, which were produced from only 104 days of data.

The model's applicability to this particular area has thus been demonstrated and the tuning and calibration of the model has resulted in acceptable predictions.

Chapter 9

CONCLUSION

A new three-dimensional tidal model for shallow waters has been developed, using transformations to map the physical body of water onto a rectangular shape which facilitates easy application of finite difference techniques. The finite difference equations have been discretised on grids in which the spacings can be varied in all three coordinate directions. This allows greater ability to accurately model tidal heights and velocities in areas which may be of significant interest or importance. For example, if the flow has large shear stresses at the sea floor and/or at the sea surface (due to wind stress), then the wave propagation analysis shows that a vertical grid, finer in those places, increases the accuracy of the numerical model. Care needs to be taken to choose a grid which keeps the finite difference approximations accurate.

The analytic solutions obtained by previous authors for linearised channel flow are discussed and these solutions are extended by the inclusion of horizontal diffusive terms and linear horizontal advective terms. A new solution has also been found for an eddy viscosity profile in the form of a “sine-squared” function. These analytic solutions form the basis for the new wave propagation analysis technique, which can be used to assess the accuracy and stability of any particular three-dimensional finite difference scheme.

The developed analysis opens new avenues of research in the area of investigating

the effect of the size and shape of the vertical eddy viscosity parameter N_η on the accuracy of a numerical model. For an appropriate N_η the relative wave speed, amplitude response and velocity error measure can be estimated and a suitable grid distribution can be chosen to minimise the errors in each of these.

There is also the possibility of developing a related analysis which uses a two-dimensional version, where the analytic solution has a horizontal boundary as in channel flow with one end closed (see Section 6.4). Such an analysis may be used to test the stability and accuracy of a numerical scheme near a land boundary as distinct from assuming the absence of such boundaries.

The accuracy of the numerical model developed has been assessed using the wave propagation analysis as well as directly comparing results against those obtained from some of the analytic solutions. The wave propagation analysis and the direct tests both show the advantage of using a variable grid in the vertical. In this way the numerical errors can be minimised and in some cases reduced markedly relative to the uniform grid case. For the direct comparisons, the “accuracy” of a scheme is judged by error measures, which are characteristically differences between the model predictions and the analytic solutions, summed for each time step and then averaged. The disadvantage of this is that the cause of the error is not indicated by the error measure and one cannot tell if the error is due to an incorrect wave speed or amplitude in the numerical model. A significant advantage of the new wave propagation analysis is that this cause is directly evident.

The numerical model is also tested against an analytic solution for a constant wind stress applied to the sea surface. When the wind stress is applied to the numerical model a suitable choice for the variable vertical grid distribution significantly improves the accuracy of the numerical model, once again emphasising the usefulness of a variable vertical grid.

The numerical model is successfully applied to the waters in the upper Spencer Gulf area of South Australia. An application area of this small size, with the given grid spacing, is almost at the limit of feasibility for the three-dimensional model, as even when the model is run with a relatively large Courant number of 0.7, the time step required, due to the size of the horizontal grid spacing, is 15 sec. A variable horizontal grid has been utilised to minimise the total number of computational elements in the model and to increase the accuracy of the numerical results at the head of the gulf. The choice of the application area is due to the location of a power station about 5 km south from Port Augusta.

When calibrated the model developed in this thesis accurately predicts tidal elevations and can provide detailed information of velocities over depth, which could be used to predict the effects of heated water discharge into the gulf stream. This model has the capability for extension to include the transportation of heat and the effect of heat on buoyancy by incorporating a variable density into the model. Further research on this aspect would yield valuable information for the field of numerical tidal prediction.

Appendix A

The Finite Difference Equations for the Transformed Y -Momentum Equation

This Appendix presents the finite difference approximations for the transformed y -momentum equation. As with the other approximations given in Chapter 4, the indexing system described in Section 3.1 forms the basis of the following difference scheme. The definitions given at the start of Chapter 4 are assumed for these approximations.

Equation (2.47) evaluated at the $(n+\frac{1}{2})^{th}$ time level and at the v or ν point of the i^{th} element in the model, for the variables at the k^{th} depth level below the surface, is

$$\begin{aligned} & \left. \frac{\partial(Hv)}{\partial t} \right|_{i,k}^{n+\frac{1}{2}} + \frac{1}{B} \left. \frac{\partial(BH\mu v)}{\partial \chi} \right|_{i,k}^{n+\frac{1}{2}} + \left. \frac{\partial(H\nu v)}{\partial \lambda} \right|_{i,k}^{n+\frac{1}{2}} + H \left. \frac{\partial(\omega v)}{\partial \eta} \right|_{i,k}^{n+\frac{1}{2}} \\ &= - \frac{gH}{B} \left. \frac{\partial \zeta}{\partial \lambda} \right|_{i,k}^{n+\frac{1}{2}} + \frac{HN_x}{L^2} \left. \frac{\partial^2 v}{\partial \chi^2} \right|_{i,k}^{n+\frac{1}{2}} + \frac{HN_\lambda}{B^2} \left. \frac{\partial^2 v}{\partial \lambda^2} \right|_{i,k}^{n+\frac{1}{2}} \\ &+ \frac{1}{H} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial v}{\partial \eta} \right) \Big|_{i,k}^{n+\frac{1}{2}} - LHf\mu \Big|_{i,k}^{n+\frac{1}{2}} . \end{aligned} \quad (A.1)$$

In the following approximations, the values of ζ^* , which in the order of calculations have been calculated using Equation (4.2), are used for the values of ζ^{n+1} , which in turn

are used in calculating H^{n+1} . The second order accurate (in space) finite difference approximations on a variably spaced grid for each term of Equation (A.1) are listed below.

(a)

$$\left. \frac{\partial(Hv)}{\partial t} \right|_{i,k}^{n+\frac{1}{2}} \approx \left(\frac{\vec{H}_i^{n+1} v_{i,k}^{n+1} - \vec{H}_i^n v_{i,k}^n}{\Delta t} \right) = HV1 \cdot v_{i,k}^{n+1} + HV2, \quad (\text{A.2})$$

where

$$HV1 = \frac{\vec{H}_i^{n+1}}{\Delta t} \quad \text{and} \quad HV2 = -\frac{\vec{H}_i^n v_{i,k}^n}{\Delta t}.$$

(b)

$$\begin{aligned} \frac{1}{B} \left. \frac{\partial(BH\mu v)}{\partial \chi} \right|_{i,k}^{n+\frac{1}{2}} &= \frac{B_m(\vec{H}_i^n + \vec{H}_i^{n+1})\vec{\mu}_{i,k}^n \tilde{v}_{i,k}^n - B_{m-1}(\vec{H}_{i1}^n + \vec{H}_{i1}^{n+1})\vec{\mu}_{i1,k}^{n+1} \tilde{v}_{i1,k}^n}{(B_{m-1} + B_m)\Delta\chi_m} \\ &= BHUV, \end{aligned} \quad (\text{A.3})$$

where the evaluation of $\tilde{v}_{i1,k}^n$ and $\vec{\mu}_{i,k}^n$ is split between the n^{th} and $(n+1)^{th}$ time level.

