ON n-COVERS OF PG(3,q) AND RELATED STRUCTURES

by

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Thesis submitted for the degree of

Doctor of Philosophy

at the University of Adelaide

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November, 1991

M'illumina d'immenso.

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INTRODUCTION

In the realm of finite projective geometry, the notion of a spread has proven to be of fundamental importance and has been the driving force behind a sizeable amount of the research carried out in this field, especially during the second half of this century. Much of the interest in spreads has arisen from the pioneering work of André in [2] and also of Bruck and Bose in [19] and [20] who demonstrated the usefulness of spreads in the construction of finite translation planes.

The simplest examples of spreads are those of the finite three dimensional projective space $PG(3,q)$. In this case a spread consists of q^2+1 pairwise skew lines. The principal topic of this thesis concerns generalisations of the spreads of $PG(3,q)$; these generalisations, referred to by Ebert in [52] and [53] as n-covers of $PG(3,q)$, are sets of lines which satisfy the property that each point of $PG(3,q)$ is incident with exactly n of these lines. Sets of lines of $PG(3,q)$ satisfying this condition have also been referred to as n-fold spreads of 1-spaces (see [71]). We note before proceeding, that in [14], Beutelspacher also speaks of n-covers of $PG(t,q)$; his definition, however, is used in quite a different context and so is not relevant to what we discuss here.

Ebert's motivation for studying n-covers of $PG(3,q)$ (see [53]) arose from considering the problem of completing partial packings of $PG(3,q)$ to packings. Given a partial packing of $PG(3,q)$, the complement of it in $PG(3,q)$ is an n-cover for some integer n. The problem of completing the partial packing then becomes one of partitioning the n-cover into n spreads. Ebert therefore defined a proper n-cover of $PG(3,q)$ to be one which cannot be so partitioned.

Our motivation is slightly different. We are more interested in studying n-covers for their own sake, that is to say our main aim in this thesis is to develop a theory of n-covers which parallels that of the spreads of $PG(3,q)$. As a part of this theory involves examining the designs which result from the n-covers, we have found it necessary to alter the definition of a proper n-cover. In this thesis, a proper n-cover of $PG(3, q)$ is one which is not the disjoint union of two n_i-covers with $n_1 + n_2 = n$. The reason for this alteration will become evident when we consider the decomposability of the aforementioned designs.

Regarding the existence of n-covers of $PG(3, q)$, they exist for all admissible values of n, that is for all n from 1 to $q^2 + q + 1$. This is an immediate consequence of the existence of packings in $PG(3, q)$, (see, for example [13] by Beutelspacher and [46] and [47] by Denniston), as any subset of n spreads in a packing constitutes an n-cover. The first infinite class of proper n-covers was discovered by Ebert and appears in [53]; the n-covers in this class are all 2-covers and exist for every odd prime power q. His construction relies on a partition of the point-set of $PG(3, q)$ by ovoids which he established in [52]. The 2-covers then result by exploiting the manner in which certain lines of $PG(3, q)$ intersect the ovoids. He also provided examples of proper 2-covers for $q = 2$, 4 and 8. Since these are constructed by means similar to those used in the case that q is odd, there is good reason for believing that they too lie in an infinite class, but this is yet to be resolved.

The chief goal of Chapter II of this thesis is to describe and explore possible alternative techniques for constructing proper n-covers of $PG(3, q)$. These mainly involve beginning with a known $(q+1)$-cover of $PG(3, q)$, from which we remove maximal sets of pairwise disjoint spreads. This either yields a partition of the $(q + 1)$-cover into spreads or it produces at least one proper n-cover with $2 \leq n \leq q+1$. The three types of $(q + 1)$-covers we consider are the set of lines of a general linear complex, the set of lines in the quadratic complex consisting of the tangent lines to an elliptic quadric and the set of lines in a long Singer orbit. Using these sets, we obtain explicit examples of proper n-covers for some small values of q. Our main result, based on a theorem of

Thas on generalised quadrangles (see Theorem 3.4.1 of [104]), is that proper n-covers exist in $PG(3,q)$, q even for some n satisfying $2 \leq n \leq q$.

In studying n-covers of $PG(3,q)$ the major obstacle to overcome is that of actually determining whether or not a given n-cover is proper. A very basic sufficient condition for an n-cover not to be proper is that it contain a spread, this condition also being necessary for n equal to 2 and 3. The question of the existence of spreads in the quadratic complex (described in the previous paragraph) and in a long Singer orbit, is indirectly addressed in part by the work of Ebert in [53] and [54]. For general linear complexes, much more is known.

The only existing examples of spreads lying in a general linear complex (also called symplectic spreads) are the following: The regular spreads for all values of q (see, for example, [51]), the Lüneburg spreads for $q = 2^{2h+1}$, $h \geq 1$ (see [90] and [91]) and finally, for q odd and non-prime, a class of spreads constructed by Kantor in [82] which gives rise to a family of Knuth semifield planes. As a supplementary fact, we prove that a subregular spread which is not also regular, is not symplectic for any value of q.

In [61], Glynn showed that two regular spreads lying in a general linear complex of $PG(3,q)$, q even intersect in either a single line or in the $q+1$ lines of a regulus. By using a different argument (which is independent of the parity of q), we prove that the result also holds for q odd. For the case that q is even, stronger results have been obtained by Bagchi and Sastry in [3]. In particular, by dealing with an equivalent problem, they implicitly proved that any spread lying in a general linear complex L_q of $PG(3,q)$, q even meets every regular spread and every Lüneburg spread lying in L_q, in at least one line. They have also implicitly classified the intersection patterns of the regular and Lüneburg spreads lying in a common general linear complex (see [4]).

By applying these techniques and results, we construct four explicit examples of proper n-covers. The first two are a proper 2-cover and a proper 3-cover of $PG(3,2)$.

The 2-cover is isomorphic to the one constructed in [53]. The set of lines of the 3-cover has appeared before in print (see [71], p.84) as a 3-fold spread of 1-spaces but it is not mentioned that it does not contain a spread of $PG(3,2)$. Thus, in this sense the proper 3-cover is new. The other two n-covers we construct are both proper 3-covers of $PG(3,3)$. The first one is a symplectic 3-cover; the second 3-cover, which is projectively distinct from the first one, is complemented by a 10-cover of $PG(3,3)$ which consists of the union of ten pairwise disjoint spreads. This shows that if d is the largest natural number for which a partial packing of $PG(3,3)$ with $13-d$ spreads is guaranteed to be completable to a packing, then d is at most two.

Central to the study of spreads is the task of classifying the projectively distinct spreads of $PG(2t+1,q)$, $t \geq 1$. Complete classifications of the spreads of $PG(3,q)$ have been achieved for $q = 2,3,4$ and more recently for $q = 5$ (see [72] and [44] for $q = 2,3$, [86] for $q = 4$ and [6], [97] and [124] for $q = 5$). However, the case for $q = 7$ is far from being completed; it is reported in [124] that the projectively distinct spreads of $PG(3,7)$ containing reguli, alone number at least fifty. In keeping with this, in Chapter III we initiate a similar classification of proper n-covers for the simplest case with $q = 2$. As is well-known, the unique 1-cover (or spread) of $PG(3,2)$ is the regular spread. We show that there is a unique proper 2-cover of $PG(3,2)$, thereby answering in the affirmative, a conjecture made by Ebert in [52]. There is currently only one known example of a proper 3-cover of $PG(3,2)$; we prove that the existence of a proper 3-cover projectively distinct from the first one is equivalent to the existence of a proper 4-cover of $PG(3,2)$. For the last three cases with $n = 5$, 6 and 7, we prove that no proper n-covers exist.

In demonstrating a variety of the results pertaining to proper 2-covers and 3-covers of $PG(3,2)$, we make repeated use of a geometric structure which is isomorphic to the unique generalised quadrangle of order 2. As an aside to the general discussion, we show

how extra blocks may be added to this structure (referred to as Sylvester's syntheme-duad structure in [104]) to embed it in $PG(3,2)$. This gives rise to an elementary proof of the classical isomorphism between the groups S_6 and $PSp(4,2)$ (see [72] and [75] for alternative proofs) and also gives a geometrical interpretation of a result which is peculiar to S_6 amongst the class of symmetric groups.

To prove the non-existence of proper n-covers of $PG(3,2)$ for $n = 5$, 6 and 7, we are led back to the problem of extending regular partial packings. This subcase of the general problem of extending partial packings has received some attention in its own right. In [47], Denniston mentioned in his discussion on cyclic packings of $PG(3,8)$ that despite making attempts, he did not succeed in constructing a regular cyclic packing, while more recently Lunardon has shown in [89] that $PG(3,q)$, q odd, never admits a regular packing. The only known regular packings are the two projectively distinct packings of $PG(3,2)$ (see [44]) and evidence, such as that already stated, suggests they may be the only ones.

In our treatment here, we exploit Bruck's representation of a regular spread of $PG(3,q)$ and adapt a technique used by Bruen and Thas in [28] and [31] (this was employed in obtaining bounds for partial spreads of $PG(3,q)$) in order to set up a correspondence between the spreads in the partial packing and blocking sets of $PG(2,q^2)$. More precisely, we consider the cross-sections of the set of lines of the spreads in the partial packing by planes of $PG(3,q^2)$ which contain none of these lines. We then show that a regular partial packing with k spreads is extendable to one with $k+1$ regular spreads if and only if one of the cross-sections is not a blocking set of its ambient plane.

Using similar means, we also consider a related problem which yields results for regular partial packings of $PG(3,2)$ more simply. Coupling these results with a partial computer verification, we arrive at the interesting conclusion that every partial packing of $PG(3,2)$ is completable to a packing.

In Chapter IV, we turn to one of the main applications of n-covers of $PG(3,q)$ which is that of constructing quasi-n-multiple designs. A balanced incomplete block design D with parameter set (v,b,r,k,λ) is said to be a quasi-n-multiple of a (possibly non-existent) design D' with parameter set (v,b',r',k,λ') if and only if $\frac{b}{b'} = \frac{r}{r'} = \frac{\lambda}{\lambda'} = n$ for some positive integer n.

The simplest examples of quasi-n-multiple designs can be constructed by taking a given design D with v varieties labelled 1 to v and a set of $n-1$ permutations from the symmetric group S_v. Each of the $n-1$ permutations, when applied to D, produces a new design isomorphic to D which has the same set of varieties but in general, different blocks. By taking the union of the set of blocks in D and the sets of blocks in the $n-1$ images of D, we obtain a quasi-n-multiple of D (possibly with repeated blocks) based on the same set of varieties as D. In the case where each permutation is the identity permutation, the resulting design is simply called an n-multiple design. This particular technique has been utilised by Jungnickel in [79] and [80] to obtain lower bounds for the numbers of quasi-2-multiples of affine and projective planes and also of biplanes and residual biplanes.

Regarding quasi-n-multiple designs, the question of irreducibility of the designs arises in a most natural way. A quasi-n-multiple design based on a design D is said to be irreducible or indecomposable if it is not the union of two quasi-n_i-multiple designs based on D with $n_1 + n_2 = n$. Various methods for testing the irreducibility of quasi-2-multiple designs exist and typically involve examining particular multigraphs associated with the designs (see, for example, [105]). Other criteria for irreducibility have also been the impetus for the construction of irreducible quasi-2-multiple designs from existing designs; the technique entails modifying the blocks of the designs in such a way that the resulting quasi-2-multiple designs satisfy these criteria (see [16], [17] and [94]). For further results on this subject, see [92] and [125].

Using n-covers of $PG(3, q)$ we are able to set up a general technique for constructing examples of quasi-n-multiple Sperner designs. The simplest of these are also quasi-n-multiple affine designs and arise from the n-cover by generalising Bruck and Bose's construction of translation planes from spreads of $PG(3, q)$. We discuss the irreducibility of the quasi-n-multiple designs in terms of what we refer to as their spectra and in particular, for the affine case, we prove that the design is irreducible if and only if the n-cover is proper.

Following the construction of the designs, we turn our attention to the problem of determining if a given quasi-n-multiple affine design arises from an n-cover. With $n = 1$, the problem has already been answered by Bruck and Bose in [19]. An affine design arises from a spread of $PG(3, q)$ if and only if it is a translation plane of dimension two over its kernel. With $n \geq 2$, by making some reasonable assumptions, we are able to associate a regular, uniform linear space $\mathcal{L}$ with each design. The problem then reduces to one of showing that $\mathcal{L}$ is a finite affine space. A key element in establishing a necessary and sufficient condition for this to happen is Buekenhout's characterisation of affine spaces of order at least four; in [38], he shows that if the subspace generated by any three non-collinear points of a finite linear space $\mathcal{L}$ is an affine plane of some fixed order $q \geq 4$, then $\mathcal{L}$ is necessarily a finite affine space. More recently, in [118], Teirlinck has given an alternative proof of this for $q \geq 5$.

The first example of a finite linear space which satisfied this condition for $q = 3$ without being an affine space, was presented by Hall in [65]. Therefore such spaces have been coined Hall triple systems. In 1965 (see [66]), Hall and Bruck noticed that Hall triple systems can be coordinatised by a type of loop known as an exponent 3 commutative Moufang loop. Thus much of the existing theory on these systems is stated in terms of the properties of their corresponding loops. In [66], however, Hall uses the Burnside group $B(3, 3)$ to give an explicit construction of the smallest Hall triple system which is not an affine space. In Section 4.3 we generalise his construction by using the Burnside group $B(3, r)$, $r \geq 3$. We also use the Burnside groups to produce a second class of Hall triple systems; we have not been able to ascertain however, if they are just affine spaces or possibly new systems.

Resolvable designs admitting more than one resolution have been studied by a number of people; see for instance [41], [55] and [120].

We conclude this Chapter by initiating a study of possible alternative resolutions of the quasi-n-multiple affine designs arising from n-covers of $PG(3, q)$, $n \geq 2$. As a first step, since each resolution class of any resolution of such a design is equivalent to a partition of the point-set of $AG(4, q)$ by affine subplanes, we consider the problem of partitioning the point-set of $AG(2t, q)$ by t-dimensional affine subspaces and give examples of several of these partitions. Few articles on this problem seem to have been published, except in the case that $q = 2$ (see [8]). Finally we discuss some connections between one of these partitions and t-spreads of $PG(2t + 1, q)$.

We have already described how a quasi-n-multiple design may be constructed from a given design by using a set of $n - 1$ permutations which act on the varieties of the design. Similarly, given two finite affine planes of the same order, we can embed one plane in the other by defining a one-to-one mapping between the two point-sets. This type of observation inspired Ostrom to develop the theory of derivation and later, general net replacement (see [81], Chapter 7, [101] and [102]). By using such a mapping it can be shown that any finite affine plane can be obtained from any other affine plane of the same order via net replacement; often, however, the replacement will be trivial in the sense that almost all lines of the first plane will be replaced with lines from the second. Thus, in applying the method of net replacement, it has usually been the case that subplanes or some other "regular" sets in a given affine plane have been taken as the lines of a replacement net. This has also usually meant that the new plane has been a translation plane if and only if the original plane was also.

In Chapter V, we consider the notion of net replacement as it was first conceived in an attempt to construct new finite affine planes from a pair of existing ones of the same order.

To introduce some clarity to the picture, we first show that with respect to any one-to-one embedding of an affine plane in another of the same order, the two planes can be uniquely decomposed into pairs of conjugate irreducible replacement nets. Two affine planes of the same order are then said to be compatible if the number of pairs of nets so determined for at least one embedding is greater than or equal to two. By replacing some but not all of the nets in one plane, we produce a series of potentially new planes. Then, by refining some connections between replacement nets and blocking sets of finite projective planes which Bruen first demonstrated in [28], we set about analysing the compatibility of affine planes of prime order and also those of order 3^2 and 5^2. (We note that for the case of planes of order 9, the computer searches by Lam *et al* completed in 1990, have verified that the only projective planes of order 9 are the four known planes namely, the Desarguesian plane, the hall plane, the dual Hall plane and the Hughes plane (see [84] for a detailed study of these planes). Hence, all finite affine planes of order 9 are also known. We include our findings however, in the hope that they may provide a suitable setting for giving an alternative proof cum verification of Lam's results.

In the final section of Chapter V, we present a general method for constructing blocking sets of finite affine planes from Rédei blocking sets of finite projective planes. By applying this method, we construct a new $(2q^t - 1)$-blocking set in the affine plane $AG(2, q^t)$ for each t and for all $q \geq 3$. Jamison and independently Brouwer and Schrijver have shown that the minimal cardinality of a blocking set in $AG(2, q^t)$ is $2q^t - 1$ (see [76] and [24]). Thus it follows that our blocking sets attain this minimum cardinality. For further details on affine blocking sets see [12] and [35].

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and, to the best of my knowledge and belief, the thesis contains no material previously published or written by another person, except where due reference is made in the text. I consent to the thesis being made available for photocopy and loan.

Martin Glen Oxenham

ACKNOWLEDGEMENTS

I wish to express my sincere thanks to Dr. L.R.A. Casse for his patience, guidance and friendship during the course of this study.

I also wish to thank my family and friends both here and overseas for their support throughout the years.

Ringrazio sinceramente Professor A. Barlotti per la sua assistenza ed amicizia durante il mio soggiorno a Firenze, Italia nel 1985. Tante grazie vanno anche a mio fratello Arrigo Bonisoli al quale sono sempre indebitato per tutto in genere (e per il piacere di caffé in particolare).

Many thanks also go to Mary Mattaliano for typing up this thesis; without her help, it might never have been submitted.

Finally, I acknowledge the financial support that I received from a Commonwealth Research Grant and an Italian Scholarship during the first few years in which this research was carried out.

CHAPTER I
PRELIMINARIES

INTRODUCTION

In this chapter we present the preliminary results which will be required in subsequent chapters.

1.1. FINITE INCIDENCE STRUCTURES

The main part of this thesis is concerned with n-covers of $PG(3,q)$ which are particular subsets of the line-set of $PG(3,q)$. Therefore, in discussing the properties of an n-cover it is convenient to take the n-cover as the line-set of an incidence structure. The notion of an incidence structure also arises in various other situations throughout this thesis, so it is convenient to present the basic definitions and results here.

Definition 1.1.1. ([44], p.1) A *finite incidence structure* is a triple $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$, where $\mathbf{P}, \mathbf{B}, \mathbf{I}$ are finite non-empty sets with $\mathbf{P} \cap \mathbf{B} = \phi$ and $\mathbf{I} \subseteq \mathbf{P} \times \mathbf{B}$. The elements of $\mathbf{P}$ are called *points* or *varieties*, those of $\mathbf{B}$ *lines* or *blocks* and those of $\mathbf{I}$ *flags.* □

Definition 1.1.2. ([44], p.300) Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure. A *chain* in S is a finite sequence $CH(x_0, x_n) = \{x_0, \ldots, x_n\}$ of elements in S such that

$$x_{i-1} \; \mathbf{I} \; x_i \qquad \text{for } i = 1 \text{ to } n.$$

The integer n is the *length* of the chain $CH(x_0, x_n)$. If $n = 0$, then the chain is called *trivial.*

The two elements x_0 and x_n are called the *supports* of the chain (see [74], p.50) and the chain is said to be *closed* if the supports are equal. □

Remark 1.1.3. ([44], p.300) If $CH(x_0, x_n)$ is a closed chain then the number of points in $CH(x_0, x_n)$ equals the number of blocks in $CH(x_0, x_n)$ and so the length of $CH(x_0, x_n)$ is even. □

Definition 1.1.4. ([44], p.300) Let $CH(x_0, x_n)$ be a chain in a finite incidence structure $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$. Then $CH(x_0, x_n)$ is called *irreducible* if

$$x_{i-1} \neq x_{i+1} \quad \text{for } i = 1 \text{ to } n-1,$$

and $x_1 \neq x_{n-1}$ if $CH(x_0, x_n)$ is closed. □

Definition 1.1.5. Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure. A subset of blocks in $\{\ell_0, \ldots, \ell_{n-1}\}$ is called an *n-lateral* if there exists a set of points in $\mathbf{P}$ $\{P_0, \ldots, P_{n-1}\}$ such that

$$\{P_0, \ell_0, P_1, \ell_1, \ldots, P_{n-1}, \ell_{n-1}\}$$

is a closed chain in S. An n-lateral is said to be *proper* if at least one of the closed chains containing it is irreducible. □

Remark 1.1.6. If in $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ two arbitrary distinct blocks in $\mathbf{B}$ meet in at most one point, then an n-lateral in S lies in exactly one closed chain as the points of the closed chain are uniquely determined. □

Definition 1.1.7. ([44], p.301) Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure. Two arbitrary elements x, y of S are said to be connected if there exists a chain in S with x and y as supports. □

Definition 1.1.8. ([44], p.301) Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure. Then the *distance* between two arbitrary elements x, y of S is defined as follows:

$$d(x, y) = \begin{cases} \infty & \text{if } x \text{ and } y \text{ are not connected,} \\ \min\{n \mid \text{some chain of length } n \text{ connects } x \text{ and } y\}. \end{cases}$$

□

Remark 1.1.9. ([44], p.301) In a substructure of S in which any two elements are connected, the distance function satisfies the axioms of a metric. □

Definition 1.1.10. ([74], p.3) Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure with $|\mathbf{P}| = v$ and $|\mathbf{B}| = b$. Label the points P_i and blocks ℓ_j arbitrarily where $i = 1$ to v and $j = 1$ to b. Then, with respect to this labelling, the *incidence matrix* of S is the matrix $A = (\alpha_{ij})$ with $\alpha_{ij} = \begin{cases} 1 & \text{if } \; P_i \mathbf{I}\, \ell_j \\ 0 & \text{if } \; P_i \not\mathbf{I}\, \ell_j. \end{cases}$. □

1.2. FINITE LINEAR SPACES

Definition 1.2.1. ([43]) A *finite linear space* is a finite incidence structure which satisfies the two axioms below:

(i) Two points are incident with a unique line.

(ii) Each line contains at least 2 points. □

Definition 1.2.2. ([74], p.5) A finite linear space is called *uniform* if each of its lines is incident with exactly k points. Dually, it is called *regular* if each point is incident with exactly r lines. □

Most of the finite linear spaces considered in this thesis are both regular and uniform. These include the finite affine and projective spaces, the Hall triple systems and the finite Sperner spaces. We note, however, in general, regularity does not necessarily imply uniformity and vice versa.

Definition 1.2.3. ([44], p.29) A *collineation* of a finite linear space $\mathcal{L}$ is a permutation of the point-set of $\mathcal{L}$ which preserves incidence between the points and lines of $\mathcal{L}$. □

Remark 1.2.4. A number of authors of papers on finite linear spaces also use the term *automorphism* instead of *collineation*. (See [43], [50] and [69] for instance.)

□

Definition 1.2.5. ([43]) A *linear subspace* $\mathcal{L}'$ of a finite linear space $\mathcal{L}$ is a subset of $\mathcal{L}$ such that any line of $\mathcal{L}$ intersecting $\mathcal{L}'$ in at least two points is contained in $\mathcal{L}'$. □

In a finite linear space, the intersection of any collection of subspaces, if it is non-empty, is also a subspace. Also, given any non-empty set X of points of $\mathcal{L}$, there is at least one subspace of $\mathcal{L}$ which contains X, namely $\mathcal{L}$, itself. These two facts lead us to:

Definition 1.2.6. ([70]) Let X be a set of points in a finite linear space $\mathcal{L}$. Then the *subspace generated by* X is defined to be the intersection of all the subspaces of $\mathcal{L}$ which contain X. $\square$

Definition 1.2.7. ([70]) Let $\mathcal{L}'$ be a subspace of a finite linear space $\mathcal{L}$ and let the least number of points required to generate $\mathcal{L}'$ be $n+1$. Then the *dimension* of $\mathcal{L}'$ is n. A subspace S' for which $\dim \mathcal{L}' = \dim \mathcal{L} - 1$ is called a *hyperplane* of $\mathcal{L}$. $\square$

For convenience, we also introduce the following term for a subspace of dimension 2 in a finite linear space.

Definition 1.2.8. Let $\mathcal{L}$ be a finite linear space. A *triangular subspace* (Δ-space) $\mathcal{L}'$ of $\mathcal{L}$ is any subspace of $\mathcal{L}$ for which $\dim \mathcal{L}' = 2$. $\square$

The terminology and ideas used in the next construction are borrowed from Chapter XI of [73] where they are used in a slightly different context.

Construction 1.2.9. Let $\mathcal{L}$ be a finite linear space and let X be a non-empty set of points of $\mathcal{L}$. Let X_1 be the set of points of $\mathcal{L}$ obtained from X as follows:

Let P, Q be two distinct points in X. Form the line $\langle P, Q\rangle$. As P, Q vary over X, we obtain a finite collection of lines. Let X_1 be the union of all the points on these lines. Then $X \subset X_1$ and X_1 is called the *1-step extension* of X.

Similarly we can define the 1-step extension X_2 of X_1; continuing in this way we can produce a sequence of 1-step extensions

$$X = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_r \subset \ldots$$

In this case, we say X_r is the *r-step extension* of X. Furthermore, since $|X_i| \leq |X_{i+1}|$ for each i and $|\mathcal{L}|$ is finite, the sequence will eventually terminate; therefore, there exists a smallest integer R for which

$$X_R = X_{R+1}. \qquad \square$$

Theorem 1.2.10. Let R be defined as in Construction 1.2.9. Then X_R is a subspace of $\mathcal{L}$ and X_R coincides with the subspace of $\mathcal{L}$ generated by X.

Proof. The proof is similar to the proofs of Lemma 11.2 and Theorem 11.3 of [73].

$\square$

Definition 1.2.11. ([118]) Let $\mathcal{L}$ be a finite linear space and let S be a set of subspaces of $\mathcal{L}$. Then $\mathcal{L}$ is said to be a *finite planar space* with respect to S if and only if each triangle in $\mathcal{L}$ lies in a unique subspace of S. The elements of S are then called *planes*.

$\square$

Remark 1.2.12. A finite linear space may give rise to more than one planar space because often there will be more than one choice for the set of subspaces S. For examples of such finite linear spaces, see [114]. $\square$

We conclude this section with two elementary lemmata.

Lemma 1.2.13. ([81], p.8, [44], pp.138-139) Let $\mathcal{L}$ be a uniform linear space with n^2 points and n points on each line. Then $\mathcal{L}$ is a finite affine plane of order n. $\square$

Lemma 1.2.14. ([44], pp.138-139) Let $\mathcal{L}$ be a uniform linear space with n^2+n+1 points and $n+1$ points on each line. Then $\mathcal{L}$ is a finite projective plane of order n. $\square$

1.3. FINITE AFFINE AND PROJECTIVE SPACES

Because of the importance of finite affine and projective spaces in numerous different settings, various characterisations of them have been established through the years; in this section, we examine several of these.

Definition 1.3.1. ([7], p.I-1). A *finite projective space* is a finite linear space which satisfies the following axioms:

(i) Each pair of distinct points is incident with a unique line.

(ii) (Veblen) If a line intersects two sides of a triangle (not at their point of intersection), then it also intersects the third side.

(iii) (Fano) Every line contains at least three points. □

It is well-known that in a finite projective space Σ every line has the same number $m+1$ of points (m is known as the *order* of Σ) and, by arguing inductively on the dimension n of Σ (see definition 1.2.7), that Σ has exactly $\dfrac{m^{n+1}-1}{m-1}$ points. Moreover, if the dimension of Σ is at least 3, then the theorem of Desargues holds in Σ. As a consequence of this, the algebraic system (a *ternary ring*) which is used to coordinatise any plane of Σ is a finite skewfield. By the celebrated theorem of Wedderburn (see [123]) each finite skewfield is itself a field. Thus the number of points on each line of Σ is $q+1$ for some prime power q, whence the order of Σ is also q. By arguing more strictly it is possible to show that every finite projective space of order q and dimension n is isomorphic to the finite projective space which can be constructed (see [44], pp.27-28) as follows:

Construction 1.3.2. ([71], p.29) Let $V = V(n+1,q)$ be an $(n+1)$-dimensional vector space over the Galois field $GF(q)$ with zero vector 0. Consider the equivalence relation on the points of $V\backslash 0$ whose equivalence classes are the one dimensional subspaces of V with the origin deleted; that is, if $P, Q \in V\backslash\{0\}$ and for some basis $P = (x_0, \ldots, x_n)$, $Q = (y_0, \ldots, y_n)$, then P is equivalent to Q if and only if $x_i = ty_i$ for all i and for some $t \in GF(q)\backslash\{0\}$.

Then the set of equivalence classes is the point-set of the n-dimensional projective space $PG(n,q)$ over $GF(q)$ in this representation. The projective subspace spanned by a set of points in $PG(n,q)$ is represented by the vector subspace of V spanned by the corresponding one dimensional subspaces of V. □

In [111], Singer uses a correspondence between this representation of $PG(n,q)$ and the finite field $GF(q^n)$ to prove that $PG(n,q)$ admits a collineation group which cyclically permutes its point-set.

The proof of Singer's Theorem is well-known. However, since the proof is constructive and a large proportion of what we do in the latter part of Chapter II relies on this construction, we include it here.

Theorem 1.3.3. (Singer's Theorem, [111]) $PG(n,q)$ admits a collineation group which cyclically permutes its point-set.

Proof. Let f be a primitive monic polynomial of degree $n+1$ over $GF(q)$ i.e.

$$f(x) = x^{n+1} - a_n x^n - \ldots - a_2 x^2 - a_1 x - a_0.$$

Let β be a zero of f. Then we can adjoin β to $GF(q)$ to construct the extension field $GF(q^{n+1}) = GF(q)(\beta)$. In this case the set $\{1, \beta, \ldots, \beta^n\}$ forms a basis for $GF(q^{n+1})$ over $GF(q)$.

In order to express a non-zero element of $GF(q^{n+1})$ as a linear combination of the basis elements, we use the fact that $f(\beta) = 0$, i.e.

$$\beta^{n+1} = a_n \beta^n + \ldots + a_1 \beta + a_0.$$

We then call two elements β^i, β^j of $GF^*(q^{n+1}) = \langle \beta \rangle$ similar if and only if

$$\beta^i \cdot \beta^{-j} = \beta^{i-j} \in GF^*(q) = \left\{1, \beta^v, \beta^{2v}, \ldots, \beta^{(q-2)v}\right\}$$

where $v = \dfrac{(q^{n+1}-1)}{(q-1)}$. It is evident that the property of similarity is an equivalence relation.

Now since each element of $GF(q^{n+1})$ is uniquely expressible as a linear combination of $1, \beta, \ldots, \beta^n$ over $GF(q)$, we can associate a coordinate vector $(x_0, x_1, \ldots, x_n)$ with each element

$$\beta^j = \sum_{i=0}^{n} x_i \beta^i.$$

It then follows that two elements β^i, β^j of $GF^*(q^{n+1})$ are similar if and only if their corresponding coordinate vectors are scalar multiples of each other where the scalar belongs to $GF^*(q)$.

Consequently the correspondence extends to one between $GF^*(q^{n+1})$ and $PG(n,q)$ in the following manner.

The points of $PG(n,q)$ $\leftrightarrow$ The equivalence classes determined by the similarity relation on the elements of $GF^*(q^{n+1})$.

The lines of $PG(n,q)$ $\leftrightarrow$ The sets of the form

$$\left\{ \lambda_1\beta^i + \lambda_i\beta^j \;\middle|\; \begin{array}{l} \lambda_1, \lambda_2 \in GF(q) \\ \lambda_1, \lambda_2 \text{ not both } 0 \end{array} \right\}$$

where β^i and β^j are representatives from distinct equivalence classes.

The incidence in $PG(n,q)$ $\leftrightarrow$ Set inclusion.

Consider the mapping ϕ where

$$\begin{array}{rccc} \phi: & GF^*(q^{n+1}) & \rightarrow & GF^*(q^{n+1}) \\ & \beta^i & \mapsto & \beta^{i+1}. \end{array}$$

Now the elements

$$\begin{array}{lcl} \beta^i & & \\ \phi(\beta^i) & = & \beta^{i+1} \\ \phi^2(\beta^i) & = & \beta^{i+2} \\ \vdots & & \\ \phi^{v-1}(\beta^i) & = & \beta^{i+v-1} \end{array}$$

are pairwise dissimilar, while β^i and $\phi^v(\beta^i) = \beta^{i+v}$ are similar. Hence ϕ permutes the equivalence classes cyclically in cycles of length $v = \dfrac{q^{n+1}-1}{q-1}$. Clearly ϕ induces a collineation Φ of $PG(n,q)$ because it permutes the point-set of $PG(n,q)$ and also maps the line $\langle\beta^i, \beta^j\rangle$ to the line $\langle\beta^{i+1}, \beta^{j+1}\rangle$.

The cyclic collineation group $\langle \Phi \rangle$ (the Singer group) therefore cyclically permutes the point-set of $PG(n,q)$ as required. $\square$

Remark 1.3.4. The collineation Φ corresponds to the homography with matrix

$$\begin{bmatrix} 0 & & & & & a_0 \\ 1 & & & & & a_1 \\ & 1 & & & & a_2 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & 1 & a_n \end{bmatrix}.$$

(This can be shown by expanding $\phi(\beta^i) = \beta^{i+1}$ as $\beta.\beta^i = \beta\Big(x_0 + x_1\beta + x_2\beta^2 + \ldots + x_n\beta^n\Big)$ and replacing β^{n+1} by $a_0 + a_1\beta + \ldots + a_n\beta^n$). In addition, where it doesn't lead to confusion, we often simply write $\langle \beta \rangle$ in place of $\langle \Phi \rangle$. $\square$

Before discussing finite affine spaces, we state:

Theorem 1.3.5. (The Dimension Theorem, [74]) Let Σ, Σ' be subspaces of $PG(n,q)$. Then

$$\dim(\Sigma \oplus \Sigma') + \dim(\Sigma \cap \Sigma') = \dim \Sigma + \dim \Sigma',$$

(where $\Sigma \oplus \Sigma'$ denotes the subspace spanned by Σ and Σ'). $\square$

Remark 1.3.6. The dimension theorem is also referred to as *Grassmann's identity* in some texts. It can also be taken as an axiom in an alternative definition of a finite projective space (see [71]). $\square$

Definition 1.3.7. ([7], p.I-2) Let Σ_{n-1} be a hyperplane of a finite projective space Σ_n of dimension n and order m. Then the linear space obtained by deleting Σ_{n-1} from Σ_n and all points and lines in it is called a *finite affine space.* It also has dimension n and by definition, its order is also m. $\square$

Remark 1.3.8. (a) If the dimension of Σ_n is at least 3 or $\Sigma_n = PG(2,q)$, then the corresponding affine space is denoted by $AG(n,q)$ where q is the order of Σ_n. $AG(n,q)$ is well-defined because, by duality, the Singer group of $PG(n,q)$ also permutes the hyperplanes of $PG(n,q)$ cyclically. Hence the construction does not depend on Σ_{n-1}.

(b) The deleted hyperplane is often referred to as the ***special hyperplane*** of the affine space and accordingly, its points are referred to as ***special points***. (See [73], p.85.)

□

In a finite affine space, we have an additional (equivalence) relation called *parallelism*. Two lines are defined to be parallel if they coincide or if the corresponding lines in the projective space meet in a special point (see [7], p.I-2). Using this relation, we can give an alternative definition of a finite affine space.

Definition 1.3.9. ([71], p.39) Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure endowed with an equivalence relation *parallelism* on its lines. Then S is a ***finite affine space*** if it satisfies the axioms below.

(i) Any two points are incident with exactly one line.

(ii) For every point P and line ℓ, there is a unique line ℓ' parallel to ℓ and containing P.

(iii) If $\langle P_1, P_2\rangle$ and $\langle P_3, P_3\rangle$ are parallel lines and P is a point on $\langle P_1, P_3\rangle$ distinct from P_1 and P_3, then there is a point P' on $\langle P, P_2\rangle$ and $\langle P_3, P_4\rangle$.

(iv) If no line contains more than two points and P_1, P_2, P_3 are distinct points, then the line ℓ_3 through P_3 parallel to $\langle P_1, P_2\rangle$ and the line ℓ_2 through P_2 parallel to $\langle P_1, P_3\rangle$ have a point P in common.

(v) Some line contains exactly $m \geq 2$ points. □

1.4. k-CAPS, OVALOIDS AND OVOIDS IN $PG(n,q)$

Definition 1.4.1. ([72], p.33) In $PG(n,q)$ a *k-cap* is a set of k points no three of which are collinear. A k-cap is *complete* if it is not contained in a $(k+1)$-cap. □

It can be shown (see [72], pp.33-34) that if a set of k points in $PG(3,q)$, $q > 2$ is a k-cap, then $k \leq q^2 + 1$. For each $q > 2$, this upper bound is attained by the elliptic quadrics and so the inequality cannot be improved. In view of this, we have:

Definition 1.4.2. ([72], p.34) In $PG(3,q)$, $q > 2$, a (q^2+1)-cap is called an *ovaloid.* □

When $q = 2$, the greatest value of k for which a k-cap exists is $k = 8$ (see [72], p.33). In this case the 8-cap is the complement of a plane in $PG(3,2)$. There also exist complete 5-caps in $PG(3,2)$, namely the elliptic quadrics (see [72], p.96); they are characterised in:

Theorem 1.4.3. (cf. [72], Theorem 18.2.1) Let $\mathcal{K}$ be a set of 5 points in $PG(3,2)$ such that no plane of $PG(3,2)$ meets $\mathcal{K}$ in more than three points. Then $\mathcal{K}$ is a complete 5-cap of $PG(3,2)$ and it consists of the 5 points of an elliptic quadric. (The proof of this theorem is essentially contained in the third paragraph of the proof of Theorem 18.2.1 in [72].) □

Definition 1.4.4. ([72], p.52) An *ovoid* in $PG(n,q)$, $n \geq 3$ or in a finite projective plane is a k-cap such that the tangent lines at each point form a hyperplane. □

In [44], p.48, it is shown that ovoids can only exist in $PG(3,9)$ or in a finite projective plane. In particular the ovoids of $PG(3,q)$ and the ovaloids of $PG(3,q)$, $q > 2$ are identical (when $q = 2$ an ovoid is a complete 5-cap of $PG(3,2)$, that is an elliptic quadric).

