TYPE I MULTIPLIER REPRESENTATIONS OF LOCALLY COMPACT GROUPS

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SUMMARY

Let G be a locally compact abelian group and ω a normalized Borel multiplier on G. We are concerned primarily with the von Neumann algebra $V(G,\omega)$ generated by the regular ω-representation ρ of G defined by

$$\rho(g)f(x) = f(g^{-1} x)\omega(g^{-1},x) ,$$

$g \in G$, almost all $x \in G$, and $f \in L^2(G)$.

The pair (G,ω) is called type I if every ω-representation of G is type I.

In Chapter II, we investigate the structure of (G,ω) in the case where G is abelian and (G,ω) is type I. In particular, we can reduce the study of such pairs to the case where G is residually finite. Furthermore, if G is separable and divisible, then (G,ω) is type I if and only if there exists a bicontinuous isomorphism from G to a group of the form $H \times H^\wedge$ where H is a closed subgroup of G, which carries ω to a multiplier on $H \times H^\wedge$ that is similar to the multiplier ω' given by

$$\omega'((x,\lambda)(y,\chi)) = \lambda(y) ,$$

$(x,\lambda)(y,\chi) \in H \times H^\wedge$.

Chapter III provides information about the maximal type I central projection e in $V(G,\omega)$ in the case where G is a discrete group. Indeed we have $e \neq 0$ if and only if there exists a subgroup H of G such that the index [G:H] is finite, the commutator H' has finite order and ω restricted to H is the trivial multiplier. We also show how e can be realized as a convolution operator on $L^2(G)$. As a consequence of this result we can prove that for G discrete, (G,ω) is type I if and only if $V(G,\omega)$ is a type I von Neumann algebra and that this occurs if and only if G has an abelian subgroup A of finite index in G such that ω restricted to A is a trivial multiplier. This result generalizes easily to assert that a locally compact group G with normalized Borel multiplier ω satisfies $V(G,\omega)$ is type $I_{\leq k}$ for some natural number k if and only if G has an open abelian subgroup A of finite index in G such that the restriction of ω to A is trivial.

Finally, in Chapter IV these results are generalized to an arbitrary locally compact group G with Borel multiplier ω. Let e be the maximal type I finite central projection in $V(G,\omega)$, then $e \neq 0$ if and only if

(i) $[G:\Delta] < \infty$

(ii) Δ' has compact closure, and

(iii) there exists a finite dimensional ω-representation of Δ, where Δ denotes the closed normal subgroup of G consisting of all those elements whose conjugacy class has compact closure. Again we can construct e as a convolution operator on $L^2(G)$ and use this to prove that the following are equivalent.

(i) All irreducible ω-representations of G are finite dimensional.

(ii) $V(G,\omega)$ is type I finite.

(iii) The following properties hold.

(a) $[G:\Delta] < \infty$

(b) Δ' has compact closure.

(c) There exists a finite dimensional ω-representation of Δ.

(d) $\cap\{\ker \pi : \pi$ is a finite dimensional (ordinary) representation of $\Delta\} = \{1\}$.

(iv) All the irreducible (ordinary) representations of the central extension G^{ω} are finite dimensional.

Mackey's normal subgroup analysis is used in conjunction with the above theorem to construct a group G and multipliers ω_t, $t \in [0,1]$ such that $V(G,\omega_t)$ is type I finite and has a non-zero type I_n part for arbitrarily large natural numbers n if t is rational, and is type II_1 if t is irrational.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

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CHAPTER I

PRELIMINARIES

This chapter is introductory in nature. It serves to define notation, to provide the background, and to introduce some of the results needed for later chapters. For more detailed properties of locally compact groups, C*-algebras and von Neumann algebras, the reader is referred to [7, 18, 34, 40, 47].

1. Locally compact abelian groups

Let G be a locally compact abelian group. By this we will always mean a topological group which is locally compact and T_0 (and hence normal). We adopt the common usage throughout of using the word separable to describe a locally compact group which is ~~metrizable~~ second countable. Group homomorphisms (isomorphisms etc.) are such strictly in the algebraic sense unless it is specifically stated otherwise. A bicontinuous isomorphism is also referred to as a topological isomorphism. Given two locally compact groups H and K, then $H \times K$ denotes their direct product (with Cartesian product topology) and H is identified (up to topological isomorphism) with the subgroup $H \times \{e\}$. Similarly for K. A subgroup H of G is called a topological direct summand of G if it is closed and there exists a closed subgroup K of G such that G and $H \times K$ are topologically isomorphic, and is called a direct summand if the same property holds for the discrete topology on G.

Let G be a locally compact abelian group. The character group (or dual) of G, denoted by $G^\wedge$, is a group whose elements are continuous homomorphisms, $G \to \mathbb{T}$ (where $\mathbb{T}$ is the group of complex numbers of modulus 1 with the induced topology) and with multiplication defined pointwise. The sets $\{\chi \in G^\wedge : \chi(C) \subseteq V\}$, where C is a compact subset of G and V is a neighbourhood of the identity in $\mathbb{T}$, is a basis of neighbourhoods for a locally compact group topology on $G^\wedge$. If S is a subset of G, then $A[G^\wedge,S] = \{\chi \in G^\wedge : \chi(s) = 1$ all $s \in S\}$ denotes the annihilator of S in $G^\wedge$. The following duality theorems and structure theorem are well known.

<u>THEOREM 1.1.</u> *Let* G *be a locally compact abelian group, then*

(i) $(G^\wedge)^\wedge$ *is topologically isomorphic with* G.

(ii) *If* S *is a subset of* G *and* K *is the smallest closed subgroup of* G *containing* S, *then* $A[G, A[G^\wedge,S]] = K$.

(iii) *If* H *and* K *are closed subgroups of* G *such that* $H \subseteq K$, *then* $A[K^\wedge,H]$ *is topologically isomorphic with* $A[G^\wedge,H]/A[G^\wedge,K]$ *and* $(K/H)^\wedge$.

(iv) *If* H *is a closed subgroup of* G, *then* H *is open (respectively compact) if* $A[G^\wedge,H]$ *is compact (respectively open.)*

PROOF. See [18, 24.8, 24.10 and 23.24].

THEOREM 1.2. *A locally compact group* G *is topologically isomorphic to* $\mathbb{R}^n \times K$, *where* K *is a locally compact abelian group containing a compact open subgroup. The integer* n *is an invariant of* G.

For a proof of this result see [18, 24.30].

The following result relates the structure of G to that of G^.

THEOREM 1.3.

(i) *The dual of a divisible locally compact abelian group is torsion free.*

(ii) *The dual of a compact totally disconnected group is torsion.*

(iii) *If* G *is a compact group, then the following are equivalent.*

(a) G *is connected.*

(b) G^ *is torsion free.*

(c) G *is divisible.*

PROOF. See [18, 24.23, 24.26, 3.5 and 24.25].

Finally, we mention some algebraic properties of discrete abelian groups.

THEOREM 1.4.

(i) *A divisible subgroup* D *of an abelian group* G *is a direct summand. Indeed the complement may be chosen to contain any subset of* G *which intersects trivially with* D.

(ii) *If* H *is a subgroup of* G *such that the order of each element in* H *is less than some fixed integer, and* G/H *is torsion free, then* H *is a direct summand of* G.

(iii) *A finite abelian group is isomorphic with a finite product* ΠH_p, *where each* H_p *is a cyclic group of prime power order.*

PROOF. See [18, A.8 and A.25] and [12, 27.5].

2. Concrete C*-algebras and von Neumann algebras

Let H be a Hilbert space with inner product $\langle \cdot,\cdot \rangle$, and let B(H) be the algebra of all bounded linear operators on H and B(H)* the Banach space dual of B(H). The uniform topology on B(H) is given by the norm $\|a\|$ $(a \in B(H))$, where

$$\|a\| = \sup_{\|\xi\| \leq 1, \xi \in H} \|a\xi\|$$

B(H) is a Banach algebra under this norm and, with the adjoint operation $a \to a^*$ (defined by the relation $\langle a\xi,\eta \rangle = \langle \xi,a^*\eta \rangle$ for all $\xi,\eta \in H$) as the involution, B(H) is a

C*-algebra. Indeed any norm closed *-subalgebra of $B(H)$ (that is, a subalgebra invariant under the action of the involution *) is a C*-algebra. For a converse we have the following.

<u>THEOREM 2.1</u>, ([40, 1.16.6]). *Let* A *be a C*-algebra, then there exists a *-isomorphism from* A *onto a uniformly closed *-subalgebra of* $B(H)$ *for some Hilbert space* H.

Given a subset A of $B(H)$ we denote by A' the commutant of A viz. $A' = \{a \in B(H) : ab = ba \text{ for all } b \in A\}$, by A'' the set $(A')'$ and by CA the centre $A \cap A' = \{a \in A : ab = ba \text{ all } b \in A\}$ of A.

A *-subalgebra A of $B(H)$ for which $A'' = A$ is called a von Neumann algebra (or W*-algebra) acting on H.

The weak operator topology on $B(H)$ is the smallest topology on $B(H)$ such that for all $\xi,\eta \in H$, the function $B(H) \to C : a \to \langle a\xi,\eta\rangle$ is continuous. Each von Neumann algebra acting on H is automatically closed with respect to this topology and conversely, a weakly closed *-subalgebra of $B(H)$ containing the identity operator on H is a von Neumann algebra (Sakai [40, 1.20.3]).

Let A be a von Neumann algebra acting on H and let V be the linear space of all continuous linear functionals on A with respect to the weak operator topology and denote by A_* the norm closure of V in $B(H)^*$. The $\sigma(A,A_*)$ topology on A, that is the smallest topology on A such that each element in

A_* is continuous, is called the σ-weak (or ultra-weak topology). A_* consists of all σ-weak continuous linear functionals (also called normal functionals) and is called the predual of A; its Banach space dual can be identified (up to isometric isomorphism) with A by means of the map $A \to (A_*)^* : a \to T_a$, where $T_a(\psi) = \psi(a)$, $a \in A$ and $\psi \in A_*$ ([37, Theorem 2.4]). Furthermore, A_* is unique in the sense that no other norm closed subspace of $B(H)^*$ has this property ([47, III.3.9]).

THEOREM 2.2, ([47, III.3.10]). *Every *-isomorphism from a von Neumann algebra to another is σ-weakly continuous.*

Suppose we have von Neumann algebras A_i, (i = 1, 2) acting on Hilbert spaces H_i, then a *-isomorphism $\Phi : A_1 \to A_2$ is said to be spatial if there exists an isometry U of H_1 onto H_2 such that

$$(a) = UaU^*, \qquad a \in A.$$

We point out at once that *-isomorphisms are not necessarily spatial ([40, page 119]).

3. Classification of von Neumann algebras

Let A be a von Neumann algebra acting on a Hilbert space H. A self adjoint element e of A is called a projection if $e^2 = e$; a projection e is called central if ae = ea for all $a \in A$ (that is $e \in CA = A \cap A'$), and abelian if eAe is an abelian algebra. Two projections p, q are said to be orthogonal if $pq = 0$.

Let p and q be two projections in A. If there exists an element u of A such that $u^*u = p$ and $uu^* = q$, then p is said to be equivalent to q and we denote this by $p \sim q$. If there exists a projection q_1 $(\leq q)$ equivalent to p, we write this by $p \quad q$ or $q \quad p$. The relation "$\sim$" satisfies the conditions of equivalence and the relation " " is reflexive and transitive.

Let p be a projection in a von Neumann algebra A. p is said to be finite, if for a projection p_1 in A, $p_1 \leq p$ and $p_1 \sim p$ imply $p_1 = p$; p is said to be purely infinite if it does not contain any non-zero finite projection; p is said to be infinite if it is not finite.

Abelian projections are finite ([40, 2.2.8]). Also one can easily see that along with finite projection, all smaller ones are also finite. A von Neumann algebra is said to be finite (respectively purely infinite etc.) if its identity is finite (respectively purely infinite etc.).

Let z_j, $j \in J$ be a family of mutually orthogonal finite central projections of a von Neumann algebra A and let $z = \sum z_j$. Suppose $z \sim p \leq z$. Then $pz_j \sim z_j$. Hence $pz_j = z_j$ $(j \in J)$, so that $p = z$. Hence z is also finite. Hence there exists a unique maximal finite central projection z_1 in A. Similarly, we can see that there exists a unique maximal purely infinite central projection z_3 in A. Set $z_2 = I - z_1 - z_3$.

A is said to be semifinite if $z_3 = 0$; properly infinite if $z_1 = 0$; properly infinite and semifinite if $z_1 = z_3 = 0$.

THEOREM 3.1, ([40, 2.2.3]). *A von Neumann algebra* A *is uniquely decomposed into a direct sum of three algebras which are finite, properly infinite and semifinite, and purely infinite, respectively.*

Finiteness, semi-finiteness can be characterized using traces. Let A be a von Neumann algebra acting on a Hilbert space H. A trace on $A^+ = \{a \in A : \langle a\xi,\xi\rangle \geq 0 \text{ all } \xi \in H\}$ is a function φ defined on A^+, with non-negative finite or infinite values, possessing the following properties.

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$, $a,b \in A^+$,

(ii) $\varphi(\lambda a) = \lambda\varphi(a)$, $\lambda \in \mathbb{C}$, $a \in A^+$ (with the convention that $0 . +\infty = 0$),

(iii) If u is a unitary operator of A, we have $\varphi(uau^{-1}) = \varphi(a)$, $a \in A^+$.

φ is said to be finite if $\varphi(a) < +\infty$ for all $a \in A^+$; φ is said to be semifinite if, for every $a \in A^+$, $\varphi(a)$ is the least upper bound of the numbers $\varphi(b)$ for the $b \in A^+$ such that $b \leq a$ and $\varphi(b) < +\infty$; φ is said to be faithful if the conditions $a \in A^+$ and $\varphi(a) = 0$, imply that $a = 0$; and φ is said to be normal if it is σ-weakly continuous.

THEOREM 3.2, ([40, 2.5.4 and 2.5.7]). *Let* A *be a von Neumann algebra. Then* A *is finite (respectively semi-*

finite) if and only if for any non-zero a *in* A^+, *there exists a normal finite (respectively semifinite) trace* φ *such that* $\varphi(a) \neq 0$. A *is properly infinite (respectively purely infinite) if and only if there is no normal finite (respectively semi-finite) trace on* A^+ *except for the identical zero trace.*

On a semi-finite von Neumann algebra, there exists a semifinite faithful trace.

A von Neumann algebra is said to be type I if every non-zero central projection contains an abelian projection; type II if it is semi-finite and does not contain any abelian projection; type III if it is purely infinite. A finite type I (respectively type II) von Neumann algebra is said to be type I_f (respectively type II_1), and a properly infinite type I (respectively type II) von Neumann algebra is said to be type I_∞ (respectively II_∞). A central projection z in a von Neumann algebra A is said to be type X (X = I, II etc.) if zA is type X.

THEOREM 3.3, ([40, 2.2.10]). *A von Neumann algebra is uniquely decomposed into a type* I, *type* II *and type* III *direct summand.*

The following diagram will help interpret these definitions and facts.

<table>
<tr><td></td><td>finite</td><td colspan="2">properly infinite</td></tr>
<tr><td>type I</td><td>I_f</td><td>I_∞</td><td></td></tr>
<tr><td>type II</td><td>II_1</td><td>II_∞</td><td></td></tr>
<tr><td>type III</td><td></td><td></td><td>III</td></tr>
<tr><td></td><td colspan="2">semi-finite</td><td>purely infinite</td></tr>
</table>

<table>
<tr><td>type I_f</td><td>type II_1</td><td>type I_∞</td><td>type II_∞</td><td>type III</td></tr>
<tr><td colspan="2">finite</td><td colspan="3">properly infinite</td></tr>
<tr><td colspan="4">semi-finite</td><td>purely infinite</td></tr>
</table>

4. Type I_f von Neumann algebras and polynomial identities

Let A be a von Neumann algebra with centre $C = \{a \in A : ab = ba \text{ all } b \in A\}$. A is said to be type I_n $(n = 1, 2, \ldots)$ if it is *-isomorphic to $C \otimes B(H)$, where H is a Hilbert space of dimension n. Note that $C \otimes B(H)$ is nothing other than the $n \times n$ matrix algebra over C.

THEOREM 4.1, ([40, 2.3.2]). *A type I_f von Neumann algebra can be decomposed as a direct sum of type I_n von Neumann algebras ($n \in \mathbb{Z}$).*

Using this result, we define a von Neumann algebra A to be type $I_{\leq k}$ if the type I_n part in A is zero whenever $n > k$.

For any natural number k, let S_k denote the standard polynomial in k non-commuting variables

$$S_k(a_1, \ldots, a_k) = \sum (-1)^{\gamma} a_{\gamma(1)} a_{\gamma(2)} \cdots a_{\gamma(k)}$$

where the sum is taken over all permutations γ of $\{1, \ldots, k\}$ and $(-1)^{\gamma}$ denotes the signature of the permutation. Let A be an algebra. We say that the identity $S_k = 0$ is satisfied identically in A (or more briefly A satisfies S_k) if

$$S_k(A) = \{0\}$$

where $S_k(A)$ denotes the set $\{S_k(a_1, \ldots, a_k) : \text{all } a_i \in A, i = \{1, \ldots, k\}$. Polynomial identities are relevant in this context because of the Amitsur and Levitski Theorem:

<u>THEOREM 4.2</u>, ([39, 1.4.1]). *For every commutative ring* R, *the algebra of* $n \times n$ *matrices* R_n *over* R *satisfies* S_{2n}.

We have a partial converse of this theorem which is due to Kaplanski.

<u>THEOREM 4.3</u>. *An* $n \times n$ *matrix algebra over a field does not satisfy* S_{2k} *for* $k < n$.

For the rest of this Section, we follow Taylor [48].

<u>THEOREM 4.4</u>. *If the von Neumann algebra* A *is not of type* $I_{\leq n}$, *then there exists a copy of* M_{n+1} *(the*

$(n+1)\times(n+1)$-*complex matrices) in* A.

To see this, note that if A is not of type $I_{\leq n}$, then there exists a set of $n + 1$ mutually orthogonal equivalent projections in A. As in Smith [44, Lemma 9.3], a copy of M_{n+1} can be constructed in A.

<u>THEOREM 4.5.</u> *Let* A *be a von Neumann algebra and* n *a natural number. Then* A *satisfies* S_{2n} *if and only if* A *is of type* $I_{\leq n}$.

<u>PROOF.</u> If A satisfies S_{2n}, then by 4.3 and 4.4, A is of type $I_{\leq n}$. Conversely, if A is type $I_{\leq n}$, then A is a direct sum of the algebras A_k, $(1 \leq k \leq n)$, where each A_k is type I_k. By 4.2, each A_k satisfies S_{2n}. Therefore A satisfies S_{2n}.

<u>PROPOSITION 4.6.</u> *Suppose* B *is a weakly dense subalgebra of the von Neumann algebra* A, *then* A *satisfies* S_{2n} *if* B *does.*

<u>PROOF.</u> Let a^i be a net in A converging weakly to a. By the linearity of S_{2n} and because multiplication in A is weakly continuous, $S_{2n}(a^i, a_2, \ldots, a_{2k})$ converges to $S_{2k}(a, a_2, \ldots, a_{2k})$. Similarly for the other variables. This proves the result.

5. <u>Representations</u>

We give a very brief outline of definitions and required results. For more details see [7, 15, 2, 30].

Let A be a Banach *-algebra. A representation of A in a Hilbert space H is a homomorphism

$$\pi : A \to B(H)$$

which is non-degenerate in the sense that $\pi(a)\xi = 0$ all $a \in A$ implies $\xi = 0$. Since a *-isomorphism from a Banach *-algebra to a C*-algebra is norm reducing, a representation of A is necessarily continuous.

The (Hilbert) dimension of H is called the dimension of π and is denoted by dim π. The kernel ker π of π is the set $\{a \in A : \pi(a) = 0\}$; and π is said to be faithful if ker $\pi = \{0\}$. Two representations π and π' are said to be equivalent if there exists a unitary operator $U : H_{\pi'} \to H_\pi$ such that $U^{-1}\pi(a)U = \pi'(a)$ for all $a \in A$. We will not usually distinguish between a representation and its equivalence class.

Given a set of representations π_j, $j \in J$, of A, define their direct sum $\oplus_{j\in J} \pi_j$ by letting A act on the Hilbert space $\oplus_{j\in J} H_j$ coordinate-wise. Conversely, if $H = \oplus_{j\in J} H_j$ and each H_j invariant under the action of A (via π), then $\pi = \oplus_{j\in J} \pi_j$, where π_j $(j \in J)$ is the restriction of π to H_j.

Suppose we have two representations π and π' of the Banach *-algebra A. Their tensor product is the representation $\pi \otimes \pi'$ whose associated Hilbert space is $H_\pi \otimes H_{\pi'}$ and is defined by

$$(\pi \otimes \pi')(a)(\xi \otimes \xi') = \pi(a)\xi \otimes \pi'(a)\xi' ,$$

$\xi \in H_\pi$, $\xi \in H_{\pi'}$, $a \in A$.

Denote by $V(\pi) = \{\pi(a) : a \in A\}''$ the von Neumann algebra generated by the operators $\pi(a)$, $a \in A$. $V(\pi)$ is just the weak closure in $B(H_\pi)$ of the complex linear span of the operators $\pi(a)$, $a \in A$.

π is called a factor (or primary) representation if the centre of $V(\pi)$ consists of scalar multiples of the identity in $V(\pi)$(that is $CV(\pi) = \mathbb{C}.I$).

PROPOSITION 5.1, ([40, 1.21.9]). *Let* π *be a representation of the Banach *-algebra* A. *The following are equivalent.*

(i) $V(\pi) = B(H_\pi)$.

(ii) *If* H *is a closed subspace of* H_π *which is invariant, that is* $\pi(a)\xi \in H$ *all* $\xi \in H$, $a \in A$, *then* $H = \{0\}$ *or* $H = H_\pi$.

If either of these equivalent conditions are satisfied, then π is called irreducible.

Suppose we have an irreducible representation π and an operator T such that $\pi(a)T = T\pi(a)$ all $a \in A$, then by 5.1, T commutes with all the projections in $B(H_\pi)$ and thus T is a scalar [15, IV.3.9]. ~~π is one dimensional~~. This is called Schur's Lemma.

We will need a way of passing from Banach *-algebras to C*-algebras. Suppose π is a representation of the Banach *-algebra A and let $a \in A$. Since π is norm reducing, $\|\pi(a)\| \leq \|a\|$, thus the supremum of $\|\pi(a)\|$ as π runs through all the irreducible representations of A is a well defined number which we denote by $\|a\|'$. Let I be the set of $a \in A$ such that $\|x\|' = 0$, which is a closed self-adjoint two-sided ideal of A. The map $a \to \|a\|'$ defines a norm on the quotient A/I. Endowed with this norm, A/I satisfies all the C*-algebra axioms except that A/I is not complete in general. The completion of A/I is a C*-algebra called the enveloping C*-algebra of A and is denoted by C*(A). The canonical map of A into B is a norm-reducing *-homomorphism whose image is dense in B. When A is a C*-algebra, we can identify A with C*(A).

<u>THEOREM 5.2</u>, ([7, 2.7.4]). *Let* A *be a Banach *-algebra with an approximate identity and* τ *the canonical map of* A *into* C*(A).

(i) *If* π *is a representation of* A, *there is exactly one representation* ρ *of* C*(A) *such that*

$\pi = \rho \circ \tau$.

(ii) *The map* $\pi \to \rho$ *is a bijection of the set of* *representations of* A *onto the set of representations of* $C^*(A)$.

(iii) π *is irreducible if and only if* ρ *is irreducible.*

(iv) $V(\pi) = V(\rho)$.

This result shows that the majority of questions concerning representations of Banach *-algebras with an approximate identity, it is enough to deal only with the C*-algebra case.

The following result ensures an adequate supply of representations.

THEOREM 5.3, ([47, I.9.23]). *Let* A *be a C*-algebra and let* a *in* A *be non-zero. Then there exists an irreducible representation* π *of* A *such that* $\pi(a) \neq 0$.

A representation π of a C*-algebra A is said to be type I (respectively finite etc.) if the von Neumann algebra $V(\pi)$ is type I (respectively finite etc.). A is called type I if $V(\pi)$ is type I for all representations π of A.

Let A be a C*-algebra. The set of equivalence classes of irreducible representations of A denoted by $A^\wedge$ is called the dual of A. The dual $A^\wedge$ is given the hull-kernel topology which is defined as follows. A subset F of $A^\wedge$ has closure

$$F^- = \{\pi \in A^\wedge : \ker \pi \supseteq \cap_{\pi' \in F} \ker \pi'\} .$$

'Type I-ness' of a separable C*-algebra (that is one that contains a countable dense subset) can be characterized using this topology. This is a deep theorem due to Glimm.

THEOREM 5.4, ([7, 9.1 and 9.5.2]). *Let* A *be a C*-algebra, then* A *is type* I *if and only if every factor representation is type* I. *If in addition* A *is separable, then* A *is type* I *if and only if* $A^\wedge$ *is a* T_0 *topological space.*

6. Multipliers and multiplier representations

Let G be a locally compact group with identity element e. A multiplier (or cocycle or factor set) on G is a Borel measurable function $\omega : G \times G \to \mathbb{T}$ (the group of complex numbers of modulus 1, with the induced topology) which satisfies

$$\omega(g,h)\omega(gh,k) = \omega(g,hk)\omega(h,k) \qquad g,h,k \in G,$$
$$\omega(e,g) = \omega(g,e) = 1 \qquad g \in G.$$

Two multipliers ω_1 and ω_2 are similar $\omega_1 \sim \omega_2$, if there exists a Borel measurable function $\gamma : G \to \mathbb{T}$ such that

$$\omega_1(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1}\,\omega_2(g,h) \qquad g,h \in G.$$

A multiplier which is similar to 1 (the constant function on $G \times G$) is said to be trivial. Every multiplier is similar to a normalized multiplier, that is one which satisfies the additional property

$$\omega(g,g^{-1}) = 1, \quad g \in G.$$

Indeed, if ω is an arbitrary multiplier, then the multiplier

$$g,h \to \omega(gh,(gh)^{-1})^{\frac{1}{2}}\ \omega(g,g^{-1})^{-\frac{1}{2}}\ \omega(h,h^{-1})^{-\frac{1}{2}}\ \omega(g,h)$$

is normalized. The square root here is taken in a Borel measurable fashion.

If ω is normalized, then using the cocycle identity, we have

$$\begin{aligned}\omega(g^{-1},h^{-1}) &= \omega(g^{-1},h^{-1})\,\omega(g^{-1}h^{-1},hg)\\ &= \omega(g^{-1},g)\,\omega(h^{-1},hg)\\ &= \omega(h^{-1},hg),\end{aligned}$$

and

$$\begin{aligned}\omega(h,g)\,\omega(g^{-1},h^{-1}) &= \omega(h,g)\,\omega(h^{-1},hg)\\ &= \omega(hh^{-1},g)\,\omega(h^{-1},h)\\ &= 1,\end{aligned}$$

that is $\omega(h,g)^{-1} = \omega(g^{-1},h^{-1})$, all $h,g \in G$.