(c)

$$\begin{aligned} \left. \frac{\partial(H\nu v)}{\partial \lambda} \right|_{i,k}^{n+\frac{1}{2}} &= \frac{H}{B} \left(2v - \mu \frac{\partial \beta}{\partial \chi} \right) \left. \frac{\partial v}{\partial \lambda} \right|_{i,k}^{n+\frac{1}{2}} + \frac{v}{B} \left(v \frac{\partial H}{\partial \lambda} - \mu H \frac{\partial B}{\partial \chi} - \frac{\partial \beta}{\partial \chi} \frac{\partial \mu H}{\partial \lambda} \right) \Big|_{i,k}^{n+\frac{1}{2}} \\ &\approx \frac{\vec{H}_i^n + \vec{H}_i^{n+1}}{B_{m-1} + B_m} \left(2v_{i,k}^n - \frac{(\vec{\mu}_{i,k}^n + \vec{\mu}_{i1,k}^n)(\beta_m - \beta_{m-1})}{2\Delta\chi_m} \right) \\ &\quad \times \left(\frac{v_{i+1,k}^n + (r_j^2 - 1)v_{i,k}^n - r_j^2 v_{i-1,k}^{n+1}}{\Delta\lambda_{j+1}(1 + r_j)} \right) \\ &\quad + \frac{2v_{i,k}^n}{B_{m-1} + B_m} \left(\frac{v_{i,k}^n (HDY^n + HDY^{n+1})}{2} \right. \\ &\quad \left. - \frac{(\vec{H}_i^n + \vec{H}_i^{n+1})(\vec{\mu}_{i,k}^n + \vec{\mu}_{i1,k}^{n+1})(B_m - B_{m-1})}{4\Delta\chi} \right. \\ &\quad \left. - \frac{(\beta_m - \beta_{m-1})HUDY}{\Delta\chi} \right) \end{aligned}$$

$$= HVV , \quad (A.4)$$

where

$$\begin{aligned} HDY^n &= \frac{(\Delta\lambda_j + a)H_{i+1}^n + c.H_i^n + r_y(\Delta\lambda_j - a.r_y)H_{i-1}^n}{a^2r_y(r_y + 1)} , \\ HUDY &= \frac{1}{4a^2r_y(r_y + 1)} \left((\Delta\lambda_j + a)(H_{i+1}^n + H_{i+1}^{n+1})(\mu_{i+1,k}^n + \mu_{i+1,k}^{n+1}) \right. \\ &\quad \left. + c(H_i^n + H_i^{n+1})(\mu_{i,k}^n + \mu_{i,k}^{n+1}) \right. \\ &\quad \left. + r_y(\Delta\lambda_j - a.r_y)(H_{i-1}^n + H_{i-1}^{n+1})(\mu_{i-1,k}^{n+1} + \mu_{i-1,k}^{n+1}) \right) , \\ a &= \frac{1}{2}(\Delta\lambda_{j-1} + \Delta\lambda_j) , \\ c &= (r_y + 1)(a(r_y - 1) - \Delta\lambda_j) , \\ r_y &= \frac{\Delta\lambda_j + \Delta\lambda_{j+1}}{\Delta\lambda_{j-1} + \Delta\lambda_j} , \\ r_j &= \frac{\Delta\lambda_{j+1}}{\Delta\lambda_j} . \end{aligned}$$

(d)

$$H \frac{\partial(\omega v)}{\partial \eta} \Big|_{i,k}^{n+\frac{1}{2}} \approx HVW1.v_{i,k-1}^{n+1} + HVW2.v_{i,k}^{n+1} + HVW3.v_{i,k+1}^{n+1} + HVW4 , \quad (A.5)$$

where

$$\begin{aligned} HVW1 &= - \left(\frac{\vec{H}_i^n + \vec{H}_i^{n+1}}{4\Delta\eta_k(r_k + 1)} \right) r_k^2 \vec{\omega}_{i,k-1}^n , \\ HVW2 &= \left(\frac{\vec{H}_i^n + \vec{H}_i^{n+1}}{4\Delta\eta_k(r_k + 1)} \right) (r_k^2 - 1) \vec{\omega}_{i,k}^n , \\ HVW3 &= \left(\frac{\vec{H}_i^n + \vec{H}_i^{n+1}}{4\Delta\eta_k(r_k + 1)} \right) \vec{\omega}_{i,k+1}^n , \\ HVW4 &= (HVW1.v_{i,k-1}^n + HVW2.v_{i,k}^n + HVW3.v_{i,k+1}^n) , \\ r_k &= \frac{\Delta\eta_k}{\Delta\eta_{k-1}} . \end{aligned}$$

(e)

$$-\frac{gH}{B} \frac{\partial \zeta}{\partial \lambda} \Big|_{i,k}^{n+\frac{1}{2}} \approx -\frac{g}{2B} \left(\vec{H}_i^n \cdot ZDY^n + \vec{H}_i^{n+1} \cdot ZDY^{n+1} \right) = DZDY , \quad (\text{A.6})$$

where

$$ZDY^n = \frac{(\Delta\lambda_j + a)\zeta_{i+1}^n + c.\zeta_i^n + r_y(\Delta\lambda_j - a.r_y)\zeta_{i-1}^n}{a^2 r_y(r_y + 1)} .$$

(f)

$$\begin{aligned} \frac{HN_\chi}{L^2} \frac{\partial^2 v}{\partial \chi^2} \Big|_{i,k}^{n+\frac{1}{2}} &\approx \frac{4N_\chi(\vec{H}_i^n + \vec{H}_i^{n+1}) \left(v_{i2,k}^n - (r_m + 1) \right) v_{i,k}^n + r_m v_{i1,k}^{n+1}}{r_m(r_m + 1)L^2(\Delta\chi_{m-1} + \Delta\chi_m)^2} \\ &= HNX , \end{aligned} \quad (\text{A.7})$$

where

$$r_m = \frac{\Delta\chi_m + \Delta\chi_{m+1}}{\Delta\chi_{m-1} + \Delta\chi_m} .$$

(g)

$$\begin{aligned} \frac{HN_\lambda}{B^2} \frac{\partial^2 v}{\partial \lambda^2} \Big|_{i,k}^{n+\frac{1}{2}} &\approx \frac{N_\lambda(\vec{H}_i^n + \vec{H}_i^{n+1}) \left(v_{i+1,k}^n - (r_j + 1) \right) v_{i,k}^n + r_j v_{i-1,k}^{n+1}}{r_j(r_j + 1)B_m^2 \Delta\lambda_j^2} \\ &= HNY . \end{aligned} \quad (\text{A.8})$$

