NEW APPROACHES TO SPACE-TIME SINGULARITIES

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Contents

ABSTRACT

The main objective of this thesis is to gain a deeper understanding of the singularities which arise in solutions of Einstein's field equations. This will involve both an indepth study of one particular solution, namely the Curzon solution, as well as the development of a whole new framework for handling singularities which occur in arbitrary space-times.

The Curzon solution is a special member of Weyl's class of metrics (the class of all static, axisymmetric, vacuum solutions). The deceptively simple appearance of the Curzon metric guaranteed that its surprisingly pathological singularity structure would remain undiscovered for many years. Chapter 1 gives an historical perspective on this solution. This is of great interest, because the early work on the subject from the late sixties onwards precisely mirrors the slow but steady growth in the understanding of singularities at large during those years.

In Chapter 2 the study of the Curzon solution begins in earnest. The analysis is initially restricted to the spacelike hypersurfaces t = constant, so that one has only to consider the behaviour of spacelike geodesics and curves which lie in them. It is possible to find all such geodesics which approach the central 'directional singularity'. Ultimately a new compactified coordinate system (for each hypersurface) is introduced, which clearly separates out the directional singularity into a ring of curvature singularities threaded by spacelike geodesics heading out to infinity.

The class of all t-varying geodesics—timelike, null and spacelike—which approach the Curzon singularity is obtained in Chapter 3. Many of these reach the singularity with finite affine parameter and finite curvature. New coordinates for the Curzon space-time are constructed which permit these geodesics to be extended, whilst still preserving all features of the spacelike hypersurfaces derived in Chapter 2. The Curzon metric can be smoothly connected with Minkowski space. Chapter 4 is a survey of the Weyl metrics at large, giving the state-of-the-art of this subject and pinpointing what remains to be done.

Finally, in Chapters 5 & 6, a framework is developed for deciding whether or not any given pseudo-Riemannian manifold $(\mathcal{M}, g)$ is singular. This is based on a new topological construction called the abstract boundary (a-boundary) of $\mathcal{M}$. Of course the 4-dimensional space-times of general relativity provide the motivation for this work, and for this special class the new scheme has a number of advantages over those already in existence, such as being more easily applied to specific examples, and not requiring that the space-time under consideration be maximally extended. Removable and directional singularities fit naturally into this framework, and are given a rigorous definition for the first time.

STATEMENT

(a) It is hereby acknowledged that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. Also, to the best of the author's knowledge and belief, the thesis contains no material which has previously been published or written by another person, except where due reference is made in the text.

(b) The author consents to the thesis being made available for photocopying and loan if applicable.

Susan M. Scott

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The research presented here was conducted at the University of Adelaide, in the Department of Mathematical Physics, during its final epoch. The author notes the passing of that Department with deep regret. It was a privilege to work under the Departmental leadership of two such eminent scientists as Prof. H.S. Green and Prof. C.A. Hurst. Their approaches to research and teaching were a shining example to us all, and at a personal level, their words of wisdom and their concern and guidance will long be remembered by the author. Special thanks are also due to two of the Departmental secretaries, Mrs. Barbara Beilby and Mrs. Mary Genovese, whose efficiency, helpfulness and senses of humour were always greatly valued.

The author wishes to express her deep gratitude to the supervisor of this research, Dr. Peter Szekeres, whom she considers the finest relativist ever produced by Australia. His originality and his undaunted tackling of difficult problems were inspirational, and have hopefully been imparted to some extent to the author. He encouraged an independence which, although difficult to assimilate at the time, has served the author well since then, and will no doubt be an attribute for life. Finally, and perhaps the most valued of all, are his ongoing friendship, support and collaboration.

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On a personal level, the encouragement of the author's parents has been unfailing. They have shown great interest in her work and its progress throughout, and have always been there when needed, both morally and materially. Their rôle in the author's education will always be remembered with fond gratitude.

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PREFACE

Much of the material incorporated in this thesis has previously been published by the author. It has appeared (or will appear) as follows:

Chapter 2

Scott, S.M. and Szekeres, P.: *General Relativity and Gravitation* **18**, 557–570 (1986). 'The Curzon singularity I: Spatial sections'

Chapter 3

Scott, S.M. and Szekeres, P.: *General Relativity and Gravitation* **18**, 571–583 (1986). 'The Curzon singularity II: Global picture'

Chapter 4

Scott, S.M.: *Proceedings of the Centre for Mathematical Analysis* **19**, 175–195 (1989). 'A survey of the Weyl metrics'

Chapters 5 & 6

Scott, S.M.: *Proceedings of the Third Hungarian Relativity Workshop*, ed. Perjés, Z.I., NOVA Science Publishers, Inc., New York (to appear 1991), 30 pages. 'When is a pseudo-Riemannian manifold non-singular?'

Minor modifications have been made to the papers which comprise Chapters 2, 3 and 4, in order that they fit better into the thesis format. The paper which forms the last two chapters (5 & 6) is identical with the original version, except that it has been split into two because of its length. As a result, the numbering of sections and figures is different in those two chapters to that in earlier chapters. Also Chapter 6 continues straight on from Chapter 5, and begins with Section 7 (not Section 1). Each of Chapters 1–4 comes complete with its own introduction at the beginning, and list of references at the end. Of course, in the case of the last two chapters, only Chapter 5 has an introduction, and the list of references appears at the end of Chapter 6.

Chapter 1

An Historical Perspective on the Curzon Solution

1.1 Introduction

In this chapter, an historical perspective on the Curzon solution is given. Although the solution has been known for many years (since 1924), investigations of its singularity structure really only began in the late sixties. It transpires that an understanding of the singularity structure is central to an understanding of the global structure of the solution. This will become apparent over the next three chapters.

The Curzon metric is presented in Section 1.2 as a special member of Weyl's class of metrics (the static, axisymmetric, vacuum solutions of Einstein's field equations). The fact that the metric possesses a singularity—the Curzon singularity—is immediately obvious. Section 1.3 is a review of past analyses of this singularity, together with the conclusions drawn from them. This includes the development of the notion of a 'directional singularity', a term which will make regular appearances throughout this thesis.

Section 1.4 is entirely devoted to a description of the most far-reaching of these past analyses, namely that of Szekeres and Morgan. They produced the first ever extension of the Curzon metric, in fact the only extension in existence prior to the one which will be given in Chapter 3. Section 1.5 is a critique of the work reviewed in Section 1.3. A comparison of the conclusions reached by the various authors provides an interesting insight into the level and growth of understanding of singularities during those years.

In Section 1.6 the relationship between the Curzon solution and Israel's theorem is considered, since it appears at the outset that the Curzon solution might be a counterexample to the theorem. Of course, a close examination of the precise statement of the theorem reveals that the Curzon solution is, in fact, excluded by two of the technical conditions. However, the exercise does raise some interesting questions about the possible topologies of event horizons, and the related issue of the nakedness of singularities.

1.2 The Curzon metric

Using cylindrical coordinates (r, z, φ) with $r \geq 0$, $z \in \mathbb{R}$ and $0 \leq \varphi < 2\pi$, the static, axisymmetric, vacuum solutions of Einstein's field equations are given by the Weyl metrics [1], [2] (see also Synge [3])

$$ds^2 = -\,e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\varphi^2 \tag{1.1}$$

where $\lambda\,(r, z)$ and $\nu\,(r, z)$ are solutions of the equations

$$\lambda_{rr} + \lambda_{zz} + r^{-1}\lambda_r = 0 \tag{1.2}$$

and

$$\nu_r = r\,({\lambda_r}^2 - {\lambda_z}^2)\;, \qquad \nu_z = 2r\lambda_r\lambda_z\;. \tag{1.3}$$

If a solution λ of Eq. (1.2) is found, then Eqs. (1.3) can be integrated to find ν. In fact Eq. (1.2) is recognised as being simply the Laplace equation in cylindrical coordinates for a φ-independent function. There is thus a straightforward method of obtaining static, axisymmetric, vacuum, general relativistic fields. Namely choose an appropriate Newtonian gravitational field and then integrate the Eqs. (1.3).

An obvious choice is the gravitational field produced by a spherically symmetric mass distribution with total mass m, which is located at the origin of the cylindrical coordinate system. So

$$\lambda = -m/R \qquad \text{where} \qquad R = \sqrt{r^2 + z^2} \tag{1.4}$$

and

$$\nu = -\frac{m^2 r^2}{2R^4} \ , \tag{1.5}$$

where the constant of integration has been set to zero to ensure that the condition of elementary flatness is satisfied along the z-axis.

This monopole solution is the so-called Curzon metric [4]. It is not equivalent to the Schwarzschild metric—the unique spherically symmetric, vacuum solution of general relativity. That solution is generated by the Newtonian potential of a constant density line mass (or rod) with total mass m and length $2m$, which is located along the z-axis with mid-point at the origin ([5], [6]).

For the Curzon solution, each metric component becomes either zero or infinite at the origin of the cylindrical coordinate system i.e. the metric is singular at $R = 0$. Further investigation of the space-time in this region is necessary to determine whether this is due to a particular property of the space-time, or if it is simply the result of a bad choice of coordinates.

1.3 Past analyses of the Curzon singularity

Gautreau and Anderson [7] calculated the invariant Kretschmann scalar

$$\alpha = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \tag{1.6}$$

for the Curzon metric and found it to be of the form

$$\alpha = \exp\left[\frac{2m}{R}\left(\frac{mr^2}{R^3} - 2\right)\right] . [\text{polynomial in } m/R] \quad . \tag{1.7}$$

For straight line trajectories $z = cr$, $c \in \mathbb{R}$ to $R = 0$, the behaviour of α was seen from Eq. (1.7) to be $\alpha \to \infty$ as $R \to 0$. However for approaches along the z-axis where $r = 0$, it was found that $\alpha \to 0$ as $R \to 0$ (see Table 1.1 and Figure 1.1). They noted

> '... there appears to be a directionality associated with the "singularity" at the origin'

and concluded at the end of the article that

> '... the singular behaviour of an invariant quantity may not always indicate the location of an intrinsic singularity, so that when examining an invariant quantity one must be sure to take into consideration the possibility of its directional behaviour'.

Stachel [8] took a different approach to the analysis of this directional singularity—he was interested in determining its size (or extent). Using the coordinate relations $r = R\sin\theta$ and $z = R\cos\theta$, the Curzon metric was put into the spherical coordinate (R, θ, φ) form

$$ds^2 = -e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dR^2 + R^2 d\theta^2) + R^2 \sin^2\theta\, e^{-2\lambda} d\varphi^2 \tag{1.8}$$

$$\text{where} \qquad \lambda = -\frac{m}{R} \qquad \text{and} \qquad \nu = -\frac{m^2 \sin^2\theta}{2R^2} \quad . \tag{1.9}$$

Table 1.1: Directional behaviour of the Kretschmann scalar at the Curzon singularity

Trajectory	$\lim_{R \to 0} \alpha$
$z = cr \ , \ c \in \mathbb{R}$	∞
$r = 0$	0

The Gautreau and Anderson analysis of the behaviour of the Kretschmann scalar α as $R \to 0^+$ along straight line trajectories to the Curzon singularity ($R = 0$).

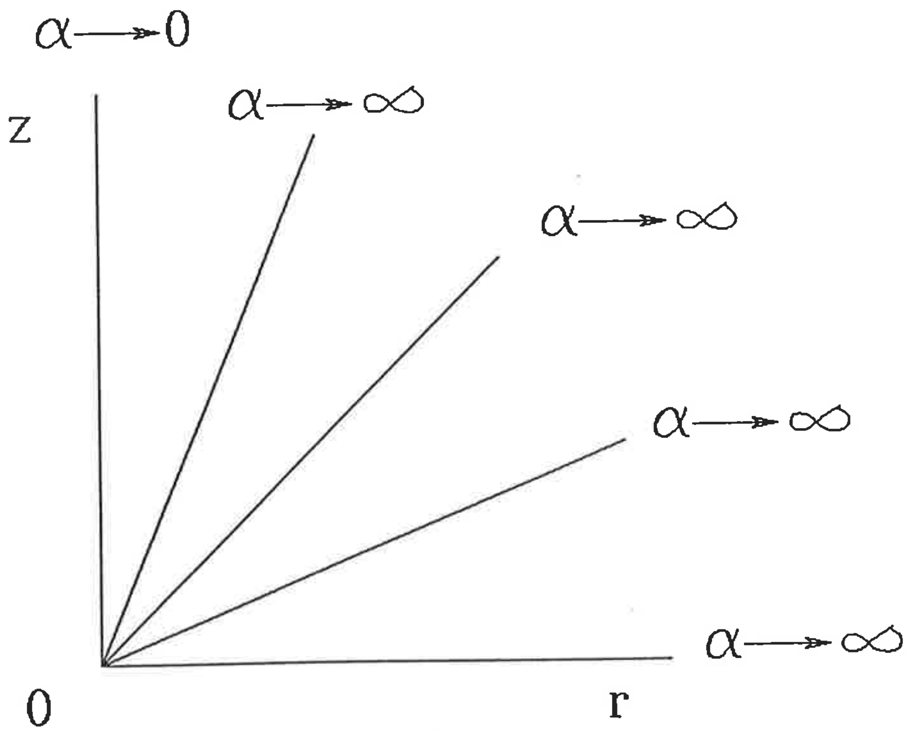

Figure 1.1 The r-z quarter-plane $r \geq 0$, $z \geq 0$. A variety of straight line trajectories to $R = 0$ are depicted, each labelled with the limiting behaviour of α as $R \to 0^+$ along that particular trajectory. The approach along the z-axis (i.e. $r = 0$) is the odd one out, since for all other approaches the Kretschmann scalar becomes singular at $R = 0$.

The area A of the two-dimensional surfaces t = constant, g_{00} = constant (so t, R = constants) was calculated. In this static space-time, these two-surfaces are invariantly characterisable gravitational equipotentials. It was found that

$$A = \iint \sqrt{g_{\theta\theta}}\sqrt{g_{\varphi\varphi}}\; d\theta\, d\varphi \tag{1.10}$$

$$= 4\pi R^2 \exp\left[\frac{2m}{R} - \frac{m^2}{2R^2}\right] \int_0^1 \exp\left[\frac{m^2 x^2}{2R^2}\right] dx \ . \tag{1.11}$$

The conclusion was that the gravitational equipotential surfaces shrink in area as the value of R decreases from infinity, until they reach a minimum. They then begin to increase in area as R decreases further, finally becoming infinite as $R \rightarrow 0^+$.

It is also pointed out that the lines $\theta = 0$ and $\theta = \pi$ in the hypersurfaces t = constant are spacelike geodesics approaching the two "regular" points of $R = 0$, which lie an infinite spatial distance from any point on these geodesics. The lines $\theta = \pi/2$, φ = constant in the hypersurfaces t = constant are also spacelike geodesics leading to singular points of $R = 0$ on the plane of symmetry. These points lie a finite distance away from any finite value of R. Stachel notes

> '... the approach to $R \rightarrow 0^+$ along different directions corresponds to an approach to different limiting points on the infinite surface $R = 0$'

and adds that

> '... there seems to be no difficulty with the criterion of the blowing up of a curvature scalar for the occurrence of real singularities in the Riemann space'.

However he reasons that there is no possibility of extending the manifold on which the metric is imposed beyond the surface $R = 0$, since at least one curvature invariant becomes infinite along almost all directions of approach to the surface (except for $\theta = 0$ and $\theta = \pi$).

It is also claimed that the hypersurface $t =$ constant should be regarded as being multiply connected, since any closed curve in it enclosing a gravitational equipotential cannot be shrunk continuously to a point. The example provided is the curve in the plane of symmetry $\theta = \pi/2$ with $R =$ constant which has the length $L = 2\pi R \exp(m/R)$. With the value of R decreasing from infinity this length reaches a minimum value of $2\pi m e$ at $R = m$, and then increases again without limit as $R \to 0^+$.

Finally, Stachel notes that Israel [9]

> '... has shown that the Schwarzschild metric is the only empty space static metric of a sufficiently regular class which can have a non-singular event horizon. Our result for the Curzon metric shows one of the alternate topological peculiarities that can occur: an event horizon of infinite area on which an invariant of the Riemann tensor becomes singular.'

In a further paper by Gautreau [10] it is conjectured:

> '... Stachel's result suggests that there might be in general a correspondence between equipotential surfaces approaching a non-zero area and directional singularities'.

He points out, however, that no general proof of this is known.

Cooperstock and Junevicus [11] extended the analysis of the Curzon singularity performed by Gautreau and Anderson [7]. Instead of using straight line trajectories to $R = 0$ as was done in [7], they considered the behaviour of α (Eq. (1.7)) along the family of curves

$$\frac{z}{m} = b\left(\frac{r}{m}\right)^n , \qquad n > 0 \ . \tag{1.12}$$

For simplicity, since the hypersurfaces $t =$ constant have the plane of symmetry $z = 0$, the constant b was assumed to be positive.

They found that as $r \to 0^+$

$$\begin{aligned} &(1) \quad \alpha \to \infty \quad \text{for } \; n > 2/3 \\ &(2) \quad \alpha \to \infty \quad \text{for } \; n = 2/3, \;\; 0 < b < (1/2)^{1/3} \\ &\qquad\; \alpha \to 0 \quad \text{for } \; n = 2/3, \;\; b \geq (1/2)^{1/3} \\ &(3) \quad \alpha \to 0 \quad \text{for } \; 0 < n < 2/3 \;\; . \end{aligned}$$

These results are summarised in Table 1.2, and illustrated in Figure 1.2.

The interesting feature of this work was the discovery of a class of curves, for example those with $2/3 < n < 1$, which are both asymptotic to the z-axis as $r \to 0^+$ and have the behaviour $\alpha \to \infty$. It was this peculiarity which led Cooperstock, Junevicus and Wilson [12] to believe that there was a deficiency in the concept of a "directional singularity", remarking that

> '... If anything, the entire concept should be referred to as a trajectory singularity rather than a directional singularity'.

In Cooperstock and Junevicus [11] they add

> '... By the criterion of Gautreau and Anderson, one might be led to call the termination point in the z-axis direction both singular and non-singular. We feel that it is more reasonable to simply call it singular along with every other termination point'.

1.4 The Szekeres and Morgan extension of the Curzon metric

Szekeres and Morgan [13] took a different interpretation of the Gautreau and Anderson [7] analysis. They thought that the regular behaviour of α along the z-axis, unlike that for all other directions of approach to $R = 0$,

> '... leads one to make the suggestion that possibly the Curzon metric "opens up" for particles approaching $R = 0$ along the z-axis, allowing them to pass on into some new region'.

Table 1.2: Behaviour of the Kretschmann scalar along curves approaching the Curzon singularity

Family of curves $\frac{z}{m} = b\left(\frac{r}{m}\right)^n \quad b, n > 0$	$\lim_{r \to 0} \alpha$
$n > 2/3$	∞
$n = 2/3, \quad 0 < b < (1/2)^{1/3}$	∞
$n = 2/3, \quad b \geq (1/2)^{1/3}$	0
$0 < n < 2/3$	0

The Cooperstock and Junevicus analysis of the behaviour of the Kretschmann scalar α as $r \to 0^+$ along the curves $z/m = b\,(r/m)^n$, where $b, n > 0$. Whether or not α becomes singular (i.e. infinite) at the Curzon singularity depends on the precise values of b and n which are chosen. The 'transition curve' is seen to be the one with $n = 2/3$ and $b = (1/2)^{1/3}$.

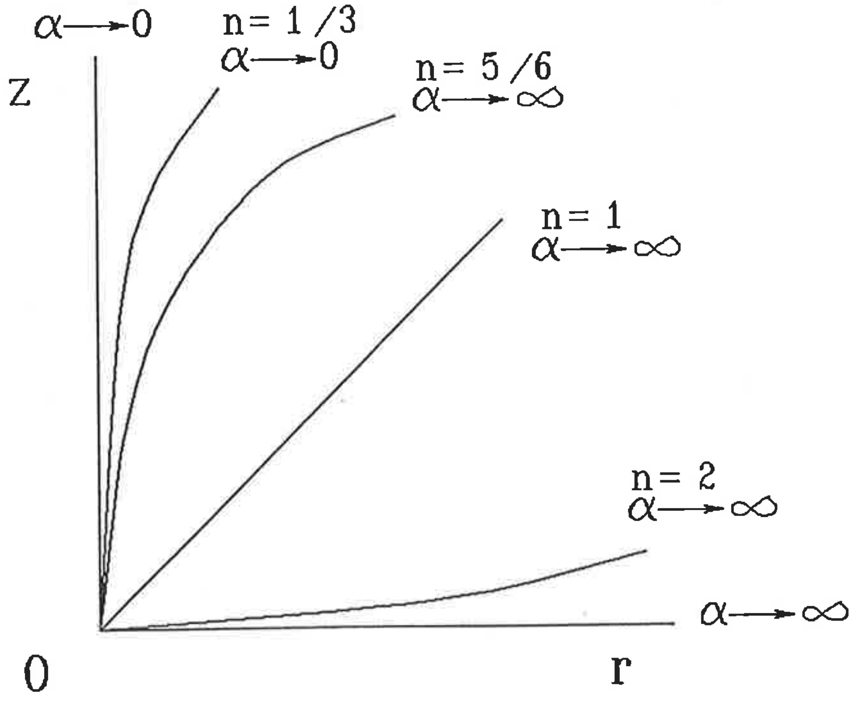

Figure 1.2 Curves of the type $z/m = b\,(r/m)^n$ in the r-z quarter-plane $r \geq 0$, $z \geq 0$. Each curve drawn is labelled with its value of n as well as the limiting behaviour of α as $r \to 0^+$ along that particular curve. Of the two curves asymptotic to the z-axis, $\alpha \to 0$ along one ($n = 1/3$), and becomes singular along the other ($n = 5/6$). This demonstrates that the behaviour of α can vary even amongst curves which all approach the singularity in *one direction*—in this case the z-axis.

In order to obtain more information about the singularity, they focussed their attention on *null geodesics* which lie in a fixed plane $\varphi =$ constant, and which approach the singularity at $R = 0$. To find such geodesics, they searched for asymptotic solutions of the relevant geodesic equations as $R \to 0$. Apart from the possibility of some of these geodesics spiralling in towards $R = 0$ (which was not considered), it was found that there are only two types of solution—either the geodesics approach the z-axis asymptotically and have the form

$$r/m \sim \rho\, e^{-m/|z|} \quad , \tag{1.13}$$

where ρ is a non-negative constant, or else they lie in the plane $z = 0$.

This led Szekeres and Morgan to conclude that

> '... The behaviour along the $r = 0$ geodesics turns out to be general rather than peculiar for it appears to be the case that almost all geodesics approach the z-axis very rapidly (in Weyl's coordinates) as $R \to 0$, the only geodesics not exhibiting this behaviour being those which lie in the $z = 0$ plane'.

Treating Eq. (1.13) as though it were the exact equation of the null geodesics near $R = 0$, they defined a "comoving" coordinate ρ, and a "retarded" time coordinate u $(z > 0)$:

$$r/m = \rho\, e^{-m/z} \tag{1.14}$$

$$t/m = u + \int_{z_0}^{z} e^{2m/z} dz + \frac{\rho^2 m}{2z^2} \quad , \quad z_0 = \text{constant} > 0 \ . \tag{1.15}$$

However, it is pointed out that the terms "comoving" and "retarded" are not exactly applicable, since Eq. (1.13) only represents the asymptotic form of the geodesics, but that they do apply in the limiting sense as $z \to 0^+$.

For approaches to $R = 0$ such that $z \to 0^+$ while ρ remains bounded, the Curzon metric transformed to the new coordinates (u, ρ, z, φ) was found to have the form

$$ds^2 = -2\,dudz + d\rho^2 + \rho^2 d\varphi^2 + O(z^{-8}e^{-2m/z})\,h_{\mu\nu}dx^\mu dx^\nu \ , \tag{1.16}$$

where $h_{\mu\nu}$ is a tensor whose behaviour is regular at $z = 0$.

So in these coordinates, the Curzon metric is completely regular as $z \to 0^+$ while ρ remains bounded, and may be connected across the plane $z = 0$ with flat space expressed in double-null cylindrical coordinates (z, u, ρ, φ)

$$ds^2 = -2\,dudz + d\rho^2 + \rho^2 d\varphi^2 \quad (z < 0) \ . \tag{1.17}$$

The resulting space-time is C^∞ at $z = 0$, but not analytic, so the connection with Minkowski space is just one of an infinite number of possible C^∞ extensions.

It is pointed out that all outward going geodesics $(z > 0)$ from $R = 0$ are both past and future complete in these coordinates. The inward going geodesics $(z > 0)$ are, however, future incomplete as they approach $z = 0$ asymptotically in the new coordinates. The suggested remedy for this situation was to replace the "retarded" time coordinate u by an "advanced" time coordinate v defined by

$$t/m = v - \int_{z_0}^{z} e^{2m/z}dz - \frac{\rho^2 m}{2z^2} \ , \quad z_0 = \text{constant} > 0 \ . \tag{1.18}$$

For approaches to $R = 0$ such that $z \to 0^+$ and ρ remains bounded, the metric transformed to the coordinates (v, ρ, z, φ) is again extendible across $z = 0$ in a similar manner to before. The lower half $(z < 0)$ of the Curzon metric can be covered and extended in an analogous fashion, by replacing z with $-z$ in the various coordinate definitions.

The plane of symmetry $z = 0$ is not, however, covered by any of these coordinate patches, and must be separately covered by, for example, the original Weyl coordinates.

The Kretschmann scalar α becomes infinite for approaches to $R = 0$ in this plane, but in the (u, ρ, z, φ) and (v, ρ, z, φ) coordinate patches this real singularity has been pushed out to $z = 0$, $\rho = \infty$ i.e. (suppressing u and v respectively) it appears as a ring placed at infinity with null geodesics threading through it.

It is concluded from this that

> '... the deceptively simple point-like appearance of $R = 0$ in the Weyl coordinates must be abandoned. Indeed by using comoving coordinates it has more the appearance of an infinite plane ($z = 0$) at which the space-time is momentarily flat'.

However, it is noted that geodesics approaching this plane at large values of ρ, have to cross a ridge of high curvature close to $z = 0$ before reaching this flat region. This is so because the invariant Kretschmann scalar α has an infinite limit along the lines $r = k|z|$ $(k \in \mathbb{R}^+)$ as $z \to 0$, and thus along the curves

$$\rho = k\,|z|/m\, e^{m/|z|}\ , \qquad u, v = \text{constant}\ , \qquad \varphi = \text{constant} \tag{1.19}$$

(see Figure 1.3).

Although Szekeres and Morgan chose to extend the Curzon space-time by various half-portions of Minkowski space, they also commented on another interesting C^∞ extension:

> '... Perhaps the most natural extension is to connect the lower half to the upper half of the Curzon metric across $z = 0$ in such a way that geodesics entering $z = 0$ from below emerge into $z > 0$ and geodesics entering from above emerge into $z < 0$. From the above discussion this is clearly possible to achieve in a C^∞ manner, and would have the advantage that particles entering $R = 0$ do not "disappear" from the external world'.

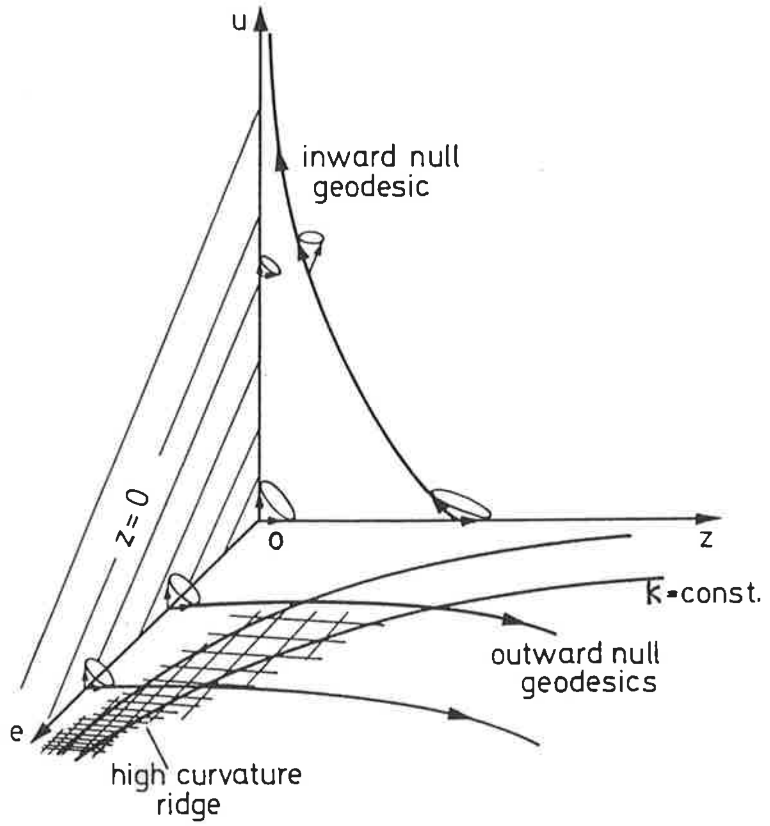

Figure 1.3 The Curzon space-time ($z > 0$) near $R = 0$ displayed in (u, ρ, z, φ) coordinates (φ is suppressed). The metric can be connected in a C^∞ manner with flat space across the plane $z = 0$. In this way the two outward null geodesics shown become past complete as well as future complete. The inward null geodesic ($\rho = 0$) is, however, still future incomplete and inextendible in these coordinates. Sections of two curves $r = kz$ (k = constant > 0) are drawn. To the left of these curves is a ridge of high curvature just before flat space is reached at $z = 0$.

1.5 Comments on past analyses of the Curzon singularity

Given that Gautreau and Anderson only found the limiting behaviour of α for straight line trajectories $z = cr$, $c \in \mathbb{R}$ and $r = 0$ to the singularity at $R = 0$, their development of the notion of *directionality* with respect to singularities was a reasonable first approximation. With the subsequent analysis of the behaviour of α along the curves given by Eq. (1.12), which was carried out by Cooperstock and Junevicus, the concept of a directional singularity was refined to that of a trajectory singularity.

In terms of the *description* of this type of singularity, the proposed label "trajectory singularity" must be the ultimate one, in the sense that it cannot be refined. After all, the limit of α as $R \to 0^+$ along a particular trajectory is unique when it exists. What was needed at that time was not a better description of the singularity, but a better understanding of what such a description indicates about the *singularity structure*, and indeed, about the global structure of the space-time.

