



The University of Adelaide
Department of Physics and Mathematical Physics

Modelling and Experiments on the Behaviour of Injection Current Modulated Multimode Semiconductor Lasers

By

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Abstract

This thesis investigates the behaviour of semiconductor lasers under large amplitude sinusoidal current modulation. As these systems have applications in optical fibre communication systems, an understanding of their behaviour is highly desirable. Semiconductor lasers are also nonlinear systems and can exhibit complicated dynamics depending upon their operating conditions. For this reason, they are also widely studied as a system in which to test, experimentally, the theory of nonlinear dynamical systems. This thesis focuses directly on the behaviour of Fabry-Perot (FP) lasers under direct modulation of the injection current. It encompasses both modelling of these devices using the laser rate equations and experimental measurements. Experimentally, the dynamics of the laser is probed via measurement of the power spectra of the output intensity and of the time average longitudinal mode spectra. These measurements are performed over a wide range of modulation frequencies and modulation amplitudes.

A main component of the research involves numerical modelling using the laser rate equations. In most previous investigations the single mode rate equations were used. However, a FP laser operates with multiple longitudinal modes. Therefore, in this thesis we use the multimode rate equations to investigate the bifurcation scenarios occurring with change in the modulation frequency and amplitude. We also investigate the effect on the dynamics of the shift in the gain peak with an increase in carrier density, resulting from the population of higher energy electronic states (referred to as band-filling)[1]. We show that both these effects contribute significantly to the predicted bifurcation scenarios leading to large deviation from the single mode predictions. In particular, we show that inclusion of multiple longitudinal modes acts to reduce the extent of bifurcations occurring under current modulation. Such predictions are more consistent with the bifurcations (or lack thereof) observed experimentally in FP semiconductor lasers[2].

Previous studies[1] on the effects of multiple longitudinal modes and band-filling on the behaviour of the time average longitudinal mode spectra variation with modulation frequency showed that, when the modulation frequency is coincident with the relaxation oscillation frequency, multimode operation ensues. Moreover, band-filling effects lead to a spectral shift of the longitudinal mode spectra towards the shorter wavelength modes[1, 36]. We demonstrate, experimentally and numerically, that this behaviour also occurs when a harmonic of the modulation frequency is coincident with the relaxation oscillation frequency.

In previous studies it was shown that a system's sensitivity to noise near a bifurcation manifests itself as structure in the power spectra of the system observables. Such structures are called noise precursors. Investigations into the dynamics of period doubling bifurcations

in a current modulated semiconductor laser have shown that in these systems the precursor to the bifurcation consists of the relaxation oscillation frequency being pulled to half the modulation frequency before the onset of the bifurcation[3, 4]. This has been observed experimentally in a (single mode) distributed feed-back semiconductor laser and explained theoretically using the single mode rate equations[4]. We show that this is also predicted by the multimode rate equations and show experimental results for a multimode FP laser.

Other bifurcations predicted by the multimode rate equations include the saddle-node bifurcations and Hopf bifurcations. We show that at a saddle-node bifurcation point, the multimode rate equations predict an abrupt mode hop to an adjacent longitudinal mode, which is usually accompanied by hysteresis. Hopf bifurcation on the other hand leads to the presence of another frequency, other than the driving frequency and its harmonics in the power spectrum of the photon density. To our knowledge neither of these two bifurcations have been previously discussed in reference to the multimode semiconductor laser rate equations.

Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or any other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Chapter 1

Introduction

Large amplitude modulation of the injection current of a semiconductor laser at microwave frequencies has applications in optical fibre communication systems[5, 6, 7, 8]. For this application, ideally we require the output intensity to exactly follow the evolution of the input injection current and therefore a linear device is desirable. A semiconductor laser is, however, like most physical systems, inherently nonlinear. Its behaviour under direct modulation of the injection current is analogous to that of a nonlinear oscillator[9, 10]. It exhibits a damped relaxation oscillation whose physical origin lies in the different decay rates of the electron and hole populations and the photon density within the semiconductor medium[6]. For applications in optical communications, large amplitude modulation of the injection current is required in order to achieve significant signal to noise ratio. Under these conditions the nonlinearities become important leading to distortion of the input signal. Therefore, knowledge and understanding of this behaviour is essential for the implementation of these devices in practical applications.

Unlike a linear oscillator, the resonant frequency of a nonlinear oscillator is not constant but depends on the amplitude and frequency of the input modulation[11]. Furthermore, if strongly driven, this resonance can become unstable, through bifurcation, leading to an undamped oscillation[12]. These types of instabilities are a very important aspect of dissipative nonlinear dynamical systems as they are responsible for the vast array of behaviours exhibited by these systems, including strange attractors and chaotic motion, as opposed to the fixed point and periodic motions of a linear system[13]. The question as to whether such behaviours are relevant to modulated semiconductor lasers has received much attention[6, 1, 14, 15, 16, 17, 2, 18, 19, 20, 21, 22, 23, 24, 9, 10, 25, 26, 27, 28, 29]. We address some aspects of this issue in this thesis.

In the majority of studies of the bifurcation behaviour exhibited by a current modulated semiconductor laser the single mode rate equations were used[16, 17, 2, 18, 19, 20, 21, 22, 23, 24, 9, 10, 25, 26, 27, 28, 29]. However, this model predicts many phenomena that have never been observed experimentally. Previous studies have shown that the in-

clusion of other important physical characteristics, such as gain saturation[16, 17], carrier diffusion[30, 15], etc. leads to a description more consistent with experimental observations. In this thesis experimental investigations into the behaviour of current modulated AlGaAs Fabry-Perot lasers are conducted. Such devices have applications in short-range communication systems[7]. This class of lasers essentially operates with multiple longitudinal modes. Existing studies using multimode rate equations have concentrated on the transient behaviour[31, 32, 33] and spectral characteristics[1, 30, 14, 34, 35]. However, to our knowledge, the implications of multimode operation on the bifurcation behaviour has not been previously studied. We use the multimode semiconductor laser rate equations to investigate the bifurcation scenarios occurring with changes in modulation frequency and modulation amplitude. Comparisons are made with experimental results.

Due to the Pauli exclusion principle, an increase in the carrier density (or active region temperature) leads to the population of higher energy states in the crystal band structure (referred to as band-filling). Since the gain coefficient, which is proportional to the stimulated emission rate, depends on the distribution of electron and hole populations within the semiconductor band structure, the spectral gain function is strongly dependent on both the carrier density and crystal temperature[36]. The effects of band-filling on the longitudinal mode spectra of a current modulated semiconductor laser were studied by Tarucha and Otsuka[1]. They showed that modulation at the relaxation oscillation frequency leads to a significant increase in the number of longitudinal modes with a spectral shift towards the shorter wavelength modes. Band-filling effects have also been considered in investigations of the transient behaviour of these devices[31, 32]. Since direct current modulation results in dynamical carrier density variation we would expect band-filling effects to also have implications for the dynamical behaviour. Despite this, band-filling effects are rarely considered in investigations of the bifurcation behaviour of semiconductor lasers. For this reason, we have performed a detailed comparison of the bifurcation structures predicted by the single mode and multimode rate equations (with and without band-filling) in order to determine to what extent these characteristics affect the nonlinear behaviour of these devices. Due to the sub-nanosecond time scales typically encountered in modulated semiconductor lasers we were unable to follow the dynamical variation of the output intensity directly with our experimental arrangement. Thus, comparisons between numerical predictions and experiment are made via time average measurements of the longitudinal mode spectra and intensity power spectrum measurements.

The organisation of this thesis is as follows. In Chapter 2 general aspects of laser operation and construction are reviewed. The semiclassical laser theory is introduced and the multimode rate equations derived. The main motivation for including a detailed discussion on this subject is to present a systematic study of the approximations made in obtaining these equations in order to have a physical understanding of their range of validity. In Chap-

ter 3 the experimental arrangement is described and the behaviour of the laser under DC operation characterised. The results of diagnostic measurements, used to determine the parameter values required for the rate equations, are also presented in this chapter. Extensive numerical studies on the bifurcation scenarios predicted by the single mode and multimode rate equations are given in Chapter 4. The longitudinal mode spectra evolution as a function of modulation frequency are also calculated and compared with experimental results. Chapter 5 discusses the approach to bifurcations. This behaviour is probed experimentally via intensity power spectra. The implications of this behaviour for the location of the bifurcation surfaces in the parameter space are also discussed. In this chapter, Langevin noise terms, describing noise induced by the discrete nature of the carrier recombination process, are also included in the simulations in order to determine the effect of noise on the dynamics. Finally, concluding remarks are made in Chapter 6.

Chapter 2

Semiconductor Laser Theory

2.1 Introduction

This chapter is devoted entirely to a description of semiconductor laser operation leading ultimately to the derivation of the rate equations; these equations will be used throughout the remainder of this thesis. The discussion given here is quite detailed in order to present a concise study of the approximations made in obtaining such a theory and to indicate possible areas of improvement. Due to the complicated nature of the semiconductor laser system, any description will represent, of necessity, a significant approximation to the actual device; a fact that has led to a diversity of models proposed to describe their operation[1, 37, 38, 15, 16, 19, 29, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61]. An understanding of the relevant physical properties of these devices is essential for distinguishing the different theoretical approaches and determining the relative advantages of each.

As a physical system the semiconductor laser involves the interaction of the electromagnetic field with the electron and hole populations of an inverted semiconductor crystal. It is a non-equilibrium many-body system and is very difficult to model from first principles; hence approximations, leading to significant simplification of the problem, are often employed[36]. A frequently used approach is to neglect the coulomb interactions between the carriers and to treat each carrier as though it were a single particle moving in a periodic potential describing the interaction with the crystal lattice[62, 63]. A further approximation is to neglect the quantum nature of the electromagnetic field and treat it classically as a solution to Maxwell's equations; the semiconductor medium on the other hand is treated quantum mechanically. This approach, known as the semiclassical theory[64, 65], is the one taken here, and leads to a set of rate equations governing the evolution of the total carrier and photon densities within the active region of the semiconductor laser. However, in the process of obtaining these equations a large number of approximations are made, not all of

which are valid under all circumstances. Thus, an understanding of the physical implications of these approximations is desirable. Moreover, extensions to the theory often lead to significant complications greatly increasing the complexity of calculations[36]. The purpose of this chapter is to introduce the approximations made in obtaining the rate equations in a systematic way and to indicate their range of validity. However, many of the issues raised in this chapter will not be addressed in the remainder of this thesis but constitute areas of further work.

This chapter is organised as follows. In section 2, we review general aspects of semiconductor laser operation, construction and physical properties. The quantum mechanical description of the semiconductor medium is described in section 3, along with a brief review of some relevant concepts from solid state physics. The interaction Hamiltonian for the interaction with the electromagnetic field is also introduced. In section 4 the semiclassical theory is discussed and laser rate equations are derived. Finally, section 5 discusses some aspects of these devices not considered by the semiclassical theory.

2.2 Semiconductor Laser Structures

2.2.1 Introduction

The first semiconductor laser (SCL) appeared in 1962, in three independent laboratories, just four years after the advent of the laser[66, 67, 68]. These earlier lasers were required to operate at cryogenic temperatures. In 1969 a significant advance was made with the development of the heterostructure; a device which allows for both light and current confinement greatly increasing the efficiency and reliability of these devices[69, 70]. Over the last three decades, the further development of SCL devices has been considerable due, largely, to the number of optoelectronic applications to which these lasers are particularly suited as a result of their small size, high efficiency and reliability and natural compatibility with existing electronic technology.

Probably the most important application is optical fibre communication systems[5, 6, 7, 8]. One of the main advantages of using SCL systems is that the laser intensity can be modulated simply by modulating the injection current: most existing optical communication systems are based on the direct detection of the light intensity. However, a large amount of research is devoted to coherent optical communications in which the optical frequency is used as the carrier and coherent detection techniques are employed[5]. The advantage of this approach is the optical bandwidth may be better utilised. This approach imposes strict requirements on the frequency stability of the lasers and has led to the development of distributed feedback (DFB) and distributed bragg reflector (DBR) lasers which incorporate a frequency selective mirror[71]. A great deal of work has also been done on linewidth

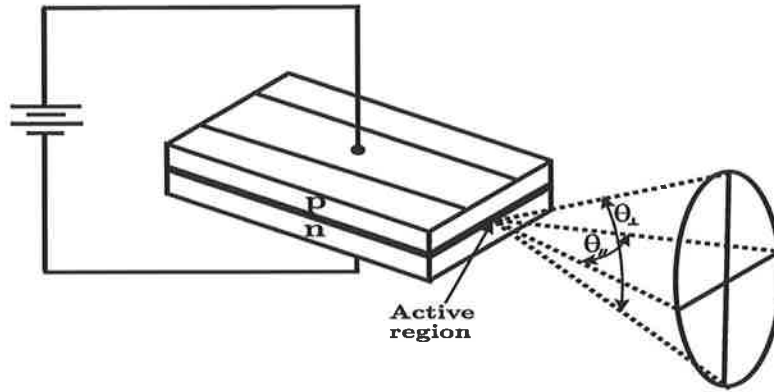


Figure 2.1: The geometry of the semiconductor laser device.

narrowing using external feedback[5, 72, 73].

Some other applications include optical memory, laser printers and copiers and optical pumping of solid state lasers. External cavity lasers are also employed as a variable frequency source for spectroscopy[74].

The essential laser structure consists of a forward biased pn junction (see Fig. 2.1). The light emission is generated in the depletion region of the junction by the recombination of a conduction band electron and a valence band hole.

In order to achieve lasing some form of optical feedback is required. The simplest means of forming cavity mirrors is simply to cleave the crystal to form a Fabry-Perot cavity. However, reflection coatings or more sophisticated reflectors such as DFB or DBR structures may be employed. Such laser structures are not investigated in this thesis and the following discussion pertains predominantly to FP lasers.

More specific aspects of device structure are discussed in the following sections.

2.2.2 Semiconductor Laser Materials

When an electron promoted to the conduction band relaxes back to the valence band both energy and momentum must be conserved in the transition. For typical semiconductor transitions, the photon wavenumbers, $k_p = 2\pi/\lambda$, are of the order of $10^7 m^{-1}$ whereas the electron wavenumbers lie in the range $0 < k_e < \pi/a$ where π/a is typically of the order of $10^{10} m^{-1}$ (a is the lattice constant). Therefore photon momenta are negligible compared with the electron momenta within the semiconductor[6]. Thus, for an indirect bandgap semiconductor, in which the conduction band minimum and the valence band maximum are separated by approximately π/a , nonradiative transitions tend to dominate. For this reason most SCL systems are constructed from direct bandgap semiconductors.

The emission wavelength of a SCL depends on the semiconductor composition. The

two most important semiconductor compounds are $AlGaAs$ and $InGaAsP$. The compounds based on $Al_{1-x}Ga_xAs$ are direct bandgap materials for $x < 0.38$ and emit wavelengths in the range $0.75 - 0.87\mu m$ depending upon the Aluminium content. The quaternary compounds $In_{1-x}Ga_xAs_yP_{1-y}$ emit wavelengths in the range $1.15 - 1.67\mu m$. These compounds are particularly important for optical communication systems since they can be made to operate at wavelengths of $1.31\mu m$ and $1.55\mu m$ corresponding, respectively, to minimum loss and dispersion for existing optical fibre systems. Another frequently used active region compound is $Ga_{0.5}In_{0.5}P$ sandwiched between $AlGaInP$ layers, which operates at approximately $670nm$ [70, 6]. The most recently developed compounds for use in semiconductor laser fabrication are the $(In_x Ga_{1-x})_y Al_{1-y}N$ compounds[75] and various II-VI semiconductor materials[76]. Both these materials have applications for blue-green and blue emitting semiconductor lasers.

Generally, the active region of conventional (bulk) lasers is of the order of $1000nm$. However, if the active region is made sufficiently narrow ($5 - 10nm$) quantum effects become important and lead to marked differences in performance. Lasers which have such a thin active layer are referred to as quantum well lasers, in the case of single layer, and multiple quantum well lasers, in the case of several layers. The main advantage of this type of laser is reduced threshold currents. They also exhibit higher modulation bandwidths and improved temperature dependence[70].

Another important laser structure is the vertical cavity surface emitting laser (VCSEL) [77]. The fundamental difference between VCSEL devices and conventional laser structures is that light emission, in a VCSEL, occurs perpendicular rather than parallel to the heterojunction. This is achieved by the use of distributed bragg reflectors mirrors to form the laser cavity. These types of lasers exhibit several desirable features including single frequency operation (arising as a result of their short cavity lengths) and very low threshold currents ($< 1 mA$) resulting from their small active region volumes[77].

In the experimental investigations conducted here, only lasers with an $AlGaAs$ active region are used. Accordingly, we restrict the following discussion to $AlGaAs$ semiconductor compounds only. Some relevant material parameters for $AlGaAs$ are given in Appendix A.

2.2.3 Heterostructures

The crystals $GaAs$ and $AlAs$ have almost identical lattice constants due to the nearly equal sizes of Ga and Al . Hence some Gallium atoms may be replaced by Aluminium atoms while still retaining the same basic lattice structure. As the content of Aluminium increases the bandgap increases and the refractive index decreases; these two factors are crucial for designing heterostructure lasers since they allow for simultaneous light and carrier confinement[78, 69].

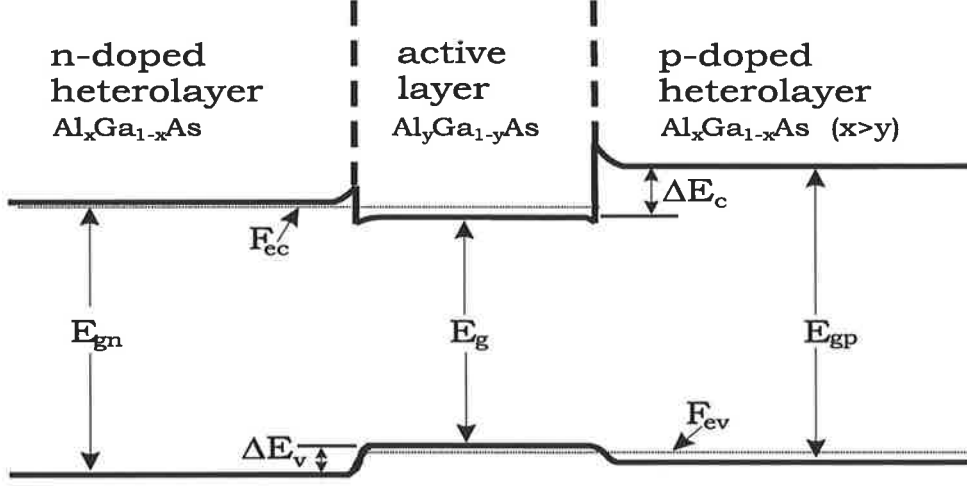


Figure 2.2: The basic heterojunction under forward bias. The symbols E_{gn} , E_{gp} and E_g correspond to the energy gaps between the conduction and valence bands for the p-doped region, n-doped region and active region, respectively. ΔE_c is the energy barrier for conduction band electrons and ΔE_v is the energy barrier for the valence band holes, for carriers within the active region. The quantities F_{ec} and F_{ev} are the quasi-Fermi levels for the conduction and valence bands, respectively[6].

The basic AlGaAs heterostructure is shown in Fig. 2.2. Application of a forward injection current results in the injection of electrons, from the n-doped barrier layer, and holes, from the p-doped barrier layer, into the active region. The difference in bandgap energy between the active region and the barrier layers results in an energy barrier ΔE_c , which prevents electrons from passing to the p-doped region, and a corresponding energy barrier ΔE_v , which prevents holes from passing through to the n-doped region. Hence carriers are confined within the active region increasing the efficiency with which they recombine to emit photons[6, 78, 79]. Though all heterostructures operate on the principles described above, more sophisticated device structures exist that use separate heterolayers for current and light confinement[79].

2.2.4 Laser Modes

Longitudinal Modes

In the simplest case the longitudinal mode structure results from the Fabry Perot (FP) cavity formed by the two cleaved facets of the crystal. For uncoated facets the Fresnel reflection coefficient (for normal incidence) is[80]

$$R = \left(\frac{n - 1}{n + 1} \right)^2 \quad (2.1)$$

where n is the refractive index of the semiconductor and the refractive index of air is approximated as 1. For GaAs, $n \approx 3.5$ yielding a reflectivity of approximately 30%. In some cases a anti-reflection or reflection coating is applied to one or both facets of the crystal in order to increase the output power or lower the threshold current.

The free spectral range (or longitudinal mode frequency spacing) of a FP cavity is [80]

$$\delta\nu = \frac{c}{2nL} \quad (2.2)$$

where c is the speed of light in vacuo, n is the refractive index of the medium and L is the length of the cavity. For a passive resonator, the photon lifetime, τ_p (also referred to as the cavity lifetime) is defined by means of the equation [70]

$$\frac{dI}{dt} = -\frac{I}{\tau_p}, \quad (2.3)$$

where I is the intensity stored in the mode. Assuming mirror losses can be considered as being uniformly distributed over the cavity τ_p is given by [70]

$$\tau_p = \frac{nL}{c[\alpha L - \ln \sqrt{R_1 R_2}]} \quad (2.4)$$

where α is the average distributed loss per unit length and R_1 and R_2 are the cavity mirror reflectivities. This parameter is important in the dynamical description of semiconductor laser operation.

Transverse and Lateral Modes

For single transverse mode operation both transverse and lateral confinement is required. Transverse confinement of the mode, in the direction perpendicular to the plane of the junction, is provided by the waveguide formed by the heterostructure barrier layers. Since these layers have a lower refractive index than the active layer, light is confined by total internal reflection at these boundaries. For such a dielectric waveguide both TE (electric field parallel to the junction) and TM modes (magnetic field parallel to the junction) may propagate. The TE mode, however, usually dominates since facet reflectivities are higher for the TE modes. Hence, SCL systems are usually strongly polarised with the electric field parallel to the junction [6].

Lateral confinement of the mode, within the plane of the junction but perpendicular to the laser axis, is achieved by gain or index guiding. The gain guiding mechanism is the result of nonuniform gain profile in the junction plane caused by localised current injection [79]. Gain guided lasers tend to operate in multiple longitudinal modes, have increased threshold currents and increased astigmatism relative to index guided lasers. The

increased astigmatism is the result of the nonuniform refractive index profile, associated with the nonuniform gain profile, resulting in varying phase and hence a curved wavefront in the lateral direction[81]. However, gain guided lasers are easier to fabricate than index guided lasers and, due to broader mode widths, generally allow higher output powers.

Index guiding essentially requires a waveguide to be fabricated within the plane of the junction and there exist an extremely large number of different structures proposed to achieve this[6]. For sufficiently narrow stripe widths index guided lasers operate predominantly in the lowest order transverse mode. Since the index guiding method, in general, achieves tighter confinement as compared to the gain guiding method these lasers usually have a much better lateral mode structure.

Due the different stripe widths in the lateral and transverse planes, the output mode tends to be highly elliptical and divergent with the largest divergence angle perpendicular to the plane of the junction (see Fig. 2.1).

2.3 Relevant Concepts from Solid State Physics

2.3.1 Introduction

In this section we first introduce some general results from solid state physics. Pertinent properties of the semiconductor compound AlGaAs are discussed.

Within a crystalline solid it is the outer shell or valence electrons that have the greatest contribution to the bonding and also the optical and transport properties of the medium. It is the behaviour of these electrons that we will be most concerned with in the following discussion; the inner shell electrons are least affected by the incorporation of the atom into the solid[63, 82].

The valence electrons have a very broad energy distribution. It is useful to calculate the dependence of electron energy, ϵ on crystal momentum, k . The graphs of energy versus momentum are referred to as the band structure of the crystal. A simplifying assumption used in the energy band calculations is to assume that the actual crystal can be represented by a perfectly periodic lattice, neglecting lattice defects and lattice vibrations[62, 63, 82]. It is convenient to describe the periodic lattice by a set of vectors $\vec{R}_i$ such that

$$\vec{R}_i = m_{i1}\vec{a}_1 + m_{i2}\vec{a}_2 + m_{i3}\vec{a}_3 \quad (2.5)$$

where $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ are the vectors describing the shortest possible independent periodic translations of the lattice and m_{ij} are the set of all integers[63, 82].

We also use the mean field approximation in which we assume that each electron moves independently of the other electrons in a static potential $V(\vec{r})$ having the periodicity

of the lattice[62]. In the mean field approximation the single particle electronic wavefunction $\phi(\vec{r})$ obeys the time independent Schrodinger equation[62]

$$H\phi = \left[-\frac{\hbar^2}{2m_o} \vec{\nabla}^2 + V(\vec{r}) \right] \phi = \epsilon\phi \quad (2.6)$$

where $V(\vec{r}) = V(\vec{r} + \vec{R})$ is a periodic function with the same periodicity as the lattice. A knowledge of the translation symmetry helps in simplifying the solution of equation 2.6 and leads to Bloch's theorem for the electronic wavefunctions[63, 82]:

$$\phi(\vec{r} + \vec{R}) = e^{i\vec{k} \cdot \vec{R}} \phi(\vec{r}) \quad (2.7)$$

which is satisfied by the following choice of ϕ

$$\phi_{\lambda k}(\vec{r}) = \frac{e^{i\vec{k} \cdot \vec{r}}}{\sqrt{\mathcal{V}}} u_{\lambda k}(\vec{r}) \quad (2.8)$$

where $\mathcal{V}$ is the crystal volume. The functions $u_{\lambda k}(\vec{r})$ are known as Bloch functions and have the periodicity of the lattice:

$$u_{\lambda k}(\vec{r}) = u_{\lambda k}(\vec{r} + \vec{R}). \quad (2.9)$$

We have labelled the eigenfunctions by a subscript λ , known as the band index, which will be required later[63, 82]. Substituting equation 2.8 in to the Schrodinger equation 2.6 we obtain the following equation for $u_{\lambda k}(\vec{r})$:

$$\left[-\frac{\mathbf{p}^2}{2m_o} + \frac{\hbar}{m_o} \vec{k} \cdot \vec{\mathbf{p}} + V(\vec{r}) \right] u_{\lambda k}(\vec{r}) = \left[\epsilon - \frac{\hbar^2 k^2}{2m_o} \right] u_{\lambda k}(\vec{r}) \quad (2.10)$$

where we have represented the electron momentum operator by $\vec{\mathbf{p}}^1$. The general solution to the Schrodinger equation 2.6 can be expressed as

$$\phi_{\lambda}(\vec{r}) = \sum_k \frac{e^{i\vec{k} \cdot \vec{r}}}{\sqrt{\mathcal{V}}} u_{\lambda k}(\vec{r}). \quad (2.11)$$

Hence, the electronic wavefunctions have the form of a linear superposition of plane waves, each modulated by a periodic function which has the same periodicity as the lattice[63].

We will find it useful to introduce a set of vectors $\vec{G}_i$, called reciprocal lattice vectors, which are defined such that[63, 82]

$$\vec{G}_i \cdot \vec{R}_j = 2\pi \eta_{ij} \quad (2.12)$$

¹We represent momentum by a bold face $\vec{\mathbf{p}}$ in order to distinguish it from the dipole moment operator to be defined later.

where η_{ij} is an integer. The set of all vectors $\vec{G}_i$ define the reciprocal lattice. The reciprocal lattice can be considered as the Fourier space associated with the crystal lattice. The utility of the reciprocal lattice is that the electron energy is a periodic function of the reciprocal lattice vectors²:

$$\epsilon(\vec{k}) = \epsilon(\vec{k} + \vec{G}). \quad (2.15)$$

Therefore, in order to describe the dependence of electron energy on momentum it suffices to use k values within the unit cell ($k < G$) of the reciprocal lattice, defined as the first Brillouin zone[63].

The band structure is determined by solving the Schrodinger equation numerically with appropriate lattice symmetry and potential functions[62]. It is important to note that each of the k states in the band structure correspond to single particle states.

2.3.2 Semiconductor Band Structure

The binary compounds GaAs and AlAs are both examples of $III - V$ compounds in which the Gallium and Aluminium atoms each have three valence electrons and Arsenic has five valence electrons. These valence electrons are shared between each of the atoms to form four tetrahedrally directed covalent bonds. However, the difference in the core charges of the respective atoms also results in an ionic contribution to the bonding[78, 82].

The basic structure of $Ga(Al)As$ is shown in Fig. 2.3(a). It is known as the Zinc Blende structure and is formed by two interpenetrating face centred cubic (FCC) lattices displaced along the body diagonal of the cubic cell by one quarter the length of the diagonal[83]³. The reciprocal lattice corresponding to the FCC lattice is body centred cubic (BCC) and is shown in Fig. 2.3(b); the first Brillouin zone is also shown along with the high symmetry points Γ , X and L [83]. The point Γ always corresponds to the zone centre ($\vec{k} = 0$) and the six points L and the six points X correspond to the wavevectors $\vec{k} = 2\pi/a(\pm 1, \pm 1, \pm 1)$ and $\vec{k} = \{2\pi/a(\pm 1, 0, 0), 2\pi/a(0, \pm 1, 0), 2\pi/a(0, 0, \pm 1)\}$, respectively[62].

The band structure of GaAs and AlAs is shown in Fig. 2.4 along the directions $\Gamma \longrightarrow L$ and $\Gamma \longrightarrow X$. GaAs is a direct bandgap semiconductor whereas AlAs is indirect. Since

²In order to show this we write the Bloch function, which has energy eigenvalue $\epsilon(k)$, as

$$\phi_k(\vec{r}) = e^{i(\vec{k} + \vec{G}) \cdot \vec{r}} e^{-i\vec{G} \cdot \vec{r}} u_k(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} u_{k+G}(\vec{r}) = \phi_{k+G}(\vec{r}) \quad (2.13)$$

where $u_{k+G}(\vec{r}) = e^{i\vec{G} \cdot \vec{r}} u_k(\vec{r})$. We may readily show that $u_{k+G}(\vec{r})$ is also a periodic function of $\vec{r}$ since

$$u_{k+G}(\vec{r} + \vec{R}) = e^{i\vec{G} \cdot \vec{r}} e^{i\vec{G} \cdot \vec{R}} u_k(\vec{r} + \vec{R}) = e^{i\vec{G} \cdot \vec{r}} u_k(\vec{r}) = u_{k+G}(\vec{r}) \quad (2.14)$$

where we have used $e^{i\vec{G} \cdot \vec{R}} = 1$ (from equation 2.12) and the periodicity property of $u_k(\vec{r})$. Since $\phi_k(\vec{r})$ and $\phi_{k+G}(\vec{r})$ are the same function then they have the same energy eigenvalue; hence $\epsilon(\vec{k}) = \epsilon(\vec{k} + \vec{G})$.

³The Zinc Blende structure can be regarded as a FCC Bravais lattice with a two point basis set $\{0, \frac{a}{4}(\hat{x}, \hat{y}, \hat{z})\}$ [82]

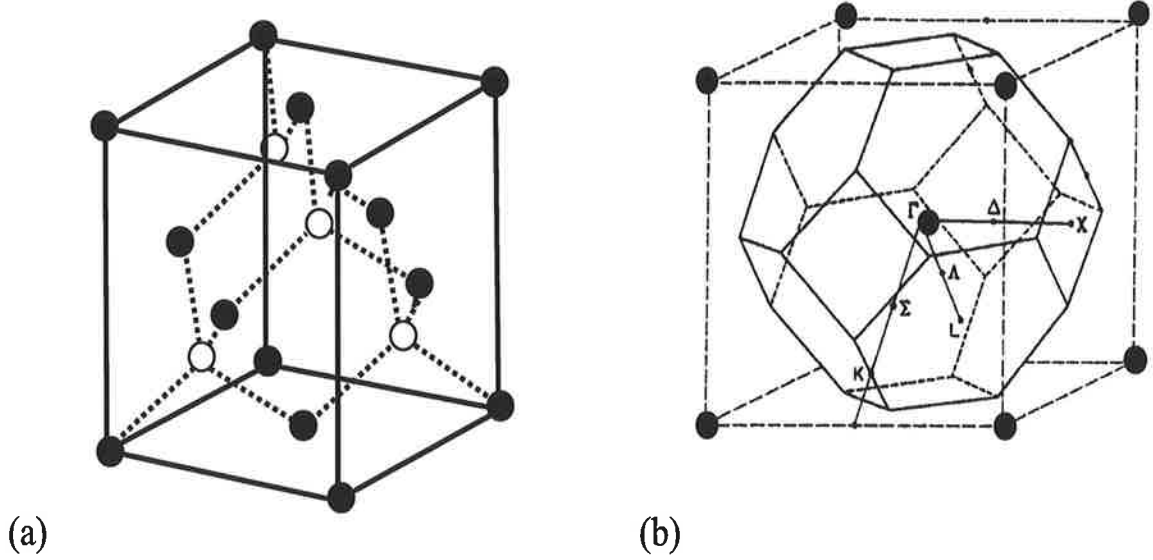


Figure 2.3: The unit cell of (a) the crystal lattice and (b) the reciprocal lattice of the Zinc Blende structure. In (b) the first Brillouin zone is also shown. After [83]

GaAs and AlAs have nearly equal lattice constants some Ga atoms may be partially replaced by Al atoms while still retaining the same basic lattice structure[69]. As the Aluminium content is increased the AlGaAs compound makes a smooth transition from a direct to an indirect bandgap semiconductor.

The Brillouin zone of the FCC lattice is highly symmetrical; a direct result of the symmetry of the crystal. This symmetry has two important consequences for the band structure[62]. Firstly, if two k vectors in the Brillouin zone can be transformed into each other under a symmetry operation of the Brillouin zone then their electron energies must be identical. Points and axes in reciprocal space which have this property are said to be equivalent. The second consequence of the crystal symmetry is that the electronic wavefunctions can be expressed such that they have definite transformation properties under a symmetry operation of the crystal. i.e. they reflect the symmetry of the point in reciprocal space to which they correspond. Due to this symmetry property it can be shown that some expectation values of operators vanish which leads to selection rules analogous to those found in atomic physics. This is important for the electric dipole transitions which govern optical emission from the semiconductor[62]⁴.

⁴The symmetry properties of the wavefunctions also have significant implications for the crystal band structure. For an excellent introduction see [62].

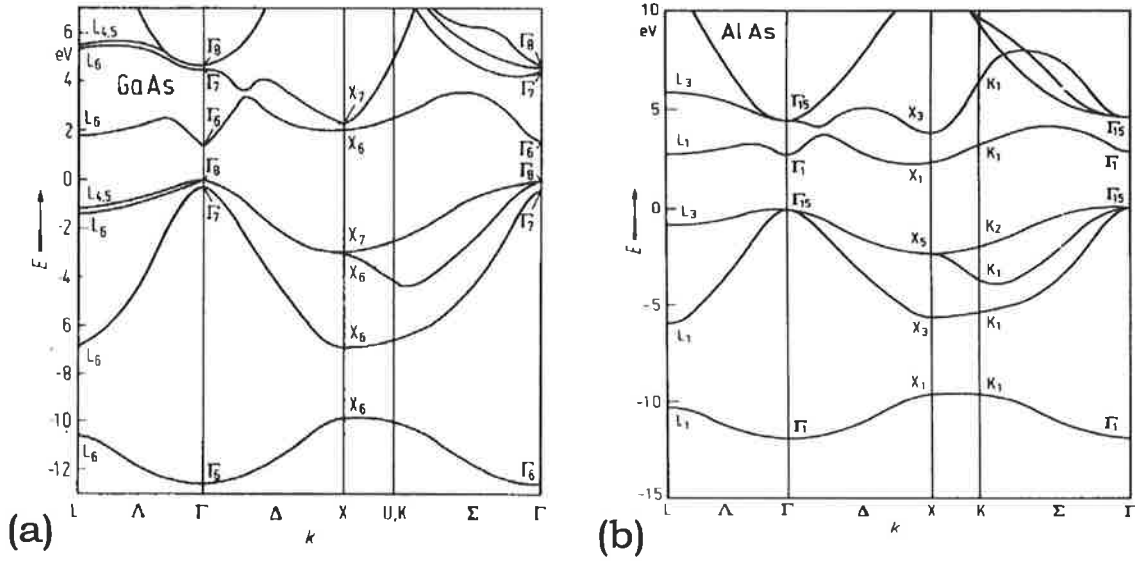


Figure 2.4: The band structure of (a) GaAs and (b) AlAs. After [84]

Effective Mass and $\vec{k} \cdot \vec{p}$ Method

In this section the $\vec{k} \cdot \vec{p}$ method for calculating semiconductor band structure is presented⁵. Many of the results used in this section require group theoretical arguments to justify them. Since these arguments are beyond the scope of this thesis relevant results will be introduced where required without justification.

The $\vec{k} \cdot \vec{p}$ method is a perturbative method which may be used to calculate the band structure about some high symmetry point in k space for which the energy eigenvalues and eigenfunctions are known. In this case we choose the zone centre Γ since it is the most relevant for optical transitions. The starting point for the $\vec{k} \cdot \vec{p}$ theory is equation 2.10 for $u_{\lambda k}(\vec{r})$:

$$\left[H_o + \frac{\hbar}{m_o} \vec{k} \cdot \vec{p} \right] u_{\lambda k}(\vec{r}) = \bar{\epsilon}_{\lambda k} u_{\lambda k}(\vec{r}) \quad (2.16)$$

where⁶

$$H_o = \frac{\mathbf{p}^2}{2m} + V(\vec{r}) \quad (2.17)$$

and

$$\bar{\epsilon}_{\lambda k} = \epsilon_{\lambda k} - \frac{\hbar^2 k^2}{2m_o}. \quad (2.18)$$

For $k = 0$, equation 2.16 reduces to $H_o u_{\lambda 0} = \epsilon_{\lambda 0} u_{\lambda 0}$, the solutions of which form a complete

⁵This section follows closely the discussion of P. Yu and M. Cardona[62]. See also [85]

⁶In order to simplify the notation, the absolute magnitude of all vector quantities, e.g. $|\vec{p}|$, are denoted by their symbol, e.g. p , throughout this thesis.

set of orthonormal basis functions. We treat the $\vec{k} \cdot \vec{p}$ term as a perturbation; hence these calculations are only valid for small k . For a non-degenerate band we can expand equation 2.16 to second order in k and solve using non-degenerate perturbation theory yielding:

$$u_{\lambda k} = u_{\lambda 0} + \frac{\hbar}{m_o} \sum_{\lambda' \neq \lambda} \frac{\vec{k} \cdot \langle u_{\lambda 0} | \vec{p} | u_{\lambda' 0} \rangle u_{\lambda' 0}}{\epsilon_{\lambda 0} - \epsilon_{\lambda' 0}} \quad (2.19)$$

and

$$\epsilon_{\lambda k} = \epsilon_{\lambda 0} + \frac{\hbar^2 k^2}{2m_o} + \frac{\hbar^2}{m_o} \sum_{\lambda' \neq \lambda} \frac{|\vec{k} \cdot \langle u_{\lambda 0} | \vec{p} | u_{\lambda' 0} \rangle|^2}{\epsilon_{\lambda 0} - \epsilon_{\lambda' 0}}. \quad (2.20)$$

It is convenient to express equation 2.20 for the band energy as

$$\epsilon_{\lambda k} = \epsilon_{\lambda 0} + \frac{\hbar^2}{2} \left[\frac{1}{m^*} \right]_{ij} k_i k_j \quad (2.21)$$

where summation over repeated indices is implied and i and j each run from 1 to 3 corresponding to the x , y and z directions respectively. The quantity $[1/m^*]$ is an effective mass tensor defined by

$$\left[\frac{1}{m^*} \right]_{ij} = \frac{1}{m_o} \left[\delta_{ij} \pm \sum_{\lambda'} \frac{2\mathcal{P}_{\lambda'i}^* \mathcal{P}_{\lambda'j}}{m_o(\epsilon_{\lambda 0} - \epsilon_{\lambda' 0})} \right] \quad (2.22)$$

where $\mathcal{P}_{\lambda'i} = \langle u_{\lambda 0} | p_i | u_{\lambda' 0} \rangle$. This equation shows that the electron effective mass is the result of coupling between electronic states in different bands via the $\vec{k} \cdot \vec{p}$ interaction. Due to the denominator $(\epsilon_{\lambda 0} - \epsilon_{\lambda' 0})$ it is the closest energy bands which have the most influence.

We will use equation 2.22 to calculate the effective mass of the lowest conduction band. In *III-V* semiconductors, the lowest conduction band couples most strongly to the highest valence band. Though it also couples to the higher conduction band states, the coupling is small due to symmetry properties of the wavefunctions. Hence a good approximation to the conduction band effective mass may be obtained by considering the valence band only.

The lowest conduction band consists of two degenerate levels. These are conventionally denoted by $|S \uparrow\rangle$ and $|S \downarrow\rangle$ in analogy with the s states of atomic physics. The arrows indicate the two spin states of the electron. There are six valence band states which, in the first instance, we assume to be degenerate. These are conventionally denoted $|X, \uparrow(\downarrow)\rangle$, $|Y, \uparrow(\downarrow)\rangle$ and $|Z, \uparrow(\downarrow)\rangle$ in analogy with the p orbitals of atomic physics. However, these states are not the result of spherical symmetry, as in the atomic case, but have the cubic symmetry of the lattice.

By symmetry arguments it can be shown that the only nonzero matrix elements are [62]

$$\langle S | \mathbf{p}_x | X \rangle = \langle S | \mathbf{p}_y | Y \rangle = \langle S | \mathbf{p}_z | Z \rangle = i\mathbf{P}. \quad (2.23)$$

	<i>Without Spin Orbit Coupling</i>	<i>With Spin Orbit Coupling</i>
Conduction Band	$l = 0, s = 1/2$ $m_l = 0 \quad m_s = \pm 1/2$ (2-fold degenerate)	$J = l + s = 1/2, m_J = \pm 1/2$ (2-fold degenerate)
Valence Band	$l = 1, s = 1/2$ $m_l = 0, \pm 1 \quad m_s = \pm 1/2$ (6-fold degenerate)	$J = 3/2, m_J = \pm 3/2, \pm 1/2$ (4-fold degenerate) $J = 1/2, m_J = \pm 1/2$ (2-fold degenerate)

Figure 2.5: Valence band splitting due to spin orbit coupling

In this case the effective mass is scalar quantity. Setting the zero of energy to lie at the top of the valence band and denoting the energy gap between the valence band maximum and conduction band minimum by ϵ_g , from equation 2.22, we obtain

$$\frac{1}{m_c^*} \approx \frac{1}{m_o} \pm \frac{2\mathbf{P}^2}{m_o^2 \epsilon_g} \quad (2.24)$$

This equation is useful since it gives an approximate relation between the conduction band effective mass and the momentum matrix elements between the conduction band and valence band states at $k = 0$ and hence can be used to obtain either quantity if the other is known (e.g. from experiment). We shall see later that the electric dipole operator matrix elements, which describe optical emission, are proportional to the momentum matrix elements.

In principle, equation 2.22 could also be used to obtain the valence band effective masses. However, we have so far neglected the spin orbit interaction, which is a relativistic effect that couples angular momentum quantum states to spin states of the electron. This interaction acts to lift the degeneracy of the valence states as shown in Fig. 2.5. The lowest valence band state, known as the split-off or spin-orbit band, corresponds to the two spin states $m_J = \pm 1/2$ of total angular momentum $J = 1/2$. The upper valence bands states are the four states corresponding to total angular momentum $J = 3/2$ and are degenerate at $k = 0$. However, the valence band also couples to the next highest conduction band which leads to a split in the degeneracy of the $J = 3/2$ states for k values greater than zero. The states $J = 3/2, m_J = \pm 3/2$ are the higher energy states and are known as the heavy hole band. The lower states, corresponding to $J = 3/2, m_J = \pm 1/2$, are known as the light hole band. The reason for these terms will become apparent shortly.

The details of the valence band calculation will not be presented. However, for

completeness we include the results[62]. Due to symmetry properties of the wavefunctions there are six non-zero momentum matrix elements between the valence band and the next highest conduction band. These are all equal in magnitude and are denoted by Q . The energy separation between the $J = 3/2$ valence bands and the next highest conduction band is denoted by ϵ'_g

The energy of the doubly degenerate spin-orbit band is

$$\epsilon_{so} = -\Delta_o + \frac{\hbar^2 k^2}{2m_o} \left[1 - \frac{2}{3} \left(\frac{\mathbf{P}^2}{m_o(\epsilon_g + \Delta_o)} + \frac{2Q^2}{(\epsilon'_g + \Delta_o)} \right) \right] = -\Delta_o + \frac{\hbar^2 k^2}{2m_{so}^*} \quad (2.25)$$

where m_{so}^* is the spin orbit effective mass and is given by the reciprocal of the quantity in the square brackets. Since the spin-orbit band is lower in energy than the other valence bands it has minimal contribution to SCL operation.

The energies for the $J = 3/2$ valence bands are

$$\epsilon_{\pm} = Ak^2 \pm [B^2 k^4 + C^2(k_x^2 k_y^2 + k_y^2 k_z^2 + k_z^2 k_x^2)]^{1/2} \quad (2.26)$$

where

$$\frac{2mA}{\hbar^2} = 1 - \frac{2}{3} \left[\left(\frac{\mathbf{P}^2}{m_o \epsilon_g} \right) + \left(\frac{2Q^2}{m_o \epsilon'_g} \right) \right] \quad (2.27)$$

$$\frac{2mB}{\hbar^2} = \frac{2}{3} \left[\left(\frac{-\mathbf{P}^2}{m_o \epsilon_g} \right) + \left(\frac{Q^2}{m_o \epsilon'_g} \right) \right] \quad (2.28)$$

$$\frac{2mC}{\hbar^2} = \frac{16\mathbf{P}^2 Q^2}{3m_o^2 \epsilon_g \epsilon'_g}. \quad (2.29)$$

For *GaAs*, the valence bands have negative curvature and hence negative effective masses. It is useful to define empty states in the valence band as holes which can be considered as charge carriers with charge $+e$. If the energy of the vacant electronic state is $-\epsilon$ then the energy of the hole state is ϵ (for the zero of energy defined at the valence band maximum). The energy of the two hole bands corresponding to $J = 3/2$ are

$$\epsilon_{hh} = -\epsilon_+ \text{ and } \epsilon_{lh} = -\epsilon_- \quad (2.30)$$

where the subscripts *hh* and *lh* refer to heavy holes and light holes respectively. These expressions define constant energy surfaces in k space. A constant energy surface for the heavy hole energy, $-\epsilon_+$, is shown in Fig. 2.6. These energy surfaces are not spherical but are referred to as “warped” spheres. The warping is the result of the cubic symmetry of the crystal and leads to different hole effective masses along different directions in k space. Often, for simplicity, the hole effective masses are assumed to be isotropic. Approximate light hole and heavy hole effective masses can be obtained by averaging equation 2.30 over

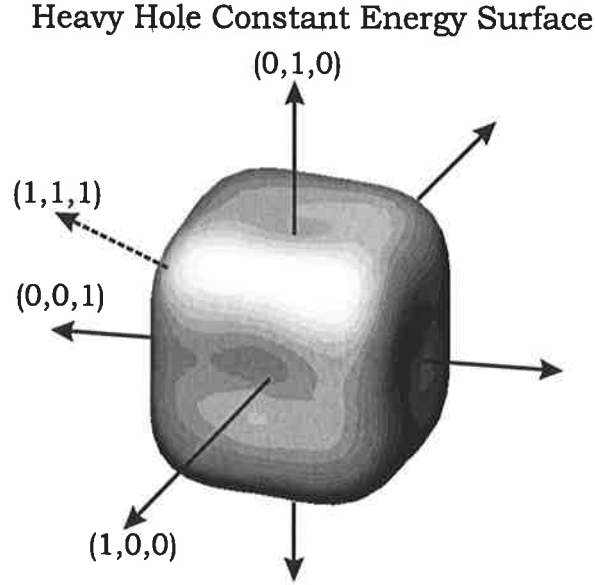


Figure 2.6: Constant energy surface for the heavy hole for an energy value of $\epsilon_{hh} = 5$.

all directions in k space[62]:

$$\frac{1}{m_{hh}^*} = \frac{1}{\hbar^2} \left[-2A + 2B \left(1 + \frac{2C^2}{15B^2} \right) \right] \quad (2.31)$$

$$\frac{1}{m_{lh}^*} = \frac{1}{\hbar^2} \left[-2A - 2B \left(1 + \frac{2C^2}{15B^2} \right) \right]. \quad (2.32)$$

The equations for the energy bands, obtained from second order perturbation theory and assuming isotropic effective masses, are symmetric parabolic bands. This approximation is equivalent to assuming the electrons and holes behave as free particles but with an effective mass given by the reciprocal of the band curvature[63]:

$$m = \hbar^2 \left(\frac{d^2 \epsilon}{dk^2} \right)^{-1}. \quad (2.33)$$

Equation 2.33 can be generalised to the case in which bands are not parabolic. In this case the effective masses are k dependent.

2.3.3 Quantum Description of the Semiconductor

In this section the quantum mechanical description of the semiconductor medium is discussed. The Hamiltonian of the system and the interaction with an external light field is

introduced⁷.

It is convenient to use the second quantised representation of a semiconductor rather than elementary quantum mechanics since the electron and hole particle number is not, in general, conserved; electron hole pairs can be created and annihilated by interaction with the radiation field and other pump and decay processes. The quantum field theory formalism utilised is summarised in Appendix B.

In the second quantised representation the wavefunction of the electron is replaced by a field operator while $\vec{r}$ and t are treated as parameters. The wavefunction $\hat{\Psi}$ is written as

$$\hat{\Psi}(\vec{r}, t) = \sum_{\lambda} \sum_{k, s_z} a_{\lambda k s_z}(t) \phi_{\lambda k s_z}(\vec{r}) \quad (2.34)$$

where $\phi_{\lambda k s_z}(\vec{r})$ are the single particle eigenfunctions for an electron in the semiconductor, and summation is over the band index $\lambda = c$ or v , momentum k and spin quantum number s_z . The operator $a_{\lambda k s_z}(t)$ is the annihilation operator for the electron in the state $\phi_{\lambda k s_z}(\vec{r})$. The Hermitean adjoint $a_{\lambda k s_z}^{\dagger}(t)$ creates an electron in the state $\phi_{\lambda k s_z}(\vec{r})$. Since electrons are Fermions the operators a and $a^{\dagger}$ obey anticommutation relations

$$[a_{\lambda k s_z}, a_{\lambda' k' s'_z}]_+ = [a_{\lambda k s_z}^{\dagger}, a_{\lambda' k' s'_z}^{\dagger}]_+ = 0 \quad (2.35)$$

$$[a_{\lambda k s_z}, a_{\lambda' k' s'_z}^{\dagger}]_+ = [a_{\lambda k s_z}^{\dagger}, a_{\lambda' k' s'_z}]_+ = 0 \quad (2.36)$$

$$[a_{\lambda k s_z}, a_{\lambda' k' s'_z}^{\dagger}]_+ = \delta_{\lambda \lambda'} \delta_{k k'} \delta_{s_z s'_z} \quad (2.37)$$

where $[A, B]_+ = AB + BA$ is the anticommutator. The operators a and $a^{\dagger}$ are analogous to harmonic oscillator annihilation and creation operators (except they obey anticommutation relations). In the following, we shall absorb the s_z summation into the summation over k for notational convenience. For Fermions there can be at most one particle in any one state; we shall denote these states by $|0_{\lambda k}\rangle$ and $|1_{\lambda k}\rangle$ for zero and one particle states respectively. The result of $a_{\lambda k}$ and $a_{\lambda k}^{\dagger}$ acting on these states is

$$a_{\lambda k} |0_{\lambda k}\rangle = a_{\lambda k}^{\dagger} |1_{\lambda k}\rangle = 0 \quad (2.38)$$

$$a_{\lambda k}^{\dagger} |0_{\lambda k}\rangle = |1_{\lambda k}\rangle \quad (2.39)$$

$$a_{\lambda k} |1_{\lambda k}\rangle = |0_{\lambda k}\rangle \quad (2.40)$$

Using these relations we can show

$$a_{\lambda k}^{\dagger} a_{\lambda k} |0_{\lambda k}\rangle = 0 \quad (2.41)$$

$$a_{\lambda k}^{\dagger} a_{\lambda k} |1_{\lambda k}\rangle = |1_{\lambda k}\rangle \quad (2.42)$$

⁷This section follows the discussion given in reference [36].

Hence the states $|0_{\lambda k}\rangle$ and $|1_{\lambda k}\rangle$ are eigenstates of the operator $a^\dagger a$ with eigenvalues equivalent to the number of particles (equals 0 or 1) in the state; hence $a^\dagger a$ is referred to as the number operator.

Often it is sufficient to consider just two bands; the highest energy valence band (heavy hole band) and the lowest conduction band. In the electron-hole representation we can define hole annihilation and creation operators

$$b_{-k}^\dagger \equiv a_{vk} \quad (2.43)$$

$$b_{-k} \equiv a_{vk}^\dagger \quad (2.44)$$

since annihilation of an electron in the valence band of momentum k is equivalent to the creation of a hole with momentum $-k$ and vice versa. We can similarly define electron operators by

$$a_k \equiv a_{ck} \quad (2.45)$$

$$a_k^\dagger \equiv a_{ck}^\dagger \quad (2.46)$$

where now when we refer to electrons we mean conduction band electrons and holes refer to vacant valence band states.

The total Hamiltonian for a semiconductor laser consists of terms describing the free carrier Hamiltonian H_o , the Coulomb interaction among carriers H_c and the interaction of the carriers with the radiation field H_I . The second quantised form of the free part of the Hamiltonian operator (which includes the interaction with the lattice potential) is given by

$$\hat{H}_o = \int d^3r \hat{\Psi}^\dagger(\vec{r}) H_o \hat{\Psi}(\vec{r}). \quad (2.47)$$

Substituting equation 2.34 for $\hat{\Psi}$ and using $H_o|\phi_{\lambda k}\rangle = \epsilon_{\lambda k}|\phi_{\lambda k}\rangle$ and the orthonormality properties of $\phi_{\lambda k}(\vec{r})$:

$$\int d^3r \phi_{\lambda k}^*(\vec{r}) \phi_{\lambda' k'}(\vec{r}) = \delta_{kk'} \delta_{\lambda\lambda'}, \quad (2.48)$$

we obtain

$$\hat{H}_o = \sum_{\lambda k k'} \int d^3r a_{\lambda k}^\dagger a_{\lambda k'} \phi_{\lambda k}^*(\vec{r}) H_o \phi_{\lambda k'}(\vec{r}) = \sum_{\lambda k} \epsilon_{\lambda k} a_{\lambda k}^\dagger a_{\lambda k}. \quad (2.49)$$

Evaluating the sum over bands and using a two band approximation yields

$$\hat{H}_o = \sum_k [\epsilon_{ck} a_k^\dagger a_k + \epsilon_{vk} (1 - b_{-k}^\dagger b_{-k})]. \quad (2.50)$$

Since the origin of energy is arbitrary we leave out the constant term $\sum_k \epsilon_{vk}$. Using $\epsilon_{vk} =$

$-\epsilon_{hk}$ and $\epsilon_{ck} = \epsilon_g + \epsilon_{ek}$ we obtain

$$\hat{H}_o = \sum_k [(\epsilon_g + \epsilon_{ek})a_k^\dagger a_k + \epsilon_{hk}b_{-k}^\dagger b_{-k}] \quad (2.51)$$

which is the free part of the semiconductor Hamiltonian in the second quantised representation.

To compute expectation values we need to choose a basis set. A convenient basis set is one made up of products of the eigenstates of the number operators

$$N_{ek} = a_k^\dagger a_k \quad (2.52)$$

$$N_{hk} = b_k^\dagger b_k \quad (2.53)$$

for the electrons and holes respectively. These are the products

$$\begin{aligned} |\{n_i\}\rangle &= |n_{ek_1}\rangle |n_{ek_2}\rangle \dots |n_{hk_1}\rangle |n_{hk_2}\rangle \dots \\ &= |n_{ek_1}, n_{ek_2}, \dots, n_{hk_1}, n_{hk_2}, \dots\rangle. \end{aligned} \quad (2.54)$$

These products give the occupancy of every single particle state (labeled by k) in the portion of the band structure under consideration. Because of the Pauli exclusion principle the n 's are either zero or one. We can also write the number operator for a many particle system

$$\hat{N} = \int \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}(\vec{r}) d^3r = \sum_k a_k^\dagger a_k \quad (2.55)$$

which gives the total number of particles.

Electric Dipole Interactions

The interaction of a semiconductor with the electromagnetic field is described by interaction Hamiltonian V_I ⁸. The complete form of V_I is quite complicated. However the main contribution to V_I arises from the electric dipole interaction:

$$V_I = \vec{p} \cdot \vec{E} \quad (2.56)$$

⁸The results presented in this section can be found in references [62] and [85].

where $\vec{p}$ is the electric dipole moment and $\vec{E}$ is the electric field⁹. The second quantised representation of the dipole moment operator is¹⁰

$$\vec{p} = \int d^3r \hat{\Psi}^\dagger(\vec{r}, t) e\vec{r} \hat{\Psi}(\vec{r}, t) = \sum_{k k' \lambda \lambda'} a_{\lambda' k'}^\dagger a_{\lambda k} \int d^3r \phi_{\lambda' k'}(\vec{r})^* e\vec{r} \phi_{\lambda k}(\vec{r}). \quad (2.57)$$

Consider the integral quantity in equation 2.57 which we denote by $\vec{\mu}_{\lambda\lambda'}$:

$$\vec{\mu}_{\lambda\lambda'}(k, k') = \int d^3r \phi_{\lambda' k'}^*(\vec{r}) e\vec{r} \phi_{\lambda k}(\vec{r}) = \langle \phi_{\lambda' k'} | e\vec{r} | \phi_{\lambda k} \rangle. \quad (2.58)$$

We refer to $\vec{\mu}_{\lambda\lambda'}$ as the dipole operator matrix element. Using $H_o |\phi_{\lambda k}\rangle = \epsilon_{\lambda k} |\phi_{\lambda k}\rangle$ and its adjoint we can show that

$$\langle \phi_{\lambda' k'} | [\vec{r}, H_o] | \phi_{\lambda k} \rangle = (\epsilon_{\lambda k} - \epsilon_{\lambda' k'}) \langle \phi_{\lambda' k'} | \vec{r} | \phi_{\lambda k} \rangle. \quad (2.59)$$

Using the identity[86]

$$[\vec{r}, H_o] = i\hbar \frac{\partial H}{\partial \vec{p}} = \frac{i\hbar \vec{p}}{m_o} \quad (2.60)$$

and equation 2.59, we obtain

$$\langle \phi_{\lambda' k'} | e\vec{r} | \phi_{\lambda k} \rangle = \frac{i\hbar e}{m_o(\epsilon_{\lambda k} - \epsilon_{\lambda' k'})} \langle \phi_{\lambda' k'} | \vec{p} | \phi_{\lambda k} \rangle. \quad (2.61)$$

Hence the electric dipole matrix elements are proportional to momentum matrix elements. The advantage of using the momentum matrix elements is that they also occur in the $\vec{k} \cdot \vec{p}$ bandstructure calculations. Writing the single particle wavefunctions $\phi_{\lambda k}(\vec{r})$ in Bloch form and using

$$\vec{p} \phi_{\lambda k}(\vec{r}) = i\hbar \vec{\nabla} (u_{\lambda k}(\vec{r}) e^{i\vec{k} \cdot \vec{r}}) = (\hbar \vec{k} + \vec{p}) u_{\lambda k}(\vec{r}) \quad (2.62)$$

we obtain for the momentum matrix elements

$$\langle \phi_{\lambda' k'} | \vec{p} | \phi_{\lambda k} \rangle = \int_{\mathcal{V}} d^3r e^{-i(\vec{k}' - \vec{k}) \cdot \vec{r}} u_{\lambda' k'}(\vec{r})^* (\hbar \vec{k} + \vec{p}) u_{\lambda k}(\vec{r}). \quad (2.63)$$

Since we are interested in transitions between different bands, the integral over the term proportional to $\hbar \vec{k}$ vanishes because $u_{\lambda' k'}(\vec{r})$ are orthogonal for $\lambda \neq \lambda'$. Because the Bloch functions are periodic functions of the lattice vectors it is convenient to rewrite the integral over the entire crystal volume $\mathcal{V}$ in equation 2.63 as the sum of integrals over all unit cells.

⁹To be consistent with the electric dipole approximation, the electric field in Equation 2.56 is time varying only; spatial dependence of the field is neglected[62].

¹⁰Classically the electric dipole moment of a collection of charges q_i at $\vec{r}_i$ from a position $\vec{R}$ is given by $\vec{p}(\vec{R}) = \sum_i q_i \vec{r}_i(\vec{R})$. The analogous quantum mechanical operator for an electron is $\vec{p} = e\vec{r}$ with expectation value $\langle \vec{p} \rangle = \int d^3r \Psi^*(\vec{r}, t) e\vec{r} \Psi(\vec{r}, t)$. This has the same form as the classical expression if we identify $\sigma = e|\Psi(\vec{r}, t)|^2$ as the charge density.

We denote the unit cell volume by ν and the crystal volume $\mathcal{V} = N\nu$ where N is the number of unit cells. Making the change of variable

$$\vec{r} \longrightarrow \vec{r}' + \vec{R}_j, \quad (2.64)$$

where $\vec{r}'$ lies within a unit cell and $\vec{R}_j$ is a lattice vector, we obtain

$$\langle \phi_{\lambda'k'} | \vec{p} | \phi_{\lambda k} \rangle = \sum_{j=1}^N e^{-i(\vec{k}' - \vec{k}) \cdot \vec{R}_j} \int_{\nu} d^3 r' e^{-i(\vec{k}' - \vec{k}) \cdot \vec{r}'} u_{\lambda'k'}(\vec{r}')^* \vec{p} u_{\lambda k}(\vec{r}'). \quad (2.65)$$

where we have used the property of the Bloch functions: $u_{\lambda k}(\vec{r} + \vec{R}_j) = u_{\lambda k}(\vec{r})$. The summation of $e^{-i(\vec{k}' - \vec{k}) \cdot \vec{R}_j}$ over all lattice vectors results in a delta function $\delta_{k'k}$ and implies optical transitions are direct transitions since $\vec{k}' = \vec{k}$, i.e. the electron's momentum is the same before and after the transition. This is called the electric dipole approximation and is equivalent to neglecting the photon momentum in comparison with typical electron momenta within the crystal[85]. It is a direct result of using equation 2.56 for the interaction energy[62]. Equation 2.65 simplifies to

$$\langle \phi_{\lambda'k} | \vec{p} | \phi_{\lambda k} \rangle = \int d^3 r u_{\lambda'k}(\vec{r})^* \vec{p} u_{\lambda k}(\vec{r}). \quad (2.66)$$

Substituting this result into equation 2.58, and using equation 2.61 we obtain for the k -dependent dipole moment matrix element

$$\vec{\mu}_{\lambda\lambda'(k)} = \frac{i\hbar e}{m_o(\epsilon_{\lambda k} - \epsilon_{\lambda'k})} \langle u_{\lambda'k} | \vec{p} | u_{\lambda k} \rangle. \quad (2.67)$$

The k dependence arises from the integral over the Bloch functions and from the k dependence of the electron and hole energies.

Substituting equation 2.67 into equation 2.57 for the dipole moment operator yields

$$\vec{p} = \sum_{\lambda'\lambda k} \vec{\mu}_{\lambda\lambda'}(\vec{k}) a_{\lambda k}^\dagger a_{\lambda'k}. \quad (2.68)$$

Evaluating the sum over bands and using a two band approximation we obtain

$$\vec{p} = \sum_k [\vec{\mu}_k a_k^\dagger b_{-k}^\dagger + \vec{\mu}_k^* b_{-k} a_k], \quad (2.69)$$

where $\vec{\mu}_k$ is dipole matrix element between the conduction and valence bands, and equations 2.43 to 2.46 have been used for the electron and hole operators.

In general we will approximate $\vec{\mu}_k$ by its value at $k = 0$. For GaAs, the momentum

matrix elements between the conduction and valence bands are given by equation 2.23

$$\langle S|\mathbf{p}_x|X \rangle = \langle S|\mathbf{p}_y|Y \rangle = \langle S|\mathbf{p}_z|Z \rangle = i\mathbf{P}.$$

Therefore, from equation 2.67

$$\frac{\mu_o}{e} \approx \frac{-\hbar\mathbf{P}}{m_o\epsilon_g} \approx 6.4\text{\AA} \quad (2.70)$$

using the values given in table A.1.

The macroscopic polarisation of the medium is obtained by performing a statistical average of equation 2.68 over all possible states $|\{n\}\rangle$ of the system[65, 36]:

$$\vec{P} = \frac{1}{V} \sum_k [\vec{\mu}_k \langle a_k^\dagger b_{-k}^\dagger \rangle + \vec{\mu}_k^* \langle b_{-k} a_k \rangle] = \sum_k [\vec{\mu}_k p_k^* + \vec{\mu}_k^* p_k] \quad (2.71)$$

where $p_k = \langle b_{-k} a_k \rangle$. Statistical summations are usually performed using the density matrix formalism[64, 36].

The interaction Hamiltonian (equation 2.56) in the second quantised representation is

$$V_I = \sum_{\lambda'\lambda k} \vec{\mu}_{\lambda\lambda'}(k) a_{\lambda k}^\dagger a_{\lambda' k} \cdot \vec{E} = \sum_k [\vec{\mu}_k a_k^\dagger b_{-k}^\dagger + \vec{\mu}_k^* b_{-k} a_k] \cdot \vec{E} \quad (2.72)$$

where the second expression pertains to the two band approximation.

Coulomb Interactions

The Coulomb term in the the Hamiltonian H_c describes the Coulomb interactions among carriers. In the second quantised representation the operator H_c is given by[36]

$$H_c = \frac{1}{2} \sum_{k,k',q} V_q (a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k + b_{k+q}^\dagger b_{k'-q}^\dagger b_{k'} b_k - 2a_{k+q}^\dagger b_{k'-q}^\dagger b_{k'} a_k) \quad (2.73)$$

where

$$V_q = \frac{1}{V} \int d^3r e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) = \frac{4\pi e^2}{\epsilon_0 V q^2} \quad (2.74)$$

is the Fourier transform of the Coulomb potential energy. The first term in equation 2.73 represents the repulsion between electrons, the second the repulsion between holes and the third the attraction between electron and holes. The Coulomb part of the Hamiltonian causes intraband transitions between different k states but leaves the total number of carriers unchanged.

Inclusion of the Coulomb interactions is relatively complicated since they involve more than one carrier and so quantum mechanical many-body techniques are required. We use a simpler, alternative approach in which we treat the Coulomb interaction as a reservoir interaction, rather than a dynamical interaction, whose primary effect is to establish thermo-

dynamic equilibrium within the bands[36]. Since the Coulomb interactions are strong they result in large carrier-carrier scattering rates (typically $10^{13}s^{-1}$) relative to other dynamical processes such as interband recombination (typically 10^9s^{-1}). This allows the electron and hole populations to rapidly reach a quasiequilibrium state (though the carrier population as a whole is not) on time scales over which the total numbers of electrons and holes vary[36]. This is a very powerful assumption and leads to a significant simplification of the laser equations[36]. We will return to this approximation later.

Due to the Coulomb interaction between carriers it is possible for an electron and hole to form a bound pair called an exciton. In bulk GaAs the energy of the lowest bound exciton state is approximately 4.2 meV which is small compared with room temperature thermal energy, $k_B T \approx 25$ meV. Hence, for bulk semiconductors, excitons are usually ionized at room temperature and thus are not important for semiconductor laser operation[36].

2.3.4 Fermi-Dirac Distribution and Density of States

In a semiconductor in equilibrium the probability of occupation of a level with momentum k and energy ϵ_k is given by the Fermi-Dirac distribution[63, 82]

$$f(\epsilon_k) = \frac{1}{\exp[\beta(\epsilon_k - \mu)] + 1} \quad (2.75)$$

where $\beta = 1/kT$ is the inverse temperature and μ is the Fermi energy. For a semiconductor in thermal equilibrium the Fermi energy lies within the bandgap ($\epsilon_g > \mu > 0$).

As mentioned in the previous section, for a semiconductor laser, which is not a system in thermal equilibrium, we assume a quasiequilibrium situation in which the carriers within each band are in thermal equilibrium. In this case, separate chemical potentials are defined for the conduction and valence band[36, 69, 70]. The probability of occupation of a state k is given by the Fermi-Dirac distribution $f_{\alpha k}$ where $\alpha = e$ for electrons and h for holes. The carrier population probabilities are then given by

$$f_{\alpha k}(\epsilon_{\alpha k}) = \frac{1}{\exp(\beta(\epsilon_{\alpha k} - \mu_{\alpha})) + 1} \quad (2.76)$$

where $\alpha = e$ for electrons and h for holes. The μ_{α} are the quasichemical potentials (or quasi-Fermi levels) and are chosen to yield the total carrier density N . In this section I wish to discuss how the μ_{α} are determined when N is known.

In the absence of doping the total carrier density is the same for electrons and holes. For a lightly doped gain region the total electron density N_e and the total hole density N_h are different. In a p-doped gain medium $N_h = N_e + N_A$ where N_A is the acceptor density and in an n-doped gain medium $N_e = N_h + N_D$ where N_D is the donor density[36]. In a heavily

doped gain medium the band structure is modified and laser transitions are possible between free carrier and k independent bound impurity states[79]. Heavily doped gain regions will not be discussed since they are rarely encountered in semiconductor lasers[36]. The following discussion pertains to intrinsic gain media. However, the results can be easily generalised to a lightly doped gain medium using the above expressions for the carrier densities[36].

The total carrier density N , for a semiconductor gain medium in quasiequilibrium, is

$$N = \frac{1}{\mathcal{V}} \sum_k f_{ek} = \frac{1}{\mathcal{V}} \sum_k f_{hk} \quad (2.77)$$

where $\mathcal{V}$ is the volume of the sample and the summation over k includes summation over all relevant k states and over spin states $s_z = \pm 1/2$. In a bulk semiconductor, in the two band model, every electron and hole state is specified uniquely by the momenta k_x , k_y and k_z and the spin component s_z .

For a finite crystal volume of $\mathcal{V} = L_x L_y L_z$, where L_x , L_y and L_z are the lengths of the semiconductor crystal in the x , y and z directions respectively, we assume periodic boundary conditions such that[82]

$$\begin{aligned} \phi_{\lambda k}(x + L_x, y, z) &= \phi_{\lambda k}(x, y, z) \\ \phi_{\lambda k}(x, y + L_y, z) &= \phi_{\lambda k}(x, y, z) \\ \phi_{\lambda k}(x, y, z + L_z) &= \phi_{\lambda k}(x, y, z). \end{aligned} \quad (2.78)$$

Using equation 2.78 and evoking Bloch's theorem we obtain the following condition

$$e^{ik_x L_x} = e^{ik_y L_y} = e^{ik_z L_z} = 1 \quad (2.79)$$

which implies that k must satisfy

$$k = 2\pi \left(\frac{m_x}{L_x} + \frac{m_y}{L_y} + \frac{m_z}{L_z} \right) \quad (2.80)$$

where m_x , m_y and m_z are integers such that $-\infty < m_i < \infty$. The sum over k states in equation 2.77 requires summing over the integers m_i .

In the limit L_x , L_y and $L_z \rightarrow \infty$ there is a continuous range of possible k values. We shall assume that the crystal is sufficiently large that we can assume a continuous range of values for k_x , k_y and k_z . This greatly simplifies the summation over k states since we can replace the sum over k by an integral over k . In order to do this we require the density of modes in k -space $\mathcal{D}(k)$ [69]. Using equation 2.80

$$\mathcal{D}(k) = \frac{1}{\text{Volume of 1 mode}} = \frac{1}{k_x k_y k_z} = \frac{\mathcal{V}}{(2\pi)^3}. \quad (2.81)$$

Hence

$$\frac{1}{V} \sum_{\mathbf{k}} \sum_{s_z} \longrightarrow \frac{2}{V} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \frac{V}{(2\pi)^3} \quad (2.82)$$

and the factor of two results from the summation over spin states. If we are integrating over a spherically symmetric function, then we can replace[36]

$$\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \longrightarrow \int_0^{\infty} 4\pi k^2 dk \quad (2.83)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. Using equation 2.82 and 2.83 equation 2.77 becomes

$$N_{\alpha} = \frac{2}{(2\pi)^3} \int_0^{\infty} dk 4\pi k^2 f_{\alpha}(k) = \int_0^{\infty} dk \mathcal{D}(k) f_{\alpha}(k) \quad (2.84)$$

where

$$\mathcal{D}(k) = \frac{k^2}{\pi^2} \quad (2.85)$$

is the electron or hole density of states for a bulk semiconductor. It gives the number of states between k and $k + dk$.

Since the Fermi-Dirac distributions are usually expressed in terms of carrier energy rather than momentum it is useful to change integration variables in equation 2.84 from k to $\epsilon_{\alpha k}$. The free carrier energy $\epsilon_{\alpha k}$, in the parabolic band approximation, is[36]

$$\epsilon_{\alpha k} = \frac{\hbar^2 k^2}{2m_{\alpha}} \quad (2.86)$$

Using this change of variables, equation 2.84 becomes

$$N_{\alpha} = \int_0^{\infty} \mathcal{D}_{\alpha}(\epsilon_{\alpha k}) f_{\alpha}(\epsilon_{\alpha k}) d\epsilon_{\alpha k} \quad (2.87)$$

where

$$\mathcal{D}_{\alpha}(\epsilon_{\alpha k}) = \frac{1}{2\pi^2} \left(\frac{2m_{\alpha}}{\hbar^2} \right)^{3/2} \sqrt{\epsilon_{\alpha k}} \quad (2.88)$$

is the energy density of states ($\alpha = e$ or h). Equation 2.87 can be easily interpreted; it says that the total carrier density is given by the integral over all carrier energies of the product of the density of states of energy $\epsilon_{\alpha k}$ and the probability that they are occupied¹¹.

Using equation 2.87 we can determine what fraction of the total hole density resides in the heavy hole band and what fraction resides in the light hole band[36]. Since the valence band is in quasiequilibrium we have the same quasi-Fermi level μ_h for both valence bands.

¹¹We would like to point out that equation 2.87 is only approximate since we are using the effective mass parabolic bands to approximate the band structure over the entire Brillouin zone. However, provided the Fermi-Dirac distributions are approximately zero in the region beyond where the parabolic band approximation does not apply the difference is not significant.