In 1955 Barlotti and Panella independently proved that all ovoids in $PG(3,q)$, q odd are elliptic quadrics, thus extending the result of Segre, namely that all $(q+1)$-arcs in $PG(2,q)$, q odd are irreducible conics. (See [7], p.V-3, and [44], p.48.)

The only known ovoids in $PG(3,2^{2h})$ are the elliptic quadrics; it is conjectured that they are the only ovoids in $PG(3,2^{2h})$. This is known to be the case when $h = 1$ and 2 (see [7], p.V-8, [72], p.35 and [99]). However, when $q = 2^{2h+1}$, $h \geq 1$, there exist other ovoids known as the *Suzuki-Tits ovoids.* These have the form:

$$\{(0,0,0,1)\} \cup \left\{(1, x, y,\ xy + x^{\sigma+2} + y^{\sigma}) | x, y \in GF(2^{2h+1})\right\}$$

where σ is the *Frobenius automorphism* which maps each x in $GF(2^{2h+1})$ to $x^{2^{h+1}}$; it follows from this that $x^{\sigma^2} = x^2$ for all x in $GF(2^{2h+1})$. (See [119].)

1.5. FINITE NETS AND NET REPLACEMENT

The definitions, theorems and other results appearing in this section (apart from Definition 1.5.9 and Theorem 1.5.10 which are original) are drawn from [102] by Ostrom.

Definition 1.5.1. Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure endowed with an equivalence relation *parallelism* on its lines. Then S is a *finite net* if $|\mathbf{P}|$ is finite and

(i) Each point belongs to exactly one line of each parallel class,

(ii) If ℓ_1, ℓ_2 are lines of different parallel classes, then ℓ_1 and ℓ_2 have exactly one point in common,

(iii) There are at least three parallel classes and at least two points on each line. $\square$

Remark 1.5.2. If instead the number of parallel classes is one or two and the remaining conditions are satisfied, then the structure is called a *trivial net.* $\square$

In a finite net $\mathcal{N}$ every line has exactly n points for some fixed integer n; this is a consequence of the fact that $\mathcal{N}$ has at least 3 parallel classes. Furthermore $\mathcal{N}$ then has exactly n^2 points. We are thus led to

Definition 1.5.3. Let $\mathcal{N}$ be a finite net. The *order* n of $\mathcal{N}$ is the number of points on each line of $\mathcal{N}$. The *degree* k of $\mathcal{N}$ is the number of parallel classes of $\mathcal{N}$. $\square$

Remark 1.5.4. A finite affine plane of order n is a finite net of order n and degree $n+1$. $\square$

Remark 1.5.5. Two trivial nets with the same numbers of points and parallel classes and with n points per line are isomorphic. $\square$

Definition 1.5.6. Let $\mathcal{N}$ be a finite net. Then $\mathcal{N}$ is said to be *replaceable* if there is a net $\mathcal{N}'$ defined on the same point-set as $\mathcal{N}$ such that each pair of points that is collinear in $\mathcal{N}$ is also collinear in $\mathcal{N}'$ and vice versa. The net $\mathcal{N}'$ is called a *replacement net* for $\mathcal{N}$ and $\mathcal{N}, \mathcal{N}'$ are called *conjugate replacement nets.* $\square$

Definition 1.5.7. Let $\mathcal{N}$ be a finite net of order n. A set T of n points in $\mathcal{N}$ is a *transversal* of $\mathcal{N}$ if no two points of T are collinear in $\mathcal{N}$. $\square$

Definition 1.5.8. Let $\mathcal{N}_1, \mathcal{N}_2$ be a pair of nets defined on the same point-set such that every line of $\mathcal{N}_1$ is a transversal of $\mathcal{N}_2$ and every line of $\mathcal{N}_2$ is a transversal of $\mathcal{N}_1$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are said to be *disjoint.* $\square$

Definition 1.5.9. Let $\mathcal{N}_1$ be a replaceable net with a replacement net $\mathcal{N}_2$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are said to be *irreducible* with respect to one another if there do not exist conjugate replacement nets $\mathcal{N}_1'$ and $\mathcal{N}_2'$ such that

(i) $\mathcal{N}_1' \subset \mathcal{N}_1$ and $\mathcal{N}_1' \neq \mathcal{N}_1$

(ii) $\mathcal{N}_2' \subset \mathcal{N}_2$ and $\mathcal{N}_2' \neq \mathcal{N}_2$. $\square$

Theorem 1.5.10. Let $\mathcal{N}$ and $\mathcal{M}$ be a pair of nets defined on the same point-set. Let $\mathcal{N}_1, \mathcal{N}_2$ be a pair of nets in $\mathcal{N}$ and $\mathcal{M}_1, \mathcal{M}_2$ be a pair of nets in $\mathcal{M}$ such that $\mathcal{N}_1, \mathcal{M}_1$ and $\mathcal{N}_2, \mathcal{M}_2$ form two pairs of conjugate replacement nets. If $\mathcal{N}_1 \cap \mathcal{N}_2 \neq \phi$ and $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \phi$, then $\mathcal{N}_1 \cap \mathcal{N}_2$ and $\mathcal{M}_1 \cap \mathcal{M}_2$ also form a pair of conjugate replacement nets.

Proof. Let P, Q be two points which lie on a line ℓ of $\mathcal{N}_1 \cap \mathcal{N}_2$. Then $\ell \in \mathcal{N}_1$ and $\ell \in \mathcal{N}_2$, so P and Q also lie on a line m_1 of $\mathcal{M}_1$ and on a line m_2 of $\mathcal{M}_2$ because for each i, $\mathcal{M}_i$ is a replacement net for $\mathcal{N}_i$.

Now $\mathcal{M}_1$ and $\mathcal{M}_2$ are both subnets of $\mathcal{M}$. Hence $m_1 = m_2$. Therefore $m_1 \in$

$\mathcal{M}_1 \cap \mathcal{M}_2$ and so P, Q also lie on a line of $\mathcal{M}_1 \cap \mathcal{M}_2$.

By symmetry the converse is also true. This $\mathcal{N}_1 \cap \mathcal{N}_2$ and $\mathcal{M}_1 \cap \mathcal{M}_2$ are conjugate replacement nets. □

Theorem 1.5.11. Let $\pi = \mathcal{N}_1 \cup \ldots \mathcal{N}_i \cup \mathcal{M}$ be a finite affine plane such that $\mathcal{N}_1, \ldots, \mathcal{N}_i$ are replaceable nets while $\mathcal{M}$ may or may not be replaceable. Let $\mathcal{N}_1', \ldots, \mathcal{N}_i'$ be any set of replacement nets for $\mathcal{N}_1, \ldots, \mathcal{N}_i$ respectively. Then

$$\pi' = \mathcal{N}_1' \cup \ldots \cup \mathcal{N}_i' \cup \mathcal{M}$$

is an affine plane. □

Theorem 1.5.11 gives rise to the method of net replacement which alters the structure of a given affine plane π by replacing one or more replaceable nets which lie in π. Often the resulting plane π' is not isomorphic to π. In the special case in which π has order n^2 and the replaceable net has degree $n+1$, the process is also known as derivation. In this case the lines of the replacement net are represented by affine Baer subplanes of π. (See [102], [103] and [81].)

1.6. HALL TRIPLE SYSTEMS AND FINITE SPERNER SPACES

Definition 1.6.1. ([70],[95]) A *Steiner triple system* is a uniform finite linear space with line size 3. □

Definition 1.6.2. ([10]) In a Steiner triple system a *symmetry (with fixed point a)* is an involutory mapping σ_a such that

$$\begin{aligned} \sigma_a : \quad & x \mapsto y \\ & a \mapsto a \end{aligned}$$

whenever $\{a, x, y\}$ is a line of the system. □

Definition 1.6.3. ([10]) A *Hall triple system* is a Steiner triple system such that each of its symmetries is a collineation of the system. □

Theorem 1.6.4. ([65], Theorem 3.1) A Steiner triple system is a Hall triple system if and only if each triangular subspace (see Definition 1.2.8) of the system is a finite affine plane of order 3. □

It is shown in [63] that each Hall triple system has 3^s points for some positive integer s which is called the *size* of the system. Moreover any two minimal generating sets of such a system have the same number of elements. Writing this number as $n+1$, the *dimension* of the system is then defined to be n. The finite affine spaces of order 3 are the classical examples of Hall triple systems; these are known as the *abelian Hall triple systems.* The smallest non-abelian Hall triple system has size 4 and dimension 3; all systems with these parameters are isomorphic to one another. It is also known that there exist exactly two non-isomorphic Hall triple systems of dimension 5 and size 6. Various other results of this nature are also known. (See [9],[10].)

We have indicated in Section 3 that there are various ways in which finite affine and projective spaces may be defined or constructed. The same is true of Hall triple systems. For example, the sufficient condition in Theorem 1.6.4 may be taken as the definition of a Hall triple system, in which case the statement of Definition 1.6.3 may be proven as a consequence. By coordinatising Hall triple systems it is possible to give a further characterisation of them. We now outline this.

Definition 1.6.5. ([10]) An *exponent 3 commutative Moufang loop* (3-CM loop) is a set $\mathcal{M}$ endowed with a commutative binary operation $\circ$ which admits a unit 1 and which satisfies the two following identities:

(i) $x^2 \circ (x \circ y) = y$,

(ii) $x^2 \circ (y \circ z) = (x \circ y) \circ (x \circ z)$. □

It is reported in [106] that in 1965, M. Hall and R.H. Bruck discovered the close ties which exist between 3-CM loops and Hall triple systems. Given a 3-CM loop with binary operation $\circ$, we can construct a Hall triple system by taking the elements of the loop as the points of the system and the triples $\{x, y, (x \circ y)^2\}$ as the lines of the system (the incidence relation being that of set inclusion). Conversely, given a Hall triple system S, we can construct a 3-CM loop $\mathcal{M}$ with binary operation $\circ$ as follows:

For any two points a and b of S, we define $a \cdot b$ to be the third point on the line containing a and b if a and b are distinct, otherwise it is equal to a.

Then the elements of $\mathcal{M}$ are the points of S and for all elements x and y of $\mathcal{M}$, we set

$$x \circ y = (e \cdot x) \cdot (e \cdot y)$$

where e is an arbitrary fixed point of S.

Two 3-CM loops constructed in this way via distinct fixed points e and e' are isomorphic. Furthermore the Hall triple systems which arise from these 3-CM loops by the technique which we described above, are all isomorphic to S.

Thus, up to isomorphism, there is a one-to-one correspondence between Hall triple systems and 3-CM loops. (See [10], [66] and [106] for further details.)

It can be readily checked that the 3-CM loop corresponding to the finite affine space $AG(n, 3)$, $n \geq 2$ is associative. Therefore we conclude the following:

Theorem 1.6.6. ([106]) A Hall triple system associated with the 3-CM loop $\mathcal{M}$ is a finite affine space if and only if $\mathcal{M}$ is associative. □

Definition 1.6.7. ([7], p.I-4) Let $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ be a finite incidence structure endowed with an equivalence relation *parallelism* on its lines. Then S is a *finite Sperner space* if it satisfies the following axioms:

(i) Any two points are incident with exactly one line.

(ii) For every point P and line ℓ, there is a unique line ℓ' parallel to ℓ and containing P.

(iii) Each line contains exactly $m \geq 2$ points for some m. □

It is readily seen by comparing this definition with Definition 1.3.9 that a finite affine space is also a finite Sperner space; it is for this reason that finite Sperner spaces are also known as (finite) *weak affine spaces.* For further details on these spaces see for instance [7], [18], [96] and [108].

1.7. BLOCKING SETS OF FINITE AFFINE AND PROJECTIVE PLANES

Definition 1.7.1. ([29]) A *blocking set* $\mathcal{B}$ in a finite projective plane is a set of points such that every line of the plane contains at least one point of $\mathcal{B}$ and such that no line of π is completely contained in $\mathcal{B}$. □

Definition 1.7.2. ([71], p.367) A blocking set $\mathcal{B}$ in a finite projective plane is said to be *minimal* or *irreducible* if no proper subset of $\mathcal{B}$ is also a blocking set. □

Theorem 1.7.3. ([71], p.367) Let $\mathcal{B}$ be a blocking set in a finite projective plane. Then $\mathcal{B}$ is irreducible if and only if for every point P of $\mathcal{B}$ there is some line ℓ such that $\ell \cap \mathcal{B} = \{P\}$. ($\ell$ is called a *tangent.*) □

Blocking sets were first studied in connection with games theory. Thus historically, a blocking set of minimal cardinality has been called a *committee.* By its definition, a committee is also an irreducible blocking set, although the converse is not true in general. For small values of q, the exact sizes and structures of committees in planes of order q are known. (See [29], [71].) More generally, we have

Theorem 1.7.4. ([29]) Let $\mathcal{B}$ be a blocking set in a finite projective plane π of order n. Then $|\mathcal{B}| \geq n + \sqrt{n} + 1$ and equality holds if and only if $\mathcal{B}$ is the point-set of a Baer subplane of π (whence n is a square). □

Definition 1.7.5. ([32]) A blocking set $\mathcal{B}$ in a finite projective plane π of order n is of *type* (n, k) if $|\mathcal{B}| = n + k$ and some line of π meets $\mathcal{B}$ in exactly k points. □

In the literature, blocking sets of type (n, k) are also often called Rédei blocking sets. However, it is claimed incorrectly in some papers (see for example [36], p.81, [34], p.57) that a blocking set of type (n, k) is irreducible. As a counter-example, given a square order (n^2) projective plane containing a Baer subplane, we can take the point-set of the subplane plus any other point as our blocking set $\mathcal{B}$. Then $\mathcal{B}$ is of type $(n^2, n+2)$ but it is not irreducible. Hence we restrict the term ***Rédei blocking set*** to mean an irreducible blocking set of type (n, k).

A number of results on Rédei blocking sets are known. Two such results are stated in Theorems 5.2.3 and 5.2.4 in Chapter V. For further results see [15], [85].

A result in a similar vein is

Theorem 1.7.6. ([37]) Let $\mathcal{B}$ be a blocking set in $\pi = PG(2, q)$ with q a square. Let $|\mathcal{B}| = q + \sqrt{q} + 1 + t$ where $0 < t < \sqrt{2q} - \sqrt{q} - \frac{1}{2q}$. Then there exists a subset T of $\mathcal{B}$ such that the points of $\mathcal{B} \backslash T$ are the points of a Baer subplane of π. $\square$

Remark 1.7.7. The statement of Theorem 1.7.6 is still valid for an arbitrary finite affine plane of square order provided that $t = 1$. (See [32].) $\square$

Definition 1.7.8. ([35]) A *blocking set* $\mathcal{B}$ in a finite affine plane is a set of points such that every line of the plane contains at least one point of $\mathcal{B}$. $\square$

Remark 1.7.9. Unlike blocking sets in a projective plane, blocking sets in an affine plane may contain lines of the plane. $\square$

In $AG(2, q)$, the number of points in a blocking set is at least $2q - 1$; there exist examples of blocking sets which attain the lower bound, so the inequality is sharp. However in some non-Desarguesian affine planes of order q, there exist blocking sets with fewer than $2q - 1$ points. (See [12], [24], [35] and [76].)

1.8. SPREADS, PACKINGS AND SWITCHING SETS IN $PG(n,q)$

Definition 1.8.1. ([109], p.23) A *partial t-spread* of $PG(n,q)$ is a set of pairwise disjoint t-dimensional subspaces of $PG(n,q)$. A *t-spread* of $PG(n,q)$ is a partial t-spread of $PG(n,q)$ such that each point of $PG(n,q)$ is incident with a unique element of the partial t-spread. □

Because the elements of a t-spread partition the point-set of $PG(n,q)$, the range of values of t for which a t-spread exists is somewhat restricted. More precisely we have

Theorem 1.8.2. ([109], p.23) $PG(n,q)$ possesses a t-spread if and only if $t+1$ divides $n+1$. □

Definition 1.8.3. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be disjoint partial t-spreads of $PG(n,q)$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be *conjugate partial t-spreads* if each point of $PG(n,q)$ is incident with an element of $\mathcal{S}_1$ if and only if it is incident with an element of $\mathcal{S}_2$. The pair of spreads is then said to be a *switching set* of $PG(n,q)$. □

Remark 1.8.4. The definition of a switching set of $PG(n,q)$ above differs slightly from that given by Bruck and Bose in [20], p.165. There, they define a switching set to be a partial t-spread with at least one conjugate partial t-spread. □

Since the 1960's much of the investigation into partial t-spreads has concentrated on the case where $n = 2t+1$. This is due to a relationship between partial t-spreads of $PG(2t+1,q)$ and finite nets which was discovered and first employed by Bruck and Bose in [19] in the construction of finite affine planes.

To construct a finite net, they begin with $PG(2t+2,q)$ and consider a partial t-spread of a hyperplane $PG(2t+1,q)$. The points of the net are then the points of $PG(2t+2,q)\backslash PG(2t+1,q)$ and the lines of the net are those $(t+1)$-dimensional subspaces of $PG(2t+2,q)$ which meet $PG(2t+1,q)$ in t-dimensional subspaces which are elements of the partial t-spread. In the case that the partial t-spread is a t-spread, the resulting net is an affine plane of order q^{t+1}. These planes are all finite translation planes and what is more, every finite translation plane can be constructed in this manner.

Finally, we note that if two partial t-spreads form a switching set, then the corresponding nets are conjugate replacement nets. Thus, given a t-spread containing a partial t-spread with a conjugate, the replacement of the partial t-spread with its conjugate is equivalent to the replacement of a net in the translation plane corresponding to the t-spread.

Definition 1.8.5. ([36], p.84) A net constructed from a partial t-spread via Bruck and Boses' construction is called a *translation net.* $\square$

Definition 1.8.6. ([44], p.220) A *t-regulus* in $PG(2t+1, q)$ is a partial t-spread with $q+1$ elements which satisfies the property that any line meeting three of its elements, meets every one of its elements. $\square$

A line meeting each element of a t-regulus is called a *transversal.* From the definition of a t-regulus it is immediate that each transversal meets each element in exactly one point. It can also be shown that through every point of each element of the t-regulus, there passes a unique transversal. Thus the transversals are pairwise skew lines. In particular when $t = 1$, the transversals of a 1-regulus (or simply regulus) constitute a partial spread conjugate to the regulus.

The existence of t-reguli is well-known; in $PG(2t+1, q)$, the non-degenerate quadrics on index $t+1$ are always covered by t-reguli. In fact, given any three disjoint t-dimensional spaces in $PG(2t+1, q)$, there is a unique t-regulus containing them. This motivates the following definition:

Definition 1.8.7. ([44], p.221) A t-spread $\mathcal{S}$ of $PG(2t+1, q)$ is said to be *regular* if and only if for any three distinct elements of $\mathcal{S}$, the t-regulus containing them also lies in $\mathcal{S}$. $\square$

Theorem 1.8.8. ([44], p.221) Let $q > 2$. Then a t-spread of $PG(2t+1, q)$ is regular if and only if the affine plane constructed from it is Desarguesian. $\square$

Remark 1.8.9. When $q = 2$, each line has exactly three points, so every t-spread of $PG(2t+1, 2)$ is regular. $\square$

The concept of regularity has been extended in [98] to t-spreads of $PG(n, q)$ via a connection which exists between a t-regulus and the classical Segre variety (see [98], p.19 for the definition of this variety). However for our purposes, it is sufficient to examine only the regular spreads of $PG(3, q)$ in more detail.

In [26], p.439, Bruck introduced a representation of regular spreads of $PG(3, q)$ via pairs of (conjugate skew) lines of $PG(3, q^2)$. Before reviewing this representation, we introduce the following terminology:

Let ℓ be a line of $PG(3, q^2)$. Then ℓ is said to be *real, imaginary* or *entirely imaginary* if ℓ has exactly $q+1$, 1 or 0 points of $PG(3, q)$ respectively. Similarly, a plane π of $PG(3, q^2)$ is *real* or *imaginary* if it has exactly q^2+q+1 or $q+1$ points of $PG(3, q)$ respectively.

Construction 1.8.10. Let α be a zero of a primitive quadratic over $GF(q)$. Then we can adjoin α to $GF(q)$ to construct the extension field $GF(q^2)$. Each element of $GF(q^2)$ can then be represented uniquely in the form $x + \alpha y$ where x and y both belong to $GF(q)$.

Now $\alpha^{q^2} = \alpha \neq \alpha^q$ because $\alpha \in GF(q^2)\backslash GF(q)$. Hence the mapping

$$\sigma : x + \alpha y \mapsto x + \alpha^q y$$

is an automorphism of $GF(q^2)$ of order two. Furthermore σ fixes $GF(q)$ elementwise.

If we now construct $PG(3, q^2)$ from $GF(q^2)$ via the technique used in construction 1.3.2, it is easily seen that σ extends to a collineation of $PG(3, q^2)$ with order two which fixes $PG(3, q)$ pointwise.

We then define two points P and Q of $PG(3, q^2)$ to be *conjugate* if $P^\sigma = Q$ (in which case $P = Q^\sigma$) and similarly we define two lines ℓ and m of $PG(3, q^2)$ to be *conjugate* if $\ell^\sigma = m$ (in which case $\ell = m^\sigma$).

Theorem 1.8.11. ([26], p.439) Let ℓ and ℓ^σ be two pairwise skew conjugate lines in $PG(3, q^2)$. Then each line joining a point P on ℓ to its conjugate P^σ on ℓ^σ meets $PG(3, q)$ in a line. Furthermore, the resulting $q^2 + 1$ lines of $PG(3, q)$ are pairwise skew and form a regular spread of $PG(3, q)$. □

Since all regular spreads of $PG(3, q)$ are projectively equivalent (see [91], p.246, lemma 4.8.2), it follows that every regular spread of $PG(3, q)$ can be represented in this way. In fact, each regular spread can be represented uniquely in this way. The following can also be proven.

Theorem 1.8.12. ([26]) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four pairwise disjoint lines in $PG(3, q)$ such that ℓ_4 is also disjoint from each line of the regulus $\mathcal{R}$ of $PG(3, q)$ defined by ℓ_1, ℓ_2 and ℓ_3. Then there is a unique regular spread containing these four lines. □

Definition 1.8.13. ([26]) Let $\mathcal{S}$ be a regular spread of $PG(3, q)$. Let $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k$ be k spreads such that $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{S}_i = (\mathcal{S}_{i-1} \backslash \mathcal{R}) \cup \mathcal{R}'$ for $i \geq 1$ where $\mathcal{R}'$ is the opposite regulus of a regulus $\mathcal{R}$ in $\mathcal{S}_{i-1}$. Then the sequence of spreads $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k$ is called a *subregular sequence of spreads of length k*. Furthermore, a spread $\mathcal{S}'$ is said to be *subregular of index k* if there exists a subregular sequence of length k ending in $\mathcal{S}'$ but none of shorter length. □

Theorem 1.8.14. ([100]) Let $\mathcal{S}_k$ be a subregular spread of index k in $PG(3, q)$ constructed from the regular spread $\mathcal{S}$. Then there exists a set W of pairwise disjoint reguli in $\mathcal{S}$ for which $\mathcal{S}_k = \mathcal{S}^W$, where $\mathcal{S}^W$ denotes the spread obtained from $\mathcal{S}$ by reversing the reguli in W. □

Remark 1.8.15. ([100]) If we reverse $(q-1)$ pairwise disjoint reguli in a regular spread of $PG(3,q)$, then the resulting spread is again regular. □

Theorem 1.8.16. ([33]) Let $\mathcal{S}$ be a regular spread of $PG(3,q)$. Let $\mathcal{S}'$ be a partial spread lying in $\mathcal{S}$ which has a conjugate partial spread $\mathcal{S}''$. If $\mathcal{S}'$ contains two distinct reguli which share at least one line, then

$$|\mathcal{S}'| \geq \frac{(q+1)(q+3)}{4}\,. \qquad \square$$

Definition 1.8.17. ([46]) A *partial packing* of $PG(3,q)$ is a set of pairwise disjoint spreads of $PG(3,q)$. A *packing* of $PG(3,q)$ is a partial packing which satisfies the condition that each line of $PG(3,q)$ lies in a (unique) spread belonging to the partial packing. □

Packings of $PG(3,q)$ exist for each q (see [46], [47], [48]). However, the only known regular packings occur in $PG(3,2)$; there are exactly two such projectively distinct packings. (See [40].) When q is odd, $PG(3,q)$ has no regular packings. This is proven in [89]. (Note: Therein, packings are referred to as parallelisms.)

1.9. PLÜCKER COORDINATES AND THE KLEIN CORRESPONDENCE

In $PG(3,q)$ a point P can be represented by a homogeneous 4-tuple (x_0, x_1, x_2, x_3). Dually, a plane π can be represented by a homogeneous 4-tuple $[y_0, y_1, y_2, y_3]$. With respect to this representation, the point P and the plane π are incident if and only if

$$(x_0, x_1, x_2, x_3) \cdot [y_0, y_1, y_2, y_3] = 0.$$

To develop a similar representation for the lines of $PG(3,q)$, it is necessary to move to $PG(5,q)$ and we do this via the notion of Plücker coordinates.

Definition 1.9.1. ([87], p.245) Let ℓ be a line in $PG(3,q)$ and let $P = (x_0, x_1, x_2, x_3)$ and $Q = (y_0, y_1, y_2, y_3)$ be two distinct points on ℓ. Then the *Plücker coordinate vector* of the line ℓ is defined to be

$$(\ell) = (p_{01}, p_{02}, p_{03}, p_{12}, p_{31}, p_{23})$$

where $p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$. Its components are called the *Plücker coordinates* of ℓ. □

At the outset, it appears that the Plücker coordinates of a line are dependent on the points chosen. However, up to scalar multiples, they are independent of the two points; this can be argued as follows:

Let $\lambda_1 P + \mu_1 Q$, $\lambda_2 P + \mu_2 Q$ be any two distinct points on ℓ with P and Q as before. Then the (i,j)th Plücker coordinate is

$$\begin{vmatrix} \lambda_1 x_i + \mu_1 y_i & \lambda_1 x_j + \mu_1 y_j \\ \lambda_2 x_i + \mu_2 y_i & \lambda_2 x_j + \mu_2 y_j \end{vmatrix} = \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix} \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

where $\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}$ is a non-zero constant independent of i and j. Hence, up to scalar multiples, the coordinates are independent of the points chosen and can also be considered as the homogeneous coordinates of a point in $PG(5,q)$.

Moreover, by direct expansion, we can prove that the coordinates satisfy the condition

$$\Omega_1((\ell)) = p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.$$

Hence, if P is a point of $PG(5,q)$ representing a line of $PG(3,q)$, then P lies on the hyperbolic quadric

$$x_0x_5 + x_1x_4 + x_2x_3 = 0.$$

Conversely, the coordinates of each point of the quadric are the Plücker coordinates of some line in $PG(3,q)$.

Definition 1.9.2. ([72], p.28, [82]) Let $\mathcal{K}$ be the hyperbolic quadric above. Then $\mathcal{K}$ is called the *Klein quadric* and the correspondence between lines in $PG(3,q)$ and points on $\mathcal{K}$ is called the *Klein correspondence.* □

Remark 1.9.3. By considering a line ℓ of $PG(3,q)$ as being the intersection of two planes, we can develop dual Plűcker coordinates for ℓ in a similar way. □

Before examining the properties of Plűcker coordinates, it is convenient to briefly discuss the polarity of $PG(5,q)$ associated with the Klein quadric $\mathcal{K}$. (These details are drawn from [71], p.111, [72], p.4 and [44], pp.41-43.)

Using the equation of $\mathcal{K}$, we can define the bilinear form

$$\Omega_2((x),(y)) = (x_0y_5 + x_5y_0) + (x_1y_4 + x_4y_1) + (x_2y_3 + x_3y_2),$$

which maps $GF^6(q) \times GF^6(q)$ to $GF(q)$. This gives rise to a polarity ρ of $PG(5,q)$ with respect to which two points (x) and (y) are conjugate if and only if $\Omega_2((x),(y)) = 0$.

The points (x) which are self-conjugate satisfy the equation $\Omega_2((x),(x)) = 0$, i.e.

$$2(x_0x_5 + x_1x_4 + x_2x_3) = 0.$$

When q is odd, it follows that the self-conjugate points are exactly the points of $\mathcal{K}$ and ρ is an ordinary polarity. However, when q is even, every point is self-conjugate and ρ is a symplectic (or null) polarity.

Theorem 1.9.4. ([87], p.246) Let ℓ_1 and ℓ_2 be two lines in $PG(3,q)$ with Plűcker coordinates $(p_{01},p_{02},p_{03},p_{12},p_{31},p_{23})$ and $(q_{01},q_{02},q_{03},q_{12},q_{31},q_{23})$ respectively; then ℓ_1 and ℓ_2 intersect if and only if

$$p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} + p_{23}q_{01} + p_{31}q_{02} + p_{12}q_{03} = 0.$$

(Note: In [72], p.4, the term on the left is called the *mutual invariant* of ℓ_1 and ℓ_2.)

Proof. Choose two points (x_0,x_1,x_2,x_3), (y_0,y_1,y_2,y_3) on ℓ_1 and two points $(z_0,z_1,z_2,z_3),(w_0,w_1,w_2,w_3)$ on ℓ_2. Then the two lines intersect if and only if the four points are coplanar. A necessary and sufficient condition for this is that

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \\ w_0 & w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

Expanding the determinant along the first two rows, we obtain the mutual invariant on the left-handside of the equation, as required.

□

Remark 1.9.5. Given two distinct lines ℓ_1 and ℓ_2 of $PG(3,q)$, with Plűcker coordinate vectors (ℓ_1) and (ℓ_2) respectively, their mutual invariant is the same as $\Omega_2((\ell_1),(\ell_2))$. Hence, by Theorem 1.9.4, the lines ℓ_1 and ℓ_2 intersect if and only if $\Omega_2((\ell_1),(\ell_2)) = 0$, that is, if and only if (ℓ_1) and (ℓ_2) are conjugate with respect to the polarity ρ associated with $\mathcal{K}$. □

Apart from providing a convenient means for investigating the intersection of lines in $PG(3,q)$, one of the main advantages of working with Plűcker coordinates is that, when proving results regarding lines of $PG(3,q)$, the proofs can often be simplified by re-interpreting the lines as points of the Klein quadric $\mathcal{K}$ and making use of the properties of $\mathcal{K}$. For this reason, we now briefly examine the representations of some common sets of lines as point-sets on $\mathcal{K}$.

If we take a set of lines $\{\ell_1,\ldots,\ell_n\}$ in $PG(3,q)$ and consider an arbitrary linear combination of their Plűcker coordinate vectors, then in general the point $PG(5,q)$ which we obtain will not necessarily lie on the Klein quadric and so will not represent a line of $PG(3,q)$. Thus, we consider only those linear combinations which do yield lines of $PG(3,q)$. Given just two lines ℓ_1 and ℓ_2, it can be shown that a non-trivial linear combination of (ℓ_1) and (ℓ_2) yields the Plűcker coordinates of a third line ℓ_3 if and only if ℓ_1 and ℓ_2 intersect one another. In this case it can also be shown that

ℓ_1, ℓ_2 and ℓ_3 are coplanar and lie in a pencil. Thus, by taking all possible linear combinations of (ℓ_1) and (ℓ_2), we obtain the $q+1$ lines of a planar pencil in $PG(3,q)$ which is represented on $\mathcal{K}$ by the generator $\langle(\ell_1),(\ell_2)\rangle$.

Given three lines ℓ_1, ℓ_2 and ℓ_3 there are several different cases to consider. The only one of interest here is that in which ℓ_1, ℓ_2 and ℓ_3 are pairwise skew. It can be shown that exactly $q+1$ lines of $PG(3,q)$ have Plücker coordinate vectors which are linearly dependent on (ℓ_1), (ℓ_2) and (ℓ_3). These $q+1$ lines comprise the unique regulus containing ℓ_1, ℓ_2 and ℓ_3. The corresponding $q+1$ points on $\mathcal{K}$ all lie in the plane defined by (ℓ_1), (ℓ_2) and (ℓ_3) and no two of them lie on a common generator of $\mathcal{K}$. Hence they are the $q+1$ points of an irreducible conic on $\mathcal{K}$.

Extending this result, it follows that the lines of a regular spread of $PG(3,q)$ are represented on $\mathcal{K}$ by the points of an elliptic quadric; the elliptic quadric is the section of $\mathcal{K}$ by a three dimensional subspace whose image with respect to the polarity ρ is a line skew to $\mathcal{K}$.

Finally, in anticipation of the next section, the image on $\mathcal{K}$ of the lines of a general linear complex of $PG(3,q)$ is the set of points of a parabolic quadric; the parabolic quadric is the section of $\mathcal{K}$ by a non-tangent hyperplane of $PG(5,q)$.

For further details of these results and a more comprehensive list of examples see [87], Chapter XV and [72], pp.29-31.

1.10. LINE COMPLEXES OF $PG(3,q)$

Definition 1.10.1. ([71], p.49) Let $GF(q)[X]$ with $X = \{x_0, \ldots, x_n\}$ denote the ring of polynomials in the $n+1$ indeterminates $x_0, \ldots, x_n$ over $GF(q)$. Then a polynomial f in $GF(q)[X]$ is called a *form* if it is homogeneous. □

Definition 1.10.2. ([71], p.49) A *variety* in $PG(n,q)$ is a subset $\mathcal{F}$ of points in $PG(n,q)$ for which there exists a finite number of forms $f_1, f_2, \ldots, f_r$ in $GF(q)[X]$ such that

$$\mathcal{F} = \Big\{\text{points } P \in PG(n,q) \mid f_1(P) = f_2(P) \ldots = f_r(P) = 0\Big\}.$$

In this case we write

$$\mathcal{F} = V(f_1, f_2, \ldots, f_r).$$

□

Definition 1.10.3. ([71], p.49) A *primal* in $PG(n,q)$ is a variety in $PG(n,q)$ which is defined by a single form f in $GF(q)[X]$. The *order* of the primal is the degree of f.□

Definition 1.10.4. ([71], p.51) Let $\mathcal{F}$ be a variety contained in a subspace Σ_r of $PG(n,q)$ and Σ_s be a subspace of $PG(n,q)$ skew to Σ_r. Then the *cone* $(\Sigma_s)\mathcal{F}$ consists of the points on the lines $\langle P, Q\rangle$ with P in Σ_s and Q in $\mathcal{F}$. □

Remark 1.10.5. ([71], p.51) Each cone $(\Sigma_s)\mathcal{F}$ is a variety. □

Definition 1.10.6. ([87], p.371) Let f and Ω_1 be two forms in $\{x_0, \ldots, x_5\}$ over $GF(q)$ such that $\Omega_1((x)) = x_0x_5 + x_1x_4 + x_2x_3$ (so that $\Omega_1((x)) = 0$ is the Klein quadric $\mathcal{K}$) and Ω_1 is not a factor of f. Then the *line complex* of $PG(3,q)$ defined by f is the set of lines of $PG(3,q)$ represented by the points of the variety

$$V(f, \Omega_1),$$

(via the Klein correspondence). □

In discussing the nature of line complexes of $PG(3,q)$, it is useful to consider the lines of the complex through each point P. The lines of the complex through P form a cone known as the *complex cone* associated with P. Furthermore, if the form f

defining the complex has degree n, then the complex cones all have order n as well. Dually, the lines of the complex lying in a given plane π form an envelope known as the *complex envelope* associated with π; the class of each complex envelope (the notion of *class* dualises that of *order*) is also n.

When the degree of f is two, the complex $\mathrm{L}_q^{(2)}$ is called a *quadratic complex.* (See [87], pp.371-372.)

LINEAR COMPLEXES

Consider the linear form

$$f = a_5x_0 + a_4x_1 + a_3x_2 + a_2x_3 + a_1x_4 + a_0x_5$$

which defines the complex L_q. Let $a = (a_0, a_1, a_2, a_3, a_4, a_5)$ and consider $\Omega_1(a)$.

(i) If $\Omega_1(a) = 0$, then a lies on the Klein quadric $\mathcal{K}$ and hence represents a line ℓ of $PG(3,q)$. Consequently, any point $x = (x_0, x_1, x_2, x_3, x_4, x_5)$ in $V(f, \Omega_1)$ represents a line of $PG(3,q)$ meeting ℓ because f is equivalent to the mutual invariant (defined in the preceding section) of a and x. Thus L_q consists of ℓ and every line meeting ℓ and is known as a *special linear complex.*

(ii) If $\Omega_1(a) \neq 0$, then a doesn't lie on the Klein quadric $\mathcal{K}$ and so does not represent a line of $PG(3,q)$.