Although it is not absolutely clear, it seems that both Gautreau and Anderson, and Cooperstock and Junevicus were inclined to still consider the Curzon singularity as a point. The former group probably thought that it *was not* an intrinsic singularity. This is implied when they point out that

> '... the singular behaviour of an invariant quantity may not always indicate the location of an intrinsic singularity',

and presumably refers to the fact that the Kretschmann scalar does not become infinite for *all* straight line trajectories to $R = 0$.

The latter group thought that $R = 0$ *was* an intrinsic singularity. Since they had found curves along which $\alpha \to \infty$ as $R \to 0^+$ and which were also asymptotic to the

z-axis, they felt that it was

> '... more reasonable to simply call (the termination point in the z-axis direction) singular along with every other termination point'.

This alone does not, of course, imply that they were thinking of $R = 0$ as a point. However, if they were actually considering it to be some sort of surface with non-zero area, then they obviously had in mind that this would consist of intrinsically singular points, each one corresponding to the termination point for a particular *direction* of approach to $R = 0$. This would have been very strange indeed in the light of their refinement of the notion of a directional singularity to that of a trajectory singularity, and it is more likely that they simply thought of it as an intrinsically singular point.

The work of Stachel, occurring between that of Gautreau and Anderson, and Cooperstock and Junevicus, was by far the most significant in terms of settling the status of the Curzon singularity as a point or otherwise. Whilst curves and trajectories to the singularity are useful tools for discovering many of its features, there are more direct means by which to determine its size (or extent). Stachel calculated the area of the two-dimensional gravitational equipotential surfaces (given by $t, R =$ constants), and found that it became infinite as $R \to 0^+$.

Relating this result to the work of Gautreau and Anderson, Stachel concludes that

> '... the approach to $R \to 0^+$ along different directions corresponds to an approach to different limiting points on the infinite surface $R = 0$'.

Given that the notion of a "trajectory singularity" was yet to be proposed by Cooperstock and Junevicus, the conclusion was a good one—in fact, one which could easily have been modified to incorporate the refinement that Cooperstock and Junevicus later made to the concept of a directional singularity.

However, it remains to be seen later in this thesis that the above conclusion is not completely correct, in the sense that approaches to $R = 0$ along *different directions* will in many cases correspond to approaches to the *same* limiting point on the infinite surface $R = 0$. Furthermore, it will also be seen that different approaches to $R = 0$ in the *same direction* (e.g. different curves asymptotic to a particular direction), can sometimes correspond to approaches to *different* limiting points on the infinite surface $R = 0$. This last remark is not really very surprising given the findings of Cooperstock and Junevicus for the z-axis direction.

In relation to the problem of determining which points on the infinite surface $R = 0$ represent intrinsic singularities, Stachel asserts that

> '... there seems to be no difficulty with the criterion of the blowing up of a curvature scalar for the occurrence of real singularities in the Riemann space'.

This is in direct contrast with Gautreau and Anderson's earlier comment on the same matter. Of course, it is now generally accepted that the singular behaviour of a curvature scalar, for instance the Kretschmann scalar, is a *sufficient condition* for the existence of a real singularity of the space-time. (For a discussion of this and other criteria, see for example [14].)

One gathers, from the above assertion, that Stachel thought of almost all limiting points on the infinite surface $R = 0$ as being real (or intrinsic) singularities. This is because he regarded each such limiting point as corresponding to a different direction of approach to $R = 0$, and was aware from Gautreau and Anderson's work that $\alpha \to \infty$ as $r \to 0^+$ in nearly all of these directions. The only exception is that $\alpha \to 0$ for an approach along the z-axis, and it is clear that he considered the limiting point corresponding to such an approach as being "regular".

Although Stachel found that the Curzon singularity at $R = 0$ is actually an infinite surface, he did not attempt to determine its *topology*. If one accepts, for the moment, his conclusion that different directions of approach to $R = 0$ correspond to approaches to different limiting points on this infinite surface, then it is difficult to imagine how the regular limiting point for an approach along the z-axis (e.g. with $z > 0$) meshes in with the intrinsically singular limiting points for all other directions of approach.

To address this question, one really needs to return from thinking about the surface area of gravitational equipotentials, to examining in more detail the behaviour and properties of curves which approach $R = 0$ along or asymptotic to the z-axis. Stachel did not do this, and thus was not in a position to conjecture about the possible topology of $R = 0$.

It's clear why he considered the manifold to be inextendible through $R = 0$. He viewed this infinite surface as being almost entirely comprised of real singularities where the Kretschmann scalar blows up—so there was obviously no hope of extension through any such point. That only left the two "regular" limiting points corresponding to approaches to $R = 0$ along $\theta = 0$ and $\theta = \pi$ respectively.

He knew that in a hypersurface $t =$ constant, $\theta = 0$ and $\theta = \pi$ are spacelike geodesics which have infinite length between any geodesic point ($z \neq 0$) and $R = 0$. In the absence of something which could be extended, there was little reason for him to expect that extensions through these two regular points of $R = 0$ would be possible.

As previously mentioned, Stachel claimed that a hypersurface $t =$ constant should be regarded as being multiply connected, since any closed curve in it enclosing a gravitational equipotential cannot be shrunk continuously to a point. This claim seems to confuse the issues of whether the gravitational equipotentials $t =$ constant,

$R =$ constant are simply connected or not, and whether the hypersurfaces $t =$ constant are simply connected or not. In any case, the concept of simple connectedness does not appear to be well understood.

A gravitational equipotential $t =$ constant, $R =$ constant has the topology of a sphere and thus, clearly, is simply connected. In a hypersurface $t =$ constant, a closed curve which does not pass through $R = 0$ can always be shrunk continuously to a point (with $R \neq 0$). There is no necessity to shrink such a curve to the particular "point" $R = 0$, which the work of Stachel would seem to suggest.

His example concerning the length of curves which lie in the plane of symmetry $\theta = \pi/2$ and have $R =$ constant, is not of direct relevance to the question of the simple connectedness of the hypersurfaces $t =$ constant. However, it may well show that the plane of symmetry itself is *not* simply connected. In order to further investigate these questions, it will be necessary to determine what meaning, if any, can be given to the statement that a curve passes through (or indeed lies in) $R = 0$.

1.6 Israel's theorem

Recall from Section 1.3 the comment by Stachel [8] that Israel [9]

> '... has shown that the Schwarzschild metric is the only empty space static metric of a sufficiently regular class which can have a non-singular event horizon. Our result for the Curzon metric shows one of the alternate topological peculiarities that can occur: an event horizon of infinite area on which an invariant of the Riemann tensor becomes singular.'

In fact, it will later be shown that *the Curzon space-time possesses a nonsingular event horizon at $R = 0$*. This means that Stachel's concludions about the space-time structure at $R = 0$ are (partially) incorrect. Furthermore, his description of Israel's work suggests that there is a problem reconciling our findings for the Curzon space-time with Israel's results.

Perhaps the phrase '... of a sufficiently regular class' would exclude the Curzon space-time from consideration. In order to find out, it is necessary to take a short diversion to examine Israel's paper [9]. In the introduction, Israel states that the conjecture which he intends to prove is the following:

> '... that Schwarzschild's solution is uniquely distinguished among all static, asymptotically flat, vacuum fields by the fact that it alone possesses a nonsingular event horizon.'

As is known from Section 1.2, the Curzon solution is certainly a static, vacuum field. It is also asymptotically flat, as can be seen by taking the limit of the metric as $R \to \infty$. This limit is

$$ds^2 = -\,dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 \tag{1.20}$$

which is simply the flat space metric expressed in cylindrical coordinates (r, z, φ).

So with the statement of the conjecture as it stands, the Curzon solution is not excluded from the class of solutions under consideration, and thus remains a counterexample. However, the statement of the theorem as given in the abstract puts a different slant on things:

> '... Among all static, asymptotically flat vacuum space-times with closed simply connected equipotential surfaces g_{00} = constant, the Schwarzschild solution is the only one which has a nonsingular infinite-red-shift surface $g_{00} = 0$.'

Whilst with this version the Curzon solution remains a member of the class of solutions under consideration, it is no longer clear that it satisfies the final criterion, namely, that it '... has a nonsingular infinite-red-shift surface $g_{00} = 0$'. The surface $g_{00} = 0$ is the infinite surface $R = 0$, and although there does exist a nonsingular infinite-red-shift surface at $R = 0$, it does not constitute the *entire* surface. So this discrepancy may well remove the Curzon solution from its status as a counterexample to Israel's theorem.

Israel used the following line element for the metric of a static space-time:

$$ds^2 = g_{\alpha\beta}(x^1, x^2, x^3)\, dx^\alpha dx^\beta - V^2 dt^2 \quad , \tag{1.21}$$

$$V = V(x^1, x^2, x^3) \quad , \qquad t = x^0 \quad , \tag{1.22}$$

where α and β run from 1 to 3.

$$V^2 = |\xi.\xi| \tag{1.23}$$

ξ is a hypersurface-orthogonal, timelike Killing vector field.

His explicit statement of the theorem was as follows:

In a static space-time, let Σ be any spatial hypersurface $t =$ constant, maximally extended consistent with $\xi.\xi < 0$. We consider the class of static fields such that the following conditions are satisfied on Σ:

(a) Σ is regular, empty, noncompact, and "asymptotically Euclidean".

(b) The equipotential surfaces $V =$ constant > 0, $t =$ constant are regular, simply connected closed 2-spaces.

(c) The invariant $R_{ABCD}R^{ABCD}$ formed from the 4-dimensional Riemann tensor is bounded on Σ.

(d) If V has a vanishing lower bound on Σ, the intrinsic geometry of the 2-spaces $V = c$ approaches a limit as $c \to 0^+$, corresponding to a closed regular 2-space of finite area.

THEOREM — The only static space-time satisfying (a), (b), (c) and (d) is Schwarzschild's spherically symmetric vacuum solution.

For the Curzon space-time, Σ is any spatial hypersurface t = constant with $R > 0$. Σ is certainly regular, empty, noncompact and asymprotically Euclidean, and thus satisfies condition (a). The equipotential surfaces V = constant > 0, t = constant are the surfaces R = constant > 0, t = constant. As discussed in Section 1.5, these surfaces are regular, simply connected closed 2-spaces, and so condition (b) is also satisfied.

However, as was seen in Section 1.3, the limit of the Kretschmann scalar α along certain approaches to $R = 0$ (in Σ) is infinite, and so condition (c) is *not* satisfied. V *has* a vanishing lower bound on Σ, but the intrinsic geometry of the 2-spaces $R = c$ *does not* approach a limit as $c \to 0^+$ corresponding to a closed regular 2-space of finite area (see Section 1.5). Thus condition (d) is also *not* satisfied.

It is now clear that the Curzon space-time is excluded by conditions (c) and (d) of the theorem, and thereby loses its status as a counterexample. Nevertheless, this discussion has highlighted the fact that it is these two conditions of the theorem which fine-tune its result, namely that the Schwarzschild solution is the only static space-time to satisfy all four conditions.

At the time of this work, a nonsingular event horizon in a static field was, presumably, always thought of as a 2-surface which *completely surrounds* a real singularity i.e. it is a regular, simply connected, closed 2-space of finite area (compare with conditions (b) and (d)). If this were not so, more care would probably have been taken in both the paraphrasing of the theorem, and the interpretation of its significance. As an example of the latter, Israel concludes the paper by saying that

> '... The result of this paper would have important astrophysical consequences if it were permissible to consider the limiting external field of a gravitationally collapsing asymmetric (non-rotating) body as static. In that case, only two alternatives would be open—either the body has to divest itself of all quadrupole and higher moments by some mechanism (perhaps gravitational radiation), or else an event horizon ceases to exist.'

As it stands this comment is incorrect as will be seen later. It can be corrected by replacing '... or else an event horizon ceases to exist' with '... or else a naked singularity will occur'. In other words, if a collapsing body retains some quadrupole and/or higher moments, the resulting singulatity will be 'visible' at some regions at infinity. However, a nonsingular event horizon may still exist which prevents the singularity from being 'visible' at other regions at infinity.

So what Israel's theorem effectively tells us is that curvature singularities of a static vacuum field will always be naked (i.e. not completely enclosed by a nonsingular event horizon), with the exception of the curvature singularity of Schwarzschild's spherically symmetric vacuum solution. Although this is a very important result, it gives no actual information about the *nakedness* of the singularity e.g. whether the singularity is partially or totally naked, and the effect of this nakedness on the 'unprotected' regions of the space-time.

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Chapter 2

The Spacelike Hypersurfaces of the Curzon Solution

2.1 Introduction

The term "directional singularity" is used in general relativity. It is applied if the limit of an invariant scalar formed from the Riemann tensor is found to depend on the direction of approach to the singularity. One of the best known examples of such directional behaviour is the Curzon singularity occurring at $R = 0$ in the Weyl metric [1]

$$ds^2 = -e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\varphi^2 \tag{2.1}$$

for a monopole potential (the Curzon solution [2])

$$\lambda = -m/R \qquad \text{and} \qquad \nu = -\frac{m^2 r^2}{2R^4} \qquad \text{where} \qquad R = \sqrt{r^2 + z^2}\ . \tag{2.2}$$

It was first noticed by Gautreau and Anderson [3] that the Kretschmann scalar

$$\alpha = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \tag{2.3}$$

tends to the value zero along the z axis, but becomes infinite for other (straight line) directions of approach to $R = 0$. A more detailed analysis encompassing a wider class of curves was carried out by Cooperstock and Junevicus [4].

This directional behaviour has been shown to be symptomatic of a subtle singularity structure [5], whereby null geodesics approaching $R = 0$ may in some cases be extended beyond the Weyl coordinate patch. These geodesics appear to thread their way through a ringlike singularity (infinite curvature), which can be reached by other null geodesics in finite affine parameter. This structure is at best suggestive. It is proposed to put it on a firmer footing in this chapter, by viewing the matter entirely from a spatial point of view. The full space-time picture will be given in the next chapter.

In Section 2.2, spacelike geodesics approaching $R = 0$ in the hypersurfaces $t =$ constant are discussed. When confined to a fixed plane $\varphi =$ constant, these geodesics turn out to have only two basic types of behaviour. Either they asymptote exponentially towards the z axis, but in an oscillatory way, or else they approach the r axis in a non-oscillatory manner. In Section 2.3 the analysis of Cooperstock and Junevicus for power-law curves $z \propto r^n$ is developed further. In particular, the critical case $n = 2/3$ discovered by them is analysed in greater detail. In addition to the behaviour of the Kretschmann scalar, the proper distance and first curvature along these curves are also considered.

These properties are used in Section 2.4 to set up a compactified coordinate system for the hypersurface $t =$ constant, in which the real singularity is clearly separated. This singularity appears as a ring which can be reached by spacelike geodesics in finite proper distance. A large class of non-extendible curves, including the oscillating geodesics, thread their way through the ring to terminate in a new region of compactified infinity. In the original Weyl coordinate system, these curves all terminated at $R = 0$. Thus the picture of a ring singularity tentatively proposed in [5] is upheld in the spatial sections.

2.2 Spacelike geodesics approaching $\mathbf{R = 0}$

The geodesic equations for the Weyl metric Eq. (2.1) have been given previously by Szekeres and Morgan [5]. Only the case of spacelike geodesics lying in a hypersurface t = constant is considered here, for which the equations are

$$r''(1 - H^2 r^{-2} e^{2\lambda}) = (1 + r'^2)\,[\,H^2 r^{-2} e^{2\lambda}\,(r'\nu_z - \nu_r + r^{-1}) + \nu_r - \lambda_r - r'\,(\nu_z - \lambda_z)\,] \tag{2.4}$$

$$\dot{\phi} = H r^{-2} e^{2\lambda} \tag{2.5}$$

and
$$e^{2(\nu-\lambda)}\,(\dot{r}^2 + \dot{z}^2) + r^2 e^{-2\lambda} \dot{\phi}^2 = 1 \tag{2.6}$$

where $' \equiv d/dz$, $\dot{} \equiv d/ds$ and s is the proper distance.

Case (a) : ϕ = constant

For these geodesics $H = 0$, and Eq. (2.4) reduces to

$$r'' = (1 + r'^2)\,[\,\nu_r - \lambda_r - r'\,(\nu_z - \lambda_z)\,] \ . \tag{2.7}$$

It is known from symmetries present in the metric that geodesics lie along both the r and z axes. Suppose there exists an asymptotic solution of Eq. (2.7) of the form

$$r \sim kz, \quad r' \sim k \quad \text{as} \quad z \to 0 \ .$$

On substituting into Eq. (2.7) the explicit Curzon forms (Eq. (2.2)) for λ and ν, and integrating once with respect to z, one finds

$$r' \sim \frac{m^2 k}{2\,(1 + k^2)^2 z^2} \ .$$

Clearly no finite, non-zero values of k are admissible, whence $k = 0$ or $k = \infty$. That is, all geodesics in this plane approaching $R=0$ are either asymptotic to the z axis or asymptotic to the r axis.

An approximate solution may be obtained for geodesics asymptotic to the z axis as $z \to 0$ by neglecting terms in $(r/z)^2$ and r'^2 in Eq. (2.7). This yields the differential equation

$$r'' = -\,(m^2/z^4 + m/z^3)\,r \,+\, (m/z^2)\,r'$$

which for $z > 0$ has the exact solution

$$r\,(z) \;=\; z e^{-m/2z} \left\{ \alpha \cos\left[(\sqrt{3}/2)\,(m/z) \right] + \beta \sin\left[(\sqrt{3}/2)\,(m/z) \right] \right\} \qquad (2.8)$$

(see Ince [6]), where α and β are constants. Thus the geodesics are asymptotic to these curves that oscillate about the z axis with exponentially decreasing amplitudes as $z \to 0$. They will be referred to as the *oscillating geodesics* (including the geodesic along the z axis).

Computer-generated solutions of the exact geodesic equation Eq. (2.7) (using a 4th-order Runge-Kutta method) reveal an interesting property of these oscillating geodesics (see Figure 2.1). Geodesics originating from a point (r_0, z_0), with z_0 and r_0/z_0 both small, reconverge many times—in fact, after every crossing of the z axis—for quite a wide range of initial slope r'_0. These points appear to have a countably infinite number of conjugate points.

A similar treatment with r rather than z as the independent variable, can be used to find geodesics asymptotic to the r axis as $r \to 0$. They have the form

$$z(r) \;=\; \gamma\, r^2 \; [\, 1 + (r/m) + \gamma^2 r^2 + O(r^3) \,] \qquad (2.9)$$

where γ is a constant, and will be called the *non-oscillating geodesics* (including the geodesic along the r axis).

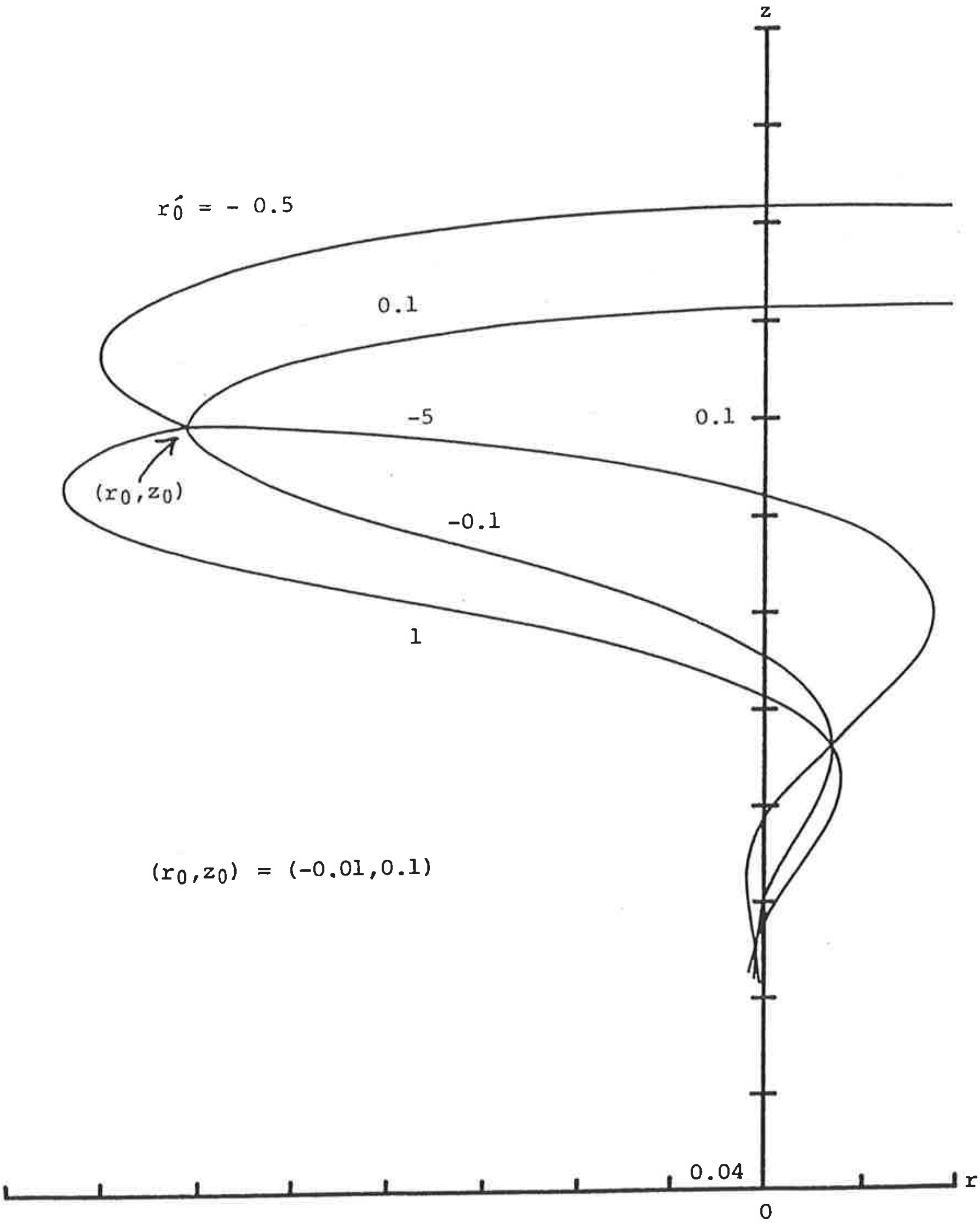

Figure 2.1 Oscillating geodesics in the r-z plane. Five geodesics are plotted from a point (r_0, z_0), generated as computer solutions of the exact geodesic equation Eq. (2.7) for a range of initial slope r_0' $(m = 1)$. The lower three geodesics focus and refocus on opposite sides of the z axis.

Case (b) : $\dot{\phi} \neq 0$

The asymptotic solutions of Eq. (2.4) with $H \neq 0$ may similarly be shown to approach the r or z axes. Eq. (2.5) and Eq. (2.6) must be used to estimate the angular dependence. The results are

(i) for geodesics approaching the z axis as $z \to 0^+$

$$r(z) \sim \left(\frac{2}{\sqrt{3}}\frac{|H|}{m}\right)^{\frac{1}{2}} z\, e^{-m/2z} \tag{2.10}$$

$$\phi(z) \sim \pm\frac{\sqrt{3}}{2}\frac{m}{z} + \beta \tag{2.11}$$

(ii) for geodesics approaching the plane $z = 0$ as $r \to 0$

$$z(r) = \gamma r^2 + O(r^3)$$

$$\phi(r) \sim \omega$$

where $\beta, \gamma, \omega =$ constants. The former class of geodesics spiral about the z axis as they approach $R = 0$.

This analysis of spacelike geodesics approaching $R = 0$ started from the assumption that they have the asymptotic form $r \sim kz$. Other possibilities, such as spiralling geodesics in the r-z plane, have not been considered. However, in retrospect, once the new picture to be developed in this chapter has been obtained, it will become clear that such possibilities cannot exist. For example, a spiralling geodesic would have to approach both a new region at infinity and the real singularity in larger and larger loops. Even if such a geodesic were to exist (which seems highly unlikely), it would approach no new points.

2.3 Analysis of curves approaching R = 0

Cooperstock and Junevicus [4] considered the limit of the Kretschmann scalar α (Eq. (2.3)) as $r \to 0^+$ along the family of curves

$$\frac{z}{m} = C\left(\frac{r}{m}\right)^n$$

(C, n = constants > 0). They found that even among the curves of this family which are asymptotic to the z axis (i.e. $n < 1$), there was a division into those for which $\alpha \to 0$ and those for which $\alpha \to \infty$. Their inclination was to lump these all together and regard them as terminating at the same "singular point". However, a different view will be taken here. It is felt that this analysis simply indicates the presence of "structure" in the Curzon singularity, and that the directional dependence of the Kretschmann scalar as $R \to 0$ indicates a crushing of the space-time in a neighbourhood of the singularity, brought about by the use of Weyl coordinates.

In order to uncover this structure, a much extended analysis of curves approaching $R = 0$ in the r-z plane will be undertaken. As well as considering the limit of the Kretschmann scalar, the proper distance and the first curvature along each curve as $R \to 0$ are also determined. Because of symmetries, it is only necessary to consider the quarter-plane $r, z \geq 0$ in what follows.

(a) Limit of the Kretschmann scalar

The Kretschmann scalar for the Curzon metric is given explicitly by

$$\begin{aligned}\alpha &= 8\,e^{2m(mr^2R^{-4}-2R^{-1})}\,[\,2\,m^4r^2R^{-12}\,(m^2-3mR+3R^2) \\ &\qquad +6\,m^2R^{-8}\,(m-R)^2\,]\ . \end{aligned} \tag{2.12}$$

For curves of the type

$$\frac{z}{m} = b\left(\frac{1}{2}\right)^{\frac{1}{3}}\left(\frac{r}{m}\right)^{n} \qquad (b, n > 0) \tag{2.13}$$

Cooperstock and Junevicus found that as $r \to 0$

(1) $\alpha \to 0$ for $0 < n < 2/3$

(2) $\alpha \to 0$ for $n = 2/3, \ b > 1$

$\alpha \to \infty$ for $n = 2/3, \ 0 < b \leq 1$

(3) $\alpha \to \infty$ for $n > 2/3$.

They actually found that $\alpha \to 0$ for $n = 2/3$ and $b = 1$, which is incorrect. This critical case is important and may be more finely split by considering the two-parameter family of curves

$$\frac{z}{m} = \left(\frac{1}{2}\right)^{\frac{1}{3}}\left(\frac{r}{m}\right)^{\frac{2}{3}} - c\left(\frac{1}{2}\right)^{\frac{2}{3}}\left(\frac{r}{m}\right)^{\frac{4}{3}} \ln\left(k\frac{r}{m}\right) \tag{2.14}$$

where $c, k =$ constants > 0. The behaviour of α along this family as $r \to 0$ is as follows:

(1) $\alpha \to 0$ for $c > 1/2$

(2) $\alpha \to \text{constant}.\, k^6$ for $c = 1/2$

(3) $\alpha \to \infty$ for $0 < c < 1/2$.

(b) Length of curves

For a curve $z(r)$ which approaches the origin, define its length L from a fixed point $\left(r_0,\ z_0 = z(r_0)\right)$ to be

$$L = \lim_{\epsilon \to 0} \int_{\epsilon}^{r_0} e^{m/R - m^2 r^2/2R^4} (1 + z'^2)^{1/2}\, dr \ ,$$

$$\text{where} \quad R = \sqrt{r^2 + z^2(r)} \ .$$

L is easily seen to be finite for the r axis geodesic and for other geodesics asymptotic to it (the non-oscillating geodesics of Section 2.2), while for oscillating geodesics asymptotic to the z axis it is infinite. This is somewhat surprising in view of the fact that timelike geodesics asymptotic to the z axis reach $R = 0$ in *finite* proper time. For the families of curves given by Eq. (2.13) and Eq. (2.14):

(1) L is finite for $n > 2/3$ and for $n = 2/3$ with $0 < b < 1$ or $b = 1$, $0 < c < 2/9$

(2) L is infinite for $0 < n < 2/3$ and for $n = 2/3$ with $b > 1$ or $b = 1$, $c \geq 2/9$.

(c) First curvature of curves as $R \to 0$

If u^μ is the unit tangent vector to the curve $z(r)$, and a^μ is the acceleration vector

$$a^\mu = u^\mu{}_{;\nu} u^\nu \quad ,$$

the first curvature β is defined by

$$\beta = g_{\mu\nu} a^\mu a^\nu$$

(see Synge [1]).

For the Curzon metric

$$\beta = \frac{e^{-2\psi}}{(1+z'^2)^2}\left[\left(-\frac{z'z''}{1+z'^2} + \psi_z z' - \psi_r z'^2\right)^2 + \left(\frac{z''}{1+z'^2} - \psi_z + \psi_r z'\right)^2\right] ,$$

where $\psi = \nu - \lambda$ is determined from Eq. (2.2). Along the r and z axes β is, of course, zero. For the families of curves given by Eq. (2.13) and Eq. (2.14), one obtains as $r \to 0$

(1) $\beta \to \infty$ for $n > 2/3$ and for $n = 2/3$ with $0 < b < 1$ or $b = 1$, $0 < c < 5/9$

(2) $\beta \to$ constant. $k^{10/3}$ for $n = 2/3$, $b = 1$, $c = 5/9$

(3) $\beta \to 0$ for $0 < n < 2/3$ and for $n = 2/3$ with $b > 1$ or $b = 1$, $c > 5/9$.

The above properties (a), (b) and (c) are all summarised in Table 2.1.

Table 2.1: Properties of spacelike curves approaching $R = 0$ in the r-z plane

Curves	L	$\lim\alpha$	$\lim\beta$	$\lim(x,y)$
non-oscillating geodesics	finite	∞	0	$(\frac{\pi}{2},0)$
$n > \frac{2}{3}$, $b > 0$	finite	∞	∞	$(\frac{\pi}{2},0)$
$n = \frac{2}{3}$, $0 < b < 1$	finite	∞	∞	$(\frac{\pi}{2},0)$
$n = \frac{2}{3}$, $b = 1$, $0 \leq c < \frac{2}{9}$	finite	∞	∞	$(\frac{\pi}{2},0)$
$n = \frac{2}{3}$, $b = 1$, $\frac{2}{9} \leq c < \frac{1}{2}$	∞	∞	∞	$(\frac{\pi}{2},0)$
$n = \frac{2}{3}$, $b = 1$, $c = \frac{1}{2}$	∞	finite	∞	$\left(\frac{\pi}{2}, -\tan^{-1}(\text{constant}.k^{-3/2})\right)$
$n = \frac{2}{3}$, $b = 1$, $\frac{1}{2} < c < \frac{5}{9}$	∞	0	∞	$(\frac{\pi}{2},-\frac{\pi}{2})$
$n = \frac{2}{3}$, $b = 1$, $c = \frac{5}{9}$	∞	0	finite	$(\frac{\pi}{2},-\frac{\pi}{2})$
$n = \frac{2}{3}$, $b = 1$, $c > \frac{5}{9}$	∞	0	0	$(\frac{\pi}{2},-\frac{\pi}{2})$
$n = \frac{2}{3}$, $b > 1$	∞	0	0	$(\frac{\pi}{2},-\frac{\pi}{2})$
$0 < n < \frac{2}{3}$, $b > 0$	∞	0	0	$(\frac{\pi}{2},-\frac{\pi}{2})$
oscillating geodesics	∞	0	0	$(-,-\frac{\pi}{2})$

All limits are taken for $R \to 0$. L is the proper distance along the curve, α the Kretschmann scalar, β the first curvature. The non-oscillating geodesics are defined by Eq. (2.9) and asymptote along the r axis, the oscillating geodesics are defined by Eq. (2.8) and asymptote along the z axis. The other curves are given by Eq. (2.13) and Eq. (2.14), the latter essentially being an extension of the former for parameter values $n = \frac{2}{3}$, $b = 1$.

