# Semigroup Methods for Degenerate Cauchy Problems and Stochastic Evolution Equations

Isna Maizurna

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SUB CRUCE LUMEN

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# Abstract

The thesis is devoted to semigroup methods for degenerate Cauchy problems and stochastic evolution equations. It consists of four parts. In Chapter 2, degenerate abstract Cauchy problems $B\frac{d}{dt}u(t) = Au(t)$, $u(0) = x$, and $\frac{d}{dt}(Bu)(t) = Au(t)$, $(Bu)(0) = x$ are investigated in a Banach space $X$, for a closed linear operator $A$ and a bounded linear operator $B$ with non–trivial kernel. The $(n,\omega)$–well–posedness of the problems is studied. Firstly, necessary and sufficient conditions for the $(n,\omega)$–wellposedness of the Cauchy problem for the inclusion $\frac{d}{dt}u(t) \in \mathcal{A}u(t)$, $u(0) = x$, where $\mathcal{A}$ is a multi–valued linear operator on $X$, are given. Then the obtained results are applied to the original degenerate problems.

In Chapter 3, the well–posedness of the degenerate Cauchy problem $B\frac{d}{dt}u(t) = Au(t)$, $u(0) = x$ in the space of distributions of exponential growth is investigated. Necessary and sufficient conditions in terms of distribution semigroups are given. The connection between $n$–times integrated semigroups and distribution semigroups is established.

In Chapter 4, the inhomogeneous abstract Cauchy problem $\frac{d}{dt}u(t) = Au(t) + \psi(t)$, $u(0) = x$ is studied. Several types of solutions of the inhomogeneous problem, namely $n$–integrated solutions, $n$–weak solutions and $K$–generalized solutions are investigated. Conditions for the existence of those solutions are given.

In Chapter 5, stochastic differential equations $dX(t) = AX(t)dt + BdW(t)$, $X(0) = \xi$ in a separable Hilbert space are considered, where $\xi$ is an $H$–valued random variable and $W(t)$ is an $H$–valued Wiener process. Conditions for the existence of a $2n$–integrated solution respectively for a weak $2n$–integrated solution are given. It is shown that both solutions possess continuous versions. The quasi–reversibility method is used for studying an ill–posed problem $dX(t) = AX(t)dt + BdW(t)$, $X(0) = \xi$.

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# Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to the thesis being made available for loan and photocopying.

Adelaide, July 5, 1999

( Isna Maizurna )

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# Chapter 1

# Introduction

Many real life processes can be mathematically described by a differential equation

$$\frac{d}{dt}u(t) = f(u(t)), \ u(0) = x.$$

Here it is assumed that the state of the system at initial time $t = 0$ is known to be $u(0) = x$ and that the complete future of the path $t \mapsto u(t)$ within the state space $X$ is uniquely defined by the initial value and the fulfilment of the differential equation for all times $t \geq 0$. In general, problems of this kind are called Cauchy problems.

It is well known that, under Lipschitz conditions on the vector field $u \mapsto f(u)$, the unique solution of the differential equation does exist for all times $t \geq 0$ and even depends continuously on the initial value $x$ in the state space $X$. Problems having these three properties, existence, uniqueness and continuous dependence of the solution with respect to the initial data, usually are called well–posed.

Unfortunately, for infinite dimensional state space $X$, Lipschitz continuity is hard to achieve, even for linear abstract Cauchy problems

$$\frac{d}{dt}u(t) = Au(t), \quad u(0) = x. \tag{1.1}$$

Typical examples are the heat equation, respectively, the Schrödinger equation, where the state spaces are function spaces and the operator $A$ is given by the Laplacians $A = \Delta$, respectively, $A = -i\Delta$.

To overcome this difficulty, the theory of semigroups has been initiated by few authors, see for example [63]. This theory allows one to deduce from properties of the operator $A$ and its resolvent, the existence of a semigroup $\{S(t) : t \in [0, \infty)\}$ of bounded linear operators. The properties of the semigroup and its generator guarantee the existence of the unique solution

$$t \mapsto u(t) = S(t)x$$

of the linear Cauchy problem (1.1), for any $x \in D(A)$. Moreover, the solution depends continuously on the initial data.

Later, the notion of a semigroup was generalized in many different directions. Lions, [35], introduced distribution semigroups in connection with generalized well-posedness of the abstract Cauchy problem (1.1). Arendt, see [1, 2], introduced the concept of an $n$–times integrated semigroup $\{V(t) : t \in [0, \infty)\}$ of linear bounded operators and its generator. If $A$ is the generator of an $n$–times integrated semigroup, then

$$t \mapsto \frac{d^n}{dt^n} V(t)x$$

is the unique solution of the Cauchy problem (1.1), for any $x \in D(A^{n+1})$.

In the present thesis we investigate several classes of Cauchy problems which are discussed in four chapters.

The first class consists of degenerate problems, which are important in fluid mechanics and diffusion processes, see [10]. Here, one often has the situation that, due to different particle sizes in some areas of the domain of interest, the differential equation is not active in certain areas. Then the mathematical model has one of the forms

$$B \frac{d}{dt} u(t) = Au(t), \quad u(0) = x \tag{1.2}$$

or

$$\frac{d}{dt}(Bu)(t) = Au(t), \quad (Bu)(0) = x, \tag{1.3}$$

where $B$ is an operator with non–trivial kernel. Accordingly, if the operator $B$ has a trivial kernel, the problem is called non–degenerate, and can be treated in the same way as the standard problem (1.1).

The second class consists of inhomogeneous problems and is of special interest in control theory, when the system is subject to an external influence, see [8]. The mathematical model can often be written as

$$\frac{d}{dt}u(t) = Au(t) + \psi(t), \quad u(0) = x, \tag{1.4}$$

where $t \mapsto \psi(t)$ is called the inhomogeneity. Accordingly, if $\psi(t) \equiv 0$ we have the standard Cauchy problem (1.1), called the homogeneous problem.

The third class is the one of stochastic differential equations of the form

$$dX(t) = AX(t)dt + BdW(t), \; X(0) = \xi, \tag{1.5}$$

in a separable Hilbert space, where $\xi$ is an $H$–valued random variable and $W(t)$ is an $H$–valued Wiener process.

In **Chapter 2** of the thesis, we consider the two degenerate abstract Cauchy problems (1.2) and (1.3) for closed linear operators $A$ and bounded linear operators $B$. We investigate the $(n, \omega)$–well–posedness of the problems. First we investigate the $(n, \omega)$–well–posedness of the Cauchy problem for the inclusion

$$\frac{d}{dt}u(t) \in \mathcal{A}u(t), \quad u(0) = x, \tag{1.6}$$

where $\mathcal{A}$ is a multi–valued linear operator on a Banach space $X$. Then results obtained are applied to the degenerate problems (1.2) and (1.3), with $\mathcal{A} := B^{-1}A$ or $\mathcal{A} := AB^{-1}$ respectively.

Degenerate problems of the form (1.2) or (1.3) in Banach spaces were considered by many authors, see [10, 21, 22, 23, 41, 42, 44, 47, 62] with the references therein. For degenerate problems (1.2) and (1.3), the operators $B^{-1}A$ or $AB^{-1}$ are multi–valued. This motivated the investigation of multi–valued linear operators in connection with abstract Cauchy problems.

Cauchy problems (1.6) have found an increasing interest, see [22, 23, 29, 42, 48, 62]. Favini and Yagi, [22, 23], studied the existence of strict solutions, which are continuously differentiable and satisfy the inclusion for all times $t \geq 0$. Under the condition that the resolvent set is contained in a certain region within the complex plane, and that for those values the resolvent satisfies some growth conditions they derive the unique existence of a strict solution. Their results even can be extended to inhomogeneous inclusions, where the solution can be represented with the variation of constants formula.

Knuckles and Neubrander, [29], investigated $n$–integrated solutions of the inclusion problem. They use Laplace transformation methods to derive results on the unique existence of $n$–integrated solutions. Their approach is strongly related to our work, since, for non degenerate Cauchy problems, there is a well known one–to–one correspondence between $n$–integrated solutions and solutions which can be represented via $n$–times integrated semigroups. This relationship has not been elaborated so far for inclusions.

Melnikova, Alshansky and Gladchenko, [42, 48], investigated the inclusion problem from the classical, strongly continuous semigroup point of view. They give conditions of Hille–Yosida type for the well–posedness of the problem and the existence of a degenerate semigroup. Here, the degeneracy of the semigroup $\{S(t) : t \in [0, \infty)\}$ refers to the property that a non–trivial subspace of the state space $X$ is contained in all kernels ker $S(t)$. Typically, this non–trivial subspace is the complement of $\mathcal{A}(0)$, the multi–valued image of the origin.

Yagi, [62], used a similar Hille–Yosida type condition to guarantee the existence of a strongly continuous semigroup on the domain of $\mathcal{A}$. This semigroup then allows even the representation of strict solutions for the inhomogeneous problem.

The main result of this chapter is the statement of a necessary and sufficient condition for the $(n, \omega)$–well–posedness of the inclusion problem (1.6) under a decomposition assumption on $X$. Using the theory of $n$–times integrated semigroups, we prove that the Hille-Yosida type condition is necessary and sufficient for the $(n, \omega)$–well–posedness of the inclusion problem (1.6) on the domain $D(\mathcal{A}^{n+1})$ of $\mathcal{A}^{n+1}$. Similar results are obtained for the degenerate equations (1.2) and (1.3).

In **Chapter 3** we consider the degenerate Cauchy problem (1.2) in a Banach space $X$, where $A$ is a closed linear operator and $B$ is a bounded linear operator in $X$.

We investigate the well–posedness of the problem in the space of distributions of exponential growth.

Distribution semigroups, introduced by Lions, [35], play an important role in connection with abstract Cauchy problems, where they allow one to obtain generalized solutions. They have been investigated by several authors, see [3, 11, 19, 33, 44, 47, 54, 58, 60]. In [35], for the non–degenerate Cauchy problem (1.1), the equivalence is shown between the well–posedness of problem (1.1), in the sense of distributions, and the existence of distribution semigroup generated by the operator $A$.

In [3] and in [58], for the non degenerate Cauchy problem, strong connections between distribution semigroups and *local* integrated semigroups have been established by Arendt, El–Mennaoui and Keyantuo, and by Tanaka and Okazawa.

Earlier, distribution semigroups for degenerate problems were considered by Melnikova and Filinkov, see [47]. Here, under the assumption that the Banach space $X$ can be decomposed into range and kernel of a power of the resolvent of the multi–valued operator, the authors showed that well–posedness of the problem in the sense of distributions is equivalent to the existence of a degenerate distribution semigroup, generated by a single–valued branch of the multi–valued operator $\mathcal{A} = B^{-1}A$.

In [11], Chazarain characterized the well–posedness of the degenerate Cauchy problem in the sense of distributions by the existence of a region in a complex plane such that for any element from that region, the resolvent exists and is polynomially bounded.

Recently obtained results in [33] are closely related, where the relationship of *local* integrated semigroups and distribution semigroups is investigated for non degenerate Cauchy problems with non–densely defined operators.

For the degenerate Cauchy problem (1.2), we study the strong connections between the well–posedness of the degenerate problem in the sense of distributions of exponential growth, the existence of a degenerate distribution semigroup of exponential growth and the existence of a degenerate global $n$–times integrated semigroup.

Firstly, we prove the equivalence between the well–posedness of the degenerate Cauchy problem in the sense of distributions of exponential growth and the existence of a degenerate distribution semigroup of exponential growth, generated by the part of a single–valued branch of the multi–valued operator $B^{-1}A$. Secondly, we show that the existence of such distribution semigroup is equivalent to the existence of a degenerate global $n$–times integrated semigroup.

In **Chapter 4** we investigate several types of solutions of the inhomogeneous problem (1.4) in a Banach space $X$. This problem was considered by many authors, e.g. [1, 6, 8, 13, 15, 19, 24, 27, 28, 31, 37, 59]. It turns out that, to obtain a unique continuously differentiable solution $t \mapsto u(t)$, which has values in the domain $D(A)$, one needs to require strong conditions on the operator $A$, on the initial value $x \in X$ and on the smoothness of $t \mapsto \psi(t)$.

For example, in addition to the Hille–Yosida condition on the operator $A$, Arendt, [1], needs to impose the condition that the function $t \mapsto \psi(t)$ is twice continuously differentiable and $Ax + \psi(0)$ is in the closure of the domain $A$, to obtain a unique solution for an initial value $x \in D(A)$. To prove a similar result, Kellerman and Hieber, [28], assume some kind of smoothness of $t \mapsto \psi(t)$. Da Prato and Sinestrari, [15], require that $t \mapsto \psi(t)$ has to be in some Sobolev space, in order to obtain a unique solution for an initial value $x \in D(A)$. In [19, 24, 31] a similar result is proved under the assumption of either continuity of $t \mapsto \psi(t)$ with $\psi(t) \in D(A)$ for all $t \geq 0$ and continuity of $t \mapsto A\psi(t)$ or of continuous differentiability of $t \mapsto \psi(t)$.

If we want to relax these assumptions, we obtain solutions in a generalized sense. Ball, [6], considered a continuous solution $t \mapsto u(t)$ involving the dual space $X^*$ of $X$, where the mapping $t \mapsto \langle u(t), v\rangle$ is absolutely continuous for all $v \in X^*$ and satisfies, for almost all $t \in [0, \infty)$, the equation

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle \psi(t), v\rangle$$

for all $v \in D(A^*)$, where $A^*$ is the adjoint operator of $A$. Such a solution is called a weak solution and can be obtained for any initial value $x \in X$ and any integrable inhomogeneity $t \mapsto \psi(t)$.

Another type of generalized solution is the continuous solution of the integrated problem, see [37, 59]. It is proved in [37] that the solution of the integrated problem

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t \psi(s)ds$$

is equivalent to the weak solution defined in [6]. Thieme, [59], showed that, in order to obtain solutions of the twice integrated problem

$$v(t) = tx + A\int_0^t v(s)ds + \int_0^t (t-s)\psi(s)ds$$

for the integrable function $t \mapsto \psi(t)$, the operator $A$ necessarily is the generator of a once integrated semigroup.

More general solutions of the inhomogeneous problem were investigated by Cioranescu and Lumer, [13]. They described continuous solutions of the $K$–generalized problem

$$\frac{d}{dt}v(t) = Av(t) + K(t)x + (K * \psi)(t)$$

for commuting operator–valued kernels $K$. For this purpose they required that $A$ is the generator of a $K$–convoluted semigroup, (see [12]).

Following the idea of Thieme, [59], we investigate the uniqueness of the $n$–times integrated solution of the inhomogeneous problem for the case that the operator $A$ generates an $n$–times integrated semigroup. We prove that, for this case, there exists a unique $n$–integrated solution of the inhomogeneous problem (1.4). For such operators $A$ we obtain solutions of the original problem for initial values from a smaller subset of $X$ under a smoothness condition on $t \mapsto \psi(t)$.

Next we study weak solutions of $n$–integrated problems. Under the same assumption on $A$, we obtain the unique solvability of the weak $n$–integrated problem. We also show that any $n$–integrated solution is an $(n-1)$–weak integrated solution.

Finally, for the case when $A$ generates a $K$–convoluted semigroup, we discuss the $K$–generalized solution of the inhomogeneous problem. We give conditions which guarantee the existence of the $K$–generalized solution. We obtain results on the solution of the original problem on a subset of $X$.

In **Chapter 5**, we consider the stochastic differential equation (1.5) in a separable Hilbert space $H$. The problem was considered in [14, 16]. In [16], Da Prato and Zabczyk investigated strong and weak solutions of the problem. Assuming that $A$ is the generator of a strongly continuous semigroup, $B$ is a bounded operator and $S(t)BB^*S^*(t)$ is of trace class for all $t \geq 0$, where $S(t)$ is the strongly continuous semigroup generated by $A$, they proved the existence of a continuous version of a weak solution.

Using the results of the previous chapter, we investigate the $n$–integrated solution and the weak $n$–integrated solution of the stochastic differential equation (1.5). Under the condition that $A$ generates an $n$–times integrated semigroup, assuming that $B$ is bounded, $V(t)BB^*V^*(t)$ is of trace class and the stochastic convolution belongs to $D(A)$ we prove that the problem (1.5) has a $2n$–integrated solution with a continuous version. Dropping the assumption that the stochastic convolution is in $D(A)$, we show that the problem has a weak $2n$–integrated solution with a continuous version.

Finally, we use the quasi–reversibility method introduced by Lattes and Lions [34], for studying an ill–posed problem (1.5) in a separable Hilbert space $H$, with unobservable initial condition $\xi$ and the operator $A$ belonging to the class $\mathcal{H}_c^+$ of self adjoint operators on Hilbert space $H$, generating basis $\{e_k\}$ for $H$, corresponding to eigenvalues $\{\lambda_k\}$ such that $-\infty < \lambda_1 \leq \lambda_2 \leq \cdots$, and $\lim_{n\to\infty} \lambda_n \to +\infty$.

It is known that if $A \in \mathcal{H}_c^+$, then the Cauchy problem (1.1) is, in general, not well–posed, and the semigroup $\{U(t) : t \in [0,\infty)\}$ generated by $A$ consists of unbounded linear operators. It was proved in [26] that, if $A \in \mathcal{H}_c^+$ then for any $\epsilon > 0$, the Cauchy problem

$$u'(t) = (A - \epsilon A^2)u(t), \; u(0) = u_0$$

is well–posed $D(A^2)$ and $(A - \epsilon A^2)$ generates a strongly continuous semigroup $\{U_\epsilon(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$.

Given an $H$–valued observable random variable $\eta$, we consider the conditional expectation $E[\xi|\eta]$, the optimal (in the mean–square sense) estimator of $\xi$ in terms of

$\eta$, such that $E(|\xi - E(\xi|\eta)|^2) < \delta$.

Assuming that there exists a solution of (1.5) for the initial value $\xi$, we approximate the solution at time $t = T$ by solutions of the problem

$$dX(t) = (A - \epsilon A^2)X(t)dt + BdW(t), \; X(0) = \xi_\delta, \tag{1.7}$$

on certain correctness classes. Thus we regularize the original problem by perturbing both the operator and the initial condition.

# Chapter 2

# Cauchy Problems for Inclusions

In this chapter we consider the following degenerate abstract Cauchy problems

$$Bu'(t) = Au(t), \ t \geq 0, \ u(0) = x, \tag{2.1}$$

and

$$\frac{d}{dt}(Bv(t)) = Av(t), \ t \geq 0, \ Bv(0) = x, \tag{2.2}$$

in a complex Banach space $X$, where $A : D(A) \subset X \to X$ is a closed linear operator and linear operator $B \in \mathcal{L}(X)$ has a non–trivial kernel. Both problems can be written in the form of a Cauchy problem for inclusion

$$u'(t) \in \mathcal{A}u(t), \quad t \geq 0, \ u(0) = x, \tag{2.3}$$

by setting $\mathcal{A} = B^{-1}A$ for the problem (2.1), and $\mathcal{A} = AB^{-1}$, $u(t) = Bv(t)$ for the problem (2.2). Classical solutions of these problems are continuously differentiable functions $t \mapsto u(t)$ with values in the domain $D(A)$ of $A$, such that the differential equations are fulfilled for all times $t \geq 0$.

It can be easily shown that each of the two degenerate abstract Cauchy problems is equivalent to the corresponding inclusion problem in the sense that solutions of one problem are – or are easily transformed to – solutions of the other problem.

The main aim of this chapter is to discuss the $(n, \omega)$–well–posedness of the degenerate problems with the help of the $n$–times integrated semigroup generated by a multi–valued operator. First, we study problem (2.3). Later, the results obtained for this problem are transformed to corresponding statements for the original degenerate problems (2.1) and (2.2).

Hence we divide the chapter into two sections. In the first section we recall some properties of multi–valued linear operators. In the second section, we give necessary and sufficient conditions for the $(n,\omega)$–well–posedness of the degenerate Cauchy problems. For the non–degenerate case, that is for a single–valued $\mathcal{A}$, the $(n,\omega)$–well–posedness on $D(\mathcal{A}^{n+1})$ of the problem (2.3) is characterized by the Miyadera-Feller-Phillips-Hille-Yosida (MFPHY) type condition

$$\|\frac{d^k}{d\lambda^k}\frac{(\lambda-\mathcal{A})^{-1}}{\lambda^n}\| \leq \frac{Mk!}{|\text{Re}\lambda-\omega|^{k+1}},$$

given that $\overline{D(A)} = X$. In this thesis we show that for multi–valued operators $\mathcal{A}$, the well–posedness on $D(\mathcal{A}^{n+1})$ implies the weaker estimate

$$\|\frac{d^k}{d\lambda^k}\frac{(\lambda-\mathcal{A})^{-1}}{\lambda^{n+1}}\| \leq \frac{Mk!}{|\text{Re}\lambda-\omega|^{k+1}}.$$

The former (MFPHY) type condition implies $(n,\omega)$–well–posedness only on a smaller subspace of $D(\mathcal{A}^{n+1})$. To obtain a characterization of the $(n,\omega)$–well–posedness, we need the assumption that a certain decomposition of the state space $X$ is valid, see Theorem 2.33.

## 2.1 Multi–Valued Linear Operators

In this section, we collect some elementary properties of multi–valued linear operators in Banach spaces, as found in the monograph [23]. Let $F$ and $G$ be subsets of the complex Banach space $X$ and $\lambda \in \mathbb{C}$. We define the addition and scalar multiplication of subsets of $X$ as usual by

$$F + G := \{f + g : f \in F, g \in G\},$$

$$\lambda F := \{\lambda f : f \in F\}.$$

**Definition 2.1** *A map* $\mathcal{A} : X \to 2^X$ *is called a multi–valued linear operator on* $X$*, if* $D(\mathcal{A}) := \{u \in X : \mathcal{A}u \neq \emptyset\}$ *is a linear manifold in* $X$ *and for any* $\lambda, \mu \in \mathbb{C}$*,* $u, v \in D(\mathcal{A})$ *holds*

$$\lambda \mathcal{A}u + \mu \mathcal{A}v \subset \mathcal{A}(\lambda u + \mu v). \tag{2.4}$$

In particular, for $\lambda = \mu = 0$ we have $0 \in \mathcal{A}0$. From this and the fact that $D(\mathcal{A})$ is a linear manifold in $X$, we conclude that $D(\mathcal{A})$ is a linear subspace of $X$. We denote by $\mathcal{M}(X)$ the set of all multi–valued linear operators on $X$.

Multi–valued linear operators have the same linearity properties as single–valued linear operators, except that homogeneity is only given for scalars not equal to zero.

**Proposition 2.2** *Let* $\mathcal{A} \in \mathcal{M}(X)$*. Then for all* $u, v \in D(\mathcal{A})$ *and all* $\lambda \in \mathbb{C}\backslash\{0\}$ *holds:*

*(i)* $\mathcal{A}u + \mathcal{A}v = \mathcal{A}(u + v)$,

*(ii)* $\lambda \mathcal{A}u = \mathcal{A}(\lambda u)$.

**Proof.** From (2.4) we obtain

(i) $\mathcal{A}(u + v) - \mathcal{A}v \subset \mathcal{A}(u + v - v) = \mathcal{A}u$, which gives $\mathcal{A}(u + v) \subset \mathcal{A}u + \mathcal{A}v$,

(ii) $\mathcal{A}(\lambda u) = \lambda(\lambda^{-1}\mathcal{A}(\lambda u)) \subset \lambda \mathcal{A}u$. ∎

For the representation of multi–valued operators we have the following result.

**Proposition 2.3** *Let* $\mathcal{A} \in \mathcal{M}(X)$*, then* $\mathcal{A}0$ *is a linear subspace of* $X$ *and for any* $u \in D(\mathcal{A})$*,* $f \in \mathcal{A}u$ *we have* $\mathcal{A}u = f + \mathcal{A}0$.

**Proof.** By Proposition 2.2 we have $\mathcal{A}0 + \mathcal{A}0 = \mathcal{A}0$ and $\lambda \mathcal{A}0 = \mathcal{A}0$. Hence $\mathcal{A}0$ is a linear subspace. Now let $u \in D(\mathcal{A})$ and $f \in \mathcal{A}u$, then

$$f + \mathcal{A}0 \subset \mathcal{A}u + \mathcal{A}0 = \mathcal{A}u.$$

On the other hand, for any $g \in \mathcal{A}u$ we have $g - f \in \mathcal{A}u - \mathcal{A}u = \mathcal{A}0$. Hence

$$g = f + (g - f) \in f + \mathcal{A}0.$$

Thus $g \in \mathcal{A}u$ implies $g \in f + \mathcal{A}0$. It means $\mathcal{A}u \subset f + \mathcal{A}0$. Therefore the equality $\mathcal{A}u = f + \mathcal{A}0$ holds. ■

In contrast to single–valued operators, multi–valued operators always have an inverse, which is in general multi–valued.

**Definition 2.4** *Let $\mathcal{A} \in \mathcal{M}(X)$. We call the operator $\mathcal{A}^{-1}$ the inverse of the operator $\mathcal{A}$, if*

$$D(\mathcal{A}^{-1}) := \operatorname{ran}(\mathcal{A}) := \bigcup_{u \in D(\mathcal{A})} \mathcal{A}u \quad \text{and} \quad \mathcal{A}^{-1}f := \{u \in D(\mathcal{A}) : f \in \mathcal{A}u\}.$$

**Proposition 2.5** *$\mathcal{A}^{-1} \in \mathcal{M}(X)$ if $\mathcal{A} \in \mathcal{M}(X)$.*

**Proof.** If $u \in \mathcal{A}^{-1}f$ and $v \in \mathcal{A}^{-1}g$ then $f \in \mathcal{A}u$ and $g \in \mathcal{A}v$, so $f + g \in \mathcal{A}(u + v)$. Hence $(u + v) \in \mathcal{A}^{-1}(f + g)$. Similarly, $u \in \mathcal{A}^{-1}f$ implies $\lambda f \in \mathcal{A}(\lambda u)$ and $\lambda u \in \mathcal{A}^{-1}(\lambda f)$. Hence $\lambda \mathcal{A}^{-1}f \subset \mathcal{A}^{-1}(\lambda f)$. Therefore $\mathcal{A}^{-1}$ is a multi–valued linear operator. ■

Some standard operations also can be defined for multi–valued operators.

**Definition 2.6** *Let $U$ be a single–valued linear operator in $X$ and $\mathcal{A} \in \mathcal{M}(X)$. We define the sum and the composition as follows.*

$$\begin{aligned} D(\mathcal{A} + U) &= D(\mathcal{A}) \cap D(U), \\ (\mathcal{A} + U)u &= \mathcal{A}u + Uu \quad \text{for} \quad u \in D(\mathcal{A} + U). \end{aligned}$$

$$\begin{aligned} D(\mathcal{A}U) &= \{u \in D(U) : Uu \in D(\mathcal{A})\}, \\ \mathcal{A}Uu &= \mathcal{A}(Uu) \quad \text{for} \quad u \in D(\mathcal{A}U). \end{aligned}$$

$$\begin{aligned} D(U\mathcal{A}) &= \{u \in D(\mathcal{A}) : \mathcal{A}u \cap D(U) \neq \emptyset\}, \\ U\mathcal{A}u &= \{Uv : v \in \mathcal{A}u \cap D(U)\} \quad \text{for} \quad u \in D(U\mathcal{A}). \end{aligned}$$

**Theorem 2.7** *The operators $\mathcal{A} + U$, $\mathcal{A}U$ and $U\mathcal{A}$ defined above are multi–valued linear operators.*

In particular, the operators $(\lambda - \mathcal{A})$, $B^{-1}A$ and $AB^{-1}$ are multi–valued linear operators. The following definition of the resolvent for multi–valued operators is of significance.

**Definition 2.8** *The set* $\rho(\mathcal{A}) := \{\lambda \in \mathbb{C} : (\lambda - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$ *is called the resolvent set of the linear multi–valued operator* $\mathcal{A}$ *and* $R(\lambda) := (\lambda - \mathcal{A})^{-1}$ *is called the resolvent of* $\mathcal{A}$*.*

It turns out that the resolvent defined in this way enjoys similar properties as in the single–valued case.

**Theorem 2.9** *For any* $\mathcal{A} \in \mathcal{M}(X)$ *the resolvent set* $\rho(\mathcal{A})$ *is an open subset of* $\mathbb{C}$*. The map* $\lambda \mapsto R(\lambda) = (\lambda - \mathcal{A})^{-1} \in \mathcal{L}(X)$ *is holomorphic on* $\rho(\mathcal{A})$*.*

**Theorem 2.10** *For any* $\mathcal{A} \in \mathcal{M}(X)$ *and* $\lambda \in \rho(\mathcal{A})$ *holds the following:*

$$(\lambda - \mathcal{A})^{-1} \subset \lambda(\lambda - \mathcal{A})^{-1} - I \subset A(\lambda - \mathcal{A})^{-1}.$$

*In particular,* $(\lambda - \mathcal{A})^{-1}\mathcal{A}$ *is single–valued on* $D(\mathcal{A})$ *and*

$$(\lambda - \mathcal{A})^{-1}\mathcal{A}u = (\lambda - \mathcal{A})^{-1}f$$

*for any* $f \in \mathcal{A}u$*.*

In general, $\mathcal{A}0 \subset X$ can be any subspace of $X$. For multi–valued operators with non–empty resolvent set it is necessarily a closed subspace, since it is the kernel of a bounded linear operator.

**Proposition 2.11 (Resolvent Identity)** *For any* $\mathcal{A} \in \mathcal{M}(X)$ *and* $\lambda, \mu \in \rho(\mathcal{A})$ *the following identity holds:*

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu).$$

**Proposition 2.12** *Let* $\mathcal{A} \in \mathcal{M}(X)$ *and* $\rho(\mathcal{A}) \neq \emptyset$*. Then for any* $\lambda \in \rho(\mathcal{A})$ *holds* $\mathcal{A}0 = \ker(\lambda - \mathcal{A})^{-1}$*.*

**Proof.** Let $f \in \ker(\lambda - \mathcal{A})^{-1}$, i.e. $(\lambda - \mathcal{A})^{-1} f = 0$, which is equivalent to $f \in (\lambda - \mathcal{A})0 = \mathcal{A}0$. ∎

Also, the notion of closedness can be carried through to multi–valued operators.

**Definition 2.13** *We say that an operator $\mathcal{A} \in \mathcal{M}(X)$ is closed if for any sequences $\{u_j\} \subset D(\mathcal{A})$ and $f_j \in \mathcal{A}u_j$ with*

$$\lim_{j\to\infty} u_j = u, \quad \lim_{j\to\infty} f_j = f$$

*we have $u \in D(\mathcal{A})$ and $f \in \mathcal{A}u$.*

An example of a closed linear multi–valued operator is the operator $B^{-1}A$ for the case that $A$ is closed and $B$ is bounded with non–trivial kernel.

**Lemma 2.14** *Let $A : D(A) \subset X \to X$ be closed and $B : X \to X$ be bounded with $\ker B \neq \{0\}$. Then $B^{-1}A : D(B^{-1}A) \to X$ is a closed multi–valued operator.*

**Proof.** Let $x_j \in D(B^{-1}A)$ with $x_j \to x$ in $X$. Let $y_j \in B^{-1}Ax_j$ with $y_j \to y$ in $X$. By the boundedness of $B$ we have

$$By_j \to By \text{ in } X.$$

Obviously $By_j = Ax_j$. By the closedness of $A$ we conclude $x \in D(A)$ and $Ax = By$. Thus $x \in D(B^{-1}A)$ and $y \in B^{-1}Ax$. This means that $B^{-1}A$ is closed and multi–valued since $\ker B \neq \{0\}$. ∎

Next, we define powers of a multi–valued linear operator.

**Definition 2.15** *Let $\mathcal{A} \in \mathcal{M}(X)$ and $n \in \mathbb{N}$. We define $\mathcal{A}^0 := \mathrm{Id}_X$, the identity on $X$, and the nth power of $\mathcal{A}$ inductively by*

$$D(\mathcal{A}^n) := \{x \in D(\mathcal{A}^{n-1}) : \text{there is a } y \in \mathcal{A}^{n-1}x \text{ with } y \in D(\mathcal{A})\},$$

*and*

$$\mathcal{A}^n x := \bigcup_{y \in \mathcal{A}^{n-1}x \cap D(\mathcal{A})} \mathcal{A}y \quad \text{for } x \in D(\mathcal{A}^n).$$

We will write shortly

$$\mathcal{A}^n x = \bigcup_{y \in \mathcal{A}^{n-1}x} \mathcal{A}y,$$

since we can set $\mathcal{A}y := \emptyset$ for elements $y \in X - D(\mathcal{A})$.

Using the induction principle we show that the $n$th power of a multi–valued operator indeed is a multi–valued operator.