Each normalized multiplier on G defines an extension G^{ω} of $\mathbb{T}$ by G. It is the set $\mathbb{T} \times G$ provided with the multiplication

$$(s,g)(t,h) = (st\omega(g,h),gh)$$

(and then because ω is normalized, $(s,g)^{-1} = (s^{-1},g^{-1})$) and a topology in which a basis of neighbourhoods of the identity is composed of the sets AA^{-1}, where A is a set of finite positive measure for the product of right Haar measures on $\mathbb{T}$ and G. This is the topology defined by Weil on groups with an invariant measure (Weil [51, Appendix 1]). It is easy to check that this topology induces on $\mathbb{T}$ (indentified with the central subgroup $\mathbb{T} \times \{e\}$ of G^{ω}) its original topology and makes G^{ω} into a topological group extension of $\mathbb{T}$ by G. Since both $\mathbb{T}$ and G are locally compact, so is G^{ω}. G^{ω} is uniquely determined, to within topological isomorphism, by the similarity class of ω. In fact if $\omega_1 \sim \omega_2$, say $\omega_1(h,g) = \gamma(h)\gamma(g)\gamma(hg)^{-1}\ \omega_2(h,g)$, then the map $\Phi : (t,g) \to (t\gamma(g),g)$ is an isomorphism of G^{ω_1} onto G^{ω_2}. By Fubini's theorem, Φ is measure preserving and it follows from the definition of the topology of these groups that Φ is bicontinuous.

Kleppner ([26, Lemma 2]) has given the following alternative definition of the topology on G^{ω}.

PROPOSITION 6.1, ([26, Lemma 2]). *The sets* $(U \times F)(U \times F)^{-1}$, *where* U *runs through a basis of neighbourhoods of* 1 *in* $\mathbb{T}$ *and* F *through the sets of positive measure in* G, *form a basis for the neighbourhoods of* (1,e) *in* G^{ω}.

Let H be a closed subgroup of G. Then ω restricted to $H \times H$ is a multiplier on H which we also denote by ω, and H^{ω}

is algebraically isomorphic to a subgroup of G^{ω}. Moreover, we have as an immediate consequence of Proposition 6.1, the following result.

PROPOSITION 6.2 ([3, Lemma 1.1]). *Let* H *be a closed subgroup of* G. *Then the inclusion map* $H^{\omega} \to G^{\omega}$ *is a bicontinuous isomorphism of* H^{ω} *with a closed subgroup of* G^{ω}.

Let G be a locally compact group and ω a Borel multiplier on G. The twisted L^1-algebra $L^1(G,\omega)$ is the space $L^1(G)$ of complex-valued integrable functions with multiplication defined by

$$f * h(x) = \int_G f(g)h(g^{-1}x)\ (g,g^{-1}x)dg$$

and with a *-operation defined by

$$f^*(x) = f(x^{-1})^{-}\ \Delta(x^{-1})$$

where $^{-}$ denotes complex conjugation and Δ is the modular function. One verifies at once that $L^1(G,\omega)$ is a Banach *-algebra possessing an approximate identity, which is determined up to isomorphic *-isomorphism by the equivalence class of ω. In general $L^1(G,\omega)$ is not a C*-algebra ([7, 13.3.6]). However Theorem 5.2 applies. The C*-completion of $L^1(G,\omega)$ is called the twisted group C*-algebra of (G,ω) and is denoted by $C^*(G,\omega)$. If ω is identically 1, then we delete all mention of it so that $C^*(G,\omega)$ becomes $C^*(G)$ which is called the group C*-algebra of G.

Let G be a locally compact group and ω a Borel multiplier on G. The construction of $C^*(G,\omega)$ is useful because its representations correspond to multiplier representations of G. A multiplier representation (or ω-representation) in a Hilbert space H_π is a map of G into the space of unitary operators $U(H_\pi)$ such that

$$g \to \pi(g)\xi \text{ is measurable, } \xi \in H_\pi,$$

$$\pi(g)\pi(h) = \omega(g,yh)\pi(gh), \quad g,h \in G.$$

The concepts of 'dimension', 'equivalence', 'direct sum', '$V(\pi)$', 'irreducibility', and 'factor' that we defined for representations of algebras apply to this situation equally as well, they are independent of the object being represented. For details see Gaal [15, IV.1, page 145]. We let the reader make the obvious definitions. However if π_i is an ω_i-representation of G (i = 1, 2), then $\pi_1 \otimes \pi_2$ (defined in a manner similar to that used for representations of algebras) is an $\omega_1\omega_2$-representation, where $\omega_1\omega_2$ is the multiplier $(g,h) \to \omega_1(g,h)\omega_2(g,h), g,h \in G$. The set of equivalence classes of irreducible ω-representations is denoted by $(G,\omega)\hat{}$. An ω-representation where the multiplier ω is identically 1 is called an ordinary representation or simply a representation of G.

Mackey has shown that $\pi \to \pi^0$ where

$$\pi^0(t,g) = t^n\pi(x), \qquad (t,g) \in G^\omega, \ n \in \mathbb{Z}$$

is a bijection between the set of equivalence classes of ω^n-representations of G and the set of classes of (ordinary) representations π^0 of G^ω such that $\pi^0(t,g) = t^n\pi^0(1,g)$, all $(t,g) \in G^\omega$ (see [26, Corollary to Theorem 1]).

The following theorem establishes the desired connection.

THEOREM 6.3, ([7, 13.3.5]). *Let* G *be a locally compact group and* ω *a Borel multiplier on* G. *For each* ω*-representation* π *of* G, *put*

$$\pi'(f) = \int_G \pi(g)f(g)dg \ ,$$

$f \in L^1(G,\omega)$, *then* $' : \pi \to \pi'$ *is a bijection between* ω*-representations of* G *and representations of the Banach* *-*algebra* $L^1(G,\omega)$.

This together with Theorem 5.2 sets up a one-to-one correspondence between ω-representations of G and the representations of $C^*(G,\omega)$. As the notation already suggests, this correspondence preserves dimension, irreducibility, equivalence, $V(\pi)$, direct sum and primaryness etc. We use it to transfer the topology on $C^*(G,\omega)\hat{}$ to $(G,\omega)\hat{}$. One can show that for G abelian and ω trivial, this topology on $(G,\omega)\hat{}$ coincides with that given for abelian groups in Section 1. Note that the above remark together with Theorem 5.3 also shows that there exist irreducible ω-representations of G for every multiplier ω. We say that the pair (G,ω) is type I (or G is ω-type I) if $C^*(G,\omega)$ is type I.

7. Moore groups and the regular representation

Let G be a locally compact group. We adopt the following notation consistently throughout. Let H be a subgroup of G, then [G:H] denotes the index of H in G, H^- the closure of H and H' the commutator subgroup of H - the subgroup of G generated by the elements $\{ghg^{-1}h^{-1} : g,h \in G\}$. If dim π is finite for all $\pi \in G^\wedge$, then G is called a Moore group (after C.C. Moore who characterized these groups in [31]). For any locally compact group G, the von Neumann kernel is the closed normal subgroup G_0 of G given by

$$G_0 = \cap\{\ker \pi : \pi \in G^\wedge \text{ and } \dim \pi < \infty\} .$$

THEOREM 7.1. *Let* G *be a locally compact group and* H *a closed subgroup of finite index in* G, *then* $H_0 = G_0$.

PROOF. Suppose $x \in H_0$ and π is a finite dimensional representation of G, then π restricted to H is a finite dimensional representation of H, so $\pi(x) = I$. It follows that $H_0 \subseteq G_0$. Since H/H_0 is maximally almost periodic and $[G/H_0 : H/H_0]$ is finite, by Heyer [19, Satz 7.2.2], G/H_0 is also maximally almost periodic, thus $H_0 = \cap\{\ker \pi : \pi$ is a finite dimensional representation of G lifted from $G/H_0\} \supseteq G_0$.

Let G be a locally compact group. We denote by G_{FC} the topological finite class group of G, that is the closed subgroup of G consisting of the elements x in G such that the set $\{g^{-1}xg : g \in G\}$ has compact closure. If $G = G_{FC}$, then G is called a (topological) FC group.

These concepts alone allow for a characterization of Moore groups.

THEOREM 7.2, (Robertson [38, Theorem 1]). *Let* G *be a locally compact group. Then* G *is a Moore group if and only if* G *satisfies the following properties.*

(i) $[G : G_{FC}] < \infty$,

(ii) $(G_{FC})'^{-}$ *is compact, and*

(iii) $G_0 = \{e\}$.

This theorem is due to Robertson. Other proofs can be found in Kaniuth [23, page 233] and Poguntke [35, Satz 3.4]. The proof is far from trivial and we omit it.

The statement of 7.2 can be changed slightly, without much difficulty, to give the following.

PROPOSITION 7.3 ([38]). *The locally compact group* G *is Moore if and only if there exists a closed subgroup* K *of* G *such that* $[G:K] < \infty$, K'^{-} *is compact and* $K_0 = \{e\}$.

Kaniuth ([23]) has shown that an SIN group, that is a locally compact group which has basis of neighbourhoods at the identity consisting of sets which are invariant under the

G-action of conjugation, is a Moore group if and only if it is type I.

Moore groups can also be characterized using the regular representation. Let G be a locally compact group. The left regular representation ρ of G in $L^2(G)$ is defined by

$$(\rho(g)f)(x) = f(g^{-1}x) ,$$

$x,g \in G$, $f \in L^2(G)$. It is clear the ρ is indeed a representation. Similarly we define the right regular representation λ

$$(\lambda(g)f)(x) = f(xg)\Delta(x)^{\frac{1}{2}} ,$$

$g,x \in G$, $f \in L^2(G)$, where in this context, Δ denotes the modular function. Denote by V(G) the von Neumann algebra $V(\rho)$ generated by the operators $\rho(g)$, $g \in G$, and by V'(G) the von Neumann algebra generated by the operators $\lambda(g)$, $g \in G$.

The following theorem relating V(G) to V'(G) is much deeper than its statement indicates.

<u>THEOREM 7.4</u>, ([46, Theorem 3]). *Let* G *be a locally compact group. Adopt the above notation, then* $V'(G) = V(G)'$ *(or equivalently* $V'(G)' = V(G)$*).*

<u>PROPOSITION 7.5</u>, (Taylor [48, Proposition 4.1]). *Let* G *be a locally compact group. Then* V(G) *is a finite von Neumann*

algebra if and only if G *is a* SIN *group.*

Thus if G is a SIN group which is type I, then V(G) is type I_f (that is both type I and finite. See Section 3.) Indeed the following is true.

THEOREM 7.6, (Kaniuth [23, Satz 3]). *A locally compact group* G *is a Moore group if and only if* V(G) *is type* I_f.

This was first proved by Kaniuth in [23]. An alternative proof appears in Taylor [48, Corollary 1 to Theorem 4]. We can combine all these results in one single statement.

THEOREM 7.7, ([23, 38]). *Let* G *be a locally compact group. The following statements are equivalent.*

(i) G *is a Moore group.*

(ii) V(G) *is type* I_f.

(iii) The following properties are satisfied.

$[G:G_{FC}] < \infty$

$(\Delta_G)'^-$ *is compact, and*

$G_0 = \{e\}$.

Let G be a locally compact group and e the maximal type I_f projection in V(G). We will be interested in conditions on G such that $e \neq 0$. Kaniuth has given such conditions.

THEOREM 7.8, (Kaniuth [23, Satz 2]). *The maximal type* I_f

central projection in V(G) *is non-zero if and only if the following conditions hold.*

(i) $[G:G_{FC}] < \infty$

(ii) $(G_{FC})'^{-}$ *is compact.*

An alternative proof of this theorem appears in Taylor [48]; he also proved the following.

THEOREM 7.9, (Taylor [48, Theorem 4 and Corollary 4]). *Suppose the maximal type* I_f *central projection* e *in* V(G) *is non-zero, then the von Neumann kernel* G_0 *is compact and* eV(G) *is spatially isomorphic to* $V(G/G_0)$.

As we pointed out earlier, for SIN groups, V(G) is type I if and only if it is type I_f. In particular this is true of discrete groups. Both Kaniuth and Thoma have given characterizations of type I discrete groups. We state both of these in the following result.

THEOREM 7.10, (Thoma [49], Kaniuth [22], Smith [45]). *Let* G *be a discrete group, then the following are equivalent.*

(i) G *is type* I.

(ii) V(G) *is type* I *(or equivalently type* I_f*).*

(iii) *The centre of* G_{FC}*, that is* $\{g \in G : gh = hg$ *all* $h \in G_{FC}\}$ *has finite index in* G.

(iv) G *is a Moore group.*

The equivalence of (i) and (iii) is due to Thoma and that between (ii) and (iii) is due to Kaniuth. An alternative proof of '(ii) is equivalent to (iii)' is given by Smith [45]. The main difficulty of this theorem lies in constructing an abelian subgroup of finite index in G. Different ways of doing this (using different but equivalent hypotheses) can be found in Isaacs and Passman [21] and Schlichting [42].

Much of this thesis consists of results generalizing the theorems of this section to locally compact groups with non-trivial multipliers, and we do this mostly using the methods developed in Smith [45] and Taylor [48].

8. Induced representations, Mackey's construction

Let G be a separable locally compact group, K a closed subgroup, ω a Borel multiplier on G and π an ω-representation of K. The induced representation $\pi\uparrow_K^G$ is an ω-representation of G defined on a Hilbert space H. Following Auslander [2], we let α be the measurable function $\alpha(g,x) = d(g.\mu)/d\mu$, $g \in G$, $x \in G/K$, (for more details see also Mackay [27] and Blattner [4]). Define H to be the space of measurable functions f from G to H_π (that is, $g \to \langle f(g),\xi\rangle$ is measurable for each $\xi \in H_\pi$) such that $f(gk) = \pi(k)^{-1}f(g)$ and $|f| \in L^2(G/K,\mu)$. The last condition needs a note of explanation; observe that $|f(gk)| = |f(g)|$ since π is unitary, so that $|f|$ is really a scalar function on G/K. Then define

$$(\pi\uparrow_K^G(g)f)(s) = f(g^{-1}s)\alpha(g,h(s))^{\frac{1}{2}}\omega(g^{-1},s)^{-1} ,$$

where h is the canonical map from G to G/K. One verifies that $\pi\uparrow_K^G$ is a unitary operator and that $g \to \pi\uparrow_K^G(g)$ is an ω-representation of G. One also verifies that $\pi\uparrow_K^G$, up to unitary equivalence does not depend on the choice of μ as the notation already suggests. The notion of induced representation is compatible with taking central extensions, as indicated in the following result. If π is an ω-representation of a group G, denote by π^0 the corresponding (ordinary) representation of the group extension G^ω.

PROPOSITION 8.1, ([3, Lemma 1.2]). *Let* K *be a closed subgroup of a separable locally compact group* G *with normalized Borel multiplier* ω. π *is an* ω*-representation of* K, *then* $(\pi\uparrow_H^G)^0$ *and* $\pi^0\uparrow_{H^\omega}^{G^\omega}$ *are equivalent.*

PROOF. It is quickly checked that the map $f \to f'$, where $f'(x) = f(1,x)$, for all f in the Hilbert space of $\pi^0\uparrow_{H^\omega}^{G^\omega}$, sets up the desired equivalence.

THEOREM 8.2, (Mackey [27, Theorem 4.1]). *Let* H *and* K *be closed subgroups of a separable locally compact group* G *with Borel multiplier* ω, *such that* $H \subseteq K$. *If* π *is an* ω*-representation of* H, *then* $\pi\uparrow_H^G$ *is equivalent to* $(\pi\uparrow_H^K)\uparrow_K^G$.

THEOREM 8.3, (Mackey [27, Theorem 10.1]). *Let* G *be a separable locally compact group with Borel multiplier* ω. *If* H *is a closed subgroup and* π_j, $j \in J$ *a collection of* ω*-representations of* H, *then* $(\oplus_{j\in J}\pi_j)\uparrow_H^G$ *is equivalent to* $\oplus_{j\in J}(\pi_j\uparrow_H^G)$.

Let G be a separable locally compact group with multiplier ω, and H a closed normal subgroup such that (H,ω) is type I (see Section 6). If $\pi \in (H,\omega)^\wedge$ and $g \in G$, then π^g will be the ω-representation of H defined by

$$\pi^g(h) = \pi(g^{-1}hg)\omega^{-1}(h,g)\omega(g,g^{-1}hg), \qquad h \in H.$$

It is easy to check that the action $g \to (\pi \to \pi^g)$ of G on $(H,\omega)^\wedge$ satisfies $\pi^{gh} = (\pi^h)^g$, all $h,g \in G$. The set $\{g \in G : \pi^g$ is equivalent to $\pi\}$ is a closed subgroup of G called the stabilizer of π and is denoted by K_π. An orbit is a subset of $(H,\omega)^\wedge$ of the form $\{\pi^g : g \in G\}$ for some irreducible ω-representation π of H.

The dual $(H,\omega)^\wedge$ has a natural Borel structure as defined in Auslander [2]. Fell [11, page 95] has observed that this Borel structure is just that generated by the topology of $(H,\omega)^\wedge$.

Now each primary ω-representation π of G determines a projection valued measure μ_π on $(H,\omega)^\wedge$ which is unique up to equivalence (two measures being equivalent if they have the same null sets), whose values are projections on the Hilbert space of the representation π, and which is a quasi-orbit in the sense that

(i) $\mu_\pi(g.A) = \mu_\pi(A)$ for all Borel sets A in $(H,\omega)^\wedge$ and $g \in G$, and

(ii) if A is an invariant Borel set (that is $g.A = A$

for all $g \in G$) then either A or its complement has μ_π-measure zero,

(that is, μ_π is G-invariant and ergodic). For details see Mackey [28, 30] and Auslander [2]. The measure μ_π is said to be transitive if it is concentrated on an orbit θ (that is $\mu_\pi((H,\omega)\hat{} - \theta) = 0$).

THEOREM 8.4, (Effros [8, Theorem 2.6]). *Let* G *be a separable locally compact group and* ω *a Borel multiplier on* G. *Let* H *be a closed normal subgroup such that* (H,ω) *is type* I. *If the* G*-orbits in* $(H,\omega)\hat{}$ *are locally closed (a set is locally closed if it is the intersection of an open and closed set), then* μ_χ *is transitive for each primary* ω*-representation* χ *of* G. *Conversely if* $(H,\omega)\hat{}$ *has no non-transitive ergodic measures, then the* G *orbits in* $(H,\omega)\hat{}$ *are locally closed.*

THEOREM 8.5, (Mackey [28, Theorem 8.1]). *Let* G, ω *and* H *be as in* Theorem 8.4. *Let* θ *be an orbit in* $(H,\omega)\hat{}$ *and* π *an element in* θ *with stabilizer* K_π. *Then the mapping* $\lambda \to \lambda\uparrow_{K_\pi}^{G}$ *sets up a one-to-one correspondence between the primary* ω*-representations of* K_π *whose restriction to* H *is a multiple of* π, *and the primary* ω*-representations* χ *of* G *such that* μ_χ *is concentrated on* θ. *Furthermore, the two von Neumann algebras* $V(\lambda)$ *and* $V(\lambda\uparrow_{K_\pi}^{G})$ *are *-isomorphic.*

THEOREM 8.6, (Mackey [28, Theorem 8.2]). *Let* G, ω *and* H *be as in* Theorem 8.4. *If* π *is an irreducible* ω*-representation* *of* H, *then there exists a multiplier* τ *of* K_π/H *and a* $\tau'\omega$-*representation* π' *of* K_π *where* τ' *is the lifting of* τ *to* K_π, *such that* $\pi'(h) = \pi(h)$ *all* $h \in H$.

THEOREM 8.7, (Mackey [28, Theorem 8.3]). *Let* G, ω, H, π, K_π, π' *and* τ *be as in* Theorem 8.6. *For each* τ^{-1}-*representation* λ *of* K_π/H, *denote by* λ' *the* $(\tau')^{-1}$-*representation of* K_π *obtained by composing* λ *with the canonical map* $K_\pi \to K_\pi/H$. *Then* $\lambda \to \lambda' \otimes \pi'$ *sets up a one-to-one correspondence (equivalent representations being identified) between the set of primary* τ^{-1}-*representations of* K_π/H *and the set of primary* ω-*representations of* K_π *which reduce on* H *to a multiple of* π. *Furthermore, the two von Neumann algebras* $V(\lambda)$ *and* $V(\lambda' \otimes \pi')$ *are *-isomorphic.*

We can summarize Theorems 8.4 to 8.7 as follows. Let G, ω, H, π, K_π, π' and τ be as in Theorem 8.6. Then

$$\lambda \to (\lambda' \otimes \pi')\uparrow_{K_\pi}^{G}$$

sets up a one-to-one correspondence between the primary τ^{-1}-representations of K_π/H and the primary ω-representations χ of G such that μ_χ is concentrated on the orbit containing π. The two von Neumann algebras $V(\lambda)$ and $V((\lambda' \otimes \pi')\uparrow_{K_\pi}^{G})$ are *-isomorphic. Furthermore, if the G-orbits in $(H,\omega)^\wedge$ are locally closed, then as π varies through $(H,\omega)^\wedge$, the above construction yields all the primary ω-representations of G.

CHAPTER II

ON THE STRUCTURE OF ω-TYPE I LOCALLY COMPACT ABELIAN GROUPS

Let ω be a normalized multiplier on a locally compact abelian group G. Baggett and Kleppner [3] have given a useful criterion for deciding when (G,ω) is type I. However not much is known about the structure of type I pairs (G,ω). In this chapter we investigate the structure of locally compact abelian groups that admit ω-type I multipliers. In particular, a complete structure theory for a certain class of such groups is given. (See Theorem 4.5, 5.11 and Corollary 5.12.)

1. Notation and elementary facts

Throughout this chapter, all groups are locally compact and abelian (this includes discrete groups), and all multipliers are normalized and Borel measurable, unless otherwise stated. We adopt the notation of Chapter I Section 1.

Let ω be a normalized multiplier on the locally compact abelian group G. Denote by $\tilde{\omega}$ the antisymmetrized form of ω, that is

$$\tilde{\omega}(k,g) = \omega(k,g)/\omega(g,k) \ ,$$

$k,g \in G$.

PROPOSITION 1.1, ([25, Proposition 1.1, Lemma 7.1 and Lemma 7.2]). *Let* G *be a locally compact abelian group and* ω *a normalized multiplier on* G. *We have*

(i) $\tilde{\omega} : G \times G \to \mathbb{T}$ *is a continuous bicharacter.*

(ii) $g \to \tilde{\omega}(\cdot,g)$ *is a continuous homomorphism from* G *to* $G^\wedge$.

(iii) $\tilde{\omega}(k,g) = 1$ *all* $k,g \in G$ *if and only if* ω *is trivial.*

The map $g \to \tilde{\omega}(\cdot,g)$ is also denoted by the symbol $\tilde{\omega}$. No confusion should arise from this ambiguity.

PROOF. (i) The bilinearity of $\tilde{\omega}$ follows from the equation

$$\begin{aligned}
\tilde{\omega}(gh,k) &= \omega(gh,k)\omega(k,gh)^{-1} \\
&= \omega(gh,k)\omega(g,h)\omega(g,h)^{-1}\omega(k,gh)^{-1} \\
&= \omega(g,hk)\omega(h,k)\omega(kg,h)^{-1}\omega(k,g)^{-1} \\
&= \omega(g,hk)\omega(h,k)\omega(g,k)\omega(g,k)^{-1}\omega(gk,h)^{-1}\omega(k,g)^{-1} \\
&= \omega(g,hk)\omega(h,k)\omega(g,k)\omega(g,kh)^{-1}\omega(k,h)^{-1}\omega(k,g)^{-1} \\
&= \tilde{\omega}(g,k)\tilde{\omega}(h,k),
\end{aligned}$$

$g,h,k \in G$. Clearly $\tilde{\omega}(k,g)$ is a measurable character in k for fixed $g \in G$. It follows that $\tilde{\omega}(k,g)$ is continuous in k for fixed g ([18, 22.19]). Similarly $\omega(k,g)$ is continuous in g for fixed k. By [26, Corollary to Lemma 1], $\tilde{\omega}$ is continuous as a function of two variables at the identity in $G \times G$. Using this, we show that $\tilde{\omega}$ is continuous. Let (g,h) be an arbitrary point of $G \times G$ and let V be a neighbourhood of 1 in $\mathbb{T}$. Let V_1 be a neighbourhood of 1 in T such that $V_1^3 \subseteq V$. Because $\tilde{\omega}$ is continuous at the identity in $G \times G$, there exists a neighbourhood U of the identity in G such that $\tilde{\omega}(U,U) \subseteq V_1$. The neighbourhood can be chosen so that $\tilde{\omega}(U,g) \subseteq V_1$ and $\tilde{\omega}(k,U) \subseteq V_1$. Now $\tilde{\omega}(kU,gU) = \tilde{\omega}(k,g)\tilde{\omega}(k,U)\tilde{\omega}(U,g)\tilde{\omega}(U,U)$ $\tilde{\omega}(k,g)V_1^3 \subseteq \tilde{\omega}(k,g)V$ and $\tilde{\omega}$ is continuous at (k,g).

(ii) Since $\tilde{\omega}(\cdot,g)$ is a measurable function of G, as in (i), it is continuous. That $\tilde{\omega} : G \to G\hat{}$ is a homomorphism follows from the bilinearity of $\tilde{\omega}$. For the continuity, we observe that the sets $P = \{X \in G\hat{} : X(C) \subseteq V\}$, where C is a compact subset of G and V a neighbourhood of the identity in $\mathbb{T}$, is a basis for the neighbourhoods at the identity in $G\hat{}$. Since $\tilde{\omega} : G \times G \to \mathbb{T}$ is continuous, if $g_0 \in W = \{g \in G : \tilde{\omega}(g) \in P\}$ is fixed and $k \in C$, there exists an open set $U \times U'$ containing (k,g_0) such that $\tilde{\omega}(U,U') \subseteq V$. But C being compact can be covered by a finite family $U_1, \ldots, U_n$ of such sets U, and if $U' = \cap_{i=1,\ldots,n} U_i'$ is the intersection of the corresponding U' sets, we have $\tilde{\omega}(C,U') \subseteq V$. Thus if $g_0 \in W$, there exists an open set U' such that $g_0 \in U' \subseteq W$, proving that W is open and $\tilde{\omega}$ continuous.

(iii) Let π be an irreducible ω-representation of G (such representations exist - see remark following Theorem I.6.3). If $\tilde{\omega} = 1$, then $\pi(g)\pi(h) = \pi(h)\pi(g)$ for all $g,h \in G$, thus by the remark following I.5.1, π is one dimensional and ω is trivial.