2.4 New coordinates for the r-z plane

The length L_K of the Killing vector orbit t, r, z constant with tangent vector $\partial/\partial\phi$ is

$$L_K = \int_0^{2\pi} re^{-\lambda}\, d\phi = 2\pi re^{-\lambda} \ .$$

This is the circumference of the circle r, z = constant lying in a spatial section. Its "radius" may be defined to be

$$\rho\,(r,z) = L_K/2\pi = re^{-\lambda} \ .$$

For the Curzon metric, the curves in the r-z plane of constant radius of revolution ρ are

$$r = \rho\, e^{-m/R} \ , \tag{2.15}$$

which as $R \to 0$ have the behaviour (for $z > 0$)

$$r \sim \rho\, e^{-m/z} \ .$$

For large R they behave as

$$r \sim \rho$$

as expected, since the space is asymptotically flat. Thus the curves

$$r = \rho\, e^{-m/z} \tag{2.16}$$

are good approximations to the curves given by Eq. (2.15), both for $z \ll 1$ and $z \gg 1$. It should, however, be realised that the real curves of constant radius of revolution are quite complex, breaking into two pieces for $\rho > me$.

The distance $L(z)$ along the curves given by Eq. (2.16), from a fixed point (r_0, z_0) to $\big(r\,(z), z\big)$, can be shown to behave as

$$L(z) \sim (z/m)^2 \, e^{m/z} \quad \text{as} \quad z \to 0$$

and

$$L(z) \sim z/m \quad \text{as} \quad z \to \infty \ .$$

As shown in [5], there is a family of geodesics *not* lying in the spacelike hypersurfaces $t =$ constant (i.e. including timelike, null and spacelike geodesics with $dt/ds \neq 0$), whose spatial projections behave like Eq. (2.16) as $z \to 0$. These were found to be a suitable set of curves on which to base a "comoving" system of coordinates. However, the geodesics lying *in* the spacelike hypersurfaces $t =$ constant and asymptotic to the z axis display oscillatory behaviour. The associated refocussing property discussed in Section 2.2, makes these geodesics unsuitable candidates for setting up such a comoving coordinate system. Instead we choose new coordinates x and y so that they are "approximately" comoving for the curves given by Eq. (2.16), both for small and for large z.

For the quarter-plane $r \geq 0$, $z \geq 0$ define

$$x = \tan^{-1}\left(\frac{r}{m}\, e^{m/z}\right) + \tan^{-1}\left(\frac{r}{m}\, e^{-(\sqrt{2}m/r)^{2/3}}\right) \tag{2.17}$$

$$y = \tan^{-1}\left(3\,\frac{z}{m} - \frac{(z/m)^2\, e^{\psi}}{[\,R^8 + 1 + \frac{1}{3}(r/m)^2 R^{-4}\,]^{1/4}}\right) \ , \tag{2.18}$$

$$\text{where} \qquad \psi = \nu - \lambda = \frac{m}{R} - \frac{m^2 r^2}{2R^4} \ .$$

The curves given by Eq. (2.16) behave as

$$x \sim \tan^{-1}\rho \ , \quad y \sim -\frac{\pi}{2} + \left(\frac{z}{m}\right)^{-2} e^{-m/z} \quad \text{for } z \to 0 \ ,$$

and

$$x \sim \tan^{-1}\rho + \tan^{-1}\left(\rho\, e^{-(\sqrt{2}/\rho)^{2/3}}\right) \ , \quad y \sim \frac{\pi}{2} - \frac{m}{3z} \quad \text{for } z \to \infty \ .$$

Therefore, the coordinates x, y are indeed comoving for these curves in the asymptotic sense that the x coordinate tends to a constant at each end, and can thus be used to label the curves ($\tan y$ acts as a proper distance parameter along the curves near these ends). Some of these curves and their metric normal curves

$$\frac{r}{m} = \left[\frac{2}{3}\left(\zeta - \frac{z}{m}\right)^3\right]^{1/2} \qquad (\zeta = \text{constant}) \tag{2.19}$$

are plotted in x-y coordinates in Figure 2.2.

Eq. (2.18) is a rather complicated expression, but has been selected after some trial and error, partly to ensure that the x, y coordinates are in one-to-one correspondence with the r, z coordinates. Arctan functions have been chosen for compactification purposes. The part of the boundary specified by

$$0 \le x \le \pi, \quad y = \pi/2$$

and

$$x = \pi, \quad 0 \le y \le \pi/2$$

represents points of spacelike infinity of the Curzon metric, spacelike geodesics in the x-y plane terminating there with infinite proper length. It is depicted in the figures by two thick lines. Since the analysis here will bring in new points at infinity, this will be referred to as the "old" spacelike infinity.

In the figures the positive r axis maps onto the segment $\pi/2 < x < \pi$ of the x axis, while the positive z axis maps onto the entire y axis $-\pi/2 < y < \pi/2$. From Eq. (2.12) it can be seen that near $R = 0$ the Kretschmann scalar α behaves as

$$\alpha \sim 48\, R^{-8}\, e^{-4\psi}\, [\, 1 + \tfrac{1}{3}\, (r/m)^2 R^{-4}\,] \ ,$$

and from Eq. (2.18) the coordinate y is seen to behave as

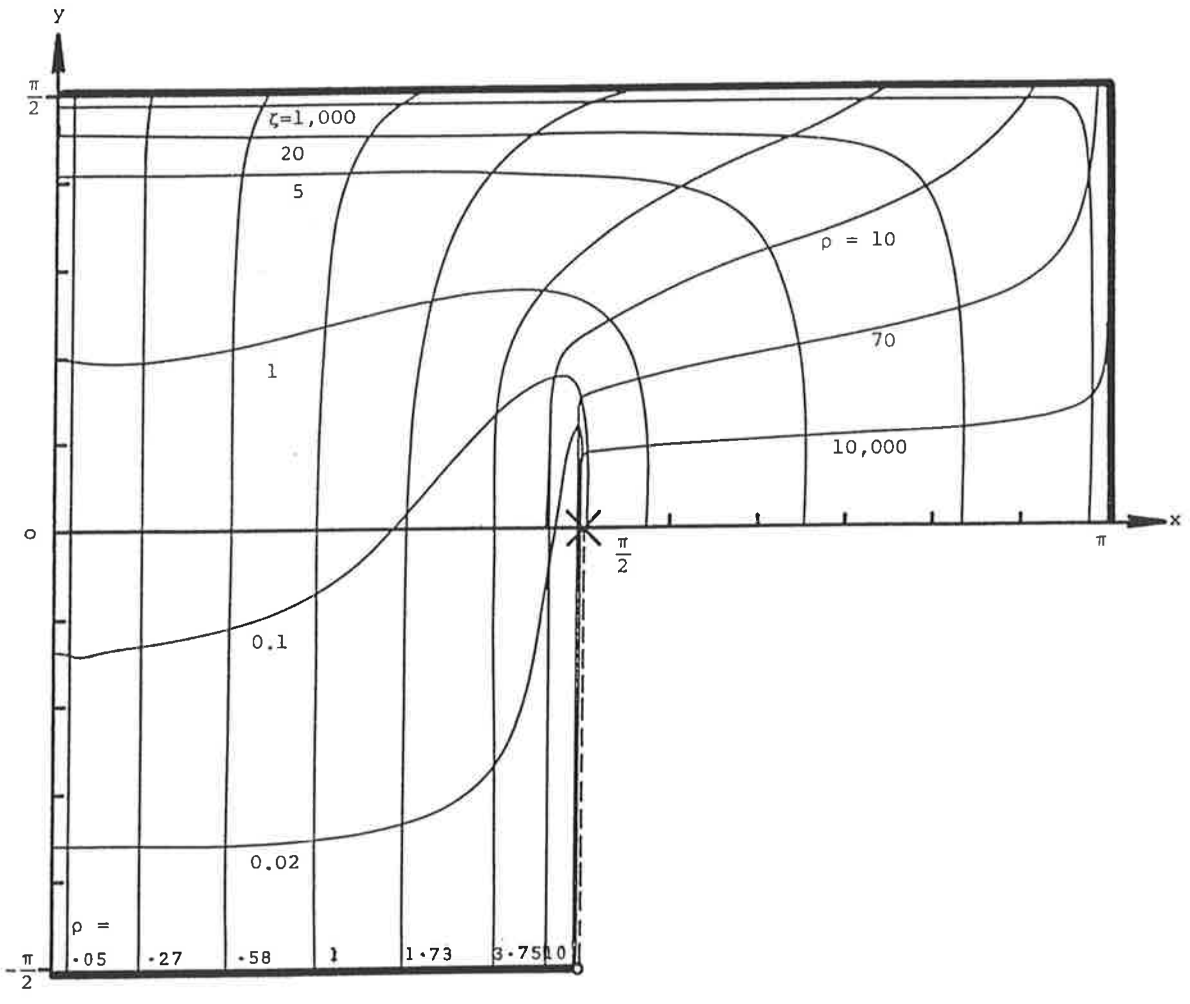

Figure2.2 The compactified r-z quarter-plane. The curves given by Eq. (2.16) are computer-plotted in x, y coordinates for a variety of values of ρ. They stretch from the "new" spacelike infinity, which is the thick line at the bottom of the diagram, to the "old" spacelike infinity along the top. Also plotted are some of their metric normal curves given by Eq. (2.19), which cut across them. The spacelike infinities are depicted by the thick lines and the small circle, the "edge" by a dashed line, and the real singularity at $x = \pi/2$, $y = 0$ by a cross.

$$y \sim \tan^{-1}\left[3\frac{z}{m} - 2\left(\frac{3}{\alpha}\right)^{1/4}\left(\frac{z}{mR}\right)^2\right] .$$

Using this approximation, one can quickly find the termination points (x, y) of the curves discussed in Section 2.3. These are listed in the last column of Table 2.1. They all terminate along the line $x = \pi/2$, $0 \geq y \geq -\pi/2$. In Figure 2.3 curves given by Eq. (2.13) with $n = 2/3$ are plotted using x, y coordinates. Those with $0 < b \leq 1$ terminate at the singularity $(\pi/2, 0)$, while those with $b > 1$ terminate at $(\pi/2, -\pi/2)$.

In the new coordinates $R = 0$ corresponds to the boundary specified by

$$0 \leq x \leq \pi/2 , \quad y = -\pi/2$$

and

$$x = \pi/2 , \quad -\pi/2 \leq y \leq 0 .$$

This boundary has the following features:

1. The point $x = \pi/2$, $y = 0$ represents a real singularity of the Curzon metric, and is depicted in the figures by a cross. The limit of the Kretschmann scalar along curves which terminate at this point is infinite, and many of these curves (including all the non-oscillating geodesics given by Eq. (2.9)) are of finite length L.

2. The line $0 \leq x < \pi/2$, $y = -\pi/2$ represents a new region of spacelike infinity of the Curzon metric. The geodesic down the z axis terminates at the point $x = 0$, $y = -\pi/2$ and is of infinite length, but although the oscillating geodesics which asymptote toward the z axis approach the line $y = -\pi/2$, they do not terminate at particular points of this line. The curves given by Eq. (2.16) which are of infinite length L do, however, terminate at points of this line, and do so with

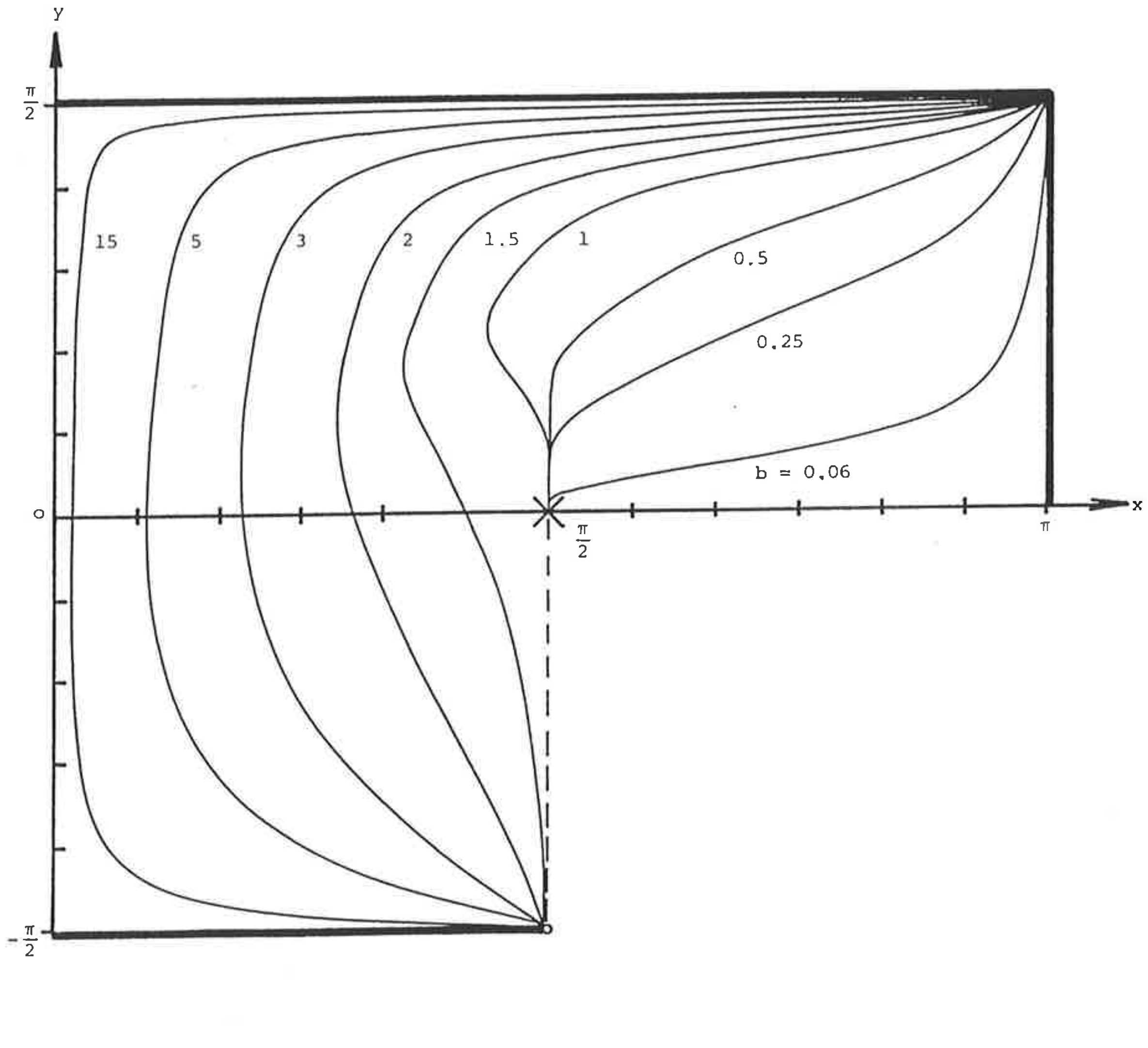

Figure 2.3 The critical Cooperstock-Junevicus curves. These are given by Eq. (2.13) with $n = 2/3$ and are computer-plotted for various values of b. For $0 < b \leq 1$ they terminate at the real singularity, while for $b > 1$ they terminate at the "new" spacelike infinity depicted by the small circle.

zero first curvature β—i.e. the fact that they are non-extendible is not due to any intrinsic "oscillatory" behaviour. The limit of the Kretschmann scalar along these and other curves terminating on this line is zero, indicating that space is asymptotically flat here. This flat spacelike infinity is of infinite extent.

3. The line $x = \pi/2$, $-\pi/2 < y < 0$ represents an "edge" of the space, and is depicted in the figures by a dashed line. There are no curves in the x-y plane that terminate on this line with zero first curvature (e.g. no asymptotic geodesics), which explains the choice of the label "edge". The curves given by Eq. (2.14) with $c = 1/2$ terminate along this edge with the behaviour

$$\tan y \to -\text{constant}.\, k^{-3/2}, \quad \alpha \to \text{constant}.\, k^6, \quad L \to \infty, \quad \beta \to \infty \ .$$

4. The point $x = \pi/2$, $y = -\pi/2$ represents a new spacelike infinity of the Curzon metric, and is depicted in the figures by a small circle. Curves ending at this point are of infinite length L, and many terminate with zero first curvature. The limit of the Kretschmann scalar along curves ending at this point is zero, indicating that flat space is also approached in this direction.

2.5 Conclusions

This analysis has been restricted to the quarter-plane $r \geq 0$, $z \geq 0$. Similar behaviour is, of course, to be found in the other quarter-planes. In Figure 2.4 the two half-spaces $z \geq 0$ and $z \leq 0$ are shown in x-y-ϕ coordinates. To represent the entire spacelike hypersurface $t =$ constant, these two patches should be joined along the plane $y = 0$, $\pi/2 \leq x \leq \pi$, thus creating a "double-sheet" in the region $0 \leq x < \pi/2$. In these coordinates the real singularity of the Curzon metric becomes a ring $x = \pi/2$,

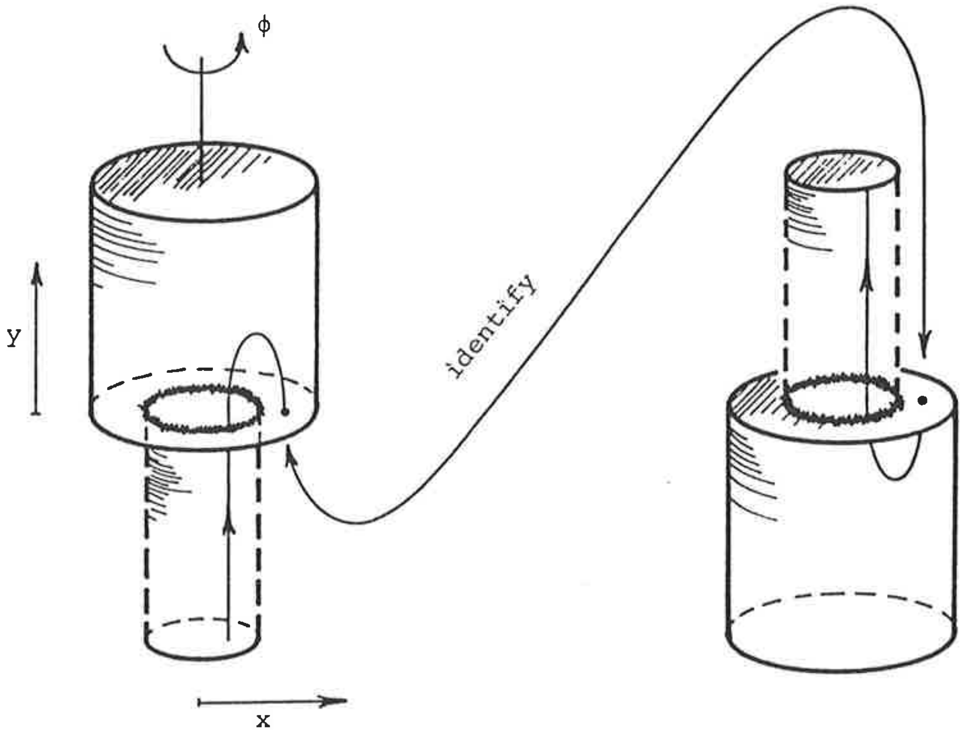

Figure 2.4 Schematic picture of a 3-D section t = constant of the Curzon solution. The two patches shown are drawn in the compactified x-y-ϕ coordinates, and correspond to $z \geq 0$ and $z \leq 0$ for a t = constant section. The "new" spacelike infinities are the free bases of the small cylinders, while the free bases and the walls of the large cylinders constitute the "old" spacelike infinities. The two patches must be joined at $y = 0$ along the parts of the bases of the large cylinders outside of and including the singularity (which is the jagged ring). One of the curves of constant radius of revolution $\rho > me$ given by Eq. (2.15) is shown starting in the left patch and ending in the right.

$y = 0$, $0 \leq \phi < 2\pi$ about the y axis, with spacelike geodesics threading through it. The spacelike geodesics given by Eq. (2.10) and Eq. (2.11) spiral through the ring and continue out to the "new" infinity.

The ring singularity has a finite "radius" in the sense that it can be reached by curves, indeed by geodesics, of finite length from the y axis (originally the z axis). Presumably there is a minimal such geodesic whose length could be used as an actual value for this radius. The circumference of this ring should, however, be regarded as being infinite. This is readily seen by plotting the curves of constant radius of revolution ρ given by Eq. (2.15). For $\rho > me$ these curves start in one sheet at $0 < x < \pi/2$, $y = -\pi/2$, wind once around the ring, and re-emerge on the other sheet. The winding becomes tighter and tighter as $\rho \to \infty$. This peculiar behaviour (finite radius but infinite circumference) is due entirely to the infinite curvature of the space at the ring.

This picture of the Curzon singularity as a ring in the spatial sections is consistent with the picture suggested in [5]. There, however, the ring was placed at an infinitely large radius. Also null geodesics not terminating at the real singularity were shown to be incomplete, requiring an extension of the Weyl coordinates. The spacelike hypersurfaces, however, are now seen to be *complete* in the original Weyl coordinates— only the *topology* of the singularity is incorrectly given by these coordinates, due to the crushing up of the ring plus new spacelike infinity into a single point. In Chapter 3 these two pictures will be linked together to provide a complete description for the entire Curzon space-time.

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Chapter 3

The Global Structure of the Curzon Solution

3.1 Introduction

In Chapter 2 it was shown that in the spacelike hypersurfaces $t =$ constant, the real singularity at $R = 0$ of the Curzon solution

$$ds^2 = -e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda}d\varphi^2 \tag{3.1}$$

where

$$\lambda = -m/R\,, \qquad \nu = -\frac{m^2r^2}{2R^4} \qquad \text{and} \qquad R = \sqrt{r^2+z^2}\,, \tag{3.2}$$

has the structure of a ring with finite radius. Spacelike geodesics approaching $R = 0$ in these hypersurfaces either terminate in finite proper distance at the ring singularity, or thread their way through the ring and are inextendible (i.e. have infinite length). In this way a "new" region of spacelike infinity appears on the other side of the ring.

All this is inherent in the original Weyl coordinates, since (for $R > 0$) they are in precise one-to-one correspondence with the new coordinates. Where they differ, is that in the new coordinates the ring singularity can *only* be approached by curves along which the curvature (as defined by the Kretschmann scalar) becomes infinite. Curves approaching $R = 0$ with finite curvature (the so-called "directional behaviour" of the

Curzon singularity [2], [3]), terminate either at the new infinity or else at an "edge" adjoining the ring singularity. In this chapter it is intended to use this picture as a basis for providing a complete description of the entire Curzon space-time.

In Section 3.2 the behaviour of *all* geodesics (timelike, null and spacelike) approaching $R = 0$ is given. There is also a review of an earlier attempt [4] to extend the Curzon metric in such a way, that some geodesics terminating at $R = 0$ in the Weyl coordinates with finite affine parameter and finite curvature become extendible. In Section 3.3 a new coordinate system for the Curzon metric is displayed. This is shown in Section 3.4 to exhibit all the features of the spatial sections derived in Chapter 2, and yet permits the relevant geodesics to be extended. As in [4], it is possible to connect the Curzon metric smoothly with Minkowski space. Physical interpretations of this curious behaviour, such as a possible collapse scenario, are discussed in Section 3.5.

3.2 Behaviour of geodesics approaching $\mathbf{R = 0}$

The behaviour of null geodesics approaching $R = 0$ was discussed by Szekeres and Morgan [4], while in Chapter 2 spacelike geodesics approaching $R = 0$ in the hypersurfaces $t =$ constant were examined in detail. Actually, not all null geodesics approaching $R = 0$ were uncovered in [4]. After a great deal of further analysis of the geodesic equations, it has been possible to obtain the behaviour of *all* geodesics (i.e. spacelike, null and timelike) approaching $R = 0$. The results are as follows:

(i) Geodesics with t $\neq$ constant and $\phi =$ constant

(a) Asymptotic to the z axis as $z \to 0^+$

$$r(z) \sim e^{-m/z}(A + Bz)$$

$$|t(z)| \sim \int_z e^{2m/u}\,du - \frac{mA^2}{2z^2} - \frac{mAB}{z} + t_0 + \frac{1}{2}\left(\frac{\epsilon}{K^2} - B^2\right) z$$

$$s(z) \sim s_0 - z/K$$

where A, B, K, t_0, s_0 are constants, $\epsilon = \pm 1$ or 0 for spacelike, timelike and null geodesics respectively, and $s(z)$ is the proper distance, proper time, or affine null parameter along the geodesic.

(b) Asymptotic to the r axis as $r \to 0^+$

$$z(r) = Ar^2\left[1 + 2\frac{r}{m} + \left(A^2 + \frac{3}{m^2}\right) r^2 + O(r^3)\right]$$

$$|t(r)| \sim t_0 + e^{m^2A^2}\int_r e^{2m/u - m^2/2u^2}\,du$$

$$s(r) \sim s_0 + \frac{1}{K} e^{m^2A^2}\int_r e^{-m^2/2u^2}\,du$$

(ii) Geodesics with $t, \phi \neq$ constant

(a) Asymptotic to the z axis as $z \to 0^+$

$$r(z) = e^{-m/z}\left[A + Bz + \frac{1}{2}\left(\frac{H^2}{K^2}\right)\left(\frac{1}{A^3}\right) z^2 - \frac{1}{2}\left(\frac{H^2}{K^2}\right)\left(\frac{B}{A^4}\right) z^3 + O(z^4)\right]$$

$$\phi(z) = \phi_0 - \left(\frac{H}{K}\right)\left(\frac{1}{A^2}\right) z + O(z^2)$$

$$|t(z)| \sim \int_z e^{2m/u}\,du - \frac{mA^2}{2z^2} - \frac{mAB}{z} + t_0$$

$$s(z) \sim s_0 - z/K$$

where H and ϕ_0 are constants.

(b) Asymptotic to the surface $z = 0$ as $r \to 0^+$

As in (i)(b) but with

$$\phi(r) \sim \phi_0 + (H/K)\, e^{m^2 A^2} \int_r u^{-2}\, e^{-m^2/2u^2 - 2m/u}\, du \quad .$$

Spacelike, timelike and null geodesics occur in each of these four classes. In the various expansions the parameter ϵ appears in lower-order terms than those given.

(iii) Spacelike geodesics in the spatial sections t = constant

These were discussed in detail in Chapter 2. The geodesics asymptotic to the z axis are complete ($s \to \infty$ as $z \to 0$) and exhibit either a slow oscillatory behaviour or a slow spiralling behaviour as $s \to \infty$, while ones asymptotic to the plane $z = 0$ terminate in finite proper distance at the real singularity.

All geodesics in classes (i) and (ii) are incomplete, since the affine parameter s approaches a finite limit s_0 as $R \to 0$. Along geodesics asymptotic to the plane $z = 0$ (classes (i)(b) and (ii)(b)), the Kretschmann scalar

$$\alpha = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

becomes infinitely large as $R \to 0$. Hence these geodesics terminate at a genuine singularity and are inextendible. On the other hand, $\alpha \to 0$ along all geodesics asymptoting to the z axis (classes (i)(a) and (ii)(a)). The affine parameters $(s - s_0)$ of these geodesics are proportional to the coordinate z.

Suppose that one adopts the following asymptotically "comoving" coordinates along the ingoing ($t \to +\infty$) null geodesics ($\varepsilon = 0$) of class (i)(a) with $B = 0$:

$$\begin{aligned}
\rho &= r\, e^{m/z} \\
v &= t - \int_z^{z_0} e^{2m/u}\, du + \frac{m\rho^2}{2z^2} \qquad (z_0 = \text{constant} > 0) \qquad (3.3) \\
u &= -2z \ .
\end{aligned}$$

Then as was shown in [4], as $z \to 0^+$ along these geodesics, the Curzon metric given by Eq. (3.1) and Eq. (3.2) becomes

$$ds^2 = -\,dudv + d\rho^2 + \rho^2 d\phi^2 + O(z^{-8}e^{-2m/z})\, h_{\mu\nu} dx^\mu dx^\nu \ , \qquad (3.4)$$

where $h_{\mu\nu}$ is a tensor whose behaviour is regular at $z = 0$.

ρ and v approach constant values along these geodesics as $z \to 0^+$, ρ being like an axial radial coordinate, v a null (advanced time) coordinate. The coordinate u is an asymptotically affine null parameter along the geodesics, and approaches a null (retarded time) coordinate as can be seen from the form of the metric (Eq. (3.4)). Clearly it is possible to match this metric smoothly (i.e. in a C^∞ way) across $u = 0$ with Minkowski space

$$ds^2 = -\ dt^2 + dx^2 + dy^2 + dz^2$$

expressed in double-null cylindrical coordinates

$$v = t + z \ , \qquad u = t - z \ , \qquad x = \rho\cos\phi \ , \qquad y = \rho \sin\phi \ . \qquad (3.5)$$

In the remainder of this chapter it will be shown that coordinates can be chosen such that *both* ingoing and outgoing null geodesics can simultaneously be extended into regions of Minkowski space, whilst still preserving the ring structure of the curvature singularity discovered in Chapter 2.