The total hole density, N_h , is then given by the sum of the light hole and heavy hole densities

$$N_h = N_{hh} + N_{lh} = (m_{hh}^{3/2} + m_{lh}^{3/2}) \left[\int_0^\infty d\epsilon \frac{1}{2\pi^2 \hbar^3} \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon - \mu_h)} + 1} \right] \quad (2.89)$$

Therefore

$$\frac{N_{lh}}{N_h} = \frac{1}{1 + \left(\frac{m_{hh}}{m_{lh}}\right)^{3/2}} \approx 6\% \quad (2.90)$$

so only 6% of holes reside in the light hole band which is small enough to neglect for our simple free carrier model[36]. In general, we know the carrier density N and we need to determine the corresponding chemical potential (quasi-Fermi level) μ_α by inverting the integral

$$N_\alpha = \left(\frac{2m_\alpha}{\hbar^2}\right)^{3/2} \frac{1}{2\pi^2} \int_0^\infty \frac{d\epsilon \sqrt{\epsilon}}{e^{\beta(\epsilon - \mu_\alpha)} + 1} \quad (2.91)$$

This equation cannot be solved analytically so, in general, μ_α is found iteratively with the necessary integrations performed numerically. A useful approximate expression for $\mu_\alpha > 0$ can be obtained by using a series representation for the Fermi-Dirac functions, a resummation and a Pade approximation resulting in[85]

$$\beta\mu_\alpha \approx \ln \bar{N}_\alpha + K_1 \ln(K_2 \bar{N}_\alpha + 1) + K_3 \bar{N}_\alpha \quad (2.92)$$

where $\bar{N}_\alpha$ is given by

$$\bar{N}_\alpha = 4N_\alpha \left[\frac{\pi\beta\hbar^2}{2m_\alpha} \right]^{3/2} \quad (2.93)$$

and $K_1 = 4.8966851$, $K_2 = 0.04496457$ and $K_3 = 0.1333760$. This expression is an excellent approximation for the range $-\infty < \beta\mu_\alpha \leq 30$.

2.4 Semiclassical Laser Theory

2.4.1 Introduction

There have been various models proposed to describe laser operation. The simplest model is one in which the light intensity is determined by energy conservation. In this case the medium gain is equated to the losses in the cavity. This model, though attractive in its simplicity, cannot account for the time dependence of the intensity, mode pulling or for treating multimode operation[65].

A more versatile model is the semiclassical theory in which the semiconductor gain medium is treated quantum mechanically and the electromagnetic field is treated classically as a solution to Maxwell's equations[64, 36]. This theory describes several important aspects

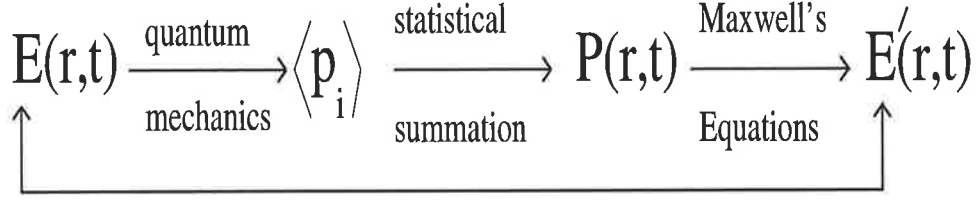


Figure 2.7: Flow diagram of the semiclassical laser theory (after [65]).

of laser operation and is adequate for many problems of interest. We use the semiclassical theory in our treatment of the semiconductor laser[65].

The laser can be also be treated by a fully quantised theory in which both the gain medium and the electromagnetic field are quantised. A fully quantised theory can explain build up from vacuum, photon statistics and can predict the laser linewidth; these things are not explained in the semiclassical theory[87].

In the self consistent, semiclassical theory the classical electric field $\vec{E}$ induces a polarisation, $\vec{p}$ within the semiconductor medium according to the laws of quantum mechanics. The macroscopic polarisation, $\vec{P}(\vec{r}, t)$ of the medium, obtained by performing a statistical summation of $\vec{p}$ over all possible state vectors, $|\{n\}\rangle$ of the system, in turn produces an electric field $\vec{E}'(\vec{r}, t)$. The solution to the semiclassical theory is obtained by demanding that the resulting electric field $\vec{E}'(\vec{r}, t)$ is equivalent to the field $\vec{E}(\vec{r}, t)$ inducing the polarisation[65]. This procedure is represented diagrammatically in Fig. 2.7.

2.4.2 Electromagnetic Field Equations

In this section the self consistent equations for the longitudinal mode field amplitudes and phases will be derived¹². Our starting point is the wave equation for the electric field

$$-\vec{\nabla}^2 \vec{E} + \left(\frac{n}{c}\right)^2 \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_o \frac{\partial^2 \vec{P}}{\partial t^2} \quad (2.94)$$

where c is the speed of light in vacuum, n is the refractive index, μ_o is the permeability and $\vec{P}$ is the medium polarisation. Since the transverse field components $E(x)$ and $E(y)$ vary slowly compared to the longitudinal component $E(z)$ (z lies along the laser axis) we can neglect the x and y derivatives¹³. Hence, we can make the approximation[65], $-\vec{\nabla}^2 \vec{E} \approx -\partial^2 E / \partial z^2$, in equation 2.94. By expanding the electric field in the normal modes of the laser cavity we

¹²The discussion given in this section is from [65].

¹³The condition $\partial^2 E / \partial z^2 \gg \partial^2 E / \partial x^2, \partial^2 E / \partial y^2$ arises because the beam width is much greater than the lasing wavelength.

can separate the spatial and time dependencies. Only discrete modes with the frequencies

$$\Omega_m = \frac{m\pi c}{nL} = \frac{K_m c}{n} \quad (2.95)$$

achieve appreciable magnitude within the cavity. Here L is the length of the cavity, K_m is the corresponding wave number and m is a large integer. In addition we assume the electric field is linearly polarised so that it can be represented by a scalar quantity. The multimode field can be written as

$$E(z, t) = \frac{1}{2} \sum_m \mathcal{E}_m(t) \exp[-i(\nu_m t + \phi_m)] U_m(z) + c.c. \quad (2.96)$$

where $U_m(z)$ determines the z dependence of the field and is given by

$$U_m(z) = \sin(K_m z) \quad (2.97)$$

for standing waves. The functions $\mathcal{E}_m(t)$ and $\phi_m(t)$ are real slowly varying amplitude and phase coefficients, respectively. We have ignored the dependence of $\mathcal{E}_m(t)$ with z which is a good approximation for highly reflecting cavity mirrors¹⁴. The optical frequency is ν and is not necessarily equal to the the passive cavity frequency Ω since the active medium introduces some dispersion.

The induced polarisation of the medium is of the form

$$P(z, t) = \frac{1}{2} \sum_m \mathcal{P}_m(t) \exp[-i(\nu_m t + \phi_m)] U_m(z) + c.c. \quad (2.98)$$

where $\mathcal{P}_m(t)$ is a complex, slowly varying component of the polarisation in the direction of $\mathcal{E}_m$. The complex nature of $\mathcal{P}_m(t)$ reflects the fact the polarisation has a different phase to the electric field; the real part is in phase with the electric field and results in dispersion due to the medium. The imaginary part is in quadrature with the field and results in gain or loss. We can obtain equations of motion for $\mathcal{E}_m(t)$ and $\phi_m(t)$ by substituting positive frequency parts of equations 2.96 and 2.98 (i.e. without complex conjugates¹⁵) into the wave equation 2.94. We assume that $\mathcal{E}_m(t)$, $\phi_m(t)$ and $\mathcal{P}_m(t)$ are slowly varying quantities relative to the optical frequency which allows us to neglect terms proportional to $\ddot{\mathcal{E}}_m$, $\ddot{\phi}_m$ and $\dot{\mathcal{E}}_m \dot{\phi}_m$. In addition, since in general the medium polarisation is much smaller than ϵE , we also neglect terms proportional to $\dot{\mathcal{P}}_m$ and $\ddot{\mathcal{P}}_m$. Substituting equations 2.96 and 2.98 into

¹⁴This is actually not true of many semiconductor laser cavities[78, 36].

¹⁵Since the wave equation (eqn. 2.94) is linear and real, then we can proceed using a complex representation for the fields, E and P ; the derived relations will also be true for the full quantities. The advantage of this approach is that complex algebra is much simpler than using trigonometric identities. This approach is used in other areas of physics, one well known example being the analysis of AC circuits.

the wave equation 2.94 and dividing through by $\exp[-i(\nu_m t + \phi_m)]U_m(z)n^2/c^2$ we obtain

$$[\Omega_m^2 - (\nu_m + \dot{\phi}_m)^2]\mathcal{E}_m - 2i\nu_m\dot{\mathcal{E}}_m = \frac{\nu_m^2 \mathcal{P}_m}{\epsilon} \quad (2.99)$$

where we have used $\epsilon = \epsilon_o n^2$. We approximate $\Omega_m^2 - (\nu_m + \dot{\phi}_m)^2 \approx 2\nu_m(\Omega_m - \nu_m + \dot{\phi}_m)$ since Ω_m and ν_m are much larger than $\dot{\phi}_m$. Equating real and imaginary parts we obtain the self-consistency equations

$$\dot{\mathcal{E}}_m(t) + \frac{1}{2\tau_p}\mathcal{E}_m(t) = -\frac{1}{2}\left(\frac{\nu_m}{\epsilon}\right)\text{Im}(\mathcal{P}_m) \quad (2.100)$$

$$\nu_m + \dot{\phi}_m = \Omega_m - \frac{1}{2}\left(\frac{\nu_m}{\epsilon}\right)\frac{\text{Re}(\mathcal{P}_m)}{\mathcal{E}_m}. \quad (2.101)$$

We have so far neglected losses from the cavity. A precise treatment involves solving Maxwell's equations with appropriate boundary conditions at the cavity mirrors. We have taken the simpler (though less precise) approach of adding a phenomenological cavity decay term $\mathcal{E}_m/2\tau_p$ to equation 2.100 in analogy with the treatment given in section 2.2.4 for the photon lifetime.

We can relate the macroscopic polarisation to the susceptibility of the medium via

$$\mathcal{P}(t) = \epsilon\chi\mathcal{E}(t) = \epsilon(\chi' + i\chi'')\mathcal{E}(t). \quad (2.102)$$

For simple dielectrics χ can be approximated by a scalar quantity independent of E and hence the polarisation vector lies in the direction of the inducing electric field (as we have assumed here). However, in general χ is tensorial in nature and a nonlinear function of the applied field $\mathcal{E}$. Substituting equations 2.102 into equations 2.100 and 2.101 gives

$$\dot{\mathcal{E}}_m(t) + \frac{\mathcal{E}_m}{2\tau_p} = \frac{1}{2}\nu_m\chi''\mathcal{E}_m(t) \quad (2.103)$$

$$\nu_m + \dot{\phi}_m = \Omega_m - \frac{1}{2}\nu_m\chi'. \quad (2.104)$$

Equation 2.104 implies that the modal oscillation frequency ν_m is shifted from the passive cavity frequency Ω_m by an amount $-\frac{1}{2}\nu\chi'$. Since we are mainly concerned with the dynamics of the modal amplitudes under current modulation we will ignore the phase equations and assume $\nu_m \approx \Omega_m$. However, we would like to point out that, when treating feedback from an external mirror, equation 2.104 has important implications and should be considered.

Our main result of this section is equation 2.103. It is convenient at this point to

introduce the laser gain g defined by means of the following equation:

$$g(\nu) = \frac{n}{c} \nu \chi'' \quad (2.105)$$

which is usually expressed in units of cm^{-1} . An alternative definition, denoted by the symbol G , is sometimes also used in the rate equations and is related to g by

$$G = cg/n. \quad (2.106)$$

We rewrite equation 2.103 using equations 2.105 and 2.106 and the modal photon density $s_m = \epsilon \mathcal{E}_m^2 / \hbar \nu$ [36]:

$$\dot{s}_m = G s_m - \frac{s_m}{\tau_p}. \quad (2.107)$$

2.4.3 Free Carrier Theory

Two Band Model

The free carrier model of a semiconductor gain medium assumes that the carriers behave as non-interacting free Fermions but with an effective mass which takes into account the interaction of the carriers with the periodic lattice potential [63]. Hence the band structure is approximated by parabolic bands, near the band edges, with an effective mass proportional to the reciprocal of the band curvature (equation 2.33) [63]. In addition, since the hole population resides predominantly in the heavy hole band, we will restrict ourselves to a two band model consisting of just the lowest conduction band and the heavy hole band [36]. The bandstructure is illustrated in Fig. 2.8 along with the relevant photon transition. The relationship between the various energy values is as follows [36]:

$$\begin{aligned} \text{conduction band: } \epsilon_{ek} &= \epsilon_g + \epsilon_{ek} = \epsilon_g + \frac{\hbar^2 k^2}{2m_e} \\ \text{valence band: } \epsilon_{hk} &= \frac{\hbar^2 k^2}{2m_h} \\ \text{photon energy: } \hbar\omega_k &= \epsilon_g + \epsilon_r = \epsilon_g + \frac{\hbar^2 k^2}{2m_r} \end{aligned}$$

where the reduced mass m_r is defined as

$$\frac{1}{m_r} = \left[\frac{1}{m_e} + \frac{1}{m_h} \right] \quad (2.108)$$

and $\epsilon_r = \epsilon_{ek} + \epsilon_{hk}$ is referred to as the reduced mass energy.

In order to achieve gain in the semiconductor medium we need to create a population inversion. This requires exciting electrons from the valence band to the conduction band;

Two-level Parabolic Band Structure

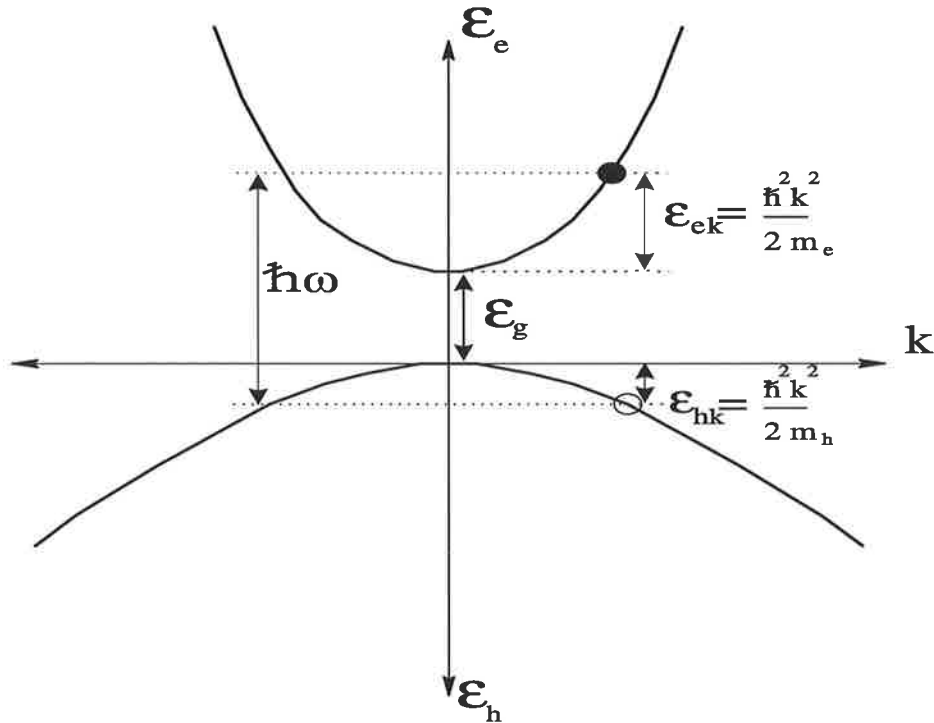


Figure 2.8: Bandstructure for the parabolic two band model.

hence a semiconductor laser is not a system in thermal equilibrium. The inverted semiconductor can, however, be approximated by a quasiequilibrium state in which the electrons in the conduction band can be considered as being in thermal equilibrium and similarly for the holes in the valence band though the total carrier population as a whole is not. Hence, separate chemical potentials are used for the conduction and valence bands[36, 70, 69].

Quasi-equilibrium is achieved by the rapid carrier-carrier scattering caused by the coulomb interactions among carriers. Provided the stimulated emission rate is not too high, the carrier-carrier scattering dominates other dynamical processes, and rapidly restores equilibrium carrier distributions within the bands even when the total carrier density has been perturbed[36]. Hence, the carrier distributions follow Fermi-Dirac distributions which depend only on temperature T and indirectly on the total carrier density N (via changes in the chemical potential).

The quasiequilibrium approximation implies that we need only consider dynamical variation of the total carrier density N rather than having to follow the evolution of the level populations on an individual k basis.

This is a very important and powerful assumption since it results in a drastic reduction in the number of variables required to describe SCL operation. Modelling coulomb interactions as reservoir interactions is an over simplification and in a more realistic model

they need to be considered explicitly. However, the free carrier theory describes many aspects of SCL operation and is considerably simpler[36].

Free Carrier Equations of Motion

In the free carrier theory the k -dependent Hamiltonians are given by¹⁶

$$H_k = \left[\epsilon_g + \frac{\hbar^2 k^2}{2m_e} \right] a_k^\dagger a_k + \frac{\hbar^2 k^2}{2m_h} b_{-k}^\dagger b_{-k} - [\mu_k a_k^\dagger b_{-k}^\dagger + \mu_k^* a_k b_{-k}] E \quad (2.109)$$

and the total Hamiltonian is

$$H = \sum_k H_k \quad (2.110)$$

where the summation over k also includes the summation over the spin variable s_z . In this section I shall derive equations of motion for the electron and hole number operator expectation values and the “dipole” operator expectation value¹⁷

$$n_{ek} = \langle a_k^\dagger a_k \rangle \quad (2.111)$$

$$n_{hk} = \langle b_{-k}^\dagger b_{-k} \rangle \quad (2.112)$$

$$p_k = \langle b_{-k} a_k \rangle. \quad (2.113)$$

In the Heisenberg picture, the equation of motion for an operator θ is

$$i\hbar \frac{d\theta}{dt} = [\theta, H]. \quad (2.114)$$

Using anticommutation relations (equations 2.36 and 2.37) for the Fermion operators and the Hamiltonian given by equation 2.109 we can obtain equations of motion for the operators in equations 2.111 to 2.113[36]:

$$\frac{d}{dt}(b_{-k} a_k) = i\omega_k b_{-k} a_k - \frac{i}{\hbar} \mu_k (a_k^\dagger a_k - b_{-k}^\dagger b_{-k} - 1) E \quad (2.115)$$

$$\frac{d}{dt}(a_k^\dagger a_k) = \frac{i}{\hbar} a_k^\dagger b_{-k} E + h.a. = \frac{d}{dt}(b_{-k}^\dagger b_{-k}). \quad (2.116)$$

The abbreviation *h.a.* stands for Hermitean adjoint. Taking the expectation values of equations 2.115 and 2.116 we have

$$\dot{p}_k = i\omega_k p_k - \frac{i}{\hbar} \mu_k (n_{ek} + n_{hk} - 1) E \quad (2.117)$$

¹⁶This section follows closely the discussion given in reference [36].

¹⁷In the quasi-equilibrium approximation the n_{ek} and n_{hk} are Fermi-Dirac distributions. However, we use this more general notation here since the following results apply to the general case. We will discuss the implications of deviations from Fermi-Dirac distributions in section 2.5.

$$\dot{n}_{ek} = \frac{i}{\hbar}(\mu_k p_k^* - \mu_k p_k)E = \dot{n}_{hk} \quad (2.118)$$

In these equations we have not included decay processes or pumping processes; both these processes are important for laser operation. The relaxation processes include carrier-carrier scattering, carrier-phonon and carrier-impurity (lattice imperfection) scattering. These effects are not included explicitly in the free carrier Hamiltonian (equation 2.109). Also we have neglected carrier recombination via spontaneous emission. It is simplest to account for these effects phenomenologically by adding appropriate pump and decay terms to equations 2.117 and 2.118:

$$\dot{p}_k = i\omega_k p_k - \frac{i}{\hbar}\mu_k(n_{ek} + n_{hk} - 1)E + \dot{p}_k|_{col} \quad (2.119)$$

$$\dot{n}_{\alpha k} = \Lambda_{\alpha k} - B_k n_{ek} n_{hk} - \gamma_{nr} n_{\alpha k} + \dot{n}_{\alpha k}|_{col} + \frac{i}{\hbar}(\mu_k p_k^* - \mu_k p_k)E \quad (2.120)$$

where $\alpha = e$ or h , $\Lambda_{\alpha k}$ is the pump rate due to injection current, γ_{nr} is the non-radiative decay constant due to capture by vacancies caused by crystal defects and

$$B_k = \frac{1}{4\pi\epsilon_0} \frac{4\omega_k^3 |\mu_k|^2 n^3}{3\hbar c^3} \quad (2.121)$$

is the spontaneous emission rate constant[36]. The spontaneous emission term $B_k n_{ek} n_{hk}$ is proportional to both the electron and hole occupation probabilities since both an electron of momentum k in the conduction band and a hole of momentum $-k$ in the valence band is required before spontaneous recombination can occur.

The terms $\dot{p}_k|_{col}$ and $\dot{n}_{\alpha k}|_{col}$ are the collision contributions. The electric dipole dephasing arises from the rapid carrier-carrier scattering rates. In the simplest approximation we can model collision contributions to the polarisation equation as a simple polarisation decay

$$\dot{p}_k|_{col} \approx -\gamma p_k. \quad (2.122)$$

For the $\dot{n}_{\alpha k}|_{col}$ we can obtain an approximate expression for the effect of collision on the $n_{\alpha k}$ by noting that the carrier-carrier scattering causes a redistribution of the carriers within the bands rather than a loss of carriers and hence should not change the total carrier density N . Since the net effect of intraband scattering is to return the electron and hole distributions to equilibrium the simplest way to model the carrier scattering terms is

$$\dot{n}_{\alpha k}|_{col} = \gamma_{\alpha}(n_{\alpha k} - f_{\alpha k}). \quad (2.123)$$

This approximation is consistent with the polarisation decay equation 2.122 provided we use

the dipole decay constant[36]

$$\gamma = \frac{1}{2}(\gamma_e + \gamma_h). \quad (2.124)$$

By summing equation 2.120 over all k we obtain the equation of motion for the total carrier density N :

$$\dot{N} = \Lambda - \gamma_{nr}N - \frac{1}{\mathcal{V}} \sum_k B_k n_{ek} n_{hk} - \frac{i}{\hbar \mathcal{V}} \sum_k (\mu_k p_k^* - \mu_k p_k) E. \quad (2.125)$$

The injection current pump term Λ is

$$\Lambda = \frac{\eta I}{e \mathcal{V}} \quad (2.126)$$

where η is the total quantum efficiency that the injected current contributes to the inversion, I is the injected current, e is the charge on an electron and $\mathcal{V}$ is the active region volume.

For scattering rates γ_α sufficiently large, such that it dominates any other process that may cause the carrier distributions to deviate from their equilibrium distributions, we can replace

$$n_{ek} \approx f_{ek} \quad (2.127)$$

$$n_{hk} \approx f_{hk}. \quad (2.128)$$

Also, since the dipole decay rate, γ is fast compared to other time-scales, the polarisation dynamics may be adiabatically eliminated to give the rate equations for the electron and hole populations. These will be derived in the next section.

2.4.4 The Rate Equation Approximation

The rate equation approximation applies to systems in which the dipole decay time $T_2 \equiv 1/\gamma$ is much smaller than times over which the carrier density or the electric field amplitude vary significantly. Under these conditions, the fast relaxing variables, p_k rapidly reach a quasisteady state determined by the instantaneous values of the slowly varying variables; hence, the fast relaxing variables are able to track the evolution of the slower variables. We can eliminate the rate equations for the fast variables by integrating their equations of motion assuming the other variables are constants; this procedure is known as adiabatic elimination[65, 64, 36].

To integrate the equation of motion for p_k (equation 2.119) we multiply by the integrating factor $\exp[(i\omega_k + \gamma)t]$ to obtain[36]

$$\frac{d}{dt} (p_k e^{(i\omega_k + \gamma)t}) = \frac{i}{\hbar} \mu_k [f_{ek} + f_{hk} - 1] E e^{(i\omega_k + \gamma)t} \quad (2.129)$$

which can be formally integrated to give

$$p_k(t) = \frac{i}{\hbar} \mu_k \int_{-\infty}^t dt' E(t') e^{(i\omega_k + \gamma)(t'-t)} [f_{ek}(t') + f_{hk}(t') - 1]. \quad (2.130)$$

If we assume that $f_{\alpha k}(t)$ and the field envelope vary little in the time T_2 , we can make the rate equation approximation, which involves evaluating $f_{\alpha k}(t)$ and the field envelope at the time t , and removing them from the integral. For simplicity, initially we consider a single mode field¹⁸

$$E(t) = \frac{1}{2} \mathcal{E}(t) e^{-i(\nu t + \phi)} + c.c. \quad (2.131)$$

where all quantities have the same meanings as those in equation 2.96. In order to evaluate equation 2.130 we require the following integrals:

$$\int_{-\infty}^t dt' e^{-(i\omega_k + \gamma)(t-t')} e^{-i(\nu t + \phi)} = \frac{e^{-i(\nu t + \phi)}}{i(\omega_k - \nu) + \gamma} \quad (2.132)$$

$$\int_{-\infty}^t dt' e^{-(i\omega_k + \gamma)(t-t')} e^{i(\nu t + \phi)} = \frac{e^{i(\nu t + \phi)}}{i(\omega_k + \nu) + \gamma}. \quad (2.133)$$

We make the rotating wave approximation which involves neglecting the second integral since, at optical frequencies, its denominator is far greater than that for the first term[64, 36, 65]. Substituting equation 2.131 into equation 2.130, and using equation 2.132 and the rate equation approximation, we obtain for the dipole moment operator[36]

$$p_k = \frac{i\mu_k}{2\hbar} \mathcal{E} [f_{ek} + f_{hk} - 1] \left\{ \frac{e^{-i(\nu t + \phi(t))}}{i(\omega_k - \nu) + \gamma} \right\}. \quad (2.134)$$

Even though we have assumed a constant field envelope and carrier distribution in the derivation of equation 2.134, this equation is still valid for time varying field and carrier populations, provided they vary little in the time T_2 . Substituting equation 2.134 into equation 2.125, and setting $n_{\alpha k} = f_{\alpha k}$, we obtain the rate equation for the carrier density N [36]:

$$\dot{N} = \frac{\eta I}{e\mathcal{V}} - \gamma_{nr} N - \frac{1}{\mathcal{V}} \sum_k B_k f_{ek} f_{hk} - \frac{\mathcal{E}^2}{2\hbar^2 \gamma \mathcal{V}} \sum_k |\mu_k|^2 [f_{ek} + f_{hk} - 1] \mathcal{L}(\omega_k - \nu) \quad (2.135)$$

where

$$\mathcal{L}(\omega_k - \nu) = \frac{\gamma^2}{\gamma^2 + (\omega_k - \nu)^2} \quad (2.136)$$

Equation 2.135 is valid for single mode fields. In general we will need to consider multimode fields of the form given by equation 2.96. In this case, the rate equation approximation

¹⁸We have not considered the z -dependence of the field, in this section, in order to be consistent with the electric dipole approximation.

is, in general, not valid since both the field amplitudes and the carrier populations will oscillate at the longitudinal mode beat frequencies. This leads to intermode coupling from nonlinearities in the polarisation resulting in cross mode saturation effects and multiwave mixing[65, 6, 36, 88, 89, 90]. In addition, if the longitudinal mode spacing approaches the carrier-carrier scattering rate, the quasiequilibrium approximation is no longer valid and leads to spectral hole burning effects[36, 89, 91, 92].

In the multimode case, one approach is to solve the coupled rate equations 2.119 and 2.120 perturbatively. This leads to a relatively complicated set of equations involving higher mode couplings[65]. Here we take the simpler (though less rigorous) approach in which we ignore the carrier population pulsations at the intermode beat frequencies and make a rate equation approximation analogous to that used in the single mode case[65]. This approach leads to a set of coupled rate equations for the mode intensities and the total carrier density. To this end, we substitute the multimode field (equation 2.96) into equation 2.130 and again make the approximation that $n_{\alpha k}$ and $\mathcal{E}$ are sufficiently slowly varying that they may be evaluated at the time t and removed from the integral. Proceeding in a similar way to the single mode case we obtain for the dipole moment operator

$$p_k = \frac{i\mu_k}{\hbar} [f_{ek} + f_{hk} - 1] \sum_m \mathcal{E}_m \left\{ e^{-i(\nu_m t + \phi_m)} \mathcal{D}_k(\nu_m) \right\}. \quad (2.137)$$

where we have defined

$$\mathcal{D}_k(\nu) = \frac{1}{i(\omega_k - \nu) + \gamma}. \quad (2.138)$$

and we have also used the rotating wave approximation. To obtain the rate equation for N we substitute equation 2.137 into equation 2.125. This results in cross terms of the form $\mathcal{E}_m \mathcal{E}_n \exp[i(\nu_m - \nu_n)t]$ which lead to pulsations of the carrier density at the intermode beat frequencies. To be consistent with the rate equation approximation we neglect these terms¹⁹. This leads to the following rate equation for the carrier density N :

$$\dot{N} = \frac{\eta I}{e\mathcal{V}} - \gamma_{nr}N - \frac{1}{\mathcal{V}} \sum_k B_k f_{ek} f_{hk} - \sum_m \mathcal{E}_m^2 \sum_k \frac{1}{2\hbar^2 \gamma \mathcal{V}} |\mu_k|^2 [f_{ek} + f_{hk} - 1] \mathcal{L}(\omega_k - \nu_m) \quad (2.139)$$

Note that the rate equation approximation, in which the polarisation dynamics are adiabatically eliminated, is consistent with the quasiequilibrium approximation in which carrier occupation probabilities are given by Fermi-Dirac distributions since both approximations involve time scales very much less than the dipole decay time. However, the quasiequilibrium approximation also requires the additional assumption that the field intensity must not be so strong that it can burn holes in the Fermi-Dirac distributions; the rate equation

¹⁹Carrier pulsations can be important for solitary laser diodes[88]. We discuss these effects further in Section 2.5.

approximation is still valid in this case[36].

2.4.5 Free Carrier Gain

In this section explicit expressions for the free carrier gain will be derived²⁰. To do this we first derive the complex susceptibility $\chi = \chi' + i\chi''$; the gain is then obtained from equation 2.105. Using equations 2.71 and 2.98 for the macroscopic polarization

$$P(t) = \frac{1}{\mathcal{V}} \sum_k [\mu_k p_k^* + \mu_k^* p_k] = \frac{1}{2} \sum_m e^{-i(\nu_m t - \phi_m)} \mathcal{P}_m(t) + c.c. \quad (2.140)$$

we can write the complex polarization amplitude $\mathcal{P}(t)$ in terms of p_k

$$\mathcal{P}(t) = 2e^{-i(kz - \nu t - \phi(t))} \frac{1}{\mathcal{V}} \sum_k \mu_k^* p_k. \quad (2.141)$$

Substituting equation 2.137 for p_k into equation 2.141 and using

$$\mathcal{P}(t) = \epsilon \chi \mathcal{E}(t) \quad (2.142)$$

we obtain the following expressions for χ :

$$\chi = \frac{1}{\epsilon \hbar \gamma \mathcal{V}} \sum_k |\mu_k|^2 [f_{ek} + f_{hk} - 1] \mathcal{L}(\omega_k - \nu) \left\{ \frac{\omega_k - \nu}{\gamma} - i \right\} \quad (2.143)$$

The laser intensity gain is then given by equation 2.105:

$$g(\nu) = \frac{\nu n}{\epsilon c \hbar \gamma \mathcal{V}} \sum_k |\mu_k|^2 [f_{ek} + f_{hk} - 1] \mathcal{L}(\omega_k - \nu) \quad (2.144)$$

We can simplify the summation over k by assuming a spherically symmetric integrand and a continuous distribution of states. As in section 2.3.4 on the density of states we replace

$$\frac{1}{\mathcal{V}} \sum_k \longrightarrow \int_0^\infty d\epsilon_r \mathcal{D}(\epsilon_r) \quad (2.145)$$

where

$$\mathcal{D}(\epsilon_r) = \frac{1}{2\pi^2} \left(\frac{2m_r}{\hbar^2} \right)^{3/2} \sqrt{\epsilon_r}. \quad (2.146)$$

²⁰The information contained in this section can be found in reference [36].

In order to determine the optical gain as a function of photon energy $\hbar\nu$ it is convenient to change the integration variable to $\hbar\omega$ where

$$\hbar\omega = \epsilon_g + \frac{\hbar^2 k^2}{2m_e} + \frac{\hbar^2 k^2}{2m_h} = \epsilon_g + \frac{\hbar^2 k^2}{2m_r} = \epsilon_g + \epsilon_r. \quad (2.147)$$

We also need to write the Fermi-Dirac distribution in terms of the variable $\hbar\omega$. The relationship between the various energy variables is shown in Fig. 2.8. We can write $\epsilon_{\alpha k}$ in terms of the reduced mass energy ϵ_r since

$$\epsilon_{\alpha k} = \frac{\hbar^2 k^2}{2m_\alpha} \frac{m_r}{m_r} = \frac{\epsilon_r m_r}{m_\alpha} = \frac{m_r}{m_\alpha} (\hbar\omega - \epsilon_g). \quad (2.148)$$

Therefore

$$f_\alpha(\hbar\omega) = \frac{1}{\exp[\beta(\frac{m_r}{m_\alpha}(\hbar\omega - \epsilon_g - \mu_\alpha))] + 1}. \quad (2.149)$$

The Lorentzian lineshape overestimates the effects of homogeneous broadening due to its slowly decaying tails. This leads to (unphysical) absorption at frequencies below the bandgap. This effect may be compensated by using a Sech rather than a Lorentzian function[36]²¹. To this end we make the replacement

$$\frac{\hbar^2 \gamma^2}{\hbar^2 \gamma^2 + (\hbar\omega - \hbar\nu)^2} \longrightarrow \text{sech} \left(\frac{\omega_k - \nu}{\gamma} \right) \quad (2.150)$$

in equation 2.144. The optical gain is then given by

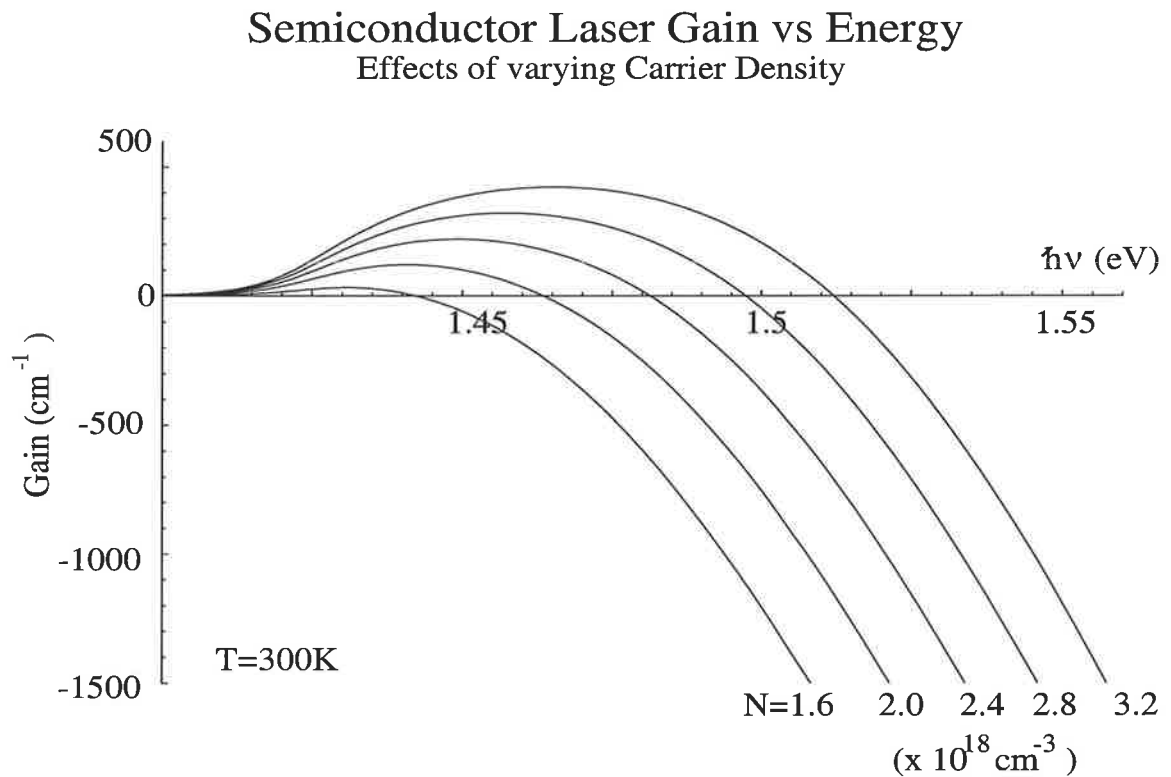
$$\begin{aligned} g(\hbar\nu) &= \frac{\nu n}{\epsilon c \hbar \gamma} \frac{1}{2\pi^2} \left(\frac{2m_r}{\hbar^2} \right)^{3/2} \int_0^\infty d(\hbar\omega) \text{sech} \left(\frac{\omega_k - \nu}{\gamma} \right) |\mu_k|^2 \\ &\times (\hbar\omega - \epsilon_g)^{1/2} [f_e(\hbar\omega) + f_h(\hbar\omega) - 1] \end{aligned} \quad (2.151)$$

Figure 2.9 shows the gain calculated using equation 2.151 for carrier densities in the range 1.6 to $3.2 \times 10^{18} \text{ cm}^{-3}$ and a temperature $T = 300 \text{ K}$. Parameters used in the calculation are given in table 2.1 and are appropriate for an undoped GaAs medium. The results of this calculation show that the position of the peak of the gain shifts with increasing carrier density. This is very different behaviour to most other types of lasers involving atomic transitions and has important consequences for the dynamics under current modulation. In Fig. 2.10(a) and (b) we plot the peak gain as a function of carrier density and the wavelength at the peak gain versus carrier density respectively; both these graphs are calculated from the data shown in Fig. 2.9. We observe that both these quantities, to a very good approximation, vary linearly with N .

²¹There exist other approaches to this problem. See for example [79].

Table 2.1: Parameter values for the free carrier model gain.

Symbol	Value	Definition
m_e	9.109×10^{-31} Kg	Free electron mass
e	1.6×10^{-19} C	charge of an electron
h	6.626×10^{-34} Js	Planck's constant
c	3×10^8 ms^{-1}	Speed of light in vacuum
k	1.380658×10^{-23} J	Boltzmann's constant
ϵ_o	8.854×10^{-12} F/m	Permittivity of the vacuum
m_c	$0.0665 m_e$	Conduction band electron effective mass
m_h	$0.5 m_e$	Valence band hole effective mass
n	3.5	refractive index
τ_e	3 ns	Carrier recombination time
T_2	10^{-13} s	Polarisation decay time
μ	$6.4 \times 10^{-10} e mC$	dipole matrix element

Figure 2.9: The free carrier gain for a temperature of $T = 300$ K. The carrier densities vary from 1.6 to $3.2 \times 10^{18} \text{ cm}^{-3}$ in steps of $0.4 \times 10^{18} \text{ cm}^{-3}$.

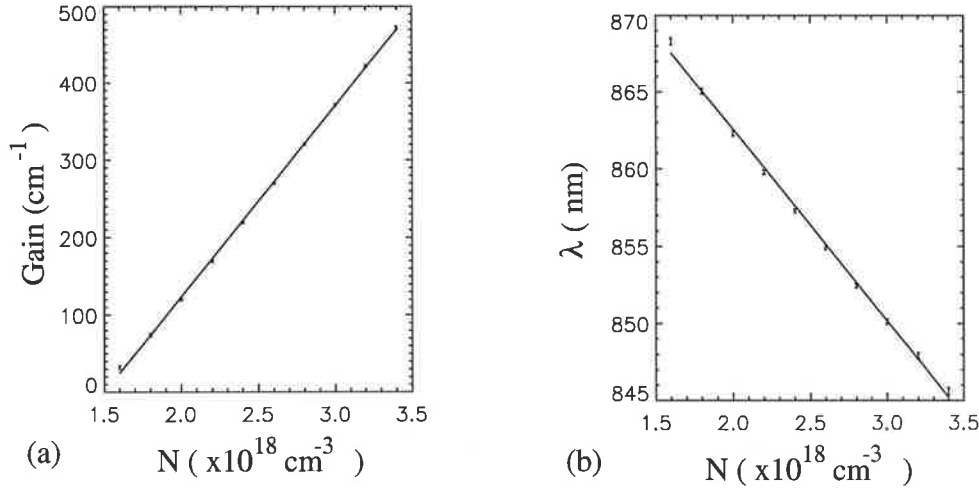


Figure 2.10: (a): The peak gain versus carrier density. (b): The wavelength at which the peak gain occurs versus carrier density. Both quantities show a linear dependence on carrier density.

The free carrier gain is also dependent upon the temperature. In Fig. 2.11 we show the gain calculated from equation 2.151 for a fixed carrier density of $N = 2 \times 10^{18} \text{ cm}^{-3}$. We observe both a decrease in the peak gain and a shift of the peak gain towards lower photon energies with an increase in temperature. Hence we would expect the lasing wavelength and the laser threshold to be temperature dependent.

The parameter values used in the above calculations do not necessarily correspond to those for the semiconductor laser used in the experimental investigations conducted in this thesis. The purpose of presenting these results is to illustrate qualitatively how the effect of changing the carrier density and the temperature modifies the semiconductor laser gain.

2.4.6 The Multimode Rate Equations

Equations 2.139 and 2.107 constitute a set of rate equations describing the evolution of the carrier density and modal intensities within the active region of a semiconductor laser. Both equations contain terms involving the SCL gain $g(\nu)$ as derived in the previous section. Integration of these equations requires numerical computation of integrals involving Fermi-Dirac distributions which need to be re-evaluated as the carrier density varies. In this section we introduce an approximation to the free carrier gain which greatly reduces the computation time involved in numerically integrating equations 2.139 and 2.107.

From Fig. 2.10(a) we observe that the peak of the gain varies nearly linearly with carrier density. A frequently used approximation to the peak gain versus carrier density is:

$$G_p = A(n - n_o) \quad (2.152)$$

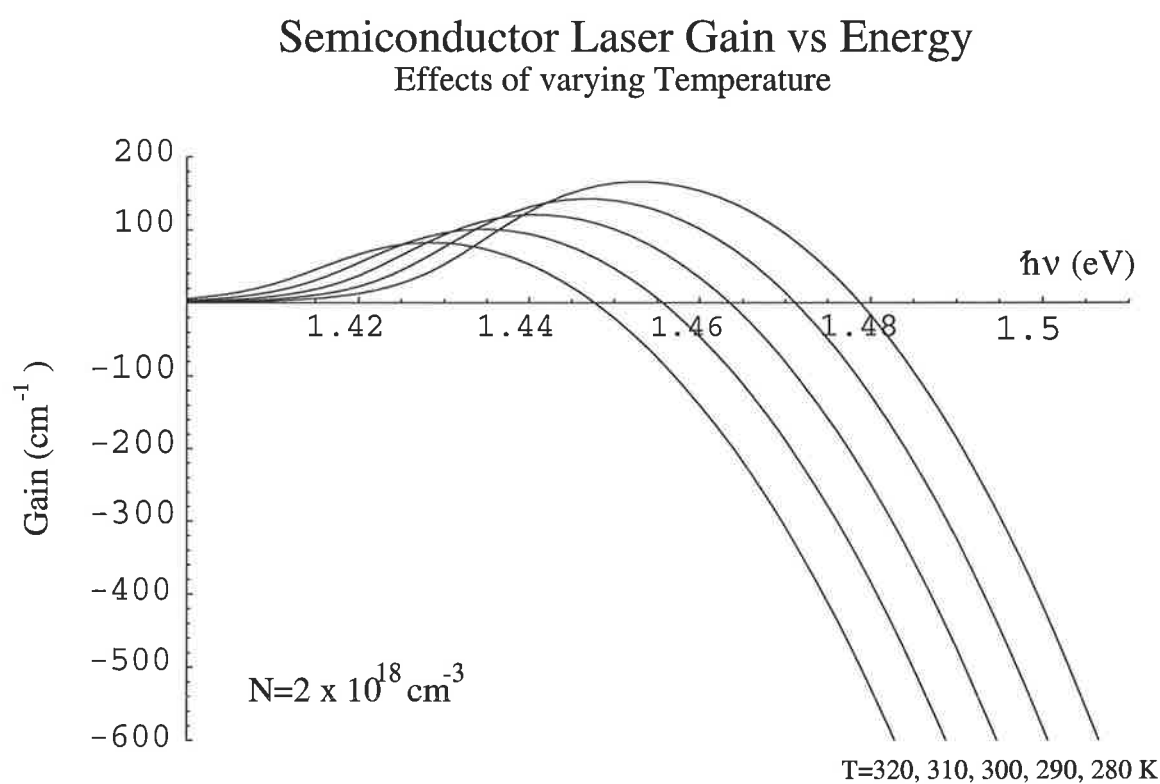


Figure 2.11: Semiconductor gain for temperatures in the range 280 to 320K. An increase in temperature results in a decrease in the maximum gain and a shift in the position of the gain peak towards lower photon energies. The dependence of the bandgap energy on temperature has been taken into account in these calculations (see Table A.1).

where A is the gain coefficient and n_o is the number of carriers required to achieve transparency. Such an approximation is suitable for a single mode model. When discussing multimode fields a quadratic approximation to the gain spectra is often used[1, 6, 34, 35, 15]:

$$G(\lambda_i) = An \left[1 - \left(\frac{2(\lambda(n) - \lambda_i)}{\Delta\lambda_g} \right)^2 \right] - An_o \quad (2.153)$$

where $\Delta\lambda_g$ is the width of the gain curve, λ_i is the wavelength of mode i and $\lambda(n)$ wavelength at the peak of the gain. The longitudinal mode wavelengths are

$$\lambda_i = \lambda_o + i\delta\lambda \quad (2.154)$$

where λ_o is the wavelength at the peak gain for carrier densities equivalent to the threshold carrier density n_{th} and $\delta\lambda$ is the longitudinal mode spacing. From Fig. 2.10(b) we observe that the position of the gain peak varies approximately linearly with carrier density. Hence, we can approximate $\lambda(n)$ by[1]:

$$\lambda(n) = \lambda_o + k[1 - n/n_{th}] \quad (2.155)$$

where k gives the spectral shift of the gain with carrier density and n_{th} is the threshold carrier density.

Since the spontaneous decay term $R_{sp} = \sum_k B_k f_{ek} f_{hk}$ involves summation over k of the product of the Fermi-Dirac functions, simplification of this term is also required. In general we use a linear approximation for the total spontaneous decay rate of carriers that includes the contributions of both radiative and non-radiative decay. Before we do this, however, it is instructive to compute the dependence of the total spontaneous emission rate on the carrier density. Parameter values used in the calculation are given in Table 2.1 and we convert the sum over k to an integral as in the gain calculation of the previous section. The results of this calculation (Figure 2.12(a)) show that the spontaneous emission rate varies approximately quadratically with carrier density for low carrier densities. A quadratic dependence of spontaneous emission rate on n is a frequently made approximation[36, 93, 29, 27]. However, as Fig. 2.12 shows, for higher carrier densities deviation from this quadratic dependence occurs. For simplicity, we make a linear approximation to the spontaneous carrier decay rate about the laser threshold carrier density, which includes the contributions of both non-radiative and radiative decay²². We denote the total carrier decay rate by

²²A linear approximation to the total spontaneous decay rate is used frequently in semiconductor rate equations. See for example [1, 19, 9, 23, 20, 16].

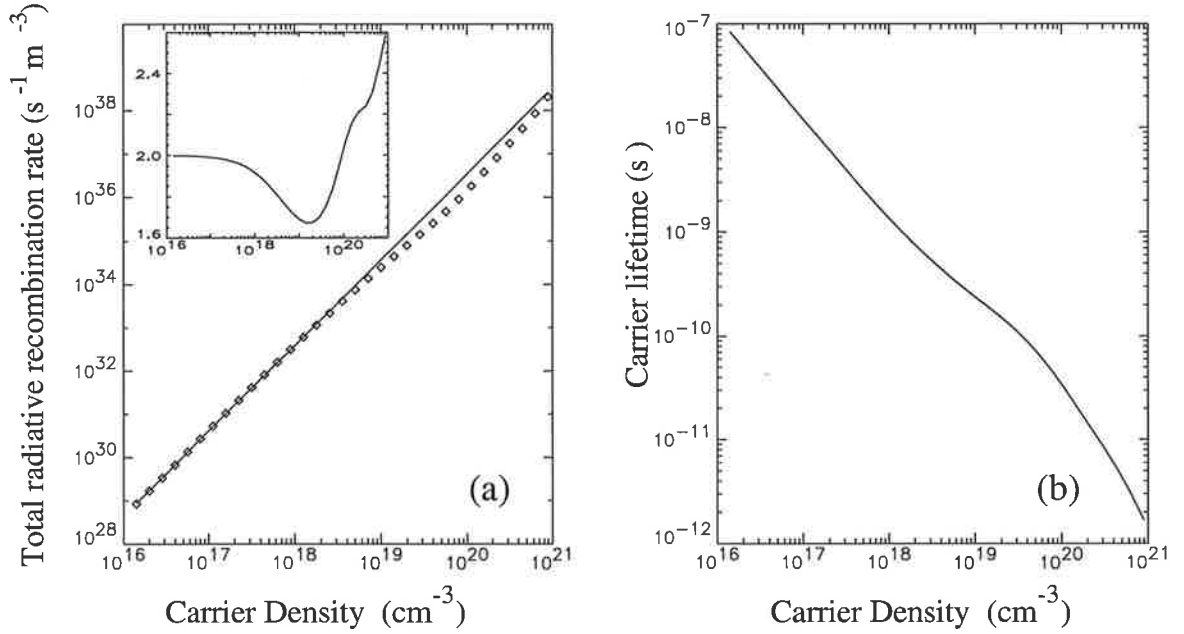


Figure 2.12: (a): The total spontaneous emission rate versus carrier density. The slope of this graph (shown in inset) shows a slope of two, for low carrier densities, indicating a quadratic dependence on the carrier density. For higher carrier densities the slope of the graph deviates significantly from two. (b): The spontaneous carrier lifetime versus carrier density.

$R(n)^{23}$. The carrier lifetime is defined as:

$$\frac{1}{\tau_e} = \left. \frac{\partial R(n)}{\partial n} \right|_{n=n_{th}} \quad (2.156)$$

In Fig. 2.12(b) we plot the derivative $\partial R_{sp}(n)/\partial n$ versus carrier density showing the contribution of radiative spontaneous decay to the carrier lifetime. For typical threshold carrier densities, τ_e is of the order of $3ns$.

Since some spontaneously emitted photons also couple into the laser mode we include a term $\beta n/\tau_e$ in the photon density rate equation where β is the fraction of spontaneous emission coupled into the laser mode[81, 94, 95] and is typically of the order of 10^{-4} to 10^{-5} [15, 16, 17, 2, 18, 19, 20, 21, 22, 23, 3, 4, 9].

Our final rate equations are[1, 34]

$$\frac{dn}{dt} = \frac{I(t)}{eV} - \frac{n}{\tau_e} - \sum_{j=-(M-1)/2}^{(M-1)/2} G_j s_j \quad (2.157)$$

²³It is understood that this term does not include the decay due to stimulated emission.

$$\frac{ds_j}{dt} = \left(G_j - \frac{1}{\tau_p}\right) s_j + \frac{\beta n}{\tau_e} \quad (2.158)$$

and will be used throughout the remainder of this thesis. For numerical purposes it is convenient to renormalise the variables n and s_j by defining $N = n/n_{th}$ and $P_j = s_j/\tilde{s}_0$ where $n_{th} = n_o + 1/A\tau_p$ is the threshold carrier density and $\tilde{s}_0 = n_{th}\tau_p/\tau_e$ ²⁴. Equations (2.157) and (2.158) become

$$\frac{dN}{dt} = \frac{1}{\tau_e} \left[I_b + m \sin(2\pi ft) - N - \frac{1}{1-\delta} \sum_{j=-(M-1)/2}^{(M-1)/2} (N\delta_j - \delta)P_j \right] \quad (2.159)$$

$$\frac{dP_j}{dt} = \frac{1}{\tau_p} \left[\left(\frac{N\delta_j - \delta}{1-\delta} \right) P_j - P_j + \beta N \right] \quad (2.160)$$

where $\delta = n_o/n_{th}$, $I_b = I_{DC}/I_{th}$, $m = I_{AC}/I_{th}$ which will be referred to as the modulation index. The quantity δ_j , embodies the gain spectral dependence, and is given by

$$\delta_j = \left\{ 1 - 4 \left(\frac{k(1-N) - j\delta\lambda}{\Delta\lambda_g} \right)^2 \right\}. \quad (2.161)$$

We note that in order to make contact with the existing literature we also use a single mode version of equations 2.157 and 2.158. Single mode models frequently use the linear gain model of equation 2.152. We obtain this relation from equation (4.11) if we substitute $k = 0$ and consider only the central mode, λ_o . In the remaining sections all single mode model calculations employ the gain model of equation (2.152). The single mode equations are obtained from (2.159) and (2.160) by setting $k = 0$ and $M = 1$:

$$\frac{dN}{dt} = \frac{1}{\tau_e} \left[I_b + m \sin(2\pi ft) - N - \frac{N - \delta}{1 - \delta} P \right] \quad (2.162)$$

$$\frac{dP}{dt} = \frac{1}{\tau_p} \left[\frac{N - \delta}{1 - \delta} P - P + \beta N \right]. \quad (2.163)$$

These equations are equivalent to those used in [16, 17, 2, 18, 19, 20, 21, 22],[3] except for the omission of the gain saturation term. We have not included gain saturation effects since, in AlGaAs lasers, they have been found to be less significant than in InGaAsP lasers[3]. However, we remark that the inclusion of gain saturation leads to an immense increase in the complexity of the problem, because the associated spectral hole burning effects[6] imply a deviation from quasiequilibrium[36]. We discuss this further in Section 2.5.

²⁴This normalisation procedure is the same as that used in references [3, 34]. See Appendix D for details.

2.4.7 Laser Noise

In order to account for random fluctuations in the photon and carrier densities, we add Langevin noise terms to the rate equations 2.159 and 2.160. Physically the fluctuations arise from the discrete nature of the carrier recombination process; they are quantum mechanical in origin. Thus, a derivation of these terms requires quantisation of the electromagnetic field[87]. For a semiconductor laser, modelling noise and decay terms is by no means a trivial problem. Complications arise, as a consequence of the semiconductor bandstructure, when deriving noise terms for the rate equations via the adiabatic elimination procedure used for eliminating the polarisation dynamics[36]. Furthermore, faithful representation of the underlying physics requires careful modelling of the pump and decay processes[96, 97, 98, 99]²⁵.

The laser is an open system; energy is input via the pumping process and lost via decay through the cavity mirrors, spontaneous emission, scattering, etc. This coupling to the environment leads to damping terms as described above and also to random fluctuations of the photon and carrier densities. There are two main approaches used for modelling damping and fluctuations in quantum systems. The first, the density matrix approach, is reminiscent of the Schrodinger picture in quantum mechanics and leads to the derivation of a master equation describing the density matrix evolution[64]. This methodology has the advantage that it is a systematic derivation and thus any approximations made can be clearly stated. In this section we use the more phenomenological based Langevin equation approach as it is compatible with the rate equation description²⁶. The procedure used here follows closely that of Marcuse[103]. This approach is, strictly speaking, applicable to a three level laser system. However, since we are concerned with general qualitative features of the noisy photon density power spectra, rather than specific technical aspects of the noise, this procedure is sufficient for our purposes.

To model the fluctuations in the carrier and photon densities, Langevin noise terms, $F_N(t)$ and $F_{P_j}(t)$ are appended to equations 2.159 and 2.160, respectively. The Langevin noise terms are assumed to be Gaussian random processes with Markovian autocorrelation functions[87, 107]

$$\langle F_\alpha(t)F_\alpha(t') \rangle = V_\alpha^2 \delta(t - t') \quad (2.164)$$

²⁵For example, amplitude noise squeezing is observed in semiconductor lasers when the injection current noise is suppressed[100, 101, 99, 102].

²⁶The density matrix approach to modelling semiconductor laser noise is used in references [97, 98] whereas the Langevin approach is used in references [36, 103, 104, 105]. A comparison of the two approaches is given in reference [106].

where $\alpha = N$ or P_j . The variances V_α are as follows[107, 3]:

$$V_N = \left\{ \frac{\frac{I(t)}{I_{th}} + N + \frac{1}{1-\delta} \sum_{j=-M/2}^{M/2} (N\delta_j + \delta) P_j}{\tau_e^2 n_{th} \mathcal{V}} \right\}^{\frac{1}{2}} \quad (2.165)$$

$$V_{P_j} = \left\{ \frac{\left(\frac{N\delta_j + \delta}{1-\delta} \right) P_j + P_j + \beta N}{\tau_p^2 \tilde{s}_o \mathcal{V}} \right\}^{\frac{1}{2}}. \quad (2.166)$$

2.5 Other Considerations

As will be apparent from the proceeding discussion, a significant number of approximations have been made in obtaining equations 2.157 and 2.158. In this section we will discuss some other factors of importance in SCL operation which are not considered in the semiclassical theory discussed thus far.

2.5.1 Coulomb Interactions

Since Coulomb interactions involve more than one carrier, quantum mechanical many-body techniques are required to analyse this phenomena. These techniques yield an infinite hierarchy of coupled differential equations involving ensemble averages of increasing numbers of carrier creation and annihilation operators[36]. This hierarchy of equations must be truncated if a solution, albeit approximate, is to be obtained. We will not discuss these procedures further. However, it is instructive to mention some modifications to the free-carrier theory which arise as a result of these higher order corrections[36].

Plasma Screening involves the weakening of the carrier-carrier Coulomb interaction due to the background carrier charges.

Bandgap Renormalisation At higher carrier densities, plasma screening leads to a lowering of the energy gap between the conduction and valence bands.

Coulomb Enhancement Inclusion of many-body corrections results in a renormalisation of the electric dipole interaction energy which leads to Coulomb enhancement of the interband transition probabilities. This in turn leads to a modification of the gain spectrum.

Furthermore, inclusion of Coulomb effects leads naturally to the collision terms which were introduced phenomenologically in Section 2.4.3. Under application of appropriate approximations these lead to the carrier decay and polarisation decay terms, equations 2.123 and 2.122, respectively. Many-body descriptions of semiconductor laser operation have been developed. These can be found in references [36, 58, 108, 59, 60, 61].

2.5.2 Linewidth Enhancement Factor (α)

The linewidth enhancement factor, α , describes the coupling between the semiconductor gain and refractive index. This coupling arises as a result of the asymmetry of the semiconductor laser gain. This can be explained as follows. Firstly, we note that the gain (or loss) is related to the imaginary part of the susceptibility χ whereas the refractive index is related to the real part of the susceptibility. The α parameter can be expressed as [109, 110]

$$\alpha = -\frac{d[\Re(\chi(n))]/dn}{d[\Im(\chi(n))]/dn} = \frac{4\pi}{\lambda} \frac{d\mu/dn}{dg/dn} \quad (2.167)$$

where, n is the carrier density, λ is the lasing wavelength, μ is the refractive index and g is the semiconductor gain. From the Kramers-Kronig relationship [63], it is known that a change in the real part of the susceptibility is accompanied by a change in the imaginary part. For a symmetric gain spectrum the symmetry of the Kramers-Kronig transform leads to a dispersion curve for the refractive index which has a zero at the gain peak (and hence the lasing wavelength). Thus, gain and index fluctuations are not strongly coupled in these laser systems and the parameter α is close to zero. However, for a semiconductor laser, the asymmetry of the gain curve leads to a dispersion curve for the refractive index that is non-zero at the lasing wavelength (gain peak). Thus for a semiconductor laser the α parameter has an unusually high value of 4-7 [110].

This high value of α leads to several physical characteristics unique to semiconductor lasers. Firstly, the linewidth enhancement factor, as its name implies, has important consequences for the laser linewidth. In conventional laser systems, the finite linewidth arises as a consequence of the phase diffusion of the electric field. However, in a semiconductor laser, fluctuations in the gain (resulting from the randomly emitted spontaneous photons) are accompanied by refractive index fluctuations. These index fluctuations in turn lead to additional phase fluctuations and hence increased linewidth. Thus, modification of the Schawlow-Townes formula, by the addition of a factor $(1 + \alpha)^2$ is required to correctly predict the semiconductor laser linewidth [102, 111, 112].

Another consequence of the α parameter is that amplitude modulation of the intensity by direct modulation of the injection current is accompanied by frequency modulation. Thus injection current modulation results in the generation of FM sidebands on the laser emission spectrum which leads to dispersion in optical fibre systems [70]. However, this coupling also provides a means of frequency modulating a single frequency laser for use in coherent optical communication systems.

The α parameter is also important in the case of delayed feedback of laser emission into the semiconductor laser cavity from an external reflector. The phase of the feedback relative to the phase of the field inside the cavity is very important in determining the

operating characteristics and variations of the phase of the electric field must be considered in the dynamical description of these devices[109, 110].

2.5.3 Device Parasitics

The high frequency limits of electronic devices are determined by their parasitic elements. In general, the parasitics of interest are those that divert high frequency components of the injection current away from the active region of the device[113]. A simplified circuit model of the device parasitics of a semiconductor laser is shown in Fig. 2.13[6]. Typically, they take the form of a resistance in series with the intrinsic diode combined with a shunt capacitance. In Fig. 2.13, L_p accounts for the inductance of the bondwire to the laser chip. It is typically the order of 1nH/mm of bondwire[6]. R_s is the series resistance of the laser chip and is generally in the range 3-10 Ω [113]. Above threshold, the input resistance of the intrinsic laser diode is considerably less than 1 Ω and can be neglected; this results from the clamping of the carrier density (see Appendix D) and thus the quasi-Fermi levels leading to an almost constant voltage for the intrinsic laser diode above threshold[6]. C_p is the parasitic chip capacitance and is strongly dependent on the device structure. In practice, shunt capacitance can arise from different sources within the laser chip and the dominant parasitic path may be difficult to determine precisely[113]. Circuit parasitics can be obtained experimentally by measuring the microwave reflection co-efficients using a network analyser[114, 115].

An alternative method to solving the rate equations by direct numerical integration is to transform the equations to an equivalent circuit model which can then be solved using circuit analysis techniques[116, 117, 118]. The main advantage of this approach is that the parasitic network can be easily interfaced to the intrinsic diode response thus allowing device-circuit interactions to be determined.

2.5.4 Carrier Diffusion

Carrier diffusion in the lateral direction, with the active region of a semiconductor laser plays a role in the damping of the relaxation oscillations[15, 14]. To describe lateral carrier diffusion a term

$$D \frac{\partial^2 n(x)}{\partial x^2} \quad (2.168)$$

is appended to the rate equation for the carrier density, where D is the diffusion coefficient, x is the lateral coordinate and $n(x)$ is the spatially dependent carrier density in the lateral direction[14]²⁷. Spatial dependence of the optical field, in the lateral direction, must also

²⁷We remark that a lateral variation in the carrier density is inconsistent with the free carrier theory for several reasons. Firstly, periodic boundary conditions are assumed in the free carrier theory and therefore large scale, non-periodic spatial variations (i.e. those not associated with a Bloch function for example) in the

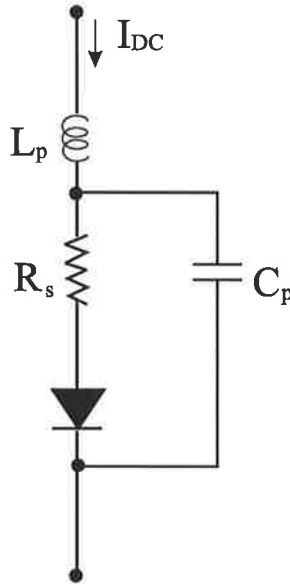


Figure 2.13: A simple equivalent circuit of semiconductor laser device parasitics.

be considered in this analysis[15]. The extent of damping resulting from carrier diffusion is strongly dependent on the laser structure[6, 14]. In general, maximum damping of the relaxation oscillation occurs for stripe widths approximately the same size as the carrier diffusion length[6].

2.5.5 Nonlinear Gain

The nonlinear dependence of the semiconductor spectral gain function on the laser intensity has implications for the spectral characteristics and dynamic behaviour of these devices. The nonlinear gain affects both the response of the laser under direct modulation of the injection current and the longitudinal mode behaviour with increasing injection current[6, 16, 17, 119, 120, 121].

The origin of the nonlinear gain in semiconductor lasers has been widely studied[122, 123, 120, 90, 92, 119, 124, 125, 88, 89, 126, 127, 128]. Two important mechanisms, spectral hole-burning and carrier heating, have been shown to have significant contributions[122, 120, 119, 124, 125, 89, 91]. The gain nonlinearity due to spectral hole burning occurs only around the frequency corresponding to the laser transition. It arises when the loss of carriers due to

carrier density are neglected. Secondly, in deriving the free-carrier gain, the carriers are described in terms of momentum rather than position. Related to this is the assumption of quasi-equilibrium distributions which implies that the carrier distributions, in terms of the Bloch states, are completely specified (statistically) for a given carrier density and temperature. We hasten to add that, though we have indicated the inconsistencies between these two descriptions, our intention is not to question either model's validity but to point out the difficulties associated with modelling these devices.

stimulated emission occurs at a faster rate than the rate at which carrier-carrier scattering can restore the quasi-equilibrium distribution. Thus, depletion of carriers in the conduction and valence band k states, corresponding to the laser transition, occurs leading to a flattening of the gain curve in this region[6, 120, 92, 89].

The gain nonlinearity resulting from carrier heating, on the other hand, appears over a wide spectral range since carrier heating affects the entire carrier populations in the conduction and valence bands. Carrier heating refers to processes which cause an increase in the temperature of the carrier distributions above the lattice temperature. One of the main causes of carrier heating is stimulated emission, since it results in the removal of carriers, with momentum k close to the band edge, that have below average kinetic energy. These carriers are replaced by current injection. However, due to the Pauli exclusion principle only those energy levels not already filled can be occupied by the injected electrons. This leads to preferential injection of carriers into higher energy levels in the bands; a process known as Pauli pump blocking. Carrier-carrier scattering subsequently drives the resulting carrier distributions towards Fermi-Dirac distributions but with a temperature higher than the lattice temperature. Thus, the combination of the removal of “cold” carriers by stimulated emission and the injection of “hot” carriers leads to an overall increase in the temperature of the carrier distribution relative to the semiconductor lattice temperature. Energy from the carrier distributions is transferred to the lattice via carrier-phonon interactions. By this means, the carrier temperature is restored to the lattice temperature. However, the effectiveness of this equilibration process depends upon the relative time-scales of carrier-phonon and carrier-carrier scattering. Thus, sufficiently high stimulated emission rates, can lead to quasi-equilibrium carrier distributions with temperatures higher than the lattice temperature, resulting in an overall change in the spectral gain function.

2.5.6 Summary

We have included, in this section, a brief introduction to several physical processes which are not modelled via the multimode rate equations 2.157 and 2.158. All these processes are expected to affect the dynamical response of the carrier and photon densities. The purpose of this section was to introduce these concepts and to point to the relevant papers; the study of implications of these processes for the dynamical response of a semiconductor laser under injection current modulation constitutes areas of further work.

Chapter 3

Semiconductor Laser Diagnostics

In this chapter the experimental arrangement is described. Two types of measurement are performed: time averaged longitudinal mode spectra and the power spectrum of the total intensity. Due to the sub-nanosecond time-scales typical for dynamical variations of the laser intensity under direct current modulation, we were not able to measure the time dependence of the intensity variations directly. Therefore, all comparisons between theory and experiment are made via comparison with the power spectra or time averaged measurements.

The characteristics of the laser under steady state (DC) operation, are presented. This includes the threshold behaviour of the total intensity, the longitudinal mode spectra variation and the intensity power spectra variation with injection current. Finally diagnostic measurements, used to determine the parameter values required for the rate equations, are described and the results of these measurements summarised.

3.1 Experimental Arrangement

The experimental arrangement, shown in Fig. 3.1, allows measurement of the power spectrum of the total intensity and the time averaged longitudinal mode spectra. A Fabry-Perot semiconductor laser with an AlGaAs active region (wavelength 830nm) was used in the experiment. The combination of a Glan polariser and a Fresnel rhomb, was used to isolate the laser from back reflections as these can alter the dynamic behaviour. Temperature stabilisation was achieved by mounting a Peltier cell, controlled by a Newport TC100 thermoelectric temperature controller, close to the semiconductor laser. An investigation into the behaviour of the laser under direct current modulation involved modulation at frequencies up to 3GHz. A microstripline circuit was utilised for this purpose. The details of its construction are given in Appendix C. We were unable to measure the frequency response of this circuit. However, though modulation at 3GHz was detected (by monitoring the output intensity fluctuations) it is likely that there is some attenuation of the input AC modulation at higher frequencies.

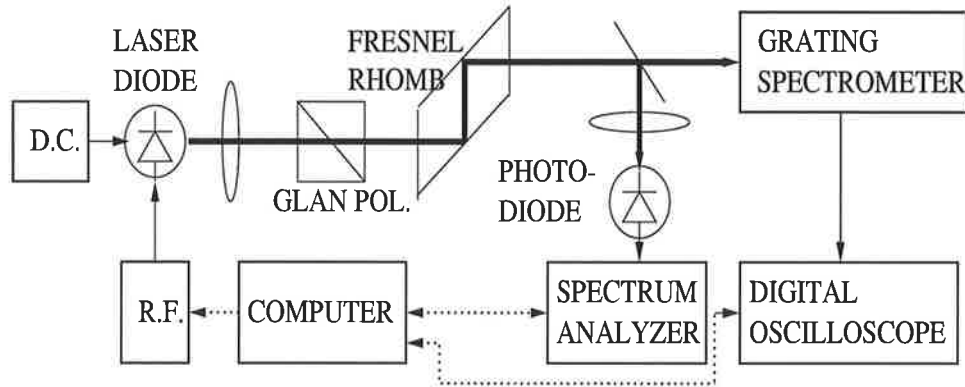


Figure 3.1: The experimental setup.

This should be taken into consideration when interpreting the experimental data.

The laser was modulated in the frequency range 5kHz-3GHz with modulation powers in the range -50dBm to 15dBm using a Rohde and Schwarz SmT03 signal generator. The DC source for the laser was provided by a Newport PCS100 precision current source.

The power spectrum of the total intensity was measured by focusing the laser output into a New Focus 1537 photodiode (bandwidth 6GHz) connected to a Tektronix 497P spectrum analyser. Optional amplification of the photodiode signal was provided by a Mini-circuits ZHL-1042J power amplifier (10MHz to 4.2GHz bandwidth). Time averaged longitudinal mode spectra were measured using a grating spectrometer with a linear photodiode array (Hamamatsu model S3923) in place of the exit slit. The output of the spectrometer was connected to a TDS540 digital oscilloscope.

In order to facilitate the collection of data GPIB data acquisition using a personal computer was employed. The spectrum analyser, signal generator and oscilloscope are all GPIB programmable instruments. Thus, the modulation frequency and power could be adjusted programmatically and the resulting power spectrum and longitudinal mode spectrum recorded. This allowed large regions of the parameter space composed of the modulation frequency and amplitude to be scanned. The National Instruments software package, Labview, was used to write the data acquisition programs.

3.2 Steady State Operation

In this section we characterise the properties of the laser under DC operation. Figure 3.2 shows the output power versus injection current for various operating temperatures in the range 15-40°C. For output powers below 3mW a linear relation between output power and injection current above the laser threshold are observed. The threshold current increases with increasing temperature, characteristic of these devices. An experimental determination

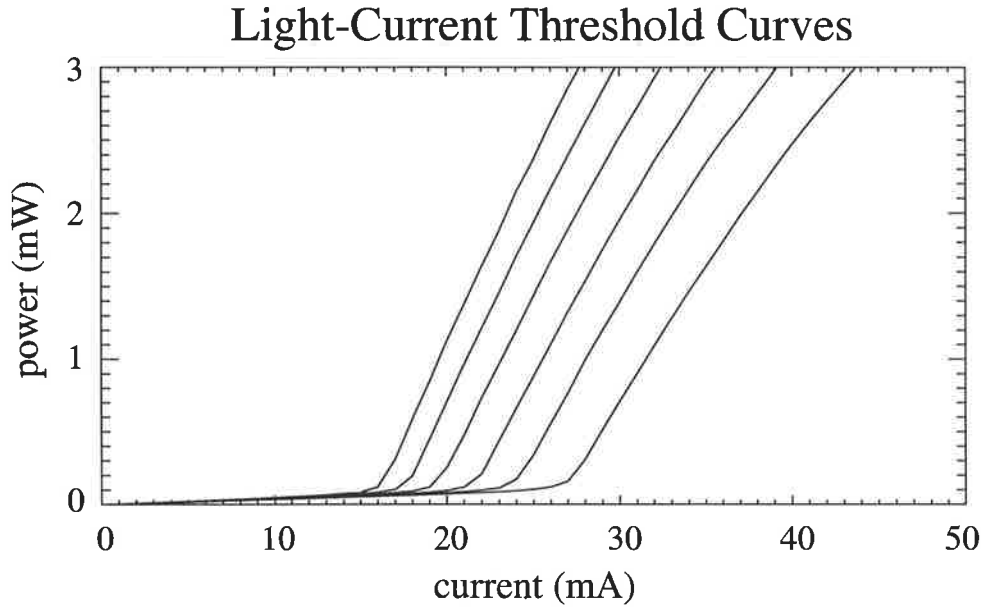


Figure 3.2: Output power versus injection current for different temperatures. From left to right the threshold curves correspond to temperatures of $T = 15, 20, 25, 30, 35$ and 40°C .

of the threshold current can be obtained by calculating the derivative, $d\log(P)/d\log(I_{DC})$, where P is the output power of the laser. This curve is strongly peaked at the threshold current thus yielding an accurate estimate of this quantity¹. Figure 3.3 shows the behaviour of $\log(P)$ versus I , in part (a), and the slope of this curve, in part (b), for an operating temperature of 25°C ; the threshold current for this temperature is 21mA.

The longitudinal mode spectra versus injection current are shown in Fig. 3.4. Generally, the laser operates predominantly in a single dominant longitudinal mode if biased sufficiently far above threshold. The lasing mode is not constant for all bias currents, however, and usually shifts to longer wavelengths with an increase in the bias current. Several explanations for this behaviour have been proposed in the literature. These include nonlinear gain and an increase in the active region temperature with bias current [6, 121, 119, 124]. Temperature changes, at a fixed bias current, also induce a change in the lasing mode. This results from three competing effects. The first effect is a shift in the modal frequencies caused by thermal expansion of the crystal². Secondly, as discussed in chapter 2, an increase in temperature results in a redistribution of the carriers within the bands according to the Fermi-Dirac distribution. This causes a shift in position and a reduction in the gain peak with increasing temperature³. If the temperature induced gain shift is sufficient an adjacent mode will eventually experience higher gain than the current lasing mode and a mode hop

¹A discussion is given in Appendix D. See also [129].