**Lemma 2.16** *$\mathcal{A}^n \in \mathcal{M}(X)$ for $\mathcal{A} \in \mathcal{M}(X)$.*

**Proof.** We prove this using the induction principle. The hypothesis holds for $n = 1$. Suppose that it holds for $n-1$, that is $\mathcal{A}^{n-1} \in \mathcal{M}(X)$. By the supposition we have that $D(\mathcal{A}^{n-1})$ is a linear manifold and for $u, v \in D(\mathcal{A}^{n-1})$ holds

$$\lambda \mathcal{A}^{n-1}u + \mu \mathcal{A}^{n-1}v \subset \mathcal{A}^{n-1}(\lambda u + \mu v).$$

We show that the hypothesis is true for $n$. First we show that $D(\mathcal{A}^n)$ is a linear manifold. Let $u, v \in D(\mathcal{A}^n)$. Then there exist $u' \in \mathcal{A}^{n-1}u$ with $u' \in D(\mathcal{A})$ and $v' \in \mathcal{A}^{n-1}v$ with $v' \in D(\mathcal{A})$. Since $D(\mathcal{A})$ is linear manifold and $u', v' \in D(\mathcal{A})$, we have $u' + v' \in D(\mathcal{A})$ and $\lambda u' \in D(\mathcal{A})$. Moreover, by supposition that $\mathcal{A}^{n-1} \in \mathcal{M}(X)$ and using Proposition 2.2, we have

$$u' + v' \in \mathcal{A}^{n-1}u + \mathcal{A}^{n-1}v = \mathcal{A}^{n-1}(u+v) \text{ and } \lambda u' \in \lambda \mathcal{A}^{n-1}u = \mathcal{A}^{n-1}u, \ \lambda \neq 0.$$

Hence there exist $(u' + v') \in \mathcal{A}^{n-1}(u+v)$ with $(u'+v') \in D(\mathcal{A})$ and $\lambda u' \in \mathcal{A}^{n-1}(\lambda u)$ with $\lambda u \in D(\mathcal{A})$. Therefore $(u+v), \lambda u \in D(\mathcal{A}^n)$. Thus $D(\mathcal{A}^n)$ is a linear manifold.

It remains to show that

$$\lambda \mathcal{A}^n u + \mu \mathcal{A}^n v \subset \mathcal{A}^n(\lambda u + \mu v)$$

for $u, v \in D(\mathcal{A}^n)$. Let $u, v \in D(\mathcal{A}^n)$, then

$$\begin{aligned} \lambda \mathcal{A}^n u + \mu \mathcal{A}^n v &= \lambda \bigcup_{u' \in \mathcal{A}^{n-1}u} \mathcal{A}u' + \mu \bigcup_{v' \in \mathcal{A}^{n-1}v} \mathcal{A}v' \\ &= \bigcup_{u' \in \mathcal{A}^{n-1}u} \lambda \mathcal{A}u' + \bigcup_{v' \in \mathcal{A}^{n-1}v} \mu \mathcal{A}v' \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{u' \in \mathcal{A}^{n-1}u, v' \in \mathcal{A}^{n-1}v} \lambda \mathcal{A}u' + \mu \mathcal{A}v' \\
&\subset \bigcup_{u' \in \mathcal{A}^{n-1}u, v' \in \mathcal{A}^{n-1}v} \mathcal{A}(\lambda u' + \mu v') \\
&= \bigcup_{(\lambda u' + \mu v') \in \mathcal{A}^{n-1}(\lambda u + \mu v)} \mathcal{A}(\lambda u' + \mu v') \\
&= \mathcal{A}^n(\lambda u + \mu v).
\end{aligned}
$$

Hence $\mathcal{A}^n \in \mathcal{M}(X)$. ∎

## 2.2 $(n, \omega)$–Well–Posedness of Cauchy Problem for Inclusions

In this section we discuss the $(n, \omega)$-well-posedness of the Cauchy problem for inclusion (2.3). In particular for the case that the multi–valued operator $\mathcal{A}$ generates a degenerate $n$–times integrated semigroup. Before we give the notion of a solution for the inclusion.

**Definition 2.17** *A function $u \in C^1([0, \infty); X) \cap C([0, \infty); D(\mathcal{A}))$ is called a solution of the Cauchy problem (2.3), if $u(0) = x$ and if it satisfies the differential inclusion of (2.3) for all $t \geq 0$.*

The $(n, \omega)$-well–posedness means that solutions grow exponentially with respect to time and can be estimated by some norm of the initial values.

**Definition 2.18** *The Cauchy problem (2.3) is called $(n, \omega)$–well-posed on a subset $E \subset D(\mathcal{A}^n)$, if for any $x \in E$ there exists a unique solution $u(\cdot)$, such that for all $\lambda \in \rho(\mathcal{A})$ there is a $C \geq 0$ with*

$$
\|u(t)\| \leq C \exp(\omega t) \|x\|_n, \quad \|x\|_n := \inf\{\|y\| : R^n(\lambda) y = x\}.
$$

Later, in Lemma 2.27, we will show that $R^n(\lambda)X = D(\mathcal{A}^n)$. In fact, $\|\cdot\|_n$ defines a norm on $D(\mathcal{A}^n)$.

**Lemma 2.19** *The map*

$$\begin{aligned} \|\cdot\|_n : D(\mathcal{A}^n) &\to \mathbb{R}, \\ x &\mapsto \|x\|_n := \inf\{\|y\| : R^n(\lambda)y = x\} \end{aligned}$$

*defines a norm in* $D(\mathcal{A}^n)$.

**Proof.** It is obvious that $\|x\|_n \geq 0$ for all $x \in D(\mathcal{A}^n)$. First we show that $\|x\|_n = 0$ if and only if $x = 0$.

$$\begin{aligned} \|x\|_n = 0 &\iff \inf\{\|y\| : R^n(\lambda)y = x\} = 0 \\ &\iff x = R^n(\lambda)0 \\ &\iff x = 0. \end{aligned}$$

Next we show the homogeneity of the map. For $\mu = 0$ is clear from above. For $\mu \neq 0$, we can calculate

$$\begin{aligned} \|\mu x\|_n &= \inf\{\|y\| : R^n(\lambda)y = \mu x\} \\ &= \inf\{\|y\| : R^n(\lambda)\mu^{-1}y = x\} \\ &= \inf\{\|\mu z\| : R^n(\lambda)z = x\} \\ &= |\mu| \cdot \|x\|_n. \end{aligned}$$

Finally, we show that the triangle inequality also holds for the map.

$$\begin{aligned} \|x_1 + x_2\| &= \inf\{\|y\| : R^n(\lambda)y = x_1 + x_2\} \\ &\leq \inf\{\|y_1 + y_2\| : R^n(\lambda)y_1 = x_1 and R^n(\lambda)y_2 = x_2\} \\ &\leq \inf\{\|y_1\| : R^n(\lambda)y_1 = x_1\} + \inf\{\|y_2\| : R^n(\lambda)y_2 = x_2\} \\ &= \|x_1\|_n + \|x_2\|_n. \end{aligned}$$

∎

**Remark 2.20** *For a single-valued linear operator* $\mathcal{A}$, *the* $\|\cdot\|_n^*$*-norm defined by*

$$\|x\|_n^* := \|x\| + \|\mathcal{A}x\| + \cdots + \|\mathcal{A}^n x\|$$

*is equivalent to the* $\|\cdot\|_n$*-norm defined in Definition 2.18.*

**Proof.** If $\mathcal{A}$ is a single–valued operator, then

$$\begin{aligned}
\|x\|_n &= \|(\lambda - \mathcal{A})^n x\| \\
&= \|\lambda^n x - \binom{n}{1}\lambda^{n-1}\mathcal{A}x + \cdots + (-1)^k\binom{n}{k}\lambda^{n-k}\mathcal{A}^k x + \cdots + \mathcal{A}^n x\| \\
&\le |\lambda^n|\|x\| + \binom{n}{1}|\lambda^{n-1}|\|\mathcal{A}x\| + \cdots + \binom{n}{k}|\lambda^{n-k}|\|\mathcal{A}^k x\| + \cdots + \|\mathcal{A}x\| \\
&\le \beta\,(\|x\| + \|\mathcal{A}x\| + \cdots + \|\mathcal{A}^n x\|)
\end{aligned}$$

for

$$\beta = \max\{|\lambda^n|, \binom{n}{1}|\lambda^{n-1}|, \cdots, \binom{n}{n-1}|\lambda|\}.$$

Hence $\|x\|_n \le \beta\|x\|_n^*$ for some $\beta > 0$. On the other hand, since $R(\lambda) = (\lambda - \mathcal{A})^{-1}$ is a bounded operator, we have

$$\|x\| = \|R^n(\lambda)(\lambda - \mathcal{A})^n x\| \le \|R^n(\lambda)\| \cdot \|(\lambda - \mathcal{A})^n x\| < C_n\|(\lambda - \mathcal{A})^n x\|,$$

where $C_n = \|R(\lambda)\|^n$. Moreover we have

$$\begin{aligned}
\|\mathcal{A}x\| &= \|(\lambda - \mathcal{A})x - \lambda x\| \le \|\lambda x\| + \|R^{n-1}(\lambda)(\lambda - \mathcal{A})^n x\| \\
&\le (|\lambda|C_n + C_{n-1})\|(\lambda - \mathcal{A})^n x\| \\
&< K_1\|(\lambda - \mathcal{A})^n x\|.
\end{aligned}$$

Similarly,

$$\|\mathcal{A}^k x\| \le K_k\|(\lambda - \mathcal{A})^n x\|$$

for $k = 2, 3, \ldots, n$. We take

$$K = \max\{C_n, K_1, K_2, \ldots, K_n\},$$

then

$$\|x\| + \|\mathcal{A}x\| + \|\mathcal{A}^2 x\| + \cdots + \|\mathcal{A}^n x\| \le (n+1)K\|(\lambda - \mathcal{A})^n x\|.$$

Therefore $\|x\|_n^* \le \alpha\|x\|_n$ for $\alpha = (n+1)K > 0$. Thus $\|\cdot\|_n$ and $\|\cdot\|_n^*$ are equivalent. ∎

The $n$–times integrated semigroups together with their generators defined below play a key role in our analysis.

**Definition 2.21** *A one–parameter family of bounded linear operators* $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ *is called an n–times integrated semigroup, if*

*(i)* $V(0) = 0$ *and for all* $s, t \in [0,\infty)$ *holds*

$$V(t)V(s) = \frac{1}{(n-1)!}\int_0^s \left((s-r)^{n-1}V(t+r) - (t+s-r)^{n-1}V(r)\right) dr.$$

*(ii)* $t \mapsto V(t)$ *is strongly continuous, i.e. for any* $x \in X$ *the map* $t \mapsto V(t)x$ *is continuous on* $[0,\infty)$.

*(iii) An n–times integrated semigroup is called exponentially bounded if, there exist* $C > 0$ *and* $\omega \in \mathbb{R}$ *such that for all* $t \in [0,\infty)$ *holds*

$$\|V(t)\| \leq C \exp(\omega t).$$

*(iv) We say the subspace*

$$N = \{x \in X : V(t)x = 0 \text{ for all } t \in [0,\infty)\}$$

*is the degeneration space of the n–times integrated semigroup* $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. *An n–times integrated semigroup is called degenerate if* $N \neq \{0\}$, *and non–degenerate otherwise.*

**Definition 2.22** *An operator* $\mathcal{A} \in \mathcal{M}(X)$ *is called the generator of the n–times integrated semigroup* $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$, *if it satisfies the relation*

$$(\lambda - \mathcal{A})^{-1}x = \int_0^\infty \lambda^n \exp(-\lambda t)V(t)x dt \qquad (2.5)$$

*for all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\lambda > \omega$ *and all* $x \in X$.

There is an equivalent characterization of a generator of $n$–times integrated semigroups (see [59]). We choose the one above, since here we are interested in the exponentially bounded $n$–times integrated semigroups. The exponential boundedness implies the existence of the Laplace transform.

In the following lemma, we show that if $\mathcal{A}$ generates an $n$–times integrated semigroup, i.e. (2.5) is satisfied, then $N = \ker(\lambda - \mathcal{A})^{-1}$. On the other hand, we have shown in Proposition 2.12 that $\ker(\lambda - \mathcal{A})^{-1} = \mathcal{A}0$. Hence the degeneration space $N = \mathcal{A}0$. This means that any degenerate $n$–times integrated semigroup has a multi–valued generator, and conversely if a multi-valued operator is the generator of a semigroup, then the semigroup is degenerate.

**Lemma 2.23** *Let $\mathcal{A} \in \mathcal{M}(X)$ be the generator of the exponentially bounded $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \geq 0\}$. Then*

$$\ker(\lambda - \mathcal{A})^{-1} = N$$

*for all $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\lambda > \omega$.

**Proof.** The inclusion $N \subset \ker(\lambda - \mathcal{A})^{-1}$ is obvious. We will show that $\ker(\lambda - \mathcal{A})^{-1} \subset N$. To this end we define for $x \in \ker(\lambda - \mathcal{A})^{-1}$ the map

$$\begin{aligned} F : (\max(0,\omega), \infty) &\to X, \\ \lambda &\mapsto \int_0^\infty \exp(-\lambda t) V(t) x dt. \end{aligned}$$

Obviously $F(\lambda) = 0$ for all $\lambda \in (\max(0,\omega), \infty)$. Hence, for all $k \in \mathbb{N}$ we obtain

$$\frac{d^k}{d\lambda^k} F(\lambda) = 0$$

for all $\lambda \in (\max(0,\omega), \infty)$. Calculating the derivatives we obtain

$$\int_0^\infty (-t)^k \exp(-\lambda t) V(t) x dt = 0$$

for all $k \in \mathbb{N}$. Hence,

$$\int_0^\infty P(t) \exp(-\lambda t) V(t) x dt = 0$$

for all real polynomials $t \mapsto P(t)$. Accordingly, for any $x^* \in X^*$ the equation

$$\int_0^\infty P(t) \langle V(t)x, x^* \rangle \exp(-\lambda t) dt = 0$$

is valid, in particular for any Laguerre polynomial $P(t)$. On the other hand, the Laguerre polynomials form a complete orthogonal system of the Hilbert space $L^2((0,\infty);\mathbb{C})$ equipped with the measure $\exp(-\lambda t)dt$, see [56]. The map $t \mapsto \langle V(t)x, x^* \rangle$ is an element of this Hilbert space, for $\lambda > 0$ large enough. Hence,

$$\langle V(t)x, x^* \rangle = 0$$

for all $t \geq 0$ and all $x^* \in X^*$. We conclude that $V(t)x = 0$ for all $t \geq 0$, which implies $x \in N$. ■

**Remark 2.24** *The generator of the degenerate $n$–times integrated semigroup defined in Definition 2.22 is uniquely determined.*

**Proof.** Suppose there are two multi–valued operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfying the relation (2.5), then we have

$$(\lambda - \mathcal{A}_1)^{-1}x = (\lambda - \mathcal{A}_2)^{-1}x, \; x \in X. \tag{2.6}$$

Let $x \in D(\mathcal{A}_1)$, consequently $x \in D(\lambda - \mathcal{A}_1)$, i.e. $x \in \text{ran}(\lambda - \mathcal{A}_1)^{-1}$. By (2.6), $x \in \text{ran}(\lambda - \mathcal{A}_2)^{-1}$. Hence $x \in D(\lambda - \mathcal{A}_2)$, i.e. $x \in D(\mathcal{A}_2)$. This shows that $D(\mathcal{A}_1) \subseteq D(\mathcal{A}_2)$. Using the same argument, we obtain $D(\mathcal{A}_2) \subseteq D(\mathcal{A}_1)$. Therefore $D(\mathcal{A}_1) = D(\mathcal{A}_2)$. Now let $\mathcal{A}_1 x \neq \mathcal{A}_2 x$ for $x \in D(\mathcal{A}_1) = D(\mathcal{A}_2)$. Then

$$(\lambda - \mathcal{A}_1)x \neq (\lambda - \mathcal{A}_2)x.$$

Applying $(\lambda - \mathcal{A}_1)^{-1}$ and using equation (2.6) we obtain $x \neq x$ which is a contradiction. Thus $\mathcal{A}_1 x = \mathcal{A}_2 x$ for all $x \in D(\mathcal{A}_1) = D(\mathcal{A}_2)$. Therefore $\mathcal{A}_1 = \mathcal{A}_2$. ■

In [1], Theorem 3.1, it is shown, for a single–valued operator $\mathcal{A}$, that

$$(\lambda - \mathcal{A})^{-1} = \int_0^\infty \lambda^n \exp(-\lambda t) V(t) dt$$

is the resolvent if and only if $V(t)V(s)$ satisfies property (i) of Definition 2.21. Since the resolvent identity also holds for multi–valued operators (see Proposition 2.11), the proof of these statements can be immediately adapted to multi–valued operators $\mathcal{A}$. Similarly, Theorem 4.1 in [1], which is proven for non–degenerate $n$–times integrated semigroups, also holds for degenerate ones.

Accordingly, we can formulate the following theorem, which was originally stated in [1], Theorem 4.1, for single–valued operators $\mathcal{A}$.

**Theorem 2.25** *Let $n \in \mathbb{N}$, $\omega \in \mathbb{R}$ and $M \geq 0$. A linear operator $\mathcal{A}$ is the generator of an $(n+1)$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$ satisfying*

$$\limsup_{h \to 0+} \frac{1}{h} \|V(t+h) - V(t)\| \leq M \exp(\omega t), \quad t \geq 0,$$

*if and only if there exists $a \geq \max\{\omega, 0\}$ such that $(a, \infty) \subset \rho(\mathcal{A})$ and*

$$\|\frac{d^k}{d\lambda^k} \frac{R(\lambda)}{\lambda^n}\| \leq \frac{Mk!}{(Re\lambda - \omega)^{k+1}}, \; k = 0, 1, 2, \ldots, \; \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda > \omega.$$

The following proposition describes the relation between the $n$–times integrated semigroup and its generator for multi–valued operators. It generalizes the corresponding properties of single–valued generators of $n$–times integrated semigroups, as in [51], Lemma 5.1, or in [1], Proposition 3.3.

**Proposition 2.26** *Let $\mathcal{A} \in \mathcal{M}(X)$ be the generator of the degenerate $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$. Then*

*(i) For all $x \in D(\mathcal{A})$ we have $V(t)x \in D(\mathcal{A})$ and $V(t)\mathcal{A}x \in \mathcal{A}V(t)x$.*

*(ii) For all $x \in D(\mathcal{A})$ holds*

$$V(t)x = \frac{t^n}{n!}x + \int_0^t V(s)\mathcal{A}x ds.$$

*(iii) For all $x \in X$ we have*

$$V(t)x \in \frac{t^n}{n!}x + \mathcal{A} \int_0^t V(s)x ds.$$

**Proof.** (i) By the definition of the generator of the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$, for all $x \in X$ and $\lambda \in \rho(\mathcal{A})$ we have

$$\begin{aligned} \int_0^\infty \exp(-\mu t) V(t) R(\lambda) x &= \mu^{-n} R(\mu) R(\lambda) x \\ &= \mu^{-n} R(\lambda) R(\mu) x \\ &= \int_0^\infty \exp(-\mu t) R(\lambda) V(t) x dt. \end{aligned}$$

The uniqueness of the Laplace transform implies that

$$V(t)R(\lambda)x = R(\lambda)V(t)x. \tag{2.7}$$

Now let $x \in D(\mathcal{A})$, substitute $y = (\lambda - \mathcal{A})x$ to (2.7), we obtain

$$V(t)x = R(\lambda)V(t)(\lambda - \mathcal{A})x.$$

Applying $(\lambda - \mathcal{A})$ to this equation, we have

$$(\lambda - \mathcal{A})V(t)x = (\lambda - \mathcal{A})R(\lambda)V(t)(\lambda - \mathcal{A})x \ni V(t)(\lambda - \mathcal{A})x.$$

Hence $V(t)x \in D(\mathcal{A})$ and $V(t)\mathcal{A}x \in \mathcal{A}V(t)x$.

(ii) For all $x \in D(\mathcal{A})$, we have

$$\begin{aligned}
&\int_0^\infty \lambda^{n+1} \exp(-\lambda t)\frac{t^n}{n!}xdt = x = R(\lambda)(\lambda - \mathcal{A})x \\
= &\int_0^\infty \lambda^{n+1} \exp(-\lambda t)V(t)xdt - \int_0^\infty \lambda^n \exp(-\lambda t)V(t)\mathcal{A}xdt \\
= &\int_0^\infty \lambda^{n+1} \exp(-\lambda t)V(t)xdt - \int_0^\infty \lambda^{n+1} \exp(-\lambda t)\int_0^t V(s)\mathcal{A}xdsdt,
\end{aligned}$$

and uniqueness of the Laplace transform implies

$$V(t)x = \frac{t^n}{n!}x + \int_0^t V(s)\mathcal{A}xds, \quad x \in D(\mathcal{A}).$$

(iii) For all $x \in X$, using $x \in (\lambda - \mathcal{A})R(\lambda)x$, we obtain

$$\begin{aligned}
&\int_0^\infty \lambda^{n+1} \exp(-\lambda t)\frac{t^n}{n!}xdt = x \in (\lambda - \mathcal{A})R(\lambda)x \\
= &(\lambda - \mathcal{A})\int_0^\infty \lambda^n \exp(-\lambda t)V(t)xdt \\
= &\int_0^\infty \lambda^{n+1} \exp(-\lambda t)V(t)xdt - \mathcal{A}\int_0^\infty \lambda^n \exp(-\lambda t)V(t)xdt \\
= &\int_0^\infty \lambda^{n+1} \exp(-\lambda t)V(t)xdt - \int_0^\infty \lambda^{n+1} \exp(-\lambda t)\mathcal{A}\int_0^t V(s)xdsdt.
\end{aligned}$$

Thus

$$V(t)x \in \frac{t^n}{n!}x + \mathcal{A}\int_0^t V(s)xds, \quad x \in X.$$

∎

The following lemma describes the range of the powers of the resolvent. It is of particular importance for the definition of the $(n, \omega)$–well–posedness above, where the norm $\|\cdot\|_n$ only is defined for elements in the range of $R^n(\lambda)$.

**Lemma 2.27** *For $\mathcal{A} \in \mathcal{M}(X)$ and $\lambda \in \rho(\mathcal{A})$ we have $D(\mathcal{A}^k) = R^k(\lambda)X$. In particular $R^k(\lambda)X$ does not depend on the choice of $\lambda \in \rho(\mathcal{A})$.*

**Proof.** Let $x \in D(\mathcal{A}^k)$. It follows that $x \in D(\mathcal{A}^i)$ for all $i \leq k$. Consequently, $(\lambda - \mathcal{A})^k x$ is a non-empty set. Let $y \in (\lambda - \mathcal{A})^k x$. Then we have

$$R^k(\lambda)y = R^k(\lambda)(\lambda - \mathcal{A})^k x = x.$$

Hence $x \in R^k(\lambda)X$. Therefore $D(\mathcal{A}^k) \subseteq R^k(\lambda)X$. On the other hand, let $x \in R^k(\lambda)X$, i.e. $x = R^k(\lambda)y$ for some $y \in X$. Equivalently we obtain

$$y \in (\lambda - \mathcal{A})^k R^k(\lambda)y = (\lambda - \mathcal{A})^k x.$$

Therefore, $(\lambda - \mathcal{A})^k x$ is not empty and $x \in D((\lambda - \mathcal{A})^k)$. Hence $x \in D(\mathcal{A}^k)$. Therefore $R^k(\lambda)X \subseteq D(\mathcal{A}^k)$. This completes the proof of $D(\mathcal{A}^k) = R^k(\lambda)X$.

Next, using the induction principle, we show that $R^k(\lambda)X$ is independent of the choice of $\lambda \in \rho(\mathcal{A})$. Let $x \in R(\lambda)X$, i.e $x = R(\lambda)y$ for some $y \in X$. Consequently, we have

$$y \in (\lambda - \mathcal{A})x = (\lambda + \mu - \mu - \mathcal{A})x = (\lambda + \mu) + (\mu - \mathcal{A})x$$

for some $\mu \in \rho(\mathcal{A})$. Thus $(\mu - \mathcal{A})$ contains $y + (\mu - \lambda)x = z \in X$. Equivalently $x = R(\mu)z$ for some $z \in X$. Hence $x \in R(\mu)X$.

Suppose that for any $x \in R^j(\lambda)X$, $x \in R^j(\mu)X$ for $\mu \in \rho(\mathcal{A})$ and $j \geq 1$.
Now let $x \in R^{j+1}(\lambda)X$. We can write $x$ as

$$\begin{aligned} x &= R^{j+1}(\lambda)y, \quad \text{for} \quad y \in X \\ &= R(\lambda)R^j(\lambda)y \\ &= R(\lambda)z, \quad \text{for} \quad z \in R^j(\lambda)X. \end{aligned}$$

From the supposition above $x \in R^j(\mu)X$ for $\mu \in \rho(\mathcal{A})$. Thus $x = R(\lambda)z$ for $z \in R^j(\mu)X$, i.e. $z \in (\lambda - \mathcal{A})x$. Equivalently,

$$z \in (\lambda - \mu + \mu - \mathcal{A})x = (\mu - \mathcal{A})x + (\lambda - \mu)x.$$

Since $x \in R^{j+1}(\lambda)X$, $x \in R^j(\lambda)X$ and hence $x \in R^j(\mu)X$ from the supposition. Thus $(\lambda - \mu)x \in R^j(\mu)X$. Therefore $(\mu - \mathcal{A})x$ contains $z - (\lambda - \mu)x = u \in R^j(\mu)X$. Conclusively $x \in R^{j+1}(\mu)X$ for $\mu \in \rho(\mathcal{A})$. This shows that $R^k(\lambda)X$ does not depend on the choice of $\lambda \in \rho(\mathcal{A})$. ∎

The following theorem is a main step towards a characterization of $(n, \omega)$–well–posedness. It shows that a (MFPHY) type condition is necessary for it. From the $(n, \omega)$–well–posedness of the Cauchy problem (2.3) on $D(\mathcal{A}^{n+1})$, we can construct an exponentailly bounded $(n+1)$–times integrated semigroup from which the (MFPHY) type condition is obtained.

**Theorem 2.28** *Let $\mathcal{A}$ be a closed linear multi–valued operator in $X$. Suppose that $\rho(\mathcal{A}) \neq \emptyset$. If the Cauchy problem (2.3) is $(n, \omega)$–well–posed on $E = D(\mathcal{A}^{n+1})$, then the condition*

$$\left\|\frac{d^k}{d\lambda^k}\frac{(\lambda - \mathcal{A})^{-1}}{\lambda^{n+1}}\right\| \leq \frac{Mk!}{(\mathrm{Re}\lambda - \omega)^{k+1}}, \quad k = 0, 1, 2, \ldots, \quad \lambda \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\lambda > \omega \tag{2.8}$$

*holds for some $M > 0$.*

**Proof.** Let $t \mapsto u(t)$ be the unique solution of (2.3) with respect to the initial data $x \in D(\mathcal{A}^{n+1})$. By the definition of the $(n, \omega)$–well–posedness of (2.3), it satisfies the estimate

$$\|u(t)\| \leq C \exp{(\omega t)} \|x\|_n. \tag{2.9}$$

We introduce a solution operator $U(t)$ on $D(\mathcal{A}^{n+1})$ by $U(t)x := u(t)$. Due to (2.9) we can extend $U(t)$ to the space $D_{n+1}$, the closure of $D(\mathcal{A}^{n+1})$ endowed with the $\|\cdot\|_n$–norm. As $U(t)x = u(t)$ is a solution of (2.3) we have

$$U'(t)x \in \mathcal{A}U(t)x = \lambda U(t)x - (\lambda - \mathcal{A})U(t)x. \tag{2.10}$$

Now we show that $R(\lambda)U(t)x$ and $U(t)R(\lambda)x$ are solutions of the problem (2.3) with initial value $R(\lambda)x, x \in D(\mathcal{A}^n)$. Let $y = R(\lambda)x \in D(\mathcal{A}^{n+1})$ for $x \in D(\mathcal{A}^n)$. Then by the well–posedness of (2.3) we have $U'(t)R(\lambda)x \in \mathcal{A}U(t)R(\lambda)x$. So $U(t)R(\lambda)x$ is a solution of the Cauchy problem satisfying the estimate

$$\|U(t)R(\lambda)x\| \leq C\exp(\omega t)\|R(\lambda)x\|_n. \tag{2.11}$$

Since

$$\{y \in X : R^{n-1}(\lambda)y = x\} \subseteq \{y \in X : R^n(\lambda)y = R(\lambda)x\},$$

we have

$$\begin{aligned} \|R(\lambda)x\|_n &= \inf\{\|y\| : R^n(\lambda)y = R(\lambda)x\} \\ &\leq \inf\{\|y\| : R^{n-1}(\lambda)y = x\} = \|x\|_{n-1}. \end{aligned}$$

So the estimate in (2.11) becomes

$$\|U(t)R(\lambda)x\| \leq C\exp(\omega t)\|x\|_{n-1}.$$

On the other hand, noting that $R(\lambda)\mathcal{A}$ is a single–valued operator on $D(\mathcal{A})$ (see Theorem 2.10), and applying $R(\lambda)$ to (2.10), we obtain

$$R(\lambda)U'(t)x = R(\lambda)\mathcal{A}U(t)x = \lambda R(\lambda)U(t)x - U(t)x, \ x \in D(\mathcal{A}^{n+1}), \tag{2.12}$$

and since $U(t)x \in D(\mathcal{A})$, we have

$$U(t)x = R(\lambda)(\lambda - \mathcal{A})U(t)x \in (\lambda - \mathcal{A})R(\lambda)U(t)x.$$

Hence

$$R(\lambda)\mathcal{A}U(t)x \in \mathcal{A}R(\lambda)U(t)x.$$

Therefore,

$$R(\lambda)U'(t)x = (R(\lambda)U(t)x)' \in \mathcal{A}R(\lambda)U(t)x \quad \text{and} \quad R(\lambda)U(0)x = R(\lambda)x.$$

Thus $R(\lambda)U(t)x$ is a solution of the inclusion problem (2.3) with respect to the initial value $R(\lambda)x \in D(\mathcal{A}^{n+1})$. The equality $R(\lambda)U(t)x = U(t)R(\lambda)x$ follows from the uniqueness of the solution. By integrating we obtain from equation (2.12)

$$R(\lambda)U(t)x - R(\lambda)x = \int_0^t \lambda R(\lambda)U(s)xds - \int_0^t U(s)xds.$$

Writing $U_1(t)x := \int_0^t U(s)xds$ for $x \in D(\mathcal{A}^{n+1})$, we have

$$U_1(t)x = -R(\lambda)U(t)x + R(\lambda)x + \lambda \int_0^t R(\lambda)U(s)xds.$$

Define $U_1$ on $D(\mathcal{A}^n)$ by

$$U_1(t)x := -U(t)R(\lambda)x + R(\lambda)x + \lambda \int_0^t U(s)R(\lambda)xds.$$

Applying $R(\lambda)$ to both sides of this equation, we have

$$R(\lambda)U_1(t)x = -R(\lambda)U(t)R(\lambda)x + R(\lambda)R(\lambda)x + \lambda R(\lambda) \int_0^t U(s)R(\lambda)xds.$$

Using the commutativity of $R(\lambda)$ and $U(t)$ we obtain $R(\lambda)U_1(t)x = U_1(t)R(\lambda)x$. Hence $U_1(t)$ is defined and commutes with $R(\lambda)$ in $D(\mathcal{A}^n)$ and satisfies the estimate

$$\|U_1(t)x\| \leq C_1 \exp(\omega t)\|x\|_{n-1}.$$

By this estimate, $U_1(t)$ can be extended to $D_n$, the closure of $D(\mathcal{A}^n)$ endowed with the $\|\cdot\|_{n-1}$-norm. In general, for $x \in D(\mathcal{A}^{n+1-k})$ we define

$$U_k(t)x := -U_{k-1}(t)R(\lambda)x + \frac{t^{k-1}}{(k-1)!}R(\lambda)x + \lambda \int_0^t U_{k-1}(s)R(\lambda)xds. \qquad (2.13)$$

Then we have

$$U_k(t)x = \int_0^t U_{k-1}(s)xds, \ k = 2, 3, \ldots, n, \ x \in D(\mathcal{A}^{n+2-k})$$

and

$$U_k(t)x = -R(\lambda)U_{k-1}(t)x + \frac{t^{k-1}}{(k-1)!}R(\lambda)x + \lambda \int_0^t R(\lambda)U_{k-1}(s)xds.$$

Similarly, we have $R(\lambda)U_k(t)x = U_k(t)R(\lambda)x$, $x \in D(\mathcal{A}^{(n+1)-k})$. Hence $U_k(t)$ is defined and commutes with $R(\lambda)$ in $D(\mathcal{A}^{(n+1)-k})$ and is satisfying the estimate

$$\|U_k(t)x\| \leq C_k t^{k-1} \exp(\omega t)\|x\|_{n-k} \leq C_k \exp(\tilde{\omega}t)\|x\|_{n-k}$$

for any real number $\tilde{\omega}$ which is large enough. Therefore, we can extend $U_k(t)$ to $D_{(n+1)-k}$, the closure of $D(\mathcal{A}^{(n+1)-k})$ endowed with the $\|\cdot\|_{n-k}$-norm. In particular, $U_n(t)$ is defined and commutes with $R(\lambda)$ on $D(\mathcal{A})$. It is satisfying the estimate

$$\|U_n(t)x\| \leq C \exp(\omega' t)\|x\|,$$

and may be continued to $\overline{D(A)}$; $U_{n+1}(t)$ is defined and commutes with $R(\lambda)$ on the whole space $X$ and satisfies

$$\|U_{n+1}(t)x\| \leq C' \exp{(\omega' t)}\|x\|. \tag{2.14}$$

Now we denote $V(t)x := U_{n+1}(t)x$. We show that $V(t)$ is the exponetially bounded $(n+1)$–times integrated semigroup with the generator $\mathcal{A}$. From the definition of $V(t) = U_{n+1}(t) = \int_0^t U_n(s)ds$ we have $V(0) = 0$. The equation (2.13) implies that for any $x \in X$ the map $t \mapsto V(t)x$ is continuous for $t \geq 0$, so property (ii) of Definition 2.21 holds. Property (iii) of Definition 2.21 follows directly from (2.14). To show that $V(t)$ satisfies property (i) of Definition 2.21 it is sufficient to show that

$$R(\lambda)x = \int_0^\infty \lambda^{n+1} \exp{(-\lambda t)} V(t)x dt \tag{2.15}$$

for all $\lambda \in \mathbb{C}$ with Re$\lambda > \omega$. Multiplying (2.13) by $\lambda^k \exp{(-\lambda t)}$ for $k = n+1$ and integrating the equation we get

$$\int_0^\infty \lambda^{n+1} \exp{(-\lambda t)} U_{n+1}(t)x dt = R(\lambda)x = \int_0^\infty \lambda^{n+1} \exp{(-\lambda t)} V(t)x dt.$$

The above equation holds for all $\lambda$ from some open set where $R(\lambda)$ exists by the assumption. Since $\lambda \mapsto R(\lambda)$ is holomorphic on $\rho(\mathcal{A})$ (see Theorem 2.9) and the map

$$\lambda \mapsto \int_0^\infty \lambda^n \exp{(-\lambda t)} V(t)x dt$$

is defined and holomorphic in $\{\lambda \in \mathbb{C} : \text{Re}\lambda > \omega\}$, the equation (2.15) can be extended analytically to the half plane $\{\lambda \in \mathbb{C} : \text{Re}\lambda > \omega\}$. Therefore $V(t)$ is the degenerate exponetially bounded $(n+1)$–times integrated semigroup generated by $\mathcal{A}$. Hence the estimate (2.8) follows by differentiating equation (2.15). ■

From the proof above, we see that $U_n(t)$ commutes with $R(\lambda)$ and is defined on $\overline{D(\mathcal{A})}$. Similarly, it can be shown that $U_n(t)$ forms a degenerate $n$–times integrated semigroup with the generator $\mathcal{A}$ on $\overline{D(\mathcal{A})}$. Thus, on this set, $R(\lambda) = (\lambda - \mathcal{A})^{-1}$ satisfies the estimates

$$\left\|\frac{d^k}{d\lambda^k} \frac{(\lambda - \mathcal{A})^{-1}}{\lambda^n}\right\| \leq \frac{Mk!}{(\text{Re}\lambda - \omega)^{k+1}}, \quad k = 0, 1, 2, \ldots, \quad \text{Re}\lambda > \omega$$

for an $M > 0$ and $\omega \in \mathbb{R}$.

