Topics in Modern Quantum Theory

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Abstract

This thesis contains three pieces of work.

The first concerns the first massive level of closed bosonic string theory. Free-field equations are derived by imposing Weyl invariance on the world-sheet. A two-parameter solution to the equation of motion and constraints is found in two dimensions with a flat linear-dilaton background. One-to-one tachyon scattering is studied in this background. The results support Dhar, Mandal and Wadia's proposal that 2d critical string theory corresponds to the $c = 1$ matrix model in which both sides of the Fermi sea are excited.

In the second, a claim regarding the effective action of four-dimensional $\mathrm{SU}(2)_L$ gauge theory is examined. Specifically, it has been proposed that at high and low temperature the effective action contains a three-dimensional Chern-Simons term whose coefficient is the chemical potential for baryon number. By performing exact calculations in a related two-dimensional theory it is demonstrated that the existence of the Chern-Simons term in four dimensions may be rather subtle.

Finally, the effective action describing the scattering of three well-separated extremal brane solutions, in 11d supergravity, with zero p_- transfer and small transverse velocities is calculated. It is proved that to obtain this action only the leading-order solution to Einstein's equations is needed. The result obtained agrees with Matrix theory. Using an interpretation of the conjecture of Maldacena the effective action can be viewed as the large-N limit of the Matrix theory description of three supergraviton scattering at leading order.

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Statement of originality and wishes regarding free distribution

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It is my wish that all copies of this thesis, especially the one deposited in the University library, be made freely available for loan and photocopying.

13 Oct 1998

Andrew Wilkins.

Acknowledgments

It goes without saying that I wouldn't have been able to complete this Ph.D. without the countless hours my supervisor Jim McCarthy has spent discussing physics with me. It has truly been an honour to work with him. We have enjoyed a very rewarding relationship, not only because of his large knowledge of physics but also because he has been a good friend. More than this, however, it is through working with him that I have gained a great love of physics. I think this must be the mark of an excellent supervisor — that the student should go out into the world brimming with excitement about his field of study.

The other person I thank is my wife Belinda. She has been a constant source of emotional and financial support.

General Introduction

It is a testament to the rapidly changing face of modern physics that this thesis should contain not one, but rather three somewhat disjoint topics [240, 241, 341].

Not too long ago, much attention was being paid to the non-perturbative realisation of the two-dimensional (2d) string called the $c = 1$ matrix model. Although the 2d string was only a toy model, it was hoped that its description through the matrix model could shed light on questions such as the dynamics of black-holes. Such a programme proved frustratingly problematic, however, because the string seemed to admit a multitude of non-perturbative extensions. Subsequently most of these have been found to be non-unitary and it was only relatively recently that a unitary extension was proposed [98].

Chapter 1 contains a reasonably self-contained introduction to perturbative bosonic string theory and the $c = 1$ matrix model. As the reader will no doubt appreciate, to make a detailed, pedagogical exposition of either of these fields would not only be an enormous undertaking, but would also be entirely unjustified in this thesis. Nevertheless, even though many interesting side-issues have been ignored, it is hoped that enough material is presented to make the original work in Chapter 2 understandable to a reader at graduate-student level.

As just mentioned, currently there is only one version [98] of the $c = 1$ matrix model that is unitary and seems to contain the entire perturbative 2d string. In [98] it was explained how the space-time black-hole is realised in the matrix model by comparing the tachyon-graviton effective dynamics with scattering processes in the matrix model. A natural extension of this led to a proposal for the map between excitations in the matrix model and the higher string states, but it could not be checked because the scattering of tachyons in backgrounds other than the black-hole had not been worked out. This is the subject of Chapter 2. First, the constraints on the first massive level of the string which result from the imposition of Weyl invariance on the world-sheet are derived. An explanation is given as to why some previous attempts to obtain these equations have failed. Then the effective dynamics of tachyons in this background is calculated. The form of the discrete state which is the 2d realisation of the string's first massive level is found by solving its equation of motion and constraints. Studying one-to-one tachyon scattering in this discrete-state background allows an identification of the discrete state in the matrix model. The results confirm the conjecture of [98].

The second topic, contained in Chapter 3, is also a calculation in two dimensions. This time though the theory is not string theory but QED with a background axial charge. It has been claimed that the four-dimensional analogue of this theory (a simplified standard model) has an effective action which contains a three-dimensional Chern-Simons term whose coefficient is the chemical potential for baryon number. The Chern-Simons term breaks the degeneracy of the gauge field's topological vacua and in the early universe it might be expected that as the W-boson "falls" from vacua to vacua baryons are created. However, a careful study of the 2d theory suggests that the appearance of the Chern-Simons term is not as straightforward as was originally thought.

The final part of this thesis contains a calculation motivated by modern string theory. Currently physicists' ideas about strings are undergoing a revolution. Not only are there many more objects in string theory than previously imagined, but the five superstring theories are now thought to be related by various duality symmetries. The "super-theory" which contains all known superstring theories is dubbed M-theory. Very little is known about the properties of M-theory except that it is naturally formulated in eleven dimensions (11d), that it has 11d supergravity as its low-energy limit and that it is the strong-coupling limit of type IIA string theory. Yet remarkably, as explained in Chapter 4, an explicit realisation of M-theory compactified along a light-like direction, called Matrix theory, has been found [22].

In order to test that Matrix theory truly describes M-theory, the scattering amplitudes of the former might be compared with those of 11d supergravity. Unfortunately supergravity is only a low-energy approximation and it was recently realised that its domain of applicability is too limited for this comparison to provide a conclusive "test" of the Matrix theory conjecture. Nevertheless, it is intriguing that in virtually all cases checked so far supergravity reproduces the Matrix-theory scattering.

Initial calculations of the three-to-three supergraviton scattering amplitude seemed to show that supergravity and Matrix theory predicted different results [113]. Given the strong correlation between the two theories thus far it was a shame that the $3 \rightarrow 3$ process disagreed. In order to try to resolve this discrepancy Chapter 5 presents an alternative calculation of the effective action of supergravity: Instead of calculating Feynman graphs the effective action is found as an expansion in the small transverse velocities and large transverse separations of the supergravitons. In the calculation all spin effects are neglected and there is no longitudinal momentum transfer. It is found that Matrix theory and supergravity do in fact agree.

Bosonic String Theory, Two Dimensions and the $c = 1$ Matrix Model

This chapter contains an introduction to the bosonic string. After formulating the action, canonical quantisation and then path-integral quantisation are performed. Critical and non-critical strings are defined. Various methods of finding the equations of motion for the space-time fields are presented. String theory with two space-time dimensions is then introduced; first its canonical quantisation and then its non-perturbative realisation through the $c = 1$ matrix model. Scattering in the matrix model is examined and the space-time tachyon identified. Finally, the proposal of Dhar, Mandal and Wadia [98] is outlined. Most of the material presented here can be found in more detail in the text by Green, Schwarz and Witten [177] and the review by Polchinski [274].

Just as a relativistic point-particle moves so as to minimise the length of its world line, the dynamics of the first-quantised string are chosen so that the area of its world-sheet is minimised. In flat space-time a simple choice for the action is then

$$\begin{aligned} S &= -\frac{1}{2\pi\alpha'}(\text{Area of world-sheet}) \\ &= -\frac{1}{2\pi\alpha'}\int_{\mathcal{M}}\sqrt{-\det \partial_a X^\mu \partial_b X_\mu}\ . \end{aligned} \tag{1.1}$$

This is called the "Nambu-Goto" action [168, 254]. The fields X^μ ($\mu = 0, \dots, d-1$), corresponding to the d space-time dimensions, live on the two-dimensional world-sheet $\mathcal{M}$ which is parameterised by σ^a ($a = 0, 1$). The quantity $\frac{1}{2\pi\alpha'}$ is the string tension since for a static string $S = -\frac{1}{2\pi\alpha'}(\text{length})\Delta t$. The action is clearly independent of the parameterisation of the world-sheet.

The Nambu-Goto action is classically equivalent to the more convenient "Polyakov" action [284]

$$S_P[X, g] = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}}\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu\ , \tag{1.2}$$

in which the metric on the world-sheet is denoted by g_{ab}. That this is classically equivalent is seen by eliminating the metric g_{ab} from the action using its equation of motion

$$g_{ab} = -2\partial_a X^\mu \partial_b X_\mu / (g^{cd}\partial_c X^\nu \partial_d X_\nu)\ . \tag{1.3}$$

Here the world-sheet metric has Lorentzian signature although for most applications it will be more convenient to use the Wick-rotated Euclidean form.

Two important symmetries of the Polyakov action are reparameterisation and Weyl invariance on the world-sheet, under which the metric varies as

$$\begin{aligned} \delta g_{ab} &= \nabla_a v_b + \nabla_b v_a \ , \quad \text{and} \\ g_{ab} &\to e^{2\sigma} g_{ab} \ , \end{aligned} \tag{1.4}$$

respectively (X^μ is a world-sheet scalar). The function σ is called the Weyl factor. It should not be confused with the world-sheet coordinates σ^a. By fixing these symmetries, the metric can be eliminated, at least classically, up to a finite dimensional "moduli space" of inequivalent metrics. The topology of the world-sheet dictates the number of modular parameters; there are none for the sphere, one for the torus and $3h - 3$ for higher genus surfaces with h handles. Heuristic arguments for this can be found in [177, Sec. 3.3] and [274, Sec. 1.8], while [7, App. B] gives a more rigorous approach. Most of the material presented here pertains to the sphere only and references are given to works concerning higher genus world-sheets. Similarly, the calculations in Chapter 2 are on the sphere only ("string tree level").

1.1 Canonical quantisation

Canonical quantisation is made much simpler by choosing the so-called conformal gauge

$$g_{ab} = e^{2\sigma}\eta_{ab} \equiv e^{2\sigma}\,\mathrm{diag}\,(-1, 1) \ , \tag{1.5}$$

(note the Lorentzian signature used throughout this section). This is analogous to choosing the gauge $\partial{\cdot}A = 0$ in electrodynamics [199] and the string version of the physical-state condition $\langle \mathrm{phys}' | \partial{\cdot}A | \mathrm{phys} \rangle = 0$ is

$$\langle \mathrm{phys}' | T_{ab} | \mathrm{phys} \rangle \equiv \langle \mathrm{phys}' | \frac{1}{\sqrt{g}} \frac{\delta S_P}{\delta g^{ab}} | \mathrm{phys} \rangle = 0 \ . \tag{1.6}$$

This constraint imposes reparameterisation invariance on the spectrum.

In the conformal gauge the action decomposes into a sum of d harmonic oscillators, the modes of which create standing waves on the string. A spectrum of excited strings is thereby built from the vacuum and each state is identified with a particle in space-time. The modes of the time-like coordinate X^0 will produce negative-norm states, but these will be removed from the physical spectrum by the condition Eq. (1.6). For later purposes, it is useful to make this construction explicitly.

Parameterise the world-sheet by the time-like σ^0 and the compact σ^1

$$0 \le \sigma^1 < 2\pi \ . \tag{1.7}$$

Performing a Fourier decomposition, each of the d fields X^μ splits into a right-moving and a left-moving half

$$X^\mu(\sigma_0,\sigma_1) = x^\mu + 2\alpha' p^\mu \sigma^0 + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\sigma^0+\sigma^1)} + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-in(\sigma^0-\sigma^1)} \tag{1.8}$$

The simplest way of constructing the spectrum is to consider just one set of movers and then take a tensor product. Equivalently, the open string

$$X^\mu(\sigma_0,\sigma_1) = x^\mu + 4\alpha' p^\mu \sigma^0 + 2i\sqrt{\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\sigma^0}\cos n\sigma^1 \ , \tag{1.9}$$

with Neumann boundary conditions $\partial_{\sigma^1} X^\mu|_{\sigma^1=0,\pi} = 0$ can be used. It is relatively simple to check that p^μ is the average space-time momentum as the notation suggests.

Canonically quantising, $[\frac{\delta S}{\delta \partial_0 X_\mu}(\sigma^1), X^\nu(\tilde{\sigma}^1)] = -i\delta(\sigma^1-\tilde{\sigma}^1)\eta^{\mu\nu}$, the modes have the following commutation relations

$$\begin{aligned} [p^\mu, x^\nu] &= i\eta^{\mu\nu} \ , \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu} \ . \end{aligned} \tag{1.10}$$

The vacuum $|0,p\rangle$, corresponding to an unexcited string moving with momentum p^μ is annihilated by all the α_n^μ for $n>0$. After normal-ordering, the modes of the energy momentum tensor are

$$\begin{aligned} L_m &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_n{\cdot}\alpha_{m-n} \ , \\ L_0 &= \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}{\cdot}\alpha_n \ , \end{aligned} \tag{1.11}$$

where $\alpha_0 = \sqrt{2\alpha'}p^\mu$. These obey the so-called "Virasoro algebra"

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \tag{1.12}$$

where the "central charge" c, takes the value

$$c = d \ , \tag{1.13}$$

for the case in hand.

1.1.1 Light-cone quantisation

The conformal gauge does not completely fix the symmetries given in Eq. (1.4), for it is possible to choose v^a and σ so that the reparameterisation cancels the Weyl rescaling. In the light-cone quantisation scheme [164] this remaining symmetry is completely fixed by choosing the light-cone "time" $X^+ \equiv X^0 + X^{d-1} = x^+ + p^+\sigma^0$. The constraint Eq. (1.6) can then be solved for the longitudinal light-cone coordinate $X^- \equiv X^0 - X^{d-1}$ in terms of the transverse coordinates

X^i. Therefore, in the light-cone gauge all string excitations are generated by the transverse oscillators α^i_n. This means all states have positive norm and the theory is unitary.

Consider the first excited state of the open string $|e,p\rangle = e_i\alpha^i_{-1}|0,p\rangle$. Now, since the underlying theory is Lorentz invariant, the light-cone quantisation should give a Lorentz invariant string theory[1]. But, under the action of a Lorentz boost the transversely polarised vector $|e,p\rangle$ acquires a longitudinal polarisation unless it is massless (the spin of a massive particle is labeled by an irrep of SO$(d-1)$ whereas a massless particle corresponds to an irrep of SO$(d-2)$). Therefore, by Lorentz invariance $|e,p\rangle$ corresponds to a massless particle.

1.1.2 Old-covariant quantisation

In a similar fashion to QED, the constraint algebra obeyed by the modes L_m forbids that $L_m|\text{phys}\rangle = 0\ \forall m$. In the old covariant quantisation method, the positive half of the spectrum annihilates physical states,

$$(L_0 - a)|\text{phys}\rangle = 0 = L_m|\text{phys}\rangle \quad \text{for} \quad m > 0\ , \tag{1.14}$$

where the factor of a comes from the normal-ordering ambiguity in L_0. The situation is further complicated by the existence of "spurious" states which have the form

$$L_{-m}|\psi\rangle \quad \text{for} \quad m > 0\ . \tag{1.15}$$

These are orthogonal to all physical states since $\langle\text{phys}|L_{-m}|\psi\rangle = \langle\text{phys}|L^\dagger_m|\psi\rangle = 0$. Therefore, if a state is both physical and spurious it is equivalent to the null-state. Such null states are modded out of the physical subspace since all amplitudes containing them vanish. Finding the physical-state space is then a problem of finding the equivalence classes

$$|\text{phys}\rangle \simeq |\text{phys}'\rangle \quad \text{if} \quad \exists \quad |\psi\rangle \quad \text{st} \quad |\text{phys}\rangle = |\text{phys}'\rangle + L_{-m}|\psi\rangle\ . \tag{1.16}$$

This method generates the same physical states as the ghost-number zero sector generated by the more modern BRST procedure and can be proved to produce only positive-norm states [58, 165, 321] for $a = 1$ and $d = 26$.

As an illustration, the first three levels of the open string are examined. They are:

1. The unexcited string $|0,p\rangle$. The only non-trivial physical state condition is $L_0 - a = 0$ which yields (recall $\frac{1}{2}\alpha_0^2 = \alpha' p^2$)

$$M^2 = -p^2 = -a/\alpha'\ . \tag{1.17}$$

Soon it will be shown that $a = 1$ so that this state is tachyonic.

[1] This is a condition for the consistent quantisation of the model.

2. The vectors $|e, p\rangle = e_\mu \alpha^\mu_{-1}|0, p\rangle$. Recall that by demanding underlying Lorentz invariance in the light-cone quantisation method this state was massless. Here, the condition $L_0 - a = 0$ gives the mass formula $M^2 = -p^2 = (1-a)/\alpha'$. Therefore $a = 1$. The physical-state condition $L_1 = 0$ implies $p{\cdot}e = 0$. Since this is a massless particle the Lorentz frame $p^\mu = (\omega, \omega, 0, \ldots, 0)$ can be chosen and thus, so far, the physical states are the $d-2$ transverse oscillations $e^\mu = (0, 0, \ldots, 1, \ldots, 0)$, and the one state $e \propto p$. At this level there is only one possible spurious state $L_{-1}|0, p\rangle = \sqrt{2\alpha'}p{\cdot}\alpha_{-1}|0, p\rangle$. Thus the physical state with $e \propto p$ is spurious and therefore null. Removing this one null state reduces the physical-state space to the $d-2$ transverse oscillations only.

3. The first massive level $|f, p\rangle = f_\mu \alpha^\mu_{-2}|0, p\rangle + f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0, p\rangle$. The $(L_0 - 1)$, L_1 and L_2 conditions give respectively

$$\begin{aligned} M^2 = -p^2 &= 1/\alpha' \ , \\ f_\mu + \sqrt{2\alpha'} f_{\mu\nu} p^\nu &= 0 \ , \\ 2\sqrt{2\alpha'} f{\cdot}p + f^\mu_\mu &= 0 \ . \end{aligned} \tag{1.18}$$

The second condition simply says that f_μ is not really an independent degree of freedom; using this in the third gives a tracelessness-type condition

$$f_{\mu\nu}(4\alpha' p^\mu p^\nu - \eta^{\mu\nu}) = 0 \ . \tag{1.19}$$

By writing the most general spurious state

$$(a_\mu L_{-1}\alpha^\mu_{-1} + bL_{-2})|0, p\rangle \ , \tag{1.20}$$

($L_{-1}L_{-1}$ can be soaked into the first term) it is easy to check that a physical state is spurious, and therefore null, if

$$f_{\mu\nu} \propto a_\mu p_\nu + a_\nu p_\mu - \tfrac{1}{3}\eta_{\mu\nu} a{\cdot}p \quad \text{with} \quad (d-26)a{\cdot}p = 0 \ . \tag{1.21}$$

If $d \neq 26$, then removing these null states removes $(d-1)$ components of $f_{\mu\nu}$, leaving a total of $(\frac{1}{2}d(d+1) - d)$ linearly independent components. Unfortunately, in this case, the physical state conditions Eq. (1.14) have failed to remove one negative-norm state. This can be seen heuristically by comparing with the light-cone quantisation in which there are only $(\frac{1}{2}d(d+1) - d - 1)$ independent components: $\frac{1}{2}(d-2)(d-1)$ from f_{ij} and $(d-2)$ from f_i. On the other hand, at $d = 26$, another null state appears. Removing this gives a spectrum containing only positive-norm states. Finally then, in the "critical dimension" $d = 26$, the first massive level contains a physical state which has $(\frac{1}{2}d(d+1) - 1 - d)$ linearly independent components, corresponding to a transverse, traceless, symmetric 2-tensor.

Putting together two copies of the above to form a closed string, it is clear that the first three levels are: a tachyon $|0, p\rangle$; a massless set of tensors $H_{\mu\nu}$ with $(d-2)^2$ components which can be decomposed under $\mathrm{SO}(d-2)$ into a traceless symmetric tensor, an antisymmetric tensor and a singlet; and a massive field $E_{\mu\nu\lambda\rho} = E_{(\mu\nu)(\lambda\rho)}$ which is traceless inside the pairs of indices and transverse on all indices (so that it contains $(\frac{1}{2}d(d+1) - 1 - d)^2$ independent components). Henceforth only closed strings will be considered in this chapter.

1.2 Vertex operators and physical-state conditions

The spectrum of states can also be built by applying vertex operators to the vacuum $|0,0\rangle$. For example, the tachyon $|0,p\rangle$ is created by the normal-ordered vertex operator $:e^{ip\cdot X}:$

$$|0,p\rangle = \, :e^{ip\cdot X}: |0,0\rangle \, . \tag{1.22}$$

(Normal ordering is defined by putting all the annihilation modes to the right, as on p. 5.) That this corresponds to the tachyon can be checked by acting with α_0^μ to obtain $\sqrt{2\alpha'}p^\mu$ and by annihilating the state with the positive frequency modes α_m. Another example is the closed-string graviton $g_{\mu\nu}(p)\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0,p\rangle$, which is created by the vertex operator

$$g_{\mu\nu}(p) : g^{ab}\partial_a X^\mu \partial_b X_\mu e^{ip\cdot X} : \, . \tag{1.23}$$

Naturally, the states these vertex operators create do not necessarily obey the physical-state conditions. Recall that these conditions are a consequence of the underlying reparameterisation and Weyl invariance of the theory. Thus, if all possible correlators $\langle\ldots\rangle$ are invariant under world-sheet reparameterisations and Weyl rescalings, the states created by the vertex operators will correspond to on-shell physical particles and the amplitudes obtained will correspond to physical processes in space-time. As will be discussed in more detail below, retaining explicit reparameterisation covariance leads to a Weyl anomaly in general. A consistent quantisation of string theory therefore requires

$$\frac{\delta}{\delta\sigma}\langle\ldots\rangle = 0 \, . \tag{1.24}$$

Much of this Chapter 2 will be dedicated to solving this equation in a weak-field expansion for strings living in backgrounds containing massless, tachyonic and massive fields.

1.3 Path-integral quantisation

The path integral is not only a convenient calculational tool, but it incorporates string interactions in a beautiful way. The partition function is

$$Z = \sum_{\text{handles}} g_{\text{str}}^{-\chi} \int \frac{[\mathrm{d}X]_g[\mathrm{d}g]_g}{\mathrm{Vol}} e^{-S_P[X,g]} \, , \tag{1.25}$$

with amplitudes being calculated by the insertion of appropriate vertex operators. The sum runs over the number of handles of the string world-sheet, g_{str} is the string coupling constant, χ is the Euler character of the world-sheet, Vol is the volume of the reparameterisation and Weyl symmetry groups and S_P is the Polyakov action. The purpose of this subsection is to provide an explanation of this formula.

String theory, as presented here, is is the theory of first-quantised strings. Strings interact by joining and splitting their world-sheets as in Fig 1.1. Locally, there is no coordinate-invariant distinction between free propagation and interaction in the scattering process depicted — all

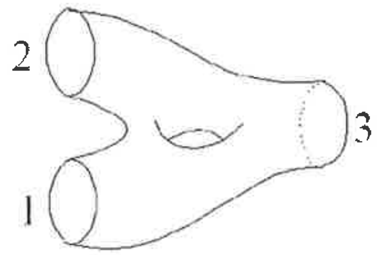

Figure 1.1 : A one-loop three-point amplitude between states $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$.

the information of whether or not an interaction occurs is contained in the topology of the world-sheet. In closed string theory, the number of boundaries corresponds to the number of incoming or outgoing string states created by vertex operators, while the number of handles corresponds to the number of loops. So the $\sum g_{\rm str}^{-\chi}$ in the partition function Eq. (1.25), is a natural way of including all possible paths (topologies) alá Feynman. The sum runs over all possible world-sheet topologies with Euler character $\chi = 2-2h$, and the factor $g_{\rm str}^{-\chi}$ weights each handle h with the appropriate factor of the string coupling constant $g_{\rm str}$.

Both X and g are dynamical variables in the Polyakov action Eq. (1.2), therefore they are integrated over in the path integral. However, as mentioned previously, the action has two symmetries classically — reparameterisation and Weyl invariance on the world-sheet. The "Vol" is meant to symbolise the modding-out by an associated volume. Care must be taken here since quantum mechanically there is an anomaly — the measures $[\mathrm{d}X]_g[\mathrm{d}g]_g$ cannot be defined so that they respect both reparameterisation invariance and Weyl symmetry. By definition, reparameterisation invariance is kept manifest so the Weyl symmetry becomes anomalous. There are two ways to deal with this anomaly: Constraints can be placed upon the parameters of the theory so that the anomaly is zero and thus the conformal mode σ decouples; or, σ can be retained and the parameters become constrained by imposing that the theory be invariant under simultaneous Weyl rescalings and shifts of σ. The former approach will be called a "critical" string and the latter a "non-critical" string. It is common to consider the conformal mode in the non-critical string as an extra space-time coordinate. Thus a non-critical string in d dimensions is equivalent to a critical string in $d+1$ dimensions; the extra dimension being provided by the conformal mode. The following sections investigate these statements in more detail.

Before moving on, though, a brief word can now be said concerning two other reparameterisation invariant terms which can be added to the Polyakov action. These are the cosmological term and the Einstein term

$$S_P \to S_P + \mu_1^2 \int \sqrt{g} + \frac{1}{4\pi}(\log g_{\rm str}) \int \sqrt{g}R \ . \qquad (1.26)$$

The first is not Weyl invariant, but both must be added for renormalisability, in general. Since $\frac{1}{4\pi}\int \sqrt{g}R = \chi$, the coefficient of the Einstein term is just $\log g_{\rm str}$, and μ_1^2 is the bare cosmological constant (its renormalised value will often be fixed to be zero).

1.4 Factorising the functional measures

A general metric g_{ab} can be obtained by performing a reparameterisation (v^a) and Weyl rescaling (σ) of a "fiducial" metric $\hat{g}_{ab}(\tau)$ that only depends on a finite number of modular parameters τ_j. In this way the functional integration over metrics can be decomposed into an integration over v^a, σ and $\hat{g}$ and, if the remainder of the path integral is reparameterisation invariant, $[d^2v]$ can be immediately canceled against the volume of the reparameterisation group. In the present case, consider only world-sheets with spherical topology [284] since, as mentioned on p. (4) the sphere has no modular parameters. The existence of a non-trivial moduli space complicates the exposition considerably and details can be found in [7, 103, 248].

Naturally, it is convenient to define the measures so that they are explicitly reparameterisation invariant. For the matter fields, the obvious choice is to use the scalar product with respect to the metric g_{ab}

$$1 = \int [\mathrm{d}^d \delta X]_g e^{-|\delta X|_g^2} \quad \text{where} \quad |\delta X|_g^2 = \int_{\mathcal{M}} \sqrt{g}\delta X^\mu \delta X_\mu \ . \tag{1.27}$$

Note that while this is reparameterisation invariant it is not Weyl invariant. Similarly, the measure for g_{ab} is induced from the reparameterisation invariant distance on the tangent space to the space of metrics at g_{ab}

$$|\delta g|_g^2 = \int_{\mathcal{M}} \sqrt{g}(g^{ac}g^{bd} + \rho g^{ab}g^{cd})\delta g_{ab}\delta g_{cd} \ , \tag{1.28}$$

where $\rho > -\frac{1}{2}$ (for positive-definiteness) but otherwise arbitrary.

Under a small reparameterisation and Weyl rescaling, the metric changes according to Eq. (1.4)

$$\begin{aligned} \delta g_{ab} &= (\nabla_a v_b + \nabla_b v_a - g_{ab}\nabla\cdot v) + g_{ab}(2\delta\sigma + \nabla\cdot v) \\ &= (P_1 v)_{ab} + 2g_{ab}\delta\sigma' \end{aligned} \tag{1.29}$$

where the variation has been decomposed into a traceless part and a part proportional to the metric. The integration over g_{ab} can then be factorised into an integration over v^a and σ with a Jacobian factor [284]

$$[\mathrm{d}g]_g = \det P_1 [\mathrm{d}^2 v]_{\hat{g}} [\mathrm{d}\sigma]_{\hat{g}} \ . \tag{1.30}$$

Now everything is reparameterisation covariant so $[\mathrm{d}^2 v]_{\hat{g}}$ can be modded-out from Eq. (1.30) immediately. Further remarks concerning the measure for σ induced from Eq. (1.28) will be made soon; for now it is clear that $[d\sigma]_{\hat{g}}$ cannot be removed since σ-dependence is hiding in the measures. A careful examination of measures and the above procedure can be found in [340].

By writing the Jacobian in terms of ghosts b^{ab} and c_a, the measure reads

$$[\mathrm{d}g]_g/\mathrm{Vol}_{\mathrm{reparam}} = \int [\mathrm{d}b]_g [\mathrm{d}c]_g e^{-\frac{1}{2\pi}\int_{\mathcal{M}} \sqrt{g} b^{ab}\nabla_a c_b} [\mathrm{d}\sigma]_{\hat{g}} \ , \tag{1.31}$$

where, since P_1 maps vectors into symmetric traceless tensors, b^{ab} is symmetric and traceless.

There is one final complication. After gauge fixing there are still the 3 conformal Killing vectors for the sphere and one for the torus. To break these gauge symmetries, the standard practice is to fix the position of 3 vertex operators on the sphere and one on the torus.

1.4.1 The critical string

Dependence on the conformal mode is still hiding in the measures for the matter fields X^μ and the ghosts. By making this explicit, conditions for overall Weyl invariance can be derived and the measure $[\mathrm{d}\sigma]_{\hat{g}}$ can be removed.

First, all σ dependence will be made manifest. Since under an infinitesimal Weyl rescaling the effective action varies as

$$\delta S_{\text{eff}}(g) = \left\langle \int_{\mathcal{M}} \sqrt{g} T_{ab} \delta g^{ab} \right\rangle_{Xbc} = \left\langle 2 \int_{\mathcal{M}} \sqrt{g} T^a{}_a \delta\sigma \right\rangle_{Xbc} , \tag{1.32}$$

the anomaly may be calculated by integrating up the trace of the energy momentum tensor. The notation $\langle \ldots \rangle_{Xbc}$ indicates that only the fields X^μ, b^{ab} and c_a are being integrated over. Explicit details are nicely presented in [136], while the original work for the spherical world-sheet was done in [284] and for world-sheets of arbitrary genus in [7].

On purely dimensional grounds, the trace looks like

$$16\pi \langle T^a{}_a \rangle = \tfrac{26-d}{3} R(g) + \mu^2 , \tag{1.33}$$

where $R(g)$ is the scalar curvature of the original world-sheet metric and μ^2 is the cosmological constant which has dimension 2. It is the coefficient $(26-d)/3$ which needs to be calculated. This is quite lengthy, but straightforward, and can be found in the above references. The matter fields X^μ contribute the factor of d while the ghosts' contribution is 26. To integrate the variation, parameterise $g_{ab} = e^{2\sigma} \hat{g}_{ab}$ and use $R = e^{-2\sigma}(\hat{R} - 2\Box\sigma)$ to obtain

$$\delta S_{eff}(\hat{g}, \sigma) = \tfrac{1}{8\pi} \int_{\mathcal{M}} \sqrt{\hat{g}} \left(\tfrac{26-d}{3} (\hat{R} - 2\Box\sigma) + \mu^2 e^{2\sigma} \right) \delta\sigma . \tag{1.34}$$

Integrating this version is easy since all the σ-dependence has been rendered explicit. Renormalising the cosmological term and neglecting the constant of integration, the result is

$$[\mathrm{d}^d X \mathrm{d}b \mathrm{d}c]_g e^{\mu_1^2 \int \sqrt{g}} = [\mathrm{d}^d X \mathrm{d}b \mathrm{d}c]_{\hat{g}} \exp\left(\frac{26-d}{24\pi} \int_{\mathcal{M}} \sqrt{\hat{g}} (\sigma\Box\sigma - \hat{R}\sigma + \mu_0 e^{2\sigma}) \right) . \tag{1.35}$$

The action written here is called the "Louiville action".

The critical string moving in R^d is therefore defined by the partition function

$$\begin{aligned} Z &= \sum_\chi g_{\text{str}}^{-\chi} \int [\mathrm{d}^d X \mathrm{d}^2 b \, \mathrm{d}c]_{\hat{g}} \exp -S_{\text{tot}} , \\ S_{\text{tot}} &= \frac{d-26}{24\pi} \int_{\mathcal{M}} \sqrt{\hat{g}} (\sigma\Box\sigma - \hat{R}\sigma + \mu_0 e^{2\sigma}) + S_P(X, g) + S_{gh}(b, c, \hat{g}), \end{aligned} \tag{1.36}$$

with the condition that all correlators be independent of σ

$$\frac{\delta\langle\ldots\rangle}{\delta\sigma} = 0 \,. \tag{1.37}$$

It is clear that this condition is solved by $d = 26$ if the vertex operators themselves are Weyl invariant. In later sections, strings moving in more interesting backgrounds will be studied by replacing $S_P(X, g)$ by a non-linear sigma model action. The backgrounds modify the $d = 26$ solution.

1.4.2 The non-critical string

The conformal mode σ, is retained in the non-critical string. Its measure $[\mathrm{d}\sigma]_{\hat{g}}$ is complicated since the norm, induced from Eq. (1.28) is

$$|\delta\sigma|_g^2 = (1 + 2\rho)\int_{\mathcal{M}} \sqrt{\hat{g}} e^{2\sigma}(\delta\sigma)^2 \,. \tag{1.38}$$

It is unknown how to proceed with the quantisation using this measure, therefore it is assumed that there exists a Jacobian $J(\sigma, \hat{g})$ which describes the transition from measure to the simpler measure with norm

$$|\delta\sigma|_{\hat{g}}^2 = \int_{\mathcal{M}} \sqrt{\hat{g}}(\delta\sigma)^2 \,. \tag{1.39}$$

Furthermore, it is postulated [87, 115] that this Jacobian, when suitably regulated, is given by the exponential of a renormalisable local action. The most general such action consistent with the reparameterisation invariance of the underlying theory is of the Liouville form,

$$[\mathrm{d}^d X \mathrm{d}b\, \mathrm{d}c\, \mathrm{d}\sigma]_g = [\mathrm{d}^d X \mathrm{d}b\, \mathrm{d}c\, \mathrm{d}\sigma]_{\hat{g}} \exp\left(\frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{\hat{g}}(\tfrac{1}{2}\sigma\Box\sigma - Q\hat{R}\sigma - \mu e^{\alpha\sigma})\right). \tag{1.40}$$

The Liouville action now contains arbitrary renormalised parameters Q, μ and α (relative to the Liouville action written previously the field σ has been scaled by α). All dependence on σ has now been written explicitly. Therefore, the non-critical string moving in R^d is defined by the partition function

$$\begin{aligned} Z &= \sum_{\chi} g_{\mathrm{str}}^{-\chi} \int [\mathrm{d}^d X \mathrm{d}b\, \mathrm{d}c\, \mathrm{d}\sigma]_{\hat{g}} \exp -S_{\mathrm{tot}} \,, \\ S_{\mathrm{tot}} &= -\frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{\hat{g}}(\tfrac{1}{2}\sigma\Box\sigma - Q\hat{R}\sigma - \mu e^{\alpha\sigma}) + S_P(X, \hat{g}) + S_{gh}(b, c, \hat{g}) \,, \end{aligned} \tag{1.41}$$

with the condition that the theory is invariant under Weyl transformations, that is, under the simultaneous shift

$$\hat{g} \to e^{\bar{\sigma}}\hat{g} \quad \text{and} \quad \sigma \to \sigma - \bar{\sigma}/\alpha \,. \tag{1.42}$$

In Chapter 2, it will be shown that this constraint implies [115] $12Q^2 + (d + 1) - 26 = 0$ and $\alpha = Q \pm \sqrt{Q^2 - 2}$ (these conditions follow from the dilaton and tachyon equations of motion respectively, see Eq. (2.40)).

1.5 Strings in more complicated backgrounds

The Polyakov action can be replaced by a non-linear sigma model

$$\begin{aligned} S[X,g] &= \frac{1}{4\pi\alpha'}\int_{\mathcal{M}}\Big(\sqrt{g}G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu g^{ab} + B_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu\epsilon^{ab} \\ &\qquad +\alpha'\sqrt{g}R\Phi(X) + \alpha'\sqrt{g}T(X)\Big)\ . \end{aligned} \tag{1.43}$$

This is the most general local action with at most two derivatives (so that it is renormalisable) that is reparameterisation invariant both on the world-sheet and in space-time[2]. Loops in the sigma model (not string loops) are counted by the string tension α', which means that to zero loops (classically), the sigma model is Weyl invariant. However, at one loop, Weyl invariance has been destroyed by the terms $\sqrt{g}R\Phi$ and $\sqrt{g}T$. It will be restored by canceling the anomaly against these two terms.

Notice that e^{-S} is made up of of exponentials of vertex operators and thus it corresponds to a coherent state of strings. This means [227] that this action describes a string living in a non-trivial background made up of a string graviton ($G_{\mu\nu}$), an antisymmetric tensor ($B_{\mu\nu}$), the dilaton (Φ) and the tachyon (T). Soon it will be shown that the Weyl invariance condition Eq (1.24) implies that the functions $G_{\mu\nu}$, $B_{\mu\nu}$, Φ and T obey the equations of motion associated with the graviton, antisymmetric tensor, dilaton and tachyon respectively.

By comparing the critical string of Eq. (1.36) with the non-critical string Eq. (1.41), it is now clear that by treating σ as an extra space-time coordinate, a non-critical string in d-dimensional Minkowsky space is equivalent to a critical string which lives in $d+1$ dimensions in a tachyonic linear-dilaton background ($T = \mu e^{\alpha\sigma}$ and $\Phi = Q\sigma$).

One final point to make is that by comparing the non-linear sigma model with the extra terms Eq. (1.26), it is clear that the string coupling constant is just the exponential of the expectation value of the dilaton

$$g_{\rm str} = \exp\langle\Phi\rangle\ . \tag{1.44}$$

1.6 Beta-functions, tachyons and weak fields

In Sec. (1.4.1) it was explained that in order that there be no Weyl anomaly for the bosonic string in an empty flat background the dimension of space-time must be 26. Weyl invariance with a background graviton field was first studied by Friedan [147, 148]. One of the achievements of his thesis was the calculation of the graviton's beta-function to two-loops using a normal coordinate expansion in the partition function. Setting the beta-function to zero, up to total-derivative terms, is equivalent to demanding Weyl invariance. This is explained in the next small section where his calculation is reproduced (on the sphere to one sigma-model loop) as an example of the β-function method.

[2]Soon it will be shown that the other possible term $\frac{1}{4\pi}\int\sqrt{g}\tilde{A}_\mu D^2 X^\mu$ can be removed by a redefinition of X^μ

The calculation can also be extended to include all the massless fields (Eq. (1.43) with $T = 0$). However, as will be explained in Sec. (1.6.2), with the inclusion of the tachyon there are contributions to the Weyl anomaly that are invisible to any finite order in the loop expansion [85]. These contributions can be obtained by using a weak-field expansion instead of the sigma-model loop expansion. A popular method of implementing the weak-field expansion is the "Wilson renormalisation group" approach which was pioneered by Banks and Martinec [23] and developed by Hughes et. al [189]. This method is only briefly summarised at the end of this subsection since it is very similar in spirit to the explicit calculations presented in Chapter 2.

1.6.1 The β-function method

As an illustration of this method, take a string moving in a gravitational field

$$S = \frac{1}{4\pi\alpha'} \int \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \ . \tag{1.45}$$

A classical solution $\bar{X}^\mu$ obeys the equations of motion

$$g^{ab} D_a \partial_b \bar{X}^\mu = 0 \ . \tag{1.46}$$

Expanding around the "background field" $\bar{X}^\mu$ in the usual fashion $X^\mu = \bar{X}^\mu + Q^\mu$ with a non-trivial metric $G_{\mu\nu}$ is inconvenient since tensors at the two points $\bar{X}$ and $\bar{X} + Q$ vary differently under space-time reparameterisations. In order to keep reparameterisation invariance manifest, the idea is to replace the displacements Q in the neighbourhood of $\bar{X}$ by a "covariant displacement" defined by the geodesic running through $\bar{X}$ to $\bar{X} + Q$. Specifically, for a small neighbourhood around $\bar{X}$ there will be a unique geodesic described by $\rho^\mu(\tau)$;

$$\frac{\mathrm{d}^2 \rho^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}\rho^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}\rho^\lambda}{\mathrm{d}\tau} = 0 \ . \tag{1.47}$$

Here the affine parameter τ is bounded $0 \le \tau \le 1$ with $\rho^\mu(0) = \bar{X}^\mu$ and $\rho^\mu(1) = \bar{X}^\mu + Q^\mu$. The covariant displacement may then be defined using the "normal coordinates"

$$s^\mu = \left. \frac{\mathrm{d}\rho^\mu}{\mathrm{d}\tau} \right|_{\tau=0} \ . \tag{1.48}$$

These coordinates are useful because geodesics are simply straight lines; for take two points $\bar{X} + Q$ and $\bar{X} + Q'$ on the same geodesic, then $\tau' = a\tau$ and so $s'^\mu = \frac{\tau}{\tau'} s^\mu$. Because of this property, the connection obeys particularly simple properties in the normal-coordinate frame (denoted by a tilde)

$$\begin{aligned} 0 &= \widetilde{\Gamma}^\mu_{\nu\lambda} \ , \\ 0 &= \partial_\rho \widetilde{\Gamma}^\mu_{\nu\lambda} + \partial_\nu \widetilde{\Gamma}^\mu_{\lambda\rho} + \partial_\lambda \widetilde{\Gamma}^\mu_{\rho\nu} \ , \end{aligned} \tag{1.49}$$

and similarly for higher symmeterised derivatives. (This is often taken as the starting point for the definition of Riemann normal coordinates rather than the above geometrical definition.)

The idea is to write the action in the normal coordinate frame using identities following from Eq. (1.49) and then covariantise. The result will be valid in all frames. For instance, since

$$\begin{aligned}
\widetilde{\bar{X}^\mu + Q^\mu} &= \rho^\mu(1) = \rho^\mu(0) + \left.\frac{\partial \rho^\mu}{\partial \tau}\right|_{\tau=1} + \tfrac{1}{2}\left.\frac{\partial^2 \rho^\mu}{\partial \tau^2}\right|_{\tau=1} + \dots \\
&= \bar{X}^\mu + s^\mu - \widetilde{\Gamma}^\mu_{\nu\lambda} s^\nu s^\lambda + \dots \\
&= \bar{X}^\mu + s^\mu \ , \\
\widetilde{R}^\mu{}_{\nu\lambda\rho} &= \partial_\lambda \widetilde{\Gamma}^\mu_{\nu\rho} - \partial_\rho \widetilde{\Gamma}^\mu_{\nu\lambda} + 0 \ \Rightarrow \ \partial_\lambda \widetilde{\Gamma}^\mu_{\nu\rho} = \tfrac{1}{3}(\widetilde{R}^\mu{}_{\nu\lambda\rho} + \widetilde{R}^\mu{}_{\rho\lambda\nu}) \ ,
\end{aligned} \tag{1.50}$$

then

$$\begin{aligned}
\partial_a \widetilde{(\bar{X}+Q)}^\mu &= \partial_a \bar{X}^\mu + \nabla_a s^\mu - \tfrac{1}{3}\widetilde{R}^\mu{}_{\alpha\nu\beta}(\bar{X})\partial_a \bar{X}^\nu s^\alpha s^\beta + O(s^3) \ , \\
T_{\mu\nu}(\widetilde{\bar{X}+Q}) &= \sum_{n=0} \frac{1}{n!}(\widetilde{\partial_{\mu_1}\dots\partial_{\mu_n} T_{\mu\nu}|_{\bar{X}}}) s^{\mu_1}\dots s^{\mu_n} \\
&= T_{\mu\nu}(\bar{X}) + \widetilde{\nabla_{\mu_1} T_{\mu\nu}} s^{\mu_1} + \left(\nabla_{\mu_1}\widetilde{\nabla_{\mu_2} T_{\mu\nu}} - \tfrac{1}{6}\widetilde{R}^\lambda_{\mu_1\mu\mu_2} T_{\lambda\nu} - \tfrac{1}{6}\widetilde{R}^\lambda_{\mu_1\nu\mu_2} T_{\mu\lambda}\right) s^{\mu_1} s^{\mu_2} \\
&\quad + O(s^3) \ .
\end{aligned} \tag{1.51}$$

Note that all the terms on the RHS have been covariantised so that the results are true in any frame. To $O(s^3)$ the action is

$$S = S(\bar{X}) + \frac{1}{4\pi}\int \left(G_{\mu\nu}(\bar{X})\nabla_a s^\mu \nabla_a s^\nu - R_{\mu\alpha\nu\beta}(\bar{X})\partial_a \bar{X}^\nu \partial_a \bar{X}^\nu s^\alpha s^\beta\right) + O(\alpha') \ , \tag{1.52}$$

(by scaling $s \to \sqrt{\alpha'} s$ the α' dependence has been made explicit).

The first term of the expanded action gives the propagator and using dimensional regularisation it is found that the only divergent contribution to the one-loop effective action is the tadpole diagram with an insertion of the second term;

$$S^{\text{div}}_{\text{eff}} \sim \frac{1}{\epsilon}\int R_{\mu\nu}(\bar{X})\partial_a \bar{X}^\nu \partial_a \bar{X}^\nu \ . \tag{1.53}$$

In general the effective action is of the form

$$S_{\text{eff}} = \int G^\epsilon_{\mu\nu}(\bar{X})\partial_a \bar{X}^\mu \partial_a \bar{X}^\nu \ , \tag{1.54}$$

where $G^\epsilon_{\mu\nu}$ depends on the dimensionful scale parameter ϵ introduced through the regularisation scheme. Demanding that S_{eff} is invariant under rigid changes of scale implies that the integral of the β-function be zero; $\int \beta_{\mu\nu}\partial_a \bar{X}^\mu \partial_a \bar{X}^\nu = 0$. However, this is too weak since all it amounts to is that the *integral* of the trace of the energy momentum tensor be zero

$$\int T^a{}_a = \int \beta_{\mu\nu}\partial_a X^\mu \partial_a X^\nu = 0 \ , \tag{1.55}$$

(note this is an operator statement). For the theory to be invariant under local scale transformations the condition

$$0 = T^a{}_a = \beta_{\mu\nu}\partial_a X^\mu \partial_a X^\nu + \nabla_\mu V_\nu \partial_a X^\mu \partial_a X^\nu \ , \tag{1.56}$$

must be imposed, where the second term has been added for generality since its integral is zero. In fact [191, 325], in order that the necessary renormalisation be produced by a local counterterm V_ν must be taken to be the gradient of the dilaton: $V_\nu = -2\nabla_\nu\Phi$, (this condition will be reproduced in Chapter 2). Therefore, the theory is conformal invariance iff

$$R_{\mu\nu}(\bar{X}) + 2\nabla_\mu\nabla_\mu\Phi = 0 \ . \tag{1.57}$$

The calculation can be extended to the whole massless sector which results in [62]

$$\begin{aligned} 0 &= R_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\Phi \ , \\ 0 &= \nabla^\mu H_{\mu\nu\lambda} - 2\nabla^\mu\Phi\, H_{\mu\nu\lambda} \ , \\ 0 &= \tfrac{26-d}{3} + \tfrac{\alpha'}{2}R - \tfrac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 2\alpha'\nabla^2\Phi - 2\alpha'(\nabla\Phi)^2 \end{aligned} \tag{1.58}$$

where $H_{\mu\nu\lambda} = \frac{1}{3}(\partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu})$.

A simple solution of these equations is

$$G_{\mu\nu} = \eta_{\mu\nu} \ , \quad B_{\mu\nu} = 0 \quad \text{and} \quad \Phi(X) = Q{\cdot}X \quad \text{with} \quad 6\alpha' Q^2 = 26 - d \ . \tag{1.59}$$

Evidently, a linear dilaton makes it possible for a string to propagate consistently in dimensions other than 26.

Finally, the β-function calculation has been extended to higher sigma-model loops (which add higher-derivative terms to the equations of motion) and string loops (see, for example, [65, 192, 193, 242, 244, 243] and [66, 143, 144, 293, 326] respectively).

1.6.2 Tachyons and the weak-field expansion

Applied as above, the β-function method does not give the correct equation of motion for the tachyon, as explained by Das and Sathiapalan [85]. The problem was that every order in the loop expansion became finite simply by making the theory one-loop finite because the tachyon vertex operator is super-renormalisable. More explicitly, consider the string in the conformal gauge in a background of tachyons [74]

$$S = \frac{1}{4\pi}\int_{\mathcal{M}} \left(\tfrac{1}{2}\partial_a X^\mu \partial_a X_\mu + \lambda T(X)\right) \ . \tag{1.60}$$

The coupling constant λ is dimensionless; all dimensionful constants have been soaked into T (and $\alpha' = 2$). Expanding around the background $\bar{X}$ yields an infinite number of vertices

$$T(X) = T(\bar{X} + Q) = \sum_{n=0}^{\infty} \frac{1}{n!} Q^{\mu_1} \ldots Q^{\mu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} T(\bar{X}) \ . \tag{1.61}$$

Now consider expanding the effective action in powers of $T(\bar{X})$. At first order an infinite number of tadpoles will be generated. The point is that because T is super-renormalisable, if these are renormalised then graphs with more insertions of $T(\bar{X})$ will be finite! Because no higher-order renormalisations are needed, the beta function is linear in the tachyon field which means that there don't seem to be any tachyon self-interactions.

To understand this, Das and Sathiapalan specialised to the case $T(X) = \cos\beta X^1$ which is the well-studied statistical-mechanics sine-Gordon model. Applying the results of [11, 260, 338] it was found that although a single renormalisation made every order in the loop expansion finite, the *sum* of the loop contributions diverged. To find the correct equation of motion, the loop expansion can be summed by employing a weak-field expansion. To second order the equation of motion in a linear-dilaton background is [85]

$$(\partial^2 - 2Q\cdot\partial + 2)T - \tfrac{1}{2}T^2 = 0 \ , \tag{1.62}$$

which will be verified in Sec. 2.2.

The Wilson renormalisation group approach is the most common method used to implement the weak-field expansion. It starts from a flat world-sheet and places constraints on the space-time fields by demanding that the theory be invariant under conformal transformations

$$\delta\sigma^a = v^a(\sigma) \quad \text{with} \quad \partial_a v_b + \partial_b v_a = \delta_{ab}\partial\cdot v \ . \tag{1.63}$$

A formal equation (the "Wilson equation") which expresses invariance under this transformation is derived [189, 299] in much the same way as a Ward identity. A short-distance cutoff is used and the equation is solved, in principle, order-by-order in the couplings. Rather than giving more details, at this stage it is more efficient to note the two important differences between this approach and the method that will be used in Chapter 2:

1. In the standard approach a flat world-sheet is used and conformal invariance is imposed, whereas Weyl invariance on a curved world-sheet is used in Chapter 2; and,

2. The "Wilson equation" is an operator statement, however, in the next section correlators are calculated explicitly.

The standard approach fails to obtain the tracelessness condition Eq. (1.19) at the first massive level. Some suggestions as to why this is so will be put forward in Chapter 2.

The weak-field expansion has the nice feature that it can easily handle all the levels of the string, unlike the β-function approach. However, the β-function method is manifestly covariant and thus background independent, while the weak-field expansion derives the equations in a particular gauge since space-time reparameterisation invariance is broken from the very start. (Actually, at low orders the equations are easily covariantisable [328], and, since the goal of Chapter 2 is to study tachyon scattering in a fixed background, it is convenient to choose a gauge anyway.) The weak-field expansion has another disadvantage, as shall be seen in the next chapter. This is that at second order things become very complicated since, generically, every field will enter every equation. It simply must be assumed that it is consistent to study a finite subset of fields. It should also be mentioned that some progress has recently been made towards forming a type of "covariant weak-field expansion" [26, 27]. However, this formulation uses the flat world-sheet and, in the critical dimension, still misses the (analogue of the) tracelessness condition at the first massive level.

1.7 A summary of the perturbative 2d string

In two space-time dimensions it is clear that a non-trivial background is needed for consistent string propagation. Such a background is provided by the flat linear-dilaton vacuum of Eq. (1.59). Canonical quantisation in this vacuum can be carried out [274, Sec. 1.6,5.6] in an analogous way to the flat empty background of Sec. 1.1. Alternatively, the physical spectrum can be found by solving the equations of motion and constraints Eq. (1.58) and Eq. (1.62). Both methods will be described here. From the outset, consider the case in which the background charge just lies along the "1" direction, $Q^\mu = (0, \sqrt{2})$. As mentioned previously, this string theory is equivalent to the one-dimensional non-critical string.

There are two static solutions to the tachyon equation Eq. (1.62) when it is linearised

$$T(X^1) = e^{QX^1} \ , \quad X^1 e^{QX^1} \ , \tag{1.64}$$

and one traveling wave

$$T(X) = e^{ip\cdot X + QX^1} \ , \tag{1.65}$$

with $p^2 = 0$, so that in 2 dimensions, (or, more specifically, in the 1d non-critical string), the tachyon is massless. The second solution of Eq. (1.64) will play a role later when the derivation of tachyon S-matrix elements is summarised.

At the next level of the string, it is found that the only solution is non-propagating and contains just one free parameter [238, 344]. This is called the "black-hole" (its form is derived below using canonical quantisation). The approach will be continued in Sec. 2.6 where it will be shown that the next level above the black-hole contains two time-dependent, but non-propagating, states.