A normalized multiplier ω on a locally compact abelian group G is called non-degenerate (or totally skew) if $\tilde{\omega}$ is an injection. Given any subset S of G, we denote by $S\omega$ the subgroup $S\omega = \{g \in G : \omega(s,g) = 1 \text{ all } s \in S\}$. Since $S\omega$ is the intersection $\cap_{s\in S}\{S\}\omega$ of closed sets, it must be closed. S is called isotropic if $S\omega \supseteq S$ and maximal isotropic if $S\omega = S$. An application of Zorn's lemma (see Hannabus [17, 1.6]) ensures that each isotropic set is contained in a maximal isotropic subgroup. The operation on sets $S \to S\omega$ is inclusion reversing and $(\cup S_\alpha) = \cap(S_\alpha\omega)$.

For any $S' \subseteq G$,

$$\begin{aligned} S\omega &= \{g \in G : \tilde{\omega}(s,g) = 1 \text{ all } s \in S\} \\ &= \{g \in G : \tilde{\omega}(g) \in A[G\hat{\ },S]\} \\ &= (\tilde{\omega})^{-1}(A[G\hat{\ },S]), \end{aligned}$$

thus $\tilde{\omega}(S\omega) = A[G\hat{\ },S] \cap \tilde{\omega}(G)$ and

$$\begin{aligned} A[G,\tilde{\omega}(S)] &= \{g \in G : \chi(g) = 1 \text{ all } \chi \in \tilde{\omega}(S)\} \\ &= \{g \in G : \omega(s,g) = 1 \text{ all } s \in S\} \\ &= S\omega\ . \end{aligned}$$

The closure of $\tilde{\omega}(G)$ is a closed subgroup in $G\hat{\ }$, thus by I.1.1(ii),

$$\tilde{\omega}(G)^{-} = A[G\hat{\ },A[G,\tilde{\omega}(G)]] = A[G\hat{\ },G\omega].$$

Since $G\omega = \ker \tilde{\omega}$, if ω is non-degenerate (that is $G\omega = \{e\}$), then the range of $\tilde{\omega}$ is dense in $G^\wedge$.

PROPOSITION 1.2. *Let* ω *be a non-degenerate multiplier on* G. *Then* $\tilde{\omega}$ *is an open map if, and only if* $\tilde{\omega}$ *is a bicontinuous isomorphism. Moreover, if* G *is separable and the range of* $\tilde{\omega}$ *is closed, then* $\tilde{\omega}$ *is a bicontinuous isomorphism.*

PROOF. If $\tilde{\omega}$ is an open map, then $\tilde{\omega}(G)$ is open and closed, hence $\tilde{\omega}$ is onto. If G is separable and $\tilde{\omega}(G) = G^\wedge$, then by [18, 5.29], $\tilde{\omega}$ is open.

The condition that $\tilde{\omega}$ is a bicontinuous isomorphism eradicates a great deal of pathology and forces some order onto the structure of the group G (for instance it must be self dual). It is not surprising then that it is equivalent to ω being non-degenerate and type I.

THEOREM 1.3, (Baggett and Kleppner [3, Theorem 3.2]). *Let* ω *be a non-degenerate multiplier on the locally compact abelian group* G, *then* (G,ω) *is type* I *if, and only if* $\tilde{\omega} : G \to G^\wedge : g \to \tilde{\omega}(\cdot,g)$ *is a topological isomorphism.*

PROOF. We give an outline of the proof of this Theorem for G separable. For more details and for a proof in the case where G is not separable, see Hannabuss [17] and Baggett and Kleppner [3].

Let G be separable. By 1.3 it is enough to prove that (G,ω) is type I if and only if $\tilde{\omega}(G)$ is closed. Let H be a maximal isotropic subgroup of G. Since ω restricted to H is trivial (1.1 (iii)), $(H,\omega)^{\wedge}$ is isomorphic to the abelian group dual $H^{\wedge}$. Suppose $\tilde{\omega}$ is a topological isomorphism. Then the G-orbit of an element in $H^{\wedge}$ is of the form $\{\tilde{\omega}(\cdot,g) : g \in G\}$ which is closed by assumption. Thus by Theorem I.8.4, Mackey's construction with H as the closed normal subgroup gives all the factor representations of G. Because ω is non-degenerate, the stabilizer of the orbit $\{\tilde{\omega}(\cdot,g) : g \in G\}$ is H, thus the ω-representations obtained using Mackey's construction are all type I.

Conversely suppose (G,ω) is type I. Again by the non-degeneracy of ω, the stabilizer of each element in $H^{\wedge}$ is just H. In particular the action of G on $(H,\omega)^{\wedge}$ is essentially free, and under these conditions, the projective version of Auslander [2, II, Proposition 3.1] asserts that there are no ergodic measures on $(H,\omega)^{\wedge}$ which are not transitive. Thus by Theorem I.8.4, the G-orbits in $(H,\omega)^{\wedge}$ are locally closed. We deduce that the subgroup $\{\tilde{\omega}(\cdot,g) : g \in G\}$ of $G^{\wedge}$ is locally closed. But we know that its closure is $G^{\wedge}$, so $\{\tilde{\omega}(\cdot,g) : g \in G\}$ is an open subgroup of $G^{\wedge}$. Open subgroups are also closed, so it must be all of $G^{\wedge}$.

PROPOSITION 1.4. *Suppose* $G\omega \subseteq S$ *for some subset* S *of* G *and* $\tilde{\omega}(G)$ *is closed (in particular this is true if* ω *is non-degenerate and type* I*), then*

$$\tilde{\omega}(S\omega) = A[G^{\wedge},S], \text{ and}$$
$$(S\omega)\omega = K ,$$

where K *is the smallest closed subgroup containing* S.

PROOF. From our earlier remarks, $\tilde{\omega}(G) = A[G^\wedge, G\omega] \supseteq A[G^\wedge, S]$, so

$$\tilde{\omega}(S\omega) = A[G^\wedge,S] \cap A[G^\wedge,G\omega] = A[G^\wedge,S], \text{ and}$$
$$(S\omega)\omega = A[G,\tilde{\omega}(S\omega)] = A[G,A[G^\wedge,S]] = K.$$

We collect this result together with some other elementary facts as a proposition.

PROPOSITION 1.5. *Let* ω *be non-degenerate and type* I *on the locally compact and abelian group* G. *Suppose* K *and* L *are closed subgroups of* G *such that* $K \subseteq L$, *then*

(i) $(K\omega)\omega = K$, $\tilde{\omega}(K\omega) = A[G^\wedge,K]$ *and* $K\omega = A[G,\tilde{\omega}(K)]$.

(ii) K *is compact (respectively open) if, and only if* $K\omega$ *is open (respectively compact).*

(iii) $(L/K)^\wedge$ *is topologically isomorphic with* $K\omega/L\omega$.

(iv) *If* S_α *is a collection of subsets in* G, *then* $(\cap S_\alpha)\omega = (\cup(S_\alpha\omega))^-$.

PROOF. (i) follows from 1.4 and earlier remarks, and (ii) follows from (i), 1.3 and I.1.1.(iv). To see (iii), use I.1.1.(iii) and observe that

$$K\omega/L\omega \cong \tilde{\omega}(K\omega)/\tilde{\omega}(L\omega) \cong A[G^\wedge,K]/A[G^\wedge,L] \cong (L/K)^\wedge ,$$

where the symbol "$\cong$" denotes topological isomorphism.

2. Some groups don't admit multipliers

Denote by $\mathbb{T}$ the circle group - the group (under multiplication) of all complex numbers of modulus one (with the induced topology), by $\mathbb{R}$ the group of real numbers, by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rational numbers and by $\mathbb{Z}(n)$ the cyclic group of order n. For a fixed prime p, let $\mathbb{Z}(p^\infty)$ be the subgroup of $\mathbb{T}$ consisting of elements whose order is a power of p, Δ_p the group of p-adic integers and Ω_p the group of p-adic numbers (see Appendix).

LEMMA 2.1. *Suppose the abelian group* G *has the property that for all* $x \in G$, *there exists an* $h \in G$ *such that* $h^2 = x$. *If* ω *is a multiplier on* G *such that* $\tilde{\omega}(x,y) = \tilde{\omega}(y,x)$ *all* $x,y \in G$, *then* ω *is trivial.*

PROOF. $\tilde{\omega}(x,y) = \omega(x,y)/\omega(y,x) = \omega(y,x)/\omega(x,y) = \omega^2(y,x)$ implies $\tilde{\omega}(x,y)^2 = 1$ all $x,y \in G$. Given any $x,y \in G$, choose $h \in G$ such that $h^2 = x$, then $\tilde{\omega}(x,y) = \tilde{\omega}(h^2,y) = \tilde{\omega}^2(h,x) = 1$. That is ω is symmetric, thus trivial (1.1 (iii)).

LEMMA 2.2, (Kleppner [25, Lemma 7.5]). *Let* G *be a discrete group with multiplier* ω. *If* $\tilde{\omega}(x,y) = \tilde{\omega}(y,x)$ for *all* $x,y \in G$, *then* ω *is similar to a multiplier lifted from a quotient* G/H *which is of exponent* 2 *(that is each element of* G/H *has order at most* 2*).*

PROOF. The multiplier $\tilde{\omega}$ is symmetric, so by 1.1 (iii) there exists a function $\gamma : G \to \mathbb{T}$ such that $\tilde{\omega}(x,y) = \gamma(x)\gamma(y)\gamma(xy)^{-1}$. Similarly, there exists a function $\tau : G \to \mathbb{T}$ such that $\omega(x,y)\omega(y,x) = \tau(x)\tau(y)\tau(xy)^{-1}$ for all $x,y \in G$. Thus $\omega(x,y)^2 = \gamma\tau(x)\gamma\tau(y)\gamma\tau(xy)^{-1}$ for all $x,y \in G$. This implies ω is similar to a multiplier ω_1 such that $\omega_1^2 = 1$. In fact $\omega_1(x,y) = (\gamma\tau(xy)\gamma\tau(x)^{-1}\gamma\tau(y)^{-1})^{\frac{1}{2}}\omega(x,y)$, where the square root is chosen in any fashion. Let $H = G\omega_1$. Since $\tilde{\omega}_1(x^2,y) = \omega_1(x,y)^2 = 1$ all $x,y \in G$, we have $x^2 \in H$ all $x \in G$, that is G/H has exponent 2. Since $\omega_1|_{H\times H}$ is trivial and $\omega_1^2 = 1$, there exists $\tau_1 : H \to \mathbb{T}$ such that $\omega_1(x,y) = \tau_1(x)\tau_1(y)\tau_1(xy)^{-1}$ and τ_1^2 is a character of H. Let τ' be an extension of τ_1^2 to G ([18, 24.12]), and define τ_2 by $\tau_2(x) = \tau'(x)^{\frac{1}{2}}$ where the square root is chosen in any fashion with the restriction $\tau_1(x) = \tau_2(x)$ all $x \in H$. Let $\omega_2(x,y) = \omega_1(x,y)\tau_2(xy)\tau_2(x)^{-1}\tau_2(y)^{-1}$, then $\omega_2^2 = 1$, $\omega_2(x,y) = \omega_2(y,x)$ all $(x,y) \in H \times G$ and $\omega_2(x,y) = 1$ all $x,y \in H$.

Let π be an irreducible ω_2-representation of G and $\{g_\alpha\}$ a set of coset representatives modulo H containing the identity of G. Define π^0 as follows

$$\pi^0(x) = \pi(x), \text{ and}$$
$$\pi^0(xg_\alpha) = \pi(x)\pi(g_\alpha), \text{ for } x \in H.$$

Clearly π^0 is a multiplier representation whose associated multiplier ω_3 is similar to ω_2 (and consequently ω). Furthermore, for $x,y \in H$,

$$\pi^0(xyg_\alpha) = \pi(xy)\pi(g_\alpha)$$

$$= \pi(x)\pi(y)\pi(g_\alpha)$$
$$= \pi^0(x)\pi^0(yg_\alpha) ,$$

thus $\omega_3(x,y) = 1$ all $(x,y) \in H \times G$, and by the symmetry of ω_3, $\omega_3(y,x) = 1$, $(x,y) \in H \times G$. From this follows that ω_3 is constant on the $H \times H$ cosets in $G \times G$. Indeed, if $g,h \in H$, then $\omega_3(gx,hy) = \omega_3(g,x)\omega_3(gx,hy) = \omega_3(g,xhy)\omega_3(x,hy) = \omega_3(x,hy) = \omega_3(x,yh)\omega_3(y,h) = \omega_3(x,y)\omega_3(xy,h) = \omega_3(x,y)$. Thus there exists a multiplier ω' on G/H whose lifting to G is similar to ω. This proves the lemma.

Part of the following result can also be obtained from [6, Lemma 2].

THEOREM 2.3. *Let G be a discrete group which is either cyclic or of the form $\mathbb{Z}(p^\infty)$ or $\mathbb{Q}$. If ω is a multiplier on G, then ω is trivial.*

PROOF. Case 1. Suppose G is cyclic. If it is finite, say $G = \mathbb{Z}(n)$, then $\tilde{\omega}(\cdot,1)$, being a character of $\mathbb{Z}(n)$, is of the form $\tilde{\omega}(p,1) = \exp[2\pi ikp/n]$; it now follows from the bilinearity of $\tilde{\omega}$ that

$$\tilde{\omega}(p,q) = \exp[2\pi ikpq/n] ,$$

$p,q \in \mathbb{Z}(n)$, $k \in \mathbb{Z}$. If G is infinite, then G is isomorphic with $\mathbb{Z}$, and by the same reasoning as above,

$$\tilde{\omega}(p,q) = \exp[2\pi ipq\alpha] ,$$

$p,q \in \mathbb{Z}$ for some $\alpha \in [0,1[$. At any rate $\tilde{\omega}$ is symmetric. The only quotient of a cyclic group which is of exponent 2 is $\mathbb{Z}(2)$ or the trivial group. Since a multiplier ω' on $\mathbb{Z}(2)$ can be assumed to be normalized, we have $1 = \omega'(-1,1) = \omega'(1,-1) = \omega'(1,1)$, that is ω' is trivial. Hence by Lemma 2.2, ω is trivial.

Case 2. The dual of $\mathbb{Z}(p^{\infty})$ is the group of p-adic integers Δ_p (see Appendix) which is torsion free. Thus the homomorphism $\tilde{\omega} : \mathbb{Z}(p^{\infty}) \to \Delta_p$ must be trivial. Hence $\tilde{\omega} = 1$ and $\tilde{\omega}$ is trivial.

Case 3. $\mathbb{Q}^{\wedge}$ is torsion free and divisible (see Appendix). If $\chi \in \mathbb{Q}^{\wedge}$, then for each $n \in \mathbb{Z}$, $x \to \chi(x/n)$, $x \in \mathbb{Q}$ is a character of $\mathbb{Q}$ which we denote by $\chi^{1/n}$ and is the only character of $\mathbb{Q}$ satisfying $(\chi^{1/n})^n = \chi$ (use the fact that $\mathbb{Q}^{\wedge}$ is torsion free). Thus if the map $\tilde{\omega} : \mathbb{Q} \to \mathbb{Q}^{\wedge}$ maps 1 to χ, then $\tilde{\omega}(\cdot,1/n) = \chi^{1/n}$, that is

$$\tilde{\omega}(s/m,t/n) = \chi\left(\frac{st}{mn}\right) ;$$

this is symmetric, $\mathbb{Q}$ is divisible, so by Lemma 2.1, ω is trivial.

LEMMA 2.4. *Let ω be a Borel multiplier on a locally compact abelian group G. If H is a dense subgroup of G such that the restriction of ω to H is trivial, then ω is trivial.*

PROOF. $\tilde{\omega} : G \times G \to \mathbb{T}$ is continuous (1.1 (i)). The restriction $\tilde{\omega} : H \times H \to \mathbb{T}$ is the trivial map, thus $\tilde{\omega} = 1$ and ω is trivial (1.1 (iii)).

COROLLARY 2.5. *A Borel multiplier on any one of the following groups:* $\mathbb{T}$, $\mathbb{R}$, Δ_p *and* Ω_p; *is trivial.*

PROOF. $\mathbb{Z}(p^\infty)$ is dense in $\mathbb{T}$, $\mathbb{Q}$ is dense in $\mathbb{R}$ and Ω_p, and $\mathbb{Z}$ is dense in Δ_p (see Appendix). Now use Lemma 2.4.

We know from I.1.3.(iii) that the dual of a torsion free discrete group (for example $\mathbb{Q}^\wedge$) is connected. The following result shows that these groups don't admit non-trivial multipliers.

PROPOSITION 2.6. *Let* G *be a compact group. If* G *is connected (or equivalently divisible* (I.1.3.(iii))*) then a Borel multiplier* ω *on* G *is trivial.*

PROOF. The homomorphism $\tilde{\omega} : G \to G^\wedge$ must be trivial because G is connected and $G^\wedge$ discrete.

3. Some useful results

THEOREM 3.1. *Let* G *be a locally compact abelian subgroup which has a compact open subgroup* K. *Suppose* ω *is a non-degenerate multiplier, then* ω *is type* I *if, and only if* G *contains a compact open maximal isotopic subgroup. Moreover, any compact maximal isotropic* H *satisfies* $\tilde{\omega}(H) = A[G^\wedge, H]$ *and* G/H *is topologically isomorphic to* $H^\wedge$.

PROOF. Suppose ω is type I. By Proposition 1.5.(ii), $K\omega$ is compact and open, so $K \cap K\omega$ is a compact open isotropic subgroup. Without loss of generality, we assume $K\omega \supseteq K$. Let H be a compact maximal isotropic subgroup, then $\tilde{\omega}(H)$ (being the continuous image of a compact set) is compact and closed; $\tilde{\omega}(G)$ is dense in $G^\wedge$, thus $\tilde{\omega}(H) = \tilde{\omega}(H\omega) = A[G^\wedge,H] \cap \tilde{\omega}(G)$ $= (A[G^\wedge,H] \cap \tilde{\omega}(G))^- = A[G^\wedge,H] \cap \tilde{\omega}(G)^- = A[G^\wedge,H]$, the fourth equality being valid because $A[G^\wedge,H]$ is open and closed. It follows that $\tilde{\omega}$ restricted to H is a continuous isomorphism onto $A[G^\wedge,H]$ so it must be open [18, 5.29]. Let A be open in G and $A^g = A \cap g^{-1}H$, $g \in G$. The set A^g is open in H, thus the set $\tilde{\omega}(A) = \cup\{\tilde{\omega}(g)\tilde{\omega}(A^g) : g \in G\}$, being the union of open sets in $G^\wedge$, is open in $G^\wedge$. The proposition now follows from 1.3.

Let G be a locally compact abelian group for each positive integer n, denote by G^n the subgroup $\{g^n : g \in G\}$ and by G_n the closed subgroup $\{g \in G : g^n = e\}$.

THEOREM 3.2. *Let ω be a non-degenerate type* I *multiplier on the separable abelian group* G, *then* ω^n *is type* I *(*n *a fixed integer) if and only if* G^n *is closed in* G.

PROOF. First observe that the proof of 1.3 extends readily to assert that a multiplier ω (which is not necessarily non-degenerate) on a (separable) group G is type I if and only if the range of $\tilde{\omega} : G \to G^\wedge$ is closed. Let ω be as in the hypothesis of the Theorem. By 1.3, G and $G^\wedge$ are topologically isomorphic. The range of $\tilde{\omega}^n$ is $(\tilde{\omega}(G))^n = (G^\wedge)^n$, thus by the above remark, ω^n is type I if and only if G^n is closed in G.

<u>THEOREM 3.3.</u> *Let* G *be a separable abelian locally compact group and* ω *a multiplier on* G. *Then* G^{ω} *is type* I *if and only if* ω^{n} *is type* I *for all* $n \in \mathbb{Z}$.

<u>PROOF</u>. Observe that $\mathbb{T}$ is a type I normal subgroup of G. Since $\mathbb{T}^{\wedge}$ is discrete, G^{ω}-orbits are closed, so by I.8.4, all factor representations of G^{ω} are obtained using Mackey's construction (I.8.5, I.8.6, I.8.7). Indeed, because $\mathbb{T}$ is central, a factor representation π of G^{ω} reduces on $\mathbb{T}$ to $m.\chi$, where m is a cardinal and χ the character $t \to t^{n}$ $(t \in \mathbb{T})$ of $\mathbb{T}$ for some $n \in \mathbb{Z}$ (see [15, IV.7.20]). Now $m.\chi'$, where $\chi'(t,x) = \chi(t)$, $((t,x) \in G^{\omega})$ is a multiplier extension of $m.\chi$ to G^{ω}, and the multiplier associated with χ' is precisely ω^{-n}. Thus by I.8.7, each π which restricts on $\mathbb{T}$ to a multiple of χ is type I if and only if ω^{-n} is type I, and G^{ω} is type I if and only if ω^{n} is type I all $n \in \mathbb{Z}$.

Suppose G, ω and H are as in Theorem 3.1 and G is a group of exponent p for some prime p (that is each element in G has order at most p). Since G is a vector space over the field of p elements and H is a subspace it admits a complement, that is a subgroup K (isomorphic to $G/H \cong H^{\wedge}$) such that $G = H \times K$. Since H is open, we actually have a topological isomorphism between G and $H \times H^{\wedge}$.

Such decompositions of G where we insist that H is compact and open cannot always be found even if we allow the additional hypothesis that ω^n is type I for all $n \in \mathbb{Z}$ (or equivalently that G^ω is type I), as the following example shows.

EXAMPLE 3.4. (i). Let H be a locally compact abelian group and $G = H \times H^\wedge$. Then ω defined by

$$\omega((x,\lambda)(y,\chi)) = \lambda(y) \ .$$

$(x,\lambda),(y,\chi) \in G$ is a non-degenerate multiplier which according to 1.3 is type I. Moreover, if G is divisible and torsion free, then ω^n is non-degenerate and type I for each $n \in \mathbb{Z}$ (Theorem 3.2).

(ii). Suppose we let H in (i) be the group of p-adic numbers Ω_p, then $H^\wedge$ is also the group of p-adic numbers (see Appendix), that is $G = \Omega_p \times \Omega_p$, in particular G is divisible and torsion free, thus ω^n is non-degenerate and type I for each $n \in \mathbb{Z}$. By Theorem 3.1, G has a maximal isotropic compact open subgroup H. We show that no such H can be a direct summand of G. For otherwise, G must be isomorphic with $H \times H^\wedge$; but the dual of H, H being compact and totally disconnected, must be torsion (I.1.3.(ii)), contradicting the torsion freeness of G.

(iii). For each $i \in \mathbb{Z}$, let K_i be the group $\mathbb{Z}(4) = \langle C : C^4 = 1\rangle$ and K_i' the subgroup $\mathbb{Z}(2) = \langle C^2\rangle$ of K_i. Define H to be the subgroup of ΠK_i consisting of elements (a_i) such that $a_i \in K_i'$ for all but a finite number of indices i; and topologize H so that $\Pi K_i'$ (with the compact cartesian product topology) becomes a compact open subgroup.

Let ω be as in (i). Clearly, G^2 is not closed in G and by Theorem 3.2, ω^2 is not type I. Furthermore, Theorem 3.3 shows that although ω is type I, G^ω is not a type I group.

Given a locally compact abelian group G and a non-degenerate type I multiplier, we can however hope for the existence of a closed isotropic subgroup H of G such that G decomposes as a product $H \times H^\wedge$. Indeed the rest of this chapter is devoted to showing that such a subgroup exists under certain additional hypothesis on G, for example if G is divisible and separable, or if the connected component of G is open.

THEOREM 3.5. (Mackey [28, 9.6]). *Let* H *and* K *be two locally compact abelian groups and* ω *a multiplier* $H \times K$. *If* ω' *is the function on* $(H \times K) \times (H \times K)$ *defined by*

$$\omega'(hk,h'k') = \omega(h,h')\omega(k,k')\omega(k,h') ,$$

$(h,k)(h',k') \in H \times K$, *then* ω' *is a multiplier which is similar to* ω.

PROOF. For $(h,k),(h',k') \in H \times K$, we have

$$\omega(hk,h'k')\omega(h,k) = \omega(h,kh'k')\omega(k,h'k'), \text{ and}$$
$$\omega(h,h')\omega(hh',kk') = \omega(h,h'kk')\omega(h',kk'),$$

thus

$$\omega(hk,h'k') = \frac{\omega(k,h'k')\omega(h,h')\omega(hh',kk')}{\omega(h,k)\omega(h',kk')}$$

$$= \frac{\omega'(hk,h'k')\gamma(hh'kk')}{\gamma(hk)\gamma(h'k')} \cdot \frac{\omega(h',k')\omega(h',k)\omega(k,h'k')}{\omega(k,h')\omega(k,k')\omega(h',kk')},$$

where γ is the function on $H \times K$ defined by $\gamma(hk) = \omega(h,k)$

Now

$$\frac{\omega(h',k')\omega(h',k)\omega(k,h'k')}{\omega(k,h')\omega(k,k')\omega(h',kk')} = \frac{\omega(h',k)\omega(kh',k')\omega(k,h')}{\omega(k,h')\omega(h'k,k')\omega(h',k)} = 1$$

Hence the result.

COROLLARY 3.6. *Let* H *and* K *be locally compact and abelian. If* ω *is a non-degenerate multiplier on* $H \times K$ *such that* $K\omega = H$, *then* ω *is similar to the multiplier* ω' *defined by*

$$\omega'((h,k)(h',k')) = \omega(h,h')\omega(k,k').$$

PROOF. Since $K\omega = H$, $\tilde{\omega}(k,h) = 1$ all $(h,k) \in H \times K$. Hence the result follows by Theorem 3.5.

If the conditions of Corollary 3.6 are satisfied, then we say that ω splits relative to H and K.

THEOREM 3.7. *Let* ω *be a non-degenerate Borel multiplier on a locally comapct group* G. *Suppose* G *has a maximal isotropic subgroup* H *which is a topological direct summand, then* ω *is type* I *if and only if it is similar to a multiplier* ω_1 *of the form*

$$\omega_1(\psi(x,\lambda),\psi(y,\chi)) = \lambda(y),$$

$(x,\lambda),(y,\chi) \in H \times H^\wedge$, *for some topological isomorphism* $\psi : H \times H^\wedge \to G$.

PROOF. Suppose such an isomorphism ψ exists. Denote by $\psi^\wedge$ its dual isomorphism, then

$$\tilde{\omega}_0 = \psi^\wedge \circ \tilde{\omega}_1 \circ \psi$$

where ω_0 is the multiplier $\omega_0((x,\lambda),(y,\chi)) = \lambda(y)$, $(x,\lambda),(y,\chi) \in H \times H^\wedge$. Clearly $\tilde{\omega}_0$ is a topological isomorphism, thus so is $\tilde{\omega}$. By 1.3, ω_1 and ω are type I.