In the following theorem, we show that if the (MFPHY) estimates hold then the Cauchy problem (2.3) is $(n, \omega)$–well–posed on a subset

$$E_1 = R^{n+1}(\lambda)\overline{D(\mathcal{A})} \subset E = D(\mathcal{A}^{n+1}) = R^{n+1}(\lambda)X.$$

Notice that the power of the denominator in the left–hand side of the (MFPHY) type condition is $n$, instead of $(n+1)$ in Theorem 2.28. In the single–valued case, a power $n$ is already sufficient for the $(n, \omega)$–well–posedness on $D(A^{n+1})$ in the case when $\overline{D(A)} = X$.

**Theorem 2.29** *Let $\mathcal{A}$ be a closed linear multi–valued operator on $X$. Suppose that the condition*

$$\|\frac{d^k}{d\lambda^k}\frac{(\lambda - \mathcal{A})^{-1}}{\lambda^n}\| \le \frac{Mk!}{(\mathrm{Re}\lambda - \omega)^{k+1}}, \quad k = 0, 1, 2, \ldots, \quad \lambda \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\lambda > \omega \tag{2.16}$$

*holds for some $M > 0$ and $\omega \in \mathbb{R}$. Then the Cauchy problem for the inclusion (2.3) is $(n, \omega)$–well–posed on $E_1 = R^{n+1}(\lambda)\overline{D(\mathcal{A})}$.*

**Proof.** If the estimate (2.16) holds for some $M > 0$ and $\omega \in \mathbb{R}$, then, by Theorem 2.25, $\mathcal{A}$ is the generator of an $(n+1)$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$ with the property

$$\limsup_{h \to 0+} \frac{1}{h}\|V(t+h) - V(t)\| \le M \exp(\omega t), \quad t \ge 0. \tag{2.17}$$

By property (ii) of Proposition 2.26, for all $x \in D(\mathcal{A})$

$$V(t)x = \frac{t^{n+1}}{(n+1)!}x + \int_0^t V(s)\mathcal{A}x ds$$

and by property (iii) of the same proposition, for all $x \in X$ holds

$$V(t)x \in \frac{t^{n+1}}{(n+1)!}x + \mathcal{A}\int_0^t V(s)x ds.$$

Hence

$$V'(t)x = \frac{t^n}{n!}x + V(t)\mathcal{A}x, \; V'(t)x \in \frac{t^n}{n!}x + \mathcal{A}V(t)x, \quad x \in D(\mathcal{A}). \tag{2.18}$$

Using the property (2.17) and closedness of $\mathcal{A}$ we can extend the inclusion in (2.18) to $\overline{D(\mathcal{A})}$. Now we take $x \in D(\mathcal{A})$ such that $\mathcal{A}x \subset D(\mathcal{A})$, i.e $x \in D(\mathcal{A}^2)$. Then from (2.18) we have

$$\mathcal{A}V'(t)x = \frac{t^n}{n!}\mathcal{A}x + \mathcal{A}V(t)\mathcal{A}x, \quad x \in D(\mathcal{A}^2),$$

$$V'(t)\mathcal{A}x \in \frac{t^n}{n!}\mathcal{A}x + V(t)\mathcal{A}^2 x \subset \frac{t^n}{(n)!}\mathcal{A}x + \mathcal{A}V(t)\mathcal{A}x, \quad x \in D(\mathcal{A}^2).$$

Therefore, $V'(t)\mathcal{A}x \in \mathcal{A}V'(t)x$ for $x \in D(\mathcal{A}^2)$ and also

$$V''(t)x = \frac{t^{n-1}}{(n-1)!}x + V'(t)\mathcal{A}x, \quad V''(t)x \in \frac{t^{n-1}}{(n-1)!}x + \mathcal{A}V'(t)x. \tag{2.19}$$

Now we show that (2.19) also holds for $x \in R(\lambda)\overline{D(\mathcal{A})}$ . Let $y \in \overline{D(\mathcal{A})}$, then there exists a sequence $y_k \in D(\mathcal{A})$ such that $y_k \to y$. Take $x_k = R(\lambda)y_k$. Then $x_k \in R(\lambda)D(\mathcal{A})$ and

$$x_k \to R(\lambda)y = x \in R(\lambda)\overline{D(\mathcal{A})}.$$

Moreover,

$$\mathcal{A}x_k = \mathcal{A}R(\lambda)y_k = -(\lambda - \mathcal{A})R(\lambda)y_k + \lambda R(\lambda)y_k,$$

and the sets $\{-(\lambda - \mathcal{A})R(\lambda)y_k\}$ contain the points $\{-y_k\}$ which converges to $-y$. Therefore, $-y_k + \lambda R(\lambda)y_k \in D(\mathcal{A})$ converges to $-y + \lambda R(\lambda)y \in \overline{D(\mathcal{A})}$. Hence,

$$V'(t)\mathcal{A}x_k \to V'(t)\mathcal{A}x,$$

as $V'(t)$ is bounded (property (2.17)). Since

$$\frac{t^{n-1}}{(n-1)!}x_k \to \frac{t^{n-1}}{(n-1)!}x$$

and $V''(t)$ is closed, we have

$$V''(t)x = \frac{t^{n-1}}{(n-1)!}x + V'(t)\mathcal{A}x, \quad x \in R(\lambda)\overline{D(\mathcal{A})}. \tag{2.20}$$

From the inclusion in (2.19), the closedness of $\mathcal{A}$, and the boundedness of $V'(t)$, we obtain

$$V''(t)x \in \frac{t^{n-1}}{(n-1)!}x + \mathcal{A}V'(t)x, \quad x \in R(\lambda)\overline{D(\mathcal{A})}.$$

Now take $x \in D(\mathcal{A})$ such that $\mathcal{A}x \subset R(\lambda)\overline{D(\mathcal{A})}$, i.e $x \in R^2(\lambda)\overline{D(\mathcal{A})}$. For such $x$ there exists $V''(t)\mathcal{A}x$. Therefore, differentiating equation (2.20), we have

$$V^{(3)}(t)x = \frac{t^{n-2}}{(n-2)!}x + V''(t)\mathcal{A}x \in \frac{t^{n-2}}{(n-2)!}x + \mathcal{A}V''(t)x, \quad x \in R^2(\lambda)\overline{D(\mathcal{A})}.$$

We continue the process and obtain

$$\begin{aligned} V^{(n+2)}(t)x &\in \mathcal{A}V^{(n+1)}(t)x, \quad x \in R^{n+1}(\lambda)\overline{D(\mathcal{A})}, \\ V^{(n+1)}(0)x &= x. \end{aligned}$$

Therefore, $V^{(n+1)}(t)x$ is a solution of the Cauchy problem (2.3) for any initial value $x \in R^{n+1}(\lambda)\overline{D(\mathcal{A})}$. Now we show that this solution is unique. Let $y(t)$ be another solution of (2.3). Then we have

$$\begin{aligned} \int_0^\infty \lambda \exp(-\lambda t) y(t) dt &= x + \int_0^\infty \exp(-\lambda t) y'(t) dt \\ &\in x + \mathcal{A} \int_0^\infty \exp(-\lambda t) y(t) dt, \end{aligned}$$

hence,

$$x \in (\lambda - \mathcal{A}) \int_0^\infty \exp(-\lambda t) y(t) dt.$$

Applying $R(\lambda)$ to this inclusion, we obtain

$$R(\lambda)x = \int_0^\infty \exp(-\lambda t) y(t) dt,$$

or

$$\int_0^\infty \lambda^{n+1} \exp(-\lambda t) V(t) x dt = \int_0^\infty \exp(-\lambda t) V^{(n+1)}(t) x dt = \int_0^\infty \exp(-\lambda t) y(t) dt.$$

Therefore, $V^{(n+1)}(t)x = y(t)$. We have shown that, if there exists a $k$–times integrated semigroup, then any solution of the Cauchy problem (2.3) coincides with the $k$–th derivative of the integrated semigroup.
Similarly to [31] and [42], from the existence of a unique solution we automatically obtain the estimate of the solution. So, the Cauchy problem (2.3) is (n,$\omega$)–well–posed on $R^{n+1}(\lambda)\overline{D(\mathcal{A})}$. ■

Now compare the obtained results with the results on the uniform well–posedness of the Cauchy problem (2.3). Here, uniformly well–posed means $(0, \omega)$–well–posed. We cite a corresponding statement in [48].

**Theorem 2.30** *Let $\mathcal{A}$ be a closed linear multi–valued operator on $X$ and let $\rho(\mathcal{A}) \neq \emptyset$. Let $X_1 := \overline{D(\mathcal{A})}$, and $\tilde{\mathcal{A}}u := \mathcal{A}u \cap X_1$ where $D(\tilde{\mathcal{A}}) := \{u \in X : \tilde{\mathcal{A}}u \neq \emptyset\}$. Then* *the following statements are equivalent:*

*(i) The Cauchy problem (2.3) is uniformly wellposed on* $D(\mathcal{A})$.

*(ii)* $D(\tilde{\mathcal{A}}) = D(\mathcal{A})$, *and the operator* $\tilde{\mathcal{A}}$ *is a single–valued and generator of a strongly continuous semigroup* $\{U(t) \in \mathcal{L}(X) : t \geq 0\}$ *on* $X_1$.

*(iii) The decomposition*

$$X = X_1 \oplus \mathcal{A}0 \tag{2.21}$$

*and* $(MFPHY)$ *type condition*

$$\|\frac{d^k}{d\lambda^k}(\lambda - \tilde{\mathcal{A}})^{-1}\| \leq \frac{Kk!}{(Re\lambda - \omega)^{k+1}}, \quad k = 0, 1, 2, \ldots, \quad \lambda \in \mathbb{C} \quad \text{with} \quad \text{Re}\lambda > \omega \tag{2.22}$$

*holds for a* $K > 0$ *and an* $\omega \in \mathbb{R}$.

*In this case for any* $x \in D(\mathcal{A})$, $u(t) = U(t)x$ *is the unique solution of (2.3).*

The decomposition (2.21) generalizes the property of the generator of a strongly continuous semigroup to be densely defined to the degenerate case. From the theorem above we have a projector $P = U(0) : X \to X_1 = \overline{D(\mathcal{A})}$ such that $X_1 = PX$, $\mathcal{A}0 = \ker P$. Due to the decomposition (2.21), the estimate (2.22) obtained for $(\lambda - \tilde{\mathcal{A}})^{-1}$ on $X_1$ may be written as the estimate for $(\lambda - \mathcal{A})^{-1}$ on $X$.

In our case, since we have not introduced a decomposition of $X$, the $(MFPHY)$ type estimates for $\frac{(\lambda-\mathcal{A})^{-1}}{\lambda^n}$ obtained in Theorem 2.28 only holds on $\overline{D(\mathcal{A})}$. On $X$ we obtained the estimates for $\frac{(\lambda-\mathcal{A})^{-1}}{\lambda^{n+1}}$. On the other hand, similar to the property that for all $x \in D(\mathcal{A})$, $\lambda R(\lambda)x \to x$, [31], in the case that $\mathcal{A}$ satisfies (2.16) with $n = 0$, we can prove the following proposition. It justifies a generalization of the decomposition (2.21).

**Proposition 2.31** *Let* $\mathcal{A} \in \mathcal{M}(X)$ *satisfy (2.16), then for all* $x \in D(\mathcal{A}^{n+1}) = R^{n+1}(\lambda)X$ *holds*

$$\lambda R(\lambda)x \to x, \quad \text{as} \quad \lambda \to \infty.$$

**Proof.** Let $x \in D(\mathcal{A}^{n+1})$, then there exists $y \in X$ such that $x = (\lambda_0 - \mathcal{A})^{-(n+1)}y$ for some $\lambda_0 \in \rho(\mathcal{A})$ and

$$\begin{aligned}
& \|\lambda(\lambda - \mathcal{A})^{-1}x - x\| = \|\lambda(\lambda - \mathcal{A})^{-1}\left((\lambda_0 - \mathcal{A})^{-1}\right)^{n+1} y - \left((\lambda_0 - \mathcal{A})^{-1}\right)^{n+1} y\| \\
= \; & \|\lambda(\lambda_0 - \lambda)^{-1}(\lambda - \mathcal{A})^{-1}\left((\lambda_0 - \mathcal{A})^{-1}\right)^{n} y - \lambda_0(\lambda_0 - \lambda)^{-1}\left((\lambda_0 - \mathcal{A})^{-1}\right)^{n+1} y\| \\
& \vdots \\
= \; & \|\lambda(\lambda_0 - \lambda)^{-(n+1)}(\lambda - \mathcal{A})^{-1}y - \cdots - \lambda_0(\lambda_0 - \lambda)^{-1}\left((\lambda_0 - \mathcal{A})^{-1}\right)^{n+1} y\| \\
\to \; & 0, \quad \text{as} \quad \lambda \to \infty.
\end{aligned}$$

■

It follows $\ker R(\lambda) \cap D(\mathcal{A}^{n+1}) = \{0\}$ and $\ker R(\lambda) \cap \overline{D(\mathcal{A}^{n+1})} = \{0\}$, i.e

$$\ker R(\lambda) \cap \overline{R^{n+1}(\lambda)X} = \{0\}.$$

We just proved that the equality above is a consequence of the (MFPHY) type condition (2.16). In the sequel we suppose more namely that the kernel and the closure of the range of the resolvent power span the whole space. Here we will assume that we have the decomposition

$$X = \overline{R^{n+1}(\lambda)X} \oplus \ker R^{n+1}(\lambda) = \overline{D(\mathcal{A}^{n+1})} \oplus \mathcal{A}^{n+1}0, \tag{2.23}$$

which generalizes (2.21) for the case of a degenerate $n$–times integrated semigroup.

**Remark 2.32** *It is shown in Lemma 2.27 that $R^{n+1}(\lambda)X$ does not depend on the choice of $\lambda \in \rho(\mathcal{A})$. Using the same technique, the same conclusion holds for $\ker R^{n+1}(\lambda)$. Therefore the decomposition (2.23) does not depend on the choice of $\lambda \in \rho(\mathcal{A})$.*

With the decomposition (2.23), we can state necessary and sufficient conditions for the $(n, \omega)$–well–posedness of the Cauchy problem (2.3) on $E = D(\mathcal{A}^{n+1})$.

**Theorem 2.33** *Let $\mathcal{A}$ be a closed linear multi–valued operator on $X$ such that $\rho(\mathcal{A}) \neq \emptyset$ and let the decomposition (2.23) hold. Then the Cauchy problem (2.3)* *is* $(n,\omega)$*-well-posed on* $E = D(\mathcal{A}^{n+1})$ *if and only if the condition*

$$\|\frac{d^k}{d\lambda^k}\frac{(\lambda-\mathcal{A})^{-1}}{\lambda^n}\| \leq \frac{Mk!}{(Re\lambda-\omega)^{k+1}}, \quad k=0,1,2,\ldots \quad \lambda\in\mathbb{C} \quad \text{with} \quad \mathrm{Re}\lambda>\omega \tag{2.24}$$

*holds for some* $M > 0$ *and* $\omega \in \mathbb{R}$.

**Proof.** Suppose the Cauchy problem (2.3) is $(n,\omega)$–well-posed on $E = D(\mathcal{A}^{n+1})$. Let $t \mapsto u(t)$ be the unique solution corresponding to the initial value $x \in D(\mathcal{A}^{n+1})$. Hence $u(t)$ satisfies the stability type estimate

$$\|u(t)\| \leq C\exp(\omega t)\|x\|_n. \tag{2.25}$$

We define the corresponding solution operator $U(t)$ by

$$U(t)x := \begin{cases} u(t), & x \in D(\mathcal{A}^{n+1}) \\ 0, & x \in \mathcal{A}0. \end{cases}$$

Due to estimate (2.25), $U(t)$ can be extended to $D_{n+1} \oplus \mathcal{A}0$, where $D_{n+1}$ is the closure of $D(\mathcal{A}^{n+1})$ endowed with the $\|\cdot\|_n$–norm. For any $x \in D_{n+1} \oplus \mathcal{A}0$, similar to the proof of Theorem 2.28, we have

$$(R(\lambda)U(t)x)' = R(\lambda)U'(t)x = \lambda R(\lambda)U(t)x - U(t)x. \tag{2.26}$$

It is shown in Theorem 2.28 that $R(\lambda)U(t)x = U(t)R(\lambda)x, x \in D(\mathcal{A}^n)$ is the solution of the problem (2.3) with the estimate

$$\|R(\lambda)U(t)x\| \leq C\exp(\omega t)\|x\|_{n-1}. \tag{2.27}$$

Similarly, $R(\lambda)U(t)x = U(t)R(\lambda)x$ for any $x \in \mathcal{A}0$. The estimate (2.27) also holds for $x \in \mathcal{A}0$. Thus $R(\lambda)U(t)x = U(t)R(\lambda)x$ for any $x \in D(\mathcal{A}^{n+1}) \oplus \mathcal{A}0$ and estimate (2.27) holds. By integrating equation (2.26) we obtain

$$R(\lambda)U(t)x - R(\lambda)x = \int_0^t \lambda R(\lambda)U(s)xds - \int_0^t U(s)xds.$$

Denoting $U_1(t)x := \int_0^t U(s)xds$ we have

$$U_1(t)x \quad = \quad -U(t)R(\lambda)x + R(\lambda)x + \lambda\int_0^t U(s)R(\lambda)xds$$

and $U_1(t)$ satisfying the estimate

$$\|U_1(t)x\| \leq C_1 \exp(\omega t)\|x\|_{n-1}.$$

Hence $U_1(t)$ is defined on $D(\mathcal{A}^n) \oplus \mathcal{A}0$. The commutativity of $U(t)$ and $R(\lambda)$ implies

$$U_1(t)R(\lambda)x = R(\lambda)U_1(t)x$$

and by the estimate above, $U_1(t)$ can be extended to the closure of $D(\mathcal{A}^n)$ endowed with the $\|\cdot\|_{n-1}$–norm and also can be defined on $\mathcal{A}^2 0$. For $k = 2, 3, \ldots, n$, writing

$$U_k(t)x := -U_{k-1}(t)R(\lambda)x + \frac{t^{k-1}}{(k-1)!}R(\lambda)x + \lambda \int_0^t U_{k-1}(s)R(\lambda)xds,$$

we have

$$U_k(t)x \quad = \quad -R(\lambda)U_{k-1}(t)x + \frac{t^{k-1}}{(k-1)!}R(\lambda)x + \lambda \int_0^t R(\lambda)U_{k-1}(s)xds.$$

Hence $U_k(t)x$ are defined for $x \in D\left(\mathcal{A}^{n+1-k}\right)$ and exponentially bounded by norm $\|x\|_{n-k}$, it can be extended to $D_{n+1-k}$ and $\mathcal{A}^k 0$. $U_n(t)$ is defined and bounded on $\overline{D(\mathcal{A})}$ and $\mathcal{A}^{n+1}0$, hence on $X$. Now denote $V(t) := U_n(t)x$. Similar to the proof of Theorem 2.28, it can be shown that $V(t)$ is an $n$–times integrated semigroup with generator $\mathcal{A}$. It is degenerate on $\mathcal{A}0$. Hence, it holds

$$R(\lambda)x = \int_0^t \lambda^{n+1} \exp(-\omega t)V(t)xdt$$

for all $\lambda \in \mathbb{C}$ with Re $\lambda > \omega$. The estimate (2.24) follows by differentiating this equation $k$–times.

Suppose that the estimate (2.24) holds. Then by Theorem 2.29, the Cauchy problem (2.3) is $(n, \omega)$–well–posed on $R^{n+1}(\lambda)\overline{D(\mathcal{A})} \subset D(\mathcal{A}^{n+1})$. By the decomposition (2.23),

$$D\left(\mathcal{A}^{n+1}\right) = R^{n+1}(\lambda)X = R^{n+1}(\lambda)\overline{R^{n+1}(\lambda)X}.$$

On the other hand, we have

$$R^{n+1}(\lambda)\overline{R^{n+1}(\lambda)X} \subset R^{n+1}(\lambda)\overline{D(\mathcal{A})},$$

and hence $D(\mathcal{A}^{n+1}) \subset R^{n+1}(\lambda)\overline{D(\mathcal{A})}$. So $R^{n+1}(\lambda)\overline{D(\mathcal{A})} = D(\mathcal{A}^{n+1})$. Therefore, the Cauchy problem for the inclusion (2.3) is $(n,\omega)$–well–posed on $D(\mathcal{A}^{n+1})$. ■

Bearing in mind Theorem 2.33, we now consider the abstract Cauchy problem (2.1). Previously, we define the resolvent $R(\lambda) := (\lambda - \mathcal{A})^{-1}$. By this definition, with the operator $\mathcal{A} = B^{-1}A$ we have

$$R(\lambda) = \left(\lambda - B^{-1}A\right)^{-1} = \left(B^{-1}B\left(\lambda - B^{-1}A\right)\right)^{-1} = (\lambda B - A)^{-1}B =: R_1(\lambda).$$

By assuming that the resolvent set

$$\rho_1(A,B) := \{\lambda \in \mathbb{C} : R_1(\lambda) = (\lambda B - A)^{-1}B \in \mathcal{L}(X)\}$$

is not empty and that the decomposition

$$X = \overline{R_1^{n+1}(\lambda)X} \oplus \ker R_1^{n+1}(\lambda) \tag{2.28}$$

is valid. Theorem 2.33 leads to the following corollary.

**Corollary 2.34** *Consider the Cauchy problem (2.1) with linear operators $A$ and $B$ such that $B^{-1}A$ is closed. Let $\rho_1(A,B) \neq \emptyset$ and the decomposition (2.28) hold. Then the Cauchy problem (2.1) is $(n,\omega)$–well–posed on $R_1^{n+1}(\lambda)X$ if and only if the condition*

$$\left\|\frac{d^k}{d\lambda^k}\frac{R_1(\lambda)}{\lambda^n}\right\| \leq \frac{Mk!}{(Re\lambda - \omega)^{k+1}}, \quad k = 0,1,2,\ldots, \quad \lambda \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\lambda > \omega \tag{2.29}$$

*holds for some $M > 0$ and $\omega \in \mathbb{R}$.*

Similarly, for the abstract Cauchy problem (2.2), we consider the resolvent $R_2(\lambda)$ for the multi–valued operator $\mathcal{A} = AB^{-1}$:

$$R(\lambda) = \left(\lambda - AB^{-1}\right)^{-1} = \left(\left(\lambda - AB^{-1}\right)BB^{-1}\right)^{-1} = ((\lambda B - A)B^{-1})^{-1} = B(\lambda B - A)^{-1}.$$

Define $R_2(\lambda) := B(\lambda B - A)^{-1}$. Using the assumption of the non–emptyness of the corresponding resolvent set

$$\rho_2(A,B) := \{\lambda \in C : R_2(\lambda) = B(\lambda B - A)^{-1} \in \mathcal{L}(X)\}$$

and assuming that $X$ can be decomposed as follows,

$$X = \overline{R_2^{n+1}(\lambda)X} \oplus \ker R_2^{n+1}(\lambda) \tag{2.30}$$

we state the following result.

**Corollary 2.35** *Consider the Cauchy problem (2.2) with linear operators $A$ and $B$ such that $AB^{-1}$ is closed. Let $\rho_2(A, B) \neq \emptyset$ and the decomposition (2.30) hold. Then the Cauchy problem (2.2) is $B-(n, \omega)$-well-posed on $R_2^{n+1}(\lambda)X$ (that is, there exists a solution $v$ such that $Bv$ is unique and $n$-stable) if and only if the condition*

$$\|\frac{d^k}{d\lambda^k}\frac{R_2(\lambda)}{\lambda^n}\| \leq \frac{Mk!}{(Re\lambda - \omega)^{k+1}}, \quad k = 0, 1, 2, \ldots, \quad \lambda \in \mathbb{C} \quad \text{with} \quad \text{Re}\lambda > \omega \tag{2.31}$$

*holds for some $M > 0$ and $\omega \in \mathbb{R}$.*

# Chapter 3

# Degenerate Abstract Cauchy Problems

In this chapter, we reconsider the degenerate first order abstract Cauchy problem (2.1):

$$Bu'(t) = Au(t), \quad t \geq 0, \quad u(0) = x$$

in a complex Banach space $X$, where $A$ and $B$ are linear operators in $X$. As in the previous chapter, we investigate the degenerate case that $\ker B \neq \{0\}$ and define the resolvent

$$R_1(\lambda) := (\lambda B - A)^{-1} B$$

and the resolvent set

$$\rho_1(A, B) := \{\lambda \in \mathbb{C} : R_1(\lambda) := (\lambda B - A)^{-1} B \in \mathcal{L}(X)\}.$$

Throughout the whole chapter, we assume that $\rho_1(A, B) \neq \emptyset$ and that the space $X$ can be decomposed as

$$X = \overline{R_1^{n+1}(\lambda)X} \oplus \ker R_1^{n+1}(\lambda). \tag{3.1}$$

Recall that this decomposition does not depend on the choice of $\lambda \in \rho_1(A, B)$ (see Remark 2.32). In order to simplify the notation, we shortly write

$$X_{n+1} := \overline{R_1^{n+1}(\lambda)X} \text{ and } K_{n+1} := \ker R_1^{n+1}(\lambda)$$

Thus, the decomposition above can be written as

$$X = X_{n+1} \oplus K_{n+1}. \tag{3.2}$$

We investigate conditions for the well–posedness of the Cauchy problem (2.1). In contrast to the previous chapter, in this chapter we do not investigate the inclusion problem. Instead, we work with single–valued branches of the corresponding multi–valued operator $\mathcal{A} := B^{-1}A$. Furthermore, here we are interested in the well–posedness in the sense of distributions of exponential growth. For this purpose, we make use of the theory of degenerate distribution semigroups and relate them to degenerate $n$–times integrated semigroups.

This chapter is divided into four sections. In Section 1 we discuss the algebraic structure of the decomposition (3.2), and introduce the single–valued branch of the multi–valued operator $B^{-1}A$. In Section 2 we collect basic properties of distribution semigroups and their generators. In Section 3 we give necessary and sufficient conditions for the well–posedness of the degenerate Cauchy problem (2.1) in the sense of distributions of exponential growth. Finally, in the last section, we relate distribution semigroups and $n$–times integrated semigroups.

## 3.1 Preliminaries

We consider the operator

$$B^{-1}A : D(B^{-1}A) \subset X \to X.$$

Since $\ker B \neq \{0\}$, this operator is multi–valued. Nevertheless, it is possible to relate single–valued operators to $B^{-1}A$.

**Lemma 3.1** *Let $H \subset X$ be a subspace with $X = H \oplus \ker B$. Then the operator $\mathcal{A}_s : D(B^{-1}A) \to X$ defined by*

$$\mathcal{A}_s x := B^{-1}Ax \cap H$$

*is linear and single–valued. Moreover, if $B^{-1}A$ is closed and $H \subset X$ is closed, then $\mathcal{A}_s$ is a closed operator.*

**Proof.** Obviously $\mathcal{A}_s \in \mathcal{M}(X)$. $\mathcal{A}_s$ is single–valued since

$$\begin{aligned}\mathcal{A}_s(0) &= B^{-1}A(0) \cap H \\ &= B^{-1}(0) \cap H \\ &= \ker B \cap H = \{0\}.\end{aligned}$$

Let $x_j \in D(B^{-1}A)$ with $x_j \to x$ in $X$, and let $\mathcal{A}_s x_j \to y$ in $X$. By the closedness of $B^{-1}A$, we see that $x \in D(B^{-1}A)$ and $y \in B^{-1}Ax$. By the closedness of $H \subset X$, we conclude $y \in H$. Thus $y = \mathcal{A}_s x$. Hence $\mathcal{A}_s$ is a closed operator. ■

Later, the decomposition (3.2) will motivate the choice of a specific subspace $H \subset X$, defining the single–valued branch of $B^{-1}A$. In the following, we discuss the algebraic structure of $K_{n+1} := \ker R_1^{n+1}(\lambda)$ in the decomposition (3.2). For this purpose, we define the $A$–associated with $\ker B$–vectors.

**Definition 3.2 (i–th $A$–associated with $\ker B$–vectors)** *Define $\tilde{K}^0 := \ker B \backslash \{0\}$ and for $i = 1, 2, 3, \ldots$*

$$\tilde{K}^i := \{x \in X : \text{there is a } y \in \tilde{K}^{i-1} \text{ with } Ay = Bx\} \backslash \{0\}.$$

*We call $\tilde{K}^i$ the set of the i–th A–associated with $\ker B$ vectors.*

It follows from the definition that for any $x_i \in \tilde{K}^i$, there exists a family of $A$–associated with $\ker B$ vectors $x_0, x_1, \ldots, x_{i-1}$ such that $x_k \in \tilde{K}^k$ with $Ax_k = Bx_{k+1}$ for $k = 0, 1, \ldots, i-1$. This family can be used to represent the resolvent operation for $x_i \in \tilde{K}^i$.