Now proceed with the canonical analysis. Roughly speaking, if the world-sheet has spherical topology, the insertion of the linear-dilaton vertex operator

$$\exp -\frac{1}{4\pi}\int \sqrt{g} R Q\cdot X \ , \tag{1.66}$$

into the path integral shifts the momentum conservation condition to $\sum p_i = -2iQ$ (this is explicitly shown in Eq. (2.26)). Accordingly, to create a circular string with space-time momentum p^μ requires a vertex operator

$$: e^{ipX + Q\cdot X} : \ . \tag{1.67}$$

The Virasoro generators are also changed

$$L_0 - 1 = \tfrac{1}{2}p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n \quad \text{and} \quad L_m = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_n\cdot\alpha_{m-n} + i(m+1)Q\cdot\alpha_m \tag{1.68}$$

where $\alpha_0^\mu = p^\mu - iQ^\mu$ and the central charge in Eq. (1.12) is $c = 2 + 3Q^2$ (and $\alpha' = \frac{1}{2}$).

The first level is thus massless, but is still conventionally called the tachyon. From Sec. 1.1, where the second level was a massless field with $(d-2)$ independent components, it might be expected that no physical excitations would remain in 2d. This counting breaks down at certain discrete momenta, however. The states that appear at these momenta are called "discrete states". As the mass level increases, more and more of these discrete states appear [53, 54, 215, 222, 345, 350].

By way of illustration, consider the open-string state $|e,p\rangle = e_\mu \alpha_{-1}^\mu |0,p\rangle$. The $L_0 - 1$ and the L_1 conditions give

$$M^2 = -p^2 = 2 \quad \text{and} \quad (p^\mu + iQ^\mu)e_\mu = 0 \ , \tag{1.69}$$

respectively. Since $Q^2 = 2$, the physical state is $e_\mu \propto p_\mu - iQ_\mu$. However, this is exactly the spurious state

$$L_{-1}|0,p\rangle = (p_\mu - iQ_\mu)\alpha_{-1}^\mu|0,p\rangle \ . \tag{1.70}$$

So, it seems that there are no physical states, as predicted by the component counting.

This is true at generic momenta but not at special points in momentum space [282]. At $p = -iQ^\mu$ the L_1 condition becomes empty and the spurious states are those with $\epsilon^\mu \propto Q^\mu$. So, there is one physical state

$$\bar{Q}_\mu \alpha_{-1}^\mu |0, -iQ^\mu\rangle \ , \tag{1.71}$$

where $\bar{Q}\cdot Q = 0$. Similarly, at $p = iQ^\mu$, there are no spurious states, so there is another physical state

$$\bar{Q}_\mu \alpha_{-1}^\mu |0, iQ^\mu\rangle \ . \tag{1.72}$$

Now take a direct product of two open strings to make a closed string. Then because of the modified vertex operator Eq. (1.67), the latter state simply corresponds to $: g^{ab}\partial_a X^0 \partial_b X^0 :$ (recall $Q^\mu = (0, \sqrt{2})$ so $\bar{Q}$ lies just along the zero direction). By comparing with Eq. (1.43) describing the string in background fields, it is clear that this insertion simply re-scales components of the flat metric G_{00}. However, this can easily be absorbed by a rescaling of X^0. The former state is more interesting; it is created by the vertex operator

$$: g^{ab}\partial_a X^0 \partial_b X^0 e^{2\sqrt{2}X^1} : \ . \tag{1.73}$$

This also corresponds to a rescaling of components of the metric, but one which diverges at infinity

$$\delta G_{00} = M e^{2\sqrt{2}X^1} \ . \tag{1.74}$$

Here M is an undetermined constant (it arises as a constant of integration in the sigma-model approach). In the coordinates

$$X^1 = X^{1\prime} - \tfrac{1}{4}\sqrt{2} M e^{2\sqrt{2}X^{1\prime}} \ , \tag{1.75}$$

the metric is $\mathrm{d}s^2 = (1 - Me^{2\sqrt{2}X^{1\prime}})\left((\mathrm{d}X^{1\prime})^2 - (\mathrm{d}X^0)^2\right)$. There is a horizon at $X^{1\prime} = -\frac{1}{4}\sqrt{2}\log M$ and the curvature blows up at $X^{1\prime} = \infty$. Thus this solution appears to resemble a black-hole (further remarks can be found in [110, 238, 327, 329, 344]). The analysis of the higher levels of the 2d string proceeds in a similar fashion.

1.8 The $c = 1$ matrix model

The $c = 1$ matrix model describes the non-perturbative behaviour of critical strings living in 2 space-time dimensions. It is derived from the non-critical string in 1 dimension. The string theory presented so far has been purely perturbative since its quantum mechanics was prescribed by the perturbative path integral. Any non-perturbative description of string theory is therefore very exciting since processes invisible to perturbation theory, such as the evolution of black-holes, can be studied.

The standard line of thought is to assume that all closed bosonic critical string theories in a particular dimension are descriptions of the one theory expanded around different vacuum configurations corresponding to the different background fields. Coupled with the fact that the non-critical string in 1D is equivalent to the 2D critical string, the matrix model is hoped to describe the non-perturbative physics of 2D strings in all backgrounds. However, until the proposal of Dhar, Mandal and Wadia (DMW) [97, 98], there was no clue as to how these other backgrounds could be seen in the matrix model. This is the reason why it is important to check their proposal, for, if it is right, it gives a prescription for creating an arbitrary space-time background in the matrix model in which to do (non-perturbative) scattering experiments.

Before specialising to 2d, consider first the partition function corresponding to a non-critical string in R^d

$$Z = \sum_{\text{handles } h} g_{\text{str}}^{-\chi} \int \frac{[\mathrm{d}^d X \mathrm{d}g]_g}{\mathrm{Vol}} e^{-\frac{1}{2}\int \sqrt{g} g^{ab}\partial_a X^\mu \partial_b X^\mu - \mu \int \sqrt{g}}\,, \tag{1.76}$$

where χ is the Euler character ($\chi = 2 - 2h$). The idea of the matrix model is to define $\int \frac{[dg]_g}{Vol}$ by discretising the world-sheet [10, 51, 52, 86, 210]:

$$Z_{\text{dis}} \equiv \sum_{\text{handles}} \sum_{\substack{\text{random} \\ \text{discretisations}}} \frac{1}{G} e^{-\mu A} g_{\text{str}}^{-\chi} \int \prod_{i,\mu} \mathrm{d}' X_i^\mu e^{-\sum_{\langle ij \rangle}(X_i - X_j)^2}\,, \tag{1.77}$$

where G is the symmetry factor of the discretisation, $\mathrm{d}'X_i^\mu$ indicates that the zero-mode of $\Box$ has been omitted, and the sum $\sum_{\langle ij \rangle}$ is over nearest neighbours. An elegant way to reproduce the sums in Eq. (1.77) is to use a Feynman sum of graphs which construct the dual lattice. The quantum theory that produces this Feynman sum is called the "matrix model". Then finally, the continuum limit is reached by taking the size of each simplex to zero and the number of simplices to infinity (the "double scaling limit"). Although the discretised version has been defined for all d, the continuum limit is only solvable for $d \leq 1$. For a review of matrix models and string theory see [160].

1.8.1 An example — pure gravity

Consider the case of $d = 0$ which corresponds to a string propagating in a non-existent space-time. For definiteness, take the simplices to be squares and fix the area of each to be $\frac{1}{2}$. The curvature at a vertex is negative (positive) when the number of incident squares is more (less) than four. For a particular "squarulation" define F to be the number of faces, E the number of edges and V the number of vertices. Then the total area $A = \frac{1}{2}F$ and $\chi = F - E + V$.

To generate all possible discretisations with given number of faces (surface area) and number of handles, the dual lattice is employed. This is just a closed ϕ^4 diagram with F vertices, V loops and E propagators. The matrix model partition function is

$$e^{Z_{MM}} = \int \mathrm{d}\phi \exp\left(-\tfrac{1}{2}\phi^2 + g\phi^4\right) . \tag{1.78}$$

The exponential of Z_{MM} has been taken because only connected graphs are of interest. In fact, as shall now be shown, in order to count the number of handles and to obtain orientable surfaces, $N \times N$ Hermitian matrices must be used rather than simply ϕ,

$$e^{Z_{MM}} = \int \mathrm{d}M_{N\times N} \exp\left(-N(\tfrac{1}{2}\mathrm{Tr}\, M^2 + g\mathrm{Tr}\, M^4)\right) . \tag{1.79}$$

The propagator, vertex and a loop are shown in Fig. 1.2 (notice the arrows on the lines originate

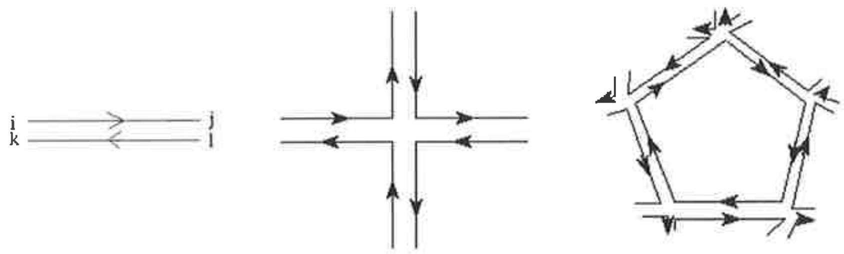

Figure 1.2 : The fundamental building blocks of the matrix model of pure gravity. The propagator $N^{-1}\delta^i_j\delta^k_l$, vertex gN and a loop $\delta^{i_1}_{i_2}\delta^{i_2}_{i_3}\dots\delta^{i_p}_{i_1} = N$.

from the Hermiticity of M and define a certain orientation of the surface). A typical graph has [318]

$$N^{-E}(Ng^{-1})^F N^V = N^\chi g^{-F} , \tag{1.80}$$

so that by considering an $N \times N$ matrix rather than just ϕ, the number of handles has been able to be counted. Comparing with Eq. (1.77), the following identifications between the matrix model and the discretised string can be made

$$N = g_{\mathrm{str}}^{-1} \quad \text{and} \quad g^2 = e^{-\mu} . \tag{1.81}$$

Note also that the symmetry factor $1/G$ has been taken care of automatically by the symmetry factor of Feynman diagrams.

Before investigating the continuum limit, it is worth mentioning that the problem can be simplified by diagonalising $M = U^\dagger DU$. After integrating out the U's the partition function becomes

$$Z_{MM} = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \Delta(\lambda) e^{-N\sum_i \left(\frac{1}{2}\lambda_i^2 + g\lambda_i^4\right)} , \tag{1.82}$$

where the induced measure includes the Vandermonde determinant

$$\Delta(\lambda) = \prod_{j<l} (\lambda_j - \lambda_l) . \tag{1.83}$$

It is important to note that this is antisymmetric in the eigenvalues λ_i, because in the $c = 1$ model such determinants arise in a similar fashion and will prepare antisymmetric (fermionic) initial and final states.

It is intuitively clear that the continuum limit corresponds to biasing the Feynman sum towards surfaces with a large number of vertices. In other words, the coupling must be tuned $g \to g_c$ so that the average total area blows up

$$\langle A \rangle = -\frac{\partial \log Z_{dis}}{\partial \mu} = 2g \frac{\partial \log Z_{MM}}{\partial g} \to \infty . \tag{1.84}$$

Therefore, the behaviour of Z as the coupling tends towards some critical value is of interest. From Eq. (1.80), the partition function can be expanded in a power series

$$Z_{MM} = N^2 Z_0 + Z_1 + N^{-2} Z_2 + \dots , \tag{1.85}$$

where Z_h is the partition function arising from surfaces of genus h. The continuum limit is not simply the $N \to \infty$ limit since this would produce only planar graphs. Instead, the "double scaling limit" is taken, in which the critical limit $g \to g_c$ is taken in conjunction with large N limit in such a way that each genus gets weighted equally. The rest of this section motivates the exact form of the double scaling limit.

Writing the non-critical string of Eq. (1.41) in terms of a partition function $Z(A)$ for fixed area $A = \int \sqrt{g}$

$$\begin{aligned} Z &= \int \mathrm{d}A e^{-\mu A} Z(A) , \\ Z(A) &= \int [\mathrm{d}\dots]_{\hat{g}} \exp\left(\frac{1}{4\pi}\int_{\mathcal{M}} Q\hat{R}\sigma + \dots\right) \delta\left(\int \sqrt{\hat{g}} e^{\alpha\sigma} - A\right) , \end{aligned} \tag{1.86}$$

the scaling dependence of $Z(A)$ on A can be determined. Under a shift $\sigma \to \sigma + \bar{\sigma}/\alpha$ for $\bar{\sigma}$ constant, the measure does not change — the only change is from the dilaton term and the delta function, giving

$$Z(A) = e^{Q\bar{\sigma}\chi/\alpha - \bar{\sigma}} Z(e^{-\bar{\sigma}} A) . \tag{1.87}$$

The choice $e^{\bar{\sigma}} = A$ gives

$$Z(A) = A^{Q\chi/\alpha - 1} Z(1) \equiv A^{\frac{1}{2}\chi(\Gamma_{str} - 2) - 1} Z(1) . \tag{1.88}$$

In the discrete case A is quantised which gives [42]

$$Z_{MM} = Z_{dis,\chi} \sim \sum_A e^{-\mu A} A^{\frac{1}{2}\chi(\Gamma_{str}-2)-1} \sim \sum_A g^A A^{\frac{1}{2}\chi(\Gamma_{str}-2)-1} \sim (g-1)^{\frac{1}{2}\chi(2-\Gamma_{str})} . \quad (1.89)$$

Therefore the partition function can be written as the sum

$$Z = \sum_\chi a_\chi \left(N(g-1)^{\frac{1}{2}(2-\Gamma_{str})} \right)^\chi . \quad (1.90)$$

Differentiating to find the average total area, it is clear that the type of divergence is the same for each genus

$$\langle A \rangle \sim \frac{1}{g-1} . \quad (1.91)$$

The continuum limit in this simple model is therefore defined as

$$(g-1) \to 0 \quad \text{and} \quad N \to \infty, \quad \text{with} \quad N(g-1)^{\frac{1}{2}(2-\Gamma_{str})} \text{ constant} . \quad (1.92)$$

This is a "double scaling limit". It appears in much the same way in the minimal models and below in the $c = 1$ model. In the $c = 1$ model, it is especially important to know that the double scaling limit really matches the continuum, since it is hoped that the matrix model will shed light on physical processes such as black-hole evolution. This matching has not been shown rigorously, however, the double scaled matrix model does agree with every amplitude that has been calculated in the continuum.

1.8.2 The $c = 1$ model and its double scaling limit

For one space-time dimension, the expansion in g_{str} of Eq. (1.77) can be generated from the partition function

$$e^{Z_{MM}(T)} = \int [\mathrm{d}M(t)] \exp\left(-N \int_0^T \mathrm{d}t \,\mathrm{Tr}\,(\dot{M}^2 - V(M)) \right) . \quad (1.93)$$

The kinetic term in the action has been obtained in the following way: The quadratic choice of kinetic term in Eq. (1.77) corresponding to the Polyakov action maps to a Gaussian propagator $\Delta(X) \sim e^{-X^2}$ in the matrix model. The leading small-momentum behaviour of the associated momentum space kinetic term $\Delta^{-1}(p) \sim e^{p^2}$, looks like $1 + p^2$. It is this that has been used, *not* the inverse of the Gaussian propagator [160]. Because the model is ultraviolet convergent, only the short distance, non-universal, behaviour is affected by this substitution. This claim is supported by comparing the calculations of the string susceptibility Γ_{str} in the continuum [217] and for large N [209]. More indirectly, it is also supported by the agreement between the double scaled correlators and those in the continuum. Similarly, the double-scaled answers do not depend on the choice of V, other than needing $V(\pm\infty) = \infty$ and, as shall soon be shown, the existence of a quadratic maximum. This is encouraging since it points to the fact that the true continuum results are being calculated rather than some lattice artifact.

Diagonalising M yields the quantum mechanics of N spinless particles [56]

$$e^{Z_{MM}(T)} = \int \prod_i [\mathrm{d}\lambda_i(t)] \Delta(\lambda_i(0)) \Delta(\lambda_i(T)) \exp\left(-N \int_0^T \mathrm{d}t \sum_i (\dot{\lambda}_i^2 - V(\lambda_i))\right) . \tag{1.94}$$

The Vandermonde determinants that come from the Jacobian at each intermediate time-step have canceled with those coming from $\mathrm{Tr}\, \dot{M}^2 = \sum \dot{\lambda}_i^2 + \sum U_{ij}^\dagger (\lambda_i - \lambda_j)^2 U_{ji}$, and only the boundary terms remain. These prepare final and initial fermionic states since Δ is antisymmetric in the λ_i. Thus the matrix model is a system of N uncoupled fermions moving in a potential V.

There still remains the double scaling limit. Motivated by the $c = 0$ case, this will consist of tuning the coupling g (contained in V) to some critical value, while simultaneously taking the large N limit such that some scaling parameter is held fixed. First notice that in the partition function Eq. (1.94) the number of eigenvalues N appears in precisely the same way that $\hbar^{-1}$ appears usually. Continue this analogy by writing the Hamiltonian $H = NH'$ where the new Hamiltonian H' is given by

$$H' = \sum_i \tfrac{1}{2} p_i^2 + V(\lambda_i) \quad \text{with} \quad [p_i, \lambda_j] = \hbar \delta_{ij} = N^{-1} \delta_{ij} . \tag{1.95}$$

Each fermion occupies a volume $2\pi\hbar = 2\pi/N$ in phase space and in the large N limit a continuous Fermi fluid appears. Consider this classical limit in more detail. The volume of the fluid is held constant since

$$N = \frac{\text{volume of occupied phase space}}{\text{volume of one fermion}} = \frac{\int \mathrm{d}p \mathrm{d}\lambda\, \theta(\epsilon_F - E'(p, \lambda))}{2\pi/N} , \tag{1.96}$$

which implies

$$1 = \int \frac{\mathrm{d}p \mathrm{d}\lambda}{2\pi} \theta(\epsilon_F - E'(p, \lambda)) . \tag{1.97}$$

In these formulae, θ is a step function, $E'(p, \lambda) = \frac{1}{2} p^2 + V(\lambda)$ and ϵ_F is the Fermi energy. The above identity, true in the large N limit, determines the Fermi energy in terms of the coupling g.

As noted previously, the large N limit suppresses all but the graphs with spherical topology. The free energy F_0, of such graphs is

$$F_0 = \lim_{N \to \infty} N^{-2} \lim_{T \to \infty} T^{-1} Z_{MM}(T) = \lim_{N \to \infty} N^{-2} N E_0' , \tag{1.98}$$

where E_0' is the total energy of the fermions in their ground state

$$E_0' = \sum_{i=1}^{N} E_i' . \tag{1.99}$$

In summary, Eq. (1.97) determines the Fermi level ϵ_F in terms of g, and the goal is to find the non-analytic behaviour of Eq. (1.98). First consider the following heuristic argument: Suppose the potential is given by $V(\lambda) = -\frac{1}{2}\lambda^2 + g\lambda^4$. Then, as g increases, the fluid moves up the potential wells as depicted in Fig. 1.3. At some critical value of g, the Fermi fluid will reach the

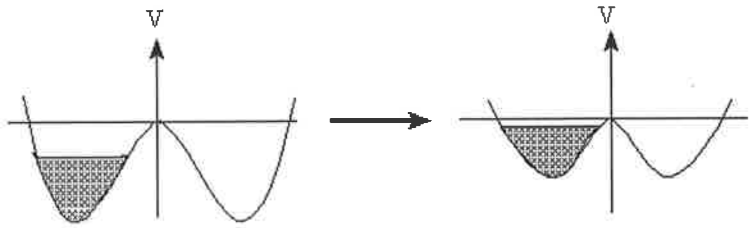

Figure 1.3 : As the coupling g increases, the Fermi level gradually reaches the top of the quadratic maximum. There is a critical point as the fluid reaches the top.

top of the quadratic maximum. At this point, the free energy diverges. Putting some body into these statements, the free energy can be explicitly calculated [56]

$$\begin{aligned} F_0 &= \int \frac{\mathrm{d}p\mathrm{d}\lambda}{2\pi}(\tfrac{1}{2}p^2 - \tfrac{1}{2}\lambda^2 + g\lambda^4)\,\theta(\epsilon_f - \tfrac{1}{2}p^2 + \tfrac{1}{2}\lambda^2 - g\lambda^4) \\ &= \frac{1}{3\pi}\int_{\sqrt{-2\epsilon_F}} \mathrm{d}\lambda\sqrt{2\epsilon_F + \lambda^2 - 2g\lambda^4}\left(\epsilon_F - \lambda^2 + 2g\lambda^4\right) \\ &\sim \epsilon_F^2 \log -\epsilon_F \ . \end{aligned} \tag{1.100}$$

In the last line, only the leading singular behaviour has been kept. This arises from the end-points of the integral which are close to the central quadratic maximum as indicated in the second line. The critical limit is $\epsilon_F \to 0$.

From the derivation, it is clear that the only important distinguishing feature of the potential was a quadratic maximum. Multiplying F_0 by N^2 to obtain the free energy suggests that the scaling parameter in the double scaling limit should be $N\epsilon_F$. This is in contrast with the $c = 0$ case, since the scaling variable does not involve a power of $g - g_c$. Instead, evaluating the singular part of Eq. (1.97) yields $g - g_c \sim -\epsilon_F \log(-\epsilon_F)$.

The double scaling limit is for the $c = 1$ matrix model is defined to be [57, 161, 179, 247, 264, 270]

$$\epsilon_F \to 0 \quad \text{and} \quad N \to \infty \quad \text{with} \quad \bar{\mu} \equiv -N\epsilon_F = \text{constant} \ . \tag{1.101}$$

As $N \to \infty$ the splitting between fermion levels is tending to zero which results in a continuous Fermi liquid. At the same time the Fermi sea is rising up to the top of the quadratic potential; $\bar{\mu}$ parameterises how close the Fermi level gets to the top. The quantity $\bar{\mu}$ is not to be confused with the cosmological constant μ; in fact, since each string diagram is weighed by $\bar{\mu}^{\chi}$ (see Eq. (1.80)), $\bar{\mu}$ is simply the inverse string coupling $g_{\text{str}} \sim \bar{\mu}^{-1}$.

The double scaling limit is easy to take in the second quantised version of the system. Here the Hamiltonian is

$$H = N\int \mathrm{d}\lambda \left(\frac{1}{2N^2}\partial_\lambda\zeta^\dagger\partial_\lambda\zeta + V(\lambda)\zeta^\dagger\zeta\right) \ , \tag{1.102}$$

where $\zeta(\lambda)$ is a second-quantised spinless field. Define $\lambda = N^{-1/2}q$ and $\zeta(\lambda) = N^{1/4}\psi(q)$, then in the double scaling limit only the the quadratic behaviour of the potential survives and the

Hamiltonian becomes

$$H = \int \mathrm{d}q \left(\tfrac{1}{2}\partial_q\psi^\dagger\partial_q\psi - \tfrac{1}{2}q^2\psi^\dagger\psi\right) , \tag{1.103}$$

which is just the non-relativistic quantum field theory describing non-interacting fermions living in an inverted harmonic oscillator potential! The fermions are filled up to the level $H = -\bar{\mu}$.

1.8.3 The space-time interpretation

Although the double-scaled matrix model is supposed to describe 2d critical string theory, the identification of the space-time states in the matrix model has been a problem fraught with difficulties and misunderstandings. It might be expected that the tachyon, which is a scalar, should be related to the bosonised fluctuations of the Fermi surface [84, 239, 271, 310]. Indeed this is so, but the relative normalisation of the wavefunctions was, however, quite difficult to calculate and it is the goal of this section to give an overview of the result. The realisation of the discrete states in the matrix model is still speculative (although it is hoped that the next chapter adds weight to DMW's proposal) and comments regarding this problem are saved till the next section.

The idea is quite easy in principle. The correlators of tachyons are calculated in the 2d string theory and compared with scattering of small bumps on top of the Fermi surface. The first part of this subsection is taken from Polchinski's lectures [274], while the second part comes from the papers of Di Francesco and Kutasov [105, 106].

On the matrix model side, only small fluctuations around the static solution will be considered. This is because the results will be compared with the predictions from perturbative string theory and this regime ($g_{\mathrm{str}} \to 0$) is described by the matrix model with the Fermi level well below the quadratic maximum ($\bar{\mu} \sim g_{\mathrm{str}}^{-1}$), or, equivalently, by small perturbations of the Fermi sea which do not cross the potential barrier. Thus, it seems safe to consider only one side of the potential since tunnelling through and washing over the barrier are both non-perturbative effects. This idea will be challenged in the next section, but for now, it will be accepted that only one side is needed in order to describe the perturbative scattering of tachyons.

The classical motion[3] of each individual fermion is governed by the single-particle Hamiltonian $H = \frac{1}{2}(p^2 - q^2)$,

$$D_t p = q \quad \text{and} \quad D_t q = p , \tag{1.104}$$

where D_t is the comoving derivative which follows the fermion of interest. This has the particularly simple solution

$$p = -|a|\sinh(t-b) \quad \text{and} \quad q = -|a|\cosh(t-b) , \tag{1.105}$$

[3]The map between the matrix model and the space-time has only been worked out at tree level. However, the wavefunction renormalisation needed to map the tachyon into the matrix model originates because the latter was introduced using, essentially, a short-distance cutoff. Presumably then, since short distance effects should not depend on the global topology of the world-sheet (g_{str} is a relevant coupling), the map is the same to all orders of perturbation theory [274].

where a and b are arbitrary constants. The "skin" of the static sea is given by $H = -\bar{\mu}$, or

$$p_{\pm}(q)_{\text{static}} = \pm\sqrt{q^2 - 2\bar{\mu}}\ . \tag{1.106}$$

In this formula, and henceforth, + (−) will stand for quantities with positive (negative) momentum p. These are, of course, incoming (outgoing) fermions, as depicted in Fig. 1.4

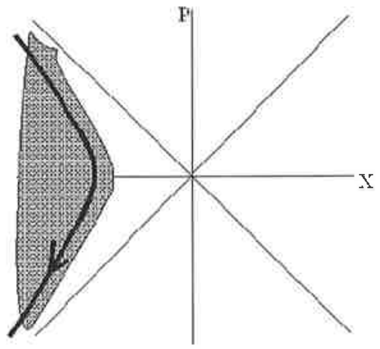

Figure 1.4 : A typical excited Fermi sea (shaded). The incoming fermions at the top move on hyperbolae until they leave the diagram at the bottom left corner. The RHS of the sea is unexcited and small pulses (perturbative string theory) never pass over to this region.

$$(-)\ =\ (p<0)\ =\ \text{outgoing} \qquad (+)\ =\ (p>0)\ =\ \text{incoming} \tag{1.107}$$

Rather than describe the motion of each individual fermion, it is much more convenient to use collective coordinates $p_{\pm}(q,t)$ which are the positions of the sea's upper and lower surfaces for a small perturbation around $p_{\pm}(q)_{\text{static}}$. The equation of motion for $p_{\pm}$ is simply that of the position p of the Fermi surface at a fixed q

$$\partial_t p_{\pm}(q,t) = q - p_{\pm}(q,t)\partial_q p_{\pm}(q,t)\ . \tag{1.108}$$

In order to find the matrix-model scalar, define a new coordinate $\tau = -\log(-q)$, an asymptotically small perturbation $\epsilon_{\pm}$, and the quantities $\pi_{\bar{S}}, \bar{S}$

$$\begin{aligned} p_{\pm}(q,t) &= \mp q \pm \frac{1}{q}(\bar{\mu} + \epsilon_{\pm}(\tau,t))\ , \\ \pi^{-1/2}(\bar{\mu} + \epsilon_{\pm}(\tau,t)) &= \pm\pi_{\bar{S}}(\tau,t) - \partial_\tau \bar{S}(\tau,t)\ . \end{aligned} \tag{1.109}$$

Written in terms of these new variables, the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \int_{p_-}^{p_+} \tfrac{1}{2}(p^2 - q^2) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \left(\tfrac{1}{6}(p_+^3 - p_-^3) - \tfrac{1}{2}q^2(p_+ - p_-)\right) \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} d\tau \left(\pi_{\bar{S}}^2 + (\partial_\tau \bar{S})^2 + e^{2\tau}O(\bar{S}^3)\right)\ . \end{aligned} \tag{1.110}$$

Coupled with the equation of motion for $p_{\pm}$, the second line of this equation implies

$$[\epsilon_{\pm}(\tau), \epsilon_{\pm}(\tau')] = \mp 2\pi i \partial_\tau \delta(\tau - \tau')\ , \tag{1.111}$$

meaning that $\bar{S}$ is a scalar with $\pi_{\bar{S}}$ as its conjugate. The Hamiltonian also shows that the self-interaction of $\bar{S}$ grows with increasing τ. This is similar to the way $g_{\rm str}$ increases as the Liouville wall is approached. The tachyon is not, however, simply given by $\bar{S}$. The matrix model gives a discrete approximation to the vertex operators and it would not come as a surprise to find a wavefunction renormalisation resulting from such a process.

With the help of an infinite number of conserved charges, the classical S-matrix can be obtained. The quantities

$$W_{mn} = e^{(m-n)t} \int_{\delta F} \frac{\mathrm{d}p\mathrm{d}q}{2\pi} (-p-q)^m (p-q)^n \ , \tag{1.112}$$

where the integral runs over static Fermi sea subtracted from the excited Fermi sea, are conserved by the equations of motion Eq. (1.104). Their Poisson brackets satisfy a classical w_∞ algebra structure [16, 83, 100, 245, 251] which is also generated by the vertex operators of the discrete modes in the 2d string [345, 350]. The relation between these two algebras is not entirely clear. Evaluating these quantities in the far past and the far future allows the incoming pulse described by $\epsilon_+(t,\tau)$ to be expressed in terms of the outgoing pulse $\epsilon_-(t,\tau)$. By Eq. (1.109) and the equation of motion for $\bar{S}$, the incoming pulse is a function of $\tau_- = t-\tau$ asymptotically, ie $\epsilon_+(t,\tau) = \epsilon_+(\tau_-)$ (and similarly $\epsilon_-(t,\tau) = \epsilon_-(\tau_+)$). After Fourier transforming

$$\epsilon_\pm(\tau_\mp) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \ \alpha_\pm(\omega) e^{-i\omega\tau_\mp} \ , \tag{1.113}$$

the incoming and outgoing modes are related as follows [249, 250, 271]

$$\begin{aligned} \alpha_+(\omega) &= \sum_{k=1}^{\infty} \frac{(\sqrt{2\pi}\bar{\mu})^{1-k}}{k!} \frac{\Gamma(1+i\omega)}{\Gamma(2-k+i\omega)} \frac{1}{\sqrt{2\pi}} \left(\frac{\bar{\mu}}{2}\right)^{i\omega} \\ &\quad \times \int \mathrm{d}\omega_1 \mathrm{d}\omega_2 \ldots \mathrm{d}\omega_k \ \alpha_-(\omega_1)\ldots\alpha_-(\omega_k)\delta(\omega_1+\ldots+\omega_k-\omega) \ . \end{aligned} \tag{1.114}$$

From the commutators of $\epsilon_\pm$, the modes obey the standard oscillator algebra

$$[\alpha_\pm(\omega), \alpha_\pm(\omega')] = \pm 2\pi\omega\delta(\omega+\omega') \ . \tag{1.115}$$

Using these two last equations, any amplitude can be calculated, for instance, the $1\to n$ amplitude is

$$\langle \omega_1, \ldots, \omega_n | \omega \rangle = \left(\frac{\sqrt{2\pi}}{\bar{\mu}}\right)^{n-1} \omega_1 \ldots \omega_n \left(\frac{\bar{\mu}}{2}\right)^{i\omega} \frac{\Gamma(1+i\omega)}{\Gamma(2-n+i\omega)} \sqrt{2\pi}\delta(\omega_1+\ldots+\omega_n-\omega) \ . \tag{1.116}$$

Now the analogous process in string theory must be computed — the tree-level scattering of tachyons in a flat linear-dilaton background.

At this point, the chapter could become swamped by the technical details of what, in the author's opinion, is a very ingenious calculation. Rather than let this happen, the computation of [105, 106] is briefly summarised and the result stated (see also [13, 118, 160, 169, 178, 228, 246, 282, 298]).

Start with the 2d critical string living in a flat linear-dilaton background with non-zero cosmological constant

$$S = \frac{1}{4\pi}\int \sqrt{g}\left(\tfrac{1}{2}\partial_a X^\mu \partial_a X^\mu + RQ\cdot X + \mu e^{\alpha X^2}\right) , \tag{1.117}$$

where $12Q^2 = 26 - d = 24$ and $\alpha = Q_2 + \sqrt{(Q_2)^2 - 2}$. If Q_μ lies purely along the "2" (Liouville) direction ($Q_\mu = (0, \sqrt{2})$), then this theory is equivalent to a non-critical string theory in the single dimension X^1. There are then two possible forms for the cosmological term; $e^{\alpha X_2}$ and $X_2 e^{\alpha X_2}$ (see Eq. (1.64)). Because of the tachyon self-interaction these two forms will mix and, unfortunately, it is the latter which dominates at weak string coupling [270] making the calculation untenable in this regime. For this reason it is convenient to "regulate" the theory by calculating with a small but non-zero Q_1.

The amplitudes of interest are N-point functions of tachyons $\langle T_{k_1} T_{k_2} \ldots T_{k_N}\rangle$, where the tachyon vertex operator is given in Eq. (1.67). Only the tree-level amplitudes are calculated. It is convenient to perform the integral over the zero mode a_0 of X_2. Using [170, Eq. 3.331.1]

$$\int_{-\infty}^{\infty} \mathrm{d}a_0 \exp\left(-\mu e^{a_0} + s a_0\right) = \mu^s \Gamma(-s) , \tag{1.118}$$

the correlators are

$$\langle T_{k_1} T_{k_2} \ldots T_{k_N}\rangle = \mu^s \Gamma(-s) \int [\mathrm{d}^2 X] e^{-S_0} \left(\int_{\mathcal{M}} e^{\alpha X^2}\right)^s T_{k_1} T_{k_2} \ldots T_{k_N} . \tag{1.119}$$

Naturally the zero mode of X_2 is to be left out of the functional integral, S_0 is the action with $\mu = 0$ and s is given by

$$\sum_{i=1}^{N} \beta(k_i) + \alpha s = 2Q . \tag{1.120}$$

For $s \in \mathbb{Z}_+$, the path integral Eq. (1.119) can be computed in terms of integrals of powers of propagators. For example, in the case of the three-point function, the three vertex operators can be inserted at fixed positions on the sphere (in order to completely break the Killing symmetries, see p. 11). Choosing these points to be $z = 0, 1, \infty$, the result is [105, 106, 119, 120]

$$\begin{aligned}\langle T_{k_1} T_{k_2} T_{k_3}\rangle &= \mu^s \Gamma(-s) \prod_{i=1}^{s} \int \mathrm{d}^2 w_i |w_i|^{-2\alpha\beta(k_1)} |1 - w_i|^{-2\alpha\beta(k_3)} \prod_{i<j} |w_i - w_j|^{-2\alpha^2} , \\ &= \left(\mu \frac{\Gamma(\frac{1}{2}\alpha^2)}{\Gamma(1 - \frac{1}{2}\alpha^2)}\right)^s \prod_{i=1}^{3} \frac{\Gamma(\frac{1}{2}\beta_i^2 - \frac{1}{2}k_i^2)}{\Gamma(1 - \frac{1}{2}\beta_i^2 + \frac{1}{2}k_i^2)} .\end{aligned} \tag{1.121}$$

At this point, the arguments become too lengthy and complicated to present here. Suffice to say that higher-point functions also give ratios of Gamma functions. Eq. (1.121) has only been calculated for integer s, but this is exactly where $\Gamma(-s)$ is ill-defined. Furthermore, the integrals only converge in certain regions in momentum space. Amazingly, DiFrancesco and Kutasov [105, 106] managed to make physical arguments that allowed the correlators to be understood for all s and by examining the poles in general amplitudes were able to extend the correlators to the whole of momentum space.

Comparing the results from the matrix model with those from the string theory, the following map between the two theories is found

$$\alpha_{\pm}(\omega)_{\text{string}} = \left(\frac{\pi}{2}\right)^{\frac{i\omega}{4}} \frac{\Gamma(\pm i\omega)}{\Gamma(\mp i\omega)} \alpha_{\pm}(\omega)_{\text{mm}} \ , \tag{1.122}$$

where $\alpha(\omega)_{\text{mm}}$ are the modes of the massless scalar $\bar{S}$ (Eq. (1.113)) and $\alpha(\omega)_{\text{string}}$ are the modes of the re-scaled tachyon $S = Te^{-\Phi}$. The ratio of gamma functions is known as the "leg-pole" factor. The first detailed investigation into the physical consequences of the leg-pole factor was by Polchinski and Natsuume [278] who demonstrated how it can convert the free fermions of the matrix model into space-time tachyons that exhibit many complicated (self)-interactions.

1.9 The proposal

To reproduce the perturbative string theory results only very small bumps on top of the Fermi sea need be considered. Tunnelling through the potential barrier and fluid washing over the wall are both non-perturbative effects. So, when working in the semi-classical limit with very small pulses, it seems safe to work with only one side of the potential. Indeed, since only one side was needed to reproduce the perturbative tachyon physics, it would seem that there would be an infinite number of non-perturbative completions of the model [249], depending on what was done with the other side. On the other hand, most of these completions were argued to be non-unitary[273], basically because fluid was lost over to the other side.

DMW [98] argued that the space-time metric must couple to the entire energy-momentum tensor of the theory. The Hamiltonian of the string theory is identical to that of the matrix model (unless the potential is modified by hand from the beginning). A generic perturbation of the Fermi fluid has total energy coming from *both* sides of the potential, so the metric must couple to this total energy.

Allowing excitations of both halves of the sea introduced another degree of freedom into the model that had not yet been utilised. There were now *two* scalar fields — the average and the difference of the bosonised fluctuations on each side of the barrier. By considering a symmetric leg-pole transform and comparing with the tachyon-graviton effective theory, DMW proposed that the tachyon, S, is the leg-pole transform of the average of the fluid fluctuations, while the total energy of the difference variable, Δ, is the mass of the black-hole.

This section will outline the proposal of DMW and leave the details for Appendix A. The extended leg-pole transform, (which, because it treats both sides identically, gets around the problem of non-unitarity mentioned above), and the idea of how the matrix model realises the presence of a discrete background, will be presented.

1.9.1 The leg-pole transformation

The formalism [99, 100, 101] uses a phase-space density of fermions $U(p,q,t)$. In the semi-classical limit this is just unity inside the sea and zero outside. Fluctuations around the static sea are denoted by $\delta U(p,q,t)$ which is, for the moment, only non-zero for the LHS of the sea. The leg-pole transform can then be written in one compact equation as an integral over phase space

$$S(x,t) = \int \mathrm{d}q\mathrm{d}p \; f(-qe^{x})\delta U(p,q,t) \; , \tag{1.123}$$

in which the function f is simply a zeroth-order Bessel function [170]

$$f(\sigma) = \frac{1}{2\sqrt{\pi}} J_0\left(2(\tfrac{2}{\pi})^{1/8}\sqrt{\sigma}\right) \; , \tag{1.124}$$

and $S = Te^{-\Phi}$ is the rescaled tachyon.[4] Eq. (1.123) is only valid asymptotically as $x \to -\infty$. Higher-order corrections to this leg-pole transformation which are of order xe^{2x} have also been calculated [102]. Here, only the asymptotic form of the transformation will be used since only bulk correlators (large negative values of x) will be calculated.

Note, so far, the fluid fluctuation has been non-zero only for $q < 0$. When excitations on both sides of the potential are allowed, DMW suggested that the symmetrical leg-pole transform should be used

$$S(x,t) = \frac{1}{\sqrt{2}} \int \mathrm{d}q\mathrm{d}p \; f(2^{1/4}|q|e^{x})\delta U(p,q,t) \; . \tag{1.125}$$

The factors of $1/\sqrt{2}$ and $2^{1/4}$ are for convenience only. This is the first part of their proposal.

1.9.2 The tachyon and higher states

Due to the particularly simple dynamics of the matrix model, it can be shown [101, 102] that a generic fluctuation can be evolved back into the distant past so that it becomes possible to write S_- explicitly in terms of S_+. This yields the following two equations

$$\begin{aligned} \frac{1}{\sqrt{2}} \sum_{\alpha=1,2} \epsilon_+^{\alpha}(\tau) &= -4\pi \int_{-\infty}^{\infty} \mathrm{d}x^- \; S_+(x^-)\partial_\tau^2 f\left(\sqrt{\bar{\mu}'}e^{\tau - x^-}\right) \; , \\ S_-(x^+) &= \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \sum_{\alpha=1,2} \int_0^{\sqrt{2}\epsilon_+^{\alpha}(\tau)} \mathrm{d}\epsilon \; f\left(\sqrt{\bar{\mu}'}(1 - (\epsilon/\bar{\mu}'))e^{-\tau + x^+}\right) \; . \end{aligned} \tag{1.126}$$

where $\bar{\mu}' = |\bar{\mu}/\sqrt{2}|$ (this varies slightly from DMW's convention $\bar{\mu}' = \sqrt{2}|\bar{\mu}|$).[5] In this formula, the small initial perturbation of the top-LHS of the Fermi sea has been parameterised by $\epsilon_+^1(\tau)$ and the bottom-RHS by $\epsilon_+^2(\tau)$. They are just the leg-pole transform of the incoming tachyon $S_+(x^-)$. Another leg-pole transform of these variables gives the outgoing tachyon $S_-(x^+)$.

[4] Comparing with Eq. (1.65) it is seen that the vertex operator for S is simply e^{ipX}.

[5] For fluctuations purely on the LHS of the sea, the variable τ was defined in the previous section; $\tau = -\log(-q)$. For the fluctuations on the other side $\tau = -\log q$.

The second part of the proposal concerns the realisation of the tachyon and the discrete states in the matrix model. As suggested by the above formula, the leg-pole of the tachyon is taken to be the sum $\frac{1}{\sqrt{2}}\sum_{\alpha=1,2}\epsilon_+^a(\tau)$. Call this variable $\phi(\tau)$,

$$\phi(\tau)=\frac{1}{\sqrt{2}}\left(\epsilon_+^1(\tau)+\epsilon_+^2(\tau)\right) . \tag{1.127}$$

The difference variable Δ,

$$\Delta(\tau)=\frac{1}{\sqrt{2}}\left(\epsilon_+^1(\tau)-\epsilon_+^2(\tau)\right) , \tag{1.128}$$

will be associated the discrete-mode backgrounds. With $\Delta=0$, the tree-level tachyon scattering amplitudes in the flat linear-dilaton background are recovered since the scenarios on either side of the potential are identical. However, extra scattering effects come into play when Δ is non-zero.

DMW considered the case when Δ was small and retained just the first non-trivial term in the expansion of S_- around $\Delta=0$. This term was further expanded in powers of ϕ — for 1-1 tachyon scattering, only the linear order is needed. This yields

$$\begin{aligned} S_-(x^+) \;=\;& \int_{-\infty}^{\infty} d\tau \int_0^{\phi(\tau)} d\epsilon\; f\left(\sqrt{\bar{\mu}'}(1-(\epsilon/2\bar{\mu}'))e^{-\tau+x^+}\right) \\ & -\frac{1}{4\sqrt{\bar{\mu}'}}\int_{-\infty}^{\infty} d\tau\Delta^2(\tau)e^{-\tau+x^+} \\ & \times\left[f'\left(\sqrt{\bar{\mu}'}e^{-\tau+x^+}\right)-\frac{\phi(\tau)e^{-\tau+x^+}}{2\sqrt{\bar{\mu}'}}f''\left(\sqrt{\bar{\mu}'}e^{-\tau+x^+}\right)+O(\phi^2)\right] \\ & +O(\Delta^4) , \end{aligned} \tag{1.129}$$

where the prime on f means derivative with respect to its argument. Notice that the first term of the $O(\Delta^2)$ part is not zero even when the incoming tachyon ϕ is set to zero! The DMW proposal interprets this extra contribution as coming from a tachyon background which has been dynamically generated by a nonzero value of Δ. The $O(\phi)$ term will therefore have to contain a part which describes tachyon scattering off this extra tachyon background. It will also contain parts which correspond to tachyons scattering off the space-time realisation of Δ.

Using the first of Eq. (1.126) the $O(\phi,\Delta^2)$ part of S_- can be written as

$$\frac{1}{2\bar{\mu}'}\int_{-\infty}^{\infty} dx^- S_+(x^-)\int_{-\infty}^{\infty} d\tau\; \Delta^2(\tau)e^{-2\tau+2x^-}\partial_\tau^2 f\left(\sqrt{\bar{\mu}'}e^{\tau-x^-}\right)f''\left(\sqrt{\bar{\mu}'}e^{-\tau+x^+}\right) . \tag{1.130}$$

Assuming that the incoming tachyon wavepacket is highly localised around a large positive value of x^-, and taking the early-time limit $x^+\to-\infty$, Bessel functions can be expanded in the small parameters e^{-x^-} and e^{x^+}. The final result of 1-1 tachyon scattering is

$$\begin{aligned} S_-(x^+\to-\infty) \;=\;& (\Delta=0 \text{ part})+(\text{part from dynamically created tachyon background}) \\ & +\sum_{n=1,m=0}^{\infty} C_{mn}e^{(n+1)x^+}\int_{-\infty}^{\infty} dx^- e^{-(m+1)x^-}S_+(x^-) , \end{aligned} \tag{1.131}$$

where the coefficient C_{nm} is given by

$$C_{mn}=\frac{(-1)^{m+n+1}}{4\pi}\left|\frac{\bar{\mu}'}{\sqrt{\pi/2}}\right|^{\frac{1}{2}(m+n)-1}\frac{2n-1}{(m!)^2 n!(n-1)!}\int_{-\infty}^{\infty} d\tau\Delta^2(\tau)e^{(m-n)\tau} . \tag{1.132}$$

By comparing with the effective tachyon field theory, as shall also be done in the next section, DMW showed:

- The $(m,n) = (0, \text{arbitrary})$ corresponded to the tachyon scattering off the dynamically created background,

$$-\frac{1}{4\sqrt{\mu'}}\int_{-\infty}^{\infty} \mathrm{d}\tau \ \Delta^2(\tau)e^{-\tau+x^+} f'\left(\sqrt{\mu'}e^{-\tau+x^+}\right) \ . \tag{1.133}$$

- The $(m,n) = (1,1)$ part was the same as tachyon scattering off a black-hole background if the "mass" parameter of the black-hole was given by

$$M = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau\Delta^2(\tau) \ . \tag{1.134}$$

They postulated that other moments of the difference variable corresponded to the parameters of the other discrete states. Using the effective field theory of tachyons in a massive-mode background derived in Eq. (2.117), it will be shown in Chapter 2 section that this is true at the next level above the black-hole.

Massive Fields and the 2D String

The first massive level of closed bosonic string theory is studied. Free-field equations are derived by imposing Weyl invariance on the world sheet. A two-parameter solution to the equation of motion and constraints is found in two dimensions with a flat linear-dilaton background. One-to-one tachyon scattering is studied in this background. The results support Dhar, Mandal and Wadia's proposal that 2D critical string theory corresponds to the $c = 1$ matrix model in which both sides of the Fermi sea are excited.

So that the reader does not have to wade back through the introductory chapter, the notation will be introduced afresh. Recall that tree-level amplitudes in closed bosonic string theory are calculated by the insertion of appropriate vertex operators into the path integral

$$Z = \int \frac{[\mathrm{d}g\mathrm{d}X]_g}{V_{\mathrm{reparam}\times\mathrm{Weyl}}} e^{-S[X^\mu, g_{ab}]} , \tag{2.1}$$

where X^μ are a collection of d scalar fields living on the string world-sheet which will be taken to have the topology of a sphere and metric g_{ab}. Latin letters $a, b, \ldots = 1, 2$ are used to indicate world-sheet indices, while the space-time indices will always be denoted by Greek letters, $\mu = 0, \ldots, d-1$. World-sheet reparameterisation invariance will be kept manifest throughout the calculation, so after fixing the conformal gauge $g_{ab} = e^{2\sigma}\delta_{ab}$ on two patches on the sphere, the partition function reads (Sec. 1.4.1)

$$Z = \int \frac{[\mathrm{d}\sigma]}{V_{\mathrm{Weyl}}} \left\{ \int [\mathrm{d}^d X]_\delta \exp\left(\frac{26-d}{24\pi} \int \sigma \Box \sigma - S[X, g] \right) \right\} , \tag{2.2}$$

with $\Box = \delta^{ab}\partial_a\partial_b$. Weyl invariance of the theory means that arbitrary correlation functions $\langle \prod_i X^{\mu_i} \rangle$ calculated with the path integral contained in the curly parentheses are independent of σ (Eq. (1.37)). Then the measure $\int [\mathrm{d}\sigma]/V_{\mathrm{Weyl}}$ can be set to unity. Notice that ghost terms have been omitted. This is because their action and measure have been taken to be independent of σ and only a simple subset of the possible physical states will be considered.[1] The ghosts can then be integrated out yielding an unimportant (σ independent) constant.

[1] It is easy to imagine adding higher-derivative terms containing ghosts to the action or the vertex operators which would make the discussion more complicated.

In the standard Wilson renormalisation group approach (p. 16), the equations of motion are obtained as an operator statement inside the path integral. In contrast, this chapter utilises a source J^μ and calculates the generating functional

$$Z[J] = e^{\frac{26-d}{24\pi}\int \sigma\Box\sigma} \int [\mathrm{d}^d X] e^{-S[X^\mu, g_{ab}] + \int J\cdot f(X)} , \tag{2.3}$$

explicitly, by employing a weak-field expansion and using a short-distance cutoff. The coupling $\int J\cdot f(X)$ will be explained soon. This approach is closely related to the one used by Brustein, Nemeschansky and Yankielowicz[59].

At each level of the string, the derivation of the linearised equations of motion follows the same pattern. First, the most general action for that level is written down. This is dictated by the requirement of space-time and world-sheet reparameterisation invariance. All gauge symmetries present in the action are fixed and redundant fields are eliminated.[2] After this fixing, the generating functional is calculated to the linear order in the fields. Naive regularisation ambiguities are removed by adding local counter-terms to the action (in practice this is usually implemented through the use of a covariant functional measure and amounts to a set of field-redefinitions). The theory is then renormalised. Finally, varying with respect to the conformal mode yields a set of "β-functions" which are set to zero to obtain the equations of motion.

There are a number of subtleties inherent in this method and these are most easily illustrated by considering the familiar scenario of a string living in a background of massless and tachyonic fields (Eq. (1.43)). Sec. 2.1 derives the linearised equations of motion for these fields by expanding around a flat, linear-dilaton background (Eq. (1.59)). Some non-linear corrections to the tachyon field equation are then considered. The linearised equations of motion for the first massive level of the string are calculated in Sec. 2.4. The first part of the chapter concludes in Sec. 2.5 with the derivation of the non-linear corrections to the tachyon field equation coming from the first massive level.

In the second part of the chapter, the discrete state remnant of the propagating massive particle is found by solving its linearised field equations in two dimensions. It is then quite an easy task to check DMW's hypothesis by studying one-to-one tachyon scattering in this background (Sec. 2.7). The results presented here were published in a more condensed form in [341].

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[2] At the massive level, it is particularly important to do this since the equations of motion are presumed to contain no extraneous gauge degrees of freedom. As mentioned previously, the standard Wilson renormalisation group approach does not produce enough constraints on the massive field. If there was some remaining symmetry, then the lack of constraints would not necessarily be a problem since the symmetry could be fixed by further constraining the field.

2.1 Massless field equations

A string living in a background of three massless space-time fields, the graviton $G_{\mu\nu}$, the anti-symmetric tensor $B_{\mu\nu}$ and the dilaton Φ, and one tachyonic field T is described by the generalised non-linear sigma model action of Eq. (1.43),

$$S[X,\sigma] = \frac{1}{4\pi}\int_{\mathcal{M}} \left(\tfrac{1}{2}\sqrt{g}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu g^{ab} + \tfrac{1}{2}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \epsilon^{ab} + \sqrt{g}R\Phi(X) + \sqrt{g}T(X)\right) . \tag{2.4}$$

Here the string tension α' has been set to 2 and $g = \det g_{ab}$. As explained in Sec. 1.6, it is well known from beta-function results that strings can consistently propagate in a flat linear-dilaton background

$$G_{\mu\nu} = \eta_{\mu\nu} , \quad \Phi(X) = Q{\cdot}X \quad \text{and} \quad B_{\mu\nu} = 0 , \tag{2.5}$$

where $12Q^2 = 26-d$. This subsection will verify that this is indeed a consistent (Weyl-invariant) string background. Then, by expanding

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad \Phi = Q{\cdot}X + \phi , \tag{2.6}$$

in which $h_{\mu\nu}$, ϕ and all other fields are considered to be $O(\lambda)$ (where λ is a small parameter), the linearised equations of motion will be obtained.