3.3 New coordinates for the Curzon metric

In Chapter 2 the following coordinates were proposed for the spatial sections $t,\ \phi\ =$ constant:

$$x\ =\ \tan^{-1}\left(\frac{r}{m}\,e^{m/z}\right)+\tan^{-1}\left(\frac{r}{m}\,e^{-(\sqrt{2}m/r)^{2/3}}\right) \tag{3.6}$$

$$y\ =\ \tan^{-1}\left(3\,\frac{z}{m}\ -\ \frac{(z/m)^2\,e^{\psi}}{[\,R^8+1+\frac{1}{3}\,(r/m)^2R^{-4}\,]^{1/4}}\right)\ , \tag{3.7}$$

$$\text{where}\qquad \psi\ =\ \nu-\lambda\ =\ \frac{m}{R}\ -\ \frac{m^2r^2}{2R^4}\quad .$$

These coordinates have the effect of compactifying the upper r-z plane ($z > 0$) into a curious double-rectangular-shaped region specified by

$$-\pi < x < \pi \qquad 0 < y < \pi/2$$

and

$$-\pi/2 < x < \pi/2 \qquad -\pi/2 < y \leq 0\ \ .$$

The real singularity occurs at the pair of boundary points $x = \pm\,\pi/2$, $y = 0$, which converts to a ring when rotated about the central y axis (thus including the ϕ coordinate). The boundary of the upper rectangle given by $-\pi\ \leq\ x\ \leq\ \pi$, $y\ =\ \pi/2$ and $x\ =\ \pm\,\pi$, $0\ \leq\ y\ \leq\ \pi/2$ is the "old" spacelike infinity of the Curzon metric (corresponding to $R\ =\ \infty$), while the boundary line of the lower rectangle given by $-\pi/2\ \leq\ x\ \leq\ \pi/2$, $y = -\pi/2$ represents the "new" spacelike infinity. This boundary line is approached in an oscillatory manner by spacelike geodesics belonging to class (iii), Section 3.2. These geodesics, which asymptote to the z axis as $z \rightarrow 0^+$ in the original Weyl coordinates, are complete.

If null and timelike curves are to be included, it is tempting to employ a compactified time coordinate

$$\tau = \tan^{-1}(t/m) \quad .$$

The region $z > 0$, $\phi =$ constant of the Curzon space-time then has the slab-like T-shaped structure of Figure 3.1, with the curvature singularity occurring along the jagged lines. However, there is an immediate problem with this picture. The ingoing geodesics of class (i)(a), Section 3.2 which approach $z = 0$ with vanishing Kretschmann scalar, all terminate at the upper edge $-\pi/2 < x < \pi/2$, $y = -\pi/2$, $\tau = \pi/2$. Furthermore, they do so with finite affine parameter and should be extendible as discussed in Section 3.2. However in these coordinates there is no hope of performing such an extension, since a whole plane of arrival has been crushed to a line.

A similar situation would occur with the Schwarzschild solution for geodesics approaching $r = 2m$, if one were to adopt a compactified time coordinate $\tau = \tan^{-1} t$. The situation there is remedied by using a null (advanced time) coordinate [5]. The same procedure could be adopted here using coordinates $v' = \tan^{-1} v$, x, y, ϕ (for v, see Eq. (3.3)). However, these coordinates crush the hypersurface $z = 0$ of the Curzon space-time to the 2-surface specified by

$$v' = -\pi/2 \qquad \pi/2 < |x| < \pi \qquad y = 0 \qquad 0 \leq \phi < \pi \quad .$$

This renders the coordinate transformation not one-to-one on this crucial hypersurface across which the upper half of the Curzon space-time $z > 0$ is joined to the lower half $z < 0$. In addition, just as in the Schwarzschild case, past incomplete geodesics emanating from $R = 0$ (i.e. those in classes (i)(a) and (ii)(a) of Section 3.2 with $t \to -\infty$ as $z \to 0^+$) will remain incomplete. The famous Kruskal double-null coordinates [6] solve this problem in the Schwarzschild case.

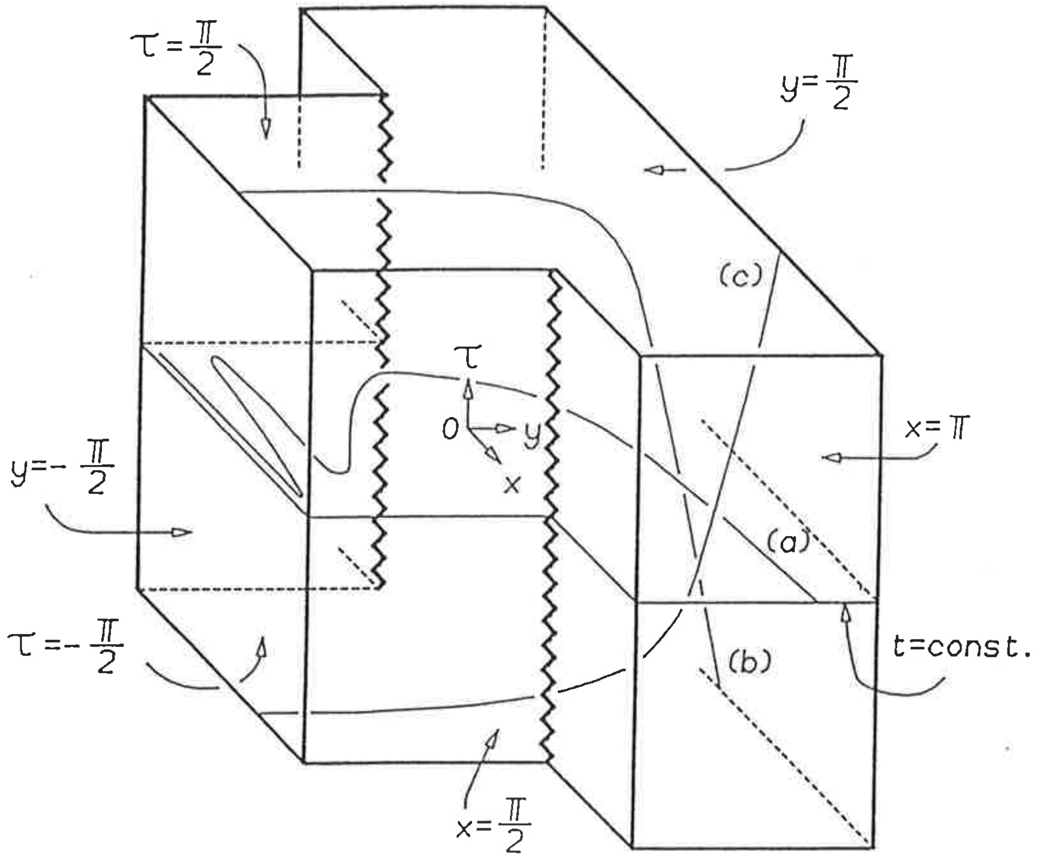

Figure 3.1 The Curzon space-time ($z > 0$, ϕ = constant) in x, y, τ coordinates. O is the origin $x = y = \tau = 0$. (a) is an oscillating spacelike geodesic in a spatial section t = constant, (b) is a future incomplete null (or timelike or spacelike) geodesic, and (c) is a past incomplete null (or timelike or spacelike) geodesic.

In the case at hand, both problems can be resolved by adopting a kind of half-advanced, half-retarded time coordinate T, and at the same time making a modification to the y coordinate whilst leaving x alone.

$$\begin{aligned} T &= \tan^{-1}\left[e^{-K}\left(\frac{t}{m}+H\right)+\frac{t}{m}\left(y+\frac{\pi}{2}\right)^3\right] \\ &\quad + \tan^{-1}\left[e^{-K}\left(\frac{t}{m}-H\right)+\frac{t}{m}\left(y+\frac{\pi}{2}\right)^3\right] \end{aligned} \tag{3.8}$$

$$\begin{aligned} Y &= \frac{\pi}{2}+\tan^{-1}\left[a\,y^3\left(x^2-\frac{\pi^2}{4}\right)^2\frac{m}{z}+3\,\frac{z}{m}\right. \\ &\quad \left. - \frac{(z/m)^2\,e^{\psi}}{[\,R^8\,(1+(t/m)^4)+1+\frac{1}{3}\,(r/m)^2R^{-4}\,]^{1/4}}\right] \end{aligned} \tag{3.9}$$

where

$$H\,(r,z) = \int_1^{z/m} e^{2/u}\,du + \tfrac{1}{2}\,(r/z)^2\,e^{2m/z}$$

$$K\,(r,z) = \left(y+\frac{\pi}{2}\right)R + \left(\frac{\tan x}{\tan y}\right)^2 .$$

x and y are given by Eq. (3.6) and Eq. (3.7), and a is a positive constant chosen preferably to be fairly small ($< \pi^{-4}$) to ensure that the coordinates are one-to-one. The coordinate ranges are $-\pi < T < \pi$, $-\pi < x < \pi$, $0 < Y < \pi$. The region specified by $\pi/2 < |\,x\,| < \pi$, $0 < Y < \pi/2$ is excluded, as may readily be shown given the shape of the x-y coordinate patch.

3.4 Features of the new coordinates

In view of the somewhat awesome complexity of the coordinate transformation just given in Section 3.3, a few comments are probably in order. Along ingoing geodesics (classes (i)(a) and (ii)(a) of Section 3.2 with $t \to \infty$), one has that as $z \to 0^+$

$$t/m + H \to \text{constant}\,, \qquad t/m - H \to \infty \ ,$$

whilst along outgoing geodesics ($t \rightarrow -\infty$) the reverse is true, namely

$$t/m + H \rightarrow -\infty \ , \qquad t/m - H \rightarrow \text{constant} \ .$$

Along such geodesics $x \rightarrow \tan^{-1} A = \text{constant}$, while

$$y \sim -\frac{\pi}{2} + \left(\frac{m}{z}\right)^2 e^{-m/z} \ .$$

Hence K approaches zero very rapidly as $z \rightarrow 0^+$, and the factors e^{-K} have no significance along the geodesics for this limit. Since

$$|t| \sim (z^2/2m)\, e^{2m/z} \ ,$$

the terms

$$\frac{t}{m}\left(y + \frac{\pi}{2}\right)^3 \rightarrow 0$$

and also make no serious contribution along the geodesics.

Thus one has that along these geodesics as $z \rightarrow 0^+$

$$T \sim \tan^{-1}[t/m + H] + \tan^{-1}[t/m - H] \ ,$$

so that for the ingoing geodesics

$$T \rightarrow \text{constant} > 0 \quad \text{as} \quad t \rightarrow \infty \ ,$$

whilst for the outgoing geodesics

$$T \rightarrow \text{constant} < 0 \quad \text{as} \quad t \rightarrow -\infty \ .$$

The coordinate $Y \sim (+\text{ve constant})\, z$ along these geodesics as $t \rightarrow \pm\infty$ respectively, and thus behaves like an affine parameter. The factor $[1 + (t/m)^4]$ has been incorporated into the last term under the arctan in the expression for Y, in order that this term does not dominate as $z \rightarrow 0^+$, since it is the first term under the arctan which gives Y the desired affine behaviour.

However, it is also desirable to retain the essential features of the spacelike hypersurfaces $t =$ constant discussed in Chapter 2. Near $R = 0$ ($z > 0$) in these hypersurfaces, the curves of constant radius of revolution A about the z axis are $r(z) \sim A\, e^{-m/z}$ (see Section 2.4). Since geodesics belonging to class (i)(a) in Section 3.2 (with $B = 0$) have the same $r(z)$, it is readily shown that these curves have the following behaviour in the new coordinates as $z \to 0^+$:

$$x \to \tan^{-1} A = \text{constant}$$

$$T \sim 2\,(t/m)/H^2 \to 0 \;\; .$$

Since $t/m =$ constant, the last term under the arctan in the expression for Y now dominates as $z \to 0^+$, so that

$$Y \sim (m^2/z^2)\, e^{-m/z} \to 0 \;\; ,$$

which is the same as the behaviour of $y + \pi/2$ along these curves near $z = 0$. Thus these curves have the form

$$T \sim 2\frac{t}{m} . 4\left(\frac{Y}{\log Y}\right)^4 \quad \text{as} \quad Y \to 0^+ \;\; .$$

The factors e^{-K} in Eq. (3.8) are important for curves in the spatial sections approaching different parts of the plane $z = 0$ ($z \to 0^+$, $r \to$ constant > 0). Along such curves $e^{-K}(t/m \pm H) \to 0$ as $z \to 0^+$, whence

$$T \sim 2\,\tan^{-1}\left[\frac{t}{m}\left(y + \frac{\pi}{2}\right)^3\right] \to 2\,\tan^{-1}\left(\frac{\pi^3}{8}\frac{t}{m}\right) \;\; .$$

Without these factors of e^{-K}, the spatial sections $t =$ constant would all fold down to the 2-surface $T = 0$, $\pi/2 < |\,x\,| < \pi$, $Y = \pi/2$, $0 \le \phi < \pi$. Because the influence of the terms $t/m + H$ and $t/m - H$ in the expression for T is removed by these factors, the equatorial plane $-\infty < t < \infty$, $r > 0$, $z = 0$, $0 \le \phi < 2\pi$ corresponds to the 3-surface $-\pi < T < \pi$, $\pi/2 < |\, x \,| < \pi$, $Y = \pi/2$, $0 \leq \phi < \pi$ in the new coordinates. This is necessary if the upper half ($z > 0$) of the Curzon space-time is to be smoothly connected with the lower half ($z < 0$) across this surface. For similar reasons geodesics belonging to classes (i)(b) and (ii)(b) in Section 3.2, which approach $R = 0$ asymptotically to the plane $z = 0$ with finite values of t, also approach the curvature singularity at $|\, x \,| = \pi/2$, $Y = \pi/2$ in the new coordinates with the full range of values of T. The relative placement of all spacelike curves in the spacelike hypersurfaces $t =$ constant is unchanged from the discussion in Chapter 2.

Finally, the appearance of the metric in these new coordinates near the surface $Y = 0$ should be considered. For $T > 0$ the coordinate transformations given by Eqs. (3.9), (3.6) & (3.8) have the asymptotic form

$$\begin{aligned} z/m &\sim a\,(\pi^3/8)\,(x^2 - \pi^2/4)^2 \tan Y \\ r/m &\sim e^{-m/z} \tan x \\ t/m &\sim \int_{z/m} e^{2/u}\, du \,-\, (m/2z^2)\tan^2 x \,-\, \cot T \;. \end{aligned} \tag{3.10}$$

Comparing this with the coordinate transformation given by Eqs. (3.3), it may be seen that the coordinates x, Y, T are related at $Y = 0$ to ρ, u, v by

$$\begin{aligned} \rho &= \tan x \\ u &= -\, a\,(\pi^3/4)\,(x^2 - \pi^2/4)^2 \tan Y \\ v &= -\cot T \;. \end{aligned} \tag{3.11}$$

This is a perfectly acceptable coordinate transformation of Minkowski space expressed in the double-null cylindrical coordinates given by Eq. (3.5), and has the effect of compactifying it to a box

$$-\pi/2 < x < \pi/2 \,, \qquad -\pi/2 < Y < \pi/2 \,, \qquad 0 < T < \pi \;.$$

Negative values of ρ are associated with points having $\phi' = \phi + \pi$ $(0 \le \phi < \pi)$. Of course the standard allowances must be made for the removable coordinate singularity at $\rho = 0$ in

$$\begin{aligned} ds^2 &= -\,dudv + d\rho^2 + \rho^2 d\phi^2 \\ &= a\,\frac{\pi^3}{4}\left(x^2 - \frac{\pi^2}{4}\right)\left[4x\,\tan Y\,dx + \left(x^2 - \frac{\pi^2}{4}\right)\sec^2 Y\,dY\right]\frac{dT}{\sin^2 T} \\ &\qquad + \sec^4 x\;dx^2 + \tan^2 x\;d\phi^2 \ . \end{aligned} \tag{3.12}$$

It is now clear that the Curzon metric expressed in coordinates x, Y, T, ϕ can be connected with the half $Y < 0$ of Minkowski space as expressed in Eq. (3.12). The junction across the surface $Y = 0$, $T > 0$ is C^∞. The surface $Y = 0$, $T < 0$ can similarly be joined to Minkowski half-space on changing the equation for t/m occurring in Eqs. (3.10) to

$$t/m \sim -\int_{z/m} e^{2/u}\,du + (m/2z^2)\tan^2 x - \cot T \ .$$

The situation is depicted in Figure 3.2. The end surfaces $Y = 0$, $T > 0$ and $Y = 0$, $T < 0$ at which the junctions with Minkowski half-spaces are made, are shown at an angle purely to emphasise the fact that these junctions must be made with *separate* Minkowski half-spaces. This also highlights the fact that the junction surfaces at $Y = 0$ are null hypersurfaces. Note, however, that Y is not a null coordinate throughout Minkowski space whilst T is, as can be seen from Eqs. (3.11) and Eq. (3.12).

3.5 Discussion and conclusions

What is to be made of this bizarre property that the Curzon metric extends smoothly to Minkowski space? There seem to be two basic outlooks on this question. One can either regard the Curzon solution as the possible *end product* of a non-spherical

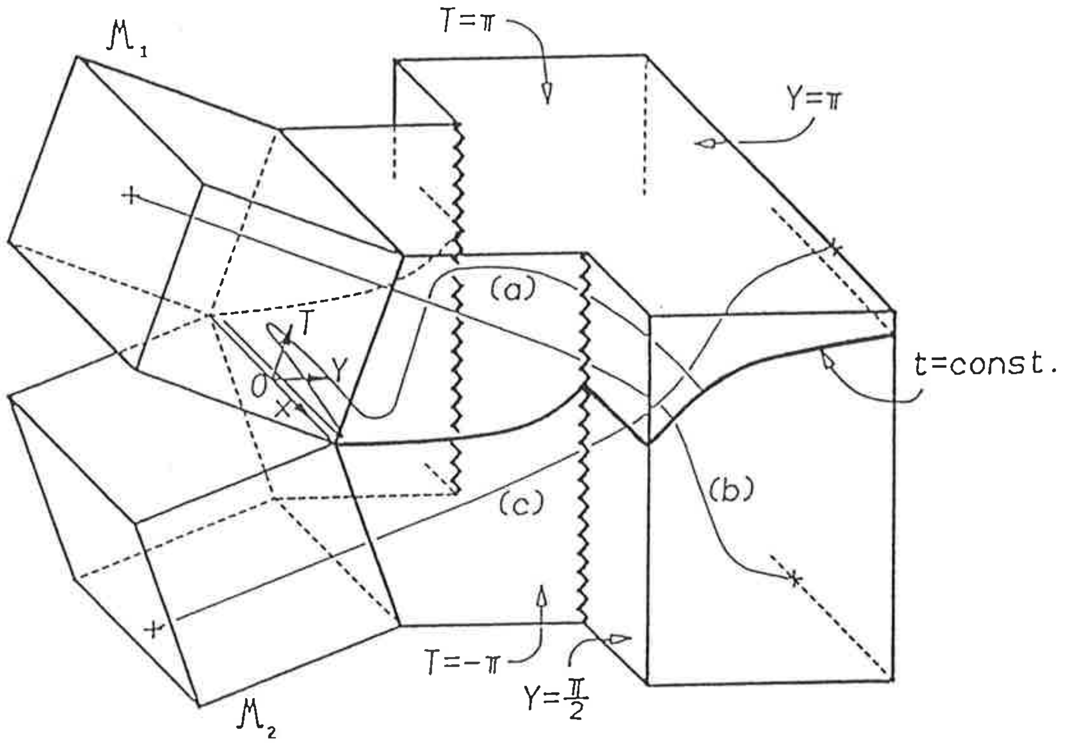

Figure 3.2 The extended Curzon space-time ($z > 0$, ϕ = constant) in x, Y, T coordinates. O is the origin $x = Y = T = 0$. The spatial sections t = constant (e.g. the surface enclosed by the bold line) bend upward (or downward) from the "new" spacelike infinity at $T = 0$, $Y = 0$. Curves (a), (b) and (c) are as in Figure 3.1. Curves (b) and (c) are now fully extended through the adjoining Minkowski half-spaces $\mathcal{M}_1$ and $\mathcal{M}_2$.

collapse, or one can view it as a *development* out of Minkowski space in much the same way as a plane-sandwich wave.

In considering the first option, it should be pointed out that cosmic censorship is by no means a settled issue in general relativity. Indeed some studies [7], [8], [9] suggest that it may not hold even in the case of spherical symmetry. Suppose that a non-rotating axisymmetric arrangement of matter collapsed in such a way as to lead to a final state represented by the Curzon solution. It is not being said that this is in fact possible, but neither can one be certain that it, or something similar, is totally impossible.

The lower left-hand part of Figure 3.2, including the lower Minkowski space ($T < 0$), is no longer relevant in this collapse situation, being replaced by the interior solution of the collapsing matter. Presumably the matter divides into two, some of it ending up in the singularity at $|x| = \pi/2$, $Y = \pi/2$ while the rest proceeds through the ring and continues to the flat region beyond $Y = 0$ (Figure 3.3). The collapse in this case results in both a naked singularity (the ring) *and* an event horizon at $Y = 0$, since events with $Y < 0$ can never be seen by observers near the Curzon infinity (i.e. near $|x| = \pi$ or $Y = \pi$).

It is felt that such a scenario may not be totally unrealistic physically since the singularity, although naked, is not a "harmful" singularity in the sense that any "light" emanated from it becomes infinitely redshifted. This is easily seen since the redshift at infinity from any particle situated on a Killing vector orbit z, r, ϕ constant is given by $e^{-\lambda}$, which clearly approaches infinity as $R \to 0$. Infinitely redshifted singularities, such as those occurring in the Friedmann cosmologies, are not regarded as genuinely naked in view of their benign redshift. Possibly the Curzon singularity may be granted a similar status.

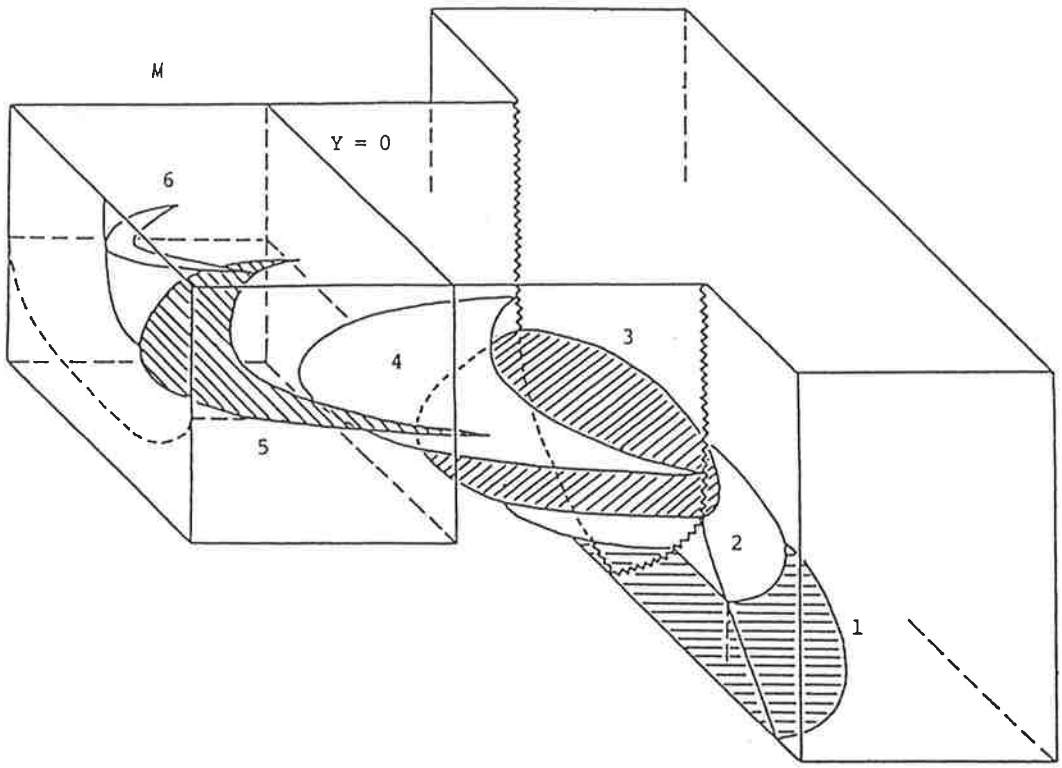

Figure 3.3 A hypothetical collapse to the Curzon solution. x, Y, T coordinates with ϕ constant are used. The collapse of half the matter is depicted as a series of six layered sections beginning with Section 1 at the bottom (which is like half a flattened sphere). A singularity develops in this section and then spreads out to form a ring. Part of the matter passes through the ring and eventually disappears over the horizon at $Y = 0$ into the Minkowski region $\mathcal{M}$.

Of course one is entitled to feel a little sceptical about this picture. After all, a non-spherical collapse would be expected to radiate gravitational waves and should not have a static exterior solution. However non-spherical collapses *can* have static exteriors [9], [10], and in any case this situation may approximate the true one with regard to its general global features—for example, the lack of infinite blueshifts across any horizons indicates that there is nothing inherently unstable in the picture given here.

The Minkowskian continuation for $Y < 0$ looks most peculiar, since positive density matter is being matched to a flat space exterior. Examples of this kind do, however, exist in the literature [11], [12]. In any case the choice of matching to Minkowski space is fairly arbitrary. It was simply the *easiest* space to match smoothly across the boundary $Y = 0$. Any other space which matches smoothly to Minkowski space (e.g. plane waves, another Curzon solution, etc.) would do just as well.

A tempting idea is to match the lower half ($z < 0$) of the Curzon space-time with the upper half ($z > 0$) across the surface $Y = 0$, in such a way that particles disappearing through the ring from the top half reappear in the lower half and vice versa. These two half-spaces must, in any case, be joined along the side walls $Y = \pi/2$, $\pi/2 \leq |x| < \pi$, $-\pi < T < \pi$ of Figure 3.2. However because of the C^∞, non-analytic nature of the matching conditions at $Y = 0$, there is nothing to convince one that this procedure has any overriding advantages.

But there is a totally different viewpoint to all this. Consider a timelike observer in Figure 3.2 beginning life in the lower Minkowski region $T < 0$. At a certain point of time she encounters the null surface $Y = 0$ and some curvature begins to develop. This is not an unfamiliar situation in general relativity. The most standard such example is the plane wave which is of Petrov type N, because that is what one must have across a piecewise C^2 junction with Minkowski space [13]. However if the matching is very smooth (C^3 or higher), then the curvature need not be type N. Here one has an extremely smooth C^∞ curvature development, and the resulting space has Petrov type I.

What is truly amazing is that the observer finds herself in a whole new world, as it were. In front of her she suddenly sees a naked ring singularity (not visible before she entered the curvature region). But it hardly need bother her since its infinite redshift makes it effectively invisible. She is, however, faced with a dilemma—to proceed through the ring (or even into it if she wishes to perish), or to continue up the diagram to reemerge in the upper Minkowski space. If the latter option is taken, the whole experience has been something similar to passing through a sandwich wave. The first option is peculiar in that as the observer passes through the ring and heads toward the Curzon spatial infinity, she sees behind her an increasingly pointlike particle of mass m. A massive particle has, in effect, been created out of nothing. Does one have here the seeds of some fantastic particle creation theory based on general relativity?

Both of these viewpoints are thought-provoking. The Curzon "particle" is *not* the isolated monopolar particle of general relativity—that role obviously belongs to the Schwarzschild solution. From a distance, however, it appears just like a particle but endowed with some higher multipole moments [14]. The detailed analysis of the singularity given here has shown it to possess a wealth of structure and physical possibilities. It would be most interesting to find out if other Weyl solutions or, even more ambitiously, stationary axisymmetric solutions show similar or other structures in their singularities.

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Chapter 4

A Survey of the Weyl Metrics

4.1 Introduction

Hermann Weyl derived his class of metrics in 1917, just one year after Einstein had presented his now-celebrated general theory of relativity to the world. With the passing of some seventy years since that initial flurry, it is perhaps time to pause and assess what progress has been made with respect to the Weyl metrics. Exactly what is known about them, and what still remains to be done?

From about twenty-five years ago an interest in the Weyl metrics developed, particularly as exterior solutions in astrophysical problems [1] and as possible final states of gravitational collapse [2], [3]. However, in addition to being of relevance to physics, they are also of interest simply because they present us with the rare opportunity of explicitly determining and investigating a large class of relativistic metrics.

The Weyl metrics are, in principle, all 'known' since there exists a precise algorithm for generating them from an infinite set of Newtonian potential functions. This procedure is given in Section 4.2. In practice, however, the global structure of only a few such solutions is well understood, and it seems that much work and new insights will be required if this situation is to change.

The member of the Weyl class which is simplest to obtain is the Curzon metric (see Section 4.2). Yet despite the ease with which it is generated, its source structure and global structure remained a mystery until the papers by Scott & Szekeres [4], [5] (see also Scott [6]) appeared in 1986. Due to space considerations their findings will not be summarised here, but nonetheless form an integral part of this subject (see Chapters 1, 2 & 3).

Since the Schwarzschild solution belongs to the Weyl class, the question naturally arises as to how it is generated. This question is investigated and answered in Section 4.3 and ultimately, of course, involves a change from Weyl coordinates to the standard Schwarzschild coordinates. Finding the relationship between the two coordinate systems is facilitated by a consideration of the general form of gravitational equipotentials of the Weyl metrics.

The Schwarzschild solution is a special member of the subclass of the Weyl metrics known as the Zipoy-Voorhees metrics. These metrics form the main focus of this survey and are discussed in Sections 4.4, 4.5 & 4.6. In Section 4.4 the metrics are specified and new coordinates more suited to their geometry are chosen to replace the original Weyl coordinates. The problem of finding sources for these metrics is discussed in Section 4.5, and the possibility of performing extensions is considered in Section 4.6.

Some general properties of the Weyl metrics are given in Section 4.7 and, in particular, the relationship between an arbitrary Weyl metric and its generating Newtonian potential is examined. The question of how flat space is generated within this framework is fully investigated, and at the end of this section there is a list of some related open problems.

In Section 4.8 a brief history of the static two-body problem of general relativity is presented, including the early controversy over the two-particle Curzon solution, as well as some much more recent developments. Section 4.9 gives a short description of a new mathematical approach to stationary, axisymmetric, vacuum space-times. It is hoped that this approach will eventually offer new insight into some of the unanswered questions related to the Weyl metrics.

The survey concludes with Section 4.10 which consists of a small but new observation by the author regarding ring singularities occurring in Weyl metrics. Before proceeding it only remains to point out that the aim of this survey was to be as comprehensive as possible within the given space constraints. There are, of course, certain omissions for which the author apologises in advance.