**Lemma 3.3** *Let $\lambda \in \rho_1(A, B)$. For any $x_i \in \tilde{K}^i$ hold the relations*

$$R_1(\lambda)x_i = -x_{i-1} - \lambda x_{i-2} - \cdots - \lambda^{i-1}x_0, \tag{3.3}$$

$$R_1^j(\lambda)x_i = (-1)^j \left( \binom{j}{0} x_{i-j} + \binom{j}{1} \lambda x_{i-j-1} + \cdots + \binom{j}{j-1} \lambda^{i-j} x_0 \right) \tag{3.4}$$

*for* $j = 1, 2, \ldots, i$. *In particular we have*

$$R_1^i(\lambda)x_i = (-1)^i x_0. \tag{3.5}$$

**Proof.** We prove (3.3) using the induction principle and noting that $Ax_k = Bx_{k+1}$, $k = 0, 1, 2, \ldots, i-1$.

For $x_1 \in \tilde{K}^1$ we have

$$\begin{aligned} R_1(\lambda)x_1 &= (\lambda B - A)^{-1}Bx_1 = (\lambda B - A)^{-1}Ax_0 \\ &= (\lambda B - A)^{-1}(-(\lambda B - A)x_0 + \lambda Bx_0) \\ &= -x_0 \end{aligned}$$

Now suppose that (3.3) holds for $i-1$, i.e.

$$R_1(\lambda)x_{i-1} = -x_{i-2} - \lambda x_{i-3} - \cdots - \lambda^{i-2}x_0.$$

Then we obtain

$$\begin{aligned} R_1(\lambda)x_i &= (\lambda B - A)^{-1}Bx_i = (\lambda B - A)^{-1}Ax_{i-1} \\ &= (\lambda B - A)^{-1}(-(\lambda B - A)x_{i-1} + \lambda Bx_{i-1}) \\ &= -x_{i-1} + \lambda(\lambda B - A)^{-1}Bx_{i-1} \\ &= -x_{i-1} + \lambda(-x_{i-2} - \lambda x_{i-3} - \cdots - \lambda^{i-2}x_0) \\ &= -x_{i-1} - \lambda x_{i-2} - \lambda^2 x_{i-3} - \cdots - \lambda^{i-1}x_0. \end{aligned}$$

Relation (3.4) is a direct consequence of (3.3). ■

The following proposition describes some properties of the set of the $i$–th $A$–associated with $\ker B$–vectors. It is proved in [47].

**Proposition 3.4** *Let* $i = 0, 1, 2, \ldots$ *and* $\rho_1(A, B) \neq \emptyset$. *Then it holds :*

*(i)* $\ker B \cap \ker A = \{0\}$ *and* $\tilde{K}^i \cap \ker A = \emptyset$ *for* $i \geq 1$.

*(ii)* $\tilde{K}^i \cap \tilde{K}^j = \emptyset$ *for all* $i \neq j$.

*(iii)* $K^i := \tilde{K}^i \cup \{0\}$ *is a closed subspace of* $X$.

*(iv) Let* $\Lambda \subset \mathbb{C}$ *be a domain containing sequences converging to infinity. If there exists* $M > 0$ *such that*

$$||R_1(\lambda)|| \leq M\frac{|\lambda^n|}{|f(\lambda)|}$$

*for* $\lambda \in \Lambda$ *with* $|f(\lambda)| \to \infty$ *for* $\lambda \to \infty$, *then for all* $x \in R_1^{n+1}(\lambda)X$ *we have*

$$\lim_{\lambda \to \infty} \lambda R_1(\lambda)x = x$$

*and*

$$X_{n+1} \cap \ker B = \{0\} \text{ and } \tilde{K}^{n+1} = \emptyset$$

*(v)* $K_{n+1} = \ker B \oplus K^1 \oplus K^2 \oplus \cdots \oplus K^n$.

Of particular interest is statement (v), since it allows to rewrite the decomposition (3.2) as

$$X = \tilde{X} \oplus \ker B \tag{3.6}$$

where

$$\tilde{X} = X_{n+1} \oplus K^1 \oplus K^2 \oplus \cdots \oplus K^n.$$

Having the decomposition (3.6) we define a single–valued branch.

**Definition 3.5** *The linear operator* $\mathcal{A}_s : D(\mathcal{A}_s) = D(B^{-1}A) \to X$ *defined by*

$$\mathcal{A}_s x = B^{-1}Ax \cap \tilde{X}$$

*is called the single–valued branch of the multi–valued operator* $B^{-1}A : D(B^{-1}A) \to X$.

With the operator $\mathcal{A}_s$ as a single–valued branch of the multi–valued operator $B^{-1}A$, the Cauchy problem for inclusion (2.3) with $\mathcal{A} = B^{-1}A$ can be written as the non–degenerate problem

$$u'(t) = \mathcal{A}_s\, u(t), \quad t \geq 0, \quad u(0) = x. \tag{3.7}$$

Since $\tilde{X} \subset X$ is a closed subspace, the single–valued branch is a closed operator if $B^{-1}A$ is a closed multi–valued operator. Later, we will assume that the operator $A$ is closed and $B$ is bounded. By Lemma 2.14, this assumption guarantees that $B^{-1}A$ is closed.

In the sequel we endow the domain

$$D(B^{-1}A) := \{x \in D(A) : \text{there is a } y \in D(B) \text{ with } Ax = By\}$$

with the norm

$$\|x\|_1 := \inf\{\|y\| : R_1(\lambda)y = x\}.$$

Since the operator $\mathcal{A}_s$ is the single–valued branch of the multi–valued operator $B^{-1}A$, the space $D(B^{-1}A)$ is homeomorphic to the space $D(\mathcal{A}_s)$ endowed with $\mathcal{A}_s$–graph norm. Furthermore we define the part of the single–valued branch as follows:

**Definition 3.6** *Let $\mathcal{A}_s$ be the single–valued branch of the multi–valued operator $B^{-1}A$. We define an operator $\tilde{\mathcal{A}}_s : D(\tilde{\mathcal{A}}_s) \to X_{n+1}$ by*

$$\tilde{\mathcal{A}}_s x := \mathcal{A}_s x \quad \text{for} \quad x \in D(\tilde{\mathcal{A}}_s) := \mathcal{A}_s^{-1}(X_{n+1}) \cap X_{n+1}.$$

*This operator is called the part of $\mathcal{A}_s$ in $X_{n+1}$.*

Another way to visualize the idea behind the definition of a part is to consider the graph of $\mathcal{A}_s$. Assuming that $\mathcal{A}_s$ is closed, its graph is closed and we can intersect it with the closed product space $X_{n+1} \times X_{n+1}$ to obtain a new closed linear operator. This new operator is exactly the part of $\mathcal{A}_s$ in $X_{n+1}$.

## 3.2 Distribution Semigroups

In the following we introduce some standard spaces of test functions. We denote by $\mathcal{D}_0$ respectively $\mathcal{E}$ the spaces of infinitely differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$ with compact support in $[0, \infty)$ respectively with any support. The sequence of test functions $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_0$ is convergent to zero in $\mathcal{D}_0$, if (i) there exists a compact set $H \subset [0, \infty)$ such that $\text{supp}(\phi_n) \subseteq H$, for all $n \in \mathbb{N}$, where $\text{supp}(\phi) = \text{clos}\{t \in [0, \infty) : \phi(t) \neq 0\}$ and (ii) for all integer $m \geq 0$, $\phi_n^{(m)} t \to 0$ uniformly for $t \in \mathbb{R}$. We say $\phi_n \to \phi$ in $\mathcal{D}_0$ if $\phi_n - \phi \to 0$ in $\mathcal{D}_0$. The convergence in the space $\mathcal{E}$ is defined in a similar way.

For a Banach space $X$ we denote by $\mathcal{D}_0'(X)$ respectively $\mathcal{E}'(X)$ the spaces of $X$–valued distributions, consisting of all linear operators $U : \mathcal{D}_0 \to X$ respectively $U : \mathcal{E} \to X$, which are continuous in the following sense: If $\phi_n \to 0$ in $\mathcal{D}_0$ respectively in $\mathcal{E}$, then $U(\phi_n) \to 0$ in $X$. For $U \in \mathcal{D}_0'(X)$ respectively $U \in \mathcal{E}'(X)$ and $\phi \in \mathcal{D}_0$ respectively $\phi \in \mathcal{E}$ we write

$$\langle U, \phi \rangle := U(\phi).$$

We say two distributions $U$ and $V$ are equal if

$$U(\phi) = V(\phi)$$

for all test functions of the corresponding space. For a real– or complex–valued distribution $U$ and for an $x \in X$ we denote by $U \otimes x$ the distribution defined by

$$(U \otimes x)(\phi) = U(\phi)x.$$

For any $U \in \mathcal{D}_0'(X)$ we define the derivative of $U$ as the distribution

$$U'(\phi) = -U(\phi').$$

For the convergence in the space of distributions we need to define the notion of bounded set in $\mathcal{D}_0$ respectively in $\mathcal{E}$. A set $\mathcal{F} \subset \mathcal{D}_0$ is bounded if for any sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ and for any sequence of real numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ we have

$$\varepsilon_n \phi_n \to 0 \text{ in } \mathcal{D}_0.$$

Consequently [19], $\mathcal{F} \subset \mathcal{D}_0$ is bounded if and only if (i) there exists a compact subset $F \subset [0, \infty)$ such that

$$\text{supp}(\phi) \subset \text{F}, \text{ for all } \phi \in \mathcal{F}$$

and (ii) for every $m \geq 0$, there exists a constant $M_m > 0$ with

$$|\phi^{(m)}(t)| \leq M_m, \text{ for all } \phi \in \mathcal{F},\ t \in F.$$

A sequence $\{U_n\}_{n\in\mathbb{N}}$ of distributions in $\mathcal{D}_0'(X)$ is said to converge to a distribution $U \in \mathcal{D}_0'(X)$ if

$$U_n(\phi) \to U(\phi)$$

uniformly on bounded subsets of $\mathcal{D}_0$. The boundedness of a subset $\mathcal{F}$ of $\mathcal{E}$ and convergence in $\mathcal{E}'(X)$ are defined in the same way.

A distribution $U \in \mathcal{D}_0'(X)$ is said to vanish in an open subset $\Omega$ of $[0, \infty)$, if $U(\phi) = 0$ for all $\phi \in \mathcal{D}_0$ with $\text{supp}(\phi) \subset \Omega$; we write shortly $U = 0$ in $\Omega$. We denote by $\Omega(U)$, the union of all open subsets of $[0, \infty)$ where $U$ vanishes. We define the support of $U$ as

$$\text{supp(U)} := [0, \infty) \backslash \Omega(\text{U}).$$

Notice that the support of $U$ is a closed subset in $[0, \infty)$.

By the Structure Theorem in ([19], Theorem 8.1.5), for any $U \in \mathcal{D}_0'(X)$ and any open bounded subset $\Omega$ of $\mathbb{R}$ with $\overline{\Omega} \subset [0, \infty)$, there exist a continuous function $f : \mathbb{R} \to X$ and a natural number $m \geq 0$ such that

$$U = f^{(m)} \text{ in } \Omega.$$

We denote by $\mathcal{S}$, the Schwartz space of rapidly decreasing functions consisting of all infinitely differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$t^j \phi^{(k)}(t) \to 0, \text{ as } |t| \to \infty,$$

for all $j$ and $k$ in $\mathbb{N} \cup \{0\}$. A sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ converges to $0 \in \mathcal{S}$ if

$$||\phi_n||_{j,k} \to 0, \text{ as } n \to \infty$$

for all integers $j, k \geq 0$, where

$$||\phi||_{j,k} = \sup_{0 \leq l \leq k} \sup_{-\infty < t < \infty} (1 + |t|)^j |\phi^{(l)}(t)|.$$

We denote $\mathcal{S}'(X)$, the space of tempered $X$–valued distributions consisting of linear operators $U : \mathcal{S} \to X$, which are continuous in the sense:

$$U(\phi_n) \to 0 \text{ in } X, \text{ whenever } \phi_n \to 0 \text{ in } \mathcal{S}.$$

Similarly, a set $\mathcal{F} \subset \mathcal{S}$ is bounded if $\varepsilon_n \phi_n \to 0$ in $\mathcal{S}$ for any sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ and for any real sequence $\varepsilon_n \to 0$. Accordingly, [19], $\mathcal{F}$ is bounded in $\mathcal{S}$ if and only if for any integers $j, k$, there exists a constant $M_{j,k} > 0$ such that

$$\|\phi\|_{j,k} \leq M_{j,k}, \text{ for all } \phi \in \mathcal{F}.$$

A sequence $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{S}'(X)$ converges to $U \in \mathcal{S}'(X)$ if

$$U_n(\phi) \to U(\phi)$$

uniformly on bounded subsets of $\mathcal{S}$. Similarly, the Structure Theorem also holds for distributions in $\mathcal{S}'(X)$ ([19],Theorem 8.2.3), i.e. for any $U \in \mathcal{S}'(X)$ there exist integers $m, r \geq 0$ and a continuous function $f : \mathbb{R} \to X$ such that $U = f^{(m)}$ in $\mathbb{R}$ and $|f(t)| = O(|t|^r)$ for $|t| \to \infty$. We denote by $\mathcal{S}'_\omega(X)$, the space of all distributions $U \in \mathcal{D}'_0(X)$ such that for an $\omega \in \mathbb{R}$ holds,

$$\exp(-\omega t)U \in \mathcal{S}'(X).$$

We call $\mathcal{S}'_\omega(X)$ the space of distributions of exponential growth.

For two distributions $U, V \in \mathcal{D}'_0(X)$ (or in $\mathcal{S}'(X)$), we define the convolution as

$$U * V := (f * g)^{(m+p)},$$

where $f$ and $g$ are the continuous functions such that $f^{(m)}$ and $g^{(p)}$ represent $U$ and $V$ respectively according to the Structure Theorem.

The following definition of distribution semigroups is of fundamental importance since we characterize the well–posedness of the degenerate Cauchy problem (2.1) in the sense of distributions with this semigroups. Originally, non–degenerate distribution semigroups were introduced by Lions [35].

**Definition 3.7 (Distribution semigroup)** *We say a distribution $Q \in \mathcal{D}'_0(\mathcal{L}(X))$ is a distribution semigroup if*

$$\langle Q, \phi * \psi\rangle = \langle Q, \phi\rangle\langle Q, \psi\rangle,$$

*for all $\phi, \psi \in \mathcal{D}_0$.*

*A distribution semigroup is called regular, if for any*

$$y \in \text{ran } Q := \{Q(\phi)x \in X : \phi \in \mathcal{D}_0, \quad x \in X\},$$

*the distribution $Qy$ is equal to a function $t \mapsto u(t)$ such that $u(t) = 0$ for $t < 0$, $t \mapsto u(t)$ is continuous for $t > 0$ and continuous from the right at $t = 0$ with $u(0) = y$.*

*A distribution semigroup is called degenerate on $Z$ if $Q(\phi)x = 0$ for all $\phi \in \mathcal{D}_0$ if and only if $x \in Z$.*

*A distribution semigroup is said to be of exponential growth if there exists $\omega \in \mathbb{R}$ such that* $\exp(-\omega t)Q \in \mathcal{S}'(\mathcal{L}(X))$.

For a real– or complex–valued distribution $E \in \mathcal{E}'$ with $E = 0$ for $t < 0$ and for any $\rho \in \mathcal{D}_0$ we have $E * \rho \in \mathcal{D}_0$, (see [35]).

**Definition 3.8** *We say $x \in D(Q(E))$ if there exists a regularizing sequence $\rho_n \in \mathcal{D}_0$, $\rho_n \to \delta$, such that*

$$\begin{aligned} Q(\rho_n)x &\to x, \\ Q(E * \rho_n)x &\to y =: Q(E)x. \end{aligned}$$

**Lemma 3.9** *[35] Let $Q$ be a distribution semigroup. Then*
*(i) For all $x \in X$, $\phi \in \mathcal{D}_0$ holds $Q(\phi)x \in D(Q(E)^m)$ for any integer $m$ and*

$$Q(E)^m Q(\phi)x = Q(E * \cdots * E * \phi)x.$$

*(ii) $D(Q(E))$ is dense in $X$.*
*(iii) If $x \in D(Q(E))$, then*

$$Q(E * \phi)x = Q(E)Q(\phi)x = Q(\phi)Q(E)x.$$

*(iv) If $x_j \in D(Q(E))$ with $x_j \to 0$ in $X$ and $Q(E)x_j \to y$ in $X$ then $y = 0$. Hence $Q(E)$ is closable. We write $\overline{Q(E)}$, the closed linear extension of $Q(E)$.*
*(v) For $E \in \mathcal{E}'$ with $E = 0$ for $t < 0$ we have*

$$\overline{Q(E)}Q(\phi)x = Q(E)Q(\phi)x = Q(E * \phi)x$$

*for $\phi \in \mathcal{D}_0$ and $x \in X$.*

**Definition 3.10** *The operator $\overline{Q(-\delta')}$ is called the generator of the distribution semigroup $Q$.*

The following proposition states some properties of a distribution semigroup $Q$. We will use these properties to prove that a distribution semigroup coincides with the solution operator of the degenerate Cauchy problem (2.1).

**Proposition 3.11** *[44] Let $Q$ be a distribution semigroup degenerate on $Z$ and $E \in \mathcal{E}'$. Then $Q(E)$ has the following properties:*
*(i) For all $\phi \in \mathcal{D}_0$ and $x \in X$ we have $Q(\phi)x \in D(Q(E))$ and*

$$Q(E)Q(\phi)x = Q(E * \phi)x.$$

*Moreover, $D(Q(E)) \cap Z = \emptyset$.*
*(ii) For all $\phi \in \mathcal{D}_0$ and $x \in D(Q(E))$ holds*

$$Q(E)Q(\phi)x = Q(\phi)Q(E)x.$$

*(iii) If* $\operatorname{ran} Q \subset \tilde{X}$ *($X = \tilde{X} \oplus Z$), then there exists the closure of $Q(E)$ and for all $\phi \in \mathcal{D}_0$ and $x \in X$ we have*

$$\overline{Q(E)}Q(\phi)x = Q(E)Q(\phi)x.$$

*(iv) If* $\operatorname{ran} Q \subset \tilde{X}$ *and $Q$ is regular, then for $x \in \overline{\operatorname{ran} Q}$ we have*

$$\overline{Q(\psi_+)}x = Q(\psi)x,$$

*where*

$$\psi_+(t) = \begin{cases} \psi(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

## 3.3 Well-Posedness in the Space of Distributions of Exponential Growth

In this section, we show that the well–posedness of the Cauchy problem (2.1) in the sense of distributions is equivalent to the existence of a degenerate distribution semigroup generated by the part of the single–valued branch $\mathcal{A}_s$ on $X_{n+1}$.

**Definition 3.12** *A distribution $U \in \mathcal{D}'_0(D(B^{-1}A)) = \mathcal{D}'_0(D(\mathcal{A}_s))$ is called a solution of the degenerate Cauchy problem (2.1) if it satisfies*

$$B\langle U, \phi' \rangle + A\langle U, \phi \rangle = -\langle \delta, \phi \rangle Bx \tag{3.8}$$

*for all $\phi \in \mathcal{D}_0$ and $x \in X$. Or equivalently*

$$P * U = \delta \otimes Bx \tag{3.9}$$

*for all $x \in X$, where $P := \delta' \otimes B - \delta \otimes A$. The solution $U$ is called degenerate on $Z \subset X$ if $U(\phi) = 0$ for all $\phi \in \mathcal{D}_0$ implies $x \in Z$.*

**Definition 3.13** *The Cauchy problem (2.1) is called well-posed in the sense of distributions if for any $x \in X$ there exists a unique solution $U \in \mathcal{D}'_0(D(\mathcal{A}_s))$, which is stable in the space of distributions, degenerate on $K_{n+1}$, and such that $U \in \mathcal{D}'_0(D(\tilde{\mathcal{A}}_s))$ for $x \in X_{n+1}$. The Cauchy problem is called well-posed in the sense of distributions of exponential growth if $U \in \mathcal{S}'_\omega(D(\mathcal{A}_s))$.*

We say that a solution $U$ is stable in the space of distributions if for any initial values $x_j \in X$ such that $x_j \to 0$ in $X$, the corresponding solutions $U_j$ tend to zero in the space of distributions.

In the following we show that the well–posedness of the Cauchy problem (2.1) in the sense of distributions with exponential growth is equivalent to the existence of a solution operator of certain equations, see equations (3.10) and (3.11) in Theorem 3.14. Furthermore, it is shown that the existence of this solution operator is a necessary and sufficient condition for the existence of a semigroup distributions of exponential growth.

**Theorem 3.14** *Let $A, B$ be linear operators in $X$ and suppose that $A$ is closed and $B$ is bounded. Assume that $\rho_1(A, B) \neq \emptyset$ and that the decomposition (3.2) holds. Then the following statements are equivalent:*

*(i) The Cauchy Problem (2.1) is well–posed in the sense of distributions of exponential growth.*

*(ii) There exists a solution operator*

$$S \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))),$$

*which is degenerate on $K_{n+1}$, such that*

$$(P * S)x = \delta \otimes Bx, \quad x \in X, \tag{3.10}$$

$$(S * \mathcal{P})x = \delta \otimes x, \quad x \in D(\mathcal{A}_s), \tag{3.11}$$

*where $\mathcal{P} = \delta' \otimes I - \delta \otimes \mathcal{A}_s$. In this case $\langle U, \phi \rangle = \langle S, \phi \rangle x$, for all $\phi \in \mathcal{D}$ and $x \in X$.*

*(iii) There exists a regular distribution semigroup of exponential growth $Q$ degenerate on $K_{n+1}$, with* ran$Q$ *dense in $X_{n+1}$ and with generator $\tilde{\mathcal{A}}_s$.*

**Proof.** First we prove that (i) implies (ii). By the definition of well–posedness of the problem (2.1), for all $x \in X$ there exists $U \in \mathcal{S}'_\omega(D(\mathcal{A}_s))$, degenerate on $K_{n+1}$, such that $U \in \mathcal{S}'_\omega(D(\tilde{\mathcal{A}}_s))$ for $x \in X_{n+1}$ and $U_j \to 0$ for any $x_j \to 0$. Since $U \in \mathcal{S}'_\omega(D(\mathcal{A}_s))$, we have $U \in \mathcal{D}'_0(D(\mathcal{A}_s))$ and $\exp(-\omega t)U \in \mathcal{S}'(D(\mathcal{A}_s))$. Now we define

$$S(\phi)x = Sx(\phi) := U(\phi), \quad \phi \in \mathcal{D}, \ x \in X.$$

We have $Sx \in \mathcal{D}'_0(D(\mathcal{A}_s))$ and $\exp(-\omega t)Sx \in \mathcal{S}'(D(\mathcal{A}_s))$ i.e. $Sx \in \mathcal{S}'_\omega(D(\mathcal{A}_s))$ for $x \in X$. Moreover, $Sx \in \mathcal{D}'_0(D(\tilde{\mathcal{A}}_s))$ and $\exp(-\omega t)Sx \in \mathcal{S}(D(\tilde{\mathcal{A}}_s))$ for $x \in X_{n+1}$, or for $x \in X_{n+1}$ we have $Sx \in \mathcal{S}'_\omega(D(\tilde{\mathcal{A}}_s))$. By the well–posedness of the problem (2.1), for any $x_j \to 0$, we have $\|S(\phi)x_j\| = \|U_j(\phi)\| \to 0$, for all $\phi \in \mathcal{D}$. This implies that $S(\phi) \in \mathcal{L}(X, D(\mathcal{A}_s))$.

Now we show that $S \in \mathcal{D}'_0(\mathcal{L}(X, D(\mathcal{A}_s)))$, i.e. that $\text{supp}(S) \subset [0, \infty)$ and for all sequences $\phi_n \in \mathcal{D}$, with $\phi_n \to 0$ holds:

$$||S(\phi_n)||_{\mathcal{L}(X,D(\mathcal{A}_s))} \to 0.$$

Consider the set $\mathcal{B} = \{Sx : ||x|| \leq c\} \subset \mathcal{D}'_0(X)$. Since for any $Sx_j \in \mathcal{B}$ and any $\epsilon_j \to 0$ we have

$$\epsilon_j Sx_j = S(\epsilon_j x_j) \to 0,$$

by the definition of a bounded subset in the space of vector–valued distributions, [19], $\mathcal{B}$ is bounded in $\mathcal{D}'_0(X)$. By, [19] Lemma 8.1.9, for any $Sx$ from a bounded subset $\mathcal{B}$ and any $\phi_j \to 0$ in $\mathcal{D}$, there exist $p \in \mathbb{N}$ and $M > 0$ such that

$$||Sx(\phi_j)|| \leq M||\phi_j||_{p,p}, \qquad \text{for all } j \in \mathbb{N},$$

where

$$||\phi||_{j,k} := \sup_{0 \leq i \leq k} \sup_{t \in \mathbb{R}} |(1 + |t|^j)\phi^i(t)|.$$

That means that $S(\phi_j)x \to 0$ uniformly in $x$ from a bounded set or equivalently

$$||S(\phi_j)||_{\mathcal{L}(X,D(\mathcal{A}_s))} \to 0.$$

Now we show that

$$S \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s)))$$

satisfies equations (3.10) and (3.11). First we show (3.10). By the Structure Theorem ([19], Theorem 8.2.3), there exist $f \in C(\mathbb{R}, \mathcal{L}(D(\mathcal{A}_s), X))$, $g \in C(\mathbb{R}, \mathcal{L}(X, D(\mathcal{A}_s)))$, $h \in C(\mathbb{R}, D(\mathcal{A}_s))$ and $p, q, r, m, n, l \geq 0$ such that $P = f^{(p)}$, $S = g^{(q)}$, $Sx = h^{(r)}$, and $||f(t)|| = O(t^m)$, $||g(t)|| = O(t^n)$, $||h(t)|| = O(t^l)$, for $t \to \infty$. Therefore,

$$\begin{aligned}
\langle P * S, \phi \rangle x &= \langle f * g^{(p+q)}, \phi \rangle x \\
&= (-1)^{p+q} \langle f * g, \phi^{(p+q)} \rangle x \\
&= (-1)^{p+q} \int \phi^{(p+q)}(t) \int f(s) g(t-s) ds dt x \\
&= (-1)^{p+q} \int f(s) ds \int g(t-s) \phi^{(p+q)}(t) dt x
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}(-1)^{p+q}\sum_{k=o}^{n} f(s_k)\Delta s_k \int g(t-s_k)x\phi^{(p+q)}(t)dt \\
&= \lim_{n\to\infty}(-1)^{p+q}\sum_{k=o}^{n} f(s_k)\Delta s_k \int g(t)x\phi^{(p+q)}(t+s_k)dt \\
&= \lim_{n\to\infty}(-1)^{p+q}\sum_{k=o}^{n} f(s_k)\Delta s_k(-1)^q\langle Sx,\phi^{(p)}(t+s_k)\rangle \\
&= (-1)^p\int f(s)ds(-1)^r\int h(t)\phi^{(p+r)}(t+s)dt \\
&= (-1)^{p+r}\int \phi^{(p+r)}(t)\int f(s)h(t-s)dsdt \\
&= \langle P * Sx,\phi\rangle.
\end{aligned}
$$

So for all $x \in X$, we have

$$\langle P * S,\phi\rangle x = \langle P * Sx,\phi\rangle = \delta\otimes Bx.$$

Now we show (3.11). From (3.10) we have

$$(P * S')x = (P * S)'x = \delta'\otimes Bx, \quad x\in X,$$

$$(P * S)\mathcal{A}_s x = \delta\otimes\mathcal{A}_s x, \quad x\in D(\mathcal{A}_s),$$

$$(P * (\delta\otimes x)) = \delta'\otimes Bx - \delta\otimes Ax, \quad x\in D(\mathcal{A}_s).$$

Hence

$$P * (S'x - S\mathcal{A}_s x - \delta\otimes x) = 0,$$

and by the uniqueness of the solution of the problem (2.1), we get

$$S'x - S\mathcal{A}_s x - \delta\otimes x = 0$$

or

$$S'x - S\mathcal{A}_s x = (S * \mathcal{P})x = \delta\otimes x, \quad x\in D(\mathcal{A}_s).$$

Now we show that (ii) implies (i). Suppose there exists a solution operator

$$S\in\mathcal{S}'_\omega(\mathcal{L}(X,D(\mathcal{A}_s)))\cap\mathcal{S}'_\omega\left(\mathcal{L}\left(X_{n+1},D(\tilde{\mathcal{A}}_s)\right)\right),$$

degenerate on $K_{n+1}$, and satisfying equations (3.10) and (3.11). By definition of $S$ we have

$$S\in\mathcal{D}'_0(\mathcal{L}(X,D(\mathcal{A}_s)))\cap\mathcal{D}'_0(\mathcal{L}(X_{n+1},D(\tilde{\mathcal{A}}_s)))$$

and

$$\exp(-\omega t)S \in \mathcal{S}'(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))).$$

Define

$$U(\phi) := S(\phi)x = Sx(\phi).$$

Then $U = Sx \in \mathcal{S}'_\omega(D(\mathcal{A}_s))$ is a solution of the problem (2.1)

$$-B\langle Sx, \phi'\rangle - A\langle Sx, \phi\rangle = \langle \delta, \phi\rangle Bx, \quad x \in X,$$

$$-\langle Sx, \phi'\rangle - \tilde{\mathcal{A}}_s\langle Sx, \phi\rangle = \langle \delta, \phi\rangle x, \quad x \in X_{n+1},$$

and $U$ is degenerate on $K_{n+1}$. Using the associative property of the convolution, for any solution $U$ we obtain

$$U = (\delta \otimes I) * U = (S * \mathcal{P}) * U = S * (\delta \otimes x) = Sx, \quad x \in X_{n+1}.$$

We need to show that the solution $U$ is also unique on $K_{n+1}$. Since $Sx = 0$ for all $x \in K_{n+1}$ and for all $\phi \in \mathcal{D}_0$ we have $Sx = 0$ on $(0, \infty)$, and since $S \in \mathcal{D}'_0(\mathcal{L}(X, D(\mathcal{A}_s)))$, we have $Sx = 0$ on $(-\infty, 0)$. These imply that supp $(Sx) = 0$, then as a corollary of the structure theorem, [19] Corollary 8.1.7, for distributions with point support $\{0\}$ we have

$$Sx = \sum_{i \leq k} \delta^{(i)} z_i, \quad z_i \in X.$$

Here $k$ is such that $\mathcal{D}^k$ is the space of $k$–times continuously differentiable functions in which the distribution $Sx$ can be extended. Since $Sx$ is a solution of (3.10), we have

$$\delta^{(k+1)} \otimes Bx + \delta^{(k)} \otimes (Bz_{k-1} - Az_k) + \cdots + \delta' \otimes (Bz_0 - Az_1) + (-Bx - Az_0) = 0,$$

and $z_k \in \ker B$, $z_{k-1} \in \tilde{K}_1$, $\cdots$, $z_0 \in \tilde{K}_k$, $-x \in \tilde{K}_{k+1}$. Therefore for $x \in \tilde{K}_i$, $1 \leq i \leq k+1 \leq n$ and its $A$–associated vectors $x_0, \ldots, x_{i-1}$ we have

$$Sx_i = -\sum_{j=0}^{i-1} \delta^{(j)} \otimes x_{i-1-j}, \tag{3.12}$$

and any solutions $U$, degenerate on $K_{n+1}$, have the same form. Thus, the solution $U$ is unique for all $x \in K_{n+1}$, and hence for all $x \in X$. Let $x_j \to 0$, then for all

$\phi \in \mathcal{D}$ we have $S(\phi) \in \mathcal{L}(X, D(\mathcal{A}_s))$ and

$$\begin{aligned}
\|U_j(\phi)\| &= \|Sx_j(\phi)\|_{\mathcal{A}_s} \\
&= \|Sx_j(\phi)\| + \|\mathcal{A}_s Sx_j(\phi)\| \\
&\leq \|S(\phi)x_j\| + \|S'(\phi)x_j\| + \|\phi(0)x_j\| \\
&= \|S(\phi)x_j\| + \|S(\phi')x_j\| + \|\phi(0)x_j\| \to 0
\end{aligned}$$

as $\|S(\phi)x_j\| \to 0$, $\|S(\phi')x_j\| \to 0$, and $\|\phi(0)x_j\| \to 0$. Therefore $Sx_j = U_j \to 0$ in $\mathcal{S}'_\omega(D(\mathcal{A}_s))$.