²See Table A.1 for the thermal expansion coefficient.

³This effect also results in the increased threshold current with increasing temperature.

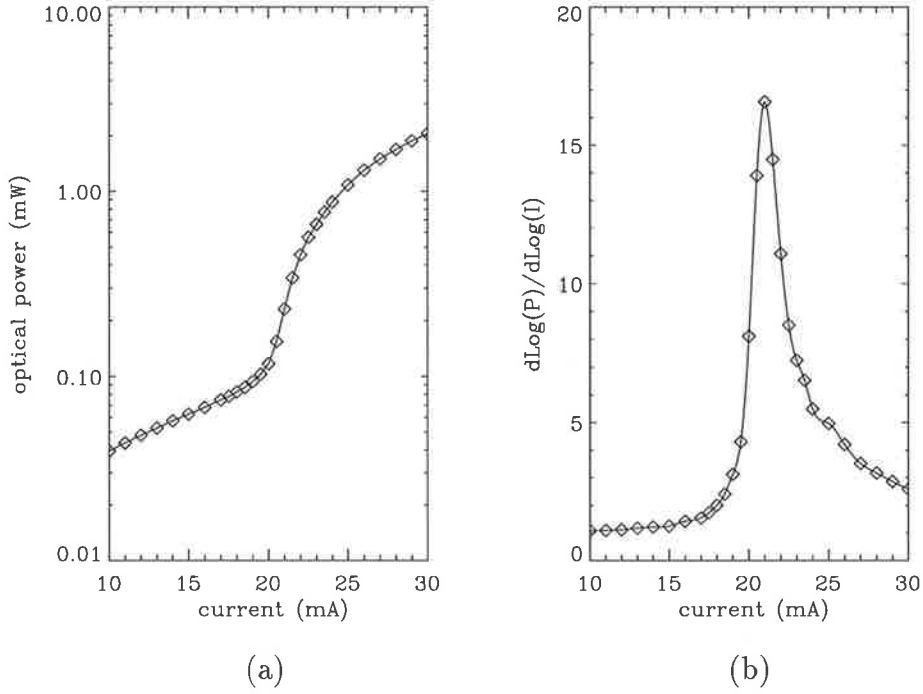


Figure 3.3: (a): Output power (log scale) versus injection current. (b): The derivative $d\log(P)/d\log(I_{DC})$ versus injection current. The position of the peak corresponds to the threshold current.

results. Associated with the gain shifts are gain induced refractive index changes. These act to change the optical cavity length and consequently the mode frequencies.

The output intensity power spectra of a semiconductor laser exhibit a peak at the relaxation oscillation frequency arising from noise induced excitation of the natural system resonance. The intensity noise spectrum as a function of injection current is shown in Fig. 3.5. The laser temperature was 25°C . In Fig. 3.6, we plot the position of the relaxation oscillation peak versus injection current, in part (a), and versus $\sqrt{P}$ in part (b). The position of the relaxation oscillation peak was detected by fitting a Lorentzian function to the peak of each of the power spectra in Fig. 3.5; the associated error bars are the errors obtained for this parameter in the fit⁴. The solid lines, in Figs. 3.6(a) and (b) are a fit to the experimental data. For Fig. 3.6(a), the solid curve is given by

$$f_{RO} = a\sqrt{I_{DC} - I_{th}} \quad (3.1)$$

where $a = 0.79 \pm 0.08 \text{ GHz}/\sqrt{\text{mA}}$ and $I_{th} = 20.3 \pm 0.8 \text{ mA}$. For Fig. 3.6(b) the solid curve is given by

$$f_{RO} = b\sqrt{P} + c \quad (3.2)$$

⁴The method used was the gradient-expansion algorithm to compute a nonlinear least squares fit to the data [130].

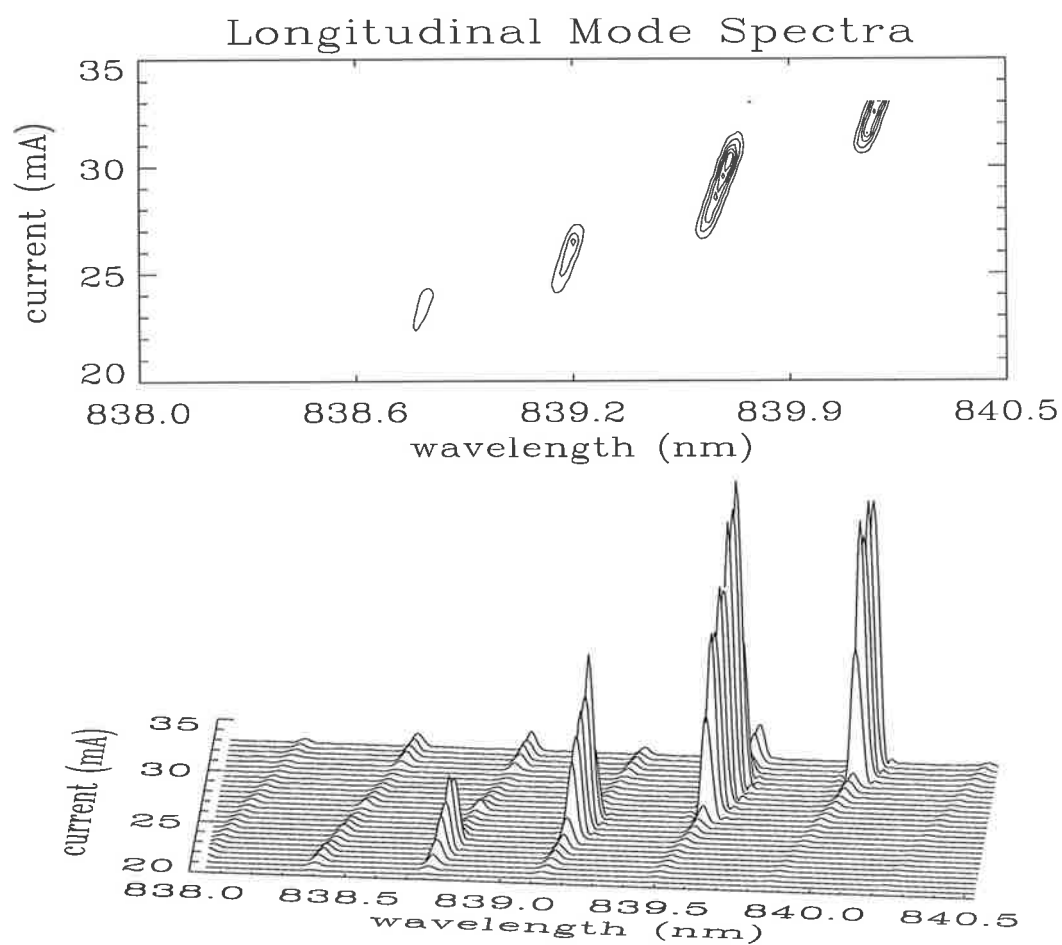


Figure 3.4: Longitudinal mode spectra versus injection current shown as a contour plot (above) and surface plot (below). The vertical axis corresponds to mode power.

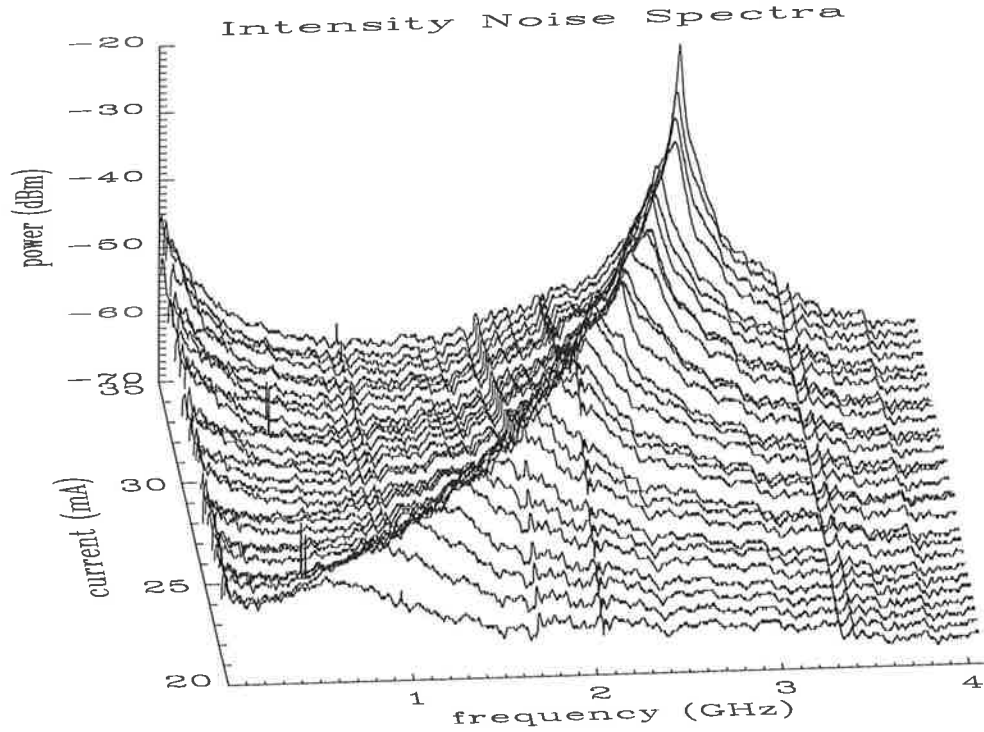


Figure 3.5: Intensity noise power spectra versus injection current showing a peak at the relaxation oscillation frequency.

where $b = 1.78 \pm 0.08 \text{ GHz}/\sqrt{\text{mW}}$ and $c = -0.10 \pm 0.08 \text{ GHz}$.

3.3 Rate Equation Parameter Values

In this section we describe simple diagnostic measurements used to determine the parameter values required for modelling via the multimode semiconductor laser rate equations (equations 2.159 and 2.160). Such measurements are necessary for quantitative comparisons since the parameter values are both device dependent (even for lasers of the same model number) and are also dependent on the operating conditions (such as the laser temperature).

3.3.1 Carrier Lifetime

The spontaneous carrier lifetime, τ_e , may be obtained by measuring the frequency response of the laser biased below threshold[131, 132, 133, 93, 134]. In this case, the decay resulting from stimulated emission may be neglected for sufficiently small bias currents. Under these conditions the rate equation for the carrier density (equation 2.157) may be approximated

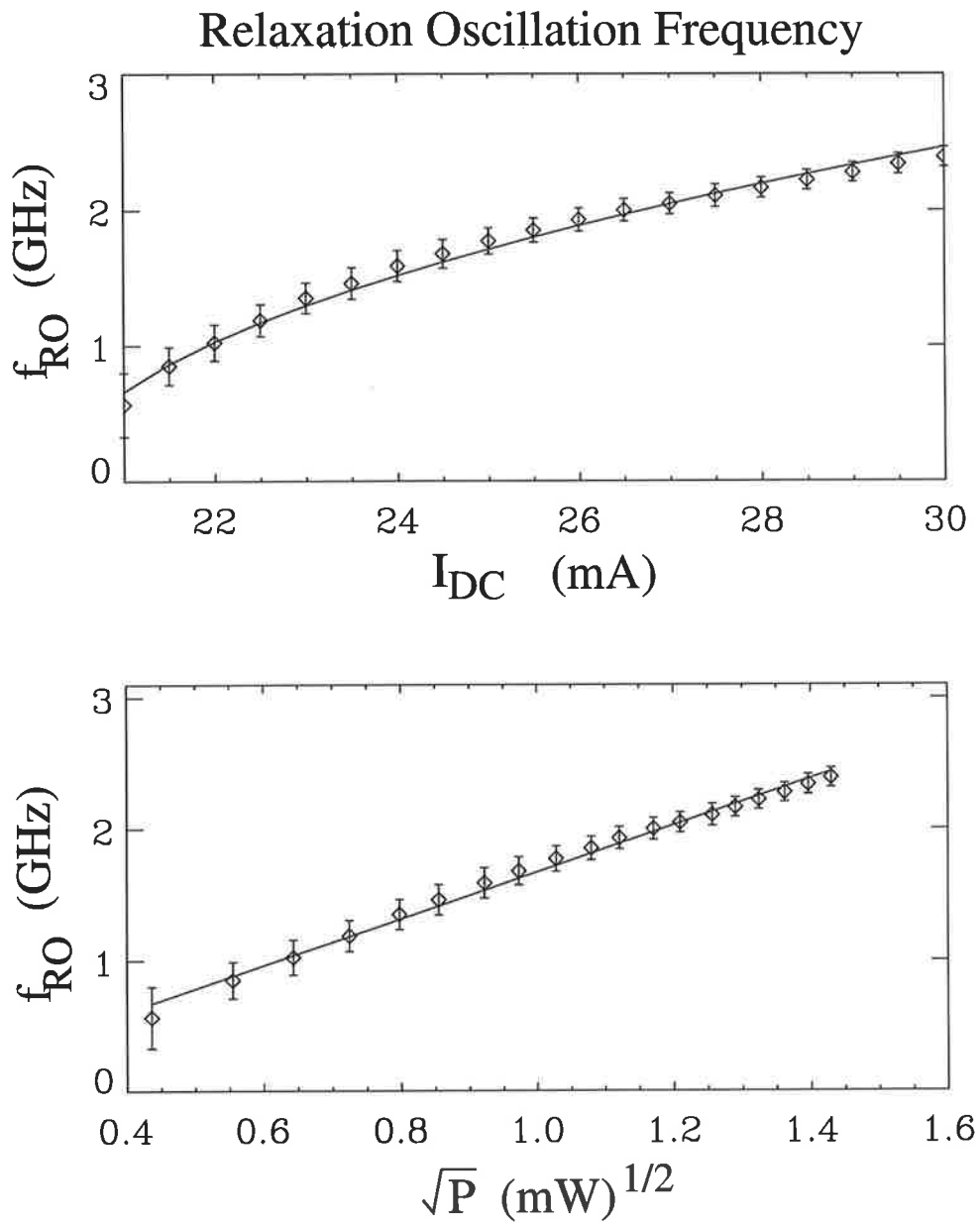


Figure 3.6: The relaxation oscillation frequency versus injection current (above) and versus the square root of the output power (below).

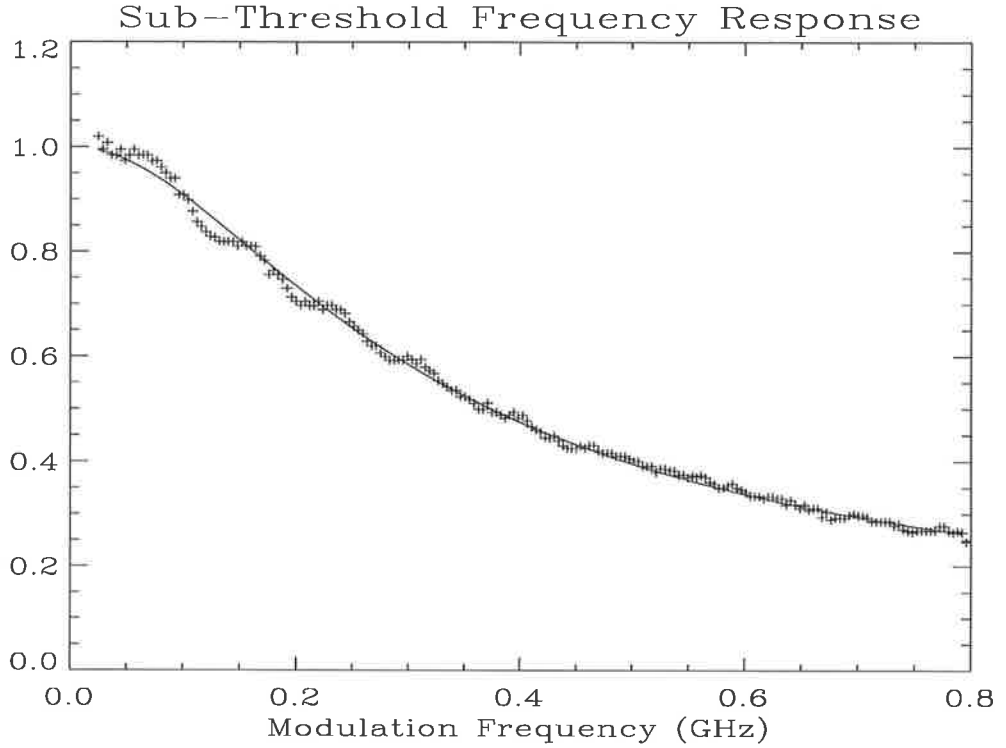


Figure 3.7: Normalised frequency response of the laser biased below threshold ($I_{DC} = 19\text{mA}$).

by

$$\dot{n} = I(t) - \frac{n}{\tau_e} \quad (3.3)$$

where $I(t) = I_{DC} + I_{AC} \sin(2\pi ft)$ is the injection current. The transfer function for equation 3.3 is a Lorentzian centred at 0:

$$\frac{|H(i2\pi f)|}{|H(0)|} = \frac{1}{\sqrt{1 + (2\pi f\tau_e)^2}}. \quad (3.4)$$

Sufficiently far below threshold, for small signal modulation, the amplitude of the output intensity variation is proportional to the carrier density variation. Thus, the above transfer function can be measured experimentally by monitoring the output intensity. An example of a measured transfer function, for $I_{DC} = 19\text{mA}$, is shown in Fig. 3.7. In order to take into account any variations of the input RF power as a function of frequency, the experimental data was divided by the measured frequency response of the signal generator. The solid curve in Fig. 3.7 represents a fit of equation 3.4 to the experimental data. This measurement was repeated for several bias currents in the range 17-19mA and the results averaged. The carrier lifetime obtained by this method was $0.78 \pm 0.05\text{ns}$.

An assumption we have made in using equation 3.3 is that the spontaneous lifetime

is independent of the carrier density. In the more general case, the inverse carrier lifetime, τ_e^{-1} , of equation 3.3 is replaced by

$$\frac{1}{\tau_e} \rightarrow A + Bn + cn^2 \quad (3.5)$$

where A is the contribution due to nonradiative transitions, Bn is the contribution due to spontaneous recombination⁵, and Cn^2 results from Auger recombination[132]. The contribution of Auger recombination is generally insignificant in AlGaAs lasers but tends to have a greater importance in InGaAsP lasers[132]. To detect the dependence of the spontaneous carrier lifetime on carrier density it is necessary to perform the measurement of Fig. 3.7 for a range of bias currents below threshold[132]. Though we were able to perform this measurement for injection currents in the range 19mA to 10mA we were unable to discern any significant change in the carrier lifetime over this current range. For bias currents below 10mA the emitted intensity was too low to be detected above the noise floor of the measurement system. Furthermore, at lower bias currents the laser diode resistance can no longer be considered to be constant and thus alters the frequency response of the sub-threshold laser[134]. A simple analysis of the rate equations, including carrier density dependent spontaneous lifetime, shows that we would expect these effects to be most significant at very low bias currents. Since this regime was not accessible with our experimental arrangement we were unable to measure the extent of these effects. In all subsequent analysis we assume a constant carrier lifetime.

3.3.2 Gain Spectra

The gain spectra, as a function of wavelength, can be obtained by measuring the emission spectrum of the laser biased below threshold, using the procedure of Hakki and Paoli[135, 136]. The sub-threshold laser can be considered as a Fabry Perot cavity in which the active medium contributes net loss or gain depending on the bias current. As the threshold is approached the gain provided by the active medium acts to partially compensate for the absorption and scattering losses and the mirror losses thus increasing the effective finesse of the cavity and consequently affecting the observed fringe visibility. Therefore, knowledge of the laser gain can be obtained from experimental observation of the visibility of the Fabry-Perot fringes in the emission spectra of the laser.

The two mirrors forming the Fabry-Perot cavity of the laser have reflectivities R_1 and R_2 and the net power attenuation constant is $(g - \alpha)$, where α is the average distributed loss constant due to absorption and scattering and g is the semiconductor gain. Upon each round trip of the cavity the light field is attenuated, according to $(g - \alpha)$ and the mirror

⁵See section 2.4.6 for a discussion of this term.

losses, and experiences a phase shift, $\delta = 4\pi nL/\lambda$, due to propagation between the mirrors⁶. It is convenient to define the round trip (de)amplification factor ($|a| < 1$)

$$a = |a|e^{i\delta} = \sqrt{R_1 R_2} e^{(g-\alpha)L} e^{i\delta} = e^{(g-\alpha_c)L} e^{i\delta} \quad (3.6)$$

where $\alpha_c = \alpha + \alpha_m$ is the total cavity loss coefficient and

$$\alpha_m = -\frac{1}{L} \ln \sqrt{R_1 R_2} \quad (3.7)$$

is the contribution due to mirror losses. To determine the transmitted field, multiple reflections from the cavity mirrors need to be considered. Thus, the transmitted field is

$$E_T = E_o \sum_{n=0}^{\infty} a^n = \frac{E_o}{1-a} \quad (3.8)$$

or in terms of intensity, $I_T \propto |E_T|^2$

$$I_T = \frac{I_o}{1 - 2|a| \cos \delta + |a|^2}. \quad (3.9)$$

For constructive interference, $\cos \delta = 1$ ($\delta = 0, 2\pi, 4\pi, \dots$) yielding the maximum transmitted intensity

$$I_{max} = \frac{I_o}{(1 - |a|)^2} \quad (3.10)$$

and for destructive interference ($\cos \delta = -1$) we obtain the minimum transmitted intensity

$$I_{min} = \frac{I_o}{(1 + |a|)^2}. \quad (3.11)$$

From equations 3.10, 3.11 and 3.6 we obtain the nett gain

$$g_{eff} = g - \alpha_c = \frac{1}{L} \ln \left(\frac{\sqrt{I_{max}} - \sqrt{I_{min}}}{\sqrt{I_{max}} + \sqrt{I_{min}}} \right). \quad (3.12)$$

Thus, measurement of $\sqrt{I_{max}}/\sqrt{I_{min}}$ for each Fabry-Perot fringe in the emission spectrum yields the gain as a function of wavelength according to 3.12.

Figure 3.8(a) shows the measured gain spectra for bias currents in the range 11 to 19 mA. We observe the curves shown in Fig. 3.8(a) to be qualitatively similar to the theoretical predictions (Fig. 2.9). Since the total width of the sub-threshold laser spectrum was several times larger than the spectral width that could be measured using the photodiode array, several spectra were measured at different grating positions for each bias current

⁶All quantities defined as in equation 2.4.

and combined. The gain spectra shown in Figure 3.8(a) were subsequently calculated by determining numerically the minimum and maximum voltages for each Fabry-Perot mode. Due to the presence of large scale fringes (with period the order of 5 or 6 Fabry-Perot modes), believed to result from the interference of reflections from the front and back surfaces of the linear photodiode array window, the minimum and maximum voltages were further smoothed with a boxcar average in order to average out the effect of these fringes. The resulting gain, $g - \alpha_c$, as a function of wavelength was then determined using equation 3.12. The cavity length, L was determined from the wavelength spacing, $\Delta\lambda$, of the Fabry-Perot fringes; from equation 2.2 for the free spectral range, L is given by

$$L = \frac{c}{2n\Delta\nu} = \frac{\lambda_o^2}{2n\Delta\lambda}. \quad (3.13)$$

Both λ_o , taken to be the position of the peak gain at the laser threshold and $\Delta\lambda$, were obtained from the emission spectrum of the laser biased below threshold. The values of these parameters are $\lambda_o = 837 \pm 4\text{nm}$ and $\Delta\lambda = 0.37 \pm 0.02\text{nm}$, respectively. The refractive index was the only parameter that was not measured. It is assumed to have the literature value⁷ of 3.5.

Care was taken, when aligning the spectrometer, that the spectra corresponding to different grating positions overlapped without discontinuities. Observation of Fig. 3.8(a) shows that this was achieved for nearly all grating positions except those for the longer wavelengths. In this region light levels are extremely low and the measurement becomes increasingly sensitive to the noise inherent in the photodiode array and associated circuitry. Towards the longer wavelengths, the gain spectra of Fig. 3.8(a) are observed to level off, corresponding to the region below the bandgap. In this region, the semiconductor medium is transparent (i.e. contributes neither gain or loss) and thus this level corresponds to the total cavity loss coefficient α_c . The value of α_c was determined to be $53 \pm 3\text{cm}^{-1}$ by averaging over all the gain spectra of Fig. 3.8(a) in this region; the dotted line in Fig. 3.8 corresponds to the value of α_c .

The peak gain as a function of the injection current, calculated from the data shown in Fig. 3.8(a), is shown in Fig. 3.8(b). The solid line indicates a linear fit to the experimental data:

$$g_{eff} = a_1 I_{DC} + b_1 \quad (3.14)$$

where $a_1 = 2.9 \pm 0.2\text{cm}^{-1}/\text{mA}$ and $b_1 = -62 \pm 3\text{cm}^{-1}$. Extrapolating this line to zero gain gives a measurement of the threshold current since at the threshold the nett gain is zero. The threshold current, from equation 3.14, is found to be $21.2 \pm 0.6\text{mA}$.

⁷See Table A.1. Note that the refractive index is not required for experimental determination of laser rate equation parameter values.

The dotted line in Fig. 3.8(b) indicates the value of α_c obtained from the data of Fig. 3.8(a). The intersection of equation 3.14 with this line gives the value of the injection current required to reach transparency, I_o . Since below threshold a linear relation between injection current and carrier density is predicted⁸, the value of the parameter δ can be obtained from the ratio of I_o/I_{th} yielding a value of $\delta = 0.15 \pm 0.07$.

The position of the peak gain as a function of injection current, also determined from the data of Fig. 3.8(a), is shown in Fig. 3.8(c). Determination of the position of the gain peak is relatively more difficult than determining the peak gain due to the flatness of the curves in this region. The error bars shown in Fig. 3.8(c) show the distance between adjacent experimental data points in Fig. 3.8(a). Only the gain spectra for the four highest injection currents were used. The solid line indicates a linear fit to the data:

$$\lambda = b_2 - a_2 I_{DC} \quad (3.15)$$

where $a_2 = -0.9 \pm 0.15 \text{ nm/mA}$ and $b_2 = 856 \pm 2 \text{ nm}$. The slope of this line is given by k/I_{th} (see equation 2.155) thus yielding a value of $k = 19 \pm 3 \text{ nm}$.

Finally, in Fig. 3.8(d), we plot a quadratic approximation (dashed line) to the measured gain spectra at 19mA (solid line). Due to the asymmetry of the measured gain the quadratic fit is only a good approximation on the shorter wavelength side of the gain spectrum. However, experimental measurement of the longitudinal mode spectra under current modulation⁹ show it is the longitudinal modes in the vicinity of the gain peak and towards the shorter wavelengths which have the greatest contribution to the laser dynamics. Therefore, from this consideration, we would expect that modes lying on the longer wavelength side of the gain to be less important. The width of the gain, $\Delta\lambda_g$, can be obtained from the intersection of the quadratic fit with the dotted line corresponding to the value of α_c . This yields a value for the gain width of $\Delta\lambda_g = 30 \text{ nm}$.

Knowledge of the total cavity loss, α_c allows us to determine the photon lifetime according to equation 2.4:

$$\tau_p = \frac{nL}{c[\alpha L - \ln \sqrt{R_1 R_2}]} = \frac{nL}{c\alpha_c L} = 2.2 \pm 0.1 \text{ ps}. \quad (3.16)$$

Using equation 3.7 we can estimate the contribution due solely to the mirror losses by estimating the value of the refractive index to be 3.5 and thus the mirror reflectivities $R_1 = R_2 = 0.32$ (see equation 2.1). This gives a value of $\alpha_m = 42 \pm 2 \text{ cm}^{-1}$. The average distributed loss, α due to absorption, scattering etc. is given by $\alpha = \alpha_c - \alpha_m = 11 \pm 4 \text{ cm}^{-1}$; this is

⁸See Appendix D

⁹We show this in Section 4.4.2.

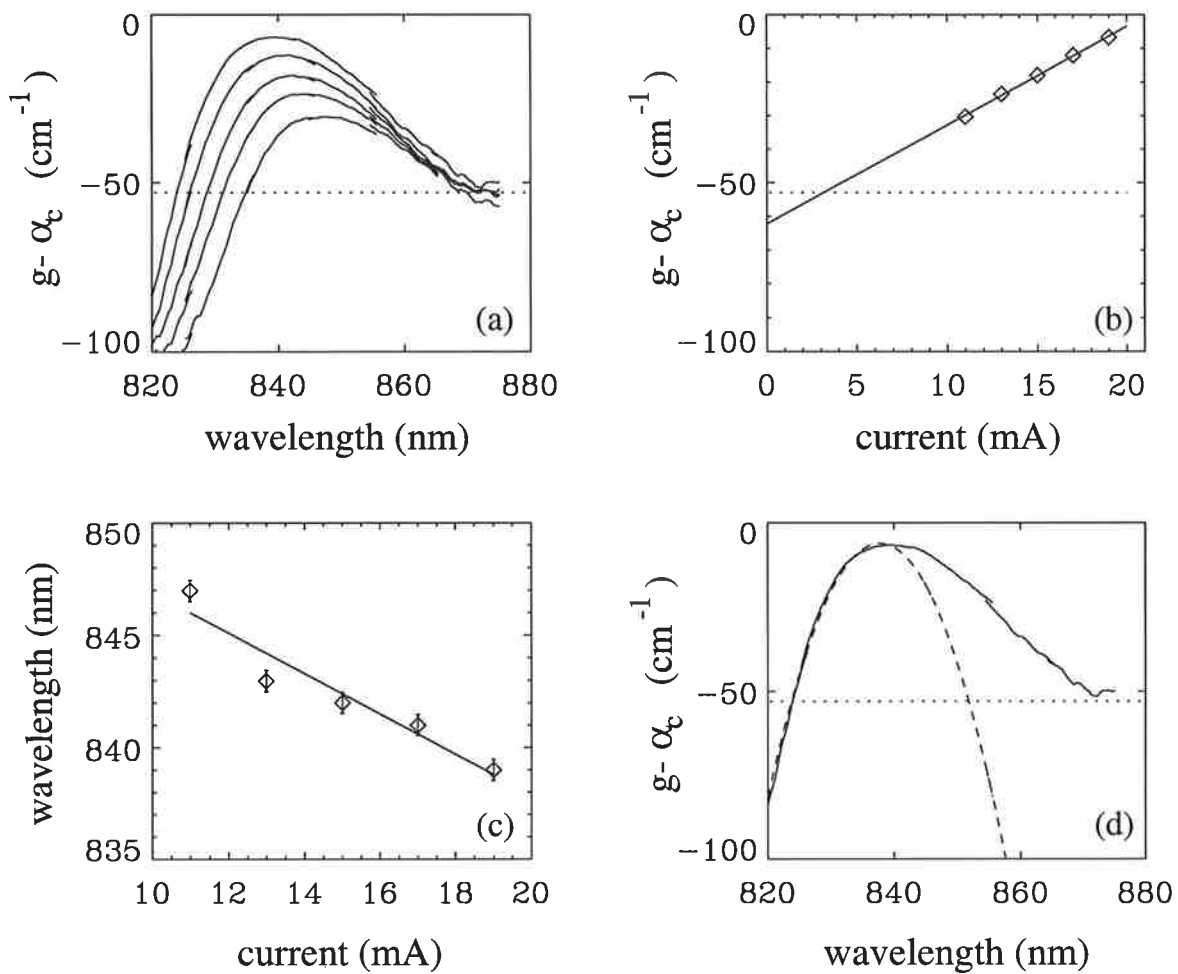


Figure 3.8: (a): Laser spectral gain function for injection currents in the range 11-19mA. (b): Peak gain as a function of the injection current. (c): Position of the peak gain as a function of the injection current. (d): Quadratic approximation to the spectral gain function corresponding to $I_{DC} = 19$ mA.

consistent with the frequently quoted value of 10cm^{-1} given in the literature¹⁰.

3.3.3 Spontaneous Emission Parameter

The spontaneous emission parameter β determines the fraction of the spontaneous emission coupled into the laser mode. Consequently it affects the laser output both above and below threshold. Several methods have been proposed to determine β experimentally. In [2], the ratio of the slopes of the intensity versus current curve far above and far below the laser threshold were used to determine β . However, this method is really only applicable to a single mode laser. In [137, 95] the power in the central longitudinal mode was measured as function of the injection current below threshold. The corresponding theoretical curves were calculated numerically from the rate equations for several different values of β . Comparison of the theoretical curves with the experimental data yields the appropriate value of β . Here we use a similar method but compare the numerical results with the experimentally measured power in the lasing mode above threshold. Using the experimental data of Fig. 3.4, we determine the power in each longitudinal mode as a function of the injection current by calculating the area under the peaks. These are plotted in Fig. 3.9 (diamond symbols) along with the total power (cross symbols), calculated by summing the modal powers. In order to compare with the numerical simulations, we plot normalised power¹¹ versus normalised injection current (I_b) in Fig. 3.9 where $I_b = 1$ corresponds to the laser threshold. The change in the laser mode above threshold, observed experimentally, is not predicted in the rate equation simulations¹². For this reason we plot only the normalised modal powers for values of the injection current between the modal transition regions. The solid curve in Fig. 3.9 corresponds to the total normalised photon density calculated using the experimentally determined parameter values. The dashed dot curves correspond to the normalised photon density in the highest mode calculated for different values of β in the range 10^{-5} to 5×10^{-5} . Figure 3.9 shows that the value of $\beta = 2 \times 10^{-5}$ yields modal powers most closely in agreement with the experimental results.

3.3.4 Summary of Results

The results obtained for the parameter values are summarised in Table 3.1. In all measurements the laser temperature was stabilised to 25°C . Three independent measurements of the threshold current were performed. These include, the location of the maximum of $d\text{Log}P/d\text{Log}I$ from the output power versus injection current measurements, determining I_{th} from the measurement of the relaxation oscillation frequency versus injection current and

¹⁰See for example [70], page 580.

¹¹The total power is normalised such that, above threshold, the power versus I_b curve has a slope of one.

¹²See Appendix D, Section D.2.

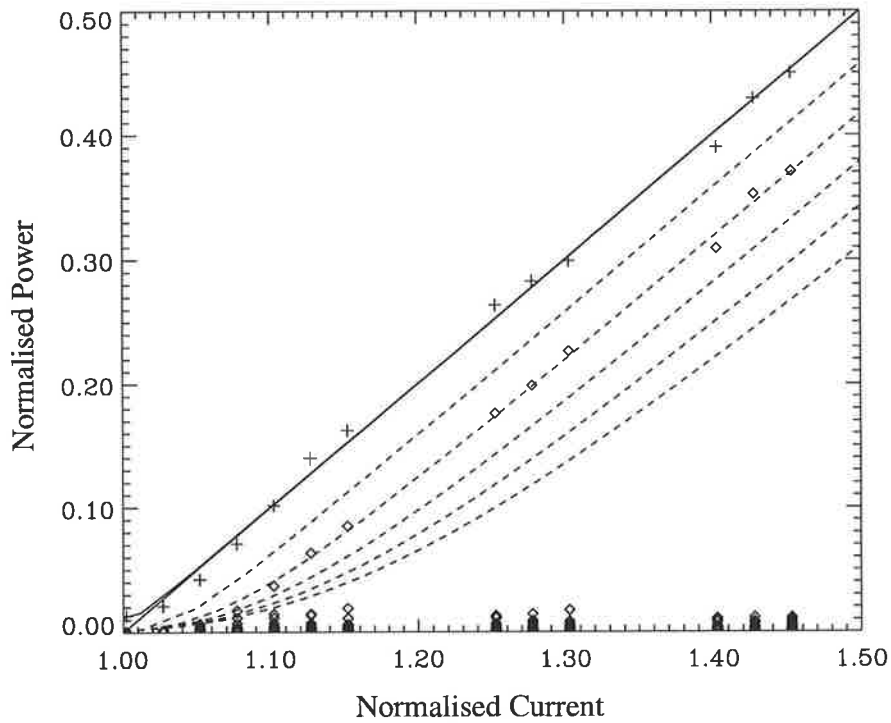
Normalised Central Mode Power for varying β 

Figure 3.9: Normalised power versus normalised current for the total power (crosses) and longitudinal mode power (diamonds), from the data of Fig. 3.4. The solid curve corresponds to the numerical predictions for the total normalised power from the multimode rate equations. The dash-dot curves correspond to the numerical predictions for the power in the dominant mode, for different values of β ; from top to bottom the curves correspond to $\beta=10^{-5}$, 2×10^{-5} , 3×10^{-5} , 4×10^{-5} and 5×10^{-5} . The value of $\beta = 2 \times 10^{-5}$ gives results most closely in agreement with experiment.

Table 3.1: Parameter values for the rate equations

Symbol	Value	Model
τ_e	$0.78 \pm 0.05\text{ns}$	both
τ_p	$2.2 \pm 0.1\text{ps}$	both
δ	0.15 ± 0.07	both
β	2×10^{-5}	both
$\Delta\lambda_g$	30nm	multimode
$\delta\lambda$	$0.37 \pm 0.02\text{nm}$	multimode
k	$19 \pm 3\text{nm}$	multimode
λ_o	$837 \pm 4\text{nm}$	multimode

from the current at which zero net gain results, obtained from measurement of the laser gain. These methods yielded values of $I_{th} = 21, 20.3 \pm 0.8$, and 21.2 ± 0.6 mA. Though all values are consistent, within experimental error, in all subsequent calculations we use the value $I_{th} = 21\text{mA}$ obtained from the power versus injection current curves since this method is the most accurate.

The measurement of the relaxation oscillation frequency versus injection current allows us to perform a cross check on the parameters τ_e , τ_p and δ , since the parameter, a of equation 3.1 is given by¹³

$$a = \frac{1}{2\pi\sqrt{\tau_e\tau_p(1-\delta)I_{th}}} \quad (3.17)$$

From the values given in Table 3.1 we obtain $a = 0.91 \pm 0.05\text{GHz}/\sqrt{(mA)}$ which is comparable to that obtained from the measurement of the relaxation oscillation frequency versus injection current, $a = 0.79 \pm 0.08$, thus confirming the consistency of the measurement technique.

The measurements described in this section, used to determine the parameter values required for the rate equations, were performed, for the most part, after extensive numerical calculations, using typical values for the laser parameters taken from the literature, were completed. For this reason, the experimentally measured parameter values are not used for many of the numerical simulations presented in the following two chapters. However, a comparison of the experimental results and the theoretical predictions, using the parameter values of Table 3.1 is given in Section 4.8.

¹³A discussion is given in Appendix D. In particular, refer to equation D.17.

Chapter 4

Bifurcation Scenarios

In this chapter we give the results of detailed numerical simulations performed using the single mode and multimode semiconductor laser rate equations. Bifurcation diagrams, time averaged longitudinal mode spectra and state diagrams in the (f, m) parameter space are calculated for each system of equations and the results compared. Though, experimentally we were unable to follow the time variations of the intensity directly, comparisons are made between the results of simulations and experiment via time-averaged measurements of the longitudinal mode spectra.

This chapter is structured as follows. In section 4.1 we introduce relevant concepts from nonlinear dynamical systems theory. We then review the field of the nonlinear dynamics of lasers with particular attention to current modulated semiconductor lasers. In section 4.2 we introduce the concept of an attractor for dissipative dynamical systems and show examples of the types of attractors typically encountered when studying numerical solutions of the semiconductor laser rate equations. A comparison of the bifurcation scenarios predicted by the single mode and multimode rate equations is presented in section 4.3. Section 4.4 discusses the predictions of the multimode rate equations for the time-average longitudinal spectra and comparison is made with the results of experiments. In section 4.5 we discuss the implications for the spontaneous emission factor on the dynamical behaviour and in section 4.6 we discuss period doubling, hysteresis and Hopf bifurcations in the multimode model in more detail. In section 4.7 we show two dimensional state diagrams of the bifurcation surfaces in the parameter space of modulation frequency and amplitude, (f, m) . Experimental measurements of period doubling regions in the (f, m) parameter space are also presented. Finally, in section 4.8 a quantitative comparison is made between the experimental results and the multimode rate equation predictions using the parameter values measured in chapter 3.

4.1 Overview

This overview has three parts. We first give a brief introduction to the field of nonlinear dynamics in order to introduce some important concepts and to put the field of nonlinear behaviour in laser systems in context. A historical overview of the field of nonlinear dynamics in lasers is given. This is not intended, by any means, to be a complete discussion but to provide a brief introduction to this active field of research. We then discuss previous work on the nonlinear dynamics exhibited by semiconductor laser devices under a variety of operating conditions. Though we are primarily interested in the behaviour under direct current modulation, we also discuss the behaviour of self-pulsing lasers and those operated in external cavity configurations in order to exemplify the rich variety of behaviour exhibited by these devices.

4.1.1 Introduction to Nonlinear Dynamics

It was recognised as early as 1892, with the work of Poincare[138], that particular Hamiltonian systems exhibit chaotic behaviour. However, the fact that many simple nonlinear systems, with only a few degrees of freedom, exhibit complicated behaviour was not widely appreciated until the advent of the computer. The increased availability of these devices has made the rapid calculation of numerical solutions possible and facilitated the graphic display of the intricate geometric structures associated with these systems. Very often in physics, complicated systems of equations are solved by linearisation. Though such methods yield useful information about a dynamical system they cannot reveal the true nature of its behaviour. The reason such methods are so often utilised is that closed form solutions of nonlinear systems are often difficult, if not impossible, to obtain. Hence it is imperative that some form of approximation is used or numerical solutions sought[139].

One very important, and perhaps surprising, aspect of nonlinear systems is that some features of the dynamics are repeated in many different systems independent of their physical properties[13]. Such behaviour is termed ‘Universality’ and is important in that it allows complicated systems to be replaced by simpler ones while still retaining essential features of the dynamics. In particular, three universal routes to chaos have been identified so far: the period doubling (Feigenbaum) route, the intermittency (Pomeau-Manneville) route and the quasiperiodic (Ruelle-Takens-Newhouse) route[140, 141]. In the period doubling route a series of period doubling bifurcations take place as a control parameter is tuned, eventually leading to chaos after an infinite number of bifurcations have occurred. Each successive period doubling scales by a certain factor which is apparently universal[142]. The intermittency route is characterised by nearly regular motion interrupted by a number of short irregular bursts. As a control parameter is varied the mean time between these bursts varies. The intermittency route is divided into three classes based on the statistics

of the time intervals between bursts: type I intermittency is associated with an inverse tangent bifurcation, type II with a Hopf bifurcation and type III with a period doubling bifurcation[141]. The quasiperiodic route arises from a series of Hopf bifurcations. The first Hopf bifurcation results in periodic motion while the second leads to motion on a two dimensional torus. Depending on whether the ratio between these two frequencies is rational or irrational, periodic or quasiperiodic motion occurs[143, 144]. A third Hopf bifurcation finally results in deterministic chaos.

Chaotic solutions are not, however, the only interesting aspect of nonlinear systems. Many exhibit a great deal of order and structure in the type of behaviour they predict. Intricate and delicate patterns that appear on smaller and smaller scales are a hallmark of these systems[145]. Though numerical simulations of the semiconductor laser rate equations predict that chaotic solutions occur in these systems we do not concentrate on these behaviours. Our main concern is the investigation of bifurcation phenomena in the rate equations. In particular, we are interested in how bifurcations occur, what the implications are for the global behaviour of the system and how the presence of noise effects a system close to a bifurcation point. Various aspects of these phenomena will be discussed in this and the following chapter.

4.1.2 Nonlinear Dynamics of Lasers

Instabilities in laser systems, in the form of spontaneously generated pulsations of the output intensity, have been observed almost since the first demonstration of laser action¹[146, 147, 148]. Attempts to explain these phenomena in a number of systems has initiated an active area of research into the nonlinear behaviour exhibited by these devices. Much of this interest stems from the fact that optical systems are ideal systems in which to study nonlinear behaviour due to their inherent controllability.

One of the simplest proposed models of laser operation, appropriate for single mode, homogeneously broadened lasers, are the Maxwell-Bloch equations[148]. These are a set of three ordinary differential equations describing the evolution of the electric field, E the population inversion, N and the medium polarisation, P . These equations were shown by Haken[149]² to be equivalent to the Lorenz equations, originally proposed as a simple model of nonlinear behaviour in fluid dynamics[152]. However, it was shown that, in general, the requirements for chaotic instability are unobtainable for most practical laser systems since a very high gain, high loss resonator is required with the ability to drive the system from ten to twenty times above threshold[147]. Indeed, Lorenz type chaos has only been observed in

¹By spontaneously generated we refer to pulsations not resulting from usual methods such as mode locking, multimode operation, saturable absorption, etc.

²See references [150, 151] for recent numerical studies of these equations.

Far-Infrared (FIR) NH_3 lasers[148].

The Maxwell-Bloch equations possess three characteristic time-scales corresponding to the decay rates, γ_E , γ_P and γ_N of the electric field, medium polarisation and population inversion, respectively[64]. In the majority of laser systems these decay rates are vastly different leading to a reduction in the number variables required to adequately describe the dynamics of these systems. This leads to a classification of lasers into three main categories[141, 146]. For class C lasers (e.g. Far-Infrared lasers) all three damping constants are of equal magnitude. Thus, adequate description requires all three Maxwell-Bloch equations. In class B lasers (e.g. Ruby, Nd:YAG, CO_2 and semiconductor lasers) the decay rate γ_P for the medium polarisation is significantly larger than the other decay rates. Thus, the polarisation follows the evolution of the other variables and therefore may be adiabatically eliminated³. The system is then fully described by two coupled rate equations for the inversion and electric field. Since the phase space is now reduced to two dimensions, such systems can only exhibit relaxation oscillations between the electric field and the population inversion; chaotic solutions are not possible. However, by the introduction of an additional degree of freedom such as external feedback or pump modulation such solutions become possible. Finally, for class A lasers (e.g. He-Ne, Ar^+ and Dye lasers) the decay rate γ_N is also considerably larger than γ_E and N can also be adiabatically eliminated from the equations of motion. Therefore, this system has a fixed point solution only[153].

Though, the instability threshold for homogeneously broadened lasers is too high to be reached in most practical laser systems, inhomogeneously broadened systems, in general, exhibit lower thresholds for instability. Most experiments on inhomogeneously broadened lasers are performed on the He-Xe laser[148]. Spontaneous pulsations in these lasers, initially investigated by Casperson in the early 1970's, were not explained until 1978[147]. The presence of inhomogeneous broadening implies that different groups of atoms interact with the field independently. This results in different saturation characteristics and different stability criterion. Though considerably more complicated than their homogeneously broadened counterparts, theoretical investigations into the dynamics exhibited by these devices has been largely successful[146, 147, 148].

The first experimental observation of chaotic laser dynamics was using a CO_2 laser with modulated resonator loss[148]. Since CO_2 lasers are class B lasers they cannot spontaneously exhibit chaotic behaviour. A period doubling route to chaos has also been observed in this system for modulated gain and optical resonator frequency[148].

Investigations into the nonlinear dynamics of multimode solid state lasers (e.g. Nd:YAG and $LiNdP_4O_{12}$ (LNP) lasers) have received much attention recently[154, 155, 156, 157]. The standing wave nature of the electromagnetic field in a Fabry-Perot resonator results in

³This is exactly the procedure used in Section 2.4.4 in the derivation of the semiconductor laser rate equations.

spatial hole-burning of the population inversion. This leads to coupling between different longitudinal modes due to cross-saturation of the laser gain⁴. The coupling induced between longitudinal modes leads to an interesting phenomena known as ‘anti-phase’ dynamics; in these systems the intensity noise power spectra for each longitudinal mode exhibit several resonant peaks but the spectrum of the total intensity exhibits only one peak corresponding to the highest relaxation oscillation frequency of the laser. This behaviour results from the fact that the phases of the longitudinal mode intensity oscillations, at all but the largest of the system eigenfrequencies, are arranged in such a way that cancellation is achieved when they are superposed to obtain the total intensity[155, 156].

Finally, we note that all the laser systems discussed so far belong to the class of dissipative nonlinear dynamical systems. However, the free-electron laser is an example of an optical system that exhibits Hamiltonian chaos[146].

4.1.3 Nonlinear Dynamics of Semiconductor Lasers

Large amplitude modulation of the injection current of semiconductor lasers at microwave frequencies has applications in optical fibre communication systems[6, 5, 7, 158, 8] and short pulse generation[1]. In the latter case, the nonlinear nature of the device is exploited in the generation of short pulses[1]. However, for applications in optical communication, ideally we require the laser intensity to exactly follow the evolution of the input modulation current and therefore a linear device is desirable. However, it is often the case that laser diodes are operated under conditions in which the nonlinearities become important leading to distortion of the input signal. Thus, it is imperative that the behaviour of these devices under such operating conditions is well understood. Indeed, investigations into the behaviour of these systems under a variety of operating conditions have been extensive[6, 5, 1, 14, 15, 16, 17, 2, 18, 19, 20, 21, 22, 23, 24, 9, 10, 25, 26, 27, 28, 29, 141, 39, 109, 159, 160].

The majority of numerical studies have been carried out using the single mode rate equations[16, 17, 2, 18, 19, 20, 21, 22, 3, 4, 23, 24, 9, 10, 25, 26, 27, 28, 29]. Under direct modulation of the injection current, these predict a rich variety of behaviours such as period doubling, period tripling, multiple spiking and hysteresis[18, 19, 20, 21, 22, 3, 24, 9, 10, 25, 26, 27]. In most of these studies a period doubling route to chaos was identified. In a recent study [25, 26], a further route to chaos via period doubling, period quadrupling and period tripling leading to chaos was identified with increase in the modulation index. The laser was modulated at approximately twice the relaxation oscillation frequency and noise fluctuations were included in the simulations. This transition does not correspond to any of the known routes to chaos. The reason for this anomalous behaviour was shown to result from the

⁴Coupling between different longitudinal modes can also be achieved by the introduction of a nonlinear frequency doubling crystal within the laser resonator[157].

coexistence of a period doubling route to chaos and a period three attractor in the noise free system. The introduction of noise obscures the hysteresis resulting from the bistable nature of these two attractors and only the period four and period three regions are evident.

Not all of the behaviours predicted in numerical simulations are observed experimentally. Only period doubling[2, 18, 22, 27] and multiple spiking[2, 18] have been observed in current modulated FP lasers where as in DFB lasers, period doubling[4, 27, 28], period tripling[27, 28] and chaos[28] have been observed. To the best of our knowledge, [28] is the only documented evidence of chaos in a directly modulated diode laser. The evidence for chaos was provided by a broad-band peak in the power spectrum of intensity fluctuations⁵. The observed route to chaos was via period doubling, period quadrupling and period tripling to chaos consistent with that demonstrated numerically in [25, 26]. In most experimental investigations, lasers with a bulk active region were used. However, in [27] an extensive investigation of the behaviour of MQW lasers under direct modulation of the injection current both with FP and DFB structures was undertaken. They observed the occurrence of period doubling in the FP laser and period doubling and period tripling in the DFB laser. This behaviour was confirmed in numerical simulations using the single mode rate equations.

The disparity between the nonlinear behaviour predicted by the single mode rate equations and experimental observations has received much attention in the literature. It has been observed that an increase in the damping of the relaxation oscillations acts to reduce the nonlinear behaviour. This arises from the fact that the relaxation oscillation is intimately connected with the occurrence of bifurcations⁶, and therefore the larger the damping the more difficult it is to drive a system to instability. In [16, 2] it was postulated that instabilities would only occur if the decay time of the relaxation oscillations exceeded the modulation period since in this case strong interferences occur between successive periods. Physical causes of damping include the fraction of spontaneous emission coupled into the laser mode (β)[34, 15, 9, 27], gain saturation[16, 17], carrier diffusion[14, 15] and carrier density dependent spontaneous lifetime[29]. Also it has been demonstrated that the presence of intrinsic laser noise acts to obscure higher order period doubling in an experiment[22].

In contrast to the extensive work on the single mode equations there are few studies of the dynamics predicted by the multimode rate equations. The emphasis in most studies has been on the transient behaviour[31, 32, 33] and spectral characteristics[1],[30, 14, 34, 35]. That FP lasers become multimode under direct current modulation is well known[30, 14, 34]. However, as shown by Tarucha and Otsuka[1], this is dependent not only upon the amplitude but also the frequency of the modulation. In particular, modulating at a frequency close to the relaxation oscillation leads to a significant increase in the number of longitudinal

⁵Though this in itself does not constitute proof of chaotic behaviour it gives a strong indication that such behaviour does exist.

⁶We explore this further in chapter 5.

modes with the spectrum shifting towards the shorter wavelength modes. This effect was shown to result from the spectral shift of the gain with carrier density⁷. Since the spectral gain shift arises from the population of higher energy states upon an increase in the carrier density such effects are referred to as band-filling[1, 36]. As direct current modulation leads to dynamical variation of the carrier density, we expect band-filling effects to be important in the dynamical description of these devices. Despite this many of the multimode rate equations models given in the literature do not include band-filling effects⁸.

Subharmonic pulsation of the intensity (period doubling) for modulation frequencies near twice the relaxation oscillation frequency has been predicted using multimode rate equations[1] and subsequently observed in a FP laser[162]. However, to the best of our knowledge, no previous investigations into the bifurcation scenarios predicted by multimode rate equations models with change in modulation frequency or amplitude, exist. Thus, in this chapter we compare the bifurcation scenarios predicted by single mode and multimode rate equations models (with and without band-filling) in order to demonstrate the importance of both multimode operation and band-filling in the dynamical description of FP semiconductor lasers.

For completeness we briefly mention studies performed on the behaviour of self-pulsing and external cavity lasers. The dynamics of self-pulsing lasers under direct current modulation show interesting behaviour[163, 164, 165, 166, 167]. In these systems, self-pulsations are induced, in a nominally stable laser, by catastrophic optical damage[165, 166]. Under direct modulation of the injection current, these lasers exhibit regions of frequency locking between the pulsation frequency and the modulation frequency. By keeping the ratio between these two frequencies fixed at a constant irrational value (usually the golden mean), and increasing the modulation amplitude, a quasiperiodic route to chaos is observed[165].

The study of the effects of optical feedback on semiconductor lasers has been largely motivated by the application of these devices in optical communication systems and laser disc applications[5]. Unintentional optical feedback into such a system is undesirable as it can lead to degradation in system performance[6]. However, by anti-reflection (AR) coating the front facet of the laser, high levels of optical feedback can lead to improved performances, specifically linewidth reduction and robust single frequency operation.

Optical feedback from an external mirror can be considered as the delayed feedback of laser emission back into the active region of the laser. This leads to the mathematical description of the laser in terms of a delay-differential equation[39]; such a system is infinite dimensional. The description of these devices is further complicated by the fact that semiconductor lasers have a unique characteristic in that there is strong coupling between

⁷See section 2.4.5.

⁸See for example [6, 33, 30, 14, 34, 35, 161, 15]. In [33, 30, 14] a Lorentzian rather than a quadratic approximation to the gain was used.

the refractive index and the laser gain[36]; the fact that the laser gain is asymmetric with respect to the peak results in an anomalous dispersion effect at the laser frequency based on the Kramers-Kronig relationship[109, 63]. This is expressed by the linewidth enhancement factor, α given by[109]

$$\alpha = -\frac{4\pi}{\lambda} \left(\frac{\partial n / \partial N}{\partial G / \partial N} \right) \quad (4.1)$$

where G is the laser gain, N is the carrier density, λ is the lasing wavelength and n is the refractive index. The unusually large value of α exhibited by semiconductor lasers leads to effects which are unique to these systems. The behaviour of external cavity semiconductor lasers are frequently modelled by the Lang-Kobayashi rate equations[39], which are appropriate for low levels of feedback. Other models have been developed for higher feedback regimes[168].

Several regimes of operation have been identified based on the level of optical feedback [72]. In regime I, corresponding to very low levels of optical feedback, the laser operates stably in a single longitudinal mode. The spectral width is broadened or narrowed relative to the solitary laser, depending on the phase of the optical feedback. In regime II, several cavity modes are possible and this regime is characterised by linewidth broadening caused by noise induced mode hopping between external cavity modes[169]. Regime III is characterised by stable, single frequency operation. Significant linewidth reduction with increased levels of optical feedback is observed in this regime independent of the phase of the feedback. With further increase of optical feedback, coherence collapse occurs (regime IV) which is characterised by a dramatic increase in the laser linewidth to several tens of Giga-Hertz with the laser operating on many external cavity modes. In this regime, there is a tremendous increase in the relative intensity noise. Coherence collapse is the result of strong instabilities, which occur due to the highly nonlinear nature of the system, and the dynamical behaviour in this regime is very complicated. The transition from III-IV has received much attention in the literature; particular attention has been given to the approach to instabilities[170, 171, 172, 173], noise characteristics[169, 73, 174, 175] and low frequency fluctuations[176, 177, 178, 179]. With further increase of feedback, stable single mode operation is achieved (regime V). Due to the high levels of optical feedback required, regime V is generally only accessible with lasers that have an Anti-Reflection coating on the front facet.

The dynamics of external cavity semiconductor lasers, especially in the coherence collapsed state, is extremely complicated and work into the behaviour of these devices is an ongoing area of research. Finally, we remark that all three routes to chaos have been observed in these systems for different operating conditions[180, 181, 182, 141].

4.2 Attractors

The purpose of this section is twofold. The first is to introduce the concept of an attractor which is extremely important for dissipative nonlinear systems. The second is to illustrate the various ways of representing data from a multivariable nonlinear dynamical system with numerical solutions from the single mode rate equations. All numerical integrations are performed using a fourth order Runge-Kutta algorithm[183]. Details of the integration procedure are included in Appendix E.

For the sake of brevity, we write the semiconductor rate equations in the general form

$$\begin{aligned}\dot{x}_1 &= Q_1(x_1, x_2, \dots, x_N, \lambda) \\ \dot{x}_2 &= Q_2(x_1, x_2, \dots, x_N, \lambda) \\ &\vdots \\ \dot{x}_N &= Q_N(x_1, x_2, \dots, x_N, \lambda)\end{aligned}\tag{4.2}$$

where λ denotes one or more control parameters. These equations can be written in vector notation as follows:

$$\dot{\vec{x}} = \vec{Q}(\vec{x}, \lambda), \quad \vec{x} \in R^N.\tag{4.3}$$

For the multimode rate equations $\vec{x} = (N, P_{-(M-1)/2}, \dots, P_0, \dots, P_{(M-1)/2}, \psi)$ where ψ is the phase of the sinusoidal injection current. We include ψ as a dynamical variable in order to convert the rate equations to an autonomous system⁹; ψ satisfies the following differential equation, $\dot{\psi} = 2\pi f$, where f is the driving frequency.

There are several ways of representing data from multivariable dynamical systems. Typical representations include time series, phase space plots, Poincare sections and power spectra[13]. In the following we show examples of these representations using solutions of the single mode rate equations (equations 2.162 and 2.163) since, in this case, the trajectories are easiest to visualise as they reside in only three dimensions. For this system the phase space (N, P, ψ) is a three dimensional periodic manifold in which ψ is the periodic variable: $\psi \in [0, 2\pi]$. We represent points in this space using the cylindrical co-ordinates $(P \cos(\psi), P \sin(\psi), N)$. Some typical solutions of the single mode rate equations are shown in Figs. 4.2 to 4.5 illustrating the different ways of representing the same data. The time series representation is simply a graph of the evolution of the dynamical variables, $x(t)$,

⁹Autonomous systems are those in which the dynamical equations have no explicit time dependence[183]. The rate equations, are a non-autonomous system (meaning explicit time dependence) for a time dependent injection current.

versus integration time. The power spectrum is given by[184]

$$S(\nu) = \frac{1}{\tau} \left| \int_0^\tau x(t) e^{i2\pi\nu t} dt \right|^2 \quad (4.4)$$

where τ is the length of the data set. The time series and power spectrum are shown in parts (a) and (b) of Figs. 4.2 to 4.5, respectively. The phase space plot, in which the system trajectories are plotted in the space of the dynamical variables, is shown in part (c).

The Poincare section is obtained by stroboscopically sampling the dynamical variables at the period, T , of the driving term[183]. This is equivalent, in the case of the single mode rate equations, to plotting the variables (N, P) in a plane, Σ , in the cylindrical phase space, defined by a fixed phase of the driving term. The three dimensional phase space and Poincare section are illustrated schematically in Fig. 4.1. Poincare section plots are shown in part (d) of Figs. 4.2 to 4.5. At this point it is convenient to introduce the Poincare map $\vec{P}(\vec{x})$ which maps points in the plane Σ onto other points in the plane Σ under the action of the flow (equations 4.3). The Poincare map is defined¹⁰

$$\vec{x}_{k+1} = \vec{P}(\vec{x}_k) \quad (4.5)$$

where $\vec{x}_k = (x_1(kT + t_o), x_2(kT + t_o), \dots, x_N(kT + t_o))$ and $\psi_o = 2\pi f t_o$ defines the Poincare plane. An analytic form for $\vec{P}(\vec{x})$ is unobtainable for most dynamical systems. However, successive iterates can be obtained numerically by integrating equations 4.3 for one period of the modulation. One of the advantages of using the Poincare map is that it reduces the dimensionality of the system by one. It is also useful for characterising the stability of periodic cycles as we will discuss in Section 5.2.1.

We are primarily interested in the long term (post transient) behaviour of solutions to equation 4.3. Long term solutions of a dissipative system, which are stable, are termed attractors[13, 185]. However, before we discuss the types of attractors typically encountered in the single mode rate equations under current modulation, we first consider the dynamical evolution for a constant normalised bias current of $I_b = 1.5$. Other parameter values used in this section are given in Table 4.1[3]. Figure 4.2(a) shows the evolution of the photon density with time. In this case we plot the transient behaviour as the system relaxes to its steady state. The observed damped oscillatory behaviour is the well known phenomenon known as relaxation oscillation[70, 6]. The power spectrum, calculated from the data in part (a), is shown in Fig. 4.2(b). A peak at 1.5GHz is observed corresponding to the resonant frequency¹¹. In Fig. 4.2(c) we plot the trajectory in phase space. Since, in this example, no AC modulation is applied we use the two dimensional phase space, (N, P) . We observe

¹⁰The index k indicates the k^{th} iterate of $\vec{P}(\vec{x})$ not the k^{th} component of the vector, $\vec{x}$.

¹¹Equation D.17 gives the value of f_{RO} as a function of the rate equation parameter values.

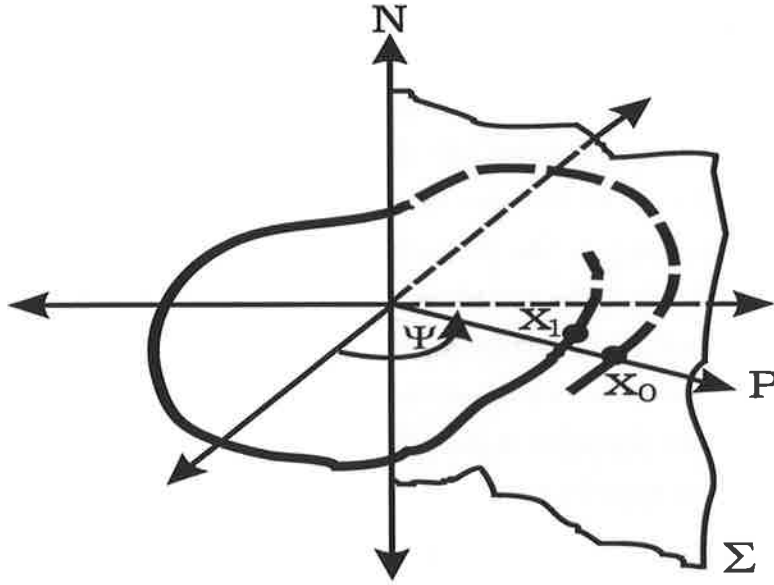


Figure 4.1: An illustration of the three dimensional cylindrical phase space for the single mode rate equations also showing a Poincare plane (Σ) and a typical trajectory.

Table 4.1: Parameter values for the single mode rate equations used in Section 4.2

Symbol	Value
τ_e	$3 \times 10^{-9} \text{ s}$
τ_p	$6 \times 10^{-12} \text{ s}$
δ	0.692
β	10^{-4}
I_b	1.5

that the trajectory spirals into a point in the (N, P) plane; thus the attractor is a fixed point. Even though no modulation is applied in this example we generate a Poincare section assuming a modulation frequency of 2GHz as an illustration of the principles (i.e. we sample the trajectory at intervals of 0.5ns). The data is shown in Fig. 4.2(d). In this case we observe that the transient behaviour manifests itself in a sampled spiral in the Poincare plane. Each consecutive iterate is rotated by $3\pi/2$ relative to the previous one. This reflects the fact that the ratio¹² $2\pi(f/f_{RO}) = 2\pi(2/1.5) = 3\pi/2$.

For the data shown in Fig. 4.3, a modulation index of $m = 0.55$ and modulation frequency of 0.4GHz is applied. We have allowed the system to reach steady state in the following examples; the transient part of the orbit is not shown. The attractor shown in Fig. 4.3 is a periodic cycle. The time series, Fig. 4.3(a), shows a spiking behaviour, within

¹²Refer to section 5.2.1 for details.

one modulation period, that is related to the relaxation oscillation behaviour observed in Fig. 4.2[23]. The period of the cycle can be ascertained from the power spectrum, Fig. 4.3(b), which shows that the lowest frequency peak occurs at 0.4GHz corresponding to the driving frequency. Hence the periodic attractor has the same period as the driving current. The phase space plot and Poincare section are shown in Fig. 4.3(c) and (d), respectively. The fact that there is only one point in the Poincare plane is also indicative of the fact that we have a periodic cycle with the same period as the driving current; in this case the periodic cycle corresponds to a fixed point of the Poincare map. Figure 4.4 also shows a periodic attractor. However, the period is four times that of the driving current in this case.

In Fig. 4.5 a chaotic attractor is shown[19]. There are several indications of chaotic behaviour. The time series appears quite random and there are broad-band features in the power spectrum. The phase space plot and Poincare section also indicate the complicated nature of the attractor. Figure 4.5(d) shows that the attractor intersects the Poincare plane at a large number of points revealing the elaborate structure of the trajectory.

4.3 Bifurcation Diagrams

In this section we show bifurcation diagrams for the single mode and multimode rate equations in order to compare the bifurcation scenarios predicted in each case. Bifurcation diagrams are calculated by stroboscopically sampling the dynamical variables at the period of the driving current, $T = 1/f$, as a control parameter is varied in small steps; this is equivalent to plotting the values of the variables in the Poincare plane. For each value of the control parameter the equations are integrated until the steady state behaviour is reached and the transients discarded. For each subsequent value of the control parameter, the last data point from the previous control parameter value is used as an initial condition. In general the modulation frequency or modulation index is varied. Diagrams are calculated for both increasing and decreasing values of the control parameter[183].

4.3.1 Comparison of Single Mode and Multimode Predictions

For comparison, bifurcation diagrams for the normalised carrier density, N , are given for both the single mode and multimode rate equations¹³. Parameter values used in this section are given in Table 4.2 and are typical for an AlGaAs semiconductor laser[16, 20, 23, 3, 1]. For AlGaAs FP lasers, spontaneous emission parameters are typically in the range 10^{-4} to 10^{-5} [15, 16, 17, 2, 18, 19, 20, 21, 22, 23, 3, 4, 9]. In this section we set $\beta = 10^{-4}$; we investigate

¹³The single mode equations are equations 2.162 and 2.163 and the multimode rate equations are equations 2.159 and 2.160 of Section 2.4.6. The single mode rate equations can be obtained from the multimode rate equations by setting $k = 0$ and the number of modes to one.

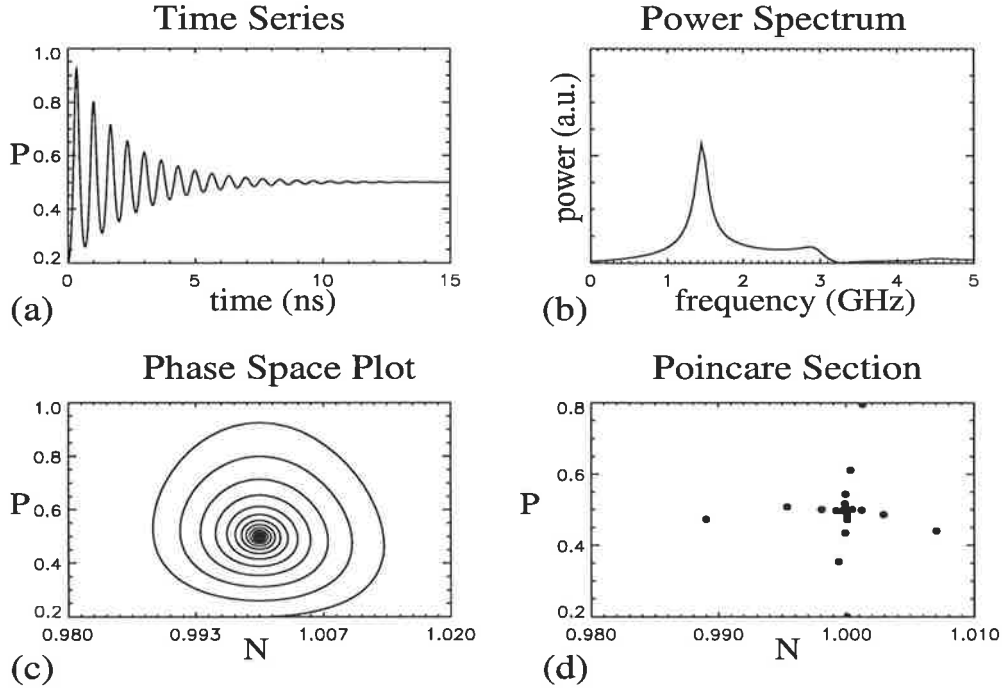


Figure 4.2: A typical solution of the single mode rate equations, in the absence of injection current modulation, showing the system relaxing to its steady state. The time series is shown in (a), the power spectrum in (b), the phase space plot in (c) and Poincare section in (d). The power spectrum, shown in (b), is plotted using a linear scale on the vertical axis.

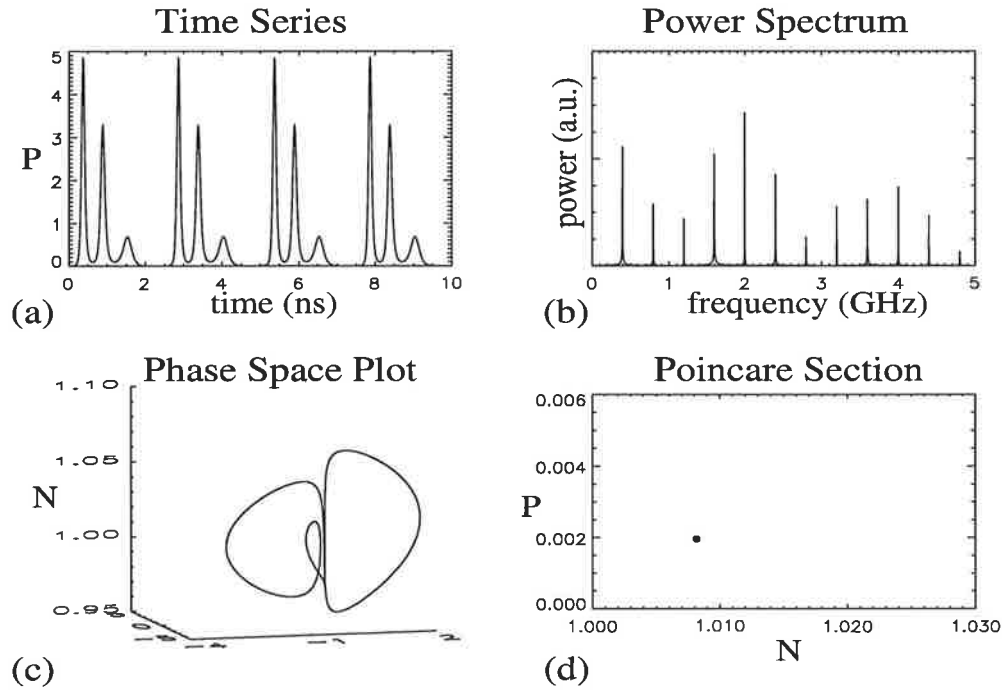


Figure 4.3: Time series (a), the power spectrum (b), the phase space plot (c) and Poincare section (d) for a periodic solution of the single mode rate equations, with the same period as the modulation period, for a modulation index of $m = 0.55$ and a modulation frequency of $f = 0.4\text{GHz}$. The power spectrum, shown in (b), is plotted using a linear scale on the vertical axis.

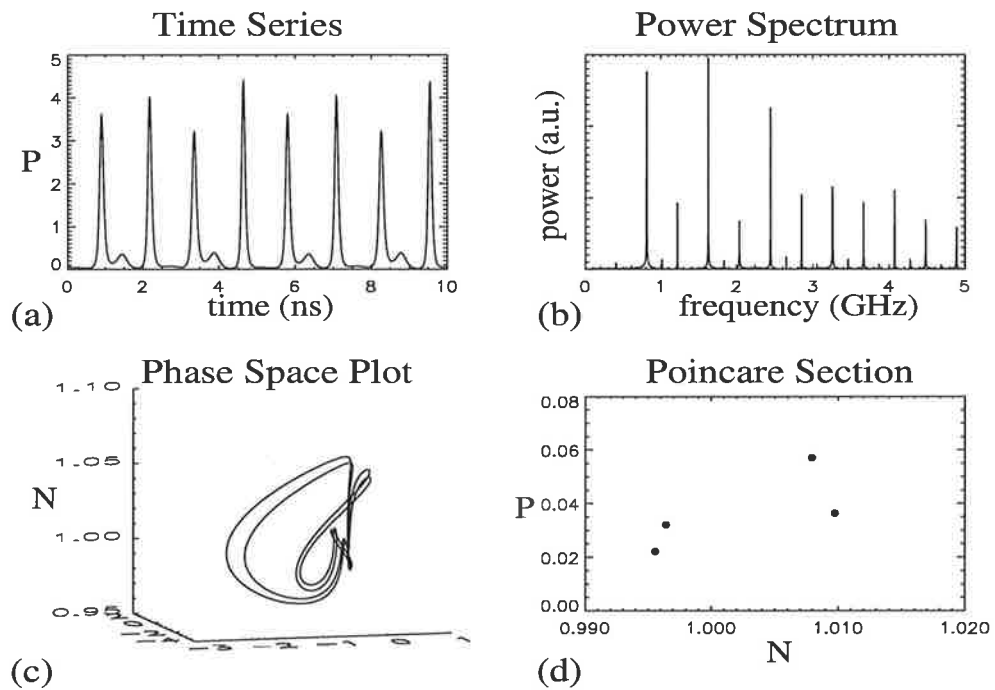


Figure 4.4: Time series (a), the power spectrum (b), the phase space plot (c) and Poincare section (d) for a periodic solution with a period four times that of the modulation period. The modulation index is $m = 0.55$ and the modulation frequency is $f = 0.82\text{GHz}$. The power spectrum, shown in (b), is plotted using a linear scale on the vertical axis.