Conversely, suppose ω is type I, then G is topologically isomorphic with $H \times H^\wedge$ (see proof of 3.1), thus we assume without loss of generality that $G = H \times H^\wedge$. Define $\rho : G \to G^\wedge$ by $[\rho(x,\lambda)](y,\chi) = \chi(x)\lambda(y)^-$, then ρ is a bicontinuous isomorphism such that $\rho(H) = A[G^\wedge,H]$ and $\rho(H^\wedge) = A[G^\wedge,H^\wedge]$. It follows that the map $\rho^{-1} \circ \tilde{\omega} : G \to G$ is a bicontinuous automorphism such that $\rho^{-1} \circ \tilde{\omega}(H) = H$ and hence $\rho^{-1} \circ \tilde{\omega}(H^\wedge) = H^\wedge$; but $\tilde{\omega}(H^\wedge\omega) = A[G^\wedge,H^\wedge] = \rho(H^\wedge) = \tilde{\omega}(H^\wedge)$, thus $H^\wedge\omega = H^\wedge$. By [1, Theorem 1], there exists a bicontinuous automorphism ψ_1 (respectively ψ_2) of H (respectively $H^\wedge$) such that $\rho^{-1} \circ \tilde{\omega}(x,\lambda) = (\psi_1(x),\psi_2(x))$ for all $(x,\lambda) \in G$. Define $\psi : G \to G$ by $\psi(x,\lambda) = (\psi_1^{-1}(x),\lambda)$, then ψ is a bicontinuous automorphism.

Since H (respectively $H^\wedge$) is maximal isotropic, the restriction of ω to H (respectively $H^\wedge$) is trivial. Thus by Theorem 3.5, we can (and do) assume that ω is similar to a multiplier ω_1 defined by

$$\omega_1((x,\lambda),(y,\chi)) = \tilde{\omega}((1,\lambda),(y,1)),$$

$(x,\lambda),(y,\chi) \in G$. Clearly

$$\begin{aligned}\omega_1(\psi(x,\lambda),\psi(y,\chi)) &= \tilde{\omega}((1,\lambda)),(\psi_1^{-1}(y),1)) \\ &= [\tilde{\omega}(\psi_1^{-1}(y),\psi_2^{-1}(1))](1,\lambda) \\ &= [\rho(y,1)](1,\lambda) \\ &= \lambda(y).\end{aligned}$$

Thus whenever G decomposes as a product $H \times H^\wedge$ with H maximal isotropic, we know precisely the form which ω takes.

A multiplier ω satisfying the hypothesis of the above theorem is called a cross multiplier.

LEMMA 3.8. *Let* H, K *and* L *be locally compact and abelian and* ω *a non-degenerate type* I *multiplier on* $H \times K \times L$ *such that the map* $\tilde{\omega} : H \times K \times L \to H^\wedge \times K^\wedge \times L^\wedge$ *satisfies* $\tilde{\omega}(H) = K^\wedge$, *then*

$$\tilde{\omega}(K) = H^\wedge,\ \tilde{\omega}(L) = L^\wedge \text{ and } L\omega = K \times H.$$

PROOF. By 1.5, $K^\wedge = \tilde{\omega}(H) = \tilde{\omega}((H\omega)\omega) = A[G^\wedge,H\omega]$, so $H\omega = A[G,A[G^\wedge,H\omega]] = A[G,K^\wedge] = H \times L$ (use I.1.1.(ii)), thus $\tilde{\omega}(H \times L) = \tilde{\omega}(H\omega) = A[G^\wedge,H] = K^\wedge \times L^\wedge$, consequently $\tilde{\omega}(L) = L^\wedge$ and $\tilde{\omega}(K) = H^\wedge$. Finally $\omega(L\omega) = A[G^\wedge,L] = H^\wedge \times K^\wedge = \tilde{\omega}(H \times K)$, that is $L\omega = H \times K$.

The following theorem allows us to deal (amongst other things) with finite products of groups.

THEOREM 3.9. *Let* G *be a locally compact abelian group with a non-degenerate and type* I *multiplier* ω, *and suppose* H *is a closed isotropic subgroup which is a topological direct summand of* G. *Then* G *is topologically isomorphic with* $H \times H^\wedge \times H\omega/H$ *and* ω *is similar to a multiplier* ω' *of the form*

$$\omega'((h,\lambda,x)(h',\lambda',x')) = \tau((h,\lambda)(h',\lambda'))\sigma(x,x'),$$

where τ *is a cross multiplier on* $H \times H^\wedge$ *and* σ *is a non-degenerate and type* I *multiplier on* $H\omega/H$.

PROOF. All isomorphisms stated in this proof are topological isomorphisms. There exists a closed subgroup K of G such that $G = H \times K$; now $\tilde{\omega}(H\omega) = A[G^\wedge,H] = A[H^\wedge \times K^\wedge,H] = K^\wedge$ is a topological direct summand of $G^\wedge$. The map $\tilde{\omega}$ is an isomorphism, thus $H\omega$ is also a topological direct summand, thus there exists closed subgroups K, L of G such that $H \times L \cong H\omega$ and $G \cong H\omega \times K$. Collecting all this information together, we have

$$G \cong H \times K \times L$$
$$H\omega \cong H \times L$$
$$L \cong H\omega/H$$
$$K \cong G/H\omega \cong H^\wedge \qquad \text{(use 1.6 (iii))}.$$

This proves the first part of the theorem about the structure of G. Now $\omega(H) = A[G^\wedge, H\omega] = A[H^\wedge \times K^\wedge \times L^\wedge, H \times L] = K^\wedge$, thus by Lemma 3.8,

$$\tilde{\omega}(K) = H^\wedge,\ \tilde{\omega}(L) = L \text{ and } L\omega = K \times H.$$

It now follows from Corollary 3.6 that ω is similar to a multiplier $\tau\sigma$ where τ is a multiplier on $K \times H$ and σ is a multiplier on L. Since

$$\tilde{\omega}|_{H\times K} = \widetilde{\tau\sigma}|_{H\times K} = \tilde{\tau}|_{H\times K}$$

is a bicontinuous isomorphism from $H \times K$ to $H^\wedge \times K^\wedge$ such that $\tilde{\tau}(H) = K^\wedge$ and $\tilde{\tau}(K) = H$, by Theorem 3.7, τ is a cross multiplier on $H \times K \cong H \times H^\wedge$. Finally,

$$\tilde{\omega}|_L = \widetilde{\tau\sigma}|_L = \tilde{\sigma}|_L$$

is a bicontinuous isomorphism from L to $L^\wedge$, thus σ is type I and non-degenerate on $L \cong H\omega/H$.

<u>COROLLARY 3.10</u>. *Let ω be a non-degenerate multiplier on the finite abelian group G, then G is isomorphic to $H \times H^\wedge$ for some subgroup H of G and ω is similar to a cross multiplier.*

<u>PROOF</u>. Clearly ω is type I. Using the structure theory of abelian groups (see I.1.4.(iii)) G must be a finite product $G = \Pi_{i=1}^{n} H_i$ where each H_i is a cyclic group of prime power order. For each i, $(H_i)\omega = \{g \in G : \tilde{\omega}(h,g) = 1 \text{ all } h \in H_i\}$. By Theorem 2.3, $\omega|_{H_i\times H_i}$ is trivial, so $(H_i)\omega \supseteq H_i$. Now

Theorem 3.9 and induction yields the required result.

COROLLARY 3.11. *Let* ω *be a non-degenerate multiplier on* $\mathbb{R}^n$, *then* n *is even (*n = 2k*),* ω *is type* I *and is a cross multiplier on* $\mathbb{R}^k \times \mathbb{R}^k$.

PROOF. $\tilde{\omega}$ is a continuous homomorphism. Clearly $\tilde{\omega}(r/s.x) = r/s\tilde{\omega}(x)$ all $r,s \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and by continuity $\tilde{\omega}(t.x) = t.\tilde{\omega}(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. Thus $\tilde{\omega}$ is a vector space homomorphism; since $\tilde{\omega}$ is injective, the range $\tilde{\omega}(\mathbb{R}^n)$ is n-dimensional and so is all of $\mathbb{R}^n$. By 1.2 and 1.3, ω is type I. The rest follows from 3.9.

Recall that every locally compact group G can be written as a product $\mathbb{R}^n \times H$, where H is a locally compact abelian group containing a compact open subgroup. The integer n is an invariant of G (see I.1.2).

PROPOSITION 3.12. *Let* ω *be a non-degenerate and type* I *multiplier on* $\mathbb{R}^n \times H$ *where* H *is a locally compact abelian group, then* ω *is similar to* $\tau\sigma$ *where* τ *is a cross multiplier on* $\mathbb{R}^n$ *and* σ *is a non-degenerate and type* I *multiplier on* H.

PROOF. By Corollary 2.5, $\mathbb{R}$ is an isotropic subgroup. It is a topological direct summand, thus by Theorem 3.9 and induction, ω is of the desired form.

4. The connected component of G as a direct summand

We saw in Section 3 that for our purposes, we can disregard direct factors of $\mathbb{R}$ in G, so we can and do assume that G has a compact open subgroup H. For the whole of this section, we suppose that G admits a non-degenerate type I multiplier ω. Let C be the connected component of G, then C must be a closed subgroup of H and is thus compact. We point out at once that C is divisible (I.1.3.(iii)); divisible subgroups of discrete abelian groups are always direct summands (I.1.4.(i)); furthermore, if C is a direct product of circle groups then C is also a topological direct summand ([18, 25.31a]). Conversely, Fulp and Griffith [13, Corollary 3.2] have shown that a connected group C which is a topological direct sum in every locally compact abelian group (with compact open subgroup) in which it occurs as the connected component of the identity must necessarily be a product of circle groups. In particular, there exists a locally compact group G whose identity component is not a topological direct summand of G. It is not clear if the same is true if G is self-dual. Below we provide some conditions on G which ensures that the connected component of the identity is a topological direct summand.

LEMMA 4.1. *Suppose we have abelian groups* $H \subseteq K \subseteq G$ *(no topology) such that* H *is divisible and* K/H *is a direct summand of* G/H, *then* K *is a direct summand of* G.

PROOF. From the hypothesis and I.1.4.(i), there exist subgroups F and M in G such that $K = H \times F$ and $G = H \times M$. Furthermore, M can be chosen to contain F. Since F is a direct summand of M, there exists a subgroup L such that $M = L \times F$. It follows that $G = H \times F \times L$ and K is a direct summand of G.

Let G be a locally compact group with compact open subgroup H, connected component C and non-degenerate type I multiplier ω. Since C_ω is open in G (1.5 (ii)), C_ω is a direct summand of G in the discrete sense if and only if it is so in the topological sense.

LEMMA 4.2. *Let G, ω and C be as above. Then C is a topological direct summand of G if and only if* C_ω *is a direct summand of* G.

PROOF. Suppose G is topologically isomorphic to $C \times K$, then $G\hat{} = C\hat{} \times K\hat{}$ and (1.5 (i)), $\tilde{\omega}(C_\omega) = A[G\hat{},C] = K\hat{}$ is a topological direct summand. The "only if" part now follows from the fact that $\tilde{\omega}$ is a topological isomorphism. The "if" part is similar.

PROPOSITION 4.3. *Let G, ω and C be as above. If any of the following properties are satisfied,*

(i) G/C *is torsion free,*

(ii) C_ω/C *is divisible,*

(iii) C_ω/C *has bounded order,*

(iv) $C\omega/C$ *is compact or* C *is open,*

(v) G *is torsion free,*

then C *is a topological direct summand of* G.

PROOF. (i). See Fulp [14, Corollary 9].

(ii). Because $C\omega/C$ is divisible, it must be a direct summand of G/C (I.1.4.(i)). Now C is divisible, so by 4.1, $C\omega$ is a direct summand of G and so by 4.2, C is a topological direct summand of G. (An alternative proof can be constructed using (i).)

(iii). Since $(G/C)/(C\omega/C) \cong G/C\omega$ which is topologically isomorphic to $C^\wedge$, is a torsion free group (I.1.3.(iii)), by I.1.4.(ii), $C\omega/C$ is a direct summand of G/C. Now proceed as in (ii).

(iv). Since $C\omega/C$ is self dual (1.5 (iii)), $C\omega/C$ compact or discrete implies $C\omega/C$ is finite and thus of bounded order. Now use (iii).

(v). Since C is divisible it is a direct summand, thus G torsion free implies G/C is torsion free. Now use (i).

<u>THEOREM 4.4</u>. *Let* G *be a locally compact abelian group and* ω *a non-degenerate and type* I *multiplier, then* G *is of the form* $\mathbb{R}^n \times K$, *where* K *contains a compact open subgroup* H (I.1.2). *Let* C *denote the connected component of* K *and* ω_1 *the restriction of* ω *to* K. *If either*

(i) $C\omega_1/C$ *is of exponent* p *(*p *- a fixed prime), or*

(ii) $C\omega_1/C$ *is compact or* C *is open,*

then G *has a maximal isotropic subgroup which is a topological direct summand. Consequently the structure of* G *and* ω *is completely determined by* Theorem 3.7.

PROOF. By Proposition 3.12, we may assume that n = 0. Since C is compact and connected, Proposition 2.6 shows that C is isotropic, thus by 3.9 and 4.3, we may assume $G = C\omega_1/C$. (i). As in 4.3 (iii), G must be finite. The result now follows from Corollary 3.10. (ii). By Theorem 3.1, G has a maximal isotropic compact open subgroup L. Since G is a vector space over the field of p-elements and L is a subspace, it admits a complement, but L is open, thus it is a topological direct summand.

5. Local direct products and divisible groups

Given a locally compact abelian group G with non-degenerate and type I multiplier ω, observe that the group $C\omega/C$, where C is the connected component of the identity, is a totally disconnected group ([18, 7.3]). We can deal with some totally disconnected groups by decomposing them as local direct products of topological p-groups. A definition of local products is as follows. Let G_i, $i \in I$ be a collection of locally compact abelian groups each with a compact open subgroup H_i. The local direct product of the G_i with respect to the compact open subgroups H_i, denoted by $LP_{i\in I}(G_i,H_i)$ is the subgroup of the full (discrete) direct product $\Pi_{i\in I}G_i$ defined by

$$\{(g_i) \in \Pi G_i : g_i \in H_i \text{ for all but finitely many } i \in I\}$$

and is topologized so that the subgroup $\Pi_{i\in I}H_i$ (with compact cartesian product topology) becomes a compact open subgroup of $LP_{i\in I}(G_i,H_i)$. (see also [18, 6.16]).

Abelian topological p-groups (or simply p-groups, p a fixed prime) and local direct product decompositions of topological groups into p-groups are dealt with in detail by Braconnier [5] and to some extent by Vilenkin [50]. The following is a brief exposition of the facts we need later.

An element x of an abelian topological group is called p-primary if

$$\lim_{n\to\infty} x^{p^n} = 1 .$$

(Braconnier's alternative equivalent definition states that x is p-primary if the homomorphism $\mathbb{Z} \to G : n \to x^n$ extends to a continuous homomorphism $\Delta_p \to G$.) An abelian topological group consisting entirely of p-primary elements is called a p-group.

PROPOSITION 5.1, ([5]). *A discrete abelian group* G *is a* p-*group if and only if each element of* G *has order a power of* p.

PROOF. The result follows immediately from the definition.

LEMMA 5.2, ([5]). *A compact abelian group* G *is a* p-*group if and only if* G^ *is a* p-*group.*

PROOF. Suppose G is compact. Let $\chi \in G\hat{}$. For each $x \in G$, by the continuity of χ, $x^{p^n} \to 1$ implies $\chi^{p^n}(x) = \chi(x^{p^n}) \to 1$, hence $\chi(G)$ is a p-group; it is also a compact subgroup of $\mathbb{T}$ and hence finite. In particular $\chi^{p^n} = 1$ some $n \in \mathbb{Z}$. Conversely suppose G is a discrete p-group, let U be an open set in G^. By the definition of the topology on G^, there exists a finite set $F = \{x_1, \ldots, x_k\} \subseteq G$ and $\varepsilon > 0$ such that

$$\{\chi \in G\hat{} : |\chi(x) - 1| < \varepsilon \text{ all } x \in F\} \subseteq U.$$

Let n_i be the smallest integer such that $x_i^{p^{n_i}} = 1$ and let $n = \max\{n_i\}$, then $\chi^{p^n} = 1$ all $x \in F$, thus $\chi^{p^n} \in U$. It follows that $\chi^{p^n} \to 1$ as $n \to \infty$.

PROPOSITION 5.3, ([5]). *Let* H *be a closed subgroup of the* p-*group* G, *then* H *and* G/H *are also* p-*groups. Conversely, if* H *is an open* p-*group in* G *and* G/H *is a* p-*group, then* G *is a* p-*group.*

PROOF. The first assertion about closed subgroups is obvious. Let $\varphi : G \to G/H$ be the canonical homomorphism. φ is continuous, so for all $x \in G$, we infer from the discreteness of G/H and Proposition 5.1 that $x^{p^n} \in H$ for some n, so $(x^{p^n})^{p^m} = x^{p^{m+n}} \to 1$ as $m \to \infty$.

<u>THEOREM 5.4</u>, ([5]). *Suppose the locally compact abelian group* G *has a compact open subgroup* H, *then* G *is a* p-*group if and only if* G^ *is a* p-*group.*

<u>PROOF</u>. Since G^/A[G^,H] is topologically isomorphic to H^ which is a p-group (Lemma 5.2) and A[G^,H] is isomorphic to the dual of G/H which is also a p-group, the result follows from Proposition 5.3.

<u>LEMMA 5.5</u>. *Let* G *be a locally compact abelian group. If* $x \in G$ *is* p-*primary and* q-*primary for two distinct primes* p *and* q, *then* $x = 1$.

<u>PROOF</u>. Let $\chi \in G^\wedge$, then by the continuity of χ, $\chi(x)^{p^n} = \chi(x^{p^n}) \to 1$, thus $\chi(x)$ is a p-adic rational, that is $\chi(x)^{p^n} = 1$ some n. Similarly, $\chi(x)^{q^m} = 1$ some m, thus $\chi(x) = 1$. This is true for all $\chi \in G^\wedge$, so $x = 1$.

<u>PROPOSITION 5.6</u>, (Braconnier [5, page 48]). *Let* G *be a compact totally disconnected group. Denote by* P *the set of rpimes* {2, 3, ...}, *then there exist closed subgroups* H_p $(p \in P)$ *of* G *such that* H_p *is a* p-*group and*

$$G = \Pi_{p \in P} H_p \;.$$

Furthermore, an element $x \in G$ *belongs to* H_p *for a prime* p *if and only if* x *is* p-*primary.*

<u>PROOF</u>. By I.1.3.(ii) G^ is a torsion group, thus by

[18, A.3], $G^\wedge$ is a weak direct product of p-groups. Now apply duality and Lemma 5.2. For the second part, let φ_p denote the canonical projection map $G \to H_p$ and let x be q-primary (for a fixed prime q), then $\varphi_p(x)$ is both p-primary and q-primary, so by Lemma 5.5, $\varphi_p(x) = 1$ whenever $p \neq q$. It follows that $x \in H_p$.

THEOREM 5.7, (Braconnier [5, page 49] and Vilenkin [50, page 86]). *Suppose* G *is a locally compact abelian group with compact open subgroup* H *such that both* G *and* $G^\wedge$ *are totally disconnected. Denote by* P *the set of primes* {2, 3, ...}, *then there exist subgroups* G_p, H_p *of* G *such that* G_p *is a* p-*group,* H_p *is a compact open subgroup of* G_p *and*

$$G = LP_{p\in P}(G_p, H_p) \qquad H = \Pi_{p\in P} H_p.$$

Elements of G_p *are uniquely characterized as those* $x \in G$ *such that* $x^{p^n} \to 1$.

PROOF. For $x \in G\backslash H$, define H^x to be the group generated by $\{x\} \cup H$. The quotient G/H is torsion because it is topologically isomorphic with $A[G^\wedge,H]^\wedge$ - the dual of a compact totally disconnected group which must be torsion (I.1.3.(ii)); thus H^x, the union of a finite number of compact cosets, is compact. Suppose we have the primary decompositions

$$H = \Pi_{p\in P} H_p \quad \text{and} \quad H^x = \Pi_{p\in P} H_p^x$$

which we may obtain by appealing to Proposition 5.6, then by the latter part of that proposition the group $G_p = \cup_{x\in G} H_p^x$ is a p-group and $G_p \cap H = H_p$. Since G_pH/H is discrete, by [18, 5.32] G_p/H_p is discrete and H_p is open in G_p. Suppose $p \neq q$ and $x \in G_p \cap G_q$, then Lemma 5.5 shows that $x = 1$. Next, suppose $x \in G\backslash H$. Since $H^x/H = \Pi_{p\in P} H_p^x/H_p$ is a finite product (recall the index of H in H^x is finite), x is a finite sum

$$\sum_{p\in P} x_p + h ,$$

with $x_p \in G_p$ and $h \in H$. This says $G = LP_{p\in P}(G_p,H_p)$. Finally, the proof that $x \in G_p$ if and only if $x^{p^n} \to 1$ is identical to the proof of the corresponding fact in Proposition 5.6.

Having set up the necessary machinery, we proceed via a few preliminary results to the main theorem.

PROPOSITION 5.8. *Suppose* G *is totally disconnected and admits a non-degenerate and type* I *multiplier* ω. *Choose a compact open maximal isotropic subgroup of* G (3.1) *and let* $G = LP_{p\in P}(G_p,H_p)$ *be a primary decomposition as in* Theorem 5.7. *Then each* G_p, $p \in P$ *admits a non-degenerate and type* I *multiplier* ω_p *such that*

$$\tilde{\omega}((g_i),(h_i)) = \Pi_{p\in P}\, \tilde{\omega}_p(g_p,h_p)$$

for all $(g_i),(h_i) \in G$ *where the product above is always finite.*

PROOF. Fix a prime q. Observe that $G\hat{} = LP_{p\in P}$

$(G_p^\wedge, A[G_p^\wedge, H_p])$ is precisely the primary decomposition of $G^\wedge$ and the image $\omega(G_q)$ consists of those $\tilde{\omega}(x)$, $x \in G$ such that $\tilde{\omega}(x)^{q^n} \to 1$ as $n \to \infty$, thus $\tilde{\omega}(G_q) = G_q^\wedge$. An appeal to Corollary 3.6 yields the desired multiplier ω_q. It remains to verify the above product formula. Let $(g_i),(h_i) \in G$ and let Q be a finite subset of P such that $g_p \in H_p$, $h_p \in H_p$ for all p in the complement of Q. Then $G = L \times K$ where $L = \Pi_{p\in Q} G_p$ and $K = LP_{p\in P\setminus Q}(G_p, H_p)$, and we can argue as above to obtain non-degenerate and type I multipliers σ and τ on L and K respectively such that

$$\tilde{\sigma} = \Pi_{p\in Q} \tilde{\omega}_p \quad \text{and} \quad \tilde{\omega} = \tilde{\sigma}\tilde{\tau}.$$

However $\tilde{\tau}((g_i),(h_i)) = 1$ since $g_p, h_p \in H_p$ all $p \in P\setminus Q$, thus $\tilde{\omega} = \tilde{\sigma}$ and the result follows.

LEMMA 5.9. *Let be a non-degenerate multiplier on* $G = \Omega_p^n$, *then*

(*i*) $n = 2k$, G *is topologically isomorphic to* $K \times K^\wedge$ *where* K *is a closed subgroup of* G *of the form* Ω_p^k *and* ω *is cross multiplier on* $K \times K^\wedge$. (*In particular,* ω *is type* I.)

(*ii*) *If* H *is a compact open maximal isotropic subgroup of* G (*and such a group always exists by* Proposition 3.1), *then* H *is of the form* $L \times L'$ *for some compact subgroups* L *and* L' *of* K *and* $K^\wedge$ *respectively.*

PROOF. (i). Since G is a finite dimensional vector space over the p-adic field Ω_p, the proof of Corollary 3.11 applies to (i).

(ii). Let φ (respectively ψ) be the (continuous) projection map $G \to K$ (respectively $G \to K^\wedge$). First we show that $\varphi(H) \subseteq H$ and $\psi(H) \subseteq H$. Since $\varphi(H) \cap \psi(H) = \{1\}$, the direct sum $L = \varphi(H) \times \psi(H)$ is a compact totally disconnected group, so its dual must be torsion (I.1.3.(ii)). Now $\tilde{\omega}$ is injective and $\tilde{\omega}(L) = A[G^\wedge, L\omega]$ is topologically isomorphic to $G/L\omega$ and $L^\wedge$ (1.5), thus $\tilde{\omega}$ restricted to K must be trivial and L isotropic. In particular $\tilde{\omega}(k,\varphi(h)) = \tilde{\omega}(\varphi(k),\psi(h))$, $\tilde{\omega}(\psi(k),\varphi(h)) = 1$ for all $h,k \in H$, thus $\varphi(h) \in H\omega = H$ whenever $h \in H$. Similarly with ψ. It is now easy to check that $H = \varphi(H) \times \psi(H)$.

LEMMA 5.10. *Suppose we have a separable locally compact abelian* p*-group which is divisible. If* ω *is a non-degenerate and type* I *multiplier on* G, *then* G *is of the form* Ω_p^n *for some integer* n.

PROOF. Since G is self-dual, it must be torsion free I.1.3.(i), thus by Rajagopalan and Soundararajan [36, Lemma 12], G is a local product

$$G = LP_{i \in I}(\Omega_p, \Delta_p)_i .$$

For G to be divisible however, the index set I must be finite, thus G is of the form Ω_p^n.

THEOREM 5.11. *Let* G *be a locally compact abelian group with compact open subgroup and non-degenerate and type* I *multiplier* ω. *Let* C *denote the component of the identity in* G. *Suppose* $C\omega/C$ *is separable and divisible, then* G *is of the form* $H \times H\hat{}$ *and* ω *is similar to a cross multiplier.*

PROOF. According to 2.6, 3.9 and 4.3 (ii), we can assume without loss of generality that $G = C\omega/C$. Let H be a compact open maximal isotropic subgroup of G. Use 5.7 to write

$$G = LP_{p \in P}(G_p, H_p), \quad H = \Pi_{p \in P} H_p .$$

Now by Proposition 5.8, for each $p \in P$, there exists a non-degenerate and type I multiplier ω_p on G_p such that

$$\tilde{\omega} = \Pi_{p \in P} \tilde{\omega}_p .$$

Observe that the hypotheses of Lemma 5.10 apply to each G_p, thus by Lemma 5.9, we can write

$$G_p = K_p \times K_p', \quad H_p = L_p \times L_p' ,$$

where K_p, K_p' (respectively L_p and L_p') are closed subgroups of G_p (respectively H_p) and K_p is a maximal isotropic subgroup in G_p (relative to the multiplier ω_p). Let

$$K = LP_{p \in P}(K_p, L_p) \quad \text{and} \quad K' = LP_{p \in P}(K_p', L_p'),$$

then $G = K \times K'$ and

$$K\omega = \{(g_i) \in G : \omega((h_i),(g_i)) = 1 \text{ all } (h_i) \in K\}$$
$$= \{(g_i) \in G : \Pi_{p\in P}\, \omega_p(h_p,g_p) = 1 \text{ all } (h_i) \in K\},$$

but K_p is maximally isotropic in G_p for all $p \in P$, thus $K\omega = K$. Now apply Theorem 3.7.

<u>COROLLARY 5.12</u>. *Let* G *be a separable locally compact abelian group. If* G *is divisible and admits a non-degenerate and type* I *multiplier* ω, *then there exists a closed subgroup* H *of* G *and a topological isomorphism from* G *to* $H \times H^{\wedge}$, *such that the image of* ω *under this isomorphism is similar to a multiplier of the form*

$$\omega_1((x,\lambda)(y,\chi)) = \lambda(y) ,$$

$(x,\lambda),(y,\chi) \in H \times H^{\wedge}$.