Next we prove that (ii) implies (iii). Let $Q \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s)))$ be a solution of (3.10) and (3.11) and $Q$ is degenerate on $K_{n+1}$. We will show that $Q$ satisfies the properties of distribution semigroup. By definition of $Q \in \mathcal{S}'_\omega(\mathcal{L}(D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s)))$ we have

$$Q \in \mathcal{D}_0(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{D}'_0(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))),$$

and

$$\exp(-\omega t)Q \in \mathcal{S}'(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))).$$

We also have that for any $F \in \mathcal{D}'_0(X_{n+1})$, $v = Q * F$ is a unique solution of the equality

$$\mathcal{P} * v = F. \tag{3.13}$$

Let $\phi, \psi \in \mathcal{D}_0$ and define $\phi_1(t) := \phi(-t)$, $\psi_1(t) := \psi(-t)$. Let $t \mapsto u(t)$ be a solution of (3.13) with $F = \phi_1 \otimes x$, $x \in X_{n+1}$, then

$$u(t) = Q * (\phi_1 \otimes x)$$

and

$$-\tilde{\mathcal{A}}_s u + u' = \phi_1 \otimes x. \tag{3.14}$$

Here $t \mapsto u(t) = Q * (\phi_1 \otimes x) = (Q * \phi_1)x$ is an infinitely differentiable function with values in $D(\tilde{\mathcal{A}}_s)$ and $u(0) = Q(\phi)x$. Similarly, let $v(t)$ and $w(t)$ be solutions of (3.13) with $F = (\psi_1 * \phi_1) \otimes x$ and $F = \psi_1 * u(0)$ respectively. That is for $x \in X_{n+1}$

$$v(t) = Q * ((\psi_1 * \phi_1) \otimes x) = (Q * \psi_1 * \phi_1)x$$

satisfies

$$\begin{aligned} -\tilde{A}_s v + v' &= (\psi_1 * \phi_1)x, \\ v(0) &= Q(\phi * \psi)x, \end{aligned} \tag{3.15}$$

and

$$w(t) = Q * (\psi_1 * u(0)) = (Q * \psi_1)u(0)$$

satisfies

$$\begin{aligned} -\tilde{A}_s w + w' &= \psi_1 * u(0), \\ w(0) &= Q(\psi)u(0) = Q(\psi)Q(\phi). \end{aligned} \tag{3.16}$$

To prove the semigroup property, i.e. property (i) of Definition 3.7, one needs to show $v(0) = w(0)$. From (3.14) we have

$$-\tilde{A}_s(u * \psi_1) + (u * \psi_1) = (\phi_1 * \psi_1)x. \tag{3.17}$$

Hence, comparing (3.15), (3.17), we get

$$v(t) = u * \psi_1 \quad \text{and} \quad v(0) = u(\psi).$$

Let $t \mapsto H(t)$ be the Heaviside function, then $H(t)u(t) \in \mathcal{D}_0'(D(\tilde{A}_s))$. Since $\tilde{A}_s$ does not depend on $t$ we have

$$-\tilde{A}_s(Hu) + (Hu)' = H(t)(-\tilde{A}_s u + u') + \delta \otimes u(0).$$

Since for all $\phi \in \mathcal{D}_0$ holds

$$H(t)(-\tilde{A}_s u + u') = (H\phi_1) \otimes x = 0,$$

we obtain

$$-\tilde{A}_s(Hu) + (Hu)' = \delta \otimes u(0) \tag{3.18}$$

and

$$-\tilde{A}_s((Hu) * \psi_1) + ((Hu) * \psi_1)' = \psi_1 \otimes u(0).$$

Therefore $w = Hu * \psi_1$ and

$$w(0) = \int_0^\infty u(t)\psi(t)dt = u(\psi) = v(0). \tag{3.19}$$

Thus for all $\phi, \psi \in \mathcal{D}_0$, we have

$$Q(\phi * \psi)x = Q(\phi)Q(\psi)x, \quad x \in X_{n+1}.$$

$Qx = Sx$ defined by (3.12) for $x \in K_{n+1}$ also satisfies the semigroup property. Since

$$Q \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))),$$

we see that $Q$ is a distribution semigroup of exponential growth. It is degenerate on $K_{n+1}$.

Now we show that $Q$ is a regular distribution semigroup. Let $z$ be a solution of (3.13) with $F = \delta \otimes y$, $y = Q(\phi)x$, i.e.

$$z = Q * (\delta \otimes y) = Qy \quad \text{and} - \tilde{\mathcal{A}}_s z + z' = \delta \otimes y.$$

We have $u(0) = y$ and by (3.18) $z(t) = Qy = H(t)u(t)$. Therefore $Qy$ is a continuous function on $(0, \infty)$. Moreover we will show that ran$Q$ is dense in $X_{n+1}$. To do this, one needs to show $\ker Q^* = 0$ in $X_{n+1}$. Note that $\tilde{\mathcal{A}}_s \in \mathcal{L}(D(\tilde{\mathcal{A}}_s), X_{n+1})$ and $Q(\phi) = S(\phi) \in \mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))$. Let $X^*_{n+1}$ and $D(\tilde{\mathcal{A}}_s)^*$ be the dual spaces of $X_{n+1}$ and $D(\tilde{\mathcal{A}}_s)$. Then $\tilde{\mathcal{A}}^*_s \in \mathcal{L}(X^*_{n+1}, D(\tilde{\mathcal{A}}_s)^*)$. Consider $Q(\phi)$ on $X_{n+1}$ and define $Q^*(\phi) = (Q(\phi))^*$, $\phi \in \mathcal{D}_0$. Then $Q^*(\phi) \in \mathcal{L}(D(\tilde{\mathcal{A}}_s)^*, X^*_{n+1})$ and $Q^* \in \mathcal{D}'_0(\mathcal{L}(D(\tilde{\mathcal{A}}_s)^*, X^*_{n+1}))$. Since $Q$ satisfies equations (3.10) and (3.11), we have

$$\begin{aligned} Q^* * (\delta' \otimes I - \delta \otimes \tilde{\mathcal{A}}^*_s) &= \delta \otimes I_{X^*_{n+1}}, \\ (\delta' \otimes I - \delta \otimes \tilde{\mathcal{A}}^*_s) * Q^* &= \delta \otimes I_{D(\tilde{\mathcal{A}}_s)^*}. \end{aligned}$$

Therefore $Q^* \in \mathcal{D}'_0(\mathcal{L}(D(\tilde{\mathcal{A}}_s)^*, X^*_{n+1}))$ is a distribution semigroup which is non–degenerate on $D(\tilde{\mathcal{A}}^*_s)$ as $Q$ is non–degenerate on $X_{n+1}$. Hence, if $z^* \in D(\tilde{\mathcal{A}}_s)^*$ is such that for all $\phi \in \mathcal{D}_0$, $x \in X_{n+1}$ holds

$$\langle Q(\phi)x, z^* \rangle = \langle x, Q^*(\phi)z^* \rangle = 0,$$

then $Q^*(\phi)z^* = 0$. This implies that $\{Q(\phi)x : \quad x \in X_{n+1}\}$ is dense in $X_{n+1}$. And since $Q(\phi)x = 0$ for all $x \in K_{n+1}$, we have $\overline{\text{ran } Q(\phi)} = X_{n+1}$.

To complete the proof, we note that $\tilde{\mathcal{A}}_s$ is the generator of a non–degenerate distribution semigroup of exponential growth $Q$, which is densely defined on $X_{n+1}$ (see

[38]). Let ${\mathcal{A}_s}^1$ be the generator of the degenerate distribution semigroup of exponential growth $Q$. Then $D({\mathcal{A}_s}^1) \subset (K_{n+1})^c = X_{n+1}$. Therefore ${\mathcal{A}_s}^1 = \tilde{\mathcal{A}}_s$ is the generator of the constructed distribution semigroup of exponential growth $Q$ on $X$.

Now we show that (iii) implies (ii). Let $Q$ be the distribution semigroup of exponential growth from (iii), then $Q \in \mathcal{S}'_\omega(\mathcal{L}(X))$, that is $Q \in \mathcal{D}'_0(\mathcal{L}(X))$ and $\exp(-\omega t)Q \in \mathcal{S}'(\mathcal{L}(X))$. Now for $\psi \in \mathcal{D}$ we consider

$$\psi_+(t) = \begin{cases} \psi(t), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Then $\psi_+ \in \mathcal{E}'_0$ and

$$\delta' * \psi_+ = \psi'_+ + \psi(0).$$

By the property (i) of Proposition 3.11 and by the definition of the generator of a distribution semigroup we have for all $x \in X_{n+1}$ and $\phi \in \mathcal{D}_0$,

$$\begin{aligned} Q(\delta' * \psi_+ * \phi)x &= Q(\psi'_+ * \phi)x + \psi(0)Q(\phi)x = Q(\delta')Q(\psi_+ * \phi)x \\ &= -\tilde{\mathcal{A}}_s Q(\psi_+ * \phi)x = -\tilde{\mathcal{A}}_s Q(\psi_+)Q(\phi)x \\ &= Q(\psi_+)Q(\delta' * \phi)x = Q(\psi_+)(-\tilde{\mathcal{A}}_s)Q(\phi)x. \end{aligned}$$

By property (iv) of Proposition 3.11 we have

$$\begin{aligned} -\tilde{\mathcal{A}}_s Q(\psi)Q(\phi)x &= -Q(\psi)\tilde{\mathcal{A}}_s Q(\phi)x \\ &= Q(\psi')Q(\phi)x + \psi(0)Q(\phi)x, \end{aligned}$$

i.e. for $y = Q(\phi)x$

$$-\tilde{\mathcal{A}}_s Q(\psi)y - Q(\psi')y = \psi(0)y, \tag{3.20}$$

$$-Q(\psi)\tilde{\mathcal{A}}_s y - Q(\psi')y = \psi(0)y. \tag{3.21}$$

Since $\tilde{\mathcal{A}}_s$ is closed and $\overline{\{Q(\phi)x\}} = X_{n+1}$, for any $y \in X_{n+1}$ we have $Q(\psi)y \in D(\tilde{\mathcal{A}}_s)$, which implies (3.20) for $y \in X_{n+1}$. Let $y_j \to 0$ in (3.20), then $\|\tilde{\mathcal{A}}_s Q(\psi)y_j\| \to 0$. This implies that $Q(\psi) \in \mathcal{L}(X, D(\tilde{\mathcal{A}}_s))$.

If $\psi_j \to 0$, for $\psi_j \in \mathcal{D}$, then $\|Q(\psi'_j)\| \to 0$ and $\|\tilde{\mathcal{A}}_s Q(\psi_j)\| \to 0$. Hence, since $Q \in \mathcal{D}'_0(\mathcal{L}(X))$, we have $Q \in \mathcal{D}'_0(\mathcal{L}(X, D(\tilde{\mathcal{A}}_s))) \subset \mathcal{D}'_0(\mathcal{L}(X, D(\mathcal{A}_s)))$. And since

$Q$ is a distribution semigroup of exponential growth, $Q \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\tilde{\mathcal{A}}_s))) \subset \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s)))$.

Thus $S$ defined equal to $Q$ on $X_{n+1}$ and $K_{n+1}$ satisfies equation (3.10) on $X = X_{n+1} \oplus K_{n+1}$. Now we need to show that $S = Q$ also satisfies equation (3.11), i.e. (3.21) for $y \in D(\mathcal{A}_s)$. Let $x \in D(Q(-\delta'))$ and $\phi_j \to \delta$. Then $Q(\phi_j)x \to x$ and $Q(\delta' * \phi_j) \to -\tilde{\mathcal{A}}_s x$.

From equation (3.21) we have

$$-Q(\psi)\tilde{\mathcal{A}}_s x = Q(\psi')x + \psi(0)x, \quad x \in D(Q(-\delta')) \tag{3.22}$$

We need to show that the equation (3.22) also holds for any $x \in D(\tilde{\mathcal{A}}_s)$. By the definition of $\tilde{\mathcal{A}}_s = \overline{Q(-\delta')}$, for any $x \in D(\tilde{\mathcal{A}}_s)$ there exists a sequence $x_j \in D(Q(-\delta'))$ such that $x_j \to x$, and $\tilde{\mathcal{A}}_s x_j \to \tilde{\mathcal{A}}_s x$. Therefore equation (3.22) holds for $x \in D(\tilde{\mathcal{A}}_s) = D(\mathcal{A}_s) \cap X_{n+1}$. Since $Q$ is degenerate on $K_{n+1}$, for $x \in K_{n+1}$ we have $Qx = \sum_{i=0}^{k} \delta^{(i)} z_i$. $Qx$ defined by (3.12) satisfies (3.11) for $x \in D(\mathcal{A}_s) \cap K_{n+1}$. Hence $Sx = Qx$ for $x \in X = X_{n+1} \oplus K_{n+1}$ satisfies (3.11) on $D(\mathcal{A}_s)$. ■

## 3.4 Connection between Integrated Semigroups and Distribution Semigroups

It is proved in the previous section, see Theorem 3.14, that the existence of a solution operator $S \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s)))$ of the equations (3.10) and (3.11) is equivalent to the existence of a regular distribution semigroup of exponential growth $Q$. This semigroup is degenerate on $K_{n+1}$, with dense range in $X_{n+1}$. It is generated by the part of the single–valued branch of $B^{-1}A$.

Next, we show that the existence of the solution operator $S$ of order $p \geq n-1$ of the equations (3.10) and (3.11), which is degenerate on $K_{n+1}$, gives a necessary and sufficient condition for the existence of a degenerate exponentially bounded $(p+2)$–times integrated semigroup with generators $A$ and $B$. Bearing in mind that for the degenerate Cauchy problem (2.1), we define the resolvent as $R_1(\lambda) := (\lambda B - A)^{-1}$, we say that $A$ and $B$ generate an $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$ if

$$(\lambda B - A)^{-1}B = \int_0^\infty \lambda^n \exp(-\lambda t)V(t)dt. \qquad (3.23)$$

**Theorem 3.15** *Let $A, B$ be linear operators in $X$, $A$ closed, $B$ bounded, and suppose that $\rho_1(A, B) \neq \emptyset$ and that the decomposition (3.2) holds for $X$. Then the existence of a solution operator, degenerate on $K_{n+1}$, of order $p \geq n-1$ of equations (3.10) and (3.11) is equivalent to the existence of a degenerate exponential (p+2)–times integrated semigroup with generators $A, B$.*

**Proof.** Suppose that there exists

$$S \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s))) \cap \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))),$$

degenerate on $K_{n+1}$, and satisfying equations (3.10) and (3.11). By [19] Lemma 8.2.2 , there exists $p \geq 0$ and a constant $M > 0$ such that

$$||S(\phi)|| \leq M||\phi||_{p,p}, \quad \text{for } \phi \in \mathcal{S},$$

where the norm $|| \cdot ||_{j,k}$ is defined by

$$||\phi||_{j,k} = \sup_{0 \leq i \leq k} \sup_t (1 + |t|^j)|\phi^{(i)}(t)|.$$

That means the distribution $S$ is of order $p$, and we can extend $S$ to $\mathcal{S}^p(\mathbb{R})$, the space of $p$–times continuously differentiable funtions with norm $||.||_{p,p}$. Let

$$\eta_p(t) = \begin{cases} \frac{t^{p+1}}{(p+1)!}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and

$$\mathcal{X}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t \leq a < 0. \end{cases}$$

Now define $\psi_{t,p}(\cdot) \in \mathcal{S}^p(\mathbb{R})$ as $\psi_{t,p}(\cdot) := \mathcal{X}(\cdot)\eta_p(t - \cdot)$. Consider

$$\begin{aligned} V(t) &= S(\psi_{t,p}(\cdot)) \\ &= \exp(-\omega \cdot)S(\exp(\omega \cdot)\mathcal{X}(\cdot)\eta_p(t - \cdot)), \end{aligned}$$

$$V(0) = \exp(-\omega\cdot)S(\exp(\omega\cdot)\mathcal{X}(\cdot)\eta_p(-\cdot)) = 0.$$

Since the map

$$t \mapsto \exp(\omega\cdot)\mathcal{X}(\cdot)\eta_p(t-\cdot) \in \mathcal{S}^p(\mathbb{R})$$

is continuous on $\mathbb{R}$, $t \to V(t) \in \mathcal{L}(X, D(\mathcal{A}_s))$. As $\mathrm{supp}(S) \subset [0,\infty)$ and

$$\mathrm{supp}\,(\exp(\omega\cdot)\mathcal{X}(\cdot)\eta_p(t-\cdot)) \subset [-a, t],$$

we have that $V(t) = 0$ for $t \leq 0$ and

$$\begin{aligned} \|V(t)\| &= \|\exp(-\omega\cdot)S(\exp(\omega\cdot)\mathcal{X}(\cdot)\eta_p(t-\cdot)\| \\ &\leq C\|\exp(\omega\cdot)\mathcal{X}(\cdot)\eta_p(t-\cdot)\|_{p,p} \\ &\leq M\exp(\omega t)(1+|t|)^{p+1}, \quad t \geq 0. \end{aligned}$$

So for $\omega' > \omega$ and for any $t \geq 0$ we have $\|V(t)\| \leq M\exp(\omega' t)$. We can approximate $\psi_{t,p}$ by $\phi_n \in \mathcal{S}$ so that

$$\|\psi_{t,p} - \phi_n\|_{p+1} \to 0.$$

Now consider equations (3.10) and (3.11) on $\phi_n$, taking the limit, we have

$$-B\langle S, \psi'_{t,p}\rangle x - A\langle S, \psi_{t,p}\rangle x = \psi_{t,p}(0)Bx, \quad x \in X,$$

$$-\langle S, \psi'_{t,p}\rangle x - \langle S, \psi_{t,p}\rangle \mathcal{A}_s x = \psi_{t,p}(0)x, \quad x \in D(\mathcal{A}_s).$$

Hence,

$$BV(t)x = B\frac{t^{p+2}}{(p+2)!}x + A\int_0^t V(s)xds, \quad x \in X, \tag{3.24}$$

$$V(t)x = \frac{t^{p+2}}{(p+2)!}x + \mathcal{A}_s\int_0^t V(s)xds, \quad x \in D(\mathcal{A}_s). \tag{3.25}$$

So for $x \in X_{n+1}$ we have $V(t)x \in X_{n+1}$ and the equations (3.24) and (3.25) hold for $X_{n+1}$ and $D(\mathcal{A}_s)$, respectively. Therefore $V(t)$ is non-degenerate $(p+2)$-times integrated semigroup on $X_{n+1}$ with generator $\tilde{\mathcal{A}}_s$. For $x \in K_{n+1}$, we define $V(t)$ as

$$V(t)x_i = -\sum_{k=0}^{i-1}\frac{t^{((p+2)-1-k)}}{((p+2)-1-k))!}x_{i-1-k}, \quad x_i \in K_i, \quad i \leq n.$$

$V(t)$ defined by this formula also satisfies equations (3.24) and (3.25). It follows that for $x \in K_{n+1}$ relation (3.23) also holds with $n+1 = p+2$. Hence $V(t)$ is degenerate exponential (p+2)–times integrated semigroup with generators $A, B$ on $X = X_{n+1} \oplus K_{n+1}$.

To prove the converse statement, let $n \in \mathbb{N}$ and $A$ be the generator of an exponentially bounded $n$–times integrated semigroup $V(t)$, $t \geq 0$. Since, in contrast to the local case, $V(t)$ is defined for all $t \geq 0$, $S \in \mathcal{S}'_\omega(\mathcal{L}(X, D(\mathcal{A}_s)))$ may be defined as $S = V^{(n)}$

$$S(\varphi) := (-1)^n \int_0^\infty \varphi^{(n)}(t)V(t)\,dt, \quad \varphi \in \mathcal{S}.$$

Taking into account the equation for $V(t)$ on $D(\tilde{\mathcal{A}}_s)$

$$V'(t)x = \frac{t^{n-1}}{(n-1)!}x + AV(t)x, \quad t \geq 0, \; x \in D(\tilde{\mathcal{A}}_s),$$

for any $\varphi \in \mathcal{S}$ we have

$$(-1)^{n+1} \int_0^\infty \varphi^{(n+1)}(t)V(t)xdt =$$

$$(-1)^n \int_0^\infty \varphi^{(n)}(t)\frac{t^{n-1}}{(n-1)!}xdt + (-1)^n\tilde{\mathcal{A}}_s \int_0^\infty \varphi^{(n)}(t)V(t)xdt, \quad x \in D(\tilde{\mathcal{A}}_s).$$

Since $A$ is closed and all other operators in this equality are bounded, it is valid for all $x \in X_{n+1}$. Integrating by parts the first term in the right–hand side of the equality, we obtain

$$\tilde{\mathcal{A}}_s S(\varphi)x = S'(\varphi)x - \varphi(0)x, \quad x \in X_{n+1}. \tag{3.26}$$

Using the commutativity of $V(t)$ and $\tilde{\mathcal{A}}_s$ on $D(\tilde{\mathcal{A}}_s)$, we obtain (3.11) on $D(\tilde{\mathcal{A}}_s)$. From (3.26) we have

$$||S(\varphi)x||_{\mathcal{A}_s} \leq ||S(\varphi)x|| + ||Q'(\varphi)x|| + |\varphi(0)|\,||x||.$$

Therefore $S(\varphi) \in \mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s))$ for any $\varphi$, and $S \in \mathcal{D}'_0(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s)))$. Moreover, exponential boundedness of $V(t)$ implies $S \in \mathcal{S}'_\omega(\mathcal{L}(X_{n+1}, D(\tilde{\mathcal{A}}_s)))$.

For $x \in K_{n+1}$ we define $Sx$ by formula (3.12). Then $S$ satisfies the equations (3.10), (3.11). ■

# Chapter 4

# Inhomogeneous Abstract Cauchy Problems

We consider the inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + \psi(t), \quad t \geq 0, \quad u(0) = x \tag{4.1}$$

on a Banach space $X$, where $A$ is a closed linear operator and $\psi(\cdot) \in L^1_{loc}((0,\infty), X)$ and $x \in X$.

By a classical solution of the inhomogeneous problem, we mean a continuously differentiable function $t \mapsto u(t)$, which takes values in $D(A)$, satisfies the differential equation (4.1) for all times $t \geq 0$ and the initial condition $u(0) = x$.

In this chapter, we investigate several types of solutions of the inhomogeneous problem: $n$–integrated solutions, $n$–weak solutions and $K$–generalized solutions. We characterize the existence and uniqueness of such solutions for the case that $A$ generates an $n$–times integrated semigroup or a $K$–convoluted semigroup. For those operators $A$, we also investigate some conditions for the existence of classical solutions.

This chapter consists of three sections. In the first section, we discuss the $n$–integrated solutions of the Cauchy problem for the case that $A$ generates an $n$–times integrated semigroup. Moreover, using the properties of the $n$–times integrated semigroup generated by $A$, we obtain conditions for the existence of classical solutions. In Section 2, we generalize the weak solutions studied by Ball, [6], to $n$–weak solutions, and present sufficient conditions for the existence of these solutions. In particular we show that any $n$–integrated solution is an $(n-1)$–weak solution of problem (4.1). In the last section, we consider the $K$–generalized solutions, in the case that $A$ generates a $K$–convoluted semigroup. For those operators $A$, we additionally give conditions which guarantee that problem (4.1) has classical solutions and 1–integrated solutions.

## 4.1 $n$–Integrated Solutions

In this section, we treat the abstract Cauchy problem (4.1) for the case when the operator $A$ is the generator of an $n$–times integrated semigroup. We recall that $n$–times integrated semigroups were defined in Chapter 2, see Definition 2.21. Since in this section we do not require the exponential boundedness of the $n$–times integrated semigroup, the Laplace transform of such semigroup does not necessarily exist. For this reason, the definition of the generator of an $n$–times integrated semigroup, Definition 2.22, is not valid. Hence we define a generator of $n$–times integrated semigroups as in [50], or in [59].

**Definition 4.1** *A closed linear operator $A$ is called the generator of an $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$ if for all $x \in D(A)$ and $Ax = y$*

$$V(t)x = \frac{t^n}{n!}x + \int_0^t V(s)yds.$$

It is proved in [59], for non–degenerate $n$–times integrated semigroups, that an $n$–times integrated semigroup is uniquely determined by its generator in the sense that if there are two $n$–times integrated semigroups generated by $A$, then the $n$–times integrated semigroups are the same (see [59], Theorem 3.6).

However, with the generator defined in Definition 4.1, properties of $n$–times integrated semigroups as in Proposition 2.26 remain valid for a single–valued operator

$A$. We cite the corresponding Proposition 2.1 from [50].

**Proposition 4.2** *Let $A$ be the generator of an $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. Then the following holds.*

*(i) For all $x \in D(A)$ and $t \in [0,\infty)$ we have $V(t)x \in D(A)$ and $AV(t)x = V(t)Ax$ and*

$$V(t)x = \frac{t^n}{n!}x + \int_0^t V(s)Axds$$

*(ii) For all $x \in X$, $\int_0^t V(s)xds \in D(A)$ and*

$$V(t)x = \frac{t^n}{n!}x + A\int_0^t V(s)xds.$$

In the sequel, we discuss $n$–integrated solutions of the inhomogeneous abstract Cauchy problem (4.1). An $n$–integrated solution is a solution of the $n$–times integrated problem.

**Definition 4.3** *A function $v(\cdot) \in C([0,\infty);X)$ is called an $n$–integrated solution of (4.1) if for all $t \in [0,\infty)$, hold $\int_0^t v(s)ds \in D(A)$ and*

$$v(t) = \frac{t^n}{n!}x + A\int_0^t v(s)ds + \int_0^t \frac{(t-s)^n}{n!}\psi(s)ds. \tag{4.2}$$

The next lemma shows that any strict solution defines an $n$–integrated solution.

**Lemma 4.4** *If $t \mapsto u(t)$ is a strict solution of (4.1) then*

$$t \mapsto v(t) = \int_0^t \frac{(t-s)^{n-1}}{n!}u(s)ds$$

*is an $n$–integrated solution of (4.1).*

**Proof.** Let $t \mapsto u(t)$ be a strict solution of the problem (4.1). We recall that a strict solution is an absolutely continuous function $u \in W^{1,1}_{loc}((0,\infty);X) \cap L^1_{loc}((0,\infty);D(A))$ satisfying the differential equation (4.1) almost everywhere (see e.g. [8]). It satisfies

$$\begin{aligned} u(t) &= x + \int_0^t Au(s)ds + \int_0^t \psi(s)ds \\ &= x + A\int_0^t u(s)ds + \int_0^t \psi(s)ds, \end{aligned} \tag{4.3}$$

due to the closedness of the operator $A : D(A) \to X$. Then we can calculate

$$\begin{aligned}
v(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} u(s) ds \\
&= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} (x + A \int_0^s u(r) dr + \int_0^s \psi(r) dr) ds \\
&= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} x ds + A \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \int_0^s u(r) dr ds \\
&\quad + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \int_0^s \psi(r) dr \\
&= \frac{t^n}{n!} x + A \int_0^t \frac{(t-s)^n}{n!} u(s) ds + \int_0^t \frac{(t-s)^n}{n!} \psi(s) ds \\
&= \frac{t^n}{n!} x + A \int_0^t v(s) ds + \int_0^t \frac{(t-s)^n}{n!} \psi(s) ds.
\end{aligned}$$

■

The following theorem shows that the existence of a unique $n$–integrated solution of problem (4.1) is equivalent to the existence of an $n$–times integrated semigroup generated by $A$ for the homogeneous case (i.e. $\psi(t) \equiv 0$). The proof of the thoerem is independent from [50], Theorem 3.1, in which the proof for the more general case (namely $n \in \mathbb{R}^+$) is sketched.

**Theorem 4.5** *Let $\psi(t) \equiv 0$. There exists a unique $n$–integrated solution $t \mapsto v(t)$ of (4.1) for any $x \in X$ if and only if $A$ generates an $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$.*

**Proof.** The proof of the sufficient condition is straight forward and follows from the properties of the $n$–times integrated semigroup. We use property (ii) of Proposition 4.2. Since $A$ is the generator of the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$, for all $x \in X$ we have

$$V(t)x = \frac{t^n}{n!} x + A \int_0^t V(s) x ds.$$

Taking $v(t) = V(t)x$, then $t \mapsto v(t)$ is an $n$–integrated solution of (4.1). The uniqueness of the solution is concluded from the uniqueness of the $n$–times integrated semigroup generated by $A$ (see [59], Theorem 3.6).

Now we prove the necessity. For $t \geq 0$ we define the map $V(t) : X \to X$ by

$$V(t)x := v(t), \quad x \in X.$$

We will show that $V(t)$ has all the properties of an $n$–times integrated semigroup as in Definition 2.21.

Since $V(t)x := v(t,x)$ is an $n$–integrated solution of (4.1) with $\psi(t) \equiv 0$, $V(t)x$ satisfies

$$V(t)x = \frac{t^n}{n!}x + A\int_0^t V(s)xds. \tag{4.4}$$

Thus, $\{V(t) : t \geq 0\}$ obviously is a one parameter family of linear operators satisfying

(i) $V(0)x = 0$, and

(ii) $V(\cdot)x \in C([0,\infty);X)$.

We show that the linear operators $\{V(t) : t \geq 0\}$ are bounded. To this end we define for $t \geq 0$ the maps

$$\begin{aligned} F_t : X &\to Y_t := C([0,t];X), \\ x &\mapsto v(\cdot). \end{aligned}$$

Those maps are linear and defined on the whole Banach space $X$. Since $Y_t$ becomes a Banach space with the supremum norm, according to the Closed–Graph–Theorem, we only have to prove closedness of the operator $F_t$ to obtain boundedness. Let $x_k \to x$ be a convergent sequence in $X$ with $F_t x_k \to y(\cdot)$ in $Y_t$. Defining $v_k := F_t x_k$ we can write

$$v_k(t) - \frac{t^n}{n!}x_k = A\int_0^t v_k(s)ds.$$

The limit, as $k \to \infty$, of the left–hand side exists and coincides with $y(t) - \frac{t^n}{n!}x$. On the other hand we have

$$\int_0^t v_k(s)ds \to \int_0^t y(s)ds,$$

as $k \to \infty$. Thus, by the closedness of the operator $A$ we obtain $y = F_t x$ and hence $F_t$ is closed and even bounded. The linear evaluation map

$$\begin{aligned} \Pi_t : C([0,t];X) &\to X, \\ y(\cdot) &\mapsto y(t) \end{aligned}$$

obviously is bounded and $V(t)x = (\Pi_t \circ F_t)x$, thus $V(t) \in \mathcal{L}(X)$.