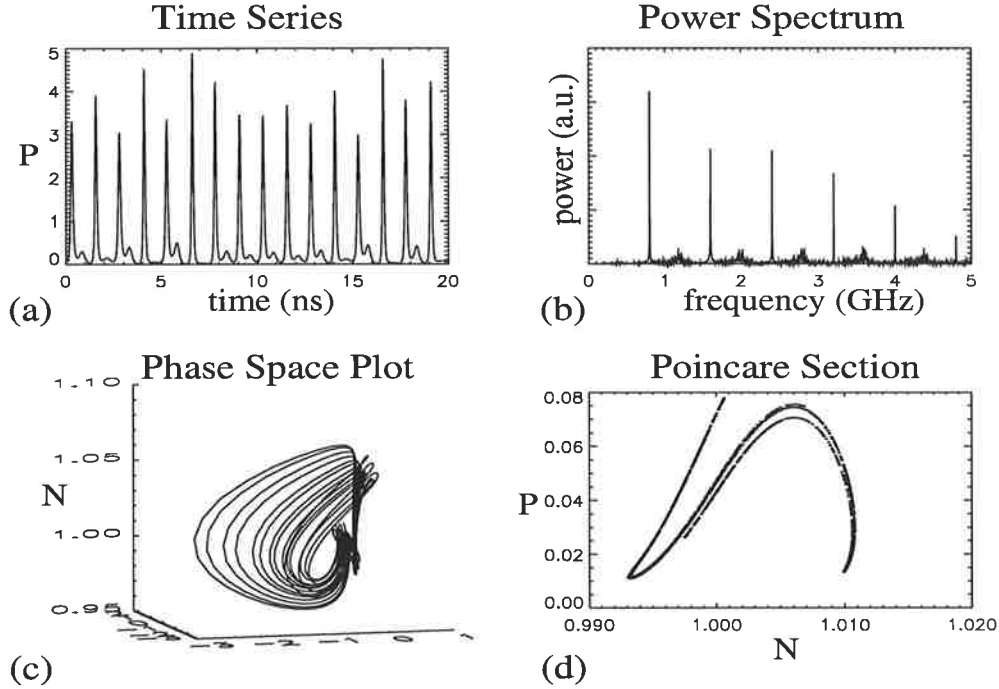


Figure 4.5: Time series (a), the power spectrum (b), the phase space plot (c) and Poincare section (d) for a chaotic solution. Parameter values are $m = 0.55$ and $f = 0.8GHz$. The power spectrum, shown in (b), is plotted using a linear scale on the vertical axis.

the effects of varying β in section 4.5. We use 25 modes in all multimode calculations since we have found this number to be sufficient for all operating conditions considered here.

Bifurcation diagrams shown in Figs. 4.6 to 4.9, correspond to the values of $I_b = 1.3, 1.5, 1.7$ and 1.9 , respectively. In each of these figures bifurcation diagrams for the single mode rate equations[19, 10, 25] are shown in the left column and those for the multimode equations in the right column. In order to show the variation with increasing modulation index, four cases are shown for each system of equations corresponding to $m = 0.25, 0.5, 0.75$ and 1.0 .

Before commenting on specific features of these bifurcation diagrams, we remark that the bifurcation scenarios predicted by the single mode rate equations and the multimode rate equations are obviously different. In particular, a period doubling route to chaos is observed in many of the single mode bifurcation diagrams but not in the multimode system.

In the single mode system, hysteresis, period doubling bifurcations and period doubling to chaos occurs[16, 2, 19, 23, 9, 10, 25, 27]. Generally, the bifurcation diagrams exhibit more complicated behaviour as the modulation index is increased. An interesting feature one can observe from these diagrams is an alternating sequence of period doubling and hysteresis regions as the modulation frequency is varied[10]. This is particularly evident in Figs. 4.7(b), 4.8(c) and 4.9(d). We also note that the observed period doubling regions

Table 4.2: Parameter values for the rate equations[16, 20, 23, 3, 1].

Symbol	Value	Model
τ_e	3×10^{-9} s	both
τ_p	6×10^{-12} s	both
δ	0.692	both
$\Delta\lambda_g$	20 nm	multimode
$\delta\lambda$	0.4 nm	multimode
k	35 nm	multimode
λ_o	830 nm	multimode
M	25	multimode

become smaller and more closely spaced for decreasing modulation frequency, in these diagrams, with the largest period doubling region occurring for modulation frequencies greater than the relaxation oscillation frequency.

The bifurcation diagrams for the multimode system are considerably less complicated than the corresponding ones for the single mode system. In particular we do not observe a period doubling route to chaos such as that observed in many of the single mode diagrams. However, in the multimode system we observe large period doubling regions for modulation frequencies near twice the relaxation oscillation frequency sometimes leading to period quadrupling. Hysteresis is also observed (see Fig. 4.9(e) for example) though it is less prevalent than in the single mode system. The multimode system also predicts the existence of a Hopf bifurcation (see Figs. 4.8(f) and 4.9(e) and (f)); we discuss this type of bifurcation in Section 4.6.3.

4.3.2 Effects of Band-filling

The differences between the predicted bifurcation behaviour for the single mode and multimode rate equations are the result of both the inclusion of multiple longitudinal modes and the non-zero value of the band-filling parameter, k . We remark that $k = 0$ is consistent with the multimode model used in [6, 33, 30, 14, 34, 35, 161, 15]. However, [33, 30, 14] use a Lorentzian rather than a quadratic approximation for the gain curve. To illustrate the importance of band-filling we show bifurcation diagrams for the multimode rate equations (Fig. 4.10) for the same parameters as in Figs. 4.8(e)-(h) (corresponding to $I_b = 1.7$) but with the band-filling parameter k equal to zero. Without band-filling, the resulting bifurcation diagrams (Fig. 4.10) resemble the single mode solutions (Figs. 4.8(a)-(d)) more closely than those for the multimode solutions with band-filling (Figs. 4.8(e)-(h)). However, Figs. 4.10(c) and (d) are distinguished by the absence of the period doubling cascade to chaos predicted by their respective single mode counterparts, Figs. 4.8(c) and (d). Therefore, the

Bifurcation Diagrams

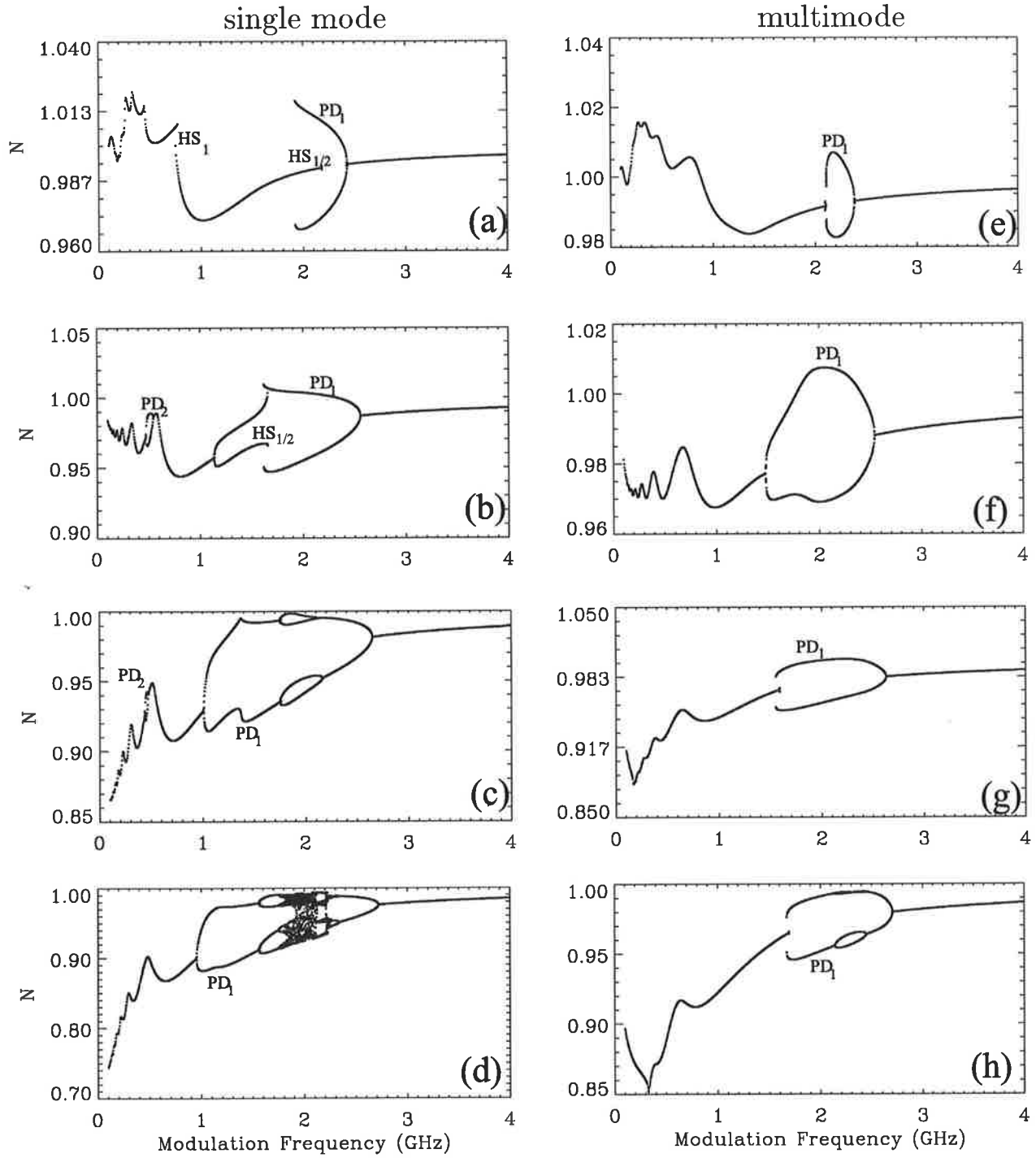


Figure 4.6: Calculated bifurcation diagrams for $I_b = 1.3$. The single mode (multimode) diagrams are shown in the left (right) column. From top to bottom diagrams correspond to modulation indices of 0.25, 0.5, 0.75 and 1.0. PD refers to period doubling, HS refers to regions of hysteresis and H indicates a Hopf bifurcation.

Bifurcation Diagrams

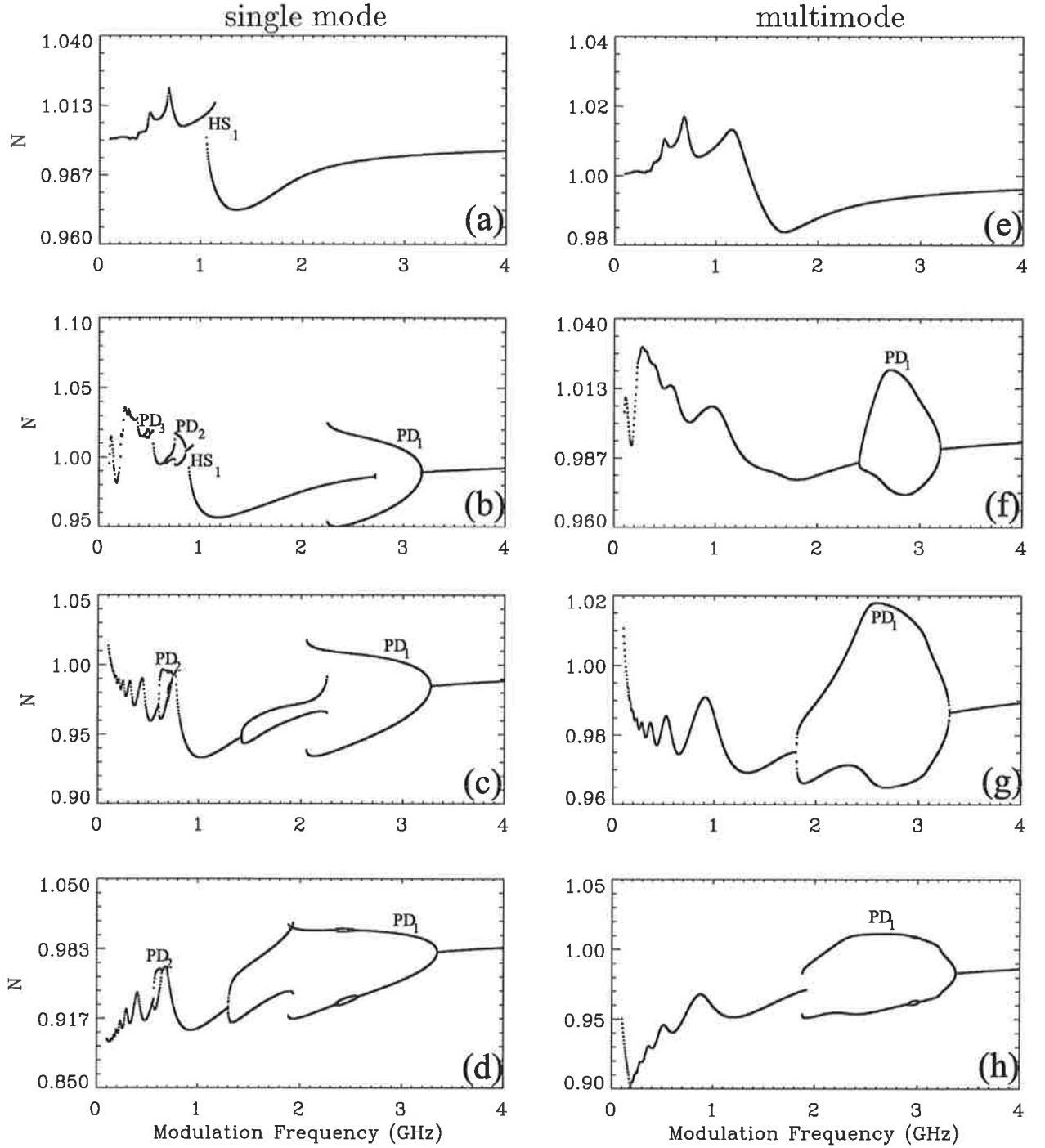


Figure 4.7: Calculated bifurcation diagrams for $I_b = 1.5$. The single mode (multimode) diagrams are shown in the left (right) column. From top to bottom diagrams correspond to modulation indices of 0.25, 0.5, 0.75 and 1.0. PD refers to period doubling, HS refers to regions of hysteresis and H indicates a Hopf bifurcation.

Bifurcation Diagrams

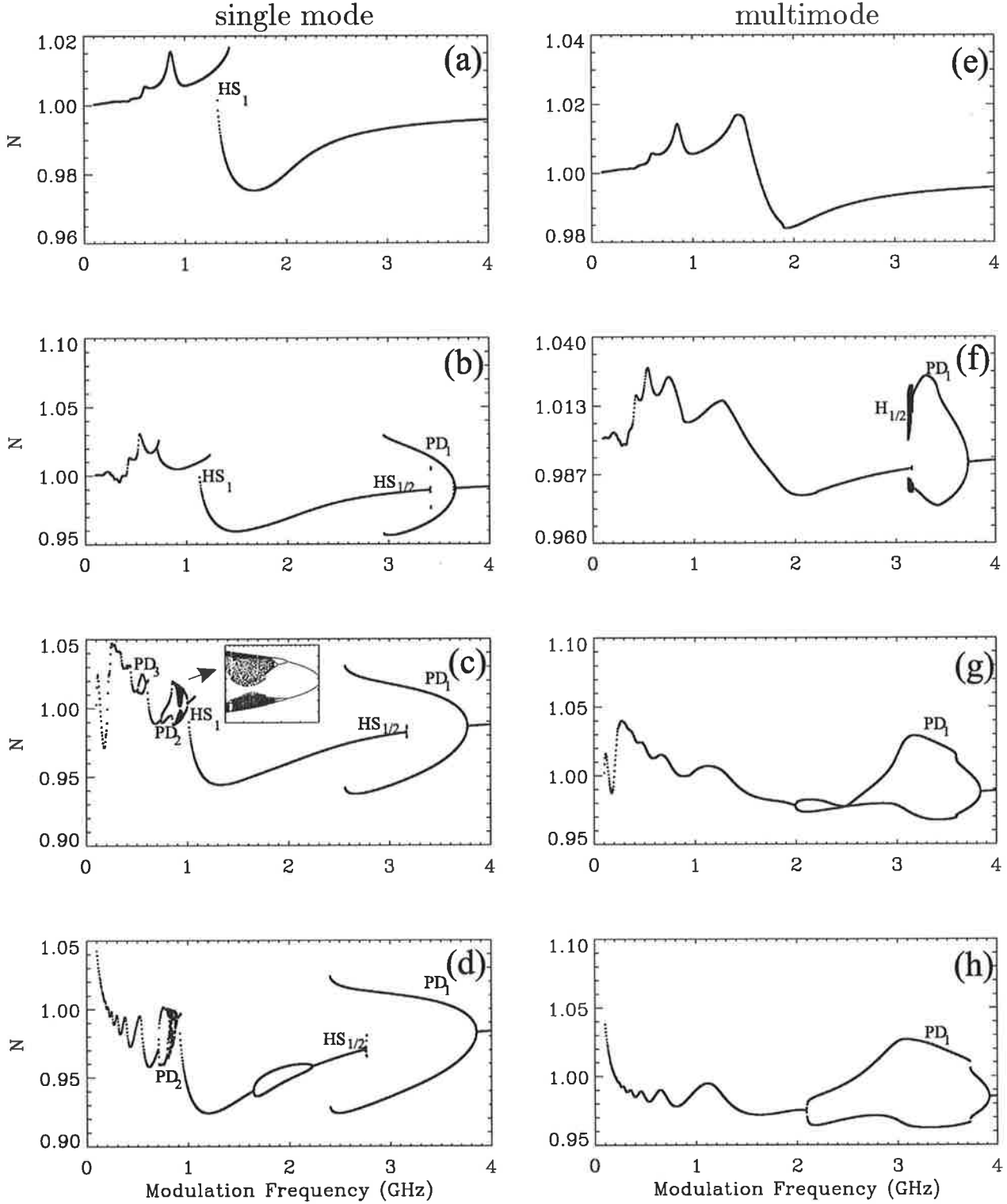


Figure 4.8: Calculated bifurcation diagrams for $I_b = 1.7$. The single mode (multimode) diagrams are shown in the left (right) column. From top to bottom diagrams correspond to modulation indices of 0.25, 0.5, 0.75 and 1.0. PD refers to period doubling, HS refers to regions of hysteresis and H indicates a Hopf bifurcation.

Bifurcation Diagrams

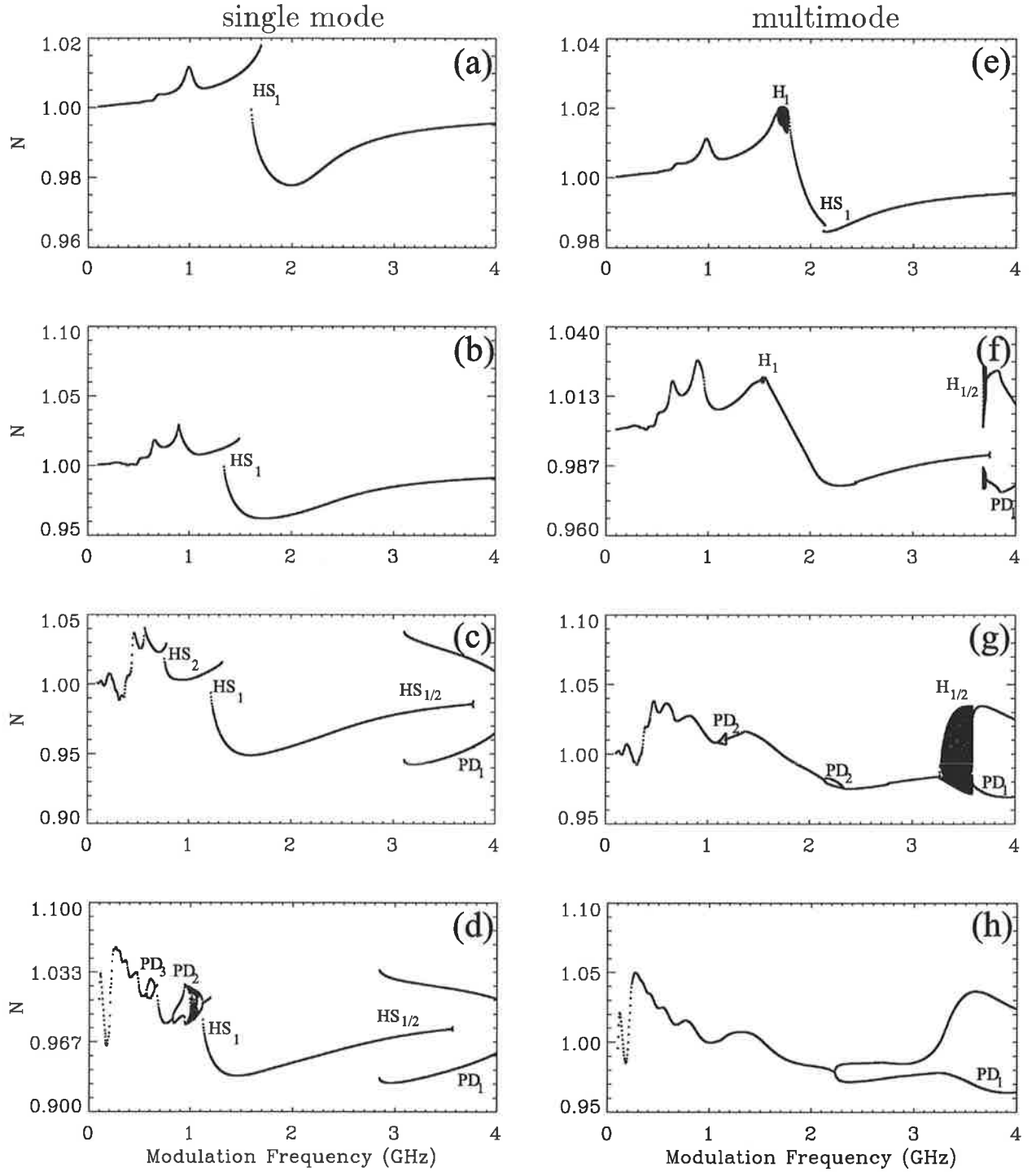


Figure 4.9: Calculated bifurcation diagrams for $I_b = 1.9$. The single mode (multimode) diagrams are shown in the left (right) column. From top to bottom diagrams correspond to modulation indices of 0.25, 0.5, 0.75 and 1.0. PD refers to period doubling, HS refers to regions of hysteresis and H indicates a Hopf bifurcation.

Bifurcation Diagrams

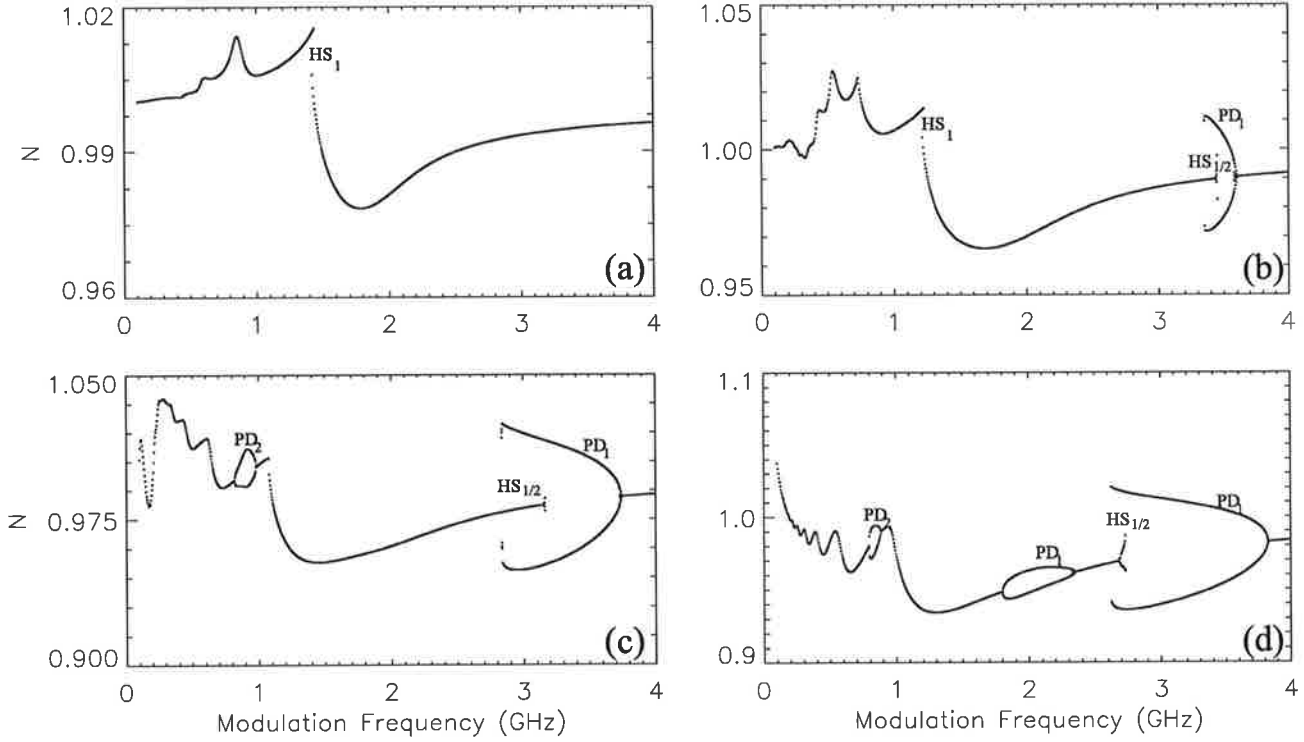


Figure 4.10: Calculated bifurcation diagrams for the multimode rate equations without band-filling effects ($k = 0$) for a bias of $I_b = 1.7$. Figures (a) to (d) correspond to modulation indices of 0.25, 0.5, 0.75 and 1.0, respectively. PD refers to period doubling, HS refers to regions of hysteresis and H indicates a Hopf bifurcation.

behaviour shown in Fig. 4.10 lies somewhere between that depicted in Figs. 4.8(a)-(d) and Figs. 4.8(e)-(h)¹⁴.

Figures 4.8 and 4.10 clearly verify that the inclusion of band-filling alters the dynamical behaviour. This effect may be understood from consideration of the gain each mode experiences over typical variations of the carrier density. In Fig. 4.11 we plot the normalised modal gains (we use the co-efficient of P_j in equations 2.159 and 2.160) versus normalised carrier density, N , for the modes corresponding to $j = 0, \pm 3$ in equation 2.154; Figs. 4.11(a) and (b) correspond to the modal gains with and without band-filling respectively. We have considered a 10% variation in N from the steady state value since numerical calculations show that typical carrier density variations are of this order. It is evident from the dissimilarity of Figs. 4.11(a) and (b) that we would expect different dynamical behaviour to be predicted from these models.

¹⁴For k -values intermediate between $k = 0$ and $350\text{\AA}$ we would expect a transition region between the two bifurcation structures depicted in Figs. 4.8(e)-(h) and Fig. 4.10.

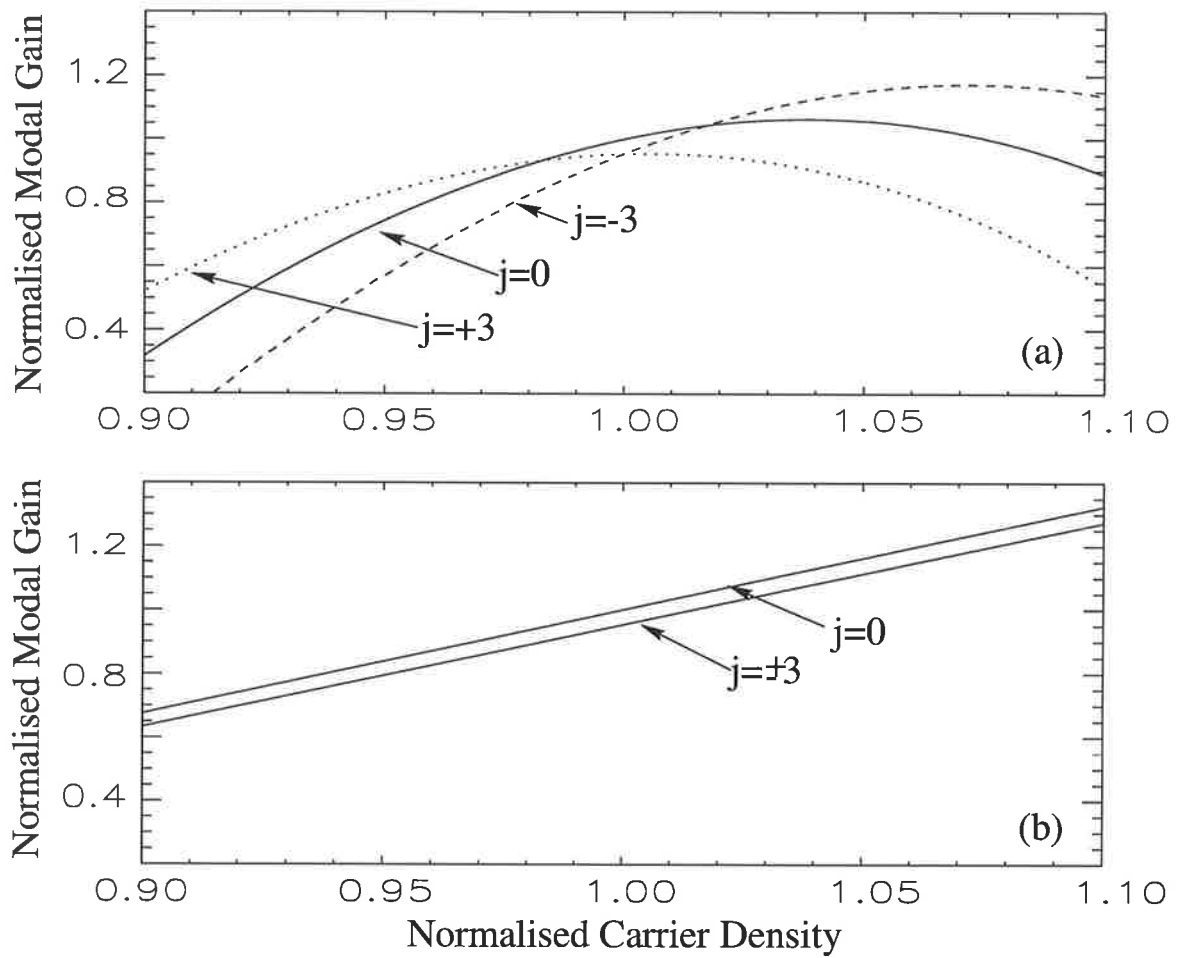


Figure 4.11: The normalised modal gain versus N for modes $0, \pm 3$. Parts (a) and (b) correspond to gain models with and without band-filling, respectively. The gain for the zeroth mode in part (b) is equivalent to the single mode gain.

4.4 Modal Behaviour

4.4.1 Numerical Results

In terms of the bifurcation behaviour, the bifurcation diagrams for each longitudinal mode show qualitatively similar behaviour to those for the carrier density and are omitted here. It is illustrative to consider, however, how the time-averaged modal photon densities vary as the modulation frequency is tuned. To calculate the time averaged photon densities for a periodic solution, we average the modal photon densities over the period of the cycle. In order to compare with experimental data we calculate the longitudinal mode spectra by multiplying the photon densities for each longitudinal mode by the spectral resolution function of the spectrometer used in experiments.

Figure 4.12 shows time averaged longitudinal mode photon density variation with modulation frequency for $I_b = 1.7$ and $m = 0.25, 0.5$ and 0.75 in parts (a), (b) and (c), respectively. For modulation frequencies near f_{RO} the spectrum becomes strongly multimode with the spectrum shifting towards the shorter wavelength modes; this phenomenon has been predicted theoretically and observed experimentally by Tarucha and Otsuka[1]. We also observe multimode operation when the harmonics of f are near f_{RO} ; most notable are the regions near $f \approx 0.85\text{GHz}$ and $f \approx 0.55\text{GHz}$ which correspond to the second and third harmonics of the driving frequency near f_{RO} , respectively. As shown in Fig. 4.12, increasing the modulation index increases the number of operating modes, when a harmonic of the modulation frequency is coincident with the relaxation oscillation frequency. However, for the higher modulation index ($m = 0.75$), the number of oscillating modes increases even for modulation frequencies not close to a resonance. Thus, the frequency dependence of the modulation behaviour is most easily resolved when the modulation index equals 0.5 for the case considered here ($I_b = 1.7$).

Figure 4.13 corresponds to the same situation as Fig. 4.12(b) except the band-filling parameter, k , is set equal to zero¹⁵. Since the modal photon densities are calculated for increasing modulation frequency only, the hysteresis region (see Fig. 4.10(b)) is not apparent. Again we observe multimode oscillation for modulation frequencies near f_{RO} , $f_{RO}/2$, $f_{RO}/3$, etc. However, the spectra are now symmetric about the central mode. Therefore, the obvious asymmetry (of Fig. 4.12) is the result of band-filling effects.

At a resonance, the excursion of the carrier density from its steady state value is largest. This is shown in Fig. 4.14 in which we plot the time dependent behaviour of the normalised carrier density, N , for a bias of $I_b = 1.7$ and modulation index of $m = 0.5$. The solid curve corresponds to a modulation frequency of $f = 1.5\text{GHz}$ and lies within the region of multimode operation of Fig. 4.12(b). The dotted curve corresponds to a modulation

¹⁵The same parameter values as in Fig. 4.10(b) are used in Fig. 4.13.

Longitudinal Mode Spectra vs Modulation Frequency

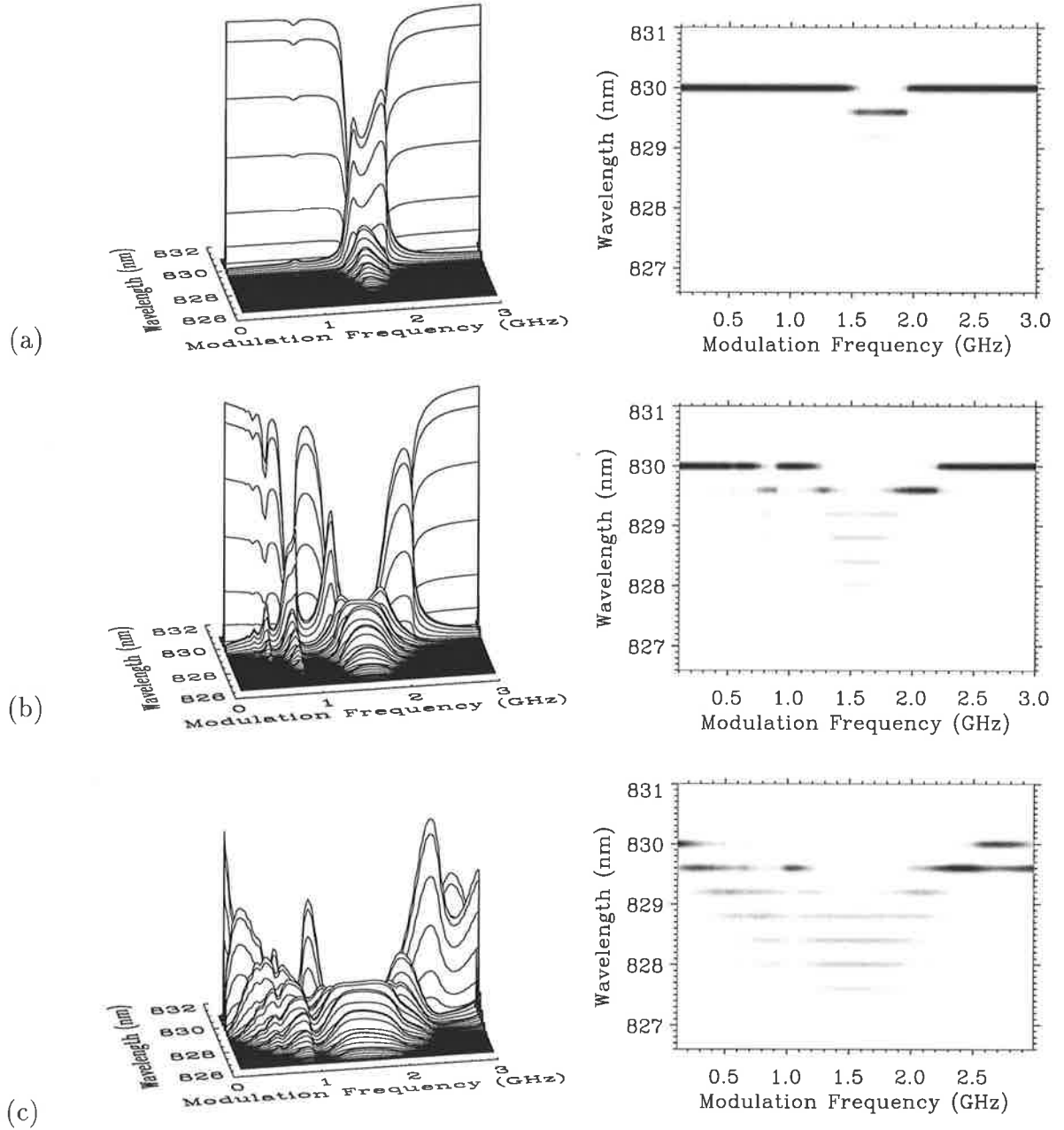


Figure 4.12: Calculated longitudinal mode spectra versus modulation frequency, shown as both a surface plot (left column) and image plot (right column), for a bias of $I_b = 1.7$. Figures (a), (b) and (c) correspond to modulation indices of 0.25, 0.5 and 0.75, respectively.

Longitudinal Mode Spectra vs Modulation Frequency

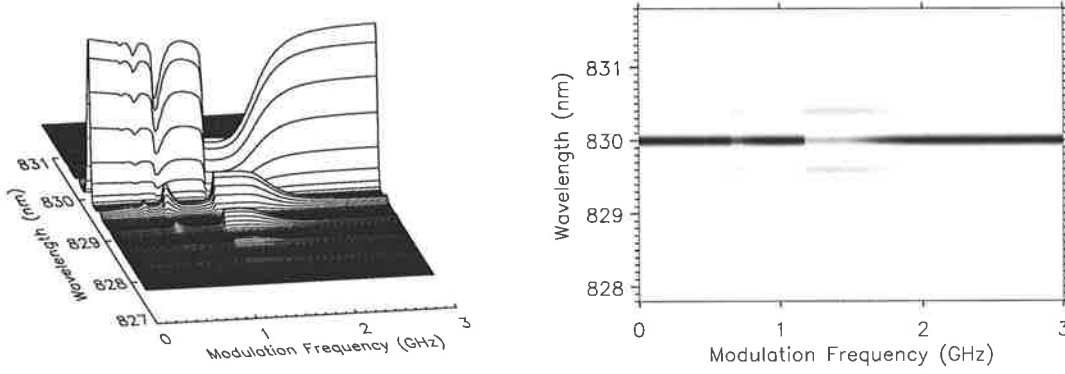


Figure 4.13: Calculated longitudinal mode spectra versus modulation frequency for the multimode model without band-filling, shown as both a surface plot (left) and image plot (right). Other parameter values are $I_b = 1.7$ and $m = 0.5$.

frequency of 2.1GHz which lies just outside the multimode region of Fig. 4.12(b). We observe that for $f = 1.5\text{GHz}$ the peak carrier density is larger than for $f = 2.1\text{GHz}$. Since the peak gain shift is proportional to N we expect that, near a resonance, the shorter wavelength modes will transiently experience higher gain and thus we expect them to exhibit increased power. However, we note that, though band-filling effects are responsible for the observed asymmetry of the longitudinal mode spectra in Fig. 4.12, the presence of multimode operation does not depend on band-filling effects alone since it is observed in both Figs. 4.12 and 4.13. Finally we note that it is a general characteristic of a resonance that the dynamical variables experience a π phase shift relative to the input modulation as the modulation frequency is tuned from 0 to ∞ ; at the resonant frequency the dynamical variables and the input modulation are $\pi/2$ out of phase. This phase shift may also play a role in the observed multimode behaviour.

In Fig. 4.15(a) and (b) we plot the modal photon densities versus time for the modulation frequencies of $f = 2.1\text{GHz}$ and $f = 1.5\text{GHz}$, respectively. Other parameter values are the same as those in Fig. 4.14. As in Fig. 4.12 we multiply the photon density for each longitudinal mode by the spectral response function of the spectrometer used in experiments to construct the longitudinal mode spectra; this allows us to conveniently plot the data in three dimensions. Data is plotted as both a surface and an image plot. The time dependent behaviour of the modal photon densities all exhibit sharp pulses spaced at the period of the modulation. In Fig. 4.15(a) we observe a single dominant longitudinal mode where as in Fig. 4.15(b) many modes oscillate. A slight “phase shift” between pulses for adjacent longitudinal modes is also observed in Fig. 4.15(b).

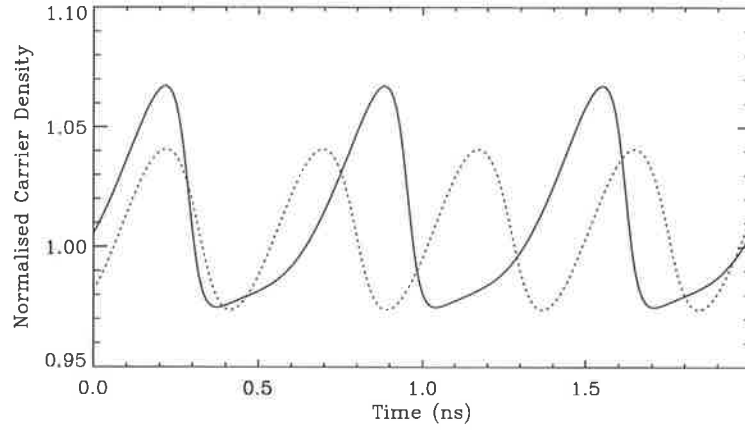


Figure 4.14: Calculated normalised carrier density versus time for the parameter values, $I_b = 1.7$ and $m = 0.5$. The solid curve corresponds to a modulation frequency of 1.5GHz and lies within the largest region of multimode operation of Fig. 4.12(b). The dashed curve corresponds to a modulation frequency of 2.1GHz and lies just beyond the largest region of multimode operation shown in Fig. 4.12(b).

Longitudinal Mode Spectra vs time

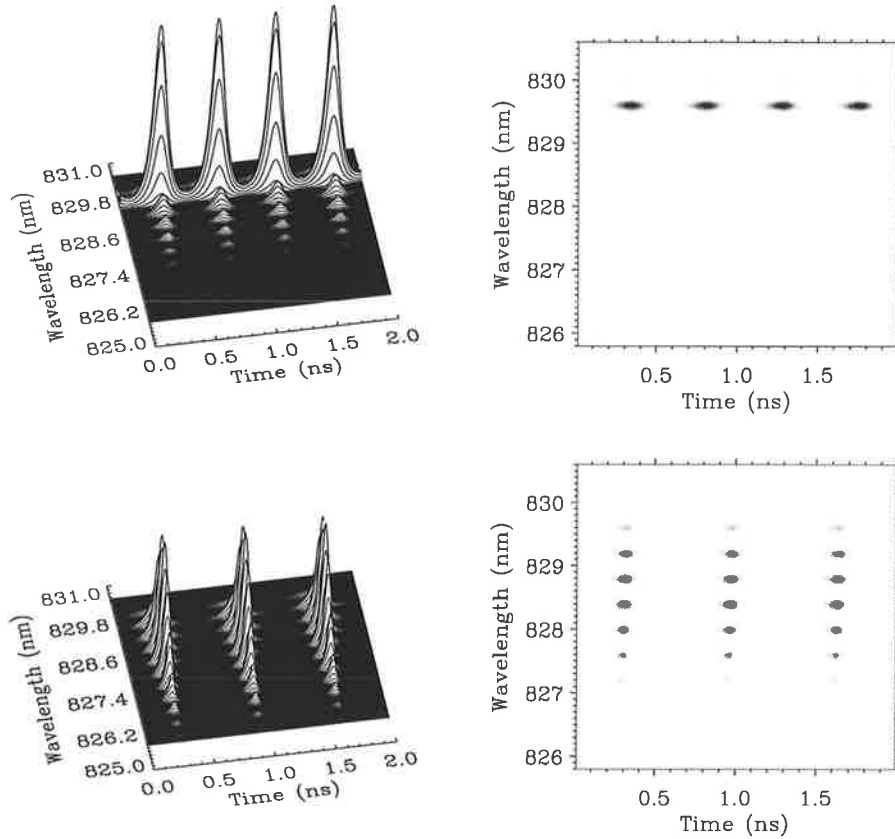


Figure 4.15: Calculated instantaneous longitudinal mode spectra versus time, for the parameter values, $I_b = 1.7$ and $m = 0.5$, shown as a surface plot (left) and image plot (right). The upper figure corresponds to a modulation frequency of 2.1GHz and the lower figure to a modulation frequency of 1.5GHz.

4.4. MODAL BEHAVIOUR

4.4.2 Experimental Results

We observe qualitatively similar behaviour to that depicted in Fig. 4.12 in experiments performed on AlGaAs FP semiconductor lasers. Using the experimental arrangement of Fig. 3.1 we record time-averaged longitudinal mode spectra as the modulation frequency is varied in steps of 0.03 GHz from 0.1 to 3GHz.

In Figs. 4.16 and 4.17 we show the longitudinal mode spectra variation with modulation frequency for an 830nm FP laser diode¹⁶. For the data shown in Fig. 4.16 the laser was biased at 29mA and modulation powers of 0dBm and 5dBm were used, in parts (a) and (b), respectively. The temperature of the laser was stabilised to 25°C. The relaxation oscillation frequency at this bias current is 2.6GHz. The region of multimode operation for modulation frequencies near f_{RO} is clearly observed in both Figs. 4.16(a) and (b)[1] as is the region for f near $f_{RO}/2$. For the higher modulation power (Fig. 4.16(b)) we can also discern the regions of multimode operation for f near $f_{RO}/3$ and $f_{RO}/4$. However, resolving these regions at this modulation power is difficult due to the overall increase in the number of operating modes over a wider range of modulation frequencies; we remark that this behaviour is qualitatively similar to that depicted in the numerical simulations (Fig. 4.12(c)). In Fig. 4.17 we show the longitudinal mode spectra versus modulation frequency for an injection current of 27mA and modulation power of 0dBm. The relaxation oscillation frequency at this injection current is 2.3GHz. In this case the regions of multimode operation (occurring when the harmonics of the driving frequency are close to the relaxation oscillation frequency) are more readily observable.

Comparison of the experimental results with Fig. 4.12 shows good qualitative agreement between theory and experiment. However, the fact that experimentally the laser becomes multimode for low modulation frequencies is not predicted by the rate equations. In experiments performed on two other types of AlGaAs FP laser diodes (wavelength 780nm) we observe this low frequency multimode effect to be much more significant and in some cases tended to obscure some of the lower frequency resonances (i.e. f near $f_{RO}/2$ and $f_{RO}/3$). In Fig. 4.18 we show the results of experimentally measured longitudinal mode spectra versus modulation frequency for a Sharp laser diode¹⁷. The laser temperature was stabilised to 25°C with a threshold current of 36mA at this temperature. In Fig. 4.18(a) the injection current was 42mA and the modulation power -3dBm and in Fig. 4.18(b) the injection current was 44mA and the modulation power was 0dBm. The relaxation oscillation frequencies, at these injection currents were 2.4GHz and 2.9GHz, respectively. In both figures we can see multimode regions for f near f_{RO} and $f_{RO}/2$. In part (b) the multimode region for f near $f_{RO}/3$ is also observable. However, at low modulation frequencies different

¹⁶This is the Mitsubishi diode laser (wavelength 830nm) used in all measurements of Chapter 3.

¹⁷model no. LT026MDC

Longitudinal Mode Spectra vs Modulation Frequency

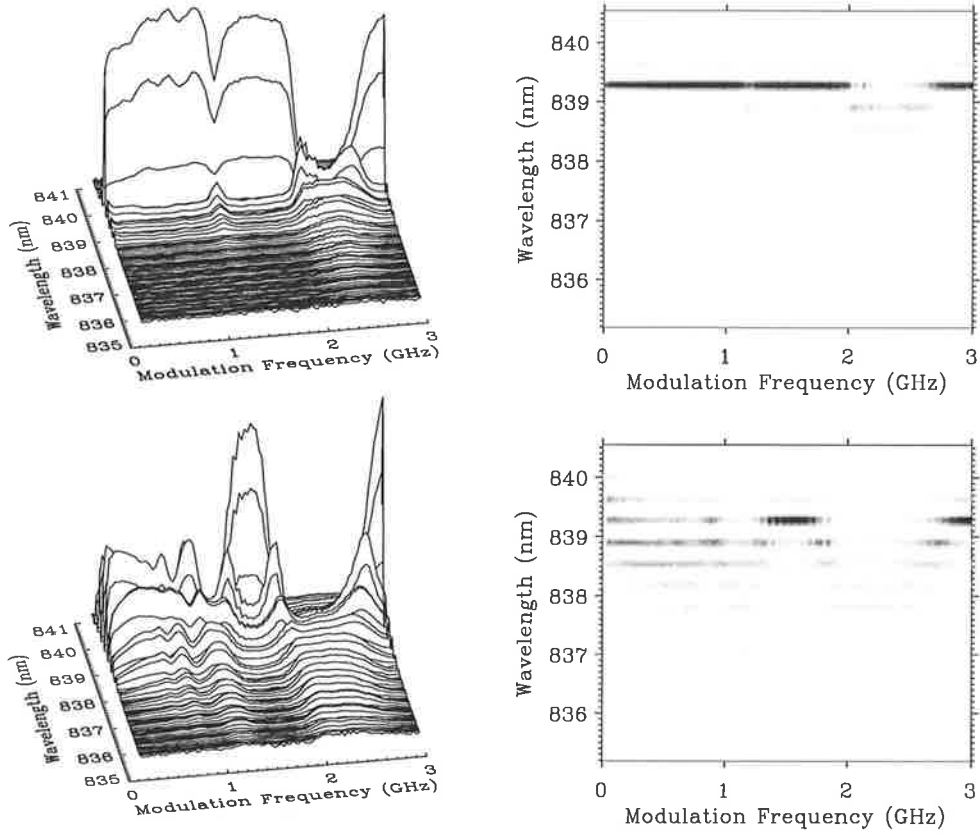


Figure 4.16: Experimental time averaged longitudinal mode spectra versus modulation frequency for the 830nm laser diode. The upper figure corresponds to a modulation power of 0dBm and the lower figure corresponds to a modulation power of 5dBm. The DC bias in both cases was 29mA.

Longitudinal Mode Spectra vs Modulation Frequency

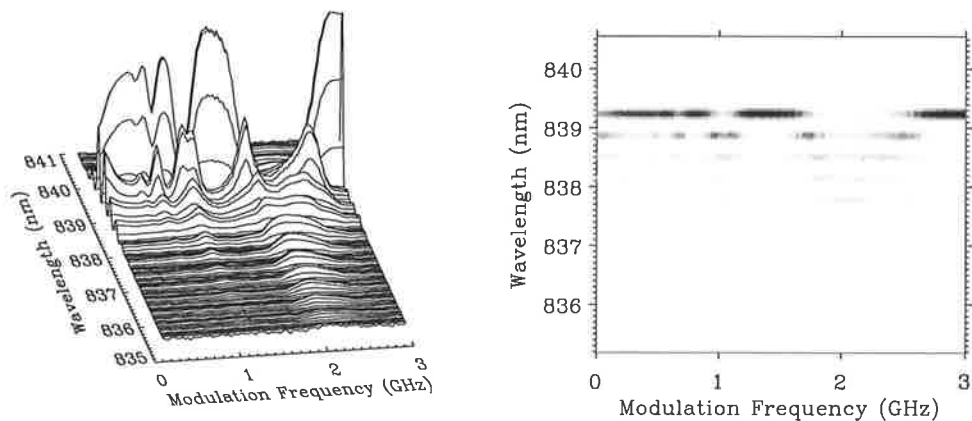


Figure 4.17: Experimental time averaged longitudinal mode spectra versus modulation frequency for the 830nm laser diode. The DC bias was 27mA and the modulation power was 0dBm.

behaviour to that predicted in the rate equation simulations is observed. In Fig. 4.19 we show the measured dependence of the longitudinal mode spectra on modulation frequency for a Mitsubishi FP laser¹⁸, for $I_{DC} = 47\text{mA}$ and modulation power of 6dBm. The threshold current for the laser temperature stabilised at 25°C was 36mA. The relaxation oscillation frequency at $I_{DC} = 47\text{mA}$ was 2.9GHz. In this case the tendency for the laser to oscillate in multiple longitudinal modes for lower modulation frequencies obscures the multimode behaviour at the higher order resonances (i.e. f near $f_{RO}/2$, $f_{RO}/3$, etc.). The multimode region for modulation frequencies near f_{RO} is still discernible. The agreement between theory and experiment is not as good for the 780nm diode lasers (Figs. 4.18 and 4.19) as the 830nm diode laser (Figs. 4.16 and 4.17). However, salient features of the predicted behaviour, namely regions of multimode operation for modulation frequencies (or its harmonics) close to the relaxation oscillation frequency are observed. We remark that for the 780nm diode lasers the relaxation oscillation resonance is not as sharply peaked as for the 830nm diode laser; thus the resonance effects become more difficult to observe.

To demonstrate that this is indeed a resonance phenomena we have included experimental power spectra of the total intensity and the corresponding time-averaged longitudinal mode spectra (Fig. 4.20) for the 830 nm laser diode. For an injection current of 27mA and modulation power of 0dBm (as in Fig. 4.17) we show the intensity power spectra and the longitudinal mode spectra for f near f_{RO} and $2f$ near f_{RO} . For comparison, we also show the longitudinal mode spectra and intensity power spectra without modulation (Fig. 4.20(a)). We observe, in Figs. 4.20(b) and (c) that the relaxation oscillation resonance has shifted to a lower frequency relative to the unmodulated laser frequency. This phenomena is observed both experimentally and in numerical simulations: regions of multimode operation are accompanied by frequency pulling of the relaxation oscillation indicating that this is essentially a nonlinear phenomena. We discuss this further in Chapter 5.

4.5 Spontaneous Emission Factor

In previous studies it was revealed that the spontaneous emission factor, β has significant influence on the damping of the relaxation oscillations and hence the nonlinear behaviour[34, 15, 9]. Furthermore, it was shown that the effect of β on the damping of the relaxation oscillations is more pronounced in the multimode model due to the larger number of modes into which spontaneous emission can couple[34, 15]¹⁹.

We have calculated the effect of β on the damping of the relaxation oscillations

¹⁸model no. ML4442N, wavelength 780nm)

¹⁹This is possibly one reason why a reduction in the number of successive period doubling bifurcations is observed in the multimode system without band-filling than in the single mode system: compare Figs 4.8(a)-(d) and Figs. 4.10(a)-(d).

Longitudinal Mode Spectra vs Modulation Frequency

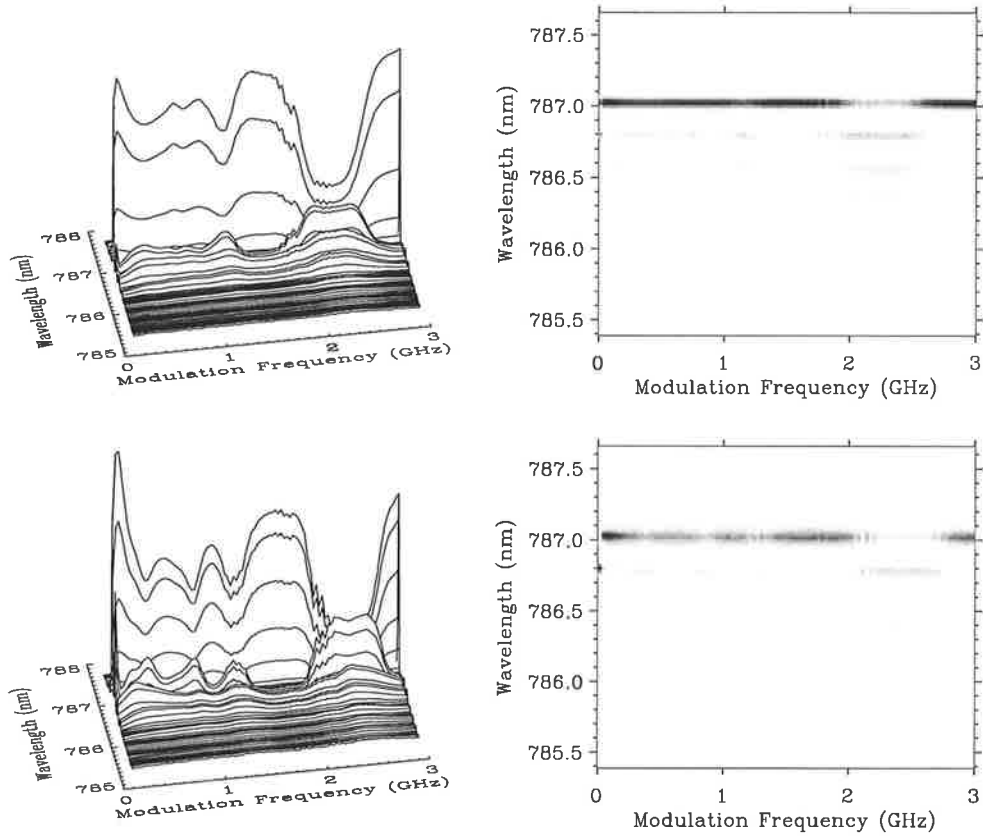


Figure 4.18: Experimental time averaged longitudinal mode spectra versus modulation frequency for the 780nm Sharp laser diode. The upper figure corresponds to an injection current of 42mA and a modulation power of -3dBm and the lower figure corresponds to an injection current of 44mA and a modulation power of 0dBm.

Longitudinal Mode Spectra vs Modulation Frequency

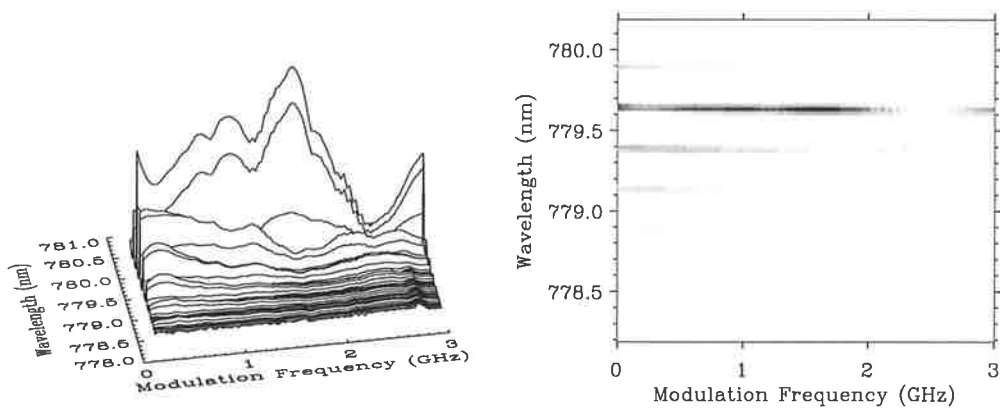


Figure 4.19: Experimental time averaged longitudinal mode spectra versus modulation frequency for the 780nm Mitsubishi laser diode, for an injection current of 47mA and a modulation power of 6dBm.

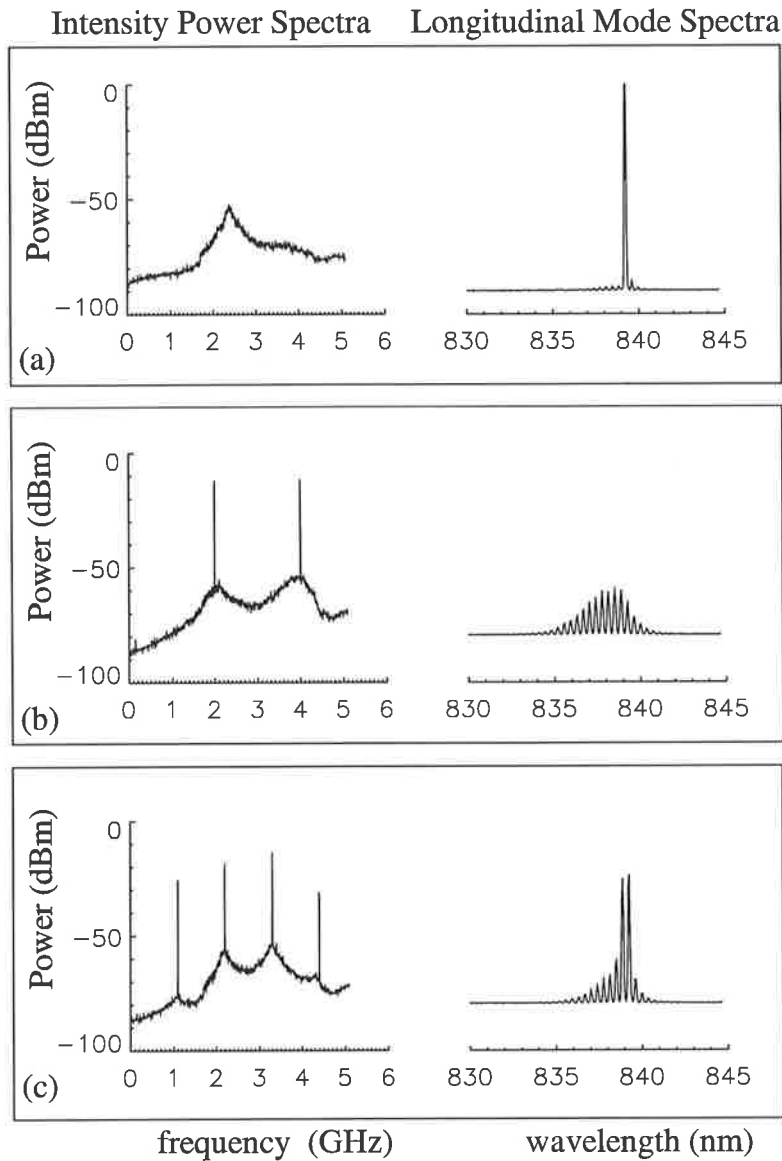


Figure 4.20: Experimental intensity power spectra (left) and corresponding time averaged longitudinal mode spectra (right). Part (a) corresponds to the laser without modulation, part (b) corresponds to modulation at the relaxation oscillation frequency for modulation frequency $f = 2\text{GHz}$ and part (c) corresponds to modulation at half the relaxation oscillation frequency for modulation frequency $f = 1.1\text{GHz}$. The laser bias is $I_{DC} = 27\text{mA}$ and modulation power is $P_{AC} = 0\text{dBm}$ as in Fig. 4.17.

explicitly from numerical solutions of the rate equations²⁰. To do this we linearise equations 4.3 about the steady state solution²¹. In the notation of equation 4.3, we can construct the Jacobian matrix of the system:

$$A = \left. \frac{\partial Q}{\partial x} \right|_{x=x_o} \quad (4.6)$$

The eigenvalues of A determine the stability of the steady state solution. We find that, for our system of equations, the matrix A has one complex conjugate pair of eigenvalues corresponding to the relaxation oscillation frequency. Close to the steady state solution, the dynamical variables evolve approximately according to

$$x(t) = e^{-\gamma t} \cos(2\pi f_{RO}t + \phi_o) \quad (4.7)$$

where $2\pi f_{RO}$ is the imaginary part and $-\gamma$ is the real part of the complex conjugate pair of eigenvalues of A . The value of γ determines the damping rate. We can also define the quality factor of a damped linear oscillator:

$$Q = \frac{\pi f_{RO}}{\gamma} \quad (4.8)$$

which gives a measure of the sharpness of the resonant peak. In the simulations a fixed bias, I_b was used and the modulation index was set to zero. The rate equations were integrated for different values of β , until the steady state solution, x_o , was obtained and the matrix A was then calculated. In Fig. 4.21 we plot the decay rate γ (left column) and the quality factor Q (right column) as a function of the spontaneous emission factor, β . In parts (a) and (b), $I_b = 1.3$ and in parts (c) and (d), $I_b = 1.7$. The solid curve corresponds to the multimode system and the dotted curve corresponds to the single mode system. We observe the decay rate, γ to increase with increasing β in all cases. However, the decay rate is much lower for the single mode system. Correspondingly, the decrease of Q with increasing β is less for the single mode system. For these parameter values, the damping at a bias of $I_b = 1.7$ is less than for $I_b = 1.3$.

To see the effect of spontaneous emission on the bifurcation behaviour, in Fig. 4.22, we show a bifurcation diagram for the same parameters as in Fig. 4.8(f) but with $\beta = 5 \times 10^{-5}$. Contrary to Fig. 4.8(f), we observe a period doubling region for f near $f_{RO}/2$, a Hopf bifurcation (labelled H_1) occurring for $f \approx 1.2\text{GHz}$ and two regions of hysteresis (marked HS_1 and $HS_{1/2}$ on the diagram). The Hopf bifurcation $H_{1/2}$ and the period doubling region for

²⁰Other parameters also contribute to the damping of the relaxation oscillations. However, we do not consider the effects of varying these parameters in this study.

²¹A more complete discussion of the linearisation procedure is given in section 5.2.1.

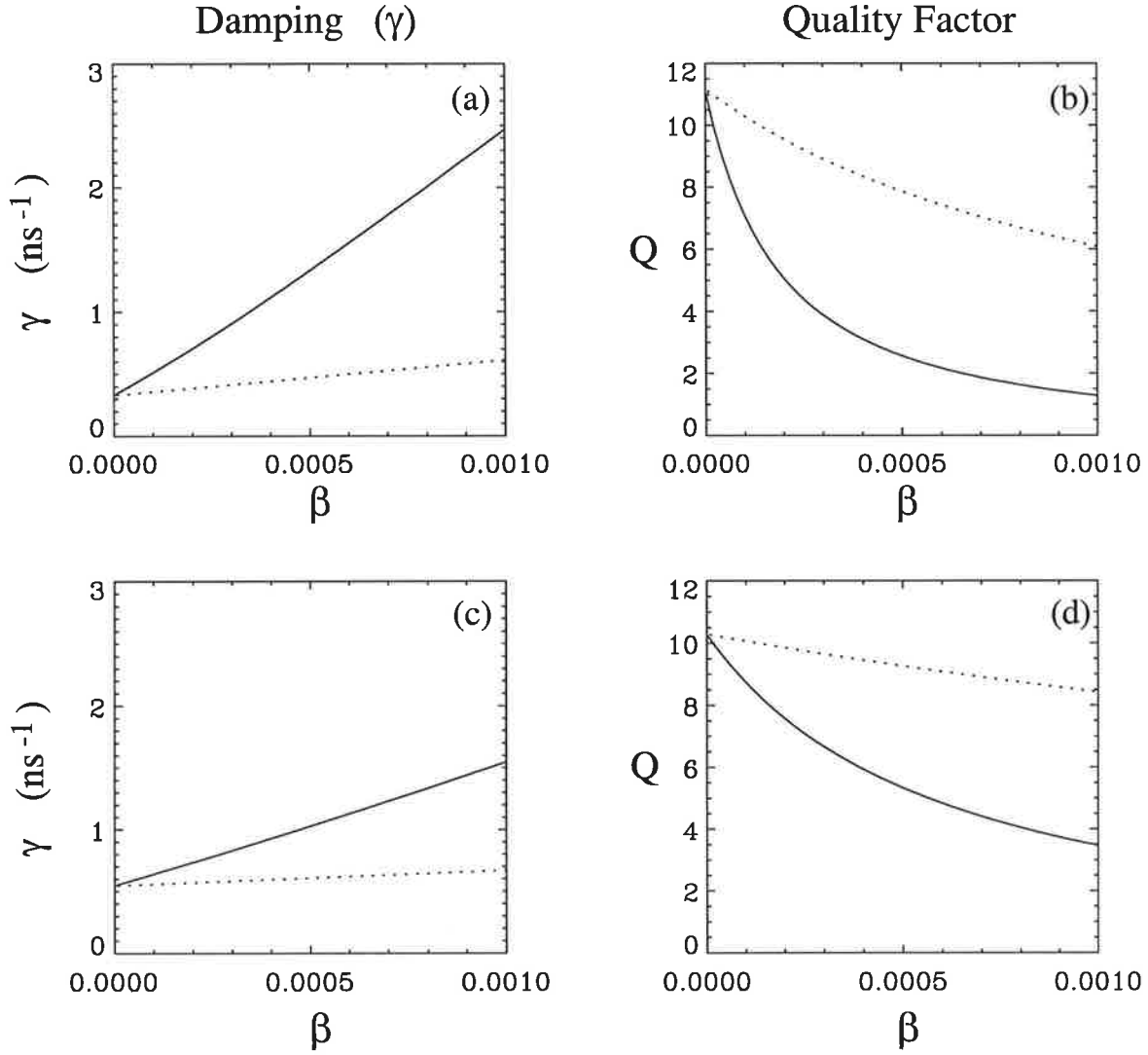


Figure 4.21: The decay rate γ (left) and quality factor Q (right) as a function of β . In parts (a) and (b), $I_b = 1.3$ and in parts (c) and (d) $I_b = 1.7$. In all cases the dotted curve corresponds to the single mode solution and the solid curve to the multimode solution.

$f > f_{RO}$ appear in both diagrams. Thus we see that, by reducing the spontaneous emission factor by a factor of two, the extent of bifurcations occurring in this system is increased. We also show the time-averaged longitudinal mode spectra in Fig. 4.22(b) calculated with the same parameter values as in Fig. 4.22(a). The qualitative features are not very different to the corresponding calculation with $\beta = 10^{-4}$ shown in Fig. 4.12(b) (for modulation frequencies up to 3GHz).

For even smaller values of β , chaotic solutions are possible in the multimode system. Fig. 4.23 shows the bifurcation diagram for the parameter values, $m = 0.75$ and $k = 350\text{\AA}$ as in Fig. 4.8(g) but with $\beta = 2 \times 10^{-5}$. We observed a period doubling cascade to chaos for modulation frequencies near $f_{RO}/2$. Separate calculations show that, for $\beta = 3 \times 10^{-5}$, this period doubling to chaos vanishes leaving only a single period doubling bifurcation. Hence multimode equations do predict chaotic solutions. However, much lower values of β relative to the single mode equations are required to achieve this.

4.6 Bifurcations

Studies on the types of bifurcations occurring in the single mode semiconductor laser rate equations have been extensive. The bifurcations occurring in the multimode system, on the other hand, have received very little attention. To our knowledge, only period doubling bifurcations have been discussed[1, 162]. In this section, we discuss period doubling, hysteresis and Hopf bifurcations in the multimode system (with band-filling) in more detail. Emphasis is on the characteristics of the resulting attractors. A discussion of how the system approaches these bifurcations is deferred until Chapter 5. In this section, $k = 350\text{\AA}$ in all calculations.

4.6.1 Period Doubling Bifurcations

Due to the sub-nanosecond timescales for dynamical variations of the system variables in a current modulated semiconductor laser it is not possible to measure the evolution of the longitudinal mode intensities directly with our experimental arrangement. Nevertheless, it is interesting to consider the time variation of these variables predicted by the multimode rate equations.

In Fig. 4.24 we show an example of a period doubled trajectory corresponding to the parameter values $I_b = 1.7$, $m = 0.75$, $f = 2.1\text{GHz}$ and $\beta = 10^{-4}$. Figure 4.24(a) shows the time dependence of the carrier density. We observe that the carrier density variation has twice the period of the input modulation. Figure 4.24(b) and (c) shows the time dependence of the longitudinal mode spectra plotted as a surface and image plot, respectively. The time dependence of the longitudinal mode spectra has an interesting structure; the output is a

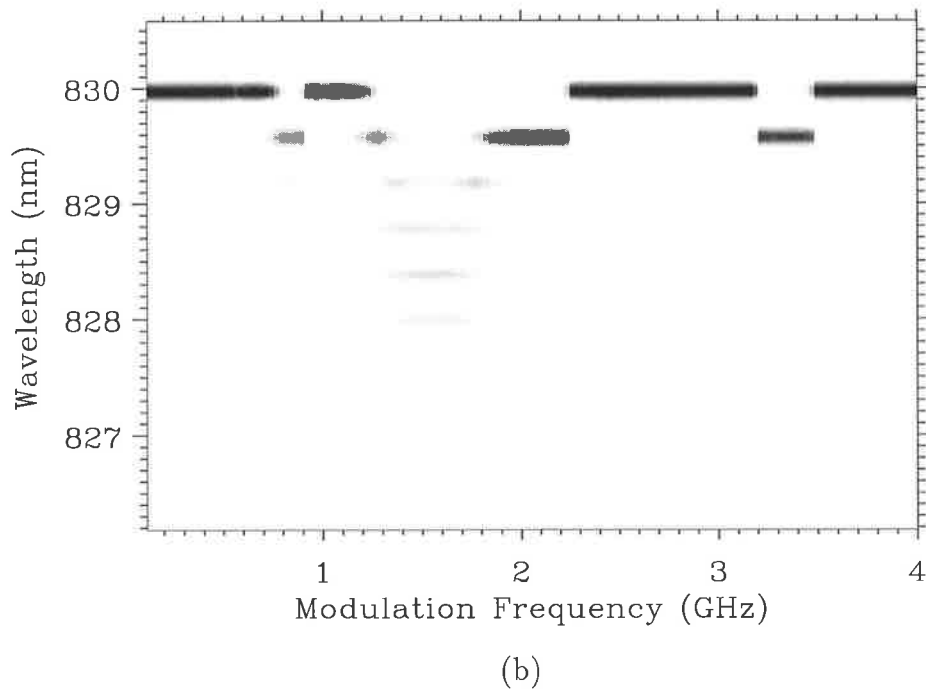
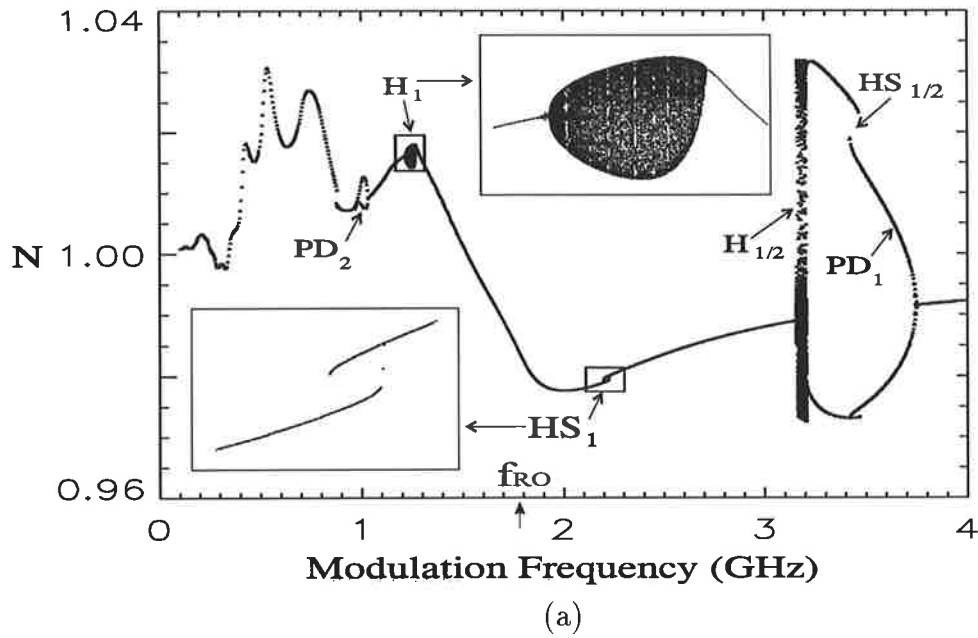


Figure 4.22: Part (a): Calculated bifurcation diagram for the same parameter values as Fig. 4.8(g) ($I_b = 1.7$, $m = 0.5$) except with $\beta = 5 \times 10^{-5}$. PD refers to period doubling, HS refers to regions of hysteresis and H indicates a Hopf bifurcation. An expanded view of the Hopf bifurcation region is shown in the upper inset and the region of hysteresis in the lower inset. Part (b) shows the calculated time averaged longitudinal mode spectra evolution with modulation frequency for the same parameter values as in (a).