Consider an arbitrary point $P(p_0, p_1, p_2, p_3)$ of $PG(3,q)$ and let $x = (x_0, x_1, x_2, x_3)$ be a second point distinct from P. Then by calculating the Plücker coordinates of the line $\langle P, x\rangle$ and using the condition that "$f(\langle P, x\rangle) = 0$", we arrive at the equation

$$\begin{aligned}
&(-\ a_3p_3 - a_4p_2 - a_5p_1)x_0\\
+\ &(\quad a_1p_3 - a_2p_2 + a_5p_0)x_1\\
+\ &(-\ a_0p_3 + a_2p_1 + a_4p_0)x_2\\
+\ &(\quad a_0p_2 - a_1p_1 + a_3p_0)x_3 = 0.
\end{aligned}$$

Thus the line $\langle P, x\rangle$ lies in the complex if and only if it lies in the plane π containing P which is represented by the above equation. Hence the lines of L_q through P

form a (planar) pencil. This type of complex is known as a *general linear complex.* Furthermore, letting $[u_0, u_1, u_2, u_3]$ be the coordinate vector of π, we have

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_5 & -a_4 & -a_3 \\ a_5 & 0 & -a_2 & a_1 \\ a_4 & a_2 & 0 & -a_0 \\ a_3 & -a_1 & a_0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

A is a non-singular skew symmetric matrix which can be used as a one-to-one mapping between the point-set of $PG(3,q)$ and the plane-set of $PG(3,q)$. It also reverses inclusion and thus induces a one-to-one mapping from the line-set of $PG(3,q)$ back to itself. The polarity defined in this way is a *null* or *symplectic polarity.* The lines of $\mathbb{L}_q$ are exactly the totally isotropic lines of the polarity. (We note that the converse is also true, namely that the set of all totally isotropic lines of a given symplectic polarity is a general linear complex.) (See [44], pp.42-51, [87], pp.371-373.)

In [49], de Resmini gives an alternative characterisation of general linear complexes; we now present this.

Definition 1.10.7. ([49]) Let S be a set of lines in $PG(3,q)$. Then S is said to have the *property* $A_{m,n}$ if there exist integers m and n with $1 < m \leq n \leq q^2+q+1$ such that

(i) $|S \cap \mathcal{P}| = 1$ or m for every (planar) pencil $\mathcal{P}$,

(ii) $|S \cap \mathcal{S}| = m$ or n for every star $\mathcal{S}$ (a star is the set of all lines through a fixed point). □

Theorem 1.10.8. ([44]) Let S be a set of lines in $PG(3,q)$. If S has the property $A_{q+1,q+1}$, then S is a general linear complex. □

In later chapters, we shall need the following miscellaneous results on general linear complexes.

Theorem 1.10.9. (a) Let L_q and L'_q be two distinct general linear complexes in $PG(3,q)$. Then L_q and L'_q intersect in the set of lines of a regular spread of $PG(3,q)$. Furthermore, there are exactly $q+1$ distinct general linear complexes containing a given regular spread of $PG(3,q)$.

(b) Let L_q be a general linear complex of $PG(3,q)$ and $\mathcal{R}$ be a regulus in $PG(3,q)$ such that $|\mathrm{L}_q \cap \mathcal{R}| \geq 3$. Then $\mathcal{R}$ lies in L_q.

Proof. For the proof of part (a), see [72], pp.6-7.

(b) To prove the result, we can re-interpret the situation on the Klein quadric $\mathcal{K}$.

L_q is represented by the section of $\mathcal{K}$ by a (non-tangent) hyperplane Σ_4, while $\mathcal{R}$ is represented by an irreducible conic C on $\mathcal{K}$, lying in a plane π. Since $|\mathrm{L}_q \cap \mathcal{R}| \geq 3$, it follows that $|\Sigma_4 \cap C| \geq 3$ and so $|\Sigma_4 \cap \pi| \geq 3$. Hence π lies completely in Σ_4 and consequently, C lies completely in $\Sigma_4 \cap \mathcal{K}$.

Returning to the original situation, we have that $\mathcal{R}$ lies completely in L_q. □

QUADRATIC COMPLEXES

Unlike linear complexes, the quadratic complexes are not simple to classify because the number of possibilities for the complex cones is greatly increased. Their study, which centres largely around a particular quartic surface called the *Kummer surface* is beyond the scope of this introduction. Thus we merely state some definitions.

Definition 1.10.10. ([87], p.372) Let $\mathrm{L}_q^{(2)}$ be a quadratic complex of $PG(3,q)$. If the complex cone through a point P is a plane-pair, then P is called a *singular point* of $\mathrm{L}_q^{(2)}$. Dually, if the complex envelope in a plane π is a point-pair, then π is called a *singular plane* of $\mathrm{L}_q^{(2)}$. □

Definition 1.10.11. ([87], p.372) Let $\mathrm{L}_q^{(n)}$ be a line complex of $PG(3,q)$. A *total point* of $\mathrm{L}_q^{(n)}$ is a point P such that every line through P is a line of $\mathrm{L}_q^{(n)}$ and a *total plane* of $\mathrm{L}_q^{(n)}$ is a plane π such that every line in π is a line of $\mathrm{L}_q^{(n)}$. □

For a classical account of line complexes see also [78].

1.11. GENERALISED QUADRANGLES

Definition 1.11.1. ([104], p.1) A *finite generalised quadrangle* is a finite incidence structure with a symmetric incidence relation, satisfying the following axioms:

(i) Each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one common line.

(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one common point.

(iii) If P is a point and ℓ is a line not incident with P, then there are a unique point Q and a unique line m for which $P \mathbf{I} m \mathbf{I} Q \mathbf{I} \ell$. □

Remark 1.11.2. Axiom (iii) above guarantees that the structure has no triangles while it always has quadrangles. □

Remark 1.11.3. Since a generalised quadrangle GQ has a symmetric incidence relation, we can interchange the roles of the points and lines to obtain a new generalised quadrangle; this is known as the *dual generalised quadrangle* of GQ. □

In the sequel, unless otherwise stated, the material comes from [104].

The first examples of finite generalised quadrangles arose through their association with certain classical groups and were recognised as such by Tits. For this reason they are referred to as the classical generalised quadrangles. A variety of other generalised quadrangles have been constructed, but for the purposes of this thesis we only need to consider the two described below.

$Q(4,q)$:

Consider a non-degenerate (parabolic) quadric Q in $PG(4,q)$. Then the points of Q together with the lines of $PG(4,q)$ lying on Q constitute a generalised quadrangle (where the incidence relation is set inclusion). It is denoted by $Q(4,q)$ and its parameters are $s = t = q$. The number of points and lines of $Q(4,q)$ is $(q+1)(q^2+1)$ in both cases.

$W(q)$:

Consider a general linear complex L_q of $PG(3,q)$. Then the points of $PG(3,q)$ together with the lines of L_q constitute a generalised quadrangle (where, as for $Q(4,q)$, the incidence relation is set inclusion). It is denoted by $W(q)$ and its parameters are $s = t = q$. It also has $(q+1)(q^2+1)$ points and lines. □

It is no coincidence that $Q(4,q)$ and $W(q)$ have the same parameters; it is because $Q(4,q)$ is isomorphic to the dual of $W(q)$. This is a consequence of the fact that the image of a general linear complex on the Klein quadric is an irreducible 4-dimensional quadric. In a general linear complex L_q of $PG(3,q)$, the $q+1$ lines of L_q through a point are coplanar and so form a pencil. Thus each point of $W(q)$ can be represented by a pencil of lines in L_q. Therefore, treating the Klein correspondence as a mapping between $W(q)$ and $Q(4,q)$, we have that a point of $W(q)$ maps to the image of the corresponding pencil on $Q(4,q)$, namely a line. In addition, two lines of $W(q)$ which meet in a point P, map to two points on the line of $Q(4,q)$ which is the image of P. Finally, two points in $W(q)$ are collinear if and only if their corresponding pencils meet in a line. Hence they map to two lines on $Q(4,q)$ which meet in a point. Therefore, the Klein correspondence induces a one-to-one inclusion reversing mapping between $W(q)$ and $Q(4,q)$, from which the result follows.

Definition 1.11.4. An *ovoid* of a generalised quadrangle GQ is a set $\mathcal{O}$ of points of GQ such that each line of GQ is incident with a unique point of $\mathcal{O}$. A *spread* of GQ is a set of lines $\mathcal{S}$ such that each point of GQ is incident with a unique line of $\mathcal{S}$. □

It is immediate from their definition that ovoids and spreads are dual concepts. In particular, spreads (ovoids) of $W(q)$ are ovoids (respectively spreads) of $Q(4,q)$. Thus it is sufficient to discuss the spreads and ovoids of $W(q)$.

The spreads of $W(q)$ are also spreads of $PG(3,q)$ lying in the corresponding general linear complex $\mathbf{L}_q$. (We describe some of these spreads in Section 2.2.)

It can also be shown that each ovoid of $W(q)$ is also an ovoid of $PG(3,q)$; in fact, $W(q)$ possesses ovoids if and only if q is even. What is more, given an arbitrary ovoid of $PG(3,q)$, q even, the mapping which interchanges the points of the ovoid with their respective tangent planes is always extendable to a symplectic polarity, the totally isotropic lines of which are simply the lines tangent to the ovoid. (See [44], p.51 and [72], p.36.) Hence every ovoid of $PG(3,q)$, q even, is also an ovoid of some isomorphic copy of $W(q)$.

We have already discussed the known ovoids of $PG(3,q)$, q even, in Section 1.4. The Suzuku-Tits' ovoids actually arise from polarities that $W(q)$ possesses when $q = 2^{2h+1}$, $h > 1$. (Note: These polarities are not polarities of $PG(3,q)$.) Each ovoid consists of the absolute points of $W(q)$. The absolute lines in each case, constitute a spread of $W(q)$ known as the Lüneburg spread.

(This spread can be constructed as follows (see [91]): Let w and $\tau_{a,b}$ be two homographies of $PG(3, 2^{2h+1})$, $h \geq 1$, whose corresponding matrices are respectively

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a^\sigma & 1 & 0 \\ ab + a^\sigma + b^{\sigma+2} & a^{\sigma+1} + b & a & 1 \end{bmatrix},$$

where a and b are elements of $GF(2^{2h+1})$ and σ is the Frobenius automorphism defined at the end of Section 1.4. Then the Lüneburg spread consists of the lines

$$\begin{aligned} g_\infty &= \{(0,s,0,t) \mid s,t \in GF(2^{2h+1})\} \\ \text{and} \quad g_{a,b} &= g_\infty^{w\tau_{a,b}},\ a,b \in GF(2^{2h+1}).\ ) \end{aligned}$$

The existence of these polarities indicate that $W(q)$ is self-dual whenever $q = 2^{2h+1}$, $h > 1$. It was conjectured for a long time that $W(q)$ was always self-dual. However, eventually Thas showed in [115] that $W(q)$ is self-dual if and only if q is even. His proof of the self-duality when q is even relies on a one-to-one mapping which is constructed between the line-set of $PG(3, q)$ and the point-set of $PG(3, q)$. Thus he also showed that the study of ovoids in $PG(3, q)$, q even, is equivalent to the study of spreads of $PG(3, q)$, q even, lying in a general linear complex.

We conclude this section with

Theorem 1.11.5. ([104], p.55, Theorem 3,4.1.i)) $W(q)$ cannot be partitioned into spreads if q is even. □

1.12. DESIGNS

Definition 1.12.1. ([16]) A *balanced incomplete block design* BIBD is a finite incidence structure with v points, b blocks, k points on each block, r blocks through each point and λ blocks through each pair of distinct points. □

Remark 1.12.2. The following relations hold between the parameters of the design:

$$\text{(i)} \qquad vr = bk$$

$$\text{(ii)} \qquad \lambda(v-1) = r(k-1).$$

Hence r and b can be uniquely calculated given v, k and λ. Therefore balanced incomplete block designs are also called (v, k, λ)-designs (see [16]). □

Definition 1.12.3. ([16]) Let D be the BIBD constructed by taking n copies ($n > 1$) of a (v, k, λ)-design. Then D is called a *n-multiple design* and has parameters $(v, nb, nr, k, n\lambda)$. □

Definition 1.12.4. ([16]) Let D be a BIBD with parameters $(v, nb, nr, k, n\lambda)$ for some integer $n > 1$. Then D is called a *quasi-n-multiple design.* □

Not all quasi-n-multiple designs are n-multiple designs although clearly the converse is true. (We shall construct examples of such designs in Chapter IV.) Thus we have

Definition 1.12.5. ([16]) Let D be a quasi-n-multiple design. Then D is said to be *irreducible* if it cannot be partitioned into two quasi-n-multiple designs on the same point-set with $n_1 + n_2 = n$. □

Definition 1.12.6. ([105], p.164) Let D be a BIBD. Then D is called *resolvable* if the block set of D can be partitioned into sets (called *resolution classes*) in such a way that each point of D lies on a unique block of each set. □

Remark 1.12.7. The finite affine planes of order n are resolvable designs with parameters $(n^2, n^2+n, n+1, n, 1)$. The resolution classes are simply the parallel classes.

□

Definition 1.12.8. ([104], p.288) A *partial geometric design* is a finite incidence structure which satisfies the following properties:

(i) Each point is incident with $1+t$ $(t \geq 1)$ blocks and each block is incident with $1+s$ $(s \geq 1)$ points.

(ii) For each given point-block pair (x, L), $x \not\mathbf{I} L$ (respectively $x\mathbf{I}L$ we have $\sum\limits_{y\mathbf{I}L}[x,y] = \alpha$ (respectively β), where $[x, y]$ denotes the number of blocks incident with x and y. (Such a design is denoted by $D(s, t, \alpha, \beta)$. □

There exists an alternative definition of a partial geometric design; it is equivalent to the first definition but ostensibly it is quite different.

Definition 1.12.9. ([22]) Let S be a finite incidence structure with v points, b blocks, r blocks through each point and k points on each block. Let N be the incidence matrix of s. Then s is called a *partial geometric design* if in addition to

$$NJ = rJ, \quad JN = kJ,$$

we have

$$NN^tN = (r + k + c - 1 - u)N + uJ$$

(where J denotes the all one matrix of the appropriate size, N^t is the transpose of N and c, u are constants). $\square$

Remark 1.12.10. The parameters from the two definitions are related as listed below:

$$r = 1 + t$$

$$k = 1 + s$$

$$u = \alpha$$

$$c = \beta - s - t - 1$$

$\square$

For further results on partial geometric designs see also [21] and [23].

CHAPTER II

n-COVERS OF PG(3,q)

INTRODUCTION

In this chapter we introduce the notion of an n-cover of $PG(3,q)$ and describe some of its properties. In [53], Ebert discusses 2-covers of $PG(3,q)$ and constructs an infinite class of proper 2-covers in the case that q is odd. His study of 2-covers was motivated by the problem of completing partial packings of $PG(3,q)$ to packings. (Rather than considering this problem here in great detail, we shall mainly examine n-covers for their own sake.) In [71], p.83, Hirschfeld speaks of "k-fold spreads of m-spaces in $PG(n,q)$". An n-fold spread of 1-spaces in $PG(3,q)$ is exactly what we define to be an n-cover. As far as can be ascertained, no other authors have published results on n-covers with n greater than one. (In [14], Beutelspacher uses the term "n-cover of $PG(m,q)$". However, these n-covers are in no way related to the n-covers considered in this thesis.)

In the latter sections we focus our attention on the construction of proper n-covers of $PG(3,q)$, proving in particular the existence of proper n-covers of $PG(3,q)$ for some n satisfying $1 < n < q+1$ and for all $q = 2^h$, $h \geq 1$.

2.1. n-COVERS OF $PG(3,q)$

Definition 2.1.1. ([53], p.262) Let $\mathcal{C}_n$ be a set of lines of $PG(3,q)$. Then $\mathcal{C}_n$ is called an *n-cover* of $PG(3,q)$ if there are exactly n lines of $\mathcal{C}_n$ through each of the points of $PG(3,q)$. □

Remark 2.1.2. In discussing properties of an n-cover $\mathcal{C}_n$ we often implicitly associate $\mathcal{C}_n$ with the finite incidence structure $S = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ where $\mathbf{P}$ is the point-set of $PG(3,q)$, $\mathbf{B}$ is the set of lines of $\mathcal{C}_n$ and $\mathbf{I}$ is the incidence relation of $PG(3,q)$ restricted to $\mathbf{P} \times \mathcal{C}_n$.

Henceforth we shall denote this incidence structure by

$$(\mathbf{P}, \mathcal{C}_n, \mathbf{I}).$$

□

Lemma 2.1.3. Let $\mathcal{C}_{n_1}$ and $\mathcal{C}_{n_2}$ be respectively an n_1-cover and an n_2-cover of $PG(3,q)$.

1: If $\mathcal{C}_{n_1}$ and $\mathcal{C}_{n_2}$ are disjoint, then $\mathcal{C}_{n_1} \cup \mathcal{C}_{n_2}$ is an (n_1+n_2)-cover of $PG(3,q)$.

2: If $\mathcal{C}_{n_1}$ is a subset of $\mathcal{C}_{n_2}$, then $\mathcal{C}_{n_2} \backslash \mathcal{C}_{n_1}$ is an $(n_2 - n_1)$-cover of $PG(3,q)$.

Proof. 1: Through each point of $PG(3,q)$ there pass n_1 lines of $\mathcal{C}_{n_1}$ and n_2 lines of $\mathcal{C}_{n_2}$. Since $\mathcal{C}_{n_1}$ and $\mathcal{C}_{n_2}$ are disjoint, there are exactly n_1+n_2 distinct lines of $\mathcal{C}_{n_1} \cup \mathcal{C}_{n_2}$ passing through each point of $PG(3,q)$. Hence $\mathcal{C}_{n_1} \cup \mathcal{C}_{n_2}$ is an (n_1+n_2)-cover.

2: The proof of this follows similarly. □

Theorem 2.1.4. Let $\mathcal{C}_n$ be an n-cover of $PG(3,q)$. Then $|\mathcal{C}_n|$ is $n(q^2+1)$.

Proof. Let $\tau(P)$ be the number of lines of $\mathcal{C}_n$ incident with the point P of $PG(3,q)$. Then counting the number of flags of $(\mathbf{P}, \mathcal{C}_n, \mathbf{I})$ in two ways we have

$$(q+1)|\mathcal{C}_n| = \sum_{P \in PG(3,q)} \tau(P).$$

Now for each point P, $\tau(P)$ equals n. Therefore

$$(q+1)|\mathcal{C}_n| = \sum_{P \in PG(3,q)} n$$

which implies that

$$(q+1)|\mathcal{C}_n| = n(q+1)(q^2+1).$$

Hence

$$|\mathcal{C}_n| = n(q^2+1).$$

□

Definition 2.1.5. Let $\mathcal{C}_n$ be an n-cover of $PG(3,q)$. Then $\mathcal{C}_n$ is a ***proper n-cover*** if $\mathcal{C}_n$ cannot be represented as the union of an n_1-cover and an n_2-cover with $n_1+n_2=n$.

□

Remark 2.1.6. This definition does not quite agree with the definition of a proper n-cover given by Ebert in [53]; he calls an n-cover proper if it is not the union of n pairwise disjoint spreads. Our definition generalises this. $\square$

Remark 2.1.7. A 1-cover or spread of $PG(3,q)$ is always proper. On the other hand a (q^2+q+1)-cover of $PG(3,q)$ contains every line of $PG(3,q)$ and so is not proper because $PG(3,q)$ always possesses at least one spread. (See Section 1.8.) $\square$

Example 2.1.8. Let $\mathcal{P}$ be a packing of $PG(3,q)$ and let S be a subset of $\mathcal{P}$ containing exactly n spreads where n satisfies $1 \leq n \leq q^2+q+1$. The spreads are pairwise disjoint and so constitute an n-cover $\mathcal{C}_n$ of $PG(3,q)$. However $\mathcal{C}_n$, by construction, is not proper. $\square$

In general it is not a simple task to determine whether or not an n-cover of $PG(3,q)$ is proper. However for $n=2$, we have:

Theorem 2.1.9. ([53]) Let $\mathcal{C}_2$ be a 2-cover of $PG(3,q)$. If $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper $(2m+1)$-lateral for some $m \geq 1$, then $\mathcal{C}_2$ is a proper 2-cover. $\square$

We conjecture that the converse to Theorem 2.1.9 is also true, but we have not been able to prove it except when $q=2$. (This proof will appear in Chapter III.) An alternative necessary and sufficient condition for a 2-cover to be proper will be given in Chapter IV.

Definition 2.1.10. Let $\mathcal{C}_n$ be an n-cover of $PG(3,q)$. Then $\mathcal{C}_n$ is said to be a *dual n-cover* if each plane of $PG(3,q)$ contains exactly n lines of $\mathcal{C}_n$. $\square$

Remark 2.1.11. The duality of n-covers of $PG(3,q)$ is an extension of the duality of spreads which is discussed by Bruen and Fisher in [27]. There they prove that all t-spreads of $PG(2t+1,q)$ are dual. We shall prove shortly that all n-covers of $PG(3,q)$ are also dual, so in the particular case that $t=n=1$, we obtain the same result. However the argument here is different to that given in [27]. $\square$

Theorem 2.1.12. Let $\mathcal{C}_n$ be an n-cover of $PG(3,q)$. Then $\mathcal{C}_n$ is dual.

Proof. Let N be the number of lines of $\mathcal{C}_n$ in an arbitrary plane of $PG(3,q)$ and for each point $P \in \pi$ let $\tau(P)$ be the number of lines of $\mathcal{C}_n$ in π which are incident with P. Then counting the number of elements of the set

$$\{(P,\ell) \mid P \text{ is a point of } \pi,\ \ell \text{ is a line of } \mathcal{C}_n \text{ lying in } \pi\}$$

in two different ways, we have

$$N(q+1) = \sum_{P\in\pi} \tau(P).$$

By Theorem 2.1.4, $\mathcal{C}_n$ has $n(q^2+1)$ lines. Hence the number of lines of $\mathcal{C}_n$ not in π is

$$\begin{aligned} n(q^2+1) - N &= \sum_{P\in\pi}(n - \tau(P)) \\ &= n(q^2+q+1) - \sum_{P\in\pi} \tau(P) \\ &= n(q^2+q+1) - N(q+1). \end{aligned}$$

Hence

$$N = n.$$

□

Remark 2.1.13: A dual n-cover $\mathcal{C}_n$ is so-called because under a correlation δ of $PG(3,q)$, $\mathcal{C}_n$ maps to $\mathcal{C}_n^\delta$ which is also an n-cover of $PG(3,q)$. By Theorem 2.2.12, it is immediate then, that correlations of $PG(3,q)$ map n-covers to n-covers. It is also evident that an n-cover $\mathcal{C}_n$ is proper if and only if $\mathcal{C}_n^\delta$ is proper for each correlation δ of $PG(3,q)$. □

2.2. GENERAL LINEAR COMPLEXES AND SYMPLECTIC SPREADS OF $PG(3,q)$

In Section 2.3 we shall construct proper n-covers of $PG(3,q)$ using several different techniques. One such technique will require the removal of spreads from a general linear complex of $PG(3,q)$. Therefore, in this section, we establish a variety of results pertaining to these spreads which we shall refer to henceforth as symplectic spreads of $PG(3,q)$. (In [82], Kantor defines a symplectic spread to be a spread of a symplectic space. We shall not define these terms here, but simply note instead that a spread of a 4-dimensional symplectic space defined over $GF(q)$ is equivalent to a spread of $PG(3,q)$ lying in a general linear complex and so, that the two definitions actually concur.) To date the only known symplectic spreads of $PG(3,q)$ are the regular spreads for all q, the Lüneburg spreads for all $q = 2^{2h+1}$, $h \geq 1$ (see Section 1.11) and several classes of spreads (including a class of spreads giving rise to Knuth semifield planes) constructed for all q odd and non-prime, by Kantor in [82].

For further details concerning the symplectic spreads of $PG(3,q)$ see [104], Section 3.4, [116] and [51], Corollary 1.

Theorem 2.2.1. Let $\mathbb{L}_q$ be a general linear complex in $PG(3,q)$, q even. Then $\mathbb{L}_q$ cannot be partitioned into spreads.

Proof. This is merely a restatement of Theorem 1.11.5. □

Theorem 2.2.2. ([3]) Let $W(q)$ be the generalised quadrangle associated with a general linear complex of $PG(3,q)$, q even. Then each ovoid of $W(q)$ meets each classical ovoid of $W(q)$ (that is each elliptic quadric and Tits' ovoid embedded in $W(q)$) in an odd number of points. In particular the intersection is non-empty. □

Remark 2.2.3. Theorem 2.2.2 is due to Bagchi and Sastry who proved it in [3] via the consideration of the binary code arising from $W(q)$. They also proved that the dimension of the code is at most $\frac{q^3}{2} + q^2 + 2$.

An interesting corollary of this is that whenever the dimension of the code attains this upper bound (as it does for $q = 2, 4$), every ovoid of $W(q)$ meets every other ovoid of $W(q)$ in an odd number of points. (See also [4] for further results.) □

Corollary 2.2.4. Let $\L_q$ be a general linear complex of $PG(3,q)$, q even. Then every spread in $\L_q$ meets every regular spread and Lüneburg spread (if $q = 2^{2h+1}$, $h \geq 1$) embedded in $\L_q$ in at least one line.

Proof. Let $\mathcal{S}$, $\mathcal{S}_R$ and $\mathcal{S}_L$ (if $q = 2^{2h+1}$, $h > 1$) be respectively an arbitrary spread, a regular spread and a Lüneburg spread embedded in $\L_q$.

Then they are also spreads of $W(q)$, the generalised quadrangle associated with $\L_q$. $W(q)$ is self-dual because q is even. Therefore to each spread above, there corresponds an ovoid of $W(q)$. $\mathcal{S}_R$ corresponds to an elliptic quadric $\mathcal{O}_E$, $\mathcal{S}_L$ corresponds to a Tits' ovoid $\mathcal{O}_T$ and $\mathcal{S}$ corresponds to an ovoid $\mathcal{O}$.

By Theorem 2.2.2, $\mathcal{O}$ intersects both $\mathcal{O}_E$ and $\mathcal{O}_T$ in at least one point. Thus invoking once more, the correspondence between the ovoids and spreads of $W(q)$, it is immediate that $\mathcal{S}$ meets both $\mathcal{S}_R$ and $\mathcal{S}_L$ in at least one line. $\square$

Remark 2.2.5. From Corollary 2.2.4, we have that for q even, two regular spreads embedded in a general linear complex, meet in at least one line. This can also be deduced from a result in [60] (see Theorem 3.3.7, p.234). However, there seems to be no mention in the literature that this result is also valid when q is odd. We now give a proof of this which is independent of the parity of q. $\square$

Theorem 2.2.6. Let $\L_q$ be a general linear complex of $PG(3,q)$. Then two regular spreads embedded in $\L_q$ intersect in either a single line or the $q+1$ lines of a regulus. In particular, they intersect in at least one line.

Proof. To prove this result, we make use of the Klein correspondence. The image of $\L_q$ on the Klein quadric $\mathcal{K}$ is a parabolic quadric $\mathcal{Q}$ which is the intersection of $\mathcal{K}$ with a hyperplane Σ_4 of $PG(5,q)$ which is not tangent to $\mathcal{K}$. The images of the two regular spreads are elliptic quadrics $\mathcal{O}_E$ and $\mathcal{O}'_E$ each of which is the intersection of $\mathcal{K}$ with a suitable 3-dimensional projective space; let these spaces be Σ_3 and Σ'_3 respectively.

Moreover, $\mathcal{O}_E$ and $\mathcal{O}'_E$ are embedded in $\mathcal{Q}$ because the two regular spreads are embedded in Ł_q. It follows then that Σ_3 and Σ'_3 both lie in Σ_4. By the dimension theorem, we have that Σ_3 and Σ'_3 intersect in a plane π.

Now each plane in Σ_3 meets $\mathcal{O}_E$ in 1 or $q+1$ points. Similarly each plane of Σ'_3 meets $\mathcal{O}'_E$ in 1 or $q+1$ points. In addition, we have

$$\pi \cap \mathcal{O}_E = \pi \cap \mathcal{Q} = \pi \cap \mathcal{O}'_E \quad (= \pi \cap \mathcal{K}).$$

Therefore, the two elliptic quadrics meet in either a single point or the $q+1$ points of an irreducible conic (which is the image on $\mathcal{K}$ of a regulus in $PG(3,q)$).

It is immediate then, via the Klein correspondence, that the two regular spreads intersect in a single line or in the $q+1$ lines of a regulus. □

Theorem 2.2.7. ([122]) Let Ł_q be a general linear complex of $PG(3,q)$. Let $\mathcal{S}$ be a spread of $PG(3,q)$ and let $\mathcal{R}$ be a regulus with opposite regulus $\mathcal{R}'$. If $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{R}'$ all lie in Ł_q, then

$$|\mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}')| = 2.$$

Proof. (We include the proof here for the sake of clarity.) Let $\mathcal{D}$ be the set of $(q+1)^2$ points lying on the lines of $\mathcal{R}$. By a routine counting argument, it can be shown that each line of $\text{Ł}_q \backslash (\mathcal{R} \cup \mathcal{R}')$ meets each $\mathcal{D}$ in a unique point.

Now let $t = |\mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}'|$. The lines of $\mathcal{S}$ not lying in $\mathcal{R} \cup \mathcal{R}'$ each meet $\mathcal{D}$ in a single point by the argument above. Therefore, the number of such lines is $[(q+1)^2 - t(q+1)]$. Hence

$$q^2 + 1 = |\mathcal{S}| = t + p(q+1)^2 - t(q+1)]$$

which, on rearrangement gives $t = 2$. □

Remark 2.2.8. From Theorem 2.2.7, it is immediate that if a symplectic spread $\mathcal{S}$ of

$PG(3,q)$ possesses a regulus $\mathcal{R}$, then the spread

$$(\mathcal{S}\backslash\mathcal{R}) \cup \mathcal{R}'$$

(where $\mathcal{R}'$ is the opposite regulus of $\mathcal{R}$) does not lie in the general linear complex Ł_q containing $\mathcal{S}$, (for if it did, we would have $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{R}'$ all lying in Ł_q with

$$|\mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}')| = q+1,$$

a contradiction). $\square$

Theorem 2.2.9. A subregular spread (of $PG(3,q)$) of index k which is not also a regular spread, is not a symplectic spread.

Proof. By Orr's characterisation of subregular spreads (see Theorem 1.8.14), the subregular spread can be obtained from a regular spread $\mathcal{S}_R$ by reversing a set of r $(1 \leq r \leq q-2)$ pairwise disjoint reguli in $\mathcal{S}_R$. Setting $V = \left\{\mathcal{R}_i\right\}_{i=1}^{r}$, (that is, the set of reguli that are to be reversed), we denote the subregular spread by $\mathcal{S}_R^V$.

Assume that $\mathcal{S}_R^V$ is symplectic and let Ł_q be a general linear complex containing it. Using the restriction on r, we have

$$\left|\mathcal{S}_R \cap \mathcal{S}_R^V\right| = q^2+1-r(q+1) \geq q+3.$$

Since a regulus has exactly $q+1$ lines, this implies that amongst the lines of $\mathcal{S}_R \cap \mathcal{S}_R^V$, there are at least four lines not lying in a common regulus. By Theorem 1.8.12, these four lines give rise to a unique regular spread. Hence $\mathcal{S}_R$ is uniquely determined by the lines in $\mathcal{S}_R \cap \mathcal{S}_R^V$.

It is convenient at this point to shift from this setting to the corresponding setting on the Klein quadric $\mathcal{K}$.

Ł_q maps to a parabolic quadric $\mathcal{Q}$ which is the intersection of $\mathcal{K}$ with a hyperplane Σ_4 of $PG(5,q)$ which is not tangent to $\mathcal{K}$, $\mathcal{S}_R$ maps to an elliptic quadric $\mathcal{O}_E$ which is the intersection of $\mathcal{K}$ with a suitable 3-dimensional projective space Σ_3 and $\mathcal{S}_R^V$ maps to a subset of points of $\mathcal{Q}$.

Moreover, we have $|\mathcal{O}_E \cap \mathcal{Q}| \geq q+3$. Hence

$$\begin{aligned} q+3 &\leq \left|(\Sigma_3 \cap \mathcal{K}) \cap (\Sigma_4 \cap \mathcal{K})\right| \\ &= \left|(\Sigma_3 \cap \Sigma_4) \cap \mathcal{K}\right| \\ &\leq \left|\Sigma_3 \cap \Sigma_4\right|. \end{aligned}$$

Any such set of $q+3$ points in $\Sigma_3 \cap \Sigma_4$ determines Σ_3 uniquely because the corresponding $q+3$ lines determine $\mathcal{S}_R$ uniquely. Thus Σ_3 lies in Σ_4 because $\Sigma_4 \cap \Sigma_3$ is a projective space which contains the projective space (Σ_3) generated by the $q+3$ points. Reinterpreting this in the original setting, it follows that $\mathcal{S}_R$ also lies in L_q because

$$\mathcal{O}_E = \Sigma_3 \cap \mathcal{K} \subseteq \Sigma_4 \cap \mathcal{K} = \mathcal{Q}.$$

Choosing the regulus $\mathcal{R}_1$ from V, we now have that $\mathcal{S}_R$, $\mathcal{R}_1$ and $\mathcal{R}_1'$ all lie in L_q and

$$\left|\mathcal{S}_R \cap (\mathcal{R}_1 \cup \mathcal{R}_1')\right| = q+1,$$

contradicting Theorem 2.2.7. Therefore $\mathcal{S}_R^V$ is not a symplectic spread. □

In Theorem 2.2.7, we stated that if a pair of opposite reguli $\{\mathcal{R}, \mathcal{R}'\}$ lies in a general linear complex L_q, then any spread $\mathcal{S}$ shares exactly two lines with $\mathcal{R} \cup \mathcal{R}'$. However, we did not mention whether it is indeed possible for a pair of opposite reguli to lie in a general linear complex. It can be shown that a pair of opposite reguli can lie in L_q if and only if q is even. A proof of the sufficiency of this condition is presented in [122]. However, the proof we present here is original.

Lemma 2.2.10. Let L_q be a general linear complex of $PG(3,q)$, q even associated with the symplectic polarity ρ. If ℓ_1 and ℓ_2 are two skew lines of L_q, then there is exactly one regulus $\mathcal{R}$ containing these lines such that

$$(\mathcal{R} \cup \mathcal{R}') \subset \mathrm{L}_q$$

where $\mathcal{R}'$ is the opposite regulus of $\mathcal{R}$.

Proof. Let P be a point on ℓ_1 and let ℓ be a second line of $\mathbb{L}_q$ through P. The plane π containing ℓ_1 and ℓ then meets $\mathbb{L}_q$ in a pencil of lines which has P as its vertex. The line ℓ_2, being skew to ℓ_1, meets π in a single point Q. The line m of the pencil which contains Q is then a line of $\mathbb{L}_q$ which contains P and meets ℓ_2. It is in fact the unique line of $\mathbb{L}_q$ containing P and meeting ℓ_2 (because the existence of a second line m' of $\mathbb{L}_q$ satisfying this property would imply that the plane containing ℓ_2, m and m' met $\mathbb{L}_q$ in a set of lines not forming a pencil – a contradiction). Thus choosing three arbitrary points on ℓ_1, there exist three uniquely determined lines m_1, m_2 and m_3 of $\mathbb{L}_q$ passing through these points and meeting ℓ_2. Again, using the fact that each plane meets $\mathbb{L}_q$ in a pencil of lines, it can be shown that these lines are pairwise skew. Hence m_1, m_2 and m_3 determine a unique regulus $\mathcal{R}'$. Now since each line of $\mathcal{R}'$ is linearly dependent on the lines m_1, m_2 and m_3 (see Sections 1.9 and 1.10) which are fixed by ρ, it follows that each line of $\mathcal{R}'$ is also fixed by ρ and so $\mathcal{R}'$ lies in $\mathbb{L}_q$.

Consider now the action of ρ on the opposite regulus $\mathcal{R}$ of $\mathcal{R}'$. Each line of $\mathcal{R}$ meets every line of $\mathcal{R}'$ so ρ fixes $\mathcal{R}$ as it fixes $\mathcal{R}'$ linewise. Therefore ρ fixes some lines of $\mathcal{R}$ and permutes the other lines in orbits of length two. If the number of orbits of length two is N, then the number of lines fixed by ρ is

$$q + 1 - 2N$$

which is odd because q is even. $\mathcal{R}$ contains at least two lines fixed by ρ, namely ℓ_1 and ℓ_2. Hence it has at least one other line fixed by ρ. It is then immediate via the same argument that we used for $\mathcal{R}'$, that $\mathcal{R}$ lies in $\mathbb{L}_q$. Thus $\mathcal{R} \cup \mathcal{R}' \subset \mathbb{L}_q$ as required. □

To conclude this section, we demonstrate a technique for constructing partial geometric designs from general linear complexes with q even. Before proving the main result, we establish some simple lemmata.

