NEW APPROACHES TO SPACE-TIME SINGULARITIES

by
Susan M. Scott
Bachelor of Science (Honours)

Department of Physics
and
Mathematical Physics

The University of Adelaide

Thesis submitted in accordance
with the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 1991

Contents

Contents i

Abstract iii

Statement iv

Acknowledgements v

Preface vi

1 An Historical Perspective on the Curzon Solution 1
- 1.1 Introduction 1
- 1.2 The Curzon metric 2
- 1.3 Past analyses of the Curzon singularity 4
- 1.4 The Szekeres and Morgan extension of the Curzon metric 8
- 1.5 Comments on past analyses of the Curzon singularity 14
- 1.6 Israel's theorem 18

2 The Spacelike Hypersurfaces of the Curzon Solution 24
- 2.1 Introduction 24
- 2.2 Spacelike geodesics approaching $\mathbf{R} = 0$ 26
- 2.3 Analysis of curves approaching $\mathbf{R} = 0$ 30
- 2.4 New coordinates for the r-z plane 34
- 2.5 Conclusions 40

3 The Global Structure of the Curzon Solution 44
- 3.1 Introduction 44
- 3.2 Behaviour of geodesics approaching $\mathbf{R} = 0$ 45
- 3.3 New coordinates for the Curzon metric 49
- 3.4 Features of the new coordinates 52
- 3.5 Discussion and conclusions 56

4 A Survey of the Weyl Metrics 63
- 4.1 Introduction 63
- 4.2 The Weyl metrics 65
- 4.3 The Schwarzschild solution 66
- 4.4 The Zipoy-Voorhees metrics 68
- 4.5 Possible sources for the Zipoy-Voorhees metrics 71
- 4.6 Possible extensions of the Zipoy-Voorhees metrics 74
- 4.7 Some general properties of the Weyl metrics 75
- 4.8 The two-particle Curzon solution 77

4.9 Recent mathematical developments 79
4.10 Ring singularities 80

5 The Abstract Boundary **84**
1 Introduction 84
2 Parametrized curves on a manifold 86
3 Enveloped manifolds and boundary sets 89
4 Limit points and C-completeness 92
5 Abstract boundaries 93
6 Pseudo-Riemannian manifolds 100

6 Definition of a Non-Singular Pseudo-Riemannian Manifold **107**
7 Extensions and regular boundary points 107
8 Boundaries and removable singularities 110
9 Directional and pure singularities 117
10 Singular pseudo-Riemannian manifolds 124

ABSTRACT

The main objective of this thesis is to gain a deeper understanding of the singularities which arise in solutions of Einstein's field equations. This will involve both an indepth study of one particular solution, namely the Curzon solution, as well as the development of a whole new framework for handling singularities which occur in arbitrary space-times.

The Curzon solution is a special member of Weyl's class of metrics (the class of all static, axisymmetric, vacuum solutions). The deceptively simple appearance of the Curzon metric guaranteed that its surprisingly pathological singularity structure would remain undiscovered for many years. Chapter 1 gives an historical perspective on this solution. This is of great interest, because the early work on the subject from the late sixties onwards precisely mirrors the slow but steady growth in the understanding of singularities at large during those years.

In Chapter 2 the study of the Curzon solution begins in earnest. The analysis is initially restricted to the spacelike hypersurfaces t = constant, so that one has only to consider the behaviour of spacelike geodesics and curves which lie in them. It is possible to find all such geodesics which approach the central 'directional singularity'. Ultimately a new compactified coordinate system (for each hypersurface) is introduced, which clearly separates out the directional singularity into a ring of curvature singularities threaded by spacelike geodesics heading out to infinity.

The class of all t-varying geodesics—timelike, null and spacelike—which approach the Curzon singularity is obtained in Chapter 3. Many of these reach the singularity with finite affine parameter and finite curvature. New coordinates for the Curzon space-time are constructed which permit these geodesics to be extended, whilst still preserving all features of the spacelike hypersurfaces derived in Chapter 2. The Curzon metric can be smoothly connected with Minkowski space. Chapter 4 is a survey of the Weyl metrics at large, giving the state-of-the-art of this subject and pinpointing what remains to be done.

Finally, in Chapters 5 & 6, a framework is developed for deciding whether or not any given pseudo-Riemannian manifold $(\mathcal{M}, g)$ is singular. This is based on a new topological construction called the abstract boundary (a-boundary) of $\mathcal{M}$. Of course the 4-dimensional space-times of general relativity provide the motivation for this work, and for this special class the new scheme has a number of advantages over those already in existence, such as being more easily applied to specific examples, and not requiring that the space-time under consideration be maximally extended. Removable and directional singularities fit naturally into this framework, and are given a rigorous definition for the first time.