4.2 The Weyl metrics

Using cylindrical coordinates (r, z, φ) where $r \geq 0$, $z \in \mathbb{R}$ and $0 \leq \varphi < 2\pi$ (with $\varphi = 0$ and $\varphi = 2\pi$ identified), the static, axisymmetric, vacuum solutions of Einstein's field equations are given by the Weyl metrics [7], [8] (see also Synge [9])

$$ds^2 = -e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\varphi^2 \tag{4.1}$$

where $\lambda(r, z)$ and $\nu(r, z)$ are solutions of the equations

$$\lambda_{rr} + \lambda_{zz} + r^{-1}\lambda_r = 0 \tag{4.2}$$

and

$$\nu_r = r\left({\lambda_r}^2 - {\lambda_z}^2\right), \qquad \nu_z = 2r\lambda_r\lambda_z . \tag{4.3}$$

If a solution λ of Eq. (4.2) is found, then Eqs. (4.3) can be integrated to find ν. In fact Eq. (4.2) is recognised as being simply the Laplace equation in cylindrical coordinates for a φ-independent function. There is thus a straightforward method of obtaining static, axisymmetric, vacuum, general relativistic fields. Namely choose an appropriate *Newtonian* gravitational field and then integrate the Eqs. (4.3).

An obvious choice is the gravitational field produced by a spherically symmetric mass distribution with total mass m, which is located at the origin of the cylindrical coordinate system. So

$$\lambda = -m/R \qquad \text{where} \qquad R = \sqrt{r^2 + z^2} \tag{4.4}$$

and

$$\nu = -\frac{m^2 r^2}{2R^4} \ . \tag{4.5}$$

This is the so-called Curzon metric [10]. Although generated by the Newtonian mass monopole it is *not* equivalent to the Schwarzschild metric, which as is well-known (Birkhoff's Theorem [11]) is the unique spherically symmetric, vacuum solution of general relativity.

4.3 The Schwarzschild solution

The Schwarzschild solution is in fact generated by the Newtonian potential of a constant density line mass (or rod) with total mass m and length $2m$, which is located along the z-axis with its mid-point at the origin. So for this important example

$$\lambda = \frac{1}{2} \ln\left(\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m}\right) \tag{4.6}$$

where

$$R_1 = \left(r^2 + (z-m)^2\right)^{1/2} \qquad \text{and} \qquad R_2 = \left(r^2 + (z+m)^2\right)^{1/2} \tag{4.7}$$

and

$$\nu = \frac{1}{2} \ln\left(\frac{(R_1 + R_2)^2 - 4m^2}{4 R_1 R_2}\right) \ . \tag{4.8}$$

Naïvely one might have been tempted to make the simple coordinate transformation

$$R = \sqrt{r^2 + z^2} \ , \qquad \tan\theta = r/z \tag{4.9}$$

from the cylindrical system (r, z, φ) used by Weyl (Weyl coordinates), to a spherical system (R, θ, φ). However this will *not* cast the metric into the familiar Schwarzschild form, since under such a coordinate transformation the rod maps to the portion of axis specified by $\theta = 0$, $0 \leq R \leq m$ and $\theta = \pi$, $0 \leq R \leq m$. So instead of producing the customary point mass, the line mass persists.

In fact any coordinate system which is one-to-one with the cartesian system (x, y, z) on an open neighbourhood of the rod can be ruled out for the same reason. A different type of coordinate transformation is needed here, and the key to finding it lies in the following observation.

For the Weyl metrics given by Eqs. (4.1), (4.2) & (4.3) the 3-metric ${}^3g_{\alpha\beta}$, ${}^3g^{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ induced on the hypersurface $t =$ constant is given by

$$
{}^3g_{\alpha\beta}\, dx^\alpha dx^\beta \;=\; e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\varphi^2 \tag{4.10}
$$

where $x^1 = r$, $x^2 = z$, $x^3 = \varphi$. Now it can be shown that

$$
{}^3g^{\alpha\beta}\, e^\lambda{}_{|\alpha\beta} \;=\; {}^3g^{\alpha\beta}\, {}^3\nabla_\beta\, {}^3\nabla_\alpha\, e^\lambda \;=\; 0 \ . \tag{4.11}
$$

Thus e^λ is an analogue of the Newtonian potential λ, and the surfaces on which it is constant may be thought of as gravitational equipotentials.

For the Schwarzschild potential λ given by Eq. (4.6),

$$
\begin{aligned}
& e^\lambda = \text{constant} \\
\Rightarrow \quad & R_1 + R_2 = c \quad (\text{a constant} > 2m) \ .
\end{aligned}
$$

These gravitational equipotentials are, of course, just the 2-surfaces $\rho =$ constant, where ρ is the radial coordinate normally used for the Schwarzschild metric. From the metric component

$$
g_{00} = -\, e^{2\lambda}
$$

the function $\rho(c)$ is readily determined to be

$$\rho(c) = \frac{1}{2}(c + 2m) \quad ,$$

and with a little more effort the coordinate transformation

$$\begin{aligned} \rho &= \frac{1}{2}(R_1 + R_2 + 2m) \\ \cos\theta &= \frac{1}{2m}(R_2 - R_1) \end{aligned} \tag{4.12}$$

is found to be the one which casts the Schwarzschild metric into its familiar form

$$ds^2 = -\left(1 - \frac{2m}{\rho}\right)dt^2 + \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad . \tag{4.13}$$

It is to be noted that this transformation from (t, r, z, φ) coordinates to $(t, \rho, \theta, \varphi)$ coordinates is a one-to-one mapping of the entire region surrounding (but not including) the line mass onto the *exterior* Schwarzschild solution $\rho > 2m$. This is not really very surprising since the Weyl metrics are *static*.

4.4 The Zipoy-Voorhees metrics

The Schwarzschild solution falls naturally into the subclass of Weyl metrics generated by the Newtonian potential of a constant density line mass (or rod) with total mass m and length $2l$, which is located along the z-axis with its mid-point at the origin. So

$$\lambda = \frac{1}{2}\frac{m}{l}\ln\left(\frac{R_1 + R_2 - 2l}{R_1 + R_2 + 2l}\right) \tag{4.14}$$

where

$$R_1 = \left(r^2 + (z - l)^2\right)^{1/2} \quad \text{and} \quad R_2 = \left(r^2 + (z + l)^2\right)^{1/2} \tag{4.15}$$

and

$$\nu = \frac{1}{2}\left(\frac{m}{l}\right)^2 \ln\left(\frac{(R_1 + R_2)^2 - 4l^2}{4\, R_1 R_2}\right) \quad . \tag{4.16}$$

This metric was first derived by Bach and Weyl [12], and is occasionally referred to as simply 'the metric of Bach and Weyl'. However it has since been discussed and investigated to a varying extent by numerous authors [13], [14], [15], [16], [17], [18], [19] and is more commonly referred to as the Voorhees metric or the Zipoy-Voorhees metric after two of them.

In fact the papers of Zipoy [14] and Voorhees [18] are particularly interesting and warrant some further discussion here. There is a common philosophy underpinning both, namely that the coordinate system chosen to express a particular Weyl metric (λ, ν) should be adapted to the symmetries of the source (or mass distribution) giving rise to the Newtonian potential λ. So for the line metrics given by Eqs. (4.14), (4.15) & (4.16) an obvious choice is the prolate spheroidal coordinate system (u, θ) defined implicitly by

$$r = l\,\sinh u\,\cos\theta \qquad z = l\,\cosh u\,\sin\theta\ , \tag{4.17}$$

where $u \geq 0$ and $-\pi/2 \leq \theta \leq \pi/2$.

If further the coordinate x is defined by $x = \cosh u$, where $x \geq 1$, then the coordinates (x, θ) form an orthogonal system whose level curves $x =$ constant and $\theta =$ constant are confocal ellipses and hyperbolas respectively, with foci at $r = 0$, $z = +l, -l$ $(x = 1,\ \theta = +\pi/2, -\pi/2)$. These coordinates are illustrated in Figure 4.1.

If further the coordinate ρ is defined by $\rho = lx$, where $\rho \geq l$, then in $(t, \rho, \theta, \varphi)$ coordinates the metric becomes

$$\begin{aligned} ds^2 \;=\; & -\,e^{2\lambda}\,dt^2 + e^{2(\nu-\lambda)}\,(\rho^2 - l^2\sin^2\theta)\left(\frac{d\rho^2}{\rho^2 - l^2} + d\theta^2\right) \\ & + e^{-2\lambda}\,(\rho^2 - l^2)\cos^2\theta\,d\varphi^2 \end{aligned} \tag{4.18}$$

where

$$\lambda = \frac{1}{2}\frac{m}{l}\ln\left(\frac{\rho - l}{\rho + l}\right) \tag{4.19}$$

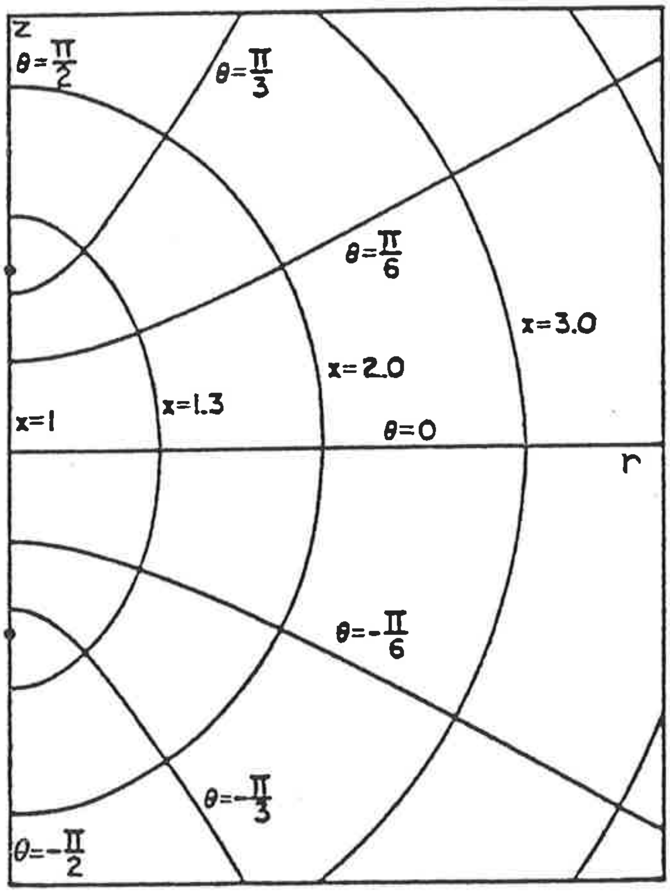


Figure 4.1 A graph showing the relationship between cylindrical coordinates (r, z) and prolate spheroidal coordinates (x, θ).

and

$$\nu = \frac{1}{2}\left(\frac{m}{l}\right)^2 \ln\left(\frac{\rho^2 - l^2}{\rho^2 - l^2 \sin^2\theta}\right) \quad . \tag{4.20}$$

The gravitational equipotentials e^λ = constant now have the particularly simple form ρ = constant ($\rho > l$), confirming that prolate spheroidal coordinates are indeed well suited to the given source. For the Schwarzschild solution ($l = m$) the metric assumes its usual form (Eq. (4.13)) by a straightforward change into $(t, \rho', \theta', \varphi)$ coordinates, where

$$\rho' = \rho + m \qquad \text{and} \qquad \theta' = \pi/2 - \theta \quad . \tag{4.21}$$

4.5 Possible sources for the Zipoy-Voorhees metrics

By examining the behaviour of a particular invariant of the Riemann tensor as $x \to 1^+$, Zipoy concludes that in all but the Schwarzschild case, $x = 1$ is comprised of curvature singularities. However this conclusion is slightly incorrect, since if $m/l > 2$ the invariant does in fact tend to *zero* as $x = 1$ is approached along either the positive z-axis or the negative z-axis (if $m/l = 2$ it tends to a finite, positive value).

The proper distance from $x_0 > 1$ to $x = 1$ along the spacelike geodesics given by $\theta = 0$, t, φ constants is found to be finite for all values of m/l. Timelike geodesics given by $\theta = 0$, φ constant reach $x = 1$ in both finite coordinate time and finite proper time for all values of m/l. However it is interesting to note that the circumference of the circles given by $\theta = 0$, t, x constants becomes infinite as $x \to 1^+$ for $m/l > 1$, and zero for $m/l < 1$.

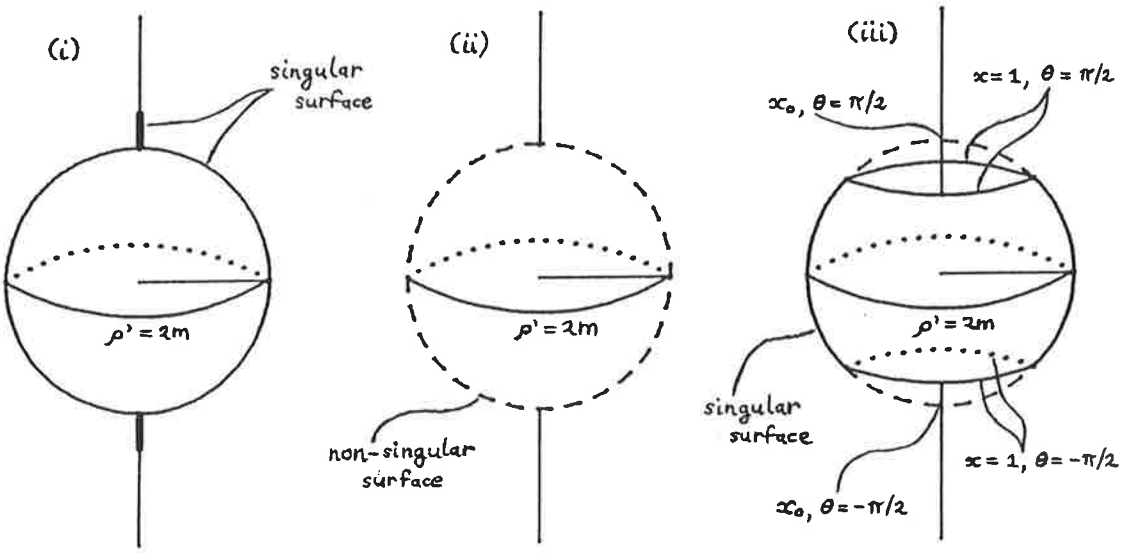
Voorhees proposed the following method for determining the *geometry* of the sources for these metrics. Assuming that all the rods are of equal mass m but have varying length $2l$, it is possible to determine the relationship $\bar{x}(x,\theta)$, $\bar{\theta}(x,\theta)$ between the prolate spheroidal coordinates $(\bar{x},\bar{\theta})$ used for the Schwarzschild solution, and those (x,θ) used for the solution generated by the rod of length $2l$.

It is then a straightforward matter to find $\rho'(x,\theta)$, $\theta'(x,\theta)$ where (ρ',θ') are the standard Schwarzschild coordinates given by Eqs. (4.21). Figure 4.2 shows how the rod $x = 1$ transforms under this change to Schwarzschild coordinates.

It is noted that for solutions with $m/l > 1$ the singular region $(x = 1)$ does not cover the entire surface $\rho' = 2m$, and indeed no curvature singularity is encountered along the axis of symmetry as $\rho' \to 2m^+$. However, consider the spacelike geodesic which in (x,θ) coordinates is given by $t =$ constant, $\theta = \pi/2$, and extends from $x = 1$, $\theta = \pi/2$ out to $x = +\infty$, $\theta = \pi/2$.

In Schwarzschild coordinates it lies along the axis of symmetry $\theta' = 0$, and extends from $\rho' = 2m$, $\theta' = 0$ out to $\rho' = +\infty$, $\theta' = 0$. But the point $(\rho' = 2m,\ \theta' = 0)$ corresponds to the point $(x_0 > 1,\ \theta = \pi/2)$ in (x,θ) coordinates. So what happens to the piece of geodesic lying between x_0 and $x = 1$?

The answer is that it maps *onto* the cap which is missing from the top of the sphere $\rho' = 2m$ in Figure 4.2 (iii)! This is an undesirable feature, and since there are further problems associated with these source representations, one concludes that the method of Voorhees yields only a very rough approximation to the source structure. The true geometry of the sources for the Zipoy-Voorhees metrics remains an open problem.



$0 < m/l < 1$ $\quad$ $l = m$ Schwarzschild solution $\quad$ $m/l > 1$

Figure 4.2 The rod of mass m and length $2l$ ($x = 1$) depicted in Schwarzschild coordinates (ρ', θ').

4.6 Possible extensions of the Zipoy-Voorhees metrics

In a more recent paper by Papadopoulos, Stewart & Witten [20], it is pointed out that the Zipoy-Voorhees metrics form the static limit of the Tomimatsu-Sato family of solutions [21], [22]. It is also noted that apart from the Schwarzschild solution ($l = m$), all metrics in the class are of Petrov type D on the axis of symmetry, and type I (or general) elsewhere. But perhaps the major revelation of the paper concerns the 'north pole' $x = 1$, $\theta = \pi/2$ and 'south pole' $x = 1$, $\theta = -\pi/2$ in solutions with $m/l \geq 2$.

In keeping with the spirit of Zipoy and Voorhees, the metric is expressed in prolate spheroidal coordinates. Then using a complex null tetrad $(m, \overline{m}, l, k)$, the Weyl tetrad components Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4 are calculated ($\Psi_1 = 0$ & $\Psi_3 = 0$). For solutions with $m/l \geq 2$, Ψ_0, Ψ_2 & Ψ_4 are infinite along $x = 1$, $-\pi/2 < \theta < \pi/2$, confirming that the rod $x = 1$ minus its endpoints (or poles) is indeed comprised of curvature singularities. However the value of each of Ψ_0, Ψ_2 & Ψ_4 at the north and south poles is found to vary according to the direction of approach to the pole.

The north and south poles are thus the locations of *directional singularities.* In an attempt to unwrap this directional behaviour, a polar-type coordinate system based on the north pole is introduced. However the attempt is unsuccessful, because the coordinate transformation maps the pole to a point. To successfully unwrap the directionality it will certainly be necessary to use a coordinate transformation which maps the pole to a higher-dimensional surface.

It can be shown that timelike geodesics lying along the axis of symmetry ($x > 1$, $\theta = \pi/2$) reach the north pole in finite proper time. Since no curvature singularity is encountered there, it is argued that an extension of the space-time is necessary. As a first step towards providing one, an extension of the *2-dimensional 'space-time'* spanned by the time coordinate t and the axis of symmetry ($x > 1$, $\theta = \pi/2$) is successfully performed.

If m/l (≥ 2) is an integer, the extension is analytic. If $n < m/l < n+1$, where n is an integer ($n \geq 2$), the extension is C^n—an analytic extension is not possible in such cases. An extension of the full 4-dimensional space-time through the north pole (or likewise the south pole) has yet to be found. It is clear however, that the ability to perform such an extension, will be intimately tied to the ability to unwrap the directionality which is present at the poles.

4.7 Some general properties of the Weyl metrics

From the preceeding discussion of the Zipoy-Voorhees metrics, and the earlier comments regarding the Curzon and Schwarzschild metrics, it is apparent that:

> *In general there is no correspondence between the geometry of the source for a Weyl metric, and the geometry of the Newtonian source from which it is generated.*

Is it at least true then, that every Weyl metric (Eq. (4.1)) is generated by a *unique* Newtonian potential $\lambda(r,z)$? At a superficial level the answer to this question is, of course, 'yes'. Since the Weyl metric coefficient g_{00} is $-e^{2\lambda}$, it is clear that two different Newtonian potentials $\lambda_1(r,z)$ and $\lambda_2(r,z)$, will certainly generate Weyl metrics which *look* different.

There is a possibility however, that if the second Weyl metric is expressed in a different coordinate system $(\bar{t}, \bar{r}, \bar{z}, \bar{\varphi})$, it could assume the same form as the first metric still expressed in Weyl coordinates. That is, different λ_1 and λ_2 might generate the *same* metric simply expressed in different coordinates. But does this actually happen in practice?

The answer lies in an interesting paper by Gautreau & Hoffman [23]. They set themselves the task of finding all Newtonian potentials $\lambda(r, z)$ which generate flat space. Obviously $\lambda = 0$ is one such potential, giving rise as it does to flat space expressed in cylindrical coordinates

$$ds^2 = -\,dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 \quad . \tag{4.22}$$

Now the Newtonian potential of a constant density line mass of infinite extent, lying along the entire z-axis is given by

$$\lambda = 2\sigma \ln r \quad , \tag{4.23}$$

where $\sigma > 0$ is the mass per unit length. If the Riemann tensor components are calculated for this subclass of the Weyl metrics, it is readily seen that they all vanish for the case $\sigma = 1/2$ $(r > 0)$. So $\lambda = \ln r$ is another potential which generates flat space.

It can be shown that there are precisely two other such potentials, namely

$$\lambda = \frac{1}{2} \ln\left(\sqrt{r^2 + z^2} - z\right) \;, \quad \lambda = \frac{1}{2} \ln\left(\sqrt{r^2 + z^2} + z\right) \quad . \tag{4.24}$$

They correspond to the Newtonian potentials of semi-infinite line masses of constant density 1/2, lying along the entire positive z-axis and entire negative z-axis respectively. So with four distinct Newtonian potentials which generate flat space, one concludes that:

> *There is not a strict* 1−1 *correspondence between the Weyl metrics and their generating Newtonian potentials* $\lambda(r, z)$.

However some questions naturally arise here. For instance, how special is the case of flat space in this context? In other words, is it true in general that a Weyl metric is generated by more than one potential? If not, then what is the class of exceptions? Also can a strict 1–1 correspondence be obtained by restricting the generating Newtonian potentials to those corresponding to mass distributions of finite extent? At the present time all of these questions remain unanswered.

4.8 The two-particle Curzon solution

No survey of the Weyl metrics would be complete without mentioning the two-particle Curzon solution, which as the name suggests, was first found by Curzon [24] (and later by Silberstein [25]). This solution is generated by the Newtonian potential $\lambda(r,z)$ of two particles (point masses) of mass m_1 and m_2. Obviously, for the mass configuration to be axisymmetric, the two particles must both lie along the z-axis at z_1 and z_2 respectively ($z_1 < z_2$). So

$$\lambda = -\frac{m_1}{\rho_1} - \frac{m_2}{\rho_2} \tag{4.25}$$

where

$$\rho_1 = \sqrt{r^2 + (z - z_1)^2} \quad \text{and} \quad \rho_2 = \sqrt{r^2 + (z - z_2)^2} \tag{4.26}$$

and

$$\nu = -\frac{1}{2} r^2 \left(\frac{{m_1}^2}{{\rho_1}^4} + \frac{{m_2}^2}{{\rho_2}^4} \right) + \frac{2 m_1 m_2}{(z_2 - z_1)^2} \left(\frac{r^2 + (z - z_1)(z - z_2)}{\rho_1 \rho_2} - 1 \right) . \tag{4.27}$$

Silberstein claimed that the existence of a *static* solution consisting only of two point masses surrounded by vacuum, indicated the incorrectness of the general theory of relativity. After all, two masses at rest in vacuum should gravitate! Einstein [26] countered that the two-particle solution is *not* purely a *vacuum* solution, and provided the following argument. Consider a small circle given by t = constant, z = constant ($z_1 < z < z_2$), r = constant, where r is small. If one takes the circumference C and radius R of this circle, it is found that in the limit as $R \to 0^+$,

$$C/R \to 2\pi\, e^{-\nu} \qquad \text{where} \qquad \nu = \nu\,(0,z) \ .$$

Now for $z_1 < z < z_2$, $\nu\,(0,z) \neq 0$ and so C/R does not approach 2π as $R \to 0^+$. Hence the space-time violates the condition of *elementary flatness* on the section of axis between the two particles, suggesting the existence of a "strut". This would explain the static nature of the solution.

However in 1968, some thirty-two years after Einstein's paper on this subject, Szekeres [27] demonstrated that static, two-body solutions *do* exist in general relativity. In his solutions, at least one of the two point masses is endowed with a multipole mass structure, which allows equilibrium to be achieved *without* the need for an intervening strut. The simplest example is that of a pure mass monopole (a Curzon particle) balanced by a mass monopole-dipole, where the mass of each particle (as represented by the monopole moment) is positive.

Another major contribution to this subject came quite recently in 1982. Using a technique to generate stationary solutions from static ones, Dietz and Hoenselaers [28] obtained from the two-particle Curzon solution, a *stationary*, axisymmetric solution representing two particles precisely balanced by their spin-spin interaction. Their solution is also a purely vacuum solution with no strut required.

The source structure for the two-particle Curzon solution is still unknown. From Section 4.2 it is known that although the Curzon metric given by Eq. (4.4) & Eq. (4.5) is generated by the Newtonian mass monopole, the Curzon solution is *not* the unique spherically symmetric, vacuum solution of general relativity. The source for the Curzon solution is a ring singularity with finite radius and infinite circumference, and the space-time has a doubled-sheeted topology inside the ring.

So without further investigation, there is no reason to expect that the source for the two-particle Curzon solution simply consists of two point masses joined by a strut. The source structure is probably considerably more complicated, and the space-time may even be extendible. A first step towards resolving these issues would presumably be to look for directional behaviour at the two particle locations: $(r = 0,\, z_1)$ and $(r = 0,\, z_2)$.

4.9 Recent mathematical developments

In a recent paper by Woodhouse & Mason [29], the ideas presented in an earlier paper by Ward [30] are developed into a geometric correspondence between the *stationary*, axisymmetric vacuum space-times and particular complex analytic objects—holomorphic vector bundles on a non-Hausdorff Riemann surface (twistor space). As a result, the solutions to the Ernst equations on space-time can be described in terms of certain free holomorphic functions on regions in the Riemann sphere (or on parts of the twistor space).

The paper discusses the effect of the action of the Geroch group on these free holomorphic functions, and also the conditions on them implied by global properties such as axis regularity and asymptotic flatness. Unfortunately the construction is, at present, tied to the use of Weyl coordinates, so that aspects of the singularity/source structure and global structure which are obscured by the use of Weyl coordinates, are difficult to address in this new framework also.

Nevertheless the construction is geometric, and it should therefore be possible to articulate it independently of the choice of such coordinates. The study of singularities would then perhaps be reducible to the study of singularities of holomorphic functions. However further work needs to be done before these ideas are able to contribute to the study of the singularities occurring in the Weyl metrics.

4.10 Ring singularities

Perhaps the most appropriate way to conclude a survey is to add a small, but new observation on the given subject—in this case the Weyl metrics. This particular observation will concern ring singularities (that is, rings comprised of curvature singularities), occurring in the hypersurfaces t = constant of the Weyl space-times. These rings are known to be a common feature throughout the entire Weyl class.

That the Weyl metrics should exhibit singularities in the form of rings is not really very surprising, since all metrics in the class are axisymmetric. So if a curvature singularity occurs at the point (r_0, z_0, φ_0) in the hypersurface t = constant $(r_0 \neq 0)$, then a curvature singularity occurs at every point (r_0, z_0, φ), where $0 \leq \varphi < 2\pi$. In other words (r_0, z_0) is a ring singularity.

What past investigators have found rather more surprising, is that these rings may have an infinite circumference. If, in addition, the ring singularity can be reached from the axis of symmetry via a finite number of spacelike geodesics, each having finite proper length, then one is indeed confronted by a highly counter-intuitive phenomenon—namely a ring having finite radius, but infinite circumference!

The Curzon metric provides the most well-known example. Although in Weyl coordinates the Curzon singularity appears as a point (at $R = 0$) exhibiting highly directional behaviour, a change to the new coordinates constructed by Scott & Szekeres unwraps the point to include, amongst other things, a ring singularity with finite radius and infinite circumference.

When past investigators of Weyl metrics have happened across an example of this phenomenon, they have tended to regard it as an exceptional case. However, the simple argument which follows will indicate that for ring singularities with finite radius occurring in Weyl metrics, the *generic* case is that the circumference of the ring is *infinite*, not finite. Note that the standard Weyl coordinates (t, r, z, φ) will be used in what follows, although the argument could proceed equally well in any other coordinate system (t, x, y, φ).

Suppose that in a Weyl space-time a curvature singularity occurs at the point $p = (t_0, r_0 \neq 0, z_0, \varphi_0)$. It is assumed that p can be reached from the axis of symmetry by a C^0 curve γ, which consists of a finite number of spacelike geodesics, each having finite proper length. So (r_0, z_0), $0 \leq \varphi < 2\pi$ is a ring singularity with finite radius which occurs in every hypersurface $t =$ constant.

Now it will generally be true that $\lambda = -\infty$ at the curvature singularity p. This means that $\lambda \rightarrow -\infty$ as p is approached from any direction. So if one considers the circle given by ($r =$ constant, z_0), where $0 < r < r_0$, then its circumference C_r is found to be $C_r = 2\pi r\, e^{-\lambda}$, and it is readily seen that as $r \rightarrow r_0$, $C_r \rightarrow +\infty$.

The details have been omitted here, but the argument can be made rigorous. Note that the fact that the ring singularity has a finite radius is *not* used to show that it has an infinite circumference. In other words, a ring singularity with an infinite radius would also have an infinite circumference, but this is not very surprising after all. It only remains to find a physical explanation of this strange phenomenon. *How* can a ring singularity with finite radius have an infinite circumference?

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Chapter 5

The Abstract Boundary — A New Boundary Construction for n-Dimensional Manifolds

1 Introduction

In general relativity, one often wishes to know whether a particular solution of Einstein's field equations is singular or not. Such a seemingly simple question has often been the cause of a great deal of confusion. Perhaps the most significant problem is that a solution usually comes packaged in one of two ways. Either it is embedded in a larger 4-dimensional manifold e.g. the Schwarzschild solution ($r > 2m$), or no embedding is given at all e.g. non-compactified Minkowski space-time.

The latter case is problematic because there is no edge to the space-time, which makes it difficult to assess whether or not singular behaviour occurs there. The former case is problematic because it provides a fixed reference. Whilst a metric may look very singular with respect to that particular embedding, it may not look singular at all with respect to another embedding e.g. the Kruskal embedding for the Schwarzschild solution. However, it has often been the case that the assessment of whether or not a solution is singular has been made relative to a given embedding.

This paper provides a new approach to the problem, our aims being to clarify the issues involved, and to provide a practical formula for judging whether or not a particular solution is singular. We have shifted the question to a wider framework, namely n-dimensional manifolds with regular metrics of arbitrary signature, since none of the techniques and ideas used are peculiar to 4-dimensional space-times.

As much flexibility as possible has been incorporated into the scheme. For instance, one may only be concerned about singular behaviour that occurs relative to a special set of curves on the manifold. This is particularly true in general relativity, where one may simply be interested in geodesics, or in curves with bounded acceleration, etc. Apart from a few basic conditions which must be complied with, there is the freedom to choose the family of curves that will be used.