Lemma 2.2.11. Let Ł_q be a general linear complex in $PG(3,q)$. Then the number of lines of Ł_q which are skew to a fixed line of Ł_q is q^3.

Proof. Let the fixed line be ℓ. Through each of the $q+1$ points of ℓ there are q other lines of Ł_q. Ł_q has $(q+1)(q^2+1)$ lines. Thus the number of lines of Ł_q skew to ℓ is

$$(q+1)(q^2+1)-(q+1)q-1=q^3.$$

□

Lemma 2.2.12. Let Ł_q be a general linear complex in $PG(3,q)$, q even. Then

(i) Each line ℓ of Ł_q lies in q^2 distinct pairs of opposite reguli that are embedded in Ł_q.

(ii) The number of distinct pairs of opposite reguli which lie in Ł_q is $\frac{1}{2}\,q^2(q^2+1)$.

Proof. (i) By Lemma 2.2.11, there are q^3 lines of Ł_q which are skew to ℓ. Each such line uniquely determines with ℓ, a pair of opposite reguli which lie in Ł_q and which contain ℓ. However the two reguli in each pair will contain between them q lines of Ł_q which are skew to ℓ. Hence the number of distinct pairs is

$$\frac{q^3}{q}=q^2.$$

(ii) Using Lemma 2.2.11, we deduce that the number of distinct pairs of skew lines in Ł_q is

$$\frac{1}{2}\,(q+1)(q^2+1)q^3.$$

Each such pair of lines uniquely determines a pair of opposite reguli (by Lemma 2.2.10) which lie in Ł_q. In addition, each such pair of opposite reguli is uniquely determined by each of the $2\times\frac{1}{2}\,q(q+1)$ pairs of skew lines contained in them. Hence the number of distinct pairs of opposite reguli lying in Ł_q is

$$\frac{1}{2}\,(q+1)(q^2+1)q^3\times\frac{1}{q(q+1)}=\frac{1}{2}\,q^2(q^2+1).$$

□

Lemma 2.2.13. Let $\mathbb{L}_q$ be a general linear complex in $PG(3,q)$, q even. Let ℓ and m be two distinct intersecting lines in $\mathbb{L}_q$. Then the number of distinct pairs of opposite reguli lying in $\mathbb{L}_q$ and containing ℓ and m is q.

Proof. To complete ℓ and m to two opposite reguli, we can begin by introducing a third line n of $\mathbb{L}_q$. For n to lie in one of the reguli, it must intersect every line of the opposite regulus. Hence assume that n meets either ℓ or m (but not both). It follows then that n is skew to either m or ℓ (otherwise ℓ, m and n would determine a plane which did not meet $\mathbb{L}_q$ in a pencil). Thus, by Lemma 2.2.10, the two skew lines determine a unique pair of reguli lying in $\mathbb{L}_q$ and by the method of proof used therein, it is immediate that ℓ, m and n all lie in the union of the two reguli. Thus to complete the proof we need to calculate how many of the pairs of opposite reguli are distinct.

Through each point of ℓ and m there pass q other lines of $\mathbb{L}_q$ and there are $2q$ points in $(\ell \cup m)\backslash(\ell \cap m)$. Hence n can be chosen $2q^2$ ways. However, in each pair of opposite reguli, the number of lines skew to either ℓ or m is $2q$. Therefore the number of distinct pairs is

$$\frac{2q^2}{2q} = q.$$

□

Lemma 2.2.14. Let $\mathbb{L}_q$ be a general linear complex in $PG(3,q)$, q even. Let $\mathcal{R}$ and $\mathcal{R}'$ be two opposite reguli lying in $\mathbb{L}_q$. Then every line of $\mathbb{L}_q$ not lying in $\mathcal{R} \cup \mathcal{R}'$, meets $\mathcal{R} \cup \mathcal{R}'$ in a single point.

Proof. This result is proven in Lemma 2.4 of [122] via a brief counting argument. □

Theorem 2.2.15. Let $\mathbb{L}_q$ be a general linear complex lying in $PG(3,q)$, q even. Define the following incidence structure $D = (\mathbf{P}, \mathbf{B}, \mathbf{I})$:

$$\mathbf{P} = \text{ the set of lines of } \L_q,$$

$$\mathbf{B} = \text{ the set of pairs of opposite reguli lying in } \L_q,$$

$$\mathbf{I} = \text{ set inclusion.}$$

Then (with respect to the notation in Definition 1.12.8) D is a partial geometric design with parameters $s = 2q + 1$, $t = q^2 - 1$, $\alpha = 4q$ and $\beta = 2q(q + 1)$.

Proof. (i) By Lemma 2.2.12, each line of $\L_q$ lies in q^2 distinct pairs of opposite reguli which also lie in $\L_q$. Hence each point of D lies in $1 + t = q^2$ blocks of D.

Each pair of opposite reguli lying in $\L_q$ contains $2(q + 1)$ distinct lines of $\L_q$. Hence each block of D contains $1 + s = 2(q + 1)$ points of D.

(ii) Let ℓ be a line of $\L_q$ and let $\mathcal{R} \cup \mathcal{R}'$ be the union of two opposite reguli lying in $\L_q$.

(a) Suppose that ℓ does not lie in $\mathcal{R} \cup \mathcal{R}'$. Then by Lemma 2.2.14, ℓ meets $\mathcal{R} \cup \mathcal{R}'$ in a single point. Let m and n be the two lines of $\mathcal{R} \cup \mathcal{R}'$ which are met by ℓ.

Now let p be an arbitrary line of $\mathcal{R} \cup \mathcal{R}'$.

If P is m or n, then $[\ell, p]$ (the number of distinct pairs of opposite reguli containing ℓ and p) is q by Lemma 2.2.13. If p is neither m nor n, then p is skew to ℓ, so by Lemma 2.2.10, $[\ell, p] = 1$. Thus

$$\begin{aligned} \sum_{p\mathbf{I}(\mathcal{R}\cup\mathcal{R}')} [\ell, p] &= \sum_{\substack{p=m \\ \text{or } p=n}} [\ell, p] + \sum_{p \neq \ell, m} [\ell, p] \\ &= 2q + 2q \\ &= 4q. \end{aligned}$$

It follows that if $(\ell, \mathcal{R} \cup \mathcal{R}')$ is a non-incident point-block pair in D, then $\alpha = \sum_{p\mathbf{I}(\mathcal{R}\cup\mathcal{R}')} [\ell, p] = 4q$.

(b) Suppose that ℓ lies in $\mathcal{R} \cup \mathcal{R}'$. Without loss of generality, let ℓ lie in $\mathcal{R}$.

Now let p be an arbitrary line of $\mathcal{R} \cup \mathcal{R}'$. If p equals ℓ, then $[\ell, p] = q^2$ by Lemma

2.2.12. If p lies in $\mathcal{R}\backslash\{\ell\}$, then p and ℓ are skew, so by Lemma 2.2.10, $[\ell,p]=1$. Finally if p lies in $\mathcal{R}'$, then ℓ and p meet in a single point, so $[\ell,p]=q$ by Lemma 2.2.13.

Thus

$$\begin{aligned}\sum_{p\mathrm{I}(\mathcal{R}\cup\mathcal{R}')}[\ell,p] &= \sum_{p=\ell}[\ell,p]+\sum_{p\in\mathcal{R}\backslash\{\ell\}}[\ell,p]+\sum_{p\in\mathcal{R}'}[\ell,p]\\ &= q^2+q+q(q+1)\\ &= 2q(q+1).\end{aligned}$$

It follows that if $(\ell,\mathcal{R}\cup\mathcal{R}')$ is an incident point-block pair in D, then $\beta = \sum_{p\mathrm{I}(\mathcal{R}\cup\mathcal{R}')}[\ell,p]=2q(q+1)$.

Therefore D is a partial geometric design with the specified parameters. □

Remark 2.2.16. The incidence matrix N of D has dimensions $v=\frac{q^3}{2}(q^2+1)(q+1)$ (by the first part of the proof of Lemma 2.2.12(ii)) and $b=\frac{q^2}{2}(q^2+1)$ (by Lemma 2.2.12(ii)). Furthermore, by the alternative definition of a partial geometric design (see Definition 1.12.9), N satisfies the matrix equation

$$NN^tN=2q(q-1)N+4qJ,$$

where J is the matrix of all ones with the same dimensions as N. The associated parameters take the values $u=4q$, $c=(q+1)(q-1)$, $r=q^2$ and $k=2(q+1)$. □

2.3. CONSTRUCTION TECHNIQUES FOR PROPER n-COVERS OF $PG(3,q)$

In [53], Ebert partitions $PG(3,q)$ into ovoids and via an examination of the lines tangent to a fixed ovoid, he obtains sets of $2(q^2+1)$ lines which constitute proper 2-covers of $PG(3,q)$ whenever q is odd. When q is even he also constructs proper 2-covers for the three cases $q=2$, 4 and 8.

The technique that we shall employ in this section to construct proper n-covers of $PG(3,q)$ involves beginning with known $(q+1)$-covers from which we remove a maximal set of pairwise disjoint spreads. Thus either we partition the $(q+1)$-cover into $(q+1)$ pairwise disjoint spreads or the set of remaining lines contains a subset of lines forming a proper n-cover for some n satisfying $2 \leq n \leq q+1$. A well-known example of a $(q+1)$-cover is the set of lines of a general linear complex in $PG(3,q)$. (When q is even this is equivalent to the set of lines tangent to an ovoid of $PG(3,q)$.)

A quadratic complex with no total points and no total planes sometimes constitutes a $(q+1)$-cover. For example, the quadratic complex comprising the lines tangent to an ovoid in $PG(3,q)$, q odd forms a $(q+1)$-cover (in this case the ovoid is necessarily and elliptic quadric (see Section 1.4); the corresponding quadratic complex has the q^2+1 points of the quadric as singular points and the q^2+1 tangent planes of the quadric as singular planes.) Besides these $(q+1)$-covers, it can be shown that the orbits of the lines of $PG(3,q)$ under the action of a Singer group all form $(q+1)$-covers except for one shorter orbit of size (q^2+1) which contains the lines of a regular spread (see [61]).

We begin by reviewing Ebert's partition of the point-set of $PG(3,q)$ by ovoids and then describe some connections between the partition and the Singer orbits of the lines of $PG(3,q)$.

In Section 1.3, we described Singer's representation of $PG(n,q)$ by a primitive element β of $GF(q^{n+1})$. Thus setting n equal to 3, we have a representation of $PG(3,q)$. The following results are established in [52] and [53]:

With respect to this representation, the set

$$\Omega_0 = \left\{\beta^{s(q+1)} \mid s = 0, 1, \ldots, q^2\right\}$$

is an ovoid of $PG(3,q)$. Then because $\langle\beta\rangle$ acts as a collineation group of $PG(3,q)$, the set

$$\Omega_i = \Omega_0 \beta^i$$

is again an ovoid for each i from 0 to q. Furthermore, these $q+1$ ovoids partition the point-set of $PG(3,q)$. (In [53], this partition is referred to as an ovoidal fibration.)

When q is odd, Ebert shows in [52], that each line of $PG(3,q)$ is tangent to exactly 0 or 2 ovoids in the partition. There are $\frac{(q^2+1)^2}{2}$ lines of the former type and $\frac{(q^2+1)(q+1)^2}{2}$ lines of the latter. By distinguishing the two types of lines, he is able to construct examples of spreads and proper 2-covers of $PG(3,q)$, q odd as follows:

(i) ([53]) Let t be an arbitrary odd integer satisfying $1 \le t \le q$. then the two lines

$$\begin{aligned} \ell &= \langle \beta^0, \beta^{t\frac{(q^2+1)}{2}} \rangle \\ \text{and} \quad \ell' &= \langle \beta^0, \beta^{(t+q+1)\frac{(q^2+1)}{2}} \rangle \end{aligned}$$

are both tangent to Ω_0 and Ω_t. In addition, they are each secant to $\frac{(q-1)}{2}$ of the other ovoids, but they are never secant to the same ovoid. Now the $q+1$ ovoids are fixed by the collineation group $\langle \beta^{q+1} \rangle$ which acts transitively on the points of each ovoid. Thus, denoting by $[\ell]$ and $[\ell']$, the respective orbits of ℓ and ℓ' under the action of $\langle \beta^{q+1} \rangle$, it is immediate that the set of lines

$$\mathcal{C}_2 = [\ell] \cup [\ell']$$

is a 2-cover of $PG(3,q)$. Furthermore $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ always contains a proper $(2m+1)$-lateral for some $m \ge 1$. Hence by Theorem 2.1.9, $\mathcal{C}_2$ is proper.

As t runs through the odd integers $1, 3, \ldots, q$, the 2-covers are pairwise disjoint because the $\frac{q+1}{2}$ pairs of lines $\{\ell, \ell'\}$ contain between them exactly the $q+1$ tangent lines of Ω_0 at the point β^0. Thus by the construction technique, it follows that each tangent line of Ω_0 lies in exactly one 2-cover. By the transitivity of $\langle \beta \rangle$ on the $q+1$ ovoids, the result extends to every line of $PG(3,q)$ tangent to one of the ovoids.

Consequently, the set of $\frac{(q^2+1)(q+1)^2}{2}$ lines tangent to two ovoids can be partitioned into $\left(\frac{q+1}{2}\right)^2$ proper 2-covers.

(ii) ([5],[54]) Let s be an arbitrary odd integer satisfying $1 \le s \le q^2$. Then there exists a positive integer d (depending on s) such that the two lines

$$\begin{aligned} \ell &= \langle \beta^0, \beta^{s(q+1)} \rangle \\ \text{and} \quad \ell' &= \beta^d(\ell^q) = \langle \beta^d, \beta^{d+sq(q+1)} \rangle \end{aligned}$$

(where ℓ^q denotes the image of ℓ under the automorphic collineation defined by $x \mapsto x^q$, $x \in GF(q^4)$) are both secant to exactly $\frac{q+1}{2}$ ovoids in the fibration but are never secant to the same ovoid. The collineation group $\langle \beta^{2(q+1)} \rangle$ fixes the $q+1$ ovoids and under its action, the point-pairs $\{\beta^0, \beta^{s(q+1)}\}$ and $\{\beta^d, \beta^{d+sq(q+1)}\}$ generate orbits of size $\frac{q^2+1}{2}$. Each point of Ω_0 lies in a unique point-pair in the first orbit and each point of Ω_d lies in a unique point-pair in the second orbit. Thus, denoting by (ℓ) and (ℓ') the respective orbits of ℓ and ℓ' under the action of $\langle \beta^{2(q+1)} \rangle$, it is immediate that the set of lines

$$(\ell) \cup (\ell')$$

is a spread of $PG(3, q)$.

When $s \neq \frac{(q^2+1)}{2}$, the integer d is uniquely determined by s. However, when $s = \frac{q^2+1}{2}$, any odd, positive integer can be taken for d. In the latter case, we have that $\left(\beta^{s(q+1)}\right)^{q^2-1} = \left(\beta^{q^4-1}\right)^{\frac{q+1}{2}} = 1$ because β is a primitive element of $GF(q^4)$ over $GF(q)$, and so $\beta^{s(q+1)}$ is an element of $GF(q^2)$. Thus, the line $\langle \beta^0, \beta^{s(q+1)} \rangle$ coincides with the set of $q+1$ elements of

$$\left\{\beta^0, \beta^{q^2+1}, \beta^{2(q^2+1)}, \ldots, \beta^{q(q^2+1)}\right\}.$$

Under the action of $\langle \beta \rangle$, this line generates the short Singer orbit (mentioned in the introduction to this section) which comprises the q^2+1 lines of a regular spread. This is the same spread which arises from this line via Ebert's construction.

As s runs through the odd integers $1, 3, \ldots, q^2$, the spreads are pairwise disjoint and the secants used to construct them account for all the lines through β^0 which are secant to exactly $\frac{q+1}{2}$ ovoids. Since $\langle \beta \rangle$ acts transitively on the ovoids in the fibration, it follows that the set of $\frac{(q^2+1)^2}{2}$ lines secant to exactly $\frac{q+1}{2}$ ovoids can be partitioned into $\frac{q^2+1}{2}$ spreads.

Combining the results outlined in parts (i) and (ii), it follows that the lineset of $PG(3,q)$, q odd can be partitioned into the union of $\frac{q^2+1}{2}$ spreads and $\left(\frac{q+1}{2}\right)^2$ proper 2-covers.

We now turn our attention to the case where q is even in [54], it is shown that in $PG(3,q)$, q even, the number of lines tangent to exactly one ovoid in the fibration is $q(q+1)(q^2+1)$, while the number of lines tangent to every ovoid in the fibration is q^2+1. The $(q+1)(q^2+1)$ lines tangent to an ovoid in the fibration form a general linear complex because q is even. Thus, there are $q+1$ distinct general linear complexes determined by the fibration. These complexes all contain the q^2+1 lines which are tangent to every ovoid in the fibration. Now by Theorem 1.10.9, two distinct general linear complexes meet in the set of lines of a regular spread and the number of distinct general linear complexes containing this spread is $q+1$. Thus the set $\mathcal{S}$ of lines tangent to every ovoid in the fibration is the set of lines of a regular spread and the $q+1$ general linear complexes determined by the ovoids account for all of the general linear complexes containing $\mathcal{S}$. Moreover, the Singer group $\langle\beta\rangle$ fixes $\mathcal{S}$ because it simply permutes the ovoids in the fibration. Therefore $\mathcal{S}$ is the short orbit which is referred to in [61]. (We note here that for q even, just as for the case of q odd, the short orbit is generated by the line $\left\{\beta^0, \beta^{q^2+1}, \beta^{2(q^2+1)}, \ldots, \beta^{q(q^2+1)}\right\}$ arising from the non-zero elements of $GF(q^2)$.) Finally, as $\langle\beta\rangle$ cyclically permutes the ovoids in the fibration while fixing $\mathcal{S}$, it is immediate that $\langle\beta\rangle$ also cyclically permutes the general linear complexes containing $\mathcal{S}$. From this we deduce:

Theorem 2.3.1. Let $\mathcal{S}_R$ be a regular spread of $PG(3,q)$, q even. Then there exists a collineation group of $PG(3,q)$ which fixes $\mathcal{S}_R$ and cyclically permutes the $q+1$ general linear complexes containing $\mathcal{S}_R$.

Proof. It was remarked after Theorem 1.8.11, that all regular spreads are projectively equivalent. Hence, there exists a collineation σ mapping $\mathcal{S}_R$ to the spread $\mathcal{S}$ defined above. It is then routine to show that $\sigma^{-1}\langle\beta\rangle\sigma$ is a collineation group of $PG(3,q)$, fixing $\mathcal{S}_R$ and cyclically permuting the general linear complexes containing $\mathcal{S}_R$. □

Remark 2.3.2. Theorem 2.3.1 is proven by different means in [122]. □

Corollary 2.3.3. Let $\mathbb{L}_q$ and $\mathbb{L}'_q$ be two general linear complexes in $PG(3,q)$, q even and let $\mathcal{S}$ and $\mathcal{S}'$ be regular spreads lying in the respective complexes. Then there exists a collineation τ which simultaneously maps $\mathbb{L}_q$ to $\mathbb{L}'_q$ and $\mathcal{S}$ to $\mathcal{S}'$.

Proof. In $PG(3,q)$, all regular spreads are projectively equivalent, so there exists a collineation σ mapping $\mathcal{S}$ to $\mathcal{S}'$. σ also maps $\mathbb{L}_q$ to a general linear complex $\sigma(\mathbb{L}_q)$ which contains $\mathcal{S}'$. $\mathbb{L}'_q$ also contains $\mathcal{S}'$ by hypothesis, therefore, by Theorem 2.3.1, there is a collineation ϕ which fixes $\mathcal{S}'$ and maps $\sigma(\mathbb{L}_q)$ to $\mathbb{L}'_q$.

Setting τ equal to $\phi\sigma$, we see that

$$\tau(\mathcal{S}) = \mathcal{S}'$$

and

$$\tau(\mathbb{L}_q) = \mathbb{L}'_q.$$

□

Corollary 2.3.4. The full collineation group $PSp(4,q)$ fixing a general linear complex of $PG(3,q)$, q even, acts transitively on the regular spreads embedded in the complex.

Proof. Let $\mathbb{L}_q = \mathbb{L}'_q$ in Corollary 2.3.3. The result is then immediate. □

PROPER n-COVERS OF $PG(3,q)$ ARISING FROM LONG SINGER ORBITS

Utilising the setting established above, we examine the orbits of the lines of $PG(3,q)$ under the action of a Singer group in the cases $q=2$ and $q=3$.

q=2: Let β be a primitive element of $GF(2^4)$ satisfying $\beta^4=\beta+1$. Then the powers of β represent the points of $PG(3,2)$ as listed below (we represent β^i by its exponent i):

1	$(0,1,0,0)$	6	$(0,0,1,1)$	11	$(0,1,1,1)$
2	$(0,0,1,0)$	7	$(1,1,0,1)$	12	$(1,1,1,1)$
3	$(0,0,0,1)$	8	$(1,0,1,0)$	13	$(1,0,1,1)$
4	$(1,1,0,0)$	9	$(0,1,0,1)$	14	$(1,0,0,1)$
5	$(0,1,1,0)$	10	$(1,1,1,0)$	$15 \equiv 0$	$(1,0,0,0)$.

By observation the point triples $\{0,5,10\}$, $\{0,1,4\}$ and $\{0,2,8\}$ are lines of $PG(3,2)$. Under the action of $\langle\beta\rangle$, the lines generate the three line orbits.

$$\begin{array}{ll}
\text{(i)} & \big\{\{0+i, 1+i, 4+i\} \mid i = 0, \ldots, 14\big\} \\
\text{(ii)} & \big\{\{0+i, 2+i, 8+i\} \mid i = 0, \ldots, 14\big\} \\
\text{(iii)} & \big\{\{0+i, 5+i, 10+i\} \mid i = 0, \ldots, 4\big\}.
\end{array}$$

These orbits are listed below in full. (N.B. These lists also appear in [71], p.84):

(i)	$\{0,1,4\}$	$\{5,6,9\}$	$\{10,11,14\}$
	$\{1,2,5\}$	$\{6,7,10\}$	$\{11,12,0\}$
	$\{2,3,6\}$	$\{7,8,11\}$	$\{12,13,1\}$
	$\{3,4,7\}$	$\{8,9,12\}$	$\{13,14,2\}$
	$\{4,5,8\}$	$\{9,10,13\}$	$\{14,0,3\}$
(ii)	$\{0,2,8\}$	$\{5,7,13\}$	$\{10,12,3\}$
	$\{1,3,9\}$	$\{6,8,14\}$	$\{11,13,4\}$
	$\{2,4,10\}$	$\{7,9,0\}$	$\{12,14,5\}$
	$\{3,5,11\}$	$\{8,10,1\}$	$\{13,0,6\}$
	$\{4,6,12\}$	$\{9,11,2\}$	$\{14,1,7\}$
(iii)	$\{0,5,10\}$		
	$\{1,6,11\}$		
	$\{2,7,12\}$		
	$\{3,8,13\}$		
	$\{4,9,14\}$		

Orbit (iii) constitutes a spread $\mathcal{S}$ of $PG(3,2)$. We showed immediately preceding Theorem 2.3.1, that $\mathcal{S}$ is regular but in this instance the result is trivial because all spreads of $PG(3,2)$ are regular.

Consider orbit (i). If it does not represent a proper 3-cover of $PG(3,2)$, then it is either the union of three pairwise disjoint spreads or it is the union of a spread and a proper 2-cover (which is disjoint from the spread). In fact orbit (i) does not contain a spread; we now show this.

Assume that orbit (i) contains a spread $\mathcal{S}$. Since $\langle\beta\rangle$ acts cyclically and hence transitively on the lines of the orbit, we can suppose without loss of generality, that $\{1,2,5\}$ is a line of $\mathcal{S}$. $\mathcal{S}$ contains a line which passes through the point 3. This line can only be $\{3,4,7\}$ or $\{3,14,15\}$. $\mathcal{S}$ also contains a line through the point 6. However there are no lines of orbit (i) which pass through 6 and which are skew to $\{1,2,5\}$ and $\{3,4,7\}$ or to $\{1,2,5\}$ and $\{3,14,15\}$. This gives us a contradiction. Hence orbit (i) contains no spread.

A similar analysis can be made of orbit (ii) to show that it does not contain a spread either. Hence we have

Theorem 2.3.5. The line orbits of $PG(3,2)$ containing 15 lines represent proper 3-covers of $PG(3,2)$. □

$\mathbf{q=3}$: This case will only be partially treated here for reasons of practicability. Despite this, we obtain some interesting results.

Let β be a primitive element of $GF(3^4)$ satisfying $\beta^4 = -\beta^3 + 1$. Then the powers of β represent the points of $PG(3,3)$ as listed below (as for the case $q = 2$, we represent β^i by i):

1 $(0,1,0,0)$	11 $(1,1,0,1)$	21 $(1,-1,0,1)$	31 $(1,0,0,1)$
2 $(0,0,1,0)$	12 $(1,1,1,-1)$	22 $(1,1,-1,-1)$	32 $(1,1,0,-1)$
3 $(0,0,0,1)$	13 $(1,-1,-1,1)$	23 $(1,-1,-1,0)$	33 $(1,-1,-1,-1)$
4 $(1,0,0,-1)$	14 $(1,1,-1,1)$	24 $(0,1,-1,-1)$	34 $(1,-1,1,0)$
5 $(1,-1,0,-1)$	15 $(1,1,1,1)$	25 $(1,0,-1,0)$	35 $(0,1,-1,1)$
6 $(1,-1,1,-1)$	16 $(1,1,1,0)$	26 $(0,1,0,-1)$	36 $(1,0,1,1)$
7 $(1,-1,1,1)$	17 $(0,1,1,1)$	27 $(1,0,-1,-1)$	37 $(1,1,0,0)$
8 $(1,1,-1,0)$	18 $(1,0,1,0)$	28 $(1,-1,0,0)$	38 $(0,1,1,0)$
9 $(0,1,1,-1)$	19 $(0,1,0,1)$	29 $(0,1,-1,0)$	39 $(0,0,1,1)$
10 $(1,0,-1,1)$	20 $(1,0,1,-1)$	30 $(0,0,1,-1)$	$40 \equiv 0$ $(1,0,0,0)$

The following four sets of points represent lines of $PG(3,3)$: $\{0,10,20,30\}$, $\{0,3,4,31\}$, $\{0,2,18,25\}$ and $\{0,5,11,19\}$. Using them, we obtain the four line orbits (under the action of $\langle\beta\rangle$)

$$\begin{array}{ll}
\text{(i)} & \big\{\{0+i,3+i,4+i,31+i\} \mid i=0,\ldots,39\big\} \\
\text{(ii)} & \big\{\{0+i,2+i,18+i,25+i\} \mid i=0,\ldots,39\big\} \\
\text{(iii)} & \big\{\{0+i,5+i,11+i,19+i\} \mid i=0,\ldots,39\big\} \\
\text{(iv)} & \big\{\{0+i,10+i,20+i,30+i\} \mid i=0,\ldots,9\big\}.
\end{array}$$

These orbits are listed below in full.

(i)	$\{0,3,4,31\}$	$\{10,13,14,1\}$	$\{20,23,24,11\}$	$\{30,33,34,21\}$
	$\{1,4,5,32\}$	$\{11,14,15,2\}$	$\{21,24,25,12\}$	$\{31,34,35,22\}$
	$\{2,5,6,33\}$	$\{12,15,16,3\}$	$\{22,25,26,13\}$	$\{32,35,36,23\}$
	$\{3,6,7,34\}$	$\{13,16,17,4\}$	$\{23,26,27,14\}$	$\{33,36,37,24\}$
	$\{4,7,8,35\}$	$\{14,17,18,5\}$	$\{24,27,28,15\}$	$\{34,37,38,25\}$
	$\{5,8,9,36\}$	$\{15,18,19,6\}$	$\{25,28,29,16\}$	$\{35,38,39,26\}$
	$\{6,9,10,37\}$	$\{16,19,20,7\}$	$\{26,29,30,17\}$	$\{36,39,0,27\}$
	$\{7,10,11,38\}$	$\{17,20,21,8\}$	$\{27,30,31,18\}$	$\{37,0,1,28\}$
	$\{8,11,12,39\}$	$\{18,21,22,9\}$	$\{28,31,32,19\}$	$\{38,1,2,29\}$
	$\{9,12,13,0\}$	$\{19,22,23,10\}$	$\{29,32,33,20\}$	$\{39,2,3,30\}$

(ii)	$\{0,2,18,25\}$	$\{10,12,28,35\}$	$\{20,22,38,5\}$	$\{30,32,8,15\}$
	$\{1,3,19,26\}$	$\{11,13,29,36\}$	$\{21,23,39,6\}$	$\{31,33,9,16\}$
	$\{2,4,20,27\}$	$\{12,14,30,37\}$	$\{22,24,0,7\}$	$\{32,34,10,17\}$
	$\{3,5,21,28\}$	$\{13,15,31,38\}$	$\{23,25,1,8\}$	$\{33,35,11,18\}$
	$\{4,6,22,29\}$	$\{14,16,32,39\}$	$\{24,26,2,9\}$	$\{34,36,12,19\}$
	$\{5,7,23,30\}$	$\{15,17,33,0\}$	$\{25,27,3,10\}$	$\{35,37,13,20\}$
	$\{6,8,24,31\}$	$\{16,18,34,1\}$	$\{26,28,4,11\}$	$\{36,38,14,21\}$
	$\{7,9,25,32\}$	$\{17,19,35,2\}$	$\{27,29,5,12\}$	$\{37,39,15,22\}$
	$\{8,10,26,33\}$	$\{18,20,36,3\}$	$\{28,30,6,13\}$	$\{38,0,16,23\}$
	$\{9,11,27,34\}$	$\{19,21,37,4\}$	$\{29,31,7,14\}$	$\{39,1,17,24\}$
(iii)	$\{0,5,11,19\}$	$\{10,15,21,29\}$	$\{20,25,31,39\}$	$\{30,35,1,9\}$
	$\{1,6,12,20\}$	$\{11,16,22,30\}$	$\{21,26,32,0\}$	$\{31,36,2,10\}$
	$\{2,7,13,21\}$	$\{12,17,23,31\}$	$\{22,27,33,1\}$	$\{32,37,3,11\}$
	$\{3,8,14,22\}$	$\{13,18,24,32\}$	$\{23,28,34,2\}$	$\{33,38,4,12\}$
	$\{4,9,15,23\}$	$\{14,19,25,33\}$	$\{24,29,35,3\}$	$\{34,39,5,13\}$
	$\{5,10,16,24\}$	$\{15,20,26,34\}$	$\{25,30,36,4\}$	$\{35,0,6,14\}$
	$\{6,11,17,25\}$	$\{16,21,27,35\}$	$\{26,31,37,5\}$	$\{36,1,7,15\}$
	$\{7,12,18,26\}$	$\{17,22,28,36\}$	$\{27,32,38,6\}$	$\{37,2,8,16\}$
	$\{8,13,19,27\}$	$\{18,23,29,37\}$	$\{28,33,39,7\}$	$\{38,3,9,17\}$
	$\{9,14,20,28\}$	$\{19,24,30,38\}$	$\{29,34,0,8\}$	$\{39,4,10,18\}$
(iv)	$\{0,10,20,30\}$	$\{3,13,23,33\}$	$\{6,16,26,36\}$	$\{9,19,29,39\}$
	$\{1,11,21,31\}$	$\{4,14,24,34\}$	$\{7,17,27,37\}$	
	$\{2,12,22,32\}$	$\{5,15,25,35\}$	$\{8,18,28,38\}$	

Orbit (iv) consists of the lines of a regular spread of $PG(3,3)$ (see the comment in the introduction to this section).

Orbit (i) is generated by the line $\{0,3,4,31\}$. By listing the elements of Ω_0, Ω_1, Ω_2 and Ω_3, it is easy to see that this line is secant to Ω_0 and Ω_3. Thus, all of the lines in orbit (i) are secant to exactly two of the ovoids because $\langle\beta\rangle$ permutes the ovoids.

Now since $q = 3$, there are $\frac{(3^2+1)^2}{2} = 50$ lines of $PG(3,3)$ which are secant to exactly two ovoids. Ten of these lines lie in orbit (iv); the other forty lie in orbit (i). Therefore by the results outlined for q odd at the beginning of this section, it follows that orbit (i) can be partitioned into four distinct spreads.

These spreads can be described as follows:

(a) $\mathcal{S}_1 = (\{0,3,4,31\}) \cup (\{2,5,6,33\})$,

(b) $\mathcal{S}_2 = (\{4,7,8,35\}) \cup (\{6,9,10,37\})$,

where in both cases ( ) denotes the orbit of the line under the action of $\langle \beta^8 \rangle$ and

(c) $\mathcal{S}_3 = \beta(\mathcal{S}_1)$,

(d) $\mathcal{S}_4 = \beta(\mathcal{S}_2)$.

In [54], Ebert shows for all q odd, that the spreads constructed in this way give rise to flag-transitive affine planes, that is, affine planes which admit collineation groups which act transitively on the incident point-line pairs of the plane. In $PG(3,3)$ there are exactly two projectively distinct spreads, namely the regular spread and the subregular spread of index one. These both give rise to flag-transitive planes (see [26], p.437 and [84]). Hence, this property does not distinguish the spreads and we are forced to examine the reguli which contain at least three lines of each spread.

It is routine to check that the lines of $\mathcal{S}_1$ which meet the line $\{0,5,11,19\}$ do not form a regulus. Similarly, the lines of $\mathcal{S}_2$ which meet the line $\{4,9,15,23\}$ do not form a regulus. Hence $\mathcal{S}_1$ and $\mathcal{S}_2$ are both subregular of index one. It follows by their definitions that $\mathcal{S}_3$ and $\mathcal{S}_4$ are also subregular of index one.

Orbits (ii) and (iii) cannot be partitioned into spreads and so are projectively distinct from orbit (i). In fact, neither of them contains even a single spread; it is not known, however, if either of them contains a proper 2-cover. Hence, they are either proper 4-covers of $PG(3,3)$ or they can be partitioned into two proper 2-covers of $PG(3,3)$. We now show that orbit (ii) does not contain a spread of $PG(3,3)$. (The proof for orbit (iii) can be argued in similar fashion.)

Assume that orbit (ii) does contain a spread $\mathcal{S}$. $\langle\beta\rangle$ acts cyclically and so transitively on the lines of orbit (ii). Hence, we can suppose, without loss of generality, that $\{0,2,18,25\}$ is a line of the spread. Since $\mathcal{S}$ is a spread, there exists a line in $\mathcal{S}$ which passes through the point 1 and which is skew to $\{0,2,18,25\}$. This line is either $\{1,3,19,26\}$ or $\{1,17,24,39\}$. Concentrating on the line $\{1,3,19,26\}$ first, we can repeat the argument with respect to the points 4, 5 and 8. In this order, the lines are uniquely determined as follows: $\{0,2,18,25\}$, $\{1,3,19,26\}$, $\{4,6,22,29\}$, $\{5,7,23,30\}$. However, there is no line in orbit (ii) which passes through the point 8 and which is skew to the four lines. A similar analysis for the line $\{1,17,24,39\}$ yields the 4 lines: $\{0,2,18,25\}$, $\{1,17,24,39\}$, $\{3,5,21,28\}$, $\{4,6,22,29\}$. Once again, there is no line in orbit (ii) which passes through the point 7 and which is skew to the four lines. Combining these results we obtain a contradiction. Therefore orbit (ii) contains no spread of $PG(3,3)$.

Although it is difficult to decompose orbits (ii) and (iii) or prove they are indecomposable, we can still use them to construct examples of proper n-covers, simply by considering their union.

The two lines $\{0,2,18,25\}$ and $\{0,5,11,19\}$ which we used to generate the orbits are secant to Ω_2 and tangent to Ω_0, Ω_1, and secant to Ω_3 and tangent to Ω_0, Ω_1 respectively. Since $\langle\beta\rangle$ permutes the ovoids, it follows that every line in the union of orbits (ii) and (iii) is tangent to exactly two ovoids and secant to exactly one ovoid. By the results stated at the beginning of the section, we have that the set of eighty lines can be partitioned into four proper 2-covers. These 2-covers are

(a) $\mathcal{C}_2' = [\{0,5,11,19\}] \cup [\{0,25,2,18\}]$

(b) $\mathcal{C}_2' = [\{0,15,17,33\}] \cup [\{0,35,6,14\}]$

where in both cases [ ] denotes the orbit of the line under the action of $\langle\beta^4\rangle$.

(c) $\mathcal{C}_2^3 = \beta(\mathcal{C}_2')$.

(d) $\mathcal{C}_2^4 = \beta^2(\mathcal{C}_2')$.