The source J^μ can be coupled to any world-sheet scalar and the theory will remain invariant under reparameterisations of the world-sheet. All choices will break space-time reparameterisation invariance so the equations derived by imposing Weyl invariance will be gauge fixed. Different couplings $\int J{\cdot}f$ will correspond to different gauges. The physics of the theory should not depend on the gauge choice, but for the purposes of this section it is convenient to choose the source term to be

$$\int J_\mu(X^\mu + 2\sigma Q^\mu) . \tag{2.7}$$

There are two reasons for this particular form. Firstly, it handles contributions from the linear part of the dilaton field exactly. Secondly, using the usual coupling $\int J{\cdot}X$ and demanding Weyl-invariance of the one-point function results in the gauge condition

$$0 = \partial_\mu\Phi + \tfrac{1}{2}\partial_\mu\eta^{\mu\nu}h_{\mu\nu} - \partial_\nu h_{\mu\nu} , \tag{2.8}$$

to $O(\lambda)$. A flat linear-dilaton background is obviously inconsistent with this gauge condition. Of course the background can be rotated to be compatible with the gauge and, since the other equations of motion are covariant, S-matrix elements will be unaffected. However, this is an unnecessary nuisance. Eq. (2.7) is not really all that exotic since it is well known from spontaneously broken theories that expanding around different points in configuration space can be advantageous. Using this analogy, the choice $\int J{\cdot}(X + 2\sigma Q)$ is equivalent to expanding around the true vacuum, while the coupling $\int J{\cdot}X$ corresponds to expanding around the unstable maxima — here there is a non-zero tadpole that runs away into the vacuum.

Now proceed with the steps outlined on p. (36).

The action: The guiding principle in writing down Eq. (2.4) is that it should be the most general action with at most two derivatives that is reparameterisation invariant both on the world-sheet and in space-time. Therefore, the term $\frac{1}{4\pi}\int\sqrt{g}\tilde{A}_\mu D^2X^\mu$ must also be considered. Here D^2 is the covariant Laplacian

$$D^2X^\mu = g^{ab}D_a\partial_bX^\mu = g^{ab}\left(\partial_a\partial_bX^\mu - \Gamma^c_{ab}(g)\partial_cX^\mu + \Gamma^\mu_{\nu\lambda}(G)\partial_aX^\nu\partial_bX^\lambda\right) . \tag{2.9}$$

In fact, the field $\tilde{A}_\mu$ can be eliminated, as will be shown below, and shall be called a "redundant field". The usual justification for not considering such fields is that they give no contribution to S-matrix elements since, to $O(\lambda)$, the on-shell condition is $D^2X^\mu = 0$.

Elimination of redundant fields: A redundant field is defined to be one that can be eliminated by making a redefinition of X^μ (and possibly other fields).[3] $\tilde{A}^\mu$ is such a field since with the action S given by Eq (2.4) it is easy to check that

$$S[X^\mu] + \frac{1}{4\pi}\int\sqrt{g}\tilde{A}_\mu D^2X^\mu = S[X^\mu - \tilde{A}^\mu(X)] + \frac{1}{4\pi}\int\sqrt{g}RQ_\mu\tilde{A}^\mu , \tag{2.10}$$

to first order in λ. Then, with the definitions

$$X'^\mu = X^\mu - \tilde{A}^\mu(X) \quad \text{and} \quad \phi' = \phi + Q\cdot\tilde{A} , \tag{2.11}$$

and the use of the "covariant measure"

$$\left[\mathrm{d}^dX\right]_{\mathrm{cov}} = \left[\mathrm{d}^dX\sqrt{\det\left(G_{\mu\nu} - \nabla_\mu\tilde{A}_\nu - \nabla_\nu\tilde{A}_\mu\right)}\right] , \tag{2.12}$$

the partition function reads

$$Z = \int[\mathrm{d}^dX]_{\mathrm{cov}}e^{-S[\phi;X]-\frac{1}{4\pi}\int\tilde{A}_\mu D^2X^\mu} = \int\left[\mathrm{d}^dX'\sqrt{\det G_{\mu\nu}(X')}\right]e^{-S[\phi';X']} , \tag{2.13}$$

to $O(\lambda)$. (The primes will be dropped in the following calculations.) This procedure would have also worked if $\eta_{\mu\nu}$, instead of $G_{\mu\nu}$, had been used in the covariant measure.

The covariant measure was used so that the the redundant field wasn't reintroduced by the Jacobian of transformation following from the change of variable. Covariant measures will be used frequently throughout this work for this reason, but also in a more general sense to elegantly encode certain field redefinitions. The additions to the action resulting from such field redefinitions must be both local world-sheet scalars and covariant under space-time diffeomorphisms. Therefore, these are also the defining characteristics of a covariant measure.

Elimination of Stückelberg fields: It is also useful to note that $\tilde{A}^\mu$ could have been absorbed directly into the metric since the action is invariant under the transformations

$$\delta\tilde{A}_\mu = \Lambda_\mu \quad \text{and} \quad \delta G_{\mu\nu} = \nabla_\mu\Lambda_\nu + \nabla_\nu\Lambda_\mu , \tag{2.14}$$

[3] The argument given here is to $O(\lambda)$ only. Since $\tilde{A}^\mu$ is also a kind of Stückelberg field, it can be eliminated to all orders in λ. The situation at the massive level is nowhere near as simple and there the redundant fields will only be eliminated to $O(\lambda)$.

where ∇_μ is the covariant space-time derivative. This makes $\tilde{A}^\mu$ look like a "Stückelberg field" — a field which is introduced in order that a massive field theory have a gauge invariance. $\tilde{A}^\mu$ is not a Stückelberg field in the true sense of the term since $G_{\mu\nu}$ is massless, however. Such fields will be encountered at the first massive level and their corresponding gauges will be fixed by setting the Stückelberg fields to zero.

Evaluating the generating functional to $O(\lambda)$: Because of the spherical topology, the Euler character is non-zero

$$\frac{1}{4\pi}\int\sqrt{g}R = -\frac{1}{2\pi}\int\Box\sigma = 2\;. \tag{2.15}$$

Also, $\Box$ has a zero-mode which means the propagator Δ satisfies

$$-\Box\tfrac{1}{4\pi}\Delta(z,z') = \delta(z-z') - V^{-1}e^{2\sigma}\;, \tag{2.16}$$

where V is the volume of the world-sheet. Then denoting the zero-modes of X^μ and $J^\mu/\sqrt{g}$ by a_0 and J_0^μ respectively, the square can be completed in the generating functional by shifting $X \to X - 2Q\sigma + \int \Delta J$

$$\begin{aligned}
Z[J] &= \exp\left(\tfrac{26-d}{24\pi}\int\sigma\Box\sigma\right)\int\left[\mathrm{d}^dX\right]\exp\Big[-\tfrac{1}{4\pi}\int\left(\tfrac{1}{2}\partial_aX^\mu\partial_aX^\nu\eta_{\mu\nu}+\sqrt{g}RQ\cdot X\right)\\
&\qquad\qquad -S_{\text{int}}(X)+\int J\cdot(X+2Q\sigma)\Big]\\
&= P[\sigma]\int\left[\mathrm{d}^dX\right]\exp\left[-\tfrac{1}{8\pi}\int\partial_aX^\mu\partial_aX^\nu\eta_{\mu\nu}+a_0\cdot\left(J_0-\tfrac{2Q}{\sqrt{V}}\right)-S_{\text{int}}(X+\bar{X})\right]
\end{aligned} \tag{2.17}$$

where

$$P[\sigma] = \exp\left(\tfrac{1}{2}\int J\Delta J + \left(\tfrac{26-d}{12}-Q^2\right)\tfrac{1}{2\pi}\int\sigma\Box\sigma\right)\;, \tag{2.18}$$

and S_{int} is the action of Eq. (2.4) with $G_{\mu\nu}$ and Φ replaced by the small fields $h_{\mu\nu}$ and ϕ respectively. In keeping with the β-function language a "background field" $\bar{X}$ has been defined

$$\bar{X}^\mu \equiv -2Q^\mu\sigma + \bar{X}_0^\mu \equiv -2Q^\mu\sigma + \int\Delta J^\mu\;. \tag{2.19}$$

Note, however, that the background field does *not* satisfy the on-shell condition $\Box\bar{X}=0$. The zero-mode part of the generating functional $a_0\cdot(J_0 - 2Q/\sqrt{V})$ simply enforces momentum conservation. At this point it is clear from Eq. (2.18) that the linear-dilaton vacuum Eq. (2.5) is indeed Weyl invariant.

Performing a weak field expansion of $\exp(-S_{\text{int}})$ and utilising the Fourier transform yields

$$\begin{aligned}
e^{-S_{\text{int}}} &= 1-\frac{1}{4\pi}\int_{\mathcal{M}}\int\mathrm{d}^dp\,e^{ip\cdot X}\Big(\tfrac{1}{2}h_{\mu\nu}(p)\partial_aX^\mu\partial_aX^\nu+\tfrac{1}{2}B_{\mu\nu}(p)\partial_aX^\mu\partial_bX^\nu\epsilon^{ab}\\
&\qquad -2\Box\sigma\phi(p)+e^{2\sigma}T(p)\Big)+O(\lambda^2)\;,
\end{aligned} \tag{2.20}$$

and reduces the problem to Gaussian integrals. These integrals are regulated using a short-distance cutoff ϵ. Denote the regularised propagator by Δ_ϵ.

Reparameterisation invariance will be imposed on the regularised propagator. It will also be be assumed that it satisfies certain "Leibnitz-like" relations. It is important to keep its form as arbitrary as possible so that the equations of motion are not some regularisation-specific rubbish. The constraint of reparameterisation invariance will be discussed soon, but first the Leibnitz-like relations will be explained. These arise if we demand that the Stückelberg fields can be absorbed at the quantum level; that is, that the gauge symmetry is not broken by quantum effects. In order to shorten the exposition for the massive level, the simpler case of $\tilde{A}$ will be written out in full here. Consider the $\tilde{A}^\mu$ and $h_{\mu\nu}$ parts of the generating functional, with flat functional measure $[\mathrm{d}^d X]$, at $O(\lambda)$,

$$\begin{aligned} Z[J]|_{O(\lambda)} \propto & \int \mathrm{d}^2 z\sqrt{g}e^{ip\bar{X}}e^{-\frac{1}{2}p^2\Delta_\epsilon(z,z)}\Big[\left(D^2\bar{X}^\mu + ip^\mu g^{ab}D_{z_1^a}\partial_{z_1^b}\Delta_\epsilon(z,z_1)\right)\tilde{A}_\mu(p) \\ & +\tfrac{1}{2}g^{ab}\left(\partial_a\bar{X}^\mu\partial_b\bar{X}^\nu + 2ip^\mu\partial_{z_1^a}\Delta_\epsilon(z,z_1)\partial_b\bar{X}^\nu + \eta^{\mu\nu}\partial_{z_1^a}\partial_{z_2^b}\Delta_\epsilon(z_1,z_2)\right. \\ & \left.-p^\mu p^\nu\partial_{z_1^a}\Delta_\epsilon(z,z_1)\partial_{z_2^b}\Delta_\epsilon(z,z_2)\right)h_{\mu\nu}(p)\Big]_{z_1=z_2=z,\ ip+J_0\sqrt{V}-2Q=0} . \end{aligned} \tag{2.21}$$

Now, under the first-order version of the variation Eq. (2.14) ($\delta A_\mu = \Lambda_\mu$ and $\delta h_{\mu\nu} = ip_\mu\Lambda_\nu + ip_\nu\Lambda_\mu$), the partition function varies as

$$\begin{aligned} Z[J]|_{O(\lambda)} \propto & \int \mathrm{d}^2 z\sqrt{g}e^{ip\bar{X}}e^{-\frac{1}{2}p^2\Delta_\epsilon(z,z)}\Big[g^{ab}\left(\tfrac{1}{2}\partial_{z^a}\Delta_\epsilon(z,z) - \partial_{z_1^a}\Delta_\epsilon(z,z_1)\Big|_{z_1=z}\right)p^2\partial_b\bar{X}^\mu \\ & +ip^2g^{ab}\left(\tfrac{1}{2}\partial_{z^a}\Delta(z,z) - \partial_{z_1^a}\Delta(z,z_1)\Big|_{z_1=z}\right)\left(\partial_{z_1^b}\Delta_\epsilon(z,z_1)\Big|_{z_1=z}\right)p^\mu \\ & +ig^{ab}\Big(D_{z_1^a}\partial_{z_1^b}\Delta_\epsilon(z,z_1)\Big|_{z_1=z} + \partial_{z_1^a}\partial_{z_2^b}\Delta_\epsilon(z_1,z_2)\Big|_{z_1=z_2=z} \\ & \qquad -D_{z^a}\left(\partial_{z_1^b}\Delta_\epsilon(z,z_1)\Big|_{z_1=z}\right)\Big)p^\mu\Big]\Lambda_\mu(p)\Big|_{ip+J_0\sqrt{V}-2Q=0} , \end{aligned} \tag{2.22}$$

where integration by parts has been used. If the generating functional is to be invariant under the classical symmetry, the regularised propagator must satisfy the relations

$$\begin{aligned} \partial_{z^a}\Delta_\epsilon(z,z) &= 2\partial_{z_1^a}\Delta_\epsilon(z,z_1)\Big|_{z_1=z} , \\ D_{z^a}\left(\partial_{z_1^b}\Delta_\epsilon(z,z_1)\Big|_{z_1=z}\right) &= D_{z_1^a}\partial_{z_1^b}\Delta_\epsilon(z_1,z_2)\Big|_{z_1=z_2=z} + \partial_{z_1^a}\partial_{z_2^b}\Delta_\epsilon(z_1,z_2)\Big|_{z_1=z_2=z} . \end{aligned} \tag{2.23}$$

For the absorption of the (true) Stückelberg fields at the massive level, the regularised propagator will have to satisfy these equations and generalisations thereof. Therefore, from this point on, it is taken that it satisfies the Leibnitz-like relations

$$D_{z^a}\left([O_zO'_{z'}\Delta_\epsilon(z,z')]|_{z=z'}\right) = [D_{z^a}\left(O_zO'_{z'}\Delta_\epsilon(z,z')\right)]|_{z=z'} + [O_zD_{z'^a}\left(O'_{z'}\Delta_\epsilon(z,z')\right)]|_{z=z'} , \tag{2.24}$$

where O_z is a polynomial in D_z and $O'_{z'}$ a polynomial in $D_{z'}$. To the author's knowledge, the literature contains no discussion of the case when these relations do not hold.

$\tilde{A}^\mu$ has been absorbed, so performing the weak-field expansion to $O(\lambda)$ and using the flat measure[4] $[\mathrm{d}^d X]$, the path integral yields

$$Z[J] = P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\Big\{\delta^d(p) - \frac{1}{4\pi}\int \mathrm{d}^2 z e^{ip\bar{X}}e^{-\frac{1}{2}p^2\Delta_\epsilon(z,z)}$$

[4]Recall that $\tilde{A}^\mu$ could have been absorbed either by using the gauge transformation Eq. (2.14), or by the change of integration variable Eq. (2.11). In the former case it is obvious that a flat functional measure can be used, while in the latter it is permissible (but not necessarily wise) to use $[\mathrm{d}^d X\sqrt{\det(\eta_{\mu\nu} - 2\partial_{(\mu}\tilde{A}_{\nu)})}] = [\mathrm{d}^d X']$.

$$\times \left[\epsilon^{-2} e^{2\sigma} T(p) - 2\Box\sigma\phi(p) + \tfrac{1}{2} \left(\partial_a \bar{X}^\mu \partial_b \bar{X}^\nu + 2ip^\mu \frac{\partial}{\partial z_1^a} \Delta_\epsilon(z, z_1) \partial_b \bar{X}^\nu \right) \epsilon^{ab} B_{\mu\nu}(p) \right.$$
$$+ \tfrac{1}{2} \left(\partial_a \bar{X}^\mu \partial_a \bar{X}^\nu + 2ip^\mu \partial_{z_1^a} \Delta_\epsilon(z, z_1) \partial_a \bar{X}^\nu + \eta^{\mu\nu} \partial_{z_1^a} \partial_{z_2^a} \Delta_\epsilon(z_1, z_2) \right.$$
$$\left. \left. - p^\mu p^\nu \left(\partial_{z_1^a} \Delta_\epsilon(z, z_1) \right)^2 \right) h_{\mu\nu}(p) \right]_{z_1 = z_2 = z} \Bigg\} . \tag{2.25}$$

In this formula det$'$ is the determinant without the zero-mode, the tachyon has been scaled by ϵ^{-2} for convenience and the "external momentum" p^μ has been defined

$$ip^\mu + J_0^\mu \sqrt{V} - 2Q^\mu = 0 . \tag{2.26}$$

Although the source is a world-sheet density, it must not vary under Weyl transformations. Thus p^μ is Weyl neutral and $\delta \bar{X}^\mu = -2Q^\mu \delta\sigma$, as prescribed by Eq. (2.19).

Removal of regularisation ambiguities: To keep the theory reparameterisation invariant, the regularised propagator must satisfy

$$\Delta_\epsilon(z, z) = -\log \epsilon^2 + 2\sigma(z) + O(\epsilon^2) . \tag{2.27}$$

The first term follows from an explicit calculation of the propagator on the sphere with a short-distance cutoff ϵ while the second can be found by making an infinitesimal Weyl rescaling of the Δ_ϵ [104, 287]. Using the Leibnitz-like relations, the first derivative at coincidence must then be

$$\frac{\partial}{\partial z^a} \Delta_\epsilon(z, z') \Big|_{z'=z} = \partial_a \sigma(z) + O(\epsilon^2) . \tag{2.28}$$

The second derivative of Δ_ϵ at coincidence is not entirely fixed by reparameterisation invariance [287]. The most general form contains the 2 arbitrary numbers γ_ϵ and γ_0 and the symmetric traceless matrix[5] M_{ab} which only contains terms with two derivatives

$$\partial_{z_1^a} \partial_{z_2^b} \Delta_\epsilon(z_1, z_2) \Big|_{z_1 = z_2 = z} = \gamma_\epsilon \delta_{ab} \epsilon^{-2} e^{2\sigma} + \tfrac{1}{2} \gamma_0 \delta_{ab} \Box\sigma + M_{ab} + O(\epsilon^2) . \tag{2.29}$$

This form is obtained by imposing that the contraction with g^{ab} be a scalar; to $O(\epsilon^2)$ the only possible terms are $\gamma_\epsilon \epsilon^{-2}$ and $\gamma_0 R$. On the superficial level this looks disastrous since the equations of motion may depend on the regularisation scheme used through the numbers γ_ϵ and γ_0 (M_{ab} drops out of the calculation at this level of the string since it is traceless). In fact, with judicious field redefinitions, or, equivalently by adding local counter-terms to the action all regularisation dependence can be soaked-up.

It is clear from the form of the generating functional Eq. (2.25) that γ_ϵ and γ_0 can be absorbed by redefining the dilaton and the tachyon

$$\begin{aligned} \phi'(p) &= \phi(p) - \tfrac{1}{4}(\gamma_0 - 1)\eta^{\mu\nu} h_{\mu\nu} \quad \text{and} \\ T'(p) &= T(p) + \gamma_\epsilon \eta^{\mu\nu} h_{\mu\nu} , \end{aligned} \tag{2.30}$$

Soon it will become obvious that the factor of $\frac{1}{4}\eta^{\mu\nu} h_{\mu\nu}$ serves to covariantise the equations of motion.

[5]Later we will argue that M_{ab} is in fact not arbitrary, but is independent of the regularisation scheme.

It is important to realise that these field redefinitions can be implemented by adding the local, world-sheet reparameterisation invariant, term

$$\frac{1}{8\pi}\int \eta^{\mu\nu} h_{\mu\nu}\Box\, \Delta_\epsilon(z,z')\big|_{z'=z} \,, \tag{2.31}$$

to the action. This is just the covariant measure of Eq. (2.12) in disguise

$$\left[\mathrm{d}^d X\right]_{\mathrm{cov}} = \left[\mathrm{d}^d X \sqrt{\det \tfrac{1}{2} G_{\mu\nu}}\right] \,. \tag{2.32}$$

This measure has been previously considered by Andreev, Metsaev and Tseytlin [12]. Regulating with the short-distance cutoff, leads to

$$[\mathrm{d}^d X]_{\mathrm{cov}} = [\mathrm{d}^d X] \exp\left(-\frac{1}{8\pi}\int \mathrm{d}^2 z \log\det G_{\mu\nu}\, \Box\Delta_\epsilon(z,z')\big|_{z'=z}\right) \,. \tag{2.33}$$

and a weak-field expansion of this new term adds Eq. (2.31) to the action. Then employing the relation Eq. (2.24)

$$\partial_a\left(\partial_b\Delta_\epsilon(z,z')\big|_{z=z'}\right) = \partial_a\partial_b\Delta_\epsilon(z,z')\big|_{z=z'} + \partial_b\partial'_a\Delta_\epsilon(z,z')\big|_{z=z'} \,, \tag{2.34}$$

all regularisation ambiguities disappear. Now it is obvious why the $\frac{1}{4}\eta^{\mu\nu}h_{\mu\nu}$ was added to the redefinition of the dilaton in Eq. (2.30) — it will covariantise the equations of motion because it can be thought of as coming from a covariant functional measure.

So, either by field redefinitions[6], or by using the covariant measure, the generating functional can be cast into the form

$$\begin{aligned} Z[J] \;=\; & P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\left\{\delta^d(p) - \frac{1}{4\pi}\int \mathrm{d}^2 z\, e^{ip\bar{X}_0} e^{(-p^2-2ip\cdot Q)\sigma}|\epsilon|^{p^2}\right. \\ & \times\left[\epsilon^{-2}e^{2\sigma}T(p) - 2\Box\sigma(\phi(p) - \tfrac{1}{4}\eta^{\mu\nu}h_{\mu\nu})\right. \\ & +\tfrac{1}{2}\left(\partial_a\bar{X}^\mu\partial_b\bar{X}^\nu + 2ip^\mu\partial_a\sigma\partial_b\bar{X}^\nu\right)\epsilon^{ab}B_{\mu\nu}(p) \\ & \left.\left. +\tfrac{1}{2}\left(\partial_a\bar{X}^\mu\partial_a\bar{X}^\nu + 2ip^\mu\partial_a\sigma\partial_a\bar{X}^\nu - p^\mu p^\nu(\partial_a\sigma)^2\right)h_{\mu\nu}(p)\right]\right\} \,. \end{aligned} \tag{2.35}$$

Recall that $\bar{X}_0^\mu$ is neutral under Weyl transformations and is defined in Eq. (2.19).

Renormalisation: Renormalisation at the linear level is trivial

$$T_R(p) = |\epsilon|^{p^2-2}T(p) \quad \text{and} \quad (h_R^{\mu\nu}(p), B_R^{\mu\nu}(p), \phi_R(p)) = |\epsilon|^{p^2}(h^{\mu\nu}(p), B^{\mu\nu}(p), \phi(p)) \,. \tag{2.36}$$

This corresponds to a minimal subtraction scheme and can be clearly implemented by adding local counter-terms to the action. For notational simplicity the subscripts R will be dropped in what follows.

Weyl invariance: Finally, the limit $\epsilon \to 0$ can be taken and the generating functional can be varied with respect to σ to yield

$$0 = \frac{\delta Z[J]}{\delta\sigma} \;=\; -P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\frac{1}{4\pi}e^{ip\bar{X}_0}e^{(-p^2-2ip\cdot Q)\sigma}\left\{e^{2\sigma}\left(2-p^2-2ip\cdot Q\right)T\right.$$

[6] in which case the tachyon and dilaton in Eq. (2.35) must be replaced by the redefined quantities T' and Φ' given by Eq. (2.30)

$$+\Big[\Box\sigma+\partial_a\sigma\partial_a\bar{X}_0^\lambda ip_\lambda-(\partial_a\sigma)^2\tfrac{1}{2}(p^2+2ip{\cdot}Q)\Big]$$
$$\times\Big[\tfrac{d-26}{3}+4(\eta_{\mu\nu}+h_{\mu\nu})Q^\mu Q^\nu+4p^2\phi-R$$
$$+8ip{\cdot}Q(\phi-\tfrac{1}{4}h)+4ip^\mu Q^\nu h_{\mu\nu}\Big]$$
$$-\tfrac{1}{2}\epsilon^{ab}\left(\partial_a\bar{X}^\mu\partial_b\bar{X}^\nu+2i\partial_a\sigma\partial_b\bar{X}^\mu p^\nu\right)3(p^\lambda+2iQ^\lambda)H_{\lambda\mu\nu}$$
$$+\partial_a\bar{X}_0^\mu\partial_a\bar{X}_0^\nu\left(-R_{\mu\nu}+2p_\mu p_\nu\Phi-ip{\cdot}Qh_{\mu\nu}+2ip_\mu Q^\lambda h_{\lambda\nu}\right)$$
$$-\Box\bar{X}_0^\mu\left(2ip_\mu(\phi-\tfrac{1}{4}h)+(ip^\nu-2Q^\nu)h_{\mu\nu}\right)\Big\}\ , \tag{2.37}$$

where $h=\eta^{\mu\nu}h_{\mu\nu}$ and

$$\begin{aligned}2R_{\mu\nu}&=p^2h_{\mu\nu}+p_\mu p_\nu h-p^\lambda p_\mu h_{\nu\lambda}-p^\lambda p_\nu h_{\mu\lambda}\ ,\\ H_{\lambda\mu\nu}&=\tfrac{1}{3}(p_\lambda B_{\mu\nu}+p_\mu B_{\nu\lambda}+p_\nu B_{\lambda\mu})\ .\end{aligned}\tag{2.38}$$

Note that the zeroth order part $\Box\sigma(\frac{26-d}{3}+4Q^2)$ (which is annihilated by p^λ) has been included. Using formulae such as

$$\begin{aligned}4(\nabla_\mu\Phi)^2&=4\nabla^\mu\Phi\nabla^\nu\Phi G_{\mu\nu}=4Q^\mu Q^\nu(\eta_{\mu\nu}+h_{\mu\nu})+8Q^\mu\partial_\mu\phi+O(\lambda^2)\ ,\\ 2\nabla_\mu\nabla_\nu\Phi&=2\nabla_\mu(Q_\nu+\partial_\nu\phi)=-2\partial_{(\mu}h_{\nu)\alpha}Q^\alpha+Q^\alpha\partial_\alpha h_{\mu\nu}+2\partial_\mu\partial_\nu\phi+O(\lambda^2)\ ,\end{aligned}\tag{2.39}$$

(∇_μ is the covariant space-time derivative), the coefficient of each linearly-independent term can be identified with zero to yield the linearised field equations (now expressed in position coordinates)

$$\begin{aligned}0&=(\nabla^2-2Q{\cdot}\nabla+2)T\ ,\\ 0&=\tfrac{d-26}{3}+4(\nabla_\mu\Phi)^2-4\nabla^2\Phi-R\ ,\\ 0&=(\nabla^\lambda-2Q^\lambda)H_{\lambda\mu\nu}\ ,\\ 0&=R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi\ ,\end{aligned}\tag{2.40}$$

and the gauge condition

$$0=2\partial_\mu(\phi-\tfrac{1}{4}h)+(\partial^\nu-2Q^\nu)h_{\mu\nu}\ . \tag{2.41}$$

These equations are correct to first order in λ. Hence, not only has the linear-dilaton background been proved to be consistent, but the β-function results of Eq. (1.58), supplemented by the tachyon equation, have been reproduced (to $O(\lambda)$). Furthermore, as promised at the end of Sec. 1.4.2, with the dilaton lying along just one direction, $\Phi(\bar{X})=Q\bar{X}^1$, the tachyon equation implies that $\alpha=Q\pm\sqrt{Q^2-2}$ for a tachyon of the form $T(\bar{X})=e^{\alpha\bar{X}^1}$.

A word can now be said about the connection of this work to the standard approach in which the beta-functions are calculated and set to zero. In Eq. (2.37) the beta functions are the coefficients of the linearly independent terms $e^{2\sigma}$, $\Box\sigma$, $\epsilon^{ab}\partial_a\bar{X}_0^\mu\partial_b\bar{X}_0^\nu$ and $\partial_a\bar{X}_0^\mu\partial_a\bar{X}_0^\nu$. There is also the gauge condition which is the coefficient of $\Box\bar{X}_0^\mu$ which is often discarded by demanding that the background field $\bar{X}_0^\mu$ is on-shell (or equivalently that $J=0$).

However, there are also "beta-functions" corresponding to terms that would have been non-local in the original action, $(\partial\sigma)^2$ and $\partial_a\sigma\partial_b\bar{X}_0^\mu$. It is not just luck that setting the standard beta-functions to zero implies that these new "non-local beta-functions" are zero too. This can be verified by writing the most general generating functional with two derivatives

$$Z = \int e^{ip\bar{X}_0} e^{\mathcal{O}\sigma} \Big[\Box\sigma A + (\partial\sigma)^2 B + \partial_a\sigma\partial_a\bar{X}_0^\mu C_\mu + \partial_a\bar{X}_0^\mu\partial_a\bar{X}_0^\nu D_{\mu\nu} + \epsilon^{ab}(\partial_a\sigma\partial_b\bar{X}_0^\mu E_\mu + \partial_a\bar{X}_0^\mu\partial_b\bar{X}_0^\nu F_{\mu\nu})\Big] , \qquad (2.42)$$

and taking the variation with respect to σ,

$$\frac{\delta Z}{\delta\sigma} = e^{ip\bar{X}_0} e^{\mathcal{O}\sigma} \Big[2(\Box\sigma + \partial_a\sigma\partial_a\bar{X}_0^\mu ip^\mu + \tfrac{1}{2}\mathcal{O}(\partial\sigma)^2)(\mathcal{O}A - B) + \Box\bar{X}_0^\mu(ip_\mu A - C_\mu) \partial_a\bar{X}_0^\mu\partial_a\bar{X}_0^\nu(-ip_\mu C_\nu + \mathcal{O}D_{\mu\nu}) + \epsilon^{ab}\partial_a\bar{X}_0^\mu\partial_b\bar{X}_0^\nu(-ip_\mu E_\nu + \mathcal{O}F_{\mu\nu})\Big] . \qquad (2.43)$$

Evidently, the non-local beta-functions corresponding to the operators $(\partial\sigma)^2$ and $\partial_a\sigma\partial_b\bar{X}^\mu$ are always derivatives of the dilaton beta-function. At the first massive level this general argument no longer holds: There are operators in $\delta Z/\delta\sigma$ which correspond to non-local terms in the original action whose coefficients are not-necessarily derivatives of other beta-functions. However, although the general argument breaks down, in practise, setting the true beta-functions and the gauge constraints equal to zero guarantees Weyl invariance of the theory.

2.2 Higher-order corrections to the tachyon field equation

Now the T^2 corrections to the generating functional of Eq. (2.35) will be discussed. The path integral is easily evaluated to give

$$Z_{TT} = P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\frac{1}{32\pi^2}\int \mathrm{d}p_1 T(p_1)T(p_2) \times\int \mathrm{d}^2z_1\mathrm{d}^2z_2 f_1(z_1)f_2(z_2)\exp\left(-p_1{\cdot}p_2\Delta_\epsilon(z_1,z_2)\right) , \qquad (2.44)$$

in which

$$f_i(z_i) = \exp\left[ip_i\bar{X}(z_i) + (2-p_i^2)(\sigma(z_i) + \log|\epsilon|)\right] , \qquad (2.45)$$

and momentum conservation has been used

$$ip_1^\mu + ip_2^\mu + J_0^\mu\sqrt{V} - 2Q^\mu = 0 . \qquad (2.46)$$

Up to derivatives on σ, the propagator can be written as

$$\Delta_\epsilon(z_-, z_+) = -\log\left(4|z_-|^2 + \epsilon^2 e^{-2\sigma(z_+)}\right) , \qquad (2.47)$$

where $z_\pm = \frac{1}{2}(z_1 \pm z_2)$. Expanding the f_i around $z_- = 0$, the integral over z_- can now be performed. Of course this expansion is only valid for small z_-. It is assumed that the integral is finite because the world-sheet is compact and that the f_i are well behaved. The integral is then dominated by small z_-.

The T^2 contribution to the tachyon field equation is the easiest to calculate and will be of use later when 1-1 tachyon scattering is discussed. The contribution has no derivatives on $\bar{X}^\mu$ and σ, and, using the integral

$$\begin{aligned}
&\int \mathrm{d}^2z_1\mathrm{d}^2z_2\, f_1(z_1)f_2(z_2)\exp\left(-p_1{\cdot}p_2\Delta_\epsilon(z_1,z_2)\right) \\
&\quad = 4\int \mathrm{d}^2z_+\mathrm{d}^2z_-\, f_1(z_+)f_2(z_+)\left(4|z_|^{-2}+\epsilon^2e^{-2\sigma(z_+)}\right)^{p_1\cdot p_2} + O(\partial\bar{X},\partial\sigma) \\
&\quad = -2\pi\int \mathrm{d}^2z\, f_1(z)f_2(z)\frac{(\epsilon^2e^{-2\sigma})^{1+p_1\cdot p_2}}{1+p_1{\cdot}p_2} + O(\partial\bar{X},\partial\sigma)\ ,
\end{aligned} \tag{2.48}$$

reads

$$Z_{TT} = -P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\int d^2z\, e^{ip\bar{X}}e^{(2-p^2)(\sigma-\log|\epsilon|)}\frac{1}{16\pi}\int \mathrm{d}p_1\frac{T(p_1)T(p-p_1)}{1+p_1{\cdot}(p-p_1)}\ . \tag{2.49}$$

$\bar{X}^\mu$ is still given by Eq. (2.19) and momentum conservation has been used to write the integrand in terms of p^μ given in Eq. (2.26).

The renormalisation of the tachyon is modified at $O(\lambda^2)$

$$|\epsilon|^{2-p^2}T_R(p) = T(p) + \frac{1}{4}\int \mathrm{d}p_1\frac{T(p_1)T(p-p_1)}{1+p_1{\cdot}(p-p_1)}\left(1-|\epsilon|^{-2-2p_1\cdot(p-p_1)}\right)\ , \tag{2.50}$$

but again this can be implemented by adding local counterterms to the action.

To $O(\lambda^2)$ the linearised equation of motion $-p^2-2iQ{\cdot}p+2=0$ for $T(p)$ can be used to simplify the denominator of Eq. (2.49)

$$1+p_1{\cdot}(p-p_1) = -\tfrac{1}{2}(2-2iQ{\cdot}p-p^2)\ . \tag{2.51}$$

Using this, the generating functional becomes

$$Z = -\frac{1}{4\pi}P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\int_{\mathcal{M}} e^{ip\bar{X}_0}e^{(2-2iQ\cdot p-p^2)\sigma}\left(T(p)-\tfrac{1}{2}\int d^dp_1\frac{T(p_1)T(p-p_1)}{2-2iQ{\cdot}p-p^2}\right)\ , \tag{2.52}$$

whereupon Weyl invariance reads

$$(\nabla^2-2Q{\cdot}\partial+2)T-\tfrac{1}{2}T^2=0\ . \tag{2.53}$$

2.3 Higher-order corrections in general

The previous section provides an illustration of why the simple-minded method employed here is not suited to finding all the field equations to quadratic order. The expansion of the f_i and higher derivative terms in the regulated propagator generate terms with arbitrary powers of $\partial_a\sigma$ and $\partial_a\bar{X}^\mu$. This means that, generically, T^2 terms will appear in every field equation. This is true for every other field too: At $O(\lambda)$ the fields stay confined within their respective levels (the graviton can only contribute to β^ϕ, $\beta^{B_{\mu\nu}}$ and $\beta^{G_{\mu\nu}}$, for example), but at the quadratic order the equation of motion and linear constraints C_i for an arbitrary field F look like

$$(\nabla^2 - M_F^2)F = O(\lambda^2) \quad \text{and} \quad C_i(F) = O(\lambda^2)\ , \tag{2.54}$$

where the $O(\lambda^2)$ parts contain *all* fields. It would thus be impossible to calculate the infinite number of $O(\lambda^2)$ corrections. Instead, with a finite number of levels containing fields F being $O(\lambda)$ and the rest containing fields $\bar{F}$ being $O(\lambda^2)$, it is *assumed* that the equations

$$(\nabla^2 - M_{\bar{F}}^2)\bar{F} = O(F^2) \quad \text{and} \quad C_i(\bar{F}) = O(F^2)\ , \tag{2.55}$$

have a solution for $\bar{F}$. Then, to $O(\lambda^2)$, the only equations that need be considered are

$$(\nabla^2 - M_F^2)F = O(F^2) \quad \text{and} \quad C_i(F) = O(F^2)\ . \tag{2.56}$$

2.4 The Massive Fields

Recall from Sec. 1.1 that the first massive level of the closed bosonic string consists of a field $E_{\mu\nu\lambda\rho} = E_{(\mu\nu)(\lambda\rho)}$ upon which the Virasoro constraints impose traceless inside pairs of indices and transversality [339]. Until now, these conditions had not been derived using the Wilson renormalisation group method. One condition, variously thought of as the tracelessness condition [189] or the extra null state that appears at the critical dimension [25], was always missing.

The same steps that were carried out at the massless level are now performed at the first massive level of the string. The most general action consistent with reparameterisation invariance, both on the world-sheet and in space-time, is gauge-fixed by eliminating all Stückelberg degrees of freedom. Field redefinitions are employed in the path-integral to simplify the action further by absorbing the redundant fields, and then to eliminate regularisation ambiguities. Renormalisation is performed and the linearised equations of motion are calculated.

The action: The most general reparameterisation invariant action with four derivatives on a curved world-sheet was systematically studied by Buchbinder *et. al* [60]. Before gauge fixing, the Lagrangian density is constructed from all possible contractions with g^{ab} and ϵ^{ab} of the objects $\partial_a X^\mu$, $D_a\partial_b X^\mu$, $D_aD_b\partial_c X^\mu$, $D_aD_bD_c\partial_d X^\mu$, R, $\partial_a R$ and $D_a\partial_b R$ (the covariant derivative is defined in Eq. (2.9)). Following [60], the Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_M \;=\;& \sqrt{g}g^{ab}g^{cd}\partial_a X^\mu\partial_b X^\nu\partial_c X^\lambda\partial_d X^\rho W_{\mu\nu\lambda\rho}(X) + \sqrt{g}Rg^{ab}\partial_a X^\mu\partial_b X^\nu W_{\mu\nu}(X)\\
&+\sqrt{g}R^2W(X)\\
&+g^{ab}i\epsilon^{cd}\partial_a X^\mu\partial_b X^\nu\partial_c X^\lambda\partial_d X^\rho\bar{W}_{\mu\nu\lambda\rho}(X) + Ri\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\bar{W}_{\mu\nu}(X)\ .\\
&+\sqrt{g}g^{ab}D^2X^\mu\partial_a X^\nu\partial_b X^\lambda A_{\mu\nu\lambda}(X) + \sqrt{g}D^2X^\mu D^2X^\nu A_{\mu\nu}(X) + \sqrt{g}RD^2X^\mu A_\mu(X)\\
&+i\epsilon^{ab}D^2X^\mu\partial_a X^\nu\partial_b X^\lambda\bar{A}_{\mu\nu\lambda}(X)\\
&+\sqrt{g}g^{ac}g^{bd}D_a\partial_b X^\mu\partial_c X^\nu\partial_d X^\lambda S^1_{\mu\nu\lambda}(X) + \sqrt{g}g^{ac}g^{bd}D_a\partial_b X^\mu D_c\partial_d X^\nu S^2_{\mu\nu}(X)\\
&+\sqrt{g}g^{ab}D_aD^2X^\mu\partial_b X^\nu S^3_{\mu\nu}(X) + \sqrt{g}D^2D^2X^\mu S^4_\mu(X) + \sqrt{g}\partial_a X^\mu\partial_b RS^5_\mu(X)
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{g}D^2RS^6(X)+g^{ac}i\epsilon^{bd}D_a\partial_bX^\mu\partial_cX^\nu\partial_dX^\lambda S^7_{\mu(\nu\lambda)}(X)\\
&+g^{ac}i\epsilon^{bd}D_a\partial_bX^\mu D_c\partial_dX^\nu S^8_{\mu\nu}(X)+i\epsilon^{ab}D_aD^2X^\mu\partial_bX^\nu S^9_{\mu\nu}(X)\\
&+i\epsilon^{ab}\partial_aX^\mu\partial_bRS^{10}_\mu(X)\ . \qquad (2.57)
\end{aligned}
$$

Eliminating the Stückelberg fields: The Lagrangian density above has many gauge symmetries. For each gauge symmetry there is an associated Stückelberg field. Fixing each Stückelberg field to zero fixes its corresponding gauge. For instance, it is evident that the action is invariant under

$$
\begin{aligned}
\delta W_{\mu\nu\lambda\rho} &= \tfrac{1}{2}\partial_{(\mu}\Lambda^1_{\nu)(\lambda\rho)}+\tfrac{1}{2}\partial_{(\lambda}\Lambda^1_{\rho)(\mu\nu)}\ ,\\
\delta A_{\mu\nu\lambda} &= \Lambda^1_{\mu(\nu\lambda)}\ ,\\
\delta S^1_{\mu\nu\lambda} &= \Lambda^1_{\nu(\lambda\mu)}+\Lambda^1_{\lambda(\nu\mu)}\ , \qquad (2.58)
\end{aligned}
$$

in which symmeterisation is indicated with round brackets, for example

$$
\Lambda_{\mu(\nu\lambda)}=\tfrac{1}{2}\Lambda_{\mu\nu\lambda}+\tfrac{1}{2}\Lambda_{\mu\lambda\nu}\ . \qquad (2.59)
$$

In this case, the field $S^1_{\mu\nu\lambda}$ can be considered a Stückelberg field and eliminated by choosing an appropriate $\Lambda^1_{\mu\nu\lambda}$. Buchbinder *et al.* approached this problem from a slightly different angle by enumerating all the possible divergences that could be added to the Lagrangian. For instance, the above symmetry is a consequence of the invariance of the action under the addition of

$$
\sqrt{g}D_a\left(g^{ab}g^{cd}\partial_bX^\mu\partial_cX^\nu\partial_dX^\lambda\Lambda^1_{\mu\nu\lambda}(\sigma)\right)\ , \qquad (2.60)
$$

to the Lagrangian. The complete list of divergences of dimension four is

$$
\begin{aligned}
&\sqrt{g}D_a\Big(g^{ab}g^{cd}\partial_bX^\mu\partial_cX^\nu\partial_dX^\lambda\Lambda^1_{\mu\nu\lambda}(\sigma)+g^{ab}g^{cd}D_b\partial_cX^\mu\partial_dX^\nu\Lambda^2_{(\mu\nu)}(X)\\
&\quad+g^{ab}D^2X^\mu\partial_bX^\nu\Lambda^3_{\mu\nu}(X)+g^{ab}D_bD^2X^\mu\Lambda^4_\mu(X)+g^{ab}\partial_bX^\mu R\Lambda^5_\mu(X)+g^{ab}\partial_bR\Lambda^6(X)\\
&\quad+i\epsilon^{ab}g^{cd}\partial_bX^\mu\partial_cX^\nu\partial_dX^\lambda\Lambda^7_{\mu\nu\lambda}(X)+i\epsilon^{ab}g^{cd}D_b\partial_cX^\mu\partial_dX^\nu\Lambda^8_{[\mu\nu]}(X)\\
&\quad+i\epsilon^{ab}D^2X^\mu\partial_bX^\nu\Lambda^9_{\mu\nu}(X)+i\epsilon^{ab}\partial_bX^\mu R\Lambda^{10}_\mu(X)\Big)\ . \qquad (2.61)
\end{aligned}
$$

The ten Stückelberg fields $S^i_{\dots}$ can be removed by the judicious choice of the ten $\Lambda^i_{\dots}$ fields. After this is completed, all that remains are the W, $\bar{W}$, A and $\bar{A}$-type fields. Gauge invariance in string theory has also been extensively investigated in [138, 139]. The analysis of Sec. 2.1 can be carried out for each Stückelberg field to show that the Leibnitz-like relations are equivalent to demanding that the gauge symmetries are not broken by quantum effects.

Removal of redundant fields: As in the massless case, the A-type fields can be shifted away by a change of variables in the path integral. Specifically, the analogue of the shift in Eq. (2.11) is

$$
X'^\mu=X^\mu-RA^\mu-A^\mu{}_\nu D^2X^\nu+RA^\mu{}_\nu Q^\nu-A^\mu{}_{\nu\lambda}\partial_aX^\nu\partial_bX^\lambda g^{ab}-\bar{A}^\mu{}_{\nu\lambda}\partial_aX^\nu\partial_bX^\lambda i\epsilon^{ab}/\sqrt{g}\ . \qquad (2.62)
$$

Denoting the massless action by S_0 and the action for the massive fields by $S_M = \frac{1}{4\pi}\int \mathcal{L}_M$, this shift induces

$$S_0(X) + S_M(X) = S_0(X') + S_M(X')|_{A=0=\bar{A}} + \frac{1}{4\pi}\int \sqrt{g}RQ_\mu\big(RA^\mu - RA^\mu{}_\nu Q^\nu \\ A^\mu{}_{\nu\lambda}\partial_a X^\nu \partial_b X^\lambda g^{ab} + \bar{A}^\mu{}_{\nu\lambda}\partial_a X^\nu \partial_b X^\lambda i\epsilon^{ab}/\sqrt{g}\big) \ , \quad (2.63)$$

to $O(\lambda)$. So, in order to completely remove the A-type fields, the W-type fields also need to be redefined

$$\begin{aligned} W' &= W + A{\cdot}Q - A_{\mu\nu}Q^\mu Q^\nu \ , \\ W'_{\mu\nu} &= W_{\mu\nu} + A_{\lambda\mu\nu}Q^\lambda \ , \\ \bar{W}'_{\mu\nu} &= \bar{W}_{\mu\nu} + \bar{A}_{\lambda\mu\nu}Q^\lambda \ . \end{aligned} \quad (2.64)$$

A covariant measure is used so that the A-type fields aren't reintroduced when changing functional integration variable

$$\left[\mathrm{d}^d X\right]_{\mathrm{cov}} = \Big[\mathrm{d}^d X \det{}^{1/2}\big(\tfrac{1}{2}G_{\mu\nu} - R\nabla_{(\mu}A_{\nu)} - \nabla_{(\mu}(A_{\nu)\lambda}D^2X^\lambda) - R\nabla_{(\mu}(A_{\nu)\lambda}Q^\lambda) \\ -\nabla_{(\mu}(A_{\nu)\lambda\rho}\partial_a X^\lambda \partial_b X^\rho) - \nabla_{(\mu}(\bar{A}_{\nu)\lambda\rho}\partial_a X^\lambda \partial_b X^\rho i\epsilon^{ab}/\sqrt{g})\big)\Big] \ . \quad (2.65)$$

Finally then, the partition function reads

$$Z = \int [\mathrm{d}^d X]_{\mathrm{cov}} e^{-S_0(X)+S_M(X)} = \int \left[\mathrm{d}^d X' \sqrt{\det \tfrac{1}{2}G_{\mu\nu}(X')}\right] e^{-S_0(X')+S'_M(X')}|_{A=0=\bar{A}} \ , \quad (2.66)$$

where the prime on S_M indicates that it is a function of the W' fields in Eq. (2.64). All primes will now be dropped.

The remaining terms in the action can be grouped together in a more compact form by using the 2-dimensional identities

$$\epsilon^{ab}\epsilon^{cd}g_{ac}g_{bd} = 2g \quad \text{and} \quad \epsilon^{ab}\epsilon^{cd} = g(g^{ac}g^{bd} - g^{ad}g^{bc}) \ , \quad (2.67)$$

($g = \det g_{ab}$). These imply the W-term in the Lagrangian can be written as

$$\begin{aligned} &2W_{\mu\nu\lambda\rho}gg^{ab}g^{cd}\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho \\ &= (W_{\mu\nu\lambda\rho} + W_{\mu\rho\lambda\nu} - W_{\mu\rho\nu\lambda} + W_{\mu\nu\lambda\rho})\, gg^{ab}g^{cd}\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho \\ &= \big(W_{\mu\nu\lambda\rho}gg^{ac}g^{bd} + W_{\mu\rho\nu\lambda}gg^{ac}g^{bd} + W_{\mu\rho\nu\lambda}g(g^{ad}g^{bc} + g^{ab}g^{cd})\big)\, \partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho \\ &= (W_{\mu\lambda\nu\rho} + W_{\mu\rho\nu\lambda})(\sqrt{g}g^{ac} + i\epsilon^{ac})(\sqrt{g}g^{bd} + i\epsilon^{bd})\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho \ , \end{aligned} \quad (2.68)$$

and similarly for the $\bar{W}$-term,

$$\begin{aligned} &2\bar{W}_{\mu\nu\lambda\rho}\sqrt{g}g^{ab}i\epsilon^{cd}\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho \\ &= (\bar{W}_{\mu\lambda\nu\rho} + \bar{W}_{\nu\rho\mu\lambda})(\sqrt{g}g^{ac} + i\epsilon^{ac})(\sqrt{g}g^{bd} + i\epsilon^{bd})\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho \ . \end{aligned} \quad (2.69)$$

Then the action at the first massive level can be written in the standard fashion

$$S_M(X^\mu, g_{ab}) = \int g^{-1/2}\big(E_{\mu\nu\lambda\rho}(X)\partial_a X^\mu \partial_b X^\nu \partial_c X^\lambda \partial_d X^\rho f^{ac}f^{bd} \\ +\sqrt{g}RE_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu f^{ab} + (\sqrt{g}R)^2 E(X)\big) \ , \quad (2.70)$$

where

$$\begin{aligned}
2E_{\mu\nu\lambda\rho} &\equiv W_{\mu\lambda\nu\rho} + W_{\mu\rho\nu\lambda} + \bar{W}_{\mu\lambda\nu\rho} + \bar{W}_{\nu\rho\mu\lambda} \, , \\
E_{\mu\nu} &\equiv W_{\mu\nu} + \bar{W}_{\mu\nu} \, , \\
E &\equiv W \, , \\
f^{ab} &\equiv \sqrt{g} g^{ab} + i\epsilon^{ab} \, . \qquad (2.71)
\end{aligned}$$

The identity Eq. (2.67) leads to the following useful symmetries of a product of two f^{ab} densities

$$\begin{aligned}
f^{ac}f^{bd} &= g g^{ac} g^{db} + i\sqrt{g}\epsilon^{ac}g^{bd} + i\sqrt{g}\epsilon^{bd}g^{ac} + g g^{ad}g^{cb} - g g^{ab}g^{cd} \\
&= \epsilon^{ad}\epsilon^{cb} + g g^{ab}g^{cd} + i\sqrt{g}(-\epsilon^{cb}g^{ad} - \epsilon^{ba}g^{cd}) + i\sqrt{g}(-\epsilon^{da}g^{bc} - \epsilon^{ab}g^{dc}) \\
&\qquad + g g^{ad}g^{cb} - g g^{ab}g^{cd} \\
&= f^{bc}f^{ad} \, , \quad \text{and} \\
0 &= f^{ac}f^{bd}g_{ab} \, . \qquad (2.72)
\end{aligned}$$

The first of these identities immediately implies that field $E_{\mu\nu\lambda\rho}$ is symmetric in pairs of indices, $E_{\mu\nu\lambda\rho} = E_{(\mu\nu)(\lambda\rho)}$.

Evaluating the generating functional to $O(\lambda)$: After Fourier transforming the space-time fields, the Gaussian integrals can be evaluated to give

$$\begin{aligned}
Z[J] &= P[\sigma] \left(\frac{1}{V} \det{}' \frac{\Box}{4\pi}\right)^{-\frac{1}{2}d} \Big\{ \delta^d(p^\mu) - \frac{1}{4\pi} \int d^2 z e^{ip\bar{X}} g^{-1/2} e^{-\frac{1}{2}p^2 \Delta_\epsilon(z,z)} \epsilon^2 \\
&\times \Big[E_{\mu\nu\lambda\rho} f^{ac} f^{bd} \Big(\partial_a \bar{X}^\mu \partial_b \bar{X}^\nu \partial_c \bar{X}^\lambda \partial_d \bar{X}^\rho + 2ip^\mu \partial_a \Delta \partial_b \bar{X}^\nu \partial_c \bar{X}^\lambda \partial_d \bar{X}^\rho \\
&\quad + 2ip^\lambda \partial_a \bar{X}^\mu \partial_b \bar{X}^\nu \partial_c \Delta \partial_d \bar{X}^\rho \\
&\quad + (\eta^{\mu\nu} \partial_a \partial'_b \Delta - p^\mu p^\nu \partial_a \Delta \partial_b \Delta) \partial_c \bar{X}^\lambda \partial_d \bar{X}^\rho + (\eta^{\lambda\rho} \partial_c \partial'_d \Delta - p^\lambda p^\rho \partial_c \Delta \partial_d \Delta) \partial_a \bar{X}^\mu \partial_b \bar{X}^\nu \\
&\quad + 4(\eta^{\mu\lambda} \partial_a \partial'_c \Delta - p^\mu p^\lambda \partial_a \Delta \partial_c \Delta) \partial_b \bar{X}^\nu \partial_d \bar{X}^\rho \\
&\quad + 2\partial_a \bar{X}^\mu (2\eta^{\nu\lambda} ip^\rho \partial_b \partial'_c \Delta \partial_d \Delta + \eta^{\lambda\rho} ip^\nu \partial_c \partial'_d \Delta \partial_b \Delta - ip^\nu p^\lambda p^\rho \partial_b \Delta \partial_c \Delta \partial_d \Delta) \\
&\quad + 2(\eta^{\mu\nu} ip^\lambda \partial_a \partial'_b \Delta \partial_c \Delta + 2\eta^{\mu\lambda} ip^\nu \partial_a \partial'_c \Delta \partial_b \Delta - ip^\mu p^\nu p^\lambda \partial_a \Delta \partial_b \Delta \partial_c \Delta) \partial_d \bar{X}^\rho \\
&\quad + \eta^{\mu\nu} \eta^{\lambda\rho} \partial_a \partial'_b \Delta \partial_c \partial'_d \Delta + 2\eta^{\mu\lambda} \eta^{\nu\rho} \partial_a \partial'_c \Delta \partial_b \partial'_d \Delta \\
&\quad - \eta^{\mu\nu} p^\lambda p^\rho \partial_a \partial'_b \Delta \partial_c \Delta \partial_d \Delta - 4\eta^{\mu\lambda} p^\nu p^\rho \partial_a \partial'_c \Delta \partial_b \Delta \partial_b \Delta - \eta^{\lambda\rho} p^\mu p^\nu \partial_c \partial'_d \Delta \partial_a \Delta \partial_b \Delta \\
&\quad + p^\mu p^\nu p^\lambda p^\rho \partial_a \Delta \partial_b \Delta \partial_c \Delta \partial_d \Delta \Big) \\
&\quad + \sqrt{g} R E_{\mu\nu} f^{ab} \big(\partial_a \bar{X}^\mu \partial_b \bar{X}^\nu + ip^\mu \partial_a \Delta \partial_b \bar{X}^\nu + ip^\nu \partial_a \bar{X}^\mu \partial_b \Delta \\
&\qquad + \eta^{\mu\nu} \partial_a \partial'_b \Delta - p^\mu p^\nu \partial_a \Delta \partial_b \Delta \big) \\
&\quad + \left(\sqrt{g} R\right)^2 E \Big] \Big\} \, . \qquad (2.73)
\end{aligned}$$

In this equation, the space-time fields have been scaled by ϵ^2 and the shorthand notation $\partial_a \Delta$ and $\partial_a \partial'_b \Delta$ has been used for the expressions $\partial_{z^a} \Delta_\epsilon(z,z')|_{z'=z}$ and $\partial_{z^a} \partial_{z'^b} \Delta_\epsilon(z,z')|_{z'=z}$ respectively.