Next we need to show that property (iii) in Definition 2.21 also holds for $V(t)$, i.e.

$$V(t)V(s)x = \frac{1}{(n-1)!}\int_0^t \left((t-r)^{n-1}V(s+r)x - (t+s-r)^{n-1}V(r)x\right) dr.$$

By the assumption of the existence of an $n$–integrated solution of (4.1), $V(\cdot)V(s)x$ is an $n$–integrated solution of (4.1) (with $\psi(t) \equiv 0$) and with initial value $x$ replaced by $V(s)x$. Let

$$z(t) = \frac{1}{(n-1)!}\int_0^t \left((t-r)^{n-1}V(s+r)x - (t+s-r)^{n-1}V(r)x\right) dr.$$

We show that the function $t \mapsto z(t)$ also is an $n$–integrated solution of (4.1) with $\psi(t) \equiv 0$ and initial value $V(s)x$. Consider

$$\begin{aligned}
& A\int_0^t z(s)ds \qquad (4.5)\\
= \; & A\int_0^t \left(\frac{1}{(n-1)!}\int_0^s \left((s-u)^{n-1}V(r+u)x - (s+r-u)^{n-1}V(u)x\right) du\right) ds\\
= \; & A\int_0^t\int_0^{s+r} \frac{(s+r-u)^{n-1}}{(n-1)!}V(u)xduds - A\int_0^t\int_0^r \frac{(s+r-u)^{n-1}}{(n-1)!}V(u)xduds\\
& -A\int_0^t\int_0^s \frac{(s+r-u)^{n-1}}{(n-1)!}V(u)xduds.
\end{aligned}$$

First, we calculate the first term of the right–hand side of (4.5)

$$\begin{aligned}
& A\int_0^t\int_0^{s+r} \frac{(s+r-u)^{n-1}}{(n-1)!}V(u)xduds \qquad (4.6)\\
= \; & A\int_r^{t+r}\int_0^s \frac{(s-u)^{n-1}}{(n-1)!}V(u)xduds\\
= \; & A\int_0^{t+r}\int_0^s \frac{(s-u)^{n-1}}{(n-1)!}V(u)xduds - A\int_0^r\int_0^s \frac{(s-u)^{n-1}}{(n-1)!}V(u)xduds\\
= \; & A\int_0^{t+r}\int_0^s \frac{u^{n-1}}{(n-1)!}V(s-u)xduds - A\int_0^r\int_0^s \frac{u^{n-1}}{(n-1)!}V(s-u)xduds\\
= \; & \int_0^{t+r} \frac{u^{n-1}}{(n-1)!}A\int_u^{t+r} V(s-u)xdsdu - \int_0^r \frac{u^{n-1}}{(n-1)!}A\int_u^r V(s-u)xdsdu\\
= \; & \int_0^{t+r} \frac{u^{n-1}}{(n-1)!}A\int_0^{t+r-u} V(s)xdsdu - \int_0^r \frac{u^{n-1}}{(n-1)!}A\int_0^{r-u} V(s)xdsdu\\
= \; & \int_0^{t+r} \frac{u^{n-1}}{(n-1)!}\left(V(t+r-u)x - \frac{(t+r-u)^n}{n!}x\right) du\\
& - \int_0^r \frac{u^{n-1}}{(n-1)!}\left(V(r-u)x - \frac{(r-u)^n}{n!}x\right) du
\end{aligned}$$

$$= \int_0^{t+r} \frac{(t+r-u)^{n-1}}{(n-1)!} V(u)xdu - \int_0^{t+r} \frac{(t+r-u)^{n-1}}{(n-1)!} \frac{u^n}{n!} xdu$$
$$- \int_0^r \frac{(r-u)^{n-1}}{(n-1)!} V(u)xdu + \int_0^r \frac{(r-u)^{n-1}}{(n-1)!} \frac{r^n}{n!} xdu.$$

The second term of the right–hand side of (4.5) can be calculated in the following way.

$$A \int_0^t \int_0^r \frac{(s+r-u)^{n-1}}{(n-1)!} V(u)xduds \tag{4.7}$$
$$= A \int_0^t \int_0^r \frac{(s+u)^{n-1}}{(n-1)!} V(r-u)xduds$$
$$= A \int_0^r V(r-u)x \int_0^t \frac{(s+u)^{n-1}}{(n-1)!} dsdu$$
$$= A \int_0^r V(r-u)x \left( \frac{(t+u)^n}{n!} - \frac{u^n}{n!} \right) du$$
$$= A \int_0^r \frac{(t+u)^n}{n!} V(r-u)xdu - A \int_0^r \frac{u^n}{n!} V(r-u)xdu$$
$$= \int_0^r \frac{(t+u)^{n-1}}{(n-1)!} V(r-u)xdu - \int_0^r \frac{(t+u)^{n-1}}{(n-1)!} \frac{(r-u)^n}{n!} xdu + \frac{t^n}{n!} S(r)x$$
$$- \frac{t^n}{n!} \frac{r^n}{n!} x - \int_0^r \frac{u^{n-1}}{(n-1)!} V(r-u)xdu + \int_0^r \frac{u^{n-1}}{(n-1)!} \frac{(r-u)^n}{n!} xdu$$
$$= \int_0^r \frac{(t+r-u)^{n-1}}{(n-1)!} V(u)xdu - \int_0^r \frac{(t+r-u)^{n-1}}{(n-1)!} \frac{u^n}{n!} xdu + \frac{t^n}{n!} S(r)x$$
$$- \frac{t^n}{n!} \frac{r^n}{n!} x - \int_0^r \frac{(r-u)^{n-1}}{(n-1)!} V(u)xdu + \int_0^r \frac{(r-u)^{n-1}}{(n-1)!} \frac{u^n}{n!} xdu.$$

Furthermore we simplify the last term of the right–hand side of (4.5) as follows.

$$A \int_0^t \int_0^s \frac{(s+r-u)^{n-1}}{(n-1)!} V(u)xdu \tag{4.8}$$
$$= \int_0^t \int_0^s \frac{(r+u)^{n-1}}{(n-1)!} V(s-u)xduds$$
$$= \int_0^t \frac{(r+u)^{n-1}}{(n-1)!} A \int_u^t V(s-u)xdsdu$$
$$= \int_0^t \frac{(r+u)^{n-1}}{(n-1)!} A \int_0^{t-u} V(s)xdsdu$$
$$= \int_0^t \frac{(r+u)^{n-1}}{(n-1)!} \left( V(t-u)x - \frac{(t-u)^n}{n!} x \right) du$$
$$= \int_0^t \frac{(t+r-u)^{n-1}}{(n-1)!} V(u)xdu - \int_0^t \frac{(t+r-u)^{n-1}}{(n-1)!} \frac{u^n}{n!} xdu.$$

Considering (4.6), (4.7) and (4.8), (4.5) becomes

$$
\begin{aligned}
&A\int_0^t z(s)ds &(4.9)\\
= &\int_0^{t+r}\frac{(t+r-u)^{n-1}}{(n-1)!}V(u)xdu-\int_0^{t+r}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu\\
&-\int_0^{r}\frac{(r-u)^{n-1}}{(n-1)!}V(u)xdu+\int_0^{r}\frac{(r-u)^{n-1}}{(n-1)!}\frac{r^n}{n!}xdu\\
&-\int_0^{r}\frac{(t+r-u)^{n-1}}{(n-1)!}V(u)xdu+\int_0^{r}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu\\
&-\frac{t^n}{n!}S(r)x+\frac{t^n}{n!}\frac{r^n}{n!}x\\
&+\int_0^{r}\frac{(r-u)^{n-1}}{(n-1)!}V(u)xdu-\int_0^{r}\frac{(r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu\\
&-\int_0^{t}\frac{(t+r-u)^{n-1}}{(n-1)!}V(u)xdu+\int_0^{t}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu\\
= &\int_0^{t+r}\frac{(t+r-u)^{n-1}}{(n-1)!}V(u)xdu-\int_0^{r}\frac{(t+r-u)^{n-1}}{(n-1)!}V(u)xdu\\
&-\int_0^{t}\frac{(t+r-u)^{n-1}}{(n-1)!}V(u)xdu-\frac{t^n}{n!}S(r)x\\
&+\left(\frac{t^n}{n!}\frac{r^n}{n!}x-\left(\int_0^{t+r}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n}xdu\right.\right.\\
&\left.\left.-\int_0^{r}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu-\int_0^{t}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu\right)\right).
\end{aligned}
$$

But by integration by parts we have

$$\int_0^{t+r}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu=\frac{(t+r)^{2n+1}}{(2n+1)!}x. \tag{4.10}$$

Similarly, we have

$$
\begin{aligned}
&\int_0^{r}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu &(4.11)\\
= &\frac{t^{n-1}}{(n-1)!}\frac{r^{n+1}}{(n+1)!}x+\frac{t^{n-2}}{(n-2)!}\frac{r^{n+2}}{(n+2)!}x\\
&+\frac{t^{n-3}}{(n-3)!}\frac{r^{n+3}}{(n+3)!}x+\cdots+t\frac{r^{2n}}{(2n)!}x+\frac{r^{2n+1}}{(2n+1)!}x
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{t}\frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu &(4.12)\\
= &\frac{r^{n-1}}{(n-1)!}\frac{t^{n+1}}{(n+1)!}x+\frac{r^{n-2}}{(n-2)!}\frac{t^{n+2}}{(n+2)!}x\\
&\frac{r^{n-3}}{(n-3)!}\frac{t^{n+3}}{(n+3)!}x+\cdots+r\frac{t^{2n}}{(2n)!}x+\frac{t^{2n+1}}{(2n+1)!}x.
\end{aligned}
$$

Now adding (4.10), (4.11) and (4.12) we have

$$\int_0^{t+r} \frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu - \int_0^{r} \frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu - \int_0^{t} \frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu = \frac{t^n}{n!}\frac{r^n}{n!}x.$$

Therefore

$$\frac{t^n}{n!}\frac{r^n}{n!}x - \left(\int_0^{t+r} \frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu - \int_0^{r} \frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu - \int_0^{t} \frac{(t+r-u)^{n-1}}{(n-1)!}\frac{u^n}{n!}xdu\right) = 0.$$

Hence, (4.5) becomes

$$\begin{aligned} A\int_0^t z(s)ds &= \int_0^{t+s} \frac{(t+s-r)^{n-1}}{(n-1)!}V(r)xdr - \frac{t^n}{n!}V(s)x \\ &\quad - \int_0^{s} \frac{(t+s-r)^{n-1}}{(n-1)!}V(r)xdr - \int_0^{t} \frac{(t+s-r)^{n-1}}{(n-1)!}V(r)xdr \\ &= z(t) + \frac{t^n}{n!}V(s)x. \end{aligned} \tag{4.13}$$

Hence $t \mapsto z(t)$ is the $n$–integrated solution of (4.1) with initial value $V(s)x$ and with $\psi(t) \equiv 0$. And by the assumption of the uniqueness of the $n$–integrated solution of (4.1) we have $z(t) = V(t)V(s)x$. Thus $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ is an $n$–times integrated semigroup.

It remains to show that the $n$–times integrated semigroup is generated by $A$. Let $x \in D(A)$ and $v(\cdot, Ax) := V(t)Ax$ the $n$–integrated solution of (4.1) with initial value $Ax$. Set

$$w(t) = \frac{t^n}{n!}x + \int_0^t V(s)Axds,\ t \geq 0.$$

Since $V(s)Ax$ is an $n$–integrated solution, we have

$$\begin{aligned} V(s)Ax &= \frac{s^n}{n!}Ax + A\int_0^s V(r)Axdr \\ &= A\left(\frac{s^n}{n!}x + \int_0^s V(r)Axdr\right) \\ &= Aw(s) \end{aligned}$$

Hence

$$\begin{aligned} w(t) &= \frac{t^n}{n!}x + A\int_0^t V(s)Axds \\ &= \frac{t^n}{n!}x + A\int_0^t w(s)ds. \end{aligned}$$

Thus $w(t)$ is an $n$–integrated solution of (4.1) with initial value $x$ and $\psi(t) \equiv 0$. The uniqueness of the solution implies $w(t) = V(t)x$. Therefore

$$V(t)x = \frac{t^n}{n!}x + \int_0^t V(s)Axds,$$

that is $A$ generates the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$.

■

In [3], the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ generated by $A$ is defined as the strongly continuous function $t \mapsto V(t) \in \mathcal{L}(X)$ satisfying

$$\int_0^t V(s)xds \in D(A)$$

and

$$V(t)x = \frac{t^n}{n!}x + A\int_0^t V(s)xds, \ \ t \in [0,\infty), \ \ x \in X.$$

In the theorem above we proved that the $n$–times integrated semigroup defined in Definition 2.21 is equivalent to the semigroup defined in [3].

Now recall the inhomogeneous Cauchy problem (4.1) for any $\psi(t) \in L^1_{loc}((0,\infty); X)$. In the following theorem, we show that $A$ generates an $n$–times integrated semigroup is necessary for the existence of a unique $n$–integrated solution of (4.1). Moreover, we give an explicit representation of the solution in terms of the $n$–times integrated semigroup generated by $A$. We need the following proposition to show the uniqueness of the solution. The proposition was proved in [59], Theorem 3.7 for the case when $A$ generates a once integrated semigroup. We use the same idea to show that it also holds for the case when $A$ generates an $n$–times integrated semigroup.

**Proposition 4.6** *Let $A$ be the generator of the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ and let $u : [0,\infty) \to X$ be a continuous function such that $\int_0^t u(s)ds \in D(A)$ and*

$$A\int_0^t u(s)ds = u(t), \quad t \in [0,\infty).$$

*Then $u \equiv 0$ in $[0,\infty)$.*

**Proof.** The function $r \mapsto \int_0^r u(s)ds$ is continuously differentiable. Consider

$$V(t-r)\int_0^r u(s)ds.$$

Using the commutativity, $AV(t)x = V(t)Ax, x \in D(A)$, by (ii) Proposition 4.2 we have

$$V'(t)x = \frac{t^{n-1}}{(n-1)!}x + V(t)Ax = \frac{t^{n-1}}{(n-1)!}x + AV(t)x.$$

Thus,

$$\begin{aligned}
&\frac{d}{dr}\left(V(t-r)\int_0^r u(s)ds\right)\\
= &-AV(t-r)\int_0^r u(s)ds - \frac{(t-r)^{n-1}}{(n-1)!}\int_0^r u(s)ds + V(t-r)u(r)\\
= &-V(t-r)A\int_0^r u(s)ds - \frac{(t-r)^{n-1}}{(n-1)!}\int_0^r u(s)ds + V(t-r)u(r)\\
= &-V(t-r)u(r) - \frac{(t-r)^{n-1}}{(n-1)!}\int_0^r u(s)ds + V(t-r)u(r)\\
= &-\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r u(s)ds.
\end{aligned}$$

Now integrating this equation and using the fact that $V(0) = 0$ we obtain

$$-\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\int_0^r u(s)dsdr = 0. \tag{4.14}$$

By differentiating equation (4.14) $(n+1)$–times we obtain $u(t) \equiv 0$ for all $t \in [0,\infty)$.

■

**Theorem 4.7** *Let $A$ be the generator of the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. Then*

$$t \mapsto w(t) = V(t)x + \int_0^t V(t-s)\psi(s)ds \tag{4.15}$$

*defines a unique $n$–integrated solution of (4.1).*

**Proof.** Since $V(t)$ is the $n$–times integrated semigroup generated by $A$, it is clear that $t \mapsto w(t)$ defined by (4.15) is continuous. Moreover $w(t) \in D(A)$, for all $t \in [0,\infty)$. We show that $w(t)$ satisfies

$$w(t) = \frac{t^n}{n!}x + A\int_0^t w(s)ds + \int_0^t \frac{(t-s)^n}{n!}\psi(s)ds,$$

for all $t \in [0, \infty)$. Using property (ii) Proposition 4.2 and closedness of $A$, we have

$$\begin{aligned}
A\int_0^t w(s)ds &= A\int_0^t V(s)xds + A\int_0^t\int_0^s V(s-r)\psi(r)drds \\
&= V(t)x - \frac{t^n}{n!}x + \int_0^t A\int_r^t V(s-r)ds\psi(r)dr \\
&= V(t)x - \frac{t^n}{n!}x + \int_0^t A\int_0^{t-r} V(s)ds\psi(r)dr \\
&= V(t)x - \frac{t^n}{n!}x + \int_0^t \left(V(t-r) - \frac{(t-r)^n}{n!}\right)\psi(r)dr \\
&= V(t)x - \frac{t^n}{n!}x \\
&\quad + \int_0^t V(t-r)\psi(r)dr - \int_0^t \frac{(t-r)^n}{n!}\psi(r)dr \\
&= w(t) - \frac{t^n}{n!}x - \int_0^t \frac{(t-r)^n}{n!}\psi(r)dr.
\end{aligned}$$

Hence, $t \mapsto w(t)$ given in formula (4.15) is an $n$–integrated solution of the inhomogeneous Cauchy problem (4.1). To show the uniqueness, suppose that there exists another $n$–integrated solution $t \mapsto w_0(t)$ of (4.1). Take $\tilde{w}(t) = w(t) - w_0(t)$, then $t \mapsto \tilde{w}(t)$ is a solution of

$$\tilde{w}(t) = A\int_0^t \tilde{w}(s)ds$$

and by Proposition 4.6, $\tilde{w}(t) = 0$. Thus $w(t) = w_0(t)$. Hence the solution is unique.

■

The following proposition helps to obtain conditions for the existence of unique classical solution of problem (4.1).

**Proposition 4.8** *Let $A$ be the generator of an $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$.*

*(i) For all $x \in D(A^{n+1})$, we have*

$$V^{(n)}(t)x = V(t)A^n x + \sum_{i=0}^{n-1} \frac{t^i}{i!}A^i x,$$

*and*

$$V^{(n+1)}(t)x = AV^{(n)}(t)x.$$

*(ii) If $\psi(t) \in D(A^{n+1})$ for all $t \in [0, \infty)$, then the function $t \mapsto v_0(t)$ given by*

$$v_0(t) := \int_0^t V(t-r)\psi(r)dr$$

*is $(n+1)$–times continuously differentiable, and*

$$v_0^{(n)}(t) = \int_0^t V(t-r)A^n\psi(r)dr + \sum_{i=0}^{n-1} \int_0^t \frac{(t-r)^i}{i!} A^i\psi(r)dr.$$

*Furthermore, we have*

$$v_0^{(n+1)}(t) = Av_0^{(n)}(t) + \psi(t).$$

**Proof.** (i) Since $x \in D(A^{n+1})$, by the property of the semigroup we have

$$\begin{aligned} V'(t)x &= AV(t)x + \frac{t^{n-1}}{(n-1)!}x \\ &= V(t)Ax + \frac{t^{n-1}}{(n-1)!}x \end{aligned}$$

and

$$\begin{aligned} V''(t)x &= V'(t)Ax + \frac{t^{n-2}}{(n-2)!}x \\ &= AV(t)Ax + \frac{t^{n-1}}{(n-1)!}Ax + \frac{t^{n-2}}{(n-2)!}x \\ &= V(t)A^2x + \frac{t^{n-1}}{(n-1)!}Ax + \frac{t^{n-2}}{(n-2)!}x. \end{aligned}$$

By differentiating $n$–times we obtain

$$V^{(n)}(t)x = V(t)A^n x + \sum_{i=0}^{n-1} \frac{t^i}{i!} A^i x$$

and

$$\begin{aligned} V^{(n+1)}(t)x &= V'(t)A^n x + \sum_{i=1}^{n-1} \frac{t^i}{i!} A^i x \\ &= AV(t)A^n x + \frac{t^{n-1}}{(n-1)!} A^n x + A \sum_{i=0}^{n-2} \frac{t^i}{i!} A^i x \\ &= A(V(t)A^n x + \sum_{i=0}^{n-1} \frac{t^i}{i!} A^i x) \\ &= AV^{(n)}(t)x. \end{aligned}$$

(ii) Differentiating $t \mapsto v_0(t)$ we have

$$v_0'(t) = \int_0^t V'(t-r)\psi(r)dr \tag{4.16}$$

and using that $\psi(t) \in D(A^{n+1}) \subset X$ and that $V(t)$ is the $n$–times integrated semigroup generated by $A$ we have

$$\begin{aligned} v_0'(t) &= \int_0^t V'(t-r)\psi(r)dr \\ &= \int_0^t (AV(t-r)\psi(r) + \frac{(t-r)^{n-1}}{(n-1)!}\psi(r))dr \\ &= A\int_0^t V(t-r)\psi(r)dr + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\psi(r)dr \\ &= \int_0^t V(t-r)A\psi(r)dr + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\psi(r)dr. \end{aligned}$$

Therefore

$$v_0'(t) = \int_0^t V(t-r)A\psi(r)dr + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\psi(r)dr.$$

Furthermore

$$\begin{aligned} v_0''(t) &= \int_0^t V'(t-r)A\psi(r)dr + \int_0^t \frac{(t-r)^{n-2}}{(n-2)!}\psi(r)dr \\ &= \int_0^t (AV(t-r)A\psi(r) + \frac{(t-r)^{n-1}}{(n-1)!}A\psi(r))dr + \int_0^t \frac{(t-r)^{n-2}}{(n-2)!}\psi(r)dr \\ &= A\int_0^t V(t-r)A\psi(r)dr + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!}A\psi(r)dr + \int_0^t \frac{(t-r)^{n-2}}{(n-2)!}\psi(r)dr \\ &= \int_0^t V(t-r)A^2\psi(r)dr + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!}A\psi(r)dr + \int_0^t \frac{(t-r)^{n-2}}{(n-2)!}\psi(r)dr. \end{aligned}$$

By continuing this process we obtain

$$v_0^{(n)} = \int_0^t V(t-r)A^n\psi(r)dr + \sum_{i=0}^{n-1}\int_0^t \frac{(t-r)^i}{i!}A^i\psi(r)dr,$$

and

$$\begin{aligned} v_0^{(n+1)}(t) &= \int_0^t V'(t-r)A^n\psi(r)dr + \sum_{i=1}^{n-1}\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}A^i\psi(r)dr + \psi(t) \\ &= A\int_0^t V(t-r)A^n\psi(r)dr + \int_)^t \frac{(t-r)^{n-1}}{(n-1)!}A^n\psi(r)dr \\ &\quad + A\sum_{i=0}^{n-2}\int_0^t \frac{(t-r)^i}{i!}A^i\psi(r) + \psi(t) \end{aligned}$$

$$= A(\int_0^t V(t-r)A^n\psi(r)dr + \sum_{i=0}^{n-1}\int_0^t \frac{(t-r)^i}{i!}A^i\psi(r)dr) + \psi(t)$$
$$= Av_0^{(n)}(t) + \psi(t).$$

■

By the above proposition, we have the following result.

**Corollary 4.9** *Let $A$ be the generator of the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ and let $\psi(t) \in D(A^{n+1})$ for all $t \geq 0$. Then for all $x \in D(A^{n+1})$ the expression*

$$t \mapsto u(t) := V(t)A^n x + \sum_{i=0}^{n-1}\frac{t^i}{i!}A^i x + \int_0^t V(t-r)A^n\psi(r)dr + \sum_{i=0}^{n-1}\int_0^t \frac{(t-r)^i}{i!}A^i\psi(r)dr$$

*defines a classical solution of the inhomogeneous abstract Cauchy problem (4.1).*

## 4.2 $n$–Weak Solutions

In this section we discuss the $n$–weak solution of the inhomogeneous abstract Cauchy problem (4.1). By $A^*$ we denote the adjoint operator of $A$.

**Definition 4.10** *A function $w(\cdot) \in C([0,\infty); X)$ is said to be an $n$–weak solution of the inhomogeneous abstract Cauchy problem (4.1) if $w(0) = 0$ and if for all $x^* \in D(A^*)$ the scalar function $t \mapsto \langle w(t), x^*\rangle$ is differentiable on $[0,\infty)$ and satisfies*

$$\frac{d}{dt}\langle w(t), x^*\rangle = \langle w(t), A^*x^*\rangle + \langle \frac{t^n}{n!}x, x^*\rangle + \langle \int_0^t \frac{(t-s)^n}{n!}\psi(s)ds, x^*\rangle. \tag{4.17}$$

**Lemma 4.11** *Let $t \mapsto v(t)$ be an $n$–integrated solution of (4.1). Then $t \mapsto v(t)$ is an $(n-1)$–weak solution of (4.1).*

**Proof.** Since $t \mapsto v(t)$ is an $n$–integrated solution of (4.1), $v(\cdot) \in C([0,\infty); X)$ and for all $x \in X$ we have

$$v(t) = \frac{t^n}{n!}x + A\int_0^t v(s)ds + \int_0^t \frac{(t-s)^n}{n!}\psi(s)ds.$$

Moreover, for all $x^* \in D(A^*)$, we have

$$\begin{aligned}\langle v(t), x\rangle &= \langle A\int_0^t v(s)ds, x^*\rangle + \langle\frac{t^n}{n!}x, x^*\rangle + \langle\int_0^t \frac{(t-s)^n}{n!}\psi(s)ds, x^*\rangle \\ &= \langle\int_0^t v(s)ds, A^*x^*\rangle + \langle\frac{t^n}{n!}x, x^*\rangle + \langle\int_0^t \frac{(t-s)^n}{n!}\psi(s)ds, x^*\rangle.\end{aligned}$$

Hence by differentiating the above equation, we obtain

$$\frac{d}{dt}\langle v(t), x\rangle = \langle v(t), A^*x^*\rangle + \langle\frac{t^{n-1}}{(n-1)!}x, x^*\rangle + \langle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\psi(s)ds, x^*\rangle.$$

Thus $t \mapsto v(t)$ is an $(n-1)$–weak solution of (4.1). ∎

We showed in Proposition 4.7 that there exists a unique $n$–integrated solution of (4.1) if the operator $A$ in the problem (4.1) generates an $n$–times integrated semigroup. In the lemma above, we proved that any $n$–integrated solution is an $(n-1)$–weak solution. Hence for those $A$, the function given by (4.15) defines a unique $(n-1)$–weak solution of (4.1).

**Corollary 4.12** *Let $A$ be the generator of the $n$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. Then $t \mapsto w(t)$ given by formula (4.15) is a unique $(n-1)$–weak solution of the inhomogeneous abstract Cauchy problem (4.1).*

**Proof.** By Proposition 4.7, the function given by (4.15):

$$t \mapsto w(t) = V(t)x + \int_0^t V(t-s)\psi(s)ds$$

defines a solution of (4.1). By the lemma above, it is an $(n-1)$–weak solution of (4.1). It remains to show that the solution is unique. The proof of the uniqueness is similar to the proof of the uniqueness of the $n$–integrated solution and we omit it. Again, it is based on Proposition 4.6. ∎

Having the result above we deduce the following result for the existence of a unique $n$–weak solution of (4.1).

**Corollary 4.13** *Let $A$ be generator of an $(n+1)$–times integrated semigroup $\{V(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. Then there exists a unique $n$–weak solution of (4.1).*

In the following discussion we connect the concepts of the $n$–weak solution and of the weak solution for the inhomogeneous abstract Cauchy problem.

**Definition 4.14** *A function $u(\cdot) \in C([0,\infty); X)$ is said to be a weak solution of the Cauchy problem (4.1) if $u(0) = x$ and if for each $x^* \in D(A^*)$ the scalar function $t \mapsto \langle u(t), x^* \rangle$ is continuously differentiable on $[0,\infty)$ and*

$$\frac{d}{dt}\langle u(t), x^* \rangle = \langle u(t), A^* x^* \rangle + \langle \psi(t), x^* \rangle.$$

**Proposition 4.15** *Let $t \mapsto u(t)$ be a weak solution of the problem (4.1). Then the function*

$$t \mapsto \int_0^t \frac{(t-s)^n}{n!} u(s) ds$$

*is an $n$–weak solution of (4.1).*

**Proof.** It is clear that $t \mapsto \int_0^t \frac{(t-s)^n}{n!}$ is continuous. Let $x^* \in D(A^*)$. Since $t \mapsto u(t)$ is a weak solution, we have

$$\frac{d}{dt}\langle u(t), x^* \rangle = \langle u(t), A^* x^* \rangle + \langle \psi(t), x^* \rangle,$$

and hence

$$\langle u(t), x^* \rangle = \langle x, x^* \rangle + \langle \int_0^t u(s) ds, A^* x^* \rangle + \langle \int_0^t \psi(s) ds, x^* \rangle.$$

Now we consider

$$\begin{aligned} & \langle \int_0^t \frac{(t-s)^n}{n!} u(s) ds, x^* \rangle \qquad (4.18) \\ = & \int_0^t \frac{(t-s)^n}{n!} \langle u(s), x^* \rangle ds \\ = & \int_0^t \frac{(t-s)^n}{n!} \left( \langle x, x^* \rangle + \langle \int_0^s u(r) dr, A^* x^* \rangle + \langle \int_0^s \psi(r) dr, x^* \rangle \right) ds \\ = & \langle \int_0^t \frac{(t-s)^n}{n!} x ds, x^* \rangle + \langle \int_0^t \frac{(t-s)^n}{n!} \int_0^s u(r) dr ds, A^* x^* \rangle \\ & + \langle \int_0^t \frac{(t-s)^n}{n!} \int_0^s \psi(r) dr ds, x^* \rangle \\ = & \langle \frac{t^{n+1}}{(n+1)!} x, x^* \rangle + \langle \int_0^t \frac{(t-s)^{n+1}}{(n+1)!} u(s) ds, A^* x^* \rangle \\ & + \langle \int_0^t \frac{(t-s)^{n+1}}{(n+1)!} \psi(s) ds, x^* \rangle. \end{aligned}$$

Differentiating (4.18) we obtain

$$\begin{aligned}\frac{d}{dt}\langle\int_0^t \frac{(t-s)^n}{n!}u(s)ds, x^*\rangle &= \langle\int_0^t \frac{(t-s)^n}{n!}u(s)ds, x^*\rangle + \langle\frac{t^n}{n!}x, x^*\rangle \\ &\quad +\langle\int_0^t \frac{(t-s)^n}{n!}\psi(s)ds, x^*\rangle.\end{aligned}$$

Hence, $t \mapsto \int_0^t \frac{(t-s)^n}{n!}u(s)ds$ is an $n$–weak solution of (4.1). ∎

## 4.3 $K$–Generalized Solutions

In this section we discuss $K$–generalized solutions of the inhomogeneous problem (4.1).

**Definition 4.16** *A function $v(\cdot) \in C([0,\infty); X)$ is called a $K$-generalized solution of (4.1) if for all $t \in [0,\infty)$ hold $\int_0^t v(s)ds \in D(A)$ and*

$$v(t) = A\int_0^t v(s)ds + K(t)x + (K * \psi)(t), \tag{4.19}$$

*where $K(t) = \int_0^t k(s)ds$, $t \mapsto k(t)$ is a continuous function on $[0,\infty)$.*

We showed in the first section of this chapter, for the case that $K(t) = \frac{t^n}{n!}$, the condition that $A$ generates an $n$–times integrated semigroup is necessary for the existence of a unique solution of (4.1) for any initial value in $X$. We study the $K$–generalized solution for the case that the operator $A$ in problem (4.1) generates a $K$–convoluted semigroup.

**Definition 4.17** *Let $A$ be a closed operator. If there exists a strongly continuous operator family $\{S_K(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ such that*
*(i) for $x \in D(A)$, $S_K(t)x \in D(A)$ and $AS_K(t)x = S_K(t)Ax$ for $t \in [0,\infty)$,*
*(ii) for $x \in X$, $\int_0^t S_K(s)xds \in D(A)$ and*

$$S_K(t)x = A\int_0^t S_K(s)xds + K(t)x, \quad t \in [0,\infty). \tag{4.20}$$

*Then we call $\{S_K(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$ the $K$-convoluted semigroup generated by $A$.*

For the homogeneous problem, it is shown in [12], Theorem 2.3 that the condition that $A$ generates a $K$–convoluted semigroup is necessary and sufficient for the existence of a unique $K$–generalized solution of (4.1). In the theorem below we show that $A$ generates a $K$–convoluted semigroup is necessary for the existence of a $K$–generalized solution of (4.1) for any $\psi(t) \in L^1_{loc}((0,\infty);X)$.

**Theorem 4.18** *Let $A$ be the generator of a $K$–convoluted semigroup $\{S_K(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. Then the function*

$$t \mapsto w(t) = S_K(t)x + \int_0^t S_K(t-s)\psi(s)ds \tag{4.21}$$

*defines a $K$–generalized solution of (4.1).*

**Proof.** Since $A$ is the generator of a $K$–convoluted semigroup $\{S_K(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$, $t \mapsto w(t)$ given by (4.21) is continuous.

We show that $t \mapsto w(t)$ defined in (4.21) satisfies (4.19):

$$\begin{aligned}
A\int_0^t w(s)ds &= A\int_0^t S_K(s)ds + A\int_0^t\int_0^s S_K(s-\tau)\psi(\tau)d\tau ds \\
&= S_K(t)x - K(t)x + \int_0^t\int_0^s S_K(s-\tau)\psi(\tau)d\tau ds \\
&= S_K(t)x - K(t)x + \int_0^t A\int_\tau^t S_K(s-\tau)ds\psi(\tau)d\tau \\
&= S_K(t)x - K(t)x + \int_0^t A\int_0^{t-\tau} S_K(s)ds\psi(\tau)d\tau \\
&= S_K(t)x - K(t)x + \int_0^t \left(S_K(t-\tau) - K(t-\tau)\right)\psi(\tau)d\tau \\
&= S_K(t)x - K(t)x + \int_0^t S_K(t-\tau)\psi(\tau)d\tau - \int_0^t K(t-\tau)\psi(\tau)d\tau \\
&= w(t) - K(t)x - (K*\psi)(t).
\end{aligned}$$

Therefore

$$w(t) = A\int_0^t w(s)ds + K(t)x + (K*\psi)(t).$$

Hence, $t \mapsto w(t)$ is a $K$–generalized solution of (4.1). ∎

**Definition 4.19** *A differential operator of infinite order*

$$P\left(\frac{d}{dt}\right) := \sum_{n=0}^{\infty} \alpha_n \frac{d^n}{dt^n}, \quad \alpha_n \in \mathbb{C}$$

*is called an ultradifferential operator of class* $(M_n)$ *if there exists a constant* $L > 0$ *such that*

$$|\alpha_n| = O\left(\frac{L^n}{M_n}\right).$$

For the case that $t \mapsto k(t)$, $(\int_0^t k(s)ds = K(t))$, is the solution of the ultradifferential equation $P(\frac{d}{dt})k(t) = \delta$, where $P$ is the ultradifferential polynomial of Gevrey class $n!^{\frac{1}{a}}$, $0 < a < 1$, it is shown in [12] that, if $A$ generates a $K$–convoluted semigroup which is also shown equivalent to the existence of a unique $K$–generalized solution of the homogeneous problem, (4.1) has a unique once–integrated solution for any intial value $x$ from a smaller subset of $X$.

**Proposition 4.20** *[12] Let* $h > L^{-1}$ *and asumme that* $A$ *generates a* $K$*-convoluted semigroup* $\{S_K(t) \in \mathcal{L}(X) : t \in [0, \infty)\}$*. Define*

$$X_h := \{x \in X : S_K(\cdot)x \in C^\infty([0,\infty), X), \|S_K^{(n)}(t)x\| = O(h^n n!^{\frac{1}{a}}), n \in \mathbb{N}\}.$$

*Then the Cauchy problem 4.1 has a unique once–integrated solution for any* $x \in X_h$.

Using this result we derive the following corollary.