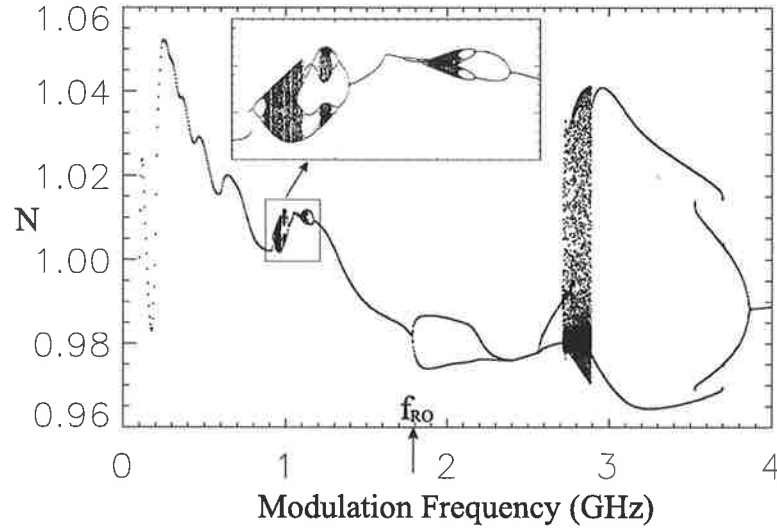


Figure 4.23: Calculated bifurcation diagram for the multimode model including band-filling for $\beta = 2 \times 10^{-5}$, $m = 0.75$ and $I_b = 1.7$,

series of pulses. However, these pulses are not evenly spaced in time and exhibit a different mode structure in alternate pulses.

Figure 4.12(c), showing the time average longitudinal mode spectra versus f , and the bifurcation diagram, Fig. 4.8(g), are both calculated using the same set of parameter values; as in Fig. 4.24 these correspond to the parameter values $I_b = 1.7$ and $m = 0.75$. Figure 4.8(g) shows that period doubling occurs for modulation frequencies in the range 2 to 3.8GHz. A comparison with Fig. 4.12(c), for this range of modulation frequencies, shows that the longitudinal mode spectrum for period doubling are modulation frequency dependent.

4.6.2 Hysteresis

Hysteresis in the single mode rate equations has been studied extensively[19, 23, 9, 25]. However, to the best of our knowledge, we present the first study of this phenomena for the multimode system. In this section we concentrate on the hysteresis region HS_1 of Fig. 4.22 (also shown in the lower inset). This region is accompanied by mode hopping between adjacent longitudinal modes. In Fig. 4.25(a) we plot the time averaged photon density for the central mode ($\lambda = \lambda_o$) and adjacent mode ($\lambda = \lambda_{-1}$); all other modes have negligible power. All parameters are identical to those of Fig. 4.22. In Fig. 4.25(a) we observe that as the modulation frequency is increased from 2.15 to 2.25GHz, an abrupt mode hop from λ_{-1} to λ_o at approximately $f = 2.225$ GHz occurs. When the modulation frequency is decreased through this region, however, we observe an abrupt transition from λ_o to λ_{-1} at approximately 2.2025 GHz; hence hysteresis occurs. For larger values of β , we no longer observe hysteresis in this region but we still observe a transition to an adjacent

Period Doubling in the Time Domain

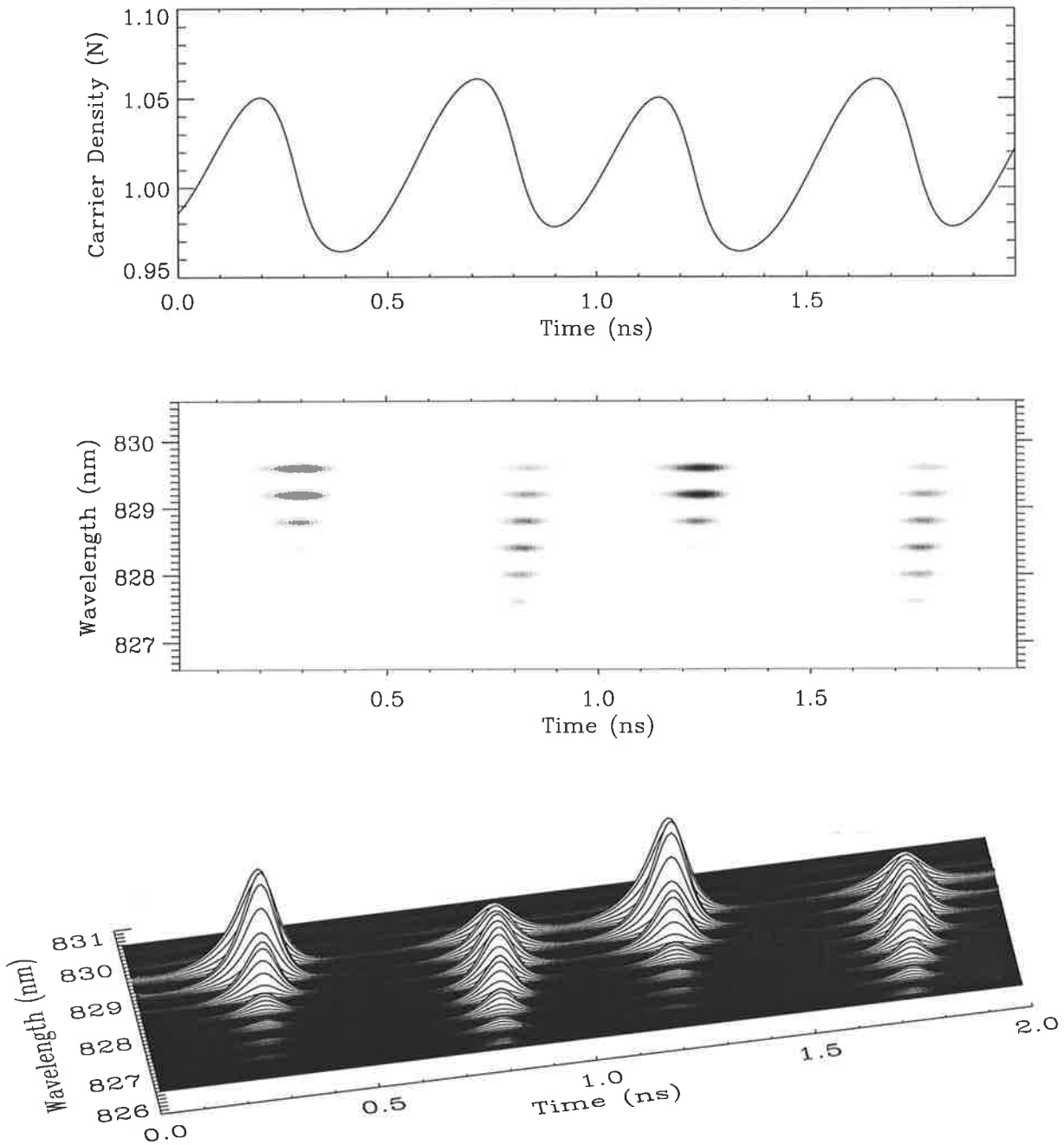


Figure 4.24: The upper figure shows the calculated time variation for the normalised carrier density showing a period doubled attractor. The modulation period is 0.476ns ($f=2.1\text{GHz}$). The lower two figures show the time variation of the instantaneous longitudinal mode spectra. Other parameter values are $I_b = 1.7$, $m = 0.75$ and $\beta = 10^{-4}$.

lasing mode. This is illustrated in Fig. 4.25(b) where we repeat the calculation of part (a) but with $\beta = 10^{-4}$.

A comparison of Figs. 4.12(b) and 4.22(b), showing the longitudinal mode spectra versus f , shows that the hysteresis region or mode transition occurs just beyond the largest region of multimode operation for increasing modulation frequency (i.e. for $f \approx 2.2\text{GHz}$). Though Figs. 4.12(b) and 4.22(b) appear similar (for modulation frequencies up to 3GHz) in Fig. 4.22(b) hysteresis occurs in the mode hopping whereas in Fig. 4.12(b) it is absent. Likewise, in the experimental data (Fig. 4.17) hysteresis is absent, though we also observe a transition to an adjacent lasing mode for modulation frequencies just beyond the largest region of multimode operation. We believe the absence of hysteresis observed experimentally is due to the amount of damping present in our laser, either the result of the size of the spontaneous emission parameter, or from some other processes such as gain saturation or carrier diffusion effects that have not been included in the present analysis. However, based on the similarity of Figs. 4.12(b), 4.22(b) and the experimental results (Fig. 4.17), it is feasible that, in a semiconductor laser exhibiting smaller damping of the relaxation oscillations, such behaviour can occur.

4.6.3 Hopf Bifurcations

It has been previously shown that periodic solutions of the single mode rate equations cannot undergo a Hopf bifurcation due to constraints imposed on the eigenvalues²². In the multimode system however, due to the larger number of degrees of freedom, these constraints no longer exist. In this section we discuss the Hopf bifurcation regions labelled H_1 and $H_{1/2}$ of Fig. 4.22.

The signature of a Hopf bifurcation is the creation of a new frequency within the system. Thus, its occurrence may be detected by the presence of another frequency, other than the modulation frequency (and its harmonics), in the power spectrum of the dynamical variables. Figure 4.26 shows the power spectrum of the total photon density (given by the sum of the modal photon densities) for $I_b = 1.7$, $m = 0.5$ and $f = 1.25\text{GHz}$, which lies within the region labelled H_1 in Fig. 4.22. Figure 4.26 clearly shows the presence of sidebands on the modulation frequency and its harmonics indicating two frequency dynamics. The time dependence of the total photon density and modal photon densities are shown in Fig. 4.27(a) and (b), respectively. Figure 4.27(a) shows a series of sharp pulses, spaced at the period of the modulation, whose amplitudes are modulated by a lower frequency. In figure 4.27(b) we plot the longitudinal mode spectra as a function of time as an image plot. We observe that the structure of the longitudinal mode spectra also changes with the low frequency modulation.

²²This is discussed in references[10, 186]. See also section 5.6.

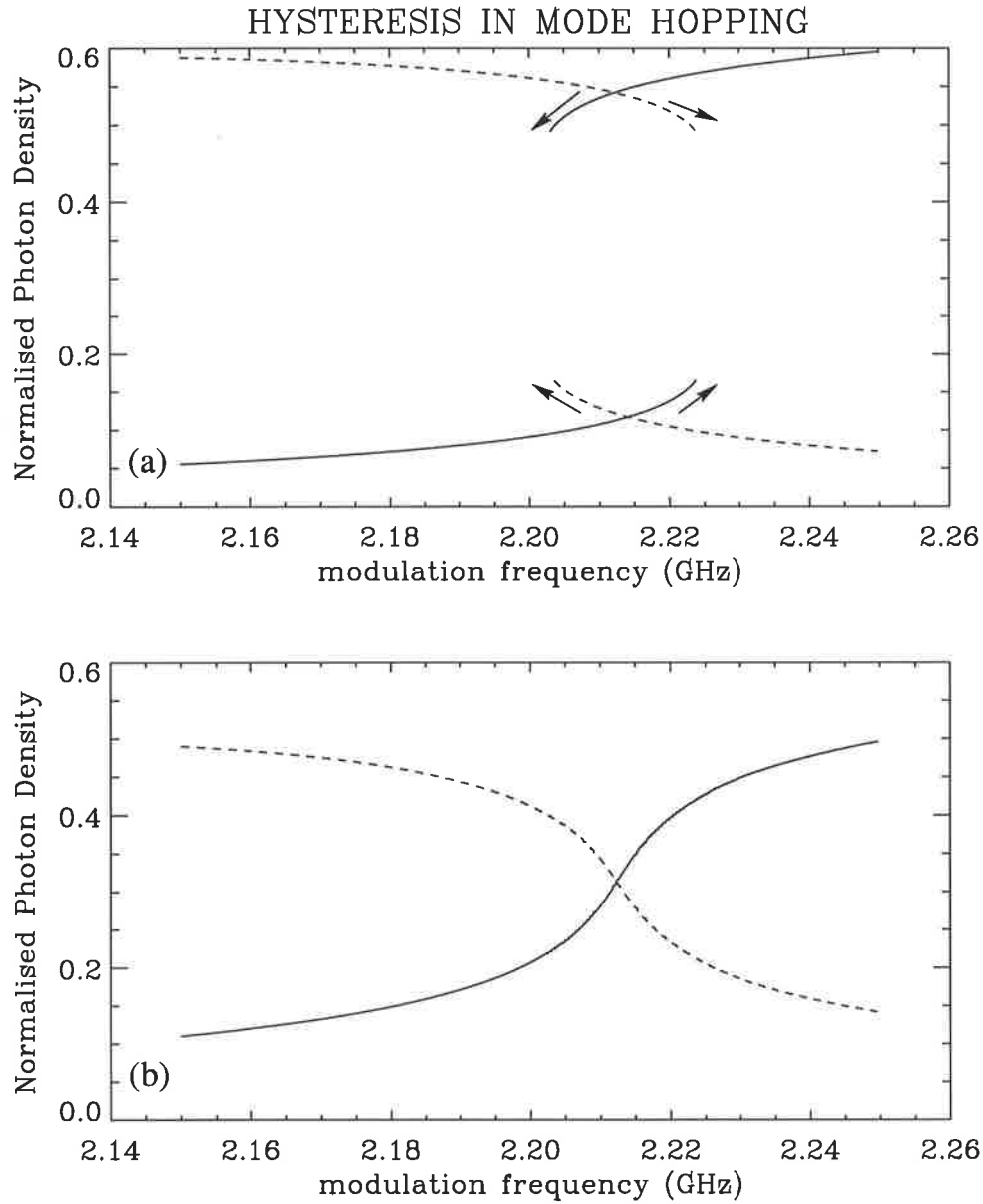


Figure 4.25: (a): Hysteresis in mode hopping as the modulation frequency is varied. The solid line corresponds to the zeroth mode and the dashed line corresponds to the adjacent shorter wavelength mode. Parameter values are identical to those used in Fig. 4.22 in the lower inset; in particular $\beta = 5 \times 10^{-5}$, $I_b = 1.7$ and $m = 0.5$. (b): Same parameters as in (a) except $\beta = 10^{-4}$. Hysteresis is not observed in the mode transition in this case.

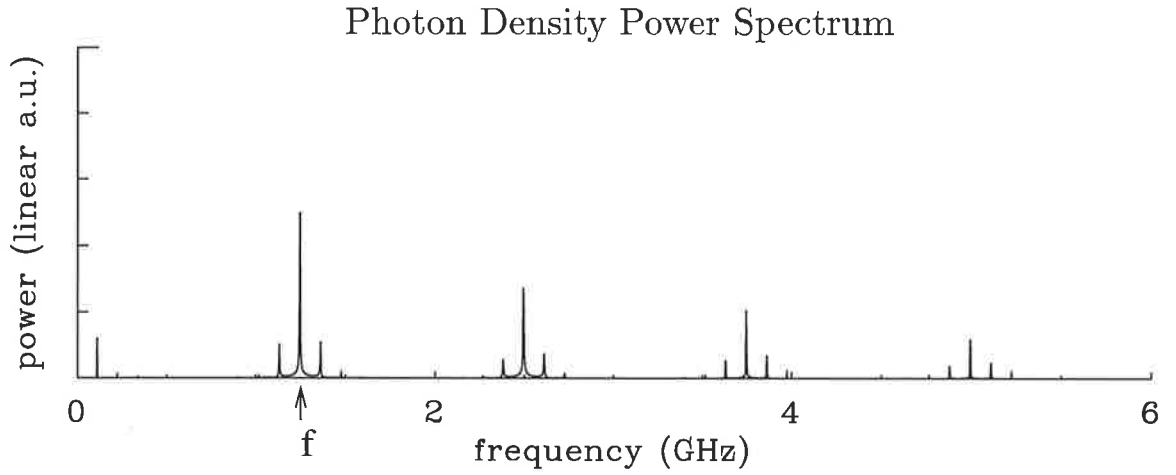


Figure 4.26: Calculated power spectrum of the total photon density for $I_b = 1.7$, $m = 0.5$, $\beta = 5 \times 10^{-5}$ and $f = 1.25\text{GHz}$. Side bands on the modulation frequency and its harmonics indicate the presence of a second frequency.

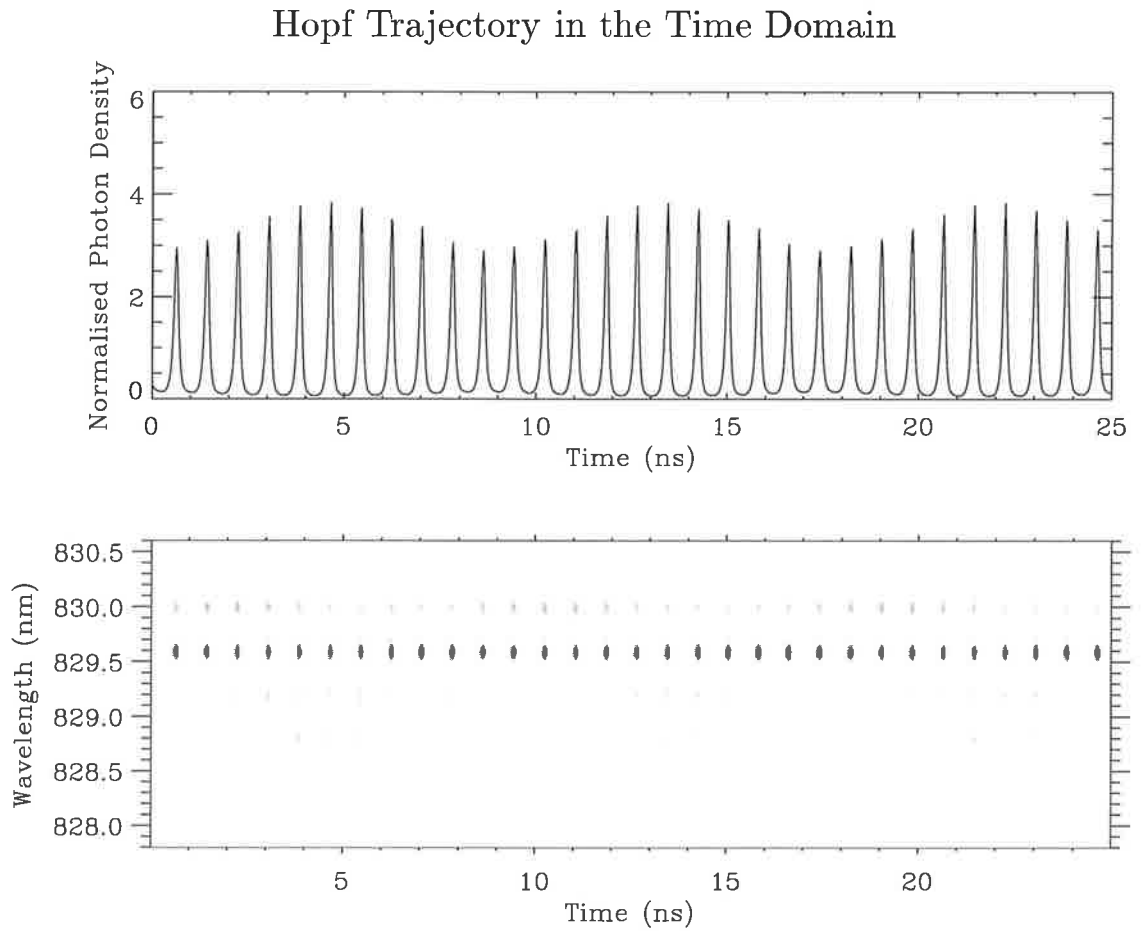


Figure 4.27: Calculated behaviour in the time domain corresponding to the power spectrum of Fig. 4.26. The total photon density is shown in the upper figure and the longitudinal mode spectra variation (shown as an image plot) is given in the lower figure.

Corresponding photon density power spectrum and time dependent behaviour are shown in Figs. 4.28 and 4.29, respectively, for the Hopf region labelled $H_{1/2}$ in Fig. 4.22. The modulation frequency was 3.18GHz. In the photon density power spectrum (Fig. 4.28) we again observe side bands on the modulation frequency and also at $f/2$ due to the period doubled orbit. The time dependent behaviour is slightly more complicated than the previous case. However, we also observe the structure of the longitudinal mode spectra to change with time.

4.7 Global Behaviour

To summarise the global behaviour, in Section 4.7.1 we present state diagrams [23, 2, 20, 21] for both single and multimode systems. In these diagrams the transition boundaries between the period one and two and period two and four solutions are plotted as a function of modulation frequency and modulation index. The boundaries for Hopf bifurcations are also shown. The bifurcation boundaries are calculated numerically by checking for period one and period two solutions of the rate equations for a square grid of modulation frequencies and amplitudes. Regions of hysteresis between period one cycles are not shown as they could not be determined by our method of calculation; however, regions of hysteresis between period one and period two solutions are included. In all calculated diagrams the resolution is 0.032 in m and 0.038GHz in f . In Section 4.7.2 we show the results of experimentally determined regions of period doubling in the (f, m) parameter space.

4.7.1 Numerical Results

In Fig. 4.30 we show two dimensional state diagrams for the single mode system (left column) and the multimode system (right column) for $I_b = 1.3, 1.5, 1.7$ and 1.9 in parts (a) to (d), respectively. All other parameters are given in Table 4.2 and the spontaneous emission factor is 10^{-4} .

In the single mode system we observe a series of period doubling regions which become smaller and more closely spaced as the modulation frequency decreases. The largest period doubling region occurs for modulation frequencies greater than the relaxation oscillation frequency with the lowest modulation index required for period doubling occurring at a modulation frequency of $f = 2f_{RO}$. The period doubling regions for modulation frequencies less than the relaxation oscillation frequency become more prevalent for larger I_b . Period four regions lying within the period two regions are also shown; some of these regions contain chaotic solutions which can be verified by comparison with the single mode bifurcation diagrams of Figs. 4.6 to 4.9. In all diagrams for the single mode system there is a hysteresis region $HS_{1/2}$ where there are co-existing period one and period two solutions.

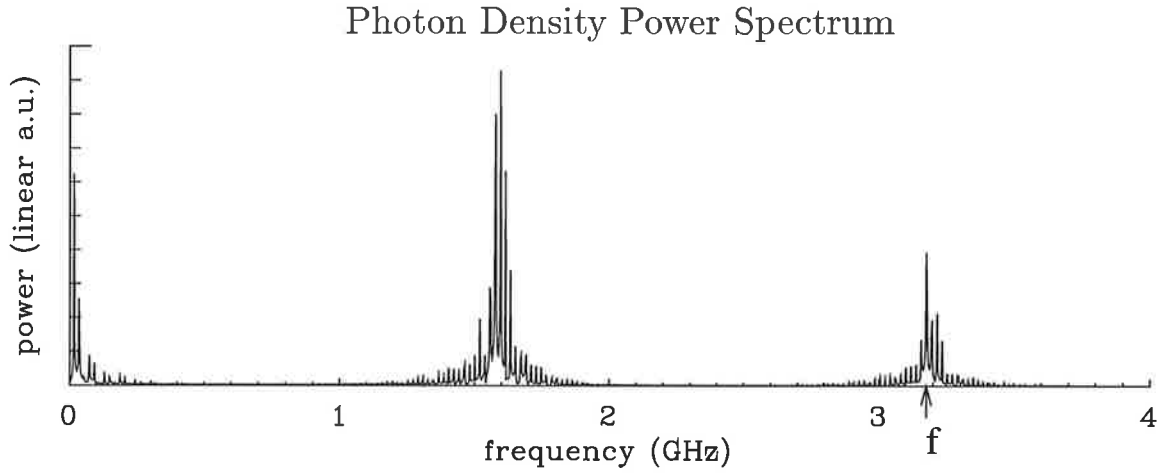


Figure 4.28: Calculated power spectrum of the total photon density for $I_b = 1.7$, $m = 0.5$, $\beta = 5 \times 10^{-5}$ and $f = 3.18 \text{ GHz}$. Side bands on the modulation frequency and its subharmonic indicate the presence of a second frequency.

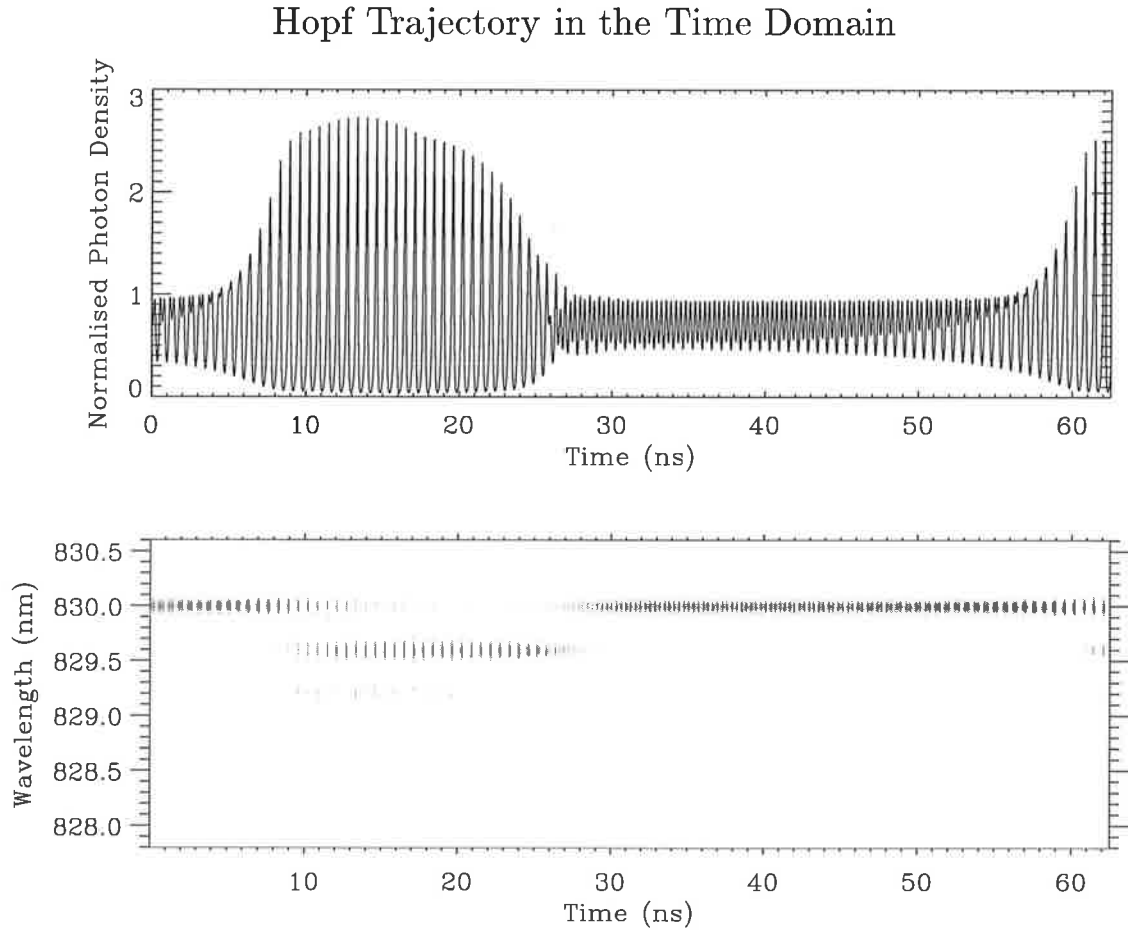


Figure 4.29: Calculated behaviour in the time domain corresponding to the power spectrum of Fig. 4.28. The total photon density is shown in the upper figure and the longitudinal mode spectra variation (shown as an image plot) is given in the lower figure.

Hysteresis regions between co-existing period one solutions and between co-existing period two solutions also exist though these are not shown in Fig. 4.30. For the $I_b = 1.3$ diagram, for modulation indices above 1.3, there is a co-existence between several different attractors (including a period three and a chaotic attractor) making the diagram too complicated to be easily interpreted in this region. Therefore we have not shown these bifurcation surfaces.

There are some qualitative similarities between the single mode and multimode solutions. In the multimode system, the lowest modulation index required for period doubling also occurs for a modulation frequency of $f = 2f_{RO}$. It is interesting to note that the minimum modulation index for period doubling is in fact lower in the multimode system than in the single mode system. The largest period doubling region (for $f > f_{RO}$) occurs in all diagrams in the multimode system. This period doubling region contains period four solutions as in the single mode case. However, in the diagrams corresponding to $I_b = 1.7$ and 1.9 it also contains a region in which a Hopf bifurcation has occurred. The general shape of the largest period doubling region is different, however, for the single mode and multimode systems. In particular, we observe that the hysteresis region, between the period one and period two solutions, is located differently in each of these systems. Only one period doubling region, for $f < f_{RO}$, is observed in the multimode state diagrams for $I_b = 1.9$ ²³. However, contrary to the single mode predictions, this region does not contain any higher period solutions. For $I_b = 1.9$ we also observe a region H_1 , which corresponds to a Hopf bifurcation boundary.

To determine the effect of band-filling on the bifurcation surfaces, in Fig. 4.31 we show state diagrams for the multimode system without band-filling, for $I_b = 1.3$ and 1.7. All other parameters are given in Table 4.2 (except k which is equal to zero). Similarly to the bifurcation diagrams of section 4.3, these diagrams show a greater resemblance to the single mode system than to the multimode system with band-filling. The period doubling regions are, however, smaller in this case and there are no period four (or higher period) solutions for $f < f_{RO}$.

4.7.2 Experimental Results

In this section we present the results of experiments used to determine the regions in the parameter space of modulation frequency and amplitude in which a period doubled attractor exists. Experimental determination of the occurrence of period doubling is accomplished by measuring the RF power, in the intensity power spectrum, at a frequency equal to half the modulation frequency. We scanned the (f, m) parameter space, by repeating the above

²³There is also a small region for $I_b = 1.7$ but it is barely observable.

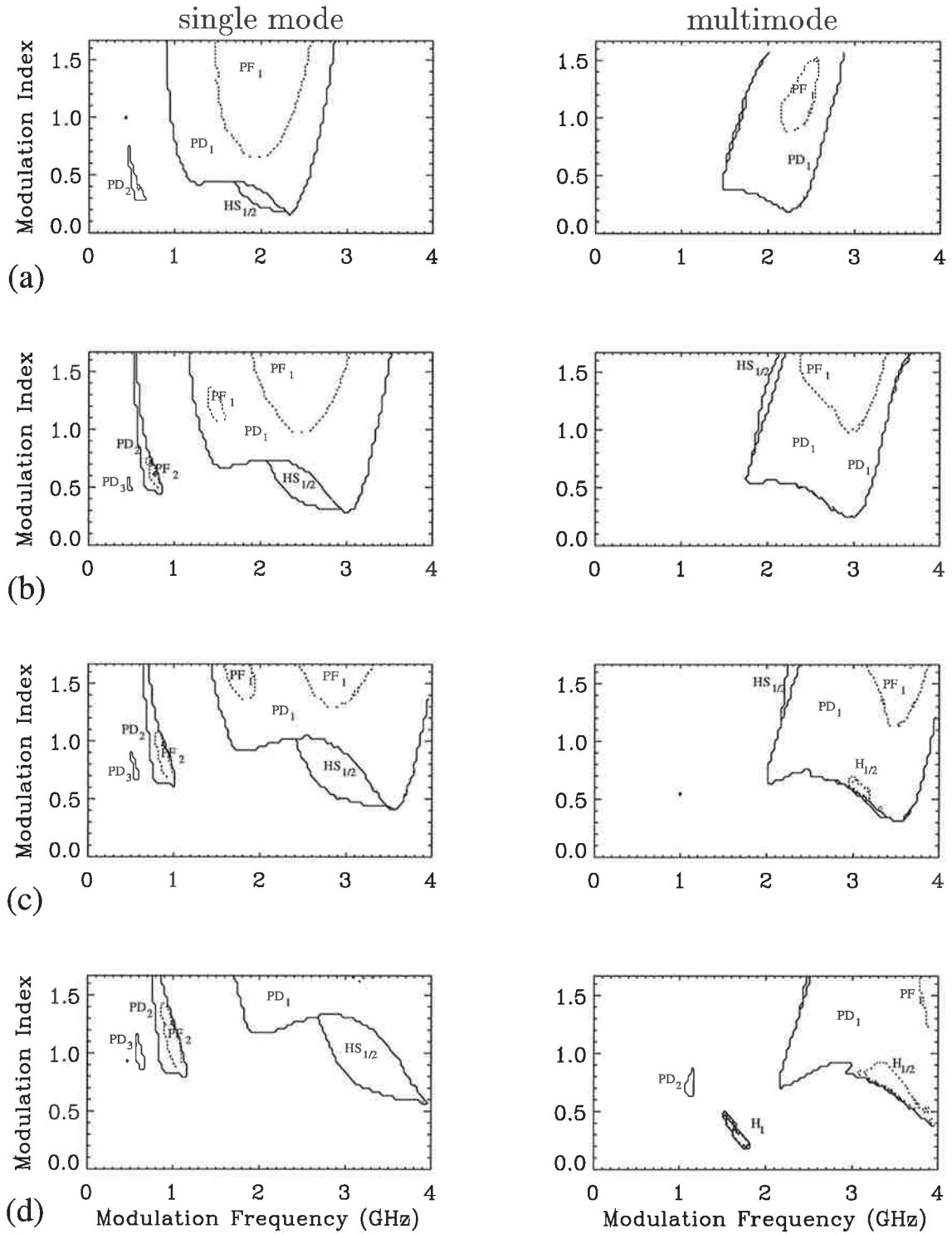


Figure 4.30: Calculated period doubling and Hopf bifurcation boundaries in the parameter space of modulation frequency and modulation index for the single mode system (left) and multimode system with band-filling (right). Parts (a) to (d) correspond to $I_b = 1.3, 1.5, 1.7$ and 1.9 , respectively. The solid lines labelled by PD indicate boundaries between period one and period two solutions. Solid lines labelled by HS indicate the boundaries between period one and period two solutions in which hysteresis is involved. Dashed lines labeled by PF indicate the boundaries between period two and period four solutions and dashed lines labeled by H indicate boundaries for Hopf bifurcations.

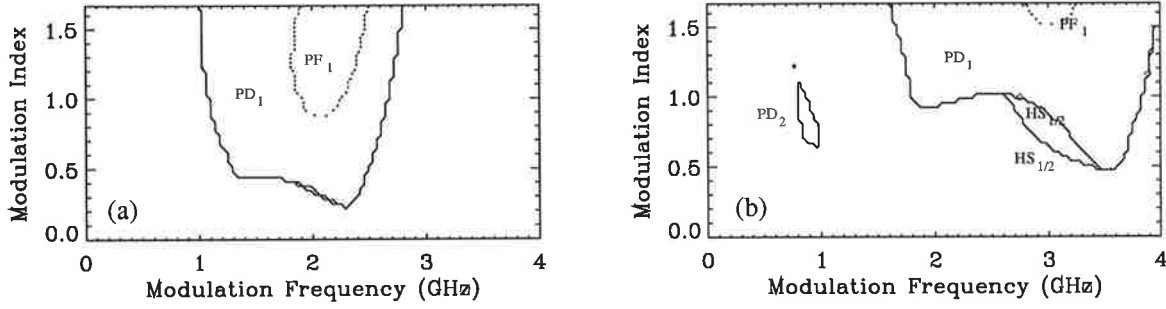


Figure 4.31: Calculated period doubling and Hopf bifurcation boundaries in the parameter space of modulation frequency and modulation index for the multimode system without band-filling ($k = 0$). Parts (a) and (b) correspond to $I_b = 1.3$ and 1.7 , respectively. The solid lines labelled by PD indicate boundaries between period one and period two solutions. Solid lines labelled by HS indicate the boundaries between period one and period two solutions in which hysteresis is involved. Dashed lines labeled by PF indicate the boundaries between period two and period four solutions.

measurement, for a finely spaced grid of modulation frequencies and powers²⁴. The results are shown in Figs. 4.32(a) to (c) for bias currents of $I_{DC} = 22, 25$ and 27 mA, respectively. Period doubling for modulation frequencies below the relaxation oscillation frequency was not detected. Nevertheless, period doubling for modulation frequencies above the relaxation oscillation frequency is observed in each case.

Experimentally, the maximum modulation power is restricted by the maximum optical power that can be output from the laser without causing damage to the facets. Furthermore, in our experimental arrangement we only had modulation frequencies up to 3 GHz and, as a consequence, were not able to access the full range of the parameter space congruent with that shown in Fig. 4.30, as we require modulation frequencies up to twice the relaxation oscillation frequency. Therefore, we cannot compare the shapes of the predicted period doubling regions with experiment for this range of bias currents. We defer this until the next section. Despite these limitations, there are qualitative comparisons that can be made between the experimental data and the numerical simulations. We observe a shift in the position of the period doubling region to both higher modulation frequencies and higher modulation powers with increasing bias current consistent with the numerical simulations. Also, experimentally we do not observe period doubling for modulation frequencies below the relaxation oscillation frequency.

²⁴The grid spacing was 0.025 GHz in modulation frequency and 0.24 dBm in modulation power.

Experimental Period Doubling Regions

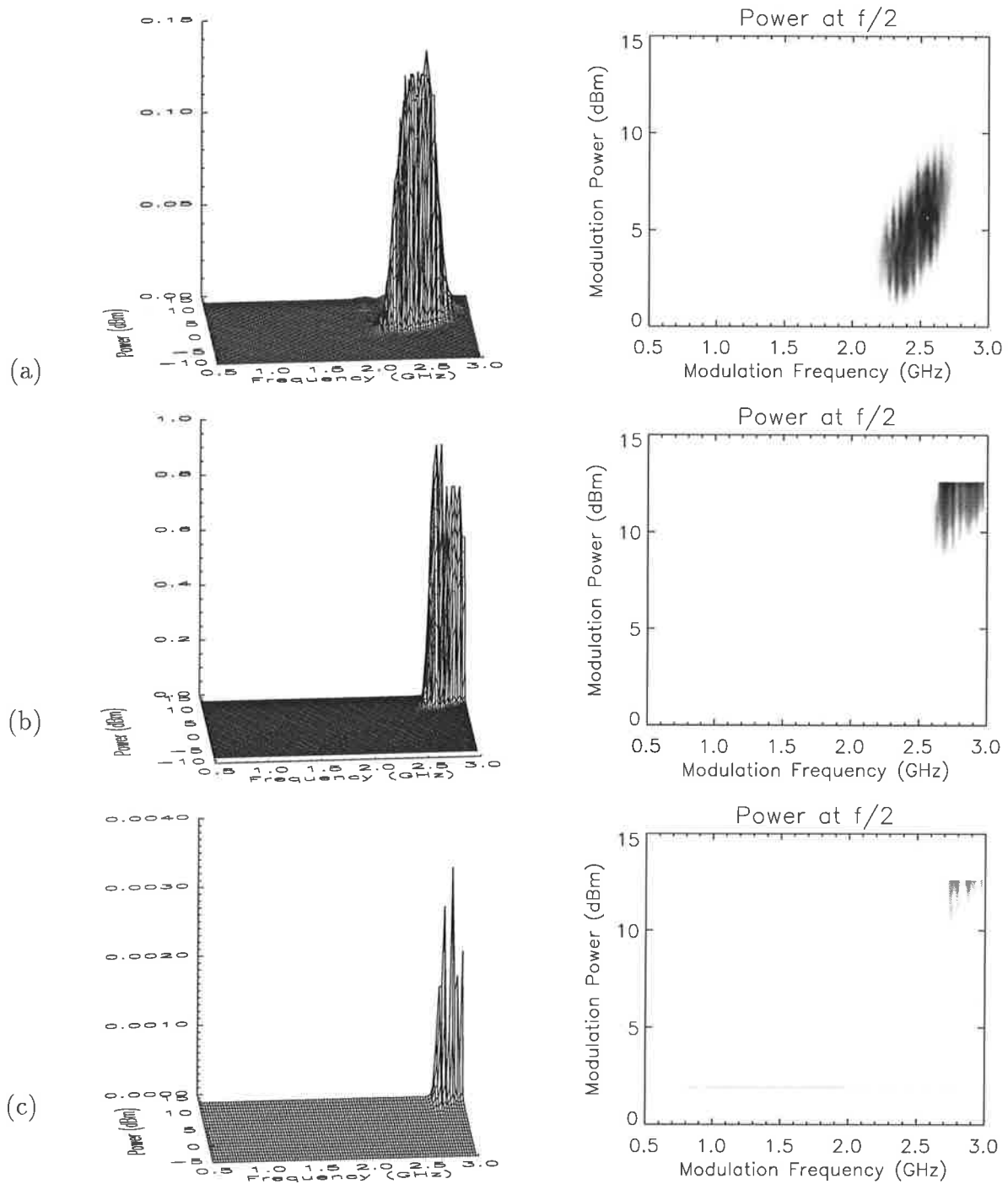


Figure 4.32: Experimentally determined regions of period doubling. Figures, from top to bottom, correspond to bias currents of 22, 25 and 27mA, respectively. Data is shown as both a surface plot (left) and filled contour plot (right).

4.8 Quantitative Comparisons

All previous numerical results so far shown in this chapter were obtained using typical parameter values for an AlGaAs semiconductor laser given in the literature (Table 4.2). We have not attempted quantitative comparisons between modelling and experiment throughout this thesis for two reasons. Firstly, the experimental determination of the laser parameters (discussed in Chapter 3) was done after extensive numerical investigations using the aforementioned parameter values were completed. Time constraints did not permit a repetition of these calculations using the experimentally measured parameter values; therefore we have not used the experimentally determined parameter values in most of the numerical calculations. Furthermore, during the course of taking measurements presented in this thesis, the laser diode aged somewhat giving different experimental results and also leading to a variation in the laser parameters. Thus, we were not able to use the experimentally determined parameters to compare numerical solutions with existing experimental data. For these reasons, in most cases, we have attempted qualitative but not quantitative comparisons between the numerical modelling using the multimode rate equations and the experimentally observed behaviour of the laser. In this section we provide a quantitative comparison between the multimode rate equation predictions and the experimental results using the experimentally measured rate equation parameter values (see Table 3.1).

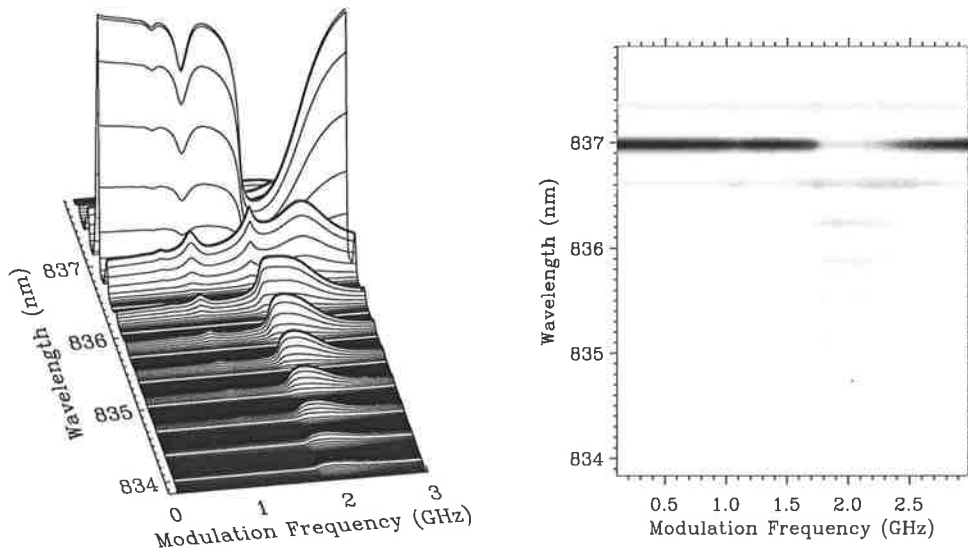
In Figs. 4.33 and 4.34 we show the measured and predicted longitudinal mode spectra evolution with modulation frequency for a DC bias of 27.5mA. For a threshold current of 21mA this corresponds to the value $I_b = 1.31$. In Figs. 4.33 and 4.34 modulation powers of -5dBm and -1.5 dBm were used, respectively. These correspond to modulation indices of 0.17 and 0.25. Despite differences in the relative intensities between different longitudinal modes, shown in the experimental and numerical data of Figs. 4.33 and 4.34, the general form of the longitudinal mode behaviour in each case is similar. In particular, the widths and positions of the multimode regions as a function of the modulation frequency are consistent. Moreover, the multimode regions, generally, have approximately the same number of longitudinal modes in both the numerical and experimental data.

In Fig. 4.35 we show measured and calculated period doubling regions in the (f, m) parameter space. The experimental injection current was 23mA corresponding to a value of $I_b = 1.095$. As in section 4.7.2, the regions of period doubling were determined experimentally by measuring the power at half the modulation frequency in the intensity power spectra. Contrary to section 4.7.2, however, the modulation index is varied linearly and the measured power at $f/2$ is plotted logarithmically. Figure 4.35(b) shows the experimental results plotted as a filled contour plot such that the darkest regions correspond to higher powers. The numerical results are shown in Fig. 4.35(a).

In general comparisons between the shapes of the numerically determined period

Longitudinal Mode Spectra vs Modulation Frequency

Numerical Results



Experimental Results

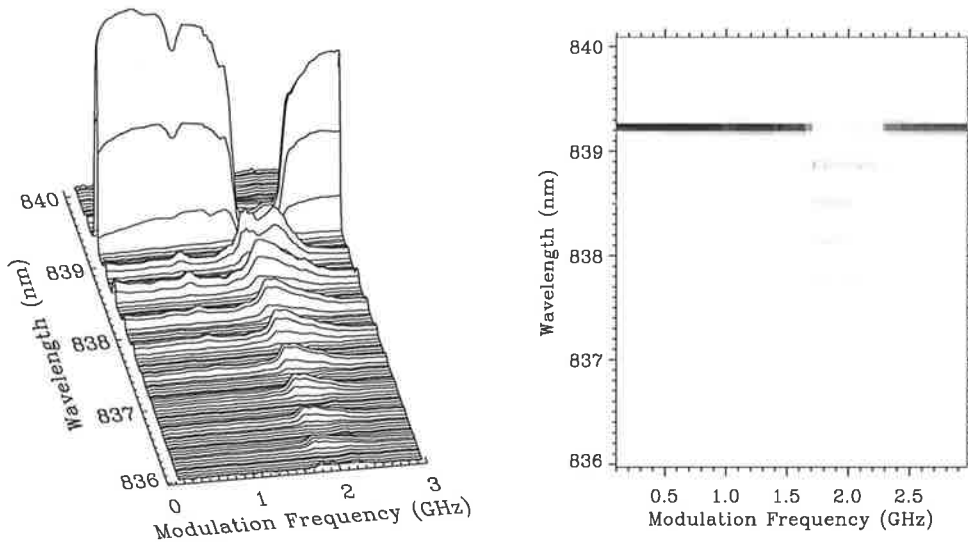
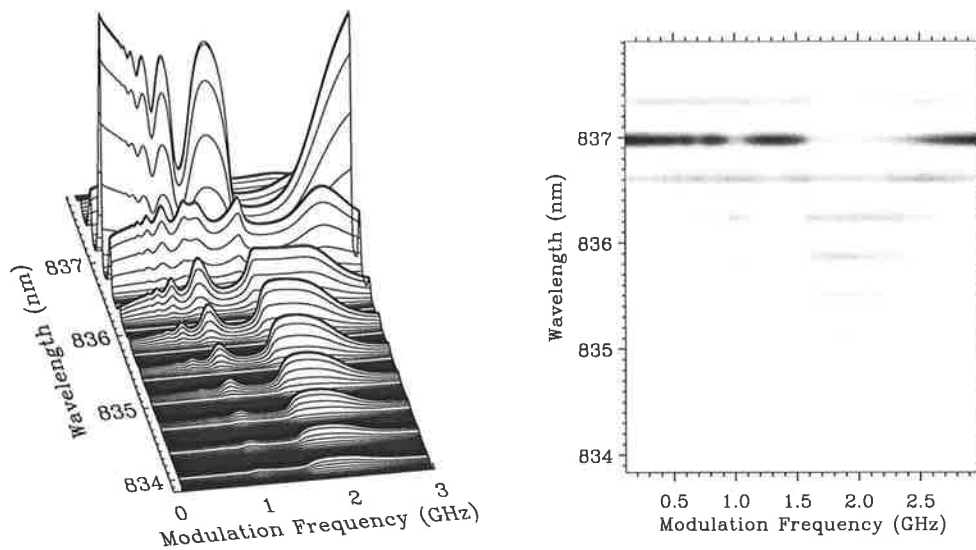


Figure 4.33: Time average longitudinal mode spectra versus modulation frequency. The numerical results are shown in the upper figure and the experimental results are shown in the lower figure. The modulation power was -5dBm.

Longitudinal Mode Spectra vs Modulation Frequency

Numerical Results



Experimental Results

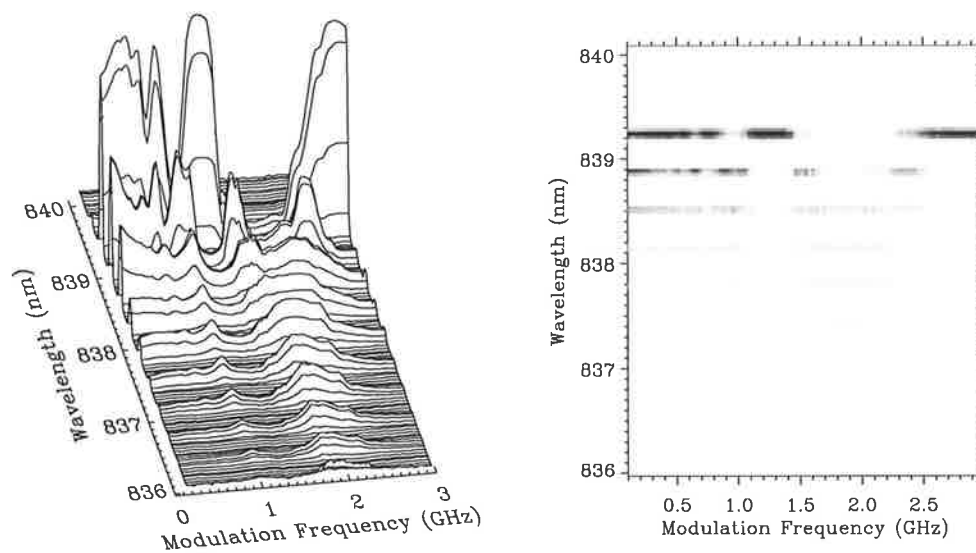


Figure 4.34: Time average longitudinal mode spectra versus modulation frequency. The numerical results are shown in the upper figure and the experimental results are shown in the lower figure. The modulation power was -1.5dBm.

doubling regions and those measured experimentally is complicated by the presence of noise in the real laser. This arises from the fact that, close to period doubling, there is enhancement of the noise at $f/2$ which acts to obscure the actual bifurcation point[3, 186]. We discuss this further in chapter 5. Such comparisons are further complicated by the fact that the frequency response of the laser diode circuit is not flat and therefore, for a constant input modulation power from the signal generator, we expect the effective modulation index to vary as a function of the modulation frequency. Thus, these effects should be taken into consideration when comparing the numerical and experimental results.

Comparison of Figs. 4.35(a) and (b) shows good agreement between theory and experiment for the lower modulation indices. Both the minimum modulation index required for period doubling and the width and position of the period doubling region as a function of the modulation frequency are consistent. Furthermore, only a single period doubling region occurs in both Figs. 4.35(a) and (b) with no further bifurcations to a period four attractor predicted numerically, consistent with the experimental observations. The main discrepancy between Figs. 4.35(a) and (b) is that experimentally the period doubled attractor disappears when the modulation index is increased. This is not predicted by the multimode rate equations indicating the absence of one, or several, physical characteristics in the underlying equations that are important under these operating conditions. Possible candidates include nonlinear gain saturation or circuit parasitics in either the drive circuit or the semiconductor device itself²⁵.

4.9 Concluding Remarks

Due to the complicated nature of the semiconductor system, any description will represent, of necessity, a significant approximation to the actual device. Therefore, the question arises as to what constitutes a reasonable approximation and the answer is always reliant upon the particular physical characteristic that is being represented. In this chapter we have performed a comparative study of the bifurcation scenarios under sinusoidal current modulation predicted by several rate equation models of varying complexity.

The simplest model, the single mode rate equations, has been widely studied. However, it is well known that this model predicts many phenomena that have never been observed experimentally. Previous analysis have shown, however, that the inclusion of other important physical characteristics, such as gain saturation, laser noise, etc. leads to a de-

²⁵Separate calculations show that an increase in the value of k (approximately double) is sufficient to cause the period doubled attractor to disappear with increasing modulation index consistent with experimental results. Thus, an underestimate of the value of k could be a possible cause of the differences in Figs. 4.35(a) and (b) for the higher modulation indices. A possible cause for underestimating k from the sub-threshold laser gain measurements of Section 3.3.2 could arise, for example, if the temperature of the active region varies for different bias currents.

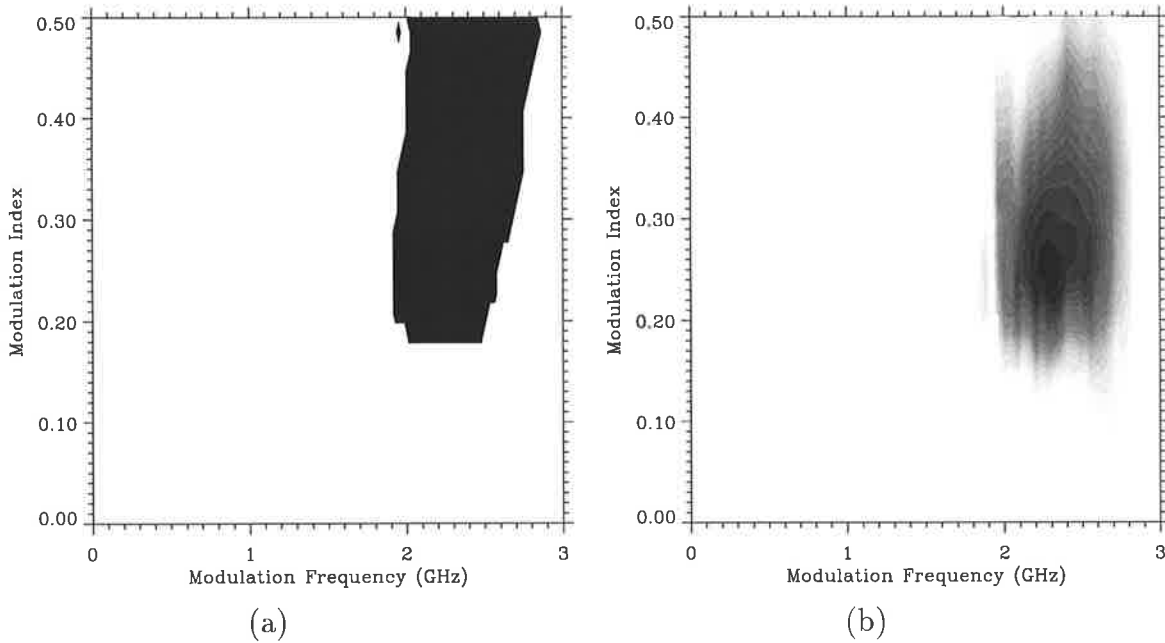


Figure 4.35: Numerical (a) and experimental (b) period doubling regions for $I_b = 1.095$ ($I_{DC} = 23\text{mA}$).

scription more consistent with experimental observations.

In our study the effects of band-filling and multiple longitudinal modes on the predicted bifurcation behaviour were considered. To our knowledge, neither of these effects have been considered in previous studies of the bifurcation scenarios of current modulated FP semiconductor lasers. It was shown that the inclusion of multiple modes acts to reduce the extent of the bifurcation phenomena. Such predictions are consistent with the bifurcations (or lack thereof) observed experimentally in FP semiconductor lasers. The importance of band-filling was also investigated. It was shown that the inclusion of band-filling results in changes in both the bifurcation behaviour predictions and the time averaged longitudinal mode behaviour. Moreover, the predictions of the multimode model with band-filling are the most closely in agreement with experiment. A distinguishing feature of the experiments was that when a harmonic of the driving frequency was coincident with the relaxation oscillation frequency an increase in the number of operating modes occurred. Such behaviour was predicted by both multimode models investigated. However, the inclusion of band-filling led to a spectral shift of the mode spectra towards shorter wavelengths in agreement with experimental observations, thus demonstrating the importance of this effect. On the contrary, in the multimode model without band-filling, the mode spectra were symmetric about the central mode.

A closer investigation into the nonlinear dynamics predicted by the multimode model with band-filling showed that band-filling has implications for the bifurcation scenarios predicted under sinusoidal modulation. We have demonstrated that for some combinations of

parameters a Hopf bifurcation is predicted. This phenomenon is distinguished by the presence of side bands on the modulation frequency and its harmonics in the power spectra of the total intensity; a signature readily detectable in an experiment. We defer experimental investigations into this phenomena until Chapter 5.

In the single mode case, hysteresis has been studied extensively. However, to our knowledge the study presented here of this phenomenon in the multimode system, is the first undertaken. We have shown that the hysteresis region is distinguished by an abrupt jump to an adjacent lasing mode, with a change in the value of the modulation frequency at which the mode hop occurs, depending on whether the frequency is increased or decreased past the bifurcation points. Though we have not observed hysteresis experimentally, we have observed a change in the laser mode in the region corresponding to this phenomenon.

Chapter 5

Approach to Bifurcations

5.1 Introduction

In this chapter, we discuss the phenomenon of bifurcations in nonlinear dissipative systems in greater detail. A bifurcation occurs when an attractor becomes unstable and the manner in which this happens affects the resulting system behaviour. It is well known that systems close to an instability are sensitive to small perturbations, such as that arising from the presence of noise for example¹; since bifurcations are synonymous with instability, a system close to a bifurcation point is extremely sensitive to external noise. In previous studies[186, 187, 188, 12, 193, 194, 195], it was shown that a system's sensitivity to fluctuations near an instability manifests itself as structure in the power spectrum. Such structures are referred to as noisy precursors since they are specific to the type of bifurcation occurring and as a consequence can be used to predict the type of instability to be encountered before it has actually occurred[186, 12]. Therefore, observation of noisy power spectra provides a means of probing the system dynamics. As a semiconductor laser is a physical device interacting with its environment, noise is an inherent property of this system. In this chapter we discuss the relevance of noisy precursors to modulated semiconductor lasers.

This chapter is organised as follows. In section 5.2 we discuss pertinent technical aspects of nonlinear systems theory. In particular, we discuss the stability properties of fixed point and periodic attractors and the techniques used to perform numerical integrations of stochastic differential equations. In section 5.3 generic properties of period doubling, saddle-node and Hopf Bifurcations are discussed. In sections 5.4, 5.5 and 5.6 we consider these bifurcations in the single mode and multimode rate equations. Finally in section 5.7 the implications for the global bifurcation behaviour are discussed.

¹See for example references [186, 187, 188, 12, 189, 190, 191, 192].

5.2 Nonlinear Differential Equations

5.2.1 Linear Stability Analysis

The phenomenon of instability is a very important aspect of nonlinear dynamical systems. Instabilities are responsible for the creation of new attractors through bifurcations and ultimately for the occurrence of irregular motions[13]. In this section we discuss the stability properties of fixed point and periodic solutions of nonlinear dissipative systems. The discussion follows closely that of [183].

Characterisation of Fixed points

Consider a dynamical system in the variable $\vec{x}$ governed by equations

$$\dot{\vec{x}} = \vec{Q}(\vec{x}, \lambda), \quad \vec{x} \in R^N \quad (5.1)$$

where λ denotes one or more control parameters. Assume we have a fixed point solution, $\vec{x}^*$; this implies that $\vec{x}^*$ satisfies

$$\vec{Q}(\vec{x}^*) = 0. \quad (5.2)$$

To determine the stability of this fixed point we first linearise equation 5.1 about the fixed point. For simplicity, initially we will consider the one dimensional case. Performing a Taylor expansion of $Q(x)$ about x^* we obtain

$$\dot{x} = Q(x) \approx Q(x^*) + \left. \frac{dQ}{dx} \right|_{x=x^*} (x - x^*) + \dots \quad (5.3)$$

We define a new variable $\xi = x - x^*$ and set $\lambda = \left. \frac{dQ}{dx} \right|_{x=x^*}$. Using equation 5.2, equation 5.3 becomes

$$\dot{\xi} = \lambda \xi. \quad (5.4)$$

This is a linear differential equation that has solution $\xi(t) = \xi(0)e^{\lambda t}$. Defining $\mu = e^{\lambda}$ we write can this equation as $\xi(t) = \xi(0)\mu^t$. Thus, the constant μ (or equivalently λ) determines the stability of the fixed point. If $|\mu| < 1$ then $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$ and hence $x \rightarrow x^*$ and so the fixed point is stable. If $|\mu| > 1$ then $\xi(t) \rightarrow \infty$ and so the fixed point is unstable. The constant μ is often referred to as the characteristic multiplier and λ as the characteristic exponent. The stability of the fixed point can also be defined in terms of the sign of the characteristic exponent, λ . If $\lambda < 0$ then $\mu < 1$ and the fixed point is stable. Similarly if $\lambda > 0$ then $\mu > 1$ and the fixed point is unstable.

Generalising this argument to an N-dimensional system we obtain the linearised

equation

$$\vec{\xi} = DQ(\vec{x}^*)\vec{\xi} \quad (5.5)$$

where $\vec{\xi} = \vec{x} - \vec{x}^*$ and $DQ_{ij} = \partial Q_i / \partial x_j$ is the Jacobian matrix of $\vec{Q}$ evaluated at the fixed point, $\vec{x}^*$. Let the eigenvalues of $DQ(\vec{x}^*)$ be λ_i and the corresponding eigenvectors $\vec{\eta}_i$. For distinct eigenvalues², the trajectories with initial conditions, $\vec{x}_o = \vec{x}^* + \vec{\xi}_o$, have solution (to first order)

$$\begin{aligned} \vec{x}(t) &= \vec{x}^* + \vec{\xi}(t) \\ &= \vec{x}^* + \exp(DQ(\vec{x}^*)t)\vec{\xi}_o \\ &= \vec{x}^* + C_1 e^{\lambda_1 t} \vec{\eta}_1 + \cdots + C_N e^{\lambda_N t} \vec{\eta}_N \end{aligned} \quad (5.6)$$

where $C_i \in \mathbb{C}$ are constants chosen to give the correct initial condition. Since the matrix $DQ(\vec{x}^*)$ is real, its eigenvalues and eigenvectors are either real or come in complex conjugate pairs. For λ_i real, C_i and $\vec{\eta}_i$ are also real and equation 5.6 shows that λ_i gives the rate of contraction ($\lambda_i < 0$) or expansion ($\lambda_i > 0$) in the direction of $\vec{\eta}_i$. For λ_i complex, we have $\lambda_{i+1} = \bar{\lambda}_i$, $\vec{\eta}_{i+1} = \bar{\vec{\eta}}_i$ and $C_{i+1} = \bar{C}_i$. Setting $\lambda_i = \alpha_i + i\beta_i$ we can write

$$\begin{aligned} C_i e^{\lambda_i t} \vec{\eta}_i + C_{i+1} e^{\lambda_{i+1} t} \vec{\eta}_{i+1} &= 2\Re[C_i e^{\lambda_i t} \vec{\eta}_i] \\ &= 2|C_i| e^{\alpha_i t} \{\cos(\beta_i t + \theta_i) \Re(\vec{\eta}_i) - \sin(\beta_i t + \theta_i) \Im(\vec{\eta}_i)\} \end{aligned} \quad (5.7)$$

where $\theta_i = \arg(C_i)$. This equation is the parametric equation of a spiral on the plane spanned by the vectors $\Re(\vec{\eta}_i)$ and $\Im(\vec{\eta}_i)$. Similarly to the case of real eigenvalues, $\Re(\lambda_i) = \alpha_i$ gives the rate of contraction ($\alpha_i < 0$) or expansion ($\alpha_i > 0$) of the spiral and the imaginary part, β_i , gives the frequency of rotation.

Combining the results for both real and complex cases we can conclude that the real part of the eigenvalues of the Jacobian matrix determine the stability of the fixed point. In particular, if $\Re(\lambda_i) < 0$ for all i then $\vec{x}^*$ is asymptotically stable. If on the other hand $\Re(\lambda_i) > 0$ for all i then $\vec{x}^*$ is unstable. If there exists i and j such that $\Re(\lambda_i) < 0$ and $\Re(\lambda_j) > 0$ then $\vec{x}^*$ is a saddle point. We shall also refer to saddle points as unstable fixed points.

In the above classification fixed points whose Jacobians have eigenvalues with zero real part have not been discussed. These are termed non-hyperbolic fixed points³ and play a special role in bifurcation phenomena.

²The case of repeated eigenvalues is discussed in [183].

³Hyperbolic fixed points, with all eigenvalues with non-zero real part, are the types of fixed points that are typically found in dynamical systems. Non-hyperbolic points do not occur as frequently; in the language of dynamical systems theory they are said to be non-generic[196].

Characterisation of Periodic Solutions

In this section the stability of limit cycles (or periodic cycles) will be discussed. In particular, we will consider a N^{th} order non-autonomous system of the form

$$\dot{\vec{x}} = \vec{Q}(\vec{x}) + \vec{g}(t) \quad (5.8)$$

where $\vec{g}(t) = \vec{g}(t + T)$ is a periodic forcing term. We assume equation 5.8 has a periodic solution, which we denote by $\vec{x}_p(t)$, of period T :

$$\vec{x}_p(t) = \vec{x}_p(t + T) \quad (5.9)$$

where $\vec{x}_o = \vec{x}_p(t_o)$ is the initial condition. Determining the stability of a periodic solution is most easily facilitated through use of the Poincare map⁴. The Poincare map, $\vec{P}_\Sigma$, samples the trajectory $\vec{x}_p$ at a rate equal to the driving frequency $f = 1/T$. Hence a periodic solution of period T will correspond to the fixed point, $\vec{x}^*$, of the Poincare map. Furthermore, the stability properties of the periodic solution and the fixed point are equivalent[183]. To determine the stability of $\vec{x}^*$ we need to determine how points near $\vec{x}^*$ behave under the influence of $\vec{P}_\Sigma$. To do this, we linearise equation 4.5⁵, $x_{k+1} = P_\Sigma(x_k)$, as in the continuous case, to obtain

$$\vec{\sigma}_{k+1} = DP(\vec{x}^*)\vec{\sigma}_k \quad (5.10)$$

where $\vec{\sigma}_k = \vec{x}_k - \vec{x}^*$ and $DP(\vec{x}^*)$ is the Jacobian matrix of $\vec{P}_\Sigma(\vec{x})$ evaluated at $\vec{x}^*$. Let the eigenvalues of $DP(\vec{x}^*)$ be $m_i \in \mathbb{C}$ with corresponding eigenvectors, $\vec{\eta}_i \in \mathbb{C}^{N-1}$ where $i = 1, \dots, N-1$. The eigenvalues m_i are also referred to as characteristic multipliers or Floquet multipliers. Since $DP(\vec{x}^*)$ is real then m_i and $\vec{\eta}_i$ are real or come in complex conjugate pairs. Assuming eigenvalues are distinct⁶ we can write (to first order)

$$\begin{aligned} \vec{\sigma}_k &= DP(\vec{x}^*)\vec{\sigma}_o \\ &= C_1 m_1^k \vec{\eta}_1 + \dots + C_{N-1} m_{N-1}^k \vec{\eta}_{N-1} \end{aligned} \quad (5.11)$$

where $C_i \in \mathbb{C}$ are constants chosen to satisfy the initial conditions. For real eigenvalues, we can see from equation 5.11, that the magnitudes of m_i , give the amount of contraction ($|m_i| < 1$) or expansion ($|m_i| > 1$) near $\vec{x}^*$ along the direction $\vec{\eta}_i$ for one iteration of the map. For m_i complex we have

$$C_{i+1} m_{i+1}^k \vec{\eta}_{k+1} = \bar{C}_i \bar{m}_i^k \vec{\eta}_i. \quad (5.12)$$

⁴Floquet Multiplier theory can also be used for this purpose[197].

⁵The index k refers to the iterates of the Poincare map, not the components of the vector x .

⁶The case of repeated eigenvalues is discussed in [183]

Setting $m_i = \gamma_i e^{i\phi_i}$ and using 5.12 gives

$$\begin{aligned} C_i m_i^k \vec{\eta}_i + C_{i+1} m_{i+1}^k \vec{\eta}_{i+1} &= 2\Re(C_i m_i^k \vec{\eta}_i) \\ &= 2|C_i| |\gamma_i|^k \{ \cos(k\phi_i + \theta_i) \Re(\vec{\eta}_i) - \sin(k\phi_i + \theta_i) \Im(\vec{\eta}_i) \} \end{aligned} \quad (5.13)$$

where $\theta_i = \arg(C_i)$. Equation 5.13 is the parametric equation of a sampled spiral on the plane spanned by the vectors $\Re(\vec{\eta}_i)$ and $\Im(\vec{\eta}_i)$. We can see from equation 5.13 that the magnitude, $|m_i| = \gamma_i$, gives the amount of expansion ($|m_i| > 1$) or contraction ($|m_i| < 1$) of the spiral for one iteration of the map. The angle of the multiplier in the complex plane, $\arg(m_i) = \phi_i$, gives the frequency of rotation.

Combining the results for real and complex multipliers we conclude that it is the magnitude of the multiplier that determines the stability. In particular, a fixed point of $\vec{P}_\Sigma(\vec{x})$, with $|m_i| < 1$ for all i is stable. Otherwise it is unstable. A fixed point for which there exist m_i and m_j such that $|m_i| > 1$ and $|m_j| < 1$ will sometimes be referred to as saddle point. In these arguments I have assumed that $|m_i| \neq 1$. Points for which at least one multiplier has magnitude equal to one are called non-hyperbolic. These points play a special role in bifurcations. Though the matrix $DP(\vec{x}^*)$ in general depends on the choice of the Poincare plane, Σ , its eigenvalues are in fact independent of Σ [183]. Thus they may be regarded as a property of the periodic cycle as well as the fixed point, $\vec{x}^*$.

If $\vec{x}_p(t)$ is a periodic solution with period mT where T is the period of the forcing term then $\vec{x}^* = \vec{x}_p(mT)$ will be a fixed point of the m^{th} iterate of the Poincare map, $\vec{P}_\Sigma^m(\vec{x})$. All the proceeding arguments can also be applied to this case by substituting $\vec{P}_\Sigma^m(\vec{x})$ for $\vec{P}_\Sigma(\vec{x})$ [183].

Summary

I wish to briefly summarise the main results of this section.

1. For an N -dimensional system governed by $\dot{\vec{x}} = \vec{Q}(\vec{x})$ the stability of a fixed point, $\vec{x}^*$, of this system is determined by the sign of the real part of the eigenvalues, $\lambda_i = \alpha_i + i\beta_i$, of the matrix $DQ(\vec{x}^*)$. In particular, a fixed point is stable if all its eigenvalues have negative real parts. When the eigenvalues of $DQ(\vec{x}^*)$ are complex then the flow of the vector field near $\vec{x}^*$ spirals about $\vec{x}_p$ and the frequency of rotation is given by the imaginary part of λ_i .
2. For an $N - 1$ dimensional system governed by the map $\vec{x}_{k+1} = \vec{P}(\vec{x}_k)$, the stability of a fixed point, $\vec{x}^*$, of this map is determined by the magnitude of the eigenvalues, $m_i = \gamma_i e^{i\phi_i}$ of $DP(\vec{x}^*)$. The fixed point is stable for all $|m_i| < 1$ i.e. for all multipliers within the unit circle. The angle of m_i in the complex plane gives the frequency of rotation. The eigenvalues m_i are called characteristic multipliers or Floquet multipliers.

Since a periodic solution corresponds to a fixed point of the Poincare map, the Poincare map can be used to determine the stability of the periodic solution.

In order to characterise the stability of periodic solutions of the rate equations the matrix $DP(x^*)$ is required. Fortunately this matrix may be obtained numerically. The method used to obtain $DP(x^*)$ is discussed in Appendix F.