Removal of regularisation ambiguities: Once again it looks as if regularisation ambiguities may be a problem, because the second derivative of the regularised propagator, given by

Eq. (2.29), contains the arbitrary numbers γ_ϵ and γ_0 and the symmetric traceless matrix M_{ab},

$$\partial_{z_1^a}\partial_{z_2^b}\Delta_\epsilon(z_1,z_2)\Big|_{z_1=z_2=z} = \gamma_\epsilon\delta_{ab}\epsilon^{-2}e^{2\sigma} + \tfrac{1}{2}\gamma_0\delta_{ab}\Box\sigma + M_{ab} + O(\epsilon^2)\ . \tag{2.74}$$

The situation is complicated further by the terms that look generically like $\partial_a\partial'_b\Delta\partial_c\Delta$ and $\partial_b\partial'_b\Delta\partial_c\partial'_d\Delta$. Because $\partial_a\partial'_b\Delta$ is of order ϵ^{-2}, the $O(\epsilon^2)$ corrections to $\partial_a\Delta$ and $\partial_a\partial'_b\Delta$ must be considered.

The ambiguities fall into two categories; those which contain the symmetric, traceless matrix M_{ab}, and those which don't. All the ambiguities of the latter type can be absorbed by employing a covariant measure similar to the one of Eq. (2.32). Since $f^{ac}f^{bd}$ is symmetric and traceless (see Eq. (2.72)), the terms in the partition function of this type are

$$\begin{aligned} g^{-1/2}\Big[&E_{\mu\nu\lambda\rho}f^{ac}f^{bd}4\eta^{\mu\lambda}\partial_a\partial'_c\Delta\left(\partial_b\bar{X}^\nu\partial_d\bar{X}^\rho + ip^\nu\partial_b\Delta\partial_d\bar{X}^\rho + ip^\rho\partial_b\bar{X}^\nu\partial_d\Delta\right.\\ &\left.+\tfrac{1}{2}\eta^{\nu\rho}\partial_b\partial'_d\Delta - p^\nu p^\rho\partial_b\Delta\partial_d\Delta\right) + \sqrt{g}RE_{\mu\nu}f^{ab}\eta^{\mu\nu}\partial_a\partial'_b\Delta\Big]\ . \end{aligned} \tag{2.75}$$

The covariant measure

$$\begin{aligned} &\left[\mathrm{d}^dX\right]_{\mathrm{cov}}\\ &= \left[\mathrm{d}^dX\sqrt{\det\left(\tfrac{1}{2}G_{\mu\nu} + RE_{\mu\nu} + 4E_{\mu\lambda\nu\rho}f^{ab}(\partial_aX^\lambda\partial_bX^\rho - \tfrac{1}{2}G^{\lambda\rho}\partial_a\partial_b\Delta_\epsilon)/\sqrt{g}\right)}\right]\ , \end{aligned} \tag{2.76}$$

removes all ambiguities from from Eq. (2.75) in an elegant fashion. This results immediately from the Leibnitz-like relations of Eq. (2.24)

$$g^{ab}\partial_a(\partial_b\Delta_\epsilon) = g^{ab}\partial_a\partial'_b\Delta + g^{ab}\partial_a\partial_b\Delta\ , \tag{2.77}$$

and by noting that $f^{ac}\partial_a\partial'_c\Delta = \sqrt{g}g^{ac}\partial_a\partial'_c\Delta$. It is important to realise that upon regulating this measure as in Eq. (2.33), the extra terms added to the action are both local and world-sheet reparameterisation invariant and thus correspond to simple field redefinitions as in Eq. (2.30) (the only place where this might fail is with $f^{ab}\partial_a\partial_b\Delta_\epsilon$ but this is local by definition).

Now consider the other class of ambiguities. These are the terms in the partition function which contain M_{ab},

$$\begin{aligned} \Big[&E_{\mu\nu\lambda\rho}f^{ac}f^{bd}\Big\{\eta^{\mu\nu}\partial_a\partial'_b\Delta\left(\partial_c\bar{X}^\lambda\partial_d\bar{X}^\rho + ip^\lambda\partial_c\Delta\partial_d\bar{X}^\rho + ip^\rho\partial_c\bar{X}^\lambda\partial_d\Delta\right.\\ &\left.+\tfrac{1}{2}\eta^{\lambda\rho}\partial_c\partial'_d\Delta - p^\lambda p^\rho\partial_c\Delta\partial_d\Delta\right) + (\mu\nu ab)\leftrightarrow(\lambda\rho cd)\Big\}\Big]\ . \end{aligned} \tag{2.78}$$

Using the symmetric-traceless property of $f^{ac}f^{bd}$ the $\partial_a\partial'_b\Delta$ becomes simply M_{ab}. To remove this ambiguity, the term

$$gE_{\mu\nu\lambda\rho}f^{ac}f^{bd}\left\{\eta^{\mu\nu}M_{ab}\partial_c\bar{X}^\lambda\partial_d\bar{X}^\rho + (\mu\nu ab)\leftrightarrow(\lambda\rho cd)\right\}\ , \tag{2.79}$$

would have to be added to the Lagrangian. However, M_{ab} has the general form

$$M_{ab} = M_1(\sigma)\partial_a\partial_b\sigma + M_2(\sigma)\partial_a\sigma\partial_b\sigma - \mathrm{trace}\ , \tag{2.80}$$

and is non-local since only $\Box\sigma$, not $\partial_a\sigma$, is local. Unless M_{ab} is zero then, Eq. (2.79) is not a local counterterm! On the other hand, if M_{ab} were zero, all terms in $Z[J]$ of the form $\eta^{\mu\nu}E_{\mu\nu\lambda\rho}$ and $\eta^{\lambda\rho}E_{\mu\nu\lambda\rho}$ would vanish. Then it would not be possible to derive a tracelessness condition on $E_{\mu\nu\lambda\rho}$!

Since Eq. (2.79) is a non-local counterterm, we are forced to suppose that M_{ab} is *regularisation independent.* Its non-vanishing is supported by one particular diffeomorphism covariant calculation [104] which yields

$$M_{ab} = \tfrac{1}{3}\left(\partial_a\partial_b\sigma + \partial_a\sigma\partial_b\sigma - \tfrac{1}{2}\delta_{ab}\Box\sigma - \tfrac{1}{2}\delta_{ab}(\partial\sigma)^2\right) . \tag{2.81}$$

We shall continue, assuming that the coefficients in M can be expanded as a power series:

$$M_i(\sigma) = \sum_{n=0}^{\infty} M_i^{(n)}\sigma^n . \tag{2.82}$$

In fact, it will be seen in that virtually all forms of M_{ab} result in the same equations of motion for $E_{\mu\nu\lambda\rho}$.

This point is worth emphasising given the discussion in Refs [189] and [60] where the traceless condition on $E_{\mu\nu\lambda\rho}$ was missed. It is now clear that the failure to obtain this constraint could be due to the use of a regularisation scheme which has $M_{ab} = 0$. Notice that this can be seen in a vertex-operator [339] calculation too — tracelessness would come from a self-contraction of $\partial\bar{X}^\mu\partial\bar{X}^\nu$, and if the regularisation was such that this was zero, no tracelessness condition would be found.

Renormalisation: Once again the minimal subtraction is trivial

$$\left(E_R^{\mu\nu\lambda\rho}, E_R^{\mu\nu}, E_R\right) = |\epsilon|^{p^2+2}\left(E^{\mu\nu\lambda\rho}, E^{\mu\nu}, E\right) . \tag{2.83}$$

The limit $\epsilon \to 0$ can now be taken.

Weyl invariance: Imposing Weyl invariance on the generating functional

$$\begin{aligned}
Z[J] \;=\; & P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\left\{\delta^d(p^\mu) - \frac{1}{4\pi}\int \mathrm{d}^2z e^{ip\bar{X}}e^{-(p^2+2)\sigma}\epsilon^2\right.\\
& \times\Big[E_{\mu\nu\lambda\rho}f^{ac}f^{bd}\Big(\partial_a\bar{X}^\mu\partial_b\bar{X}^\nu\partial_c\bar{X}^\lambda\partial_d\bar{X}^\rho + 2ip^\mu\partial_a\sigma\partial_b\bar{X}^\nu\partial_c\bar{X}^\lambda\partial_d\bar{X}^\rho\\
& +2ip^\lambda\partial_a\bar{X}^\mu\partial_b\bar{X}^\nu\partial_c\sigma\partial_d\bar{X}^\rho\\
& +(\eta^{\mu\nu}M_{ab} - p^\mu p^\nu\partial_a\sigma\partial_b\sigma)\partial_c\bar{X}^\lambda\partial_d\bar{X}^\rho + (\eta^{\lambda\rho}M_{cd} - p^\lambda p^\rho\partial_c\sigma\partial_d\sigma)\partial_a\bar{X}^\mu\partial_b\bar{X}^\nu\\
& +4(\eta^{\mu\lambda}\partial_a\partial_c\sigma - p^\mu p^\lambda\partial_a\sigma\partial_c\sigma)\partial_b\bar{X}^\nu\partial_d\bar{X}^\rho\\
& +2\partial_a\bar{X}^\mu(2\eta^{\nu\lambda}ip^\rho\partial_b\partial_c\sigma\partial_d\sigma + \eta^{\lambda\rho}ip^\nu M_{cd}\partial_b\sigma - ip^\nu p^\lambda p^\rho\partial_b\sigma\partial_c\sigma\partial_d\sigma)\\
& +2(\eta^{\mu\nu}ip^\lambda M_{ab}\partial_c\sigma + 2\eta^{\mu\lambda}ip^\nu\partial_a\partial_c\sigma\partial_b\sigma - ip^\mu p^\nu p^\lambda\partial_a\sigma\partial_b\sigma\partial_c\sigma)\partial_d\bar{X}^\rho\\
& +\eta^{\mu\nu}\eta^{\lambda\rho}M_{ab}M_{cd} + 2\eta^{\mu\lambda}\eta^{\nu\rho}\partial_a\partial_c\sigma\partial_b\partial_d\sigma\\
& -\eta^{\mu\nu}p^\lambda p^\rho M_{ab}\partial_c\sigma\partial_d\sigma - 4\eta^{\mu\lambda}p^\nu p^\rho\partial_a\partial_c\sigma\partial_b\sigma\partial_b\sigma - \eta^{\lambda\rho}p^\mu p^\nu M_{cd}\partial_a\sigma\partial_b\sigma\\
& +p^\mu p^\nu p^\lambda p^\rho\partial_a\sigma\partial_b\sigma\partial_c\sigma\partial_d\sigma\Big)
\end{aligned}$$

$$-2\Box\sigma E_{\mu\nu} f^{ab}\left(\partial_a \bar{X}^\mu \partial_b \bar{X}^\nu + ip^\mu \partial_a\sigma\partial_b\bar{X}^\nu + ip^\nu\partial_a\bar{X}^\mu\partial_b\sigma\right.$$
$$\left. +\eta^{\mu\nu}\partial_a\partial_b\sigma - p^\mu p^\nu \partial_a\sigma\partial_b\sigma\right)$$
$$+ 4(\Box\sigma)^2 \, E]\Big\} \;, \tag{2.84}$$

is extremely messy. (Notice that all regularisation ambiguities, excepting M_{ab}, have been removed.) As a preliminary, by inspection it is clear that for $Q^\mu = 0$, the following are sufficient conditions for Weyl invariance

$$\begin{aligned}
0 &= (\nabla^2 - 2)E_{\mu\nu\lambda\rho} \;, \\
0 &= \eta^{\mu\nu} E_{\mu\nu\lambda\rho} = \eta^{\lambda\rho}E_{\mu\nu\lambda\rho} \;, \\
0 &= \nabla^\mu E_{\mu\nu\lambda\rho} = \nabla^\lambda E_{\mu\nu\lambda\rho} \;, \\
0 &= \eta^{\mu\lambda}E_{\mu\nu\lambda\rho} - E_{\nu\rho} \;, \\
0 &= \eta^{\mu\nu}E_{\mu\nu} - 8E \;.
\end{aligned} \tag{2.85}$$

Evidently $E_{\mu\nu}$ and E are just traces of $E_{\mu\nu\lambda\rho}$. $E_{\mu\nu\lambda\rho}$ is transverse and traceless inside the pairs of indices as expected. These conditions are actually also the necessary conditions for Weyl invariance. The outline of the derivation of the equations of motion (with $Q^\mu \neq 0$) will now be given with special attention being paid to any appearances of M_{ab}.

Hindsight makes the calculation a great deal easier. All that is needed are the β functions corresponding to the operators

$$\begin{aligned}
&\partial_a\partial_b\bar{X}_0^\mu\partial_c\partial_d\bar{X}_0^\nu \;, \quad \partial_a\bar{X}_0^\mu\partial_b\partial_c\bar{X}_0^\nu\partial_d\bar{X}_0^\rho \;, \quad \partial_a\bar{X}_0^\mu\partial_b\bar{X}_0^\nu\partial_c\bar{X}_0^\lambda\partial_d\bar{X}_0^\rho \;, \\
&\partial_a\partial_b\partial_c\partial_d\sigma \quad \text{and} \quad \partial_a\partial_b\sigma\partial_c\bar{X}_0^\mu\partial_d\bar{X}_0^\nu \;,
\end{aligned} \tag{2.86}$$

where the world-sheet indices will be contracted with g^{ab} or ϵ^{ab}. To the author's knowledge, calculating these particular β functions in this particular order is the most efficient way of finding the equations of motion. Note that only a very few of the many (54 in total) β functions are needed and that all the ones used correspond to local operators. The following is not meant to be an explicit derivation of the β functions — that would require many pages of tedious algebra which would not be illuminating in the least. Rather the results are summarised so that the enthusiastic reader may verify their own calculations.

- The term of the form $\partial_a\partial_b\bar{X}_0^\mu\partial_c\partial_d\bar{X}_0^\nu$ in $\delta Z/\delta\sigma$ is

$$4(E_{\mu\nu} - \eta^{\lambda\rho}E_{\lambda\mu\rho\nu})f^{ab}\partial_a\partial_c\bar{X}^\mu\partial_b\partial_c\bar{X}^\nu + M_1^{(0)}\eta^{\lambda\rho}(E_{\mu\nu\lambda\rho} + E_{\lambda\rho\mu\nu})\Box\bar{X}^\mu\Box\bar{X}^\nu \;. \tag{2.87}$$

This immediately implies the two equations

$$\begin{aligned}
0 &= E_{\mu\nu} - \eta^{\lambda\rho}E_{\lambda\mu\rho\nu} \;, \\
0 &= M_1^{(0)}\eta^{\lambda\rho}(E_{\mu\nu\lambda\rho} + E_{\lambda\rho\mu\nu}) \;.
\end{aligned} \tag{2.88}$$

- The coefficient of $\partial_a\bar{X}_0^\mu\partial_b\partial_c\bar{X}_0^\nu\partial_d\bar{X}_0^\rho$ can be simplified considerably after using the above two equations. It then reads

$$2f^{ab}\partial_a\bar{X}_0^\mu\partial_b\bar{X}_0^\nu\Box\bar{X}_0^\lambda\Big\{iM_1^{(0)}\eta^{\alpha\beta}p^\nu E_{\mu\lambda\alpha\beta} + iM_1^{(0)}\eta^{\alpha\beta}p^\mu E_{\alpha\beta\nu\lambda} - 2ip^\alpha W_{\lambda\mu\alpha\nu} - 2ip^\alpha W_{\alpha\mu\lambda\nu}$$

$$
\begin{aligned}
&\qquad +4Q^\alpha E_{\lambda\mu\alpha\nu}+4Q^\alpha E_{\mu\alpha\lambda\nu}\}\\
&+\tfrac{1}{2}\partial_a\bar{X}^\mu\partial_b\bar{X}_0^\nu\partial_c\partial_d\bar{X}_0^\lambda\left\{f^{ac}f^{bd}(iM_1^{(0)}\eta^{\alpha\beta}p^\lambda E_{\mu\nu\alpha\beta}-4ip^\alpha E_{\mu\nu\alpha\lambda}+8Q^\alpha E_{\mu\nu\alpha\lambda})\right.\\
&\qquad\left.+f^{ca}f^{db}(iM_1^{(0)}\eta^{\alpha\beta}p^\lambda E_{\alpha\beta\mu\nu}-4ip^\alpha E_{\alpha\lambda\mu\nu}+8Q^\alpha E_{\alpha\lambda\mu\nu})\right\}\ .
\end{aligned}
\tag{2.89}
$$

These then imply

$$
\begin{aligned}
0 &= M_1^{(0)}\eta^{\mu\nu}p_\alpha E_{\mu\nu\lambda\rho}-4(p^\nu+2iQ^\nu)E_{\alpha\nu\lambda\rho}\ ,\\
0 &= (p^\alpha+2iQ^\alpha)E_{\mu\nu\alpha\lambda}+(p^\alpha+2iQ^\alpha)E_{\alpha\lambda\mu\nu}\ .
\end{aligned}
\tag{2.90}
$$

Tracing the first of these and using the second of Eq. (2.88) gives

$$
0=(p^\alpha+2iQ^\alpha)\eta^{\mu\nu}E_{\mu\nu\alpha\lambda}\ , \tag{2.91}
$$

while tracing the second with $\eta^{\lambda\mu}$ and using the first of Eq. (2.88) gives

$$
0=(p^\alpha+2iQ^\alpha)(E_{\alpha\mu}+E_{\mu\alpha})\ . \tag{2.92}
$$

• After using all the equations derived so far, setting to zero the coefficient of $\partial_a\bar{X}_0^\mu\partial_b\bar{X}_0^\nu\partial_c\bar{X}_0^\lambda\partial_d\bar{X}_0^\rho$ implies that

$$
0=(p^2+2ip\cdot Q+2)E_{\mu\nu\lambda\rho}\ . \tag{2.93}
$$

• The coefficient of $\Box\Box\sigma$ is simply

$$
8E-2\eta^{\mu\nu}E_{\mu\nu}+\tfrac{1}{2}M_1^{(0)}\eta^{\mu\nu}\eta^{\lambda\rho}E_{\mu\nu\lambda\rho}+\eta^{\mu\lambda}\eta^{\nu\rho}E_{\mu\nu\lambda\rho}=8E-\eta^{\mu\nu}E_{\mu\nu}\ , \tag{2.94}
$$

where both parts of Eq. (2.88) have been used.

• The final β function to calculate is the one corresponding to the operator $\partial_a\partial_b\sigma\partial_c\bar{X}_0^\mu\partial_d\bar{X}_0^\nu$. This calculation is the longest one but after repeated use of the equations derived thus far and with the assumption that $M_1^{(0)}\neq 0$, the following is obtained

$$
0=2p^\mu(p^\nu+2iQ^\nu)E_{\mu\nu\lambda\rho}-\tfrac{1}{2}(p^2+2)M_1^{(0)}\eta^{\mu\nu}E_{\mu\nu\lambda\rho}+2(M_1^{(1)}-M_2^{(0)})\eta^{\mu\nu}E_{\mu\nu\lambda\rho}\ . \tag{2.95}
$$

Upon subtracting this from the contraction of p^α with the first of Eq. (2.90), the result is simply

$$
0=(2M_2^{(0)}+M_1^{(0)}-2M_1^{(1)})\eta^{\mu\nu}E_{\mu\nu\lambda\rho}\ . \tag{2.96}
$$

If M_{ab} is such that

$$
2M_2^{(0)}+M_1^{(0)}-2M_1^{(1)}\neq 0 \quad\text{and}\quad M_1^{(1)}\neq 0\ , \tag{2.97}
$$

then the field $E_{\mu\nu\lambda\rho}$ is traceless! Using Eq. (2.90) it is then also transverse. In the special case where $M_1^{(0)}=0$ then $E_{\mu\nu\lambda\rho}$ is still transverse, and the β function equation is simply

$$
0=(M_1^{(1)}-M_2^{(0)})\eta^{\mu\nu}E_{\mu\nu\lambda\rho}=(M_1^{(1)}-M_2^{(0)})\eta^{\mu\nu}E_{\lambda\rho\mu\nu}\ . \tag{2.98}
$$

So, provided the multiplier is non-zero, tracelessness is still obtained.

In summary, Weyl invariance of the generating functional to $O(\lambda)$ with a flat, linear-dilaton background is equivalent to

$$\begin{aligned}
0 &= (\nabla^2 - 2Q\cdot\nabla - 2)E_{\mu\nu\lambda\rho} \ , \\
0 &= \eta^{\mu\nu}E_{\mu\nu\lambda\rho} = \eta^{\lambda\rho}E_{\mu\nu\lambda\rho} \ , \\
0 &= (\nabla^\mu - 2Q^\mu)E_{\mu\nu\lambda\rho} = (\nabla^\lambda - 2Q^\lambda)E_{\mu\nu\lambda\rho} \ , \\
0 &= \eta^{\mu\lambda}E_{\mu\nu\lambda\rho} - E_{\nu\rho} \ , \\
0 &= \eta^{\mu\nu}E_{\mu\nu} - 8E \ ,
\end{aligned} \tag{2.99}$$

providing the second derivative of the propagator at coincidence

$$\partial_{z_1^a}\partial_{z_2^b}\Delta_\epsilon(z_1,z_2)\Big|_{z_1=z_2=z} = \gamma_\epsilon\delta_{ab}\epsilon^{-2}e^{2\sigma} + \tfrac{1}{2}\gamma_0\delta_{ab}\Box\sigma + M_{ab} + O(\epsilon^2) \ , \tag{2.100}$$

does not have a transverse, traceless part M_{ab}, given in general by

$$M_{ab} = \partial_a\partial_b\sigma\sum_{n=0} M_1^{(n)}\sigma^n + \partial_a\sigma\partial_a\sigma\sum_{n=0} M_2^{(n)}\sigma^n - \text{trace} \ , \tag{2.101}$$

with the specific form

$$0 = 2M_2^{(0)} + M_1^{(0)} - 2M_1^{(1)} \ . \tag{2.102}$$

If this equation is true, then it is possible that traceless condition would be missing.

As a final note, it is easy to imagine that if M_{ab} contained sufficiently high powers of σ then the β functions corresponding to (non-local) operators with many powers of σ would only contain the terms $\eta^{\mu\nu}E_{\mu\nu\lambda\rho}$ and $\eta^{\lambda\rho}E_{\mu\nu\lambda\rho}$. It would thereby be possible to arrange things so that the traceless condition originated from these β functions rather than the local ones described above. For instance, if Eq. (2.102) held but

$$M_{ab} = \sigma^n\partial_a\partial_b\sigma - \tfrac{1}{2}\delta_{ab}\sigma^n\Box\sigma \ , \tag{2.103}$$

for some large n then using Eq. (2.90) and Eq. (2.93) the partition function becomes simply

$$\int d^2z e^{ip\cdot\bar{X}}E_{\mu\nu\lambda\rho}f^{ac}f^{bd}\Big(M_{ab}\eta^{\mu\nu}\partial_c\bar{X}^\lambda\partial_d\bar{X}^\rho + \tfrac{1}{2}M_{ab}M_{cd}\eta^{\mu\nu}\eta^{\lambda\rho} + (\mu\nu ab)\leftrightarrow(\lambda\rho cd)\Big) \ . \tag{2.104}$$

The unimportant prefactor has been omitted and for simplicity consider the case $Q^\mu = 0$. Then the coefficient of the term $\sigma^{2n-1}\partial_a\partial_b\sigma\partial_c\partial_d\sigma f^{ac}f^{bd}$ is

$$\eta^{\mu\nu}\eta^{\lambda\rho}E_{\mu\nu\lambda\rho} = 0 \ . \tag{2.105}$$

Similarly, the coefficient of $\sigma^{n-1}\partial_a\partial_b\sigma\partial_c X^\lambda\partial_d X^\rho$ is

$$\begin{aligned}
0 &= \eta^{\mu\nu}E_{\mu\nu\lambda\rho}f^{ac}f^{bd} + \eta^{\mu\nu}E_{\lambda\rho\mu\nu}f^{ca}f^{db} \\
&= \eta^{\mu\nu}(E_{\mu\nu\lambda\rho} - E_{\lambda\rho\mu\nu})(gg^{ad}g^{cb} - gg^{ab}g^{cd} + i\sqrt{g}\epsilon^{ac}g^{bd} + i\sqrt{g}\epsilon^{bd}g^{ac}) \\
&\quad + \eta^{\mu\nu}(E_{\mu\nu\lambda\rho} + E_{\lambda\rho\mu\nu})gg^{ac}g^{bd}
\end{aligned} \tag{2.106}$$

and tracelessness is still obtained.

2.5 Massive-field corrections to the tachyon field equation

This section calculates the quadratic correction to the tachyon equation of the form $TE_{\mu\nu\lambda\rho}$. There are two ways which such a term can be obtained. One comes from the covariant measure Eq. (2.76)

$$\left[\mathrm{d}^d X\right]_{\mathrm{cov}} = [\mathrm{d}^d X]\exp\left(-\frac{1}{4\pi}\int \eta^{\mu\nu}(RE_{\mu\nu} + 4e^{-2\sigma}E_{\mu\lambda\nu\rho}f^{ab}(\partial_a X^\lambda \partial_b X^\rho - \tfrac{1}{2}\eta^{\lambda\rho}\partial_a\partial_b\Delta_\epsilon))\Box\Delta_\epsilon\right) , \tag{2.107}$$

which, using $\Box\Delta_\epsilon = -2\gamma_\epsilon\epsilon^{-2}e^{2\sigma} + O(\partial\sigma)$ gives

$$\begin{aligned}-\frac{1}{16\pi^2}\int\left[\mathrm{d}^d X\right]\exp\left(-\tfrac{1}{8\pi}\int \partial_a X^\mu \partial_a X^\nu \eta_{\mu\nu} + a_0\cdot(J_0 - \tfrac{2Q}{\sqrt{V}})\right)\int \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \mathrm{d}^d p_1 \mathrm{d}^d p_2 \\ \times e^{i(p_1(X+\bar X)(z_1) + p_2(X+\bar X)(z_2))}e^{2\sigma(z_1)}T(p_1) \\ \times 4e^{-2\sigma(z_2)}\eta^{\mu\lambda}E_{\mu\nu\lambda\rho}(p_2)f^{ab}\left(\partial_a X^\nu(z_2)\partial_b X^\rho(z_2) - \tfrac{1}{2}\eta^{\nu\rho}\partial_a\partial_b\Delta\right)(2\gamma_\epsilon\epsilon^{-2}e^{2\sigma}) ,\end{aligned} \tag{2.108}$$

while the other is simply from the expansion of $e^{-S_{\mathrm{int}}}$

$$\begin{aligned}\frac{1}{16\pi^2}\int\left[\mathrm{d}^d X\right]\exp\left(-\tfrac{1}{8\pi}\int \partial_a X^\mu \partial_a X^\nu \eta_{\mu\nu} + a_0\cdot(J_0 - \tfrac{2Q}{\sqrt{V}})\right)\int \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \mathrm{d}^d p_1 \mathrm{d}^d p_2 \\ \times e^{i(p_1(X+\bar X)(z_1) + p_2(X+\bar X)(z_2))}e^{2\sigma(z_1)-2\sigma(z_2)}T(p_1) \\ \times E_{\mu\nu\lambda\rho}(p_2)f^{ac}f^{bd}\partial_a X^\mu(z_2)\partial_b X^\nu(z_2)\partial_c X^\lambda(z_2)\partial_d X^\rho(z_2) .\end{aligned} \tag{2.109}$$

All other terms give derivatives on σ or $\bar X^\mu$ which will contribute to other field equations, as explained in Sec. 2.2. In the same way, the result of performing these Gaussian integrals is particularly simple because all terms containing derivatives on σ or $\bar X^\mu$ can be dropped.

It should not come as a surprise that the contributions from the covariant measure exactly cancel the regularisation dependent parts in Eq. (2.109), leading to a total contribution of

$$\begin{aligned}\frac{1}{16\pi^2}\int \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \mathrm{d}^d p_1 e^{i(p_1\bar X_1 + p_2\bar X_2)}e^{2\sigma_1 - 2\sigma_2}T(p_1)E_{\mu\nu\lambda\rho}(p_2) \\ p_1^\mu p_1^\nu p_1^\lambda p_1^\rho(\partial_{z_2^a}\Delta(z_1,z_2))^4 e^{-\frac{1}{2}p_1^2\Delta(z_1,z_1) - \frac{1}{2}p_2^2\Delta(z_2,z_2) - p_1\cdot p_2\Delta(z_1,z_2)} ,\end{aligned} \tag{2.110}$$

with momentum conservation $ip_1 + ip_2 + J_0^\mu\sqrt{V} - 2Q^\mu = ip_1 + ip_2 + ip = 0$.

In Sec. 2.2, the TT term was simplified by performing the integral over z_-. The same idea is used here. Up to derivatives on σ, the first derivative of the propagator is

$$(\partial_a\Delta(z_1,z_2))^2 = \frac{4|z_1 - z_2|^2}{(|z_1 - z_2|^2 + \epsilon^2 e^{-2\sigma})^2} . \tag{2.111}$$

The analogous integral to Eq. (2.48) is

$$\begin{aligned}&\int \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 f_1(z_1) f_2(z_2)\left((\partial_a\Delta_\epsilon(z_1,z_2))^2\right)^2 \exp(-p_1\cdot p_2\Delta_\epsilon(z_1,z_2)) \\ &= 1024\int \mathrm{d}^2 z_+ \mathrm{d}^2 z_- f_1(z_+) f_2(z_+)|z_-|^4\left(4|z_-|^2 + \epsilon^2 e^{-2\sigma(z_+)}\right)^{-4+p_1\cdot p_2} + O(\partial\bar X, \partial\sigma) \\ &= 64\pi\frac{e^{(2+2p_1\cdot p_2)(\sigma - \log|\epsilon|)}}{(3 - p_1\cdot p_2)(2 - p_1\cdot p_2)(1 - p_1\cdot p_2)} + O(\partial\bar X, \partial\sigma) .\end{aligned} \tag{2.112}$$

Employing this result yields

$$
\begin{aligned}
Z_{TE} &= P[\sigma]\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\int d^2z e^{ip\bar{X}}e^{(2-p^2)(\sigma-\log|\epsilon|)} \\
&\quad\times\frac{2}{\pi}\int \mathrm{d}p_1\frac{E_{\mu\nu\lambda\rho}(p-p_1)p_1^\mu p_1^\nu p_1^\lambda p_1^\rho T(p_1)}{(3-p_1(p-p_1))(2-p_1(p-p_1))(1-p_1(p-p_1))}\ . \qquad (2.113)
\end{aligned}
$$

After renormalisation, which again can be implemented by adding local counterterms to the action, the denominators of Eq. (2.113) can be simplified by substituting in the mass-shell relations

$$
(-p^2-2iQ{\cdot}p+2)T(p)+O(\lambda^2)=0=(p^2+2iQ{\cdot}p+2)E_{\mu\nu\lambda\rho}(p)+O(\lambda^2)\ , \qquad (2.114)
$$

which imply

$$
1-p_1{\cdot}(p-p_1)=\tfrac{1}{2}(2-p^2-2ip{\cdot}Q)\ . \qquad (2.115)
$$

In summary, the generating functional is

$$
\begin{aligned}
Z &= -\frac{P[\sigma]}{4\pi}\left(\frac{1}{V}\det{}'\frac{\Box}{4\pi}\right)^{-\frac{1}{2}d}\int d^2z e^{ip\bar{X}_0}e^{(2-2iQ\cdot p-p^2)\sigma}\left\{T(p)-\tfrac{1}{2}\int \mathrm{d}^dp_1\frac{T(p_1)T(p-p_1)}{2-2iQ{\cdot}p-p^2}\right. \\
&\quad\left.-32\int \mathrm{d}^dp_1\frac{E_{\mu\nu\lambda\rho}(p-p_1)p_1^\mu p_1^\nu p_1^\lambda p_1^\rho T(p_1)}{(2+(2-2iQ{\cdot}p-p^2))\,(1+(2-2iQ{\cdot}p-p^2))\,(2-2iQ{\cdot}p-p^2)}\right\} \qquad (2.116)
\end{aligned}
$$

The denominators in the TE contribution can be expanded around the points in momentum space where $2-2ip{\cdot}Q-p^2=0$. Imposing Weyl invariance, the field equation is finally obtained:

$$
(\nabla^2-2Q{\cdot}\nabla+2)T-\tfrac{1}{2}T^2-16E^{\mu\nu\lambda\rho}\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\rho T=0\ . \qquad (2.117)
$$

This equation is valid up to $O(E^2)$ with a flat linear-dilaton background. For simplicity all higher derivative terms in the expansion of the TE term around the tachyon on-shell condition $2-2ip{\cdot}Q-p^2=0$ have been left out. Sec. 2.7 explains why these terms have no bearing on the validity of DMW's proposal.

2.6 The "first massive" discrete state in two dimensions

The higher-mode generalisation of the black-hole Eq. (1.74) will now be derived by finding the general solution to the equations (2.99) in two dimensions.

The Minkowsky metric $\eta^{\mu\nu}=\mathrm{diag}\,(\eta^{TT},\eta^{XX})=\mathrm{diag}\,(-1,1)$ is used. Transversality implies

$$
0=(2Q^T+\partial_T)E_{\ldots(\mu 0)\ldots}+(2Q^X-\partial_X)E_{\ldots(\mu 1)\ldots}\ , \qquad (2.118)
$$

where the dots mean there may, or may not, be indices in the slots, and the round brackets indicate that a symmetric pair of indices is being considered, ie $E_{\mu\nu\lambda\rho}=E_{(\mu\nu)(\lambda\rho)}$. Specialising this equation gives

$$
\begin{aligned}
0 &= (2Q^T+\partial_T)E_{(\mu 1)(\nu 0)}+(2Q^X-\partial_X)E_{(\mu 1)(\nu 1)}\ , \\
0 &= (2Q^T+\partial_T)E_{(\mu 0)(\nu 1)}+(2Q^X-\partial_X)E_{(\mu 1)(\nu 1)}\ , \qquad (2.119)
\end{aligned}
$$

and the difference of these equations is

$$0 = (2Q^T + \partial_T)(E_{(\mu 1)(\nu 0)} - E_{(\mu 0)(\nu 1)}) \ . \tag{2.120}$$

In the same way tracelessness also implies

$$\begin{aligned} 0 &= (2Q^T + \partial_T)E_{(\mu 0)(\nu 0)} + (2Q^X - \partial_X)E_{(\mu 0)(\nu 1)} \ , \\ 0 &= (2Q^T + \partial_T)E_{(\mu 0)(\nu 0)} + (2Q^X - \partial_X)E_{(\mu 1)(\nu 0)} \ , \end{aligned} \tag{2.121}$$

which yields

$$0 = (2Q^X - \partial_X)(E_{(\mu 0)(\nu 1)} - E_{(\mu 1)(\nu 0)}) \ . \tag{2.122}$$

The general solution of these two equations is $E_{(\mu 0)(\nu 1)} - E_{(\mu 1)(\nu 0)} = A\exp(2Q^X X - 2Q^T T)$, but by the equation of motion $A = 0$ and so

$$E_{(\mu 0)(\nu 1)} = E_{(\nu 1)(\mu 0)} \ . \tag{2.123}$$

Coupled with tracelessness, the following is also true

$$E_{(00)(00)} = E_{(00)(11)} = E_{(10)(10)} \ . \tag{2.124}$$

Therefore, all components are either equal to $E_{(00)(00)}$ or $E_{(10)(00)}$ and $E_{\mu\nu\lambda\rho}$ can be parameterised as

$$E_{\mu\nu\lambda\rho} = \begin{cases} E_+ & \text{if } \mu+\nu+\lambda+\rho = \text{even} \\ E_- & \text{if } \mu+\nu+\lambda+\rho = \text{odd} \end{cases} \ , \tag{2.125}$$

where tracelessness now reads

$$\begin{aligned} 0 &= (2Q^T + \partial_T)E_+ + (2Q^X - \partial_X)E_- \ , \\ 0 &= (2Q^T + \partial_T)E_- + (2Q^X - \partial_X)E_+ \ . \end{aligned} \tag{2.126}$$

Combining these two equations produces a second-order differential equation

$$\left((2Q^X - \partial_X)^2 - (2Q^T + \partial_T)^2\right) E_\pm = 0 \ , \tag{2.127}$$

which may be subtracted from the equation of motion to yield a simpler first-order uncoupled equation

$$(Q\cdot\partial - 4Q^2 - 2)E_\pm = 0 \ . \tag{2.128}$$

The solution can be written in terms of an arbitrary function f in the following fashion

$$E_\pm = e^{\frac{2Q^2+1}{Q^X}X} f(Q^T X - Q^X T) \ . \tag{2.129}$$

There is also another solution $E_\pm = g(Q^T X - Q^X T)$, but this is excluded because it clearly does not satisfy the equation of motion. It is then a matter of solving Eq. (2.126) and the equation of motion. The solution can be written in the form

$$E_\pm = Ae^{\left(2Q^X + \frac{1}{Q^X+Q^T}\right)X - \left(2Q^T - \frac{1}{Q^X+Q^T}\right)T} \pm Be^{\left(2Q^X + \frac{1}{Q^X-Q^T}\right)X - \left(2Q^T + \frac{2}{Q^X-Q^T}\right)T} \ , \tag{2.130}$$

in which A and B are arbitrary constants of integration. In the non-generic case of $Q^X \pm Q^T = 0$ the term which contains the potential infinity must be dropped from the solution. Thus, for generic background charge, the background solution of $E_{\mu\nu\lambda\rho}$ is a time-dependent, two-parameter solution.

2.7 Tachyon scattering in a two-dimensional massive linear-dilaton background

The parameterisation of space-time used in Sec. 1.9 (lower case x) differs from the rest of this chapter (upper case X) by

$$x^\mu = X^\mu/\sqrt{2} . \tag{2.131}$$

Taking the dilaton to lie purely in the X^1 direction gives $Q^x = \sqrt{2}Q^X = 2$ and

$$E_\pm = Ae^{5x+t} \pm Be^{5x-t} , \tag{2.132}$$

with the tachyon field equation taking the form

$$(\nabla_x^2 - 2\nabla_x\Phi\cdot\nabla_x + 4)T - T^2 - OE^{\mu\nu\lambda\rho}O'_{\mu\nu\lambda\rho}T = 0 . \tag{2.133}$$

The operators O and $O'_{\mu\nu\lambda\rho}$ originate from expanding the denominators of the TE term in Eq. (2.116) and their particular form will not be important. Scaling by the string coupling, $T = e^\Phi S$, the massless field S obeys

$$\partial_+\partial_- S = \tfrac{1}{2}e^\Phi S^2 + \tfrac{1}{2}OE^{\mu\nu\lambda\rho}e^{-\Phi}O'_{\mu\nu\lambda\rho}e^\Phi S . \tag{2.134}$$

It is this equation which will be solved to find string theory's prediction for 1-1 tachyon scattering.

Following [98, 278], one-to-one scattering is calculated in the following way: First, the discrete-mode background Eq. (2.132) is put into place and fixed. Then a small tachyon background S_0 is allowed to develop

$$\partial_+\partial_- S_0 = \tfrac{1}{2}OE^{\mu\nu\lambda\rho}e^{-\Phi}O'_{\mu\nu\lambda\rho}e^\Phi S_0 . \tag{2.135}$$

Finally, the field S is expanded around this background and the first non-trivial order is retained

$$\partial_+\partial_- S = e^{-2x}S_0 S + Ae^{5x+t}S + Be^{5x-t}S . \tag{2.136}$$

All constants, in particular, the ones coming from the operators O and O', have been absorbed into A and B. The first term describes tachyon scattering off the dynamically created tachyon background, whilst the other terms are scattering off the fixed discrete-mode background.

Integrating this equation to first order in the small background S_0, with the boundary condition

$$S(x,t) \to S_+(x^-) \quad \text{as} \quad t \to -\infty , \tag{2.137}$$

yields

$$\begin{aligned} S_-(x^+) \;&=\; \int_{-\infty}^{\infty} du^- e^{-u^-} S_+(u^-) \int_{-\infty}^{x^+} du^+ e^{u^+} S_0(u^+,u^-) \\ &\quad + Ae^{3x^+} \int_{-\infty}^{\infty} du^- e^{-2u^-} S_+(u^-) + Be^{2x^+} \int_{-\infty}^{\infty} du^- e^{-3u^-} S_+(u^-) , \end{aligned} \tag{2.138}$$

where, once again, constants have been absorbed. As was mentioned in Sec. 1.9, DMW showed that the first term, with the background given by the dynamically created Eq. (1.133),

$$S_0(u^+,u^-) = -\frac{1}{4\sqrt{\mu'}}\int_{-\infty}^{\infty} \mathrm{d}\tau\ \Delta^2(\tau)e^{-\tau+x^+} f'\left(\sqrt{\mu'}e^{-\tau+x^+}\right) , \tag{2.139}$$

was exactly the $(m,n) = (0, \text{arbitrary})$ part of the general solution Eq. (1.132). Now it is clear that Eq. (1.132) also predicts exactly the right behaviour for scattering off the discrete-state background if the constants A and B are identified as

$$A \propto \int_{-\infty}^{\infty} \mathrm{d}\tau\ \Delta^2(\tau)e^{\tau} \quad \text{and} \quad B \propto \int_{-\infty}^{\infty} \mathrm{d}\tau\ \Delta^2(\tau)e^{-\tau} . \tag{2.140}$$

So, the "first massive" discrete state occupies the $(m,n) = (2,1)$ and $(m,n) = (1,2)$ positions as expected. Thus, DMW's proposal has been checked to the next order above the black-hole.

2.8 Conclusions

By imposing Weyl invariance on the generating functional of closed bosonic string theory, the linearised equation of motion and constraints for the first massive level in a flat linear-dilaton background have been derived.

This is the first time that the correct equations have been obtained using a variant of the Wilson renormalisation group method. Two possible reasons for the failure of other attempts to obtain these equations were given: Weyl invariance on a curved world-sheet is not equivalent to conformal invariance on a flat world-sheet; or, a "sick" regularisation of the kind Eq. (2.102) was used.

The effective dynamics of tachyons in a first-massive, flat, linear-dilaton vacuum was worked out. The coupling of the tachyon to the massive field involved an infinite number of derivatives.

In two space-time dimensions, the constraints on the "first massive" level were solved to find a two-parameter time-dependent solution.

One-to-one tachyon scattering in this discrete-state background was studied and the results agree with the prediction made by Dhar, Mandal and Wadia's [98] representation of the matrix model.

It would be a thankless task to derive field equations for the higher states using the approach of this chapter. However, from the analysis of Secs. 2.2, 2.3 and 2.5 it seems likely that *all* the massive fields couple to tachyon. Moreover, a straightforward generalisation of Eq. (2.138) for tachyon scattering in an arbitrary discrete-state background

$$\mathcal{E} \sim e^{ax+bt} , \tag{2.141}$$

gives

$$S_-(x^+) \sim e^{\frac{1}{2}(a+b)x^+} \int_{-\infty}^{\infty} \mathrm{d}u^- e^{-\frac{1}{2}(a-b)u^-} S_+(u^-) . \tag{2.142}$$

Therefore, if the prediction Eq. (1.132) is correct, then the classical solutions (vertex operators) of all the higher discrete states must be of the form

$$\tfrac{1}{2}(a+b) = n+1 \quad \text{and} \quad \tfrac{1}{2}(a-b) = m+1 \; . \tag{2.143}$$

This is exactly the spectrum of states which follows from a BRST analysis [54, 222]. One puzzle remains — according to DMW's formula Eq. (1.132), the charges (such as M, A and B) at fixed $(n-m)$ are related. Thus, for example, the charge of any state with $n=m$ should be proportional to the mass of the black-hole at $n=1=m$. Presumably this is a consequence of the equivalence of the w_∞ symmetries in the matrix model (generated by the charges of Eq. (1.112)) and the string theory and will be investigated in future work.

Induced Chern-Simons Terms

It has been proposed that the effective action of four-dimensional $SU(2)_L$ gauge theory at high and low temperature contains a three-dimensional Chern-Simons term whose coefficient is the chemical potential for baryon number. This claim is examined by performing exact calculations in a related two-dimensional theory. The results demonstrate that the existence of the Chern-Simons term in four-dimensions may be rather subtle.

3.1 Background Material

This introductory section begins with a review of the Chern-Simons action in three dimensions and one dimension. Then, in Sec. (3.1.2), a small comment is made about the so-called "global SU(2) anomaly" in four dimensions. Finally, Sec. (3.1.3) discusses how the anomaly causes baryon number non-conservation and how the amplitude for such a process can be made appreciable by introducing a large fermion density.

3.1.1 Chern-Simons terms

The Chern-Simons (CS) action [73] in three dimensions is

$$S_{\mathrm{CS}} = \int \mathrm{d}^3\boldsymbol{x}\ \epsilon^{ijk}\mathrm{tr}\left(F_{ij}A_k - \tfrac{2}{3}gA_iA_jA_k\right) \ , \tag{3.1}$$

where the field strength is given, as usual, by

$$F_{ij} = \partial_i A_j - \partial_j A_i + g[A_i, A_j] \ . \tag{3.2}$$

The spatial indices are $i = 1, 2, 3$, the Yang-Mills (YM) coupling is denoted by g and the gauge field A_i is in the adjoint representation of some gauge group which is henceforth taken to be SU(2). Space has a Euclidean signature.

The action is invariant under the gauge transformation

$$A_i \to A_i^U = U^{-1}A_iU + \tfrac{1}{g}U^{-1}\partial_iU \ , \tag{3.3}$$

if the element of the gauge group U asymptotically tends to infinity

$$U(\boldsymbol{x}) \stackrel{|\boldsymbol{x}|\to\infty}{\longrightarrow} 1 \ , \tag{3.4}$$

and is "small"; that is, if it has no winding number. However, with the assumption of asymptotic spatial uniformity Eq. (3.4), the points at spatial infinity can be identified, $R^3 \to S^3$. This means that group elements are maps from S^3 to S^3 (the group manifold of SU(2)) and therefore they can have non-trivial winding number (the homotopy group $\pi_3(S^3)$ is non-trivial). Under such a "large" gauge transformation [96]

$$S_{\rm CS} \to S_{\rm CS} - \frac{16\pi^2}{g^2}\tilde{N} \ , \tag{3.5}$$

where the integer $\tilde{N}$ is the winding number of the gauge transformation. For correlation functions of observables to be gauge invariant, $e^{i\mu S_{CS}}$ must be too (at the moment the role of μ is simply to isolate the CS term from others that may appear in the total action, for instance $\int \frac{1}{2}{\rm tr}F^2$ — later μ will be the chemical potential for fermion number). This places the quantisation condition on μ:

$$8\pi\mu/g^2 \in \mathbb{Z} \ . \tag{3.6}$$

Similarly, in one dimension, the CS action for a U(1) gauge field $A(x)$ is

$$S_{CS} = \int_0^R {\rm d}x \ A(x) \ . \tag{3.7}$$

Here, the boundary condition Eq. (3.4) has been replaced with the equivalent statement that the space is compact. Performing the large gauge transformation $A(x) \to A(x) + \frac{1}{e}\partial\frac{2\pi\tilde{N}x}{R}$, the action shifts as

$$S_{CS} \to S_{CS} + \frac{2\pi}{e}\tilde{N} \ . \tag{3.8}$$

Again μ must be quantised in order that $e^{i\mu S_{CS}}$ be invariant.

No more facts about Chern-Simons theory will be needed here. However, it is a fascinating field and further comments can be found in [96]. Among its many interesting features, there are two that the author finds particularly notable. Firstly, $S_{\rm CS}$ describes a topological field theory because it makes no reference to a spacetime metric. This has been used by Witten [343] to elucidate the connection between quantum field theory and knot invariants. Secondly, adding $S_{\rm CS}$ to the standard YM action provides a mechanism for mass generation [96, 95, 203, 300, 313] (this can be seen by a calculation of the propagator — the mass $\sim \mu$).

3.1.2 The global SU(2) anomaly in four dimensions

As noticed by Witten [342], $SU(2)_L$ gauge theory with an odd number of fermion doublets is inconsistent in four dimensions. This is called the "global SU(2) anomaly" and comes about in the following way.

Since the homotopy group $\pi_4(S^3) = \mathbb{Z}_2$, there is a non-trivial class of large SU(2) transformations in four dimensions. Now consider calculating the effective action for an SU(2) doublet of Dirac fermions. This is

$$\int [\mathrm{d}\bar{\psi}\mathrm{d}\psi] e^{\bar{\psi} i \not{D} \psi} = \det i\not{D} \ , \tag{3.9}$$

where the RHS is formally just the product of the eigenvalues of the Hermitian operator $i\not{D}$. Since Dirac fermions can have a gauge invariant bare mass, Pauli-Villars regularisation can be used to define this formal product. Therefore, the RHS is gauge invariant.

By conjugating the right-handed Weyl doublet that lies inside the Dirac doublet, the latter can be decomposed into two left-handed Weyl doublets. Explicitly, the Lagrangian density

$$\mathcal{L} = \bar{\psi} i \not{D}_r \psi = \psi_L^\dagger i \bar{\sigma} \cdot D_r \psi_L + \psi_R^\dagger i \sigma \cdot D_r \psi_R \ , \tag{3.10}$$

where $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ ($\boldsymbol{\sigma}$ are the Pauli matrices) and r indicates the representation that the fermions belong to. Since $\sigma^2 \psi_R^*$ transforms as a left-handed fermion [267, Sec. 19.4], define the spinor $\psi_L' \equiv \sigma^2 \psi_R^*$. Then

$$\mathcal{L} = \psi_L^\dagger i \bar{\sigma} \cdot D_r \psi_L + \psi_L'^\dagger i \bar{\sigma} \cdot D_{\bar{r}} \psi_L' \ . \tag{3.11}$$

The second set of left-handed fermions transform under the conjugate representation $\bar{r}$ with representation matrices $T_{\bar{r}}^a = -(T_r^a)^*$. However, for SU(2) the conjugate representation is unitarily equivalent to the original; $T_{\bar{r}}^a = U T_r^a U^\dagger$. Therefore an SU(2) doublet of Dirac fermions is equivalent to two left-handed (Weyl) doublets.

By definition, integration with a single left-handed doublet gives

$$\int [\mathrm{d}\bar{\psi}_L \mathrm{d}\psi_L] e^{\bar{\psi}_L i \not{D} \psi_L} = \sqrt{\det i\not{D}} \ . \tag{3.12}$$

Now there is a choice for the sign in front of the square-root. The eigenvalues come in pairs since $\not{D}\gamma^5 = -\gamma^5 \not{D}$ and so to calculate the square-root, one eigenvalue out of each pair must be chosen and then the product of these taken. By calculating the index [15] of a related five dimensional theory, Witten was able to study the flow of the eigenvalues as the gauge field was adiabatically varied from A_μ ($\mu = 0, \dots, 3$) to A_μ^U where U is the element of SU(2) which winds at infinity. He found that an *odd* number of pairs of eigenvalues changed sign under this gauge transformation. Therefore

$$\sqrt{\det i\not{D}(A_\mu)} = -\sqrt{\det i\not{D}(A_\mu^U)} \ , \tag{3.13}$$

and the effective action was not gauge invariant. This sign-change immediately implies $Z = \int [\mathrm{d}A] \sqrt{\det i\not{D}} \exp(-S_{YM}) = 0$, so all correlation functions are ill-defined; being 0/0. However, with an *even* number of $\mathrm{SU}(2)_L$ doublets, there is no inconsistency.