**Corollary 4.21** *Suppose that all the assumptions of Proposition 4.20 hold. Then the function given by*

$$t \mapsto w(t) := P\left(\frac{d}{dt}\right)\left(S_K(t)x + \int_0^t S_K(t-s)\psi(s)ds\right)$$

*defines the unique once–integrated solution of the inhomogeneous problem (4.1) for any initial value* $x \in X_h$.

**Proof.** Consider $Aw(t)$ for $w(t)$ given by

$$w(t) = P\left(\frac{d}{dt}\right)\left(S_K(t)x + \int_0^t S_K(t-s)\psi(s)ds\right).$$

Then

$$\begin{aligned}
A\int_0^t w(s)ds &= AP\left(\frac{d}{dt}\right)\left(\int_0^t S_K(s)xds + \int_0^t\int_0^s S_K(s-\tau)\psi(\tau)d\tau ds\right) \\
&= P\left(\frac{d}{dt}\right)A\int_0^t S_K(s)xds + P\left(\frac{d}{dt}\right)\int_0^t A\int_0^s S_K(s-\tau)\psi(\tau)d\tau ds \\
&= P\left(\frac{d}{dt}\right)(S_K(tx - K(t)x) + P\left(\frac{d}{dt}\right)\int_0^t \psi(\tau)A\int_\tau^t S_K(s-\tau)dsd\tau \\
&= P\left(\frac{d}{dt}\right)(S_K(t)x - K(t)x) + P\left(\frac{d}{dt}\right)\left(\int_0^t A\int_0^{t-\tau} S_K(s)ds\psi(\tau)d\tau\right) \\
&= P\left(\frac{d}{dt}\right)(S_K(t)x - K(t)x) \\
&\quad +P\left(\frac{d}{dt}\right)\left(\int_0^t (S_K(t-\tau) - K(t-\tau))\,\psi(\tau)d\tau\right) \\
&= P\left(\frac{d}{dt}\right)\left(S_K(t)x + \int_0^t S_K(t-\tau)\psi(\tau)d\tau\right) - P\left(\frac{d}{dt}\right)K(t)x \\
&\quad -P\left(\frac{d}{dt}\right)\int_0^t K(t-\tau)\psi(\tau)d\tau \\
&= w(t) - x - \int_0^t \psi(s)ds.
\end{aligned}$$

Therefore

$$w(t) = P\left(\frac{d}{dt}\right)\left(S_K(t)x + \int_0^t S_K(t-s)\psi(s)ds\right)$$

is a once–integrated solution of the problem (4.1). ■

In [43], it is shown that, if $A$ generates a $K$–convoluted semigroup, the homogeneous Cauchy problem, problem (4.1) with $\psi(t) \equiv 0$, has a classical solution for any initial value $x \in D(P(A))$ where $P$ is the ultradifferential polynomial operator of class $(M_n)$ and $t \mapsto k(t)$ is the solution of the ultradifferential equation $P(\frac{d}{dt})k(t) = 0$ with the initial condition

$$\sum_{n=0}^{\infty} k^{(n)}(0) \sum_{p=n+2}^{\infty} \alpha_p A^{p-n-1} = 0.$$

**Theorem 4.22** *[43] Let $t \mapsto k(t)$ be a solution of the differential equation $P(\frac{d}{dt})k(t) =$ $0$, $t \geq 0$ with the initial condition*

$$\sum_{n=0}^{\infty} k^{(n)}(0) \sum_{p=n+2}^{\infty} \alpha_p A^{p-n-1} = 0,$$

*where the coefficients of the ultradifferential operator* $P(\frac{d}{dt})$ *of class* $(M_n)$ *satisfy the condition*

$$|\frac{\alpha_n}{\alpha_m \alpha_{n-m}}| \leq C, \quad \text{for all } n \in \mathbb{N} \quad \text{with } n > m$$

*for some* $C > 0$. *Let* $A$ *generate the* $K$*-convoluted semigroup* $\{S_K(t) \in \mathcal{L}(X) : t \in [0,\infty)\}$. *Then* $t \mapsto U(t)x := P(\frac{d}{dt})S_K(t)x$ *is a solution of the homogeneous Cauchy problem for any* $x \in X$ *with* $Ax \in D(P(A))$.

**Corollary 4.23** *Under the assumptions of Theorem 4.22 we have that*

$$t \mapsto w(t) = P\left(\frac{d}{dt}\right)\left(S_K(t)x + \int_0^t S_K(t-s)\psi(s)ds\right)$$

*is a solution of the inhomogeneous problem (4.1) for any* $x \in X$ *and* $t \mapsto \psi(t)$ *with* $Ax, A\psi(t) \in D(P(A))$.

# Chapter 5

# Stochastic Differential Equations

In this chapter, we consider the stochastic differential equation

$$dX(t) = AX(t)dt + BdW(t), \ X(0) = \xi, \tag{5.1}$$

in a separable Hilbert space $H$, where $A : D(A) \subset H \to H$ is a closed linear operator and $B : H \to H$ is a bounded linear operator, $W(\cdot)$ is an $H$–valued Wiener process in a probability space $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\xi$ is an $H$–valued random variable. By $H$–valued random variable, we understand an $H$–valued mapping $\xi : \Omega \to H$ which is measurable from $(\Omega, \mathcal{F})$ to $(H, \mathcal{B}(H))$, where $\mathcal{B}(H)$ is the smallest $\sigma$–field containing all closed (or open) subsets of $H$. A stochastic process $X(t)$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if, for any $t \geq 0$, $X(t)$ is $\{\mathcal{F}_t\}$–measurable. We call a stochastic process $X(t)$ independent if the $\sigma$–fields $\{\sigma(X(t))\}_{t\geq 0}$ are independent. A stochastic process $X(t)$ is called $H$–valued predictable if $X : [0, \infty) \times \Omega \to H$ (or $X : [0, T] \times \Omega \to H$) is $\mathcal{P}_\infty$–measurable (respectively $\mathcal{P}_T$–measurable), where $\mathcal{P}_\infty$ is a $\sigma$–field generated by sets of the form:

$$(s, t] \times F, \ 0 \leq s < t, \ F \in \mathcal{F}_s \text{ and } \{0\} \times F, \ F \in \mathcal{F}_0,$$

and $\mathcal{P}_T$ is the restriction of $\mathcal{P}_\infty$ to $[0, T]$. We say an operator $L$ is of trace class if its trace is bounded.

We denote by $L^2(\Omega; H)$ the Banach space of all $H$–valued square integrable mappings endowed with the norm

$$\|X\|_2 := \left(E[\|X\|^2]\right)^{\frac{1}{2}},$$

and by $C_W([0,T]; H)$ the Banach space of all mappings $X : [0,T] \to L^2(\Omega; H)$, that are continuous and adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, endowed with the norm

$$\|X(\cdot)\|_{C_W([0,T];H)} := \sup_{t\in[0,T]} \left(E[\|X(t)\|^2]\right)^{\frac{1}{2}}.$$

Let furthermore $\{e_k\}_{k\in\mathbb{N}}$ be a complete orthonormal system in $H$ and $\{\beta_k(\cdot)\}_{k\in\mathbb{N}}$ be a sequence of independent real Brownian motions on $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that, for all $y \in H$, we can write

$$\langle W(t), y\rangle = \sum_{k=1}^{\infty} \beta_k(t)\langle e_k, y\rangle.$$

## 5.1 $n$–Integrated Solutions and Weak $n$–Integrated Solutions

In this section, we discuss $n$–integrated solutions and weak $n$–integrated solutions of problem (5.1). We give conditions for the existence of solutions and the existence of continuous versions of solutions.

Assume that the operator $A$ in problem (5.1) generates an exponentially bounded $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$. We consider the stochastic convolution

$$W_n(t) := \int_0^t V_n(t-s)BdW(s) = \sum_{k=1}^{\infty}\int_0^t V_n(t-s)Be_k d\beta_k(s). \tag{5.2}$$

We show that the sequence in (5.2) is convergent in $L^2(\Omega, H)$.

**Proposition 5.1** *Let $A$ be a generator of an $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ and assume that the operator*

$$L_t x := \int_0^t V_n(s)BB^*V_n^*(s)x ds, \quad x \in H \tag{5.3}$$

*is of trace class. Then for all $t > 0$, the series in (5.2) is convergent in $L^2(\Omega; H)$ to a Gaussian random variable $W_n(t)$ with mean zero and covariance operator $L_t$. Moreover, $W_n(\cdot)$ belongs to $C_W([0,T]; H)$ for any $T > 0$.*

**Proof.** First, we show that the series in (5.2) is convergent in $L^2(\Omega; H)$. Let $t > 0$. Using the independence of $\{\beta_k\}$ and Ito's isometry, for $m, p \in \mathbb{N}$ we have

$$\begin{aligned} E\left[\left\|\sum_{k=m+1}^{m+p}\int_0^t V_n(t-s)Be_k d\beta_k(s)\right\|^2\right] &= E\left[\sum_{k=m+1}^{m+p}\left\|\int_0^t V_n(t-s)Be_k d\beta_k(s)\right\|^2\right] \\ &= \sum_{k=m+1}^{m+p} E\left[\left\|\int_0^t V_n(t-s)Be_k d\beta_k(s)\right\|^2\right] \\ &= \sum_{k=m+1}^{m+p}\int_0^t \|V_n(t-s)Be_k\|^2 ds. \end{aligned}$$

Since $L_t$ is of trace class,

$$\begin{aligned} \text{Tr } L_t &= \sum_{k=1}^{\infty}\langle L_t e_k, e_k\rangle \\ &= \sum_{k=1}^{\infty}\langle \int_0^t V_n(s)BB^*V_n^*(s)e_k ds, e_k\rangle \\ &= \sum_{k=1}^{\infty}\int_0^t \langle V_n(s)BB^*V_n^*(s)e_k, e_k\rangle ds \\ &= \sum_{k=1}^{\infty}\int_0^t \langle B^*V_n^*(s)e_k, B^*V_n^*(s)e_k\rangle ds \\ &= \sum_{k=1}^{\infty}\int_0^t \langle V_n(s)Be_k, V_n(s)Be_k\rangle ds \\ &= \sum_{k=1}^{\infty}\int_0^t \|V_n(s)Be_k\|^2 ds < \infty, \end{aligned}$$

and we have $\|W_n(t)\|_{L^2(\Omega;H)} < \infty$. We conclude the series in (5.2) is convergent to $W_n(t)$ in $L^2(\Omega; H)$. Now we show that $W_n(t)$ is also a Gaussian random variable. We calculate

$$\begin{aligned} & E\left[\langle W_n(t), h\rangle\langle W_n(t), y\rangle\right] \\ = \ & E\left[\langle\sum_{k=1}^{\infty}\int_0^t V_n(t-s)Be_k d\beta_k(s), h\rangle\langle\sum_{k=1}^{\infty}\int_0^t V_n(t-s)Be_k d\beta_k(s), y\rangle\right] \end{aligned}$$

$$= E\left[\left(\sum_{k=1}^{\infty}\int_0^t V_n(t-s)Bd\beta_k(s)\right)^2 \langle e_k, h\rangle\langle e_k, y\rangle\right]$$

$$= \sum_{k=1}^{\infty}\int_0^t \|V_n(t-s)B\|^2 ds\langle e_k, h\rangle\langle e_k, y\rangle$$

$$= \langle\int_0^t \|V_n(t-s)B\|^2 h ds, y\rangle = \langle L_t h, y\rangle.$$

Hence $W_n(t)$ is a Gaussian random variable. Finally we show that $W_n(\cdot) \in C_W([0,T]; H)$ for any $T > 0$. We only need to show that $W_n(\cdot) : [0,T] \to L^2(\Omega; H)$ is continuous.

$$\begin{aligned}
&\left(E\left[\|W_n(t) - W_n(s)\|^2\right]\right)^{1/2} \\
= &\left(E\left[\left\|\sum_{k=1}^{\infty}\int_0^t V_n(t-\sigma)Be_k d\beta_k(\sigma) - \sum_{k=1}^{\infty}\int_0^s V_n(s-\sigma)Be_k d\beta_k(\sigma)\right\|^2\right]\right)^{1/2} \\
= &\left(E\left[\left\|\sum_{k=1}^{\infty}\int_s^t V_n(t-\sigma)Be_k d\beta_k(\sigma) + \sum_{k=1}^{\infty}\int_0^s (V_n(t-\sigma) - V_n(s-\sigma))Be_k d\beta_k(\sigma)\right\|^2\right]\right)^{1/2} \\
\leq &\left(E\left[\left\|\sum_{k=1}^{\infty}\int_s^t V_n(t-\sigma)Be_k d\beta_k(\sigma)\right\|^2\right]\right)^{1/2} \\
&+ \left(E\left[\left\|\sum_{k=1}^{\infty}\int_0^s (V_n(t-\sigma) - V_n(s-\sigma))Be_k d\beta_k(\sigma)\right\|^2\right]\right)^{1/2} \\
= &\left(\sum_{k=1}^{\infty}\int_s^t \|V_n(t-\sigma)Be_k\|^2 d\sigma\right)^{1/2} + \left(\sum_{k=1}^{\infty}\int_0^s \|(V_n(t-\sigma) - V_n(s-\sigma))Be_k\|^2 d\sigma\right)^{1/2}.
\end{aligned}$$

By the assumption that $L_t$ is of trace class and by the Lebesgue dominated convergence theorem, the series in the right–hand side converge to zero, as $s \to t$. Therefore $W_n(\cdot) \in C_W([0,T]; H)$. ∎

**Corollary 5.2** *Assume that $K(\cdot)x \in C([0,T]; H)$ for any $x \in H$, and that the linear operator*

$$L_t^K x := \int_0^t K(s)BB^*K^*(s)x ds, \quad x \in H,$$

*is of trace class. Then for all $t > 0$ the series*

$$W_K(t) = \int_0^t K(t-s)BdW(s) = \sum_{k=1}^{\infty}\int_0^t K(t-s)Be_k d\beta_k(t)$$

*is convergent on $L^2(\Omega; H)$ to a Gaussian random variable $W_K(t)$ with mean zero and covariance operator $L_t^K$. Moreover $W_K(\cdot)$ belongs to $C_W([0,T]; H)$ for any $T > 0$.*

By Corollary 5.2, we have

$$\int_0^t \frac{(t-s)^n}{n!} BdW(s)$$

is a Gaussian random variable. We define the $n$–integrated solution of problem (5.1) as follows.

**Definition 5.3** *An $H$–valued predictable process $X(t)$, $t \in [0,T]$ is said to be an $n$–integrated solution of the problem (5.1) if $\int_0^t X(s)ds \in D(A)$ for all $t \geq 0$, $P_T$–almost surely, and*

$$X(t) = \frac{t^n}{n!}\xi + A\int_0^t X(s)ds + \int_0^t \frac{(t-s)^n}{n!} BdW(s). \tag{5.4}$$

**Proposition 5.4** *Let $A$ be the generator of an $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ and assume that*

$$\int_0^t V_n(t-s)BdW(s) \in D(A), \ t > 0.$$

*Then*

$$X(t) = V_n(t)\xi + \int_0^t V_n(t-s)BdW(s) \tag{5.5}$$

*is an $n$–integrated solution of the stochastic differential equation (5.1).*

**Proof.** Since $V(t)$ is the $n$–times integrated semigroup generated by $A$ and by the assumption that

$$\int_0^t V_n(t-s)BdW(s) \in D(A), \ t > 0,$$

we have

$$\int_0^t \left(\int_0^s V_n(s-r)BdW(r)\right) ds \in D(A),$$

and hence

$$\int_0^t X(s)ds = \int_0^t V_n(s)\xi ds + \int_0^t \left(\int_0^s V_n(s-r)BdW(r)\right) ds \in D(A).$$

Moreover, taking into account that

$$\int_0^t V_n(t-s)BdW(s) \in D(A),\ t>0,$$

and using the properties of the $n$–times integrated semigroup $V_n(t)$ we obtain

$$\begin{aligned}
X(t) &= V_n(t)\xi + \int_0^t V_n(t-s)BdW(s) \\
&= \frac{t^n}{n!}\xi + A\int_0^t V_n(s)xds \\
&\quad + \int_0^t \frac{(t-s)^n}{n!}BdW(s) + \int_0^t \left(A\int_0^{t-s} V_n(r)dr\right) BdW(s) \\
&= \frac{t^n}{n!}\xi + A\int_0^t V_n(s)xds + \int_0^t \frac{(t-s)^n}{n!}BdW(s) \\
&\quad + \int_0^t \left(A\int_s^t V_n(r-s)dr\right) BdW(s).
\end{aligned}$$

Using the closedness of $A$ and the stochastic Fubini theorem, we have

$$\begin{aligned}
X(t) &= \frac{t^n}{n!}\xi + A\int_0^t V_n(s)xds + \int_0^t \frac{(t-s)^n}{n!}BdW(s) \\
&\quad +A\int_0^t \left(\int_0^r V_n(r-s)BdW(s)\right) dr.
\end{aligned}$$

Changing the variable, we obtain

$$\begin{aligned}
X(t) &= \frac{t^n}{n!}\xi + A\int_0^t V_n(s)xds + \int_0^t \frac{(t-s)^n}{n!}BdW(s) \\
&\quad +A\int_0^t \left(\int_0^s V_n(s-r)BdW(r)\right) ds \\
&= \frac{t^n}{n!}\xi + A\int_0^t \left(V_n(s)x + \int_0^s V_n(s-r)BdW(r)\right) ds \\
&\quad + \int_0^t \frac{(t-s)^n}{n!}BdW(s) \\
&= \frac{t^n}{n!}\xi + A\int_0^t X(s)ds + \int_0^t \frac{(t-s)^n}{n!}BdW(s).
\end{aligned}$$

Hence $X(t)$ given by (5.5) is an $n$–integrated solution of (5.1). ■

However, the solution $X(t)$ does not necessarily have a continuous version. The purpose of our next discussion is to find conditions under which the solutions have continuous versions.

Let $A$ be the generator of an exponentially bounded $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$. Hence $A$ is also the generator of exponentially bounded $(n+j)$–times integrated semigroups $\{V_{n+j}(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ for $j = 1, 2, \ldots$. In particular, $A$ generates an exponentially bounded $2n$–times integrated semigroup $\{V_{2n}(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$. It is shown in [36], that those semigroups satisfy the relation

$$V_{2n}(t+s) = V_n(t)V_n(s) + \sum_{j=0}^{n-1} \frac{1}{j!}\left(s^k V_{2n-j}(t) + t^k V_{2n-j}(s)\right). \tag{5.6}$$

Define

$$W_{2n}(t) := \int_0^t V_{2n}(t-s)BdW(s) = \sum_{k=1}^{\infty} \int_0^t V_{2n}(t-s)Be_k d\beta_k(s). \tag{5.7}$$

We showed in Proposition 5.1 that $W_{2n}(t)$ is a Gaussian random variable with the law $\mathcal{N}(0, L_t^{2n})$, where

$$L_t^{2n} x := \int_0^t V_{2n}(s)BB^*V_{2n}^*(s)x ds, x \in H,$$

given that $L_t^{2n}$ is of trace class. Our next task is to show that $W_{2n}(t)$ has a continuous version.

**Theorem 5.5** *Assume that there is $\alpha \in (0, \frac{1}{2})$ and $T \in (0,\infty)$ such that*

$$\int_0^t s^{-2\alpha} \mathrm{Tr}\left[V_n(s)BB^*V_n^*(s)\right] ds = C_{\alpha,T}^n < \infty, \tag{5.8}$$

*and for $j = 0, 1, 2, \ldots, n-1$,*

$$\int_0^T s^{-2\alpha} \mathrm{Tr}\left[V_{2n-j}(s)BB^*V_{2n-j}^*(s)\right] ds = C_{\alpha,T}^{2n-j} < \infty. \tag{5.9}$$

*Then $W_{2n}(t)$ defined by (5.7) has a continuous version.*

**Proof.** Using (5.6) we can write

$$\begin{aligned} W_{2n}(t) &= \int_0^t V_{2n}(t-\sigma+\sigma-s)BdW(s) \\ &= \int_0^t V_n(t-\sigma)V_n(\sigma-s)BdW(s) \\ &\quad + \int_0^t \left(\sum_{j=0}^{n-1} \frac{1}{j!}(\sigma-s)^j V_{2n-j}(t-\sigma) + (t-\sigma)^j V_{2n-j}(\sigma-s)\right) BdW(s). \end{aligned}$$

Using the factorization formula

$$\frac{\pi}{\sin(\pi\alpha)} = \int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}d\sigma, \quad \alpha \in (0,1),\ 0 \le s \le t,$$

we obtain

$$\begin{aligned}
& W_{2n}(t) \\
= & \frac{\sin(\pi\alpha)}{\pi}\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}d\sigma \int_0^t V_n(t-\sigma)V_n(\sigma-s)BdW(s) \\
& +\frac{\sin(\pi\alpha)}{\pi}\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}d\sigma \sum_{j=0}^{n-1}\int_0^t \frac{1}{j!}\left((\sigma-s)^j V_{2n-j}(t-\sigma)\right) BdW(s) \\
& +\frac{\sin(\pi\alpha)}{\pi}\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}d\sigma \int_0^t \sum_{j=0}^{n-1}\frac{1}{j!}(t-\sigma)^j V_{2n-j}(\sigma-s)BdW(s).
\end{aligned}$$

By the stochastic Fubini theorem, we have

$$\begin{aligned}
& W_{2n}(t) \qquad (5.10)\\
= & \frac{\sin(\pi\alpha)}{\pi}\int_0^t V_n(t-\sigma)(t-\sigma)^{\alpha-1}\int_0^\sigma V_n(\sigma-s)(\sigma-s)^{-\alpha}BdW(s)d\sigma \\
& +\frac{\sin(\pi\alpha)}{\pi}\sum_{j=0}^{n-1}\frac{1}{j!}\int_0^t V_{2n-j}(t-\sigma)(t-\sigma)^{\alpha-1}\int_0^\sigma (\sigma-s)^{j-\alpha}BdW(s)d\sigma \\
& +\frac{\sin(\pi\alpha)}{\pi}\sum_{j=0}^{n-1}\frac{1}{k!}\int_0^t (t-\sigma)^{j+\alpha-1}\int_0^\sigma V_{2n-j}(\sigma-s)(\sigma-s)^{-\alpha}BdW(s)d\sigma.
\end{aligned}$$

Writing

$$U_n(\sigma) = \int_0^\sigma V_n(\sigma-s)(\sigma-s)^{-\alpha}BdW(s),$$

and for $j = 0,1,2,\ldots,n-1$

$$U_j(\sigma) = \int_0^\sigma (\sigma-s)^{j-\alpha}BdW(s)$$

$$U_{2n-j}(\sigma) = \int_0^\sigma V_{2n-j}(\sigma-s)(\sigma-s)^{-\alpha}BdW(s),$$

and

$$P_n(t) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t V_n(t-\sigma)(t-\sigma)^{\alpha-1}U_n(\sigma)d\sigma$$

$$P_{2n-j}(t) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t \frac{1}{j!}V_{2n-j}(t-\sigma)(t-\sigma)^{\alpha-1}U_j(\sigma)d\sigma$$

$$P_j(t) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t \frac{1}{j!}(t-\sigma)^{j+\alpha-1}U_{2n-j}(\sigma)d\sigma,$$

we can write $W_{2n}(t)$ as

$$W_{2n}(t) = P_n(t) + \sum_{j=0}^{n-1} P_{2n-j}(t) + \sum_{j=0}^{n-1} P_j(t).$$

Similar to the proof of Proposition 5.1 we can show that $U_n(\sigma)$ is a Gaussian random variable $\mathcal{N}(0, S_\sigma^n)$ for all $\sigma \in [0,T]$, where

$$S_\sigma^n x := \int_0^\sigma s^{-2\alpha} V_n(s) BB^* V^*(s) x ds.$$

Accordingly, for all $j = 0, 1, 2, \ldots, n-1$, $U_{2n-j}(\sigma)$ and $U_j(\sigma)$ are Gaussian random variables $\mathcal{N}(0, S_\sigma^{2n-j})$ and $\mathcal{N}(0, S_\sigma^j)$ respectively, where

$$S_\sigma^{2n-j} x := \int_0^\sigma s^{-2\alpha} V_{2n-j} BB^* V_{2n-j}^*(s) x ds$$

$$S_\sigma^j x := \int_0^t s^{2j-2\alpha} BB^* x ds.$$

By (5.8), for any $m > 0$, there exists a constant $D_{m,\alpha}^n$ such that for all $\sigma \in [0,T]$ holds

$$E\left[\|U_n(\sigma)\|^{2m}\right] \le D_{m,\alpha}^n \sigma^m,$$

and by (5.9) for $j = 0, 1, 2, \ldots, n-1$, there exist constants $D_{m,\alpha}^{2n-j}$ and $D_{m,\alpha}^j$ such that for all $\sigma \in [0,T]$ holds

$$E\left[\|U_{2n-j}(\sigma)\|^{2m}\right] \le D_{m,\alpha}^{2n-j} \sigma^m,$$

$$E\left[\|U_j(\sigma)\|^{2m}\right] \le D_{m,\alpha}^j \sigma^m.$$

This implies

$$\int_0^T E\left[\|U_n(\sigma)\|^{2m}\right] d\sigma \le \frac{D_{m,\alpha}^n}{m+1} T^{m+1},$$

and for $j = 0, 1, 2, \ldots, n-1$

$$\int_0^T E\left[\|U_{2n-j}(\sigma)\|^{2m}\right] d\sigma \le \frac{D_{m,\alpha}^{2n-j}}{m+1} T^{m+1},$$

$$\int_0^T E\left[\|U_j(\sigma)\|^{2m}\right] d\sigma \le \frac{D_{m,\alpha}^j}{m+1} T^{m+1}.$$

Therefore $U_n(\cdot)\omega$, $U_{2n-j}(\cdot)\omega$ and $U_j(\cdot)\omega$ are in $L^{2m}([0,T];H)$ for almost all $\omega \in \Omega$ and $j = 0, 1, 2, \ldots, n-1$. Moreover, by Hölder's inequality and taking into account the exponential boundedness of $V_i(t)$ we have

$$\begin{aligned} \|P_n(t)\| &\leq \frac{M_T}{\pi}\left(\int_0^t (t-\sigma)^{\alpha-1\frac{2m}{2m-1}} d\sigma\right)^{\frac{2m-1}{2m}} \|U_n\|_{L^{2m}([0,T];H)} \\ &= \frac{M_T}{\pi}\left(\frac{2m-1}{2m\alpha-1}\right)^{\frac{2m-1}{2m}} t^{\alpha-\frac{1}{2m}} \|U_n\|_{L^{2m}([0,T];H)}, \end{aligned}$$

where $M_T = \sup_{t\in[0,T]} \|V_n(t)\|$. Accordingly, for all $j = 0, 1, 2, \ldots, n-1$ holds

$$\begin{aligned} \|P_{2n-j}(t)\| &\leq \frac{M_T^j}{\pi j!}\left(\int_0^t (t-\sigma)^{\alpha-1\frac{2m}{2m-1}} d\sigma\right)^{\frac{2m-1}{2m}} \|U_j\|_{L^{2m}([0,T];H)} \\ &= \frac{M_T^j}{\pi j!}\left(\frac{2m-1}{2m\alpha-1}\right)^{\frac{2m-1}{2m}} t^{\alpha-\frac{1}{2m}} \|U_j\|_{L^{2m}([0,T];H)}, \end{aligned}$$

where $M_T^j = \sup_{t\in[0,T]} \|V_{2n-j}(t)\|$, and furthermore

$$\begin{aligned} \|P_j(t)\| &\leq \frac{1}{\pi j!}\left(\int_0^t (t-\sigma)^{j+\alpha-1\frac{2m}{2m-1}} d\sigma\right)^{\frac{2m-1}{2m}} \|U_{2n-j}\|_{L^{2m}([0,T];H)} \\ &= \frac{1}{\pi j!}\left(\frac{2m-1}{2mj+2m\alpha-1}\right)^{\frac{2m-1}{2m}} t^{j+\alpha-\frac{1}{2m}} \|U_{2n-j}\|_{L^{2m}([0,T];H)}. \end{aligned}$$

Hence $P_n(\cdot)\omega \in C([0,T];H)$ for almost all $\omega \in \Omega$ and for all $j = 0, 1, 2, \ldots, n-1$, $P_{2n-j}(\cdot)\omega, P_j(\cdot)\omega \in C([0,T];X)$ for almost all $\omega \in \Omega$. Thus

$$W_{2n}(\cdot)\omega = \left(P_n + \sum_{j=0}^{n-1} P_{2n-j} + \sum_{j=0}^{n-1} P_j\right)(\cdot)\omega \in C([0,T];H)$$

for almost all $\omega \in \Omega$ and (5.10) defines a continuous version of $W_{2n}(t)$. ∎

**Corollary 5.6** *Let all the assumptions of Theorem 5.5 hold. Moreover, assume that $W_{2n}(t)$ defined in (5.7) belongs to $D(A)$. Then*

$$X(t) = V_{2n}(t)\xi + W_{2n}(t)$$

*is a $2n$–integrated solution of problem (5.1) which has a continuous version.*

**Definition 5.7** *An $H$–valued predictable process $X(t)$ is said to be a weak $n$–integrated solution of (5.1) if the trajectories of $X(\cdot)$ are $P$–almost surely Bochner integrable* *and if for all $\nu \in D(A^*)$, $t \in [0,T]$ holds*

$$\langle X(t), \nu\rangle = \langle \frac{t^n}{n!}\xi, \nu\rangle + \langle \int_0^t X(s)ds, A^*\nu\rangle + \langle \int_0^t \frac{(t-s)^n}{n!} BdW(s), \nu\rangle, \tag{5.11}$$

*$P$-almost surely.*

**Proposition 5.8** *Let $A$ be the generator of an $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H); t \in [0,\infty)\}$. Then*

$$X(t) = V_n(t)\xi + \int_0^t V_n(t-s)BdW(s)$$

*is a weak $n$–integrated solution of (5.1).*

**Proof.** Without loss of generality assume that $\xi = 0$. We show that equation (5.11) is satisfied by

$$W_n(t) = \int_0^t V_n(t-s)BdW(s).$$

Fix $t \in [0,T]$ and let $\nu \in D(A^*)$. Note that

$$\int_0^t \langle A^*\nu, W_n(s)\rangle ds = \int_0^t \langle A^*\nu, \int_0^t \mathcal{X}_{[0,s]}(r)V_n(s-r)BdW(r)\rangle ds.$$

Hence by the stochastic Fubini theorem, we have

$$\begin{aligned}
\int_0^t \langle A^*\nu, W_n(s)\rangle ds &= \int_0^t \langle A^*\nu, \int_0^t \mathcal{X}_{[0,s]}(r)V_n(s-r)BdW(r)\rangle ds \\
&= \int_0^t \langle \int_0^t \mathcal{X}_{[0,s]}(r)B^*V_n^*(r-s)A^*\nu ds, dW(r)\rangle \\
&= \int_0^t \langle \int_r^t B^*V_n^*(s-r)A^*\nu ds, dW(r)\rangle.
\end{aligned}$$

Since $A$ generates an $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$, $A^*$ generates the $n$–times integrated semigroup $\{V_n^*(t) \in \mathcal{L}(H^*) : t \in [0,\infty)\}$, where $H^*$ is the dual space of $H$. Since $A^*$ and $V_n^*(t)$ commute, using the properties of the $n$–times integrated semigroup $\{V_n^*(t) \in \mathcal{L}(H^*) : t \in [0,\infty)\}$ we obtain

$$\begin{aligned}
\int_0^t \langle A^*\nu, W_n(s)\rangle ds &= \int_0^t \langle \int_r^t \mathcal{X}_{[0,s]}(r)B^*A^*V_n^*(r-s)\nu ds, dW(r)\rangle \\
&= \int_0^t \langle \int_r^t \left( B^*\frac{d}{ds}V^*(s-r)\nu - B^*\frac{(s-r)^{n-1}}{(n-1)!}\nu \right) ds, dW(r)\rangle \\
&= \int_0^t \langle B^*V^*(t-r)\nu - B^*\frac{(t-r)^n}{n!}\nu, dW(r)\rangle \\
&= \langle \nu, \int_0^t V(t-r)BdW(r)\rangle - \langle \nu, \int_0^t \frac{(t-r)^n}{n!}BdW(r)\rangle.
\end{aligned}$$

Therefore $W_n(t) = \int_0^t V_n(t-s)BdW(s)$ is a weak $n$–integrated solution of (5.1). ■

**Corollary 5.9** *Let $A$ be the generator of an $n$–times integrated semigroup $\{V_n(t) \in \mathcal{L}(H); t \in [0,\infty)\}$ and all the assumptions in Theorem 5.5 hold. Then*

$$X(t) = V_{2n}(t)\xi + \int_0^t V_{2n}(t-s)BdW(s)$$

*is a weak $2n$–integrated solution of (5.1) which has a continuous version.*

## 5.2 Quasi–Reversibility Method for Ill–posed Stochastic Differential Equations

In this section, we denote by $\mathcal{H}_c^+$, the class of self adjoint operators on a separable Hilbert space $H$ generating an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $H$, corresponding to real eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $-\infty < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \cdots < \infty$ and $\lim_{k\to\infty} \lambda_k = +\infty$.