5.2.2 Stochastic Differential Equations

In order to account for the random fluctuations in photon and carrier density we need to include Langevin noise terms, $F(x, t)$, in the rate equations[36, 103, 198, 107]. This leads to a set of nonlinear stochastic differential equations of the form

$$\begin{aligned}\dot{x}_1 &= Q_1(x_1, x_2, \dots, x_N, \lambda) + F_1(x_1, x_2, \dots, x_N, t) \\ \dot{x}_2 &= Q_2(x_1, x_2, \dots, x_N, \lambda) + F_2(x_1, x_2, \dots, x_N, t) \\ &\vdots \\ \dot{x}_N &= Q_N(x_1, x_2, \dots, x_N, \lambda) + F_N(x_1, x_2, \dots, x_N, t).\end{aligned}\tag{5.14}$$

The Langevin forces are assumed to be Gaussian random processes with zero mean: $\langle F_i(t) \rangle = 0$ [64, 107]. In the Markov assumption the auto-correlation and cross-correlation functions are proportional to Dirac delta functions[64, 87, 36]:

$$\begin{aligned}\langle F_i(t)F_i(t') \rangle &= V_i^2 \delta(t - t') \\ \langle F_i(t)F_j(t') \rangle &= r_{ij} V_i V_j \delta(t - t')\end{aligned}\tag{5.15}$$

where V_i^2 are the variances of $F_i(t)$ and r_{ij} is the correlation coefficient.

In the rate equations the variances depend on the dynamical variables[107]. This leads to a difficulty in the mathematical interpretation of equations 5.14 due to the fact that the noise function has no correlation time. Since any fluctuation in $F(t)$ results in a fluctuation in x , it is not clear at which value of x , $F(t)$ should be evaluated. i.e. before or after or at some point during the fluctuations[199]⁷. For our purposes, however, it is sufficient to approximate the delta function in equation 5.15 by a function of finite width since, when performing numerical integrations, it is necessary to consider the stochastic functions $F_i(t)$ over discrete time intervals, Δt , where Δt is the integration time step. To this end, we

⁷This issue cannot be resolved mathematically and therefore requires an additional specification. There are two such specifications: the Ito and Stratonovich definitions[199]. The Stratonovich approach is the more physical approach and leads to an additional “noise-induced” drift term. The Ito definition, on the other hand, requires new rules of integration[199]. The approach adopted here makes use of the Stratonovich definition.

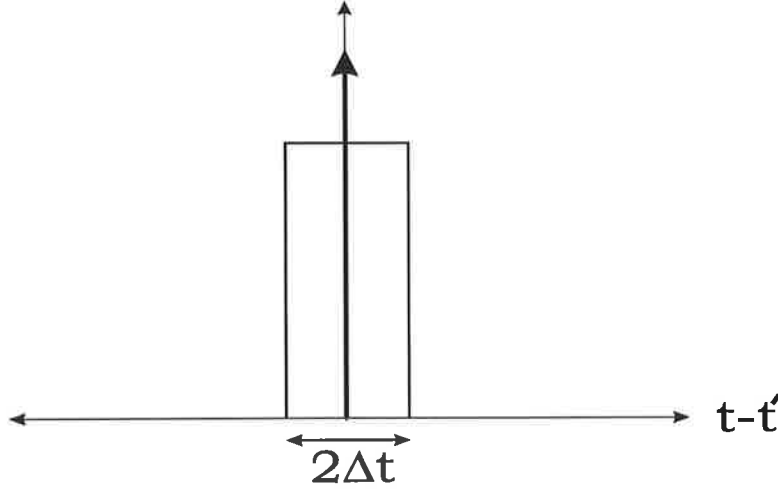


Figure 5.1:

approximate the auto-correlation function by[107]

$$\langle F_i(t)F_i(t') \rangle = \begin{cases} \frac{V_i^2}{\Delta t} & \text{for } |t - t'| < \Delta t \\ 0 & \text{for } |t - t'| > \Delta t \end{cases} \quad (5.16)$$

This function is illustrated schematically in Fig. 5.1. Thus, the stochastic function F_i , in any given time interval Δt , is given by[107]

$$F_i = \frac{V_i}{\sqrt{\Delta t}} X_i \quad (5.17)$$

where X_i is a Gaussian random variable of zero mean and unit variance. The random variables X_i are generated by means of the following equation[184]

$$X_i = \sin(2\pi y) \sqrt{-2 \ln(x)}, \quad (5.18)$$

where x and y are random variables uniformly distributed on the unit interval. As the integration time step, Δt (on the order of 10^{-12} s) is much less than a typical relaxation oscillation period (the order of 10^{-9} s), it ensures that the noise spectrum is approximately white in the region of interest (i.e. < 5 GHz)[3].

5.3 Bifurcations and Transients

5.3.1 Nonlinear Oscillators

Before discussing specific features of the bifurcations predicted by the semiconductor laser rate equations we first digress and discuss the relation between the single mode rate equations and the general class of nonlinear oscillators. As previously discussed, for small amplitude modulation the laser behaves approximately like a damped linear oscillator with a resonant frequency f_{RO} given by equation D.17. A forced damped linear oscillator obeys a differential equation of the form[200]

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = F(t). \quad (5.19)$$

For small damping ($\gamma \ll \omega_o$), such an oscillator has a resonant frequency equal to ω_o . When driven by a sinusoidal driving force this system exhibits a resonant peak centred at ω_o ; this is illustrated in Fig. 5.3(a). The position of the resonant peak is independent of the strength of the driving force, $F(t)$.

For large amplitude modulation, the linearised description is no longer valid and nonlinear effects come into play. By defining the variables N and P in terms of new variables x and y such that

$$N(t) = y(t) + 1 - (1 - \delta)e^{-x(t)} \quad (5.20)$$

$$P(t) = \beta(1 - \delta)e^{x(t)} - \beta(1 - \delta) \quad (5.21)$$

we can convert the coupled rate equations for N and P into a single differential equation in the variable x of the form⁸

$$\ddot{x} + \gamma(x)\dot{x} + f(x)x = F(t) \quad (5.22)$$

where now both the damping, $\gamma(x)$, and the ‘force’, $f(x)$ are nonlinear functions of x . We plot the function $f(x)$ (solid curve) in Fig. 5.2 where the linear ‘force’ (dashed curve) is also plotted for comparison. Figure 5.2 shows that, in the vicinity of x_o , a linear approximation to $f(x)$ suffices leading to behaviour similar to a damped linear oscillator in this regime; thus we expect a peak in the response curve centred at the resonant frequency of the linearised system. For larger values of x , the nonlinear laser ‘force’ is obviously very different from the linear ‘force’. This gives rise to the general characteristic of nonlinear oscillators that the resonant frequency is dependent upon the frequency and amplitude of the sinusoidal driving force[201]. In this case we observe a leaning over of the resonance curves eventually resulting in hysteresis (Fig. 5.3(b))[201].

It is instructive to make an analogy between linear springs and hard and soft springs.

⁸See appendix D, section D.1 for details. See also references [9, 24, 10].

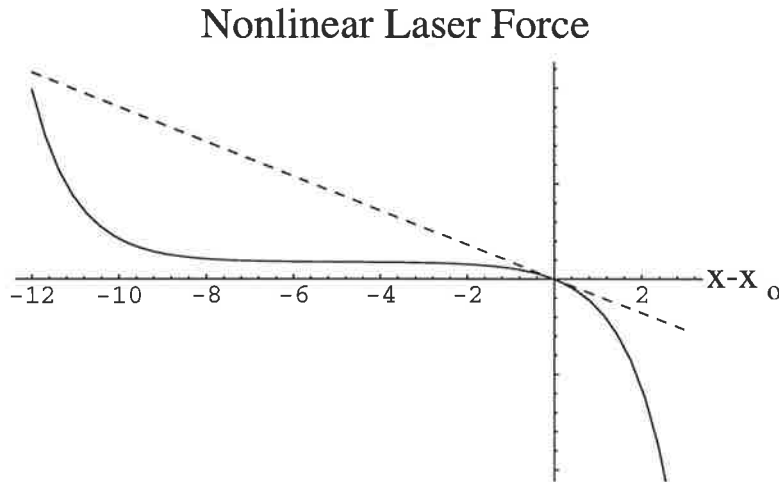


Figure 5.2: The nonlinear ‘force’ for the single mode laser rate equations (solid curve). The dashed curve corresponds to the ‘force’ for the linearised equations.

In hard springs the natural frequency is tuned to a higher frequency as the modulation amplitude is increased[202]. The corresponding behaviour for a soft spring is that the natural frequency is tuned to a lower frequency as the modulation amplitude is increased[202]; hard spring behaviour is illustrated in the resonance curves of Fig. 5.3[201]. Previous studies[3, 9] of the semiconductor laser rate equations show that the behaviour is similar to that of a soft spring: the natural frequency is tuned to a lower frequency.

Another property of the resonances of a nonlinear oscillator is that they may become undamped as a control parameter is tuned. Such behaviour is called a bifurcation and results in the genesis of a new attractor. This important difference between linear and nonlinear systems is responsible for the vast array of behaviour exhibited by nonlinear systems, including strange attractors and chaotic motion, as opposed to the fixed point and periodic attractors of a linear system[196, 13]. Thus, the behaviour of the resonances of a nonlinear system has implications for the bifurcation scenarios. We discuss this in the remainder of this chapter.

5.3.2 Overview

Qualitative changes in the topology of trajectories in the phase space as one or more control parameters are tuned are termed bifurcations. In this section we discuss the types of bifurcations that occur in general nonlinear systems. We will be concerned only with local bifurcations that depend on one control parameter⁹. The occurrence of these types of bifurcations may be detected by performing a local linear stability analysis, such as that

⁹Such bifurcations are termed codimension-1 bifurcations[196].

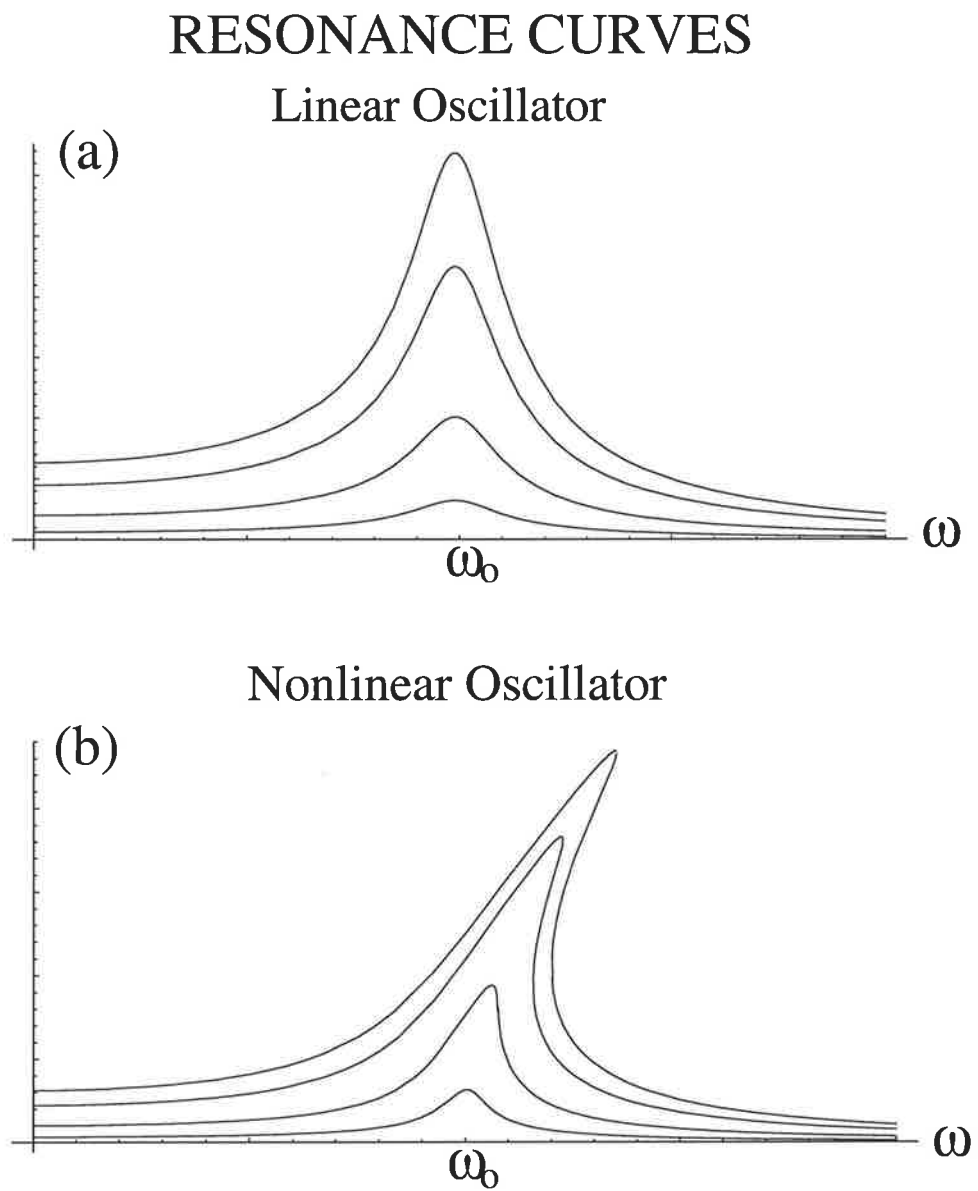


Figure 5.3: Resonance curves with increasing modulation amplitude for (a) a linear oscillator and (b) hard spring nonlinear oscillator.

described in the previous section. A bifurcation occurs when an attractor loses stability. For a limit cycle, this implies a multiplier exits the unit circle, whereas for a fixed point a pair of multipliers cross the imaginary axis into the positive half plane[183]. In both cases, at the bifurcation point, the solution becomes non-hyperbolic.

We are primarily interested in bifurcations from a periodic attractor. The way in which the multipliers exit the unit circle determines the type of bifurcation that occurs and consequently the character of the resulting system behaviour. In this section three types of bifurcations are discussed: period doubling, saddle-node and Hopf bifurcations. These are summarised as follows[12]:

1. If a single real multiplier exits the unit circle at -1 a period doubling bifurcation occurs. After the bifurcation the resulting attractor has twice the period of the original.
2. If a pair of complex multipliers exit the unit circle at $e^{\pm i\theta}$ a Hopf bifurcation occurs. This usually results in two frequency dynamics.
3. If a single real multiplier exits the unit circle at +1 a saddle-node bifurcation occurs. This is typically accompanied by a sudden change in the systems response and hysteresis¹⁰.

For small deviations from the stable fixed point, $\vec{x}^*$, such that a linearised description is valid, the multipliers determine the transient behaviour as well as the stability of $\vec{x}^*$. It is the largest $|m_i|$ that have the most influence, since the transient behaviour governed by these takes the longest to decay (see equation 5.11). This observation is particularly important for noise driven systems and especially for systems near an instability[187, 12]. For a deterministic system, such as that described by equations 5.1, once the steady state is reached the system remains there for all time. However, for a noise driven system, one consequence of the presence of noise is that it continually perturbs the system from its stable state. Therefore, the transient behaviour contributes significantly to the observed power spectrum[12]. This observation is particularly relevant to systems close to an instability since in this case there will exist one (or several) multipliers close to the unit circle. Equation 5.11 shows that the closer $|m_i|$ is to the unit circle the longer the transient orbit takes to decay and consequently contributes more to the observed power spectrum. Therefore, we expect that, close to an instability, the system's sensitivity to fluctuations will manifest itself as structure in the power spectrum[186, 187, 12]. Since the different types of bifurcations result from different behaviour of the multipliers the resulting structure observed in the power spectrum will be different in each case. Thus, it should be possible to predict the type of instability to be encountered before the onset of the bifurcation[187]. Such structures in

¹⁰We note that two other types of bifurcations, the transcritical and pitchfork bifurcations, are possible for this case. However, these rely on additional constraints such as the presence of symmetry[13].

the power spectrum are referred to as noisy precursors since they act as a precursor to the bifurcation[12].

In the above discussion we have considered the noise as a perturbation to the deterministic system. One fundamental assumption we have made is that the noise does not alter the contribution of the unperturbed dynamics[12]. In other words, we assume that the noise level is small enough such that the linearised equations 5.11 constitute a good approximation to the full nonlinear system in the presence of noise. As the noise level is increased, nonlinear effects will eventually dominate the behaviour and the above picture will be modified. Such nonlinear effects have been observed in some systems[193]. However, for our purposes a linearised description is adequate.

In the following sections the relationship between the eigenvalues of $DP(x^*)$ and the transient behaviour, both for the Poincare map and for the original flow, are elucidated. There are three main cases of interest: period doubling, Hopf and saddle-node bifurcations. The following discussion is based on the description given in [12].

5.3.3 Period Doubling Bifurcations

The first case corresponds to systems in which the largest eigenvalue, which we denote m_1 , is real and negative such that $|m_1| < 1$. Furthermore, we assume that $|m_1| \gg |m_i|$ for all $i \neq 1$. Since $|m_1|$ is the largest eigenvalue the decay time of the dynamics in the direction η_1 will be longest. Thus, close to the fixed point, the dynamics collapses to a one dimensional manifold along the direction η_1 . In this case the dynamics can be approximated by the one dimensional linearised map

$$\sigma_{k+1} = m_1 \sigma_k = m_1^k \sigma_o \quad (5.23)$$

where $\sigma_k = x_k - x^*$. Due to the negative value of m_1 successive iterates oscillate in sign as they decay to x^* . Therefore, close to x^* , the sequence $x_0, x_1, x_2, \dots$ asymptotically approach x^* in an alternating manner, along the direction $\vec{\eta}_1$ in the Poincare plane, as illustrated schematically in Fig. 5.4. Thus, the transient part of the orbit is a damped periodic oscillation of period $2T$. Accordingly, the noisy precursor for this instability are peaks in the power spectrum centred at frequencies $nf/2$, for n an odd integer, and $f = 1/T$ where T is the period of the attractor. This is illustrated in Fig. 5.5. As the bifurcation point is approached $|m_1|$ gets closer and closer to one and therefore, according to equation 5.23, the transient orbit takes an increasingly longer time to decay; at the bifurcation point the decay time tends to infinity. Thus, we expect that the noisy precursors centred at frequencies $nf/2$ will become higher and sharper as the bifurcation point is approached. Quantitative investigations of the width and height of the noisy precursor peak in the approach to a period doubling bifurcation can be found in reference[187, 203].

Loss of stability of $x_p(t)$ (or equivalently x^*) occurs when m_1 exits the unit circle

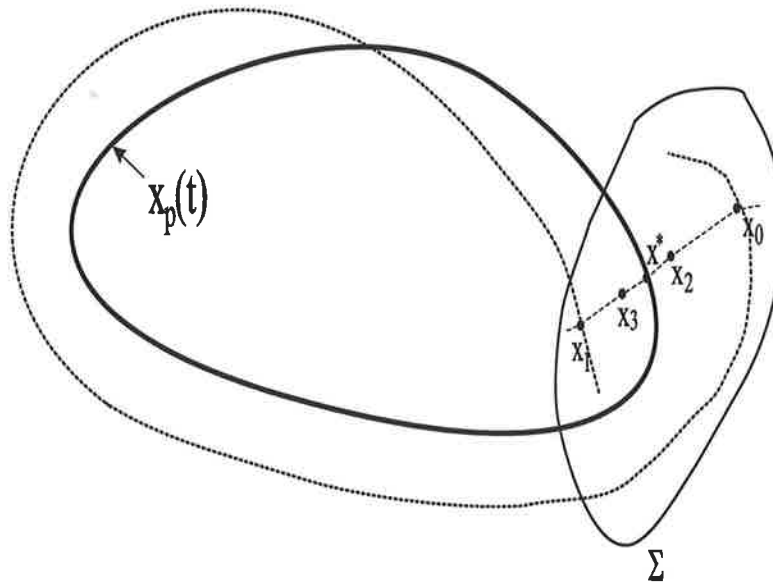


Figure 5.4: Transient approach to the steady state for a negative real eigenvalue.

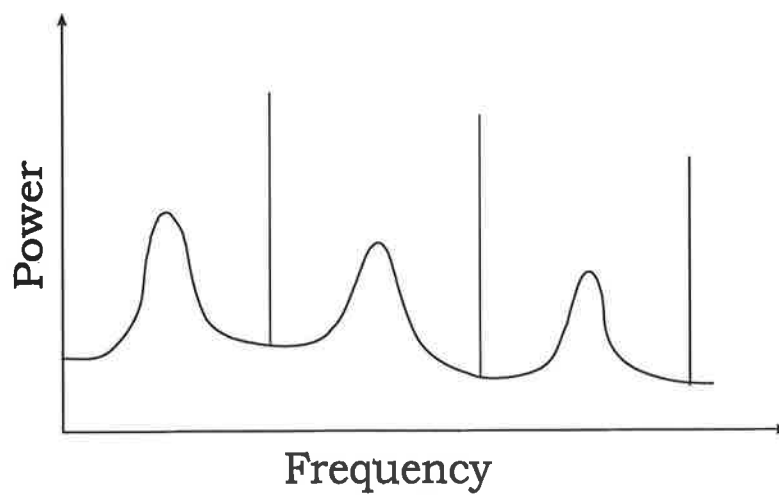


Figure 5.5: Noisy precursor peaks in the power spectrum for a period doubling bifurcation showing resonant peaks lying midway between the harmonics of f .

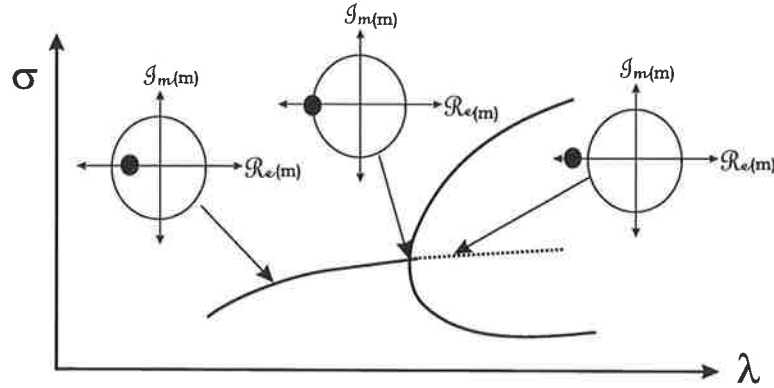


Figure 5.6: Schematic bifurcation diagram showing typical behaviour for a period doubling bifurcation. Solid (dotted) curves indicate a stable (unstable) attractor.

at -1 as a control parameter is tuned; this results in a period doubling bifurcation. After the bifurcation the resulting attractor has period $2T$. The original period T attractor still exists but is unstable. This is illustrated schematically in Fig. 5.6 where a typical bifurcation diagram for the period doubling bifurcation is shown.

5.3.4 Hopf Bifurcations

Consider now the case of complex conjugate eigenvalues $m_1 = \bar{m}_2 = \gamma e^{i\theta}$ where $\gamma < 1$. According to equation 5.13, the transient behaviour is a sampled spiral in the Poincare plane such that γ gives the amount of contraction of the spiral for one iteration of the linearised map. The angle θ of the multipliers in the complex plane gives the frequency of rotation, meaning each successive iterate is rotated by θ radians from the previous one¹¹. The angle θ is related to the transient oscillation frequency for the flow, for trajectories close to the periodic solution, $\vec{x}_p(t)$. It is easiest to understand this relationship by visualising the behaviour in the three dimensional case. Equation 5.11 describes how trajectories close to $x_p(t)$ behave and in general they will be twisted around it as illustrated in Fig. 5.7. The number of twists per period T corresponds to the ratio of the transient frequency to the driving frequency. The transient frequency constitutes a rotation in the direction θ as shown in Fig. 5.7 whereas the driving frequency corresponds to a rotation in the ψ direction. Since after one period $\psi = 2\pi$, the angle θ can be expressed as

$$\frac{\theta}{2\pi} = \frac{f_{RO}}{f} \text{mod}(1). \quad (5.24)$$

¹¹Strictly speaking this is only true if the eigenvectors $\Re(\eta_1)$ and $\Im(\eta_1)$ form an orthonormal basis. Otherwise, the trajectories are given by equation 5.13.

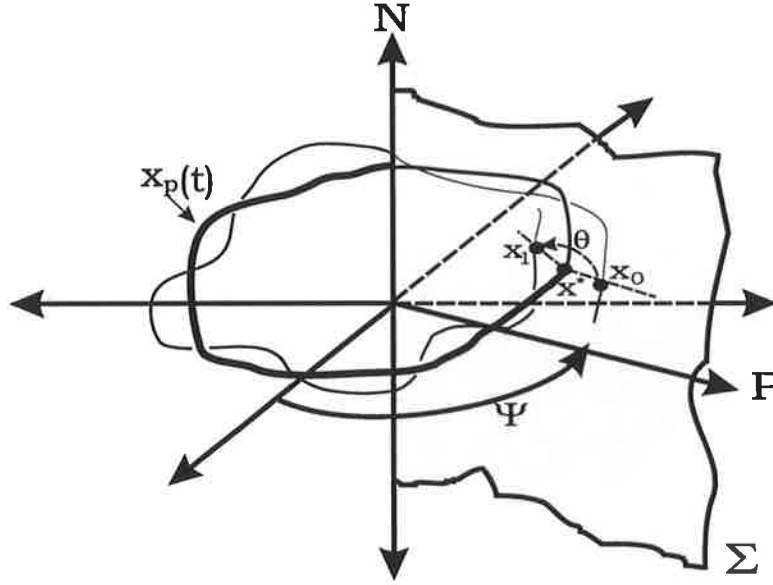


Figure 5.7:

Hence the eigenvalue, m_1 , can be expressed as

$$m_1 = \gamma \exp(i\theta) = \gamma \exp\left(\frac{i2\pi f_{RO}}{f}\right). \quad (5.25)$$

Therefore, we relate the angle θ of the largest eigenvalues in the complex plane to the transient frequency for the flow. However, from θ we cannot determine the absolute resonant frequency, f_{RO} , due to the $\text{mod}(1)$ in equation 5.24¹²

We would like to point out that though we have introduced a transient frequency f_{RO} for the nonlinear system it is not, strictly speaking, mathematically definable[11]. For a nonlinear system there is mixing between all frequencies present. Therefore, for a noise driven system, we would observe resonances at not only f_{RO} but also $nf \pm f_{RO}$ in the power spectrum (Fig. 5.9). However, the concept of a resonance is an important and useful construct as it allows us to gain some insight into the global bifurcation behaviour of the system[11]. We will return to this in section 5.7.

If the complex conjugate pair of eigenvalues in Fig. 5.8 exit the unit circle the system undergoes a Hopf bifurcation. This results in two frequency dynamics, either quasi-periodic ($\frac{2\pi}{\theta}$ irrational) or frequency locked ($\frac{2\pi}{\theta}$ rational).

¹²This is related to the Nyquist sampling theorem. If the modulation frequency is less than half of the relaxation frequency then a neighbouring trajectory will complete at least one complete twist about the stable periodic trajectory before cutting the Poincare plane. The Poincare map contains no information about the number of full twists completed but only the fraction of a twist completed as it only contains information about frequencies less than $1/T$. Therefore the absolute frequency cannot be determined for $f < f_{RO}/2$.

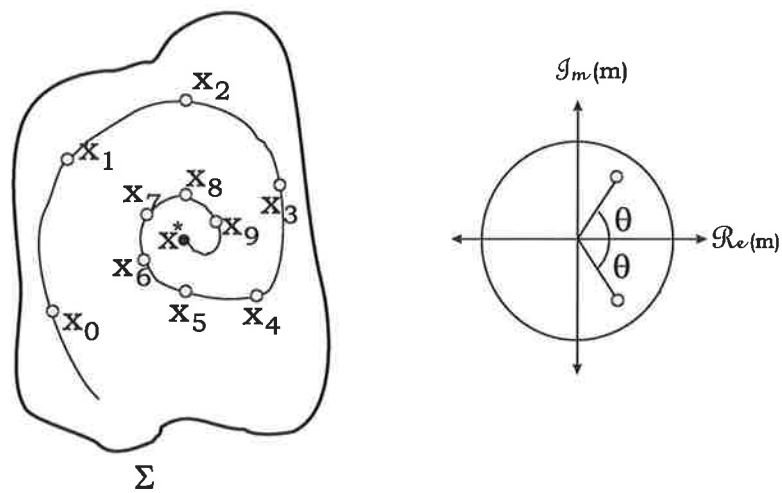


Figure 5.8: Transient approach to the steady state for complex conjugate eigenvalues

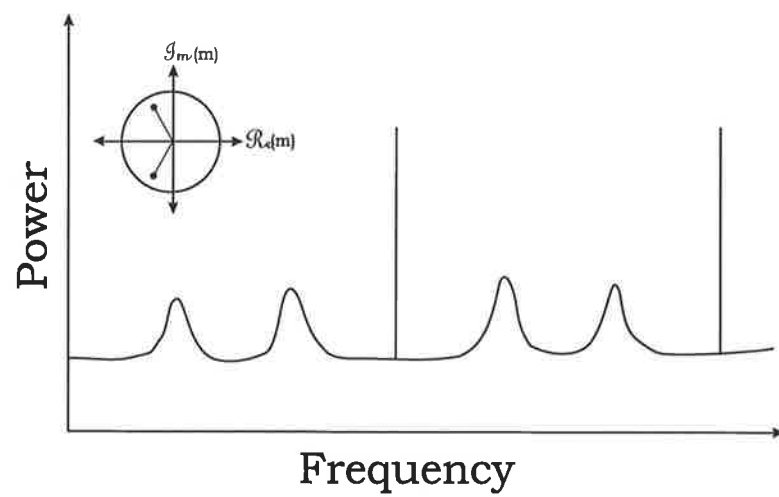


Figure 5.9: Noise precursor for a Hopf bifurcation.

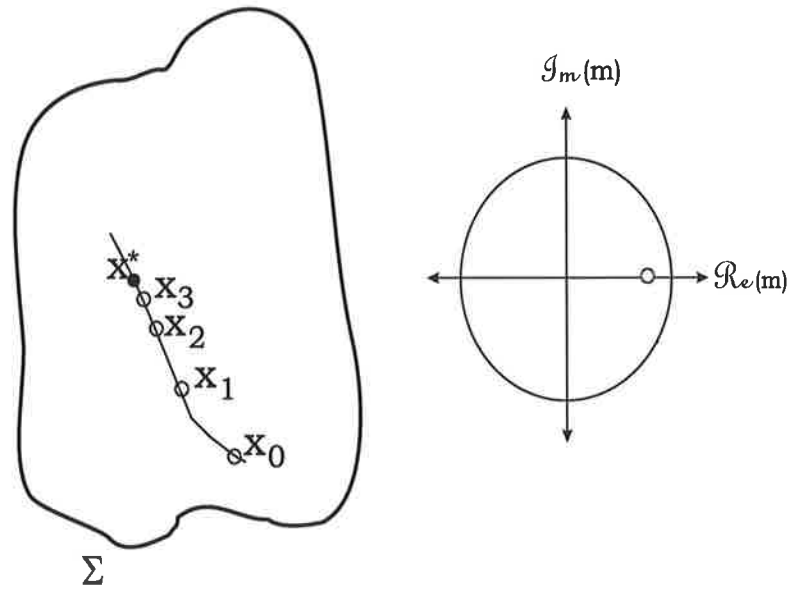


Figure 5.10: Transient approach to the steady state for real positive eigenvalues

5.3.5 Saddle-node Bifurcations

The last case of interest is that of a real, positive eigenvalue, m_1 such that $1 > |m_1| \gg |m_i|$ for all $i \neq 1$. Again, close to $\bar{x}^*$, the transient behaviour is determined by the largest eigenvalue m_1 . The transient trajectory in the Poincaré plane is illustrated in figure 5.10. In this case successive iterates approach the fixed point along the direction $\vec{\eta}_1$. This corresponds to a transient oscillation of the flow of period T/n where n is an integer. Consequently, the noisy precursor for this system corresponds to peaks centred on the modulation frequency and its harmonics. This is illustrated in Fig. 5.11.

If the eigenvalue m_1 exits the unit circle at $+1$ then a saddle node bifurcation occurs in which a saddle point collides with a stable fixed point and they both disappear. This type of bifurcation usually involves a sudden jump in the systems response and hysteresis. When the original attractor becomes unstable and disappears the system abruptly makes a transition to a new stable state. The hysteresis occurs due to the bistable nature of these attractors. An example of this is illustrated in figure 5.12¹³.

5.3.6 Discussion

The three cases outlined above occur in many dissipative nonlinear systems. In the following sections we discuss the occurrence of noisy precursors for period doubling, saddle-node and Hopf bifurcations in the semiconductor laser rate equations, including Langevin noise terms.

¹³The resonance curves of Fig. 5.3(b) are also examples of this behaviour.

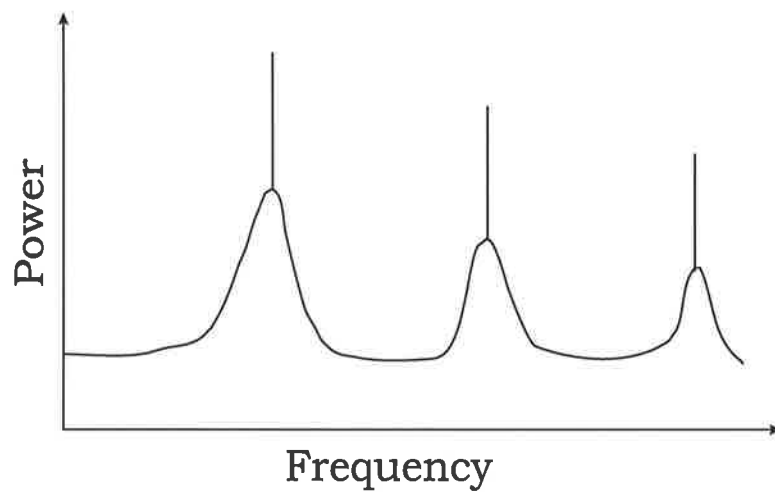


Figure 5.11: Noise precursor for a saddle-node bifurcation.

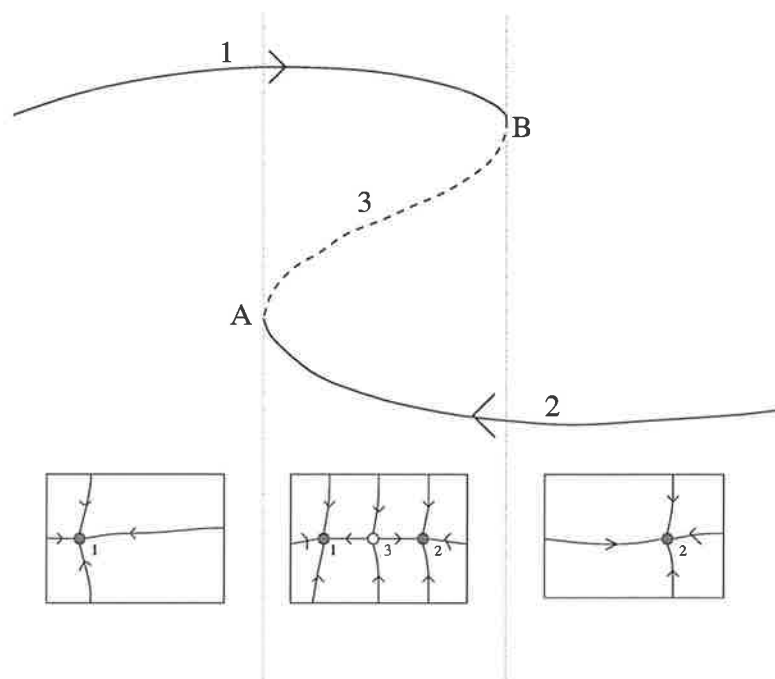


Figure 5.12: Saddle node bifurcations: an example. At bifurcation point B the stable node, corresponding to branch (1) of the bifurcation diagram, collides with the saddle point (3) leaving only the stable point (2). At bifurcation point A the stable node (2) collides with the unstable point (3) to leave only (1). We remark that more complex diagrams are possible. However the bifurcation behaviour shown at points A and B is typical of saddle-node bifurcations.

As previously discussed, for small modulation amplitudes the semiconductor laser behaves as a damped linear oscillator. From the arguments presented in section 5.2.1, a resonant frequency f_{RO} implies the existence of a pair of complex conjugate pair of multipliers. In the following sections we will show how these complex multipliers evolve with increasing modulation index for the cases in which a bifurcation eventually results. The corresponding intensity noise power spectra, showing the evolution of f_{RO} , are also shown and where relevant compared with experimental results.

5.4 Period Doubling Bifurcations

5.4.1 Single Mode System

The approach to period doubling bifurcations in current modulated semiconductor lasers has been discussed in previous studies[3, 4, 204, 195]. These systems were found to follow a virtual Hopf precursor[186, 12]. As an introduction to this phenomenon we show two examples of this behaviour in the single mode rate equations. In Fig. 5.13 we show the bifurcation diagram for the parameter values $I_b = 1.5$ and $f = 2\text{GHz}$ as m is varied from 0.01 to 1.0^{14} . Other parameter values are given in Table 4.2. A period doubling bifurcation occurs at $m = 0.73$. In Fig. 5.14 we plot the results of numerical calculations of the eigenvalues of the linearised Poincare map as a function of m for the same parameter values as in Fig. 5.13; Fig. 5.14(a) shows the trajectories of the eigenvalues in the complex plane, part (b) shows the magnitudes of the eigenvalues versus m and parts (c) and (d) show the real and imaginary parts of the eigenvalues versus m , respectively. As discussed in Section 5.3, a period doubling bifurcation occurs when a multiplier exits the unit circle at -1. Figure 5.14(a) shows that the eigenvalues, initially a complex conjugate pair, circle about the origin to the left before colliding on the negative real axis. They then separate and one exits the unit circle at -1; the value of m at which this occurs corresponds to the period doubling bifurcation point of Fig. 5.13.

In Fig. 5.15 we show the corresponding photon density power spectrum calculated for the same parameter values as Figs. 5.13 and 5.14. The Langevin noise terms are included in the simulations. We observe that, as the modulation index increases, the relaxation oscillation frequency, initially at 1.51GHz (corresponding to the small signal value given by equation D.17), is tuned to $f/2$ where it becomes amplified and sharpens considerably. This behaviour is given the name of Virtual Hopf Phenomenon since the precursor for period doubling is preceded by the precursor for the Hopf instability though the system does not undergo a Hopf bifurcation[186].

¹⁴This is a repetition of the calculation of Fig. 3(b) of reference [3].

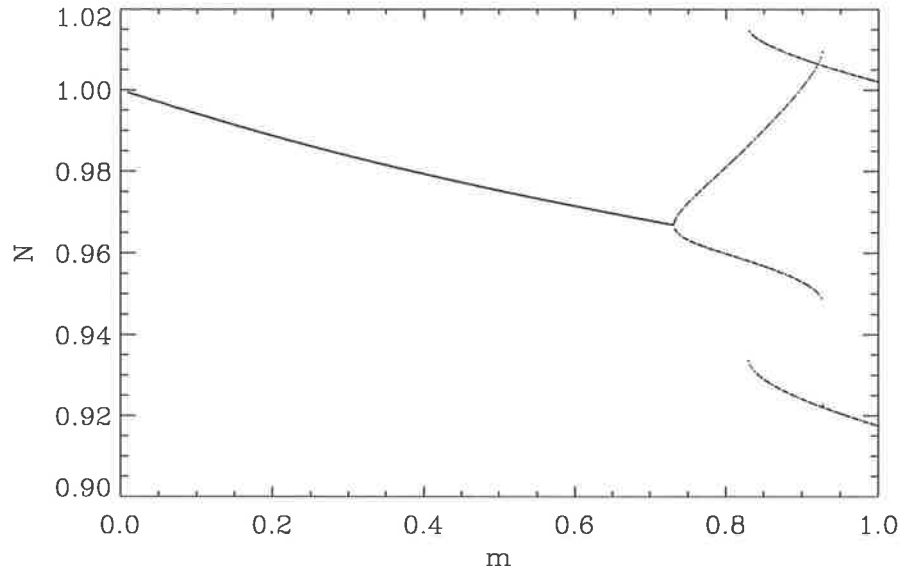


Figure 5.13: Calculation bifurcation diagram for the single mode rate equations for $I_b = 1.5$, $f = 2\text{GHz}$ and $\beta = 10^{-4}$.

To check the validity of the linearised description in the presence of noise we have made a quantitative comparison of the position of the resonant peak in the noisy power spectrum as a function of m with the numerical calculation of the eigenvalues via equation 5.24. In order to make quantitative comparisons accurate determination of the position of the resonant peak is required. To achieve this power spectra were calculated with 32768 spectral components and 16 spectra were averaged in order to reduce the variance[184]. The results of this calculation are shown in Fig. 5.16(a) for frequencies in the range 0 to 1.7GHz. Laser parameter values used in this calculation are identical to those used in the calculation of the data shown in Fig. 5.15. To determine the position of the resonant peak a Lorentzian was fitted to the data and its maximum located. The results are plotted in Fig. 5.16(b). The error bars correspond to the errors obtained in determining the Lorentzian peak position in the fit. The solid curve in Fig. 5.16(b) corresponds to the relaxation oscillation frequency calculated according to equation 5.24 from the eigenvalue calculation of Fig. 5.14. We observe that within errors, the results of these two calculations are in agreement confirming the validity of the linearised description for small perturbations from the periodic solution. Close to the bifurcation point ($m = 0.73$) there is a small difference between the position of the noise precursor and the frequency calculated from the angle of the multiplier. However, it is not clear whether this is due to a deviation from the linearised description in this region or an error in the location of the maximum due to the shape of the noise precursor.

A virtual Hopf precursor also occurs for the next largest period doubling region

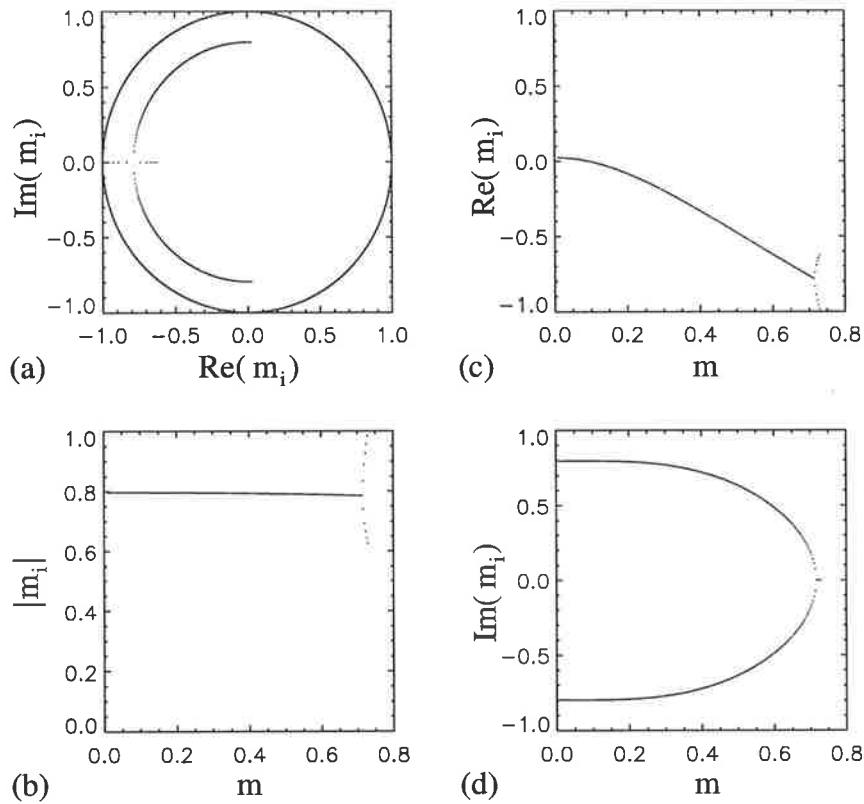


Figure 5.14: Multiplier evolution as a function of modulation index for $I_b = 1.5$, $f = 2\text{GHz}$ and $\beta = 10^{-4}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m .

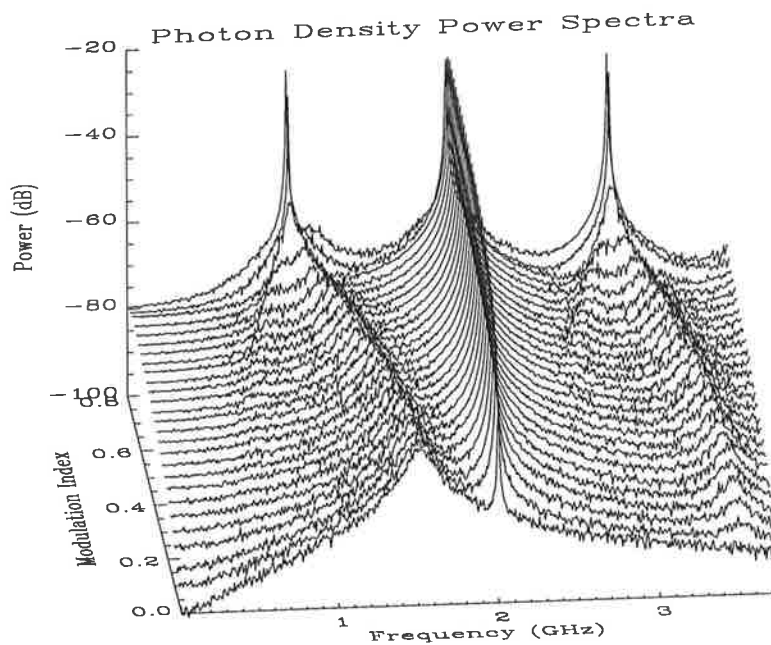
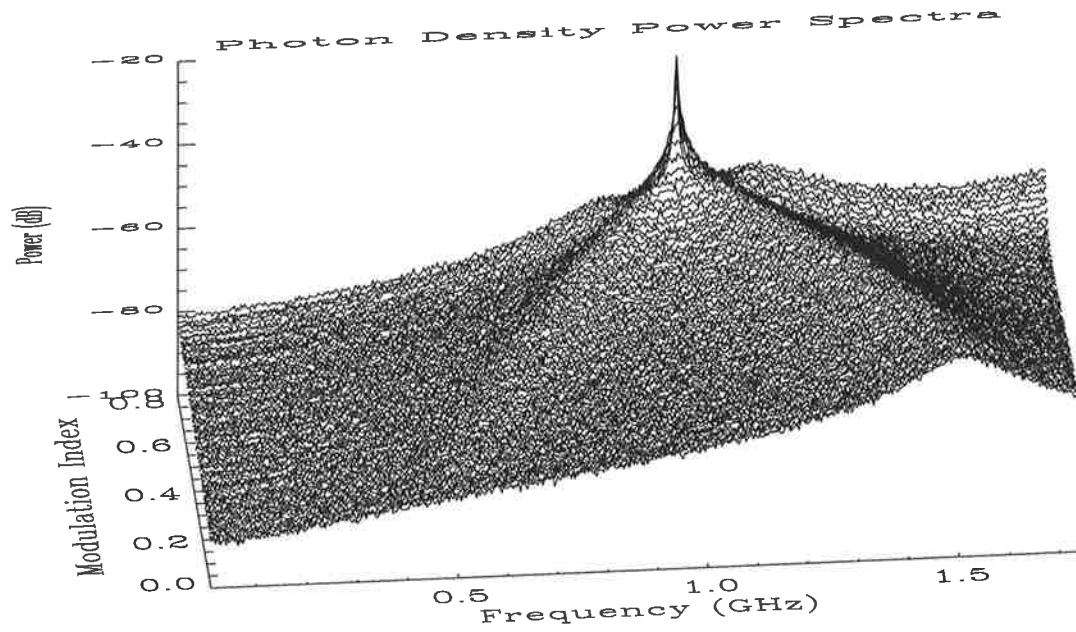
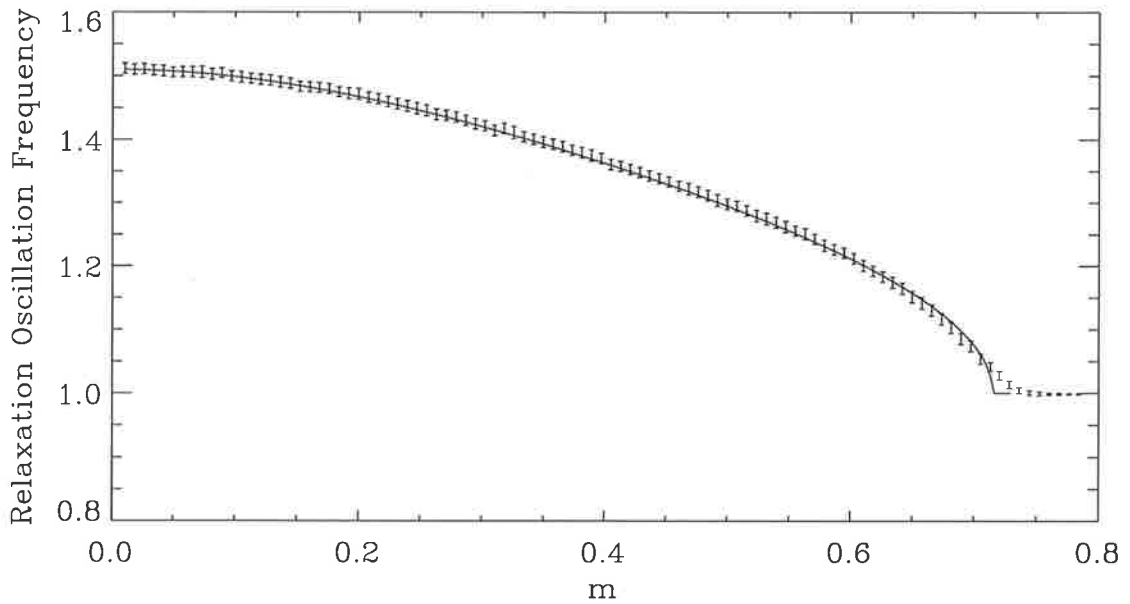


Figure 5.15: Calculated photon density power spectra for $I_b = 1.5$, $f = 2\text{GHz}$ and $\beta = 10^{-4}$. Langevin noise terms are included in the rate equations.



(a)



(b)

Figure 5.16: (a): Calculated photon density power spectra for $I_b = 1.5$, $f = 2\text{GHz}$ and $\beta = 10^{-4}$, including Langevin noise terms. (b): position of the relaxation oscillation frequency versus modulation index calculated from the data in part (a). The solid curve corresponds to the relaxation oscillation frequency calculated according to equation 5.24 from the data in Fig. 5.14.

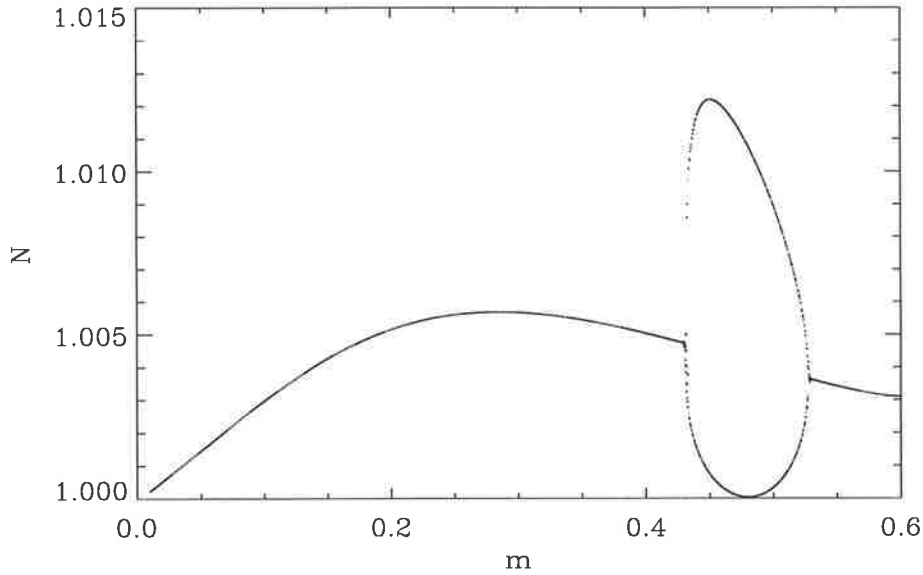


Figure 5.17: Calculated bifurcation diagram for the single mode rate equations for $I_b = 1.5$, $f = 0.85\text{GHz}$ and $\beta = 10^{-4}$.

shown in Fig. 4.30(b) (i.e. for f near $f_{RO}/2$). In Fig. 5.17 the bifurcation diagram for the parameter values $I_b = 1.5$, $f = 0.85\text{GHz}$ and m increasing from 0.01 to 0.6 is shown; a period doubling bifurcation occurs at $m = 0.43$. The bifurcation diagram shows that the period doubling bifurcation occurs suddenly rather than subtly as it does in the previous example. In this case a subcritical period doubling bifurcation occurs[13] as opposed to a supercritical bifurcation (illustrated in Fig. 5.6). The corresponding multiplier evolution is shown in Fig. 5.18; we observe the eigenvalues to circle towards the left before colliding on the negative real axis. At this point they separate and one exits the unit circle at -1. Again we find that the value of m at which this occurs corresponds to the period doubling bifurcation point of Fig. 5.17. The corresponding photon density power spectra are shown in Fig. 5.19. We observe that the relaxation oscillation frequency, initially at 1.51GHz for small m , is tuned to $3f/2$. At this point, enhancement of the power at frequencies of $nf/2$ occurs, where n is an odd integer. For increasing m these peaks become amplified and eventually undamped at the period doubling bifurcation point.

5.4.2 Multimode System

Numerical Results

In the single mode case there is only one pair of eigenvalues that determine all stability properties of the periodic attractors. The high dimensionality of the multimode system

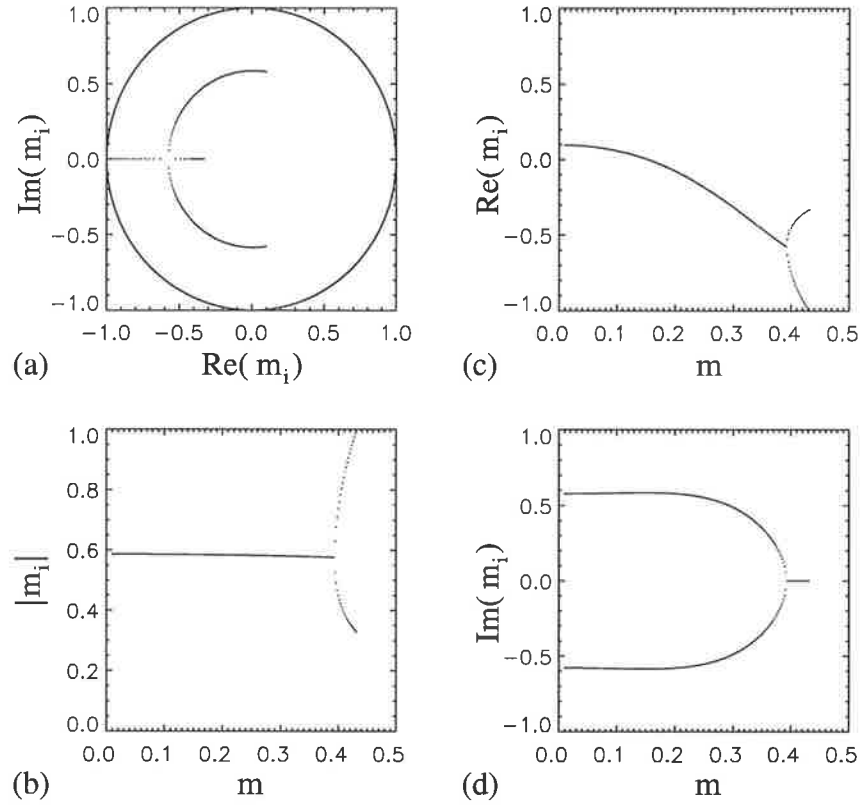


Figure 5.18: Multiplier evolution as a function of modulation index for $I_b = 1.5$, $f = 0.85\text{GHz}$ and $\beta = 10^{-4}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m .

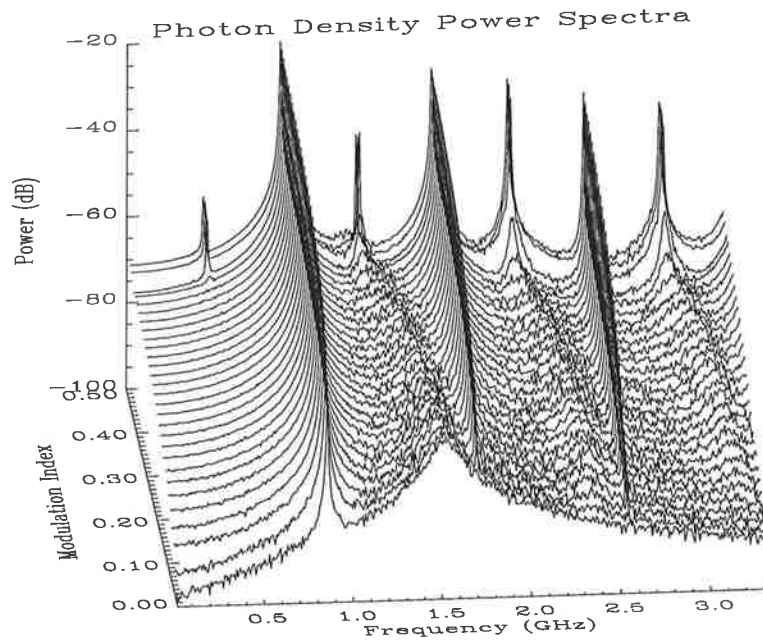


Figure 5.19: Calculated photon density power spectra for $I_b = 1.5$, $f = 0.85\text{GHz}$ and $\beta = 10^{-4}$. Langevin noise terms are included in the rate equations.

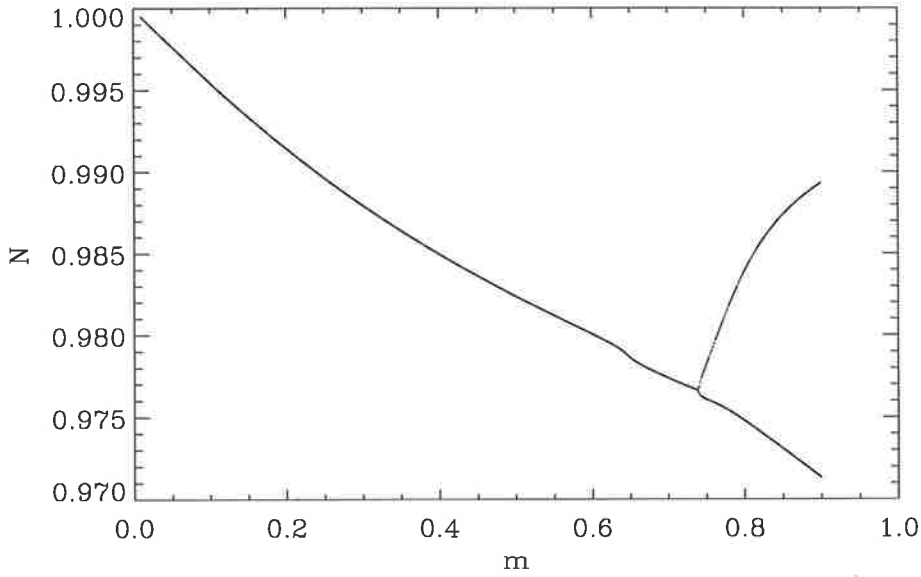


Figure 5.20: Calculated bifurcation diagram for the multimode rate equations for $I_b = 1.7$, $f = 2.4\text{GHz}$ and $\beta = 10^{-4}$. Langevin noise terms are included in the rate equations.

potentially could lead to extremely complicated behaviour. Despite this we find that, though in many respects the single mode and multimode systems predict quite different behaviour, the approach to period doubling bifurcations are qualitatively similar. A bifurcation diagram for the parameter values $I_b = 1.7$, $f = 2.4\text{GHz}$, $\beta = 10^{-4}$ and m tuned from 0.01 to 0.9 is shown in Fig. 5.20. All other parameter values are given in Table 4.2. A period doubling bifurcation occurs for $m = 0.74$. The trajectories of the multipliers in the complex plane are shown in Fig. 5.21(a). The magnitude of the multipliers, $|m_i|$, are shown in Fig. 5.21(b) and the real and imaginary parts versus m in (c) and (d) respectively. We observe that there is only one complex conjugate pair (corresponding to the relaxation oscillation frequency), all other eigenvalues being real. As m is increased this complex conjugate pair circle to the left about the origin before colliding on the negative real axis. At this point they separate and one exits the unit circle at -1 signifying a period doubling bifurcation. As mentioned this is qualitatively similar to the single mode prediction; thus the multimode semiconductor laser rate equations also undergo the virtual Hopf phenomenon. The evolution of the noise driven power spectra for the total photon density with increasing m is shown in Fig. 5.22. Similarly to the single mode predictions we observe that the relaxation oscillation frequency initially at 1.78GHz (corresponding to the small signal value given by equation D.17) is pulled to $f/2$ at which point it rapidly increases in amplitude and sharpens considerably.

A virtual Hopf precursor also occurs for the next largest period doubling region (i.e. f near $f_{RO}/2$). Figure 5.23 shows the behaviour of the multipliers for the parameter values

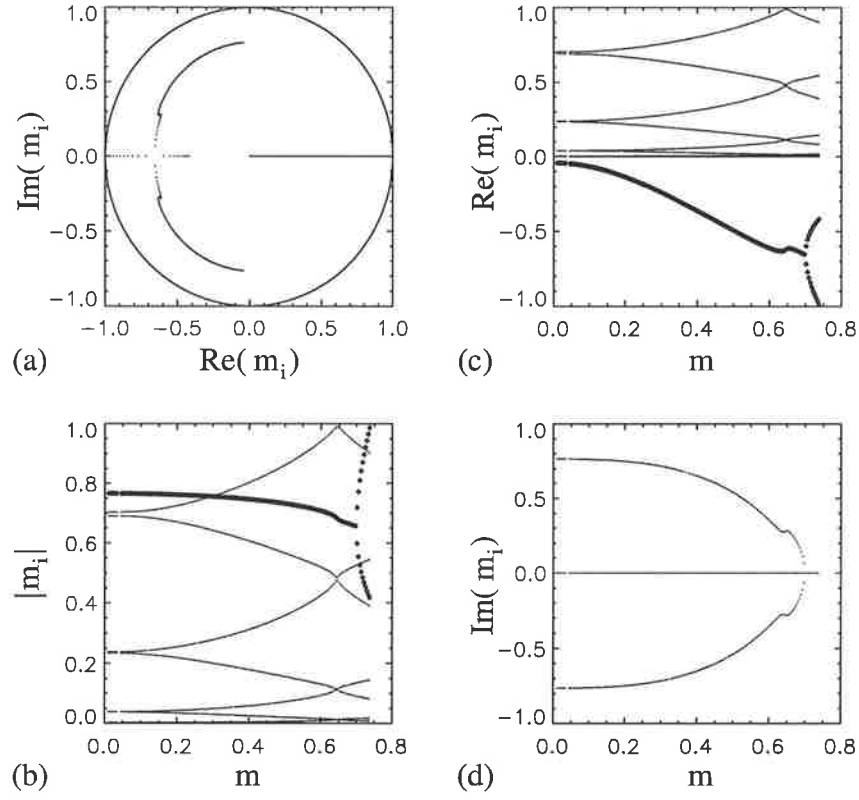


Figure 5.21: Multiplier evolution as a function of modulation index for $I_b = 1.7$, $f = 2.4\text{GHz}$ and $\beta = 10^{-4}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m . In parts (b) and (c) large symbols indicate complex conjugate multipliers.

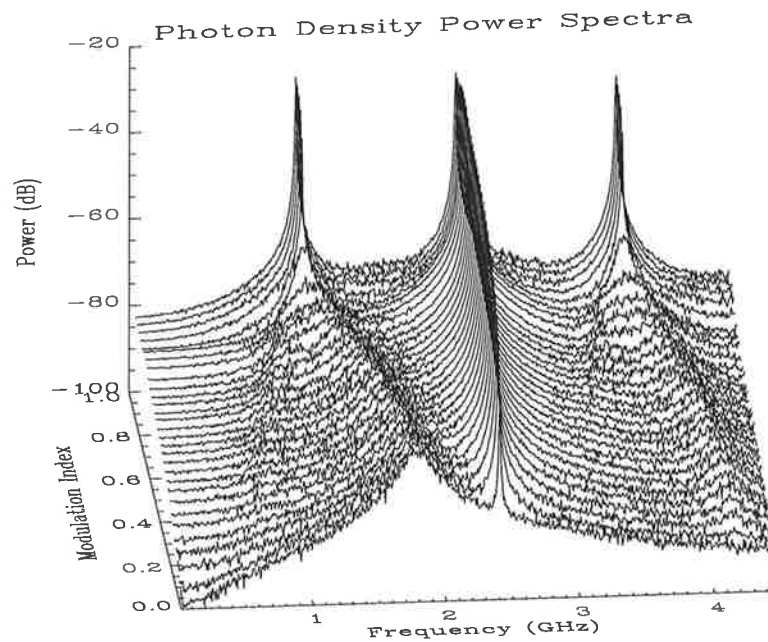


Figure 5.22: Calculated photon density power spectra for the multimode rate equations for $I_b = 1.7$, $f = 2.4\text{GHz}$ and $\beta = 10^{-4}$. Langevin noise terms are included in the rate equations.

$I_b = 1.7$, $f = 0.98\text{GHz}$, $\beta = 5 \times 10^{-5}$ and m tuned from 0.01 to 0.5. The trajectories of the multipliers in the complex plane are shown in Fig. 5.23(a). As in the previous cases the complex conjugate multipliers circle to the left about the origin before colliding on the negative real axis followed by the subsequent exit of a multiplier from the unit circle at -1. The power spectra are shown in Fig. 5.24 for m tuned from 0.01 to 0.8. We observe that the relaxation oscillation resonance, initially at 1.78GHz is pulled to $3f/2$. This is followed by the rapid enhancement of the power at frequencies given by $nf/2$, for n an odd integer, signifying a period doubling bifurcation¹⁵. As m is increased still further the peaks at $nf/2$ eventually disappear and a stable period one cycle remains.

In Figs. 5.25 and 5.26 we show the multiplier evolution and the noisy power spectra variation with increasing m , respectively, for the the same parameters as in Figs. 5.23 and 5.24 but with $\beta = 10^{-4}$. In this case, due to larger damping of the relaxation oscillations, a period doubling bifurcation does not occur. However, we do observe behaviour similar to the virtual Hopf phenomenon. Figure 5.25(a) shows that the complex conjugate pair of multipliers circle to the left about the origin before colliding on the negative real axis. In this case this pair of multipliers also separate after the collision; however, instead of the most negative multiplier exiting the unit circle it eventually starts to head back towards the origin as m is increased. The noisy power spectra are shown in Fig. 5.26. As we would expect from the eigenvalue calculations, we observe that the relaxation oscillation resonance, initially at 1.78GHz , is pulled to $3f/2$. At this point enhancement of the power at frequencies $nf/2$ occurs (for n an odd integer) though not as large as in Fig. 5.24. Therefore, in this case we observe frequency pulling of the relaxation oscillation and enhancement of the power at frequencies of $nf/2$ in the absence of a period doubling bifurcation in the noiseless system. Thus, care must be taken when utilising the power spectrum of a noise driven system as a means of detecting a period doubling bifurcation.

Experimental Results

Previous experimental results of the Virtual Hopf phenomenon in a modulated semiconductor laser have been published[4]. In the investigation of [4] a (single mode) DFB laser was used. In this section we present experimental results of the Virtual Hopf Phenomenon in a current modulated FP laser. In this section results for the 830nm laser are presented as this laser exhibits the strongest relaxation oscillation resonance and is therefore the easiest to drive to instability. The temperature of the laser was stabilised to 25°C .

We show two examples of a period doubling bifurcation with increasing modulation index. In Fig. 5.27 we show the evolution of the intensity power spectrum under increased

¹⁵In this case we also observe the noise precursor for a period four attractor though a secondary period doubling bifurcation does not take place in this instance. This is manifest as small peaks centred at frequencies $1/4$ and $3/4$ between the harmonics of f .

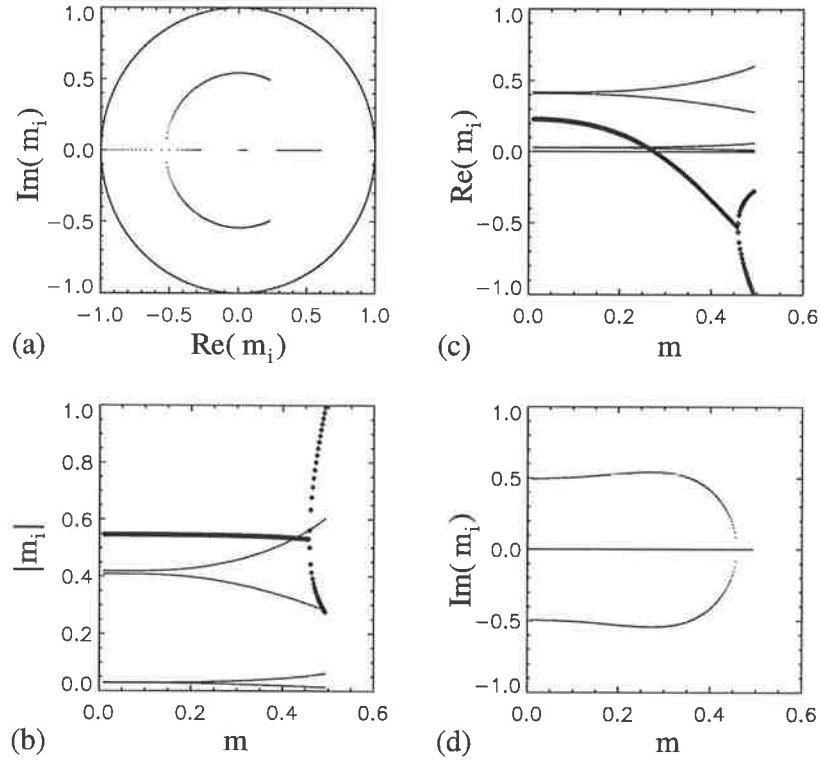


Figure 5.23: Multiplier evolution as a function of modulation index for $I_b = 1.7$, $f = 0.98\text{GHz}$ and $\beta = 5 \times 10^{-5}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m . In parts (b) and (c) large symbols indicate complex conjugate multipliers.

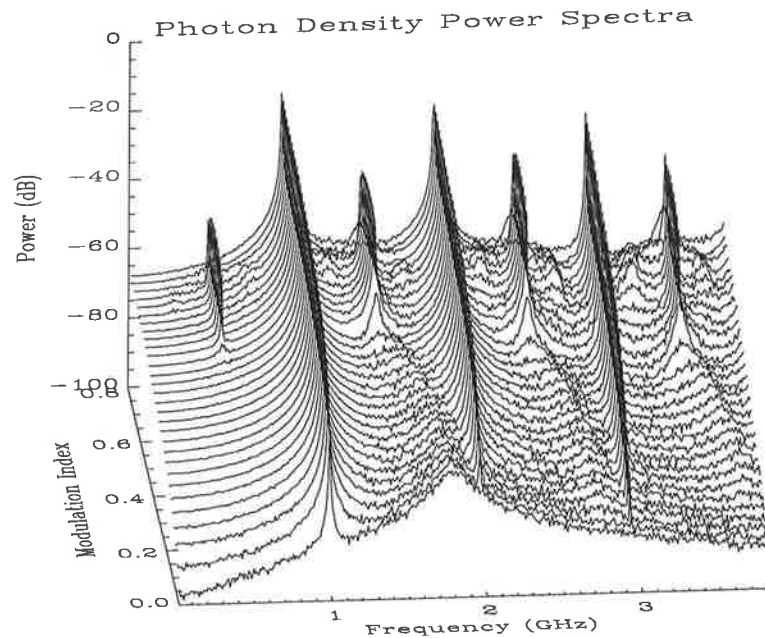


Figure 5.24: Calculated photon density power spectra for the multimode rate equations for $I_b = 1.7$, $f = 0.98\text{GHz}$ and $\beta = 5 \times 10^{-5}$. Langevin noise terms are included in the rate equations.

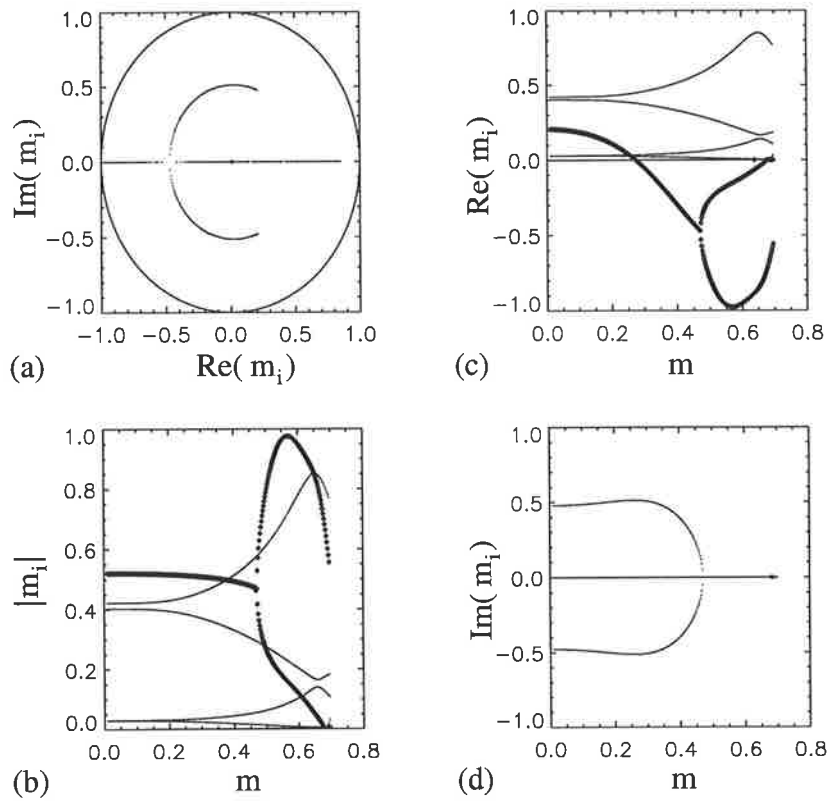


Figure 5.25: Multiplier evolution as a function of modulation index for $I_b = 1.7$, $f = 0.98\text{GHz}$ and $\beta = 10^{-4}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m . In parts (b) and (c) large symbols indicate complex conjugate multipliers.