3.1.3 Baryon-number violation

In order to study baryon-number violation in the standard model, the following simplifications are made: Neglect both the U(1) admixture (so all the W-bosons will have the same mass) and the Yukawa couplings of the fermions to the Higgs doublet. However, retain the 12 left-handed (Weyl) fermion SU(2) doublets and their minimal coupling to the W-boson A^μ. The fermionic part of the action is therefore simply

$$S_f = \int \mathrm{d}t\mathrm{d}^3\boldsymbol{x} \sum_{i=1}^{12} \bar{\psi}_L^i i \not{D} \psi_L^i \ . \tag{3.14}$$

Since there are an even number of fermions, there is no global SU(2) anomaly. However, in the presence of the $\mathrm{SU}(2)_L$ gauge fields, it is well-known [2, 31, 199, 267, 351] that the U(1) current

$$J^\mu = \sum_{i=1}^{12} \bar{\psi}_L^i \gamma^\mu \psi_L^i \ , \tag{3.15}$$

is anomalous

$$\partial_\mu J^\mu = -12\frac{g^2}{16\pi^2}\tfrac{1}{2}\mathrm{tr}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = -12\frac{g^2}{16\pi^2}\partial_\mu\epsilon^{\mu\nu\lambda\rho}\mathrm{tr}(A_\nu F_{\lambda\rho} - \tfrac{2}{3}gA_\nu A_\lambda A_\rho) \ . \tag{3.16}$$

The prefactor "12" comes from the 12 species of left-handed fermions and note the appearance of the CS Lagrangian on the RHS.

In four-dimensional Euclidean space there are classical solutions of the YM field equations which are localised in space and time. These are called instantons and the integer "instanton-number" is given by

$$n = \frac{g^2}{32\pi^2}\int \mathrm{d}\tau\mathrm{d}^3x\,\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} \ . \tag{3.17}$$

The first explicit example of such was found in [30, 283] for the case of $n = 1$. For this case, by integrating Eq. (3.16) between a "time" in the far past and one in the far future, the anomaly implies that the baryon + lepton number

$$B + L = \int \mathrm{d}^3\boldsymbol{x}\ J^0 \ , \tag{3.18}$$

will not be conserved but will change by

$$\Delta(B + L) = 12 \ . \tag{3.19}$$

The semi-classical amplitude for this process is $\exp(-8\pi^2/g^2)$ [319]. The connection to a process in Minkowsky space was made in [61, 202] by noting that the Euclidean path-integral calculation is just finding the tunnelling between gauge configurations of different winding number; the $n = 1$ solution of [30, 283] can be arranged to connect $A^\mu|_{\tilde{N}}$ with that of $A^\mu|_{\tilde{N}+1}$.

The amplitude $\exp(-8\pi^2/g^2)$ is very small in phenomenological theories. Moreover, the gauge field is just as likely to tunnel in one direction as the other since the energy of the gauge configurations is periodic in the winding number [202]. So on average there will be no more baryons created than anti-baryons. There are, however, ways to increase the amplitude, such as high temperatures or energies.

Another method of increasing the amplitude is to introduce a high density of fermions. Much work has centered around the possibility that the process of integrating-out the fermions in the theory at finite temperature with non-zero density of fermions μ induces a three-dimensional CS term into the effective action. Assuming the effective action is of the form

$$S_{\text{eff}} = S_{\text{YM}} + \mu S_{\text{CS}} \ , \tag{3.20}$$

this was studied as a model for baryogenesis in the early universe [291, 292] (some related articles are [4, 18, 226]), and the tunnelling amplitude as a function of μ was found [107, 219]. The basic idea is that the μS_{CS} term biases the the periodic vacuum structure of the YM term and with a large μ the tunnelling amplitude will be markedly increased. Also, with an appropriate choice for the sign of μ, more baryons or anti-baryons would be created depending on the theorists whim.

3.2 The claim and a counter-example

The starting point is the simplified standard model of the previous section at non-zero temperature. The effective action is

$$e^{-S_{eff}} = \int [\mathrm{d}\bar{\psi}_L \mathrm{d}\psi_L] \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x} \left(-\tfrac{1}{2}\mathrm{tr}F^2 + \sum_i \bar{\psi}_L^i i \not{D} \psi_L^i \right) \right] \ . \tag{3.21}$$

β is the inverse temperature and the space is Euclidean (see [116, 180, 208] for further related comments regarding finite temperature field theory). The gamma matrices are Hermitian and satisfy

$$[\gamma^\nu, \gamma^\lambda]_+ = 2\delta^{\nu\lambda} \ , \tag{3.22}$$

and $\gamma^5 = -\gamma_0\gamma_1\gamma_2\gamma_3$ is also Hermitian. The index ν shall still be taken to run from zero (imaginary time) to three; $\nu = 0, \ldots, 3$. A non-zero density of fermions is included by introducing a chemical potential μ for the particle-number charge Eq. (3.18). The Dirac operator is then

$$\not{D} = \not{\partial} + ig\not{A} + \mu\gamma^0 \ . \tag{3.23}$$

The claim of [258, 288, 294, 332] is that at high temperature

$$S_{\text{eff}} \overset{\beta\to 0}{\longrightarrow} \mu\beta \int_0^\beta \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x} \ \epsilon_{ijk}\mathrm{tr}\left(A_i F_{jk} - \tfrac{2}{3} g A_i A_j A_k \right) + \ldots \ . \tag{3.24}$$

The dots can, in principle, contain sub-leading terms in the expansion around $\beta = 0$ as well as terms such as the YM action (some more heuristic arguments for the existence the μS_{CS} term can be found in [107, 219, 291, 292]). At low temperatures, massive fermions are considered [288] and the effective action is expanded in powers of $1/m$;

$$S_{\text{eff}} \overset{\beta\to\infty}{\longrightarrow} \mu \int_0^\beta \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x} \ \epsilon_{ijk}\mathrm{tr}\left(A_i F_{jk} - \tfrac{2}{3} g A_i A_j A_k \right) + \ldots \ , \tag{3.25}$$

where now the dots also contain terms of $O(1/m)$ (further comments regarding the low temperature case are also made in [294]). It should also be noted that the calculations of [258, 259] have indicated that the situation is more subtle than previously expected.

That a *three*-dimensional CS term is induced into the action at high-temperature may seem a little odd initially. However, it is well-known [14, 225] that the high-temperature limit of a four-dimensional relativistic quantum field theory is effectively three dimensional. This is simply because taking the high-temperature limit in a boson propagator

$$\text{propagator} \sim \sum_{n=-\infty}^{\infty} \int \mathrm{d}^3\boldsymbol{p} \, \frac{1}{(2\pi T n)^2 + \boldsymbol{p}^2 + m^2} \,, \tag{3.26}$$

picks out the $n = 0$ term which reduces the theory to a three-dimensional one. (For fermions the situation is even more drastic — replacing the "n" by "$n + \frac{1}{2}$" shows that, naively, fermions decouple entirely.)

Furthermore, it seems extremely unlikely that the term would appear in S_{eff} simply because μ is *real* and so $e^{-\mu S_{CS}}$ is not gauge invariant! In fact, this argument is too naive, because — although the term is not gauge invariant by itself — it is still possible that the entire effective action may be invariant [93, 94, 135, 145, 146, 294]. Simple examples of this phenomenon will be presented later.

Since μ is real, only the real part of the effective action $\log\det \not{D}\not{D}^\dagger$ needs to be calculated in order to find the μS_{CS} term. The standard way [107, 288, 294] of obtaining this is to "vectorise" the model by adding $\bar{\psi}^i_R (i\not{D})^\dagger \psi^i_R$ to the action. Equivalently, six of the twelve left-handed doublets could be passed into six right-handed anti-particle doublets and then joined with the remaining left-handed fermions to yield a total of six Dirac spinors. This is nothing but the reverse of the procedure found in Sec. (3.1.2). The resultant action is

$$S = \int \bar{\psi} i \left(\not{\partial} + ig\not{A} + \mu\gamma^0\gamma^5 \right) \psi \,. \tag{3.27}$$

The chiral charge has been passed into a quasi-conserved axial charge, and purely for ease of notation, the number of Dirac fermions has been set to one rather than six. As expected, the operator $i\not{D}$ is now Hermitian.

The two-dimensional analogue of the action Eq. (3.27) provides a suitably simplified testing ground in which to check the claim. But, before describing this model, a calculation will be presented in four dimensions that produces no μS_{CS} term at low temperature. The results of this chapter have been published in [241].

In order to compare with Eq. (3.25) a gauge-invariant mass term $\bar{\psi} i m \psi$ is added to the Lagrangian. In perturbation theory, the coefficient of $\mu A^a_\nu A^a_\lambda$ in the effective action is

$$\Gamma^{\nu\lambda 0}(p, m, T) = g^2 \int_k \mathrm{tr}\gamma^\nu \Delta(k, m) i\gamma^0\gamma^5 \Delta(k, m)\gamma^\lambda \Delta(k + p, m) \,. \tag{3.28}$$

Here $\Delta(k, m)$ is the propagator of a Dirac fermion with mass m and the integral over momentum space is $\int_k = \beta^{-1} \sum_n \mathrm{d}^3\boldsymbol{k}$ for nonzero temperature. The corresponding graph is shown in Fig. 3.1. By expanding in powers of $1/m$, the form of the result is fixed purely by dimensional analysis ($\Gamma^{\lambda\delta 0}$ has dimensions of inverse-length) and symmetry requirements (the CS term contains ϵ_{ijk});

$$\Gamma^{\nu\lambda 0}(p, m, T\sim 0) = C\epsilon^{0\nu\lambda\rho} p_\rho + O(p^2/m) \,, \tag{3.29}$$

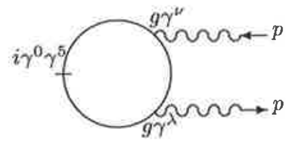

Figure 3.1 : The diagram with two gauge-field legs and one insertion of the operator $i\bar{\psi}\gamma^0\gamma^5\psi$. This is the coefficient of the $\mu A_\lambda A_\delta$ in the effective action.

where C is mass independent. Indeed, by contracting with $\mu A^a_\nu A^a_\lambda$, the CS term $\epsilon_{ijk}A_i\partial_j A_k$ appears. Some sort of regularisation is needed because the integral is linearly divergent. Pauli-Villars regularisation [265], in which a massive spinor χ, is added into the path integral

$$Z = \lim_{M\to\infty}\int[\mathrm{d}\bar{\psi}\mathrm{d}\psi\mathrm{d}\bar{\chi}\mathrm{d}\chi]e^{-S(\bar{\psi},\psi,A,m)+S(\bar{\chi},\chi,A,M)} , \tag{3.30}$$

is manifestly gauge invariant. Thereby

$$\Gamma^{\nu\lambda 0}_{\mathrm{PV}}(p,m,T\sim 0) \equiv \lim_{M\to\infty}\left[\Gamma^{\nu\lambda 0}(p,m,T\sim 0) - \Gamma^{\nu\lambda 0}(p,M,T\sim 0)\right] . \tag{3.31}$$

However, C is mass-independent, so

$$\Gamma^{\nu\lambda 0}_{\mathrm{PV}}(p,m,T\sim 0) = 0 + O(m^{-1}) , \tag{3.32}$$

which is in apparent contradiction to [258, 288, 294, 332].

In light of this result, and the subtlety of gauge invariance, it is clear that the problem needs more study. Fortunately, there is a two-dimensional analogue of this system in which calculations can be made more simply. This helps to clarify the situation and suggests that only when there is an infrared divergence (massless fermions at zero temperature) does the μS_{CS} term appear. In all other regimes the existence of the term is forbidden by choice of a gauge-invariant regulator.

3.3 The two-dimensional model

The two-dimensional equivalent of the vectorised theory of Eq. (3.27) has action

$$S = \int_{\mathcal{M}}\bar{\psi}i\not{D}\psi \quad \text{and} \quad \not{D} = \not{\partial} + m + \mu\gamma^0\gamma^5 + ie\not{A} . \tag{3.33}$$

$\mathcal{M}$ is a flat, Euclidean, 2D spacetime with coordinates (τ, x) where the τ direction is compact

$$0 \le \tau \le \beta . \tag{3.34}$$

The indices $(\nu, \lambda, \ldots)$ associated with spacetime run from zero (imaginary time) to one; $\nu = 0, 1$. All gamma matrices are traceless and Hermitian and satisfy

$$\left[\gamma^\nu, \gamma^\lambda\right]_+ = 2\delta^{\nu\lambda} \quad \text{and} \quad \gamma_5 = -i\gamma_0\gamma_1 . \tag{3.35}$$

A mass term has been included for generality at this point. It will serve to IR regulate the theory at zero temperature. The chemical potential μ for the axial charge $Q_5 = \int \bar{\psi}\gamma^0\gamma^5\psi$ is real. Finally,

$$Z[A,\mu,\bar{\eta},\eta] = \int [\mathrm{d}\bar{\psi}\mathrm{d}\psi]\, e^{-S+\int \bar{\eta}\psi+\bar{\psi}\eta}\ , \tag{3.36}$$

is the generating functional.

Recently, there has been some interest over the proper definition of Euclidean fermions. By studying the continuous rotation of spinors from Minkowsky to Euclidean space, van Nieuwenhuizen and Waldron [333, 335] have argued that $\bar{\psi} = \psi^\dagger\gamma^5$. In App. B it is shown that both Eq. (3.36) and

$$Z = \int [\mathrm{d}\psi^\dagger \mathrm{d}\psi] e^{-\int \psi^\dagger\gamma^5 \not{D}\psi}\ , \tag{3.37}$$

give a path-integral representation of the partition function $\mathrm{Tr}e^{-\beta H}$. The action in this latter form is SO(4) invariant and implies the correct equations of motion. Moreover, it has the advantage that the Dirac operator for massive fermions is Hermitian due to the extra γ^5 in the inner product. However, here the conventional form of the path integral (Eq. (3.36) with $\bar{\psi} = \psi^\dagger$) will be used.

It is only necessary to consider the U(1) theory. Under a gauge transformation, the fields vary as

$$\begin{aligned} A_\nu &\to A_\nu - ie^{-1}e^{i\theta}\partial_\nu e^{-i\theta} \\ \psi &\to e^{i\theta}\psi\ . \end{aligned} \tag{3.38}$$

When θ is well defined on $\mathcal{M}$ the transformation is called "small", while if only $e^{i\theta}$, but not θ itself, is well-defined the transformation is called "large". An example of a large gauge transformation is

$$\theta(x,\tau) = 2\pi\tilde{N}\tau/\beta\ , \quad \text{for} \quad \tilde{N} \in \mathbb{Z}\ . \tag{3.39}$$

This shifts A_0 by a constant

$$A_0 \to A_0 - 2\pi\tilde{N}/e\beta\ . \tag{3.40}$$

In this context, the CS term is given by Eq. (3.7), and

$$\mu S_{\mathrm{CS}} = \mu \int_{\mathcal{M}} A^1\ . \tag{3.41}$$

The next section presents some perturbative calculations which suggest that this term does not appear in the effective action. Then, in Sec. 3.5 the effective action will be studied nonperturbatively.

3.4 Perturbative calculations

Three perturbative calculations are described. It is found that no μS_{CS} term is induced into the effective action except for the case of massless fermions at zero temperature. This is attributed to an infrared divergence which makes the result ill-defined.

3.4.1 The one-point function defined by Pauli-Villars regularisation

Since μ is constant, it is efficient to put it into the propagator

$$\Delta(k) = \frac{-i}{i\not{k} + m + \mu\gamma^0\gamma^5} = \frac{-i}{i\not{k} + m - i\mu\gamma^1} . \tag{3.42}$$

The second equality holds in two dimensions because of the identity $\gamma^\nu\gamma^5 = -i\epsilon^{\nu\lambda}\gamma^\lambda$ and shows that a constant μ simply shifts the momentum in the loop. Expanding the path integral in powers of A, the coefficient of the linear term is the superficially linearly divergent one-point function

$$\Gamma^\nu(m,T,\mu) = \int_k \operatorname{tr} e\gamma^\nu \frac{\tilde{\not{k}} + im}{\tilde{k}^2 + m^2} \quad \text{where} \quad \tilde{k}_1 \equiv k_1 - \mu . \tag{3.43}$$

Pauli-Villars regularisation can be used to regulate this expression;

$$\Gamma^\nu_{\rm PV}(m) \equiv \lim_{M\to\infty} \left[\Gamma^\nu(m) - 2\Gamma^\nu(M) + \Gamma^\nu(2M-m)\right] . \tag{3.44}$$

Since the momentum integral is now finite all dependence on μ can be shifted away. As in the 4d case, by using Pauli-Villars, which is a gauge-invariant regularisation, no $\mu S_{\rm CS}$ term is induced into the effective action. The result is clearly independent of temperature.

It is well known that, without proper regularisation, the order of integration cannot be changed in divergent integrals. Γ^1 provides a simple example of this phenomena for it is quite possible to go further and explicitly evaluate the momentum-space integral (Γ^0 is zero by symmetric summation or integration). The mass term in the numerator of Eq. (3.43) gets killed by $\operatorname{tr}\gamma^\mu = 0$. Performing the k^1 integral first gives

$$\begin{aligned}
\Gamma^1(m,T) &= e\int_{k_0}\int_{-\Lambda-\mu}^{\Lambda-\mu} d\tilde{k}_1 \frac{\tilde{k}_1}{\tilde{k}_1^2 + m^2 + k_0^2} \\
&= e\int_{k_0} \log\left((\Lambda-\mu)^2 + m^2 + k_0^2\right) - \log\left((\Lambda+\mu)^2 + m^2 + k_0^2\right) \\
&\xrightarrow{\Lambda\to\infty} \int_{k_0} 0 = 0 .
\end{aligned} \tag{3.45}$$

However, performing the k_0 summation first yields

$$\begin{aligned}
\Gamma^1(m,T) &= \beta^2 e \int dk_1 \frac{\tilde{k}_1}{\beta\sqrt{\tilde{k}_1^2 + m^2}} \pi\tanh\left(\pi\beta\sqrt{\tilde{k}_1^2 + m^2}\right) \\
&= 2e\pi\mu .
\end{aligned} \tag{3.46}$$

The same result is obtained at zero temperature. Of course all answers are mass independent so Pauli-Villars regularisation yields zero.

3.4.2 The "Adler" argument

An alternative treatment is to expand the path integral in powers of both μ and A rather than putting μ in the propagator. The correlation-function of interest is the superficially logarithmically divergent two-point function

$$\Gamma^{\nu 0}(m,T) = -\int_k \mathrm{tr}\frac{m-i\not{k}}{k^2+m^2}i\gamma^0\gamma^5\frac{m-i\not{k}}{k^2+m^2}e\gamma^\nu \ . \tag{3.47}$$

This method has the advantage that μ can be promoted to be the zeroth component of a chiral gauge field which is non-constant in spacetime. The momentum p, flowing into the associated Feynman diagram shown in Fig. 3.2 will then be nonzero, and only after calculating will the limit $p^\nu \to 0$ be taken.

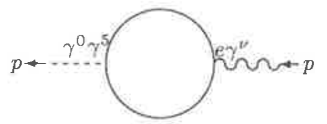

Figure 3.2 : The diagram with one gauge-field leg and one leg that corresponds to the insertion of $i\mu(x,\tau)\gamma^0\gamma^5$. The momentum flowing into the diagram is nonzero.

With nonzero p^ν, the form of the correlator is completely fixed [3] by gauge invariance, parity, Lorentz covariance and the anomaly. Under parity

$$\psi \to \gamma^0\psi \quad \text{and} \quad \bar{\psi} \to \bar{\psi}\gamma^0 \ , \tag{3.48}$$

so the term containing the new chiral gauge field $\bar{\psi}iB_\lambda\gamma^\lambda\gamma^5\psi$ is parity invariant only if B^λ is a pseudo vector. Therefore

$$\Gamma^{\nu\lambda}(p,m,T) = -\int_k \mathrm{tr}\frac{m-i(\not{k}-\not{p})}{(k-p)^2+m^2}i\gamma^\lambda\gamma^5\frac{m-i\not{k}}{k^2+m^2}e\gamma^\nu \ , \tag{3.49}$$

has one vector index and one pseudo-vector index. At zero temperature the most general expression with the correct Lorentz structure is therefore

$$\Gamma^{\nu\lambda}(p,m,T\!=\!0) = Y(p^2,m^2)\epsilon^{\nu\lambda} + Z(p^2,m^2)p_\rho\epsilon^{\rho(\nu}p^{\lambda)} \ . \tag{3.50}$$

The parentheses indicate symmeterisation $(\nu\lambda) = \frac{1}{2}\nu\lambda + \frac{1}{2}\lambda\nu$. Gauge invariance implies

$$p_\nu\Gamma^{\nu\lambda} = 0 \quad \Rightarrow \quad p_1\Gamma^{10} = -p_0\Gamma^{00} \quad \Rightarrow \quad Y = -\tfrac{1}{2}p^2 Z \ , \tag{3.51}$$

whence

$$\Gamma^{\nu\lambda}(p,m,T\!=\!0) = (-\tfrac{1}{2}p^2\epsilon^{\nu\lambda} + p_\rho\epsilon^{\rho(\nu}p^{\lambda)})Z \ . \tag{3.52}$$

Z has dimensions of p^{-2} so corresponds to a convergent Feynman integral.

Z can be determined by using the well-known result for the chiral anomaly [199, 267, 351]

$$-\epsilon^{\nu\lambda}p_\lambda p^2 Z = p_\lambda\Gamma^{\nu\lambda}(p,m,T\!=\!0) = -2e\pi\epsilon^{\nu\lambda}p_\lambda\left(1-\frac{4m^2}{\sqrt{p^2(p^2+4m^2)}}\mathrm{Arth}\frac{p^2}{\sqrt{p^2(p^2+4m^2)}}\right) \ . \tag{3.53}$$

Here $\mathrm{Arth}(x)$ is the inverse hyperbolic tangent, the expansion of which is $\mathrm{Arth}(x) = x + O(x^3)$ for small x. For vanishing momentum, the massive case therefore tends to zero

$$p^2 Z(p^2, m\!\neq\!0) \overset{p\to 0}{\longrightarrow} 0 \ . \tag{3.54}$$

This, coupled with the dimension of Z implies $Z \propto m^{-2} + O(p^2)$, and so

$$\Gamma^{\nu\lambda}(p=0, m\neq 0, T=0) = 0 \; . \tag{3.55}$$

On the other hand, the massless case has

$$p^2 Z(p^2, m=0) = 2e\pi \; . \tag{3.56}$$

which implies

$$\Gamma^{10}(p, m=0, T=0) = \frac{2e\pi p_0^2}{p_0^2 + p_1^2} \; . \tag{3.57}$$

Interestingly, this is ambiguous in the zero-momentum limit

$$\Gamma^{10}(m=0, T=0) \to \begin{cases} 0 & p_0 \to 0 \text{ then } p_1 \to 0 \\ 2e\pi & p_1 \to 0 \text{ then } p_0 \to 0 \; . \end{cases} \tag{3.58}$$

This is attributed to the IR divergence contained in the two-point function of Eq. (3.47) for $m = 0 = T$.

The IR problem for massless fermions is alleviated by heating the system to a nonzero temperature. Because the heat bath breaks Lorentz invariance, the general form of $\Gamma^{\nu\lambda}$ is much more complicated as it can depend upon the normal vector in the p^0 direction, n^ν. It is

$$\Gamma^{\nu\lambda} = A\epsilon^{\mu\nu} + Bp_\rho\epsilon^{\rho(\nu}p^{\lambda)} + Cn_\rho\epsilon^{\rho(\nu}p^{\lambda)} + Dp_\rho\epsilon^{\rho(\nu}n^{\lambda)} + En_\rho\epsilon^{\rho(\nu}n^{\lambda)} \; . \tag{3.59}$$

However, the problem at infinitesimal p^ν is simple because it is fixed by gauge invariance. This reads, as in Eq. (3.51),

$$p_1\Gamma^{10} = -p_0\Gamma^{00} \; . \tag{3.60}$$

Consider Γ^{00}; it is given by the expression

$$\begin{aligned} \Gamma^{00}(p, m=0, T) &= ie\int_k \mathrm{tr}\frac{(\not k - \not p)\gamma^0\gamma^5\not k\gamma^0}{(k-p)^2k^2} \\ &= -2e\int_k \frac{k_0(p_1-k_1) + k_1(p_0-k_0)}{(k-p)^2k^2} \; , \end{aligned} \tag{3.61}$$

where the trace over the gamma matrices, $\mathrm{tr}\gamma^\nu\gamma^\lambda\gamma^5 = 2i\epsilon^{\nu\lambda}$, has been taken. Assuming symmetric summation, $\Gamma^{00} = 0$ if $p_0 = 0$;

$$\Gamma^{00}(p_0=0, T) = -2e\sum_{k_0}\int \mathrm{d}k_1 \frac{k_0(p_1 - 2k_1)}{(k+p)^2k^2} = \sum_{k_0} k_0 f(k_0^2, p_1) = 0 \; . \tag{3.62}$$

There are no IR problems since k_0 is quantised. Therefore[1] $p_1\Gamma^{10}(p,T) = p_0(\ldots)$ where the dots do not diverge as $p_0 \to 0$. Now, since p_0 is quantised, it cannot be taken to zero smoothly. The philosophy adopted here is that the physical result is obtained by *first* setting p_0 to zero and *then* taking the limit $p_1 \to 0$. Thus, at finite temperature $\Gamma^{10}(p=0, T\neq 0) = 0$, and there is no μS_{CS} term.

[1] Although it is not needed for the argument here, Γ^{01} can be calculated exactly [281] and is again given by Eq. (3.57).

3.4.3 Pauli-Villars and the two-point function

The IR problem also occurs when naively applying Pauli-Villars regularisation to the two-point function at zero temperature. The two-point function with zero external momentum is defined to be

$$\Gamma_{\rm PV}^{\nu\lambda} \equiv \lim_{M\to 0} -ie\int \mathrm{d}^2k\left(\frac{m-i\not k}{k^2+m^2}\gamma^\lambda\gamma^5\frac{m-i\not k}{k^2+m^2}\gamma^\nu - \frac{M-i\not k}{k^2+M^2}\gamma^\lambda\gamma^5\frac{M-i\not k}{k^2+M^2}\gamma^\nu\right) . \qquad (3.63)$$

Cross-multiplying to remove the divergence and taking the trace $\mathrm{tr}\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^5 = 0$ yields

$$\begin{aligned}\Gamma_{\rm PV}^{\nu\lambda} &= \lim_{M\to 0} ie\int \mathrm{d}^2k\left(\frac{M^2}{(k^2+M^2)^2} - \frac{m^2}{(k^2+m^2)^2}\right)\mathrm{tr}\gamma^\lambda\gamma^5\gamma^\nu \\ &\quad + \lim_{M\to 0} ie\int \mathrm{d}^2k\frac{2k^2m^2+m^4-2k^2M^2-M^4}{(k^2+m^2)^2(k^2+M^2)^2}\mathrm{tr}\not k\gamma^\lambda\not k\gamma^5\gamma^\nu . \end{aligned} \qquad (3.64)$$

For $m=0$ there is an IR divergence. Forgetting this for the moment, and using the integral

$$\int \mathrm{d}^2k\, k_\alpha k_\beta f(k^2) \propto \delta_{\alpha\beta} , \qquad (3.65)$$

the traces may be evaluated

$$\mathrm{tr}\gamma^\nu\gamma^\lambda\gamma^5 = 2i\epsilon^{\nu\lambda} \quad \text{and} \quad \mathrm{tr}\gamma^\alpha\gamma^\lambda\gamma^\alpha\gamma^\nu\gamma^5 = 0 , \qquad (3.66)$$

to yield

$$\Gamma_{\rm PV}^{\nu\lambda} = \lim_{M\to 0} 2e\epsilon^{\nu\lambda}\int \mathrm{d}^2k\left(\frac{m^2}{(k^2+m^2)^2} - \frac{M^2}{(k^2+M^2)^2}\right) . \qquad (3.67)$$

Each integral is clearly independent of the mass and using $\int (k^2+1)^2 = \pi$, this implies, in contradiction to the null result obtained using the one-point function,

$$\Gamma_{\rm PV}^{10}(m, T\!=\!0) = \begin{cases} 0 & m \neq 0 \\ 2e\pi & m = 0 . \end{cases} \qquad (3.68)$$

However, the non-zero result occurs only because the IR divergence has made the result somewhat arbitrary. In this situation a natural prescription might be to define the massless theory as the limit of the massive one:

$$\Gamma_{\rm PV}^{10}(m, T\!=\!0) = 0 \qquad \forall m . \qquad (3.69)$$

In conclusion, by demanding that the calculation be gauge invariance, no $\mu S_{\rm CS}$ term is induced into the effective action, except at zero temperature for massless fermions. The appearance of the term in this regime is due to an IR divergence which makes the calculation ill-defined.

3.5 Nonperturbative results

The partition function can also be calculated directly to all orders in μ by functional methods. To make the eigenvalue problem well-defined $\mathcal{M}$ is chosen to be the torus with $0 \le \tau \le \beta$ and

$0 \leq x \leq R$. Make the Hodge decomposition on the background gauge field

$$eA_\nu = \partial_\nu \sigma + \epsilon_{\nu\lambda}\partial_\lambda \rho + h_\nu \ . \tag{3.70}$$

The fields σ and ρ are well defined on $\mathcal{M}$ and h_ν is constant.

If non-trivial winding sectors were of interest then the decomposition would take the form

$$eA_\nu = \partial_\nu \sigma + \epsilon_{\nu\lambda}\partial_\lambda \rho + h_\nu + \tilde{A}_\nu \ , \tag{3.71}$$

where $\tilde{A}^\mu$ can be chosen to be

$$\tilde{A}_0 = -\frac{2\pi\tilde{N}}{\beta R}x \quad \text{and} \quad \tilde{A}_1 = 0 \ . \tag{3.72}$$

$\tilde{N}$ is the winding number of the configuration and this choice is possible because the difference

$$A_0(\tau,x) - A_0(\tau,x+R) = \frac{1}{e}\partial_0\left(\frac{2\pi\tilde{N}}{\beta}\tau\right) \ , \tag{3.73}$$

is zero up to the large gauge transformation Eq. (3.39). In this case, the fermions would have to obey twisted boundary conditions

$$\psi(\tau,x+R) = e^{2\pi i\tilde{N}\tau/\beta}\psi(\tau,x) \quad \text{and} \quad \psi(\tau+\beta,x) = -\psi(\tau,x) \ , \tag{3.74}$$

where, once again, the large gauge transformation is evident. However, in order to find whether it is possible that the μS_{CS} term

$$\mu\int_{\mathcal{M}} A_1 = \mu R\beta h_1/e \ , \tag{3.75}$$

appears in the effective action it is only necessary to consider $\tilde{N} = 0$.

The toy model differs from the Schwinger model [302] on the torus only by the μ term. However, by using the identity $\gamma^0\gamma^5 = -i\gamma^1$ the μ can be shifted into h_1. The form of the generating functional is well known [49, 150, 151, 152, 185, 201, 289, 303] and details of the derivation can be found in App. C

$$Z[A,\bar{\eta},\eta] = \exp\left(\int \bar{\eta}e^{-i\sigma-\gamma^5\rho}\Delta_1 e^{i\sigma-\gamma^5\rho}\eta + \frac{1}{\pi}\int \rho\Box\rho\right)\det i\not{D}_1 \ . \tag{3.76}$$

Here $\not{D}_1 = \not{\partial} + i\not{h} - i\mu\gamma^1$ and has associated propagator $i\Delta_1$. The determinant of this operator can be calculated using zeta-function regularisation [185, 303] and the result written in terms of a theta function and Dedekind's eta function [9, 47, 295, 297]

$$\begin{aligned} \det i\not{D}_1 &= \left|\frac{1}{\eta(iR/\beta)}\Theta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0,iR/\beta)\right|^2 \\ &\equiv \left|q^{1/24}\prod_{m=1}^{\infty}(1-q^m)\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\theta)^2}e^{2\pi i(n+\theta)\phi}\right|^2 \ . \end{aligned} \tag{3.77}$$

In this formula $\theta = -\beta h^0/2\pi$ and $\phi = \frac{1}{2} + \frac{R(h^1-\mu)}{2\pi}$ and the parameter $q = e^{-2\pi R/\beta}$.

The effective action when the gauge field is in a nontrivial winding sector is also well known [108, 109, 153, 295, 297]. A non-zero chemical potential for the conserved *electric* charge has also been considered [8, 296]. In this case the Dirac operator is no longer Hermitian and the phase in the partition function leads to interesting results.

The partition function is clearly invariant under small gauge transformations since $e^{i\sigma}\eta$ and its conjugate are invariant. It is also invariant under large gauge transformations in the x and τ directions

$$\begin{aligned} x \text{ direction:} \quad & \delta h^1 = \tfrac{2\pi \tilde{N}}{R} \quad \text{and} \quad \bar{\eta} \to \bar{\eta} e^{2\pi i \tilde{N} x/R} \ , \\ \tau \text{ direction:} \quad & \delta h^0 = \tfrac{2\pi \tilde{N}}{\beta} \quad \text{and} \quad \bar{\eta} \to \bar{\eta} e^{2\pi i \tilde{N} \tau/\beta} \ . \end{aligned} \tag{3.78}$$

The first transformation changes the summand in Eq. (3.77) by a phase which is then canceled by the mod-squared. The second transformation can be soaked up by relabeling the index of summation.

The cylindrical limit of the generating functional must now be taken. It is possible that there are some subtleties associated with this limit. In order to look for these, a brief sortie is made into the one-dimensional world where all expressions are particularly simple.

3.5.1 The determinant of the Dirac operator on a one-dimensional manifold

Because of its simplicity, the one-dimensional case has been well studied [80, 93, 94, 135]. Here though, the emphasis is placed on finding any nontrivialities in the circle → line limit. Although this subsection is relevant to the exposition, it is hoped that the reader will not forget that its aim is simply to illustrate that the continuum limit of the 2d case contains no subtleties.

Start with the operator

$$D = i\partial + eA(t) + iM \ , \tag{3.79}$$

where $-\pi R \le t \le \pi R$. A mass term iM has been included for generality, and it will serve to IR regulate the theory. On the circle the eigenvectors are

$$\psi_\lambda = \exp\left[i\left(\lambda t - e\int^t A \right) - Mt \right] \ . \tag{3.80}$$

The boundary conditions then imply $\lambda_n = \mathcal{A} + (n/R)$ where

$$\mathcal{A} \equiv \begin{cases} \frac{e}{2\pi R}\int A - iM & \text{periodic} \\ \frac{1}{2R} + \frac{e}{2\pi R}\int A - iM & \text{antiperiodic} \ . \end{cases} \tag{3.81}$$

If $M \neq 0$ there are no zeromodes, however, if $M = 0$ there is a possibility of one zeromode depending on the value of $\int A$.

The product of eigenvalues needs regularisation. A non-gauge-invariant way to proceed is to calculate det $D(i\partial + iM)^{-1}$. This leads to a sine in the periodic case and a cosine for antiperiodic boundary conditions [135]. An alternative is zeta-function regularisation [185, 303][2] where the

[2] A nice review for the case of the Dirac operator can be found in [94].

following formal manipulations are made

$$\det D = \exp\log\prod_n \lambda_n = \exp\sum_n \log\lambda_n = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\sum_n \lambda_n^{-s}\Big|_{s=0}\right) , \tag{3.82}$$

and the zeta function $\zeta_\lambda(s) = \sum_n \lambda_n^{-s}$ is calculated by analytic continuation. For instance [170, Sec. 9.53]

$$\begin{aligned} \sum_{n=0}^{\infty}(n+x)^{-s}\Big|_{s=0} &= \tfrac{1}{2} - x , \\ \frac{\mathrm{d}}{\mathrm{d}s}\sum_{n=0}^{\infty}(n+x)^{-s}\Big|_{s=0} &= \log\Gamma(x) - \tfrac{1}{2}\log 2\pi . \end{aligned} \tag{3.83}$$

For the case in hand

$$\begin{aligned} \det D &= \exp\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(-(\mathcal{A})^{-s} + \sum_{n=0}^{\infty}(\tfrac{n}{R}+\mathcal{A})^{-s} + \sum_{n=0}^{\infty}(-\tfrac{n}{R}+\mathcal{A})^{-s}\right)\right]_{s=0} \\ &= 2\pi e^{-\pi i \mathcal{A}R} e^{-\pi i/2} \frac{1}{\mathcal{A}R\Gamma(\mathcal{A}R)\Gamma(-\mathcal{A}R)} \\ &= 1 - e^{-2\pi i \mathcal{A}R} , \end{aligned} \tag{3.84}$$

which is invariant under the large gauge transformations $A \to A - 2\pi\tilde{N}/eR$.

Consider expanding the effective action $S_{\text{eff}} = -\log\det D$ in powers of A. To quadratic order

$$S_{\text{eff}} = -\log\left(1 \pm e^{-2\pi RM}\right) \pm \frac{e^{-2\pi RM}}{1 \pm e^{-2\pi RM}} ie\int A \mp \frac{e^{-2\pi RM}}{(1 \pm e^{-2\pm RM})^2}\tfrac{1}{2}e^2\int A\int A + O(A^3) , \tag{3.85}$$

where the upper (lower) sign corresponds to the antiperiodic (periodic) case. There are three important things to notice:

- Written this way, the ζ-function regularisation looks like a non-local result. This is because the eigenvalues depend only $\int A$. However, if it is only the constant part of A that matters, all non-local products can be made to look local

$$\left(\int A\right)^n = (2\pi R)^{n-1}\int A^n|_{A=\text{constant}} . \tag{3.86}$$

- Despite the effective action being gauge invariant as a whole, each individual term in its expansion in small A is not.

- The expansion in small A is ill-defined for the periodic massless case. This is because for $A = 0$ there is a zeromode which must be removed

$$\det{}'_{\text{periodic}} D = \frac{1 - e^{-ie\int A}}{ie\int A} . \tag{3.87}$$

  The same problem crops up in perturbation theory, where there are IR divergent terms such as $\sum_n \frac{1}{n}$.

Out of interest, the derivative expansion, which is local, will be compared with this result. To compute the expansion, the heat-kernel method is used. This has the disadvantage that only the real part of the effective action, $\log\det DD^\dagger$, can be calculated, because the heat kernel is then quadratic in derivatives. Defining $D_0 = i\partial + A$, and expanding to quadratic order in D_0 with antiperiodic boundary conditions gives

$$\begin{aligned}
\log\det DD^\dagger &= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon}\mathrm{Tr}e^{-\epsilon DD^\dagger} \\
&= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon} e^{-\epsilon M^2}\frac{1}{2\pi}\int \mathrm{d}t \sum_n e^{-(2n+1)it/R} e^{-\epsilon D_0^2} e^{(2n+1)it/R} \\
&= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon} e^{-\epsilon M^2}\frac{1}{2\pi}\int \mathrm{d}t \sum_n e^{-\epsilon\left(\frac{2n+1}{2R} - D_0\right)^2} \\
&= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon} e^{-\epsilon M^2}\frac{1}{2\pi}\int \mathrm{d}t \sum_n e^{-\epsilon(2n+1)^2/4R^2} \\
&\qquad \times \left\{1 - \epsilon D_0^2 + 2\epsilon^2\frac{(2n+1)^2}{4R^2}D_0^2 + O(D_0^4)\right\}
\end{aligned} \tag{3.88}$$

The first term in curly parentheses is an unimportant divergent constant. Dropping this, performing the integral over ϵ, and using the representation for the hyperbolic tangent

$$\tanh\frac{\pi x}{2} = \frac{2x}{\pi}\sum_n \frac{1}{(2n+1)^2 + x^2}\,, \tag{3.89}$$

the real part of the effective action to $O(D_0^2)$ can be deduced

$$\begin{aligned}
2\Re S_{\text{eff}} &= -\frac{2R}{\pi}\int D_0^2\left\{\left(1 + 2MR\frac{\mathrm{d}}{\mathrm{d}(2MR)}\right)\sum_n \frac{1}{(2n+1)^2 + (2MR)^2}\right\} \\
&= -\frac{2\pi Re^2}{(e^{\pi MR} + e^{-\pi MR})^2}\int A^2 + O(D_0^4)\,.
\end{aligned} \tag{3.90}$$

Only for A being constant does this agree with the zeta-function result. (ζ-function regularisation does, however, have the advantage that it gives a prescription for the imaginary part of S_{eff}.) With periodic boundary conditions, it is clear from the first line of Eq. (3.90) that the massless case is ill-defined because of the zero mode mentioned previously.

Taking the limit to the line of the zeta-function results gives (mod$2\pi i$)

$$S_{\text{eff}} \longrightarrow \begin{cases}
2\pi RM + ie\int A & M < 0 \text{ (antiperiodic)} \\
2\pi RM + ie\int A + \pi i & M < 0 \text{ (periodic)} \\
-\log\left(1 + e^{-ie\int A}\right) & M = 0 \text{ (antiperiodic)} \\
-\log\left(1 - e^{-ie\int A}\right) + \log\left(ie\int A\right) & M = 0 \text{ (periodic)} \\
0 & M > 0\,.
\end{cases} \tag{3.91}$$

The M-dependent normalisation of $e^{-S_{\text{eff}}}$ is physically unimportant. If the limit of the massless periodic case had been taken without first removing the zeromode the effective action would not have had an expansion in small A; it is only when the compact theory is properly IR regulated that the noncompact effective action can be properly defined. In the two-dimensional model, antiperiodicity of the fermions in the time direction at nonzero temperature will provide the necessary IR regulator.

Eq. (3.91) can be compared with the expression obtained from det $D(i\partial+iM)^{-1}$. The Green's function for $i\partial + iM$ with $M \neq 0$ is

$$\begin{aligned} G(x-y) &= \int \frac{\mathrm{d}k}{2\pi} \frac{e^{ik(x-y)}}{-k+iM} \\ &= \begin{cases} ie^{-M(x-y)}\left[\theta(M)\theta(x-y) - \theta(-M)\theta(y-x)\right] & \text{for } x-y \neq 0 \\ \frac{1}{2} i \,\mathrm{sgn} M & \text{for } x-y=0\,. \end{cases} \end{aligned} \tag{3.92}$$

where θ is a step function. Expanding the effective action in powers of A, the step functions kill all terms but the linear one, resulting in

$$\begin{aligned} S_{\text{eff}} &= -\mathrm{Tr}\log(1+eAG) \\ &= -\int \mathrm{d}x \left\{ A(x)G(0) + \tfrac{1}{2}\int \mathrm{d}x' A(x)G(x-x')A(x')G(x'-x) + \ldots \right\} \\ &= -\tfrac{1}{2} i \,\mathrm{sgn} M \int_{-\infty}^{\infty} \mathrm{d}x A(x)\,. \end{aligned} \tag{3.93}$$

Because there are no large gauge transformations on the line this is gauge invariant. It it differs from the zeta function result of $-i\theta(-M)\int A$. This is an example of the well-known principle that the imaginary part of the effective action can be defined in many ways (see [19] for a review).

Finally, the continuum limit of the derivative expansion Eq. (3.90) can be easily taken. For nonzero mass the answer is clearly zero. In the massless case,

$$2\Re S_{\text{eff}} = -\tfrac{1}{2}\pi R e^2 \int A^2\,, \tag{3.94}$$

which of course agrees with zeta function result if AR is held constant as $R \to \infty$. This is surely the most physically sensible prescription since it allows for a non-zero constant mode for the gauge field.

Before closing this subsection, there is one more interesting facet of the derivative expansion which can be mentioned: If the heat-kernel analysis of Eq. (3.88) is performed on the line the result is independent of A regardless of the value of M. For

$$\begin{aligned} \log\det DD^\dagger &= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon} e^{-\epsilon M^2} \int \frac{\mathrm{d}k}{2\pi}\, e^{ikx} e^{-\epsilon(-\partial^2+2iA\partial+(i\partial A+A^2))} e^{ikx} \\ &= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon}\frac{1}{\sqrt{\epsilon}} e^{-\epsilon M^2} \int \frac{\mathrm{d}k}{2\pi}\, e^{-k^2} e^{-2\sqrt{\epsilon}kD_0 - \epsilon D_0 D_0} \\ &= \int_0^\infty \frac{\mathrm{d}\epsilon}{\epsilon}\frac{1}{\sqrt{4\pi\epsilon}} e^{-\epsilon M^2}\,, \end{aligned}$$

where the last line follows[3] by expanding the exponential in powers of ϵ. This is presumably

[3] It also should be noted that according to [50], it seems that the result in the continuum should be the same as that on the circle, up to a multiplicative constant. However, a term was dropped in their calculation (the last term in their Eq. (3.15)) which is nonzero at nonzero ϵ. Therefore, their conclusions do not hold at large ϵ. This changes the last conclusion in the appendix of [241].

another manifestation of the point that the continuum limit is ill-defined when there are zero-modes.

3.5.2 The cylindrical, or high-temperature, limit

The two-dimensional model shares many of the features of the one-dimensional case. In particular, the determinant (3.77) of $i\not{D}_1$ obtained by zeta-function regularisation is nonlocal in the gauge field. Moreover, each term in the expansion of the effective action $S_{\text{eff}} = \log\det i\not{D}_1$ in powers of $h^\lambda = \frac{e}{R\beta}\int A^\lambda$ is not gauge invariant despite S_{eff} being gauge invariant as a whole. However, there is one subtlety that the two-dimensional model does not have — in the process of taking the continuum limit in 1d, the zero-mode had to be handled with care. In the 2D model there are no IR problems because the fermions are antiperiodic along the time direction and therefore its continuum limit is trivial.

For example, at large R (the limit to the cylinder) or small β (high temperature), the parameter q is exponentially small. Setting $\theta = 0$ in order to simplify the formulae the effective action can be expanded in powers of the gauge field (h^λ is held constant as per the prescription given on p. 77)

$$\begin{aligned} S_{\text{eff}} &= -\log\det i\not{D}_1 + \ldots \\ &= -\sqrt{q}\left(e^{iR(h_1-\mu)} + e^{-iR(h_1-\mu)}\right) + \ldots \\ &= 2e\sin(R\mu)e^{-\pi R/\beta}\frac{1}{\beta}\int A^1 + \ldots . \end{aligned} \tag{3.95}$$

Here, the μS_{CS} term has been extracted. Of course, the term by itself is not gauge invariant. Taking the high temperature limit it is evident that there is no μS_{CS} term of the form Eq. (3.24) according to zeta-function regularisation.

3.6 Conclusions

A variety of calculations in a 2d toy model have been performed in order to investigate the claim [258, 288, 294, 332] that the effective action of four-dimensional SU$(2)_L$ gauge theory at high and low temperature contains a three dimensional Chern-Simons term whose coefficient is the chemical potential for baryon number.

Because the chemical potential is real, the contentious term is not gauge invariant by itself. As has been shown though, this does not rule out its appearance in the effective action. In both one and two (and presumably four) dimensions, it was demonstrated that extra non-local terms can restore the gauge invariance.

Nevertheless, all the perturbative calculations performed at nonzero temperature gave no such term. The only regime were it could possibly be induced was at zero temperature with massless fermions. This was attributed to an ambiguity brought about through an infrared divergence and was investigated from a number of different angles.

The nonperturbative effective action does contain the term but it is exponentially suppressed as the radius of the spacelike circle tends to infinity. This is quite different from the proposed behaviour since there the spacelike circle is infinite from the very start.

How then, did the authors of [288] obtain a nonzero result? The regularisation scheme was to subtract off the zero-temperature, zero-μ result. The same calculation can be performed exactly in 2d. The one-point function of Eq. (3.43) can be written in the form

$$\Gamma^1(m,T,\mu) = \int \mathrm{d}k_1 \oint_C \frac{\mathrm{d}z}{2\pi i} \left(\frac{k_1-\mu}{-z^2+(k_1-\mu)^2+m^2} \right) \tanh \tfrac{1}{2}\beta z \ , \tag{3.96}$$

where the contour of integration is shown in Fig. 3.3(a). Using partial fractions, expressing tanh

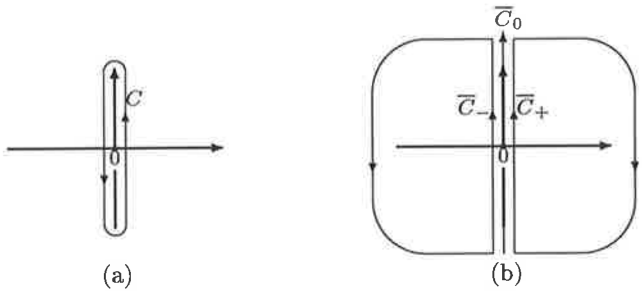

Figure 3.3 : Contours of integration in the z-plane. (a) The contour C encircles the imaginary axis, and (b) contour $\overline{C}_0$ passes up the imaginary axis and $\overline{C}_+$ ($\overline{C}_-$) encircles the RHS (LHS) of the plane.

in terms of exponentials and completing the contours $\overline{C}_\pm$ leads to

$$\begin{aligned}\Gamma^1(m,T,\mu) \ = \ & \int \mathrm{d}k_1 \frac{k_1-\mu}{\omega} \left[-\oint_{\overline{C}_+} \frac{\mathrm{d}z}{2\pi i} \left(\frac{1}{z+w} - \frac{1}{z-w} \right) \frac{1}{1+e^{\beta z}} \right. \\ & \left. -\oint_{\overline{C}_-} \frac{\mathrm{d}z}{2\pi i} \left(\frac{1}{z+w} - \frac{1}{z-w} \right) \frac{1}{1+e^{-\beta z}} + \int_{\overline{C}_0} \frac{\mathrm{d}z}{2\pi i} \left(\frac{1}{z+w} - \frac{1}{z-w} \right) \right] ,\end{aligned} \tag{3.97}$$

where $\omega = \sqrt{(k_1-\mu)^2+m^2}$ and the various contours are shown in Fig. 3.3(b). The first two give a μ-independent result for finite temperature since they are convergent and the result is

$$\Gamma^1(m,T,\mu) = 2e\pi\mu + \Gamma^1(m,0,0) \ . \tag{3.98}$$

Thus, by following [288] a μS_{CS} term is obtained. This is in contrast to Pauli-Villars regularisation which gave no μS_{CS} term

This regularisation scheme might be justified by casting it into a Pauli-Villars-like form

$$Z = \lim_{M\to\infty} \int [\mathrm{d}\bar\psi \mathrm{d}\psi \mathrm{d}\bar\chi \mathrm{d}\chi] \exp\left[-S(\bar\psi,\psi,A,m,T,\mu) + S(\bar\chi,\chi,A,M,T=0,\mu=0) \right] \ . \tag{3.99}$$

In the second action the spinor fields χ are defined over the plane. The gauge field must be the same in both actions. Presumably it is extended periodically to the plane in the second action. The second action also has no axial charge. A standard argument shows that there are no new divergences introduced by insertions of the charge of a conserved current. In the present case Q_5 is the charge of an anomalous current so this argument must be re-examined. Clearly it is somewhat uncertain as to whether this scheme can be implemented as a gauge-invariant regularisation to all orders in perturbation theory. In contrast, the regularisation schemes used in this chapter are gauge invariant and implementable to all orders.

Thus it seems that the Chern-Simons term whose coefficient is the chemical potential for baryon number is not induced into the effective action. However, this does not necessarily mean that the periodic structure of the gauge-field vacuum cannot be biased by some nonperturbative effect. The theory with nontrivial winding sectors has been well-studied [108, 109, 153, 295, 297]. It would therefore be quite possible and interesting to calculate matrix elements corresponding to the 2d analogue of baryogenesis in the early universe.

Matrix Theory

M-theory is currently regarded as the best candidate for a "theory of everything". When it is compactified along a light-like direction M-theory has a realisation as a supersymmetric quantum mechanics model with U(N) symmetry called "Matrix theory". This is described in Sec. 4.3. Before detailing Matrix theory some introductory material concerning superstrings (Sec. 4.1) and the various background solutions of their low energy dynamics called p-branes (Sec. 4.2) is given.

4.1 A lightning tour of the superstring

Most of the results presented here are arrived at by making analogies with the bosonic string studied in Chapter 1. The discussion is necessarily very streamlined and many more interesting facets of superstrings can be found in the text [177] and presumably in the forthcoming monograph by Polchinski [277].

4.1.1 The NSR action and worldsheet supersymmetry

In order to include space-time fermions into string theory, extensions of the bosonic string described in Chapter 1 must be considered. One such is to add to the Polyakov action of Eq. (1.2) d 2-component Majorana spinor fields ψ^μ which transform in the vector representation of the Lorentz group SO($d-1$,1)

$$S = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} \left(\partial_a X^\mu \partial_a X_\mu - i\bar{\psi}^\mu \gamma^a \partial_a \psi_\mu\right) \ . \tag{4.1}$$

Here γ^a are two-dimensional gamma matrices $\{\gamma^a, \gamma^b\} = -2\eta^{ab} = -2\,\mathrm{diag}\,(-1,1)$ and in the Majorana representation $\bar{\psi} = \psi^\dagger\gamma^0 = \psi^{\mathrm{T}}\gamma^0$. In a similar fashion to the spinning particle, the zero-modes of ψ^μ generate a d-dimensional Clifford algebra and will thus create states which can be identified with space-time fermions (more will be said on this later). This string theory is called the NSR string.