The central idea of the scheme, is that all possible embeddings of the given pseudo-Riemannian manifold into other n-dimensional manifolds must be compared. On the basis of these comparisons, each boundary point belonging to such an embedding is classified into one of six categories, three of which are non-singular, and three singular. This allows a precise definition of a removable singularity and a directional singularity to be formulated for the first time.

In order to elucidate the various new concepts and definitions that are introduced, a number of examples, including some from general relativity, will be given throughout. The final section of the paper contains our definition of a non-singular pseudo-Riemannian manifold, together with a discussion of its relationship to past definitions. It would be a natural progression from such a point to consider the question of an optimal boundary for a pseudo-Riemannian manifold. However, such considerations are beyond the scope of the current paper, and will appear elsewhere. Finally, whenever $\mathcal{M}, \widehat{\mathcal{M}}, \widetilde{\mathcal{M}}$, etc. occur throughout the text, they will always denote n-dimensional, connected, Hausdorff C^∞ manifolds.

2 Parametrized curves on a manifold

The concept of a curve on a manifold is integral to any discussion regarding singularities of pseudo-Riemannian manifolds. Although geodesics are often used in this context, we will include a much wider range of curves, called parametrized curves. A subclass of these curves may then be selected, according to the purpose that one has in mind.

Definition 2.1 *If $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $a < b$, then we will refer to $[a, b)$ as a* half-open interval. *A* parametrized curve *$\gamma(t)$ on a manifold $\mathcal{M}$ is a continuous map $\gamma : I \to \mathcal{M}$, where I is a half-open interval $[a, b)$.*

Definition 2.2 *A parametrized curve $\gamma(t)$ on $\mathcal{M}$ will be said to be* non-intersecting, *if for any t_1 and t_2 in I such that $t_1 \neq t_2$, $\gamma(t_1)$ and $\gamma(t_2)$ are different points of $\mathcal{M}$.*

Definition 2.3 *A non-intersecting, parametrized curve $\gamma(t)$ on $\mathcal{M}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a, b)$, is* equivalent *to another non-intersecting, parametrized curve $\gamma'(t')$ on $\mathcal{M}$ given by $\gamma' : I' \to \mathcal{M}$ where $I' = [a', b')$, iff $\{\gamma(t) : t \in I\} = \{\gamma'(t') : t' \in I'\}$ and $\gamma(a) = \gamma'(a')$. We will write that $\gamma(t) \sim \gamma'(t')$.*

If we let $\mathcal{F}$ denote the family of all non-intersecting, parametrized curves on $\mathcal{M}$, then it is readily seen that $\sim$ is a proper equivalence relation on $\mathcal{F}$. A particular equivalence class under $\sim$ will be denoted by $[\gamma(t)]$, where $\gamma(t)$ is an arbitrarily chosen member of that class.

It follows from the preceding definitions that if two non-intersecting, parametrized curves $\gamma(t)$ and $\gamma'(t')$ are equivalent, then there exists a continuous, strictly monotone increasing function $s : [a,b) \to [a',b')$ such that $\gamma' \circ s = \gamma$. Clearly $s(a) = a'$, and as $t \to b^-$, $s(t) \to b'^-$. We will say that $\gamma'(t')$ is obtained from $\gamma(t)$ by the *change of parameter s.*

Definition 2.4 *Let $\gamma(t)$ be a curve in $\mathcal{F}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a,b)$, and let $\gamma'(t')$ be another curve in $\mathcal{F}$ given by $\gamma' : I' \to \mathcal{M}$ where $I' = [a',b')$. $\gamma'(t')$ will be said to be a* subcurve *of $\gamma(t)$ iff $a \leq a' < b' \leq b$ and $\gamma(t)\,|_{[a',b')} = \gamma'(t')$.*

Lemma 2.5 Let $\gamma(t)$ be a curve in $\mathcal{F}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a,b)$, and let $\gamma'(t')$ be a subcurve of $\gamma(t)$ given by $\gamma' : I' \to \mathcal{M}$ where $I' = [a',b')$.

(i) If $b' < b$, then $\gamma'(t')$ has finite parameter range.

(ii) If $b' = b$ and $\gamma(t)$ has finite parameter range, then $\gamma'(t')$ has finite parameter range.

(iii) If $b' = b$ and $\gamma(t)$ has infinite parameter range, then $\gamma'(t')$ has infinite parameter range.

Definition 2.6 *Let $\gamma(t)$ be a curve in $\mathcal{F}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a,b)$, and let $\gamma'(t')$ be another curve in $\mathcal{F}$ given by $\gamma' : I' \to \mathcal{M}$ where $I' = [a',b')$. $\gamma(t)$ will be said to be* extendible to $\gamma'(t')$ *if there exists an i and j in I', where $a' \leq i < j < b'$, and a k in I, where $a \leq k < b$, such that $\gamma'(t')\,|_{[i,j)} \sim \gamma(t)\,|_{[k,b)}$. Note that i and k will not be unique here. We will also say that $\gamma(t)$ is* extendible, *when there exists a curve $\gamma'(t')$ in $\mathcal{F}$ such that $\gamma(t)$ is extendible to $\gamma'(t')$.*

It is a straightforward exercise to show that if the curve $\gamma(t)$ is extendible, then every other curve in $[\gamma(t)]$ is also extendible. So it is valid to speak about equivalence classes of curves being extendible or otherwise.

Now the equivalence class $[\gamma(t)]$ will contain curves with finite half-open interval $I' = [a', b')$ (i.e. b' is finite), as well as curves with infinite half-open interval $I'' = [a'', b'')$ (i.e. $b'' = +\infty$). This is an undesirable feature since it means, amongst other things, that a curve may be extendible even though it has infinite parameter range. In order to avoid these situations, we introduce some restrictions on $\mathcal{F}$.

Let $\mathcal{C} \subset \mathcal{F}$ be a family of non-intersecting, parametrized curves on $\mathcal{M}$, and let $\gamma(t)$ be a curve in $\mathcal{C}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a, b)$. We will always require that all subcurves of $\gamma(t)$ are also members of $\mathcal{C}$. The equivalence relation $\sim$ is naturally induced on $\mathcal{C}$ from $\mathcal{F}$, and as before, we will denote an equivalence class of curves in $\mathcal{C}$ by $[\gamma(t)]$.

Definition 2.7 *It will be said that* $\mathcal{C}$ *has the* bounded parameter property *(b.p.p.) if for any curve* $\gamma(t)$ *in* $\mathcal{C}$*, either all members of the equivalence class* $[\gamma(t)]$ *have finite parameter range, or all members have infinite parameter range.*

$\mathcal{C}$ will always be assumed to have the bounded parameter property in what follows, and a manifold $\mathcal{M}$ with such a $\mathcal{C}$ will be denoted by $(\mathcal{M}, \mathcal{C})$. Three well-known examples are given below.

Examples

(i) $\mathcal{M}$ is a manifold with affine connection. $\mathcal{C}_g$ is the set of all non-intersecting geodesics with affine parameter.

(ii) $\mathcal{M}$ is a manifold with affine connection. $\mathcal{C}_{gap}$ is the set of all non-intersecting, continuous, piecewise C^1 curves with generalized affine parameter [1].

(iii) $\mathcal{M}$ is a manifold with Lorentzian metric. $\mathcal{C}_{ns}$ is the set of all non-intersecting, timelike and null geodesics with affine parameter.

Theorem 2.8 Suppose that we have an $(\mathcal{M}, \mathcal{C})$. Let $\gamma(t)$ be a curve in $\mathcal{C}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a, b)$, and let $\gamma'(t')$ be another curve in $\mathcal{C}$ given by $\gamma' : I' \to \mathcal{M}$ where $I' = [a', b')$. If $\gamma(t)$ is extendible to $\gamma'(t')$, then $\gamma(t)$ has finite parameter range.

Proof

Since $\gamma(t)$ is extendible to $\gamma'(t')$,

$\exists\ i, j \in I',\ k \in I,$ where $a' \leq i < j < b'$ and $a \leq k < b$,

such that $\gamma'(t')\,|_{[i,j)} \sim \gamma(t)\,|_{[k,b)}$.

Now $\gamma(t)\,|_{[k,b)}$ is a subcurve of $\gamma(t)$, and $\gamma'(t')\,|_{[i,j)}$ is a subcurve of $\gamma'(t')$.

So both are contained in $\mathcal{C}$.

Since $j < b'$, j is finite.

Thus by the bounded parameter property, b is also finite.

QED

3 Enveloped manifolds and boundary sets

In order to formulate a definition of a singularity-free pseudo-Riemannian manifold, we need to consider the behaviour of the metric near the "fringes" or "boundaries" of the manifold. However, these two words really only make sense when $\mathcal{M}$ sits in some larger manifold $\widehat{\mathcal{M}}$. This notion is made precise in the following definition, thus enabling boundary points and boundary sets to be defined later in the section.

Definition 3.1 *An* enveloped manifold *is a pair of manifolds $\mathcal{M}$ and $\widehat{\mathcal{M}}$ and a C^∞ embedding $\varphi : \mathcal{M} \to \widehat{\mathcal{M}}$. This will be denoted by $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. We note that since both manifolds have the same dimension n, $\varphi(\mathcal{M})$ is an open submanifold of $\widehat{\mathcal{M}}$. We will also refer to the enveloped manifold as an* envelopment *of $\mathcal{M}$ by $\widehat{\mathcal{M}}$, and $\widehat{\mathcal{M}}$ will be called the* enveloping manifold.

There are two different methods of setting up an enveloped manifold, both of which will be used in the examples which follow here, and in later sections.

Method 1. We start with a manifold $\widehat{\mathcal{M}}$, and choose one of its connected, open subsets $\mathcal{M}$. This subset has a natural C^∞ differentiable structure induced from $\widehat{\mathcal{M}}$, with which it becomes an open submanifold of $\widehat{\mathcal{M}}$. Trivially, the map $i : \mathcal{M} \to \widehat{\mathcal{M}}$ given by $i(p) = p$ is a C^∞ embedding. So $(\mathcal{M}, \widehat{\mathcal{M}}, i)$ is an enveloped manifold.

Examples

(i) $\widehat{\mathcal{M}} = \mathbb{R}^2$

$\mathcal{M} = \mathbb{R}^2 -$ a Cantor set

(ii) $\widehat{\mathcal{M}} = \mathbb{R}^2$

$\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$

Note that where an enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ has been set up using this method, we will usually refer to $\varphi(\mathcal{M})$ simply as $\mathcal{M}$.

Method 2. We start with a manifold $\mathcal{M}$, and look for another manifold $\widehat{\mathcal{M}}$ into which it can be C^∞ embedded as an open submanifold. Then $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is an enveloped manifold.

Examples

(iii) $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$

$\widehat{\mathcal{M}} = \mathbb{R}^2$

$\varphi : \mathcal{M} \to \widehat{\mathcal{M}}$

$(x, y) \mapsto (x, x + y + 2)$

(iv) $\mathcal{M}$ and φ as in Example (iii)

$\widehat{\mathcal{M}} = \mathbb{R}^2 - \{(x, y) : x^2 + y^2 \leq 1\}$

In Examples (ii), (iii) and (iv) above, the manifold $\mathcal{M}$ is the same. Although the enveloping manifold is the same in Examples (ii) and (iii), the C^∞ embedding differs. In Example (iv), the enveloping manifold itself is different. These three examples thus give three different enveloped manifolds.

Definition 3.2 *A* boundary point *p of an enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is a point in the topological boundary of $\varphi(\mathcal{M})$ i.e. a point p in $\widehat{\mathcal{M}} - \varphi(\mathcal{M})$ such that every open neighbourhood U of p in $\widehat{\mathcal{M}}$ has non-empty intersection with $\varphi(\mathcal{M})$. A* boundary set *$B \subseteq \widehat{\mathcal{M}} - \varphi(\mathcal{M})$ is a connected set of such boundary points, and we will use the notation $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi, B)$ for an enveloped manifold with a particular boundary set B.*

There will often be an infinite number of boundary points associated with any particular enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. Only when $\varphi(\mathcal{M}) = \widehat{\mathcal{M}}$ will there be no boundary points at all. As the next example will illustrate, there are cases where precisely one boundary point exists.

Example

(v) $\widehat{\mathcal{M}} = \mathbb{R}^2$

$\mathcal{M} = \mathbb{R}^2 - \{O\}$ where O is the origin

The only boundary point p is O.

The only boundary sets B are $\emptyset$ and $\{O\}$.

4 Limit points and $\mathcal{C}$-completeness

Definition 4.1 *Let $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ be an enveloped manifold, and $\gamma(t)$ a non-intersecting, parametrized curve on $\mathcal{M}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a, b)$. We will say that a point $p \in \widehat{\mathcal{M}}$ is a* limit point *of $\gamma(t)$ if, in the half-open interval I, there exists an increasing sequence of numbers $t_i \to b^-$ such that $(\varphi \circ \gamma)(t_i) \to p$ (meant in the usual topological sense). If, in addition, $(\varphi \circ \gamma)(t_i) \to p$ for every such sequence $\{t_i\}$, then p will be said to be the* endpoint *of $\gamma(t)$.*

It is clear from the definition that limit points and endpoints p of curves on $\mathcal{M}$ either lie in $\varphi(\mathcal{M})$ itself, or are boundary points lying in $\widehat{\mathcal{M}} - \varphi(\mathcal{M})$. In the former case we will simply say that $\gamma(t)$ has a limit point or an endpoint in $\mathcal{M}$, namely $\varphi^{-1}(p)$. Some curves will have no limit points at all, and others may have infinitely many. Of course a curve with an endpoint has a unique limit point.

Now if $\gamma(t)$ and $\gamma'(t')$ are two curves in $\mathcal{F}$, and $\gamma(t) \sim \gamma'(t')$, then a limit point of $\gamma(t)$ will also be a limit point of $\gamma'(t')$, and if $\gamma(t)$ has the endpoint p, then $\gamma'(t')$ also has the endpoint p. So we may speak about the limit points of the equivalence class of curves $[\gamma(t)]$, and where appropriate, say that $[\gamma(t)]$ has the endpoint p. Also if $\gamma(t)$ is extendible to another curve in $\mathcal{F}$, then it is clear that $\gamma(t)$ has an endpoint in $\mathcal{M}$.

Definition 4.2 *Let $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi, B)$ be an enveloped manifold with boundary set B, and let $\gamma(t)$ be a curve on $\mathcal{M}$ belonging to $\mathcal{F}$. It will be said that $\gamma(t)$* approaches *B if it has at least one limit point in $\widehat{\mathcal{M}}$, and* all *its limit points in $\widehat{\mathcal{M}}$ lie in B.*

Definition 4.3 *Given an $(\mathcal{M}, \mathcal{C})$, we will say that $\mathcal{M}$ is $\mathcal{C}$-*complete *iff every equivalence class of curves in $\mathcal{C}$ with finite parameter range has a limit point in $\mathcal{M}$, and has no limit points in any $\widehat{\mathcal{M}} - \varphi(\mathcal{M})$, where $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is an envelopment of $\mathcal{M}$.*

As the following two examples demonstrate, $\mathcal{C}$-completeness does not guarantee that every equivalence class of curves in $\mathcal{C}$ with finite parameter range is extendible.

Examples

(i) $(\mathcal{M},\mathcal{C})$ where $\mathcal{C}$ consists of the curve $\gamma(t)$ with finite parameter range, together with all of its subcurves. If $\gamma(t)$ has an endpoint in $\mathcal{M}$, then $\mathcal{M}$ is $\mathcal{C}$-complete. However $\gamma(t)$ is clearly not extendible.

(ii) $(\mathcal{M},\mathcal{C})$ where $\mathcal{M} = \mathbb{R}^2$ and $\mathcal{C}$ contains the curve $\gamma(t)$ given by $\gamma : [0,1) \to \mathbb{R}^2$ where $\gamma(t) = (t, \sin(1-t)^{-1})$. This curve has a finite parameter range and infinitely many limit points in $\mathcal{M}$. It also lies in a bounded region of $\mathbb{R}^2$. So whether or not $\mathcal{M}$ is $\mathcal{C}$-complete will depend on the other curves belonging to $\mathcal{C}$. However, irrespective of this, $\gamma(t)$ is not extendible to any curve in $\mathcal{C}$ (or in $\mathcal{F}$ for that matter), because it doesn't have an endpoint in $\mathcal{M}$.

The converse is clearly true, namely if every equivalence class of curves in $\mathcal{C}$ with finite parameter range is extendible, then $\mathcal{M}$ is $\mathcal{C}$-complete.

5 Abstract boundaries

The set of all boundary points of a particular manifold $\mathcal{M}$ is often enormously large, since there may exist an infinite number of envelopments of $\mathcal{M}$, each having an infinite number of boundary points. It is therefore desirable to reduce it to a more manageable size, by adopting some suitable process for making identifications between boundary points. This is what we proceed to do.

Definition 5.1 *If $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi, B)$ and $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi', B')$ are two enveloped manifolds with boundary sets B and B' respectively, then we say that B* covers *B' if for every open neighbourhood U of B in $\widehat{\mathcal{M}}$ there exists an open neighbourhood U' of B' in $\widehat{\mathcal{M}}'$ such that*

$$\varphi \circ \varphi'^{-1}\left(U' \cap \varphi'(\mathcal{M})\right) \subseteq U.$$

If B consists of just a single point p, we will simply say that p covers B'. Similarly, if B' consists of a single point p', we say that B covers p'. If both consist of just a single point, then we say that p covers p'. For the latter case it can be established that if p' is a limit point of a particular curve on $\mathcal{M}$ (belonging to $\mathcal{F}$), then p is also.

Theorem 5.2 Let p be a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$, and let p' be a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$. Suppose that p covers p'. If p' is a limit point of a curve $\gamma(t)$ in $\mathcal{F}$ given by $\gamma : I \to \mathcal{M}$ where $I = [a, b)$, then p is also a limit point of $\gamma(t)$.

Proof

Since p' is a limit point of $\gamma(t)$, in the half-open interval I there exists an increasing sequence of numbers $t_i \to b^-$ such that $(\varphi' \circ \gamma)(t_i) \to p'$.

Let U be an open neighbourhood of p in $\widehat{\mathcal{M}}$.

Since p covers p', there exists an open neighbourhood U' of p' in $\widehat{\mathcal{M}}'$ such that

$$\varphi \circ \varphi'^{-1}\left(U' \cap \varphi'(\mathcal{M})\right) \subseteq U.$$

Now $\exists \; n \in \mathbb{N}$ s.t. $\forall \; i > n, \; (\varphi' \circ \gamma)(t_i) \in U' \cap \varphi'(\mathcal{M})$

$\Rightarrow \; \forall \; i > n, \; (\varphi \circ \gamma)(t_i) \in U$.

Thus p is also a limit point of $\gamma(t)$.

QED

Now suppose that B and B' are boundary sets of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. If B' is a subset of B (possibly consisting of a single point p'), then it is clear that B covers B' (B covers p'). Conversely, for two different enveloped manifolds $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ and $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$, it is possible for a single boundary point of $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ to cover a boundary set of $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$ consisting of infinitely many points.

<u>Example</u>

(i) $\widehat{\mathcal{M}} = \mathbb{R}^n$

$\mathcal{M} = \mathbb{R}^n - \{O\}$ where O is the origin

r is the usual radial coordinate

$$i : \mathcal{M} \to \widehat{\mathcal{M}} \qquad \qquad \varphi : \mathcal{M} \to \widehat{\mathcal{M}}$$

$$p \mapsto p \qquad \qquad p(r) \mapsto p(r+1)$$

$i(\mathcal{M}) = \mathcal{M} = \mathbb{R}^n - \{O\}$

$\varphi(\mathcal{M}) = \mathbb{R}^n - B^n(0,1)$ where $B^n(0,1)$ is the unit ball $(0 \le r \le 1)$

The only boundary point of $(\mathcal{M}, \widehat{\mathcal{M}}, i)$ is the origin O.

However $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ has an infinite number of boundary points, namely the unit sphere $S^n(0,1)$.

$S^n(0,1)$ covers O and O covers $S^n(0,1)$.

Covering is a partial ordering on the set of all boundary sets of a manifold $\mathcal{M}$:

$$(\mathcal{M}, \widehat{\mathcal{M}}, \varphi, B) \qquad B \text{ covers } B$$

$$(\mathcal{M}, \widehat{\mathcal{M}}_1, \varphi_1, B_1), \quad (\mathcal{M}, \widehat{\mathcal{M}}_2, \varphi_2, B_2), \quad (\mathcal{M}, \widehat{\mathcal{M}}_3, \varphi_3, B_3)$$

If B_1 covers B_2 and B_2 covers B_3, then B_1 covers B_3.

Definition 5.3 *Given two enveloped manifolds* $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi, B)$ *and* $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi', B')$ *with boundary sets* B *and* B' *respectively, we will say that* B *and* B' *are* equivalent *iff* B *covers* B' *and* B' *covers* B. *This will be denoted by* $B \sim B'$. *We will also use the notation* $p \sim B'$, $B \sim p'$ *and* $p \sim p'$ *where appropriate.*

We have thus imposed an equivalence relation $\sim$ on the set of all boundary points of the manifold $\mathcal{M}$. If we now make identifications between boundary points according to whether or not they are equivalent, then we will greatly reduce the size of the set to something more manageable. This is what we set out to do at the start of the section.

Definition 5.4 *If* p *is a boundary point of an enveloped manifold* $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$, *then the equivalence class (under* $\sim$*) of boundary points of* $\mathcal{M}$ *to which* p *belongs will be denoted by* $[p]$. *This will be referred to as an* abstract boundary point *of* $\mathcal{M}$. *The set of all such abstract boundary points will be denoted by* $\mathcal{B}(\mathcal{M})$, *and called the* abstract boundary *of* $\mathcal{M}$.

In Example (i) we saw that the boundary point O covers the boundary set $S^n(0,1)$, consisting of infinitely many points. Since the reverse is also true, we have that $O \sim S^n(0,1)$. Now if $p \in S^n(0,1)$, it may be seen that O covers p, but that p does not cover O. So O and p are not equivalent boundary points, which means that $[O]$ and $[p]$ are different abstract boundary points of $\mathcal{M}$. In such a situation we would say that $[O]$ covers $[p]$.

In general there will be no relationship at all between two abstract boundary points $[p]$ and $[q]$ of $\mathcal{M}$. However, as the example indicates, there may be cases where $[p]$ covers $[q]$ or $[q]$ covers $[p]$, but both cannot be true if $[p]$ and $[q]$ are different abstract boundary points. It is clear that covering is also a partial ordering on the set of all abstract boundary points of $\mathcal{M}$.

Now there are further ways of reducing the size of the abstract boundary $\mathcal{B}(\mathcal{M})$ of $\mathcal{M}$. If $\mathcal{M}$ is paired with a family $\mathcal{C}$ of curves on $\mathcal{M}$ i.e. $(\mathcal{M},\mathcal{C})$ (see Section 2), then we can confine ourselves to thinking about boundary behaviour which occurs relative to $\mathcal{C}$. This motivates the following definition.

Definition 5.5 *Suppose that we have an $(\mathcal{M},\mathcal{C})$, and an envelopment of $\mathcal{M}$ by $\widehat{\mathcal{M}}$, namely $(\mathcal{M},\widehat{\mathcal{M}},\varphi)$. Let p be a boundary point of $(\mathcal{M},\widehat{\mathcal{M}},\varphi)$. Then we will say that p is a* $\mathcal{C}$-boundary point *iff it is a limit point of some curve in the family $\mathcal{C}$.*

We note that p has only to be a limit point of a curve in $\mathcal{C}$, as opposed to an endpoint. This means that some $\mathcal{C}$-boundary points will not be approached by any curve in $\mathcal{C}$, although they will, of course, be approached by curves in $\mathcal{F}$. An example of such behaviour will be given in the following section, after we have introduced a metric on $\mathcal{M}$.

The set of all $\mathcal{C}$-boundary points of a manifold $\mathcal{M}$ is normally considerably smaller than the set of all boundary points of $\mathcal{M}$, because the family $\mathcal{C}$ of curves on $\mathcal{M}$ is much smaller than the family $\mathcal{F}$. The size of this set can be further reduced by identifying its elements according to whether or not they are equivalent (see Definition 5.3).

Definition 5.6 *Given an $(\mathcal{M},\mathcal{C})$, an equivalence class (under $\sim$) of $\mathcal{C}$-boundary points of $\mathcal{M}$ will be called an* abstract $\mathcal{C}$-boundary point *of $\mathcal{M}$. The set of all such abstract $\mathcal{C}$-boundary points will be called the* abstract $\mathcal{C}$-boundary *of $\mathcal{M}$, and denoted by $\mathcal{B}_{\mathcal{C}}(\mathcal{M})$.*

Corollary 5.7 Given an $(\mathcal{M},\mathcal{C})$, $\mathcal{B}_{\mathcal{C}}(\mathcal{M}) \subseteq \mathcal{B}(\mathcal{M})$.

Proof

Let p be a boundary point of the enveloped manifold $(\mathcal{M},\widehat{\mathcal{M}},\varphi)$.

Let p' be a $\mathcal{C}$-boundary point of the enveloped manifold $(\mathcal{M},\widehat{\mathcal{M}}',\varphi')$.

Suppose that $p \sim p'$.

Since p' is a $\mathcal{C}$-boundary point, it is a limit point of some curve $\gamma(t)$ in $\mathcal{C}$.

By Theorem 5.2, p is also a limit point of $\gamma(t)$.

Thus p is a $\mathcal{C}$-boundary point of $\mathcal{M}$.

It follows that two equivalent boundary points of $\mathcal{M}$ are either both $\mathcal{C}$-boundary points, or are both not $\mathcal{C}$-boundary points.

QED

In general, $\mathcal{B}_\mathcal{C}(\mathcal{M})$ will be a much smaller set than $\mathcal{B}(\mathcal{M})$. So from our starting point in this section, namely the set of all boundary points of $\mathcal{M}$, we have made some very significant reductions to obtain the abstract $\mathcal{C}$-boundary of $\mathcal{M}$. We note that covering is also a partial ordering on $\mathcal{B}_\mathcal{C}(\mathcal{M})$.

Definition 5.8 *Suppose that we have an* $(\mathcal{M}, \mathcal{C})$*, and an envelopment of* $\mathcal{M}$ *by* $\widehat{\mathcal{M}}$*, namely* $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$*. Let* p *be a* $\mathcal{C}$*-boundary point of* $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$*. Then we will say that* p *is a* point at infinity *iff it is* not *a limit point of any curve in* $\mathcal{C}$ *with finite parameter range.*

This means that all curves in $\mathcal{C}$ with p as a limit point have infinite parameter range. No ambiguity occurs here, because if $\gamma(t)$ is one such curve with infinite parameter range, then by the b.p.p., all curves in $\mathcal{C}$ belonging to $[\gamma(t)]$ also have infinite parameter range. All subcurves of $\gamma(t)$ are members of $\mathcal{C}$, and by Lemma 2.5 (iii), any such subcurve with p as a limit point also has infinite parameter range. Similarly, if $\gamma(t)$ is equivalent to a subcurve of some other curve in $\mathcal{C}$, then that curve has infinite parameter range too.

The question which naturally arises here, is whether the definition of a point at infinity can be successfully transferred from $\mathcal{C}$-boundary points of $\mathcal{M}$, to abstract $\mathcal{C}$-boundary points of $\mathcal{M}$. The answer to this question is provided by the following corollary to Theorem 5.2.

Corollary 5.9 Suppose that we have an $(\mathcal{M},\mathcal{C})$. Let p be a $\mathcal{C}$-boundary point of the enveloped manifold $(\mathcal{M},\widehat{\mathcal{M}},\varphi)$, and let p' be a $\mathcal{C}$-boundary point of the enveloped manifold $(\mathcal{M},\widehat{\mathcal{M}}',\varphi')$. If $p \sim p'$ and p is a point at infinity, then p' is also a point at infinity.

Proof

Since p' is a $\mathcal{C}$-boundary point of $\mathcal{M}$, it is a limit point of some curve $\gamma(t)$ in $\mathcal{C}$.

Suppose that $\gamma(t)$ has finite parameter range.

By Theorem 5.2, p is also a limit point of $\gamma(t)$.

So p is a limit point of a curve in $\mathcal{C}$ with finite parameter range.

But p is a point at infinity.

It follows that all curves in $\mathcal{C}$ with p' as a limit point have infinite parameter range.

Thus p' is a point at infinity.

QED

This means that every equivalence class $[p]$ of $\mathcal{C}$-boundary points of $\mathcal{M}$ either consists entirely of points at infinity, or has no such points. So we will speak about an abstract $\mathcal{C}$-boundary point of $\mathcal{M}$ being a point at infinity or otherwise. This then provides a natural way of separating the abstract $\mathcal{C}$-boundary $\mathcal{B}_{\mathcal{C}}(\mathcal{M})$ into two disjoint subsets.

6 Pseudo-Riemannian manifolds

Definition 6.1 *A* C^k metric g *on a manifold* $\mathcal{M}$ *(where* $k \in \mathbb{N} \cup \{0\}$*) is a real-valued, second rank covariant, symmetric and non-degenerate* C^k *tensor field on* $\mathcal{M}$*. The pair* $(\mathcal{M}, g)$ *will denote a manifold* $\mathcal{M}$ *with a* C^k *metric* g*, and will be called a* C^k pseudo-Riemannian manifold.

Note that since g is non-degenerate and continuous on $\mathcal{M}$, its signature is constant over $\mathcal{M}$. Our discussion will encompass all signatures, except where explicitly stated otherwise. When g is positive definite it will be referred to as a C^k *Riemannian metric*, and the pair $(\mathcal{M}, g)$ will be called a C^k *Riemannian manifold.* When g is Lorentzian and $\mathcal{M}$ is 4-dimensional, the pair $(\mathcal{M}, g)$ will be called a C^k *space-time.*

We recall from Example (i) of Section 2 that if $\mathcal{M}$ is a manifold with affine connection, the set $\mathcal{C}_g$ of all non-intersecting geodesics with affine parameter satisfies the properties required of a family $\mathcal{C}$ of curves on $\mathcal{M}$. Of course for a C^k pseudo-Riemannian manifold $(\mathcal{M}, g)$, where $k \geq 1$, the C^{k-1} pseudo-Riemannian connection will always be the particular affine connection which is chosen. Also, since $\mathcal{C}_g$ is such an important class of curves on $\mathcal{M}$, it will henceforth be assumed that it is contained in any family $\mathcal{C}$.