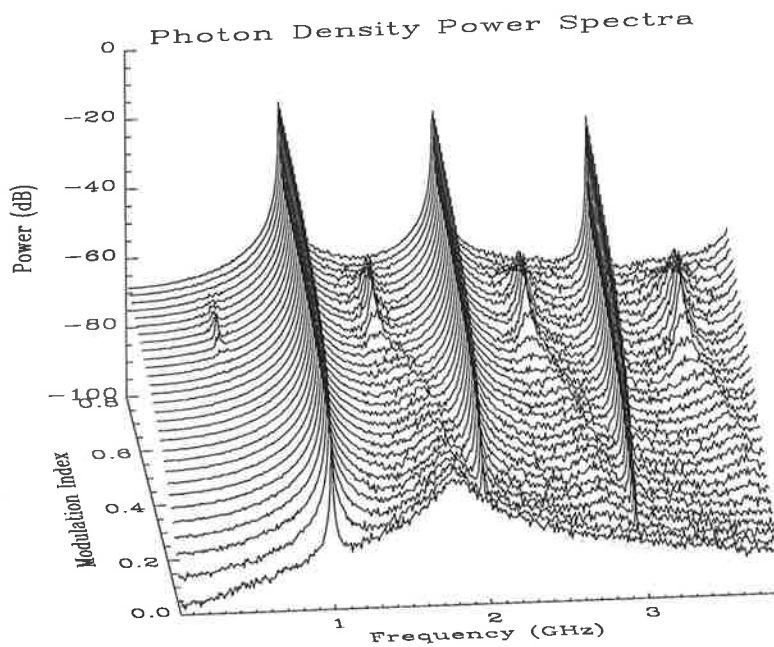


Figure 5.26: Calculated photon density power spectra for the multimode rate equations for $I_b = 1.7$, $f = 0.98\text{GHz}$ and $\beta = 10^{-4}$. Langevin noise terms are included in the rate equations.

modulation power for a modulation frequency of 2.9GHz and a DC bias of 27mA. Figure 5.27 shows that the laser follows a virtual Hopf precursor: the relaxation oscillation frequency, initially at 2.3GHz is pulled to half the modulation frequency. Then a rapid enhancement of the power at frequencies $nf/2$ (for n an odd integer) is observed accompanied by significant spectral narrowing of the noisy precursor peaks. Similarly in Fig. 5.28, for a DC bias of 24mA and modulation frequency of 2.4GHz, a virtual Hopf route to period doubling is also observed as the modulation power is increased¹⁶.

Experimentally we do not observe period doubling for modulation frequencies below the relaxation oscillation frequency. However, we do observe enhancement of the power at frequencies $nf/2$ (for n an odd integer) for suitable combinations of modulation frequency and amplitude. Figure 5.29 shows the intensity power spectrum evolution with increasing modulation power for a DC bias of 27mA and modulation frequency of 1.25GHz. We observe that with increasing modulation power the relaxation oscillation, initially at approximately 2.3GHz, is pulled to $3f/2$. At this point we see enhancement of the power at frequencies $nf/2$. Similar behaviour is shown in Fig. 5.30 for a higher DC bias of 29mA and modulation frequency of 1.3GHz. Therefore, experimentally we observe behaviour qualitatively similar to Fig. 5.26 indicating that the laser is close to but does not undergo a period doubling bifurcation for modulation frequencies below the relaxation oscillation frequency.

5.5 Saddle-node Bifurcations

In this section we discuss the noisy precursor for saddle-node bifurcations. Contrary to period doubling bifurcations we show numerically that the occurrence of saddle-node bifurcations is qualitatively different in the single mode and multimode systems.

However, before discussing specific details of saddle-node bifurcations we first consider experimentally how the laser behaves when modulated at frequencies slightly below the relaxation oscillation frequency. Figure 5.31 shows the evolution of the intensity power spectra with increasing modulation power for a modulation frequency of 1.5GHz and DC bias of 27mA. We observe that the relaxation oscillation frequency is again pulled to a lower frequency. However, in this case f_{RO} is pulled to the driving frequency resulting in noise bumps centred at the modulation frequency and its harmonics. We remark that this is similar to the noisy precursor for the saddle-node bifurcation as discussed in section 5.3.5. We discuss in this section under what circumstances we would expect to observe a saddle-node bifurcation from this system.

¹⁶The effective modulation index is increased exponentially between adjacent spectra in the experimental data rather than linearly as it is in the numerical simulations due to the use of dBm rather than voltage for the modulation amplitude.

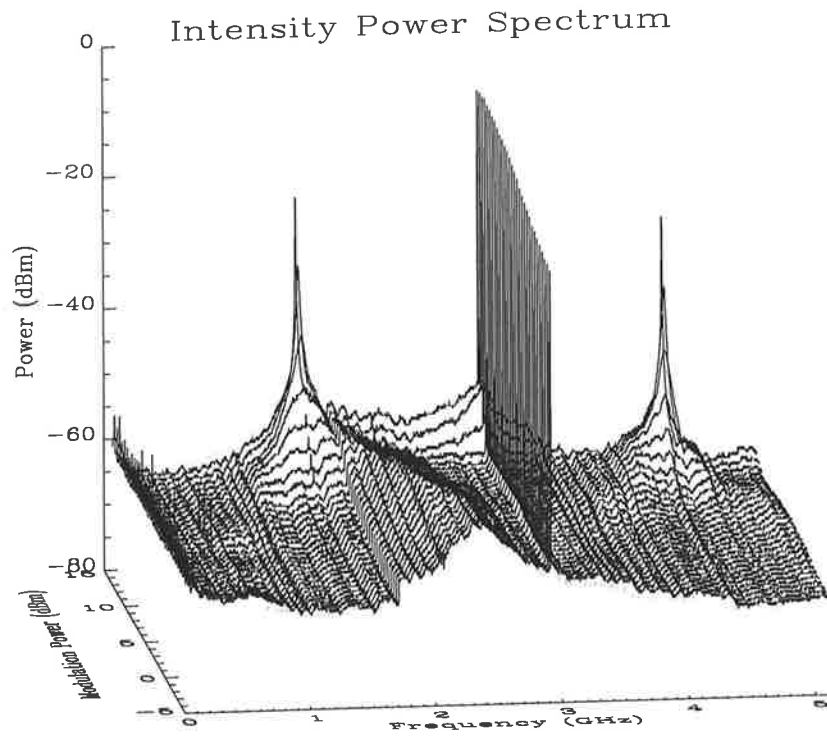


Figure 5.27: Experimental intensity power spectrum evolution with increasing modulation power for $I_{DC} = 27\text{mA}$ and $f = 2.9\text{GHz}$.

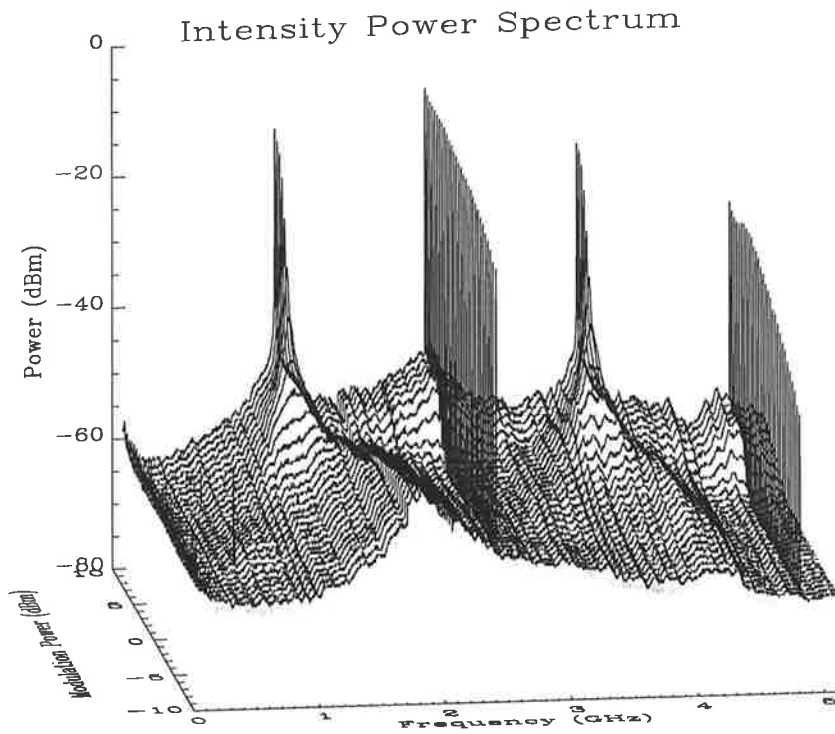


Figure 5.28: Experimental intensity power spectrum evolution with increasing modulation power for $I_{DC} = 24\text{mA}$ and $f = 2.4\text{GHz}$.

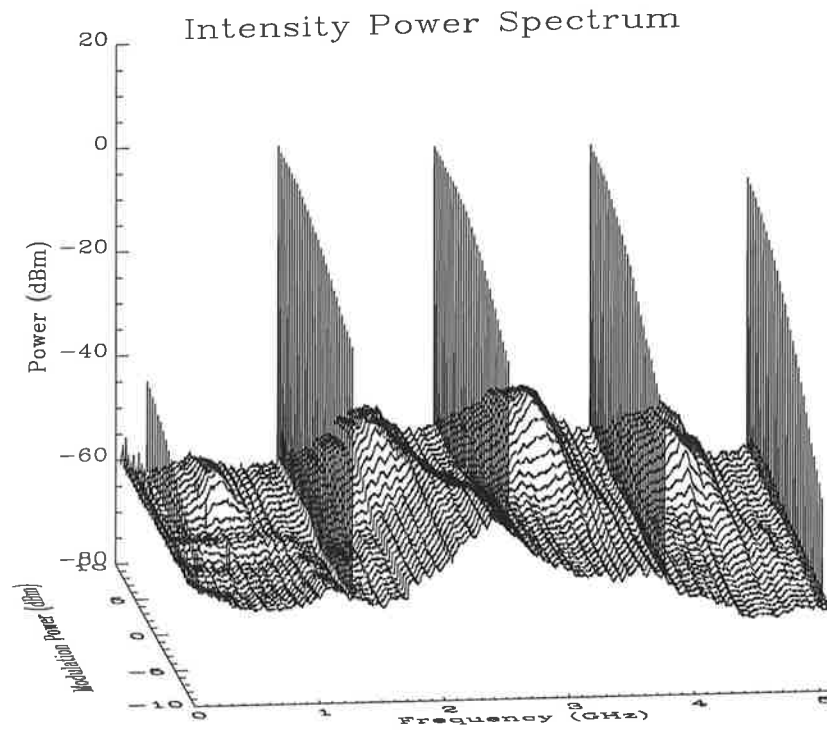


Figure 5.29: Experimental intensity power spectrum evolution with increasing modulation power for $I_{DC} = 27\text{mA}$ and $f = 1.25\text{GHz}$.

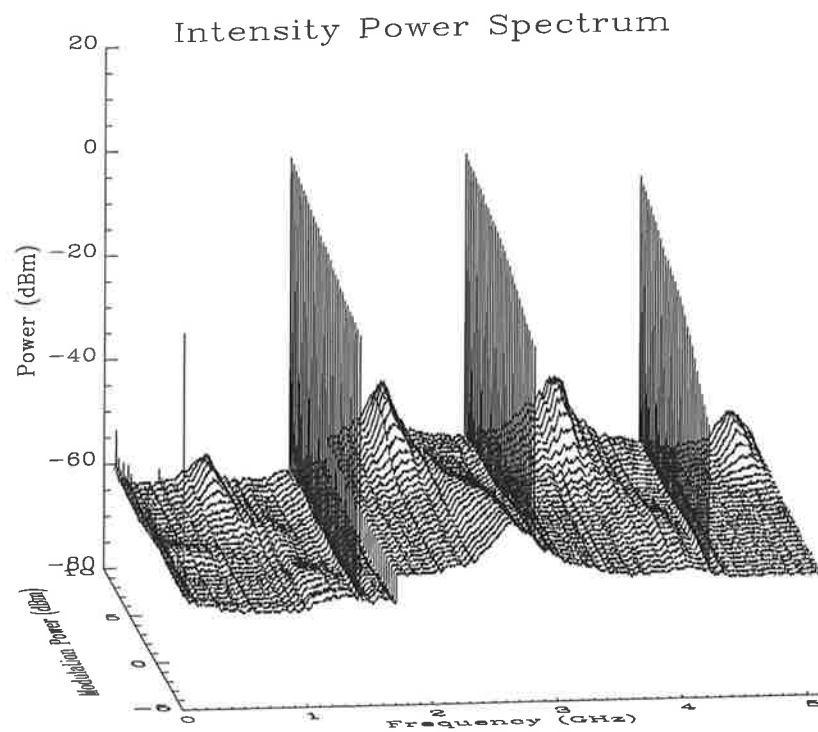


Figure 5.30: Experimental intensity power spectrum evolution with increasing modulation power for $I_{DC} = 29\text{mA}$ and $f = 1.3\text{GHz}$.

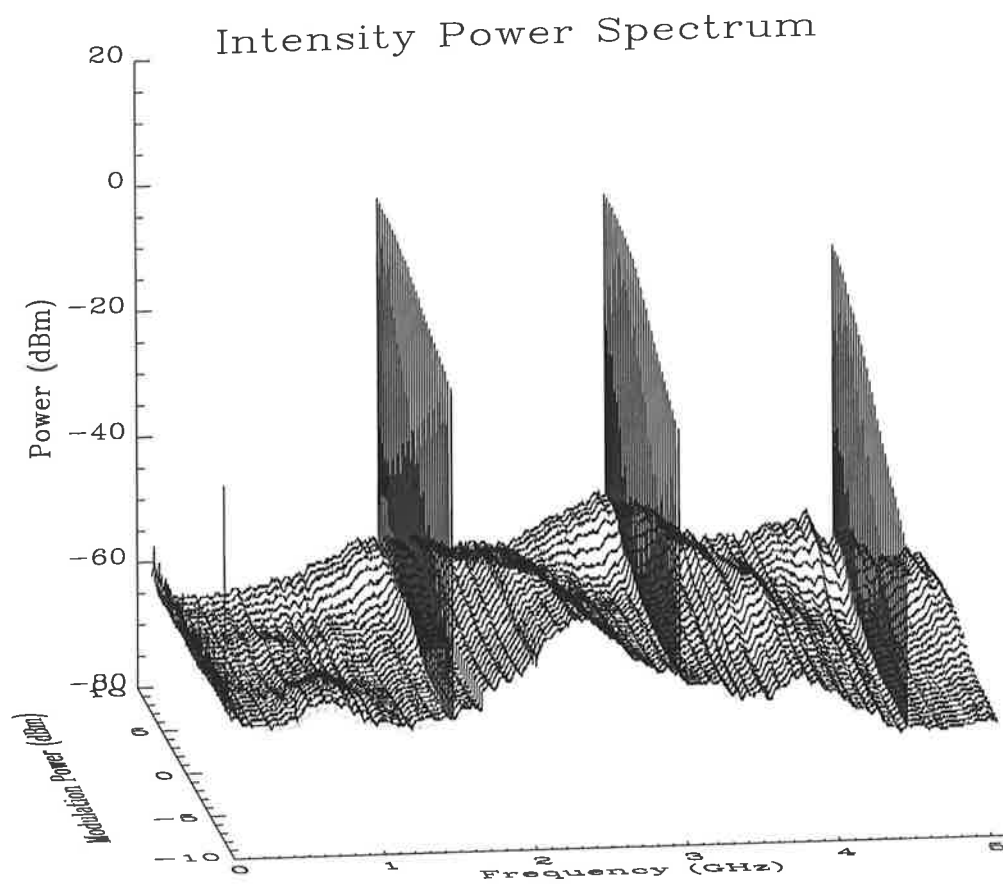


Figure 5.31: Experimental intensity power spectrum evolution with increasing modulation power for $I_{DC} = 27\text{mA}$ and $f = 1.5\text{GHz}$.

5.5.1 Single Mode System

In Figure 5.32 we show a bifurcation diagram for the single mode rate equations, computed using the parameter values $I_b = 1.7$, $f = 1.3\text{GHz}$ and $\beta = 10^{-4}$. For increasing m we observe a sudden jump in the system's response at $m = 0.41$ whereas for decreasing m this jump occurs at $m = 0.265$. At these two points a saddle node bifurcation occurs in which a stable periodic cycle collides with a saddle point and both are annihilated. For m values lying between the bifurcation points the stable periodic cycles co-exist. In order to fully understand the behaviour near a saddle-node bifurcation it is necessary to compute the multiplier evolution for both increasing and decreasing values of the control parameter. The multiplier evolution is shown in Figs. 5.33 and 5.34 for increasing and decreasing m respectively. The behaviour in each case is qualitatively similar; the eigenvalues, initially a complex conjugate pair, circle about the origin towards the right until they collide on the positive real axis. They then separate and one exits the unit circle at $+1$. At this point the stable periodic cycle disappears completely and the system abruptly makes a transition to a new stable state (compare Fig. 5.32). This manifests itself in the appearance of a new pair of complex conjugate eigenvalues that continue to rotate to the left about the origin as m is varied. The bifurcation point occurs at a different value of m depending on whether m is increased or decreased past the bifurcation points.

The corresponding intensity power spectra evolution for m varied from 0.6 to 0.01 is shown in Fig. 5.35. At the saddle-node bifurcation point, an abrupt decrease in the power over the entire spectrum occurs and peaks corresponding to the relaxation oscillation frequency appear at frequencies slightly above the harmonics of the modulation frequency. As the modulation index is decreased still further, these resonant peaks move to higher frequencies and the higher harmonics decrease in amplitude. Eventually only a single resonant peak is observed positioned at the small signal relaxation oscillation frequency of 1.78GHz.

5.5.2 Multimode System

In Fig. 5.36 we show a bifurcation diagram as m is varied for a bias of $I_b = 1.7$ and modulation frequency of $f = 2\text{GHz}$. A jump in the system's response occurs at a modulation index of $m = 0.34$ for increasing m and at $m = 0.32$ for decreasing m . The corresponding multiplier evolution for increasing m is shown in Fig. 5.37. Part (a) shows the multiplier trajectories in the complex plane, (b) shows the magnitudes of the multipliers versus m and (c) and (d) show the real and imaginary parts versus m respectively. In this case we observe that it is the largest real eigenvalue which exits the unit circle at $+1$ and causes the saddle-node bifurcation; the complex conjugate pair do not take part in the bifurcation. Instead they circle to the left about the origin and at the bifurcation point disappear. A new pair of complex eigenvalues appear, corresponding to the new periodic cycle, that continue to circle

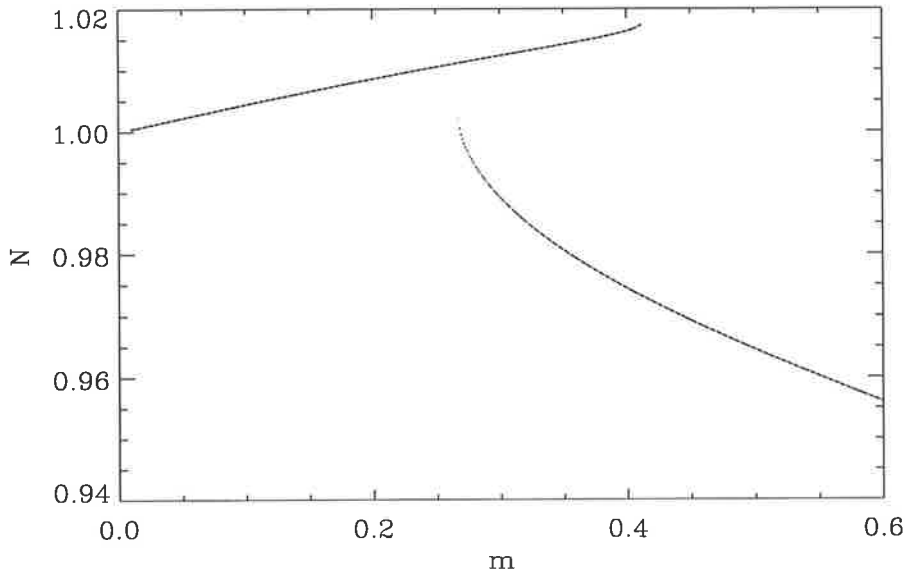


Figure 5.32: Calculated bifurcation diagram for the single mode rate equations for $I_b = 1.7$, $f = 1.3\text{GHz}$ and $\beta = 10^{-4}$.

to the left about the origin as m is increased. The corresponding power spectra, for increasing m are shown in Fig. 5.38. Though the relaxation oscillation frequency is pulled to a lower frequency it is not pulled to the driving frequency contrary to the single mode predictions. This is because, in this case, the saddle-node bifurcation occurs for modulation frequencies slightly higher than the relaxation oscillation frequency as opposed to slightly lower as it does in the single mode case. Thus, we observe, that for saddle-node bifurcations, the higher dimensionality of the multimode system gives rise to qualitatively different behaviour from the single mode system.

As discussed in the section 4.6.2, hysteresis in the multimode system is accompanied by mode hopping between adjacent longitudinal modes. Figure 5.39 shows the time averaged photon density for the central mode ($\lambda = \lambda_0$) and the adjacent mode ($\lambda = \lambda_{-1}$). As the modulation index is increased we observe an abrupt transition from λ_0 to λ_{-1} at approximately $m = 0.34$ corresponding to the saddle node bifurcation point. As m is decreased through this region we observe an abrupt transition from λ_{-1} to λ_0 at approximately $m = 0.32$; hence hysteresis occurs.

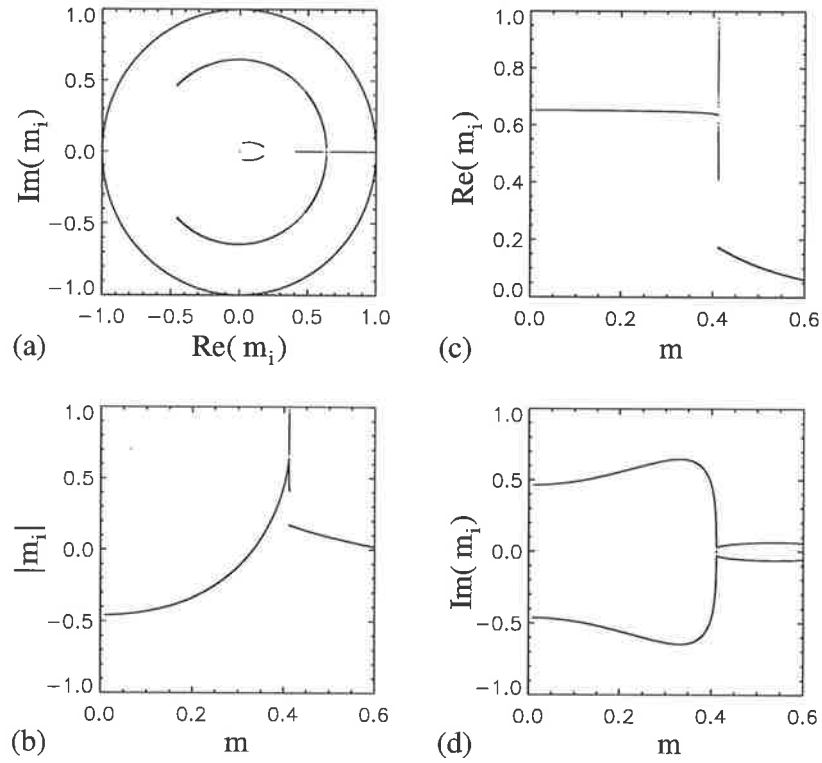


Figure 5.33: Multiplier evolution as a function of modulation index for increasing m . Parameter values are $I_b = 1.7$, $f = 1.3\text{GHz}$ and $\beta = 10^{-4}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m .

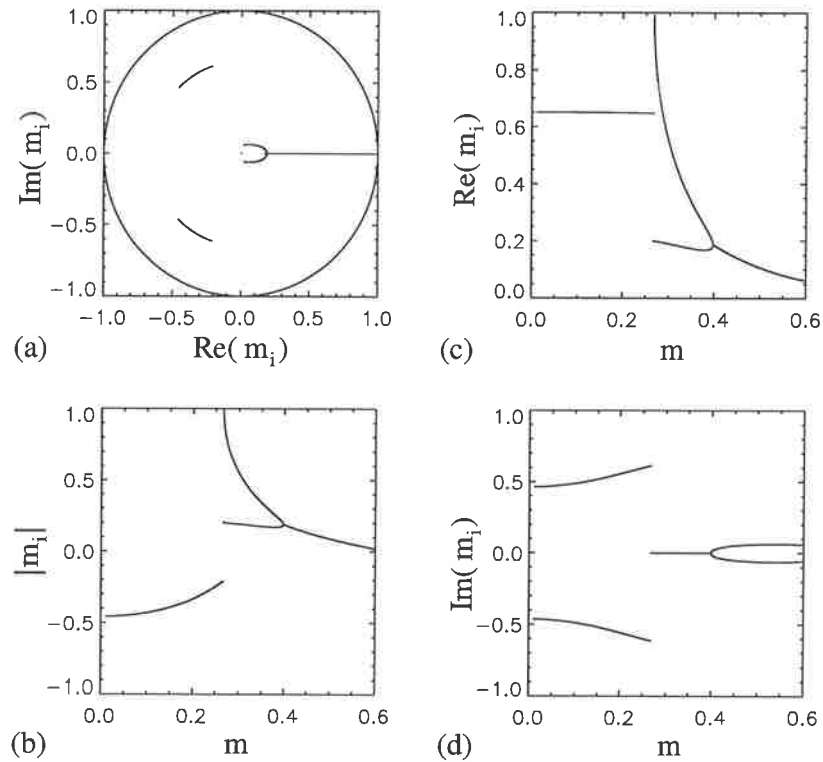


Figure 5.34: Multiplier evolution as a function of modulation index for decreasing m . Parameter values are $I_b = 1.7$, $f = 1.3\text{GHz}$ and $\beta = 10^{-4}$: Parts (a), (b) and (c) as in Figure 5.33.

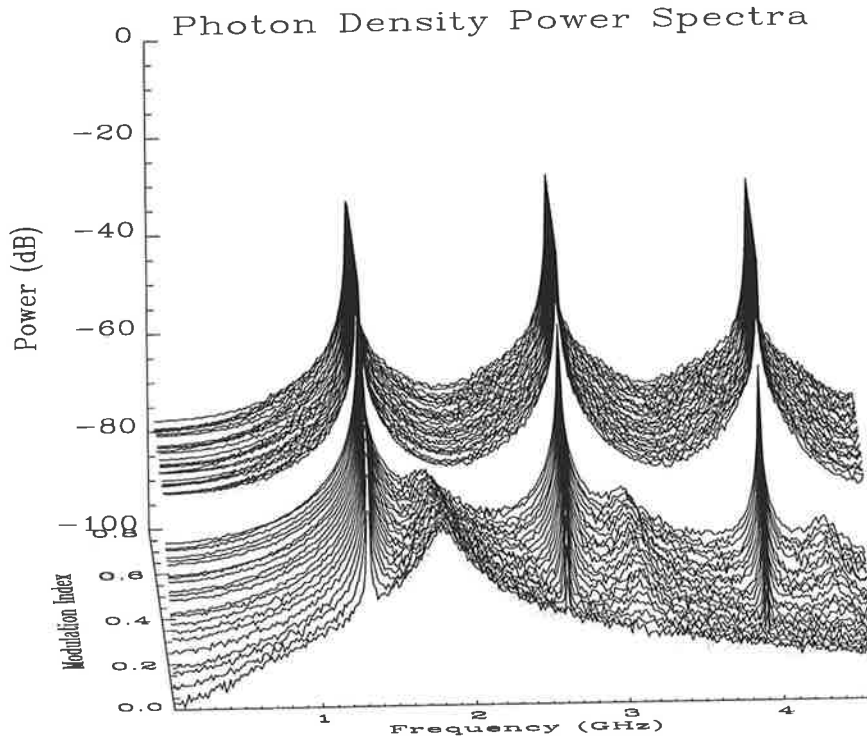


Figure 5.35: Calculated photon density power spectra for decreasing m , including Langevin noise terms. Parameter values are $I_b = 1.7$, $f = 1.3\text{GHz}$ and $\beta = 10^{-4}$.

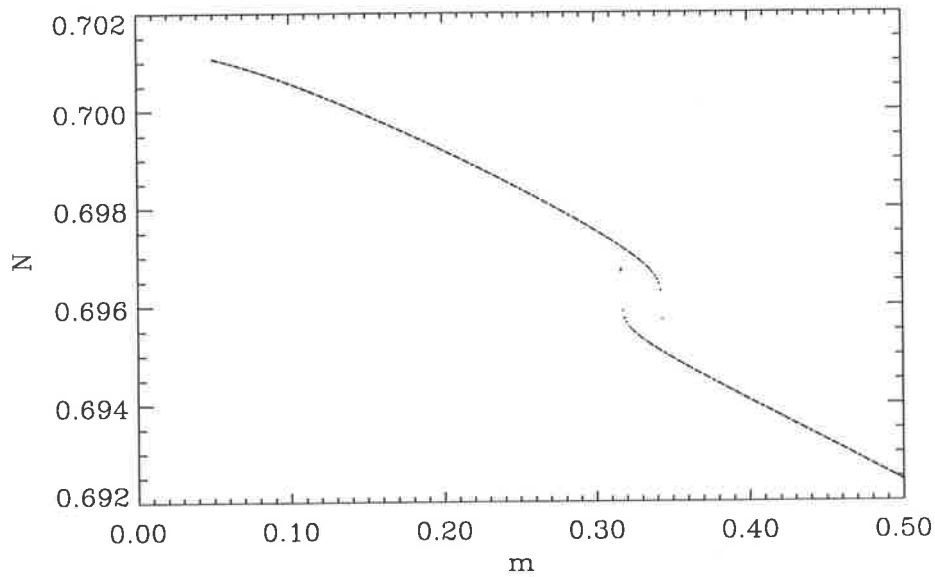


Figure 5.36: Calculated bifurcation diagram for the multimode rate equations for $I_b = 1.7$, $f = 2\text{GHz}$ and $\beta = 5 \times 10^{-5}$.

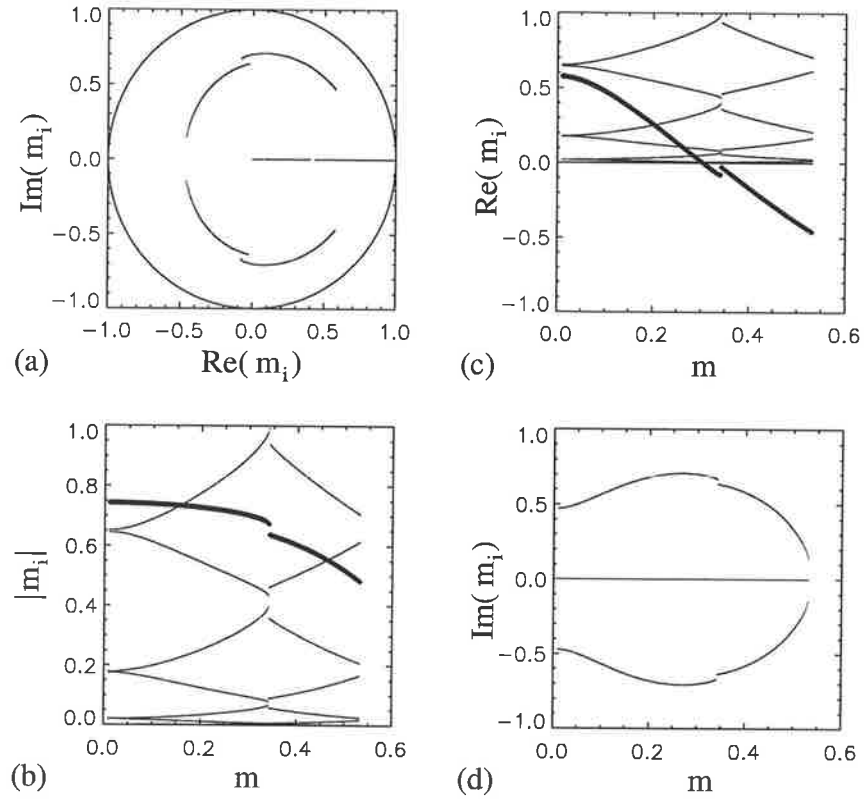


Figure 5.37: Multiplier evolution as a function of modulation index for $I_b = 1.7$, $f = 2\text{GHz}$ and $\beta = 5 \times 10^{-5}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m . In parts (b) and (c) large symbols indicate complex conjugate multipliers.

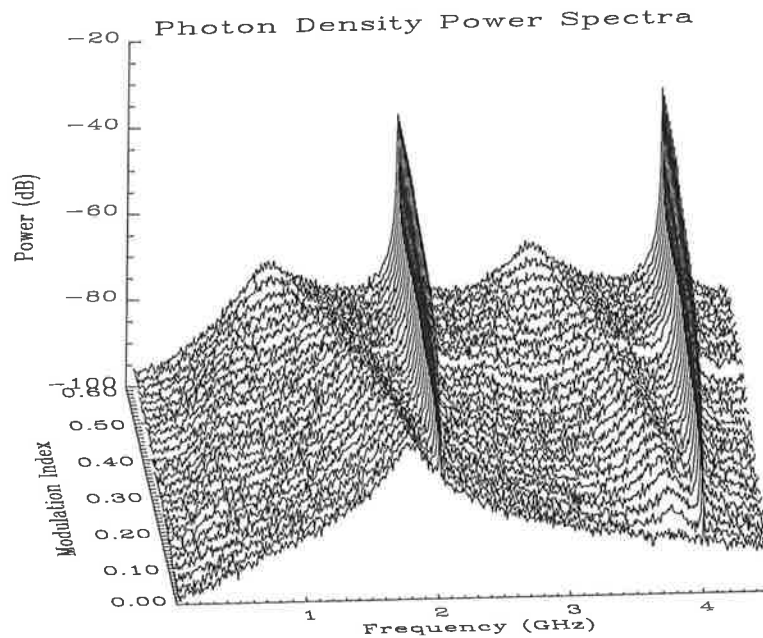


Figure 5.38: Calculated photon density power spectra for the multimode rate equations for $I_b = 1.7$, $f = 2\text{GHz}$ and $\beta = 5 \times 10^{-5}$, including Langevin noise terms.

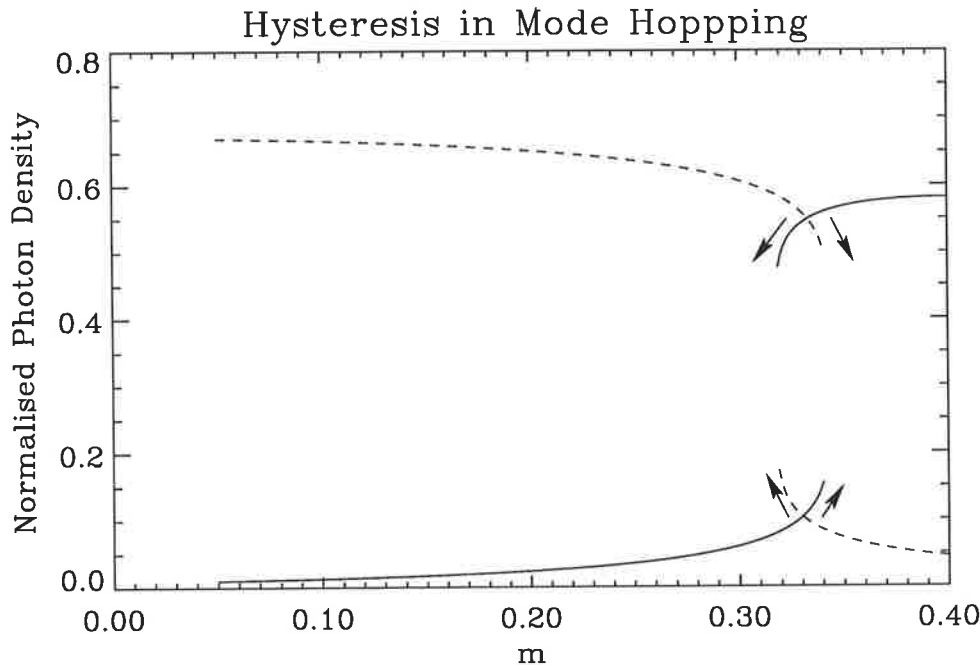


Figure 5.39: Calculated bifurcation diagram for the same parameters as in Fig. 5.36 showing hysteresis in mode hopping. The dashed curve corresponds to the central mode (under constant injection) and the solid curve corresponds to the adjacent shorter wavelength mode.

5.6 Hopf Bifurcations

5.6.1 Numerical Results

Hopf bifurcations cannot occur from periodic solutions of the single mode rate equations due to constraints imposed on the eigenvalues by the dissipative nature of the system. The proof of this assertion is as follows[186, 195, 10]. Firstly, it can be shown that the single mode rate equations, which for the purposes of this discussion we write as

$$\dot{\vec{x}} = \vec{Q}(\vec{x}), \quad (5.26)$$

are strictly dissipative, meaning that volumes in phase space are contracted everywhere. From Liouville's Theorem, we have that volumes in phase space are contracted under action of a vector field if and only if the divergence of the vector field is negative. To show this is the case for the single mode rate equations we make the transformations[10]

$$y = \ln N \quad (5.27)$$

$$x = \ln P \quad (5.28)$$

which are allowed since the photon density and carrier density are always positive. The divergence of equations 5.26, in terms of the variables x and y is then given by

$$\text{div}\{Q(x, y)\} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = -\frac{I(t)e^{-y}}{\tau_e} - \frac{\delta e^{-y}e^x}{\tau_p(1-\delta)} - \frac{\beta e^y e^{-x}}{\tau_p}. \quad (5.29)$$

Thus, the divergence is always negative provided $I(t) > 0$. Since $I(t) = I_b + m \sin(2\pi ft)$ we require¹⁷ $I_b > m$. To demonstrate the impossibility of Hopf bifurcations from periodic solutions, $x_p(t)$, of the single mode rate equations, it is necessary to consider the following relation[186]

$$\Pi_i m_i = \exp \left[\int_0^T \text{div}\{Q(x_p(t))\} dt \right] \quad (5.30)$$

where m_i are the Floquet multipliers of $x_p(t)$. Equation 5.30 has a simple physical interpretation: the lefthand side gives the factor by which volumes in phase space contract or expand under one iteration of the Poincare map whereas the righthand side is the rate of change of the phase space volume under action of the flow integrated over the period of the cycle[12]. For a strictly dissipative system equation 5.30 imposes the following constraint on the multipliers:

$$0 < m_1 m_2 \dots < 1. \quad (5.31)$$

The single mode rate equations have only two multipliers. Thus, Hopf bifurcations are excluded for this system since they require $|m_1| = |\bar{m}_2| = 1$. i.e. a complex conjugate pair to exit the unit circle which violates equation 5.31. This observation has important consequences for the bifurcation behaviour predicted by these equations since the only means by which a stable periodic solution can become unstable is if a multiplier exits the unit circle at +1 or -1, i.e., via a saddle-node or a period doubling bifurcation, respectively. Furthermore, a positive real eigenvalue implies, from equation 5.24, that the relaxation oscillation frequency is coincident with the fundamental or harmonic of the driving frequency. Therefore, in the single mode system, the relaxation oscillation frequency must be pulled to the modulation frequency and harmonics before a saddle-node bifurcation can occur. By similar arguments, the relaxation oscillation frequency must be pulled to $nf/2$ (for n and odd integer), before a period doubling bifurcation can occur. We discuss the consequences of this for the global bifurcation behaviour in the next section.

The presence of band-filling leads to difficulties in proving that the multimode rate equations are strictly dissipative i.e. volume contracting in all regions of phase space. However, regardless of the outcome of such a calculation, equation 5.30 no longer precludes Hopf bifurcations for the multimode system since the number of multipliers is greater than two.

¹⁷Of course, the divergence can also be negative for $m > I_b$. However, in order to prove this explicitly, the values of the variables x and y must be known

Indeed, as discussed in the previous chapter, Hopf bifurcations do occur in this system. In Fig. 5.40 we show the multiplier evolution corresponding to the parameter values $I_b = 1.7$, $f = 1.3\text{GHz}$, $\beta = 5 \times 10^{-5}$ and m tuned from 0.01 to 0.365. Figure 5.40(a) shows that a pair of complex multipliers circle to the right about the origin before they subsequently exit the unit circle giving rise to a Hopf bifurcation. The magnitudes of the multipliers versus m are shown in Fig. 5.40(b) and the real and imaginary parts are shown in parts (c) and (d), respectively; the Hopf bifurcation occurs for $m = 0.365$. The corresponding intensity power spectra for the noise driven system are shown in Fig. 5.41. In this instance we observe that the relaxation oscillation frequency is pulled towards the driving frequency. However, before it reaches this point it becomes undamped resulting in multiple side-bands on the modulation frequency and its harmonics; the signature of a Hopf bifurcation.

5.6.2 Experimental Results

Figure 5.42 shows experimental intensity noise power spectra for a modulation frequency of 2.6GHz and a bias current of 39mA (corresponding to $I_b = 1.86$). The temperature of the laser¹⁸ was biased to 25°C. For the spectrum shown in Fig. 5.42(a) a modulation current of 0.02mA (rms) was used. The relaxation oscillation resonant peak (at $\approx 2.9\text{GHz}$) and its second harmonic are clearly visible as is the peak at the modulation frequency (2.6GHz); no other features are present in the power spectrum. For a modulation current of 2mA (fig. 5.42(b)), side-bands, spaced at intervals of the difference frequency ($f_{RO} - f$), are observed on the modulation frequency. A small peak at ($f_{RO} - f$) is also observed. As this spectrum is qualitatively similar to the power spectra shown in Fig. 5.41, for the modulation indices above the critical value required for Hopf bifurcation, we believe this constitutes experimental evidence of the occurrence of a Hopf bifurcation. There are further factors supporting this conclusion. Firstly, we observe this behaviour for modulation frequencies slightly below the relaxation oscillation frequency which is consistent with the numerical predictions (see Fig. 5.41). Secondly, we only observe this behaviour for high bias currents. This is also consistent with the numerical predictions of the multimode rate equations since, as the state diagrams of Fig. 4.30 and the bifurcation diagrams of Figs. 4.6 to 4.9 show, Hopf bifurcations are more prevalent for higher values of I_b . One significant difference between the numerical results (Fig. 5.41) and the experimental results (Fig. 5.42) is that the relaxation oscillation frequency is pulled towards the modulation frequency by only a small amount in the experiments whereas it is pulled quite strongly towards the modulation frequency in the numerical simulations. However, such a discrepancy could result from the differences in the effective experimental parameter values from those used in the numerical simulations,

¹⁸The 830nm laser diode (used for the measurements of Chapter 3) was used for the power spectral measurements shown in Fig. 5.42.

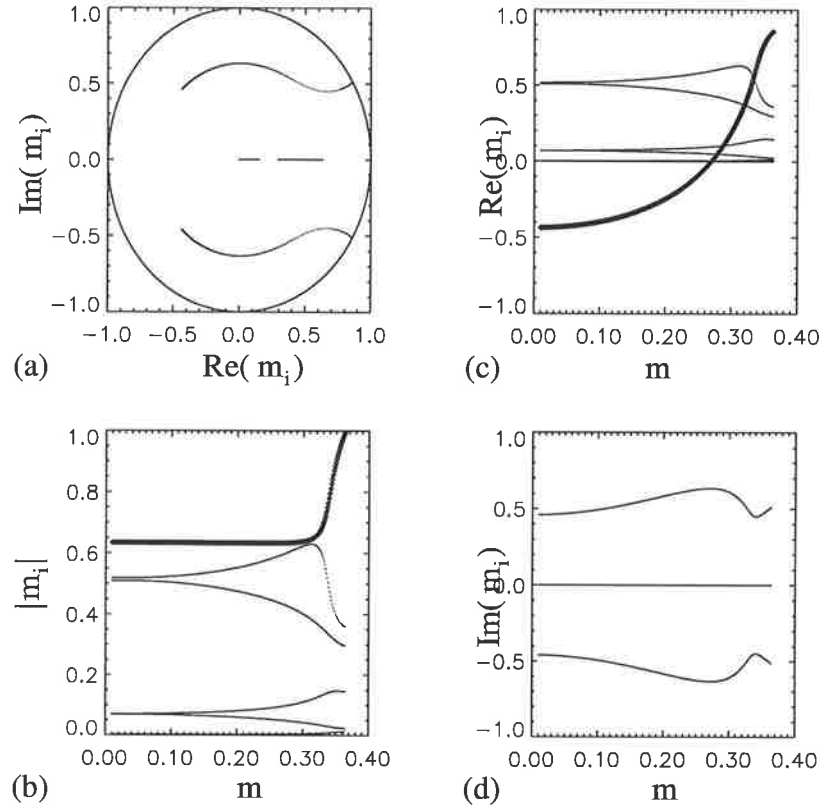


Figure 5.40: Multiplier evolution as a function of modulation index for $I_b = 1.7$, $f = 1.3\text{GHz}$ and $\beta = 5 \times 10^{-5}$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus m , (c) the real part of the multipliers versus m and (d) the imaginary part of the multipliers versus m . In parts (b) and (c) large symbols indicate complex conjugate multipliers.

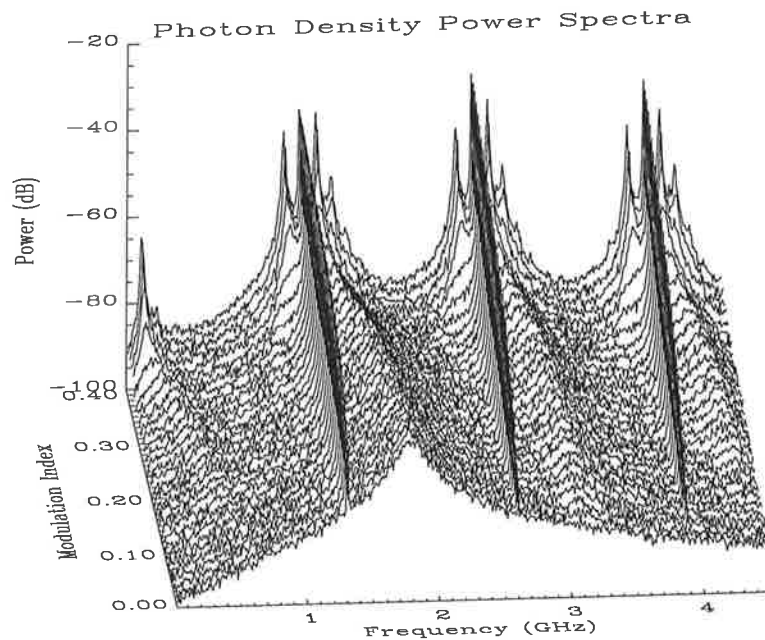


Figure 5.41: Calculated photon density power spectra for the multimode rate equations for $I_b = 1.7$ and $f = 1.3\text{GHz}$, including Langevin noise terms.

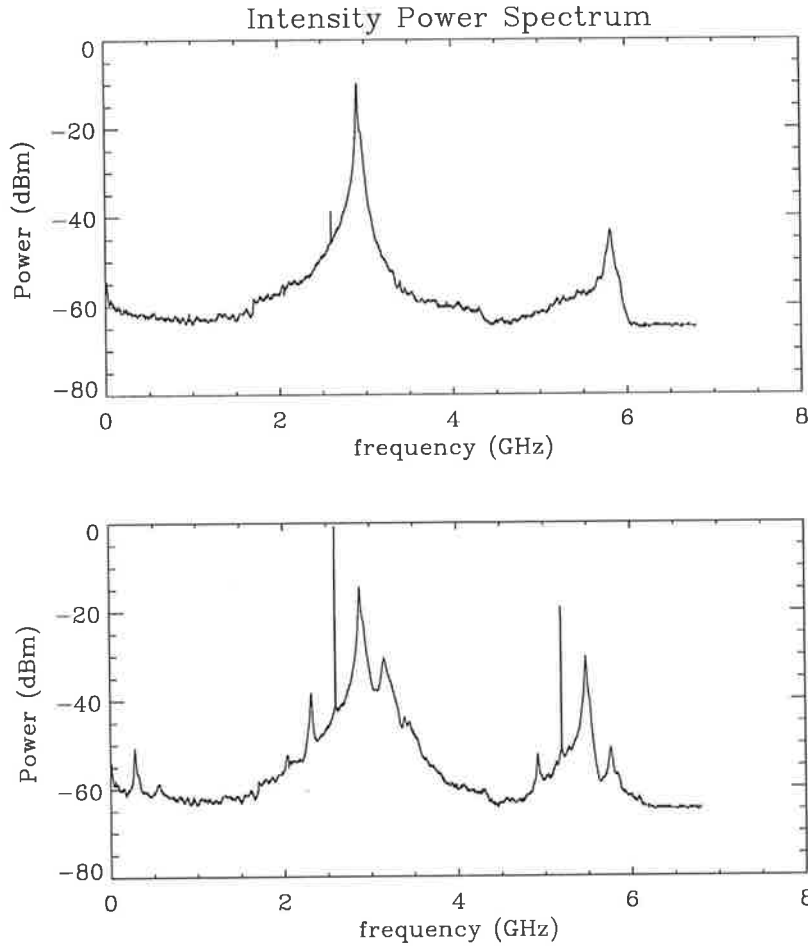


Figure 5.42: (a): Experimental intensity power spectra for a modulation frequency of 2.6GHz, DC bias of 39mA and a modulation current of 0.02mA (rms). (b): Same as (a) except the modulation current was 2mA (rms).

leading to a difference in the effective damping rate. Since the damping rate, along with the relaxation oscillation frequency, is intimately related to the location, and consequently the trajectory, of the multipliers within the unit circle, these properties will determine the extent of pulling of the relaxation oscillation frequency as the modulation frequency or amplitude is varied.

5.7 Global Behaviour

In this chapter we have so far discussed how bifurcations are approached in the single mode and multimode rate equations as the modulation index is increased. It was shown that the

relaxation oscillation resonance plays an active role in the occurrence of bifurcations. In this section we will discuss the implications of this behaviour for the location of the bifurcation surfaces in the parameter space of the modulation frequency and amplitude.

5.7.1 Single Mode System

As previously discussed, for small amplitude modulation or for small perturbations from the steady state solution, the semiconductor laser rate equations behave as a damped linear oscillator with a characteristic resonant frequency f_{RO} . When nonlinear effects are considered, however, we find that the resonant frequency is no longer constant but depends on the size of the perturbation and may become unstable, through bifurcation, leading to an undamped oscillation. Since it is the eigenvalues of the linearised Poincare map that determine the stability properties of periodic attractors, in this section we consider how these evolve with modulation frequency for different modulation amplitudes.

We first consider the multipliers of the linearised system (see appendix D for details). There is only one periodic attractor for a linear oscillator whose multipliers can be calculated analytically. They can either be calculated directly from the coupled rate equations for N and P (equations D.5 and D.6) using the methods of section 5.2.1 or from the second order linearised differential equation for P (equation D.12)[200]. For a linear system, the relationship between the Floquet multipliers $m_{\pm}$, which describe the transient behaviour in the Poincare plane, and the eigenvalues $\lambda_{\pm}$ for the flow is given by[183]

$$m_{\pm} = e^{\lambda_{\pm}/f}, \quad (5.32)$$

where f is the modulation frequency. The eigenvalues $\lambda_{\pm}$, for the single mode rate equations are given in equation D.14. For $\gamma \ll f_{RO}$, the multipliers are, to a very good approximation, given by

$$m_{\pm} \approx e^{\gamma/(2f)} e^{\pm i2\pi f_{RO}/f} \quad (5.33)$$

where γ and f_{RO} are given in equations D.15 and D.17, respectively¹⁹. This equation is independent of the modulation amplitude and the only dependence on the modulation frequency arises from the $1/f$ in the exponent of the right hand side of equation 5.33. In Fig. 5.43 we plot the multipliers, $m_{\pm}$ as a function of modulation frequency; parameter values used in the calculation are given in Table 4.2. Considering the form of equation 5.33 it is not surprising we obtain a chirped signal whose amplitude decreases exponentially as the modulation frequency tends to zero. Fig. 5.43(a) shows that the multipliers spiral outwards from the origin in the complex plane, as the modulation frequency is increased. When the ratio between the

¹⁹It is therefore no coincidence that the exponent of the second exponential of equation 5.33 (being the angle of the multipliers in the complex plane) is equivalent to equation 5.24.

resonant frequency and the modulation frequency is integral then, from equation 5.33, both multipliers have zero imaginary part and lie on the positive real axis, whereas when the ratio f_{RO}/f is equal to $n/2$, where n is an odd integer, then they lie on the negative real axis.

As the modulation index is increased the nonlinear effects lead to a modification of the behaviour shown in Fig. 5.43. In Fig. 5.44 we plot the multipliers versus modulation frequency for a modulation index of $m = 0.25$; all other parameters as in Fig. 5.43. The eigenvalues were calculated numerically from the full nonlinear rate equations. In this case we observe similar behaviour to that shown in Fig. 5.43. However, when the eigenvalues collide on the real axis in some cases they separate, move in opposite direction along the real axis, and then recombine; since the angle of the multiplier in the complex plane determines the frequency ratio between the modulation frequency and the resonant frequency then this behaviour implies that this ratio remains constant as the modulation frequency is tuned. Thus, we observe frequency pulling of the relaxation oscillations when a harmonic of the modulation frequency is tuned close to the relaxation oscillation frequency or alternatively when the relaxation oscillation frequency lies midway between the harmonics of the modulation frequency. The multiplier calculation of Fig. 5.44 corresponds to the bifurcation diagram, Fig. 4.8(a). Due to the resolution of the calculation, Fig. 5.44 does not show a multiplier exiting the unit circle at $+1$ at the saddle-node bifurcation point corresponding to the hysteresis region HS_1 of Fig. 4.8(a). However, an abrupt transition is observed at $f = 1.4GHz$ in Fig. 5.44 resulting from this bifurcation. As the eigenvalues are calculated for increasing modulation frequency only we do not observe the effects of hysteresis.

In Fig. 5.45 we show the multiplier evolution as a function of modulation frequency for $m = 0.5$ corresponding to the bifurcation diagram, Fig. 4.8(b). In this case the intervals over which the eigenvalues are real are larger and more prevalent. As the modulation index is increased still further, eventually one of these multipliers may exit the unit circle at $+1$ or -1 giving rise to a saddle-node or period doubling bifurcation, respectively. For a modulation index of $m = 0.5$ as yet no period doubling bifurcations have occurred for $f < f_{RO}$; this is consistent with the behaviour depicted in the bifurcation diagram of Fig. 4.8(b). However, two saddle-node bifurcations, at $f = 0.7GHz$ and $f = 1.25GHz$, have occurred.

In Fig. 5.46 we plot on a number line the modulation frequencies corresponding to the situation of a harmonic coincident with the relaxation oscillation frequency, i.e.

$$f = f_{RO}, \frac{f_{RO}}{2}, \frac{f_{RO}}{3}, \dots, \quad (5.34)$$

and also for the relaxation oscillation frequency lying midway between the harmonics of f :

$$f = 2f_{RO}, \frac{2f_{RO}}{3}, \frac{2f_{RO}}{5}, \dots \quad (5.35)$$

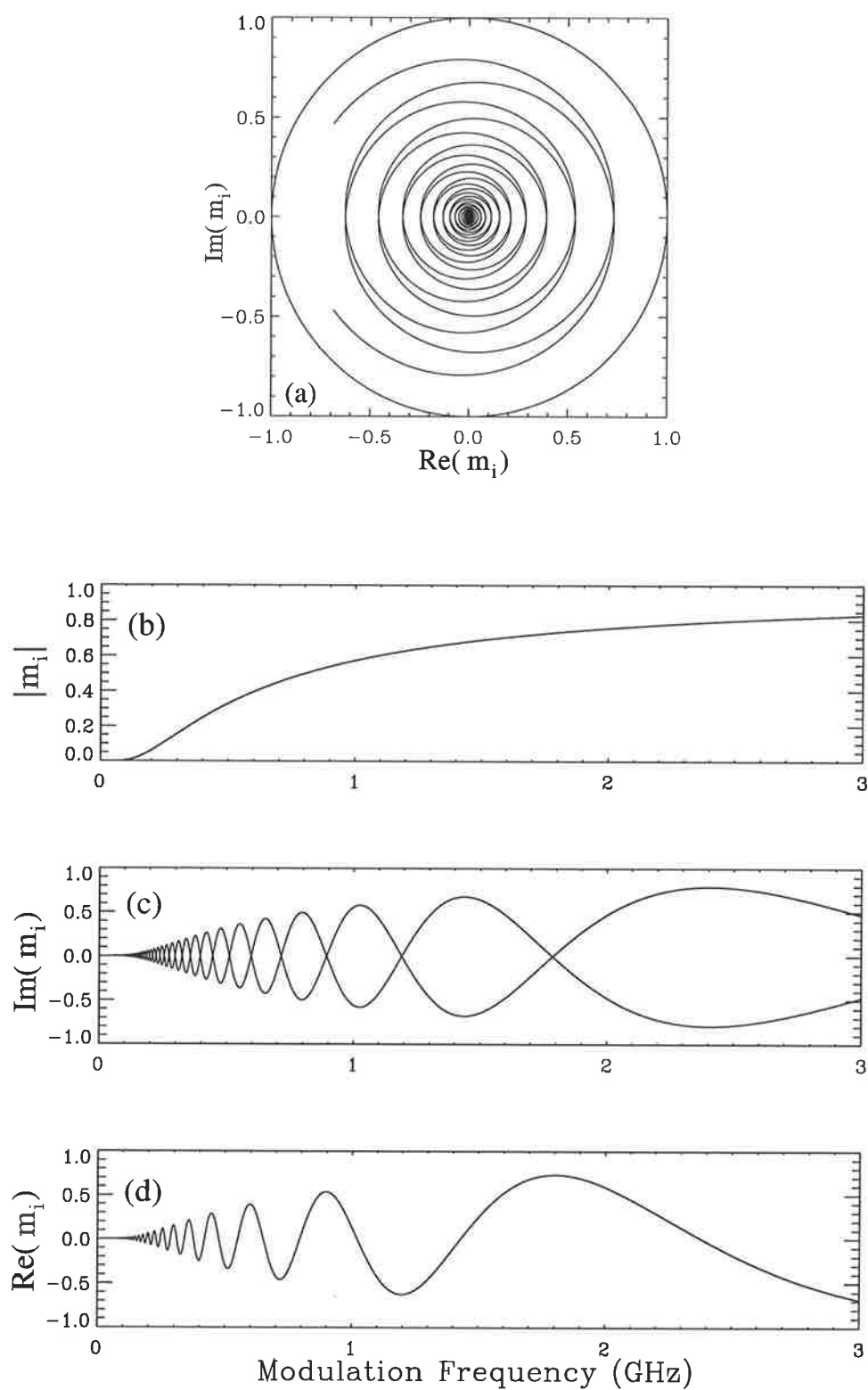


Figure 5.43: Multipliers corresponding to the linearised single mode rate equations for $I_b = 1.7$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus f , (c) the imaginary part of the multipliers versus f and (d) the real part of the multipliers versus f

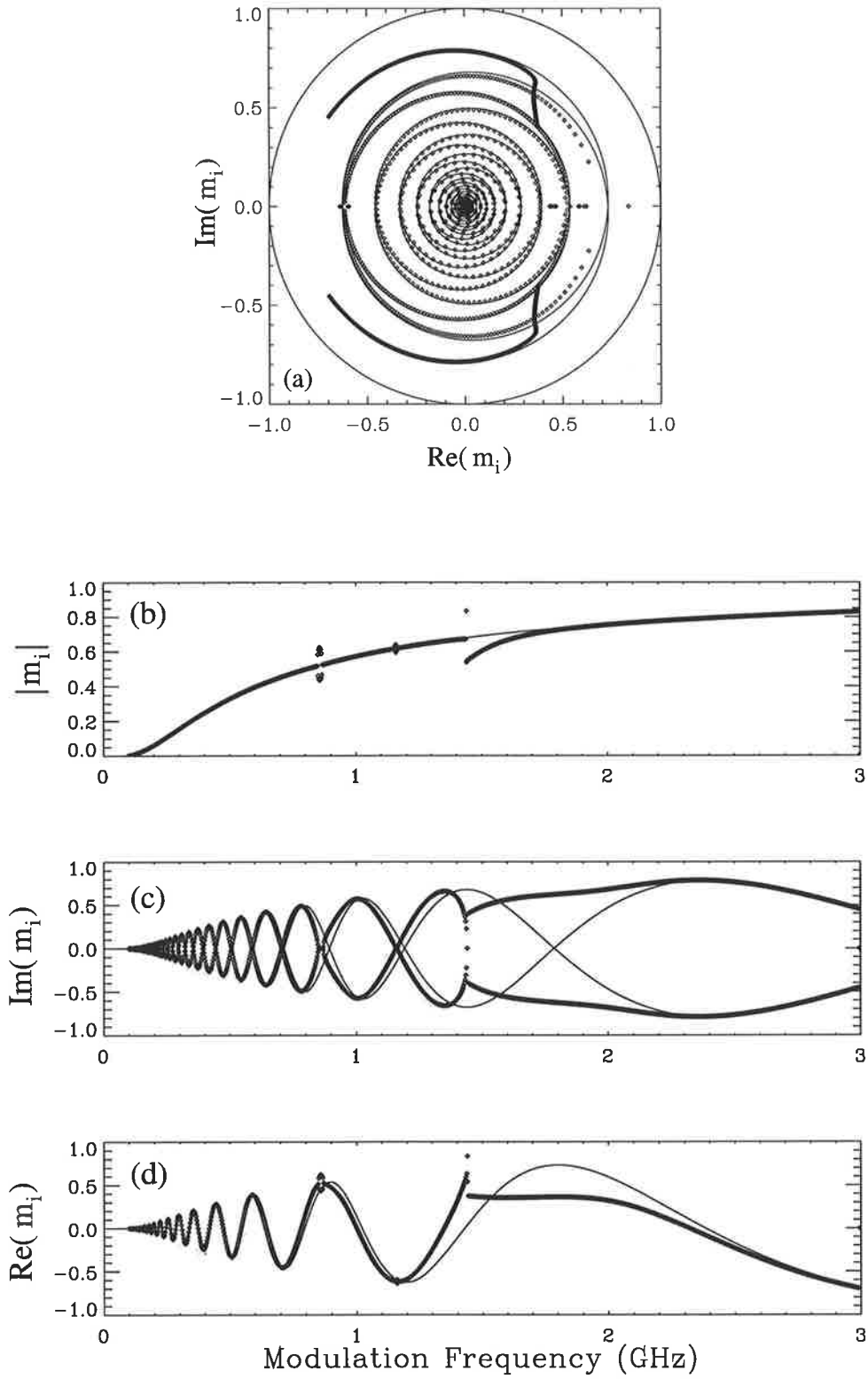


Figure 5.44: Multipliers corresponding to the single mode rate equations for $I_b = 1.7$ and $m = 0.25$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus f , (c) the imaginary part of the multipliers versus f and (d) the real part of the multipliers versus f . The solid line corresponds to the multipliers of the linearised system.

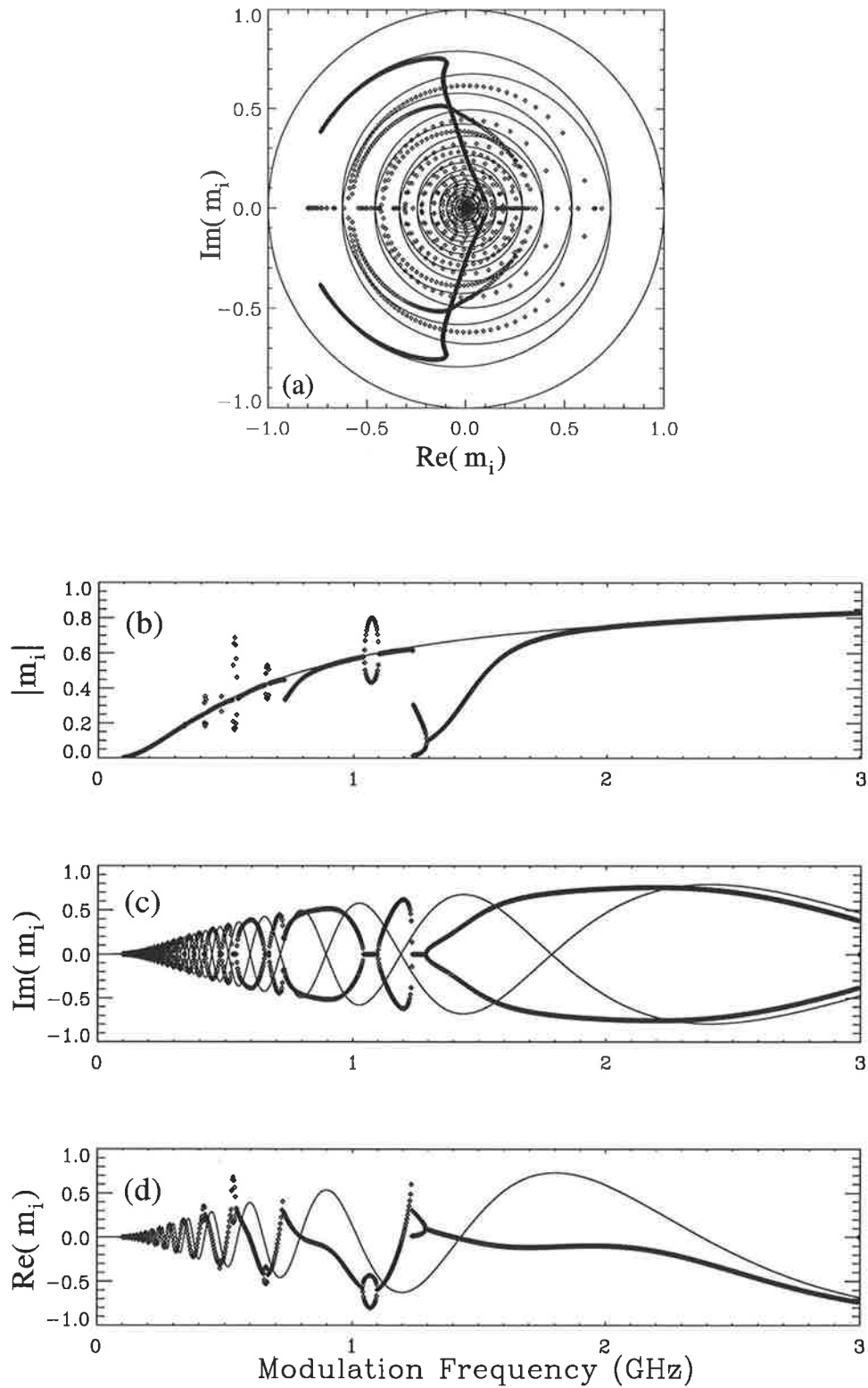


Figure 5.45: Multipliers corresponding to the single mode rate equations for $I_b = 1.7$ and $m = 0.5$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus f , (c) the imaginary part of the multipliers versus f and (d) the real part of the multipliers versus f . The solid line corresponds to the multipliers of the linearised system.

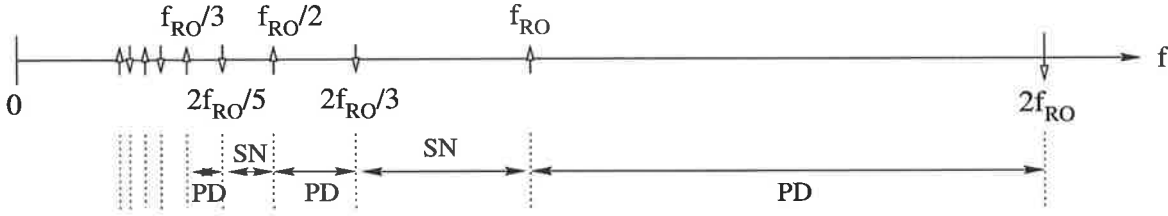


Figure 5.46:

The pattern that emerges is very similar to the bifurcation sequence observed in the bifurcation diagrams of section 4.3 and state diagrams of section 4.7, consisting of an alternating sequence of period doubling and saddle node bifurcations that become more closely spaced as f decreases[11, 10]. Given that the relaxation oscillation frequency must be pulled to a harmonic of f or midway between a harmonic of f before a bifurcation can occur we expect that these bifurcations will occur close to the resonances of the linear system. Thus the relaxation oscillation frequency serves as a landmark in the search for bifurcations in the parameter space[3, 10].

The classification of general one dimensional oscillators based on the ideas above has been discussed in reference [11]²⁰. However, in this study the torsion number, n , which gives a measure of the twisting of the local flow about a periodic orbit, was used to classify the bifurcation structures in the parameter space. This quantity has the advantage over the eigenvalues of measuring the total number of twists during one period rather than just the fraction of a twist completed. Therefore, it allows a unique labelling of bifurcations of periodic orbits without reference to the resonant frequency of the linearised system. However, the torsion number has not yet been defined for higher than three dimensions[11]. For this reason we have not used this quantity in calculations in order to compare the single mode and multimode cases.

5.7.2 Multimode Equations

In this section we show the results of calculations of the multipliers as a function of modulation frequency, performed using the multimode rate equations with band-filling ($k = 350 \text{ \AA}$); other parameter values are given in Table 4.2. For small modulation index ($m = 0.01$) the evolution of the complex conjugate pair of multipliers (shown in Fig. 5.47) is, qualitatively, very similar to that for the linearised single mode equations. Again the complex conjugate pair of multipliers spiral outwards from the origin of the complex plane as the modulation frequency is increased. The real multipliers are all positive and tend to zero as the modulation

²⁰However, for those with a symmetric potential (e.g. Duffing's oscillator) there will be modification of the above picture due to the occurrence of symmetry breaking transformations[11, 205, 206].

frequency tends to zero.

With increasing modulation index, the behaviour of the multipliers, as a function of the modulation frequency, becomes increasingly more complicated. In Fig. 5.48 we show the multiplier evolution for a modulation index of $m = 0.25$. The trajectories of the multipliers in the complex plane are shown in Fig. 5.48(a); as in the previous examples the multipliers spiral outwards from the origin but experience significant distortion of their trajectories relative to the small signal case (Fig. 5.47). In Figs. 5.48(b) and (c), the real and imaginary parts of the multipliers versus modulation frequency are shown, respectively. Figures 5.48(b) and (c) show that, when the complex conjugate pair of multipliers have negative real part, their trajectories resemble those of the corresponding single mode calculation (Fig. 5.44). However, when all multipliers shown in Fig. 5.48 have positive real part, complicated behaviour ensues. Before discussing this behaviour, we note that the eigenvalue calculation of Fig. 5.48 corresponds to the same parameter values as those used in the calculation of the longitudinal mode spectra versus modulation frequency shown in Fig. 4.12(a). For convenience, we re-plot the data of Fig. 4.12(a) in Fig. 5.48(d) in order to compare results. There are two points to be ascertained from this comparison. Firstly, in the region of multimode operation, shown in Fig. 5.48(d), the multipliers exhibit complicated trajectories. The fact that the multiplier trajectories deviate significantly from the small signal case (Fig. 5.47) in this region, indicates that the multimode operation is essentially a nonlinear process. A comparison of Figs. 5.48(c) and (d) also shows that for $f \approx 1.9\text{GHz}$ a crossing of the real multipliers occurs; this corresponds to the abrupt mode hop between adjacent modes in Fig. 5.48(d).

For completeness, we also include the multiplier evolution versus modulation frequency for $m = 0.5$ (Fig. 5.49). As in the previous example, the trajectories of the multipliers in the complex plane are shown in Fig. 5.49(a) and the real and imaginary parts versus f are shown in Fig. 5.49(b) and (c), respectively. Since this multiplier calculation corresponds to the same parameter values as those used in the calculation of the longitudinal mode spectra versus f shown in Fig. 4.12(b) we re-plot this data in fig. 5.49(d) in order to compare results. As in the previous example, the region in which complicated behaviour of the multipliers occurs, corresponds to regions of multimode operation. Furthermore, for the largest region of multimode operation (centred at approximately 1.5GHz) there exists a large region in which all multipliers lie on the real axis. This implies that the relaxation frequency is coincident with the harmonics of the driving frequency in this region. Therefore, this region of multimode is accompanied by frequency pulling of the relaxation oscillation frequency confirming that this is both a nonlinear and resonant phenomena. Similar behaviour to the previous example also occurs in regard to the abrupt mode hop between adjacent modes which occurs at $f \approx 2\text{GHz}$; we also observe crossing of the real multipliers at this frequency.

The behaviour of the complex conjugate multipliers, as they approach the negative

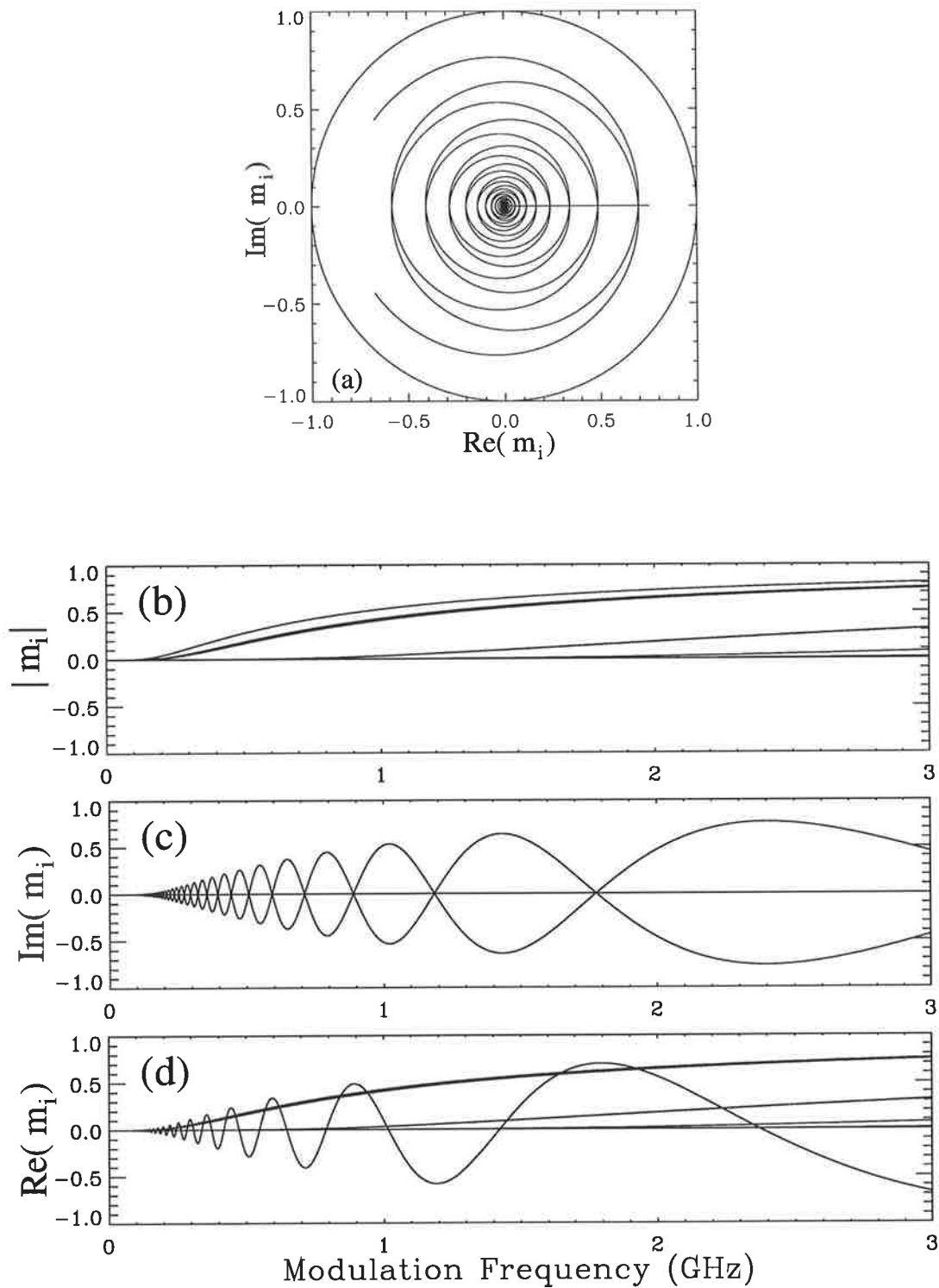


Figure 5.47: Multipliers corresponding to the multimode mode rate equations for $I_b = 1.7$ and $m = 0.01$: (a) the trajectories of the multipliers in the complex plane, (b) the magnitude of the multipliers versus f , (c) the imaginary part of the multipliers versus f and (d) the real part of the multipliers versus f

real axis, is less complicated. Figs. 5.49(b) and (c) also show that, after the complex conjugate pair of multipliers collide on the negative real axis, there are large regions of the modulation frequency in which both “complex” multipliers are pulled in opposite directions along the negative real axis; these regions correspond to frequency pulling of the relaxation oscillation frequency at frequencies lying midway between the harmonics of the modulation frequency. This also leads to enhancement of the height of the relaxation oscillation resonant peak in these regions.