This action has worldsheet "supersymmetry" under which [157]

$$\delta X^\mu \ = \ \bar{\epsilon}\psi^\mu \ ,$$

$$\delta\psi^\mu = -i\gamma^a\partial_a X^\mu\epsilon \,, \tag{4.2}$$

where ϵ is a constant Grassmann-odd Majorana spinor (to show invariance of S the Majorana fermion identity $\bar{\chi}\psi = \bar{\psi}\chi$, must be used). The commutator of two supersymmetry transformations is $[\delta_1, \delta_2]X^\mu = 2(\bar{\epsilon}_1\gamma^a\epsilon_2)i\partial_a X^\mu$ (for Majorana fermions $\bar{\epsilon}_1\gamma^a\epsilon_2 = -\bar{\epsilon}_2\gamma^a\epsilon_1$). Note that the RHS of this equation contains the translation operator. The same commutator acting on the fermions gives a similar result on shell. Denoting the generators of the transformation by Q_α, (so that $\delta X^\mu = [\bar{\epsilon}Q, X^\mu]$ for instance) the $N = 1$ supersymmetry algebra follows

$$\{Q_\alpha, Q_\beta\} = 2p_a(\gamma^a)_{\alpha\beta} \,. \tag{4.3}$$

4.1.2 Mode expansions for the fermions

In the basis

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \,, \tag{4.4}$$

the upper and lower components of ψ

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \,, \tag{4.5}$$

obey the equations of motion

$$(\partial_1 + \partial_0)\psi_-^\mu = 0 = (\partial_1 - \partial_0)\psi_+^\mu \,. \tag{4.6}$$

Thus the ψ_- and ψ_+ describe right- and left-moving modes respectively

$$\psi_- = \psi_-(\sigma^0 - \sigma^1) \quad \text{and} \quad \psi_+ = \psi_+(\sigma^0 + \sigma^1) \,, \tag{4.7}$$

and are called the chiral halves of the closed string (the chirality operator $\gamma^0\gamma^1$ acts as $\gamma^0\gamma^1\psi_\pm = \mp\psi_\pm$). For open strings the surface terms that arise through varying the Lagrangian to obtain the Euler-Lagrange equations must be set to zero by choosing appropriate boundary conditions. Further remarks concerning open strings will be made later; in this section only closed strings are considered.

Spinors can be either periodic or antiperiodic over the interval $0 \leq \sigma^1 \leq 2\pi$. Periodic boundary conditions are called *Ramond* (R) boundary conditions [286], while antiperiodic [256, 257] are called *Neveu-Schwarz* (NS), so

$$\begin{aligned}
(\mathrm{R}) \qquad \psi_-^\mu &= \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\sigma^0-\sigma^1)} \,, \\
(\mathrm{NS}) \qquad \psi_-^\mu &= \sum_{r\in\frac{1}{2}\oplus\mathbb{Z}} b_r^\mu e^{-ir(\sigma^0-\sigma^1)} \,, \\
(\mathrm{R}) \qquad \psi_+^\mu &= \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-in(\sigma^0+\sigma^1)} \,, \\
(\mathrm{NS}) \qquad \psi_+^\mu &= \sum_{r\in\frac{1}{2}\oplus\mathbb{Z}} \tilde{b}_r^\mu e^{-in(\sigma^0+\sigma^1)} \,.
\end{aligned} \tag{4.8}$$

Corresponding to the different pairings of left and right moving modes, there are four closed-string sectors that are referred to as NS-NS, NS-R, R-NS and R-R.

4.1.3 Quantisation, the GSO projection and particle content

Consider just one chiral half of the closed string. Because of the half-integer moding in the NS sector, it is possible to choose a unique nondegenerate ground state annihilated by the positive frequency modes

$$d^\mu_m|0,p\rangle_{NS} = 0 = \alpha^\mu_m|0,p\rangle_{NS} \qquad \forall m > 0\ . \tag{4.9}$$

However, in the R sector $d/2$ of the zero-modes must be chosen to annihilate $|0,p\rangle_R$ while the other $d/2$ generate a total of $2^{d/2}$ Ramond ground states (here d is assumed to be even). These form a spinor representation of the d dimensional Poincaré algebra as mentioned earlier. This representation decomposes into two halves; one of which has an even number, and the other an odd number of creation zero-modes acting on $|0,p\rangle_R$ corresponding to a decomposition into left- and right-handed chiral spacetime fermions.

The physical states must not only be annihilated by half of the Virasoro generators but also by half the modes of the supersymmetry generator. In contrast to the bosonic string, the NS sector has $a_{NS} = \frac{1}{2}$ while the R sector has $a_R = 0$. In both sectors the critical dimension is found to be 10.

Thus, just as in the bosonic string, the lowest state in the NS sector is a tachyon $|0,p\rangle_{NS}$. Using light-cone quantisation, the next state, $b^i_{-1/2}|0,p\rangle_{NS}$, is a massless vector representation of SO(8) (called $\mathbf{8_v}$) and so on.

In 10 dimensions, the R ground state consists of two **16**'s. The physical-state conditions pick out half these leaving just $\mathbf{8_s}\oplus\mathbf{8_c}$ (the **s** and **c** correspond to the even- or odd-number of fermionic modes acting on $|0,p\rangle_R$ as discussed above). Since $a_R = 0$ these are massless fermions of SO(8).

Massive physical states are built in both sectors by acting with positive modes and demanding that the results be reparameterisation invariant and form supersymmetry multiplets.

Unlike the case of the bosonic string, this is not the end of the story. Most of the material presented in Chapter 1 explicitly assumed that the world-sheet had spherical topology. Recall that for higher genus surfaces the metric cannot be totally gauged away through reparameterisations and Weyl rescalings; after modding-out these symmetries a finite dimensional space of inequivalent metrics remains. This space is called "moduli space". The torus, for example, has one complex modular parameter. However, its moduli space is not the whole complex plane because the torus has certain symmetries. For instance, it is invariant under a twist of one of its cycles by 2π. Fermions can be either periodic or antiperiodic around the two cycles. Consider the toric case where the fermions have antiperiodic boundary conditions around both cycles.

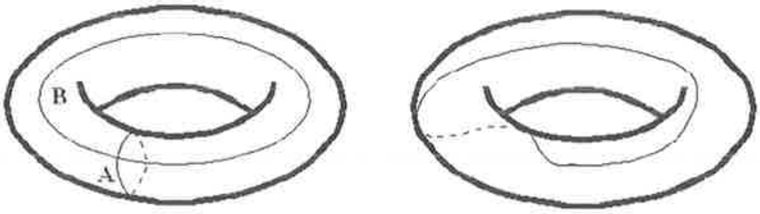

Figure 4.1 : The two cycles of the torus. The second figure shows the B cycle after a twist of angle 2π has been performed around the A cycle.

After a modular transformation that twists around the "A" cycle (see Fig. 4.1), the fermions become *periodic* around the "B" cycle since they pick up two signs — one from traversing each cycle. This example illustrates that different sectors will mix with each other under modular transformations (except the periodic-periodic sector which remains invariant). In order to retain modular invariance (the invariance of the Riemann surfaces under twistings of their cycles) a symmetric combination of the four sectors must be taken; the modular invariant partition function is

$$Z = \mathrm{Tr}(1+(-1)^F)e^{-2\pi\tau H_{NS}} - \mathrm{Tr}(1\pm(-1)^F)e^{-2\pi\tau H_R}\ . \tag{4.10}$$

Here H_{NS} (H_R) is the Hamiltonian with anti-periodic (periodic) spatial boundary conditions and F is the fermion number operator (therefore $\mathrm{Tr}(-1)^F e^{-2\pi\tau H_R}$ is the partition function with periodic boundary conditions in both directions, for instance). In the R sector there is a choice of sign since $\mathrm{Tr}(-1)^F e^{-2\pi\tau H_R} = 0$ because of the zero mode (but once a choice is made the theory has been defined). The combination Eq. (4.10) corresponds to projecting out states with odd fermion number. This is the "GSO projection" [162, 163].

One of the most important consequences of the GSO projection is that it kills the fermionic vacuum $|0,p\rangle_{NS}$. Thus the physical state-space of the superstring does not include a tachyon! The lowest state of the NS sector is therefore the massless vector boson $\mathbf{8_v}$

$$b^i_{-1/2}|0,p\rangle_{NS}\ . \tag{4.11}$$

In the R sector, one of the $\mathbf{8}$'s must be projected out. It is entirely arbitrary which one is chosen since the two differ only by a spacetime parity redefinition. However, in the closed string two choices must be made, one for each chiral half. The "IIA" string is defined by taking the opposite choice in each half while the "IIB" string has the same choice. The massless sectors of the closed superstring are thus

$$\begin{array}{ll} \text{Type IIA:} & (\mathbf{8_v}\oplus\mathbf{8_s})\otimes(\mathbf{8_v}\oplus\mathbf{8_c}) \\ \text{Type IIB:} & (\mathbf{8_v}\oplus\mathbf{8_s})\otimes(\mathbf{8_v}\oplus\mathbf{8_s})\ . \end{array} \tag{4.12}$$

Now concentrate on the IIA case only. The various products give

$$\text{NS-NS}\quad \mathbf{8_v}\otimes\mathbf{8_v} = \mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35} = \Phi\oplus B_{\mu\nu}\oplus G_{\mu\nu}\ ,$$

$$\begin{aligned}
\text{R-R} \quad \mathbf{8_s} \otimes \mathbf{8_c} &= \mathbf{8_v} \oplus \mathbf{56_t} = A_\mu \oplus A_{\mu\nu\lambda} \ , \\
\text{NS-R} \quad \mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c} \ , \\
\text{R-NS} \quad \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s} \ . \qquad (4.13)
\end{aligned}$$

Here Φ, $B_{\mu\nu}$, $G_{\mu\nu}$, A_μ and $A_{\mu\nu\lambda}$ are the dilaton, antisymmetric tensor, graviton, one-form gauge field and three-form gauge field respectively. The NS-R and R-NS sectors are fermionic and contain two dilatinos and gravitinos of opposite chirality.

It is a remarkable fact that after performing the GSO projection the resultant theory has local *spacetime* supersymmetry [162, 163]. This property is not at all obvious, however, it is possible to reformulate the NSR string so that it becomes manifest. This reformulation is called the "Green-Schwarz" string [174, 175] which will not be discussed here since it takes the exposition too far afield.

4.1.4 IIA supergravity

The low-energy effective action of IIA string theory is the "IIA supergravity action", the bosonic part of which is given by

$$\begin{aligned}
S_{IIA}^{\text{string}} &= m_{\text{str}}^8 \int \mathrm{d}t\mathrm{d}^9y\,\sqrt{-g^s}\,\Big\{e^{-2\Phi}(R^s + 4\nabla_\mu^s\Phi\nabla_s^\mu\Phi - \tfrac{1}{12}e^{-\Phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}) - \tfrac{1}{4}e^{3\Phi/2}F_{\mu\nu}F^{\mu\nu} \\
&\quad -\tfrac{1}{48}F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho} - \tfrac{1}{2304}(\sqrt{-g^s})^{-1}\epsilon^{\mu_0\ldots\mu_9}B_{\mu_0\mu_1}F_{\mu_2\ldots\mu_5}F_{\mu_6\ldots\mu_9}\Big\} \ , \qquad (4.14)
\end{aligned}$$

where R^s is the Ricci scalar, and the various field strengths are

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \ , \\
H_{\mu\nu\lambda} &= \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} \ , \\
F_{\mu\nu\lambda\rho} &= \partial_\mu A_{\nu\lambda\rho} + A_\mu H_{\nu\lambda\rho} + (-1)^P \times \text{cyclic permutations} \ . \qquad (4.15)
\end{aligned}$$

The superscripts "string" and "s" are to indicate that the action has been written in a certain "frame" called the "string frame". The quantity $m_{\text{str}} = (\alpha')^{-1/2}$ is the string mass. This frame makes it evident that the 10d Newton constant κ_{10}^2 scales as

$$\kappa_{10}^2 \sim m_{\text{str}}^{-8} g_{\text{str}}^2 \ , \qquad (4.16)$$

since $g_{\text{str}} = e^\Phi$.

The action Eq. (4.14) follows from an analysis of the β functions calculated at string tree level (which motivates the $e^{-2\Phi}$ that premultiplies the NS-NS part of the action) and at lowest order in the tension α' [62]. At higher energies it therefore receives higher-derivative corrections corresponding to higher orders in α'.

It is convenient to work in the so-called "Einstein frame" in which the Einstein term in the action has no prefactor containing the dilaton. The action in this frame is obtained by making the field redefinition

$$g_{MN}^e = e^{-\Phi/2} g_{MN}^s \ , \qquad (4.17)$$

and looks like

$$
\begin{aligned}
S_{IIA}^{\text{Einstein}} &= \frac{1}{2\kappa_{10}^2}\int \mathrm{d}t\mathrm{d}^9y\,\sqrt{-g^e}\,\Big(R^e - \tfrac{1}{2}\nabla_\mu^e\Phi\nabla_e^\mu\Phi - \tfrac{1}{4}e^{3\Phi/2}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{12}e^{-\Phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \\
&\quad -\tfrac{1}{48}e^{\Phi/2}F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho} - \tfrac{1}{2304}(\sqrt{-g^e})^{-1}\epsilon^{\mu_0\ldots\mu_9}B_{\mu_0\mu_1}F_{\mu_2\ldots\mu_5}F_{\mu_6\ldots\mu_9}\Big)\ . \qquad (4.18)
\end{aligned}
$$

Since it is this action that will be exclusively used from now on, the scripts "Einstein" and "e" will be dropped.

4.2 D0-branes

This section considers in detail a solution of the equations of motion of IIA supergravity which describes a point-like object. Solutions of supergravities have been the subject of much investigation[1] and go under the general title of "p-branes"; that is, objects with p spatial dimensions which are infinitely extended (or wrapped around a compact dimension).

Many p-branes belong to shortened multiplets of the supersymmetry algebra. Such states are termed "BPS states". As will be discussed they also saturate their "Bogomol'nyi bound" — a lower bound on a particle's mass in terms of its charge(s) which is dictated by supersymmetry. A key property of BPS states is that their degeneracy with a given set of charges is independent of the value of the string coupling constant (and all other moduli of the theory too) [349]. For consider changing g_{str} adiabatically. If the total number of states does not change discontinuously then the shortened multiplet cannot transform into a long multiplet simply because the latter contains too many states. Note that for this argument to be valid the spectrum must keep away from the continuum (otherwise state counting is not well-defined) and the mass of the BPS state must be strictly less than the total mass of any other set of BPS states carrying the same total charge (otherwise a decay could occur). This independence on the moduli has been studied in detail in [69, 70, 306, 307]. It is the reason why p-branes are so important — they provide information about the strong-coupling, low-energy limit of string theory.

In the final subsection of this section "Dp-branes" are discussed. These are surfaces on which open strings can end; the "D" stands for "Dirichlet boundary conditions". The p-branes share the same properties as the Dp-branes and are thus thought of [275] as an effective field theory description of the latter. A review of this subject can be found in Ref. [276].

[1] Some recent reviews are [125, 128, 314]. At the risk of quoting a set of measure zero, the following are some early references on the subject [1, 36, 37, 63, 64, 77, 127, 130, 131, 132, 133, 134, 176, 182, 188, 213, 315, 322] while some more recent articles that discuss modern issues such as dualities are [126, 128, 200, 229, 230, 231, 232, 233, 234, 280, 323].

4.2.1 The superparticle solution of IIA supergravity

A solution to the field equations of Eq. (4.18) is [133, 187]

$$\mathrm{d}s^2 = \tilde{f}^{-7/8}\mathrm{d}t^2 + \tilde{f}^{1/8}\mathrm{d}y\cdot\mathrm{d}y\ , \quad e^{\Phi} = \tilde{f}^{3/4}\ , \quad A_0 = \tilde{f}^{-1} - 1\ , \tag{4.19}$$

where $\tilde{f}$ is a time-independent harmonic function and all other fields are zero. This subsection will verify that Eq. (4.19) is indeed a solution, and that it corresponds to a number of static electrically charged massive "point-particles" which are extremal in the sense that their masses are equal to their charges. These point-particles are dubbed "0-branes".

Labeling 10d spacetime with coordinates[2] $x^\mu = (t, y^a)$, the positions of the particles by x_i^μ and their masses/charges by $\tilde{Q}_i$, the source action is

$$S_{\text{source}} = -\sum_i \tilde{Q}_i \int \mathrm{d}\tau\, e^{-3\Phi/4}\sqrt{-g_{\mu\nu}(x_i)\frac{\mathrm{d}x_i^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x_i^\nu}{\mathrm{d}\tau}} + \sum_i \tilde{Q}_i \int \mathrm{d}\tau\, A_\mu(x_i)\frac{\mathrm{d}x_i^\mu}{\mathrm{d}\tau}\ . \tag{4.20}$$

Parameterising the particles' worldlines by $\tau = t$, the field equations are

$$\begin{aligned}
0 &= \partial_\nu(\sqrt{-g}e^{3\Phi/2}F^{\nu\mu}) + 2\kappa_{10}^2 \sum_i \tilde{Q}_i \frac{\mathrm{d}x_i^\mu}{\mathrm{d}t}\delta(x - x_i)\ , \\
0 &= \Box\Phi - \tfrac{3}{8}e^{3\Phi/2}\sqrt{-g}F^2 + \tfrac{3}{2}\kappa_{10}^2 \sum_i \tilde{Q}_i e^{-3\Phi/4}\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x_i^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_i^\nu}{\mathrm{d}t}}\delta(x - x_i)\ , \\
0 &= -\tfrac{1}{2}g_{\mu\nu}(R - \tfrac{1}{2}(\nabla\Phi)^2 - \tfrac{1}{4}e^{3\Phi/2}F^2) + R_{\mu\nu} - \tfrac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \tfrac{1}{2}e^{3\Phi/2}F_{\mu\lambda}F_\nu{}^\lambda \\
&\quad + 2\kappa_{10}^2 \sum_i \tilde{Q}_i e^{-3\Phi/2}\tfrac{1}{2}\frac{\mathrm{d}x_{i\mu}}{\mathrm{d}t}\frac{\mathrm{d}x_{i\nu}}{\mathrm{d}t}\frac{1}{\sqrt{-g_{\mu\nu}\frac{dx_i^\mu}{dt}\frac{dx_i^\nu}{dt}}}\delta(x - x_i)\ .
\end{aligned} \tag{4.21}$$

These have solution Eq. (4.19) with

$$\Box_\perp \tilde{f} + 2\kappa_{10}^2 \sum_i \tilde{Q}_i \delta(y - y_i) = 0 \quad \text{and} \quad \frac{\mathrm{d}y_i^a}{\mathrm{d}t} = 0\ . \tag{4.22}$$

Here $\Box_\perp$ is the Laplacian in nine-dimensional space. Poison's equation is solved by

$$\tilde{f}(y - y_i) = 1 + \frac{30\kappa_{10}^2}{32\pi^4}\sum_i \tilde{Q}_i \frac{1}{|y - y_i|^7}\ . \tag{4.23}$$

The constant has been added so that the metric Eq. (4.19) is asymptotically flat.

4.2.2 Charge quantisation

In a straightforward extension of the Dirac quantisation of electric charge in four dimensions, the charges $\tilde{Q}_i$ are found to be quantised [255, 320]. In the four dimensional case, it was the presence of a Dirac monopole, which is the magnetic dual of the electric charge, that led to the

[2]Here there is an unfortunate clash of notation. Previously "a" was a worldsheet index. For the rest of this thesis the script "a" will be used exclusively as a 9-dimensional spatial index.

quantisation condition. Here, the dual is an object with six spatial dimensions and is called a 6-brane. The 6-brane solution of IIA supergravity is very similar to the 0-brane solution presented above [314].

Place the 6-brane at position $x^1 = x^2 = x^3 = 0$ so that the 2-form field-strength is closed $dF = 0$ everywhere except on this hypersurface. The spatial surface surrounding $x^1 = x^2 = x^3 = 0$ is $S^2 \times R^6$. The vector potential corresponding to this arrangement will not be well defined on the entire sphere. Rather, it must be defined on two patches of S^2 and on the overlap the two forms will differ by a gauge transformation.

It is not necessary to find these two gauge fields explicitly. However, the answer is quite simple in this instance. The field strength is given by

$$F = \frac{g}{\mathrm{Vol}(S^2)} \mathrm{d}\Omega_2 \ , \tag{4.24}$$

where $\mathrm{d}\Omega_2$ is the volume form in two dimensions ($\mathrm{d}\Omega_2 = \sin\phi \, \mathrm{d}\theta \mathrm{d}\phi$ in spherical harmonics) and g is the magnetic charge/unit 6-brane volume

$$g = \int_{S^2} F \ . \tag{4.25}$$

The field strength is closed $dF = 0$ everywhere except at the brane $x^1 = x^2 = x^3 = 0$. Call the gauge field defined on the northern hemisphere A^+ and on the southern hemisphere A^-. It is easy to solve the equation $\mathrm{d}A^\pm = \frac{g}{\mathrm{Vol}(S^2)} \mathrm{d}\Omega_2$ — in spherical harmonics the result is

$$A^\pm = \frac{g}{\mathrm{Vol}(S^2)} \left(\pm \frac{1}{2} \frac{\mathrm{Vol}(S^2)}{\mathrm{Vol}(S^1)} - \cos\phi \right) \mathrm{d}\theta \ . \tag{4.26}$$

The constant $\pm\frac{1}{2}\mathrm{Vol}(S^2)/\mathrm{Vol}(S^1)$ comes from imposing

$$\mathrm{Vol}(S^2) = \int_{S^2_+ \cup S^2_-} \mathrm{d}\Omega_2 = \frac{\mathrm{Vol}(S^2)}{g} \left(\int_{S^2_+} \mathrm{d}A^+ + \int_{S^2_-} \mathrm{d}A^- \right) = \frac{\mathrm{Vol}(S^2)}{g} \int_{S^1} (A^+ - A^-) \ , \tag{4.27}$$

the last sign coming from a change of orientation.

The phase factor of the point-particle's wavefunction associated with a rotation of the electrically-charged particle around the 6-brane should not depend on which vector potential is used, A^+ or A^-. Therefore

$$e^{i\tilde{Q} \oint A^+} = e^{i\tilde{Q} \oint A^-} \ , \tag{4.28}$$

or

$$\tilde{Q} \int_{\partial S^2_+} A^+ + \tilde{Q} \int_{\partial S^2_-} A^- = 2\pi \mathbb{Z} \ , \tag{4.29}$$

which finally yields the quantisation condition

$$\tilde{Q} g = 2\pi \mathbb{Z} \ . \tag{4.30}$$

4.2.3 Preserved supersymmetry

The solution Eq. (4.19) does not break all the spacetime supersymmetry despite the fact that the fermionic sector, antisymmetric two-form and the three-form are all set to zero. The amount of remaining supersymmetry can be calculated by demanding that the variation of the gravitino and dilatino vanish (the bosonic fields are invariant under supersymmetry when the fermions are set to zero) in the specific background. Explicitly, for type IIA supergravity in the background Eq. (4.19) the variations are [133]

$$\begin{aligned}\delta\psi_\mu &= \left[D_\mu + \tfrac{1}{64}e^{3\Phi/4}\left(\Gamma_\mu{}^{\nu_1\nu_2} - 14\delta_\mu^{\nu_1}\Gamma^{\nu_1}\right)\Gamma^{11}F_{\nu_1\nu_2}\right]\epsilon\,,\\ \delta\lambda &= \tfrac{\sqrt{2}}{4}\left[(\Gamma^\mu\partial_\mu\phi)\Gamma^{11}\epsilon + \tfrac{3}{8}e^{3\Phi/4}\Gamma^{\nu_1\nu_2}F_{\nu_1\nu_2}\epsilon\right]\,.\end{aligned}\tag{4.31}$$

Here ψ_μ and λ are the gravitino and dilatino respectively, ϵ is a 32-component Weyl spinor, Γ^μ are the 10d Dirac matrices, $D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu A_1A_2}\Gamma^{A_1A_2}$ is the covariant derivative with $\omega_{\mu A_1 A_2}$ being the Lorentz spin connection and $\Gamma^{\mu_1\mu_2\cdots\mu_n}$ being the antisymmeterised product of Γ-matrices, and $\Gamma^{11} = \Gamma^0\Gamma^1\ldots\Gamma^9$. Indices "$A$" have been flattened with the vielbein, for example $\Gamma^\mu(x) = e^\mu_A(x)\Gamma^A$. Substituting for the dilaton, metric and one-form yields

$$\begin{aligned}\delta\lambda &= \tfrac{\sqrt{2}}{4}\left(\tilde{f}^{-1}\Gamma^a\partial_a f\right)\tfrac{3}{4}\left(\Gamma^0+\Gamma^{11}\right)\epsilon\,,\\ \delta\psi_0 &= \left(\partial_0 + \tfrac{7}{32}(f^{-1}\Gamma^a\partial_a f)(\Gamma^0+\Gamma^{11})\right)\epsilon\,,\\ \delta\psi_a &= \left(\partial_a - \tfrac{7}{16}\Gamma^0\Gamma^{11}f^{-1}\partial_a f\right)\epsilon + \tfrac{1}{64}(\Gamma_a\Gamma^b - \Gamma^b\Gamma_a)f^{-1}\partial_b f(1-\Gamma^0\Gamma^{11})\epsilon\,.\end{aligned}\tag{4.32}$$

In these three equations all Γ matrices live in flat space and the "flat" index $A = (0, a)$. The "Killing spinor" which solves $0 = \delta\lambda = \delta\psi_0 = \delta\psi_a$ is $\epsilon = \epsilon_0 \log f^{7/16}$ with

$$(\Gamma^0 + \Gamma^{11})\epsilon_0 = 0\,.\tag{4.33}$$

Therefore almost all of the local supersymmetry is broken, but a rigid remnant remains since $\mathrm{tr}\,(\Gamma^0+\Gamma^{11}) = \frac{1}{2}\cdot 32$: the purely bosonic 0-brane solution Eq. (4.19) "preserves half" of the supersymmetry!

This is an example of a "BPS state". These states form smaller, "shortened" supermultiplets. They are also characterised by special values of mass and charge. This can be seen by studying the general form of the supersymmetry algebra [309]. For consider a theory which has N real supersymmetry generators Q_α. Acting on a single-particle state at rest, the algebra takes the form [75, 183]

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}(m, \tilde{Q}, \{g\})\,.\tag{4.34}$$

Here $f_{\alpha\beta}$ is a real symmetric matrix, m denotes the rest mass of the particle, $\tilde{Q}$ the various charges carried by the particle and $\{g\}$ the coordinates labeling the moduli space of the theory. The positivity of the LHS implies a bound on the mass, called the "Bogomol'nyi bound" in terms of the charges and moduli [48, 159, 76, 285, 349]

$$m \geq |f(\tilde{Q}, \{g\})|\,.\tag{4.35}$$

If $f_{\alpha\beta}$ has no zero eigenvalues then by taking an appropriate linear combination of the Q_α, it can be diagonalised. Performing a rescaling the algebra takes the form

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta} \ . \tag{4.36}$$

This is the N-dimensional Clifford algebra and so the single particle states are $2^{N/2}$-dimensional multiplets.

The more interesting case is where $f_{\alpha\beta}$ has $(N-M)$ zero eigenvalues. The algebra can then be brought into the form

$$\{Q_\alpha, Q_\beta\} = \begin{cases} \delta_{\alpha\beta} & \text{for} \quad 1 \le \alpha, \beta \le M \ , \\ 0 & \text{for} \quad \alpha \text{ or } \beta > M \ . \end{cases} \tag{4.37}$$

An irreducible representation of dimension $2^{M/2}$ can be formed by taking all the states to be annihilated by Q_α for $\alpha > M$. Moreover, because of the zero eigenvalues the Bogomol'nyi bound is saturated. The solution Eq. (4.19) forms a shortened representation and it too saturates its Bogomol'nyi bound. This might have been expected from the equality between its mass and charge seen in Eq. (4.20).

4.2.4 Uplifting to eleven dimensions

It is now well known that the low energy IIA supergravity of Eq. (4.18) is a Kaluza-Klein reduction of 11d supergravity compactified on a spacelike circle [67, 158, 195]. The relation between the ten and eleven dimensional metrics is

$$\mathrm{d}s_{11}^2 = e^{-\Phi/6}\mathrm{d}s_{10}^2 + e^{4\Phi/3}(\mathrm{d}x^{11} + A_\mu \mathrm{d}x^\mu)^2 \ , \tag{4.38}$$

where x^{11} is compactified with radius

$$R_s = \tilde{M}^{-1}\sqrt{e^{4\Phi/3}} = \tilde{M}^{-1} g_{\mathrm{str}}^{2/3} \ , \tag{4.39}$$

in which the eleven-dimensional Planck mass has been denoted by $\tilde{M}$. What is described in this subsection has been termed a "consistent oxidation" (oxidation is the opposite of reduction) which means that the 10d solution is a perfectly good, albeit specific, solution of the 11d theory [314]; namely, one for which no fields depend on x^{11}.

The action in the bulk is simply the Einstein action

$$S_E = \frac{1}{2\kappa_{10}^2}\frac{1}{2\pi R_s}\int \mathrm{d}^{11}x\sqrt{-g}R \equiv \frac{1}{2\kappa_{11}^2}\int \mathrm{d}^{11}x\sqrt{-g}R \ , \tag{4.40}$$

where the quantities on the RHS are 11-dimensional (the 11d Newton constant $\kappa_{11}^2 \sim \tilde{M}^{-9}$). There are also terms coming from antisymmetric 2-form and the RR 3-form and the fermions, but these fields will never be excited away from zero in what follows. Capital letters $M, N, \ldots$ will be used to denote 11d indices.

The RR 1-form in 10d is lifted up to the 11-μ component of the 11d metric. Therefore its charge will lift to the 11^{th} component of momentum. This must be quantised in units of $1/R_s$ so the discussion will now be specialised to the case where

$$\tilde{Q} = N/R_s \ . \tag{4.41}$$

Note that this quantisation is seemingly quite different from the Dirac quantisation detailed in Sec. 4.2.2 in that it occurs without needing the existence of a magnetically-charged object. However, by lifting the dual 6-brane to 11d, it has been shown [55] that Eq. (4.41) is consistent with the Dirac quantisation Eq. (4.30)[3].

The expectation is therefore that the source action will lift to the standard massless-particle action with constant $p^{11} = \tilde{Q}$ [190, 194, 324, 346]. This is worth checking, however, since the omission of spin might have introduced extra subtleties. Consider, then, the standard massive-particle action for one particle

$$S_m = m\int \mathrm{d}\tau\sqrt{-g_{MN}\frac{\mathrm{d}x^M}{\mathrm{d}\tau}\frac{\mathrm{d}x^N}{\mathrm{d}\tau}} = m\int \mathrm{d}\tau\sqrt{-e^{-\Phi/6}g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - e^{4\Phi/3}\left(\frac{\mathrm{d}x^{11}}{\mathrm{d}\tau} + A_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)^2} \ . \tag{4.42}$$

In the Kaluza-Klein scheme [207, 216] used here x^{11} is a cyclic variable and thus p^{11} is constant

$$p^{11} = -\frac{m}{\sqrt{-g_{MN}\mathrm{d}_\tau x^M \mathrm{d}_\tau x^N}}e^{4\Phi/3}(\frac{\mathrm{d}x^{11}}{\mathrm{d}\tau} + A_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}) \equiv \tilde{Q} \ . \tag{4.43}$$

Solving this expression for $(\mathrm{d}x^{11} + A_\mu \mathrm{d}x^{11})/\mathrm{d}\tau$ yields

$$e^{4\Phi/3}\left(\frac{\mathrm{d}x^{11}}{\mathrm{d}\tau} + A_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)^2 = \frac{-\tilde{Q}^2e^{-\Phi/6}}{m^2e^{4\Phi/3}+\tilde{Q}^2}g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \ . \tag{4.44}$$

In the limit $m \to 0$ with p^{11} fixed, these two expressions result in $(\dot{} \equiv \frac{d}{d\tau})$

$$\begin{aligned} \frac{m}{\sqrt{-g_{MN}\dot{x}^M\dot{x}^N}} &= \frac{\tilde{Q}e^{-7\Phi/12}}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \ , \\ e^{4\Phi/3}\left(\dot{x}^{11} + A_\mu\dot{x}^\mu\right)^2 &= -e^{-\Phi/6}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \ . \end{aligned} \tag{4.45}$$

The equations of motion read

$$\begin{aligned} 0 \ = \ & \frac{m}{\sqrt{-g_{MN}\dot{x}^M\dot{x}^N}}\tfrac{1}{2}\partial_\mu(e^{-\Phi/6}g_{\nu\lambda})\dot{x}^\nu\dot{x}^\lambda - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{me^{-\Phi/6}}{\sqrt{-g_{MN}\dot{x}^M\dot{x}^N}}g_{\mu\nu}\dot{x}^\nu\right) \\ & + \frac{m}{\sqrt{-g_{MN}\dot{x}^M\dot{x}^N}}\tfrac{1}{2}(\partial_\mu e^{4\Phi/3})\left(\dot{x}^{11} + A_\nu\dot{x}^\nu\right)^2 + \frac{me^{4\Phi/3}}{\sqrt{-g_{MN}\dot{x}^M\dot{x}^N}}(\partial_\mu A_\nu)\dot{x}^\nu\left(\dot{x}^{11} + A_\lambda\dot{x}^\lambda\right) \\ & - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{me^{4\Phi/3}}{\sqrt{-g_{MN}\dot{x}^M\dot{x}^N}}A_\mu\left(\dot{x}^{11} + A_\nu\dot{x}^\nu\right)\right) \ , \end{aligned} \tag{4.46}$$

[3]Moreover, by assuming that the strong coupling limit of IIA is an 11d theory (see Sec. 4.3.1) and utilising the T-duality between IIA and IIB [309] all the charges in IIA and IIB may be fixed in terms of the 11d Newton constant κ_{11}^2. Alternative derivations of the same result using the SL(2,$\mathbb{Z}$) symmetry of IIB [309] can also be made [88, 129, 301].

and upon substitution of Eq. (4.45) they reduce to exactly those generated by the source action Eq. (4.20). Therefore the source in 11d is described by Eq. (4.42) in the massless limit with the constraint that the momentum of the particle in the eleventh direction p^{11}, is held equal to the mass (=charge); $p^{11} = \tilde{Q}$.

Since 0-branes are BPS states they will live in a $2^{32/2} = 256$ dimensional representation of the supersymmetry algebra. The supergraviton multiplet in 11d supergravity is also a BPS state (it contains the graviton (44 components), the 3-form (84 components) and the gravitino (128 components)) and carries the same quantum numbers as the oxidised 0-brane. Thus they are identified.

Finally, using the map Eq. (4.38), the solution Eq. (4.19) is oxidised to

$$\mathrm{d}s_{11}^2 = (\tilde{f} - 2)\mathrm{d}t^2 + \tilde{f}(dx^{11})^2 - 2(f-1)\mathrm{d}t\mathrm{d}x^{11} + \mathrm{d}y\cdot\mathrm{d}y \ . \tag{4.47}$$

4.2.5 Coupling to R-R fields

The solitonic point-particle clearly carries R-R charge $\tilde{Q}$. Similarly all the other type-IIA p-branes studied in the literature also carry R-R charge. However, elementary string states are neutral under R-R gauge transformations. This can be seen by computing a three-point function involving any two elementary string states and an R-R gauge field [149, 276]. Following Polchinski, a heuristic argument is that the vertex operator for the R-R states, which is a product of spin fields ($V = S_\alpha \tilde{S}_\beta \leftrightarrow \mathbf{8_s} \otimes \mathbf{8_c}$), can be decomposed into antisymmetric tensors of SO(9,1)

$$V = \sum_{n \text{ even}} S_\alpha \tilde{S}_\beta \, (\Gamma^{\mu_1 \ldots \mu_n} C)_{\alpha\beta} \, F^{(n)}_{\mu_1 \ldots \mu_n}(x) \ . \tag{4.48}$$

Then, using the identity

$$\Gamma^\mu \Gamma^{\mu_1 \ldots \mu_n} = \Gamma^{\mu\mu_1 \ldots \mu_n} + (\delta^{\mu\mu_1}\Gamma^{\mu_2 \ldots \mu_n} + \text{permutations}) \ , \tag{4.49}$$

and the Virasoro conditions which imply that V satisfies the Dirac equation, the equations

$$dF^{(n)} = 0 = d^* F^{(n)} \ , \tag{4.50}$$

result. Moreover, the condition

$$F^{(n)} = \pm^* F^{(10-n)} \tag{4.51}$$

follows from the identity

$$\epsilon^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_{10-n}} \Gamma_{\nu_1 \ldots \nu_{10-n}} \propto \Gamma^{11} \Gamma^{\mu_1 \ldots \mu_n} \ , \tag{4.52}$$

(the choice of sign depends on convention and is unimportant here). These properties motivate that the $F^{(2)}$ and $F^{(4)}$ are the field strengths for the R-R 1-form and 3-form respectively. It follows that R-R amplitudes involving elementary strings vanish at zero momentum, so strings do not carry R-R charges.

4.2.6 Dp-branes

Dp-branes are hyperplanes on which strings can end [78, 220] ("D" stands for "Dirichlet boundary conditions"). They share the same properties as the p-branes described above and are thus identified as alternative, more stringy, description of the same object.

The spectrum for a bosonic string with mixed Dirichlet/Neumann boundary conditions in a flat empty background can be computed in the same way as the purely Neumann case of Sec. 1.1: Consider a collection of parallel hyper-planar Dp-branes where the coordinates X^μ can be decomposed into coordinates in the direction of the branes $X^{\hat{\mu}}$ ($\hat{\mu} = 0, \ldots, p$) and those perpendicular to the branes X^I ($I = p+1, \ldots, d-1$). A string stretched between two branes living at positions x_0^I and x_1^I has the boundary conditions

$$X^I(\sigma^0, 0) = x_0^I \quad \text{and} \quad X^I(\sigma^0, \pi) = x_1^I \ . \tag{4.53}$$

The mode expansion for this string therefore takes the form

$$X^I(\sigma) = x_0^I + (x_1^I - x_0^I)\frac{\sigma^1}{\pi} + 2\sqrt{\alpha'}\sum_{n\neq 0}\frac{\alpha_n^I}{n}e^{-in\sigma^0}\sin n\sigma^1 \ , \tag{4.54}$$

which can be compared with the Neumann case of Eq. (1.9).

The massless states of this scenario will be of interest later on. The mass formula is

$$m^2 = \left(\frac{x_1 - x_0}{2\pi\alpha'}\right)^2 + \frac{\text{Level} - 1}{\alpha'} \ . \tag{4.55}$$

Therefore massless states only exist when both ends of the string lie on the same Dp-brane (the tachyon is neglected for it will be projected out in the supersymmetric case). These massless states are of two types

$$\alpha_{-1}^{\hat{\mu}}|p; ii\rangle \quad \text{and} \quad \alpha_{-1}^I|p; ii\rangle \ . \tag{4.56}$$

The state $|p; ii\rangle$ describes a string stretched from brane "i" to brane "i" moving with momentum $p^{\hat{\mu}}$ (of course the boundary conditions enforce $p^I = 0$). The first state in Eq. (4.56) is therefore a massless vector $A_{ii}^{\hat{\mu}}$ propagating on the brane while the second is a set of transverse scalars ϕ_{ii}^I which will describe the oscillations of the brane about its "planar" ground state.

This is not the end of the story, however, for when the branes are very close (compared with $\sqrt{\alpha'}$) there are more light fields. These come from strings stretching between two branes; the light states are

$$\alpha_{-1}^{\hat{\mu}}|p; ij\rangle \quad \text{and} \quad \alpha_{-1}^I|p; ij\rangle \ . \tag{4.57}$$

Once again these correspond to a massless vector and a set of scalars. Thus, in total, the light degrees of freedom of a set of N parallel Dp-branes is

$$A_{ij}^{\hat{\mu}} \quad \text{and} \quad \phi_{ij}^I \ , \tag{4.58}$$

where $i, j = 1, \ldots, N$.

The effective action for the massless degrees of freedom for one Dp-brane (which is not necessarily planar) can be obtained from a β-function analysis and the result is [220]

$$S_p = -T_p \int \mathrm{d}^{p+1}\xi \, e^{-\Phi} \sqrt{\det(G_{\hat{\mu}\hat{\nu}} + B_{\hat{\mu}\hat{\nu}} + 2\pi\alpha' F_{\hat{\mu}\hat{\nu}})} \ . \tag{4.59}$$

This is the "Born-Infeld" action. Coordinates on the brane are labeled by $\xi^{\hat{\mu}}$, T_p is a dimensionful constant, and $G_{\hat{\mu}\hat{\nu}}$ and $B_{\hat{\mu}\hat{\nu}}$ are the pullbacks of the metric and antisymmetric tensor to the brane. $F_{\hat{\mu}\hat{\nu}}$ is the gauge field for the U(1) potential $A^{\hat{\mu}}$ that lives on the brane. When $p = 0$ this action nothing but the first term of the 0-brane source action found in Eq. (4.20) (note that the latter is in the Einstein frame while Eq. (4.59) is formulated in the string frame).

For N coincident branes it was shown above that the gauge field and the transverse scalars become $N \times N$ matrices. The gauge field belongs to the adjoint of U(N) [272, 347]. Performing a derivative expansion around flat space-time of the action describing this scenario gives

$$S_p = \int \mathrm{d}^{p+1}\xi \, \mathrm{Tr} \left(F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} + D_{\hat{\mu}}\phi^I D^{\hat{\mu}}\phi^I + [\phi^I, \phi^J][\phi^I, \phi^J] \right) \ . \tag{4.60}$$

Here $D_{\hat{\mu}}\phi^I = \partial_{\hat{\mu}}\phi^I + [A_{\hat{\mu}}, \phi^I]$ is the covariant derivative. The action is the dimensional reduction of U(N) Yang-Mills theory from $d = 25 + 1$ to $d = p + 1$!

The above arguments can be applied to the superstring. The following is a summary of some pertinent results:

- Dp-branes couple to the R-R gauge fields. In the simplest case where $F_{\hat{\mu}\hat{\nu}} = 0 = B_{\hat{\mu}\hat{\nu}}$ the contribution to the Dp-brane action from the $p+1$-form R-R gauge field $A_{[p+1]}$ is

$$Q_p \int \mathrm{d}^{p+1}\xi \, A_{[p+1]} \ , \tag{4.61}$$

  which, for the case $p = 0$, can be compared with the last term in the 0-brane source action Eq. (4.20). Not only do the Dp-branes carry R-R charges, but they comprise a complete set of electric and magnetic R-R sources. For instance, IIA has field strengths of rank 2,4,6,8 (with n and $10 - n$ being dual as in Eq. (4.51)) which are the curls of potentials of rank 1,3,5,7 (see Eq. (4.13)) which couple to Dp-branes with $p = 0, 2, 4, 6$[4].

- By considering a collection of Dp-branes with different p's, it is found that, for exactly the same reason as the p-brane case detailed above, they obey the Dirac-quantisation condition

$$Q_p Q_{6-p} = 2\pi \ . \tag{4.62}$$

  But this time only the *minimum* quantum number appears on the RHS. Objects with higher charge are thus bound states of Dp-branes. A stable bound state of 2 D0-branes has been proved to exist [311] and it is assumed here that higher bound states also exist.

- A single Dp-brane preserves half of the rigid supersymmetries of the theory — it is a BPS state [171, 275]. Bound states of Dp-branes are also BPS states [40].

[4] There is also a D8-form which implies the existence of $F^{(10)}$ [35].

- The effective low-energy theory of the NS-NS light degrees of freedom is the reduction of U(N) super-Yang-Mills theory (which has 16 real supersymmetries) from $d = 9+1$ to $p+1$ dimensions [17, 79, 90, 91, 204, 323, 331, 347]. The R-R gauge fields also couple to the U(N) gauge fields and to the curvature [34, 41, 121, 172, 173, 221].

There has been a colossal amount of interest in the subject of D-branes[5]. For more details the reader is referred to the reviews [276, 309].

4.3 Matrix theory

Matrix theory [22, 317] is conjectured to describe M-theory compactified along a light-like direction. This section will describe M-theory, the general construction of a theory compactified along a light-like direction, and Matrix theory.

4.3.1 M-theory

As was demonstrated in the previous section, IIA supergravity is a Kaluza-Klein reduction of 11d supergravity compactified on a space-like circle of radius given by Eq. (4.39). The bound states of D0-branes of the IIA string are identified with the 11d supergravitons. In terms of the string coupling the radius of compactification is simply

$$R_s = \tilde{M}^{-1} g_{\mathrm{str}}^{2/3} \,, \tag{4.63}$$

where $\tilde{M}$ is the 11d Planck mass. Note that as the string coupling tends to infinity eleven large dimensions appear and the masses of the D0-branes become small (recall the mass formula Eq. (4.41) mass $= N/R_s$ is stable under changes in g_{str}). Conversely, in the perturbative string-theory regime the radius is small but the D0-branes still exist and they become very heavy.

It is then natural to conjecture that the type IIA string can *also* be described as a compactified 11d theory. This 11d theory is called "M-theory" [324, 346]. Moreover, this 11d theory is conjectured to be Lorentz invariant. In a nut-shell

> M-theory is an 11d Lorentz invariant theory which is the strong coupling limit of IIA string theory and whose low-energy behaviour is described by 11d supergravity.

M-theory's elegance lies in the fact that it unifies the 5 consistent superstrings and the many dualities relating them under one umbrella. For more information, the reader is referred to the reviews [20, 44, 309, 336] (and to the 750-odd citations to Witten's paper [346]!)

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[5] At present Polchinski's seminal paper [275] has some 600 citations.

4.3.2 Light-like compactification

Define the light-cone coordinates

$$x^{\pm} = x^{11} \pm t \ . \tag{4.64}$$

Consider the case where x^- is periodic; typically this might be to IR regulate a theory, the zero-mode in the "minus" direction then being integrated out. This section will detail some general properties of theories defined on such a compact space.

All momenta in the minus direction will be quantised in units of the inverse radius of compactification

$$p_- = N/R_- \ . \tag{4.65}$$

Moreover, since there are no tachyons, $p_- \geq 0$. Of course p_- is also conserved. The system then splits up into an infinite number of sectors characterised by N. For instance, a scattering process with total $p_- = N/R_-$ will contain at most N particles each with $p_- = 1/R_-$. The scattering will be described by an multi-body Hamiltonian H_N.

H_N in the large-N limit defines the uncompactified theory. For consider a process with a number, n, of n_i-particle states scattering into a number m, of m_j-particle states with total $p_- = N/R_-$ [309]. Then

$$\sum_{i=1}^{n} n_i = \sum_{j=1}^{m} m_j = N \ . \tag{4.66}$$

The amplitude can be calculated using H_N. Now take the limits

$$N, R_-, n_i, m_j \to \infty \quad \text{with} \quad p_-^i = n_i/R_- \text{ and } k_-^j = m_j/R_- \text{ both fixed} \ . \tag{4.67}$$

In this limit the scattering process in question becomes an amplitude with $(m+n)$ external legs. Physically what is happening is that the n_i "partons" with $p_- = 1/R_-$ are becoming infinitely numerous and are thereby building up a state in the continuum. In the next section these partons are D0-branes (recall they carry one unit of momentum in their 11d description).

The construction just described is called the "discrete light-cone quantisation" (DLCQ) of a theory. The rest of this section is devoted to describing the construction of the DLCQ Hamiltonian in the sector N, H_N^{DLCQ}.

Compactification along a light-like direction can be defined as an infinite boost of the more usual case where a space-like direction is compact [304, 308]. For consider a theory which is compactified along the x^{11} direction with radius R_s

$$(x^{11}, t) \sim (x^{11}, t) + (2\pi R_s, t) \ . \tag{4.68}$$

Lorentz boost to a new (primed) frame

$$\begin{aligned} t' &= \frac{1}{\sqrt{1-v^2}}(t - v x^{11}) \ , \\ x' &= \frac{1}{\sqrt{1-v^2}}(x^{11} - vt) \ , \end{aligned} \tag{4.69}$$

which is moving at relative velocity v given by

$$v = \sqrt{1 - \frac{4R_s^2}{R_-^2}} \, . \tag{4.70}$$

In the boosted frame

$$\left(x^{11}, t\right)' \sim \left(x^{11}, t\right)' + \frac{2\pi R_s}{\sqrt{1-v^2}}(1, -v) \, . \tag{4.71}$$

In the limit as $R_s \to 0$

$$\left(x^{11}, t\right)' \sim \left(x^{11}, t\right)' + (\pi R_-, -\pi R_-) \, . \tag{4.72}$$

In the light-cone coordinates Eq. (4.64), it is clear that the infinitely boosted theory is compactified along the minus direction; $x^- \sim x^- + 2\pi R_-$.

Similar statements hold for the momentum. In the unprimed frame

$$p^{11} = \frac{N}{R_s} \, , \tag{4.73}$$

while in the boosted frame

$$\begin{aligned} p^{11\prime} &= \frac{1}{\sqrt{1-v^2}}(p^{11} - vp^0) \, , \\ p^{0\prime} &= \frac{1}{\sqrt{1-v^2}}(p^0 - vp^{11}) \, . \end{aligned} \tag{4.74}$$

As $R_s \to 0$, p_- becomes quantised

$$\begin{aligned} p_-' &= \tfrac{1}{2}(p^0 + p^{11})' = \frac{R_s}{2R_-}(p^0 + p^{11}) + O(R_s) \\ &= \frac{R_s}{2R_-}p^{11}\left(1 + \sqrt{1 + (\boldsymbol{p}^2 + m^2)/(p^{11})^2}\right) + O(R_s) \\ &= \frac{N}{R_-} + O(R_s) \, . \end{aligned} \tag{4.75}$$

Similarly, for a massless particle the light-cone energy can be written

$$\begin{aligned} p_+' &= \tfrac{1}{2}(p^0 - p^{11})' = \frac{R_-}{2R_s}p^{11}\left(\sqrt{1 + \boldsymbol{p}^2/p_{11}^2} - 1\right) + O(R_s) \\ &= \frac{R_-}{2N}\boldsymbol{p}^2 + O(R_s) \, . \end{aligned} \tag{4.76}$$

Evidently, p_- acts as a kind of light-cone mass [43].

Finally, following [304, 308], the Hamiltonian for the DLCQ of a theory will be derived via the infinite boost discussed above. It will thus be described as a limit of the Hamiltonian of the same theory compactified along a spacelike direction. The spacelike compactified theory is dubbed the "auxiliary theory". The following arguments are general but in the next subsection the theory in question will be M-theory and thus the "auxiliary theory" will be nothing but IIA string theory.

Denote the DLCQ Hamiltonian by H_N^{DLCQ}. Let it depend on a mass parameter M_N, the radius of compactification R_- and a number of dimensionless parameters (coupling constants, etc) $\{g\}$. Similarly denote the Hamiltonian in the auxiliary theory by $H_N^{\rm KK}$. This auxiliary theory is compactified along a spacelike direction of radius r and the particles live in the sector $p^{11} = N/r$. It has the same dimensionless parameters $\{g\}$. The (10d) rest mass of the particles in the auxiliary theory will be subtracted away so that the generator of time translations is $H_N^{\rm KK} + M$. Then in the limit $r \to 0$ with R_- fixed, the second line of Eq. (4.74) yields

$$H_N^{\rm DLCQ}(M, R_-, \{g\}) = \frac{R_-}{r}\left(H_N^{\rm KK}(M, r, \{g\}) + M - p^{11}\right) = \frac{R_-}{r} H_N^{\rm KK}(M, r, \{g\}) \ . \tag{4.77}$$

On dimensional grounds

$$\frac{R_-}{r} H_N^{\rm KK}(M, r, \{g\}) = H_N^{\rm KK}(\tfrac{R_-}{r} M, \tfrac{r}{R_-} r, \{g\}) \ . \tag{4.78}$$

Defining $r = \sqrt{RR_-}$ the result is

$$H_N^{\rm DLCQ}(M, R_-, \{g\}) = \lim_{R \to 0} H_N^{\rm KK}(M\sqrt{R_-/R}, R, \{g\}) \ . \tag{4.79}$$

This is now considered to be the definition of a theory compactified along a light-like direction [45, 46, 186, 304, 308].

4.3.3 Matrix theory

The definition Eq. (4.79) shall now be used to derive the Hamiltonian of DLCQ M-theory. The auxiliary theory is M-theory compactified along a spacelike direction. This depends on the parameter R which will be identified with the R_s used in previous sections. By definition, the auxiliary theory is nothing but IIA theory.