(h)

$$\frac{1}{H} \frac{\partial}{\partial \eta} \left(N_\eta \frac{\partial v}{\partial \eta} \right) \Big|_{i,k}^{n+\frac{1}{2}} \approx HNN1.v_{i,k-1}^{n+1} + HNN2.v_{i,k}^{n+1} + HNN3.v_{i,k+1}^{n+1} + HNN4 , \quad (\text{A.9})$$

where

$$HNN1 = \frac{r_k^2}{DENOM} \left((2 - r_k) \vec{N}_{i,k}^n + r_k \vec{N}_{i,k-1}^n \right) ,$$

$$HNN2 = \frac{-1}{DENOM} \left(\vec{N}_{i,k+1}^n - (r_k + 1)(r_k^2 - 3r_k + 1) \vec{N}_{i,k}^n + r_k^3 \vec{N}_{i,k-1}^n \right) ,$$

$$HNN3 = \frac{1}{DENOM} \left(\vec{N}_{i,k+1}^n - (2r_k - 1) \vec{N}_{i,k}^n \right) ,$$

$$HNN4 = \left(HNN1.\mu_{i,k-1}^n + HNN2.\mu_{i,k}^n + HNN3.\mu_{i,k+1}^n \right) ,$$

$$DENOM = (\Delta\eta_k)^2(r_k + 1)(\vec{H}_i^{n+1} + \vec{H}_i^n) ,$$

and the vertical eddy viscosity parameter $N(\eta) = N_\eta$ can be defined in any desired form. See Equations (2.39) and (2.40) and Chapter 4 for more detail.

(i)

$$\begin{aligned} L H f \mu \Big|_{i,k}^{n+\frac{1}{2}} &\approx L(\vec{H}_i^{n+1} + \vec{H}_i^n) \vec{f}_i^n \left(\frac{\Delta \lambda_j (\mu_{i1+1,k}^{n+1} + \mu_{i+1,k}^n) + \Delta \lambda_{j+1} (\mu_{i1,k}^{n+1} + \mu_{i,k}^n)}{\Delta \lambda_j + \Delta \lambda_{j+1}} \right) \\ &= L H F U . \end{aligned} \quad (\text{A.10})$$

Substitution of Equations (A.2) to (A.10) in Equation (A.1) yields a set of equations for any particular element of the body of water, in the form

$$P_{i,k}^{n+1} \mu_{i,k-1}^{n+1} + Q_{i,k}^{n+1} \mu_{i,k}^{n+1} + R_{i,k}^{n+1} \mu_{i,k+1}^{n+1} = S_{i,k}^n , \quad (\text{A.11})$$

where

$$\begin{aligned} P_{i,k}^{n+1} &= H V W 1 - H N N 1 , \\ Q_{i,k}^{n+1} &= H V 1 + H V W 2 - H N N 2 , \\ R_{i,k}^{n+1} &= H V W 3 - H N N 3 , \\ S_{i,k}^n &= -H V 2 - B H U V - H V V - H V W 4 + D Z D Y \\ &\quad + H N X + H N Y + H N N 4 + L H F U . \end{aligned}$$

This set of equations holds for $k = 1, 2, \dots, K - 1$, where $K + 1$ is the number of grid points along the η -axis ($k = K$ at the sea floor), so the equations for the solution of μ at the next time level are implicit in the η -direction.

At the sea surface the second order accurate finite difference approximation for an irregular grid is

$$\frac{\partial v}{\partial \eta} \Big|_{\eta=0} = a_0 v_{i,0} + b_0 v_{i,1} + c_0 v_{i,2} = D_{i,0} , \quad (\text{A.12})$$

where

$$a_0 = \frac{-r_1 - 2}{\Delta\eta_0(r_1 + 1)} , \quad b_0 = \frac{r_1 + 1}{\Delta\eta_0 r_1} \quad \text{and} \quad c_0 = \frac{-1}{\Delta\eta_0 r_1(r_1 + 1)} . \quad (\text{A.13})$$

The surface boundary condition of Equation (2.56) is used to obtain a value for $D_{i,0}$, giving

$$D_{i,0} = \frac{-\vec{H}_i \tau_{s\lambda}}{\rho N_\eta(0)} .$$

Rearranging Equation (A.12) in terms of $v_{i,2}$ at the $(n+1)^{th}$ time level and substituting this into Equation (A.11) with $k = 1$ gives

$$(P_{i,1}^{n+1} - \frac{a_0 R_{i,1}^{n+1}}{c_0})v_{i,0}^{n+1} + (Q_{i,1}^{n+1} - \frac{b_0 R_{i,1}^{n+1}}{c_0})v_{i,1}^{n+1} = S_{i,1}^n - \frac{D_{i,0}^{n+1} R_{i,1}^{n+1}}{c_0} . \quad (\text{A.14})$$

which is in the same form as Equation (A.11) if P , Q , R and S are defined by

$$\begin{aligned} P_{i,0}^{n+1} &= 0 , \\ Q_{i,0}^{n+1} &= P_{i,1}^{n+1} - r_1(r_1 + 2)R_{i,1}^{n+1} , \\ R_{i,0}^{n+1} &= Q_{i,1}^{n+1} + (r_1 + 1)^2 R_{i,1}^{n+1} , \\ S_{i,0}^n &= S_{i,1}^n + \Delta\eta_0 r_1(r_1 + 1)D_{i,0}^{n+1} R_{i,1}^{n+1} . \end{aligned}$$

If a “no-slip” condition applies on the sea floor, then Equation (A.11) with $k = K - 1$, simplifies to

$$P_{i,K-1}^{n+1}v_{i,K-2}^{n+1} + Q_{i,K-1}^{n+1}v_{i,K-1}^{n+1} = S_{i,K-1}^n . \quad (\text{A.15})$$

Thus Equations (A.11) for $k = 1, 2, \dots, K - 2$, Equation (A.14) and Equation (A.15) form a tri-diagonal set of equations (K equations and K unknowns as $v_{i,K} = 0$) which can be implicitly solved to give all the v velocity values over depth for the grid element indexed by i , at the new time level.

If a “slip” condition is required at the sea floor then the second order accurate finite difference approximation for an irregular grid is

$$\left. \frac{\partial v}{\partial \eta} \right|_{\eta=1} = a_K v_{i,K} + b_K v_{i,K-1} + c_K v_{i,K-2} = D_{i,K} , \quad (\text{A.16})$$

where

$$a_K = \frac{-2r_{K-1} - 1}{\Delta\eta_{K-1}(r_{K-1} + 1)} , \quad b_K = \frac{r_{K-1} + 1}{\Delta\eta_{K-1}} \quad \text{and} \quad c_K = -\frac{r_{K-1}^2}{\Delta\eta_{K-1}(r_{K-1} + 1)} , \quad (\text{A.17})$$

$D_{i,K}$ is evaluated using Equation (2.53), giving

$$D_{i,K} = \frac{-\vec{H}_i \tau_{b\lambda}}{\rho N_\eta(1)} ,$$

and where $\eta = 1$ or $k = K$ represents the boundary between the turbulent and laminar flows, some small distance above the physical sea floor.