As might be expected, the union of orbits (ii) and (iii) contains spreads of $PG(3,3)$ even though the orbits themselves do not. The spreads we list below were found by hand in the following way. We chose an arbitrary line from the union and searched for a spread through it. Then in the subsequent steps, we chose a line not in the existing spreads and searched for a spread through it disjoint from the others. In all, we found five pairwise disjoint spreads; these are:

(a) $\mathcal{S}_5 =$ $\{0,2,18,25\}$ $\{6,27,32,38\}$ $\{12,19,34,36\}$
$\{1,17,24,39\}$ $\{7,14,29,31\}$ $\{13,20,35,37\}$
$\{3,5,21,28\}$ $\{8,10,26,33\}$
$\{4,9,15,23\}$ $\{11,16,22,30\}$

(b) $\mathcal{S}_6 = \beta^2(\mathcal{S}_5)$

(c) $\mathcal{S}_7 = \beta^{20}(\mathcal{S}_5)$

(d) $\mathcal{S}_8 =$ $\{0,7,22,24\}$ $\{4,10,18,39\}$ $\{14,19,25,33\}$
$\{1,6,12,20\}$ $\{5,26,31,37\}$ $\{16,21,27,35\}$
$\{2,23,28,34\}$ $\{8,15,30,32\}$
$\{3,9,17,38\}$ $\{11,13,29,36\}$

(e) $\mathcal{S}_9 =$ $\{0,16,23,38\}$ $\{4,11,26,28\}$ $\{10,15,21,29\}$
$\{1,22,27,33\}$ $\{5,13,34,39\}$ $\{12,14,30,37\}$
$\{2,17,19,35\}$ $\{6,8,24,31\}$
$\{3,18,20,36\}$ $\{7,9,25,32\}$

The nine spreads $\mathcal{S}_i$ and the spread arising from orbit (iv) are pairwise disjoint and so constitute a partial packing of $PG(3,3)$. We have already mentioned that $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ are all subregular of index one (see the discussion following the definitions of these spreads). By similar means, we can show that $\mathcal{S}_5$, $\mathcal{S}_6$, $\mathcal{S}_7$ and $\mathcal{S}_8$ are also subregular of index one. However $\mathcal{S}_9$ is regular, as is the spread arising from orbit

(iv) (to check this, it is sufficient to show that the Plűcker coordinate vectors of the lines of $\mathcal{S}_9$ all lie in a 3-dimensional subspace Σ_3 of $PG(5,3)$. Since the ten coordinate vectors also represent a spread of $PG(3,3)$, it follows by the discussion at the end of Section 1.9, that $\mathcal{S}_9$ is regular). The remaining lines of $PG(3,3)$ form a proper 3-cover of $PG(3,3)$. We now prove this.

Theorem 2.3.6. The complement of the ten spreads in $PG(3,3)$ is a proper 3-cover $\mathcal{C}_3$ of $PG(3,3)$. Its lines are:

$\{4,6,22,29\}$	$\{3,8,14,22\}$	$\{25,30,36,4\}$
$\{13,15,31,38\}$	$\{24,26,2,9\}$	$\{28,33,39,7\}$
$\{16,18,34,1\}$	$\{5,10,16,24\}$	$\{30,35,1,9\}$
$\{21,23,39,6\}$	$\{8,13,19,27\}$	$\{32,37,3,11\}$
$\{25,27,3,10\}$	$\{9,14,20,28\}$	$\{33,38,4,12\}$
$\{32,34,10,17\}$	$\{12,17,23,31\}$	$\{35,0,6,14\}$
$\{33,35,11,18\}$	$\{18,23,29,37\}$	$\{36,1,7,15\}$
$\{27,29,5,12\}$	$\{19,24,30,38\}$	$\{37,2,8,16\}$
$\{0,5,11,19\}$	$\{20,25,31,39\}$	$\{15,20,26,34\}$
$\{2,7,13,21\}$	$\{21,26,32,0\}$	$\{17,22,28,36\}$

Proof. Assume $\mathcal{C}_3$ is not a proper 3-cover. Then it contains at least one spread $\mathcal{S}$. This spread contains either $\{0,5,11,19\}$, $\{0,21,26,32\}$ or $\{0,6,14,35\}$ because there is a unique line in the spread through 0.

Suppose $\mathcal{S}$ contains $\{0,6,14,35\}$. Then the line of $\mathcal{S}$ through the point 1 is either $\{1,16,18,34\}$ or $\{1,7,15,36\}$. If it is $\{1,7,15,36\}$, then the lines $\{11,32,37,3\}$, $\{9,24,26,2\}$ and $\{4,12,33,38\}$ are the unique lines of $\mathcal{C}_3$ through the points 11, 9 and 4 respectively, which are skew to $\{0,6,14,35\}$, $\{1,16,18,34\}$ and to each other. However no line of $\mathcal{C}_3$ through the point 5 is skew to all of these lines. Thus $\mathcal{S}$ does not contain $\{1,7,15,36\}$. By similar reasoning, $\mathcal{S}$ does not contain $\{1,16,18,34\}$ either. Thus $\mathcal{S}$ does not contain $\{0,6,14,35\}$.

Repeating this type of argument on the other lines through 0, we find that $\mathcal{S}$ doesn't contain any of the lines through 0. This gives us a contradiction. Thus no such spread $\mathcal{S}$ exists and $\mathcal{C}_3$ is proper. □

Remark 2.3.7. (a) By Theorem 2.3.6, the partial packing with ten spreads which we have constructed cannot be completed to a partial packing with eleven spreads. It therefore has deficiency 3. We conclude from this that if d is the largest deficiency of a partial packing of $PG(3,3)$ such that the partial packing is guaranteed to be completable, then $d \leq 2$.

(b) In the next section, we construct another proper 3-cover of $PG(3,3)$ from a general linear complex. The lines of that 3-cover through each point are coplanar, since the lines of the complex through each point lie in a planar pencil. However, the proper 3-cover constructed here contains the lines $\{0, 6, 14, 35\}$, $\{0, 5, 11, 19\}$, $\{0, 21, 26, 32\}$, which possess no common transversal but pass through the point 0. Therefore, the two proper 3-covers are projectively distinct. □

PROPER n-COVERS OF $PG(3,q)$ ARISING FROM GENERAL LINEAR COMPLEXES

We now use the results established in Section 2.2 to construct proper n-covers from general linear complexes in $PG(3,q)$. We begin with:

Theorem 2.3.8. Let $Ł_3$ be a general linear complex in $PG(3,3)$ and let $\mathcal{S}$ be a regular spread embedded in $Ł_3$. Then the lines in $Ł_3 \backslash \mathcal{S}$ form a proper 3-cover of $PG(3,3)$.

Proof. Assume $Ł_3 \backslash \mathcal{S}$ does not form a proper 3-cover of $PG(3,3)$. Then $Ł_3 \backslash \mathcal{S}$ contains at least one spread $\mathcal{S}'$ because $Ł_3 \backslash \mathcal{S}$ is a 3-cover.

In $PG(3,3)$ every spread is either regular or subregular of index 1 (see [72], p.56). By Theorem 2.2.9, a subregular spread of index 1 cannot lie in a general linear complex. Hence $\mathcal{S}'$ must be a regular spread and so $\mathcal{S}$ and $\mathcal{S}'$ are two disjoint regular spreads lying in a general linear complex. However, by Theorem 2.2.6 two regular spreads in a general linear complex always intersect in at least one line. Hence we have a contradiction and so $\mathbb{L}_3 \backslash \mathcal{S}$ is a proper 3-cover of $PG(3,3)$. $\square$

Using the same representation of $PG(3,3)$ that we used in considering the line orbits generated by a Singer group, we can represent the lines of the general linear complex (corresponding to the symplectic polarity with matrix

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

over $GF(3)$) as the sets:

$\{0,4,31,3\}$	$\{1,9,35,30\}$	$\{0,2,18,25\}$	$\{26,2,9,24\}$
$\{8,12,39,11\}$	$\{5,13,39,34\}$	$\{0,39,36,27\}$	$\{37,14,12,30\}$
$\{16,20,7,19\}$	$\{9,17,3,38\}$	$\{4,6,22,29\}$	$\{37,7,27,17\}$
$\{24,28,15,27\}$	$\{13,21,7,2\}$	$\{4,7,8,35\}$	$\{5,15,25,35\}$
$\{32,36,23,35\}$	$\{17,25,11,6\}$	$\{3,16,15,12\}$	$\{5,22,20,38\}$
$\{0,20,10,30\}$	$\{21,29,15,10\}$	$\{3,23,13,33\}$	$\{6,13,30,28\}$
$\{4,14,24,34\}$	$\{25,33,19,14\}$	$\{1,18,16,34\}$	$\{39,9,29,19\}$
$\{8,28,18,38\}$	$\{29,37,23,18\}$	$\{19,28,31,32\}$	$\{10,34,32,17\}$
$\{12,32,22,2\}$	$\{33,1,27,22\}$	$\{26,8,33,10\}$	$\{11,20,23,24\}$
$\{16,36,26,6\}$	$\{37,5,31,26\}$	$\{1,31,11,21\}$	$\{14,21,36,38\}$

By observation, line orbit (iv) (on page 71) consisting of the lines of a regular spread of $PG(3,3)$ is contained in this set of lines. Hence by removing the lines of orbit (iv), we have an explicit example of a proper 3-cover of $PG(3,3)$. Note: The general linear complex used here was also used by Ebert in [53] to construct proper 2-covers of $PG(3,3)$. He showed that the lines in the first two columns constitute a proper 2-cover of $PG(3,3)$ because they contain between them the lines $\{4,24,34,14\}$, $\{14,19,33,25\}$, $\{25,6,11,17\}$, $\{17,9,38,3\}$ and $\{3,0,31,4\}$ which form a proper 5-lateral in the corresponding incidence structure; the result is then immediate by Theorem 2.1.9. However, it was not mentioned that the other 20 lines also form a proper 2-cover. We now prove this.

Theorem 2.3.9. The lines of $\mathbb{L}_3$ which appear in the last two columns of the list form a proper 2-cover of $PG(3,3)$.

Proof. The set of lines forms a 2-cover of $PG(3,3)$ because it is the complement of a 2-cover in $\mathbb{L}_3$. To show it is proper we can either demonstrate a set of 5 lines forming a proper 5-lateral in the corresponding incidence structure or we can argue as follows:

If the 2-cover is not proper then it is the union of two distinct spreads. These spreads are either regular or subregular of index 1 because these are the only types of spreads in $PG(3,3)$. However, by Theorem 2.2.9, a subregular spread cannot lie in a general linear complex. Hence the two spreads are regular. This implies by Theorem 2.2.6, that the two spreads intersect in at least one line – a contradiction. Hence the 2-cover is proper. $\square$

A similar analysis of a general linear complex $\mathbb{L}_5$ of $PG(3,5)$ also seems plausible as a recent paper by Baker et al reports that all projectively distinct spreads of $PG(3,5)$ are known (see [6]). However, no general results on the existence of proper n-covers (with $n > 1$) embedded in a general linear complex are known currently for odd $q > 3$. In contrast, for q even we have:

Theorem 2.3.10. Let $\mathbb{L}_q$ be a general linear complex in $PG(3,q)$, q even. Then there exists a proper n-cover of $PG(3,q)$ embedded in $\mathbb{L}_q$ for some n satisfying $2 \leq n < q+1$.

Proof. First we note that a general linear complex is never a proper $(q+1)$-cover because it always contains a spread (see the introductory remarks to Section 2.2).

Hence any proper n-cover embedded in $\mathbb{L}_q$ satisfies $n < q + 1$.

Second, by Theorem 2.2.1, $\mathbb{L}_q$ cannot be partitioned into $q + 1$ disjoint spreads because q is even. Thus every maximal set of pairwise disjoint spreads in $\mathbb{L}_q$ contains at most $q - 1$ spreads. Hence by removing a maximal set of pairwise disjoint spreads from $\mathbb{L}_q$, we are left with a set C of lines which forms a cover of $PG(3, q)$ and contains no spread of $PG(3, q)$. Thus set C is either a proper n-cover of $PG(3, q)$ for some $n \geq 2$ or it contains a proper n-cover of $PG(3, q)$ for some $n \geq 2$. Hence the result is established. $\square$

Remark 2.3.11. Theorem 2.3.10 can also be established by removing a regular spread or a Lűneburg spread (for $q = 2^{2h+1}$, $h \geq 1$) from $\mathbb{L}_q$. Then by Corollary 2.2.4, the resulting set of lines does not contain a spread of $PG(3, q)$. The first proof, however, is in some ways better, because it does not rely on the nature of the spreads. $\square$

Corollary 2.3.12. There exists a proper 2-cover of $PG(3, 2)$.

Proof. By Theorem 2.3.10, there exists a proper n-cover of $PG(3, 2)$ lying in a general linear complex for some n satisfying $2 \leq n < 3$. Clearly from this, $n = 2$ is the only possibility and so $PG(3, 2)$ possesses a proper 2-cover. $\square$

Remark 2.3.13. This is the 2-cover discovered by Bruen and Ott. It was first mentioned in [52] and then again later in [53] where it was conjectured to be the unique proper 2-cover of $PG(3, 2)$ up to projective equivalence. In Chapter III, we shall prove that this is so. $\square$

CHAPTER III
REGULAR PARTIAL PACKINGS OF PG(3,q) AND PROPER n-COVERS OF PG(3,2)

3.1. THE UNIQUENESS OF THE PROPER 2-COVER OF $PG(3,2)$

We commented, in Section 2.3, that the proper 2-cover of $PG(3,2)$ constructed in Corollary 2.3.12 was first discovered by Bruen and Ott. Ebert, in [53] conjectured that all proper 2-covers of $PG(3,2)$ are projectively equivalent. In this section, we show that this is indeed the case.

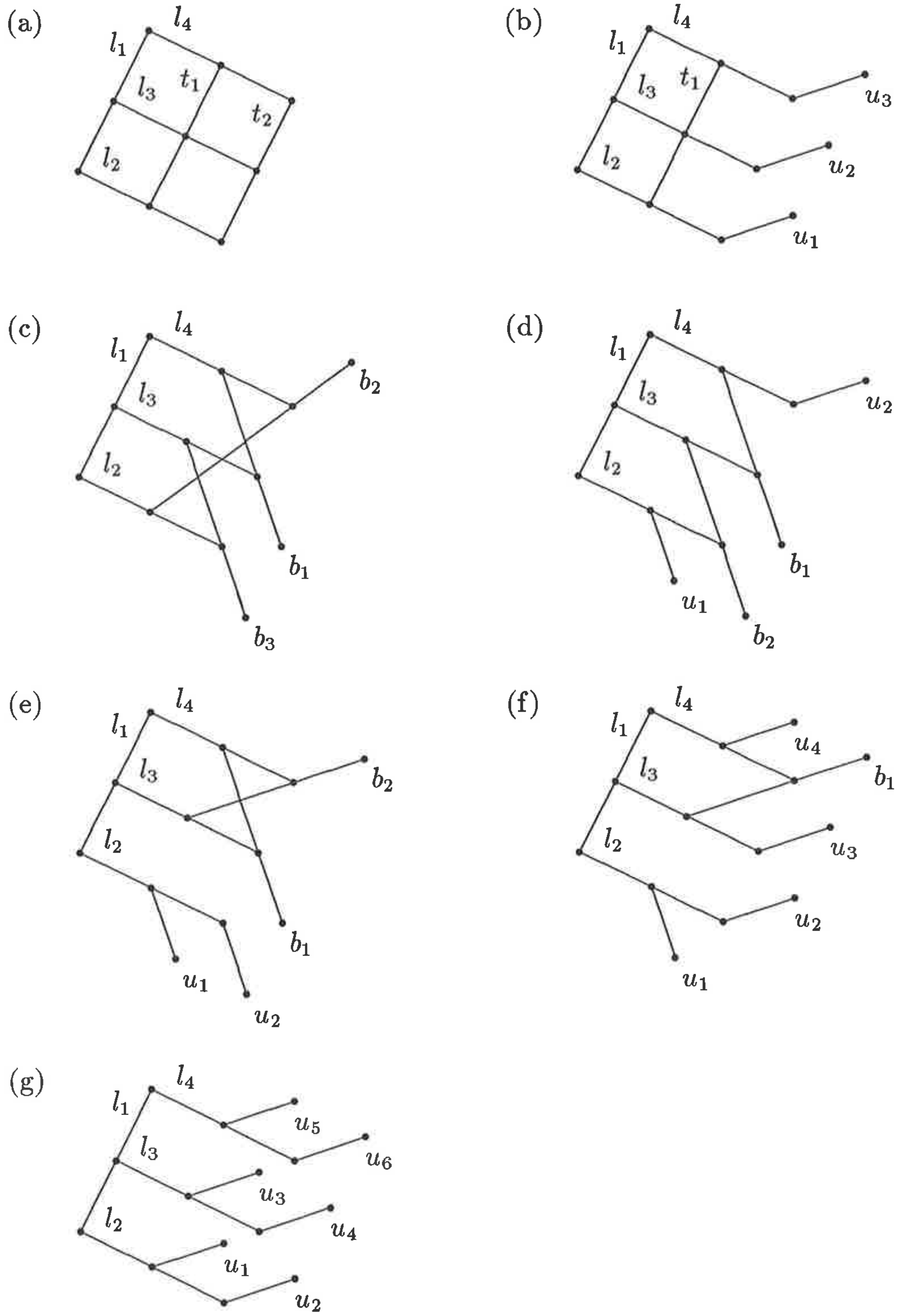
To commence, we discuss the structure of the n-laterals arising from a 2-cover of $PG(3,2)$.

Lemma 3.1.1. Let $\mathcal{C}_2$ be a 2-cover of $PG(3,2)$ with associated incidence structure $(\mathbf{P},\mathcal{C}_2,\mathbf{I})$. If, for every proper n-lateral in $(\mathbf{P},\mathcal{C}_2,\mathbf{I})$, n is even, then $\mathcal{C}_2$ is the union of two spreads.

Proof. Let ℓ_1 be a line of $\mathcal{C}_2$ and let ℓ_2, ℓ_3 and ℓ_4 be the lines of $\mathcal{C}_2$ which meet ℓ_1 in single points. By Theorem 2.1.12, $\mathcal{C}_2$ is dual and so no two of the lines ℓ_2, ℓ_3, ℓ_4 can be coplanar with ℓ_1. Thus they are pairwise skew.

Let S denote the set of these four lines. Since every 2-cover of $PG(3,2)$ contains a set of lines isomorphic to S, to establish the theorem, it is sufficient to examine those lines of $PG(3,2)$ which meet the lines of S and which would lie in a potential 2-cover containing S.

There are essentially seven distinct cases to consider and these are illustrated in Figure 3.1. (The lines marked t, b and u represent a transversal, a bisecant and a unisecant respectively.)



(a)
l_4
l_1
l_3
t_1
t_2
l_2
(b)
l_4
l_1
l_3
t_1
u_3
l_2
u_2
u_1
(c)
l_4
l_1
b_2
l_3
l_2
b_1
b_3
(d)
l_4
l_1
u_2
l_3
l_2
b_1
u_1
b_2
(e)
l_4
l_1
l_3
b_2
l_2
b_1
u_1
u_2
(f)
l_4
l_1
u_4
b_1
l_3
l_2
u_3
u_2
u_1
(g)
l_4
l_1
u_5
l_3
u_6
u_3
l_2
u_4
u_1
u_2

Figure 3.1a

Case (a):

If S has two transversals, then the nine points lying on the lines ℓ_2, ℓ_3 and ℓ_4 are covered twice. Assume these six lines lie in a 2-cover $\mathcal{C}_2$. Then the remaining four lines of $\mathcal{C}_2$ cover the remaining six points of $PG(3,2)$ exactly twice. Since the intersections of the four lines determine at most $\binom{4}{2} = 6$ distinct points, the four lines cover the six points twice if and only if they are coplanar. Thus $\mathcal{C}_2$ contains four coplanar lines.

However, by Theorem 2.1.12, each plane of $PG(3,2)$ contains exactly two lines of $\mathcal{C}_2$. This gives us a contradiction, so no such $\mathcal{C}_2$ exists.

Case (b):

Assume the eight lines lie in a 2-cover $\mathcal{C}_2$. If two of the lines $u_i, u_j, i \neq j$ intersect then $\mathcal{C}_2$ would contain a proper 5-lateral, which would contradict the hypothesis of the theorem. Hence the lines u_1, u_2, u_3 are pairwise skew. It follows immediately from this that $\{\ell_1, t_1, u_1, u_2, u_3\}$ is a spread lying in $\mathcal{C}_2$. Therefore $\mathcal{C}_2$ is the union of two spreads.

Case(c):

Assume the seven lines lie in a 2-cover $\mathcal{C}_2$. Nine points of $PG(3,2)$ are covered twice by these lines. Of the remaining six points, three of them are covered once by the lines. Let these be P_1, P_2, P_3. The other three points Q_1, Q_2, Q_3 lie on none of the seven lines.

If P_1, P_2, P_3 lie on a common line of $\mathcal{C}_2$, then the remaining two lines of $\mathcal{C}_2$ cover the points Q_1, Q_2 and Q_3 twice. This implies that the two lines coincide – a contradiction. Hence P_1, P_2, P_3 do not lie on a common line of $\mathcal{C}_2$.

If exactly two of the P_i's lie on a common line of $\mathcal{C}_2$, then this line contains one of the points Q_i. This leaves the other two points Q_j and Q_k to be covered twice by two distinct lines of $\mathcal{C}_2$. This is not possible. Therefore no two of the P_i's lie on a common line of $\mathcal{C}_2$.

The final possibility is that the three remaining lines of $\mathcal{C}_2$ each contain exactly one of the points P_1, P_2 or P_3. These three lines then each contain two of the three points Q_1, Q_2, Q_3. Thus the three lines are coplanar. As in Case (a), this contradicts Theorem 2.1.12.

Hence no such $\mathcal{C}_2$ exists.

Cases (d) and (e):

Assume the eight lines lie in a 2-cover $\mathcal{C}_2$. If any two of the lines b_1, b_2, u_1, u_2 met in a point, then $\mathcal{C}_2$ would contain a trilateral or a proper 5-lateral, which would contradict the hypothesis of the theorem. Thus the four lines are pairwise skew. It follows from this, that $\{\ell_1, b_1, b_2, u_1, u_2\}$ is a spread lying in $\mathcal{C}_2$ and so $\mathcal{C}_2$ is the union of two spreads.

Cases (f) and (g):

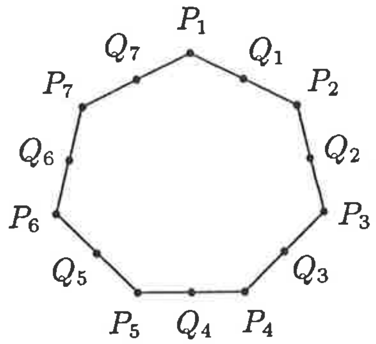
In both cases, the points on the bisecant (Case (f)) and the unisecants cannot all be distinct because $PG(3,2)$ has only fifteen points. Thus at least two of these lines meet which implies that the set of lines contains either a trilateral or a proper 5-lateral. Hence the sets of lines in (f) and (g) cannot be embedded in a 2-cover satisfying the hypotheses of the lemma.

In conclusion, whenever the set of lines lies in a 2-cover, the 2-cover is the union of two spreads. □

Remark 3.1.2. By Lemma 3.1.1, it is immediate that the incidence structure associated with a proper 2-cover $\mathcal{C}_2$ of $PG(3,2)$ contains at least one proper n-lateral with n odd. By Theorem 2.1.12, every 2-cover is dual. Therefore $\mathcal{C}_2$ cannot contain a trilateral. Hence $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 5-lateral, a proper 7-lateral or a proper 9-lateral if $\mathcal{C}_2$ is proper. □

Lemma 3.1.3. Let $\mathcal{C}_2$ be a proper 2-cover of $PG(3,2)$. If $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 7-lateral, then it contains a proper 5-lateral.

Proof. Let the points on the lines of the proper 7-lateral be labelled in the manner below:



Since the 7-lateral is proper, all the points P_i with $i = 1$ to 7 are distinct and the seven lines are distinct. Hence, there are two cases to consider, namely the case where the points Q_i with $i = 1$ to 7 are all distinct and the case where at least two of the points Q_i and Q_j coincide.

Assume first that the points Q_i with $i = 1$ to 7 are all distinct. Then the points P_i and Q_i with $i = 1$ to 7 account for 14 of the 15 points of $PG(3,2)$. Let R be the remaining point and ℓ_1 and ℓ_2 be the two lines of $\mathcal{C}_2$ which pass through R. ℓ_1 and ℓ_2 cannot contain any point of the form P_i because each such point already lies on two lines of $\mathcal{C}_2$. Hence ℓ_1 and ℓ_2 each contain two points of the form Q_i.

Without loss of generality, we can suppose that ℓ_1 contains the point Q_1 and so that $\ell_1 = \{R, Q_1, Q_i\}$ for some $i = 2$ to 7.

Q_i cannot be either of the points Q_2 or Q_7 because if it were, then the lines $\langle Q_1, P_2\rangle$, $\langle P_2, Q_2\rangle$ and $\langle Q_1, Q_2\rangle = \ell_1$ or the lines $\langle Q_1, P_1\rangle$, $\langle P_1, Q_7\rangle$ and $\langle Q_1, Q_7\rangle = \ell_1$ respectively would be the lines of a trilateral in $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$, contradicting the duality of $\mathcal{C}_2$.

If Q_i is Q_4 or Q_5, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains the proper 5-lateral

$$\{\langle Q_1, P_2\rangle, \langle P_2, P_3\rangle, \langle P_3, P_4\rangle, \langle P_4, Q_4\rangle, \langle Q_4, Q_1\rangle = \ell_1\}$$

or

$$\{\langle Q_1, P_1\rangle, \langle P_1, P_7\rangle, \langle P_7, P_6\rangle, \langle P_6, Q_5\rangle, \langle Q_5, Q_1\rangle = \ell_1\}$$

respectively.

Suppose $Q_i = Q_3$. Then the line ℓ_2 contains exactly two of the points in the set $\{Q_2, Q_4, Q_5, Q_6, Q_7\}$. Because $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ cannot contain a trilateral, it is routine to show that the only pairs of points which could lie on ℓ_2 are $\{Q_2, Q_4\}$, $\{Q_2, Q_5\}$, $\{Q_2, Q_6\}$, $\{Q_2, Q_7\}$, $\{Q_4, Q_6\}$, $\{Q_4, Q_7\}$, $\{Q_5, Q_7\}$.

If $\ell_2 = \{R, Q_2, Q_4\}$, then every point of $PG(3,2)$ would lie on a pair of distinct lines of $\mathcal{C}_2$ except for Q_5, Q_6 and Q_7. Hence $\{Q_5, Q_6, Q_7\}$ would be a line of $\mathcal{C}_2$, in which case $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ would contain the trilateral $\{\langle Q_7, Q_6\rangle, \langle Q_6, P_7\rangle, \langle P_7, Q_7\rangle\}$, contradicting the duality of $\mathcal{C}_2$. Hence ℓ_2 is not $\{R, Q_2, Q_4\}$. Similarly, we can discount the possibility $\{R, Q_2, Q_7\}$.

If $\ell_2 = \{R, Q_2, Q_5\}$, $\{R, Q_2, Q_6\}$, $\{R, Q_4, Q_6\}$ or $\{R, Q_4, Q_7\}$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains (respectively) the proper 5-lateral:

$$\{\langle Q_5, P_5\rangle, \langle P_5, P_4\rangle, \langle P_4, P_3\rangle, \langle P_3, Q_2\rangle, \langle Q_2, Q_5\rangle\}$$
$$\{\langle Q_6, P_7\rangle, \langle P_7, P_1\rangle, \langle P_1, P_2\rangle, \langle P_2, Q_2\rangle, \langle Q_2, Q_6\rangle\}$$
$$\{\langle R, Q_6\rangle, \langle Q_6, P_7\rangle, \langle P_7, P_1\rangle, \langle P_1, Q_1\rangle, \langle Q_1, R\rangle\}$$

or

$$\{\langle Q_7, P_7\rangle, \langle P_7, P_6\rangle, \langle P_6, P_5\rangle, \langle P_5, Q_4\rangle, \langle Q_4, Q_7\rangle\}.$$

Finally, if $\ell_2 = \{R, Q_5, Q_7\}$, then every point of $PG(3,2)$ lies on two distinct lines of $\mathcal{C}_2$, except for the three points Q_2, Q_4 and Q_6. It follows then that $\{Q_2, Q_4, Q_6\}$ is a line of $\mathcal{C}_2$, in which case

$$\{\langle Q_2, P_2\rangle, \langle P_2, P_1\rangle, \langle P_1, P_7\rangle, \langle P_7, Q_6\rangle, \langle Q_6, Q_2\rangle\}$$

is a proper 5-lateral of $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$.

Thus if $Q_i = Q_3$, then $\mathcal{C}_2$ contains a proper 5-lateral. Similarly, if $Q_i = Q_6$, then $\mathcal{C}_2$ contains a proper 5-lateral. This concludes the first case.

Now assume that the points Q_i with $i = 1$ to 7 are not all distinct. Without loss of generality, we can suppose that Q_1 coincides with one of the other points Q_i.

Since the 7-lateral is proper, the point Q_1 cannot coincide with Q_2 or Q_7.

If $Q_1 = Q_3$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ would contain the trilateral

$$\{\langle Q_1, P_2\rangle, \langle P_2, P_3\rangle, \langle P_3, Q_1\rangle\},$$

which would contradict the duality of $\mathcal{C}_2$. Hence $Q_1 \neq Q_3$. Similarly $Q_1 \neq Q_6$.

If $Q_1 = Q_4$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains the proper 5-lateral

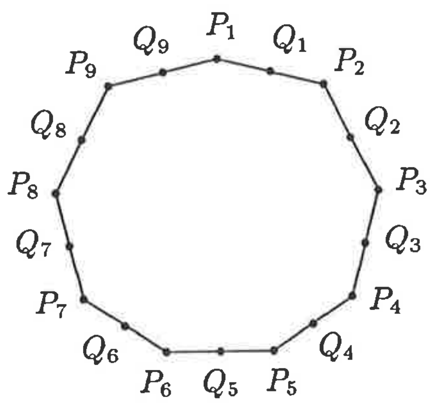
$$\{\langle Q_1, P_1\rangle, \langle P_1, P_7\rangle, \langle P_7, P_6\rangle, \langle P_6, P_5\rangle, \langle P_5, Q_1\rangle\}.$$

Similarly, if $Q_1 = Q_5$ then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 5-lateral. This concludes the second case.

By the results of cases one and two, a proper 7-lateral of $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ gives rise to a proper 5-lateral of $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$. □

Lemma 3.1.4. Let $\mathcal{C}_2$ be a proper 2-cover of $PG(3,2)$. If $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 9-lateral, then it contains a proper 5-lateral.

Proof. Let the points on the lines of the proper 9-lateral be labelled in the manner shown:



Since the 9-lateral is proper, the points P_i with $i = 1$ to 9 are all distinct. Hence at least two of the points Q_i with $i = 1$ to 9 coincide, otherwise $PG(3,2)$ would have at least 18 distinct points – a contradiction. Without loss of generality, we can suppose that $Q_1 = Q_i$ for some $i \neq 1$.

Since the 9-lateral is proper, the point Q_1 cannot coincide with Q_2 or Q_9.

If $Q_1 = Q_3$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ would contain the trilateral

$$\{\langle Q_1, P_2\rangle, \langle P_2, P_3\rangle, \langle P_3, Q_1\rangle\},$$

which would contradict the duality of $\mathcal{C}_2$. Thus $Q_1 \neq Q_3$. Similarly $Q_1 \neq Q_8$.

If $Q_1 = Q_4$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains the proper 7-lateral

$$\{\langle Q_1, P_1\rangle, \langle P_1, P_9\rangle, \langle P_9, P_8\rangle, \langle P_8, P_7\rangle, \langle P_7, P_6\rangle, \langle P_6, P_5\rangle, \langle P_5, Q_1\rangle\}.$$

Hence $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ also contains a proper 5-lateral by Lemma 3.1.3. Similarly, if $Q_1 = Q_7$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 5-lateral.

Finally, if $Q_1 = Q_5$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains the proper 5-lateral

$$\{\langle Q_1, P_2\rangle, \langle P_2, P_3\rangle, \langle P_3, P_4\rangle, \langle P_4, P_5\rangle, \langle P_5, Q_1\rangle\}.$$

Similarly, if $Q_1 = Q_6$, then $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 5-lateral.

Every case has been covered. Therefore, if $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ has a proper 9-lateral, then it also has a proper 5-lateral. □

Lemma 3.1.5. Let $\mathcal{C}_2$ be a 2-cover of $PG(3,2)$ such that $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ contains a proper 5-lateral. Then the vertices of the 5-lateral are the five points of an elliptic quadric of $PG(3,2)$.

Proof. Since the 5-lateral is embedded in $\mathcal{C}_2$, each plane of $PG(3,2)$ meets the 5-lateral in at most two lines. As two intersecting lines of the 5-lateral are incident with these vertices, it is immediate that each plane meets the set of five vertices in at most three points.

Assume that three of the vertices are collinear on the line ℓ. By taking a fourth vertex V from the set, we can form the plane $\langle \ell, V \rangle$ which meets the vertex set in at least four points. This contradicts what we proved in the first paragraph. Hence no three of the vertices are collinear.

Thus the set of vertices forms a 5-cap of $PG(3,2)$ with each plane meeting the 5-cap in at most three points. Hence by Theorem 1.4.3 the vertex set is the point-set of an elliptic quadric of $PG(3,2)$. □

We are now in a position to determine the structure of a proper 2-cover $\mathcal{C}_2$ of $PG(3,2)$. By Lemmata 3.1.3 and 3.1.4, $\mathcal{C}_2$ possesses five lines which form a proper 5-lateral of $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ and by Lemma 3.1.5, the vertices of the 5-lateral are the points of an elliptic quadric of $PG(3,2)$. Thus, we may assign homogeneous coordinates to the vertices in the following way:

$$\begin{aligned}
P_1 &= (1\ 0\ 0\ 0)\\
P_2 &= (0\ 1\ 0\ 0)\\
P_3 &= (0\ 0\ 1\ 0)\\
P_4 &= (0\ 0\ 0\ 1)\\
P_5 &= (1\ 1\ 1\ 1).
\end{aligned}$$

The other five points on the five lines of $\mathcal{C}_2$ already constructed become:

$$\begin{aligned}
Q_1 &= (1\ 1\ 0\ 0)\\
Q_2 &= (0\ 1\ 1\ 0)\\
Q_3 &= (0\ 0\ 1\ 1)\\
Q_4 &= (1\ 1\ 1\ 0)\\
Q_5 &= (0\ 1\ 1\ 1),
\end{aligned}$$

while the five remaining points of $PG(3,2)$ are:

$$R_1 = (1\ 0\ 1\ 0)$$
$$R_2 = (0\ 1\ 0\ 1)$$
$$R_3 = (1\ 1\ 0\ 1)$$
$$R_4 = (1\ 0\ 0\ 1)$$
$$R_5 = (1\ 0\ 1\ 1).$$

Thus far, we have five lines of $\mathcal{C}_2$ already constructed. Since we have already constructed a proper 2-cover of $PG(3,2)$ in Corollary 2.3.12, we know that this set of five lines can be extended to a proper 2-cover in at least one way. Each of the points Q_i with $i = 1$ to 5 lies on exactly one of these lines. A second line through the point Q_1, if it is to lie in $\mathcal{C}_2$, cannot pass through any of the points P_i with $i = 1$ to 5 because these points already lie on two distinct lines of $\mathcal{C}_2$. In addition, it cannot pass through Q_2 or Q_5, otherwise $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$ would contain a trilateral, contradicting the duality of $\mathcal{C}_2$. There are only seven lines through Q_1 and so it is routine exercise to verify that the unique line satisfying these criteria is $\{Q_1, R_2, R_4\}$.

Repeating this argument for each point Q_i, we find that the unique (second) line through Q_i satisfying the criteria is:

$$\{Q_i, R_{i+1}, R_{i+3}\}$$

with the subscripts reduced modulo 5 to a number between 1 and 5 inclusive. These five lines plus the original five lines account for all the lines of $\mathcal{C}_2$ and so $\mathcal{C}_2$ is uniquely determined by any proper 5-lateral of $(\mathbf{P}, \mathcal{C}_2, \mathbf{I})$. The lines of $\mathcal{C}_2$ are therefore:

$$\{Q_1, P_1, P_2\} \quad \{Q_1, R_2, R_4\}$$
$$\{Q_2, P_2, P_3\} \quad \{Q_2, R_3, R_5\}$$
$$\{Q_3, P_3, P_4\} \quad \{Q_3, R_4, R_1\}$$
$$\{Q_4, P_4, P_5\} \quad \{Q_4, R_5, R_2\}$$
$$\{Q_5, P_5, P_1\} \quad \{Q_5, R_1, R_3\}.$$

The points Q_i with $i = 1$ to 5 are linearly independent and therefore comprise the points of an elliptic quadric $\mathcal{Q}$ of $PG(3,2)$. It is immediate then, that the lines of the 2-cover are all tangent to $\mathcal{Q}$ and so also lie in the general linear complex defined by $\mathcal{Q}$.