Definition 6.2 $(\mathcal{M}, g, \mathcal{C})$ *will denote a* C^k *pseudo-Riemannian manifold* $(\mathcal{M}, g)$*, where* $k \geq 1$*, together with a family* $\mathcal{C}$ *of curves on* $\mathcal{M}$ *which satisfies the properties given in Section 2, and contains* $\mathcal{C}_g$ *as a subset. In addition, it will be assumed that if* $\gamma(t)$ *is a curve in* $\mathcal{C}$ *which is equivalent to a curve in* $\mathcal{C}_g$*, then* $\gamma(t)$ *is itself a member of* $\mathcal{C}_g$*.* $(\mathcal{M}, g, \mathcal{C})$ *will be referred to as a* triple.

The last condition simply ensures that the geodesics are only parametrized by affine parameters. Referring back to Example (ii) of Section 2, we observe that $\mathcal{C}_{gap}$ would be a suitable choice for $\mathcal{C}$, since the generalized affine parameter is an affine parameter on all geodesics. Of course $\mathcal{C}_{gap}$ is a much larger family than $\mathcal{C}_g$, and one would only use it if interested in the behaviour of the C^k metric g with respect to a much wider class of curves than geodesics. The family $\mathcal{C}_{ns}$ given in Example (iii) of Section 2 is not a suitable choice for $\mathcal{C}$, since it does not contain the family $\mathcal{C}_g$.

Requiring that $\mathcal{C}$ has the subset $\mathcal{C}_g$ guarantees (for the first time) that every point of the manifold $\mathcal{M}$ is located on at least one curve belonging to $\mathcal{C}$. In fact, infinitely many geodesics pass through each point p of $\mathcal{M}$. In addition, each p is the endpoint of an infinite number of geodesics with finite affine parameter range. Thus, disregarding the fact that p is not actually a $\mathcal{C}$-boundary point of $\mathcal{M}$, there is no sense in which it can be thought of as a point at infinity. We now provide two examples of manifolds with Lorentzian metrics in order to illustrate some of the points made in Sections 4 & 5.

Examples

(i) The Curzon space-time

Let $\widehat{\mathcal{M}}$ be the 4-dimensional manifold $\mathbb{R}^4$. Using standard Euclidean coordinates (t, x, y, z) on $\widehat{\mathcal{M}}$, let $\mathcal{M}$ be the open submanifold of $\widehat{\mathcal{M}}$ specified by $z > 0$. The set $B = \{ p \in \widehat{\mathcal{M}} \, : \, x = y = z = 0 \}$ is clearly a boundary set of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$.

The Curzon metric g on $\mathcal{M}$ (for $x^2 + y^2 \neq 0$) is given below. Following standard practice, it is expressed in cylindrical polar coordinates (t, r, z, φ) where $r = \sqrt{x^2 + y^2} > 0$ and $\varphi = \tan^{-1} y/x$ $(0 \leq \varphi < 2\pi)$.

$$ds^2 = -e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2e^{-2\lambda}d\varphi^2$$

where $\lambda = -\dfrac{m}{R}$, $\nu = -\dfrac{m^2r^2}{2R^4}$, $R = \sqrt{r^2+z^2}$ and $m > 0$ is a constant.

This is a C^∞ metric on $\mathcal{M}$ (for $r > 0$). Also, if the metric is re-expressed in Euclidean coordinates, then it extends across the axis $r = 0$ in a C^∞ manner. Thus g is a C^∞ metric on the whole of $\mathcal{M}$. The Curzon space-time $(\mathcal{M}, g)$ is analysed in depth and maximally extended in two papers [2], [3] by Scott & Szekeres (see also Scott [4]).

En route to performing this extension, $\mathcal{M}$ is twice re-embedded in $\widehat{\mathcal{M}}$. The first C^∞ embedding used is $\psi : (t, r, z, \varphi) \mapsto (\tan^{-1} t/m,\ r'(r,z),\ z'(r,z),\ \varphi)$, where the somewhat complicated functions r' and z' can be found in [2]. In Euclidean coordinates (t, x, y, z), the points $(t, 0, 0, z)$ lying on the axis $r = 0$ are mapped by ψ to $(\tan^{-1} t/m,\ 0,\ 0,\ z'(z))$.

The topological boundary of $\psi(\mathcal{M})$ in $\widehat{\mathcal{M}}$ is a connected, compact subset of $\widehat{\mathcal{M}}$ (see Fig. 6.1). The set $B' = B_1 \cup B_2$, where $B_1 = \{ p' \in \widehat{\mathcal{M}} : p' = (t, r, -\pi/2, \varphi)$ where $-\pi/2 < t < \pi/2,\ 0 < r \le \pi/2,\ 0 \le \varphi < 2\pi \}$ and $B_2 = \{ p' \in \widehat{\mathcal{M}} : p' = (t, x = 0, y = 0, z = -\pi/2)$ where $-\pi/2 < t < \pi/2 \}$ is a connected subset of this boundary (i.e. B' is a boundary set of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \psi)$). It can be shown that the boundary set B of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$ covers the boundary set B' of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \psi)$. However B' does not cover B, so the two boundary sets are not equivalent.

Suppose that we are interested in the family $\mathcal{C}_g$ of geodesics on $\mathcal{M}$. In the above-mentioned papers it was shown that B' is approached by a large class of oscillating spacelike geodesics. In fact every boundary point p' belonging to B' is a limit point

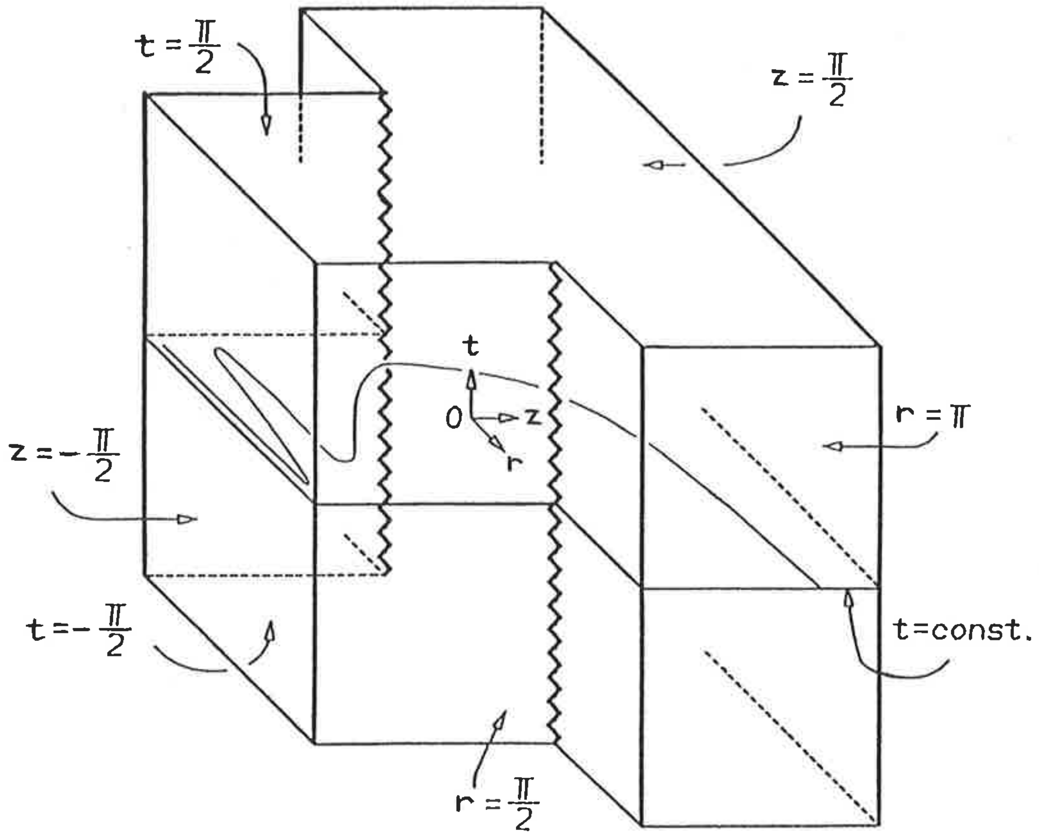

Figure 6.1 The Curzon space-time $(\mathcal{M}, g)$ is C^∞ embedded by ψ into the manifold $\widehat{\mathcal{M}} = \mathbb{R}^4$. The angular coordinate φ has been suppressed, but points to the right of the diagram (i.e. $r > 0$) have the constant angle $\phi < \pi$, and points to the left of the diagram (i.e. $r < 0$) have the constant angle $\phi + \pi$. An oscillating spacelike geodesic in one of the hypersurfaces $t =$ constant is depicted.

of infinitely many such geodesics, and is thus a $\mathcal{C}_g$-boundary point. However, apart from points belonging to the set B_2, no p' in B' is the endpoint of a curve in $\mathcal{C}_g$. Also, since no p' in B' is a limit point of a geodesic on $\mathcal{M}$ with finite affine parameter, the boundary set B' is comprised of points at infinity.

(ii) The Misner example

Let $\mathcal{M}$ be the 2-dimensional manifold $S^1 \times \mathbb{R}^1$. Using coordinates (t, ψ) on $\mathcal{M}$, where $t \in \mathbb{R}$, $0 \leq \psi < 2\pi$, a Lorentzian metric g on $\mathcal{M}$ is given by

$$ds^2 = 2\, dt\, d\psi + t\, d\psi^2 \,.$$

Clearly g is a C^∞ metric on $\mathcal{M}$, and so $(\mathcal{M}, g)$ is a C^∞ pseudo-Riemannian manifold. This example is due to Misner [5].

Suppose that we are again interested in the family $\mathcal{C}_g$ of geodesics on $\mathcal{M}$. The portion of $\mathcal{M}$ containing $t = 0$ is depicted in Figure 6.2. The vertical lines on the cylinder (i.e. $\psi =$ constant) are null geodesics which are complete on the infinite cylinder. Null cones are drawn along one such geodesic. Null geodesics also lie in $t = 0$, where they circle round and round infinitely many times, but these are all either past-incomplete or future-incomplete. Only the non-intersecting portions of these geodesics are members of $\mathcal{C}_g$.

It can be shown that other geodesics (null, timelike and spacelike) execute infinite spirals as they approach $t = 0$ from either above or below. However, each such geodesic approaches $t = 0$ with finite affine parameter, and thus is either past-incomplete or future-incomplete. So $(\mathcal{M}, g)$ is a geodesically incomplete C^∞ pseudo-Riemannian manifold. It is true though, that every geodesic in $\mathcal{C}_g$ with finite affine parameter range either has an endpoint in $\mathcal{M}$, or executes an infinite spiral to $t = 0$ and thus has infinitely many limit points in $\mathcal{M}$, namely every point on $t = 0$. So although the Misner example is geodesically incomplete, it is $\mathcal{C}_g$-complete.

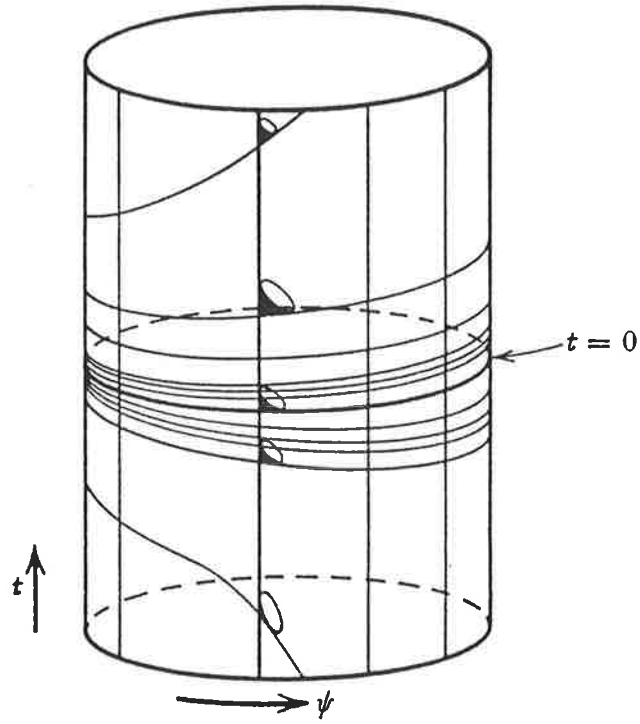

Figure 6.2 The Misner example

Chapter 6

Definition of a Non-Singular Pseudo-Riemannian Manifold

7 Extensions and regular boundary points

Suppose that $(\mathcal{M}, g)$ is a C^k pseudo-Riemannian manifold, and that $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is an envelopment of $\mathcal{M}$ by $\widehat{\mathcal{M}}$. This could be written in shortened form as $(\mathcal{M}, g, \widehat{\mathcal{M}}, \varphi)$. The C^∞ embedding $\varphi : \mathcal{M} \to \widehat{\mathcal{M}}$ induces a C^k metric on the open submanifold $\varphi(\mathcal{M})$ of $\widehat{\mathcal{M}}$, and where there is no risk of ambiguity, this will also be denoted by g. With this convention established, we now define an extension of a C^k pseudo-Riemannian manifold $(\mathcal{M}, g)$.

Definition 7.1 *A C^l* extension *of a C^k pseudo-Riemannian manifold $(\mathcal{M}, g)$, where $l \in \mathbb{N}$ and $1 \leq l \leq k$, is an envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ of $\mathcal{M}$ by a C^l pseudo-Riemannian manifold $(\widehat{\mathcal{M}}, \widehat{g})$ such that $\widehat{g}\,|_{\varphi(\mathcal{M})} = g$.*

Trivially, $(\mathcal{M}, g)$ is a C^k extension of itself. The signature of g on $\mathcal{M}$ and the signature of $\widehat{g}$ on $\widehat{\mathcal{M}}$ are the same. Any boundary point p of $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is regular in the sense that it is simply a point belonging to a manifold $\widehat{\mathcal{M}}$ on which there is a C^l metric $\widehat{g}$. This raises the question of whether we can form a notion of regularity for boundary points of arbitrary envelopments (as opposed to extensions) of $(\mathcal{M}, g)$. This motivates the following definition.

Definition 7.2 *Suppose that $(\mathcal{M}, g)$ is a C^k pseudo-Riemannian manifold and that p is a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. Then we will say that p is* C^l regular for g, *if there exists an open submanifold $\widetilde{\mathcal{M}}$ of $\widehat{\mathcal{M}}$ with $\varphi(\mathcal{M}) \subset \widetilde{\mathcal{M}}$, $p \in \widetilde{\mathcal{M}}$ and a C^l metric $\tilde{g}$ on $\widetilde{\mathcal{M}}$ such that $(\widetilde{\mathcal{M}}, \tilde{g})$ is a C^l extension of $(\mathcal{M}, g)$.*

Theorem 7.3 Suppose that $(\mathcal{M}, g, \mathcal{C})$ is a triple, and that p is a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. If p is C^l regular for g, then it is a $\mathcal{C}$-boundary point of $\mathcal{M}$.

Proof

If the boundary point p is C^l regular for g, then there exists a C^l extension $(\widetilde{\mathcal{M}}, \tilde{g})$ of $(\mathcal{M}, g)$ such that $\widetilde{\mathcal{M}}$ is an open submanifold of $\widehat{\mathcal{M}}$, $\varphi(\mathcal{M}) \subset \widetilde{\mathcal{M}}$ and $p \in \widetilde{\mathcal{M}}$.

An infinite number of geodesics on $\widetilde{\mathcal{M}}$ pass through p.

Let $\gamma'(t)$ be one such geodesic given by $\gamma' : I' \to \widetilde{\mathcal{M}}$ where $I' = [a, c)$, which satisfies the following four conditions:

(i) t is an affine parameter along γ'

(ii) $\gamma'(t)$ is non-intersecting

(iii) $\gamma'(b) = p$ for some $b \in (a, c)$

(iv) $\gamma'(t) \in \varphi(\mathcal{M}) \quad \forall \ t \in [a, b)$

Now define the curve $\gamma : I \to \mathcal{M}$, where $I = [a, b)$, by $\gamma(t) \equiv (\varphi^{-1} \circ \gamma')(t)$.

Clearly $\gamma \in \mathcal{C}_g \subseteq \mathcal{C}$.

$\gamma(t)$ has the endpoint p.

Thus p is a $\mathcal{C}$-boundary point of $\mathcal{M}$.

QED

If a boundary point of a triple $(\mathcal{M}, g, \mathcal{C})$ is not a $\mathcal{C}$-boundary point, then we will call it a *non-$\mathcal{C}$-boundary point.* Since such points are not limit points of any curves in our chosen family $\mathcal{C}$, they are of no direct interest to us, and will not undergo any further classification. Theorem 7.3 tells us that boundary points of $\mathcal{M}$ which are C^l regular for g must be $\mathcal{C}$-boundary points. Points at infinity are also $\mathcal{C}$-boundary points by definition. The following corollary to Theorem 7.3 establishes the intuitively obvious result that a point at infinity cannot cover a boundary point which is C^l regular for g.

Corollary 7.4 Suppose that $(\mathcal{M}, g, \mathcal{C})$ is a triple. Let p be a $\mathcal{C}$-boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$, and let p' be a $\mathcal{C}$-boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$. If p is C^l regular for g and p' is a point at infinity, then p' does not cover p.

Proof

The proof follows directly on from that of Theorem 7.3.

By Lemma 2.5(i), $\gamma(t)$ has finite parameter range.

If p' were to cover p, then by Theorem 5.2, p' would be a limit point of $\gamma(t)$.

This would mean that p' was a limit point of a curve in $\mathcal{C}$ with finite parameter range, and thus could not be a point at infinity.

QED

Since a point at infinity cannot cover a boundary point which is C^l regular for g, such points cannot be equivalent. In particular, this means that a point at infinity cannot simultaneously be C^l regular for g, and vice versa. From Corollary 5.9, we know that the label of 'a point at infinity' can be successfully transferred to abstract $\mathcal{C}$-boundary points of $\mathcal{M}$. In the light of Corollary 7.4, one wonders whether the same might be possible for the label 'C^l regular for g'. Unfortunately, this is not the case, as the following example demonstrates.

Example

(i) $\widehat{\mathcal{M}} = \mathbb{R}$

$\mathcal{M} = (0,1)$

$$i : \mathcal{M} \to \widehat{\mathcal{M}} \qquad\qquad \varphi : \mathcal{M} \to \widehat{\mathcal{M}}$$

$$x \mapsto x \qquad\qquad x \mapsto y = x^{1/2}$$

$i(\mathcal{M}) = \mathcal{M} = (0,1) \qquad \varphi(\mathcal{M}) = \mathcal{M} = (0,1)$

$x = 0$ is a boundary point of $(\mathcal{M}, \widehat{\mathcal{M}}, i)$

$y = 0$ is a boundary point of $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$

$x = 0$ covers $y = 0$ and $y = 0$ covers $x = 0$

i.e. $x = 0 \sim y = 0$

A C^∞ metric g on $\mathcal{M}$ is given by $ds^2 = dx^2$.

So $(\mathcal{M}, g)$ is a C^∞ Riemannian manifold.

In fact, using the coordinate x, g is a C^∞ metric on the whole of $\widehat{\mathcal{M}}$.

So the C^∞ Riemannian manifold $(\widehat{\mathcal{M}}, g)$ is a C^∞ extension of $(\mathcal{M}, g)$.

Thus $x = 0$ is C^∞ regular for g.

The induced C^∞ metric on $\varphi(\mathcal{M})$ is given by $ds^2 = 4y^2\, dy^2$.

However, $y = 0$ is not C^l regular for g for any $l \geq 1$, since the metric becomes degenerate as $y \to 0^+$.

8 Boundaries and removable singularities

In Definition 3.2, boundary points and boundary sets were defined with respect to a particular envelopment of $\mathcal{M}$, say $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. However, we have not yet given a definition of the entire boundary for such an envelopment, and will proceed to do so now. This boundary should not be confused with the abstract boundary $\mathcal{B}(\mathcal{M})$ of $\mathcal{M}$, and the abstract $\mathcal{C}$-boundary $\mathcal{B}_{\mathcal{C}}(\mathcal{M})$ of $\mathcal{M}$ (see Section 5), both of which are independent of particular envelopments of $\mathcal{M}$.

Definition 8.1 *Let $\mathcal{E}$ denote the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. The* boundary $\mathcal{B}(\mathcal{E})$ *of this envelopment of $\mathcal{M}$ is the topological boundary of $\varphi(\mathcal{M})$ in $\widehat{\mathcal{M}}$. That is, $\mathcal{B}(\mathcal{E})$ is the set consisting of all boundary points of $\mathcal{E}$. Unlike a boundary set B of $\mathcal{E}$, the boundary $\mathcal{B}(\mathcal{E})$ is not always a connected subset of $\widehat{\mathcal{M}}$.*

Our classification of the constituent boundary points of such a boundary is only partially complete. As yet, the only $\mathcal{C}$-boundary points of a triple $(\mathcal{M}, g, \mathcal{C})$ which have received a further classification, are those which are either C^l regular for g, or points at infinity. Such boundary points would normally be considered to be non-singular with respect to the C^k metric g on $\mathcal{M}$ and the chosen family $\mathcal{C}$ of curves. Since non-$\mathcal{C}$-boundary points have no bearing on whether or not the manifold $\mathcal{M}$ is $\mathcal{C}$-complete, it would seem reasonable to also classify them as being non-singular.

Definition 8.2 *Suppose that $(\mathcal{M}, g, \mathcal{C})$ is a triple, and that $\mathcal{E} \equiv (\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is an envelopment of $\mathcal{M}$ by $\widehat{\mathcal{M}}$. We will say that a boundary point p of $\mathcal{E}$ is C^l* non-singular, *if it is either C^l regular for g, a point at infinity, or a non-$\mathcal{C}$-boundary point. Otherwise we will say that it is C^l* singular. *A boundary set B of $\mathcal{E}$ will be said to be C^l* non-singular, *if it consists entirely of C^l non-singular boundary points.*

So by definition, the C^l non-singular boundary points are already further classified into three types. We now embark on a classification of the C^l singular boundary points, which coincidentally, will also be into three types.

Definition 8.3 *Suppose that $(\mathcal{M}, g, \mathcal{C})$ is a triple, and that p is a C^l singular boundary point of the envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ of $\mathcal{M}$. We will say that p is a C^l* removable singularity, *if there exists a C^l non-singular boundary set B' of another envelopment $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$ of $\mathcal{M}$, such that B' covers p. Otherwise, we will say that p is a C^l* essential singularity.

A removable singularity occurs when non-singular boundary points of an envelopment of $\mathcal{M}$ are squashed together in another envelopment. This type of singular point is not really a problem since, as the name suggests, it can be 'removed' by switching to another envelopment of $\mathcal{M}$. On the other hand, essential singularities pose a more serious problem, because their presence usually signifies that the triple $(\mathcal{M}, g, \mathcal{C})$ is, in some sense, inherently singular. They will be discussed further in the following sections.

Examples

(i) In Example (i) of Section 7, the boundary point $y = 0$ of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ is not C^l regular for g, for any $l \geq 1$. Taking the family $\mathcal{C}$ of curves to be $\mathcal{C}_g$, $y = 0$ is certainly a $\mathcal{C}_g$-boundary point of $\mathcal{M}$. However, it is not a point at infinity. It follows that $y = 0$ is a C^l singular boundary point (for all $l \geq 1$). Now the boundary point $x = 0$ of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$ is C^∞ regular for g. Since $y = 0$ is covered by the C^∞ non-singular boundary set $\{x = 0\}$, it is a C^∞ removable singularity.

(ii) The Schwarzschild solution

Let $\widehat{\mathcal{M}}$ be the 4-dimensional manifold $S^2 \times \mathbb{R}^2$. Using coordinates (t, r, θ, φ) on $\widehat{\mathcal{M}}$, let $\mathcal{M}$ be the open submanifold of $\widehat{\mathcal{M}}$ specified by $r > 2m$ (where $m > 0$).

The familiar Schwarzschild metric g on $\mathcal{M}$ is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2).$$

This is a C^∞ Lorentzian metric on $\mathcal{M}$, and so the pair $(\mathcal{M}, g)$ is a C^∞ space-time. We will take the family $\mathcal{C}$ of curves on $\mathcal{M}$ to simply be $\mathcal{C}_g$, so that we have the triple $(\mathcal{M}, g, \mathcal{C}_g)$.

Each point p of $\widehat{\mathcal{M}}$ given by $(t, 2m, \theta, \varphi)$ is clearly a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$. However no such point is C^l regular for g (for any $l \geq 1$), since the metric component g_{rr} becomes infinite as $r \to 2m^+$. A particular boundary point $(t_0, 2m, \theta_0, \varphi_0)$ is the endpoint of spacelike geodesics on $\mathcal{M}$ given by $t = t_0$, $\theta = \theta_0$ and $\varphi = \varphi_0$. Such geodesics reach the boundary point $(t_0, 2m, \theta_0, \varphi_0)$ in finite proper distance. We conclude from this that every point $(t, 2m, \theta, \varphi)$ is a C^l singular boundary point of $\mathcal{M}$ (for all $l \geq 1$).

Now Kruskal [6] re-embedded $\mathcal{M}$ in $\widehat{\mathcal{M}}$ in the following manner. Using coordinates $(t', x', \theta', \varphi')$ on $\widehat{\mathcal{M}}$, the C^∞ embedding ψ which he used is given by

$$(t, r, \theta, \varphi) \mapsto \big(\, t' = \tfrac{1}{2}\sqrt{r - 2m}\; e^{r/4m}\left(e^{t/4m} - e^{-t/4m}\right),$$
$$x' = \tfrac{1}{2}\sqrt{r - 2m}\; e^{r/4m}\left(e^{t/4m} + e^{-t/4m}\right),\ \ \theta' = \theta\,,\ \ \varphi' = \varphi\,\big).$$

$\psi(\mathcal{M}) = \{\, (t', x', \theta', \varphi') \in \widehat{\mathcal{M}} : x' > |t'|\,\}$. This set is labelled as region I in Figure 8.1. The induced C^∞ metric on $\psi(\mathcal{M})$ is as follows:

$$ds^2 = F^2(t', x')\,(-\,dt'^2 + dx'^2) + r^2(t', x')\,(d\theta'^2 + \sin^2\theta' \, d\varphi'^2)$$

where r is determined implicitly by the equation

$$(t')^2 - (x')^2 = -\,(r - 2m)\, e^{r/2m},$$

and F is given by $\quad F^2 = 16\, m^2/r \; e^{-r/2m}\,.$

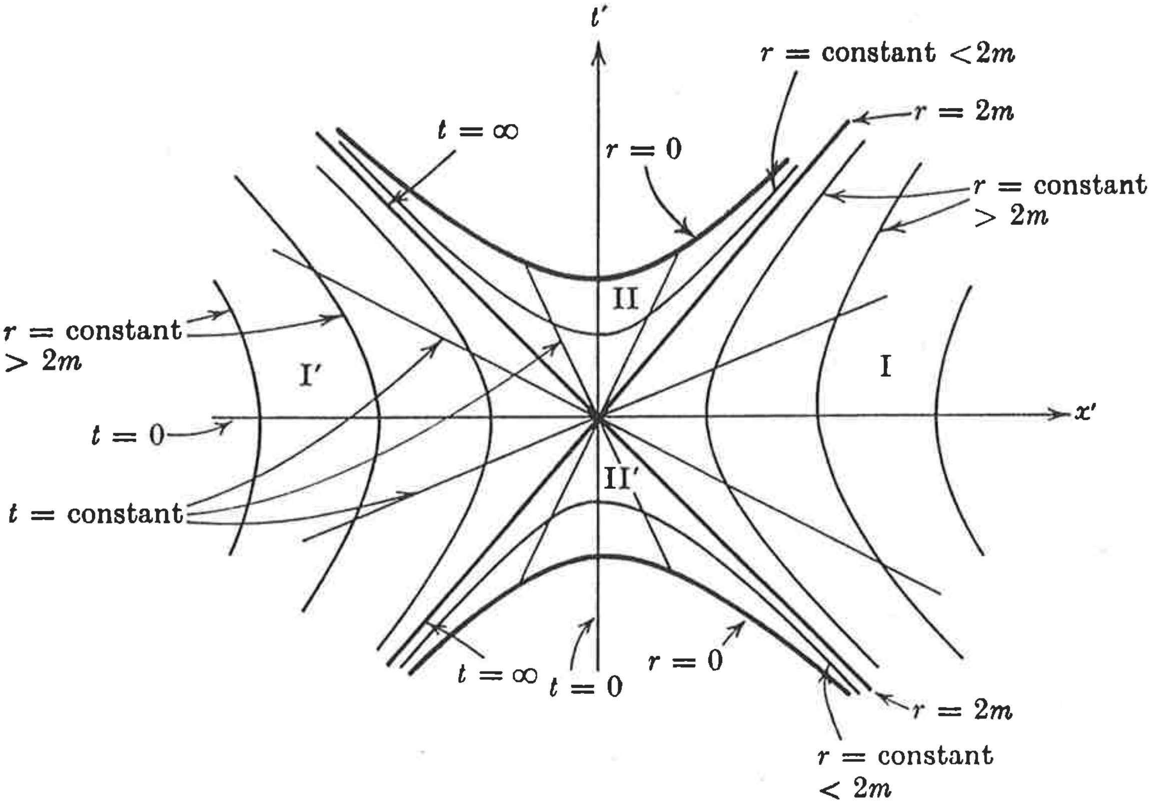

Figure 8.1 The Kruskal re-embedding ψ of the Schwarzschild solution in $S^2 \times \mathbb{R}^2$. Coordinates $(t', x', \theta', \varphi')$ are used, but the angular coordinates θ' and φ' are suppressed. $\psi(\mathcal{M})$ is region I, and $\widetilde{\mathcal{M}}$ is regions I, II, I' and II'.

Now let $\widetilde{\mathcal{M}}$ denote the open submanifold of $\widehat{\mathcal{M}}$ consisting of all points $(t', x', \theta', \varphi')$ such that $(t')^2 - (x')^2 < 2m$. $\widetilde{\mathcal{M}}$ is represented in Figure 8.1 by regions I, II, I' and II'. It can be shown that the functions $F^2(t', x')$ and $r^2(t', x')$ on $\psi(\mathcal{M})$ can be extended analytically to all of $\widetilde{\mathcal{M}}$, and are everywhere positive on $\widetilde{\mathcal{M}}$. So the induced C^∞ metric on $\psi(\mathcal{M})$ given above, can be extended to a C^∞ metric $\tilde{g}$ on $\widetilde{\mathcal{M}}$. In other words, $(\widetilde{\mathcal{M}}, \tilde{g})$ is a C^∞ extension of $(\mathcal{M}, g)$.