Rearranging Equation (A.16) in terms of $v_{i,K-2}$ and substituting this into Equation (A.11) with $k = K - 1$ gives

$$(Q_{i,K-1}^{n+1} - \frac{b_K P_{i,K-1}^{n+1}}{c_K}) v_{i,K-1}^{n+1} + (R_{i,K-1}^{n+1} - \frac{a_K P_{i,K-1}^{n+1}}{c_K}) v_{i,K}^{n+1} = S_{i,K-1}^n - \frac{D_{i,K}^{n+1} P_{i,K-1}^{n+1}}{c_K} , \quad (\text{A.18})$$

which is in the same form as Equation (A.11) if P , Q , R and S are defined by

$$\begin{aligned} P_{i,K}^{n+1} &= Q_{i,K-1}^{n+1} + \frac{(r_{K-1} + 1)^2}{r_{K-1}^2} P_{i,K-1}^{n+1} , \\ Q_{i,K}^{n+1} &= R_{i,K-1}^{n+1} - \left(\frac{2r_{K-1} + 1}{r_{K-1}^2} \right) P_{i,K-1}^{n+1} , \\ R_{i,K}^{n+1} &= 0 , \\ S_{i,K}^n &= S_{i,K-1}^n + \left(\frac{\Delta\eta_{K-1}(r_{K-1} + 1)}{r_{K-1}^2} \right) D_{i,K}^{n+1} R_{i,K-1}^{n+1} . \end{aligned}$$

Thus with a “slip” condition Equations (A.11) for $k = 1, 2, \dots, K - 1$, Equation (A.14) and Equation (A.18) form the set of equations to be solved for the values of v at the new time level and at each depth level, for the grid element indexed by i .

This is one more unknown, $v_{i,K}$ and one more equation in the tri-diagonal set of equations than for the “no-slip” condition and also introduces the problem of representing correctly the bottom shear stress τ_b .

The tri-diagonal set of equations, with either the “slip” or the “no-slip” condition on the sea floor, can easily and efficiently be solved using the algorithm developed by Thomas (1949).

Appendix B

Analytic Continuations for the Hypergeometric Function

The analytic continuations found in Abramowitz and Stegun (1972) and Erdélyi et al. Vol. 1 (1953) for the Hypergeometric functions can be used to formulate the following relationships, which are used to evaluate the analytic solutions in part (g) of Chapter 5.

The Hypergeometric function can be evaluated by the following power series expansion, which will converge rapidly if the argument ξ has a modulus less than one.

$$F(a, b; c; \xi) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\xi^n}{n}, \quad (\text{B.1})$$

where $()_n$ is the Pochhammer's symbol, which is defined by

$$\begin{aligned} (a)_0 &= 1, \\ (a)_1 &= a, \\ (a)_n &= a(a+1)(a+2)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \end{aligned} \quad (\text{B.2})$$

and Γ is the Gamma function.

For $\xi = \frac{1}{2} + \frac{i}{2} \cot\left(\frac{\pi(\eta-d)}{b}\right)$ and $n_1, n_2 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{C_a + i(\delta - f_a)b^2}{a\pi^2}}$ then the first linearly independent solution, $\xi F(n_1 + 1, n_2 + 1; 2; \xi)$ can be evaluated as follows.

If $\cot\left(\frac{\pi(\eta-d)}{b}\right) < -0.5$ **then**

$$\xi F(n_1 + 1, n_2 + 1; 2; \xi) = \xi(1 - \xi) F^*\left(\frac{1}{2} - \frac{n_1}{2}, 1 + \frac{n_1}{2}; 2; 4\xi(1 - \xi)\right). \quad (\text{B.3})$$

If $\cot\left(\frac{\pi(\eta-d)}{b}\right) > 0.5$ **then**

$$\xi F(n_1 + 1, n_2 + 1; 2; \xi) = \xi(1 - \xi)(1 - 2\xi)^{n_1-1} F^*\left(\frac{1}{2} - \frac{n_1}{2}, 2 + \frac{n_1}{2}; 2; \frac{4\xi(1 - \xi)}{(1 - 2\xi)^2}\right). \quad (\text{B.4})$$

Otherwise Equation (B.1) can be used to evaluate $\xi F(n_1 + 1, n_2 + 1; 2; \xi)$ as $|\xi| < 1$.

$F^*(a, b; c; \xi)$ is used to show that another transformation is used to evaluate the function. This transformation is defined by

$$\begin{aligned} F^*(a, b; c; \xi) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-\xi)^{-a} F\left(a, 1-c+a; 1+a-b; \frac{1}{\xi}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-\xi)^{-b} F\left(b, 1-c+b; 1-a+b; \frac{1}{\xi}\right). \end{aligned} \quad (\text{B.5})$$

Similarly for the second linearly independent solution.

If $\cot\left(\frac{\pi(\eta-d)}{b}\right) < -0.5$ **then**

$$(1 - \xi) F(-n_1, -n_2; 2; 1 - \xi) = \xi(1 - \xi)(1 - 2\xi)^{n_1-2} F^*\left(1 + \frac{n_1}{2}, \frac{3}{2} + \frac{n_1}{2}; 2; \frac{4\xi(\xi - 1)}{(2\xi - 1)^2}\right). \quad (\text{B.6})$$

If $\cot\left(\frac{\pi(\eta-d)}{b}\right) > 0.5$ **then**

$$(1 - \xi) F(-n_1, -n_2; 2; 1 - \xi) = \xi(1 - \xi) F^*\left(1 + \frac{n_1}{2}, \frac{1}{2} - \frac{n_1}{2}; 2; 4\xi(1 - \xi)\right). \quad (\text{B.7})$$

Otherwise $(1 - \xi) F(-n_1, -n_2; 2; 1 - \xi)$ can be evaluated by using Equation (B.1).

In order to analytically solve the equations in Chapter 5, the derivatives of the two linearly independent solutions must be available.

If $\cot\left(\frac{\pi(\eta-d)}{b}\right) < -0.5$ **then**

$$\frac{\partial}{\partial \eta} \xi F(n_1 + 1, n_2 + 1; 2; \xi) = \frac{i}{2} F^*\left(-\frac{n_1}{2}, \frac{1}{2} + \frac{n_1}{2}; 1; 4\xi(1 - \xi)\right). \quad (\text{B.8})$$

If $-0.5 < \cot\left(\frac{\pi(\eta-d)}{b}\right) < 0.5$ **then**

$$\frac{\partial}{\partial \eta} \xi F(n_1 + 1, n_2 + 1; 2; \xi) = \frac{i}{2} F(1 + n_1, -n_1; 1; \xi) . \quad (\text{B.9})$$

Else if $\cot\left(\frac{\pi(\eta-d)}{b}\right) > 0.5$ **then**

$$\frac{\partial}{\partial \eta} \xi F(n_1 + 1, n_2 + 1; 2; \xi) = \frac{i}{2} (1 - 2\xi)^{n_1} F^*\left(-\frac{n_1}{2}, \frac{1}{2} - \frac{n_1}{2}; 1; \frac{4\xi(1 - \xi)}{(1 - 2\xi)^2}\right) . \quad (\text{B.10})$$