<u>PROOF</u>. By I.1.2 and 3.12, we can assume without loss of generality that G has a compact open subgroup. Since G is self dual by I.1.3.(i) G is torsion free, thus by 4.3 (v), 2.6 and 3.9, it must be of the form $C \times C^{\wedge} \times C\omega/C$. Hence $C\omega/C$ is separable and divisible and the result follows from Theorem 5.11.

Finally, we remark that Theorem 3.9, Proposition 2.6 together with [1, Theorem 2] show that the problem of the structure of ω-type I locally compact abelian groups has been reduced to that of 'residual' groups (see [1, page 597]).

CHAPTER III[1]

MULTIPLIER REPRESENTATIONS OF DISCRETE GROUPS

Throughout this chapter, we fix a discrete group G and a normalized multiplier ω on G. Recall that $L^2(G)$ denotes the Hilbert space of square summable complex valued functions on G with scalar product $\langle \ , \ \rangle$, $B(L^2(G))$ the space of bounded linear operators on $L^2(G)$ and $U(L^2(G))$ the subspace of unitary operators in $B(L^2(G))$.

We denote by λ (respectively ρ) the right (respectively left) ω (respectively ω^{-1}) - representation of G given by $\rho,\lambda : G \to U(L^2(G))$, where

$$(\lambda(x)f)(g) = \omega(g,x)f(gx)$$
$$(\rho(x)f)(g) = \omega(x^{-1},g)f(x^{-1}g),$$

$f \in L(G), x,g \in G$. To make sense of this definition, we observe that

$$\begin{aligned}\lambda(x)[\lambda(y)f](g) &= \lambda(x)\omega(g,y)f(gy) \\ &= \omega(g,x)\omega(gx,y)f(gxy) \\ &= \omega(g,xy)\omega(x,y)f(gxy) \\ &= \omega(x,y)[\lambda(xy)f](g),\end{aligned}$$

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[1]The results contained in this chapter have appeared in [20].

all $f \in L(G)$, $g \in G$, hence $\lambda(x)\lambda(y) = \omega(x,y)\lambda(xy)$, and similarly $\omega(x,y)\rho(x)\rho(y) = \rho(xy)$.

Let $V(G,\omega)$ (respectively $V'(G,\omega)$) denote the Von Neumann algebra generated by ρ (respectively λ), that is the weak closure in $B(L^2(G))$ of the complex linear span of $\{\rho\ g) : g \in G\}$ (respectively $\{\lambda(g) : g \in G\}$). (See Chapter I, Section 2 and Section 7.)

The aim of this chapter is to investigate how various statements about the maximal central type I projection in $V(G,\omega)$ are reflected in the structure of the group G and the multiplier ω. This leads to a characterization of ω-type I discrete groups. The corresponding problem for ordinary representations, that is if we assume ω to be trivial, has been successfully dealt with in Thoma [49], Kaniuth [22], Smith [45], Formanek [10] and Schlichting [41].

Our methods resemble more closely those found in Smith; these are of a more elementary and algebraic nature than Thoma's and Kaniuth's "E(G)-Methoden".

1. A representation of elements in $V(G,\omega)$

To begin, we construct a way to represent elements of $V(G,\omega)$ as sequences in $L^2(G)$. For each $x \in G$, denote by φ_x the characteristic function of $\{x\} \subset G$. The set $\{\varphi_x : x \in G\}$ is an orthonormal basis for $L^2(G)$. We have

$$\rho(x)\varphi_e = \varphi_x = \lambda(x)^{-1}\varphi_e.$$

For $a \in B(L\ (G))$, let $a_{x,y} = \langle a(\varphi_y),\varphi_x\rangle$ and $a_x = a_{x,e}$, $x,y \in G$. The numbers $a_x, x \in G$ are called the coefficients of a.

<u>LEMMA 1.1</u>. (Kleppner [24, Lemma 1]). *Let G be a discrete group, ω a normalized multiplier on G, λ the right regular ω-representation of G and ρ the left regular ω^{-1}-representation of G. Suppose a ∈ $B(L^2(G))$,*

(i) $a\lambda(x) = \lambda(x)a$ *all* $x \in G$ *if and only if* $a_{x,y} = \omega(x,y^{-1})a_{xy^{-1}}$, *all* $x,y \in G$.

(ii) $a\rho(x) = \rho(x)a$ *all* $x \in G$ *if and only if* $a_{x,y} = \omega(y^{-1},x)a_{y^{-1}x}$ *all* $x,y \in G$.

<u>PROOF</u>. Suppose $a\lambda(x) = \lambda(x)a$ all $x \in G$, then

$$\begin{aligned} a_{x,y} &= \langle a(\varphi_y),\varphi_x\rangle \\ &= \langle a(\lambda(y^{-1})\varphi_e),\varphi_x\rangle \\ &= \langle a(\varphi_e),\lambda(y)\lambda(x^{-1})\varphi_e\rangle \\ &= \omega(x,y^{-1})a_{xy^{-1}}, \end{aligned}$$

all $x,y \in G$. Conversely if $a \in B(L^2(G))$ satisfies

$a_{x,y} = \omega(x,y^{-1})a_{xy^{-1}}$ all $x,y \in G$, then

$$\langle \lambda(y)a(\varphi_e),\varphi_x\rangle = \langle a(\varphi_e),\lambda(y^{-1})\lambda(x^{-1})\varphi_e\rangle$$
$$= \omega(x,y)a_{x,y} = a_{xy^{-1}}$$
$$= \langle a(\varphi_{y^{-1}}),\varphi_x\rangle$$
$$= \langle a\lambda(y)(\varphi_e),\varphi_x\rangle$$

for all $x,y \in G$, hence $a\lambda(y)(\varphi_e) = \lambda(y)a(\varphi_e)$ all $y \in G$. It follows that $a\lambda(y)(\varphi_h) = \omega(y,h^{-1})a\lambda(yh^{-1})(\varphi_e) = \lambda(y)\lambda(h^{-1})a(\varphi_e) = \lambda(y)a(\lambda_h)$, and hence that $\lambda(y)a = a\lambda(y)$ all $y \in G$. The second statement follows from a similar calculation.

PROPOSITION 1.2. *Let* G,ω,λ *and* ρ *be as in* Lemma 1.1. *If* $a \in B(L^2(G))$, *then*

(i) *If* $\lambda(x)a = a\lambda(x)$ *all* $x \in G$, *then*

$$a = \sum_{g\in G} a_g\rho(g), \text{ and}$$

(ii) *If* $\rho(x)a = a\rho(x)$ *all* $x \in G$, *then*

$$a = \sum_{g\in G} a_g\lambda(g^{-1}),$$

where the summation is to be interpreted in the sense of weak operator convergence in $B(L^2(G))$.

PROOF. Suppose $\lambda(x)a = a\lambda(x)$ all $x \in G$, then

$$a(\varphi_y) = \sum_{x\in G} a_{x,y}\varphi_x$$

$$= \sum_{x\in G} \omega(x,y^{-1})a_{xy^{-1}}\varphi_x$$

$$= \sum_{g\in G} \omega(gy,y^{-1})a_g\varphi_{gy}$$

$$= \sum_{g\in G} \omega(y^{-1},g^{-1})a_g\rho(gy)(\varphi_e)$$

$$= \sum_{g\in G} a_g\rho(g)(\varphi_y).$$

Thus the operators a and $\sum_{g\in G} a_g\rho(g)$ agree on an orthonormal basis in $L^2(G)$. It follows that they are equal. This proves part (i). Part (ii) is proved similarly.

PROPOSITION 1.3. (Kleppner [24, Theorem 1]). *Let* G,ω,ρ *and* λ *be as in* Lemma 1.1. *Let* $a \in B(L^2(G))$, *then*

(i) $\lambda(x)a = a\lambda(x)$ *all* $x \in G$ *if and only if* $a \in V(G,\omega)$

(ii) $\rho(x)a = a\rho(x)$ *all* $x \in G$ *if and only if* $a \in V'(G,\omega)$

(iii) *The commutant* $V(G,\omega)'$ *of* $V(G,\omega)$ *is equal to* $V'(G,\omega)$. *Hence* V and V' *have a common centre* $V \cap V'$.

PROOF. (i) Since

$$\begin{aligned}\rho(x)\lambda(y)f(g) &= \omega(x^{-1},g)\lambda(g)f(x^{-1}g)\\ &= \omega(x^{-1},g)\omega(x^{-1}g,y)f(x^{-1}gy)\\ &= \omega(x^{-1},gy)\omega(g,y)f(x^{-1}gy)\\ &= \omega(g,y)\rho(x)f(gy)\\ &= \lambda(y)\rho(x)f(g)\ ,\end{aligned}$$

all $x,y,g \in G, f \in L^2(G)$, we see that ρ and λ commute. It follows that each $\lambda(x)$ $(x \in G)$ commutes with all finite complex linear combinations of the operators $\{\rho(g) : g \in G\}$. But these linear combinations form a dense subspace of $V(g,\omega)$; furthermore, multiplication is weakly continuous in $V(G,\omega)$, consequently any a in $V(G,\omega)$ commutes with each $\lambda(x)$ $(x \in G)$.

Conversely, if $\lambda(x)a = a\lambda(x)$, then by 1.2(i), a belongs to the von Neumann algebra generated by the operators $\rho(g)$, $g \in G$.

(ii) The proof of this part is similar to the above.

(iii) If $a \in V(G,\omega)'$, then in particular $a\rho(g) = \rho(g)a$ all $g \in G$ and thus by (ii), $a \in V'(G,\omega)$. The remaining assertion follows from the definition of the centre (see definition the following I.2.1).

We arrange some of this information into a single statement.

THEOREM 1.4. *Let* G *be a discrete group,* ω *a normalised multiplier on* G *and* λ *the right regular* ω*-representation of* G.

(i) *Suppose* $a \in B(L^2(G))$, *then* $a \in V(G,\omega)$ *if and only if* $a\lambda(g) = \lambda(g)a$ *all* $g \in G$. *This occurs if and only if* $a_{x,y} = \omega(x,y^{-1})a_{xy^{-1}}$, *all* $x,y \in G$.

(ii) *If* $a \in V(G,\omega)$, *then* $a = \sum_{g \in G} a_g\rho(g)$ *in the sense of weak operator convergence. This decomposition is unique, that is if* $a = \sum_{g \in G} a'_g\rho(g)$, then $a'_g = a_g$ all $g \in G$.

PROOF. Except for the uniqueness, all these assertions follow immediately from 1.1, 1.2 and 1.3. Suppose we have $a = \sum_{g \in G} a'_g\rho(g)$, then $a(\varphi_e) = \sum_{g \in G} a'_g\varphi_g$ and $a_x = \langle a(\varphi_e),\varphi_x\rangle = \sum_{g \in G} a'_g(\varphi_g,\varphi_x) = a'_x$ all $x \in G$.

Since we are interested in central projections in $V(G,\omega)$, we need a method of determining when an element $a \in V(G,\omega)$ belongs to the centre of $V(G,\omega)$ in terms of the coefficients a_g $(g \in G)$ of a.

PROPOSITION 1.5. (Kleppner [24, page 557]). *Let* G,ω λ *and* ρ *be as in* Lemma 1.1. *If* $a \in V(G,\omega)$, *then* a *is in the centre* $CV(G,\omega) = \{b \in V(G,\omega) : cb = bc$ *all* $c \in V(G,\omega)\}$ *if and only if*

$$\omega(x,y)a_{y^{-1}xy} = a_x\omega(y,y^{-1}xy) ,$$

all $x,y \in G$.

PROOF. Consider the set of equivalent statements :
$a \in CV$: $a\rho(x) = \rho(x)a$ all $x \in G$;

$$\sum_{g \in G} a_g \rho(g)\rho(x) = \sum_{g \in G} a_g \rho(x)\rho(g), \text{ all } x \in G;$$

$$\sum_{x \in G} a_{yx^{-1}} \rho(y)\omega(x^{-1},xy^{-1}) = \sum_{x \in G} a_{x^{-1}y}\rho(y)\omega(y^{-1}x,x^{-1}),$$

all $y \in G$; $a_{yx^{-1}}\omega(y,x^{-1}) = a_{x^{-1}y}(x^{-1},y)$ all $x,y \in G$;
$\omega(x,y)a_{y^{-1}xy} = a_x\omega(y,y^{-1}xy)$ all $x,y \in G$. The last equivalence was obtained by a change of variable.

PROPOSITION 1.6 (Kleppner [24, Theorem 2 and Lemma 6]).
Let G *be a discrete group and* ω *a normalized multiplier.*

(i) *For* $a \in V(G,\omega)$, *the map* $g \to a_g$ *is in* $L^2(G)$.

(ii) $$(ab)_x = \sum_{g \in G} a_y b_{y^{-1}x}\omega(y^{-1},x)$$
$$= \sum_{z \in G} a_{xz^{-1}} b_z \omega(x,z^{-1}).$$

(iii) $\langle a_g, b_g \rangle = (ab^*)_e$, *and* $(a^*)_g = a_{g^{-1}}$.

(iv) *The map* $\phi : V(G,\omega) \to C : a \to a_e$ *is a finite faithful normal trace on* $V(G,\omega)^+$ *- the set of positive elements in* $V(G,\omega)$.

PROOF. (i) follows from (iii) by letting a = b. For (ii), we have

$$(ab)_x = (ab)_{x,e} = \sum_{z\in G} a_{x,z} b_{z,e}$$

$$\sum_{z\in G} a_{xz^{-1}} b_z \omega(x,z^{-1}).$$

The other formula follows by a change of variable. The first part of (iii) follows from (ii) and the second part from the equation $(a^*)_g = \langle a^*(\varphi_e),\varphi_g\rangle = \langle \varphi_e, a\varphi_g\rangle = \langle a\varphi_g,\varphi_e\rangle^- = a_{e,g}{}^- = a_{g^{-1}}{}^- \omega(e,g^{-1})^- = a_{g^{-1}}{}^-$. The proof of (iv) is as follows: Clearly ϕ is finite and normal. If $a \geq 0$, then $a = bb^*$ for some $b \in V(G,\omega)$; now $0 = \phi(a) = a_e = (bb^*)_e = (\|b_g\|_{L^2(G)})^2$ if and only if $a = 0$. Thus ϕ is faithful. For the invariance we have $\phi(ab) = (ab)_e = \langle a_g,(b^*)_g\rangle = \langle b_g,(a^*)_g\rangle = (ba)_e = \phi(ba)$.

Thus by I.3.2, $V(G,\omega)$ is a finite von Neumann algebra (this is not the case when G is non-discrete, see IV.4.1), thus $V(G,\omega)$ has no type III, type II_∞ and type I_∞ direct summands. By I.3.3 $V(G,\omega)$ is the direct sum of a type I_f von Neumann algebra and a type II_1 von Neumann algebra. Indeed by I.4.1 there exist central orthogonal projections $e, e_1, e_2, \ldots$ in $V(G,\omega)$ with $(\sum_{n=1}^\infty e_n) + e = I$ (the identity operator) and such that $e_n V(G,\omega)$ is type I_n and $eV(g,\omega)$ is type II_1.

Let G be a discrete group, ω a normalized multiplier on G and H a subgroup of G. For each $a \in V(G,\omega)$, let supp(a) denote the support of a defined by $\mathrm{supp}(a) = \{g \in G : a_g \neq 0\}$. Denote by $V(H,\omega)$ the von Neumann algebra $V(H,\omega|_{H\times H})$, where $\omega|_{H\times H}$ is the restriction of ω to H.

<u>THEOREM 1.7</u>. *If* H *is a subgroup of* G, *the set*

$K = \{a \in V(G,\omega) : \mathrm{supp}(a) \subseteq H\}$ *is a weak operator closed* $*$*-subalgebra of* $V(G,\omega)$ *(and hence is a von Neumann algebra), and there exists a (normal)* $*$*-isomorphism* $' : V(H,\omega) \to K$. *Moreover, the coefficients of* $a \in V(H,\omega)$ *are preserved under* $'$.

PROOF. Using the representation $a = \sum_{g \in H} a_g \rho(g)$ of elements in K, we easily see that K is a $*$-subalgebra of $V(G,\omega)$. To see that K is weakly closed, let $\{a^i : i \in I\} \subseteq K$ be a net converging weakly to an element $a \in V(G,\omega)$. In particular $\langle a^i(\varphi_e),\varphi_x\rangle \to \langle a(\varphi_e),\varphi_x\rangle$; but for $x \notin H$, $\langle a^i(\varphi_e),\varphi_x\rangle = 0$ all i, hence $a_x = \langle a(\varphi_e),\varphi_x\rangle$ is also equal to 0.

Let S be a set of coset representatives modulo H. A $*$-isomorphism $' : V(H,\omega) \to K$ can be defined as follows

$$a'f(x) = \frac{(af_g)(h)}{\omega(h,g)} ,$$

$f \in L^2(G)$; where $g \in S$, $h \in H$ are two elements such that $x = hg$, and f_g denotes the function $h \to f(hg)\omega(h,g)$, $h \in H$.

Note that

$$\begin{aligned} \omega(g,h)f_{gx}(h) &= f(hgx)\omega(h,gx)\omega(g,x) \\ &= f(ghx)\omega(hg,x)\omega(h,g) \\ &= f_x(hg)\omega(h,g) \\ &= \lambda(g)f_x(h), \end{aligned}$$

that is $\omega(g,h)f_{gx} = \lambda(g)f_x$. Hence if $hg = h'g'$, $h,h' \in H$,

$g, g' \in G$, then

$$\frac{af_g(h)}{\omega(h,g)} = \frac{af_{h^{-1}h'g}(h)}{\omega(h,h^{-1}h'g)}$$

$$= \frac{a[\lambda(h^{-1}h')f_{g'}](h)}{\omega(h^{-1}h',g')\ (h,h^{-1}h'g')}$$

$$= \frac{(\lambda(h^{-1}h')(af_{g'})](h)}{\omega(h',g')\omega(h,h^{-1}h')}$$

$$= \frac{(af_{g'})(hh^{-1}h')\omega(h,h^{-1}h')}{\omega(h',g')\omega(h,h^{-1}h')}$$

$$= \frac{(af_{g'})(h')}{\omega(h',g')} \quad .$$

This shows that the definition of ' does not depend on the choice of the transversal S. Moreover, we have

$$\|a'f\|_{L^2(G)} = \sum_{g\in S,h\in H}|(af_g)(h)|^2$$

$$= \sum_{g\in S}(\|af_g\|_{L^2(H)})^2$$

$$\leqslant \|a\|^2 \sum_{g\in S,h\in H}|f_g(h)|^2$$

$$= \|a\|^2(\|f\|_{L^2(G)})^2$$

$a \in V(H,\omega)$, $f \in L^2(G)$. Hence $a' \in B(L^2(G))$ for all $a \in V(H,\omega)$. The next step is to show that the map ' indeed maps into K. We do this by calculating coefficients. Let $h \in H$, $g \in S$ and $x,y \in G$ be such that $x = hg$,

$$[(\varphi_y)_g](h) = \begin{cases} yg^{-1}\omega(g,h^{-1}), & y^{-1}g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$a'_{xy} = \langle a'(\varphi_y), \varphi_x \rangle = a'(\varphi_y)(x)$$

$$= a[(\varphi_y)_g](h)/\omega(h,g)$$

$$= \begin{cases} [a(\varphi_{yg^{-1}})](h)\omega(g,y^{-1})/\omega(h,g) & \text{if } yg^{-1} \in H, \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} a_{hgy^{-1}}\omega(h,gy^{-1})\omega(g,y^{-1})/\omega(h,g) \text{ if } y \in Hg = Hx, \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} a_{xy^{-1}}\omega(x,y^{-1}) & \text{if } xy^{-1} \in H, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $a'_x = a'_{x,e} = \begin{cases} a_x & x \in H. \\ 0 & \text{otherwise} \end{cases}$ Also $a'_{x,y} = a'_{xy^{-1}}\omega(x,y^{-1})$,

hence $a' \in K$ for all $a \in V(H,\omega)$. Since multiplication and involution in $V(G,\omega)$ and K are defined wholly in terms of the coefficients a_x, respectively a'_x (which are preserved under '), we see that ' is indeed a *-homorphism from $V(H,\omega)$ into K.

Suppose $a' = 0$ for some $a \in V(H,\omega)$, then $a'_x = 0$ all $x \in G$, so in particular, $a_x = 0$ all $x \in H$ which implies $a = 0$, thus the map ' is injective. To show that ' is onto, we construct a right inverse " for ', that is a map " : $K \to V(H,\omega)$ such that $' \circ '' = id_K$. If $f \in L^2(H)$, we denote by f° the function

$$f^\circ(g) = \begin{cases} f(g) & g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Define " by $a''f = a(f^\circ)|_H$, $f \in L^2(H)$, $a \in K$. Since

$$\begin{aligned} \|a''f\|_{L^2(H)} &= \|af^\circ|_H\|_{L^2(H)}. \\ &\leq \|af^\circ\|_{L^2(G)} \\ &\leq \|a\| \ \|f^\circ\|_{L^2(G)} = \|a\| \ \|f\|_{L^2(H)} , \end{aligned}$$

as well as $a''_{x,y} = \langle a[(\varphi_y)^\circ], \varphi_x\rangle = a_{x,y} = \omega(x,y^{-1})a_{xy^{-1}} = \omega(x,y^{-1})\, a''_{xy^{-1}}$ for $x,y \in H$, we see that " maps into $V(H,\omega)$. That $' \circ '' = id_K$ follow by comparing coefficients. The asserted normality follows from I.2.2.

In view of this Theorem and because we are interested only in the type structure of $V(G,\omega)$, we will identify $V(H,\omega)$ with the weakly operator closed *-subalgebra $\{a \in V(G,\omega) : supp(a) \subseteq H\}$ whenever H is a subgroup of G.

2. Discrete finite class groups

Recall from Chapter I Section 7 that a (discrete) group G is called a FC group (finite class) if the set $\{g^{-1}xg : g \in G\}$ is finite for each $x \in G$; and for an arbitrary (discrete group) G, G_{FC} denotes the subgroup of all elements x in G such that $\{g^{-1}xg : g \in G\}$ is a finite set. Clearly G_{FC} is a FC group. For subgroups H and K, let [H,K] be the subgroup of G generated by $\{h^{-1}k^{-1}hk : h \in H, k \in K\}$. The group [H,H] is denoted by H'. Itis a non-trivial theorem of Neumann ([32, Theorem 5.1]) that for finite class groups G, the commutator G' is locally finite (that is, each finitely generated subgroup of G' is finite). We need this result in the form of the following lemma.

LEMMA 2.1 (Neumann [33, Lemma 4.1]). *Let* G *be an* FC *group,* H *a subgroup of finite index in* G *such that* $|H'| < \infty$, *then* $|G'| < \infty$.

PROOF. Let S be a set of coset representatives modulo H. If $s,t \in S$ and $g,h \in H$, then by using the equalities $[a,bc]$ $[a,c][a,b]^c$, $[ab,c] = [a,c]^b[b,c]$ (where $[a,b]$ denotes $a^{-1}b^{-1}ab$ and a^b denotes $b^{-1}ab$), we obtain

$$[gs,ht] = [g,t]^s[g,h]^{ts}[s,t][s,h]^t ,$$

so we see that G' is generated by elements of the form [s,h],[s,t], [g,h] and their conjugates. The commutator H' is finite by hypothesis, so there are only finitely many elements of the form [g,h]. Since S is a finite set, there are only finitely many elements of the form [s,t]. To see that there are only finitely many elements of the form [s,h], note that $[s,h] = s^{-1}s^h$, but G is a FC group, so we have the desired property. We have shown that G' is finitely generated. The result now follows because G' is locally finite.

3. The ω-finite class group and the ω-centre of G

Let G be a discrete group and ω a normalized multiplier on G. The concepts of finite class group, centre and centralizer of G have their natural analogues for the pair (G,ω). We define these and investigate some of their properties.

Let H be a subgroup of G. For each $x \in H$, the ω-centralizer of x in H is defined by

$$C_{\omega,H}(x) = \{g \in C_H(x) : \omega(x,g) = \omega(g,x)\},$$

and the ω-centre of H by

$$Z_\omega(H) = \cap_{x\in H} C_{\omega,H}(x)$$

$$= \{g \in Z(H) : \omega(x,g) = \omega(g,x) \text{ all } x \in H\},$$

where $C_H(x)$ denotes the (ordinary) centralizer of x in H and Z(H) the centre of H.

The ω-finite class group of G is defined by

$$\Delta = \Delta_G = \{x \in G : [G : C_{\omega,G}(x)] < \infty\}$$

Note that if ω is a trivial multiplier, then $\Delta = G_{FC}$. Because $C_{\omega,G}(x) \subseteq C(x)$ each x, we have $\Delta \subseteq G_{FC}$. In particular Δ is a FC group. The following proposition will justify the above definition.

PROPOSITION 3.1. *Let* G *be a discrete group,* ω *a normalized multiplier on* G *and* ρ *the left regular* ω^{-1}*-representation of* G. *Let* $x \in G$, *then*

(i) $C_\omega(x) = C_{\omega,G}(x) = \{g \in G : \rho(x)\rho(g) = \rho(g)\rho(x)\}$.

(ii) $C_\omega(x)$ *and* Δ *are subgroups of* G.

PROOF. (i). Suppose $g \in C_\omega(x)$, that is $g \in C(x)$ and $\omega(x,g) = \omega(g,x)$, then $\rho(g)\rho(x)\omega(g,x) = \rho(gx) = \rho(xg) = \rho(x)\rho(g)$ $\omega(x,g)$, so $\rho(x)\rho(g) = \rho(g)\rho(x)$. Conversely, suppose $g \in G$ satisfies $\rho(x)\rho(g) = \rho(g)\rho(x)$, then $\rho(gx)/\omega(g,x) = \rho(xg)/\omega(x,g)$, so in particular,

$$\frac{[\rho(gx)\varphi_e](y)}{\omega(g,x)} = \frac{[\rho(xg)\varphi_e](y)}{\omega(x,g)},$$

for all $y \in G$, where φ_e denotes the characteristic function of $\{e\}$. Thus

$$\frac{\omega(x^{-1}g^{-1},y)\psi_e(x^{-1}g^{-1}y)}{\omega(g,x)} = \frac{\omega(g^{-1}x^{-1},y)\varphi_e(g^{-1}x^{-1}y)}{\omega(x,g)}.$$

Now if we let $y = gx$, this expression simplifies to $\omega(g,x)^{-1} = \omega(g^{-1}x^{-1},gx)\varphi_e(g^{-1}x^{-1}y)/\omega(x,g)$, hence $\varphi_e(g^{-1}x^{-1}y)$ cannot be zero, but the only way this can occur is if $gx = xg$. This shows that $g \in C(x)$ and from the above expression, $\omega(x,g) = \omega(g,x)$. This completes the proof of part (i).