The points $(t' = 0, x' = 0, \theta', \varphi')$ in $\widehat{\mathcal{M}} - \psi(\mathcal{M})$ are boundary points of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \psi)$. Each such point $(0, 0, \theta_0, \varphi_0)$ is C^∞ regular for g, and covers the boundary points $(t, r = 2m, \theta_0, \varphi_0)$ of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$. This means that the C^l singular boundary points $(t, 2m, \theta, \varphi)$ of $(\mathcal{M}, \widehat{\mathcal{M}}, i)$ are, in fact, C^∞ removable singularities.

Theorem 8.4 Suppose that we have a triple $(\mathcal{M}, g, \mathcal{C})$, and that p' is a boundary point of the enveloped manifold $\mathcal{E}' \equiv (\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$. If p' is a C^l removable singularity, then every C^l non-singular boundary set B which covers p' contains at least one boundary point which is C^l regular for g.

Proof

If p' is a C^l removable singularity, then it is a C^l singular boundary point of $\mathcal{E}'$.

So p' is a $\mathcal{C}$-boundary point of $\mathcal{M}$ which is not a point at infinity.

This means that it is a limit point of some curve $\gamma(t)$ in $\mathcal{C}$ given by $\gamma : I \to \mathcal{M}$, where $I = [a, b)$, and b is finite.

So in the half-open interval I, there exists an increasing sequence of numbers $t_i \to b^-$ such that $(\varphi' \circ \gamma)(t_i) \to p'$.

Since p' is a C^l removable singularity, there exists a C^l non-singular boundary set B of an enveloped manifold $\mathcal{E}' \equiv (\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$, such that B covers p'.

Now $(\varphi \circ \gamma)(t_i) \in \varphi(\mathcal{M})$, $\forall\ i \in \mathbb{N}$.

Let $A \equiv \{ (\varphi \circ \gamma)(t_i) : i \in \mathbb{N} \} \subseteq \varphi(\mathcal{M})$.

So $A \cap B = \emptyset$.

Define $U \equiv \widehat{\mathcal{M}} - \bar{A}$.

Since $\bar{A}$ is closed, U is open.

Suppose that $\bar{A} \cap B = \emptyset$.

It follows that $B \subseteq U$.

B covers p', so there exists an open neighbourhood U' of p' in $\widehat{\mathcal{M}}'$ such that

$$\varphi \circ \varphi'^{-1} \left(U' \cap \varphi'(\mathcal{M}) \right) \subseteq U.$$

Since the sequence $(\varphi' \circ \gamma)(t_i) \to p'$,

$\exists\ n \in \mathbb{N}$ s.t. $\forall\ i > n$, $(\varphi' \circ \gamma)(t_i) \in U' \cap \varphi'(\mathcal{M})$

$\Rightarrow\ \forall\ i > n$, $(\varphi \circ \gamma)(t_i) \in U$

$\Rightarrow\ A \cap U \neq \emptyset$.

But this is a contradiction, since $U = \widehat{\mathcal{M}} - \bar{A}$.

It follows that $\bar{A} \cap B \neq \emptyset$.

Let $p \in \bar{A} \cap B$.

Since $A \cap B = \emptyset$, $p \in \bar{A} - A$.

This implies that there exists an increasing infinite subsequence $\{t_{i_k} : k \in \mathbb{N}\}$ of the sequence $\{t_i : i \in \mathbb{N}\}$ such that $(\varphi \circ \gamma)(t_{i_k}) \to p$.

Thus p is a limit point of the curve $\gamma(t)$.

Since p is a C^l non-singular boundary point of $\mathcal{E}$, and is a limit point of a curve in $\mathcal{C}$ with finite parameter range, it must be C^l regular for g.

QED

9 Directional and pure singularities

Definition 9.1 *Suppose that $(\mathcal{M}, g, \mathcal{C})$ is a triple, and that p is a boundary point of the enveloped manifold $\mathcal{E} \equiv (\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$. We will say that p is a C^l* directional singularity, *if it is a C^l essential singularity which covers either a point at infinity, or a boundary point which is C^l regular for g. We will say that p is a C^l* pure singularity, *if it is a C^l essential singularity which is not a C^l directional singularity.*

So a directional singularity is a singular boundary point which covers either a point at infinity or a regular boundary point, and yet is not itself covered by any non-singular boundary set. This means that it might, for instance, cover two non-intersecting, non-singular boundary sets of a particular envelopment of $\mathcal{M}$. It might also be equivalent to a boundary set of another envelopment of $\mathcal{M}$ which contains pure singularities as well as regular boundary points and/or points at infinity. There are quite a few possibilities.

On the other hand, pure singularities have been stripped clean of any connection with regular boundary points and points at infinity, since they neither cover them, nor are covered by them. Their presence indicates that a triple $(\mathcal{M}, g, \mathcal{C})$ is inherently singular—a notion which will be made precise in the following section.

Examples

(i) A directional singularity

$$\widehat{\mathcal{M}} = \mathbb{R} \qquad\qquad i : \mathcal{M} \to \widehat{\mathcal{M}}$$

$$\mathcal{M} = (0, 2\pi) \qquad\qquad x \mapsto x$$

$$i(\mathcal{M}) = \mathcal{M} = (0, 2\pi)$$

$x = 0$ and $x = 2\pi$ are the only boundary points of $(\mathcal{M}, \widehat{\mathcal{M}}, i)$.

$\widehat{\mathcal{M}}' = S^1$

Let (r, θ) be a standard polar coordinate patch on the manifold $\mathbb{R}^2$, where $r > 0$ and $0 < \theta < 2\pi$.

Let (r, θ') be another polar coordinate patch on $\mathbb{R}^2$, where $r > 0$, $0 < \theta' < 2\pi$ and θ' starts from $\theta = \pi$ and increases in the same direction as θ.

$\widehat{\mathcal{M}}'$ will be represented as the C^∞ submanifold of $\mathbb{R}^2$ given by

$\{(1, \theta) : 0 < \theta < 2\pi\} \cup \{(1, \theta') : 0 < \theta' < 2\pi\}$.

Now define a C^∞ embedding $\varphi' : \mathcal{M} \to \widehat{\mathcal{M}}'$ as follows:

$$
\begin{array}{lll}
\text{for} & \pi/4 < x < 7\pi/4 & x \mapsto (1\,,\, \theta = x)\,, \\
\text{for} & 0 < x < \pi/2 & x \mapsto (1\,,\, \theta' = x + \pi)\,, \\
\text{and for} & 3\pi/2 < x < 2\pi & x \mapsto (1\,,\, \theta' = x - \pi)\,.
\end{array}
$$

The point $(1, \theta' = \pi)$ in $\widehat{\mathcal{M}}'$ is the only boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$.

$(1, \pi)$ covers the boundary point $x = 0$ and the boundary point $x = 2\pi$ of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$.

A C^∞ metric g on $\mathcal{M}$ is given by $ds^2 = \frac{1}{16}\,(1 + \tan^2 x/4)^2\, dx^2$.

So $(\mathcal{M}, g)$ is a C^∞ Riemannian manifold.

In fact, g is a C^∞ metric on the open submanifold $(-2\pi, 2\pi)$ of $\widehat{\mathcal{M}}$.

Thus the boundary point $x = 0$ is C^∞ regular for g.

Now suppose that $\mathcal{C} \equiv \mathcal{C}_g$.

It can be shown that all geodesics on $\mathcal{M}$ which approach the boundary point $x = 2\pi$ have infinite affine parameter range.

Thus $x = 2\pi$ is a point at infinity.

The induced C^∞ metric on $\varphi'(\mathcal{M})$ is as follows:

$$\text{for } \pi/4 < \theta < 7\pi/4 \qquad ds^2 = \tfrac{1}{16}(1+\tan^2\theta/4)^2\, d\theta^2$$

$$\text{for } \pi < \theta' < 3\pi/2 \qquad ds^2 = \tfrac{1}{16}(1+\tan^2(\theta'-\pi)/4)^2\, d\theta'^2$$

$$\text{and for } \pi/2 < \theta' < \pi \qquad ds^2 = \tfrac{1}{16}(1+\tan^2(\theta'+\pi)/4)^2\, d\theta'^2\,.$$

It may be seen that as $\theta' \to \pi^+$, $g_{\theta'\theta'} \to \frac{1}{16}$.

However, as $\theta' \to \pi^-$, $g_{\theta'\theta'} \to +\infty$.

So the boundary point $(1,\pi)$ of $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$ is certainly not C^l regular for g, for any $l \geq 1$.

Since it covers the boundary point $x = 0$ which is C^∞ regular for g, $(1,\pi)$ is a $\mathcal{C}_g$-boundary point which is not a point at infinity.

It is thus a C^l singular boundary point of $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$.

Now $(1,\pi)$ is not a C^l removable singularity, since it is not covered by a C^l non-singular boundary set of any other envelopment of $\mathcal{M}$.

It follows that the boundary point $(1,\pi)$ is a C^l directional singularity which covers both a point at infinity, and a boundary point which is C^∞ regular for g.

(ii) The Curzon space-time

(see Example (i) of Section 6)

We recall that the boundary set B of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, i)$ consists of points p which, in Euclidean coordinates (t,x,y,z), are of the form $(t,0,0,0)$. It may be seen from the Curzon metric component $g_{tt} = -e^{-2m/R}$, where $R = \sqrt{x^2+y^2+z^2}$, that for any curve on $\mathcal{M}$ (i.e. an element of $\mathcal{F}$) which has a point p as its endpoint, $g_{tt} \to 0^-$ as $R \to 0^+$ along the curve. So p is not C^l regular for g, for any $l \geq 1$.

It was shown in [3] that each p is the endpoint of timelike, null and spacelike geodesics on $\mathcal{M}$ with finite affine parameter range. This means that p is a C^l singular boundary point of $(\mathcal{M}, \widehat{\mathcal{M}}, i)$. It was also shown that the limit of the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ along any such geodesic is infinite. This implies that, for $l \geq 2$, p cannot be covered by any C^l non-singular boundary set of another envelopment of $\mathcal{M}$. That is, p is a C^l essential singularity for $l \geq 2$.

Now referring back to Section 6, we recall that every boundary point p' belonging to the boundary set B' of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \psi)$ is a point at infinity. In fact every such boundary point is covered by precisely one boundary point p in the boundary set B of $(\mathcal{M}, \widehat{\mathcal{M}}, i)$. In particular, the boundary points $(\tan^{-1} t_0/m,\ 0 < r \leq \pi/2,\ z = -\pi/2,\ \varphi)$, where t_0 is a constant, and the boundary point $(\tan^{-1} t_0/m,\ x = 0,\ y = 0,\ z = -\pi/2)$ belonging to B', are all covered by the boundary point $(t_0,\ x = 0,\ y = 0,\ z = 0)$ belonging to B. It follows that every p in B is a C^l directional singularity, for all $l \geq 2$. Finally, it can be shown that each p is equivalent to a boundary set of $(\mathcal{M}, \widehat{\mathcal{M}}, \psi)$ which consists entirely of points at infinity, non-$\mathcal{C}_g$-boundary points $(\tan^{-1} t_0/m,\ r = \pi/2,\ -\pi/2 < z < 0,\ \varphi)$, and C^l $(l \geq 2)$ pure singularities $(\tan^{-1} t_0/m,\ r = \pi/2,\ z = 0,\ \varphi)$.

(iii) The Schwarzschild solution

We recall from Example (ii) of Section 8 that $(\widetilde{\mathcal{M}}, \tilde{g})$ is a C^∞ extension of the Schwarzschild space-time $(\mathcal{M}, g)$. Now consider the points $(t', x', \theta', \varphi')$ of $\widehat{\mathcal{M}}$ which satisfy the relation $(t')^2 - (x')^2 = 2m$, where $t' > 0$. The set B of all such points is clearly a boundary set of the enveloped manifold $(\widetilde{\mathcal{M}}, \widehat{\mathcal{M}}, i)$. Now for any curve on $\widetilde{\mathcal{M}}$ (i.e. an element of $\mathcal{F}$) which has a point $p \in B$ as its endpoint,

$\widetilde{g}_{t't'} \to -\infty$ and $\widetilde{g}_{x'x'} \to +\infty$ as p is approached along the curve. So p is not C^l regular for g, for any $l \geq 1$. We will take the family $\mathcal{C}$ of curves on $\widetilde{\mathcal{M}}$ to simply be $\mathcal{C}_g$, so that we have the triple $(\widetilde{\mathcal{M}}, \widetilde{g}, \mathcal{C}_g)$.

It is well known that each $p \in B$ is the endpoint of timelike and null geodesics on $\widetilde{\mathcal{M}}$ with finite affine parameter range, and thus is not a point at infinity. So p is a C^l singular boundary point of $\widetilde{\mathcal{M}}$ (for all $l \geq 1$). Furthermore, these are the only geodesics on $\widetilde{\mathcal{M}}$ for which p is a limit point, which means that p itself does not cover any boundary points which are points at infinity. It is also well known that the limit of the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ along every geodesic on $\widetilde{\mathcal{M}}$ with p as its endpoint is infinite. This implies that, for $l \geq 2$, p cannot be covered by any C^l non-singular boundary set of another envelopment of $\widetilde{\mathcal{M}}$. It also implies that p itself does not cover any boundary point which is C^l regular for $\widetilde{g}$, where $l \geq 2$. It follows that each boundary point $p \in B$ is a C^l pure singularity, for all $l \geq 2$.

The classification of boundary points of a triple $(\mathcal{M}, g, \mathcal{C})$ is now complete, with each boundary point belonging to precisely one of six final categories. The process of determining the relevant category for any particular boundary point is illustrated in Figure 9.1, which also incorporates the intermediate classifications that have been used. On the other hand, we have not yet completed our classification of abstract $\mathcal{C}$-boundary points of $(\mathcal{M}, g, \mathcal{C})$. Corollary 5.9 established that the label 'point at infinity' can be used for abstract $\mathcal{C}$-boundary points. However, Example (i) of Section 7 and Example (i) of Section 8 demonstrate that it is not possible, in general, to use either the label 'C^l regular for g' or the label 'C^l removable singularity' for these points. The following theorem provides two further labels which can be used.

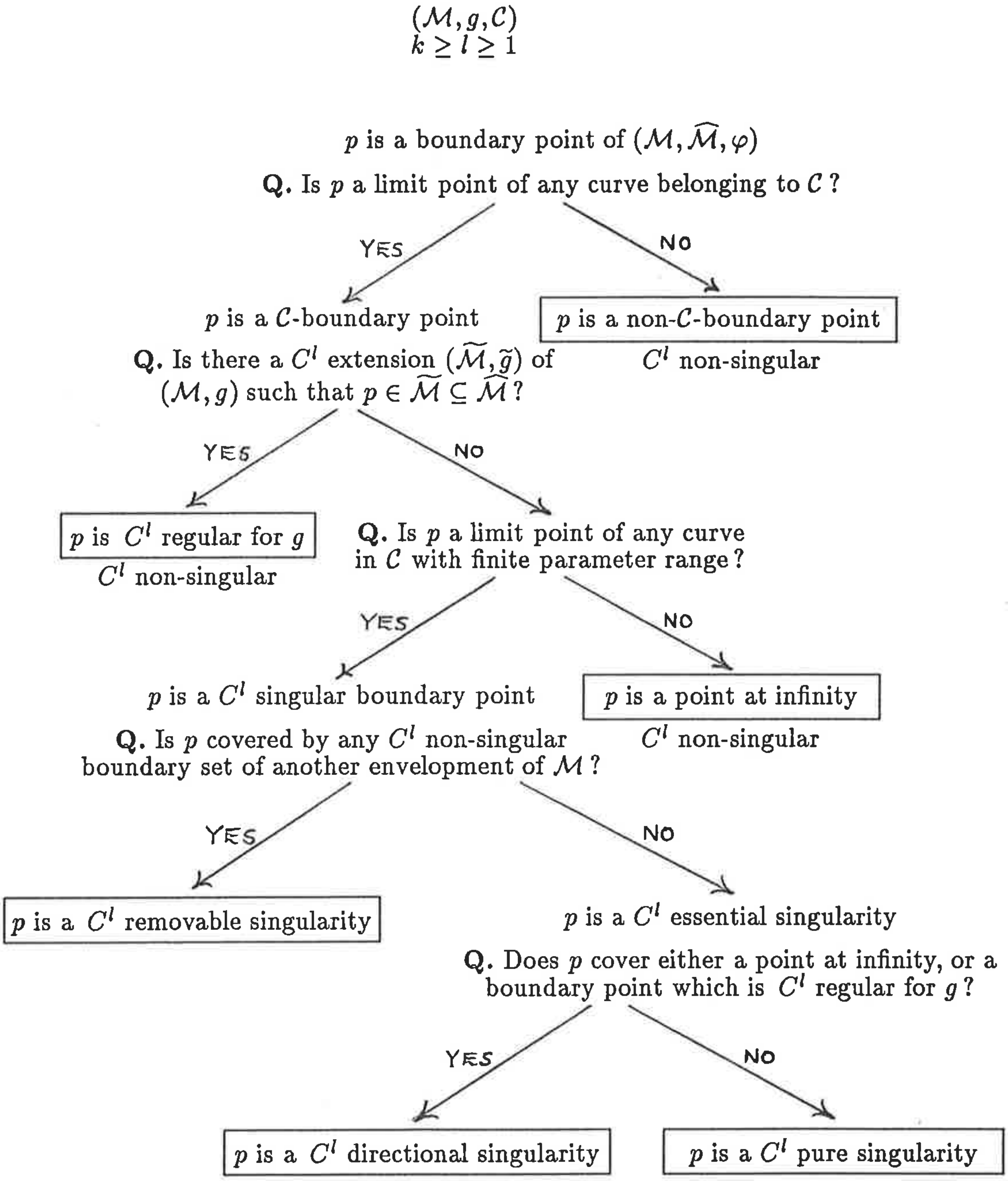

Figure 9.1 Given a triple $(\mathcal{M}, g, \mathcal{C})$ and an $l \in \mathbb{N}$, where $1 \leq l \leq k$, this is a classification of a boundary point p of the envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ of $\mathcal{M}$ by $\widehat{\mathcal{M}}$. The six possible final categories appear in boxes.

Theorem 9.2 Suppose that $(\mathcal{M}, g, \mathcal{C})$ is a triple. Let p be a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$, and let p' be a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$. If $p \sim p'$ and p is a C^l directional singularity, then p' is also a C^l directional singularity. If p is a C^l pure singularity, then p' is also a C^l pure singularity.

Proof

Let p be a C^l essential singularity

$\Rightarrow$ p is a $\mathcal{C}$-boundary point of $\mathcal{M}$.

Since $p \sim p'$, by Corollary 5.7, p' is also a $\mathcal{C}$-boundary point of $\mathcal{M}$.

Since p is not a point at infinity, by Corollary 5.9, p' is not a point at infinity either.

Suppose that p' is C^l regular for g

$\Rightarrow$ p is covered by the C^l non-singular boundary set $\{p'\}$.

But p is not a C^l removable singularity.

It follows that p' is a C^l singular boundary point.

Similarly, p' cannot be a C^l removable singularity, since this would again imply that p is a C^l removable singularity.

Thus p' is also a C^l essential singularity.

Now if p covers either a point at infinity, or a boundary point which is C^l regular for g, then so does p', and vice versa.

This means that p and p' are either both C^l directional singularities, or are both C^l pure singularities.

QED

Consider an abstract $\mathcal{C}$-boundary point $[p]$ of a triple $(\mathcal{M}, g, \mathcal{C})$. If p is a point at infinity, then $[p]$ is a point at infinity. If p is a C^l directional singularity, then by Theorem 9.2, we are entitled to call $[p]$ a C^l *directional singularity.* Similarly, if p is a C^l pure singularity, we can also call $[p]$ a C^l *pure singularity.* The only remaining possibilities are that p is either C^l regular for g, or is a C^l removable singularity.

Now we have seen in Example (i) of Section 7 and Example (i) of Section 8, that it is possible for a boundary point which is C^l regular for g to be equivalent to a C^l removable singularity. So we will say that $[p]$ is C^l *regular for* g, if at least one member of that particular equivalence class of $\mathcal{C}$-boundary points is C^l regular for g. Otherwise, if all members of the equivalence class are C^l removable singularities, then $[p]$ will also be called a C^l *removable singularity.* This completes our classification of abstract $\mathcal{C}$-boundary points of a triple $(\mathcal{M}, g, \mathcal{C})$ into five final categories. The process of determining the relevant category for any particular abstract boundary point is illustrated in Figure 9.2.

10 Singular pseudo-Riemannian manifolds

The underlying philosophy of this paper, is that in order to say when a particular triple $(\mathcal{M}, g, \mathcal{C})$ is singular, one needs to consider *all* possible envelopments of $\mathcal{M}$ by other n-dimensional manifolds $\widehat{\mathcal{M}}$. It is not sufficient to base our assessment on one given envelopment of $\mathcal{M}$, since we have seen that a singular boundary point of such an envelopment may be covered by a non-singular boundary set of another envelopment of $\mathcal{M}$, or may itself cover regular boundary points or points at infinity. Only a pure singularity cannot be 'removed' or 'separated' by switching to other envelopments of $\mathcal{M}$, and thus signifies that the triple is inherently singular.

Definition 10.1 *A triple* $(\mathcal{M}, g, \mathcal{C})$ *will be said to be* C^l singular (*where* $1 \leq l \leq k$), *iff the abstract* $\mathcal{C}$*-boundary* $\mathcal{B}_{\mathcal{C}}(\mathcal{M})$ *of* $\mathcal{M}$ *contains a* C^l *pure singularity. Otherwise it will be said to be* C^l non-singular, *or* C^l singularity-free.

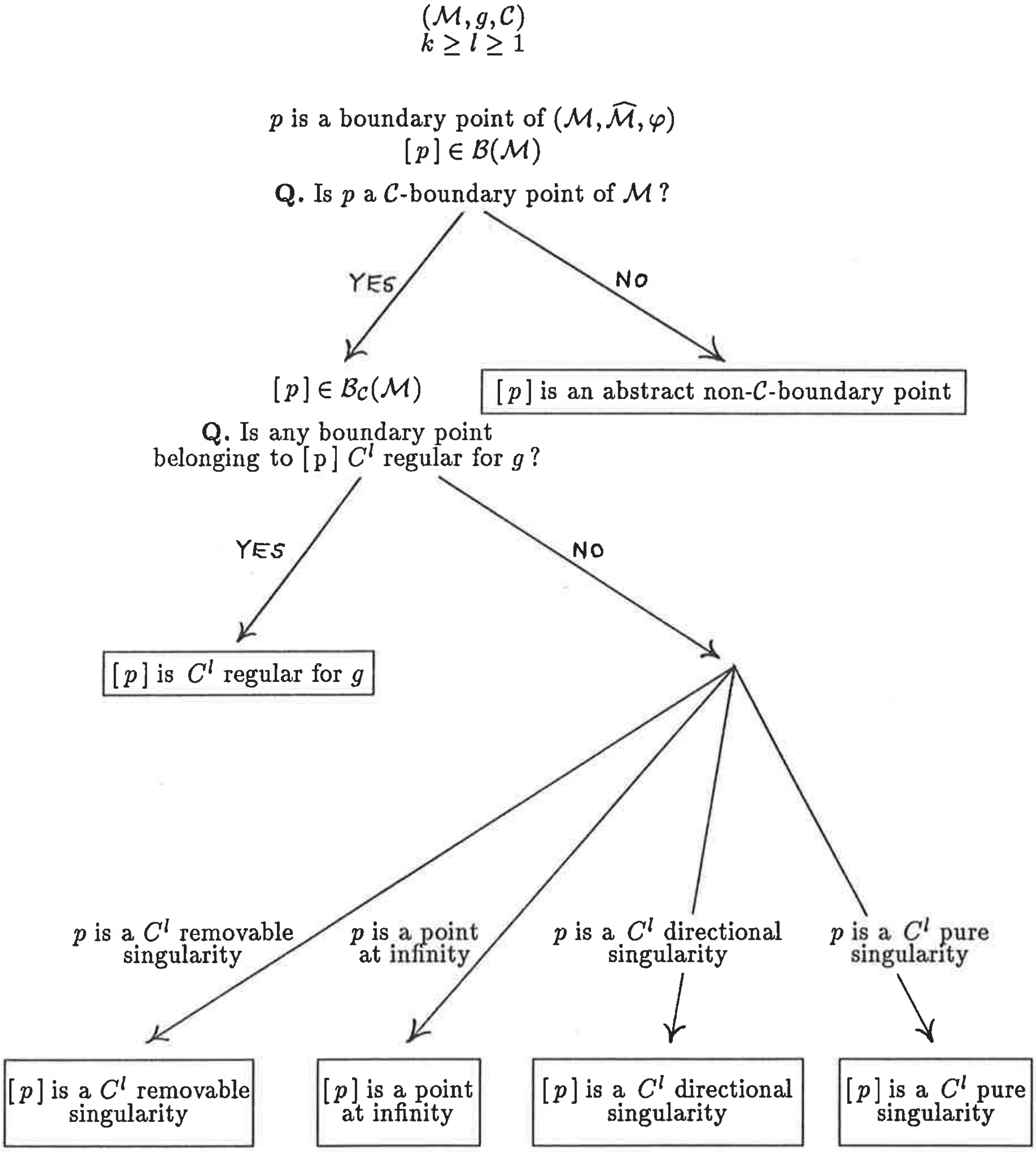

Figure 9.2 Given a triple $(\mathcal{M}, g, \mathcal{C})$ and an $l \in \mathbb{N}$, where $1 \leq l \leq k$, this is a classification of an abstract boundary point $[p]$ belonging to $\mathcal{B}(\mathcal{M})$. The six possible final categories appear in boxes.

There are two elements of choice in this whole scheme. If we begin with simply a C^k pseudo-Riemannian manifold $(\mathcal{M}, g)$, we must first choose the family $\mathcal{C}$ of curves on $\mathcal{M}$ that will be used. From Section 6 onwards it has been assumed that $\mathcal{C}_g$ is always a subset of $\mathcal{C}$. Just what other curves are also included in $\mathcal{C}$ depends entirely on the particular interests of the user, and the purpose that they have in mind. Varying the choice of $\mathcal{C}$ may change the status of a triple $(\mathcal{M}, g, \mathcal{C})$ from non-singular to singular, and vice versa. We are also free to choose an l between 1 and k. Clearly, cases will arise where a triple $(\mathcal{M}, g, \mathcal{C})$ is both C^l singular and $C^{l'}$ non-singular, where $1 \leq l' < l \leq k$. So the choice of l is an important factor.

Theorem 10.2 If a triple $(\mathcal{M}, g, \mathcal{C})$ is $\mathcal{C}$-complete, then no boundary point of any envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ of $\mathcal{M}$ is either C^l regular for g or C^l singular $(1 \leq l \leq k)$.

Proof

Let p be a boundary point of the enveloped manifold $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$.

Now suppose that p is either C^l regular for g or C^l singular.

It is thus a limit point of a curve $\gamma(t)$ in $\mathcal{C}$ with finite parameter range.

But since $(\mathcal{M}, g, \mathcal{C})$ is $\mathcal{C}$-complete, $\gamma(t)$ has no limit points in $\widehat{\mathcal{M}} - \varphi(\mathcal{M})$.

This is a contradiction.

It follows that $(\mathcal{M}, g, \mathcal{C})$ has no boundary points which are either C^l regular for g or C^l singular.

QED

Corollary 10.3 A triple $(\mathcal{M}, g, \mathcal{C})$ which is $\mathcal{C}$-complete, is C^l singularity-free.

The reverse of Corollary 10.3 is not true in general. If a triple $(\mathcal{M}, g, \mathcal{C})$ is $\mathcal{C}$-complete, then any boundary point of an envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \varphi)$ of $\mathcal{M}$, must either be a point at infinity or a non-$\mathcal{C}$-boundary point. However, a boundary point of a triple $(\mathcal{M}, g, \mathcal{C})$ which is C^l singularity-free may, in addition, be a C^l removable singularity, a C^l directional singularity or C^l regular for g.

Example

(i) $\widehat{\mathcal{M}} = \mathbb{R}^4$

(t, x, y, z) are Euclidean coordinates on $\widehat{\mathcal{M}}$.

A C^∞ metric η on $\widehat{\mathcal{M}}$ is given by $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$.

$(\widehat{\mathcal{M}}, \eta)$ is Minkowski space-time.

Let $\mathcal{C} \equiv \mathcal{C}_g$, so that we have the triple $(\widehat{\mathcal{M}}, \eta, \mathcal{C}_g)$.

Let $\mathcal{M} = \{ p \in \widehat{\mathcal{M}} : -1 < t, x, y, z < 1\}$.

$(\mathcal{M}, \eta|_{\mathcal{M}}, \mathcal{C}_g|_{\mathcal{M}})$ is a triple.

$\mathcal{E} \equiv (\mathcal{M}, \widehat{\mathcal{M}}, i)$.

Every boundary point p in the boundary $\mathcal{B}(\mathcal{E})$ of $\mathcal{E}$ is clearly C^∞ regular for $\eta|_{\mathcal{M}}$.

$\mathcal{B}(\mathcal{E})$ is also a boundary set of $\mathcal{E}$ in this particular case.

If p' is a boundary point of another envelopment $(\mathcal{M}, \widehat{\mathcal{M}}', \varphi')$ of $\mathcal{M}$, then it is covered by the C^∞ non-singular boundary set $\mathcal{B}(\mathcal{E})$.

Thus it cannot be a C^l pure singularity $(1 \leq l \leq \infty)$.

It follows that the triple $(\mathcal{M}, \eta|_{\mathcal{M}}, \mathcal{C}_g|_{\mathcal{M}})$ is C^∞ singularity-free.

However it is not $\mathcal{C}_g|_{\mathcal{M}}$-complete, since $\mathcal{B}(\mathcal{E})$ consists entirely of boundary points which are C^∞ regular for $\eta|_{\mathcal{M}}$.

In summary then, a triple $(\mathcal{M}, g, \mathcal{C})$ which is $\mathcal{C}_g$-complete is not always geodesically complete e.g. see Example (ii) of Section 6. However, a triple which is geodesically complete, is also $\mathcal{C}_g$-complete. If the triple $(\mathcal{M}, g, \mathcal{C})$ is singularity-free, it is not necessarily $\mathcal{C}$-complete, but the reverse is always true (see Corollary 10.3). A significant advantage of Definition 10.1 over past definitions, is that a triple $(\mathcal{M}, g, \mathcal{C})$ does not have to be maximally extended in order to be singularity-free.

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