Similarly

If $\cot\left(\frac{\pi(\eta-d)}{b}\right) < -0.5$ **then**

$$\frac{\partial}{\partial \eta} (1 - \xi) F(-n_1, -n_2; 2; 1 - \xi) = -\frac{i}{2} (2\xi - 1)^{-n_1 - 1} F^*\left(\frac{1}{2} + \frac{n_1}{2}, 1 + \frac{n_1}{2}; 1; \frac{4\xi(\xi - 1)}{(2\xi - 1)^2}\right) . \quad (\text{B.11})$$

If $-0.5 < \cot\left(\frac{\pi(\eta-d)}{b}\right) < 0.5$ **then**

$$\frac{\partial}{\partial \eta} (1 - \xi) F(-n_1, -n_2; 2; 1 - \xi) = -\frac{i}{2} F(-n_1, 1 + n_1; 1; 1 - \xi) . \quad (\text{B.12})$$

Else if $\cot\left(\frac{\pi(\eta-d)}{b}\right) > 0.5$ **then**

$$\frac{\partial}{\partial \eta} (1 - \xi) F(-n_1, -n_2; 2; 1 - \xi) = -\frac{i}{2} F^*\left(\frac{1}{2} + \frac{n_1}{2}, -\frac{n_1}{2}; 1; 4\xi(1 - \xi)\right) . \quad (\text{B.13})$$

Appendix C

Code for Series Evaluation of Complex Functions

This Appendix lists the Fortran code for the complex functions needed to evaluate the analytic solutions derived in Chapter 5.

On the computer system used in this thesis, a number of the functions requiring evaluation for complex arguments were not available as library routines. Consequently Fortran code for these functions was produced to augment the available library routines. The formula for the series evaluation of these functions has been obtained from Abramowitz and Stegun (1972).

The routines have been checked by using known function values for complex arguments given in tables in Abramowitz and Stegun but obviously the routines are only appropriate for complex values for which the series, used for evaluation, converge. These routines have also been used to produce the graphs given in Chapter 5 which by their “normal” shape suggest that the code for the functions is correct.

The functions evaluate Bessel, Gamma, Riemann Zeta, Hypergeometric, Legendre functions and also the two linearly independent solutions which are used in part (g) of Chapter 5 (the analytic continuations used by these two functions are given in

Appendix B). There are also functions for the integrals of the Hypergeometric and Legendre functions and the derivatives of the two linearly independent solutions used in part (g) of Chapter 5.

Double Complex Function CJ0(CZ)

C Evaluate Bessel Function J0 with a complex argument

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

```
Cterm = Dcmplx(1D0,0D0)
Csum = Cterm
k = 0
10 Continue
  k = k + 1
  Cterm = -Cterm*CZ*CZ/(4*k*k)
  Csum = Csum + Cterm
  If (Abs(Cterm) .Gt. 1E-20) Goto 10
CJ0 = Csum
Return
End
```

Double Complex Function CJ1(CZ)

C Evaluate Bessel Function J1 with a complex argument

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

```
Cterm = CZ/2
Csum = Cterm
k = 0
10 Continue
  k = k + 1
  Cterm = -Cterm*CZ*CZ/(4*k*(k+1))
  Csum = Csum + Cterm
  If (Abs(Cterm) .Gt. 1E-20) Goto 10
CJ1 = Csum
Return
End
```

Double Complex Function CY0(CZ)

C Evaluate Bessel Function Y0 with a complex argument

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

```

Common /Cy0d/ Ter, Pi

Data Ter,Pi / -0.5772 15664 90153 28606 06512D0,
&              3.1415 92653 58979 32384 62643D0 /

```

External Clog

```

Cterm = Dcmplx(1D0,0D0)
Term = Ter
Csum = Term*Cterm
k = 0
10 Continue
  k = k + 1
  Term = Term + 1D0/k
  Cterm = -Cterm*CZ*CZ/(4*k*k)
  Csum = Csum + Term*Cterm
  If (Abs(Term*Cterm) .Gt. 1E-20) Goto 10
  CY0 = 2*(Clog(CZ/2)*CJ0(CZ) - Csum)/Pi
  Return
End

```

Double Complex Function CY1(CZ)

C Evaluate Bessel Function Y1 with a complex argument

```

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

```

Common /Cy1d/ Term, Pi

```

Data Term,Pi / -0.5772 15664 90153 28606 06512D0,
&              3.1415 92653 58979 32384 62643D0 /

```

External Clog

```

Term1 = Term
Term2 = 1D0 + Term1
Cterm = Dcmplx(1D0,0D0)
Csum = (Term1+Term2)*Cterm
k = 0
10 Continue
  k = k + 1
  Term1 = Term1 + 1D0/k
  Term2 = Term2 + 1D0/(k+1)
  Cterm = -Cterm*CZ*CZ/(4*k*(k+1))
  Csum = Csum + (Term1+Term2)*Cterm
  If (Abs((Term1+Term2)*Cterm) .Gt. 1E-20) Goto 10
  CY1 = (2*Clog(CZ/2)*CJ1(CZ) - Cz*Csum/2 - 2/CZ)/Pi
  Return
End

```

Double Complex Function CJv(CZ,V)

C Evaluate Bessel Function Jv with a complex argument
 C and v is a non-integer

Implicit Double Precision (A-B,D-H,O-Z)
 Implicit Double Complex (C)

External Clog

```
Cterm = Dcmplx(1D0/Gamma(V+1),0D0)
Csum = Cterm
k = 0
10 Continue
  k = k + 1
  Cterm = -Cterm*CZ*CZ/(4*k*(V+k))
  Csum = Csum + Cterm
  If (Abs(Cterm) .Gt. 1E-20) Goto 10
  CJv = Exp(Clog(CZ/2)*V)*Csum
  Return
End
```

Double Precision Function Gamma(V)

C Evaluate Gamma Function for real argument

Implicit Double Precision (A-H,O-Z)

Double Precision C(26)

Common /Gammad/ C

	Data C / 1D0,	0.57721	56649	015329D0,	-0.65587	80715	202538D0,
&		-0.04200	26350	340952D0,	0.16653	86113	822915D0,
&		-0.04219	77345	555443D0,	-0.00962	19715	278770D0,
&		0.00721	89432	466630D0,	-0.00116	51675	918591D0,
&		-0.00021	52416	741149D0,	0.00012	80502	823882D0,
&		-0.00002	01348	547807D0,	-0.00000	12504	934821D0,
&		0.00000	11330	272320D0,	-0.00000	02056	338417D0,
&		0.00000	00061	160950D0,	0.00000	00050	020075D0,
&		-0.00000	00011	812746D0,	0.00000	00001	043427D0,
&		0.00000	00000	077823D0,	-0.00000	00000	036968D0,
&		0.00000	00000	005100D0,	-0.00000	00000	000206D0,
&		-0.00000	00000	000054D0,	0.00000	00000	000014D0,
&		0.00000	00000	000001D0			

```
Vt = V
k = 0
Do 10, While (Vt .Gt. 1D0)
  Vt = Vt - 1D0
  k = k + 1
10 Continue
Sum = 0D0
Term = Vt
```

```

      Do 20, j = 1, 26
        Sum = Sum + C(j)*Term
        Term = Term*Vt
20    Continue
      Gamma = 1/Sum
      Do 30, j = 1, k
        Gamma = Gamma*(j-1+Vt)
30    Continue
      Return
      End