(ii). Suppose $h,g \in C_\omega(x)$, then

$$\begin{aligned}\rho(h^{-1}g)\rho(x) &= \rho(h)^{-1}\rho(g)\rho(x)\omega(h^{-1},g)^{-1}\\ &= \rho(h)^{-1}\rho(x)\rho(g)\omega(h^{-1},g)^{-1}\\ &= \rho(x)\rho(h)^{-1}\rho(g)\omega(h^{-1},g)^{-1}\\ &= \rho(x)\rho(h^{-1}g),\end{aligned}$$

hence $h^{-1}g \in C_\omega(x)$ and $C_\omega(x)$ is a subgroup of G. Since $C_\omega(x^{-1}y) \supseteq C_\omega(x) \cap C_\omega(y)$, we have

$$\begin{aligned}[G : C_\omega(x^{-1}y)] &\leqslant [G : C_\omega(x) \cap C_\omega(y)]\\ &\leqslant [G : C_\omega(x)][G : C_\omega(y)]\\ &< \infty,\end{aligned}$$

whenever $x,y \in \Delta$, hence Δ is a subgroup of G and this proves (ii).

The following theorem points out the significance of Δ.

THEOREM 3.2. *Let* G *be a discrete group,* ω *a normalized multiplier on* G *and* Δ *the* ω*-finite class group of* G. *If* $V(G,\omega)$ *has a non-zero maximal type* I *part, then*

(i) $[G : \Delta] < \infty$, *and*

(ii) $|\Delta'| < \infty$.

The idea of the proof of (i) comes from Smith [44, Theorem 9.4], and the proof of (ii) is similar to that of Smith [45, Theorem 1]. We need the following lemmas.

LEMMA 3.3. *Let V be a type* I_n *von Neumann algebra, then for each non-zero irreducible representation* π of V, we have $\pi(V) = M_n(\mathbb{C})$ *(the space of* n × n *matrices over* $\mathbb{C}$*). Hence the dimension of* π *equals* n.

PROOF. Let S_{2n} denote the standard polynomial of 2n variables (see Chapter I Section 4). By I.4.5, $S_{2n}(V) = \{0\}$ and $S_{2n}(\pi(V)) = \pi(S_{2n}(V) = \{0\}$. Let H_π denote the Hilbert space of π. It follows from I.5.1 and the irreducibility of π that the weak operator closure of $\{\pi(a) \text{ " } a \in V\}$ in $B(H_\pi)$ equals $B(H_\pi)$, thus by I.4.6, $S_{2n}(B(H_\pi)) = 0$. We conclude (using I.4.3) that the dimension of π is not greater than n.

V is an n x n matrix algebra over the centre CV of V (see Chapter I Section 4), that is each $a \in V$ can be written

$$a = \sum_{1\le i,j\le n} \alpha_{ij}c_{ij} ,$$

where $\alpha_{ij} \in CV$ and c_{ij} denotes the n x n matrix whose only non-zero entry is the i,jth entry which consists of the identity operator in CV. Fix $j,k \in \{1, \dots , n\}$. Since $c_{ij}c_{jk}c_{k\ell} = c_{i\ell}$, all $1 \le i,j \le n$, $\pi(c_{jk}) = 0$ implies $\pi(c_{i\ell}) = 0$ all $1 \le i,j \le n$ and hence $\pi(V) = \{0\}$ which is a contradiction. Thus $\pi(c_{jk}) \ne 0$ all j,k. Now suppose we have complex numbers β_{ij}, $1 \le i,j \le n$ such that $\sum_{ij}\beta_{ij}\pi(c_{ij}) = 0$, then for any fixed k,ℓ $(1 \le k,\ell \le n)$,

$$\sum_{i,j}\beta_{ij}\pi(C_{kk}C_{ij}C_{\ell\ell}) = \beta_{k\ell}\pi(C_{k\ell}) = 0,$$

this implies $\beta_{k\ell} = 0$; in other words, the $\pi(c_{k\ell})$, $1 \leq k,\ell \leq n$ form a basis for $\pi(V)$, thus $\pi(V) = M_n(\mathbb{C}) = B(H_\pi)$.

LEMMA 3.4. *Let* G *be a discrete group and* ω *a normalized multiplier on* G. *Suppose* $CV(G,\omega)$ *denotes the centre of* $V(G,\omega)$, *then* $CV(G,\omega) \subseteq V(\Delta,\omega)$.

PROOF. Let $a \in CV(G,\omega)$ and $x \in G$ such that $a_x \neq 0$, then by 1.7, it suffices to show that $x \in \Delta$. From 1.5, $\omega(x,y)\, a_{y^{-1}xy} = a_x(y,y^{-1}xy)$, thus $C(x) = C_\omega(x)$, but by 1.6(i), $[G : C(x)] < \infty$, hence $x \in \Delta$.

LEMMA 3.5. *Suppose* G *is a discrete* FC *group,* ω *a normalized multiplier on* G, ρ *the left regular* ω*-representation of* G *and* H *a subgroup of* G. *Suppose there exists a projection* e *in* CV *such that both* eV *and* eV° *are type* I_n (n *a fixed integer*), *where* V *and* V° *denote the von Neuman algebras* $V(G,\omega)$ *and* $V(H,\omega)$ *respectively. If* $K = \{x \in G : \rho(h)\rho(x) = \rho(x)\rho(h)$ *all* $h \in H\} = \bigcap_{h\in H} C_{\omega,G}(h)$ *is the* ω*-centralizer of* H *in* G, *then* $e(\rho(k)\rho(g) - \rho(g)\rho(k)) = 0$ *for every* $k \in K$ *and* $g \in G$. *Consequently* $[K,G]$ *is finite.*

PROOF. Let π be a non-zero irreducible representation of eV. Since $\pi(ea)$ is non-zero for some $a \in V$, $\pi(e)\pi(ea) = \pi(ea) \neq 0$, so $\pi(eI) = \pi(e) \neq 0$, but $eI \in eV^\circ$, thus π restricted to eV° is non-zero. By Lemma 3.3, $\pi(eV^\circ) = M_n(C) = \pi(eV)$.

If $k \in K$, then $e\rho(k)$ centralizes eV°, so $\pi(e\rho(k))$ centralizes $\pi(eV^\circ) = \pi(eV)$, in particular $\pi(e\rho(k))$ commutes with $\pi(e\rho(g))$ for every $g \in G$, that is $\pi[e(\rho(k)\rho(g) - \rho(g)\rho(k))] = 0$. By I.5.3, $e\rho(k)\rho(g) = e\rho(g)\rho(k)$. If we write $e = \sum_{g\in G} e_g \rho(g)$ (1.4), then

$$\sum_{x\in G} e_x \rho(x)\rho(g)\rho(k) = \sum_{x\in G} e_x \rho(x)\rho(k)\rho(g)$$

that is

$$\sum_{y\in G} e_{yk^{-1}g^{-1}} \omega(g,k)^{-1}\omega(gk,y^{-1})\rho(y) =$$

$$\sum_{y\in G} e_{yg^{-1}k^{-1}} \omega(k,g)\omega(kg,y^{-1})^{-1}\rho(y).$$

Equating the coefficients (using the uniqueness of decomposition) and letting $x = kg$ gives $|e_{kgk^{-1}g^{-1}}| = |e_1|$. By Proposition 1.6(i), $\sum|e_g|^2 < \infty$, so the set $\{kgk^{-1}g^{-1} : k \in K, g \in G\}$ must be finite. Since this set generates $[K,G]$ and is contained in the locally finite group G', it follows that $[K,G]$ is finite.

PROOF OF THEOREM 3.2. (i) Let e_n be a non-zero central projection in $V(G,\omega)$ such that $e_n V(G,\omega)$ is type I_n. Since $e_n V(G,\omega)$ is a matrix algebra over its centre (I.4), it is of dimension at most n^2 over $Ce_n V(G,\omega) = e_n CV(G,\omega) \subseteq CV(G,\omega) \subseteq V(\Delta)$ (Lemma 3.4). Hence if $g_1, \ldots, g_{n^2+1}$ are $n^2 + 1$ elements of G belonging to distinct cosets of Δ, then there exist elements $c_1, \ldots, c_{n^2+1} \in V(\Delta,\omega)$ such that

$$\sum_{i=1}^{n^2+1} c_i(e_n\rho(g_i)) = \sum_{i=1}^{n^2+1} c_i e_n \rho(g_i) = 0.$$

with not all $(c_i e_n)\rho(g_i) = 0$, but $c_i e_n \in V(\Delta,\omega)$, so this cannot happen since the sum

$$V(\Delta,\omega)\rho(g_1) \oplus \ldots \oplus V(\Delta,\omega)\rho(g_{n^2+1}).$$

is a direct sum. This shows $[G : \Delta] \leq n^2 < \infty$.

(ii) Again we use the standard polynomial in 2n variables S_{2n} (I.4). Suppose $V(G,\omega)$ has a non-zero type I part, then there exists a central projection $e_n \neq 0$ such that $e_nV(G,\omega)$ is type I_n. We have $S_{2n}(e_nV(G,\omega)) = \{0\}$, hence $S_{2n}(e_nV(\Delta,\omega)) = \{0\}$. It follows that $e_nV(\Delta,\omega)$ is type $I_{\leq n}$ (I.4.5)

Now let e_n be a non-zero central projection in $V(\Delta,\omega)$ such that $e_nV(\Delta,\omega)$ is type I_n. Using I.4.5,

$$S_{2n}(e_nV(\Delta,\omega)) = 0, \quad S_{2n-}\,(e_nV(\Delta,\omega)) \neq 0.$$

Since the polynomial is multilinear and $V(\Delta,\omega)$ is generated by the elements $\rho(g)$, $g \in \Delta$, there exist $g_1, \ldots, g_{2n-2} \in \Delta$ with $S_{2n-2}(e_n\rho(g_1), \ldots, e_n\rho(g_{2n-2})) \neq 0$. Let H be the normal subgroup of Δ generated by the elements $g_1, \ldots, g_{2n-2}$ and their conjugates. Since Δ is a finite class group, H is finitely generated. Moreover, $S_{2n}(e_nV(H,\omega)) = 0$ and $S_{2n-}\,(e_nV(H,\omega) \neq 0$ so by I.4.5, $e_nV(H,\omega)$ has a nonzero type I_n direct summand (chosen to be maximal $eV(H,\omega)$ for some $e \in Ce_nV(H,\omega)$. We wish to show that $e \in CV(\Delta,\omega)$. Since H is normal in Δ, for any $k \in \Delta$, the automorphism $V(H,\omega) \to V(H,\omega) : a \to \rho(k^{-1})a\rho(k)$ leaves the type I_n summand $eV(H,\omega)$ fixed, that is $\rho(k)^{-1}e\rho(k)V(H,\omega)$, so by the uniqueness of e, $\rho(k^{-1})e\rho(k) = e$. It follows that $e \in CV(\Delta,\omega)$. Since $e \leq e_n$,

$eV(\Delta,\omega)$ is clearly also type I_n, and thus by Lemma 3.5, $K' < \infty$, where $K = \{x \in \Delta : \rho(h)\rho(x) = \rho(x)\rho(h)$ all $h \in H\}$. Since H is finitely generated $[\Delta : K] < \infty$ and thus the result $|\Delta'| < \infty$ follows from Lemma 2.1.

Theorem 3.2 provides us with a subgroup of G of finite index in G whose commutator subgroup is finite provided $V(G,\omega)$ has a non-zero type I part. In the next chapter, we construct such a subgroup for arbitrary locally compact groups, G using the results of Taylor [48]. However, since Taylor's work depends on the results known for discrete groups (and since our proofs are of an elementary nature), we feel justified in including them here.

4. The type I part of $V(G,\omega)$

Let G be a discrete group and ω a normalized multiplier on G. Recall that the group extension G^ω of T by G is the group whose underlying set is T x G and with multiplication $(s,x)(t,y) = (st\omega(x,y),xy),(s,x)(t,y) \in G^\omega$. Usually G^ω is endowed with the Weil topology (I.6). However, for the remainder of this chapter, we give G^ω the discrete topology. Whenever H is a subgroup of G, we identify H^ω in the obvious way with a subgroup of G^ω.

PROPOSITION 4.1. *Let* H *be a subgroup of the discrete group* G *and* ω *a normalized multiplier on* G. *Adopt the notation of* Section 3, *then*

(i) $C_{G^\omega}(t,x) = (C_\omega(x))^\omega$ *for all* $(t,x) \in G^\omega$,

(ii) $[G^{\omega} : H^{\omega}] = [G : H]$,

(iii) $(G^{\omega})_{FC} = \Delta^{\omega}$, *and*

(iv) $(Z_{\omega}(G))^{\omega} = Z(G^{\omega})$.

PROOF. (i) Suppose $(s,y) \in C_{G^{\omega}}(t,x)$, then $(st\omega(x,y),xy) = (st\omega(y,x),yx)$, thus $y \in C_{\omega}(x)$ and $(t,y) \in (C_{\omega}(x))^{\omega}$. Conversely if $(t,y) \in (C_{\omega}(x))$, then $y \in C_{\omega}(x)$, so $(s,x)(t,y) = (t,y)(s,x)$.

(ii) The map $xH \to (t,x)H^{\omega}$ sets up a one-to-one correspondence between the H cosets in G and the H^{ω} cosets in G^{ω}.

(iii) By definition, each $x \in G$ belongs to Δ if and only if $[G : C_{\omega}(x)] < \infty$. By (i) this occurs if and only if $[G^{\omega} : C_{G^{\omega}}(t,x)] < \infty$ all $(t,x) \in \Delta^{\omega}$. But this last statement is equivalent to $(t,x) \in (G^{\omega})_{FC}$.

(iv) Fix $x \in G$, then $(t,x) \in (Z_{\omega}(G))^{\omega}$ if and only if $x \in Z(G)$ and $\omega(x,y) = \omega(y,x)$ all $y \in G$, that is if and only $(t,x)(s,y) = (s,y)(t,x)$ all $(s,y) \in G^{\omega}$.

Recall that for any discrete group G, G_0 denotes the von Neumann kernel of G. (For a definition see Chapter I Section 7.)

LEMMA 4.2. *Let* G *be a discrete group and* ω *a normalized multiplier on* G. *Suppose that* H *is a subgroup of finite* *index in* G *such that* $|H'| < \infty$ *then there exists a subgroup* K *of* G *such that* $[G : K] < \infty$ *and* $K' = G_0 = \cap\{L' : [G : L] < \infty\}$.

PROOF. If $[G : L] < \infty$ then, since the characters of L/L' separate points ([18, 22.17]), we have $G_0 = L_0 \subseteq L'$, hence $G_0 \subseteq \cap \{L' : [G : L] < \infty\}$. Since $|H'| < \infty$, and H/H_0 is maximally almost periodic, by I.7.3 and I.7.10, H/H_0 is type I and H has a subgroup K containing H_0 of finite index in H such that K/H_0 is abelian. Since $[G : K] < \infty$ and $K' \subseteq H_0 = G_0$, the result follows.

The next lemma is a key lemma.

LEMMA 4.3. *If there exists a subgroup* H *of* G *such that* $[G : H] < \infty$ $|H'| < \infty$ *and* $\omega^n|_{H\times H}$ *is trivial for some* n, *then there exists a subgroup* K *of* G *such that* $[G : K] < \infty$, $G_0 = K'$ *and* $(G^\omega)_0 = (K^\omega)'$.

PROOF. By Lemma 4.2, we can assume without loss of generality that $H' = G_0$. For some map $\gamma : H \to T$, $\omega^n(x,y) = \gamma(x)\gamma(y)/\gamma(xy)$ all $x,y \in H$. An easy calculation shows that

$$(H^\omega)' = \{[\omega(x,y)\omega(x^{-1},y^{-1})\omega(xy,x^{-1}y^{-1}),xyx^{-1}y^{-1}] : x,y \in H\}.$$

Since

$$[\omega(x,y)\omega(x^{-1},y^{-1})\omega(xy,x^{-1}y^{-1})]^n = \gamma(xyx^{-1}y^{-1})^{-1},$$

we have $(H^\omega)' < \infty$. By Lemma 4.2, G^ω has a subgroup M $\subset H^\omega$ such that $[G^\omega : M] < \infty$ and $M' = (G^\omega)_0$. Let L be the image of the projection

$M \to G : (t,x) \to x$. L is a subgroup of G with the property $M \subseteq L^\omega$, hence $[G^\omega : L^\omega] < \infty$; furthermore, $(L^\omega)' = M'$ and thus $K = L \cap H$ has the desired properties.

The following theorem characterizes explicitly the maximal type I central projection in $V(G,\omega)$. It is the main result of this chapter.

THEOREM 4.4. *Let* G *be a discrete group with normalized multiplier* ω *and let* e *(respectively* e_n, $n = 1, \ldots, n \neq \infty$) *be the maximal type* I *(respectively type* I_n) *central projections in* $V(G,\omega)$, *then*

$$\sum_{n<\infty} e_n = e$$

and the following are equivalent.

(a) $e \neq 0$

(b) *there exists a subgroup* H *of* G *such that* $[G : H] < \infty$ *and* $|H'| < \infty$ *and* $\omega|_{H \times H}$ *is trivial,*

(c) $[G : \Delta] < \infty$, $|\Delta'| < \infty$ *and* G *admits a finite dimensional* ω*-representation.* (Δ *denotes the* ω*-finite class group of* G.)

Suppose $e \neq 0$, *then there exists a* 1*-dimensional* ω*-representation* γ *of* G, *such that*

(i) $\pi|_{G_\infty} = (\dim \pi).\ \gamma$, *for all finite dimensional* ω*-representations* π *of* G.

(ii) $$e = \frac{1}{|G_0|} \sum_{g \in G_0} \gamma(g)\rho(g).$$

PROOF. Suppose $e \neq 0$, then $e_n \neq 0$ for some integer n. Lemma 3.3 ensures that an irreducible representation τ of $e_n V(G,\omega)$ will give rise to a finite dimensional ω-representation $\pi : g \to \tau(e_n\rho(g^{-1}))^*$ of G, where A* denotes the adjoint of A. Together with Theorem 3.2, this yields (c).

Suppose we have (c). Let π be a finite dimensional ω-representation of G. After taking determinants, $\omega^n(x,y)\det \pi(xy) = \det \pi(x) \det \pi(y)$, and we see than ω^n is trivial, so by Theorem 3.2 and Lemma 4.3, there exists a group K such that $[G : K] < \infty$ $G_0 \supset K'$, $|G_0| < \infty$ and $(G^\omega)_0 = (K^\omega)'$. Let π be an irreducible finite dimensional ω-representation of K, then $\pi^\circ : (\lambda,x) \to \lambda\pi(x)$ is a finite dimensional representation of K^ω (I.6), so $\pi^\circ(1,x)(1,y)(1,x)^{-1}(1,y)^{-1}) = \pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} = I$ all $x,y \in K$, but π is irreducible so it is one dimensional (see remark following I.5.1), consequently $\omega|_{K\times K}$ is trivial. This implies (b).

Finally suppose (b) is true. Since $\omega|_{H\times H}$ is trivial, H has a one-dimensional ω-representation, hence by inducing (I.8), we see that G admits a finite dimensional ω-representation. Let π be such a representation. Lemma 4.3 is applicable, so as in the preceding paragraph, $\pi(x)\pi(y) = \pi(y)\pi(x)$ all $x,y \in K$, and $G_0 = K'$, where K is the subgroup we obtain from Lemma 4.3, consequently

$$\pi(xyx^{-1}y^{-1}) = \omega(y,x)\omega(y^{-1},x^{-1})\omega(yx,y^{-1}x^{-1}). \quad \text{I.}$$

Since the lefthand side of this expression is independent of the way we express $xyx^{-1}y^{-1}$ as a commutator, and since the righthand side depends only on the dimension of π, the function $\gamma : xyx^{-1}y^{-1} \to \omega(y,x)\omega(y^{-1},x^{-1})\omega(yx,y^{-1}x^{-1})$, $x,y \in G$, extends to a well defined ω-representation of G_o that satisfies (i).

To complete the proof of this theorem, we must show that our current assumptions lead to the statement (a) and (ii). Let

$$f = \frac{1}{|G_o|} \sum_{g \in G_o} \gamma(g)\rho(g).$$

By Proposition 1.5, f is central in $V(G,\omega)$. Since $g \to \gamma(g)\rho(g)$ is an ordinary representation of K, $f^2 = f$. We claim that $fV(G,\omega)$ is abelian. Suppose $a,b \in V(G,\omega)$ and π is a finite dimensional ω-representation of G, then

$$(fba)_z = \frac{1}{|K'|} \sum_{z \in K'} \gamma(z)(ba)_{z^{-1}x}\omega(z^{-1},x).$$

$$= \frac{1}{|K'|} \sum_{z \in K'} \sum_{z \in K} \gamma(z) a_y b_{z^{-1}xy^{-1}} \omega(z^{-1}x,y^{-1})\omega(z^{-1},x)$$

$$= \frac{1}{|K'|} \sum_{z \in K'} \sum_{y \in K} a_y b_{z^{-1}xy^{-1}} \pi(x)\pi(y)\pi(z^{-1}xy^{-1})^{-1}$$

Similarly,

$$(abf)_x = \frac{1}{|K'|} \sum_{z \in K'} \sum_{z \in K} a_y b_{y^{-1}xy^{-1}} \pi(x)\pi(y)\pi(y^{-1}xz^{-1})^{-1}$$

Since K' is normal in G and $xy^{-1}x^{-1}y \in K'$, $\{y^{-1}xz^{-1} : z \in K'\} = \{z^{-1}xy^{-1} : z \in K'\}$. It follows that $(fa)(fb) = abf = fba = (fb)(fa)$.

If we write $fV(G,\omega)$ as a module direct sum

$$fV(K,\omega)\rho(g_1) \oplus \ldots \oplus fV(K,\omega)\rho(g_k)$$

for some set of coset representatives $g_1, \ldots, g_k$ modulo K, then by representing $fV(G,\omega)$ as right multiplication on itself, $fV(G,\omega)$ is a matrix algebra over the abelian algebra $fV(K,\omega)$ and thus by I.4.2 and I.4.5 is type I. This proves (a).

By looking at the irreducible representations of $e_nV(G,\omega)$, whenever this is non-zero, and using I.5.3, we see that $eV(G,\omega)$ has enough finite dimensional representations to separate the points of $eV(G,\omega)$. However π is a non-zero finite dimensional representation of $eV(G,\omega)$, $g \to \pi(\rho(g^{-1}))^*$ is a finite dimensional ω-representation of G, hence by (i), $\pi(\rho(g)\gamma(g)) = I$ all $g \in G_0 = K'$, thus

$$\pi(f) = \frac{1}{|G_0|} \sum_{g \in G_0} \pi(\rho(g)\gamma(g)) = I,$$

from which we obtain $\pi(e - f) = \pi(e) - \pi(f) = 0$. Since π is arbitrary, we conclude that $e = f$. This completes the proof of Theorem 4.4.

Note that for ω trivial, this theorem reduces to results due to Formanek [10, Theorem 2] and Schlichting [41, Satz 1]. The following theorem is a consequence of 4.4.

<u>THEOREM 4.5</u>. *Let* G *be a discete group and* ω *a normalized multiplier on* G, *then the following are equivalent.*

(i) $V(G,\omega)$ *is type* I *(or equivalently type* I_f*),*

(ii) every ω-representation of G *is type* I,

(iii) G *has an abelian subgroup* A *of finite index in* G *such that for all* $x,y \in A$, $\omega(x,y) = \omega(y,x)$

(iv) G^{ω} *(with discrete topology) is type* I.

To prove this theorem we introduce the notion of a twisted group algebra. For a discrete group G with normalized multiplier ω, the twisted group algebra $A(G,\omega)$ of G consists of finitely supported measures on G with multiplication.

$$(\mu\nu)_x = \sum_{y \in G} \mu_y \nu_{y^{-1}x} \, (y^{-1},x),$$

and *-operation

$$(\mu^*)_x = (\mu_{x^{-1}})^{-},$$

$\mu,\nu \in A(G,\omega)$, $x,y \in G$, where μ_x $(x \in G)$ denotes the measure of the set $\{x\}$. Each ω^{-1}-representation π of G extends naturally to a representation

$$\mu \to \sum_{g \in G} \mu_g \pi(g)$$

of $A(G,\omega)$ which, when no confusion may arise, we shall denote by the same letter. Note that the map

$$A(G,\omega) \to V(G,\omega) \; : \; \mu \to \rho(\mu) = \sum_{g \in G} \mu_g \rho(g),$$

where ρ denotes the regular ω^{-1}-representation of G, is a *-monomorphism with range $\{a \in V(G,\omega) : \text{supp}(a) \text{ is finite}\}$. In particular, $(A(G,\omega))$ is weakly dense in $V(G,\omega)$.

PROOF OF THEOREM 4.5. Suppose (i) is true then by the proof Theorem 4.4, there exists a group K such that $[G : K] < \infty$, $K' = G_\infty$ and $\omega|_{K\times K}$ is trivial. Since the maximal type I central projection of $V(G,\omega)$ is the identity, by Theorem 4.4,

$$I = \frac{1}{|G_\infty|} \sum_{g \in G_\infty} \gamma(g)\rho(g) ,$$

hence by the uniqueness (1.4.(ii)), $G_\infty = \{e\}$ and thus K is abelian. From $\omega|_{K\times K}$ trivial, we can now conclude that $\omega(x,y) = \omega(y,x)$ all $x,y \in K$. This proves (i) implies (iii).

Suppose (iii) is true. Let H be a subgroup as described in (iii), and π an ω^{-1}-representation of G. If $g_1, \ldots, g_n$ is a set of coset representatives modulo H, then

$$A(G,\omega) = A(H,\omega)\rho(g_1) \oplus \ldots \oplus A(H,\omega)\rho(g_n)$$

where the direct sum is a module direct sum. Again, by representing $A(G,\omega)$ by right multiplication on itself, we can embed $A(G,\omega)$ in the n x n matrices over $A(H,\omega)$. By hypothesis, $A(H,\omega)$ is abelian and so $A(G,\omega)$ satisfies a polynomial identity (I.4.2). Since the von-Neumann algebra generated by π is the weak closure of $\pi(A(G,\omega))$ by I.4.5 and I.4.6, π is type I. This proves (iii) implies (ii).

The implication (ii) implies (i) is trivial.

Suppose we have $[G : A] < \infty$ and $\omega(x,y) = \omega(y,x)$ all $x,y \in A$, then $[G^\omega : A^\omega] < \infty$ (4.1 (ii)) and A^ω is abelian, so by the equivalence of (i) and (ii), G^ω is type I. Conversely, if G^ω is type I, then an abelian subgroup of finite index in G^ω projects onto an abelian subgroup of finite index in G on which ω is symmetric, so (G,ω) is type I. Thus (iii) and (iv) are equivalent.

COROLLARY 4.6. *The subgroup* A *in* 4.5 (iii) *may be taken to be* $Z_\omega(\Delta)$.

PROOF. G^ω type I implies $[G^\omega : Z((G^\omega)_{FC})] < \infty$ (I.7.10). Thus $[G : Z_\omega(\Delta)] = [G^\omega : (Z_\omega(\Delta))^\omega] = [G^\omega : Z(\Delta^\omega)] = [G^\omega : Z((G^\omega)_{FC})] < \infty$ using Proposition 4.1.