The lines of $PG(3,2)$ which complete $\mathcal{C}_2$ to a general linear complex are:

$$\{Q_1, R_3, P_4\}$$
$$\{Q_2, R_4, P_5\}$$
$$\{Q_3, R_5, P_1\}$$
$$\{Q_4, R_1, P_2\}$$
$$\{Q_5, R_2, P_3\}$$

and these form a spread of $PG(3,2)$. Hence, we have shown that every proper 2-cover of $PG(3,2)$ can be constructed by the technique used in Corollary 2.3.12. This now enables us to prove:

Theorem 3.1.6. The proper 2-covers of $PG(3,2)$ are all projectively equivalent.

Proof. Let $\mathcal{C}_2$ and $\mathcal{C}_2'$ be two proper 2-covers of $PG(3,2)$. By the argument above $\mathcal{C}_2$ can be constructed by deleting a (regular) spread $\mathcal{S}$ from a general linear complex L_2. $\mathcal{C}_2'$ can be similarly constructed from L_2' and $\mathcal{S}'$.

By Corollary 2.3.3, there exists a collineation σ mapping L_2 to L_2' and $\mathcal{S}$ to $\mathcal{S}'$. Hence

$$\sigma(\mathrm{L}_2 \backslash \mathcal{S}) = \mathrm{L}_2' \backslash \mathcal{S}';$$

that is,

$$\sigma(\mathcal{C}_2) = \mathcal{C}_2'.$$

$\square$

3.2. ON A REPRESENTATION OF THE PROPER 2-COVER OF $PG(3,2)$ AND A CLASSICAL GROUP ISOMORPHISM

Having shown in the previous section that all proper 2-covers of $PG(3,2)$ are projectively equivalent, we now focus our attention on finding the full collineation group G of the proper 2-cover. The method we employ to find G uses a structure which is isomorphic to the unique generalised quadrangle of order 2. As by-products of this determination of G, we find a new proof of the classical isomorphism between $PSp(4,2)$ and S_6, and we give a geometric representation of the subgroups of index 6 in S_6. This representation illustrates a property peculiar to S_6; this property is stated in:

Theorem 3.2.1. ([75], 5.5. Satz) Let S_n be the symmetric group on n letters.

(i) If $n \neq 6$, then there is exactly one conjugacy class of subgroups of index n in S_n. Each such subgroup is the stabiliser of a letter and so is isomorphic to S_{n-1}.

(ii) If $n = 6$, then there are exactly two conjugacy classes of subgroups of index 6 in S_6. Each of the six subgroups from one class is the stabiliser of a letter, while each of the six subgroups from the second class acts 2-transitively on the six letters. The twelve subgroups are all isomorphic to S_5. □

Remark 3.2.2. The construction of the subgroups from the second class (when $n = 6$) is given in [112], p.301. Also, the structure of the outer automorphisms of S_6 which give rise to the second conjugacy class, is studied in [77]. □

Definition 3.2.3 and details of Construction 3.2.4 to follow, can be found in [104], Chapter 6.

Definition 3.2.3. A *duad* is an unordered pair $ij = ji$ of distinct integers from among the integers 1 to 6. A *syntheme* is a set of these duads $\{ij, k\ell, mn\}$ for which i, j, k, ℓ, m and n are all distinct. □

Construction 3.2.4. Define the finite incidence structure $W = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ as below:

1) **P** is the set of all duads.

2) **B** is the set of all synthemes.

3) **I** is set inclusion.

The incidence structure W is attributed in [104] to J.J. Sylvester who first published it in [113] in 1844. It is a routine exercise to show that W is a finite generalised quadrangle of order 2. W has exactly six ovoids, each of which has the form

$$Q_i = \{ia \mid a \in \mathcal{I}\backslash\{i\}\},$$

for some i from 1 to 6, where $\mathcal{I}$ is the set $\{1, 2, 3, 4, 5, 6\}$. W also possesses a polarity, therefore it follows that W also has exactly six spreads. One spread of W can be written in the form

$$R_1 = \{1i, (i-1)(i+1), (i-2)(i+2)\}$$

where i runs from 1 to 6 and (k) means k is to be reduced modulo 5 to an integer between 2 and 6 inclusive. The other 5 spreads can be obtained by applying the transpositions $(1j)$ to R_1, for each $j = 2$ to 6 and defining

$$R_j = (1j)R_1.$$

□

Now, by the construction of W, it is immediate that W admits S_6 as a collineation group of W. In fact, S_6 is the full collineation group of W. In [104], this is stated without proof; we present a proof here.

Theorem 3.2.5. Let H be the full collineation group of W. Then H is isomorphic to S_6.

Proof. By the remarks above, S_6 is isomorphic to a subgroup H' of H. In what follows, we identify H' with S_6.

Let α be an arbitrary collineation of W. Consider the action of α on the ovoid $\mathcal{O}_1$ of W. $\mathcal{O}_1$ is mapped by α to a second ovoid $\mathcal{O}_i$ for some i. Thus for each j in $\mathcal{I}\backslash\{1\}$

(where $\mathcal{I}$ is as before), there is a unique k in $\mathcal{I}\backslash\{i\}$ such that $\alpha(1j) = ik$. Hence α induces a permutation ϕ from $\mathcal{I}$ to $\mathcal{I}$, where ϕ is defined as below:

$$\begin{aligned}\phi: \quad & 1 \mapsto i \\ & j \mapsto k \quad \text{when } \alpha(1j) = ik.\end{aligned}$$

By construction, α acts on the points of $\mathcal{O}_1$ in the same way that ϕ does. Therefore, to complete the proof, it suffices to show that α is also equivalent to ϕ for all points not on the ovoid.

Let ab be a point of $W\backslash\mathcal{O}_1$. Then a and b are both different from 1. Rewrite $\mathcal{I}$ as $\{1, a, b, c, d, e\}$. The point ab lies on the three lines tangent to $\mathcal{O}_1$, namely

$$\begin{aligned}\ell_1 &= \{ab, 1c, de\} \\ \ell_2 &= \{ab, 1d, ce\} \\ \ell_3 &= \{ab, 1e, cd\}.\end{aligned}$$

The lines ℓ_1, ℓ_2 and ℓ_3 map, under α, to three lines which have the point $\alpha(ab)$ in common and which are tangent to $\mathcal{O}_i$. Since $1c \in \mathcal{O}_1$, we have

$$\begin{aligned}\alpha(\ell_1) &= \{\alpha(ab), \alpha(1c), \alpha(de)\} \\ &= \{\alpha(ab), i\phi(c), \alpha(de)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\alpha(\ell_2) &= \{\alpha(ab), i\phi(d), \alpha(ce)\} \\ \alpha(\ell_3) &= \{\alpha(ab), i\phi(e), \alpha(cd)\}.\end{aligned}$$

Let $\alpha(ab) = st$. Since $\alpha(\ell_1)$, $\alpha(\ell_2)$ and $\alpha(\ell_3)$ are also lines of W, it follows that s, t, i, $\phi(c)$, $\phi(d)$ and $\phi(e)$ are all distinct. Thus

$$\mathcal{I} = \{i, s, t, \phi(c), \phi(d), \phi(e)\}.$$

However,

$$\mathcal{I} = \phi(\mathcal{I}) = \{i, \phi(a), \phi(b), \phi(c), \phi(d), \phi(e)\}.$$

Hence, comparing the two sets we have immediately that $\{s,t\} = \{\phi(a),\phi(b)\}$, implying that $\alpha(ab) = \phi(a)\phi(b)$. It follows that α is also equivalent to ϕ for all points of $W\backslash\mathcal{O}_1$ and so α is an element of S_6 and H is S_6. $\square$

In Section 1.11, we stated that the points of $PG(3,q)$ together with the lines of a general linear complex of $PG(3,q)$ form a generalised quadrangle $W(q)$. It is shown at the beginning of Chapter 6 in [104], that every generalised quadrangle of order 2 is isomorphic to $W(2)$ and so in particular, that W is isomorphic to $W(2)$. The proof of the general case in [104] involves showing that every point of a generalised quadrangle of order 2 is regular and from this the result is deduced. However, we shall show that W is isomorphic to $W(2)$ by embedding W into $PG(3,2)$. To do this, we need the following definition:

Definition 3.2.6. A *triplet* is a set of three duads $\{ij, jk, ki\}$, for which i, j and k are all distinct. $\square$

Construction 3.2.7. Define the finite incidence structure $\Sigma_2 = (\mathbf{P}, \mathbf{B}, \mathbf{I})$ as follows:

1) $\mathbf{P}$ is the set of all duads.

2) $\mathbf{B}$ is the set of all synthemes and all triplets.

3) $\mathbf{I}$ is set inclusion.

As a consequence of the definition of W, W is naturally embedded in Σ_2, which we now show is isomorphic to $PG(3,2)$. $\square$

Theorem 3.2.8. Σ_2 is isomorphic to $PG(3,2)$.

Proof. It is immediate that axioms (i) and (iii) of Definition 1.3.1 are satisfied; we now prove that axiom (ii) also holds.

Let ab, cd and ef represent the vertices of a triangle. Now a, b, c, d, e, f cannot all be distinct because the three duads would then form a syntheme, forcing the points to be collinear. Similarly, at least four of them must be distinct otherwise they would form a triplet. This implies that the set of vertices is equivalent to one of the following possibilities:

(i) $\{ab, ad, af\}$ with a, b, d and f all distinct.

(ii) $\{ab, bc, cd\}$ with a, b, c and d all distinct.

(iii) $\{ab, bc, de\}$ with a, b, c, d and e all distinct.

In each case there are only three points on the sides of the triangle besides the vertices. Hence it is sufficient to check that these points are collinear. For case (i), the points are bd, bf, df which form a triplet, for case (ii), the points are ac, ef and bd which form a syntheme and for case (iii), the points are ac, cf, af which form a triplet.

Hence Σ_2 is isomorphic to a finite projective space. The dimension and the order of the space are immediate from the number of points Σ_2. $\square$

Theorem 3.2.9. The lines of W form a general linear complex of Σ_2.

Proof. Let ab be a point of Σ_2 and let the elements of $\mathcal{I}\backslash\{a, b\}$ be c, d, e and f. Then the three lines of W through ab are

$$\{ab, cd, ef\}$$
$$\{ab, ce, df\}$$
$$\{ab, de, cf\}.$$

These lines are coplanar in Σ_2 because they have the point ab in common and $\{cd, df, cf\}$ as a transversal.

Consider first, a pencil P of lines with vertex V. By the above, the lines of W through V are coplanar in a plane π. The plane containing P either meets π in a line or it coincides with π. If the planes meet in a single line, then this line belongs to both W and P because V lies on the line. If they coincide then P coincides with the pencil of lines of W through V. Hence each pencil of lines in Σ_2 meets W in 1 or 3 lines.

Now consider a star of lines with vertex U. There are exactly three lines of W

through U. Hence each star of Σ_2 meets W in exactly 3 lines.

It follows that W has the property $A_{3,3}$ (see Definition 1.10.7). Therefore, by Theorem 1.10.8, the lines of W form a general linear complex of Σ_2. $\square$

It is apparent from Theorem 3.2.9, that W is isomorphic to $W(2)$ because all general linear complexes of $PG(3,2)$ are projectively equivalent. We can now use this relationship to prove the classical isomorphism existing between $PSp(4,2)$ and S_6. It is proven directly in [75] (see 9.21 Satz), while in [72] it is proven via an examination of the packings of $PG(3,2)$ (see Theorem 17.5.5). However, the proof below is more elementary.

Theorem 3.2.10. The projective symplectic group $PSp(4,2)$ is isomorphic to S_6.

Proof. The full collineation group of a general linear complex in $PG(3,2)$ is $PSp(4,2)$ (see [72], p.6). In Theorem 3.2.5, we showed that S_6 is the full collineation group of W. Also, each collineation of W extends to a collineation of Σ_2. Thus, because the lines of W form a general linear complex of Σ_2 by Theorem 3.2.9, we have that $PSp(4,2)$ is isomorphic to S_6. $\square$

Theorem 3.2.11. $PSp(4,2)$ acts transitively on the ovoids and spreads of the generalised quadrangle $W(2)$.

Proof. As all general linear complexes of $PG(3,2)$ are projectively equivalent, we can consider the action of $PSp(4,2)$ (or S_6) on W.

By the form of the ovoids of W, it follows that the permutation (ij) in S_6 maps $\mathcal{O}_i$ to $\mathcal{O}_j$ while fixing W and Σ_2. Hence $PSp(4,2)$ (or S_6) is transitive on the ovoids of W.

It is immediate from the definition of the spreads that every spread of W lies in the orbit of R_1 under the action of S_6. Hence $PSp(4,2)$ (or S_6) is transitive on the spreads in W. $\square$

Theorem 3.2.12. Let $\mathcal{O}_i$ (for $i \in 1, \ldots, 6$) be an ovoid in W. Then the stabiliser in S_6 of $\mathcal{O}_i$ is a subgroup of index 6 in S_6 and is the stabiliser of the symbol i.

Proof. Let $\mathcal{O}_i^{S_6}$ be the orbit of $\mathcal{O}_i$ under S_6 and let $G_{\{\mathcal{O}_i\}}$ be the stabiliser of $\mathcal{O}_i$. From Theorem 3.2.11, the ovoids of W all lie in a single orbit. Hence, by the orbit-stabiliser theorem, we have:

$$|S_6| = \left|G_{\{\mathcal{O}_i\}}\right| \left|\mathcal{O}_i^{S_6}\right|.$$

That is:

$$6! = \left|G_{\{\mathcal{O}_i\}}\right|.6.$$

Thus, the stabiliser in S_6 of $\mathcal{O}_i$ has 5! elements and is of index 6 in S_6. Furthermore, each of the 5! permutations of S_6 which fix the letter i, also fixes $\mathcal{O}_i$. Hence, the stabiliser in S_6 of $\mathcal{O}_i$ is the stabiliser of the symbol i. □

Theorem 3.2.13. Let R_i (for $i \in 1, \ldots, 6$) be a spread in W. Then the stabiliser in S_6 of R_i, is a subgroup of index 6 in S_6 and acts 2-transitively on the symbols 1 to 6.

Proof. As in the proof of Theorem 3.2.12, the stabiliser in S_6 of R_i is a subgroup of index 6 in S_6. Now consider the action of the permutations (12) and (34) on the spreads of W. (12) interchanges R_1 with R_2, R_3 with R_6 and R_4 with R_5, while (34) interchanges R_1 with R_6, R_2 with R_3 and R_4 with R_5. The stabiliser of a symbol j for $j = 1$ to 6 contains at least one of the permutations (12),(34). Hence none of the stabilisers of a symbol j fixes the spread R_i. Consequently, the stabiliser in S_6 of R_i is a subgroup in the second conjugacy class of S_6 (as defined in Theorem 3.2.1). This implies that the stabiliser of R_i acts 2-transitively on the symbols 1 to 6. □

Remark 3.2.14. Theorems 3.2.12 and 3.2.13 show that the stabilisers in S_6 of the ovoids of W are the six subgroups of index 6 from the first conjugacy class and that the stabilisers in S_6 of the spreads of W are the six subgroups of index 6 from the second class. Moreover, by Theorem 3.2.1, these 12 subgroups are all isomorphic to S_5. □

Theorem 3.2.15. Let $\mathcal{C}_2$ be a proper 2-cover of $PG(3,2)$. Then the full collineation group G of $\mathcal{C}_2$ is isomorphic to S_5 and acts transitively on the points of $PG(3,2)$.

Proof. By Theorem 3.1.6, all proper 2-covers of $PG(3,2)$ are projectively equivalent and can be constructed by deleting a spread $\mathcal{S}$ from a general linear complex of $PG(3,2)$. Now because a proper 2-cover uniquely determines the general linear complex containing it, it follows that a collineation of $PG(3,2)$ fixes the 2-cover if and only if it fixes the general linear complex and the spread $\mathcal{S}$.

Also, the spreads of the generalised quadrangle $W(2)$ are exactly the spreads of $PG(3,2)$ lying in the corresponding general linear complex. Hence the full collineation group G of a proper 2-cover of $PG(3,2)$ is isomorphic to the stabiliser in S_6 of a spread of W. Thus G is isomorphic to S_5 by Theorem 3.2.13 and Remark 3.2.14. Furthermore, this subgroup of S_6 acts 2-transitively on the symbols 1 to 6. This implies that the subgroup acts transitively on the duads and so, equivalently, G acts transitively on the points of $PG(3,2)$. □

3.3. REGULAR PARTIAL PACKINGS OF $PG(3,q)$ AND BLOCKING SETS OF $PG(2,q^2)$

In this section, we digress briefly to examine several relationships which exist between regular partial packings of $PG(3,q)$ and blocking sets of $PG(2,q^2)$. The technique which we employ in this examination makes use of Bruck's representation of a regular spread of $PG(3,q)$ via a pair of skew conjugate lines of $PG(3,q^2)$. (See Section 1.8.)

While packings of $PG(3,q)$ are known to exist for all q (see [13], [46], [47], [72]), there are only two known regular packings and these are both packings of $PG(3,2)$. Lunardon has recently proven in [89] that no regular packings of $PG(3,q)$ exist for q odd and evidence suggests that the same is true for q even, $q > 2$. In the sequel, we assume the definitions, constructions and theorems of Section 1.8.

Definition 3.3.1. Let $\mathcal{P} = \left\{\mathcal{S}_i\right\}_{i=1}^{k}$, $k < q^2 + q + 1$ be a partial packing of $PG(3, q)$ and let S be the set of points of $PG(3, q^2)$

$$\{P \mid P \text{ lies on a line of } \mathcal{S}_i \text{ for at least one } \mathcal{S}_i \in \mathcal{P}\}.$$

Then a *transection* of $\mathcal{P}$ is defined to be the intersection of S with an imaginary plane π such that π does not meet $PG(3, q)$ in the $q + 1$ points of a line of $\mathcal{S}_i$, $i = 1$ to k. Given such a plane π, we denote the corresponding transection by $\mathcal{P}_\pi$. □

Remark 3.3.2. Let $\mathcal{P}$ be a partial packing $\left\{\mathcal{S}_i\right\}_{i=1}^{k}$ of $PG(3, q)$. Then two arbitrary lines belonging to spreads $\mathcal{S}_i$ and $\mathcal{S}_j$, $i \neq j$, are either skew or they meet in a real point because $\mathcal{S}_i$ and $\mathcal{S}_j$ are disjoint. Also, for an arbitrary line ℓ of $PG(3, q)$ which does not belong to any $\mathcal{S}_i$, there are exactly $q + 1$ lines of each $\mathcal{S}_i$ which meet ℓ. Thus it follows that each transection $\mathcal{P}_\pi$ of $\mathcal{P}$ contains exactly

$$k(q^2 - q) + (q + 1)$$

points and the line of $PG(3, q)$ containing the $q+1$ real points of $\mathcal{P}_\pi$ contains no further point of $\mathcal{P}_\pi$. □

Lemma 3.3.3. Let $\mathcal{P} = \left\{\mathcal{S}_i\right\}_{i=1}^{k}$ be a partial packing of $PG(3, q)$. If there exists an entirely imaginary line n of $PG(3, q^2)$ which is skew to every line belonging to the union of the spreads in $\mathcal{P}$, then $\mathcal{P}$ can be extended to a partial packing containing $k + 1$ spreads by adjoining a regular spread $\mathcal{S}$.

Proof. Since n is skew to every line m of every spread $\mathcal{S}_i$, it is immediate that n^σ is skew to $m^\sigma = m$ for each such m. Furthermore, n and n^σ do not intersect because they are both entirely imaginary. Thus n and n^σ can be used to construct a regular spread $\mathcal{S}$ of $PG(3, q)$ (via Bruck's construction) which is disjoint from each spread $\mathcal{S}_i$ and so extends $\mathcal{P}$. □

Theorem 3.3.4. Let $\mathcal{P} = \left\{\mathcal{S}_i\right\}_{i=1}^{k}$, $k < q^2 + q + 1$ be a partial packing of $PG(3, q)$. Then $\mathcal{P}$ can be extended to a partial packing with $k + 1$ spreads by adjoining a regular spread if and only if there exists a transection $\mathcal{P}_\pi$ of $\mathcal{P}$ which does not contain a line and which is not a blocking set of π.

Proof. ($\Rightarrow$) Assume that $\mathcal{P}$ is extendable in the manner described and let $\mathcal{S}$ denote the regular spread adjoined to $\mathcal{P}$.

As noted in Remark 3.3.2, the lines of $\mathcal{S}$ can only meet the lines of the spreads in $\mathcal{P}$ in real points. Hence the two conjugate skew lines n and n^σ from which $\mathcal{S}$ was constructed, are skew to every line of every spread in $\mathcal{P}$.

Let π be a plane through n. If π were real, then n^σ would also lie in π, from which it would follow that n and n^σ intersected each other. Since n and n^σ are skew, we thus conclude that π is an imaginary plane. Furthermore, the unique real line of π lies in $\mathcal{S}$ and consequently does not lie in any spread of $\mathcal{P}$. Therefore, π gives rise to a transection of $\mathcal{P}$.

Now consider $\mathcal{P}_\pi$. By construction $n \cap \mathcal{P}_\pi = \phi$. Therefore $\mathcal{P}_\pi$ cannot be a blocking set of π nor can it contain all the points of a line. Thus $\mathcal{P}_\pi$ satisfies the criteria in the statement of the theorem.

($\Leftarrow$) Assume that there exists a transection $\mathcal{P}_\pi$ which is not a blocking set of π and which does not contain a line.

It is immediate that π contains a line n with $n \cap \mathcal{P}_\pi = \phi$. In addition n is entirely imaginary because the only real points of π lie in $\mathcal{P}_\pi$. Hence the result follows by Lemma 3.3.3. $\square$

Example 3.3.5. Let $\mathcal{S}$ be an arbitrary spread of $PG(3, q)$ and let π be an imaginary plane whose real line does not lie in $\mathcal{S}$. Then π gives rise to the transection $\mathcal{P}_\pi$ where $\mathcal{P}$ is the partial packing consisting solely of the spread $\mathcal{S}$. $\mathcal{P}_\pi$ consists of $q^2 + 1$ points; $q + 1$ of these are real and lie on a line ℓ while the other $q^2 - q$ points are necessarily imaginary and do not lie on ℓ (see Remark 3.3.2).

It is immediate then, that the largest number of points of $\mathcal{P}_\pi$ which could be collinear is $q^2 - q + 1$. Thus $\mathcal{P}_\pi$ does not contain a line. Also, $\mathcal{P}_\pi$ is not a blocking set of π because a blocking set of π has at least $q^2 + q + 1$ points (see Theorem 1.7.4).

Hence, by Theorem 3.3.4, there exists a regular spread $\mathcal{S}'$ of $PG(3, q)$ which is disjoint from $\mathcal{S}$. □

Example 3.3.5 does not establish a very deep result but it serves to exhibit how Theorem 3.3.4 may be applied in practice. Unfortunately, it is difficult to obtain any general results on the extendability of partial packings of arbitrary size because comparatively little is known about the structure of blocking sets, except when the number of points in the blocking set is close to the minimum number of points that such a set can have. The problem is further complicated by a result of Berardi and Eugeni which states that in $PG(2, q^2)$, there exists a blocking set of size N for each N satisfying

$$q^2 + q + 1 \leq N \leq q^4 - q,$$

(which covers all the admissible values for the size of a blocking set in $PG(2, q^2)$). (See [11].) However, if the partial packing is regular, then we can alleviate this difficulty to a certain extent by using a similar technique. This gives a necessary condition for a regular partial packing to be extendable which, in the special case that $q = 2$, also turns out to be sufficient. We now outline this technique.

Let $PG(3, q)$ be embedded in $PG(3, q^2)$ and let $\mathcal{P}$ be a regular partial packing of $PG(3, q^2)$ consisting of k spreads. Let W be the set (of lines) obtained by taking the union of the pairs of conjugate skew lines of $PG(3, q^2)$ which are used to construct the spreads. The lines within each pair are skew by the construction and the lines between pairs are also skew because the spreads are disjoint. Hence W forms a partial spread of $PG(3, q^2)$ consisting of $2k$ entirely imaginary lines. Thus, if $\mathcal{P}$ can be extended to a regular partial packing with $k + 1$ spreads, then W can be extended to a partial $2k + 2$

spread of $PG(3, q^2)$ consisting of entirely imaginary lines.

When $q > 2$, the converse is not true in general. The real lines in $PG(3, q^2)$, $q > 2$ contain at least two distinct pairs of conjugate, imaginary points. Through these four points, it is then possible to construct two pairs of conjugate, entirely imaginary lines such that the lines in one pair are skew to the lines in the other pair. Thus the four lines are pairwise skew but the resulting spreads have at least one line in common. However, when $q = 2$, there is exactly one pair of conjugate, imaginary points on each real line of $PG(3, 2^2)$. Hence the argument is reversible in this case.

Thus to investigate the extendability of regular partial packings, we can gain some information by investigating the extendability of the partial spreads W. We now formalise this notion.

Definition 3.3.6. Let W be a partial spread of $PG(3, q^2)$ consisting of entirely imaginary lines and such that $W^\sigma = W$. Then W is called a σ-partial spread of $PG(3, q^2) \backslash PG(3, q)$. □

Remark 3.3.7. Since the lines of a σ-partial spread of $PG(3, q^2) \backslash PG(3, q)$ are entirely imaginary and W is fixed by σ, the lines of W fall into conjugate pairs. In particular, we have that $|W|$ is even. In addition, W gives rise to $\frac{1}{2}\,|W|$ regular spreads of $PG(3, q)$ which intersect pairwise in 0, 1, 2 or $q + 1$ lines. □

Definition 3.3.8. Let W be a σ-partial spread of $PG(3, q^2) \backslash PG(3, q)$, with $|W| < 2(q^2 + q + 1)$ and let S be the set of points of $PG(3, q^2)$

$$\{P \mid P \text{ lies on a line of } W\}.$$

Then a *hypertransection* of W is defined to be a set of the form

$$\{\pi \cap S\} \cup \{P \text{ is a real point of } \pi\}$$

where π is an arbitrary imaginary plane of $PG(3, q^2)$ which contains no line of W. Given such a plane π, we denote the corresponding hypertransection by W_π. □

Remark 3.3.9. For an arbitrary hypertransection W_π, we have that $|W_\pi| = |W|+q+1$. $\square$

Theorem 3.3.10. Let W be a σ-partial spread of $PG(3,q^2)\backslash PG(3,q)$. Then W is extendable to a σ-partial spread with $|W|+2$ lines if and only if there exists a hypertransection W_π of W which does not contain a line and which is not a blocking set of π.

Proof. ($\Rightarrow$) Suppose that W is extendable to a σ-partial spread W' with $|W|+2$ lines. Let ℓ be one of the two lines of $W'\backslash W$. Let π be a plane through ℓ. We have already shown in the proof of Theorem 3.3.4 that such a plane is imaginary because it contains ℓ which is an entirely imaginary line. In addition, π contains no line of W because ℓ lies in π and each line in W is skew to ℓ. Thus π gives rise to the hypertransection W_π of W.

Now by construction, ℓ and W_π both lie in π and ℓ does not intersect W_π. Thus W_π does not contain a line and is not a blocking set of π; it follows that W_π is a hypertransection satisfying the criteria in the statement of the theorem.

($\Leftarrow$) Suppose there is a hypertransection W_π which contains no line and which is not a blocking set of π. It is then immediate that there is at least one line ℓ of which has no point in common with W_π. The line ℓ is therefore skew to every line in W and is also entirely imaginary because every real point π lies in W_π. Finally since $W^\sigma = W$ and ℓ is entirely imaginary, we have that ℓ^σ is skew to every line in W and also to ℓ. Hence $W \cup \{\ell, \ell^\sigma\}$ is a σ-partial spread of $PG(3,q^2)\backslash PG(3,q)$ with $|W|+2$ lines. $\square$

Definition 3.3.11. Let W be a σ-partial spread of $PG(3,q^2)\backslash PG(3,q)$ and let W_π be a hypertransection of W. Then two imaginary points P and Q of W_π are said to be *associate points* if $P = m \cap \pi$ and $Q = m^\sigma \cap \pi$ for some line m of W. $\square$

Remark 3.3.12. Two associate points P and Q can only be conjugate points if they lie on the real line ℓ of π (if this were not the case, then π would be fixed by σ because σ would fix the two distinct lines ℓ and $\langle P, Q\rangle$. π would then be real, giving a contradiction). $\square$

Lemma 3.3.13. Let W be a σ-partial spread of $PG(3,q^2)\backslash PG(3,q)$ and let W_π be a hypertransection of W. Then two associate points of W_π which are not conjugate, never lie on a line containing a real point.

Proof. Let $P = m \cap \pi$ and $Q = m^\sigma \cap \pi$ be two such associate points in W_π (where m is a line of W).

Assume that P and Q lie on a line ℓ containing a real point R. Under the action of σ, π is mapped to a plane $\pi^\sigma \neq \pi$. Thus $P^\sigma \neq P$ and $Q^\sigma \neq Q$. However $R^\sigma = R$. Therefore ℓ^σ meets ℓ in exactly the real point R.

Now by construction, ℓ meets both m and m^σ. In addition, $P^\sigma \in m^\sigma$ and $Q^\sigma \in (m^\sigma)^\sigma = m$, so ℓ^σ also meets both m and m^σ. Since ℓ and ℓ' meet in the point R, it follows that m and m^σ are coplanar and therefore meet in a point; this contradicts the fact that m and m^σ are mutually skew, as they come from W.

Hence P and Q do not lie on a line containing a real point. $\square$

Lemma 3.3.14. Let W be a σ-partial spread of $PG(3,q^2)\backslash PG(3,q)$. If $\frac{1}{2}\ |W| < q^2+q+1$, then there exists a hypertransection W_π of W such that no imaginary point of W_π lies on the unique real line of π.

Proof. Let S denote the set of lines of $PG(3,q)$ which lie in the union of the $\frac{1}{2}\ |W|$ regular spreads arising from the lines of W. Since $\frac{1}{2}\ |W| < q^2+q+1$, there exists at least one line of $PG(3,q)$ which does not lie in any of the spreads. Let ℓ be such a line and denote by π, an arbitrary imaginary plane containing ℓ.

Assume that ℓ contains a point of the form $P = m \cap \pi$ where $m \in W$. Then

$P^\sigma = (m \cap \pi)^\sigma = (m \cap \ell)^\sigma = m^\sigma \cap \ell$. It follows that ℓ lies in the regular spread constructed from m and m^σ – a contradiction. Thus no such point P exists. Moreover, π cannot then contain any line of W. Thus π gives rise to hypertransection W_π of W which satisfies the criteria in the statement of the lemma. □

In Section 1.7, we stated a result by Bruen and Silverman on blocking sets in $PG(2,q)$ (see Theorem 1.7.6). Combining this with Lemma 3.3.13 and Theorem 3.3.10 we can prove the following result on the extendability of σ-partial spreads.

Theorem 3.3.15. Let W be a σ-partial spread of $PG(3,q^2)\backslash PG(3,q)$ with $|W| \leq q^2+t$ where t satisfies $0 \leq t < q\sqrt{2} - q - \frac{1}{2q^2}$. Then W can be extended to a σ-partial spread W' with $|W'| = |W| + 2$.

Proof. Since $\frac{1}{2}\,|W| < q^2+q+1$, there exists a hypertransection W_π of W, no imaginary point of which lies on the unique real line of π (see Lemma 3.3.14). (To establish the result, we need to show that W_π does not contain a line and that it is not a blocking set; having done this, the result will follow by applying Theorem 3.3.10.)

Assume that W_π contains a line ℓ. By the preceding comment, ℓ intersects the real line in a single real point. Therefore, by Lemma 3.3.13, ℓ cannot contain a pair of associate points. We have then that W has at least $2q^2$ lines because W_π has at least $(q^2+1)-1$ pairs of associate points.

On the other hand, W has at most q^2+t lines where t satisfies $0 \leq t < q\sqrt{2} - \frac{1}{2q^2}$. It is routine to show that $q\sqrt{2} - q - \frac{1}{2q^2} < q^2$ for all prime powers and so W has strictly less than $2q^2$ lines. These two inequalities for $|W|$ are contradictory. Hence W_π does not contain a line.

Assume then, that W_π is a blocking set of π. By Theorem 1.7.6, there exists a subset T of W_π which is the set of points of a Baer subplane of π. Now, when $q = 2$, we have $t = 0$ and so $T = W_\pi$. Then trivially, T contains the three real points of π.

When $q > 2$, we have

$$(\sqrt{2}-1)q - \frac{1}{2q^2} < (q-1).$$

Thus t is at most $q-1$. Hence T contains at least 2 real points of π because at most t real points could be removed from W_π to form T. Then, since T is the set of points of a Baer subplane, all of the real points lie in T because the only points of W_π on the real line are the real points. Thus T contains all the real points of π regardless of the value of q.

T therefore, has q^2 imaginary points. If no two of these were associates, then there would be at least $2q^2$ imaginary points in W_π implying that W had at least $2q^2$ lines. We showed earlier in the proof that this cannot be. Thus, T contains a pair of associate points. By Lemma 3.3.13, the line n containing these points meets the real line in an imaginary point. However, n is a line of the Baer subplane, hence n meets the real line in a real point because these are the only points of T which lie on the real line. This gives us a contradiction. Hence W_π is not a blocking set.

In summary, W_π contains no line of π and is not a blocking set of π. Therefore, by Theorem 3.3.10, W can be extended to a σ-partial spread with $|W|+2$ lines. $\square$

Corollary 3.3.16. Let $S = \left\{\mathcal{S}_i\right\}_{i=1}^{k}$ with $k \leq 2$, be a regular partial packing of $PG(3,2)$. Then S is extendable to a regular partial packing with $k+1$ spreads.

Proof. Embed $PG(3,2)$ in $PG(3,4)$ and let W be the σ-partial spread of $PG(3,4)\backslash PG(3,2)$ representing S. W has $2k$ ($\leq 2^2$) lines, therefore by Theorem 3.3.15, W can be extended to a σ-partial spread W' with $|W'| = |W| + 2$.

Now by the discussion after Example 3.3.5, the lines of $W'\backslash W$ give rise to a spread $\mathcal{S}'$ of $PG(3,2)$ which is disjoint from the other k spreads. Hence $\left\{\mathcal{S}_i\right\}_{i=1}^{k} \cup \{\mathcal{S}'\}$ is a regular partial packing with $k+1$ spreads. $\square$

Remark 3.3.17. We have attempted to prove the result in Corollary 3.3.16 with $k = 3$

theoretically, but without success. However, we have verified by computer, that the result does hold with $k = 3$. In fact, the result holds for all k satisfying $1 \leq k \leq 6$, so that a regular partial packing of $PG(3, 2)$ is always completable to a regular packing of $PG(3, 2)$. We summarise our findings in the table below:

Table 3.3a

Size of (regular) partial packing of $PG(3, 2)$	Number of spreads disjoint to every spread in the partial packing
1	30
2	15
3	7 or 8
4	3 or 6
5	2
6	1

The last entry in the table is there simply for completeness, as the complement of a 6-cover in the lineset of $PG(3, 2)$ (a 7-cover) is automatically a 1-cover (spread). The result for a (regular) partial packing of size 5 is a consequence of the uniqueness of the proper 2-cover of $PG(3, 2)$. It can be shown that the complement of a proper 2-cover of $PG(3, 2)$ is not the union of five pairwise disjoint spreads (this was first noted by Ebert in [53]). Hence the complement of a (regular) partial packing of size 5 is a set of ten lines which is the union of two disjoint spreads. Since the two spreads are uniquely determined, the result follows. $\square$

3.4 ON THE CLASSIFICATION OF THE PROPER n-COVERS OF $PG(3,2)$, $n \geq 3$

In Sections 3.1 and 3.2, we established the projective equivalence of the proper 2-covers of $PG(3,2)$ and determined the nature of the full collineation group of such a 2-cover. To complete the picture, we deal in this section, with proper n-covers of $PG(3,2)$ with $n \geq 3$. We begin with the three simplest cases, namely those for which $n = 5$, 6 or 7.

Theorem 3.4.1. *$PG(3,2)$ contains no proper 7-cover.*

Proof. This is immediate because a 7-cover of $PG(3,2)$ contains every line of $PG(3,2)$ and $PG(3,2)$ has at least one spread. Therefore a 7-cover cannot be proper. □

Theorem 3.4.2. *$PG(3,2)$ contains no proper 6-cover.*

Proof. Let $\mathcal{C}_6$ be a 6-cover of $PG(3,2)$. Then the complement of $\mathcal{C}_6$ in the lineset of $PG(3,2)$ is a spread $\mathcal{S}$. By Corollary 3.3.16, there is at least one spread $\mathcal{S}'$ disjoint from $\mathcal{S}$. Hence $\mathcal{S}' \subseteq \mathcal{C}_6$ and so $\mathcal{C}_6$ is not proper. □

Theorem 3.4.3. *$PG(3,2)$ contains no proper 5-cover.*

Proof. Let $\mathcal{C}_5$ be a 5-cover of $PG(3,2)$. Then the complement of $\mathcal{C}_5$ in the lineset of $PG(3,2)$ is a 2-cover $\mathcal{C}_2$.

If $\mathcal{C}_2$ is the union of two disjoint spreads, then by Corollary 3.3.16, there is a third spread $\mathcal{S}$ disjoint from both of these spreads. Hence $\mathcal{S} \subseteq \mathcal{C}_5$ and so $\mathcal{C}_5$ is not proper.