Consider the stochastic differential equation (5.1):

$$dX(t) = AX(t)dt + BdW(t), \;\; X(0) = \xi,$$

where $\xi$ is an unobservable $H$–valued random variable that is, $\xi$ is $\mathcal{G}$–measurable for some $\sigma$–field $\mathcal{G} \supset \mathcal{F}$. Let $B \in \mathcal{L}(H)$ and $A \in \mathcal{H}_c^+$.

**Definition 5.10** *An $H$–valued predictable process $X(t) \in C_W([0,T];H)$ is called a solution of the problem (5.1) if $\int_0^t X(s)ds \in D(A)$ for all $t \ge 0$, $P$–almost surely, and*

$$X(t) = \xi + A\int_0^t X(s)ds + BW(t). \tag{5.12}$$

If $A \in \mathcal{H}_c^+$, then the Cauchy problem

$$u'(t) = Au(t), \;\; u(0) = u_0$$

is, in general, not well–posed, and the semigroup $\{U(t) : t \in [0, \infty)\}$ generated by $A$ consists of unbounded linear operators. It was proved in [26] that, if $A \in \mathcal{H}_c^+$, then for any $\epsilon > 0$, the Cauchy problem

$$u'(t) = (A - \epsilon A^2)u(t), \; u(0) = u_0$$

is well–posed on $D(A^2)$, and $(A - \epsilon A^2)$ generates a strongly continuous semigroup $\{U_\epsilon(t) \in \mathcal{L}(H) : \; t \in [0, \infty)\}$.

Suppose that $\xi \in L^2(\Omega, \mathcal{G}, \tilde{P})$, where $\tilde{P}|_{\mathcal{F}} = P$. We assume that there exists a solution of problem (5.1) for the initial value $\xi$.

Let $\eta$ be an observable $H$–valued random variable, i.e. $\eta$ is $\mathcal{F}$–measurable. Consider $\xi_\delta := E[\xi|\eta]$, the optimal (in the mean-square sense) estimator of $\xi$ in terms of $\eta$, such that $E[||\xi - \xi_\delta||^2] < \delta$. We approximate solutions of the problem (5.1) at $t = T$ by solutions of the problem

$$dX_{\epsilon,\delta}(t) = (A - \epsilon A^2)X_{\epsilon,\delta}(t)dt + BdW(t), \; X_{\epsilon,\delta}(0) = \xi_\delta. \tag{5.13}$$

**Definition 5.11** *A linear operator $R_\epsilon(t) : H \to L^2(\Omega; H)$, $\epsilon > 0$, is called the regularizing operator (or regularizer) of problem (5.1) for $0 < t \leq T$ if*
*(a) the operator $R_\epsilon(t)$ is a continuous linear operator on $H$,*
*(b) there is a continuous function $\epsilon(\cdot) : [0, \infty) \to [0, \infty)$, $\delta \mapsto \epsilon(\delta)$ with $\epsilon(0) = 0$ such that*

$$R_{\epsilon(\delta)}(t)\xi_\delta \to X(t), \text{ as } \delta \to 0.$$

Thus, we have introduced two parameters, $\epsilon > 0$ and $\delta > 0$, the former refering to a perturbation of the operator $A$, the latter referring to a perturbation of the initial value. If there is a coupling mechanism $\epsilon = \epsilon(\delta)$ relating them in a way that the solution of problem (5.1) can be approximated by solutions of the problem (5.13), then we call $R_{\epsilon(\delta)}(t)$ a regularizer and the approximation is a regularization. We introduce, for some constants $M_1$, $M_2$, $\tau > 0$ and fixed $T > 0$, two correctness classes, that are families of stochastic processes

$$\mathcal{M}_1 := \{t \mapsto X(t) : E\left[||AX(T)||^2\right] = E\left[\sum_{k=1}^{\infty} (\lambda_k \exp(\lambda_k T)\xi_k)^2\right] \leq M_1\},$$

and

$$\mathcal{M}_2 := \{t \mapsto X(t) : E\left[\|X(T+\tau)\|^2\right] = E\left[\sum_{k=1}^{\infty} \exp\left(2\lambda_k(T+\tau)\xi_k^2\right] \le M_2, \tau > 0\right\}.$$

If $X(\cdot) \in \mathcal{M}_1$ or $X(\cdot) \in \mathcal{M}_2$, then it holds, for a constant $M \ge 0$,

$$E\left[\sum_{k=1}^{\infty} \exp\left(2\lambda_k T\right)\xi_k^2\right] \le M. \tag{5.14}$$

**Assumption 5.12**

*(a) For any $s \ge 0$, the linear operators $U_\epsilon(s)BB^*U_\epsilon^*(s)$ are of trace class, and for all $t \ge 0$ it holds*

$$\int_0^t \text{Tr}\ \left[U_\epsilon(s)BB^*U_\epsilon^*(s)\right] ds < \infty,$$

*uniformly with respect to $\epsilon$.*

*(b) For $s \in [0,T]$, the linear operators $U(s)BB^*U^*(s)$ are of trace class and*

$$\int_0^T \text{Tr}\ \left[U(s)BB^*U^*(s)\right] ds < \infty.$$

**Assumption 5.13** *For any $s \in [0,T]$, the linear operators $AU(s)BB^*U^*(s)A^*$ are of trace class and*

$$\int_0^T \text{Tr}\ \left[AU(s)BB^*U^*(s)A^*\right] ds < \infty.$$

**Assumption 5.14** *For any $s \in [0,T+\tau]$, the linear operators $U(s)BB^*U^*(s)$ are of trace class and*

$$\int_0^{T+\tau} \text{Tr}\ \left[U(s)BB^*U^*(s)\right] ds < \infty.$$

Since $(A - \epsilon A^2)$ generates a strongly continuous semigroup, if part (a) of Assumption 5.12 holds, then by Theorem 2.2.2 [14], for any $\xi \in H$, the stochastic differential equation (5.13) has a unique solution $X_\epsilon(t)$.

**Theorem 5.15** *Let $A$ be in $\mathcal{H}_c^+$ and suppose that Assumptions 5.12 and 5.13 hold. Then the operator*

$$\begin{aligned} R_\epsilon(t)\xi := X_{\epsilon,\delta}(t) &= \sum_{k=1}^{\infty} \exp\left((\lambda_k - \epsilon\lambda_k^2)t\right)\xi_{\delta k} e_k \\ &+ \sum_{k=1}^{\infty} \int_0^t \exp\left((\lambda_k - \epsilon\lambda_k^2)(t-s)\right) Be_k d\beta_k(s), \end{aligned} \tag{5.15}$$

*where* $\xi_{\delta k} := \langle \xi_\delta, e_k \rangle$, *is a regularizer for the problem (5.1) at* $t = T$. *On the correctness class* $\mathcal{M}_1$, *for some constant* $C$ *independent of* $\delta$, *there holds*

$$E\left[\|X(T) - X_{\epsilon,\delta}(T)\|^2\right] \le C|\ln\delta|^{-\frac{2}{3}}.$$

**Proof.** The solution $X(t)$ at $t = T$ of (5.1) can be formally written as

$$X(T) = \sum_{k=1}^{\infty} \exp(\lambda_k T)\xi_k e_k + \sum_{k=1}^{\infty} \int_0^T \exp(\lambda_k(T-s))Be_k d\beta_k(s),$$

where $\xi_k = \langle \xi, e_k \rangle$. Let $X_\epsilon(t)$ be a solution of (5.13) for initial value $\xi$. Then $X_\epsilon(t)$ at $t = T$ is of the form

$$X_\epsilon(T) = \sum_{k=1}^{\infty} \exp\left((\lambda_k - \epsilon\lambda_k^2)T\right)\xi_k e_k + \sum_{k=1}^{\infty} \int_0^T \exp\left((\lambda_k - \epsilon\lambda_k^2)(T-s)\right)Be_k d\beta_k(s),$$

Consider the error

$$\begin{aligned} E\left[\|X(T) - X_{\epsilon,\delta}(T)\|^2\right] &\le E\left[\|X(T) - X_\epsilon(T)\|^2\right] + E\left[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2\right] \\ &\quad + 2E\left[\|X(T) - X_\epsilon(T)\| \, \|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|\right]. \end{aligned}$$

Using the Hölder inequality we have

$$\begin{aligned} & E\left[\|X(T) - X_\epsilon(T)\| \, \|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|\right] \\ \le & \left(E\left[\|X(T) - X_\epsilon(T)\|^2\right]\right)^{\frac{1}{2}} \left(E\left[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2\right]\right)^{\frac{1}{2}}. \end{aligned}$$

Moreover,

$$\begin{aligned} & 2\left(E\left[\|X(T) - X_\epsilon(T)\|^2\right]\right)^{\frac{1}{2}} \left(E\left[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2\right]\right)^{\frac{1}{2}} \\ \le & \; E\left[\|X(T) - X_\epsilon(T)\|^2\right] + E\left[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2\right]. \end{aligned}$$

Hence

$$E\left[\|X(T) - X_{\epsilon,\delta}(T)\|^2\right] \le 2\left(E\left[\|X(T) - X_\epsilon(T)\|^2\right] + E\left[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2\right]\right).$$

Firstly, we estimate $E[\|X(T) - X_\epsilon(T)\|^2]$, where

$$\begin{aligned} X(T) - X_\epsilon(T) &= \sum_{k=1}^{\infty} \exp(\lambda_k T)\left(1 - \exp(-\epsilon\lambda_k^2 T)\right)\xi_k e_k \\ &\quad + \sum_{k=1}^{\infty} \int_0^T \exp(\lambda_k(T-s))\left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right)Be_k d\beta_k(s). \end{aligned}$$

Using the same argument as above, $E\left[\|X(T)-X_\epsilon(T)\|^2\right]$ can be written as

$$E\left[\|X(T)-X_\epsilon(T)\|^2\right] \le 2(\triangle_1+\triangle_2),$$

where

$$\triangle_1 = E\left[\left\|\sum_{k=1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right]$$

and

$$\triangle_2 = E\left[\left\|\sum_{k=1}^{\infty}\int_0^T \exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right) Be_k d\beta_k(s)\right\|^2\right].$$

$\triangle_1$ can be estimated in the following way

$$\begin{aligned}
\triangle_1 &= E\left[\left\|\sum_{k=1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right] \\
&= E\left[\left\|\sum_{k=1}^{N}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right.\right. \\
&\quad \left.\left. + \sum_{k=N+1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right] \\
&\le 2(\triangle_1'+\triangle_1''),
\end{aligned}$$

where

$$\triangle_1' = E\left[\left\|\sum_{k=1}^{N}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right],$$

and

$$\triangle_1'' = E\left[\left\|\sum_{k=N+1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right].$$

By (5.14), the estimate of $\triangle_1'$ becomes

$$\begin{aligned}
\triangle_1' &= E\left[\left\|\sum_{k=1}^{N}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right] \\
&= E\left[\sum_{k=1}^{N}\exp\left(2\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)^2\xi_k^2\right] \\
&\le \max_{k=1,\ldots,N}\left(\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)^2\right) E\left[\sum_{k=1}^{N}\exp\left(2\lambda_k T\right)\xi_k^2\right] \\
&\le M\left(1-\exp\left(-\epsilon\lambda_N^2 T\right)\right)^2.
\end{aligned}$$

For the class $\mathcal{M}_1$, we obtain the estimate of $\triangle_1''$ as

$$\begin{aligned}
\triangle_1'' &= E\left[\left\|\sum_{k=N+1}^{\infty} \exp(\lambda_k T)\left(1-\exp(-\epsilon\lambda_k^2 T)\right)\xi_k e_k\right\|^2\right] \\
&= E\left[\sum_{k=N+1}^{\infty} \exp(2\lambda_k T)\left(1-\exp(-\epsilon\lambda_k^2 T)\right)^2 \xi_k^2\right] \\
&= E\left[\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k{}^2}\left(\lambda_k \exp(\lambda_k T)\xi_k\right)^2 \left(1-\exp(-\epsilon\lambda_k^2 T)\right)^2\right] \\
&\leq \frac{M_1}{\lambda_{N+1}^2}.
\end{aligned}$$

Choose $N = N(\epsilon)$ such that $\lambda_N^2 \leq \epsilon^{-s} \leq \lambda_{N+1}^2$, $0 < s < 1$. Then

$$\triangle_1' \leq M(1-\exp(-\epsilon\epsilon^{-s}T))^2 = M(1-\exp(-\epsilon^{1-s}T))^2 \leq \tilde{M}\epsilon^{2(1-s)},$$

and $\triangle_1'' \leq \tilde{M}\epsilon^s$. Thus, with a constant $\tilde{M} \geq M_1$, which is independent of $\epsilon$, we have

$$\triangle_1 \leq 2\tilde{M}\left(\epsilon^{2(1-s)} + \epsilon^s\right).$$

Similarly, $\triangle_2$ can be written as

$$\begin{aligned}
\triangle_2 &= E\left[\left\|\sum_{k=1}^{\infty}\int_0^T \exp(\lambda_k(T-s))\left(1-\exp(-\epsilon\lambda_k^2(T-s))\right)Be_k d\beta_k(s)\right\|^2\right] \\
&\leq 2(\triangle_2' + \triangle_2''),
\end{aligned}$$

for

$$\triangle_2' = E\left[\left\|\sum_{k=1}^{N}\int_0^T \exp(\lambda_k(T-s))\left(1-\exp(-\epsilon\lambda_k^2(T-s))\right)Be_k d\beta_k(s)\right\|^2\right],$$

and

$$\triangle_2'' = E\left[\left\|\sum_{k=N+1}^{\infty}\int_0^T \exp(\lambda_k(T-s))\left(1-\exp(-\epsilon\lambda_k^2(T-s))\right)Be_k d\beta_k(s)\right\|^2\right].$$

Using Assumption 5.12 (b), we obtain the estimate of $\triangle_2'$.

$$\triangle_2' = E\left[\left\|\sum_{k=1}^{N}\int_0^T \exp(\lambda_k(T-s))\left(1-\exp(-\epsilon\lambda_k^2(T-s))\right)Be_k d\beta_k(s)\right\|^2\right]$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} \int_0^T \| \exp(\lambda_k(T-s)) \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right) Be_k\|^2 ds \\
&= \max_{k=1,\dots,N} \left( \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right)^2 \right) \sum_{k=1}^{N} \int_0^T \| \exp(\lambda_k(T-s)) Be_k\|^2 ds \\
&= \max_{k=1,\dots,N} \left( \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right)^2 \right) \int_0^T \mathrm{Tr}\ [U(T-s)BB^*U^*(T-s)]\, ds \\
&= K(1-\exp(-\epsilon\lambda_N^2 T))^2 \\
&\leq \tilde{K}\epsilon^{2(1-s)}.
\end{aligned}
$$

And by Assumption 5.13, we obtain

$$
\begin{aligned}
\triangle_2'' &= E\left[ \left\| \sum_{k=N+1}^{\infty} \int_0^T \exp(\lambda_k(T-s)) \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right) Be_k d\beta_k(s) \right\|^2 \right] \\
&= \sum_{k=N+1}^{\infty} \int_0^T \left\| \exp(\lambda_k(T-s)) \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right) Be_k \right\|^2 ds \\
&= \sum_{k=N+1}^{\infty} \int_0^T \frac{1}{\lambda_k^2} \left(\lambda_k \exp(\lambda_k(T-s)) Be_k\right)^2 \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right)^2 ds \\
&= \max_{k\geq N+1} \left( \frac{1}{\lambda_k^2} \left(1 - \exp(-\epsilon\lambda_k^2(T-s))\right)^2 \right) \sum_{k=N+1}^{\infty} \int_0^T \|\lambda_k \exp(\lambda_k(T-s)) Be_k\|^2 ds \\
&\leq \frac{1}{\lambda_{N+1}^2} \int_0^T \mathrm{Tr}\ [AU(s)BB^*U^*(s)A^*] ds \\
&\leq \frac{K}{\lambda_{N+1}^2} \leq \tilde{K}\epsilon^s,
\end{aligned}
$$

where the constant $\tilde{K} \geq 0$ does not depend on $\epsilon$. Hence the estimate for $\triangle_2$ becomes

$$
\triangle_2 \leq 2(\triangle_2' + \triangle_2'') \leq 2\tilde{K}(\epsilon^{2(1-s)} + \epsilon^s).
$$

Therefore

$$
E[\|X(T) - X_\epsilon(T)\|^2] = 2(\triangle_1 + \triangle_2) \leq C\left(\epsilon^{2(1-s)} + \epsilon^s\right).
$$

Now consider $E[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2]$, where

$$
X_\epsilon(T) - X_{\epsilon,\delta}(T) = \sum_{k=1}^{\infty} \exp((\lambda_k - \epsilon\lambda_k^2)T)(\xi_k - \xi_{\delta k})e_k.
$$

$$
E[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2] = E\left[ \left\| \sum_{k=1}^{\infty} \exp((\lambda_k - \epsilon\lambda_k^2)T)(\xi_k - \xi_{\delta k}) \right\|^2 \right]
$$

$$
\begin{aligned}
&= E\left[\sum_{k=1}^{\infty}\exp\left(2(\lambda_k-\epsilon\lambda_k^2)T\right)\|\xi_k-\xi_{\delta k}\|^2\right]\\
&\leq E\left[\max_{k\in\mathbb{N}}\left(\exp\left(2(\lambda_k-\epsilon\lambda_k^2)T\right)\right)\sum_{k=1}^{\infty}\|\xi_k-\xi_{\delta k}\|^2\right]\\
&\leq \delta\max_{k\in\mathbb{N}}\left(\exp\left(2(\lambda_k-\epsilon\lambda_k^2)T\right)\right)\\
&\leq \delta\exp\left(\frac{T}{2\epsilon}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
E[\|X(T)-X_{\epsilon,\delta}(T)\|^2] &\leq 2\left(E[\|X(T)-X_\epsilon(T)\|^2]+E[\|X_\epsilon(T)-X_{\epsilon,\delta}(T)\|^2]\right)\\
&\leq C'\left(\epsilon^{2(1-s)}+\epsilon^s+\delta\exp\left(\frac{T}{2\epsilon}\right)\right).
\end{aligned}
$$

Obviously, the best choice of $s\in(0,1)$ is $s=2/3$. Then we obtain, with a constant $C''\geq 0$, which neither does depend on $\epsilon$ nor on $\delta$,

$$
E[\|X(T)-X_{\epsilon,\delta}(T)\|^2]\leq C''\left(\epsilon^{2/3}+\delta\exp\left(\frac{T}{2\epsilon}\right)\right).
$$

Set $\epsilon(\delta):=-\frac{T}{\ln\delta}$, then

$$
\begin{aligned}
E[\|X(T)-X_{\epsilon,\delta}(T)\|^2] &\leq C''\left(\left(-\frac{1}{\ln\delta}\right)^{2/3}+\delta\exp\left(-\frac{1}{2}\ln\delta\right)\right)\\
&\leq C\left(-\frac{1}{\ln\delta}\right)^{2/3}=C|\ln\delta|^{-\frac{2}{3}}
\end{aligned}
$$

for $\delta>0$ small enough and the proof is finished. ∎

**Theorem 5.16** *Let $A\in\mathcal{H}_c^+$ and suppose that Assumptions 5.12 and 5.14 hold. Then the operator*

$$
\begin{aligned}
R_\epsilon(t)\xi\equiv X_{\epsilon,\delta}(t) &= \sum_{k=1}^{\infty}\exp\left((\lambda_k-\epsilon\lambda_k^2)t\right)\xi_{\delta k}e_k\\
&\quad+\sum_{k=1}^{\infty}\int_0^t\exp\left((\lambda_k-\epsilon\lambda_k^2)(t-s)\right)Be_k d\beta_k(s)
\end{aligned}
\tag{5.16}
$$

*is a regularizer for the problem (5.1) at $t=T$. On the correctness class $\mathcal{M}_2$, for any $s\in(0,1)$, there exists a constant $C>0$ independent of $\delta$ such that, for $\delta>0$ small enough, holds*

$$
E\left[\|X(T)-X_{\epsilon,\delta}(T)\|^2\right]\leq C|\ln\delta|^{2(s-1)}.
$$

**Proof.** Similar to the proof of Theorem 5.15, $E\left[\|X(T)-X_{\epsilon,\delta}(T)\|^2\right]$ can be estimated as

$$E\left[\|X(T)-X_{\epsilon,\delta}(T)\|^2\right] \le 2\left(E\left[\|X(T)-X_\epsilon(T)\|^2\right]+E\left[\|X_\epsilon(T)-X_{\epsilon,\delta}(T)\|^2\right]\right),$$

where

$$X_\epsilon(T)=\sum_{k=1}^{\infty}\exp\left((\lambda_k-\epsilon\lambda_k^2)T\right)\xi_k e_k+\sum_{k=1}^{\infty}\int_0^T \exp\left((\lambda_k-\epsilon\lambda_k^2)(T-s)\right)Be_k d\beta_k(s)$$

is the solution of (5.13) with initial value $\xi$ at $t=T$, and

$$\begin{aligned} X(T)-X_\epsilon(T) &= \sum_{k=1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k \\ &\quad +\sum_{k=1}^{\infty}\int_0^T \exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right)Be_k d\beta_k(s), \end{aligned}$$

$$X_\epsilon(T)-X_{\epsilon,\delta}(T)=\sum_{k=1}^{\infty}\exp\left((\lambda_k-\epsilon\lambda_k^2)T\right)(\xi_k-\xi_{\delta k}).$$

By the same argument, we have

$$E\left[\|X(T)-X_\epsilon(T)\|^2\right]\le 2(\triangle_1+\triangle_2),$$

where

$$\triangle_1=E\left[\left\|\sum_{k=1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right],$$

and

$$\triangle_2=E\left[\left\|\sum_{k=1}^{\infty}\int_0^T \exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right)Be_k d\beta_k(s)\right\|^2\right].$$

Furthermore, $\triangle_1\le 2(\triangle_1'+\triangle_1'')$, where

$$\triangle_1'=E\left[\left\|\sum_{k=1}^{N}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right],$$

and

$$\triangle_1''=E\left[\left\|\sum_{k=N+1}^{\infty}\exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right].$$

Notice that from the proof of Theorem 5.15, the estimate of $\triangle_1'$ does not depend on $\mathcal{M}_i$, that is

$$\triangle_1' \le \left(1 - \exp\left(-\epsilon\lambda_N T\right)\right)^2.$$

For the class $\mathcal{M}_2$, we obtain the estimate of $\triangle_1''$.

$$\begin{aligned}
\triangle_1'' &= E\left[\left\|\sum_{k=N+1}^{\infty} \exp\left(\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)\xi_k e_k\right\|^2\right] \\
&= E\left[\sum_{k=N+1}^{\infty} \exp\left(2\lambda_k T\right)\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)^2 \xi_k^2\right] \\
&\quad +E\left[\sum_{k=N+1}^{\infty} \frac{1}{\exp\left(2\lambda_k \tau\right)} \exp\left(2\lambda_k(T+\tau)\right)\xi_k^2\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)^2\right] \\
&\le \max_{k\ge N+1}\left(\frac{1}{\exp\left(2\lambda_k\tau\right)}\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)^2\right) E\left[\sum_{k=N+1}^{\infty} \exp\left(2\lambda_k(T+\tau)\right)\xi_k^2\right] \\
&\le \frac{M_2}{\exp\left(2\lambda_{N+1}\tau\right)}.
\end{aligned}$$

For $N = N(\epsilon)$ such that $\lambda_N^2 \le \epsilon^{-s} \le \lambda_{N+1}^2$ with $0 < s < 1$, we have

$$\triangle_1' \le \tilde{M}\epsilon^{2(1-s)},$$

and

$$\triangle_1'' \le \frac{M_2}{\exp\left(2\lambda_{N+1}\tau\right)} \le \frac{M_2}{\exp\left(2\epsilon^{-s}\tau\right)} \le \tilde{M}\epsilon^{2(1-s)}$$

for a constant $\tilde{M} > 0$. Here we just use that $x \mapsto \exp(x)$ grows faster than any power $x \mapsto x^k$. Hence

$$\triangle_1 \le 2(\triangle_1' + \triangle_1'') \le 2\tilde{M}\epsilon^{2(1-s)} \le \tilde{M}'\epsilon^{2(1-s)}.$$

Similarly, $\triangle_2 \le 2(\triangle_2' + \triangle_2'')$, where

$$\triangle_2' = E\left[\left\|\sum_{k=1}^{N}\int_0^T \exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right) Be_k d\beta_k(s)\right\|^2\right],$$

and

$$\triangle_2'' = E\left[\left\|\sum_{k=N+1}^{\infty}\int_0^T \exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right) Be_k d\beta_k(s)\right\|^2\right].$$

Also notice that as in the proof of Theorem 5.15, the estimate $\triangle_2'$ does not depend on $\mathcal{M}_i$, which is

$$\triangle_2' \leq \tilde{K}\epsilon^{2(1-s)}.$$

Now using Assumption 5.14, we estimate $\triangle_2''$ .

$$\begin{aligned}
&\triangle_2'' \\
&= E\left[\left\|\sum_{k=N+1}^{\infty}\int_0^T \exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right)Be_k d\beta_k(s)\right\|^2\right] \\
&= E\left[\sum_{k=N+1}^{\infty}\int_0^T \left\|\exp\left(\lambda_k(T-s)\right)\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right)Be_k\right\|^2 ds\right] \\
&= E\left[\sum_{k=N+1}^{\infty}\int_0^T \frac{1}{\exp\left(2\lambda_k\tau\right)}\left(\exp\left(\lambda_k(T+\tau-s)\right)Be_k\right)^2\left(1-\exp\left(-\epsilon\lambda_k^2(T-s)\right)\right)^2 ds\right] \\
&\leq \max_{k\geq N+1}\left(\frac{1}{\exp\left(2\lambda_k\tau\right)}\left(1-\exp\left(-\epsilon\lambda_k^2 T\right)\right)^2\right) \\
&\quad\times E\left[\sum_{k=N+1}^{\infty}\int_0^{T+\tau}\left(\exp\left(\lambda_k(T+\tau-s)\right)Be_k\right)^2 ds\right] \\
&\leq \frac{1}{\exp\left(2\lambda_{N+1}\tau\right)}E\left[\sum_{k=N+1}^{\infty}\int_0^{T+\tau}\left(\exp\left(\lambda_k(T-s)\right)Be_k\right)^2 ds\right] \\
&\leq \frac{1}{\exp\left(2\lambda_{N+1}\tau\right)}\int_0^{T+\tau}\text{Tr}\left[U(s)BB^*U^*(s)\right]ds \\
&\leq \frac{M'}{\exp\left(2\lambda_{N+1}\tau\right)} \leq \hat{M}'\epsilon^{2(1-s)}.
\end{aligned}$$

Thus

$$\triangle_2 \leq 2\left(\triangle_2' + \triangle_2''\right) \leq \left(\tilde{K} + \hat{M}'\right)\epsilon^{2(1-s)} \leq K'\epsilon^{2(1-s)}.$$

Therefore,

$$\begin{aligned}
E[\|X(T) - X_\epsilon(T)\|^2] &\leq 2(\triangle_1 + \triangle_2) \leq 2\left(\tilde{M}\left(\epsilon^{2(1-s)}\right) + K'\left(\epsilon^{2(1-s)}\right)\right) \\
&\leq \hat{K}\epsilon^{2(1-s)}.
\end{aligned}$$

As in the proof of Theorem 5.15 we have

$$E[\|X_\epsilon(T) - X_{\epsilon,\delta}(T)\|^2] \leq \delta\exp\left(\frac{T}{2\epsilon}\right).$$

Hence,

$$E[\|X(T) - X_\epsilon(T)\|^2] \leq 2\left(E[\|X(T) - X_\epsilon(T)\|^2] + E[\|X_\epsilon(T) - X_\epsilon(T)\|^2]\right)$$

$$\leq C'\left(\epsilon^{2(1-s)} + \delta \exp\left(\frac{T}{2\epsilon}\right)\right).$$

Setting $\epsilon(\delta) = -\frac{T}{\ln \delta}$ we obtain, for $\delta > 0$ small enough,

$$E[\|X(T) - X_{\epsilon,\delta}(T)\|^2] \leq C|\ln \delta|^{-2(1-s)}, \; s \in (0,1)$$

and the proof is finished. ■

# List of Symbols

| | |
|---|---|
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{C}$ | set of complex numbers |
| $\mathbb{N}$ | set of natural numbers |
| $X$ | real or complex Banach space |
| $A$ | linear operator |
| $D(A)$ | domain of $A$ |
| $\mathcal{L}(X)$ | space of linear bounded operators on $X$ |
| $2^X$ | power set of $X$ |
| $\mathcal{M}(X)$ | set of multi–valued linear operators $\mathcal{A} : X \to X$ |
| $\mathcal{A}$ | multi–valued linear operator |
| $\mathcal{A}^{-1}$ | inverse of the multi–valued operator $\mathcal{A}$ |
| $\mathcal{A}^n$ | $n$–th power of the multi–valued operator $\mathcal{A}$ |
| $\mathcal{A}_s$ | single–valued branch of the multi–valued operator $\mathcal{A}$ |
| $\tilde{\mathcal{A}}_s$ | part of single–valued branch of the multi–valued operator $\mathcal{A}$ |
| $\text{ran}(\mathcal{A})$ | range of $\mathcal{A}$ |
| $\ker(\mathcal{A})$ | kernel of $\mathcal{A}$ |
| $\rho(\mathcal{A})$ | resolvent set of $\mathcal{A}$ |
| $R(\lambda)$ | the resolvent of $\mathcal{A}$ |
| $(X_1)^c$ | complement of the subspace $X_1$ of $X$ |
| $\text{Re}\lambda$ | the real part of the complex number $\lambda$ |
| $\frac{d^k}{d\lambda^k}$ | the $k$–th derivative with respect to $\lambda$ |
| $X_1 \oplus X_2$ | algebraic direct sum of two subspaces of $X$ |
| $\overline{Y}$ | the closure of $Y$ |
| $\rho_1(A, B)$ | resolvent set of $B^{-1}A$ |

| | |
|---|---|
| $\rho_2(A,B)$ | resolvent set of $AB^{-1}$ |
| $R_2(\lambda)$ | resolvent operator of $AB^{-1}$ |
| $\tilde{K}^i$ | $i$–th $A$–associated with ker$B$–vectors |
| $\binom{j}{k}$ | binomial coefficient |
| $K^i$ | $\tilde{K}^i \cup \{0\}$ |
| $\mathcal{D}_0$ | space of infinitely differentiable function with support in $[0,\infty)$ |
| $\mathcal{E}$ | space of infinitely differentiable function with any support |
| $\phi^{(m)}(t)$ | $\frac{d^m}{dt^m}\phi(t)$ |
| $\mathcal{D}'_0(X)$ | space of $X$–valued distributions on $\mathcal{D}_0$ |
| $\mathcal{E}'(X)$ | space of $X$–valued distributions on $\mathcal{E}$ |
| $\mathcal{S}$ | Schwartz space |
| $\mathcal{S}^p$ | Schwartz space of $p$–times continuously differentiable function on $\mathbb{R}$ |
| $\mathcal{S}'(X)$ | space of tempered $X$–valued distributions |
| $\mathcal{S}'_\omega(X)$ | space of tempered $X$–valued distributions of exponential growth |
| $\delta$ | Dirac distribution |
| $\mathcal{L}(X_1,X_2)$ | space of bounded linear operators from $X_1$ to $X_2$ |
| $A^*$ | adjoint operator of $A$ |
| $X^*$ | dual space of $X$ |
| $C([0,\infty);X)$ | space of continuous functions from $[0,\infty)$ to $X$ |
| $C^1([0,\infty);X)$ | space of continuously differentiable functions from $[0,\infty)$ to $X$ |
| $L^1_{loc}((0,\infty);X)$ | space of locally integrable functions |
| $W^{11}_{loc}((0,\infty);X)$ | Sobolev space of locally integrable distributions with locally integrable derivatives |
| $C^\infty([0,\infty);X)$ | space of infinitely continuously differentiable functions |
| $H$ | real or complex separable Hilbert space |
| $(\Omega,\mathcal{F},P)$ | probability space |
| $L^2(\Omega;H)$ | Banach space of $H$–valued square integrable mappings from $\Omega$ to $H$ |
| $\{e_k\}_{k\in\mathbb{N}}$ | complete orthonormal basis in $H$ |
| $\{\beta_k\}_{k\in\mathbb{N}}$ | sequence of independent real Brownian motions in $H$ |
| $E$ | expectation |
| Tr $A$ | trace of $A$ |

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