COROLLARY 4.7 ([3, Lemma 3.1]). *If* G *is a discrete abelian group, then* $V(G,\omega)$ *is type* I *if and only if* $Z_\omega(G) = G\omega = \{g \in G : \omega(g,x) = \omega(x,g) = 1$ *all* $x \in G\}$ *has finite index in* G.

PROOF. The 'if' part follows trivially from Theorem 4.5. Suppose $V(G,\omega)$ is type I, then for each $x \in G$, $C_\omega(x)$ contains the group $Z_\omega(\Delta)$ which according to Theorem 4.5 has finite index in G, thus $\Delta = G$ and $[G : Z_\omega(G)] < \infty$.

COROLLARY 4.8. *If* G *is a discrete abelian group, then either* $V(G,\omega)$ *is type* I *or* $V(G,\omega)$ *is type* II_1.

PROOF. Suppose the maximal type I part of $V(G,\omega)$ is non-zero, then by Theorem 4.4, there exists a subgroup H such that $[G : H] < \infty$ and $\omega|_{H\times H}$ is trivial. Since H is automatically abelian by Theorem 4.5, $V(G,\omega)$ is type I.

Theorem 4.4 is somewhat awkward for practical use because it is generally difficult to establish if a group and multiplier satisfy conditions 4.4(b) or 4.4(c). One would like to replace these conditions by $[G : \Delta] < \infty$, $|\Delta'| < \infty$ and ω^n is trivial for some n. The following example shows this cannot be done.

EXAMPLE 4.9. Let G be the discrete group H x H', where $H = \Pi_{j=1}^{\infty} Z(2)$ and $H' = \oplus_{j=}^{\infty} Z(2)$ and define

$$\omega((a_j,b_j),(a_j',b_j')) = \exp\left[\frac{\pi i}{2} \sum_{j=i}^{\infty} (a_j b_j' - a_j' b_j')\right],$$

$(a_j,b_j),(a_j',b_j') \in G$. By 4.7 and 4.8, $V(G,\omega)$ is type II_1, despite the fact that G and ω satisfy $[G : \Delta] < \infty$, $|\Delta'| < \infty$ and $\omega^2 = 1$.

EXAMPLE 4.10. Let G = Z x Z and

$$\omega((m,n)(m',n')) = \exp[2\pi i\alpha(mn' - m'n)].$$

Here $V(G,\omega)$ is type I if α is rational and type II_1 if α is irratinal. (Use 4.7 and 4.8).

EXAMPLE 4.11. Let p be a prime and let G be the group with finitely many generators $b, a_1, a_2 \ldots$ and the defining relations

$$b^p = a_1^p = a_2^p = \cdots = a_n^p = 1$$

$$a_i b = b a_i \qquad i = 1,2, \ldots$$

$$a_{i+k} \; a_i = b a_i a_{i+k} \qquad i,k = 1,2, \ldots .$$

It is clear that the commutator subgroup of G coincides with the centre of G and is equal to the finite cyclic group $\langle b \rangle$. This implies that there is a bound on the size of the conjugacy classes of G.

Let ω be the multiplier

$$(b^{s_0} a_1^{s_1} \ldots a_n^{s_n}, b_0^{s_0'} a_1^{s_1'} \ldots a_n^{s_n'}) = \exp\left[\frac{2\pi i}{p}(s_1 s_2' - s_1' s_2)\right] .$$

The elements of the form $b^{s_0} a_3^{s_3} a_4^{s_4} \ldots a_n^{s_n}$ form a normal subgroup H of G that is contained in Δ, hence $[G : \Delta] < \infty$; also $|\Delta'| \le |G'| = p$, so by Theorem 4.4, $V(G,\omega)$ has a non-zero maximal type I part. However, because $Z(\Delta) \cap H \subseteq Z(H) = \langle b \rangle$, the group $Z(\Delta)$ and consequently $Z_\omega(\Delta)$ has infinite index in G, thus by Theorem 4.5, $V(G,\omega)$ is not type I. We have shown that Corollary 4.8 does not necessarily hold for non-abelian groups.

The following theorem is an application of 4.5 to induced characters.

THEOREM 4.12. *Let* G *be a discrete group and* N *a normal subgroup of* G. *If* λ *is a character of* N, *that is a* 1-*dimensional representation of* N, *denote by* $K_\lambda = \{g \in G : gng^{-1}n^{-1} \in \ker \lambda$ *all* $n \in N\}$ *the stabilizer of* λ, *then the von Neumann algebra* V_λ *generated by the induced representation* $\lambda\uparrow_N^G$ *is type* I *if, and only if* K_λ *contains a subgroup* A *such that* $[K_\lambda : A] < \infty$ *and* $A' \subseteq \ker \lambda$.

PROOF. Let $P \subseteq K_\lambda$ be a set of coset representatives modulo N (including the identity of K_λ). For each $x \in K_\lambda$ denote by α_x the unique element in P such that $xN = \alpha_x N$. Let ψ be the function $G \to N : x \to \alpha_x^{-1}x$. The relations $\psi(n) = n$, $\psi(xn) = \psi(x)n$ and $\psi(nx) = \psi(x)x^{-1}nx$, for $x \in H$, $n \in N$ are evident. It is clear that $\tilde{\lambda} : x \to \lambda(\psi(x))$ is an extension of λ to K_λ and that the multiplier $\sigma(x,y) = \lambda(\psi(xy))\lambda(\psi(x)\psi(y))^{-1}$ associated with this extension can be factored through K_λ/N to yield a multiplier ω on K_λ/N. We may assume without loss of generality that ω is normalized. By [24, Theorem 7], the von Neumann algebra V_λ generated by the induced representation $\lambda\uparrow_N^G$ is *-isomorphic to $V(K_\lambda/N,\omega)$, thus by 4.5 we deduce that V_λ is type I if, and only if there exists a subgroup A in K_λ such that both (i) $[K_\lambda : A] < \infty$ and (ii) $A' \subseteq N$, $\sigma(x,y) = \sigma(y,x)$ *all* $x,y \in A$, hold. It remains to show that (ii) is equivalent to $A' \subseteq \ker \lambda$. Suppose (ii) is true. Let $x,y \in A$. Since $x^{-1}y^{-1}xy \in N$, $\psi(xy) = \psi(yxx^{-1}y^{-1}xy) = \psi(yx)x^{-1}y^{-1}xy$, but from $\sigma(x,y) = \sigma(y,x)$ follows $\lambda(\psi(yx)) = \lambda(\psi(xy))$ so that $\lambda(x^{-1}y^{-1}xy) = 1$. Conversely if $A' \subseteq \ker \lambda \subseteq N$, then for $x,y \in A$,

$\lambda(xy) = \lambda(\psi(yxx^{-1}y^{-1}xy)) = \lambda(\psi(yx))\ \lambda(x^{-1}y^{-1}xy) = \lambda(\psi(yx))$ and thus $\sigma(x,y) = \sigma(y,x)$. This completes the proof.

CHAPTER IV[2]

GROUPS WITH FINITE DIMENSIONAL IRREDUCIBLE MULTIPLIER REPRESENTATIONS

Let G be a locally compact group and ω a normalized multiplier on G. Recall that V(G) denotes the von Neumann algebra generated by the left regular (ordinary) representation of G (I.7). In Chapter III we defined and investigated the structure of the von Neumann algebra $V(G,\omega)$ in the case when G is discrete. In the more general situation when G is locally compact (but not necessarily discrete), the definitions are similar. Indeed, the map $\rho : G \to U(L^2(G))$, defined by

$$\rho(g)f(x) = \omega(g^{-1},x)f(g^{-1}x)$$

almost everywhere in x, all $g \in G$ and $f \in L^2(G)$, is a ω^{-1}-representation of G and generates a von Neumann algebra also denoted by $V(G,\omega)$. ρ is called the left regular ω^{-1}-representation of G.

In this chapter, we determine necessary and sufficient conditions on G such that the maximal type I_f central projection in $V(G,\omega)$ is non-zero (respectively the identity operator in $V(G,\omega)$), and construct this projection explicitly as a convolution operator on $L^2(G)$. This extends the results of Chapter III.

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[2]After finishing this thesis, I received a preprint entitled "The type structure of multiplier representations which vanish at Infinity" by E. Kaniuth and G. Schlichting, which contains results that are similar to those presented in this chapter.

For the case of ordinary representations, this problem has been successfully dealt with by Kamith [23] and Taylor [48].

We point out that it is no longer the case as it was for G discrete (see III.1.6(iv) and I.3.2) that $V(G,\omega)$ is necessarily finite. For an example see 4.1. It is for this reason that our methods allow a characterization of the type I_f part, but not the type I part in $V(G,\omega)$.

Recall that for a locally compact group G, G_o denotes the von Neumann kernel

$$G_o = \cap\{\ker \pi : \pi \in G\hat{} \text{ and } \dim \pi < \infty\}.$$

and G_{FC} denotes the topological finite class group of G which consists of all elements of G that belong to a relatively compact conjugacy class (I.7). If ω is a normalized Borel multiplier on G, then G^ω denotes the central extension of G (I.6). For a subset H of G, H^- denotes the closure of H in G, and if H is also a subgroup, H' denotes the commutator subgroup of H.

1. Preliminaries

Throughout this section, G will denote a locally compact group and ω a Borel multiplier on G.

LEMMA 1.1. *Let* A *be a subset of* G^{ω} *and let* $h : G^{\omega} \to G$ *denote the canonical projection, then*

(i) A^{-} *is compact if and only if* $h(A)^{-}$ *is compact,*

(ii) *if* A^{-} *is compact, then* $h(A)^{-} = h(A^{-})$.

PROOF. A^{-} compact implies $h(A^{-})$ compact, but $h(A^{-}) \supseteq h(A)$, hence $h(A)^{-}$ is compact and $h(A^{-}) \supseteq h(A)^{-}$. Conversely, if $h(A)^{-}$ is compact, then by [18, 5.24], $h^{-1}(h(A)^{-})$ is compact; but $h^{-1}(h(A)^{-}) \supset h^{-1}h(A) \supset A$, hence A^{-} is compact and $h^{-1}(h(A)^{-}) \supseteq A^{-}$, that is $h(A)^{-} = hh^{-1}(h(A)^{-}) \supseteq h(A)^{-}$. This completes the proof.

COROLLARY 1.2. *Let* G *be a locally compact group and* ω *a normalized Borel multiplier.*

(i) $(G_{FC})^{\omega} = (G^{\omega})_{FC}$.

(ii) *If one of* $(G_{FC})'^{-}$ *and* $((G^{\omega})_{FC})'^{-}$ *is compact, then so is the other and* $(G_{FC})' = h[((G^{\omega})_{FC})'^{-}]$.

PROOF. For (i), let A equal the conjugacy class of some $(t,x) \in G^{\omega}$ and in (ii), let $A = ((G^{\omega})_{FC})'$. Now use Lemma 1.1.

Combining 1.2 with I.7.8 shows that the maximal type I_f central projection in V(G) is non-zero if and only if the maximal type I_f central projection in $V(G^{\omega})$ is non-zero. This follows also from Taylor [48, Proposition 5.2].

Let E_n, $n \in \mathbb{Z}$ be the maps on $L^2(G^\omega)$ given by

$$E_n f(t,x) = t^n \int_{\mathbb{T}} s^{-n} f(s,x)\, ds,$$

for almost all $(t,x) \in G^\omega$, $f \in L^2(G^\omega)$. (Compare this with Kleppner [24, page 563].) Clearly $E_n L^2(G^\omega)$ consists of all those functions $f \in L^2(G^\omega)$ such that $f(t,x) = t^n f(1,x)$ for almost all $(t,x) \in G^\omega$. It follows that the E_n, $n \in \mathbb{Z}$ are mutually orthogonal idempotents. For $n \in \mathbb{Z}$, E_n is just convolution by the measure obtained if you multiply the measure on G^ω supported on $\mathbb{T}$ which restricts to Haar measure on $\mathbb{T}$, with the character χ_n of $\mathbb{T}$ given by $\chi_n(t) = t^n$, $t \in \mathbb{T}$. It is easy to check that E_n commutes with the right and left regular representation of G^ω, so by I.7.4, E_n is in the centre of $V(G^\omega)$.

<u>THEOREM 1.3.</u> *With the above notation, the* E_n, $n \in \mathbb{Z}$ *are mutually orthogonal central projections in* $V(G^\omega)$ *and*

(i) *the three von Neumann algebras* $E_n V(G^\omega)$, $V(G,\omega^n)$ *and* $V(\chi_{-n}\uparrow_{\mathbb{T}}^{G^\omega})$ *- the von Neumann algebra generated by the induced representation* $\chi_{-n}\uparrow_{\mathbb{T}}^{G^\omega}$ *- are spatially isomorphic,*

(ii) $\sum_{n\in\mathbb{Z}} E_n = I$ *(the identity operator).*

<u>PROOF.</u> (i) Let τ denote the left regular representation of G^ω and ρ_n, $n \in \mathbb{Z}$, the left regular ω^n-representation of G. Observe that the representation space of $\chi_{-n}\uparrow_{\mathbb{T}}^{G^\omega}$ is just $E_n L^2(G^\omega)$

and that $E_n\tau = \chi_{-n}\uparrow_{\mathbb{T}}^{G^\omega}$. It follows that $E_nV(G^\omega) = V(\chi_{-n}\uparrow_{\mathbb{T}}^{G^\omega})$.

Denote by i the injection map $i : G \to G^\omega$ and by ϕ the map

$$\phi : E_nL^2(G) \to L^2(G) : f \circ i.$$

(We insist that Haar measure on G^ω be scaled so that the measure of $\mathbb{T}$ is 1.) Since Haar measure on G^ω is the product of Haar measure on G and Haar measure on $\mathbb{T}$, we have

$$\int_{G^\omega} |f(t,x)|^2 d(t,x) = \int_G \left(\int_{\mathbb{T}} f(1,x)|^2 \, dt\right) dx$$

$$= \int_G |\phi(f)(x)|^2 \, dx,$$

thus ϕ is an isometry. The spatial monomorphism

$$E_nV(G^\omega) \to B(L^2(G)) : T \to \phi \circ T \circ \phi^{-1}$$

is weakly continuous and maps $E_n\tau(1,x)$ to $\rho_n(x)$. Since $t^n E_n\tau(1,x) = E_n\tau(t,x)$, the von Neumann algebra generated by $\{E_n\tau(\lambda,x) : x \in G\}$ is precisely $E_nV(G^\omega)$. Thus any element in $E_nV(G^\omega)$ is the limit of operators T with the property $\phi(T) \in V(G,\omega^n)$, thus by the weak continuity of ϕ, $\phi(T) \in V(G,\omega^n)$. In other words, the range of ϕ is a subset of $V(G,\omega^n)$. By Sakai [40,1.16.2], the range of ϕ is weakly closed; it contains the operators $\phi \circ E_n\tau(1,x) \circ \phi^{-1} = \rho_n(x)$, $x \in G$ and therefore must equal $V(G,\omega^n)$.

(ii) From the theory of characters, the direct sum $\oplus_{n\in\mathbb{Z}} \chi_n$ is the regular representation of $\mathbb{T}$, hence by the non-separable version of I.8.2, we see that the induced representation

$$(\oplus_{n\in\mathbb{Z}} \chi_n) \uparrow_{\mathbb{T}}^{G^{\omega}}$$

is the regular representation of G^{ω}. Since inducing commutes with taking direct sums (non-separable version if I.8.3), we have

$$\oplus_{n\in Z}(\chi_n \uparrow_{\mathbb{T}}^{G^{\omega}}) = (\oplus_{n\in\mathbb{Z}} \chi_n) \uparrow_{\mathbb{T}}^{G^{\omega}}.$$

Thus $V(G^{\omega}) = \oplus_{n\in\mathbb{Z}} V(\chi_n \uparrow_{\mathbb{T}}^{G^{\omega}})$,
which is the required result.

LEMMA 1.4. ([3]). *Let* π *be a* ω*-representation of the locally compact group* G. *Denote by* p-ker π *the closed normal subgroup*.

$$\{x \in G : \pi(x) = \gamma(x).I, \textit{ for some } \gamma(x) \in \mathbb{T}\}$$

of G. *Then* ω *is similar to a multiplier lifted from* G/p-ker π.

PROOF. Let ψ be the canonical homomorphism from the unitary group $U(H_\pi)$, where H_π denotes the Hilbert space associated with π, to the quotient $U(H_\pi)/\mathbb{T}$. According to Feldman and Greenleaf [9], this map has Borel transversal, that is a Borel map α, back from $U(H_\pi)/\mathbb{T}$ to $U(H_\pi)$, taking the identity element in $U(H_\pi)/\mathbb{T}$ to the identity element in $U(H_\pi)$ such that $\psi \circ \alpha$ is the identity map. From this we obtain the commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{\pi} & U(H_\pi) & \xrightarrow{\psi} & U(H_\pi)/\mathbb{T} \\ & \pi' \searrow & & \psi \uparrow \downarrow \alpha & \\ & & & & U(H_\pi) \end{array}$$

where π' is the map $\alpha \circ \psi \circ \pi$. Clearly π' is a multiplier representation of G associated with a multiplier ω' say, and since $\psi \circ \pi = \psi \circ \pi'$, ω' is similar to ω. From the definition of π', it follows that ω' is constant on p-ker π cosets in $G \times G$, that is, it is lifted from a multiplier on G/p-ker π.

LEMMA 1.5. *Let* G *be a locally compact group with normalized Borel multipler* ω. *Suppose the maximal type* I_f *central projection in* $V(G)$ *(or equivalently in* $V(G^\omega)$) *is non-zero, and that* G *admits a finite dimensional* ω*-representation* π, *then* $K = h((G^\omega)_\circ)$ *is compact (where* h *is the canonical projection* $G^\omega \to G$) *and* ω *is similar to a multiplier which is lifted from* G/K.

PROOF. That K is compact follows from I.7.9. Suppose $(t,x) \in (G^\omega)_\circ$, then since $(s,y) \to s\pi(y)$, $(s,y) \in G^\omega$, is a finite dimensional (ordinary) representation of G^ω, we have $I = t\pi(x)$, that is $x \in$ p-ker π. The result now follows from Lemma 1.4.

2. The main theorems

THEOREM 2.1. *Let G be a locally compact group with normalized Borel multiplier ω. Then the following three conditions are equivalent.*

(i) *The maximal type I_f central projection e in V(G) (or equivalently, the maximal type I_f central projection in $V(G^{\omega})$) is non-zero and there exists a finite dimensional ω-representation π of G.*

(ii) *The maxmimal type I_f central projection d in $V(G,\omega)$ is non-zero.*

(iii) $[G : G_{FC}] < \infty$, $(G_{FC})'^{-}$ *is compact and G admits a finite dimensional ω-representation π.*

PROOF. (i) and (iii) are obviously equivalent by I.7.8. Suppose (ii) is true and let π be a finite dimensional representation of $dV(G,\omega)$. If we compose representation with the projection $V(G,\omega) \to dV(G,\omega) : a \to da$, we obtain a representation π of $V(G,\omega)$. By I.6.3, the left regular ω^{-1}-representation ρ of G corresponds to a representation ρ' of $L^1(G,\omega^{-1})$ which we know to be faithful ([7,13.3.6]). Thus ρ' is a *-monomorphism of $L^1(G,\omega^{-1})$ to $V(G,\omega)$ and $\pi \circ \rho'$ is a representation of $L^1(G,\omega^{-1})$. Again by I.6,3, this representation corresponds to a ω^{-1}-representation of G and it is easy to check that this ω^{-1}-representation is just $g \to \pi(d\rho(g))$, $g \in G$. Hence $g \to \pi^*(d\rho(g^{-1}))$, where * denotes the Hilbert space adjoint, is a finite dimensional ω-representation of G. On the other hand, by Theore 1.3, the maximal type I_f central projection in $V(G^\omega)$ is non-zero. This shows (ii) implies (i). That (i) implies (ii) will be proved together with Theorem 2.3.

LEMMA 2.2. *Suppose* K *is a compact normal subgroup of* G *and* ω *is lifted from a multiplier* ω' *on* G/K, *then* K *is also a compact normal subgroup of* G^ω *and* G^ω/K *is topologically isomorphic to* $(G/K)^{\omega'}$.

PROOF. Let ψ be the monomorphism $G^\omega \to (G/K)^{\omega'} : (t,x) \to (t,xK)$. The kernel of this map is $\{(1,k) : k \in K\}$, that is the image of K in G^ω, thus the induced map from G^ω/K to $(G/K)^{\omega'}$ is an isomorphism. This isomorphism preserves the Haar measures on these groups; they uniquely define the topologies on these groups, thus G^ω/K and $(G/K)^{\omega'}$ are also topologically isomorphic.

THEOREM 2.3. *Suppose* G *is a locally compact group with Borel multiplier* ω. *Suppose the maximal type* I_f *central projection* d *in* $V(G,\omega)$ *is non-zero. We assume (using 1.5 and 2.1) that* ω *is lifted from a multiplier* ω' *of* G/K, *where* K *is the compact normal subgroup* $K = h((G^\omega)_\circ)$ (*h being the canonical projection* $G^\omega \to G$). *Then* d *is the operator* $L^2(G) \to L^2(G)$ *defined by*

$$df(x) = \int_K f(k^{-1}x)d\lambda(k) ,$$

almost all $x \in G$, $f \in L^2(G)$, *where* λ *is Haar measure on* G *normalized such that* $\lambda(K) = 1$. *Furthermore, for each* $n \in Z$, d *is the maximal type* I_f *central projection in* $V(G,\omega^n)$ *and* $dV(G,\omega^n)$

is isomorphic to $V(G/K,(\omega')^n)$.

PROOF. First we give a proof, as promised, of the statement '(i) implies (ii)' of Theorem 2.1. Let $\alpha : L^2(G) \to L^2(G)$ be defined by

$$\alpha f(x) = \int_K f(k^{-1}x)\, d\lambda(k),$$

almost all $x \in G$, $f \in L^2(G)$. The proof that α is a central idempotent in $V(G,\omega^n) \simeq E_n V(G^\omega)$ (and hence in $V(G^\omega)$), and that $\alpha V(G,\omega^n)$ and $V(G/K,(\omega')^n)$ are spatially isomorphic is similar to the proof of the corresponding facts about E_n in Theorem 1.3. Since

$$V(G,\omega^n) \simeq V(G/K,(\omega')^n) \oplus V(G/K,(\omega')^n)^\perp$$

(where $\perp$ denotes an orthogonal ~~complement~~ subalgebra), we have by Theorem 1.3,

$$V(G^\omega) \simeq V((G/K)^{\omega'}) \oplus V((G/K^{\omega'})^\perp$$

but $(G/K)^{\omega'}$ and $G^\omega/(G^\omega)_\circ$ are topologically isomorphic (2.2) and $V((G^\omega)/(G^\omega)_\circ)$ is spatially isomorphic to the maximal type I_f direct summand of $V(G^\omega)$ (I.7.9). In particular we have $d = \alpha \neq 0$

Now assume $d \neq 0$, then by "(ii) implies (i)" of Theorem 2.1, the maximal type I_f central projection in $V(G^\omega)$ is non-zero, thus by the same argument as above, we reach the desired conclusion.

COROLLARY 2.4. *Suppose the maximal type* I_f *central projection* d *in* $V(G,\omega)$ *is non-zero, and let* $n \in \mathbb{Z}$, *then the following equations obtain.*

$$G_\circ = h[((G^\omega)_\circ] = h[\cap \{\ker \pi : \pi \text{ is a finite dimensional representation of } G^\omega \text{ such that } \pi|_{\mathbb{T}}(t) = t^n\}]$$

$$= \{g \in G : \text{there exists } \gamma(g) \in \mathbb{T} \text{ such that } \pi(g) = \gamma(g)I \text{ for all finite dimensional } \omega^n\text{-representations of } G\},$$

where $h : G^\omega \to G$ *denotes the canonical projection.*

PROOF. Let $K = h[(G^\omega)_\circ]$ and denote the last two sets in the above equality by H and L respectively. That $K \subseteq H \subseteq L$ is clear from the definitions and the property that a ω^n-representation π of G extends to an ordinary representation π' of G^ω such that $\pi'|_{\mathbb{T}}(t) = t^n$. Using the proof of Lemma 1.5, we assume that ω is lifted from a multiplier on G/L. The finite dimensional representations of $dV(G,\omega^n)$ separate the points of $dV(G,\omega^n)$, hence $\pi^\circ(\dot{g}) = I$ if and only if $\pi^*(d\rho(g^{-1})) = \pi^*(d)$ for all such representations π^*, where π° denotes the ω^n-representation $g \to \pi^*(d\rho(g^{-1}))$ and ρ is the regular ω^n-representation of G. This happens if and only if $\rho(g)d = d$. If $g \in K$, then clearly $\rho(g)d = d$. Conversely, suppose $g \notin K$ and $\rho(g)d = d$. Let U be a neighbourhood of gK in G/K of finite Haar measure, not containing K. Let ψ' be the characteristic function of U and ψ be a lifting to G, then $\psi \in dL^2(G)$, is continuous at g and satisfies $\psi(gK) = \{1\}$, $\psi(K) = \{0\}$. Since

$\rho(g)d = \rho(g)$, we have

$$\omega(g^{-1},x)\psi(g^{-1}x) = \psi(x),$$

almost all $x \in G$. Substituting $g = x$ into this formula gives $0 = |\psi(1)| = |\psi(g)| = 1$ which is a contradiction. We have shown that $\pi^\circ(g) = I$ if and only if $g \in K$, thus

$$K = \cap \{\ker \pi^\circ : \pi \text{ is a finite dimensional representation of } dV(G,\omega^n)\}$$
$$\supseteq L$$

If we let $n = 0$, we obtain the remainder of the corollary, that is $G_0 = L$.

<u>LEMMA 2.5</u>. (Moore [31], Lemma 4.1). *Let* G *be a locally Compact group with normalized Borel multiplier* ω. *Each irreducible* ω*-representation of* G *is finite dimensional if and only if every* ω*-representation of* G *is finite.*

<u>PROOF</u>. The if part is clear because an irreducible ω-representation is finite if and only if it is finite dimensional. Let A be the twisted group C*-algebra $C^*(G,\omega)$ (see I.6). By Theorem I.6.3, it is sufficient to prove the corresponding statement of the Lemma for representations of A. If I is a two sided primitive ideal of A, then I is the kernel of an irreducible representation of A. According to our hypothesis, π has dimension n for some n. Since π induces a homomorphism for A/I to the $n \times n$ matrices, A/I satisfies the polynomial identity S_{2n} (I.4). Let F_n be the set

of primitive ideals I such that A/I satisfies the polynomial identity S_{2n}, then by assumption $F = \cup F_n$ is the primitive ideal space of A. For each subset K of F, the kernel I(K) of K is $\cap I$ ($I \in K$). The closure of K in the kernel-hull topology on F is $K^- = \{J \in F : J \supseteq I(K)\}$. We show that F_n is closed in this kernel-hull topology. Since $S_{2n}(A) \subseteq I$ for each $I \in F_n$, the polynomial S_{2n} is satisfied in $A/I(F_n)$. Moreover, if $J \in F_n^-$ then $S_{2n}(A) \subseteq I(F_n) \subseteq J$, hence $J \in F_n$. Thus F_n is closed.