If, on the other hand, $\mathcal{C}_2$ is a proper 2-cover, then as we proved in Section 3.1, $\mathcal{C}_2$ can be obtained by removing a spread $\mathcal{S}$ from a general linear complex L_2, that is $\mathcal{C}_2 = \mathrm{L}_2 \backslash \mathcal{S}$. Hence $\mathcal{S} \subseteq \mathcal{C}_5$ and so $\mathcal{C}_5$ is not proper.

Therefore, combining the two facts, we have that no 5-cover of $PG(3,2)$ is proper.

□

Before we consider the question of the existence of a proper 4-cover of $PG(3,2)$, we need to examine the spreads of $PG(3,2)$ which are disjoint from a given proper 2-cover of $PG(3,2)$. To facilitate this, we shall use a list (List 3.4b compiled below) of the 56 (regular) spreads of $PG(3,2)$. Since we have already constructed a list of the lines of $PG(3,2)$ in Section 2.3 via the Singer group (corresponding to the primitive polynomial $\beta^4 = \beta + 1$ of degree 4 over $GF(2)$), we can again make use of it; so let the lines of $PG(3,2)$ be numbered as indicated:

List 3.4a

1	$\{0,5,10\}$	10	$\{3,8,13\}$	19	$\{4,11,13\}$	28	$\{3,10,12\}$
2	$\{0,6,13\}$	11	$\{4,9,14\}$	20	$\{1,7,14\}$	29	$\{5,12,14\}$
3	$\{0,7,9\}$	12	$\{1,12,13\}$	21	$\{4,6,12\}$	30	$\{2,4,10\}$
4	$\{0,1,4\}$	13	$\{6,8,14\}$	22	$\{2,3,6\}$	31	$\{4,5,8\}$
5	$\{0,2,8\}$	14	$\{2,9,11\}$	23	$\{7,8,11\}$	32	$\{9,10,13\}$
6	$\{0,3,14\}$	15	$\{3,4,7\}$	24	$\{1,2,5\}$	33	$\{5,7,13\}$
7	$\{0,11,12\}$	16	$\{2,13,14\}$	25	$\{10,11,14\}$	34	$\{5,6,9\}$
8	$\{1,6,11\}$	17	$\{8,9,12\}$	26	$\{3,5,11\}$	35	$\{6,7,10\}$.
9	$\{2,7,12\}$	18	$\{1,3,9\}$	27	$\{1,8,10\}$		

Now each spread of $PG(3,2)$ contains five lines. Therefore we can represent each spread of $PG(3,2)$ as a set of five numbers (from between 1 and 35 inclusive). Thus using the labelling, the 56 spreads are:

List 3.4b

$\mathcal{S}_1$	$\{1,8,9,10,11\}$	$\mathcal{S}_{29}$	$\{4,10,14,29,35\}$
$\mathcal{S}_2$	$\{1,12,13,14,15\}$	$\mathcal{S}_{30}$	$\{4,16,17,26,35\}$
$\mathcal{S}_3$	$\{1,8,15,16,17\}$	$\mathcal{S}_{31}$	$\{4,13,14,28,33\}$
$\mathcal{S}_4$	$\{1,9,13,18,19\}$	$\mathcal{S}_{32}$	$\{4,9,10,25,34\}$
$\mathcal{S}_5$	$\{1,10,14,20,21\}$	$\mathcal{S}_{33}$	$\{5,19,20,28,34\}$
$\mathcal{S}_6$	$\{1,11,12,22,23\}$	$\mathcal{S}_{34}$	$\{5,12,15,25,34\}$
$\mathcal{S}_7$	$\{1,16,18,21,23\}$	$\mathcal{S}_{35}$	$\{5,8,11,28,33\}$
$\mathcal{S}_8$	$\{1,17,19,20,22\}$	$\mathcal{S}_{36}$	$\{5,18,19,29,35\}$
$\mathcal{S}_9$	$\{2,15,17,24,25\}$	$\mathcal{S}_{37}$	$\{5,20,21,26,32\}$
$\mathcal{S}_{10}$	$\{2,9,11,26,27\}$	$\mathcal{S}_{38}$	$\{5,11,12,26,35\}$
$\mathcal{S}_{11}$	$\{2,11,23,24,28\}$	$\mathcal{S}_{39}$	$\{5,8,15,29,32\}$
$\mathcal{S}_{12}$	$\{2,14,15,27,29\}$	$\mathcal{S}_{40}$	$\{5,18,21,25,33\}$
$\mathcal{S}_{13}$	$\{2,17,20,26,30\}$	$\mathcal{S}_{41}$	$\{6,14,21,27,33\}$
$\mathcal{S}_{14}$	$\{2,14,20,28,31\}$	$\mathcal{S}_{42}$	$\{6,8,17,30,33\}$
$\mathcal{S}_{15}$	$\{2,18,23,29,30\}$	$\mathcal{S}_{43}$	$\{6,12,14,31,35\}$
$\mathcal{S}_{16}$	$\{2,9,18,25,31\}$	$\mathcal{S}_{44}$	$\{6,21,23,24,32\}$
$\mathcal{S}_{17}$	$\{3,12,13,26,30\}$	$\mathcal{S}_{45}$	$\{6,9,19,27,34\}$
$\mathcal{S}_{18}$	$\{3,10,21,24,25\}$	$\mathcal{S}_{46}$	$\{6,12,23,30,34\}$
$\mathcal{S}_{19}$	$\{3,12,22,25,31\}$	$\mathcal{S}_{47}$	$\{6,17,19,24,35\}$
$\mathcal{S}_{20}$	$\{3,8,10,29,30\}$	$\mathcal{S}_{48}$	$\{6,8,9,31,32\}$
$\mathcal{S}_{21}$	$\{3,16,21,26,27\}$	$\mathcal{S}_{49}$	$\{7,16,18,31,35\}$
$\mathcal{S}_{22}$	$\{3,13,19,24,28\}$	$\mathcal{S}_{50}$	$\{7,20,22,31,32\}$
$\mathcal{S}_{23}$	$\{3,19,22,27,29\}$	$\mathcal{S}_{51}$	$\{7,10,11,24,35\}$
$\mathcal{S}_{24}$	$\{3,8,16,28,31\}$	$\mathcal{S}_{52}$	$\{7,15,16,27,34\}$
$\mathcal{S}_{25}$	$\{4,22,23,29,32\}$	$\mathcal{S}_{53}$	$\{7,13,18,30,33\}$
$\mathcal{S}_{26}$	$\{4,17,22,25,33\}$	$\mathcal{S}_{54}$	$\{7,10,20,30,34\}$
$\mathcal{S}_{27}$	$\{4,16,23,28,34\}$	$\mathcal{S}_{55}$	$\{7,11,22,27,33\}$
$\mathcal{S}_{28}$	$\{4,9,13,26,32\}$	$\mathcal{S}_{56}$	$\{7,13,15,24,32\}$

(This list was compiled by hand using the fact that each line of $PG(3,2)$ lies in exactly 8 distinct spreads of $PG(3,2)$, coupled with the fact that each pair of distinct lines lies in exactly 2 distinct spreads of $PG(3,2)$. Since every spread of $PG(3,2)$ is regular (see Remark 1.8.9), these results can be demonstrated routinely using Bruck's technique for constructing regular spreads. (See Construction 1.8.10.))

With respect to this labelling of the lines, the set of lines of the proper 2-cover $\mathcal{C}_2$ which we constructed in Section 3.1 is represented by the set

$$\{4, 7, 11, 13, 22, 23, 24, 28, 32, 33\}.$$

Hence referring to List 3.4b, we have that the spreads of $PG(3,2)$ disjoint from $\mathcal{C}_2$ are

$$\mathcal{S}_3, \mathcal{S}_5, \mathcal{S}_{12}, \mathcal{S}_{13}, \mathcal{S}_{16}, \mathcal{S}_{20}, \mathcal{S}_{21}, \mathcal{S}_{34}, \mathcal{S}_{36}, \mathcal{S}_{43}, \mathcal{S}_{45}.$$

From amongst these eleven spreads, there are exactly twenty pairs of disjoint spreads. These pairs are listed below:

List 3.4c

$\{\mathcal{S}_3, \mathcal{S}_{16}\}, \{\mathcal{S}_3, \mathcal{S}_{36}\}, \{\mathcal{S}_3, \mathcal{S}_{43}\}, \{\mathcal{S}_3, \mathcal{S}_{45}\}, \{\mathcal{S}_5, \mathcal{S}_{16}\}, \{\mathcal{S}_5, \mathcal{S}_{34}\},$
$\{\mathcal{S}_5, \mathcal{S}_{36}\}, \{\mathcal{S}_5, \mathcal{S}_{45}\}, \{\mathcal{S}_{13}, \mathcal{S}_{34}\}, \{\mathcal{S}_{13}, \mathcal{S}_{36}\}, \{\mathcal{S}_{13}, \mathcal{S}_{43}\}, \{\mathcal{S}_{13}, \mathcal{S}_{45}\},$
$\{\mathcal{S}_{16}, \mathcal{S}_{20}\}, \{\mathcal{S}_{16}, \mathcal{S}_{21}\}, \{\mathcal{S}_{20}, \mathcal{S}_{34}\}, \{\mathcal{S}_{20}, \mathcal{S}_{43}\}, \{\mathcal{S}_{20}, \mathcal{S}_{45}\}, \{\mathcal{S}_{21}, \mathcal{S}_{34}\},$
$\{\mathcal{S}_{21}, \mathcal{S}_{36}\}, \{\mathcal{S}_{21}, \mathcal{S}_{43}\}.$

It is evident from this last list, that for each spread $\mathcal{S}$ disjoint from $\mathcal{C}_2$ other than $\mathcal{S}_{12}$, there is at least one other spread disjoint from $\mathcal{C}_2$ which is also disjoint to $\mathcal{S}$. It is also readily verifiable that the lines of $\mathcal{C}_2 \cup \mathcal{S}_{12}$ are the lines of the general linear complex $\mathrm{Ł}_2$ containing $\mathcal{C}_2$.

Using this, we can now prove the following result about proper 4-covers of $PG(3,2)$.

Theorem 3.4.4. If $PG(3,2)$ contains a proper 4-cover $\mathcal{C}_4$, then the complement of $\mathcal{C}_4$ is a proper 3-cover of $PG(3,2)$.

Proof. The complement of $\mathcal{C}_4$ in the lineset of $PG(3,2)$ is a 3-cover $\mathcal{C}_3$.

Assume $\mathcal{C}_3$ is the union of three pairwise disjoint spreads. By Remark 3.3.17, there is at least one spread $\mathcal{S}$ disjoint from each of the three spreads. Hence $\mathcal{S} \subseteq \mathcal{C}_4$ which implies that $\mathcal{C}_4$ is not proper – a contradiction.

Thus, assume $\mathcal{C}_3$ is the union of a proper 2-cover $\mathcal{C}_2$ and a spread disjoint $\mathcal{S}$ from $\mathcal{C}_2$. If $\mathcal{S} \cup \mathcal{C}_2$ is the set of lines of a general complex $\mathbb{L}_2$, then there exist two other general linear complexes $\mathbb{L}'_2$ and $\mathbb{L}''_2$ containing $\mathcal{S}$. Furthermore the 2-covers $\mathbb{L}'_2\backslash\{\mathcal{S}\}$ and $\mathbb{L}''_2\backslash\{\mathcal{S}\}$ are both distinct from each other and are also disjoint from $\mathcal{S}$ and $\mathcal{C}_2$. Hence $\mathcal{C}_4 = \left(\mathbb{L}'_2\backslash\{\mathcal{S}\}\right) \cup \left(\mathbb{L}''_2\backslash\{\mathcal{S}\}\right)$ and so $\mathcal{C}_4$ is the union of two disjoint (proper) 2-covers of $PG(3,2)$ – a contradiction. If $\mathcal{S} \cup \mathcal{C}_2$ is not the set of lines of a general linear complex of $PG(3,2)$, then by the remark in the discussion immediately preceding the statement of this theorem, there is at least one spread $\mathcal{S}'$ disjoint from $\mathcal{S}$ and $\mathcal{C}_2$. Hence $\mathcal{S}' \subseteq \mathcal{C}_4$ which implies that $\mathcal{C}_4$ is not proper – again a contradiction.

We deduce from this that $\mathcal{C}_3$ is neither the union of three pairwise disjoint spreads nor the union of a proper 2-cover and a spread (disjoint from the 2-cover). Hence $\mathcal{C}_3$ is a proper 3-cover. □

To date, no examples of proper 4-covers of $PG(3,2)$ are known. However, by Theorem 3.4.4, the question of the existence of proper 4-covers will be settled once all projectively distinct proper 3-covers are known. We now discuss the results of our investigation of proper 3-covers of $PG(3,2)$.

In Section 2.3 we constructed two proper 3-covers of $PG(3,2)$ each of which consisted of the lines of a so-called long Singer orbit. Theorem 3.4.5 below shows how proper 3-covers can also be constructed using spreads and proper 2-covers.

Theorem 3.4.5. Let $\mathcal{C}_2$ be a proper 2-cover of $PG(3,2)$ and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two disjoint spreads of $PG(3,2)$ which are also disjoint from $\mathcal{C}_2$. Then the complement of

$\mathcal{C}_2 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ in the lineset of $PG(3,2)$ is a proper 3-cover $\mathcal{C}_3$ of $PG(3,2)$.

Proof. We first note that there do exist spreads which satisfy the hypothesis above (see the discussion after Theorem 3.4.3 and in particular List 3.4c). The complement of $\mathcal{C}_2 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ is a 3-cover $\mathcal{C}_3$ of $PG(3,2)$. If $\mathcal{C}_3$ is the union of three pairwise disjoint spreads, then $\mathcal{C}_2$ is complemented by five pairwise disjoint spreads. However, it was noted in Remark 3.3.17, that this is impossible. Thus $\mathcal{C}_3$ is not the union of three pairwise disjoint spreads.

Thus assume $\mathcal{C}_3$ is the union of a proper 2-cover $\mathcal{C}_2'$ and a spread $\mathcal{S}_3$ disjoint from $\mathcal{C}_2'$. Let $\mathbb{L}_2$ and $\mathbb{L}_2'$ be the general linear complexes of $PG(3,2)$ in which $\mathcal{C}_2$ and $\mathcal{C}_2'$ respectively are embedded. By Theorem 1.10.9, $\mathbb{L}_2$ and $\mathbb{L}_2'$ intersect in a regular spread and through this spread there is a third general linear complex $\mathbb{L}_2''$. Now by construction, $\mathcal{C}_2$, $\mathcal{C}_2'$ and $\mathbb{L}_2''$ are pairwise disjoint and contain amongst them every line of $PG(3,2)$. Similarly $\mathcal{C}_2$, $\mathcal{C}_2'$ and $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ are pairwise disjoint and contain amongst them every line of $PG(3,2)$. Thus $\mathbb{L}_2'' = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. This contradicts Theorem 2.2.1 (and also Theorem 2.2.6 and Corollary 2.2.4), which states that a general linear complex cannot be partitioned into spreads.

Hence $\mathcal{C}_3$ is neither the union of three spreads nor the union of a spread and a proper 2-cover and so is a proper 3-cover of $PG(3,2)$. $\square$

Given Theorem 3.4.5, we might expect that there would exist at least two projectively distinct proper 3-covers from amongst those that can be constructed using a proper 2-cover $\mathcal{C}_2$ and the twenty pairs of disjoint spreads which are also disjoint from $\mathcal{C}_2$. (In the discussion following Theorem 3.4.3 the number of such pairs of spreads was established for the particular proper 2-cover which we constructed in Section 3.1 and therefore is the same number for any proper 2-cover of $PG(3,2)$ by Theorem 3.1.6). However they can all be shown to be projectively equivalent to the proper 3-covers of $PG(3,2)$ constructed from long Singer orbits. In the remainder of this chapter, we shall establish this result.

Let $\mathcal{C}_3^*$ denote the first proper 3-cover of $PG(3,2)$ which we constructed in Section 2.3 from a long Singer orbit. With respect to list 3.4a, the lines of $\mathcal{C}_3^*$ are numbered

$$4, 6, 7, 12, 15, 16, 17, 22, 23, 24, 25, 31, 32, 34, 35.$$

Thus consulting List 3.4b, we can show that there are exactly six distinct spreads of $PG(3,2)$ disjoint from $\mathcal{C}_3^*$, namely:

$$\begin{aligned}
\mathcal{S}_1 &= \{1, 8, 9, 10, 11\}\\
\mathcal{S}_4 &= \{1, 9, 13, 18, 19\}\\
\mathcal{S}_5 &= \{1, 10, 14, 20, 21\}\\
\mathcal{S}_{10} &= \{2, 9, 11, 26, 27\}\\
\mathcal{S}_{20} &= \{3, 8, 10, 29, 30\}\\
\mathcal{S}_{35} &= \{5, 8, 11, 28, 33\}.
\end{aligned}$$

From these six spreads we obtain five pairs of disjoint spreads $\{\mathcal{S}_4, \mathcal{S}_{20}\}$, $\{\mathcal{S}_{20}, \mathcal{S}_{10}\}$, $\{\mathcal{S}_{10}, \mathcal{S}_5\}$, $\{\mathcal{S}_5, \mathcal{S}_{35}\}$ and $\{\mathcal{S}_{35}, \mathcal{S}_4\}$. Furthermore, for each pair $\{\mathcal{S}_i, \mathcal{S}_j\}$, the complement of $\mathcal{C}_3^* \cup \mathcal{S}_i \cup \mathcal{S}_j$ in the lineset of $PG(3,2)$ is a proper 2-cover $\mathcal{C}_2$ of $PG(3,2)$ (this can be deduced from the results of Table 3.3a as follows: If $\mathcal{C}_2$ were the union of two disjoint spreads $\mathcal{S}_k$ and $\mathcal{S}_m$, then $\{\mathcal{S}_i, \mathcal{S}_j, \mathcal{S}_k, \mathcal{S}_m\}$ would be a regular partial packing of $PG(3,2)$. From Table 3.3a, there is at least one spread disjoint from each of the four spreads, hence $\mathcal{C}_3^*$ contains a spread – a contradiction). Thus $\mathcal{C}_3^*$ itself arises from the construction technique established by Theorem 3.4.5 and it arises from five distinct sets of proper 2-covers and spread pairs.

Lemma 3.4.6. Let $\langle\beta\rangle$ be the Singer group from which $\mathcal{C}_3^*$ arises. Then $\langle\beta\rangle$ acts transitively on the proper 2-covers of $PG(3,2)$ disjoint from $\mathcal{C}_3^*$ and $\langle\beta^5\rangle$ fixes each such 2-cover.

Proof. First consider the spread $\mathcal{S}_4$. Writing the lines of $\mathcal{S}_4$ out in full we get:

$$\mathcal{S}_4 = \{\{0,5,10\},\{2,7,12\},\{6,8,14\},\{1,3,9\},\{4,11,13\}\}.$$

Now applying the first five elements of $\langle\beta\rangle$ to $\mathcal{S}_4$ we have that $\beta(\mathcal{S}_4) = \mathcal{S}_{20}$, $\beta^2(\mathcal{S}_4) = \mathcal{S}_{10}$, $\beta^3(\mathcal{S}_4) = \mathcal{S}_5$, $\beta^4(\mathcal{S}_4) = \mathcal{S}_{35}$ and $\beta^5(\mathcal{S}_4) = \mathcal{S}_4$.

Hence

$$\begin{aligned}
\beta(\{\mathcal{S}_4,\mathcal{S}_{20}\}) &= \{\mathcal{S}_{20},\mathcal{S}_{10}\}\\
\beta^2(\{\mathcal{S}_4,\mathcal{S}_{20}\}) &= \{\mathcal{S}_{10},\mathcal{S}_5\}\\
\beta^3(\{\mathcal{S}_4,\mathcal{S}_{20}\}) &= \{\mathcal{S}_5,\mathcal{S}_{35}\}\\
\beta^4(\{\mathcal{S}_4,\mathcal{S}_{20}\}) &= \{\mathcal{S}_{35},\mathcal{S}_4\}\\
\beta^5(\{\mathcal{S}_4,\mathcal{S}_{20}\}) &= \{\mathcal{S}_4,\mathcal{S}_{20}\}.
\end{aligned}$$

It is immediate that $\langle\beta\rangle$ acts transitively on the pairs of disjoint spreads and $\langle\beta^5\rangle$ fixes each pair of disjoint spreads. Hence, since $\langle\beta\rangle$ fixes $\mathcal{C}_3^*$, $\langle\beta\rangle$ also acts transitively on the proper 2-covers of $PG(3,2)$ disjoint from $\mathcal{C}_3^*$ and $\langle\beta^5\rangle$ fixes each such 2-cover.

□

Having now investigated the action of $\langle\beta\rangle$ on the proper 2-covers of $PG(3,2)$ disjoint from $\mathcal{C}_3^*$, we are in a position to find the order of the full collineation group G of $\mathcal{C}_3^*$.

By the orbit-stabiliser theorem, we have

$$|G| = \left|\mathcal{C}_2^G\right| \left|G_{\mathcal{C}_2}\right|$$

for each proper 2-cover disjoint from $\mathcal{C}_3^*$. By Lemma 3.4.6 $|\mathcal{C}_2^G| = 5$, because $\langle\beta\rangle$ acts transitively on these 2-covers and $\langle\beta\rangle < G$. Hence to find $|G|$, we need only find $|G_{\mathcal{C}_2}|$ for one such 2-cover $\mathcal{C}_2$.

Without loss of generality, we can choose $\mathcal{C}_2$ to be the complement of $\mathcal{C}_3^* \cup \mathcal{S}_4 \cup \mathcal{S}_{20}$.

Thus:

$$\begin{aligned}
\mathcal{C}_3^* &= \big\{\{0,1,4\},\{0,3,14\},\{0,11,12\},\{1,12,13\},\{3,4,7\},\\
&\quad \{2,13,14\},\{8,9,12\},\{2,3,6\},\{7,8,11\},\{1,2,5\},\\
&\quad \{10,11,14\},\{4,5,8\},\{9,10,11\},\{5,6,9\},\{6,7,10\}\big\},\\
\mathcal{S}_4 &= \big\{\{0,5,10\},\{2,7,12\},\{6,8,14\},\{1,3,9\},\{4,11,13\}\big\},\\
\mathcal{S}_{20} &= \big\{\{0,7,12\},\{1,6,11\},\{3,8,13\},\{5,12,14\},\{2,4,10\}\big\},\\
\text{and}\quad \mathcal{C}_2 &= \big\{\{0,6,13\},\{0,2,8\},\{4,9,14\},\{2,9,11\},\{1,7,14\},\\
&\quad \{4,6,12\},\{3,5,11\},\{1,8,10\},\{3,10,12\},\{5,7,13\}\big\}.
\end{aligned}$$

Because we have already found the full collineation group of $\mathcal{C}_2$ in terms of a subgroup of the symmetric group S_6, it is convenient to move from this setting to the setting in Section 3.2 involving duads, synthemes and triplets.

The correspondence below, between the points above and the duads of Section 3.2, is incidence preserving, as can be checked directly. (Note: To avoid confusing the numbers, the points which we have been representing by the exponents of the powers of β, will again be written in full as β^i .)

List 3.4d

$\beta^0 \leftrightarrow (56)$	$\beta^5 \leftrightarrow (15)$	$\beta^{10} \leftrightarrow (16)$
$\beta^1 \leftrightarrow (35)$	$\beta^6 \leftrightarrow (14)$	$\beta^{11} \leftrightarrow (26)$
$\beta^2 \leftrightarrow (13)$	$\beta^7 \leftrightarrow (46)$	$\beta^{12} \leftrightarrow (25)$
$\beta^3 \leftrightarrow (34)$	$\beta^8 \leftrightarrow (24)$	$\beta^{13} \leftrightarrow (23)$
$\beta^4 \leftrightarrow (36)$	$\beta^9 \leftrightarrow (45)$	$\beta^{14} \leftrightarrow (12)$.

Under this correspondence, we have:

$$\begin{aligned}
\mathcal{C}_2 &= \big\{(56)(14)(23),(56)(13)(24),(36)(45)(12),(13)(45)(26),(35)(46)(12),\\
&\quad (36)(14)(25),(34)(15)(26),(35)(24)(16),(34)(16)(25),(15)(46)(23)\big\}\\
\mathcal{S}_4 &= \big\{(56)(15)(16),(13)(46)(25),(14)(24)(12),(35)(34)(45),(36)(26)(23)\big\}\\
\text{and}\quad \mathcal{S}_{20} &= \big\{(56)(46)(45),(35)(14)(26),(34)(24)(23),(15)(25)(12),(13)(36)(16)\big\}.
\end{aligned}$$

The five lines of the regular spread $\mathcal{S}$ which completes $\mathcal{C}_2$ to a general linear complex of $PG(3,2)$, are

$$(56)(34)(12), (13)(46)(25), (35)(14)(26), (36)(15)(24), (45)(16)(23).$$

As was shown in Section 3.2, the full collineation of $\mathcal{C}_2$ is the subgroup of index 6 in S_6 which fixes $\mathcal{S}$. Using the group theory computer package "Cayley", we have calculated the twelve subgroups of index 6 in S_6 and so by trial and error, we have found that the subgroup required is

$$\langle (1625), (1436) \rangle.$$

Theorem 3.4.7. With respect to this representation, the stabiliser $G_{\mathcal{C}_2}$ of $\mathcal{C}_2$ in G is the cyclic group generated by the permutation (146253).

Proof. Since the subgroup $\langle (1625), (1436) \rangle$ of S_6 is the full subgroup of $\mathcal{C}_2$, the stabiliser $G_{\mathcal{C}_2}$ of $\mathcal{C}_2$ in G (fixing $\mathcal{C}_3^*$) is the subgroup of $\langle (1625), (1436) \rangle$ which fixes $\mathcal{S}_4$ and $\mathcal{S}_{20}$ (as a pair).

It is routine to show that the permutations (12)(36)(45) and (234)(165) do fix $\mathcal{S}_4$ and $\mathcal{S}_{20}$ as a pair. Hence $\langle (12)(36)(45), (234)(165) \rangle$ is a subgroup of $G_{\mathcal{C}_2}$.

Also from Lemma 3.4.6, we have that 3 divides $\left| G_{\mathcal{C}_2} \right|$ because $G_{\mathcal{C}_2}$ has a subgroup of order 3 isomorphic to $\langle \beta^5 \rangle$.

Once again, using "Cayley", all subgroups of $\langle (1625), (1436) \rangle$ with order divisible by 3 have been calculated. Exactly two contain the elements (12)(36)(45) and (234)(165). One is the subgroup generated by these elements. The other also contains the generator (12)(35)(46), but this element does not fix $\mathcal{S}_4$ and $\mathcal{S}_{20}$ as a pair. Therefore

$$\begin{aligned} G_{\mathcal{C}_2} &= \langle (12)(36)(45), (234)(165) \rangle \\ &= \langle (146253) \rangle. \end{aligned}$$

□

Corollary 3.4.8. The full collineation group G of $\mathcal{C}_3^*$ has order 30 and can be represented by the subgroup of $PGL(4,2)$ which is generated by the elements

$$\beta \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \rho \equiv \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Proof. In the discussion immediately after Lemma 3.4.6, we showed that $\left|\mathcal{C}_2^G\right| = 5$. By Theorem 3.4.7, $\left|G_{\mathcal{C}_2}\right| = 6$. Hence by the orbit-stabiliser theorem, we have

$$\begin{aligned} |G| &= \left|\mathcal{C}_2^G\right| \left|G_{\mathcal{C}_2}\right| \\ &= 5.6 \\ &= 30. \end{aligned}$$

By construction, $\mathcal{C}_3^*$ admits the Singer group $\langle\beta\rangle$ and by Remark 1.3.4, β corresponds to the element

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

of $PGL(4,2)$.

In the alternative representation of $PG(3,2)$ via duads, synthemes and triplets we also established that $\mathcal{C}_3^*$ admits the subgroup $\langle\rho\rangle$ of S_6 where $\rho = (146253)$. To find the element of $PGL(4,2)$ corresponding to ρ, we can avail ourselves of the correspondence between the points of $PG(3,2)$ as powers of β and as duads, indicated in List 3.4d. Hence we have:

$$\begin{aligned}
&(1\,0\,0\,0) = \beta^0 \leftrightarrow (56); \quad \rho(56) = (32) \leftrightarrow \beta^{13} = (1\,0\,1\,1) = \rho(1\,0\,0\,0)\\
&(0\,1\,0\,0) = \beta^1 \leftrightarrow (35); \quad \rho(35) = (13) \leftrightarrow \beta^{2} = (0\,0\,1\,0) = \rho(0\,1\,0\,0)\\
&(0\,0\,1\,0) = \beta^2 \leftrightarrow (13); \quad \rho(13) = (41) \leftrightarrow \beta^{6} = (0\,0\,1\,1) = \rho(0\,0\,1\,0)\\
&(0\,0\,0\,1) = \beta^3 \leftrightarrow (34); \quad \rho(34) = (16) \leftrightarrow \beta^{10} = (1\,1\,1\,0) = \rho(0\,0\,0\,1)\\
&(1\,1\,1\,1) = \beta^{12} \leftrightarrow (25); \quad \rho(25) = (53) \leftrightarrow \beta^{1} = (0\,1\,0\,0) = \rho(1\,1\,1\,1).
\end{aligned}$$

It is then routine to check that $\rho \equiv \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Now since ρ has order 6, it does not lie in $\langle\beta\rangle$ because 6 does not divide 15. Thus the cosets $\langle\beta\rangle$ and $\rho\langle\beta\rangle$ of $\langle\beta\rangle$ in G contain between them 30 elements of G and so $G = \langle\beta, \rho\rangle$ as required. $\square$

Remark 3.4.9. We showed in Lemma 3.4.6, that β^5 fixes each proper 2-cover disjoint from $\mathcal{C}_3^*$. Hence $\beta^5 \in \langle\rho\rangle$. Now β^5 has order 3 and ρ^2 is the unique element of $\langle\rho\rangle$ with order 3. Thus, $\beta^5 = \rho^2$. $\square$

Before proving the main result for this latter part of the section, we first re-examine the proper 3-covers of $PG(3,2)$ arising from the technique described in Theorem 3.4.5.

Consider the proper 2-cover $\mathcal{C}_2$ constructed in Section 3.1. We have already shown that there are exactly twenty pairs of disjoint spreads also disjoint from $\mathcal{C}_2$ and these are presented in List 3.4c. Hence twenty distinct proper 3-covers of $PG(3,2)$ can be constructed from $\mathcal{C}_2$ and these spread pairs. We have in fact constructed all of these and by consulting List 3.4b, we have shown that each such 3-cover is disjoint from exactly five pairs of disjoint spreads. The results are listed in Table 3.4a below.

Since every proper 2-cover is projectively equivalent to $\mathcal{C}_2$, it follows that every proper 3-cover arising from the technique used in Theorem 3.4.5 is disjoint from exactly five pairs of disjoint spreads. This property can be used to characterise the 3-covers. We now establish this.

Table 3.4a

Pairs of disjoint spreads also disjoint from $\mathcal{C}_2$	Pairs of disjoint spreads also disjoint from the resulting 3-cover
$\{\mathcal{S}_3, \mathcal{S}_{16}\}$	$\{\mathcal{S}_3, \mathcal{S}_{11}\}, \{\mathcal{S}_3, \mathcal{S}_{16}\}, \{\mathcal{S}_{11}, \mathcal{S}_{26}\}, \{\mathcal{S}_{16}, \mathcal{S}_{56}\}, \{\mathcal{S}_{26}, \mathcal{S}_{56}\}$
$\{\mathcal{S}_3, \mathcal{S}_{36}\}$	$\{\mathcal{S}_3, \mathcal{S}_{25}\}, \{\mathcal{S}_3, \mathcal{S}_{36}\}, \{\mathcal{S}_{25}, \mathcal{S}_{35}\}, \{\mathcal{S}_{35}, \mathcal{S}_{56}\}, \{\mathcal{S}_{36}, \mathcal{S}_{56}\}$
$\{\mathcal{S}_3, \mathcal{S}_{43}\}$	$\{\mathcal{S}_3, \mathcal{S}_{31}\}, \{\mathcal{S}_3, \mathcal{S}_{43}\}, \{\mathcal{S}_6, \mathcal{S}_{31}\}, \{\mathcal{S}_6, \mathcal{S}_{56}\}, \{\mathcal{S}_{43}, \mathcal{S}_{56}\}$
$\{\mathcal{S}_3, \mathcal{S}_{45}\}$	$\{\mathcal{S}_3, \mathcal{S}_{45}\}, \{\mathcal{S}_3, \mathcal{S}_{55}\}, \{\mathcal{S}_{27}, \mathcal{S}_{55}\}, \{\mathcal{S}_{27}, \mathcal{S}_{56}\}, \{\mathcal{S}_{45}, \mathcal{S}_{56}\}$
$\{\mathcal{S}_5, \mathcal{S}_{16}\}$	$\{\mathcal{S}_5, \mathcal{S}_{11}\}, \{\mathcal{S}_5, \mathcal{S}_{16}\}, \{\mathcal{S}_{11}, \mathcal{S}_{50}\}, \{\mathcal{S}_{16}, \mathcal{S}_{31}\}, \{\mathcal{S}_{31}, \mathcal{S}_{50}\}$
$\{\mathcal{S}_5, \mathcal{S}_{34}\}$	$\{\mathcal{S}_5, \mathcal{S}_{34}\}, \{\mathcal{S}_5, \mathcal{S}_{56}\}, \{\mathcal{S}_6, \mathcal{S}_{31}\}, \{\mathcal{S}_6, \mathcal{S}_{56}\}, \{\mathcal{S}_{31}, \mathcal{S}_{34}\}$
$\{\mathcal{S}_5, \mathcal{S}_{36}\}$	$\{\mathcal{S}_5, \mathcal{S}_{25}\}, \{\mathcal{S}_5, \mathcal{S}_{36}\}, \{\mathcal{S}_{25}, \mathcal{S}_{51}\}, \{\mathcal{S}_{31}, \mathcal{S}_{36}\}, \{\mathcal{S}_{31}, \mathcal{S}_{51}\}$
$\{\mathcal{S}_5, \mathcal{S}_{45}\}$	$\{\mathcal{S}_5, \mathcal{S}_{45}\}, \{\mathcal{S}_5, \mathcal{S}_{55}\}, \{\mathcal{S}_{31}, \mathcal{S}_{44}\}, \{\mathcal{S}_{31}, \mathcal{S}_{45}\}, \{\mathcal{S}_{44}, \mathcal{S}_{55}\}$
$\{\mathcal{S}_{13}, \mathcal{S}_{34}\}$	$\{\mathcal{S}_{11}, \mathcal{S}_{26}\}, \{\mathcal{S}_{11}, \mathcal{S}_{34}\}, \{\mathcal{S}_{13}, \mathcal{S}_{34}\}, \{\mathcal{S}_{13}, \mathcal{S}_{56}\}, \{\mathcal{S}_{26}, \mathcal{S}_{56}\}$
$\{\mathcal{S}_{13}, \mathcal{S}_{36}\}$	$\{\mathcal{S}_{11}, \mathcal{S}_{36}\}, \{\mathcal{S}_{11}, \mathcal{S}_{53}\}, \{\mathcal{S}_{13}, \mathcal{S}_{25}\}, \{\mathcal{S}_{13}, \mathcal{S}_{36}\}, \{\mathcal{S}_{25}, \mathcal{S}_{53}\}$
$\{\mathcal{S}_{13}, \mathcal{S}_{43}\}$	$\{\mathcal{S}_{11}, \mathcal{S}_{43}\}, \{\mathcal{S}_{11}, \mathcal{S}_{50}\}, \{\mathcal{S}_{13}, \mathcal{S}_{31}\}, \{\mathcal{S}_{13}, \mathcal{S}_{43}\}, \{\mathcal{S}_{31}, \mathcal{S}_{50}\}$
$\{\mathcal{S}_{13}, \mathcal{S}_{45}\}$	$\{\mathcal{S}_{11}, \mathcal{S}_{28}\}, \{\mathcal{S}_{11}, \mathcal{S}_{45}\}, \{\mathcal{S}_{13}, \mathcal{S}_{45}\}, \{\mathcal{S}_{13}, \mathcal{S}_{55}\}, \{\mathcal{S}_{28}, \mathcal{S}_{55}\}$
$\{\mathcal{S}_{16}, \mathcal{S}_{20}\}$	$\{\mathcal{S}_{11}, \mathcal{S}_{20}\}, \{\mathcal{S}_{11}, \mathcal{S}_{53}\}, \{\mathcal{S}_{16}, \mathcal{S}_{20}\}, \{\mathcal{S}_{16}, \mathcal{S}_{25}\}, \{\mathcal{S}_{25}, \mathcal{S}_{53}\}$
$\{\mathcal{S}_{16}, \mathcal{S}_{21}\}$	$\{\mathcal{S}_{11}, \mathcal{S}_{21}\}, \{\mathcal{S}_{11}, \mathcal{S}_{28}\}, \{\mathcal{S}_{16}, \mathcal{S}_{21}\}, \{\mathcal{S}_{16}, \mathcal{S}_{55}\}, \{\mathcal{S}_{28}, \mathcal{S}_{55}\}$
$\{\mathcal{S}_{20}, \mathcal{S}_{34}\}$	$\{\mathcal{S}_{20}, \mathcal{S}_{34}\}, \{\mathcal{S}_{20}, \mathcal{S}_{56}\}, \{\mathcal{S}_{25}, \mathcal{S}_{34}\}, \{\mathcal{S}_{25}, \mathcal{S}_{35}\}, \{\mathcal{S}_{35}, \mathcal{S}_{56}\}$
$\{\mathcal{S}_{20}, \mathcal{S}_{43}\}$	$\{\mathcal{S}_{20}, \mathcal{S}_{31}\}, \{\mathcal{S}_{20}, \mathcal{S}_{43}\}, \{\mathcal{S}_{25}, \mathcal{S}_{43}\}, \{\mathcal{S}_{25}, \mathcal{S}_{51}\}, \{\mathcal{S}_{31}, \mathcal{S}_{51}\}$
$\{\mathcal{S}_{20}, \mathcal{S}_{45}\}$	$\{\mathcal{S}_{20}, \mathcal{S}_{45}\}, \{\mathcal{S}_{20}, \mathcal{S}_{55}\}, \{\mathcal{S}_{22}, \mathcal{S}_{25}\}, \{\mathcal{S}_{22}, \mathcal{S}_{55}\}, \{\mathcal{S}_{25}, \mathcal{S}_{45}\}$
$\{\mathcal{S}_{21}, \mathcal{S}_{34}\}$	$\{\mathcal{S}_{21}, \mathcal{S}_{34}\}, \{\mathcal{S}_{21}, \mathcal{S}_{56}\}, \{\mathcal{S}_{27}, \mathcal{S}_{55}\}, \{\mathcal{S}_{27}, \mathcal{S}_{56}\}, \{\mathcal{S}_{34}, \mathcal{S}_{55}\}$
$\{\mathcal{S}_{21}, \mathcal{S}_{36}\}$	$\{\mathcal{S}_{21}, \mathcal{S}_{25}\}, \{\mathcal{S}_{21}, \mathcal{S}_{36}\}, \{\mathcal{S}_{22}, \mathcal{S}_{25}\}, \{\mathcal{S}_{22}, \mathcal{S}_{55}\}, \{\mathcal{S}_{36}, \mathcal{S}_{55}\}$
$\{\mathcal{S}_{21}, \mathcal{S}_{43}\}$	$\{\mathcal{S}_{21}, \mathcal{S}_{31}\}, \{\mathcal{S}_{21}, \mathcal{S}_{43}\}, \{\mathcal{S}_{31}, \mathcal{S}_{44}\}, \{\mathcal{S}_{43}, \mathcal{S}_{55}\}, \{\mathcal{S}_{44}, \mathcal{S}_{55}\}$

Theorem 3.4.10. Let $\mathcal{C}_3$ be a proper 3-cover of $PG(3,2)$. If there exist exactly five pairs of disjoint spreads also disjoint from $\mathcal{C}_3$ then $\mathcal{C}_3$ is projectively equivalent to $\mathcal{C}_3^*$.