In summary, both Figs. 5.48 and 5.49 show that the existence of many real multipliers in the multimode system leads to a deviation from the behaviour exhibited by the single mode system. The increased dimensionality of the multimode system thus leads to a change in the sequence of bifurcations that occur as the modulation frequency is varied.

5.8 Summary

One of the most interesting and fundamental properties of dissipative nonlinear systems is the occurrence of bifurcations. Bifurcations are responsible for the creation of new attractors and can, in some cases, lead ultimately to the occurrence of chaotic motion.

The study conducted in the previous chapter of the location of the period doubling bifurcation surfaces in the (f, m) parameter space showed that they exist in ‘islands’ which reduce in size and separation as the modulation frequency is reduced (see Fig. 4.30). Furthermore, the bifurcation diagrams for the single mode system (Figs. 4.6 to 4.9) showed that the period doubling bifurcations are interleaved with saddle-node bifurcations. In the multimode system both saddle-node and Hopf bifurcations can occur between the period doubling bifurcations. In this chapter, a study of the approach to these bifurcation surfaces, as the modulation amplitude is increased from zero, was conducted.

Two different types of calculations, the Floquet multipliers and noisy power spectra, were utilised for this purpose. The Floquet multipliers are the eigenvalues of the linearised Poincare map, $P(x)$. The Poincare map is often employed as means of determining the stability of a periodic attractor as it corresponds to a fixed point, x^* of $P(x)$. The utility of the method lies in the fact that the linearised Poincare map can be obtained numerically. At a bifurcation point an attractor becomes unstable; thus the Floquet multipliers provide a means of directly detecting when a bifurcation occurs.

The second type of calculation, the noisy power spectra, are useful for comparing with experimental data. Noise is an inherent property of all physical systems that cannot be eliminated. Close to an instability the noise becomes amplified in the power spectra of the system observables. Therefore, experimental observation of the power spectra provides a means of probing the system dynamics in the approach to a bifurcation. Indeed, it can even be used to predict the type of bifurcation to be encountered before it has actually occurred.

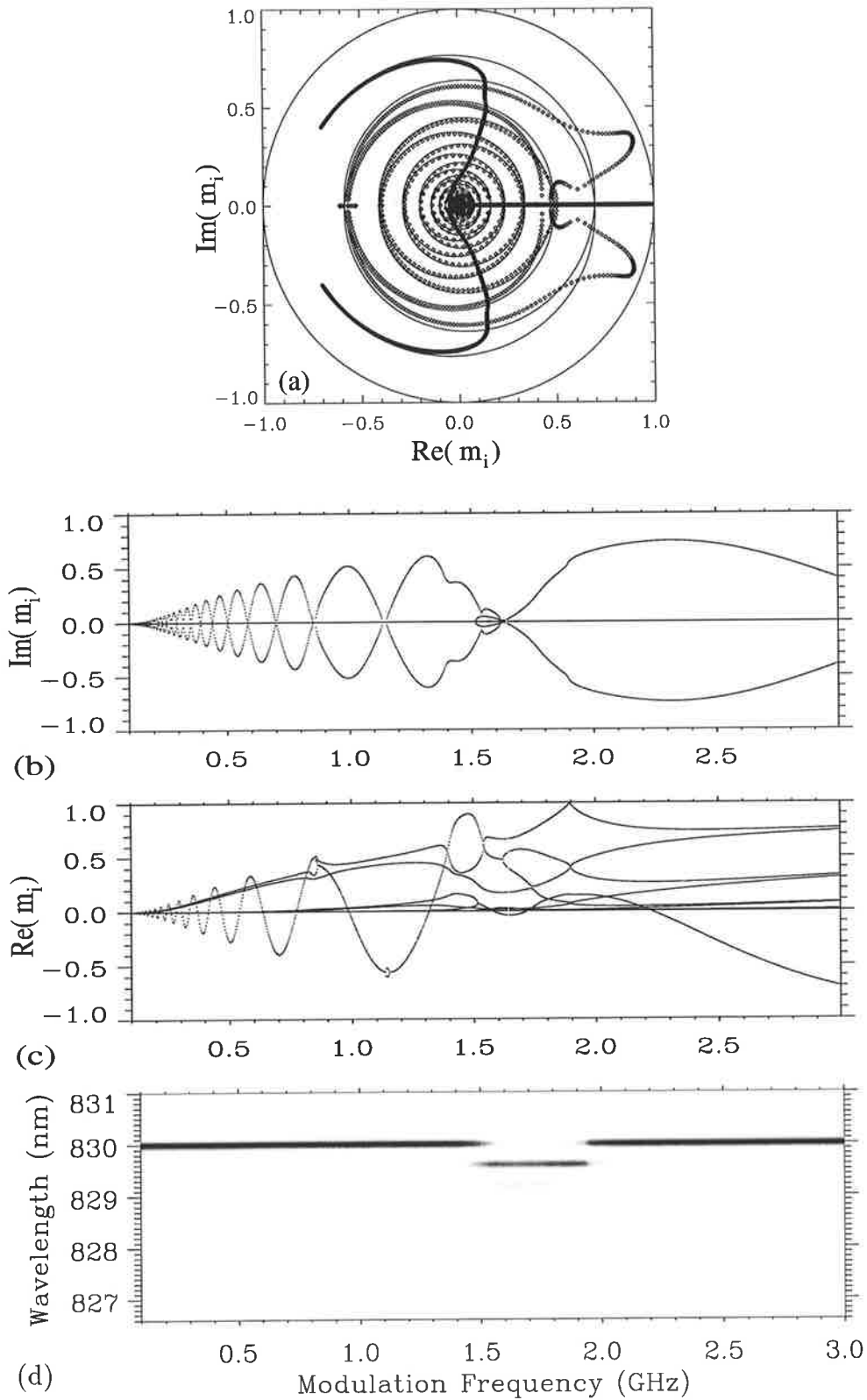


Figure 5.48: Multipliers corresponding to the multimode mode rate equations for $I_b = 1.7$ and $m = 0.25$: (a) the trajectories of the multipliers in the complex plane, (b) the imaginary part of the multipliers versus f , (c) the real part of the multipliers versus f and (d) Time average longitudinal mode spectra versus f . In part (a), the solid line corresponds to the multipliers of the linearised system.

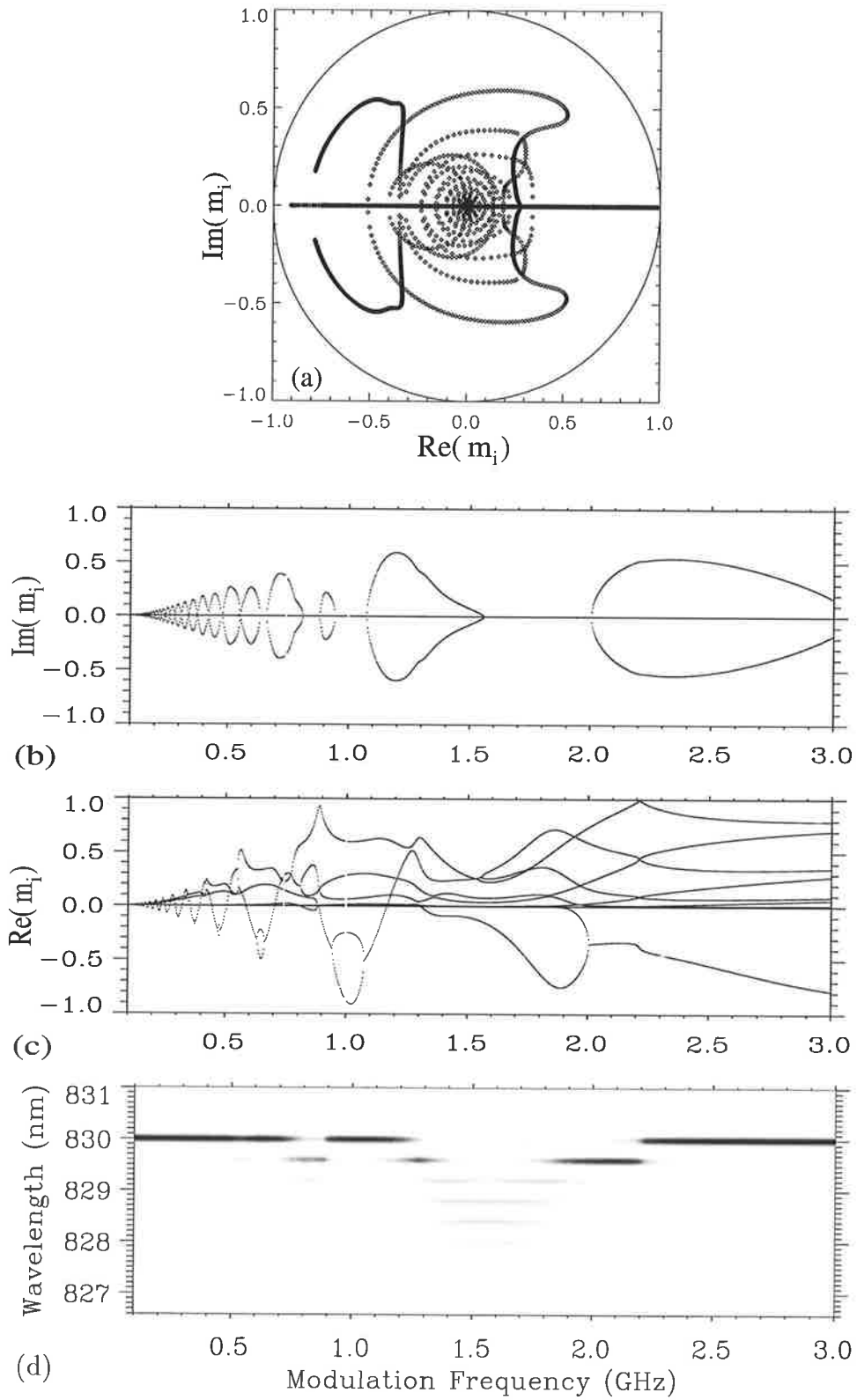


Figure 5.49: Multipliers corresponding to the multimode mode rate equations for $I_b = 1.7$ and $m = 0.5$: (a) the trajectories of the multipliers in the complex plane, (b) the imaginary part of the multipliers versus f , (c) the real part of the multipliers versus f and (d) Time average longitudinal mode spectra versus f .

The information gained from these two types of calculations are not unrelated. The Floquet multipliers govern not only the stability of the fixed point, x^* , but also the evolution of points in the Poincare plane close to x^* . In other words, since by definition a stable fixed point implies nearby trajectories converge to it, the Floquet multipliers describe the transient approach to the fixed point. Furthermore, as the Poincare map is derived from the rate equations, it also contains some information about the transient behaviour of the vector field for trajectories close to the periodic attractor. However, by virtue of the fact that the Poincare map samples the trajectories at discrete time intervals, some information is lost. In particular, if the transient behaviour of the flow is a damped oscillatory motion, then from the Nyquist sampling theorem, this transient frequency can only be determined if it is less than twice the sampling frequency. The noisy power spectra also give information about the transient frequencies. This arises from the fact that the noise fluctuations act to continually perturb the system from its stable behaviour. Thus, the transient behaviour, with which the system relaxes back to the steady state, contributes to the observed power spectrum. Therefore, calculation of both the Floquet Multipliers and noisy power spectra provide a means of probing the behaviour of the resonances of a system as a control parameter is tuned, and also the role they play in bifurcations.

Using the methods outlined above, the approach to period doubling, saddle-node and Hopf bifurcations in the semiconductor laser rate equations, was studied. The single mode rate equations possess only a single pair of multipliers which govern the stability properties of periodic attractors. Therefore, there is only one resonant frequency of this system. Furthermore, due to constraints imposed on the multipliers resulting from the dissipative properties of the system, the only possible means by which a periodic attractor can become unstable is if a single multiplier exits the unit circle at $+1$ or -1 corresponding to a saddle-node or period doubling bifurcation, respectively; Hopf bifurcations are excluded for this system²¹. Since a real multiplier implies that the ratio between the modulation frequency and the relaxation oscillation frequency is integral or half integral, the corresponding behaviour in the noisy power spectra is that the relaxation resonance is pulled to the harmonics modulation frequency for a saddle-node bifurcation whereas for a period doubling bifurcation it is pulled to midway between the harmonics of the modulation frequency (in the interest of simplifying the following discussion we subsequently refer to the frequencies lying midway between the harmonics of f as half-harmonics). In either case, the relaxation oscillation frequency is always pulled to a lower frequency as the modulation index is increased (for $f < 2f_{RO}$). A saddle-node bifurcation leads to the collision, and subsequent annihilation, of the fixed point in the Poincare plane and an unstable saddle point; after the bifurcation point the system makes a transition to an existing stable attractor. Period doubling bifurcations are

²¹Strictly speaking for $m < I_b$.

qualitatively different. At a period doubling bifurcation point, the relaxation oscillation frequency becomes undamped leading to periodic motion with double the period of the original attractor.

The above observations, taken collectively, lead to an understanding of the bifurcation sequence shown in the bifurcation diagrams (Figs. 4.6 to 4.9) of the previous chapter. Since it is easiest to drive the system to instability when the above criterion for the relaxation oscillation frequency in reference to the modulation frequency are satisfied, bifurcations are most likely to occur close to (but slightly below) the resonances of the linearised system. The alternating sequence of period doubling and saddle-node bifurcations thus arises as the harmonics and the half-harmonics of the modulation frequency pass through the relaxation oscillation frequency as the modulation frequency is increased. Bifurcations occur more easily when the lower harmonics and half-harmonics are near f_{RO} (i.e. for the higher modulation frequencies) since these correspond to shorter modulation periods and therefore considerably less decay of the relaxation oscillation will have occurred within one modulation period than for lower modulation frequencies. Finally we note that these observations are relevant to all one-dimensional nonlinear oscillators. However, some modifications arise for nonlinear oscillators with symmetric potentials due to the occurrence of symmetry breaking bifurcations.

Despite the increased dimensionality of the multimode rate equations some features present in the single mode rate equations also apply to this system. This is largely due to the fact that there is generally only a single pair of complex multipliers for this system, corresponding to the relaxation oscillation frequency. In particular the period doubling bifurcations for the multimode system also follow a virtual Hopf sequence in which the relaxation oscillation is pulled to the half-harmonics of f before the onset of the bifurcation. As the bifurcation point is approached considerable enhancement of the noise at half-harmonics of f in the photon density power spectrum occurs. These so-called noise precursors become undamped, at the bifurcation point, leading to a period doubled attractor. This behaviour was confirmed in experiments performed on a multimode FP semiconductor laser. The corresponding Floquet multiplier calculations showed that the complex multiplier pair circle to the left about the origin in the complex plane before colliding on the negative real axis. They then separate and one multiplier exits the unit circle at -1 giving rise to a period doubling bifurcation.

Saddle-node bifurcations in the multimode system were shown to result from the exit of a real multiplier from the unit circle at +1 rather than involving the complex conjugate pair of multipliers as in the period doubling bifurcation. Therefore they are unaffected by the relaxation oscillation frequency and do not follow the same pattern as the single mode system. An interesting feature of this bifurcation is that it is accompanied by hysteresis and mode hopping between the central longitudinal (under DC operation) and the adjacent

shorter wavelength mode.

Due to the higher dimensionality of the multimode system, the same constraints on the multipliers do not apply therefore allowing Hopf bifurcations. Calculation of the multiplier evolution as a function of modulation index, for a modulation frequency slightly below the relaxation oscillation frequency, shows that the complex conjugate pair circle to the right about the origin before they subsequently exit the unit circle giving rise to a Hopf bifurcation. The corresponding noisy power spectra show that the relaxation oscillation frequency is pulled towards the modulation frequency. However, before it reaches this point it becomes undamped resulting in multiple side-bands on the modulation frequency and its harmonics. Experimentally we have also observed this behaviour in a modulated FP semiconductor laser when operated at high bias currents (near twice the laser threshold current).

In summary, like the single mode semiconductor laser, the multimode semiconductor laser also exhibits behaviour analogous to a one dimensional damped nonlinear oscillator. This results from that fact that it only has a single resonant frequency. However, more detailed investigations reveal that deviations from this behaviour arise due to the increased dimensionality of this system.

Chapter 6

Conclusion

There are many existing studies of current modulated semiconductor lasers due to the application of these devices in optical fibre communication systems. The study of these devices is also of interest from the nonlinear dynamics viewpoint due to the interesting variety of behaviour exhibited by these systems under different operating conditions. In previous studies of the numerical solutions of the semiconductor laser rate equations, chaotic behaviour under direct modulation of the injection current was predicted. However, in contrast to other pump modulated laser systems, such as CO₂ or Nd:YAG lasers, the transition to chaotic behaviour is rarely observed experimentally in these systems. This disparity between theory and experiment arises, in part, due to the complicated nature of the semiconductor system. Indeed, as previous studies have shown the existence of chaotic solutions (and other nonlinear behaviour) can depend sensitively on the particular physical characteristics of these devices included in the simulation models. Thus careful consideration of the relevant physical properties of these devices is essential if an accurate description of their behaviour is to be obtained.

In Chapter 4 a detailed study of the bifurcation scenarios predicted by a multimode semiconductor laser rate equation model, which includes the effects of band-filling, was conducted. Though both multimode operation and band-filling are well known characteristics of FP semiconductor lasers, to the best of our knowledge, this is the first study of the effect of these characteristics on the predicted bifurcation scenarios under injection current modulation. As shown in Chapter 4, a comparison of the bifurcation scenarios predicted by the multimode rate equations with those of a typical single mode rate equation model, revealed several important differences. Firstly, the inclusion of multiple longitudinal modes lead to a reduction in the extent of the bifurcations that occurred as the modulation frequency was varied. In particular, a period doubling route to chaos was shown not to occur in the multimode system when it was predicted by the single mode rate equations under the same operating conditions. Such predictions are consistent with experimental observations since period doubling to chaos under direct modulation of the injection current has never been

observed to occur in these systems.

A further difference in the bifurcation behaviour of the single mode and multimode rate equation models lies in the occurrence of Hopf bifurcations in the multimode system; as shown in previous studies, these types of bifurcations cannot occur from periodic solutions of the single mode system. A Hopf bifurcation results in the creation of a new frequency and can, therefore, be detected, experimentally, by monitoring the power spectrum of the laser intensity, as the amplitude or frequency of the modulation current is varied. Experimental evidence of this phenomenon, in an injection current modulated FP semiconductor laser, was presented in Section 5.6.2.

In both Chapters 4 and 5, hysteresis in the multimode rate equations with band-filling was studied. It was shown, in numerical simulations, that the occurrence of hysteresis as the modulation frequency or amplitude is varied is accompanied by mode hopping between adjacent longitudinal modes. Experimentally, we have not observed hysteresis in mode hopping in our particular semiconductor laser. However, an abrupt mode hop to an adjacent longitudinal mode was observed in the experiments in the region in which we would expect to hysteresis to occur. As we have shown in numerical simulations, an increase in the damping of the relaxation oscillations is sufficient to remove hysteresis in the mode hopping. Thus, we suggest that hysteresis would occur in the mode hopping region in a FP semiconductor laser exhibiting less damping of the relaxation oscillations.

The behaviour of the time averaged longitudinal mode spectra under injection current modulation, was investigated experimentally over a wide range of modulation frequencies, and compared with the results of numerical simulations. Good agreement between theory and experiment was obtained when band-filling effects were included in the rate equations. A distinguishing feature, observed both numerically and experimentally, was that, when a harmonic of the driving frequency was coincident with the relaxation oscillation frequency, an increase in the number of operating longitudinal modes occurred accompanied by a spectral shift towards shorter wavelengths.

In Chapter 3, simple diagnostic measurements used to obtain the parameter values for the multimode semiconductor laser rate equations, were performed using an 830nm semiconductor laser. These experimentally determined parameter values were subsequently used in the multimode rate equations in simulations of the longitudinal mode spectral behaviour with varying modulation frequency. These predictions showed good quantitative agreement with the experimental longitudinal mode spectra, measured under the same operating conditions. The experimentally determined parameters were also used to calculate a state diagram, for $I_b = 1.095$, showing the bifurcation surfaces of the first and second period doubling bifurcations. Only a single period doubling region, for modulation frequencies above the relaxation oscillation frequency, was predicted in the simulation with no further period doubling bifurcations to period four occurring. Furthermore, no other bifurcations

(e.g. Hopf or saddle-node bifurcations) were predicted. In the corresponding experimental investigation, only a single period doubling region was detected. The width (as a function of modulation frequency) and position of this period doubling region was consistent with the numerical predictions. However, experimentally, the period doubled trajectory disappeared for higher modulation amplitudes which was not shown in the simulations.

In Chapter 5 the approach to bifurcations was studied in more detail using Floquet multiplier theory and calculations of total photon density power spectra using noise driven rate equations. We showed that a virtual Hopf approach to period doubling was predicted by the multimode semiconductor rate equations; band-filling effects were included in these calculations. The virtual Hopf Phenomenon was also demonstrated experimentally in a FP semiconductor laser.

More detailed investigations into the bifurcation behaviour predicted by the multimode rate equations with band-filling revealed several important differences from the single mode behaviour. Previous studies have shown that a single mode semiconductor laser can be viewed as a one dimensional nonlinear oscillator with a potential very similar to the Toda oscillator. In the investigations of Chapter 5, we showed that, despite the increased dimensionality, the multimode semiconductor laser rate equations also exhibit behaviour analogous to a one dimensional oscillator, largely due to the existence of a single resonant frequency. However, the presence of a large number of Floquet multipliers in the multimode system also led to differences in the predicted bifurcation behaviour. In particular, it was shown that saddle-node bifurcations are caused by a real multiplier exiting the unit circle at $+1$ and were not related to the relaxation oscillation resonance of the multimode system. The occurrence of Hopf bifurcations, as previously mentioned, is another difference between the single mode and multimode systems.

In conclusion, we have conducted detailed experimental and numerical investigations into the behaviour of injection current modulated FP semiconductor lasers. We have found that both band-filling effects and the multimode nature of the device have direct consequences for the dynamical behaviour.

Appendix A

Material Properties of AlGaAs

In this Appendix we tabulate relevant material properties of GaAs and $\text{Al}_{1-x}\text{Ga}_x\text{As}$ (Table A.1). In compiling this table, we used data from the following sources:

1. Landolt-Börnstein: Numerical Data and Functional Relationships in Science and Technology, Volume 22: Semiconductors, 1987[84]
2. Properties of Aluminium Gallium Arsenide, editor: S. Adachi, 1993[207]
3. Semiconducting and other Major Properties of Gallium Arsenide, editor: J. S. Blakemore, 1987[83]
4. GaAs, AlAs and $\text{Al}_x\text{Ga}_{1-x}\text{As}$: Material Parameters for use in Research and Device Applications, editor: J. S. Blakemore, 1987[208]

Table A.1:

ENERGY GAP				COMMENTS
$GaAs$	$E_g(T) = 1.519 - 4.408 \times 10^{-4}T^2/(T + 204)$			T in K
$Al_xGa_{1-x}As$	$E_g(x) = 1.424 + 1.247x$, $x < 0.45$			T=300K
Temp. dependence	$dE_g/dT = (-3.95 - 1.15x) \times 10^{-4}eV/K$			
CONDUCTION BAND EFFECTIVE MASS				COMMENTS
$GaAs$	$0.067m_o$ $0.063\ m_o$			T=0K T=300K
$Al_xGa_{1-x}As$	$0.067 + 0.083x$			
VALENCE BAND EFFECTIVE MASS				COMMENTS
Type of hole		$GaAs$	$Al_xGa_{1-x}As$	Superscript sph indicates a spherically averaged effective mass[62]
heavy hole	m_{hh}/m_o [100]	0.33 ± 0.05	$0.33 + 0.18x$	
	m_{hh}/m_o [111]	0.77 ± 0.12	$0.77 + 0.32x$	
	m_{hh}^{sph}/m_o	0.5	$0.5+0.25x$	
light hole	m_{lh}/m_o [100]	0.09 ± 0.004	$0.090 + 0.090x$	
	m_{lh}/m_o [111]	0.077 ± 0.003	$0.077 + 0.073x$	
	m_{lh}^{sph}/m_o	0.08	$0.08 + 0.08x$	
VALENCE BAND PARAMETERS ($GaAs$)				COMMENTS
A	-6.98(45)			parameters refer to equations (2.27)-(2.29).
B	-4.5(2)			
C	6.2(110)			
$k.p$ INTERACTION PARAMETERS ($GaAs$)				COMMENTS
E_p	25.0(5) eV			$E_p = 2P^2/m_o$ for P in eqn. 2.23
E'_p	5(1) eV			
REFRACTIVE INDEX ($GaAs$)				COMMENTS
n_∞	$3.225(1 + 4.5 \times 10^{-3}T)$			T in K
n	$[7.10 + 3.78/(1 - 0.18(h\nu)^2)]^{1/2}$, $h\nu$ in eV			$T = 300K$
LATTICE PARAMETERS				COMMENTS
Parameter	$GaAs$	$Al_{1-x}Ga_xAs$	Unit	
lattice constant (a)	5.653	$5.6533 + 0.0078x$	$\text{\AA}$	
crystal density (ρ)	5.360	$5.35 - 1.6x$	g/cm^3	
Thermal exp. (α)	6.4	$6.4 - 1.2x$	$\times 10^{-6}K$	

Appendix B

Field Quantisation

Field quantisation is a general procedure applied to the quantisation of classical fields. This leads to a formulation of quantum mechanics in terms of generation and annihilation operators that is particularly suited for many-body systems. Formally this quantisation procedure can also be applied to the one-particle Schrodinger field and is often referred to as second quantisation in this regard. The following discussion can be found in reference [85].

B.1 Classical Langrangian Field Theory

The generalisation of Lagrangian and Hamiltonian point mechanics to the continuum case requires the introduction of the Lagrangian functional:

$$L[\phi_j] = \int d^3r \mathcal{L} \left(\phi_j, \frac{\partial \phi_j}{\partial x_j}, \frac{\partial \phi_j}{\partial t}, r, t \right). \quad (\text{B.1})$$

The Langrangian field equations are obtained from Hamilton's principle:

$$\delta \int_{t_0}^{t_1} L dt = 0 \quad (\text{B.2})$$

where the variation δ acts on the fields. Hamilton's principle leads to the field equations:

$$\frac{\delta \mathcal{L}}{\delta \phi_j} - \frac{\partial}{\partial t} \frac{\delta \mathcal{L}}{\delta \frac{\partial \phi_j}{\partial t}} - \sum_i \frac{\partial}{\partial x_j} \frac{\delta \mathcal{L}}{\delta \frac{\partial \phi_j}{\partial x_i}} = 0. \quad (\text{B.3})$$

Formally we can apply Langrangian field theory to the single particle Schrodinger field ψ . In this case, the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} - H_{sch} \psi = 0 \quad (\text{B.4})$$

where

$$H_{sch} = -\frac{\hbar^2 \nabla^2}{2m} + V(r) \quad (\text{B.5})$$

can be obtained from the following Lagrangian

$$\mathcal{L} = \psi^* (i\hbar \frac{d}{dt} - H_{sch}) \psi \quad (\text{B.6})$$

as may be verified by evaluating equation B.3. Variation with respect to ψ^* and ψ yield the Schrodinger equation and its complex conjugate, respectively.

The canonical momentum, $\pi_i(r, t)$ for a field $\phi_i(r, t)$ is defined as

$$\pi_i = \frac{\delta \mathcal{L}}{\delta(\frac{\partial \phi_i}{\partial t})} \quad (\text{B.7})$$

and the Hamiltonian density is defined as

$$h = \sum_i \pi_i \frac{\partial \phi_i}{\partial t} - \mathcal{L} \quad (\text{B.8})$$

where all variables must be expressed in terms of π and ϕ . For the Schrodinger field ψ the conjugate momentum is

$$\pi = i\hbar \psi^* \quad (\text{B.9})$$

and the Hamiltonian density is

$$h = \frac{1}{i\hbar} \pi H_{sch} \psi. \quad (\text{B.10})$$

B.2 Field Quantisation

The transition from classical to quantum mechanics is achieved by replacing the classical variables r and p by operators $\hat{r}$ and $\hat{p}$ that fulfill the commutation relations

$$[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij} \quad (\text{B.11})$$

$$[\hat{r}_i, \hat{r}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad (\text{B.12})$$

In direct analogy, field quantisation is achieved by replacing the fields $\phi_i \longrightarrow \hat{\phi}_i$, $\pi_i \longrightarrow \hat{\pi}_i$ and the Hamiltonian $H \longrightarrow \hat{H}$. The following commutation relations are imposed on the fields:

$$[\hat{\phi}_i(r, t), \hat{\pi}_j(r', t)]_{\pm} = i\hbar \delta_{ij} \delta(r - r') \quad (\text{B.13})$$

$$[\hat{\phi}_i(r, t), \hat{\phi}_j(r', t)]_{\pm} = [\hat{\pi}_i(r, t), \hat{\pi}_j(r', t)]_{\pm} = 0 \quad (\text{B.14})$$

where $[A, B]_{\pm} = AB \pm BA$. Both commutators occur in nature. The sign of the commutator affects the statistics of the quantised field; the minus sign is applicable to Bose-Einstein statistics whereas the plus sign is appropriate for Fermi-Dirac statistics. Applying this procedure to the Schrodinger field we introduce the field operators $\hat{\psi}$ and $\hat{\pi} = i\hbar\hat{\psi}^{\dagger}$ and apply the Fermion anti-commutation relations (which we write in terms of $\hat{\psi}^{\dagger}$ and $\hat{\psi}$)

$$[\hat{\psi}(r, t), \hat{\psi}^{\dagger}(r', t)] = \delta(r - r') \quad (\text{B.15})$$

$$[\hat{\psi}^{\dagger}, \hat{\psi}^{\dagger}] = [\hat{\psi}, \hat{\psi}] = 0. \quad (\text{B.16})$$

The Hamiltonian density is

$$\hat{h} = \frac{1}{i\hbar} \hat{\pi}(r) H_{sch} \hat{\psi}(r) = \hat{\psi}^{\dagger}(r) H_{sch} \hat{\psi}(r) \quad (\text{B.17})$$

which yields the Hamiltonian operator of a non-interacting electron Fermion field

$$\hat{H} = \int d^3r \hat{\psi}^{\dagger}(r) H_{sch} \hat{\psi}(r). \quad (\text{B.18})$$

This procedure can be extended to the case of interacting Fermion systems; however, the starting point is the N-particle Schrodinger Hamiltonian[36]. As only non-interacting Fermion systems are considered here we do not consider this case further. It is customary to expand the field operator into the eigenfunctions of the single particle Schrodinger Hamiltonian,

$$H_{sch} \phi_n = \epsilon_n \phi_n, \quad (\text{B.19})$$

such that

$$\hat{\psi}(r, t) = \sum_n \hat{a}_n(t) \phi_n(r). \quad (\text{B.20})$$

However, these eigenfunctions could be any convenient complete set such as the Bloch functions. The operator $\hat{a}$ and its Hermitean adjoint counterpart, $\hat{a}^{\dagger}$, are referred to as annihilation and creation operators, respectively. We may derive commutation relations for the operators $\hat{a}$ and $\hat{a}^{\dagger}$:

$$[\hat{a}_n, \hat{a}_m^{\dagger}] = \delta_{nm} \quad (\text{B.21})$$

$$[\hat{a}^{\dagger}, \hat{a}^{\dagger}] = [\hat{a}, \hat{a}] = 0. \quad (\text{B.22})$$

In terms of $\hat{a}$ and $\hat{a}^{\dagger}$ the single particle Hamiltonian is

$$\hat{H} = \sum_n \epsilon_n \hat{a}_n^{\dagger} \hat{a}_n. \quad (\text{B.23})$$

Appendix C

Laser Diode Drive Circuit

This appendix describes the construction of the microstripline circuit used to achieve modulation of the laser diode injection current at frequencies up to 3GHz. The circuit diagram, which includes both the AC and DC inputs, is shown in Fig. C.1. The AC circuit (enclosed by a box in Fig. C.1) consists of a 10nf (chip) blocking capacitor and a 50Ω chip resistor. These elements are situated within a 50Ω microstripline. The circuit board material was RT Duroid with a dielectric constant of $\epsilon_r = 2.35$ and a thickness of $h = 0.8\text{mm}$. The casing of the laser (which is electrically connected to the laser diode anode) was connected to ground.

To determine the width of the microstripline required to achieve 50Ω we followed the approach of Collins[209]. The characteristic impedance is given by

$$Z_o = \sqrt{\frac{\mu_o \epsilon_o}{\epsilon_e}} \frac{1}{C_a} \quad (\text{C.1})$$

where C_a is the capacitance per unit length of an air-filled line and ϵ_e is an effective dielectric constant. To a very good approximation, the capacitance per unit length of a strip of width W and height h above a ground plane with air dielectric is given by

$$C_a(W, h) = \begin{cases} (2\pi\epsilon_o) / \ln \left[\frac{8h}{W} + \frac{W}{4h} \right] & \frac{W}{h} \leq 1 \\ \epsilon_o \left[\frac{W}{h} + 1.393 + 0.667 \ln \left(\frac{W}{h} + 1.444 \right) \right] & \frac{W}{h} > 1 \end{cases} \quad (\text{C.2})$$

The following equation, proposed by Schneider and Hammerstad[209], can be used to determine the effective dielectric constant of a microstripline with an isotropic substrate:

$$\epsilon_e = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \left(1 + \frac{12h}{W} \right)^{-1/2} + F(\epsilon_r, h) \quad (\text{C.3})$$

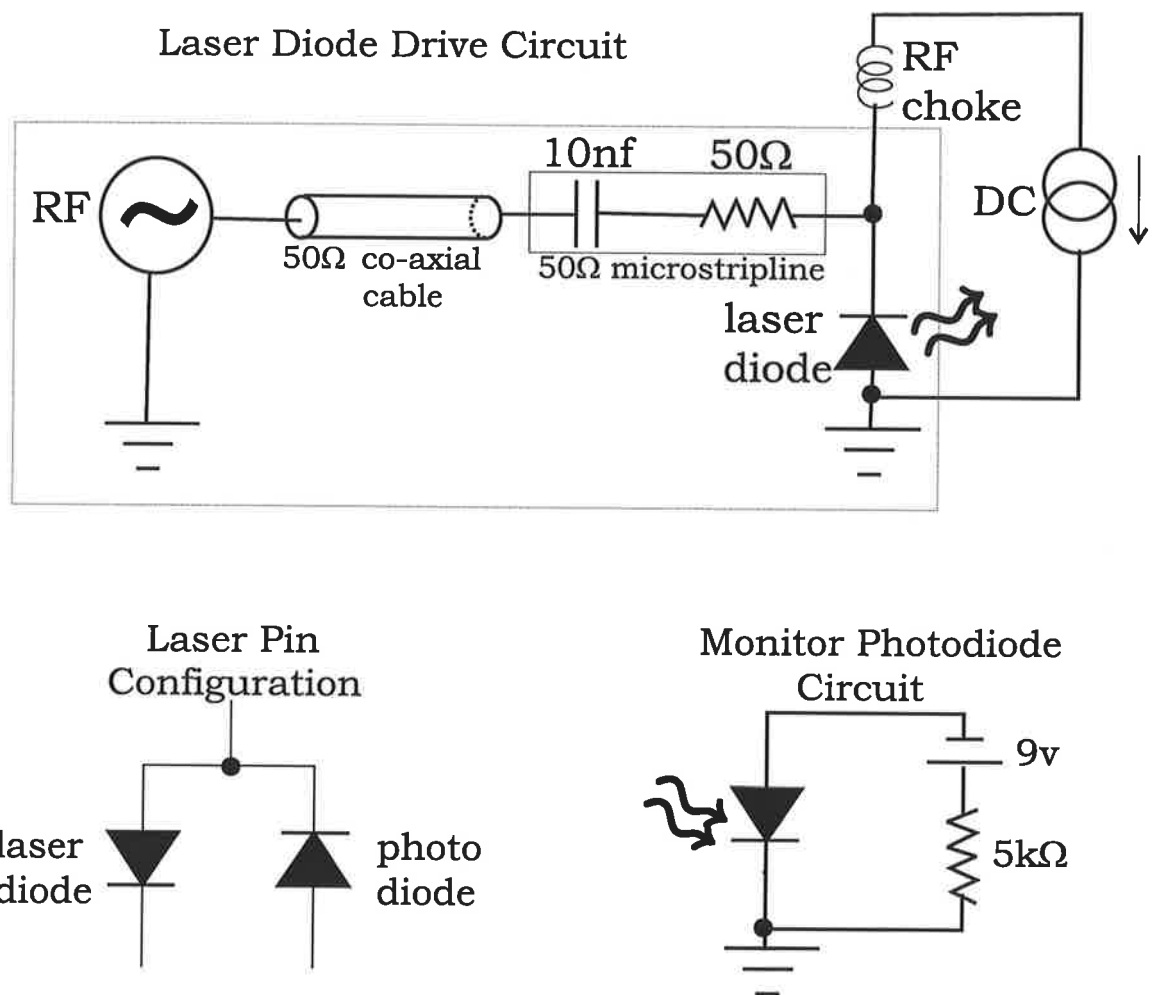


Figure C.1: The laser diode drive circuit (above) showing both AC and DC inputs. The laser diode and monitor photodiode pin connections (below left) and monitor photodiode circuit (below right) are also shown.

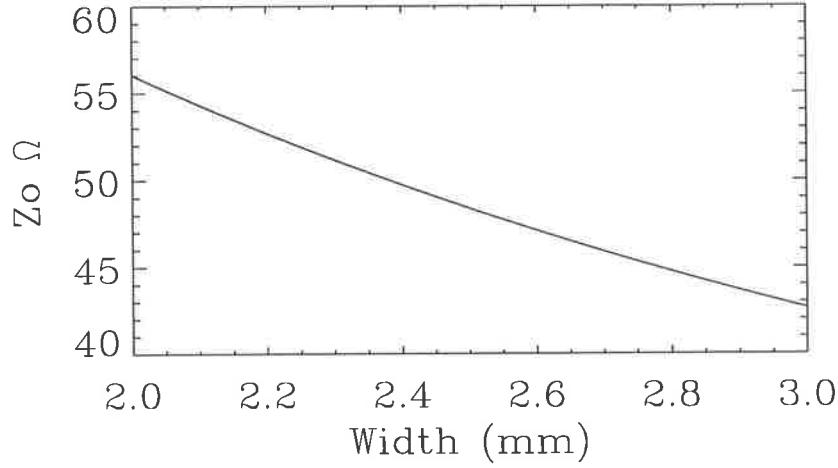


Figure C.2: Characteristic impedance of a microstripline as a function of the strip width for a dielectric medium of height, 0.8mm and dielectric constant, 2.35.

where

$$F(\epsilon_r, h) = \begin{cases} 0.02(\epsilon_r - 1) \left(1 - \frac{W}{h}\right)^2 & \frac{W}{h} < 1 \\ 0 & \frac{W}{h} > 1 \end{cases} \quad (\text{C.4})$$

We have neglected the effects of the finite thickness of the microstripline. The characteristic impedance as a function of strip width, W , for a height, $h = 0.8\text{mm}$ and a dielectric constant, $\epsilon_r = 2.35$ is shown in Fig. C.2. A strip width of 2.4mm yields an impedance of 50Ω as required.

The circuit was enclosed in a brass box of dimensions $27 \times 35 \times 8\text{mm}$. An SMA connector, mounted on the side of the box, was used to connect to the RF signal generator. The Peltier cell, used for temperature stabilisation, was attached to the lid of the box which was also in thermal contact with the casing of the laser diode. The thermistor, used to provide the feedback required for temperature stabilisation, was also attached to the lid of the box within 3mm of the laser diode casing.

Appendix D

Steady State Solutions and Linearised Rate Equations

In this appendix the steady state solutions for the single mode and multimode rate equations are discussed. An approximate linearised equation is derived in each case and compared to the results of numerical simulations using the full rate equations.

D.1 Single Mode Rate Equations

Steady State Solutions

The single mode rate equations are given as follows

$$\frac{dn}{dt} = \frac{I(t)}{e\mathcal{V}} - \frac{n}{\tau_e} - A(n - n_o)s \quad (\text{D.1})$$

$$\frac{ds}{dt} = A(n - n_o)s - \frac{s}{\tau_p} + \frac{\beta n}{\tau_e} \quad (\text{D.2})$$

Steady state solutions can be derived by setting the time derivatives of equations D.1 and D.2 to zero and considering a constant injection current. Neglecting the contribution due to spontaneous emission ($\beta = 0$), the steady state solutions are

$$\left. \begin{aligned} n &= \frac{\tau_e I_{DC}}{e\mathcal{V}} \\ s &= 0 \end{aligned} \right\} I < I_{th} \quad (\text{D.3})$$

Table D.1: Parameter values for the rate equations[16, 20, 23, 3, 1].

Symbol	Value	Model
I_b	1.7	both
τ_e	3×10^{-9} s	both
τ_p	6×10^{-12} s	both
δ	0.692	both
$\Delta\lambda_g$	200 Å	multimode
$\delta\lambda$	4 Å	multimode
k	350 Å	multimode
λ_o	8300 Å	multimode
M	25	multimode

and

$$\left. \begin{aligned} n &= n_{th} = n_o + \frac{1}{A\tau_p} \\ s &= \tilde{s}_o \left(\frac{I}{I_{th}} - 1 \right) \end{aligned} \right\} I > I_{th} \quad (\text{D.4})$$

where $I_{th} = eVn_{th}/\tau_e$ and $\tilde{s}_o = n_{th}\tau_p/\tau_e$. For numerical purposes it is convenient to normalise the variables n and s by defining $N = n/n_{th}$ and $P = s/\tilde{s}_o$ [3]. Equations D.1 and D.2 become

$$\frac{dN}{dt} = \frac{1}{\tau_e} \left[I_b + m \sin(2\pi ft) - N - \frac{N - \delta}{1 - \delta} P \right] \quad (\text{D.5})$$

$$\frac{dP}{dt} = \frac{1}{\tau_p} \left[\frac{N - \delta}{1 - \delta} P - P + \beta N \right] \quad (\text{D.6})$$

where $\delta = n_o/n_{th}$ and we have also defined the quantities $m = I_{AC}/I_{th}$ and $I_b = I_{DC}/I_{th}$.

The steady state solutions of equation D.5 and D.6, in the case of finite β , are as follows

$$P = \frac{1}{2} \left[\beta\delta - 1 + I_b + \sqrt{4(\beta - 1)I_b + (1 - \beta\delta + I_b)^2} \right] \quad (\text{D.7})$$

$$N = \frac{1}{2(\beta - 1)} \left[\beta\delta - 1 - I_b + \sqrt{4(\beta - 1)I_b + (1 - \beta\delta + I_b)^2} \right] \quad (\text{D.8})$$

In this case the solutions are valid for all values of I_b . A comparison of the steady state solutions, equations D.3 and D.4 and equations D.7 and D.8 are shown in Fig. D.1. Parameter values used are given in Table D.1. Figure D.1 shows that the carrier density (and therefore the laser gain) is clamped above threshold with the laser power increasing linearly with injection current in this region.

In Fig. D.2(a) we plot equation D.7 on a Log scale using the parameter values in

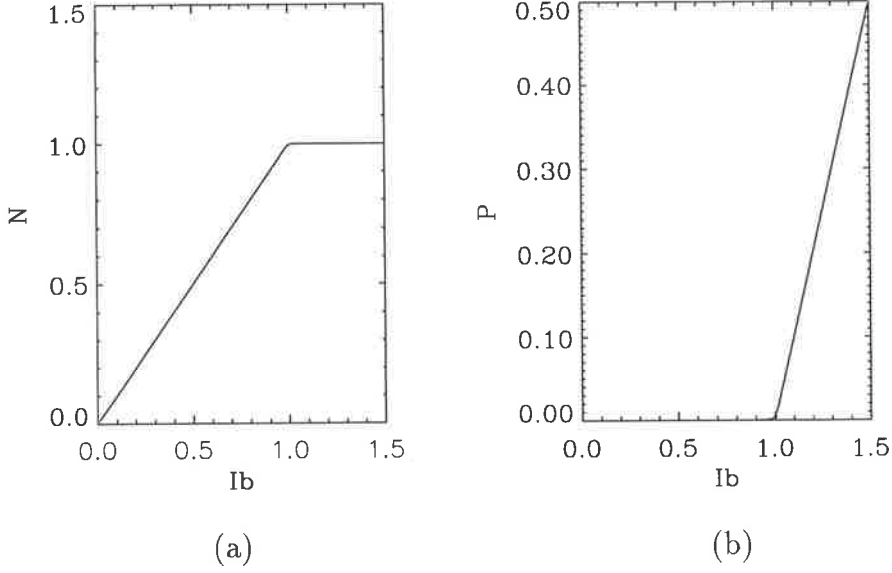


Figure D.1: Steady state solutions of the single mode rate equations as a function of I_b for the normalised carrier density (a) and normalised photon density (b).

Table D.1. It can be shown that the slope of this curve is steepest at the laser threshold¹, $I_b = 1$. We plot, in Fig. D.2(b), the derivative $d\text{Log}(P)/d\text{Log}(I_b)$ versus I_b . The sharpness of the maximum, which occurs at $I_b = 1.0$, allows accurate determination of the threshold current. This is particularly useful for determining the laser threshold from experimental power versus current curves[129].

Linearised Equations

For small deviations from the steady state solutions we can derive an approximate linear equation governing the evolution of N and P [70, 69, 78, 79]. The linear equations have the form of a damped linear oscillator with a characteristic resonant frequency, f_{RO} . Initially we shall ignore the effects of the spontaneous emission (i.e. set $\beta = 0$). Defining the variables

$$\Delta N = N - 1 \quad (\text{D.9})$$

$$\Delta P = P - I_b + 1, \quad (\text{D.10})$$

which correspond to the deviation of N and P from their steady state values ($\beta = 0$) and the variable $i(t) = m \sin(2\pi ft)$, we can obtain separate differential equations for ΔN and

¹The peak does not occur, in fact, exactly at $I_b = 1$. However for small values of β , such as those that typically occur in semiconductor lasers, the approximation is a very good one as can be observed from Fig. D.2(b).

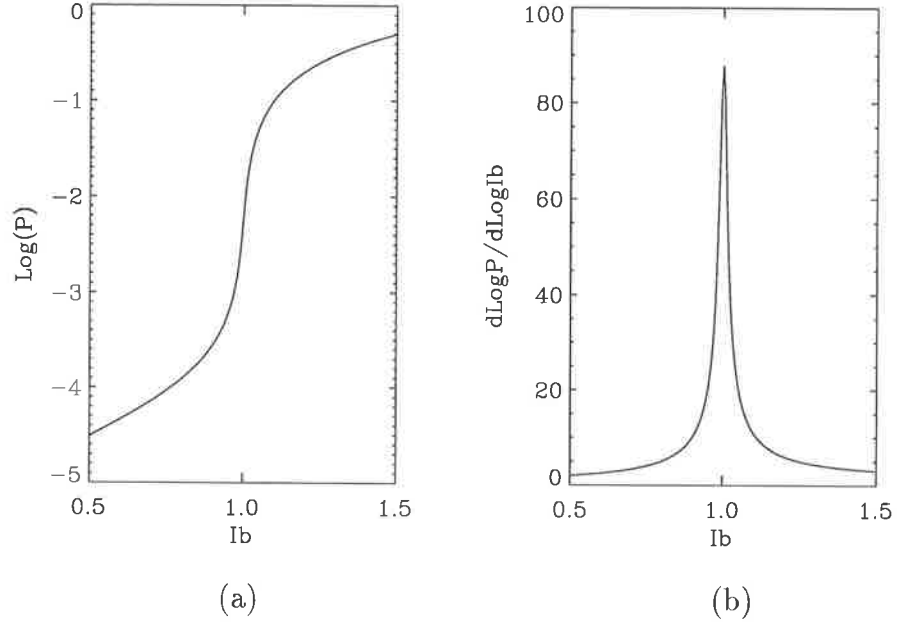


Figure D.2: (a): Total Photon density (log scale) versus I_b . (b): The derivative $d\log(P)/d\log(I_p)$ versus I_b . The position of the peak corresponds to the threshold current, $I_b = 1$.

ΔP . Substituting these variables into the rate equations D.5 and D.6 we obtain

$$\frac{d^2\Delta P}{dt^2} + \frac{1}{\tau_e} \left(\frac{I_b - \delta}{1 - \delta} \right) \frac{d\Delta P}{dt} + \frac{(I_b - 1)\Delta P}{(1 - \delta)\tau_e\tau_p} = \frac{(I_b - 1)i(t)}{(1 - \delta)\tau_e\tau_p} \quad (\text{D.11})$$

$$\frac{d^2\Delta N}{dt^2} + \frac{1}{\tau_e} \left(\frac{I_b - \delta}{1 - \delta} \right) \frac{d\Delta N}{dt} + \frac{(I_b - 1)\Delta N}{(1 - \delta)\tau_e\tau_p} = \frac{1}{\tau_e} \frac{di(t)}{dt}. \quad (\text{D.12})$$

In obtaining these equations we have neglected terms of order $\Delta N \Delta P$; thus these equations constitute a good approximation provided ΔN and ΔP are small with respect to the steady state values of N and P . Equations D.11 and D.12 correspond to the equation of a forced damped linear oscillator of the form

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = F(t). \quad (\text{D.13})$$

The homogeneous form of equation D.13 has solution

$$x(t) = x(0) \exp \left[\left(\frac{\gamma}{2} \pm \frac{1}{2} \sqrt{\gamma^2 - 4\omega_o^2} \right) t \right] \quad (\text{D.14})$$

Since the damping term,

$$\gamma = \frac{1}{\tau_e} \left(\frac{I_b - \delta}{1 - \delta} \right), \quad (\text{D.15})$$

is, for typical values of I_b , much smaller than

$$\omega_o = \sqrt{\frac{1}{\tau_e \tau_p} \left(\frac{I_b - 1}{1 - \delta} \right)} \quad (\text{D.16})$$

then the the natural frequency, or the relaxation oscillation frequency, is to a good approximation given by [31, 3]

$$f_{RO} = \frac{\omega_o}{2\pi} \approx \frac{1}{2\pi} \sqrt{\frac{I_b - 1}{(1 - \delta)\tau_e \tau_p}}. \quad (\text{D.17})$$

Thus, the relaxation oscillation frequency depends on the square root of the deviation of the injection current from the threshold value. The driving term in the equation for ΔP is proportional to $i(t)$ where as for ΔN it is the derivative of $i(t)$ which appears. Therefore, for a sinusoidal driving current, the respective forcing terms for ΔN and ΔP will always be $\pi/2$ out of phase and hence ΔN and ΔP will also be $\pi/2$ out of phase.

Equation D.17 is a good approximation for small β . Though it is possible to derive an analytic expression for f_{RO} in the case of finite β , the results are algebraically complicated and will not be given here. However, the solutions for both $\beta = 0$ (solid curve) and $\beta = 10^{-4}$ (dashed curve) are shown in Fig. D.3. Thus, we observe that equation D.17 is a very good approximation; the effect of finite β is only evident close to the laser threshold ($I_b = 1$).

Nonlinear Oscillator

By defining the variables N and P in terms of new variables x and y such that [9, 24, 10]

$$N(t) = y(t) + 1 - (1 - \delta)e^{-x(t)} \quad (\text{D.18})$$

$$P(t) = \beta(1 - \delta)e^{x(t)} - \beta(1 - \delta) \quad (\text{D.19})$$

we can convert the coupled rate equations, D.5 and D.6, for N and P into a single differential equation in the variable x of the form

$$\ddot{x} + \gamma(x)\dot{x} + f(x)x = F(t) \quad (\text{D.20})$$

where

$$\gamma(x) = \frac{-1 + \beta}{\tau_e} - \frac{\beta e^x}{\tau_e} - \frac{1}{e^x (\tau_p(1 - \delta))} \quad (\text{D.21})$$

$$f(x) = -\frac{\beta e^x}{\tau_e \tau_p} + \frac{-1 + \beta}{e^x (\tau_e \tau_p (\delta - 1))} + \frac{1 - 2\beta + \beta\delta - I_b}{\tau_e \tau_p (\delta - 1)} \quad (\text{D.22})$$

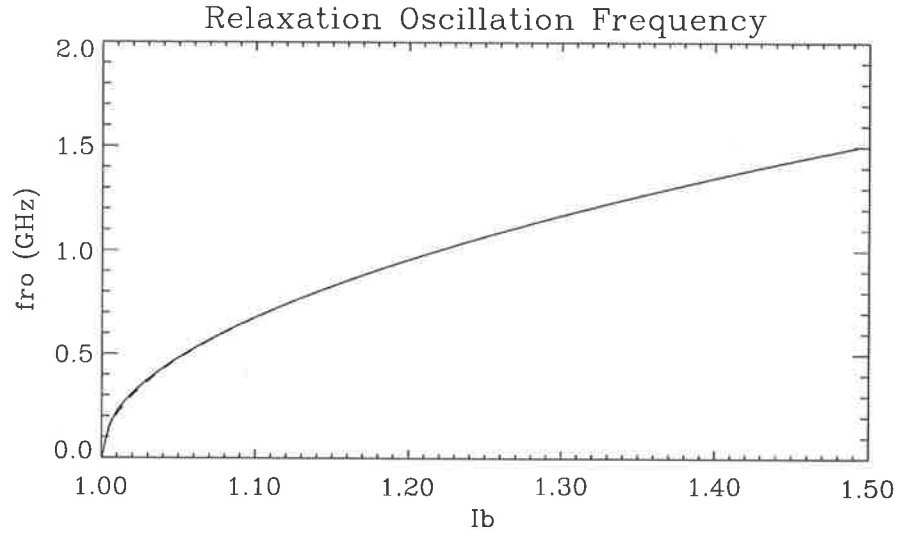


Figure D.3: The relaxation oscillation frequency as a function of I_b for the single mode rate equations. The dashed curve corresponds to equation D.17 and the solid curve corresponds to the exact resonant frequency when the effects of finite β and the damping of the relaxation oscillations are considered.

and

$$F(t) = \frac{m \sin(2 f \pi t)}{\tau_e \tau_p (1 - \delta)}. \quad (\text{D.23})$$

In this case, both the damping, $\gamma(x)$, and the force, $f(x)$ are nonlinear functions of x . The force term is in fact similar to that of the Toda nonlinear oscillator [3, 10, 210, 211]. It is plotted in Fig. D.4. The linear force is also plotted for comparison. The corresponding potential functions, given by

$$V(x) = - \int f(x) dx, \quad (\text{D.24})$$

are plotted in Fig. D.5. The laser ‘force’ is both highly nonlinear and asymmetric. In a linear oscillator the natural frequency is independent of the amplitude of oscillation. However, in a nonlinear oscillator the natural frequency is a function of the amplitude of oscillation [201, 202]. This has important consequences for the dynamics of a modulated semiconductor laser. It is instructive to make an analogy between hard and soft springs. In hard springs we find that the natural frequency is tuned to a higher frequency as the amplitude is increased. The corresponding behaviour in a soft spring is that the natural frequency is tuned to a lower frequency as the amplitude is increased. This type of behaviour usually results in hysteresis [201].

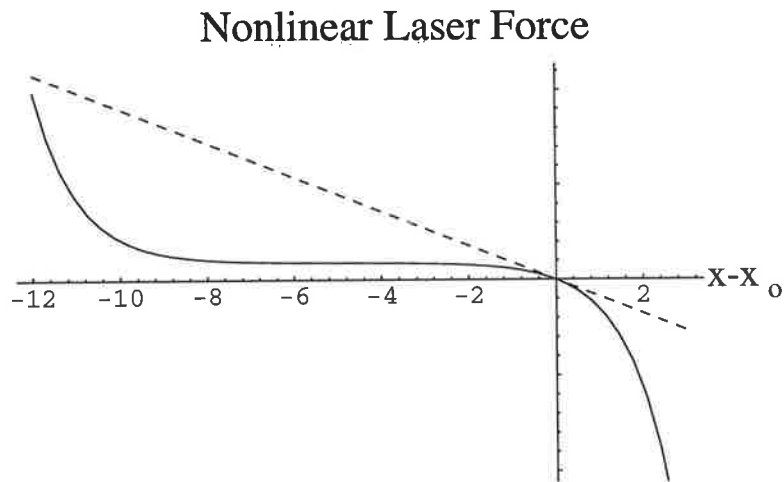


Figure D.4: The linear (dashed line) and nonlinear (solid line) laser 'force'.

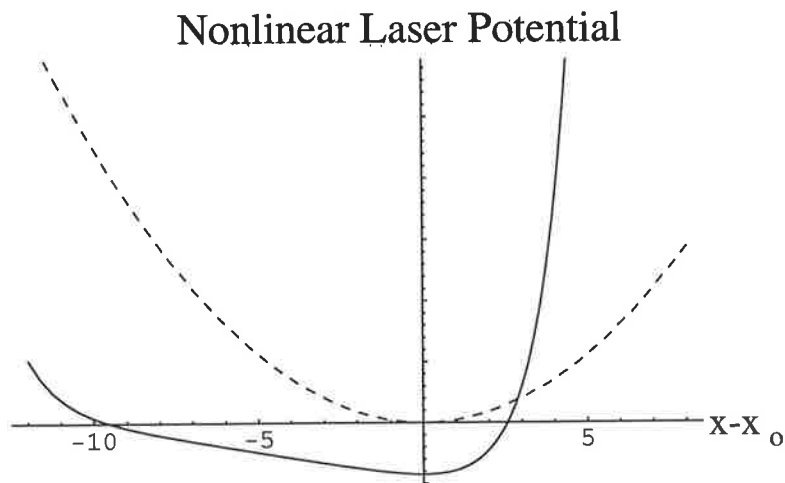


Figure D.5: The linear (dashed line) and nonlinear (solid line) laser potential. The linear potential is quadratic as is consistent with a harmonic oscillator. The nonlinear potential is similar to that for the Toda nonlinear oscillator.

D.2 Multimode Equations

Analytic results for the multimode rate equations are difficult to obtain. Therefore, in this section recourse to numerical solutions will be made. Due to the high dimensionality of the multimode system it is convenient to use the general method of linearising a system of nonlinear differential equations about their steady state solutions². For brevity we write the multimode equations in the form

$$\dot{x} = f(x) \quad (\text{D.25})$$

where $x = (N, P_{-(m-1)/2}, \dots, P_{(m-1)/2})$. The steady state solutions, x_o , are obtained by setting the left hand side of equation D.25 equal to zero:

$$f(x_o) = 0. \quad (\text{D.26})$$

If the contribution due to spontaneous emission is neglected ($\beta = 0$), the following (stable) steady state solutions are obtained

$$\left. \begin{array}{l} N = I_b \\ P_j = 0, \forall j \end{array} \right\} I_b < 1 \quad (\text{D.27})$$

and

$$\left. \begin{array}{l} N = 1 \\ P_o = I_b - 1 \\ P_{j \neq o} = 0 \end{array} \right\} I_b > 1. \quad (\text{D.28})$$

These are in fact identical to the single mode solutions. In the the case of finite β it is not possible to obtain analytic solutions to equations D.26. However, the steady state solutions can be calculated numerically by integrating equations D.25 for a constant current ($m = 0$) and neglecting the transient part of the orbit. The steady state solutions as a function of I_b , for the parameter values given Table D.1, are shown in Fig. D.6. In Fig. D.6(b) the total normalised photon density, $P = \sum_i P_i$ (solid line), and the modal photon densities P_i (dot-dashed lines), are plotted. The approximate solutions, equations D.27 and D.28, are also plotted for comparison (dotted line).

Similarly to the single mode case, to a good approximation, the laser threshold occurs when the slope of the curve $\text{Log}(P)$ versus $\text{Log}(I_b)$ is maximum. In Fig. D.7(a) we show the data of Fig. D.6(b) plotted on a Log scale and the corresponding derivative $d\text{Log}(P)/d\text{Log}(I_b)$ is shown in Fig. D.7(b). The approximation is not as good as the single mode case due to the fact that the spontaneous emission factor, β , has greater influence as a result of the larger number of longitudinal modes into which the spontaneous emission can couple[34, 15].

²This methodology is discussed in detail in section 5.2.1

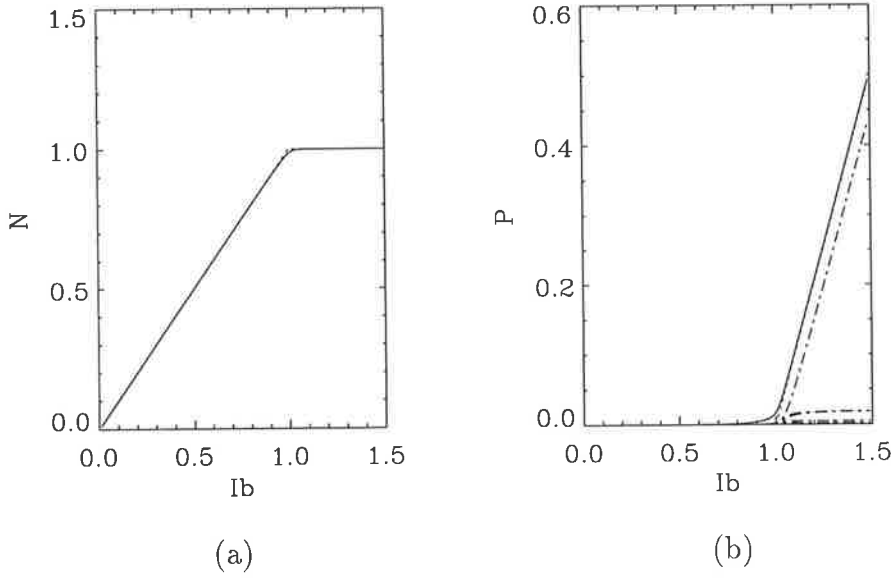


Figure D.6: Steady state solutions of the multimode rate equations as a function of I_b for the normalised carrier density, (a) and normalised total photon density, (b). The dashed curve corresponds to equations D.27 and D.28. The solid curve corresponds to $P = \sum_i P_i$ and the dot-dashed curves correspond P_i obtained from numerical integration of the multimode rate equations. A total of 25 modes were used in the calculation.

However, the deviation of the position of the peak of the curve shown in Fig. D.7(b) is less than two percent. Thus, for experimental purposes, determination of the threshold current by this means is still a viable technique.

Linearised Equations

The nonlinear equations D.25 can be linearised about the steady state solution³. Defining $\Delta x = x - x_o$ we obtain the following differential equation for Δx

$$\frac{d\Delta x}{dt} = A\Delta x \quad (\text{D.29})$$

where

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=x_o} \quad (\text{D.30})$$

is the Jacobian matrix of the system evaluated at the steady state solution. The evolution of the variables Δx , for small deviations from the steady state, is governed by the eigenvalues of A . For the multimode system of equations we find only one complex pair of eigenvalues corresponding to the relaxation oscillation resonance. The relaxation oscillation frequency

³This method is discussed in detail in section 5.2.1. See also [183].

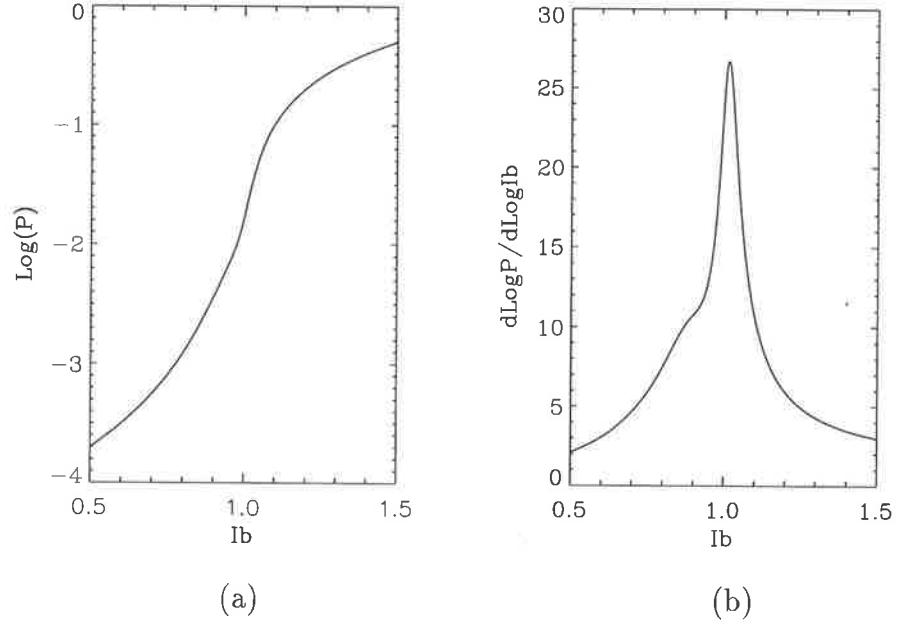


Figure D.7: (a): Total Photon density (log scale) versus I_b . (b): The derivative $d\log(P)/d\log(I_p)$ versus I_b . The position of the peak corresponds to the threshold current, $I_b = 1$.

is given by

$$f_{RO} = \frac{1}{2\pi} \Im(\lambda) \quad (\text{D.31})$$

where λ is a complex eigenvalue of A . In Fig. D.8 we plot the relaxation oscillation frequency, calculated according to equations D.30 and D.31, using the data shown in Fig. D.6(b). For comparison, we also plot the relaxation oscillation frequency calculated from equation D.17 (dashed curve). For finite β , equation D.17 is a reasonably good approximation if the laser is biased sufficiently far above threshold (see Fig. D.8). However, the agreement is not as close as in the single mode system again due to the fact that the spontaneous emission has a greater significance as a result of the large number of longitudinal modes into which the spontaneous emission can couple.

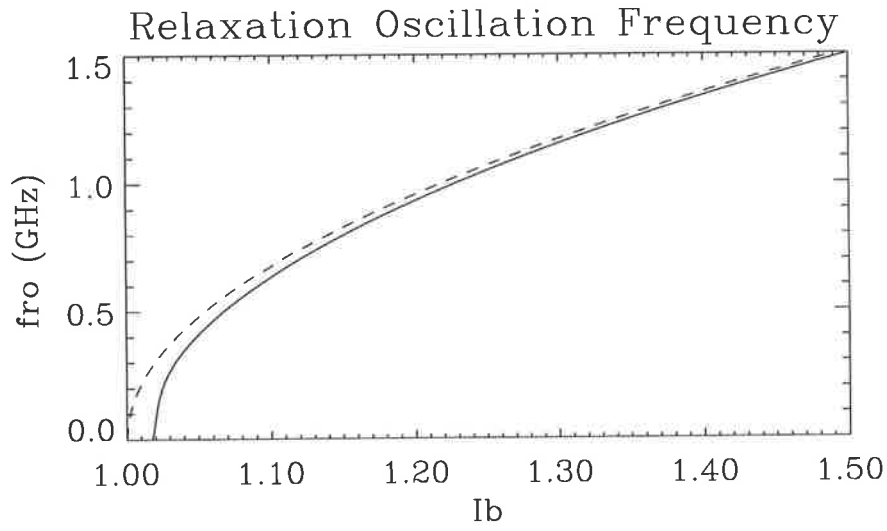


Figure D.8: The relaxation oscillation frequency as a function of I_b for the multimode rate equations. The dashed curve corresponds to equation D.17 and the solid curve corresponds to the exact resonant frequency, calculated numerically from the linearised equations D.29, showing the effects of finite β and the damping of the relaxation oscillations.

210 APPENDIX D. STEADY STATE SOLUTIONS AND LINEARISED RATE EQUATIONS

Appendix E

Numerical Integration

Numerical integration is performed using a fourth order Runge-Kutta algorithm[183]. Runge-Kutta formulae for an autonomous system of the form

$$\begin{aligned}\dot{x}_1 &= Q_1(x_1, x_2, \dots, x_N, \lambda) \\ \dot{x}_2 &= Q_2(x_1, x_2, \dots, x_N, \lambda) \\ &\vdots \\ \dot{x}_N &= Q_N(x_1, x_2, \dots, x_N, \lambda)\end{aligned}\tag{E.1}$$

are given, in component form, by

$$\begin{aligned}K_1(i) &= hQ_i(\vec{x}_k) \\ K_1(i) &= hQ_i(\vec{x}_k + \vec{K}_1/3) \\ K_1(i) &= hQ_i(\vec{x}_k - \vec{K}_1/3 + \vec{K}_2) \\ K_1(i) &= hQ_i(\vec{x}_k + \vec{K}_1 - \vec{K}_2 + \vec{K}_3) \\ x_{k+1}(i) &= x_k(i) + (K_1(i) + 3K_2(i) + 3K_3(i) + K_4(i))/8\end{aligned}\tag{E.2}$$

where i is an integer corresponding to the i^{th} vector component; i runs from 1 to N where N is the dimension of the system. The integration step size is h .

Appendix F

Variational Equation

In order to characterise the stability of periodic solutions of the rate equations the matrix $DP(x^*)$ is required. Fortunately this matrix may be obtained numerically. In this Appendix we introduce the variational equation and discuss how $DP(x^*)$ is obtained¹. The discussion follows closely that of [183].

We are primarily interested in non-autonomous systems of the form

$$\dot{x} = Q(x) + g(t) \quad (\text{F.1})$$

where $g(t) = g(t + T)$ is an additive forcing term of period T . Given that Equations F.1 have a periodic solution, $x_p(x_o, t)$, such that $x_p(x_o, t) = x_p(x_o, t + T)$ we can write

$$\dot{x}_p(x_o, t) = Q(x_p(x_o, t)) + g(t) \quad (\text{F.2})$$

which has initial condition, $x_p(x_o, t_o) = x_o$. We have included the dependence of x_p on x_o explicitly in order to indicate the functional dependence of the trajectory on the initial condition. Differentiating equation F.2 with respect to x_o and using the chain rule we obtain²

$$D_{x_o} \dot{x}_p(x_o, t) = D_x Q(x_p(x_o, t), t) D_{x_o} x_p(x_o, t) \quad (\text{F.3})$$

where $D_{x_o} x_p(x_o, t_o)$ is equal to I , the identity matrix. If we define $\Phi(x_o, t) := D_{x_o} x_p(x_o, t)$ then equation F.3 is a differential equation for Φ :

$$\dot{\Phi}(x_o, t) = D_x Q(x_p(x_o, t)) \Phi(x_o, t) \quad (\text{F.4})$$

with initial condition $\Phi(x_o, t_o) = I$. Equation F.4 is the variational equation. It is a time

¹We drop the vector notation in this Appendix.

²The notation D_x indicates the Jacobian matrix of partial derivatives where differentiation is with respect to the vector x .

varying, matrix-valued, linear differential equation which depends on the trajectory, $x_p(x_o, t)$; thus, equations F.1 and F.4 must be solved simultaneously. Though $g(t)$ does not appear explicitly in equation F.4 it is implicit in $x_p(x_o, t)$.

In order to calculate numerical solutions to the variational equation it is generally appended to the original system of equations to obtain the following combined system of differential equations:

$$\begin{Bmatrix} \dot{x} \\ \dot{\Phi} \end{Bmatrix} = \begin{Bmatrix} Q(x, t) \\ D_x Q(x, t) \Phi \end{Bmatrix} \quad (\text{F.5})$$

with initial condition

$$\begin{Bmatrix} x(0) \\ \Phi(0, t_o) \end{Bmatrix} = \begin{Bmatrix} x_o \\ I \end{Bmatrix}. \quad (\text{F.6})$$

The solution of the variational equation, $\Phi(x_o, t) = D_{x_o} x_p(x_o, t)$, gives the Jacobian matrix of the trajectory, $x_p(x_o, t)$, at time t differentiated with respect to the initial condition $x_o = x_p(0)$. We can use the variational equation to obtain the matrix $DP(x^*)$ which we require to determine the stability of a periodic solution. To see this we note that if $x_p(x_o, t)$ is a periodic solution of period T then for points on this particular trajectory the Poincare map is given by

$$P(x) := x_p(x, T). \quad (\text{F.7})$$

Therefore

$$D_x P(x^*) = D_x x_p(x^*, T) \equiv \Phi(x^*, T). \quad (\text{F.8})$$

where $\Phi(x^*, t)$ is a solution of the variational equation F.4. Therefore, in order to obtain the matrix $DP(x^*)$, we need to integrate the system of equations F.5 for time T . The matrix $DP(x^*)$ depends on the particular point $x^* = x_p(t)$ of the periodic cycle at which it is evaluated. However, its eigenvalues are independent of the choice of Σ and therefore can be considered a unique property of the periodic solution[183].

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