```

Double Precision Function Rz(n)

C Calculate the (Riemann Zeta Function) - 1D0

Implicit Double Precision (A-H,O-Z)

Double Precision A(42)

Common /Rzd/ A

```

      Data (A(i),i=1, 40) /
&      0.20205 69031 59594 28540D0, 0.64493 40668 48226 43647D0,
&      0.03692 77551 43369 92633D0, 0.08232 32337 11138 19152D0,
&      0.00834 92773 81922 82684D0, 0.01734 30619 84449 13971D0,
&      0.00200 83928 26082 21442D0, 0.00407 73561 97944 33938D0,
&      0.00049 41886 04119 46456D0, 0.00099 45751 27818 08534D0,
&      0.00012 27133 47578 48915D0, 0.00024 60865 53308 04830D0,
&      0.00003 05882 36307 02049D0, 0.00006 12481 35058 70483D0,
&      0.00000 76371 97637 89976D0, 0.00001 52822 59408 65187D0,
&      0.00000 19082 12716 55394D0, 0.00000 38172 93264 99984D0,
&      0.00000 04769 32986 78781D0, 0.00000 09539 62033 87280D0,
&      0.00000 01192 19925 96531D0, 0.00000 02384 50502 72773D0,
&      0.00000 00298 03503 51465D0, 0.00000 00596 08189 05126D0,
&      0.00000 00074 50711 78984D0, 0.00000 00149 01554 82837D0,
&      0.00000 00018 62659 72351D0, 0.00000 00037 25334 02479D0,
&      0.00000 00004 65662 90650D0, 0.00000 00009 31327 43242D0,
&      0.00000 00001 16415 50173D0, 0.00000 00002 32831 18337D0,
&      0.00000 00000 29103 85044D0, 0.00000 00000 58207 72088D0,
&      0.00000 00000 07275 95984D0, 0.00000 00000 14551 92189D0,
&      0.00000 00000 01818 98965D0, 0.00000 00000 03637 97955D0,
&      0.00000 00000 00454 74738D0, 0.00000 00000 00909 49478D0/
      Data A(41),A(42) /
&      0.00000 00000 00454 74738D0, 0.00000 00000 00227 37368D0/

```

```

      If (n .Gt.42) Then
        Rz = A(42)
        Do 10, i = 1, n-42
          Rz = Rz/2
10      Continue
      Else
        Rz = A(n)
      Endif
      Return

```

End

Double Complex Function CGamma(CV)

C Evaluate Gamma Function for complex argument

Implicit Double Precision (A-B,D-H,O-Z)

Implicit Double Complex (C)

Double Precision D(26)

Common /Cgammad/ D

```
Data D / 1D0, 0.57721 56649 015329D0, -0.65587 80715 202538D0,
&          -0.04200 26350 340952D0,  0.16653 86113 822915D0,
&          -0.04219 77345 555443D0, -0.00962 19715 278770D0,
&          0.00721 89432 466630D0, -0.00116 51675 918591D0,
&          -0.00021 52416 741149D0,  0.00012 80502 823882D0,
&          -0.00002 01348 547807D0, -0.00000 12504 934821D0,
&          0.00000 11330 272320D0, -0.00000 02056 338417D0,
&          0.00000 00061 160950D0,  0.00000 00050 020075D0,
&          -0.00000 00011 812746D0,  0.00000 00001 043427D0,
&          0.00000 00000 077823D0, -0.00000 00000 036968D0,
&          0.00000 00000 005100D0, -0.00000 00000 000206D0,
&          -0.00000 00000 000054D0,  0.00000 00000 000014D0,
&          0.00000 00000 000001D0
```

CVt = CV

k = 0

Do 10, While (Dble (CVt) .Gt. 1D0)

CVt = CVt - 1D0

k = k + 1

10 Continue

Csum = Dcmplx(0D0,0D0)

CTerm = CVt

Do 20, j = 1, 26

Csum = Csum + D(j)*CTerm

CTerm = CTerm*CVt

20 Continue

CGamma = 1D0/Csum

Do 30, j = 1, k

CGamma = CGamma*(j-1+CVt)

30 Continue

Return

End

Double Complex Function CaGamma(CV)

C Evaluate Gamma Function in an alternate manner for complex argument

Implicit Double Precision (A-B,D-H,O-Z)

Implicit Double Complex (C)

```

Common /CaGammad/ Term, Pi

Data Term,Pi / 0.5772 15664 90153 28606 06512D0,
&              3.1415 92653 58979 32384 62643D0 /

CVt = CV
k = 0
Do 10, While (Dble (CVt) .Gt. 2D0)
    CVt = CVt - 1D0
    k = k + 1
10 Continue
    CVtM1 = CVt - 1D0
    CSum = Dcmplx(0D0,0D0)
    Cterm = CVtM1*CVtM1
    Ca = CVtM1*CVtM1
    Csum = Rz(2)*Cterm/2
    n = 2
20 Continue
    n = n + 2
    Cterm = Cterm*CVtM1
    Ca = -Rz(n-1)*Cterm/(n-1)
    Cterm = Cterm*CVtM1
    Cb = Rz(n)*Cterm/n
    Csum = Csum + (Ca + Cb)
    If (Abs(Ca) .Gt. 1E-20) Goto 20
    CaGamma = Exp(CVtM1*(1D0-Term)+CSum)/CVt
    Do 30, j = 1, k
        CaGamma = CaGamma*(j-1+CVt)
30 Continue
Return
End

```

Double Complex Function CHyp(Ca,Cb,Cc,Cz)

C Calculates the Hypergeometric Function $F(a,b;c;z)$
C for $a = Ca$, $b = Cb$, $c = Cc$ and $z = Cz$

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

```

Cterm = Dcmplx(1D0,0D0)
Csum = Cterm
k = 0
10 Continue
    Cterm = Cterm*Cz*(Ca+k)*(Cb+k)/(k+1)/(Cc+k)
    k = k + 1
    Csum = Csum + Cterm
    If (Abs(Cterm) .Gt. 1E-20) Goto 10
CHyp = Csum
Return
End

```

Double Complex Function CIntHyp(Ca,Cb,Cz0,Cz1)

C Calculates the Integral of the Hypergeometric Function $F(a,b;1;z)$
 C for $Cz0 < z < Cz1$, $a = Ca$ and $b = Cb$