Writing the parameters of the auxiliary M-theory in terms of those of type IIA string theory, the DLCQ M-theory Hamiltonian can be found. The following equations are meant to demonstrate proportionalities only — all constants of proportionality are unimportant at this stage. IIA string theory has one dimensionless coupling $g_{\rm str}$, and the mass parameter $m_{\rm str} = (\alpha')^{-1/2}$. The radius of compactification is related to the coupling constant by Eq. (4.63); $\tilde{M} R_s = g_{\rm str}^{2/3}$. (The tilde on $\tilde{M}$ means that it belongs to the auxiliary theory, eg from Eq. (4.79) $\tilde{M} = M\sqrt{R_-/R_s}$ where M is the 11d Planck mass of the DLCQ theory.) Another relation between 11d and 10d quantities is found by equating the IIA supergravity action Eq. (4.18) with its 11d counterpart Eq. (4.40) [309, 346]. As was seen in Eq. (4.16), $\kappa_{10}^{-2} = m_{\rm str}^8 g_{\rm str}^{-2}$ while in the 11d theory $\kappa_{11}^{-2} = \tilde{M}^9$. This yields $\tilde{M}^9 R_s = m_{\rm str}^8 g_{\rm str}^{-2}$. Combining these two equations gives

$$m_{\rm str} = \tilde{M}^{3/2} R_s^{1/2} \ , \quad g_{\rm str} = (\tilde{M} R_s)^{3/2} \ . \tag{4.80}$$

In terms of the auxiliary M-theory and then the DLCQ M-theory parameters, the parameters of the 10d theory are then

$$\begin{aligned} m_{\rm str} &= \tilde{M}^{3/2} R_s^{1/2} = M^{3/2} R_-^{3/4} R_s^{-1/4} \ , \\ g_{\rm str} &= \tilde{M}^{3/2} R_s^{3/2} = M^{3/2} (R_- R_s)^{3/4} \ . \end{aligned} \tag{4.81}$$

In summary, the Hamiltonian for the DLCQ of M-theory has been related via an infinite boost to a Hamiltonian for M-theory compactified along a spacelike circle. This latter theory is by definition IIA string theory with the various parameter relations given in Eq. (4.81). Substituting these relations in the definition Eq. (4.79) gives the Hamiltonian for the DLCQ of M-theory

$$H_N^{\mathrm{DLCQ}}(M,R_-) = \lim_{R_s\to 0} H_N^{\mathrm{IIA}}(m_{\mathrm{str}} = M^{3/2}R_-^{3/4}R_s^{-1/4}, g_{\mathrm{str}} = M^{3/2}(R_-R_s)^{3/4})\ . \tag{4.82}$$

Taking the limit $R_s \to 0$, it is evident that the Hamiltonian for the DLCQ of M-theory is the IIA Hamiltonian with $m_{\mathrm{str}} \to \infty$ and $g_{\mathrm{str}} \to 0$.

In this limit the (10d) masses of the D0-branes tend to infinity (see Eq. (4.41)). Since the "partons" described in Sec. 4.3.2 (which have $p_- = 1/R_-$) are D0-branes, the DLCQ of M-theory is simply described by the nonrelativistic quantum mechanics of N D0-branes in the weakly coupled massless sector of IIA string theory! As was discussed in the last section, the corresponding action is the dimensional reduction of U(N) super-Yang-Mills from $d = 9+1$ to $d = 0+1$. Therefore, M-theory in the sector with $p_- = N/R_-$ is exactly described by supersymmetric quantum mechanics with 16 real supersymmetries and U(N) symmetry! This realisation of M-theory is called "Matrix theory" [22, 317].

A pessimist would take the view that Matrix theory is just a kinematical limit of M-theory; in this way it would be no more special than the low-energy 11d supergravity limit, or the small radius IIA limit or the limits which are the other string theories (these limits are predicted by taking various duals of IIA). The view taken here is that M-theory is defined through its DLCQ and the uncompactified theory can be obtained by taking the $N, R_- \to \infty$ limit with N/R_- fixed.

Clearly it is possible that all the limits ($R_s \to 0$, $N, R_- \to \infty$) may be taken in various different ways and so it is not entirely obvious that the line of argument presented here will in fact eventually produce what we think of as M-theory in 11 large dimensions [309]. This means, for example, that 11d supergravity may not correlate with the low-energy limit of finite-N Matrix theory. An alternative way to view the problem [43, 186] is to note that the tree approximation to 11d supergravity (which will be calculated in Chap. 5) compactified down to 10d is valid for distances larger than the 10d Newton constant κ_{10}^2;

$$r > G_{10}^{1/8} \sim M^{9/8}/R_s^{1/8}\ . \tag{4.83}$$

This tends to infinity as $R_s \to 0$. However, the length scales of interest in M-theory are fixed in units of M^{-1}. Thus as $R_s \to 0$ the classical scattering in supergravity does not necessarily agree with any prediction coming from low-energy Matrix theory: On the Matrix-theory side the loop corrections come as higher powers of N, as is standard in YM theory, and it is only the limit $N \to \infty$ that should agree with supergravity. Exact results are very difficult to obtain in this limit[6]. Therefore, if supergravity does agree with a finite N Matrix theory calculation

[6]The recent "Maldacena conjecture" [235, 348] provides a handle on this limit. Further comments will be made in Sec. 5.6

a nonrenormalisation theorem must have to come into effect. Perhaps is it surprising then that the two theories agree in virtually[7] all cases calculated thus far.

Before moving on to the next chapter where an explicit calculation involving D0-branes (supergravitons) is presented, the following are some tests/achievements of Matrix theory (see the reviews [20, 43, 44, 336] for more details).

- Matrix theory contains 2-branes and 5-branes [22, 24, 39, 68, 90, 323]. These are built from D0-branes and upon compactification give the various branes of string theory discussed on p. (94). Moreover, the one-loop gravitational potential between various Matrix theory objects has been found to agree with the predictions from supergravity [6, 21, 22, 28, 29, 32, 71, 72, 122, 167, 205, 211, 212, 223, 224, 237, 236, 268, 279]. At low order in velocities, the spin effects are also the same in the two theories [184, 218, 252, 269]. Some more detailed remarks will be made in the next chapter.
- The elucidation of black-hole entropy and the identification of their microstates has been a long-standing problem in physics since Bekenstein and Hawking. This and related issues are discussed within the framework of modern physics in the review [266].
- M-theory provides an umbrella under which the various string dualities may be studied. Dualities within, and compactifications of, Matrix theory are discussed in [5, 38, 142, 155, 156, 206, 290, 304, 305, 308, 312, 316].
- Lorentz invariance in Matrix theory is not at all manifest but some progress has been made [22, 91, 92, 140, 228].

[7] Some discrepancies seem to have been found in curved spaces and on orbifolds [33, 123, 124, 154].

D0-brane scattering

The effective action describing the scattering of three well-separated extremal brane solutions, in 11d supergravity, with zero p_- transfer and small transverse velocities is calculated. It is proved that to obtain this action only the leading-order solution to Einstein's equations is needed. The result obtained agrees with Matrix theory. Finally, using an interpretation of the conjecture of Maldacena the effective action can be viewed as the large-N limit of the Matrix theory description of three supergraviton scattering at leading order.

Matrix theory [22] proposes that M-theory is described by the maximally supersymmetric quantum mechanics of U(N) matrices in the large N limit. As was outlined in Chapter 4, it has also been argued [317] that Matrix theory at finite N corresponds to the DLCQ of M-theory. An important piece of supporting evidence for these conjectures is that the low-energy scattering of supergravitons in 11d supergravity agrees with the analogous processes in Matrix theory.

However, because the finite-N Matrix theory describes M-theory [304, 308] at distances short compared with the scales at which the supergravity approximation is valid there is no apriori reason why the two should agree. When they do agree it is because of the existence of a non-renormalisation theorem in the Matrix theory [46, 112, 114, 186, 262, 263]. So far the two have agreed for quite a variety of cases: the scattering of two D0-branes without spin and without longitudinal momentum transfer [22, 29, 28, 32, 122], with spin but no longitudinal momentum transfer [184, 218, 252, 269], and with longitudinal momentum transfer [21, 279]; the long-range potential between membranes and anti-membranes [6, 72], membranes and gravitons [6], D0-branes and other branes [71, 224], transverse 5-branes [223], 8-branes and other branes [268], and D0-branes and elementary strings [167]; and, more recently, the long-range potential between general objects of Matrix theory [205]. The scattering of D-brane probes in the background of near-BPS D-branes has also been studied [236, 237] and the general properties of the Matrix theory effective action have been investigated [211, 212].

In this chapter the supergravity side of the three graviton scattering process, without spin and longitudinal momentum transfer, is studied. An initial investigation of this case by Dine and Rajaraman (DR) suggested that for particular $3 \to 3$ processes supergravity disagreed with Matrix theory [113]. Here, by deriving the first few orders of the general supergravity effective action as an expansion small velocities and large separations of the D0-branes (supergravitons), it is shown that DR's effective action is incorrect. This chapter forms a much expanded version of [240].

This result has also been reported by Okawa and Yoneya [261] who performed a similar calculation, and, furthermore, calculated the two-loop effective action of Matrix theory in the eikonal approximation. The calculation presented here has the advantage that it is significantly more simple than their calculation. This simplification is achieved by proving that only the solution to the *linearised* Einstein equations is needed to obtain the supergravity effective action to the desired order.

Dine has subsequently traced the error in [113] to the derivation of the supergravity effective action from the S-matrix [111]. Other works that discuss the Dine-Rajaraman problem are [137, 141, 337].

Before proceeding with the calculation, a summary of DR's result will be given. An exact expression for the effective action is calculable in the limit where one graviton (labeled by "3") is far from the other two (called "1" and "2")

$$r_{13} \sim r_{23} \gg r_{12} . \tag{5.1}$$

By power counting the appropriate Feynman diagrams it was seen that the Matrix theory effective action did *not* contain a term of the form

$$\frac{v_3^4 v_{12}^2}{r_{13}^7 r_{12}^7} , \tag{5.2}$$

where v_3^a is the transverse velocity of the distant graviton and v_{12}^a is the relative transverse velocity of the other two. However, an analysis of the tree-level supergravity graphs showed that the S-matrix of supergravity *did* contain such a term. DR then went on to argue that the term also appeared in the supergravity effective action. By explicit calculation of the latter it is shown here that this is incorrect; rather the effective action looks like

$$S_{\text{eff}}^{\text{sugra}} \sim \frac{v_{12}^2 v_{13}^2 (v_{23}^2 - v_{13}^2 - v_{12}^2)}{r_{12}^7 r_{13}^7} + (1 \leftrightarrow 2) , \tag{5.3}$$

which does not contain a term of the form Eq. (5.2).

5.1 Notation, the light-cone and the infinite boost

As was outlined in the Chapter 4, the low energy dynamics of IIA string theory is governed by the IIA supergravity action. In the Einstein frame this action is [62]

$$\begin{aligned} S_{IIA} &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}t \mathrm{d}^9\vec{y}\sqrt{-g}\left(R - \tfrac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - \tfrac{1}{4}e^{3\Phi/2}F_{\mu\nu}F^{\mu\nu}\right) \\ &\quad + \text{boundary term} , \end{aligned} \tag{5.4}$$

where $g_{\mu\nu}$ is the metric in ten-dimensional space-time, R is its curvature, Φ is the dilaton and $F_{\mu\nu}$ the field strength for the R-R one-form A_μ. Space-time indices $\mu, \nu = 0, \ldots, 9$ and for convenience later on, space will be labeled by lower-case letters $a, b = 1, \ldots, 9$. The other massless fields are an antisymmetric 2-form and an R-R 3-form, and fermions. These will always be omitted in what follows since they will never be excited away from their vacuum values (zero). Similarly, as shall be seen below, only the metric and the R-R 1-form are included in source action. By truncating the theory in this fashion, all spin effects are thereby neglected. Finally, the boundary term in Eq. (5.4) is necessary in order that the action gives the correct equations of motion. Its precise form is discussed in detail in Sec. 5.4.

The action can be lifted into eleven dimensions using the relation

$$\mathrm{d}s_{11}^2 = e^{-\Phi/6}\mathrm{d}s_{10}^2 + e^{4\Phi/3}(\mathrm{d}x^{11} + A_\mu \mathrm{d}x^\mu)^2 \ , \tag{5.5}$$

where x^{11} is compactified with radius $R_s = M^{-1}\sqrt{e^{4\Phi/3}}$. This was discussed in Sec. 4.2.4. In the bulk the action becomes simply the Einstein action

$$S_E = \frac{1}{2\kappa_{10}^2}\frac{1}{2\pi R_s}\int \mathrm{d}^{11}x\sqrt{-g}R \equiv \frac{1}{2\kappa_{11}^2}\int \mathrm{d}^{11}x\sqrt{-g}R \ , \tag{5.6}$$

where the quantities on the RHS are 11-dimensional. Capital letters $M, N, \ldots$ will be used to denote 11d indices.

In 11d a solution to the field equations is given by Eq. (4.47)

$$\mathrm{d}s_{11}^2 = (\tilde{f}-2)\mathrm{d}t^2 + \tilde{f}(dx^{11})^2 - 2(f-1)\mathrm{d}t\mathrm{d}x^{11} + \mathrm{d}\vec{y}\cdot\mathrm{d}\vec{y} \ , \tag{5.7}$$

where $\tilde{f}$ is a harmonic function of the transverse coordinates $\vec{y}$ given in Eq. (4.23)

$$\tilde{f}(\vec{y}) = 1 + \frac{30\kappa_{10}^2}{32\pi^4}\sum_{i=1}^{N_c}\tilde{Q}_i\frac{1}{|\vec{y}-\vec{y}_i|^7} \ . \tag{5.8}$$

The 10d reduction of this solution describes N_c massive charged point-particles (bound states of D0-branes in string language) with mass = charge = $\tilde{Q}_i$ located at transverse positions $\vec{y}_i$, which are the bosonic parts of a Kaluza-Klein reduction of N_c supergravitons moving with momentum $p_i^{11} = \tilde{Q}_i$. In Sec. 4.2.4 it was seen that the source action of such a particle is the massless limit of the standard massive point-particle action with the constraint that the momentum in the eleventh direction is held constant equal to $\tilde{Q}$.

It is calculationally more convenient to express everything in light-cone coordinates $x^\pm = x^{11} \pm t$ and infinitely boost so that the theory becomes compactified along the light-like direction x^-

$$x^- \sim x^- + 2\pi R_- \ . \tag{5.9}$$

This procedure was detailed in Sec. 4.3.2.

In these coordinates, the gravity action is of course, unchanged

$$S_E = \frac{1}{2\kappa_{11}^2}\int \mathrm{d}^{11}x\sqrt{-g}R + \text{ boundary term} \ . \tag{5.10}$$

Expressed in light-cone coordinates the static solution is

$$\mathrm{d}s_{11}^2 = \mathrm{d}x^+\mathrm{d}x^- + f\mathrm{d}x^-\mathrm{d}x^- + \mathrm{d}\vec{y}\cdot\mathrm{d}\vec{y}\ . \tag{5.11}$$

Here f is a new harmonic function which is not asymptotically flat

$$f = \tilde{f} - 1 = \frac{30\kappa_{10}^2}{32\pi^4}\sum_{i=1}^{N_c}\tilde{Q}_i\frac{1}{|\vec{y}-\vec{y}_i|^7} = \frac{15\kappa_{11}^2}{32\pi^5 R_s}\sum_{i=1}^{N_c}\frac{N_i}{R_s}\frac{1}{|\vec{y}-\vec{y}_i|^7}\ . \tag{5.12}$$

Under the infinite boost g_{--} is the only metric component that changes[1]

$$\mathrm{d}x^-\mathrm{d}x^- = \frac{R_s^2}{R_-^2}\mathrm{d}x'^-\mathrm{d}x'^-\ . \tag{5.13}$$

Thus, in the boosted frame the static solution is given by Eq. (5.11) with

$$f \ \rightarrow\ \frac{R^2}{R_-^2}f = \frac{15\kappa_{11}^2}{32\pi^5 R_-}\sum_{i=1}^{N_c}Q_i\frac{1}{|\vec{y}-\vec{y}_i|^7}\ , \tag{5.14}$$

with $Q_i = N_i/R_-$.

After the infinite boost it is $p_- = N/R_-$ which must be held constant in the source action. A convenient way of implementing this constraint is to utilise the Routhian [29, 166]

$$S_{\mathrm{source}} = \lim_{m\to 0}\left(p_- v^- - S_m\right)\ , \tag{5.15}$$

where S_m is defined in Eq. (4.42). This is part way between a Lagrangian and a Hamiltonian. Parameterising the particle's worldline with x^+, the Noether charge p_- for S_m is simply

$$p_- = \frac{m}{\sqrt{-g_{MN}\frac{dx^M}{dx^+}\frac{dx^N}{dx^+}}}\left(g_{-+} + g_{-a}v^a + g_{--}v^-\right) = N/R_-\ , \tag{5.16}$$

where v^a and v^- are the velocities in the a and $-$ directions respectively. Evidently, as the massless limit is taken the denominator must tend to zero. An expression for v^- in terms of the other velocities and the metric is thus obtained. The second term of Eq. (5.15) clearly vanishes in the massless limit so the source action for the i^{th} particle is

$$S_i = Q_i\int \mathrm{d}x^+\frac{1}{g_{--}}\left\{g_{+-} + g_{-a}v_i^a - \sqrt{(g_{+-} + g_{-a}v_i^a)^2 - g_{--}(g_{++} + 2g_{+a}v_i^a + g_{ab}v_i^a v_i^b)}\right\}\ , \tag{5.17}$$

with $Q_i = N/R_-$. Naturally, all metric components are to be evaluated at the transverse positions $\vec{y}_i^a(x^+)$.

In summary then, the total action is

$$S = \frac{1}{2\kappa_{11}^2}\int \mathrm{d}^{11}x\sqrt{-g}R + \text{boundary term} + \sum_{i=1}^{N_c}S_i\ , \tag{5.18}$$

[1]The infinite boost of supergravity solutions was studied extensively in [196, 197].

The aim is to solve Einstein's equations ($2\pi R_-$ comes from the integration over x^-)

$$R_{MN} - \tfrac{1}{2}g_{MN}R + \frac{2\kappa_{11}^2}{2\pi R_-}T_{MN} = 0 \ , \tag{5.19}$$

order-by-order in the small velocities (V) and inverse-separations (L^{-1}) of the D-particles. Alternatively (and, as shall be shown later, equivalently), Einstein's equations can be solved order-by-order in κ_{11}^2 expanding around the flat-space metric

$$g_{MN} = \eta_{MN} = \begin{pmatrix} \eta_{++} & \eta_{+-} & \eta_{+b} \\ \eta_{-+} & \eta_{--} & \eta_{-b} \\ \eta_{a+} & \eta_{a-} & \eta_{ab} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_{ab} \end{pmatrix} . \tag{5.20}$$

This second method is more concise and it is the one that will be presented here. The particles must follow geodesics, and so the auxiliary condition

$$\nabla_M T^{MN} = 0 \ , \tag{5.21}$$

must also be solved. Once these equations have been solved to the desired order, the metric obtained will be substituted back into the action to yield an approximate effective action as an expansion in small V and L^{-1}. The ultimate aim is to find the effective action to $O(V^6/L^{14})$.

5.2 The first-order solution

Define

$$g_{MN} = \eta_{MN} + \kappa_{11}^2 \overset{2}{g}_{MN} + O(\kappa_{11}^4) \ . \tag{5.22}$$

This section will find $\overset{2}{g}_{MN}$. Some relevant formulae are collected in App. D. There it is shown that in the gauge

$$g^{MN}\Gamma^P{}_{MN} = 0 \ , \tag{5.23}$$

where $\Gamma^P{}_{MN}$ are the Christoffel symbols of the metric g_{MN}, the Ricci tensor is

$$\begin{aligned} R_{MN} \ = \ & -\tfrac{1}{2}g^{PQ}\partial_P\partial_Q g_{MN} + \tfrac{1}{4}\partial_M g^{PQ}\partial_N g_{PQ} - \partial_P g_{Q(M}\partial_{N)}g^{PQ} \\ & -\tfrac{1}{2}g^{PX}g^{QY}\partial_Q g_{XM}\partial_P g_{YN} + \tfrac{1}{2}g^{PX}g^{QY}\partial_X g_{MQ}\partial_P g_{YN} \ . \end{aligned} \tag{5.24}$$

Expanding to $O(\kappa_{11}^2)$ yields

$$R_{MN} = -\tfrac{1}{2}\kappa_{11}^2\Box_\perp \overset{2}{g}_{MN} + O(\kappa_{11}^4) \ , \tag{5.25}$$

where $\Box_\perp = \delta^{ab}\partial_a\partial_b$ is the Laplacian in the transverse space. The expansion of the energy-momentum tensor

$$T_{MN} = \frac{1}{\sqrt{-g}}\frac{\delta S_{\rm source}}{\delta g^{MN}} \ , \tag{5.26}$$

to $O(\kappa_{11}^2)$ is just as simple

$$\begin{aligned} T_{++} &= -\tfrac{1}{8}\textstyle\sum_i Q_i\vec{v}_i^{\,4}\delta_i & T_{+-} &= \tfrac{1}{4}\textstyle\sum_i Q_i\vec{v}_i^{\,2}\delta_i & T_{+a} &= \tfrac{1}{4}\textstyle\sum_i Q_i v_i^a\vec{v}_i^{\,2}\delta_i \\ T_{--} &= -\tfrac{1}{2}\textstyle\sum_i Q_i\delta_i & T_{-a} &= -\tfrac{1}{2}\textstyle\sum_i Q_i v_i^a\delta_i & T_{ab} &= -\tfrac{1}{2}\textstyle\sum_i Q_i v_i^a v_i^b\delta_i \ . \end{aligned} \tag{5.27}$$

In these formulae $\delta_i = \delta^9(\vec{y} - \vec{y}_i)$. Therefore, at this order, solving Einstein's equations simply reduces to finding a Green's function f;

$$-\Box_\perp f = \frac{1}{\pi R_-} \sum_{i=1}^{N_c} Q_i \delta_i \ , \tag{5.28}$$

(the annoying factor of $2\pi R_-$ will thus also be soaked-up). This is well-known in arbitrary dimensions, and in $d = 9$ is

$$f(\vec{y} - \vec{y}_i) \equiv \sum_{i=1}^{N_c} f_i = \frac{15}{32\pi^5 R_-} \sum_{i=1}^{N_c} Q_i \frac{1}{|\vec{y} - \vec{y}_i|^7} \ . \tag{5.29}$$

Thus

$$\overset{2}{g}_{MN} = \begin{pmatrix} \overset{2}{g}_{++} & \overset{2}{g}_{+-} & \overset{2}{g}_{+b} \\ \overset{2}{g}_{-+} & \overset{2}{g}_{--} & \overset{2}{g}_{-b} \\ \overset{2}{g}_{a+} & \overset{2}{g}_{a-} & \overset{2}{g}_{ab} \end{pmatrix} = \sum_{i=1}^{N_c} f_i \begin{pmatrix} \frac{1}{4}\vec{v}_i^4 & -\frac{1}{2}\vec{v}_i^2 & -\frac{1}{2}v_i^b \vec{v}_i^2 \\ -\frac{1}{2}\vec{v}_i^2 & 1 & v_i^b \\ -\frac{1}{2}v_i^a \vec{v}_i^2 & v_i^a & v_i^a v_i^b \end{pmatrix} \ . \tag{5.30}$$

Evidently, at zeroth order in the velocities the static solution Eq. (5.11) is reproduced.

In [240] this was called the "independently boosted" metric since it is simply the superposition of N_c metrics, each one corresponding to a single centre moving transversely with constant velocity. For, take the metric corresponding to a single particle at rest;

$$g_{MN} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & f & 0 \\ 0 & 0 & \delta^{ab} \end{pmatrix} \ . \tag{5.31}$$

To first order in the transverse velocity v^a, a Lorentz boost takes

$$\delta_v x^+ = -\vec{v}\cdot\vec{y} \ , \quad \delta_v x^- = \vec{v}\cdot\vec{y} \ , \quad \delta_v \vec{y} = -\tfrac{1}{2}x^+\vec{v} + \tfrac{1}{2}x^-\vec{v} \ . \tag{5.32}$$

The boost to use is not the full Lorentz boost, but rather a boost which preserves light-cone time x^+. This subgroup of the Lorentz group is called the Galilean group. In order to fix x^+, a compensating rotation in the 11-v plane through angle $\theta = |\vec{v}|$ must be made. To first order this is

$$\delta_\theta x^+ = \frac{\theta}{|\vec{v}|}\vec{v}\cdot\vec{y} \ , \quad \delta_\theta x^- = \frac{\theta}{|\vec{v}|}\vec{v}\cdot\vec{y} \ , \quad \delta_\theta \vec{y} = -\frac{\theta}{2|\vec{v}|}x^+\vec{v} - \frac{\theta}{2|\vec{v}|}x^-\vec{v} \ . \tag{5.33}$$

Taking this diagonal combination and integrating it, the expression for a finite transverse boost with velocity v^a is found

$$\begin{aligned} x'^+ &= x^+ \ , \\ x'^- &= x^- + 2\vec{v}\cdot\vec{y} - \vec{v}^2 x^+ \ , \\ \vec{y}' &= \vec{y} - x^+\vec{v} \ . \end{aligned} \tag{5.34}$$

Taking the primed coordinates as those of the static frame, the metric in the constant velocity frame is simply

$$g_{MN} = \eta_{MN} + f \begin{pmatrix} \frac{1}{4}\vec{v}^4 & -\frac{1}{2}\vec{v}^2 & -\frac{1}{2}v^b\vec{v}^2 \\ -\frac{1}{2}\vec{v}^2 & 1 & v^b \\ -\frac{1}{2}v^a\vec{v}^2 & v^a & v^a v^b \end{pmatrix} \ . \tag{5.35}$$

A linear combination of such metrics gives the "independently boosted" metric of Eq. (5.30).

Before substituting this solution into the action some remarks will be made about the higher-order corrections to the metric. It is then proved that in order to obtain the effective action to $O(V^6/L^{14})$ only the independently boosted metric is needed.

5.3 The form of higher-order solutions

5.3.1 Acceleration

If it were necessary to continue the process in the previous section to higher orders the first thing that would have to be checked is that the solution derived is consistent with the geodesic equation $\nabla_M T^{MN} = 0$. Since the moving D-particles produce a non-trivial metric, the geodesic equation will reduce to expressions for the accelerations in terms of the velocities of the particles. It is of use to know the form of these equations at $O(\kappa_{11}^2)$.

The geodesic equation reads

$$0 = \partial_M T^{MN} + \Gamma^M{}_{MP} T^{PN} + \Gamma^N{}_{MP} T^{MP} = \partial_M T^{MN} + \tfrac{1}{2} g^{MN} \partial_P g_{MN} T^{PN} + \Gamma^N{}_{MP} T^{MP} \ . \quad (5.36)$$

At $O(\kappa_{11}^2)$ the second term disappears entirely because $\eta^{MN} \overset{2}{g}_{MN} = 0$ so that $g^{MN}\partial_P g_{MN} = O(\kappa_{11}^4)$. Concentrate on the case where N is a spatial index, $N = a$. By substituting $\overset{2}{g}_{MN}$ into the energy-momentum tensor Eq. (5.26) the relevant components of the energy-momentum tensor are (for more details see App. D)

$$\begin{aligned} T^{+a} &= \tfrac{1}{2} \sum_i Q_i v_i^a \delta_i + \tfrac{1}{4}\kappa_{11}^2 \sum_{i,j} Q_i f_j v_i^a (\vec{v}_i - \vec{v}_j)^2 + O(\kappa_{11}^4) \ , \\ T^{ab} &= \tfrac{1}{2} \sum_i Q_i v_i^a v_i^b \delta_i + \tfrac{1}{4}\kappa_{11}^2 \sum_{i,j} Q_i f_j v_i^a b_i^b (\vec{v}_i - \vec{v}_j)^2 + O(\kappa_{11}^4) \ . \end{aligned} \quad (5.37)$$

The result that will now be proved is that

$$\dot{v}_i = O(\kappa_{11}^2 V^4/L^7) \ . \quad (5.38)$$

For note that T^{+a} and T^{ab} are almost identical so the first term in the geodesic equation reads

$$\partial_M T^{Ma} = \tfrac{1}{2} \sum_{ij} Q_i (\partial_+ + v_i^b \partial_b) \left(v_i^a (1 + \tfrac{1}{2}\kappa_{11}^2 f_j (\vec{v}_i - \vec{v}_j)^2) \delta_i \right) \ . \quad (5.39)$$

The factor of δ_i can be pulled out since $(\partial_+ + v_i^b \partial_b)\delta_i = 0$. Similarly, $(\partial_+ + v_i^a \partial_a) f_j = (v_i - v_j)\cdot\partial f_j$, so that this first term is

$$\partial_M T^{Ma} = \tfrac{1}{2} \sum_{ij} Q_i \delta_i \left\{ \dot{v}_i^a + \tfrac{1}{2}\kappa_{11}^2 v_i^a (\vec{v}_i - \vec{v}_j)^2 (v_i - v_j)\cdot\partial f_j + \tfrac{1}{2}\kappa_{11}^2 v_i^a f_j \partial_+ (\vec{v}_i - \vec{v}_j)^2 \right\} \ . \quad (5.40)$$

Although the exact expressions are somewhat messy, by consulting Eq. (5.27) and Eq. (5.30), it becomes clear that the last term in the geodesic equation is $O(V^4)$,

$$\Gamma^a{}_{MN} T^{MP} = \partial_M h_{aP} T^{MP} - \tfrac{1}{2}\partial_a h_{MP} T^{MP} = O(\kappa_{11}^2 V^4/L^7) \ . \quad (5.41)$$

Combining this with Eq. (5.40) results in Eq. (5.38). The other components of the geodesic equation give constraints on $\dot{v}^a$ that are consistent with the above (for instance $N = -$ is simply $(v_i - v_j)\cdot\dot{v} = 0$ to $O(\kappa_{11}^2)$). Of course the relation $\dot{v} \propto V^4$ is well-known for it is just the "flatness of moduli space" [17, 214, 330].

5.3.2 The scaling behaviour of the $O(\kappa_{11}^4)$ part of the metric

Now expand to the next order $g_{MN} = \eta_{MN} + \kappa_{11}^2 \overset{2}{g}_{MN} + \kappa_{11}^4 \overset{4}{g}_{MN} + O(\kappa_{11}^6)$. Each component of $\overset{4}{g}_{MN}$ depends on a particular power of V and L. This scaling behaviour will now be derived. Using the general form of the Ricci tensor Eq. (5.24), Einstein's equations at $O(\kappa_{11}^4)$ read

$$\begin{aligned} -\tfrac{1}{2}\Box_\perp \overset{4}{g}_{MN} &= -\tfrac{1}{2}\overset{2}{g}{}^{PQ}\partial_P\partial_Q\overset{2}{g}_{MN} + \tfrac{1}{4}\partial_M\overset{2}{g}{}^{PQ}\partial_N\overset{2}{g}_{PQ} - \partial_P\overset{2}{g}_{Q(M}\partial_{N)}\overset{2}{g}{}^{PQ} \\ &\quad +\tfrac{1}{2}\partial_Q\overset{2}{g}_{PM}\partial^P\overset{2}{g}{}^Q{}_N - \tfrac{1}{2}\partial_P\overset{2}{g}_{QM}\partial^P\overset{2}{g}{}^Q{}_N - \frac{1}{\kappa_{11}^2\pi R_-}\overset{2}{T}_{MN} \ . \end{aligned} \tag{5.42}$$

The indices on $\overset{2}{g}$ and $\overset{4}{g}$ have been raised using the flat metric η^{MN}, for example $g^{MN} = \eta^{MN} - \kappa_{11}^2\overset{2}{g}{}^{MN} + O(\kappa_{11}^4)$. Naturally, $\overset{2}{T}_{MN}$ is the $O(\kappa_{11}^2)$ part of the energy-momentum tensor. Firstly by utilising the Green's function Eq. (5.29) it is clear that $\overset{4}{g}$ is $O(L^{-14})$. Also, by explicitly substituting $\overset{2}{g}$ and using that $\dot{V} \sim V^4$, the powers of V and L contained in the various components of $\overset{4}{g}$ are

$$\overset{4}{g}_{MN} = \begin{pmatrix} \overset{4}{g}_{++} & \overset{4}{g}_{+-} & \overset{4}{g}_{+b} \\ \overset{4}{g}_{-+} & \overset{4}{g}_{--} & \overset{4}{g}_{-b} \\ \overset{4}{g}_{a+} & \overset{4}{g}_{a-} & \overset{4}{g}_{ab} \end{pmatrix} \sim \begin{pmatrix} V^6/L^{14} & V^4/L^{14} & V^5/L^{14} \\ V^4/L^{14} & V^2/L^{14} & V^3/L^{14} \\ V^5/L^{14} & V^3/L^{14} & V^4/L^{14} \end{pmatrix} . \tag{5.43}$$

Moreover, in this calculation the acceleration terms are always higher-order. Evidently $\overset{4}{g}$ is $O(V^2)\times\overset{2}{g}$. Although the fact is not needed here, this progression continues to higher order; $\overset{2n}{g}_{MN}\sim O(V^{2n-2})\times\overset{2}{g}$.

5.3.3 Scaling properties of the higher-order terms

Finally, it is now quite easy to find the general scaling properties of the higher-order terms in g_{MN}. The gauge Eq. (5.23) has the appealing feature that at $O(\kappa_{11}^{2n})$

$$R_{MN}|_{\kappa_{11}^{2n}} = -\tfrac{1}{2}\kappa_{11}^{2n}\Box_\perp \overset{2n}{g}_{MN} + \Big(\text{ terms involving } \overset{m}{g}_{MN} \text{ for } m < 2n\Big) \ . \tag{5.44}$$

Similarly, due to the κ_{11}^2 which multiplies T_{MN} in the Einstein equation, the energy-momentum tensor involves only $\overset{m}{g}_{MN}$. Einstein's equations

$$\Box_\perp \overset{2n}{g}_{MN} = \text{ lower-order terms} \ , \tag{5.45}$$

can therefore be solved formally by the Green's function Eq. (5.29). This means that at each order another power of Q_i/L^7 is introduced, so

$$\overset{2n}{g}_{MN} = O(Q^n/L^{7n}) \ . \tag{5.46}$$

5.4 The boundary term

Standard arguments (see for example [334]) imply that in addition to the usual $\frac{1}{2\kappa_{11}^2}\int\sqrt{-g}R$ term, the Einstein action must also include a boundary term;

$$S_E = \frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}}\sqrt{-g}R + \frac{1}{2\kappa_{11}^2}\int_{\partial\mathcal{M}}\sqrt{h}K \ . \tag{5.47}$$

The quantity K which is integrated over the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$ with induced metric h_{MN} is the extrinsic curvature

$$K = \nabla_M n^M - n^M n_N \nabla_M n^N \ . \tag{5.48}$$

Here n_M is the unit normal to the boundary. The extra term must be included in general because when deriving Einstein's equations a boundary term appears

$$\delta\int_{\mathcal{M}}\sqrt{g}R = \int_{\mathcal{M}}\sqrt{-g}(R_{MN} - \tfrac{1}{2}g_{MN}R)\delta g^{MN} + \int_{\mathcal{M}}\sqrt{-g}\nabla^M B_M \ , \tag{5.49}$$

where the vector B_M is

$$B_M = \nabla^N(\delta g_{MN}) - g^{NP}\nabla_M(\delta g_{NP}) \ . \tag{5.50}$$

Assuming that δg_{MN} is zero on the boundary this extra contribution is canceled by the extrinsic curvature term:

$$\delta S_E = \int_M (R_{MN} - \tfrac{1}{2}g_{MN}R)\delta g^{MN} \quad \text{for} \quad \delta g^{MN}\Big|_{\partial\mathcal{M}} = 0 \ . \tag{5.51}$$

In this construction it is assumed that a unit normal can be constructed. If the boundaries are light-like then this is impossible. In the case at hand this problem is circumvented by never[2] using the vacuum solution η_{MN} but always considering the corrected metric $g_{MN} = \eta_{MN} + \kappa_{11}^2 \overset{2}{g}_{MN} + \ldots$.

5.5 The effective action to $O(V^6/L^{14})$

Now that the preliminaries have been dispensed with the following result can be proved:

Result: The effective action to $O(V^6/L^{14})$ is found using the solution $g_{MN} = \eta_{MN} + \kappa_{11}^2 \overset{2}{g}_{MN}$ only; no higher-order solutions to Einstein's equations are needed.

Proof: To obtain the effective action, the solution g_{MN} is substituted into the action Eq. (5.18). Expand

$$g_{MN} = \eta_{MN} + \kappa_{11}^2 \overset{2}{g}_{MN} + \overset{>}{g}_{MN} \equiv \hat{g}_{MN} + \overset{>}{g}_{MN} \ . \tag{5.52}$$

Then, because $\overset{2n}{g} \sim O(L^{-7n})$

$$S[g] = S[\hat{g}] + \frac{\delta S}{\delta g}\bigg|_{\hat{g}}[\overset{>}{g}] + \text{ higher order in } L^{-1} \ . \tag{5.53}$$

[2] This problem is easy to overcome; a small regulating parameter ϵ^2 can be introduced into the $++$ and $--$ components of the metric so the boundaries become non-null. At the end of the calculation $\epsilon^2 \to 0$.

Now because of the boundary term, the second term is simply

$$\left.\frac{\delta S}{\delta g}\right|_{\hat{g}}[\overset{>}{g}] = \frac{1}{2\kappa_{11}^2}\left\{R_{MN}[\hat{g}] - \tfrac{1}{2}\hat{g}_{MN}R[\hat{g}] + 2\kappa_{11}^2 T_{MN}[\hat{g}]\right\}\overset{>}{g}^{MN} . \tag{5.54}$$

However, by construction the term in curly parentheses is zero up to $O(\overset{>}{g})$. The scaling behaviour of the corrections was written in Eq. (5.43). By multiplying this with $\overset{>}{g}^{MN}$ (to this order $\overset{4}{g}^{MN}$ can be used) gives

$$\left.\frac{\delta S}{\delta g}\right|_{\hat{g}}[\overset{>}{g}] = O(V^8) . \tag{5.55}$$

There is one final subtlety and that is in writing Eq. (5.54) it has been assumed that $\overset{>}{g}$ vanishes on the boundary. This is clearly true at spatial infinity but not at the x^+ and x^- boundaries. The latter is trivial since the coordinate x^- is cyclic so the contribution from $x^- = 0$ exactly cancels with that from $x^- = 2\pi R_-$. The former will certainly add terms to the action but these will be total derivatives in the Lagrangian:

$$\int_{x^+=\infty} n_M B^M(\overset{>}{g}) - \int_{x^+=-\infty} n_M B^M(\overset{>}{g}) = \int_{\mathcal{M}} \frac{d}{dx^+}B^+(\overset{>}{g}) . \tag{5.56}$$

Thus the result is proved. The effective action can now be calculated. For ease of notation drop the "hat" from $\hat{g}_{MN}$ since it is the only metric that will be used from now on.

5.5.1 The source action

Substituting g_{MN} into the source action yields

$$\begin{aligned}
S_{\text{source}}[g] &= \sum_k \int \mathrm{d}x^+ \tfrac{1}{2}Q_k\vec{v}_k^2 + \tfrac{1}{8}\kappa_{11}^2 \sum_{i,k}\int \mathrm{d}x^+ Q_k f_i(\vec{v}_i - \vec{v}_k)^4 \\
&\quad + \tfrac{1}{16}\kappa_{11}^4 \sum_{i,j,k} Q_k \int f_i f_j \left\{(\vec{v}_i - \vec{v}_k)^2(\vec{v}_j - \vec{v}_k)^4 + \vec{v}_k^4(\vec{v}_j^2 - \vec{v}_i^2 - 2\vec{v}_j\!\cdot\!\vec{v}_k + 2\vec{v}_i\!\cdot\!\vec{v}_k)\right\} \\
&= \sum_k \int \mathrm{d}x^+ \tfrac{1}{2}Q_k\vec{v}_k^2 + \frac{15}{16}\sum_{i,k}\frac{Q_iQ_k}{M^9R_-}\int \mathrm{d}x^+ \frac{(\vec{v}_i - \vec{v}_k)^4}{|\vec{y}_i - \vec{y}_k|^7} \\
&\quad + \left(\frac{15}{16}\right)^2 4\sum_{i,j,k}\frac{Q_iQ_jQ_k}{M^{18}R_-^2}\int \mathrm{d}x^+ \frac{(\vec{v}_i - \vec{v}_k)^4(\vec{v}_j - \vec{v}_k)^2}{|\vec{y}_i - \vec{y}_k|^7|\vec{y}_j - \vec{y}_k|^7} + O(\kappa_{11}^6 V^8/L^{21}) .
\end{aligned} \tag{5.57}$$

To obtain the last term in the second line note that the terms inside the curly parentheses must be symmetric in i and j. Also the form of the Green's function f_i given by Eq. (5.29) has been used, the 11d Planck mass M has been defined

$$\kappa_{11}^2 = 16\pi^5/M^9 , \tag{5.58}$$

in keeping with the notation of [29, 32], and the whole expression evaluated at $\vec{y} = \vec{y}_k$. The potential divergences resulting from this last step are removed through the regularisation of the point-particle

$$f_i \to \frac{15}{32\pi^5 R_-}\frac{1}{((\vec{y} - \vec{y}_i)^2 + \epsilon^2)^{7/2}} . \tag{5.59}$$

Only at the very end of the calculation has the limit $\epsilon \to 0$ be taken.

5.5.2 The boundary action

The contribution to the effective action from the boundary term is also reasonably straightforward to calculate. As mentioned in the proof above the contributions from the $x^{\pm}$ boundaries can be neglected; only the spatial boundary needs to be considered,

$$S_{\text{bdy}}[g] = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}x^- \mathrm{d}x^+ \int_{S^8} K \ . \tag{5.60}$$

The spatial boundary has been chosen to be the eight-sphere S^8 of radius r. The volume form on S^8 will be determined by the induced measure. Since n^N is a unit normal (with the metric g)

$$n_N \nabla_M n^N = \tfrac{1}{2} g_{NP} \nabla_M n^N n^P = \tfrac{1}{2} \nabla_M (1) = 0 \ , \tag{5.61}$$

and K reduces to

$$K = \nabla_M n^M = \partial_M n^M + \frac{1}{\sqrt{-g}} n^M \partial_M \sqrt{-g} \ . \tag{5.62}$$

From Eq. (5.30) is is clear that

$$\sqrt{-g} = 1 + O(\kappa_{11}^4/L^{14}) \ . \tag{5.63}$$

Since at the end of the calculation the limit $r \to \infty$ will be taken and $\int_{S^8} \sim \int r^8 \mathrm{d}\Omega_8$, simple power counting implies that all terms in K of $O(L^{-n})$ where $n \geq 9$ will give a vanishing result. In particular this means that $\sqrt{-g}$ is effectively unity. All such higher order terms are dropped for the rest of this subsection.

First the induced measure will be calculated and then the extrinsic curvature. Transform from Cartesian coordinates (x) into spherical (s) coordinates

$$x^M = x^M(x^+, x^-, r, \theta) \equiv x^M(s) \ . \tag{5.64}$$

Let $\hat{P}$ denote indices *without the r component.* The induced measure includes the Jacobian

$$\begin{aligned} \sqrt{-\det \frac{\partial x^M}{\partial s^{\hat{P}}} \frac{\partial x^N}{\partial s^{\hat{Q}}} g_{MN}(x)} &= \sqrt{-\det \left(\eta_{\hat{P}\hat{Q}}(s) + \kappa_{11}^2 \overset{2}{g}_{\hat{P}\hat{Q}}(s) \right)} \ , \\ &= \sqrt{1 + \eta^{\hat{P}\hat{Q}} \overset{2}{g}_{\hat{P}\hat{Q}}(s)} \sqrt{-\det \eta_{\hat{P}\hat{Q}}(s)} \ . \end{aligned} \tag{5.65}$$

All terms of $O(\kappa_{11}^4)$ have been dropped as explained above. However, upon noting that

$$\eta^{\hat{P}\hat{Q}} \overset{2}{g}_{\hat{P}\hat{Q}}(s) = \eta^{MN} \overset{2}{g}_{MN}(x) = 0 \ , \tag{5.66}$$

the induced measure reduces to the standard form

$$\int_{S^8} = \int r^8 \mathrm{d}\Omega_8 \ . \tag{5.67}$$

The second step is to calculate the extrinsic curvature. Only the first term ($\partial_N n^N$) contributes since

$$n^N (1/\sqrt{g}) \partial_N \sqrt{g} \sim r^{-14} \ . \tag{5.68}$$

The M^{th} component of the $\hat{P}^{\text{th}}$ tangent vector to $S^8 \times R \times [0, 2\pi R_-]$ is

$$t_{\hat{P}}^{M} = \frac{\partial x^M}{\partial s^{\hat{P}}} \ . \tag{5.69}$$

Solving the equation $0 = n^N g_{MN} t_{\hat{P}}^M$ to $O(\kappa_{11}^2)$ gives

$$n^N = \frac{\delta_a^N y^a - \kappa_{11}^2 \eta^{NM} \overset{2}{g}_{aM} y^a}{\sqrt{y^a y^b \eta_{ab} - \kappa_{11}^2 y^a y^b \overset{2}{g}_{ab}}} \tag{5.70}$$

whereupon

$$\begin{aligned} K &= \frac{8}{r} + \kappa_{11}^2 \left(\frac{5 y^a y^b \overset{2}{g}_{ab}}{r^3} - \frac{\eta^{ab} \overset{2}{g}_{ab}}{r} + \frac{1}{2r^3} y^a y^b y^c \partial_a \overset{2}{g}_{bc} - \frac{1}{r} y^a \eta^{NM} \partial_N \overset{2}{g}_{aM} \right) \\ &= \frac{8}{r} + \frac{15}{32\pi^5 R_-} \kappa_{11}^2 \frac{1}{r^8} \sum_i Q_i \left(\frac{3}{2} \frac{(\vec{y} \cdot \vec{v}_i)^2}{r^2} - \vec{v}_i^2 \right) , \end{aligned} \tag{5.71}$$

where all terms which tend too rapidly to zero as $r \to \infty$ have been dropped. The volume of the unit S^8 is $32\pi^4/105$ so integrating over x^- and this sphere produces

$$S_{\text{bdy}}[g] = -\frac{5}{21} \int \mathrm{d}x^+ \sum_i \tfrac{1}{2} Q_i \vec{v}_i^2 \ . \tag{5.72}$$

5.5.3 The gravity action in the bulk

All that remains is to calculate

$$S_{\text{bulk}} = \frac{1}{2\kappa_{11}^2} \int \sqrt{-g} R \ . \tag{5.73}$$

The calculation is straightforward but tedious. Taking a trace of the Ricci tensor Eq. (5.24) yields

$$R = -\tfrac{1}{2} g^{PQ} g^{MN} \partial_P \partial_Q g_{MN} - \tfrac{1}{4} g^{MN} \partial_M g^{PQ} \partial_N g_{PQ} - \tfrac{1}{2} g^{MN} \partial_P g_{QM} \partial_N g^{PQ} \ . \tag{5.74}$$

The effective action can then be found as an expansion in κ_{11}^2, or equivalently V^2. Notice that since $\eta^{MN} \overset{2}{g}_{MN} = 0$

$$\sqrt{-g} = 1 + O(\kappa_{11}^4) \ . \tag{5.75}$$

Also, by inspection of Eq. (5.74)

$$R = \kappa_{11}^4 R^{(4)} + \kappa_{11}^6 R^{(6)} + O(\kappa_{11}^8) \ . \tag{5.76}$$

Therefore to $O(\kappa_{11}^4)$

$$S_{\text{bulk}} = \frac{1}{2\kappa_{11}^2} \int \left(\kappa_{11}^4 R^{(4)} + \kappa_{11}^6 R^{(6)} \right) \ . \tag{5.77}$$

Calculating $R^{(4)}$ gives

$$\begin{aligned} R^{(4)} &= \tfrac{1}{2} \overset{2}{g}^{MN} \Box_\perp \overset{2}{g}_{MN} + \tfrac{1}{4} \partial_a \overset{2}{g}^{PQ} \partial_a \overset{2}{g}_{PQ} + \tfrac{1}{2} \eta^{MN} \partial_P \overset{2}{g}_{QM} \partial_N \overset{2}{g}^{PQ} \\ &= \tfrac{1}{4} \overset{2}{g}^{MN} \Box_\perp \overset{2}{g}_{MN} - \tfrac{1}{2} \overset{2}{g}^{PQ} \eta^{MN} \partial_P \partial_N \overset{2}{g}_{QM} + \text{total derivative} \ . \end{aligned} \tag{5.78}$$

Here Leibnitz and the gauge condition at $O(\kappa_{11}^2)$

$$0 = g^{MN} g_{PQ} \Gamma^Q{}_{MN}\Big|_{\kappa_{11}^2} = \eta^{MN}\partial_M \overset{2}{g}_{PN} - \tfrac{1}{2}\eta^{MN}\partial_P \overset{2}{g}_{MN} = \eta^{MN}\partial_M \overset{2}{g}_{PN} \ , \tag{5.79}$$

has been used. Also, the superscript "2" has been restored onto $\overset{2}{g}$ to emphasise that upon integration the total derivative terms give zero, or a total time derivative, as discussed in the previous subsection. Therefore

$$\begin{aligned}\frac{1}{2\kappa_{11}^2}\int \kappa_{11}^4 R^{(4)} &= \pi R_- \kappa_{11}^2 \int dx^+ d^9\vec{y}\left(\tfrac{1}{4} g^{MN}\Box_\perp g_{MN}\right) \\ &= -\frac{\kappa_{11}^2}{16}\int dx^+ d^9\vec{y} \sum_{i,k} Q_i f_k \delta_i \,(\vec{v}_i - \vec{v}_k)^4 \\ &= -\frac{1}{2}\frac{15}{16}\int dx^+ \sum_{i,k} \frac{Q_i Q_k}{M^9 R_-}\frac{(\vec{v}_i - \vec{v}_k)^4}{|\vec{y}_i - \vec{y}_k|^7} \ . \end{aligned}\tag{5.80}$$

The situation is more complicated for $R^{(6)}$, however. It and its integral over the transverse space were calculated using a MATHEMATICA program. Of course, to obtain a closed-form expression for the Lagrangian would require the solution to the integral

$$I^{ij}_{ab} = \int d^d\vec{y} \frac{(y-y_i)_a (y-y_j)_b}{|\vec{y}-\vec{y}_{k_1}|^{\alpha_1}|\vec{y}-\vec{y}_{k_2}|^{\alpha_2}|\vec{y}-\vec{y}_{k_3}|^{\alpha_3}} \ . \tag{5.81}$$

For the purpose of this chapter, which is to check Dine and Rajaraman's supergravity calculation, the integral need not be solved in its full generality. It is possible to find a solution in the parameter range of DR, namely when one of the three particles is very distant from the other two,

$$|\vec{y}_3 - \vec{y}_1| \sim |\vec{y}_3 - \vec{y}_2| \equiv |\vec{y}_3| \gg |\vec{y}_1 - \vec{y}_2| \ . \tag{5.82}$$

In this regime it is possible to extract the coefficient of

$$\frac{1}{|\vec{y}_1 - \vec{y}_2|^7 |\vec{y}_3|^7} \ , \tag{5.83}$$

in order to compare with DR's Matrix theory answer Eq. (5.2). Expanding the integrand of Eq. (5.81) in powers of $1/|\vec{y}_3|$ (the precise details can be found in App. E) yields

$$\begin{aligned}&\int d^d\vec{y} \frac{(y-y_i)_a (y-y_j)_b}{|\vec{y}-\vec{y}_1|^{\alpha_1}|\vec{y}-\vec{y}_2|^{\alpha_2}|\vec{y}-\vec{y}_3|^{\alpha_3}} \\ &= \frac{\Gamma(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}-\frac{d}{2})}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})}\pi^{d/2}\frac{1}{|\vec{y}_1-\vec{y}_2|^{\alpha_1+\alpha_2-d}}\frac{1}{|\vec{y}_3|^{\alpha_3}} \\ &\quad\times\Big\{ y_1^a y_1^b B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}, \tfrac{d}{2}-\tfrac{\alpha_2}{2}+2) + y_2^a y_2^b B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+2, \tfrac{d}{2}-\tfrac{\alpha_2}{2}) \\ &\quad + (y_1^a y_2^b + y_1^b y_2^a) B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+1, \tfrac{d}{2}-\tfrac{\alpha_2}{2}+1) + y_i^a y_j^b B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}, \tfrac{d}{2}-\tfrac{\alpha_2}{2}) \\ &\quad - (y_i^a y_1^b + y_1^a y_j^b) B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}, \tfrac{d}{2}-\tfrac{\alpha_2}{2}+1) - (y_i^a y_2^b + y_2^a y_j^b) B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+1, \tfrac{d}{2}-\tfrac{\alpha_2}{2})\Big\} \\ &\quad + \frac{\Gamma(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}-\frac{d}{2}-1)}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})}\frac{\pi^{d/2}}{2}\frac{1}{|\vec{y}_1-\vec{y}_2|^{\alpha_1+\alpha_2-d-2}}\frac{1}{|\vec{y}_3|^{\alpha_3}}\delta^{ab} B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+1, \tfrac{d}{2}-\tfrac{\alpha_2}{2}+1) \ ,\end{aligned}\tag{5.84}$$

with corrections being higher order in $1/|\vec{y}_3|$.

From the result Eq. (5.84) it is clear that the $O(V^6/L^{14})$ term will have, just like the $O(V^4/L^7)$ term, contributions from two different sorts of terms: those that have only powers of v_i in the numerator; and the "polarisation" terms whose numerators contain $v_i{\cdot}\vec{y}_j$. The term that Dine and Rajaraman calculated is one of the former and dropping all the polarisation-type terms the $R^{(6)}$ contribution reads

$$S_{\text{bulk}}|_{\kappa_{11}^4} = \left(\frac{15}{16}\right)^2 4\frac{Q_1Q_2Q_3}{M^{18}R_-^2}\int dx^+ \frac{7v_{12}^4(v_{13}^2+v_{23}^2)+7v_{12}^2(v_{13}^4+v_{23}^4)-16v_{12}^2v_{13}^2v_{23}^2}{|\vec{y}_1-\vec{y}_2|^7|\vec{y}_3|^7}\,, \tag{5.85}$$

where $v_{ik}^a = v_i^a - v_k^a$.