We define for each closed subset K of F and for every representation π of A, $P_\pi(K)$ to be the projection onto $H_\pi(K) = \{x \in H_\pi : \pi(a)x = 0 \text{ all } a \in I(K)\}$. According to [16 : Theorem 1.9], $K \to P_\pi(K)$ extends to a countably additive projection valued measure on the Borel sets of F with $P_\pi(K)$ in the centre of the von Neumann algebra $V(\pi)$ generated by π. Since $F = \cup F_n$, $H_\pi = \sum(H_\pi(F_n) - H_\pi(F_{n-1}))$. We define a subrepresentation π_n of π by restricting π to the invariant subspace $H_\pi(F_n) - H_\pi(F_{n-1})$. Then $\pi = \sum\pi_n$, and $\pi_n(a) = 0$ if $a \in I(F_n)$. Since S_{2n} is satisfied in $A/I(F_n)$ as noted above, the algebra $\pi_n(A)$ and hence its weak closure $V(\pi_n)$ satisfies $S_{2n} = 0$. Now any von Neumann algebra satisfying this identity is finite (I.4.5). Moreover $V(\pi)$ is the direct sum of the $V(\pi_n)$ and since the direct sum of finite algebra is finite (I.3), $V(\pi)$ is finite as desired.

THEOREM 2.6. *Let* G *be a locally compact group and* ω *a normalized Borel multiplier on* G. *The following are equivalent:*

(i) $V(G,\omega)$ *is type* I_f.

(ii) All irreducible ω-representation of G *are finite dimensional.*

(iii) The following conditions hold.

(a) $[G : G_{FC}] < \infty$,

(b) $(G_{FC})'^{-}$ *is compact,*

(c) G *admits a finite dimensional ω-representation,*

(d) $G_o = \{e\}$.

(iv) G^{ω} *is a Moore group, that is all its irreducible (ordinary) representations are finite dimensional.*

PROOF. (i) implies (iii). By Theorem 2.1, $[G : G_{FC}] < \infty$, $(G_{FC})'^{-}$ is compact and G admits a finite dimensional ω-representation. Further by 2.3 and 2.4, V(G,ω) type I_f implies

$$\{e\} = K = h((G^{\omega})_o) = G_o$$

(iii) implies (i). By 2.1, the maximal type I_f central projection in V(G,ω) is non-zero, so using 2.3 and 2.4, we obtain V(G,ω) is type I.

(ii) implies (iii). Let A = C*(G,ω) be the twisted group C*-algebra of G. Since each representation of A is finite dimensional, π(A) is contained in the compact operators for each irreducible representation π of A, A is CCR (or liminal), thus by Dixmier [7,5.5.2], A is type I, consequently V(G,ω) is type I. To see that V(G,ω) is finite, apply Lemma 2.5.

(iii) implies (iv). If π is a finite dimensional ω-representation of G, then the n-fold tensor product π ⊕ ... ⊕π is a finite dimensional

ω^n-representation of G, hence $V(G,\omega^n)$ is type I_f for each $n \in \mathbb{Z}$ (recall that we know (i) and (iii) to be equivalent) and by Theorem I.3, $V(G^\omega)$ is type I_f. It follows from I.7.6 that G^ω is a Moore group.

(iv) implies (ii). If π is an irreducible ω-representation of G, then $(t,x) \to t\pi(x)$ being an (ordinary) representation of G^ω, must be finite dimensional by assumption.

Before we proceed to examples, we need one other result, a result obtained using Theorem 4.5 of Chapter III, which generalizes Moore [31, Theorem 1] and Taylor [48, Theorem 2].

THEOREM 2.7. *Let* G *be a locally compact group and* ω *a normalized Borel multiplier on* G. *The following are equivalent.*

(i) $V(G,\omega)$ *is type* $I_{\leq k}$ *for some integer* k, *that is non-zero maximal type* I_n *central projections occur in* $V(G,\omega)$ *only if* $n \leq k$.

(ii) G *has an open abelian subgroup* H *of finite index in* G *such that the restriction of* ω *to* H *is trivial.*

(iii) The irreducible ω*-representations of* G *are of dimension at most* k *for some integer* k.

PROOF. (i) implies (iii). We follow the proof of Taylor [48, Theorem 2]. From I.4.5 we know that $S_{2k} = 0$ is satisfied in $V(G,\omega)$. Suppose π is an irreducible ω-representation of G, then $\pi' : g \to \pi^*(g^{-1})$ is an irreducible ω^{-1}-representation of G. Now, as in the proof of 2.1, the left regular representation of

$L^1(G,\omega^{-1})$ is a *-monomorphism into $V(G,\omega)$. Hence $L^1(G,\omega^{-1})$ also satisfies S_{2k}. It follows that $V(\pi')$ satisfies S_{2k}, consequently π' and π are of dimension at most k.

(iii) implies (ii). Following the proof of Moore [31, Theorem 1], we consider the underlying group G_d of G (with discrete topology) and the corresponding twisted group algebra $A = A(G_d,\omega)$ (see definition following III.4.5). To each ω^{-1}-representation π of G, there corresponds an algebra representation π° of $A : \mu \to \sum_{g\in G} \mu_g \rho(g)$. First we show that all representations π° obtained from irreducible ω^{-1}-representations π of G separate the points of A.

The twisted group algebra A acts on $L^1(G,\omega^{-1})$ as follows

$$\mu.f(x) = \sum_{g\in \mathrm{Supp}\ \mu}\mu_g f(g^{-1}x)\omega(g^{-1},x),$$

almost all $x \in G$, $\mu \in A$, $f \in L^1(G,\omega^{-1})$. Given a non-zero μ in A, it is clear that $\mu.f \neq 0$ for some $f \in L^1(G,\omega^{-1})$. Let π' be an irreducible representation of $L^1(G,\omega^{-1})$ such that $\pi'(\mu.f) \neq 0$ (I.5.3). According to I.6.3, there exists an irreducible ω^{-1}-representation π of G such that

$$\langle \pi'(f)\xi,\eta\rangle = \int_G \langle \pi(x)\xi,\eta\rangle f(x)\ dx$$

all $\xi,\eta \in H_\pi, f \in L^1(g,\omega^{-1})$. Using the invariance of Haar measure, we obtain

$$\langle \pi'(\mu.f)\xi,\eta\rangle = \sum \mu_g \int_G \langle \pi_x\xi,\eta\rangle\, \omega(g^{-1},x)f(g^{-1}x)\ dx$$

$$= \sum \mu_g \int_G \langle \pi_{gx}\xi,\eta\rangle \omega(g,x)f(x) \quad dx$$

$$= \sum \mu_g \int_G \langle \pi_g\pi_x\xi,\eta\rangle f(x) \quad dx$$

Hence $0 \neq \pi'(\mu.f) = \pi^\circ(\mu)\pi'(f)$. It follows that $\pi^\circ(\mu) \neq 0$. This shows that A has a separating family of representations of degree at most k. We now infer, from the fact that $S_{2k}(\pi(A)) = \pi(S_{2k}(A)) = 0$ for each such π, that $S_{2k} = 0$ is satisfied in A.

Let π be any irreducible ω^{1}-representation of G_d and π° the corresponding representation of A. By I.4.6 and I.5.1, S_{2k} is satisfied in $B(H_\pi)$ and we conclude that the dimension of π is at most k. It now follows from III.4.5. that G has an abelian subgroup K of finite index in G such that $\omega|_{K\times K}$ is symmetric. The closure H of K is an open abelian group and if $\sigma = \omega|_{H\times H}$, then the map $\tilde{\sigma} : H \to H\hat{}$ defined by $\tilde{\sigma}(g)(h) = \sigma(h,g)/\sigma(g,h)$ is continuous (II.1.1), so it follows that $\omega(g,h) = \omega(h,g)$ for all $h,g \in H$ and consequently (II.1.1), $\omega|_{H\times H}$ is trivial.

(ii) implies (i). Let $g_1, \ldots, g_k$ be a complete set of representatives modulo H, then

$$A(G,\omega) = \oplus_{i=1}^{k} A(H,\omega)\rho(g)$$

is a matrix algebra over the abelian algebra $A(H,\omega)$, then as in the proof of III.4.5, $V(G,\omega)$ is type $I_{\leq k}$.

4. EXAMPLES

Let G be a locally compact group with Borel multiplier ω. In Chapter III we saw that for G discrete, (G,ω) is type I if and only if G^{ω} is type I and this occurs if and only if $V(G,\omega)$ is type I; and in this chapter, this has been generalized to assert that for G locally compact, the following are equivalent. (i) G has only finite dimensional ω-representations, (ii) G^{ω} is a Moore group, and (iii) $V(G,\omega)$ is type I_f.

Example II.3.4(iii) shows that (G,ω) type I does not necessarily imply that G^{ω} is type I and Mackey ([29, Section 7]) constructs a non-type I group G such that V(G) is type I.

EXAMPLE 4.1. Let G be the group $\mathbb{R} \times \mathbb{R}$ and ω the multiplier

$$\omega(x,y)(x',y')) = e^{i\alpha xy'},$$

$(x,y),(x',y') \in \mathbb{R} \times \mathbb{R}$, where α is an irrational number. By II.1.3, $V(G,\omega)$ is type I. However, all irreducible ω-representations of G are infinite dimensional [3, Theorem 3.3], thus by 2.1, $V(G,\omega)$ is not finite.

EXAMPLE 4.2. Let G and ω be as in III.4.9, then $V(G,\omega)$ is type II_1, yet G satisfies the first two conditions of 2.1(iii), thus G admits no finite dimensional ω-representations.

Now we expand on Example 4.10 of Chapter III. For each $t = e^{2\pi i\alpha} \in \mathbb{T}$, $(\alpha \in [o,2\pi[)$, we obtain a multiplier ω_t on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$\omega_t((m,n),(m',n')) = t^{mn'},$$

$(m,n),(m',n') \in \mathbb{Z} \times \mathbb{Z}$. Theorems II.2.3 and II.3.5 show that (up to similarity) all multiplier on $\mathbb{Z} \times \mathbb{Z}$ are of this form. We say that t is rational (respectively irrational) if α is rational (respectively irrational). As in Example III.4.10, we see that $V(\mathbb{Z} \times \mathbb{Z},\omega_t)$ is type I if t is rational and type II_1 if t is irrational. Suppose α is rational, say $\alpha = p/q$ with p and q relatively prime. We wish to know more about the ω-representations of G and to this end, we utilize Hannabuss [17,Theorem 4.1] which says that the irreducible ω-representations of G are all induced from characters of a maximal isotropic subgroup (that is a subgroup H such that $H\omega = H$; see definition following II.1.2). Now clearly the subgroup $H = \{(m,nq) : m,n \in \mathbb{Z}\}$ of $G = \mathbb{Z} \times \mathbb{Z}$ is maximal isotropic and G/H is isomorphic to $\mathbb{Z}(q)$ thus the ω_t-representations of $\mathbb{Z} \times \mathbb{Z}$ are all of dimension q.

<u>EXAMPLE 4.3</u>. Let Δ_p (p a fixed prime) be the group of p-adic integers (for details see Appendix) and G the group $\mathbb{Z} \times \mathbb{Z} \times \Delta_p$ with multiplication

$$(a,b,x)\ (z',b',x') = (a + a',b + b',x + x' + \psi(ab')),$$

$(a,b,x),(a',b',x') \in G$, where $\psi : \mathbb{Z} \to \Delta_p$ is the canonical injection of $\mathbb{Z}$ onto a dense subgroup of Δ_p (see Appendix). We topoligize G so that Δ_p becomes a compact open subgroup. With this topology, G becomes a locally compact separable topological group. For each $t \in \mathbb{T}$, we define a multiplier

$$\sigma_t = \omega_t \circ k,$$

where $k : G \to \mathbb{Z} \times \mathbb{Z}$ is the canonical homomorphism $k(m,n,x) = (m,n)$. Given an irreducible ω_t-representation π of $\mathbb{Z} \times \mathbb{Z}$, denote by π' the σ_t- representation of G obtained by composing π with k.

Since Δ_p is compact, the abelian group dual $\Delta_p\hat{}$ is discrete, thus by I.8.4, every quasi-orbit on $\Delta_p\hat{}$ is transistive and all the irreducible σ_t-representations of G can be constructed using I.8.5, I.8.6 and I.8.7.

Identify the abelian group dual $\Delta_p\hat{}$ of Δ_p with the subgroup $\Lambda = \{s \in \mathbb{T} : s = \exp[2\pi ik/p^n], k,n \in \mathbb{Z}\}$ of $\mathbb{T}$ using the correspondence $\Lambda \times \Delta_p \to \mathbb{T} : (s,x) \to s^x$, where $x \to s^x$, $s \in \Lambda$ is the continuous extension from $\mathbb{Z}$ to Δ_p of the homomorphism $\mathbb{Z} \to \Lambda :$ $n \to s^n$ (see Appendix).

Let $t \in \mathbb{T}$ and $s \in \Lambda$, then the stabilizer of s in G is all of G. Therefore all the σ_t-representations of G are given by I.8.7. Suppose we have a σ_t-representation γ of G which reduces to a multiple of s on Δ_p (recall that s is viewed as a representation of Δ_p). First we define an extension of s to G as follows

$$s' : G \to \mathbb{T} : (a,b,x) \to s^x.$$

The multiplier associated with this extension is precisely

$$((a,b,x)(a',b',x')) \to \frac{s'((a,b,x)(x',b',x'))}{s'(a,b,x)s'(a',b',x')}$$

$$= s^{ab'} = \sigma_s((a,b,x),(a',b',x')).$$

Thus by I.8.7, γ must be of the form $\pi's'$, where π is an $\omega_{ts^{-1}}$-representation of $\mathbb{Z} \times \mathbb{Z}$. (Note that $\sigma_{ts^{-1}}\sigma_s = \sigma_t$ as required.) As s ranges through Λ, we obtain all irreducible σ_t representations of G.

Observe that ts^{-1} is rational if and only if t is rational. Hence the irreducible σ_t-representations of G, $t \in \mathbb{T}$ are all infinite dimensional if t is irrational and are all finite dimensional if t is rational. It follows from 2.1 and 2.6 that $V(G,\sigma_t)$ is type I_f if t is rational and the type I_f part in $V(G,\sigma_t)$ is zero if t is irrational.

Furthermore, if t is rational, as noted earlier, by varying s, the irreducible $\omega_{ts^{-1}}$ -representations of $\mathbb{Z} \times \mathbb{Z}$ can be chosen to be of arbitrary dimension.

The same is true of the σ_t-representations of G. Thus, although $V(G,\omega_t)$ is type I_f, Theorem 2.7 shows that $V(G,\omega_t)$ has a non-zero maximal type I_n part for arbitrary large n. This phenomenon does not occur in the case where G is discrete (combine III.4.5 and 2.7).

APPENDIX

Since there is no representation of the p-adic integers Δ_p, p-adic numbers Ω_p and $\mathbb{Q}^\wedge$ available for reference, which is suitable for our purposes, we use this Appendix to set forth such a representation, notation and some of the properties of these groups.

1. Δ_p and Ω_p

The maps $\mathbb{Z}(p^{r+n}) \to \mathbb{Z}(p^r) : x \to x(\text{mod } p^r)$ (p a fixed prime) form an inverse system of (discrete) groups. Let $\Delta_p = \underleftarrow{\mathbb{Z}(p^r)}$, $r = 1,2, \ldots$; thus Δ_p is the closed subgroup of the compact group $\Pi_{r=1}^{\infty} \mathbb{Z}(p^r)$ consisting of sequences (x_n) such that $x_n = x_{n+1}(\text{mod } p^n)$.

Δ_p is a topological ring (under pointwise multiplication) and each ideal $p^r\Delta_p = \{x + \ldots + x : p^r\text{-times}, x \in \Delta_p\}$ is closed (recall Δ_p is compact) and is the kernel of the homomorphism $\Delta_p \to \mathbb{Z}(p^r) : (x_n) \to x_r$; hence $p^r\Delta_p$ is open and the quotient $\Delta_p/p^r\Delta_p$ is isomorphic to $\mathbb{Z}(p^r)$. In fact the groups $p^r\Delta_p$ form a neighbourhood base at 0.

The homeomorphism $\psi : \mathbb{Z} \to \Delta_p : m \to (m(\text{mod } p^n))$ is an injection. Let $x = (x_n) \in \Delta_p$. Since $\psi(x_r) - x \in p^r\Delta_p$, $\lim_{n\to\infty}\psi(x_n) = x$; thus $\psi(\mathbb{Z})$ is dense in Δ_p. Similarly $p^r\mathbb{Z}$ is dense in $p^r\Delta_p$. Furthermore, if $(m,p) = 1$, it is not hard to see that $\Delta_p \to \Delta_p$: $x \to mx$ is a continuous (and hence topological) automorphism. Note that the action of $\mathbb{Z}$ on Δ_p is compatible with the ring multiplication in Δ_p, that is $mx = x + \ldots + x = (m)x$, in particular $p^r x = \psi(p^r)x = (0, \ldots , 0, p^r x_1, p^r x_2 \ldots)$ (r zeros).

PROPOSITION 1.1. (Serre [43,II Proposition 2]). *(i) For an element of* Δ_p *(respectively* $\mathbb{Z}(p^r)$*) to be invertible, it is necessary and sufficient that it is not divisbile by* p.

(ii). If U *denotes the group of invertible elements of* Δ_p*, every element of* Δ_p *can be written uniquely in the form* $p^n u$ *with* $u \in U$ *and* $n \geq 0$. *(An element of* U *is called a unit).*

PROOF. It is sufficient to prove (i) for $\mathbb{Z}(p^r)$; the case of Δ_p will follow. Now if $x \in \mathbb{Z}(p^r)$ does not belong to $p\mathbb{Z}(p^r)$, its image in $\mathbb{Z}(p)$ is not zero, thus invertible; hence there exist $y,z \in \mathbb{Z}(p^r)$ such that $xy = 1 - pz$, hence

$$xy(1 + pz + \ldots + p^{r-1}z^{r-1}) = 1,$$

which proves that x is invertible.

On the other hand, if $x = (x_n) \in \Delta_p$ is not zero, there exists a largest integer r such that x_r is zero; then $x = p^r u$ with u not divisible by p, hence $u \in U$ by (i). The uniqueness of the decomposition is clear.

PROPOSITION 1.2. *Each closed subgroup of* Δ_p *is also an ideal in* Δ_p *and all non-zero ideals in* Δ_p *are of the form* $p^r\Delta_p$ *for some integer* $r \geq 0$.

PROOF. Let I be a non-zero ideal in Δ_p. Since $\cap\, p^r\Delta_r = \{0\}$, there exists a largest integer r such that $p^r\Delta_p \supseteq I$; let $x \in I \subseteq p^r\Delta_p$ be such that $x \notin p^{r+1}\Delta_p$, thus $x = p^r u$ where u is not divisible by p (and is therefore a unit). It follows that $p^r 1 = p^r uu^{-1} \in Iu^{-1} \subseteq I$, $I \supseteq p^r\Delta_p$ and so $I = p^r\Delta_p$.

Let H be a closed subgroup of Δ_p. If $x \in H$, then $nx = \psi(n).\, x \in H$ all $n \in \mathbb{Z}$. Since $\psi(\mathbb{Z})$ is dense in Δ_p and the ring multiplication in Δ_p is continuous, $yx \in H$ all $y \in \Delta_p$, thus H is an ideal.

Next, we define the field of p-adic numbers to be the field of fractions of Δ_p. Since elements $x, x' \in \Delta_p$ are uniquely representable in the form $x = p^m u, x' = p^{m'}u'$, we have $x/x' = p^{m-m'}(u(u')^{-1}))$, so one sees immediately that $\Omega_p = \Delta_p\,[p^{-1}]$ and every non-zero element of Ω_p can be written uniquely in the form $p^n u$ with $n \in \mathbb{Z}$, $u \in U$. Addition and multiplication in Ω_p are are follows: given two elements in Ω_p, one can write them without loss of generality in the form $p^m x$, $p^{m+s}y$, where $s \geq 0$, $m \in \mathbb{Z}$, $x,y \in U$; their product is $p^{2m+s}xy$ and their sum $p^m(x + p^s y)$. Endow Ω_p with the topology such that Δ_p becomes an open subgroup. With this topology Ω_p becomes a locally compact topological field containing $\mathbb{Q}$ as a dense subfield. Finally, Ω_p is metrizable, a convenient metric being $d(p^m x, p^n y) = 2^{-\min(m,n)}$, $m,n \in \mathbb{Z}, x,y \in U$; the restriction of d to Δ_p is also a metric.

Observe that Δ_p and Ω_p are torsion free and Ω_p is divisible.

2. $\Delta_p\hat{}$ and $\Omega_p\hat{}$

Embedd $\mathbb{Z}(p^\infty)$ (respectively $\mathbb{Z}(p^r)$) the natural way onto a subgroup $\Lambda = \{\exp(2\pi ik/p^r) : k,r \in \mathbb{Z}\}$ (respectively Λ_r) of $\mathbb{T}$. Let Λ (respectively Λ_r) have the discrete topology. For fixed $\lambda \in \Lambda$, the map $\mathbb{Z} \to \Lambda : n \to \lambda^n$ extends to a continuous homomorphism $\Delta_p \to \Lambda : \lambda \to \lambda^x$ defined as follows: if $\lambda \in \Lambda_r$ $\lambda^x = \lambda^{x_r}$, where $x = (x_n)$. It is easy to check that λ^x is well defined.

Suppose $\chi \in \Delta_p\hat{}$, $\chi \neq 1$, then because $A = \{x \in \Delta_p : |\chi(x) - 1| < \sqrt{3}\}$ is an open set, there exists an integer r such that $p^r\Delta_p \subseteq A$. It follows from elementary considerations that $\chi(p^r\Delta_p) = \{1\}$. Let r be the smallest integer with this property. This gives rise to the following commutative diagram of continous maps

$$\begin{array}{ccc} \Delta_p & \xrightarrow{\chi} & \mathbb{T} \\ h\downarrow & & \uparrow\chi' \\ \Delta_p/p^r\Delta_p & \xrightarrow{\theta} & \mathbb{Z}(p^r) \end{array}$$

where h is the canonical projection, θ the isomorphism $(x_n)p^r\Delta_p \to x_r$ and χ' is necessarily of the form $\chi'(a) = \lambda^a$, $a \in Z(p^r)$ for some $\lambda \in \Lambda_r \subseteq \Lambda$. It follows that $\chi(x) = \lambda^x$. Converseley, for each $\lambda \in \Lambda_r \subseteq \Lambda$, $x \to \lambda^x$ is a character of Δ_p. Summing up, we have $\Delta_p\hat{} = \Lambda$, $A[\Delta_p\hat{}, p^r\Delta_p] = \Lambda_r$ and by the Pontryagin duality theorem [18,24.8] a pairing $(\lambda,x) = \lambda^x$, $\lambda \in \Lambda$, $x \in \Delta_p$.

The method to find $\Omega_p\hat{}$ is similar; we claim that $\Omega_p\hat{}$ is topologically isomorphic to Ω_p, the pairing being

$$(p^s x, p^r y) = \exp\ [2\pi i (xy)_{-r-s}/p^{r+s}],$$

$r,s \in \mathbb{Z}, x,y \in U$. First we note that for each $p^s x$, the map $p^r y \to (p^s x, p^r y)$ is a (continuous) character of Ω_p. To show that each character is of this form, let $\chi \in \Omega_p\hat{}$ be non-zero. As before, we select the smallest integer r (possibly negative) such that $\chi(p^r \Delta_p) = \{1\}$. For notional convenience, whenever $p^r x \in \Omega_p$, we deem $x_n = 0$ for $n \leq 0$. The epimorphism $\theta : \Omega_p \to \Lambda : p^s x \to \exp[2\pi i x_{r-s}/p^{r-s}]$ has kernel $p^r \Delta_p$. Again we construct a commutative diagram of continuous maps

$$\begin{array}{ccc} \Omega_p & \xrightarrow{\chi} & \mathbb{T} \\ \downarrow h & & \uparrow \\ \Omega/p^r\Delta_p & \xrightarrow{\theta} & \Lambda \end{array}$$

and using the previous correspondence $\Lambda\hat{} \cong \Delta_p$, we see that $\chi(p^s x) = [\theta \circ h(p^s x)]^y = (p^s x, p^{-r} y)$ for some $y \in U \subseteq \Delta_p$. Finally, it follows easily from the definition of $\Omega_p\hat{}$ that the isomorphism $\Omega_p \to \Omega_p\hat{} : p^r x \to (p^r x, \cdot)$ is bicontinuous.

3. The dual of $\mathbb{Q}$

Let $\chi \in \psi\hat{}$. For positive integers n, let $\alpha_n = \chi(1/n!)$, then $(\alpha_n)^n = \alpha_{n-1}$, $n = 2,3, \ldots$ Conversely, given a sequence $\{\alpha_n\} \subseteq \mathbb{T}$ such that $(\alpha_n)^n = \alpha_{n-1}$, $n \geq 1$, then $\chi(m/n!) = (\alpha_n)^m$ defines a character of $\mathbb{Q}$ ([18,25.5]). Thus $\mathbb{Q}\hat{}$ is the projective limit of the groups $T_n = \mathbb{T}$, $n = 1, \ldots$; the mappings being $T_n \to T_{n+1} : \alpha \to \alpha^n$. One of course has to check that the dual topology on $\mathbb{Q}\hat{}$ agrees with the inverse limit topology of the T_n.

Suppose $\{\alpha_n\} \in \mathbb{Q}^\wedge$ and $\alpha_n^{\ k} = 1$, $n = 1, \ldots$. Then for $n = 1,2, \ldots , 1 = (\alpha_{nk}^{\ k})^n = \alpha_{nk}^{\ nk} = \alpha_{nk-1}$, thus $\alpha_n = 1$ all n, so $\mathbb{Q}^\wedge$ is torsion free. We divide elements in $\mathbb{Q}^\wedge$ as follows. Given $\{\alpha_n\} \in \mathbb{Q}^\wedge$, let $\beta_n = \alpha_{n+m}^{\binom{n+m}{n}}$ (m fixed) where $\binom{n+m}{n}$ (m fixed) where $\binom{n+m}{n} = (n + m)!/n!m!$. The sequence $\{\beta_n\}$ is again in $\mathbb{Q}^\wedge$, that is $\beta_n^{\ n} = \alpha_{n+m}^{\binom{n+m}{n}n} = \alpha_{n-1}^{\binom{n-1+m}{n-1}} = \beta_{n-1}$. Also $(\beta_n)^{m!} = \alpha_{n+m}^{\frac{n+m!}{n!}} = \alpha_n$. We have shown that if $\chi \in \mathbb{Q}^\wedge$ corresponds to $\{\alpha_n\}$, then $\lambda = \{\beta_n\}$ is the unique (because is torsion free) element in $\mathbb{Q}^\wedge$ such that $\lambda^{m!} = \chi$.

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