Implicit Double Precision (A-B,D-H,0-Z)
 Implicit Double Complex (C)

```

      Cterm = Cz1
      Csum  = Cterm
      k     = 0
10  Continue
      Cterm = Cterm*Cz1*(Ca+k)*(Cb+k)/(k+2)/(1+k)
      k     = k + 1
      Csum  = Csum + Cterm
      If (Abs(Cterm) .Gt. 1E-20) Goto 10
      Cterm = Cz0
      Csum  = Csum - Cterm
      k     = 0
20  Continue
      Cterm = Cterm*Cz0*(Ca+k)*(Cb+k)/(k+2)/(1+k)
      k     = k + 1
      Csum  = Csum - Cterm
      If (Abs(Cterm) .Gt. 1E-20) Goto 20
      CIntHyp = Csum
      Return
      End
  
```

Double Complex Function CHypA(Ca,Cb,Cc,Cz)

C Calculates the Hypergeometric Function $F(a,b;c;z)$
 C using one of the linear transformations for F

Implicit Double Precision (A-B,D-H,0-Z)
 Implicit Double Complex (C)

External Clog

```

      CHypA = CGamma(Cc)*CGamma(Cb-Ca)/CGamma(Cb)/CGamma(Cc-Ca)
&          *Exp(-Clog(-Cz)*Ca)
&          *CHyp(Ca,1-Cc+Ca,1+Ca-Cb,1/Cz)
&          + CGamma(Cc)*CGamma(Ca-Cb)/CGamma(Ca)/CGamma(Cc-Cb)
&          *Exp(-Clog(-Cz)*Cb)
&          *CHyp(Cb,1-Cc+Cb,1-Ca+Cb,1/Cz)
      Return
      End
  
```

Double Complex Function CHypB(Ca,Cb,Cc,Cz)

C Calculates the Hypergeometric Function $F(a,b;c;z)$
 C using one of the linear transformations for F

```

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

```

```

External Clog

```

```

CHypB = CGamma(Cc)*CGamma(Cc-Ca-Cb)/CGamma(Cc-Ca)/CGamma(Cc-Cb)
&      *Exp(-Clog(Cz)*Ca)
&      *CHyp(Ca,Ca+1-Cc,Ca+Cb-Cc+1,1-1/Cz)
&      + CGamma(Cc)*CGamma(Ca+Cb-Cc)/CGamma(Ca)/CGamma(Cb)
&      *Exp(Clog(Cz)*(Ca-Cc)+Clog(1-Cz)*(Cc-Ca-Cb))
&      *CHyp(Cc-Ca,1-Ca,Cc+1-Ca-Cb,1-1/Cz)
Return
End

```

```

Double Complex Function CU1(X,S)

```

```

C  Calculates the first solutions for Nz = Sin()*Sin()
C  X should be Cot(Pi*(z-c)/b)
C  S should be Sigma*b*b/a/Pi/Pi

```

```

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

```

```

External Clog

```

```

Ci = Dcmplx(0D0,1D0)
Ca = 5D-1 + Sqrt(25D-2 - Ci*S)
Cc = Dcmplx(2D0,0D0)
Cn = Dcmplx(1D0,X)/2
If (X .Lt. -0.5) Then
    CU1 = Cn*(1-Cn)*CHypA(1-Ca/2,(1+Ca)/2,Cc,4*Cn*(1-Cn))
Endif
If ((X .Ge. -0.5) .And. (X .Le. 0.5)) Then
    CU1 = Cn*CHyp(Ca,1-Ca,Cc,Cn)
Endif
If (X .Gt. 0.5) Then
    Cu1 = Cn*(1-Cn)*Exp(Clog(1-2*Cn)*(Ca-2))*
&      CHypA(1-Ca/2,(3-Ca)/2,Cc,4*Cn*(Cn-1)/(1-2*Cn)/(1-2*Cn))
Endif
Return
End

```

```

Double Complex Function CDU1(X,S)

```

```

C  Calculates the derivative of first solutions for Nz = Sin()*Sin()
C  X should be Cot(Pi*(z-c)/b)
C  S should be Sigma*b*b/a/Pi/Pi

```

```

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

```

External Clog

```

Ci = Dcmplx(0D0,1D0)
Ca = 5D-1 + Sqrt(25D-2 - Ci*S)
Cc = Dcmplx(1D0,0D0)
Cn = Dcmplx(1D0,X)/2
If (X .Lt. -0.5) Then
    CDU1 = CHypA((1-Ca)/2,Ca/2,Cc,4*Cn*(1-Cn))
Endif
If ((X .Ge. -0.5) .And. (X .Le. 0.5)) Then
    CDU1 = CHyp(Ca,1-Ca,Cc,Cn)
Endif
If (X .Gt. 0.5) Then
    CDU1 = Exp(Clog(1-2*Cn)*(Ca-1))*
&    CHypA((1-Ca)/2,1-Ca/2,Cc,4*Cn*(Cn-1)/(1-2*Cn)/(1-2*Cn))
Endif
CDU1 = CDU1*Ci/2
Return
End

```

Double Complex Function CU2(X,S)

C Calculates the second solutions for $Nz = \sin() \cdot \sin()$
C X should be $\cot(\pi \cdot (z-c)/b)$
C S should be $\sigma \cdot b \cdot b/a/\pi/\pi$

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

External Clog

```

Ci = Dcmplx(0D0,1D0)
Ca = 5D-1 - Sqrt(25D-2 - Ci*S)
Cc = Dcmplx(2D0,0D0)
Cn = Dcmplx(1D0,-X)/2
If (X .Lt. -0.5) Then
    CU2 = Cn*(1-Cn)*Exp(Clog(1-2*Cn)*(Ca-2))*
&    CHypA(1-Ca/2,(3-Ca)/2,Cc,4*Cn*(Cn-1)/(1-2*Cn)/(1-2*Cn))
Endif
If ((X .Ge. -0.5) .And. (X .Le. 0.5)) Then
    CU2 = Cn*CHyp(Ca,1-Ca,Cc,Cn)
Endif
If (X .Gt. 0.5) Then
    CU2 = Cn*(1-Cn)*CHypA(1-Ca/2,(1+Ca)/2,Cc,4*Cn*(1-Cn))
Endif
Return
End

```

Double Complex Function CDU2(X,S)

C Calculates the derivative of second solutions for $Nz = \sin() \cdot \sin()$
C X should be $\cot(\pi \cdot (z-c)/b)$

C S should be $\text{Sigma} \cdot b \cdot b / a / \text{Pi} / \text{Pi}$

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

External Clog

```
Ci = Dcmplx(0D0,1D0)
Ca = 5D-1 - Sqrt(25D-2 - Ci*S)
Cc = Dcmplx(1D0,0D0)
Cn = Dcmplx(1D0,-X)/2
If (X .Lt. -0.5) Then
  CDU2 = Exp(Clog(1-2*Cn)*(Ca-1))*
&   CHypA((1-Ca)/2,1-Ca/2,Cc,4*Cn*(Cn-1)/(1-2*Cn)/(1-2*Cn))
Endif
If ((X .Ge. -0.5) .And. (X .Le. 0.5)) Then
  CDU2 = CHyp(Ca,1-Ca,Cc,Cn)
Endif
If (X .Gt. 0.5) Then
  CDU2 = CHypA((1-Ca)/2,Ca/2,Cc,4*Cn*(1-Cn))
Endif
CDU2 = -CDU2*Ci/2
Return
End
```