5.5.4 The total effective action

Summing the contributions from the source, the boundary and the bulk actions gives the following effective action for 3 gravitons

$$\begin{aligned} S_{\text{eff}} = \int dx^+ \Bigg\{ &\sum_{k=1}^{3} \tfrac{16}{21}\tfrac{1}{2}Q_k v_k^2 + \frac{15}{32}\sum_{i,k=1}^{3} \frac{Q_iQ_k}{M^9R_-}\frac{v_{ik}^4}{r_{ik}^7} \\ &- \left(\frac{15}{16}\right)^2 32\frac{Q_1Q_2Q_3}{M^{18}R_-^2}\left(\frac{v_{12}^2v_{13}^2(v_{23}^2-v_{13}^2-v_{12}^4)}{r_{12}^7r_3^7} + (1\to 2)\right)\Bigg\}\,. \end{aligned} \tag{5.86}$$

Here r_{ij} and $\vec{v}_{ij}$ are the transverse separation and relative velocity of the particles i and j. The $O(V^6/L^{14})$ term is valid up to polarisation terms which have numerators of the form $\vec{v}_{ij}{\cdot}(\vec{y}_k-\vec{y}_l)$, and in the regime where particle number "3" is very distant from the other two particles

$$r_{13} \sim r_{23} \equiv r_3 \gg r_{12}\,. \tag{5.87}$$

The careful reader will notice the unusual contribution at $O(V^2)$. A factor of $\frac{1}{2}$ comes from the source term while the boundary action gives $-\frac{5}{21}\frac{1}{2}$. The latter contribution was not included in [240] where the boundary term was simply used to cancel the total-derivative term. This boundary term contribution seems like an overcounting and it is likely that it must be subtracted in some way. The $O(V^4/L^7)$ term is standard. Most pertinently, however, is that by examining the numerator of the $O(V^6/L^{14})$ term it is clear that there is *no* contribution of the form

$$\frac{v_3^4v_{12}^2}{r_{13}^7r_{12}^7}\,. \tag{5.88}$$

The DR problem is thereby solved.

5.6 The large-N limit

Before concluding, a speculative comment regarding the large-N limit of Matrix theory is made. It has been suggested [20, 43] that the discrepancy between Matrix theory and supergravity may vanish on taking the large-N limit, because then Matrix theory should become M-theory in 11 large dimensions. However, only through the recent work of Maldacena [235] has it been possible to deal with this limit. In [198, 235] brane configurations were studied in the limit where the field theory on the brane decouples from the bulk, and it was observed that when the number of branes N, becomes large, the curvature of spacetime around the brane becomes small (for earlier discussions in the conformal case see [181] and references therein). However, for small curvatures branes are well described by extremal black-hole type solutions of the associated supergravity. Thus we are led to the following conjecture

Conjecture: In the large N limit of Matrix theory, supergravitons are described by D0-brane solutions of IIA supergravity.

This leads immediately to the following trivial resolution of DR problem.

Resolution: Since D0-branes are BPS states which can be identified with Kaluza-Klein supergraviton modes of 11d supergravity [275, 346], their leading order scattering amplitudes will be proportional to those of point particles in 11d supergravity. Therefore, leading order supergraviton amplitudes calculated using the large N limit of Matrix theory *are* those of 11d supergravity.

The calculation of this chapter may therefore be considered evidence for this conjecture.

5.7 Conclusions

This chapter has provided a resolution of the DR problem. In particular, it was shown that the classical supergravity effective action for the scattering of 3 gravitons where one graviton is relatively distant from the other two, without spin and without longitudinal momentum transfer, was of the form

$$S_{\text{eff}}^{\text{sugra}} \sim \frac{v_{12}^2 v_{13}^2 (v_{23}^2 - v_{13}^2 - v_{12}^2)}{r_{12}^7 r_{13}^7} + (1 \leftrightarrow 2) \; , \tag{5.89}$$

at $O(V^6/L^{14})$. This shows that, in agreement with Matrix theory, there is *no* term of the form $v_3^4 v_{12}^2 / r_{13}^7 r_{12}^7$.

The calculation began with the well-known solution of IIA supergravity that describes a number of static D0-branes (or, more precisely, a number of bound states of static D0-branes). Using the Kaluza-Klein relation this solution, its corresponding source, and IIA supergravity was lifted into eleven dimensions. A DLCQ of the theory was then formed by infinitely boosting around the spacelike circle. The source then corresponded to the massless limit of the standard massive point-particle action with the constraint that p_- be held constant.

Because p_- was held constant for each particle the calculation included no longitudinal momentum transfer. Also, by only considering the bosonic part of the source and gravity actions all spin effects were neglected.

Einstein's equations were then solved order-by-order in the small velocities and large separations of the supergravitons. Once the solution had been found to the desired order it was substituted back into the $S_{\text{gravity}} + S_{\text{source}}$ to obtain the effective action. In this regard there is significant overlap with [261] where the same result is obtained using very similar methods. However, it was proved here that to obtain the effective action to $O(V^6/L^{14})$ only the $O(\kappa_{11}^2)$ corrections to the metric are needed. Thus the calculation is substantially simpler than [261] where certain combinations of the metric at $O(\kappa_{11}^4)$ were calculated. This simplification was allowed only because the standard Einstien boundary term was included in the calculation.

Finally, motivated by the work of Maldacena, it was conjectured that the large-N limit of Matrix theory is exactly described by 11d supergravity.

Many extensions of this work could be considered. One, which is under active investigation, is to calculate the $O(V^8/L^{21})$ contributions to the effective action. By an extension of the proof in Sec. 5.5 only the $O(\kappa_{11}^4)$ terms of the metric are needed. Some of these have been already calculated in [261], however, of course, finding the effective action requires much more tedious work than just solving Einstein's equations. A further complication at this order is the important role played by acceleration. However, despite the difficulties, the $O(V^8/L^{21})$ terms are of interest for it is expected that here the supergravity–Matrix theory correspondence breaks down [124]. Another path for future research which is being studied is to include spin. It is widely suspected that supersymmetry and O(8) invariance in the transverse space should fix all the "spin" terms relative to the "scalar" term already calculated. More amplitudes with longitudinal momentum transfer could also be studied. Finally, it would be very nice to formulate a general proof that supergravity is (or isn't) the low-energy limit of Matrix theory.

Appendix A: The phase-space density formalism and the leg-pole transformation

This appendix reviews DMW's paper [98]. The formalism uses a Fermi fluid density $U(p,q,t)$ and so first it will be necessary to review a few facts concerning this object. The leg-pole transform of Sec. 1.8.3 is then re-written in terms of a generic fluctuation δU and it is seen that because of the particularly simple fermion dynamics (in the semi-classical limit), it is possible to write S_- as an explicit function of S_+. The proposal can then be stated precisely.

By employing the single-particle Hamiltonian $H(p,q) = \frac{1}{2}(p^2 - q^2)$, the total Hamiltonian Eq. (1.103) can be written in terms of the phase-space density [99, 100, 101] of fermions $U(p,q,t)$

$$\begin{aligned} H &= \int \frac{\mathrm{d}p\mathrm{d}q}{2\pi} H(p,q)U(p,q,t) \ , \\ U(p,q,t) &= \int_{-\infty}^{\infty} \mathrm{d}\lambda e^{-ip\lambda}\psi^\dagger(q - \tfrac{1}{2}\lambda, t)\psi(q + \tfrac{1}{2}\lambda, t) \ . \end{aligned} \tag{A.1}$$

This satisfies the equation of motion

$$(\partial_t + p\partial_q + q\partial_p)U(p,q,t) = 0 \ . \tag{A.2}$$

The W_∞ algebra can also be written in terms of U; the generators are [81, 82, 83]

$$W_{mn} = e^{(n-m)t} \int \frac{\mathrm{d}p\mathrm{d}q}{2\pi} (-p-q)^m (p-q)^n U(p,q,t) \ . \tag{A.3}$$

In the semi-classical limit, it is clear that U must satisfy

$$U^2(p,q,t) = U(p,q,t) \ , \tag{A.4}$$

(the quantum constraint is much more complicated [99]). A generic perturbation of the fluid around its vacuum configuration δU must still conserve fermion number, therefore

$$\int \frac{\mathrm{d}p\mathrm{d}q}{2\pi} \delta U(p,q,t) = 0 \ . \tag{A.5}$$

Now the leg-pole transform will be written in terms of this new variable δU.

The leg-pole transform in terms of the scalar fields is

$$S_-(x^+) = \int_{-\infty}^{\infty} \mathrm{d}\tau \ \kappa(\tau)\bar{S}_-(x^+ - \tau) \ , \tag{A.6}$$

where the kernel is

$$\kappa(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{i\omega\tau} \left(\frac{\pi}{2}\right)^{-i\omega/4} \frac{\Gamma(-i\omega)}{\Gamma(i\omega)} = \frac{\partial}{\partial\tau} J_0(z) \ , \tag{A.7}$$

J_0 is the standard zeroth-order Bessel function [170] and $z = 2(\frac{2}{\pi})^{1/8}e^{\tau/2}$. Then, since the derivative of the scalar $\bar{S}_-$ is simply related to the excitation of the Fermi sea, ϵ_- via Eq. (1.109)

$$-2\partial\bar{S}_-(x^+) = \frac{1}{\sqrt{\pi}}(\epsilon_-(x^+) - \bar{\mu}) \ , \tag{A.8}$$

the outgoing tachyon can be written as

$$S_-(x^+) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\tau \ J_0\left(2(\tfrac{2}{\pi})^{1/8}e^{\frac{1}{2}(x-\tau)}\right)\epsilon_-(t+\tau) \ . \tag{A.9}$$

Finally, writing the fluctuation of the Fermi surface ϵ_- in terms of a small change in the matrix-model momentum p away from the static situation

$$p(q,t) - p_0(q) = q + q^{-1}(\bar{\mu} + \epsilon_-(x^+)) + \sqrt{q^2 - 2\mu} \approx \frac{\epsilon_-(x^+)}{q} \ , \tag{A.10}$$

and using the variable $q = -e^{-\tau}$, the following is obtained

$$S_-(x^+) = \int_{-\infty}^{\bar{\mu}} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p \ f(-qe^x)\delta U_{\mathrm{out}}(p,q,t) \ , \tag{A.11}$$

where the function f is defined

$$f(\sigma) = \frac{1}{2\sqrt{\pi}}J_0\left(2(\tfrac{2}{\pi})^{1/8}\sqrt{\sigma}\right) \ . \tag{A.12}$$

There is a similar analysis for S_+. Both can be written as in Eq. (1.123)

$$S(x,t) = \int \mathrm{d}q\mathrm{d}p \ f(-qe^x)\delta U(p,q,t) \ , \tag{A.13}$$

which is only valid as $x \to -\infty$ (there are $O(xe^{2x}$ corrections [102])

It was noted [101, 102] that because the dynamics of the fermions was so simple, S_- could be written as an explicit function of S_+ in the following way: The equation of motion for the perturbed fluid

$$(\partial_t + p\partial_q + q\partial_p)\delta U(p,q,t) = 0 \tag{A.14}$$

can be solved to find the integral surface where δU is constant;

$$\delta U(p,q,t) = \delta U(p',q',t') \quad \text{where} \quad (p' \pm q')e^{\mp t'} = (p \pm q)e^{\mp t} \ . \tag{A.15}$$

This can be used in Eq. (A.13) to change variables from (p,q) to (p',q') with t and t' being fixed for the purposes of the change of variables. Since the measure $(\mathrm{d}p\mathrm{d}q)$ and the Fermi surface $(p^2 - q^2 = \text{constant})$ are invariant under this transformation, the tachyon field is simply

$$S(x,t) = \int \mathrm{d}p\mathrm{d}q \ f(-Q(t)e^x)\delta U(p,q,t_0) \ , \tag{A.16}$$

where

$$Q(t) = q\cosh(t - t_0) + p\sinh(t - t_0) \ . \tag{A.17}$$

Using the equation of motion the RHS is independent of t_0, so the limit $t_0 \to -\infty$ can be taken. Now the steps leading from Eq. (A.9) to Eq. (A.13) can be reversed to write S in terms of the small perturbation ϵ. A slight degree of care is needed because now the kernel depends on the momentum p as well as q and t. Before doing this, however, fluid fluctuations on both sides of the potential will be introduced.

When excitations on both sides of the potential are allowed, DMW suggested that the symmetrical leg-pole transform should be used

$$S(x,t) = \frac{1}{\sqrt{2}} \int \mathrm{d}q\mathrm{d}p \; f(2^{1/4}|q|e^{x})\delta U(p,q,t) \; . \tag{A.18}$$

The factors of $1/\sqrt{2}$ and $2^{1/4}$ are for convenience only. Making the same arguments as above, the phase-space variables can be transformed so that the fluctuation only enters as a boundary condition

$$S(x,t) = \frac{1}{\sqrt{2}} \int \mathrm{d}p\mathrm{d}q f(2^{1/4}|Q(t)|e^{x})\delta U(p,q,t_0) \; . \tag{A.19}$$

Now the fluid at time $t \to -\infty$ can be parameterised in terms of *two* variables $\epsilon_+^a(\tau)$, one on either side of the potential $a = 1,2$. Reversing the steps between Eq. (A.9) and Eq. (A.13) and then taking the limit $t \to \pm\infty$ yields Eq. (1.126)

$$\begin{aligned} S_+(x^-) &= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \mathrm{d}\tau \sum_{\alpha=1,2} \epsilon_+^a(\tau) \; f\left(\sqrt{|\bar{\mu}'/2|}e^{\tau - x^-}\right) \; , \\ S_-(x^+) &= \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \sum_{\alpha=1,2} \int_0^{\sqrt{2}\epsilon_+^a(\tau)} \mathrm{d}\epsilon \; f\left(\sqrt{\bar{\mu}'}(1 - (\epsilon/|\mu'|))e^{-\tau + x^+}\right) \; . \end{aligned} \tag{A.20}$$

Appendix B: Derivation of the path integral

In this appendix, a derivation of the path-integral representation of the partition function

$$Z = \mathrm{tr} e^{-\beta H} \ , \tag{B.1}$$

is presented. For free fermions in two dimensions the Hermitian Hamiltonian H can be expressed in terms of the Heisenberg Dirac spinor fields $\psi(x,t)$ and their conjugate momenta $\pi^\psi = i\bar{\psi}\gamma_M^0 = -i\psi^\dagger$ in the usual fashion

$$H = -i \int \mathrm{d}x \ \bar{\psi}(\gamma_M^1 \partial_x + m)\psi = -\int \mathrm{d}x \ \pi^\psi \gamma_M^0 (\gamma_M^1 \partial_x + m)\psi \ , \tag{B.2}$$

where the traceless gamma matrices in Minkowsky space γ_M, satisfy $\{\gamma_M^\mu, \gamma_M^\nu\} = 2\,\mathrm{diag}\,(-1,1)$. It is of particular interest to check that this partition function is equivalent to the path integral of Eq. (3.37) that contains van Nieuwenhuizen and Waldron's [333, 335] conjugate spinor $\bar{\psi} = \psi^\dagger \gamma^5$.

Expanding the fields in the usual basis spinors [199, Sec 2.2] the Hamiltonian is

$$H = \int \frac{\mathrm{d}k}{2\pi} \frac{m}{\omega_k} \ \omega_k \left(b^\dagger(k) b(k) + d^\dagger(k) d(k) \right) \quad \text{with} \quad \omega_k \equiv \sqrt{k^2 + m^2} \ . \tag{B.3}$$

The non-zero anticommutation relations are

$$\left\{ b(k), b^\dagger(q) \right\} = 2\pi \frac{\omega_k}{m} \delta(k-q) = \left\{ d(k), d^\dagger(q) \right\} \tag{B.4}$$

To simplify notation denote all the modes $b(k)$ and $d(k)$ by a_i. The Hamiltonian and anticommutation relations can then be written

$$H = \sum_i \omega_k a_i^\dagger a_i \quad \text{and} \quad \left\{ a_i^\dagger, a_j \right\} = 2\pi \frac{\omega_k}{m} \delta_{ij} \ , \tag{B.5}$$

where $\sum_i = \int \mathrm{d}k \sum_{b,d} \frac{m}{2\pi\omega_k}$.

Now proceed with the path-integral quantisation in the standard fashion. It is convenient to use coherent states as the complete set of states. The vacuum is defined by $a_i|0\rangle = 0 \ \forall i$. The Fock space is built from the tensor product of the one-particle states. A general state will be denoted by

$$|i_1, i_2, \ldots, i_n\rangle = a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_n}^\dagger |0\rangle \ . \tag{B.6}$$

Denote the state with all oscillators excited by $|\psi\rangle = \prod_n a_n^\dagger |0\rangle$. Then, by the Grassmann nature of the oscillators, $a_i^\dagger |\psi\rangle = 0$. Coherent states $|\theta\rangle$ and $|\bar{\theta}\rangle$ have the property that

$$\begin{aligned} a_i|\theta\rangle &= \theta_i|\theta\rangle \qquad & a_i^\dagger|\bar{\theta}\rangle &= \bar{\theta}_i|\bar{\theta}\rangle \ , \\ \langle\bar{\theta}|a_i^\dagger &= \langle\bar{\theta}|\bar{\theta}_i \qquad & \langle\theta|a_i &= \langle\theta|\theta_i \ . \end{aligned} \tag{B.7}$$

It is clear that

$$|\theta\rangle = e^{\sum_i a_i^\dagger \theta_i}|0\rangle \qquad |\bar{\theta}\rangle = e^{\sum_i a_i \bar{\theta}_i}|\psi\rangle \ ,$$
$$\langle\bar{\theta}| = \langle 0|e^{\sum_i \bar{\theta}_i a_i} \qquad \langle\theta| = \langle\psi|e^{\sum_i \theta_i a_i^\dagger} \ . \tag{B.8}$$

To reduce notation define the vector notation $\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta} = \sum_j \bar{\theta}_j\theta_j$. The states $|\theta\rangle$ form a complete and orthogonal basis. The orthogonality relations are

$$\langle\theta|\theta\rangle = \langle\bar{\theta}|\bar{\theta}\rangle = 0 \ , \quad \langle\theta|\bar{\theta}\rangle = e^{\boldsymbol{\theta}\cdot\bar{\boldsymbol{\theta}}} \quad \text{and} \quad \langle\bar{\theta}|\theta\rangle = e^{\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta}} \ , \tag{B.9}$$

and completeness takes the form

$$1 = \int \prod_i (\mathrm{d}\bar{\theta}_i \mathrm{d}\theta_i)\ |\theta\rangle e^{-\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta}} \langle\bar{\theta}| \ . \tag{B.10}$$

To show completeness first note that

$$\langle i_1, i_2, \ldots, i_n|\theta\rangle = [(-1)^n]\theta_{i_n}\ldots\theta_{i_2}\theta_{i_1} \ , \tag{B.11}$$

(the -1 factors only appear if $|0\rangle$ is fermionic). Then, using the identity

$$\int \prod_j (\mathrm{d}\bar{\theta}_j \mathrm{d}\theta_j)\ \theta_{i_n}\ldots\theta_{i_1}\bar{\theta}_{i_1}\ldots\bar{\theta}_{i_n} e^{-\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta}} = 1 \ , \tag{B.12}$$

it follows that

$$\langle\{i\}| \int \prod_j (\mathrm{d}\bar{\theta}_j \mathrm{d}\theta_j)\ |\theta\rangle e^{-\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta}} \langle\bar{\theta}|\{i'\}\rangle = \langle\{i\}|\{i'\}\rangle \ , \tag{B.13}$$

thus proving the result.

Inserting two completeness relations (one with a set of states called $|\theta^0\rangle$ and the other with $|\theta^N\rangle$) into the partition function yields

$$\begin{aligned} Z &= \sum_{\{i\}} \langle\{i\}|e^{-\beta H}|\{i\}\rangle \ , \\ &= \int \prod_j \mathrm{d}\bar{\theta}_j^N \mathrm{d}\theta_j^0\ \langle\bar{\theta}^N|e^{-\beta H}|\theta^0\rangle \int \prod_k \mathrm{d}\theta_k^N \mathrm{d}\bar{\theta}_k^0 e^{-\bar{\boldsymbol{\theta}}^N\cdot\boldsymbol{\theta}^N - \bar{\boldsymbol{\theta}}^0\cdot\boldsymbol{\theta}^0} \sum_{\{i\}} \langle\{i\}|\theta^N\rangle\langle\bar{\theta}^0|\{i\}\rangle \ . \end{aligned} \tag{B.14}$$

Completeness of the $|\{i\}\rangle$ basis can be used to simplify the last term (here $|i|$ denotes the number of excited oscillators in state $|\{i\}\rangle$ and F is the fermion number operator)

$$\begin{aligned} \sum_{\{i\}} \langle\{i\}|\theta^N\rangle\langle\bar{\theta}^0|\{i\}\rangle &= \sum_{\{i\}} (-1)^{|i|}\langle\bar{\theta}^0|\{i\}\rangle\langle\{i\}|\theta^N\rangle = \sum_{\{i\}} \langle\bar{\theta}^0|\{i\}\rangle\langle\{i\}|(-1)^F|\theta^N\rangle \\ &= \langle\bar{\theta}^0| - \theta^N\rangle = e^{-\bar{\boldsymbol{\theta}}^0\cdot\boldsymbol{\theta}^N} \ . \end{aligned} \tag{B.15}$$

In the expression for the partition function, the integration over $\bar{\theta}^0$ just yields the delta function $\delta(\boldsymbol{\theta}^0 + \boldsymbol{\theta}^N)$, giving the result

$$Z = -\int \prod_j \mathrm{d}\bar{\theta}_j^N \mathrm{d}\theta_j^0\ e^{\bar{\boldsymbol{\theta}}^N\cdot\boldsymbol{\theta}^0} \langle\bar{\theta}^N|e^{-\beta H}|\theta^0\rangle \ . \tag{B.16}$$

This final matrix element can be calculated by splitting the imaginary time into N small time-steps and inserting completeness at each stage. Using the general matrix element

$$\langle\bar{\theta}|e^{-\epsilon H}|\theta\rangle = e^{\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta}}\left(1-\epsilon\sum_i \omega_k\bar{\theta}_i\theta_i\right) \equiv \exp\bar{\boldsymbol{\theta}}\cdot(1-\epsilon\omega_k)\boldsymbol{\theta}\ , \tag{B.17}$$

the "sum over all paths" becomes

$$\begin{aligned}\langle\bar{\theta}^N|e^{-\beta H}|\theta^0\rangle &= \lim\int\prod_{l=1}^{N-1}\left(\prod_j \mathrm{d}\bar{\theta}_j\mathrm{d}\theta_j\right)^l \\ &\times\exp\left\{-\sum_{l=1}^{N-1}\left(\bar{\boldsymbol{\theta}}^l\cdot\boldsymbol{\theta}^l+\bar{\boldsymbol{\theta}}^l\cdot(1-\epsilon\omega_k)\boldsymbol{\theta}^{l-1}\right)+\boldsymbol{\theta}^N\cdot\boldsymbol{\theta}^{N-1}\right\}\ .\end{aligned} \tag{B.18}$$

In this formula the index l stands for the l^{th} time-slice. The limit is the usual one: $N\to\infty$, $\epsilon\to 0$ while keeping $N/\epsilon=\beta$. The first term in the exponential comes from completeness while the second comes from Eq. (B.17). The final term is important since upon substitution of this formula into (B.16) and integration over $\bar{\boldsymbol{\theta}}^N$, it is seen immediately that the fermions obey antiperiodic boundary conditions $\boldsymbol{\theta}^N=-\boldsymbol{\theta}^0$. Denoting an interpolating function $\boldsymbol{\theta}(\tau_l)=\boldsymbol{\theta}^l$ and taking the limit, the expected result is obtained

$$Z=\int_{antiperiodic}\left[\prod_j \mathrm{d}\bar{\theta}_j\mathrm{d}\theta_j\right]e^{-S}\quad\text{where}\quad S=\int_0^\beta \mathrm{d}\tau\,(\bar{\boldsymbol{\theta}}\cdot\dot{\boldsymbol{\theta}}+\omega_k\bar{\boldsymbol{\theta}}\cdot\boldsymbol{\theta})\ . \tag{B.19}$$

Of immediate importance is that the first term in the action is anti-Hermitian while the last two terms, coming from a Hermitian Hamiltonian, are Hermitian. It seems problematic, therefore, to cast this action into a form which is a Hermitian functional of relativistic fields as written in [333, 335].

It is possible, however, to check that the path integral of equation (B.19) is formally the same as the path integral Eq. (3.37)

$$Z=\int\left[\mathrm{d}\psi^\dagger\mathrm{d}\psi\right]e^{-\int\bar{\psi}(\partial\!\!\!/+m)\psi} \tag{B.20}$$

Upon expanding in a complete set $\boldsymbol{\theta}=\sum_n\boldsymbol{\theta}_n e^{(2n+1)\pi i\tau/\beta}$, Eq. (B.19) yields the formal result

$$Z=\det\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+\omega_k\right)^2=\prod_{n,k}\left(\frac{(2n+1)\pi i}{\beta}+\omega_k\right)^2=\prod_{n,k}\left(\frac{(2n+1)^2\pi^2}{\beta^2}+\omega_k^2\right)\ . \tag{B.21}$$

The determinant is squared because there are two oscillators b and d. Now consider finding the determinant of the Dirac operator $\gamma^5(\partial\!\!\!/+m)$. This is simply the product of the eigenvalues $\lambda_{n,k}$

$$\gamma^5(\partial\!\!\!/+m)\psi_{n,k}=\lambda_{n,k}\psi_{n,k}\ . \tag{B.22}$$

There is a double degeneracy of eigenvalues since γ^5 commutes with the Dirac operator. Squaring the eigenvalue equation and expanding in modes yields

$$\lambda_{n,k}^2=\frac{(2n+1)^2\pi^2}{\beta^2}+k^2+m^2\ . \tag{B.23}$$

This shows that Eq. (3.37) is indeed a path-integral representation of the partition function $\mathrm{Tr}e^{-\beta H}$.

Appendix C: The determinant of an invertible, Hermitian Dirac operator

Following [9, 47, 49, 150, 151, 152, 185, 201, 289, 295, 297, 303], this appendix will detail the calculation of the generating functional

$$Z[\eta,\bar{\eta}] = \int [\mathrm{d}\bar{\psi}\mathrm{d}\psi] \exp\left(-\int \bar{\psi} \not{D} \psi + \int (\bar{\eta}\psi + \bar{\psi}\eta)\right) = e^{\int \bar{\eta}\Delta\eta} \det i\not{D} \ , \tag{C.1}$$

where $i\Delta$ is the inverse of the Dirac operator, which is given by

$$\not{D} = \not{\partial} + i\not{\partial}\sigma + i\gamma_\mu \epsilon^{\mu\nu}\partial_\nu \rho + i\not{h} + i\mu\gamma^0\gamma^5 \ . \tag{C.2}$$

This operator is both invertable and anti-Hermitian so the calculation is the simplest it can possibly be (for the non-invertable and non-Hermitian cases see [19, 108, 109, 153, 295, 297] in addition to the above references). Using the identity $\gamma^\nu\gamma^5 = i\gamma_\mu\epsilon^{\mu\nu}$, the Dirac operator becomes

$$\not{D} = \not{\partial} + i\not{\partial}\sigma - \gamma_5 \not{\partial}\rho + i\not{c} \ , \tag{C.3}$$

where $c^1 = h^1 - \mu$.

Consider the following transformations

$$\begin{aligned} \psi_r &= e^{ir\sigma + r\gamma^5\rho}\psi & \text{and} \quad \bar{\psi}_r &= \bar{\psi} e^{-ir\sigma + r\gamma^5\rho} \ , \\ \eta_r &= e^{ir\sigma - r\gamma^5\rho}\eta & \text{and} \quad \bar{\eta}_r &= \bar{\eta} e^{-ir\sigma - r\gamma^5\rho} \ , \\ \not{D}_r &= e^{ir\sigma - r\gamma^5\rho}\not{D} e^{-ir\sigma - r\gamma^5\rho} & \text{and} \quad \Delta_r &= e^{ir\sigma + r\gamma^5\rho}\Delta e^{-ir\sigma + r\gamma^5\rho} \ , \end{aligned} \tag{C.4}$$

where $0 \leq r \leq 1$. Upon reaching $r = 1$, the Dirac operator is much reduced

$$\not{D}_1 = \not{\partial} + i\not{c} \ . \tag{C.5}$$

The idea is to write the generating functional in terms of the rotated fields ψ_1 and $\bar{\psi}_1$, thereby introducing a Jacobian, but also reducing the problem of finding $\det i\not{D}$ to that of calculating $\det i\not{D}_1$

$$Z[\eta_0,\bar{\eta}_0] = \int [\mathrm{d}\bar{\psi}_r \mathrm{d}\psi_r] J_r \exp\left(-\int \bar{\psi}_r i\not{D}_r \psi_r + \int (\bar{\eta}_r\psi_r + \bar{\psi}_r\eta_r)\right) = e^{\int \bar{\eta}_r\Delta_r\eta_r} J_r \det i\not{D}_r \ . \tag{C.6}$$

It is clear that $\bar{\eta}_r\Delta_r\eta_r = \bar{\eta}_0\Delta_0\eta_0$ so expressing that the partition function be independent of r yields the first-order differential equation for J_r

$$0 = \frac{\mathrm{d}J_r}{\mathrm{d}r} \det i\not{D}_r + J_r \frac{\mathrm{d}}{\mathrm{d}r} \det i\not{D}_r \ . \tag{C.7}$$

This is easily solved to give

$$J(1) = \exp\left(-\int_0^1 \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\log\det i\not{D}_r\right) = \exp\left(2\int_0^1 \mathrm{d}r\ \mathrm{Tr}\gamma^5\rho\right) . \tag{C.8}$$

Substituting $r = 1$, the generating functional becomes

$$Z[\bar{\eta}_0, \eta_0] = e^{\int \bar{\eta}_1 \Delta_1 \eta_1} J(1) \det i\not{D}_1 . \tag{C.9}$$

The Jacobian must be regulated since the infinite-dimensional trace $\mathrm{Tr}\gamma^5\rho$ is ill-defined. Fujikawa's regularisation scheme is chosen

$$\mathrm{Tr}\gamma^5\rho \equiv \lim_{\epsilon\to 0+} \mathrm{Tr}\gamma^5\rho e^{-\epsilon \not{D}^\dagger \not{D}} . \tag{C.10}$$

This is gauge invariant under the U(1) gauge transformations given by Eq. (3.38). The RHS is basically the heat-kernel and its calculation is well-known (see also Sec. 3.5.1). On the plane

$$\begin{aligned}
\mathrm{Tr}\gamma^5\rho &\equiv \lim_{\epsilon\to 0}\int \mathrm{d}^2x\ \mathrm{tr}\gamma^5\rho \int \frac{\mathrm{d}^2k}{(2\pi)^2} e^{-ikx} e^{-\epsilon \not{D}^\dagger \not{D}} e^{ikx} , \\
&= \lim_{\epsilon\to 0}\int \mathrm{d}^2x\ \mathrm{tr}\gamma^5\rho \int \frac{\mathrm{d}^2k}{(2\pi)^2} e^{-\epsilon k^2 + \epsilon 2ik\cdot\partial - i\epsilon Z\cdot k - \epsilon(Z\cdot\partial + Y)} , \\
&= \lim_{\epsilon\to 0}\int \mathrm{d}^2x\ \mathrm{tr}\gamma^5\rho \left\{\frac{1}{\epsilon}\int e^{-k^2} + \left(\tfrac{1}{2}\partial\cdot Z - \tfrac{1}{4}Z^2 - Y\right)\int e^{-k^2} + O(\epsilon)\right\} ,
\end{aligned} \tag{C.11}$$

where, in the second line $\not{D}^\dagger\not{D} = -\partial^2 + Z\cdot\partial + Y$ and in the third, k^μ has been scaled by $1/\sqrt{\epsilon}$. A small calculation gives $\frac{1}{2}\partial\cdot Z - \frac{1}{4}Z^2 - Y = \gamma^5\Box\rho$. Substituting this into the above expression yields

$$\mathrm{Tr}\gamma^5\rho = \frac{1}{2\pi}\rho\Box\rho \tag{C.12}$$

so that

$$J(1) = e^{\frac{1}{\pi}\rho\Box\rho} . \tag{C.13}$$

The calculation on the torus it is slightly more complicated due to the integrals being replaced by sums. However, it is clear that the heat-kernel depends only on the short-distance structure of the manifold and is insensitive to any global properties. This was demonstrated clearly and explicitly in [50] where the additional terms on the torus for small ϵ were of the form $e^{-R^2/\epsilon}$. Thus, Eq. (C.13) holds in the compact case too.

The eigenmodes of $i\not{D}_1$ on the torus are periodic in the x direction and anti-periodic in the τ direction. Since γ^5 anticommutes with $i\not{D}_1$ the eigenvalues come in $\pm$ pairs. Squaring the eigenvalue equation $i\not{D}_1\psi_{n,m} = \lambda_{n,m}\psi_{n,m}$ yields

$$(\partial + ic)^2\psi_{n,m} = -\lambda_{n,m}^2\psi_{n,m} . \tag{C.14}$$

Expanding in modes that satisfy the appropriate boundary conditions immediately implies

$$\lambda_{n,m}^2 = \left(\frac{(2n+1)\pi}{\beta} + c^0\right)^2 + \left(\frac{2m\pi}{R} + c^1\right)^2 . \tag{C.15}$$

The determinant of $i\not{D}_1$ is simply the formal product of the eigenvalues and can be [9, 47, 295, 297] expressed in terms of a theta function [170, 253] and Dedekind's eta function

$$\det i\not{D}_1 = \left| \frac{1}{\eta(iR/\beta)} \Theta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (0, iR/\beta) \right|^2 , \tag{C.16}$$

where $\theta = -\beta h^0/2\pi$ and $\phi = \frac{1}{2} + \frac{R(h^1-\mu)}{2\pi}$ and the parameter $q = e^{-2\pi R/\beta}$.

This completes the determination of the effective action for the system with invertible, Hermitian Dirac operator given in Eq. (C.1).

Appendix D: Some useful formulae

Formulae concerning the Ricci tensor, the source action Eq. (5.17) and its energy-momentum tensor are derived.

__The Ricci tensor__. Work in the gauge

$$g^{MN}\Gamma^{P}{}_{MN} = 0 \ , \tag{D.1}$$

where $\Gamma^{P}{}_{MN} = \frac{1}{2}g^{PQ}(\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN})$ is the Christofel symbol for the metric. Lowering the index P the gauge condition becomes

$$2g^{MN}\partial_M g_{PN} - g^{MN}\partial_P g_{MN} = 0 \ . \tag{D.2}$$

Now consider the individual terms in the Ricci tensor

$$R_{MN} = \partial_P\Gamma^{P}{}_{MN} + \Gamma^{P}{}_{MN}\Gamma^{Q}{}_{PQ} - \partial_N\Gamma^{P}{}_{MP} - \Gamma^{P}{}_{MQ}\Gamma^{Q}{}_{PN} \ . \tag{D.3}$$

With the gauge condition the second term beomes

$$\Gamma^{P}{}_{MN}\Gamma^{Q}{}_{PQ} = -\tfrac{1}{2}\partial_P g^{PQ}(\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN}) \ . \tag{D.4}$$

The first two terms in R_{MN} are then

$$\partial_p\Gamma^{P}{}_{MN} + \Gamma^{P}{}_{MN}\Gamma^{Q}{}_{PQ} = \tfrac{1}{2}g^{PQ}(\partial_P\partial_M g_{QN} + \partial_P\partial_N g_{QM} - \partial_P\partial_Q g_{MN}) \ . \tag{D.5}$$

The third and fourth terms are

$$\begin{aligned}
-\partial_N\Gamma^{P}{}_{MP} &= -\tfrac{1}{2}\partial_N g^{PQ}\partial_M g_{PQ} - \tfrac{1}{2}g^{PQ}\partial_N\partial_M g_{PQ} \ , \\
-\Gamma^{P}{}_{MQ}\Gamma^{Q}{}_{PN} &= \tfrac{1}{4}\partial_M g^{PQ}\partial_N g_{PQ} - \tfrac{1}{2}g^{PX}g^{QY}\partial_Q g_{XM}\partial_P g_{YN} + \tfrac{1}{2}g^{PX}g^{QY}\partial_X g_{MQ}\partial_P g_{YN} \ ,
\end{aligned} \tag{D.6}$$

Taking a derivative of the gauge condition yields

$$0 = \partial_P g_{Q(N}\partial_{M)}g^{PQ} + g^{PQ}\partial_P\partial_{(M}g_{N)Q} - \tfrac{1}{2}\partial_{(M}g^{PQ}\partial_{N)}g_{PQ} - \tfrac{1}{2}g^{PQ}\partial_M\partial_N g_{PQ} \ . \tag{D.7}$$

Here the terms have been rearranged so as to make the symmeterisation, which is weighted $(MN) = \frac{1}{2}MN + \frac{1}{2}NM$, unambigious. Substituting this into the Ricci tensor yields

$$\begin{aligned}
R_{MN} &= -\tfrac{1}{2}g^{PQ}\partial_P\partial_Q g_{MN} + \tfrac{1}{4}\partial_M g^{PQ}\partial_N g_{PQ} - \partial_P g_{Q(M}\partial_{N)}g^{PQ} \\
&\quad -\tfrac{1}{2}g^{PX}g^{QY}\partial_Q g_{XM}\partial_P g_{YN} + \tfrac{1}{2}g^{PX}g^{QY}\partial_X g_{MQ}\partial_P g_{YN} \ .
\end{aligned} \tag{D.8}$$

This is in a form suitable for expansion around $g = \eta$.

The source action. It is convenient to define the short-hand notation

$$\begin{aligned}
\mathcal{S}_i &= \sqrt{(g_{+-} + g_{-a}v_i^a)^2 - g_{--}(g_{++} + 2g_{+a}v_i^a + g_{ab}v_i^a v_i^b)} \ , \\
\mathcal{N}_i &= g_{+-} + g_{-a}v_i^a \ , \\
\mathcal{D}_i &= g_{++} + 2g_{+a}v_i^a + g_{ab}v_i^a v_i^b \ , \\
\delta_i &= \delta(y - y_i(x^+)) \ ,
\end{aligned} \tag{D.9}$$

where y_i^a is the transverse position of the i^{th} particle. Expanding the metric around flat space

$$g_{MN} = \eta_{MN} + h_{MN} \ , \tag{D.10}$$

the small analogues of Eq. (D.9) can be defined

$$N_i = h_{+-} + h_{-a}v_i^a \quad \text{and} \quad D_i = h_{++} + 2h_{+a}v_i^a + h_{ab}v_i^a v_i^b \ . \tag{D.11}$$

Expanding the source action for particle k gives

$$\begin{aligned}
S_k &= Q_k \int \mathrm{d}x^+ \left(\tfrac{1}{2}v_k^2 - \tfrac{1}{2}N_k v_k^2 + \tfrac{1}{2}D_k + \tfrac{1}{8}h_{--}v_k^4 \right. \\
&\qquad \left. + \tfrac{1}{2}N_k^2 v_k^2 - \tfrac{1}{2}N_k D_k - \tfrac{3}{8}h_{--}N_k v_k^4 + \tfrac{1}{4}h_{--}D_k v_k^2 + \tfrac{1}{16}h_{--}^2 v_k^6\right) + O(h^3) \ .
\end{aligned} \tag{D.12}$$

Evidently, in the background $g = \eta$, the particles have the standard point-particle action with mass $= Q$.

The energy-momentum tensor. The expansion of the energy-momentum tensor in small h_{MN} will be derived. Define

$$\Theta^{MN} = \sum_i \frac{\delta S_i}{\delta g_{MN}} \ , \tag{D.13}$$

so that

$$T_{MN} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{MN}} = -g_{MP}g_{NQ}T^{PQ} = -g_{MP}g_{NQ}\Theta^{PQ}/\sqrt{-g} \ , \tag{D.14}$$

(the negative sign comes from differentiating g_{MN} with respect to g^{MN}). Varying the source action yields

$$\begin{aligned}
\Theta^{++} &= \sum_i \frac{\frac{1}{2}Q_i\delta_i}{\mathcal{S}_i} \ , \\
\Theta^{+-} &= \sum_i \frac{\frac{1}{2}Q_i\delta_i}{g_{--}}\left(1 - \frac{\mathcal{N}_i}{\mathcal{S}_i}\right) \ , \\
\Theta^{+a} &= \sum_i \frac{\frac{1}{2}Q_i v_i^a \delta_i}{\mathcal{S}_i} \ , \\
\Theta^{--} &= \sum_i \frac{Q_i\delta_i}{g_{--}}\left(\frac{\frac{1}{2}\mathcal{D}_i}{\mathcal{S}_i} - \frac{\mathcal{N}_i - \mathcal{S}_i}{g_{--}}\right) \ ,
\end{aligned}$$

$$\begin{aligned}
\Theta^{-a} &= \sum_i \frac{\frac{1}{2}Q_i v_i^a \delta_i}{g_{--}}\left(1-\frac{\mathcal{N}_i}{\mathcal{S}_i}\right) , \\
\Theta^{ab} &= \sum_i \frac{\frac{1}{2}Q_i v_i^a v_i^b \delta_i}{\mathcal{S}_i} .
\end{aligned} \tag{D.15}$$

In terms of the $O(h)$ parameters N_i and D_i

$$\begin{aligned}
\frac{1}{\mathcal{S}_i} &= 1 - N_i + \tfrac{1}{2}h_{--}v_i^2 + O(h^2) , \\
\frac{1}{g_{--}}\left(1-\frac{\mathcal{N}_i}{\mathcal{S}_i}\right) &= -\tfrac{1}{2}v_i^2 + N_i v_i^2 - \tfrac{1}{2}D_i - \tfrac{3}{8}h_{--}(v_i^2)^2 + O(h^2) , \\
\frac{1}{g_{--}}\left(\frac{\frac{1}{2}\mathcal{D}_i}{\mathcal{S}_i} - \frac{\mathcal{N}_i - \mathcal{S}_i}{g_{--}}\right) &= \tfrac{1}{8}(v_i^2)^2 - \tfrac{3}{8}N_i(v_i^2)^2 + \tfrac{1}{4}D_i v_i^2 + \tfrac{1}{8}h_{--}(v_i^2)^3 + O(h^2) .
\end{aligned} \tag{D.16}$$

To this order then, the components of the energy-momentum tensor are

$$\begin{aligned}
\Theta^{++} &= \tfrac{1}{2}\sum_i Q_i\delta_i(1 - N_i + \tfrac{1}{2}h_{--}v_i^2) , \\
\Theta^{+-} &= -\tfrac{1}{4}\sum_i Q_i\delta_i(v_i^2 - 2N_i v_i^2 + D_i + \tfrac{3}{4}h_{--}v_i^4) , \\
\Theta^{+a} &= \tfrac{1}{2}\sum_i Q_i v_i^a\delta_i(1 - N_i + \tfrac{1}{2}h_{--}v_i^2) , \\
\Theta^{--} &= \tfrac{1}{8}\sum_i Q_i\delta_i(v_i^4 - 3N_i v_i^4 + 2D_i v_i^2 + h_{--}v_i^6) , \\
\Theta^{-a} &= -\tfrac{1}{4}\sum_i Q_i v_i^a\delta_i(v_i^2 - 2N_i v_i^2 + D_i + \tfrac{3}{4}h_{--}v_i^4) , \\
\Theta^{ab} &= \tfrac{1}{2}\sum_i Q_i v_i^a v_i^b\delta_i(1 - N_i + \tfrac{1}{2}h_{--}v_i^2) .
\end{aligned} \tag{D.17}$$

In Sec 5.2 the following solution for h_{MN} is found

$$h_{MN} = \begin{pmatrix} h_{++} & h_{+-} & h_{+b} \\ h_{-+} & h_{--} & h_{-b} \\ h_{a+} & h_{a-} & h_{ab} \end{pmatrix} = \kappa_{11}^2 \sum_i Q_i f_i \begin{pmatrix} \frac{1}{4}v_i^4 & -\frac{1}{2}v_i^2 & -\frac{1}{2}v_i^b v_i^2 \\ -\frac{1}{2}v_i^2 & 1 & v_i^b \\ -\frac{1}{2}v_i^a v_i^2 & v_i^a & v_i^a v_i^b \end{pmatrix} + O(\kappa_{11}^4) . \tag{D.18}$$

This can be substituted into the energy momentum tensor. Using

$$\begin{aligned}
-N_i + \tfrac{1}{2}h_{--}v_i^2 &= \kappa_{11}^2\sum_j Q_j f_j \tfrac{1}{2}(v_i - v_j)^2 + O(\kappa_{11}^4) , \\
-2N_i v_i^2 + D_i + \tfrac{3}{4}h_{--}v_i^4 &= \kappa_{11}^2\sum_j Q_j f_j \tfrac{1}{4}(v_i - v_j)^2(3v_i^2 + v_j^2 - 2v_i{\cdot}v_j) + O(\kappa_{11}^4) , \\
-3N_i v_i^2 + 2D_i + h_{--}v_i^4 &= \kappa_{11}^2\sum_j Q_j f_j (v_i - v_j)^2(v_i^2 + \tfrac{1}{2}v_j^2 - v_i{\cdot}v_j) + O(\kappa_{11}^4) ,
\end{aligned} \tag{D.19}$$

the components of T_{MN} read

$$\begin{aligned}
T_{++} &= -\tfrac{1}{8}\sum_i Q_i v_i^4\delta_i - \tfrac{1}{8}\kappa_{11}^2\sum_{i,j} Q_i Q_j f_j \delta_i (v_i - v_j)^2 v_i^2(v_i^2 - \tfrac{1}{2}v_j^2 - v_i{\cdot}v_j) , \\
T_{+-} &= \tfrac{1}{4}\sum_i Q_i v_i^2\delta_i + \tfrac{1}{16}\kappa_{11}^2\sum_{i,j} Q_i Q_j f_j \delta_i (v_i - v_j)^2(v_i^2 - v_j^2 - 2v_i{\cdot}v_j) ,
\end{aligned}$$

$$
\begin{aligned}
T_{+a} &= \tfrac{1}{4}\sum_i Q_i v_i^a v_i^2 \delta + \tfrac{1}{16}\kappa_{11}^2 \sum_{i,j} Q_i Q_j f_j \delta_i (v_i - v_j)^2 \left((3v_i^2 - v_j^2 - 2v_i \cdot v_j) v_i^a - 2v_i^2 v_j^a \right) \ , \\
T_{--} &= -\tfrac{1}{2}\sum_i Q_i \delta_i + \tfrac{1}{4}\kappa_{11}^2 \sum_{i,j} Q_i Q_j f_j \delta_i (v_i - v_j)^2 \ , \\
T_{-a} &= -\tfrac{1}{2}\sum_i Q_i v_i^a \delta_i + \tfrac{1}{4}\kappa_{11}^2 \sum_{i,j} Q_i Q_j f_j \delta_i (v_i - v_j)^2 v_j^a \ , \\
T_{ab} &= -\tfrac{1}{2}\sum_i Q_i v_i^a v_i^b \delta_i - \tfrac{1}{4}\kappa_{11}^2 \sum_{i,j} Q_i Q_j f_j \delta_i (v_i - v_j)^2 (v_i^a v_i^b - v_i^a v_j^b - v_j^a v_i^b) \ .
\end{aligned}
\tag{D.20}
$$

Appendix E: The three-particle integral

The aim here is to evaluate the following integral

$$I^{ij}_{ab} = \int \mathrm{d}^d y \frac{(y-y_i)_a (y-y_j)_b}{|y-y_{k_1}|^{\alpha_1} |y-y_{k_2}|^{\alpha_2} |y-y_{k_3}|^{\alpha_3}} , \tag{E.1}$$

This is possible in the specific parameter range considered by Dine and Rajaraman. Specifically, only 3 particles are considered and two of those particles are relatively close compared with the third;

$$|y_3 - y_1| \sim |y_3 - y_2| \equiv |y_3| \gg |y_1 - y_2| . \tag{E.2}$$

Their term was of the form $|y_3|^7 |y_1 - y_2|^7$ and it shall now be shown how to extract this behaviour.

Consider first the easier integral

$$I = \int \mathrm{d}^d y \frac{1}{|y-y_1|^{\alpha_1} |y-y_2|^{\alpha_2} |y-y_3|^{\alpha_3}} . \tag{E.3}$$

Because the denominators in I^{ij}_{ab} come from f_i's, all the coefficients $\alpha \geq 7$. To extract the leading large $|y_3|$ behaviour, the following steps are performed. Consider the contribution from the region $|y| > |y_3|$, this is roughly

$$\int_{|y|>|y_3|} \mathrm{d}^d y \frac{1}{|y|^{\alpha_1+\alpha_2}} \frac{1}{|y-y_3|^{\alpha_3}} \sim |y_3|^{d-\alpha_1-\alpha_2-\alpha_3} , \tag{E.4}$$

where the integral has been completed to all y and any IR problems have been regularised as in Eq. (5.59). Thus the contribution from this region is negligible. The contribution from $|y| < |y_3|$ is obtained by expanding

$$\frac{1}{|y-y_3|^{\alpha_3}} = \frac{1}{|y_3|^{\alpha_3}} + O(|y_3|^{-\alpha_3-1}) . \tag{E.5}$$

which yields

$$I \sim \frac{1}{|y_3|^{\alpha_3}} \int_{|y|<|y_3|} \mathrm{d}^d y \frac{1}{|y-y_1|^{\alpha_1} |y-y_2|^{\alpha_2}} . \tag{E.6}$$

Again the integral can be extended over all space since the corrections will be

$$\int_{|y|>|y_3|} \mathrm{d}^d y \frac{1}{|y-y_1|^{\alpha_1} |y-y_2|^{\alpha_2}} \sim |y_3|^{d-\alpha_1-\alpha_2} . \tag{E.7}$$

This yields

$$I = \frac{\Gamma(\frac{\alpha_1}{2} + \frac{\alpha_2}{2} - \frac{d}{2})}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})} \pi^{d/2} B(\tfrac{d}{2} - \tfrac{\alpha_1}{2}, \tfrac{d}{2} - \tfrac{\alpha_2}{2}) \frac{1}{|y_1 - y_2|^{\alpha_1+\alpha_2-d}} \frac{1}{|y_3|^{\alpha_3}} + O(|y_3|^{-\alpha_3-1}) , \tag{E.8}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta-function. Note that if $y_1 = y_2$ the integral

$$\int \mathrm{d}^d y \frac{1}{|y-y_1|^{\alpha_1+\alpha_2}|y-y_3|^{\alpha_3}} \sim |y_3|^{d-\alpha_1-\alpha_2-\alpha_3} , \tag{E.9}$$

is of higher order. On the other hand

$$\int \mathrm{d}^d y \frac{1}{|y-y_1|^{\alpha_1+\alpha_3}|y-y_2|^{\alpha_2}} \sim |y_1-y_2|^{d-\alpha_1-\alpha_2-\alpha_3} . \tag{E.10}$$

Even though these contributions are larger in magnitude than $|y_3|^{\alpha_3}|y_1-y_2|^{d-\alpha_1-\alpha_2}$, they shall be neglected since they are of a different structure to Dine and Rajaraman's term.

Now it is easy to extend the steps to the integral of Eq. (E.1). This yields

$$\begin{aligned}
I^{ij}_{ab} &= \int \mathrm{d}^d y \frac{(y-y_i)_a (y-y_j)_b}{|y-y_1|^{\alpha_1}|y-y_2|^{\alpha_2}|y-y_3|^{\alpha_3}} \\
&= \frac{\Gamma(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}-\frac{d}{2})}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})}\pi^{d/2}\frac{1}{|y_1-y_2|^{\alpha_1+\alpha_2-d}}\frac{1}{|y_3|^{\alpha_3}} \\
&\quad \times \Big\{ y_1^a y_1^b B(\tfrac{d}{2}-\tfrac{\alpha_1}{2},\tfrac{d}{2}-\tfrac{\alpha_2}{2}+2) + y_2^a y_2^b B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+2,\tfrac{d}{2}-\tfrac{\alpha_2}{2}) \\
&\qquad + (y_1^a y_2^b + y_1^b y_2^a) B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+1,\tfrac{d}{2}-\tfrac{\alpha_2}{2}+1) + y_i^a y_j^b B(\tfrac{d}{2}-\tfrac{\alpha_1}{2},\tfrac{d}{2}-\tfrac{\alpha_2}{2}) \\
&\qquad - (y_i^a y_1^b + y_1^a y_j^b) B(\tfrac{d}{2}-\tfrac{\alpha_1}{2},\tfrac{d}{2}-\tfrac{\alpha_2}{2}+1) - (y_i^a y_2^b + y_2^a y_j^b) B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+1,\tfrac{d}{2}-\tfrac{\alpha_2}{2}) \Big\} \\
&\quad + \frac{\Gamma(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}-\frac{d}{2}-1)}{\Gamma(\frac{\alpha_1}{2})\Gamma(\frac{\alpha_2}{2})}\frac{\pi^{d/2}}{2}\frac{1}{|y_1-y_2|^{\alpha_1+\alpha_2-d-2}}\frac{1}{|y_3|^{\alpha_3}}\delta^{ab} B(\tfrac{d}{2}-\tfrac{\alpha_1}{2}+1,\tfrac{d}{2}-\tfrac{\alpha_2}{2}+1) ,
\end{aligned} \tag{E.11}$$

with corrections being higher order in $1/|y_3|$.

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