Double Complex Function CPv(X,CV)

C Calculates the Legendre Function Pv(X) for real X and complex v

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

```
Cterm = Dcmplx(1D0,0D0)
Csum = Cterm
Z = (1D0-X)/2
k = 0
10 Continue
  k = k + 1
  Cterm = Cterm*Z*(k-1-CV)*(k+CV)/(k*k)
  Csum = Csum + Cterm
If (Abs(Cterm) .Gt. 1E-20) Goto 10
CPv = Csum
Return
End
```

Double Complex Function CQv(X,CV)

C Calculates the Legendre Function Qv(X) for real X and complex v

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

```

Common /CQvd/ Pi

Data Pi / 3.1415 92653 58979 32384 62643D0 /

CQv = Pi*(CPv(X,CV)*Cos(Pi*CV) - CPv(-X,CV))/2/Sin(Pi*CV)
Return
End

Double Complex Function CIntPv(X0,X1,CV)

C Calculates the Integral of the Legendre Function Pv(X) from X0 to X1
C for real X0 and X1 and complex v

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

Z      = (1D0-X1)/2
Cterm  = Dcmplx(Z,0D0)
Csum   = Cterm
k      = 0
10 Continue
    Cterm = Cterm*Z*(-CV+k)*(CV+1+k)/(k+1)/(k+2)
    k     = k + 1
    Csum  = Csum + Cterm
    If (Abs(Cterm) .Gt. 1E-20) Goto 10
    Z     = (1D0-X0)/2
    Cterm  = Dcmplx(Z,0D0)
    Csum   = Csum - Cterm
    k     = 0
20 Continue
    k     = k + 1
    Cterm = Cterm*Z*(-CV-1+k)*(CV+k)/(k*(k+1))
    Csum  = Csum - Cterm
    If (Abs(Cterm) .Gt. 1E-20) Goto 20
    CIntPv = -2*Csum
    Return
End

Double Complex Function CIntQv(X0,X1,CV)

C Calculates the Integral of the Legendre Function Qv(X) from X0 to X1
C for real X0 and X1 and complex v

Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)

Common /CIntQvd/ Pi

Data Pi / 3.1415 92653 58979 32384 62643D0 /

CIntQv = Pi*( CIntPv(X0,X1,CV)*Cos(Pi*CV)
&          + CIntPv(-X0,-X1,CV)          )/2/Sin(Pi*CV)

```

```
Return
End
```

```
Double Complex Function Clog(Cz)
```

```
C If the complex log of a number has an imaginary part "close to"
C the value -pi then add 2*i*pi to the original log value.
```

```
Implicit Double Precision (A-B,D-H,O-Z)
Implicit Double Complex (C)
```

```
Common /Clogd/ Pi
```

```
Data Pi /3.1415 92653 58979 32384 62643D0 /
```

```
Clog = Log(Cz)
If ( (Dimag(Clog)+Pi) .Lt. 1D-18 )
& Clog = Clog + 2*Dcplx(0D0,1D0)*Pi
Return
End
```

Appendix D

The Input File used for the Spencer Gulf Application

The following gives the input file for the model application described in Chapter 8. It defines various parameters, sets certain options and initialises some of the data. Some of the other parameters, like the number of depth levels, determine the size of the arrays in the program are set as constants at the top of the program and are fixed at compile time.

This is the input data used when the program is started. After the duration of this run the program (if requested) dumps all necessary data onto the file "fort.7", which is expected as input if the response to the first question is "No", that is the program is to not start afresh but to continue on from where it left off. This was a precaution to segment the calculations so that in the event of a computer crash or system maintenance only the previous section would need to be rerun. The input data used for a restart is listed after this input data.

Yes	! Initial Run
Yes	! Use the 1st Sample Routine
Yes	! Use the 2nd Sample Routine
Variable	! Variable or Even Vertical Grid spacing
Fine	! at sea surface => Fine-Coarse-Fine Grid
3.0	! Kappa Value
Externally	! Zeta prescribed Externally on Open Boundary

[illegible]

```

11111
1111
111
111
111      111
111      11111
111      111111
111      111111
111 1111111
111 1111111
111 1111111
111 1111111
1111 111111
111 111111
111111111
11111111
1111111
11 111
111111
11111
1111 111
111111111
11111111111
111111111111
111111111111
111111111111
11111 111111
111111 1111 11
1111111 1 1111
11111111 1 1111
111111111 111
111111111111
111 11111111
111111
222222
-32.73194 137.75000
      4
      4 4
      5 5
      5
7      6
7
6
6
6
6 6.5
      7 6.5
      6.5 7
      7 6
      8 6.5
      8 7
      8 7
      8 9
      8 9
9
10
7 9

```

```

! 2's signify the Open Boundary
-32.73194 137.85000 ! Lat and Long of Bottom
! Corners of the 15x75
! Grid.

```

```

! Depths of each Active Element

```

236

```

12 17.7 17.5 17.3 17 7
7 12 17.7 17.5 17 17
6      ! The Number of Terms in the Sinusoidal Input
13.9430356 14.9589314 15.0410686 28.9841042 30.0 30.0821373 ! Frequencies
48.050 0.2501 73.375 0.0988 73.375 0.4704 196.017 0.5793
255.173 0.6197 255.173 0.1778 ! Phases and Amplitudes for each
48.050 0.2501 73.375 0.0988 73.375 0.4704 196.017 0.5793
255.173 0.6197 255.173 0.1778 ! Open Boundary Element.
48.050 0.2501 73.375 0.0988 73.375 0.4704 196.017 0.5793
255.173 0.6197 255.173 0.1778
48.050 0.2501 73.375 0.0988 73.375 0.4704 196.017 0.5793
255.173 0.6197 255.173 0.1778
48.050 0.2501 73.375 0.0988 73.375 0.4704 196.017 0.5793
255.173 0.6197 255.173 0.1778
48.050 0.2501 73.375 0.0988 73.375 0.4704 196.017 0.5793
255.173 0.6197 255.173 0.1778
120.00 ! Length of Run
6.0 ! Time between Samples (Samp)
0.25 ! Time between Samples (Samp2)

```

This is the input data used for a restart run. The program expects to find all the necessary data it needs to continue calculations on the file "fort.7". After completion of this run the file "fort.7" is overwritten with the new dumped data.

```

No      ! Initial Run
Yes      ! 1st Sample Routine
Yes      ! 2nd Sample Routine
120.00 ! Length of Run
6.0 ! Time between Samples (Samp)
0.25 ! Time between Samples (Samp2)

```

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