Proof. First we note that for each pair of disjoint spreads $\{\mathcal{S}', \mathcal{S}''\}$, with $\mathcal{S}'$ and $\mathcal{S}''$ also disjoint from $\mathcal{C}_3$, the complement of $\mathcal{C}_3 \cup \mathcal{S}' \cup \mathcal{S}''$ in the lineset of $PG(3,2)$ is a 2-cover $\mathcal{C}_2$ of $PG(3,2)$; moreover, by the findings presented in Table 3.3a, $\mathcal{C}_2$ is proper because $\mathcal{C}_3$ cannot be complemented by four pairwise disjoint spreads. Hence the hypothesis is equivalent to the supposition that there are exactly five proper 2-covers disjoint from $\mathcal{C}_3$.

We now compute the number of such 3-covers by counting the number of disjoint (2-cover)-(3-cover) pairs in two ways. If N is the number of such 3-covers, then the number of disjoint (2-cover)-(3-cover) pairs is

$$5N$$

because each 3-cover is disjoint from exactly five 2-covers.

However, since the proper 2-covers of $PG(3,2)$ are all projectively equivalent by Theorem 3.1.6, the number of distinct proper 2-covers is the same as the size of the orbit of a single proper 2-cover $\mathcal{C}_2$ under the action of $PGL(4,2)$. Hence, by the orbit-stabiliser theorem, this number is

$$\frac{|PGL(4,2)|}{|S_5|} = \frac{|PGL(4,2)|}{120},$$

because $(PGL(4,2))_{\mathcal{C}_2}$ is isomorphic to S_5 by Theorem 3.2.15.

In addition, by the results stated in the commentary immediately preceding this theorem, each proper 2-cover is disjoint from exactly twenty pairs of disjoint spreads and for each such pair of disjoint spreads, the proper 3-cover complemented by the spread pair and the 2-cover, is disjoint from exactly five pairs of disjoint spreads (see Table 3.4a). Equivalently each such 3-cover is disjoint from exactly five distinct proper 2-covers.

It follows then that the number of disjoint (2-cover)-(3-cover) pairs is

$$20 \cdot \frac{|PGL(4,2)|}{120} = \frac{|PGL(4,2)|}{6}.$$

Equating the two numbers we have

$$5N = \frac{|PGL(4,2)|}{6},$$

which implies that the number of such 3-covers is

$$N = \frac{|PGL(4,2)|}{30}.$$

Now we have already noted in the discussion before Lemma 3.4.6 that $\mathcal{C}_3^*$ is disjoint from exactly five pairs of disjoint spreads, therefore $\mathcal{C}_3^*$ and every proper 3-cover projectively equivalent to $\mathcal{C}_3^*$ is counted amongst these $\frac{|PGL(4,2)|}{30}$ proper 3-covers.

However, by the orbit-stabiliser theorem and Corollary 3.4.8, the number of proper 3-covers projectively equivalent to $\mathcal{C}_3^*$ is

$$\frac{|PGL(4,2)|}{|\langle \beta, \rho \rangle|} = \frac{|PGL(4,2)|}{30}.$$

Hence the set of proper 3-covers disjoint from exactly five pairs of disjoint spreads is precisely the orbit of $\mathcal{C}_3^*$ under the action of $PGL(4,2)$. The result is now immediate.

□

The following corollary was established implicitly by the proof of Theorem 3.4.10:

Corollary 3.4.11. Let $\mathcal{C}_3$ be a 3-cover constructed via the technique employed in Theorem 3.4.5. Then $\mathcal{C}_3$ is projectively equivalent to $\mathcal{C}_3^*$. □

Remark 3.4.12. We have shown so far that every proper 3-cover of $PG(3,2)$ which has been constructed in this thesis is projectively equivalent to $\mathcal{C}_3^*$, except for $\mathcal{C}_3^{**}$ which consists of the set of lines of the second Singer orbit. However, it can be verified using List 3.4b, that there are precisely five pairs of disjoint spreads also disjoint from $\mathcal{C}_3^{**}$, namely

$$\{\mathcal{S}_3, \mathcal{S}_{32}\}, \{\mathcal{S}_{32}, \mathcal{S}_6\}, \{\mathcal{S}_6, \mathcal{S}_{48}\}, \{\mathcal{S}_{48}, \mathcal{S}_{51}\}, \{\mathcal{S}_{51}, \mathcal{S}_3\}.$$

Theorem 3.4.10 then implies that $\mathcal{C}_3^{**}$ is projectively equivalent to $\mathcal{C}_3^*$. What is more, the full collineation group of $\mathcal{C}_3^{**}$ is the same group as that of $\mathcal{C}_3^*$. □

In summary, we have established the following results regarding the existence of proper n-covers of $PG(3,2)$, $n \geq 2$.

(i) For $n = 5$, 6 or 7, there exist no proper n-covers.

(ii) For $n = 4$, there exist no known examples of n-covers and their existence depends on the existence of a proper 3-cover which is projectively distinct from the known proper 3-covers.

(iii) For $n = 3$, all known proper n-covers are projectively equivalent and consist of the lines of a long Singer orbit.

(iv) For $n = 2$, the proper n-covers are all projectively equivalent and consist of the lines of a general linear complex minus a spread.

CHAPTER IV
QUASI-n-MULTIPLE DESIGNS AND n-COVERS OF PG(3,q)

INTRODUCTION

In 1964, Bruck and Bose in [19] demonstrated a technique for constructing finite affine planes from t-spreads of $PG(2t+1, q)$. They proved that each such plane is a finite translation plane and conversely that each finite translation plane can be constructed in this way.

This construction has since been generalised by using s-spreads of $PG(t, q)$ with $s + 1$ necessarily dividing $t + 1$. The resulting balanced incomplete block designs are examples of finite Sperner spaces (see Section 1.6).

In this chapter, we begin by further generalising this construction to produce quasi-n-multiple Sperner designs (that is quasi-n-multiples of Sperner designs) and we discuss the reducibility of these designs in terms of the n-covers from which they are constructed (when $n = 1$, the designs are equivalent to the Sperner spaces arising from 1-spreads of $PG(4t - 1, q)$).

When a single n-cover is used in the construction, the resulting design is a quasi-n-multiple affine design and, as is the case for all designs arising from the construction, it turns out to be resolvable in at least one way. In the latter sections, we examine the problem of embedding such designs into $AG(4, q)$ and we also initiate a study of possible alternative resolutions of those designs which can be so embedded.

4.1. QUASI-n-MULTIPLE SPERNER DESIGNS

Construction 4.1.1. Let Σ_{4t} be a finite $4t$-dimensional projective space and let Σ_{4t-1}

be a fixed hyperplane of Σ_{4t}. Now by Theorem 1.8.2, Σ_{4t-1} can be partitioned into $\frac{q^{4t}-1}{q^4-1}$ 3-dimensional projective spaces Σ_3^i (where i goes from 1 to $\frac{q^{4t}-1}{q^4-1}$) because 4 divides $4t$.

Let C_n^i be an n-cover of the space Σ_3^i for each i. We now define an incidence structure S in the following manner:

The points of S are the points of $\Sigma_{4t}\backslash\Sigma_{4t-1}$.

Each block of S is a plane of Σ_{4t} which meets Σ_{4t-1} in exactly the $q+1$ points of a line lying in one of the n-covers C_n^i.

The incidence relation is the incidence relation of Σ_{4t} restricted to the sets of points and planes defined above.

The incidence structure S which we have just constructed is in fact a design. We now prove this in:

Theorem 4.1.2. *S is a BIBD with parameters $v = q^{4t}$, $b = nq^{4t-2}\,\frac{(q^{4t}-1)}{(q^2-1)}$, $r = n\,\frac{(q^{4t}-1)}{(q^2-1)}$, $k = q^2$ and $\lambda = n$.*

Proof. The points of S are the points of $\Sigma_{4t}\backslash\Sigma_{4t-1}$ which is equivalent to a $4t$-dimensional affine space. Hence the number of points in S is $v = q^{4t}$.

The blocks of S are the planes of Σ_{4t} not lying in Σ_{4t-1} which meet Σ_{4t-1} in a line of one of the n-covers C_n^i. Hence $k = q^2$. Also through each line in the union of the $\frac{q^{4t}-1}{q^4-1}$ n-covers there pass $\frac{q^{4t}}{q^2} = q^{4t-2}$ distinct planes of Σ_{4t} which do not lie in Σ_{4t-1}. Finally as each n-cover has $n(q^2+1)$ distinct lines, we conclude that the number of blocks in S is

$$n.(q^2+1).q^{4t-2}.\,\frac{(q^{4t}-1)}{(q^4-1)} = nq^{4t-2}\,\frac{(q^{4t}-1)}{(q^2-1)}\,.$$

Each point of $\Sigma_{4t}\backslash\Sigma_{4t-1}$ defines a unique plane (not lying in Σ_{4t-1}) with each line in the union of the $\frac{q^{4t}-1}{q^4-1}$ n-covers. It follows that each point of S lies in

$$n(q^2+1).\,\frac{(q^{4t}-1)}{(q^4-1)} = n\,\frac{(q^{4t}-1)}{(q^2-1)}$$

distinct blocks.

Finally, given a pair of arbitrary distinct points in $\Sigma_{4t} \backslash \Sigma_{4t-1}$, the line ℓ passing through them will meet Σ_{4t-1} in a single point. This point will lie on n distinct lines of one of the n-covers and these will each define with ℓ, a plane of Σ_{4t} not lying in Σ_{4t-1}. Therefore each pair of distinct points of S, lies in n distinct blocks of S. □

Remark 4.1.3. (a) Each design (which will be denoted by $D(n)$ from this point on) is resolvable. Each resolution (or parallel) class can be taken as the set of blocks corresponding to the set of planes not lying in Σ_{4t-1}, which meet Σ_{4t-1} in a fixed line of one of the n-covers C_n^i.

(b) None of the blocks of any $D(n)$ is repeated and two distinct blocks intersecting in at least two distinct points meet in exactly q points. Moreover, this set of q points is independent of the choice of the blocks through the two points. □

Example 4.1.4. Let the points of $PG(4,2)$ (with respect to homogeneous coordinates $(x_0, x_1, x_2, x_3, x_4)$ over $GF(2)$) be labelled as shown below:

P_1	(1 0 0 0 0)	Q_1	(1 1 0 0 0)	R_1	(1 0 1 0 0)
P_2	(0 1 0 0 0)	Q_2	(0 1 1 0 0)	R_2	(0 1 0 1 0)
P_3	(0 0 1 0 0)	Q_3	(0 0 1 1 0)	R_3	(1 1 0 1 0)
P_4	(0 0 0 1 0)	Q_4	(1 1 1 0 0)	R_4	(1 0 0 1 0)
P_5	(1 1 1 1 0)	Q_5	(0 1 1 1 0)	R_5	(1 0 1 1 0)
A_1	(0 0 0 0 1)	A_7	(1 0 1 0 1)	A_{12}	(1 1 1 0 1)
A_2	(1 0 0 0 1)	A_8	(1 0 0 1 1)	A_{13}	(1 1 0 1 0)
A_3	(0 1 0 0 1)	A_9	(0 1 1 0 1)	A_{14}	(1 0 1 1 1)
A_4	(0 0 1 0 1)	A_{10}	(0 1 0 1 1)	A_{15}	(0 1 1 1 1)
A_5	(0 0 0 1 1)	A_{11}	(0 0 1 1 1)	A_{16}	(1 1 1 1 1)
A_6	(1 1 0 0 1)				

The points P_i, Q_i, R_i are the fifteen points of Σ_3, the 3-dimensional projective space with equation $x_4 = 0$ embedded in $PG(4,2)$ and their labelling corresponds to the labelling given in Section 3.1, immediately after Lemma 3.1.5. The sixteen points A_i are the points of the 4-dimensional affine space constructed from $PG(4,2)$ by removing Σ_3.

Then, following Construction 4.1.1, we arrive at the resolvable design $D(2)$

$\{1,2,3,6\}$	$\{1,2,15,16\}$	$\{1,10,12,14\}$
$\{4,7,9,12\}$	$\{3,6,11,14\}$	$\{2,9,11,13\}$
$\{5,8,10,13\}$	$\{4,7,10,13\}$	$\{3,5,7,16\}$
$\{11,14,15,16\}$	$\{5,8,9,12\}$	$\{4,6,8,15\}$
$\{1,3,4,9\}$	$\{1,6,8,10\}$	$\{1,7,13,15\}$
$\{2,6,7,12\}$	$\{2,3,5,13\}$	$\{2,4,10,16\}$
$\{5,10,11,15\}$	$\{4,12,14,15\}$	$\{3,8,11,12\}$
$\{8,13,14,16\}$	$\{7,9,11,16\}$	$\{5,6,9,14\}$
$\{1,4,5,11\}$	$\{1,9,13,14\}$	$\{1,5,12,16\}$
$\{2,7,8,14\}$	$\{2,10,11,12\}$	$\{2,8,9,15\}$
$\{3,9,10,15\}$	$\{3,4,8,16\}$	$\{3,7,10,14\}$
$\{6,12,13,16\}$	$\{5,6,7,15\}$	$\{4,6,11,13\}$
$\{1,7,8,11\}$		
$\{2,4,5,14\}$		
$\{3,12,13,15\}$		
$\{6,9,10,16\}$		

where the numbers are the subscripts of the affine points A_i and the 2-cover used is the proper 2-cover constructed in Section 2.3 (see Corollary 2.3.12). $\square$

As was mentioned in the introduction to this chapter, these designs are quasi-n-multiple Sperner designs. We shall examine the reducibility of the designs, but before doing so we need to establish the following lemma:

Lemma 4.1.5. Let $D(n)$ be a quasi-n-multiple Sperner design arising from Σ_{4t-1} via

Construction 4.1.1. If $D(n)$ contains a subdesign $D(m)$, then each n-cover involved in the construction contains an m-cover.

Proof. Consider the blocks of $D(m)$ through a fixed point P of $D(m)$.

Each such block corresponds to a plane of Σ_{4t} meeting Σ_{4t-1} in a line in the union of the n-covers. Let the set of lines defined by the blocks of $D(m)$ through the point P be $\mathcal{C}$.

Since there are $m\ \frac{(q^{4t}-1)}{(q^2-1)}$ blocks in $D(m)$ containing P, the set $\mathcal{C}$ contains $m\ \frac{(q^{4t}-1)}{(q^2-1)}$ lines.

Let R be an arbitrary point of Σ_{4t-1}. Each line in Σ_{4t} has at least three points, so we can choose a point Q on the line through P and R with $Q \neq P, R$. Q is also a point of $D(m)$. Therefore there are exactly m blocks of $D(m)$ containing P and Q. Each block gives rise to a line of $\mathcal{C}$ through R. Thus there are exactly m lines of $\mathcal{C}$ through R. Furthermore, if Σ_3^i is the unique three dimensional space (in the partition of Σ_{4t-1}) which contains R, then these m lines lie in Σ_3^i. As R is arbitrary we conclude that the subset of the lines $\mathcal{C}_m^i$ of $\mathcal{C}$ which lie in a given Σ_3^i (belonging to the partition of Σ_{4t-1}) satisfies the property that through each point of Σ_3^i there pass exactly m lines of $\mathcal{C}_m^i$. This implies that $\mathcal{C}_m^i$ is an m-cover of Σ_3^i. Finally $\mathcal{C}_m^i$ is contained in $\mathcal{C}_n^i$ because $\mathcal{C}$ lies in the union of the n-covers. Thus each n-cover contains an m-cover. $\square$

Definition 4.1.6. Let $\mathcal{C}_n$ be an n-cover of $PG(3,q)$. The *spectrum of* $\mathcal{C}_n$ (denoted $SPEC(\mathcal{C}_n)$) is the set of all integers $m \leq n$ for which there exists an m-cover embedded in $\mathcal{C}_n$. $\square$

Remark 4.1.7. In general, if $\mathcal{C}_{n_1}$ and $\mathcal{C}_{n_2}$ are disjoint n_1 and n_2-covers, then

$$SPEC(\mathcal{C}_{n_1}) \cup SPEC(\mathcal{C}_{n_2}) \cup \left\{r + s \mid r \in SPEC(\mathcal{C}_{n_1}),\ s \in SPEC(\mathcal{C}_{n_2})\right\} \subseteq SPEC\left(\mathcal{C}_{n_1} \cup \mathcal{C}_{n_2}\right).$$

Equality can hold; for example if $\mathcal{C}_{n_1}$ and $\mathcal{C}_{n_2}$ are disjoint spreads of $PG(3,q)$ then the sets on the left-handside and right-handside are both equal to $\{1,2\}$. However, it does not hold in general; for example, if we take a general linear complex $\mathbb{L}_3$ in $PG(3,3)$, we can let $\mathcal{C}_{n_1}$ be the proper 2-cover lying in $\mathbb{L}_3$ (which was constructed by Ebert in [53]) and $\mathcal{C}_{n_2}$ be the proper 2-cover which is the complement of $\mathcal{C}_{n_1}$ in $\mathbb{L}_3$. Then

$$SPEC(\mathcal{C}_{n_1}) \cup SPEC(\mathcal{C}_{n_2}) \cup \left\{r+s \mid r \in SPEC(\mathcal{C}_{n_1}),\ s \in SPEC(\mathcal{C}_{n_2})\right\} = \{2,4\}.$$

However in Chapter 2 we showed that $\mathbb{L}_3$ can also be decomposed into the disjoint union of a spread and a proper 3-cover. (See the discussion after Theorem 2.3.8.) Hence

$$SPEC(\mathcal{C}_{n_1} \cup \mathcal{C}_{n_2}) = \{1,2,3,4\}.$$

$\square$

The spectrum of an n-cover is useful for several reasons. Firstly if $\mathcal{C}_n$ is an n-cover of $PG(3,q)$ then $|SPEC(\mathcal{C}_n)|$ gives us a way of measuring how close $\mathcal{C}_n$ is to being proper. That is, the smaller the value of $|SPEC(\mathcal{C}_n)|$, the closer $\mathcal{C}_n$ is to being proper. Secondly it can be used in determining whether or not a a given design $D(n)$ is irreducible. More precisely we have:

Theorem 4.1.8. Let $D(n)$ be a design arising via Construction 4.1.1 with respect to the n-covers $\{\mathcal{C}_n^i\}$. Then $D(n)$ is irreducible if and only if

$$\bigcap_i SPEC(\mathcal{C}_n^i) = \phi \quad \text{or} \quad \{n\}.$$

Proof. $(\Rightarrow)$ Let the design be irreducible.

Assume that

$$\bigcap_i SPEC(\mathcal{C}_n^i) \neq \phi \quad \text{nor} \quad \{n\}.$$

Then there exists an integer s lying in the intersection of the spectra and satisfying $0 < s < n$. This implies that each n-cover $\mathcal{C}_n^i$ can be divided into distinct s and

$(n-s)$-covers. If we construct the corresponding designs $D(s)$ and $D(n-s)$, we have that

$$D(s) \cup D(n-s) = D(n).$$

Therefore $D(n)$ is reducible, a contradiction. Consequently $\bigcap_i SPEC(\mathcal{C}_n^i) = \phi$ or $\{n\}$.

($\Leftarrow$) Let $\bigcap_i SPEC(\mathcal{C}_n^i) = \phi$ or $\{n\}$.

Assume the design $D(n)$ is reducible. Then there exists an integer s satisfying $0 < s < n$, such that $D(n)$ is the union of two subdesigns $D(s)$ and $D(n-s)$. By Lemma 4.1.4, this implies that each n-cover contains an s and an $(n-s)$-cover. Hence

$$\bigcap_i SPEC(\mathcal{C}_n^i) \supseteq \{s, n-s\}, \quad \text{a contradiction.}$$

Hence the design is irreducible. □

Remark 4.1.9. When $t = 1$, Σ_{4t-1} is a 3-dimensional projective space. Therefore there is a single n-cover $\mathcal{C}_n$ involved in the construction which implies that $\cap SPEC(\mathcal{C}_n) \neq \phi$. This leads to:

Corollary 4.1.10. Let $D(n)$ be constructed via an n-cover $\mathcal{C}_n$ of $PG(3,q)$. Then $D(n)$ is irreducible if and only if $\mathcal{C}_n$ is proper. (That is $SPEC(\mathcal{C}_n) = \{n\}$.) □

The problem of decomposing block designs has been investigated by several authors (for example see [39], [88] and the references listed therein), but principally in the cases where $\lambda = 2, 3$. The general consensus is that decomposing block designs for values of $\lambda > 2$ is computationally inefficient (that is NP-complete). It therefore seems reasonable to expect that the problem of decomposing n-covers will also be computationally inefficient. Notwithstanding this, to decompose a quasi-n-multiple Sperner design $D(n)$ with $\lambda = n > 3$, it is probably more efficient to decompose the n-covers from which $D(n)$ is constructed and then apply Theorem 4.1.8. However for $\lambda = n = 2$ the problem can be solved efficiently via analysis of certain graphs defined by the design. We conclude this section with a discussion of these graphs and their connections with the decomposition of the designs on which they are defined.

Definition 4.1.11. ([88]) Let D be a BIBD with b blocks. The *associated multigraph* of D has b vertices (one for each block of D); two vertices of the graph are connected by exactly n edges where n is the number of distinct pairs of points lying in the intersection of the two blocks corresponding to the two vertices. □

Definition 4.1.12. (See for example [17].) Let D be a BIBD with v points. The *adjacency multigraph* $A(P)$ of a point P of D has $v-1$ vertices (one for each point of D other than P); two vertices of $A(P)$ are connected by exactly n edges where n is the number of blocks in D containing P which also contain the two points corresponding to the vertices. □

Theorem 4.1.13. ([88]) Let D be a BIBD with $\lambda = 2$. Then D is irreducible if and only if its associated multigraph contains an odd cycle. □

Theorem 4.1.14. (See for example [17].) Let D be a BIBD with $\lambda = 2$. Then D is irreducible if its adjacency multigraph $A(P)$ (for some point P) contains an odd cycle. □

Remark 4.1.15. The converse of Theorem 4.1.14 does not hold in general. For a discussion on why this does not hold plus a summary of the known techniques for testing the reducibility of quasi-multiple designs, see [105], Section 4.6. □

We mentioned in Chapter II after Theorem 2.1.9 that we would give a necessary and sufficient condition for a 2-cover of $PG(3,q)$ to be proper. Having set up the required machinery to do this, we can now state:

Theorem 4.1.16. Let $\mathcal{C}_2$ be a 2-cover of $PG(3,q)$ and $D(2)$ be the quasi-2-multiple affine design arising from $\mathcal{C}_2$. Then $\mathcal{C}_2$ is proper if and only if the associated multigraph of $D(2)$ contains an odd cycle.

Proof. By Corollary 4.1.10, $\mathcal{C}_2$ is proper if and only if $D(2)$ is irreducible and by Theorem 4.1.13, $D(2)$ is irreducible if and only if its associated multigraph contains an odd cycle. Hence the result follows. $\square$

Remark 4.1.17. By a similar line of reasoning we also have that $\mathcal{C}_2$ is proper if the adjacency multigraph $D(2)$ (for some point P) contains an odd cycle. $\square$

Example 4.1.18. Let $D(2)$ be the quasi-2-multiple affine design which we constructed in Example 4.1.4 from the unique proper 2-cover $\mathcal{C}_2$ of $PG(3,2)$. Suppose that we did not know that $\mathcal{C}_2$ was proper. It is a simple exercise to show that $D(2)$ contains the blocks

$$\begin{gathered}\{1,2,3,6\}\\ \{1,2,15,16\}\\ \{1,7,13,15\}\\ \{1,9,13,14\}\\ \{1,3,4,9\}.\end{gathered}$$

These give rise to an odd cycle in both the associated multigraph and the adjacency multigraph $A(1)$ of $D(2)$. Hence by Theorem 4.1.16 and Remark 4.1.17 respectively, $\mathcal{C}_2$ is a proper 2-cover. $\square$

For the case of 2-covers in $PG(3,2)$ it is arguably simpler to determine directly whether or not a 2-cover is proper. However in $PG(3,q)$ with $q > 2$, an analysis of the multigraphs of the design may be more efficient than a direct examination of the 2-cover.

4.2. CHARACTERISING FINITE AFFINE SPACES AND EMBEDDINGS OF QUASI-MULTIPLE AFFINE DESIGNS

In the previous section we constructed examples of quasi-n-multiple Sperner designs from n-covers of $PG(3,q)$. In the particular case where a single n-cover was used, we obtained a quasi-n-multiple affine design. The point-set of this design was the point-set of $AG(4, q)$ and the blockset was a subset of the set of planes of $AG(4, q)$. We shall refer to such an embedding of a design in $AG(4, q)$ as a natural embedding.

In this section we consider the problem of reversing the construction. That is, we find conditions that guarantee that a quasi-n-multiple affine design ($n \geq 2$) can be naturally embedded in $AG(4, q)$, $q \geq 3$. This in effect, does reverse the construction because a quasi-n-multiple affine design (without repeated blocks) naturally embedded in $AG(4, q)$ does give rise to at least one n-cover of the 3-dimensional projective which completes $AG(4, q)$ to $PG(4, q)$.

To do this we associate a uniform, regular linear space with the design and then determine when the space is affine. Thus we begin with a discussion of uniform, regular linear spaces. (For the basic definitions and terminology see Section 1.2.)

Definition 4.2.1. Let $\mathcal{L}$ be a finite linear space. Then the *triangle bound* ($\triangle$-bound) of $\mathcal{L}$ is

$$\max\{|\mathcal{L}_\triangle| \;\big|\; \mathcal{L}_\triangle \text{ is a } \triangle\text{-subspace of } \mathcal{L}\}.$$

(see Definition 1.2.8). □

Theorem 4.2.2. Let $\mathcal{L}$ be a finite uniform linear space of order $n \geq 3$. If $\mathcal{L}$ has $\triangle$-bound n^2 then each $\triangle$-subspace of $\mathcal{L}$ is either a projective plane of order $n-1$ or an affine plane of order n.

Proof. Let $\{P_1, P_2, P_3\}$ be a triangle $\triangle$ in $\mathcal{L}$. Since we have $n \geq 3$, $\triangle$ is not a subspace of $\mathcal{L}$. Thus consider the 1-step extension of $\triangle$. It contains the line $\langle P_2, P_3 \rangle = \{P_2, P_3, \ldots, P_i, \ldots, P_{n+1}\}$ and the point P_1.

Therefore the 1-step extension $\triangle$, is not a subspace of $\mathcal{L}$ because the lines $\langle P_1, P_i \rangle$, $i = 4, \ldots, n+1$ do not lie in $\triangle_1$ but do lie in the subspace generated by $\triangle$. Now the lines $\langle P_1, P_i \rangle$, $i = 2, \ldots, n+1$ do lie in the 2-step extension of $\triangle$, so by counting the number of points lying on these lines we find that $\triangle_2$ has at least $n(n-1)+1 = (n-1)^2+(n-1)+1$

points.

If $\triangle_2$ has exactly this number of points then it is a projective plane of order $n-1$ by Lemma 1.2.14 because each of its lines has exactly n points.

If $\triangle_2$ has more than $n(n-1)+1$ points then there is a point P_{n+2} of $\triangle_2$ which is not collinear with P_1 in $\triangle_2$. Thus $\triangle_2$ is not a subspace of $\mathcal{L}$. The 3-step extension of $\triangle$ contains the line $\langle P_1, P_{n+2}\rangle$ and so has at least $n-1$ more points than $\triangle_2$ because P_1 is the only point lying on the line which also lies in $\triangle_2$. Hence $\triangle_3$ has at least $(n-1)+n(n-1)+1 = n^2$ points. It follows that $\triangle_3$ has exactly n^2 points by the triangle bound of $\mathcal{L}$ and is the subspace of $\mathcal{L}$ generated by $\triangle$. By Lemma 1.2.13, $\triangle_3$ is an affine plane of order n because each of its lines contains exactly n points. □

Corollary 4.2.3. Let $\mathcal{L}$ be a finite, uniform linear space of order $n \geq 3$ and with $\triangle$-bound n^2. Then $\mathcal{L}$ is a planar space with respect to the set of its triangular subspaces.

Proof. To establish that $\mathcal{L}$ is a planar space we need to show that a triangle $\triangle$ cannot lie in more than one triangular subspace.

Thus let $\triangle$ be a triangle lying in the triangular subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$. Then $\mathcal{L}_1 \cap \mathcal{L}_2$ is a subspace of $\mathcal{L}$ containing $\triangle$ and so is either a projective plane of order $n-1$ or an affine plane of order n.

By a well-known theorem of Bruck (see [73], p.81), a projective plane of order m may only be embedded in a projective plane of order n if $n = m^2$ or $m^2 + m \leq n$. Thus a projective plane of order $n-1$ cannot be embedded in a projective plane of order n or therefore in an affine plane of order n. It follows from this that, if $\mathcal{L}_1 \cap \mathcal{L}_2$ is a projective plane of order $n-1$, then so are $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence, we have $\mathcal{L}_1 = \mathcal{L}_2$. Otherwise, $\mathcal{L}_1 \cap \mathcal{L}_2$ is an affine plane of order n^2, in which case it again follows that $\mathcal{L}_1 = \mathcal{L}_2$. Hence $\mathcal{L}$ is planar. □

Remark 4.2.4. The classical planar spaces are the affine and projective spaces with respect to their sets of triangular subspaces. These are special in the sense that their planes are all isomorphic. Such planar spaces known as π-spaces have been studied by Delandtsheer in [42]. Only a few examples of π-spaces are known. They include the affine and projective spaces already mentioned, Hall triple systems (see Sections 1.6 and 4.3), Steiner systems $S(3,n,v)$ (in which the lines are the point-pairs and the blocks are the planes) and a class of spaces in which each plane is a degenerate projective plane. Relevant to the study of π-spaces is the following result by Teirlinck:

Theorem 4.2.5. ([117]) Let $\mathcal{L}$ be a finite planar linear space in which each plane is either affine or projective. Then every plane is affine or every plane is projective. □

Corollary 4.2.6. Let $\mathcal{L}$ be a finite uniform linear space of order $n \geq 3$ and with triangle bound n^2. Then, with respect to its set of triangular subspaces, $\mathcal{L}$ is a planar space in which every plane is an affine plane of order n.

Proof. By Corollary 4.2.3, $\mathcal{L}$ is a planar space with respect to its triangular subspaces. Theorem 4.2.2 implies that each triangular space is either an affine plane of order n or a projective plane of order $n-1$. Since $\mathcal{L}$ has triangle bound n^2, it has at least one triangular subspace which is an affine plane of order n. Hence by Theorem 4.2.5 every triangular subspace of $\mathcal{L}$ is an affine plane of order n. □

Remark 4.2.7. From Corollary 4.2.6, it is immediate that we can define a parallelism relation locally on the triangular subspaces of a finite uniform linear space $\mathcal{L}$ of order n and with Δ-bound n^2. The results of work done by Buekenhout show that if n is at least 4, then the local parallelisms extend to a parallelism of the whole space in which case $\mathcal{L}$ is a finite affine space. When n is 3, this is no longer true in general. Hall produced the first counter-example of this in the form of a Steiner triple system S in which each Δ-subspace is an affine plane of order 3 while S is not an affine space. (We construct two infinite classes of Hall triple systems, as they have become known, in the next section.) However, when $|\mathcal{L}|$ is 3^4 we can give a simple criterion for determining whether or not $\mathcal{L}$ is an affine space because there are exactly two non-isomorphic Hall triple systems on 3^4 points. $AG(4,3)$ is the first and is generated by five points, while the other, denoted by L_3, is generated by four non-coplanar points. Hence summarising these results we have:

Theorem 4.2.8. Let $\mathcal{L}$ be a finite uniform linear space of order $n \geq 3$ and with Δ-bound n^2. If $n \geq 4$, then $\mathcal{L}$ is a finite affine space (see [38]). If $n = 3$, $|\mathcal{L}| = 3^4$ and $\mathcal{L}$ is not generated by any set of four points then $\mathcal{L}$ is the finite affine space $AG(4,3)$. (See [65].) □

Remark 4.2.9. (1) In [118], Teirlinck establishes Buekenhout's result via a different argument, but only for $n \geq 5$.
(2) A set of arguments similar to those used in this section, can be used to show that a finite, uniform linear space of order n and with Δ-bound $n^2 - n + 1$ is a finite projective space.
(3) The converse of Theorem 4.2.8 also holds trivially. □

In the sequel we use this characterisation of affine spaces, in particular of $AG(4,q)$, $q \geq 3$ to discuss the natural embedding of quasi-multiple affine designs in $AG(4,q)$.

It is well-known that a finite affine plane can be embedded naturally in $AG(4,q)$ if and only if it can be constructed from a spread of the hyperplane which completes $AG(4,q)$ to $PG(4,q)$, in which case the plane is a finite translation plane of degree 2 over its kernel. Hence the question of embedding a quasi-1-multiple affine design in $AG(4,q)$ is completely settled. (See [19] and [91] for details of this.)

When n is at least 2 and $q = 2$, it is not obvious whether a given quasi-n-multiple affine design is embeddable naturally in $AG(4,2)$ because the planes of $AG(4,2)$ cannot be easily characterised as triangular subspaces. More precisely, each plane of $AG(4,2)$ has four points while each triangular subspace has exactly three points (the three points of the triangle). Thus it is simpler in this case to compare a quasi-n-multiple affine design on 16 points with the designs constructed from existing n-covers of $PG(3,2)$. Note however, that the complete determination of those designs still depends on the classification of the proper 3 and 4-covers of $PG(3,2)$.

Therefore, for the remainder of this section, we assume that $n \geq 2$ and $q \geq 3$.

Quasi-multiple affine designs have been studied by a number of authors; in [80], Jungnickel constructed quasi-2-multiple affine designs from existing affine designs of any order by permuting the points (and therefore also the blocks) of the designs and then adjoining the two block sets. These designs are all reducible and often contain repeated blocks. Quasi-2-multiple affine designs on 9 points have been studied by Morgan in [94] and Mathon and Rosa in [92]. Of special interest is entry (30) on p.314 of [92], which has no repeated blocks and so as a consequence, satisfies the property that the two blocks intersect in 0, 1 or 2 points. Other papers such as [16], [17] give general techniques for constructing irreducible quasi-2-multiple designs from existing designs with $\lambda = 1$. (Also see [105], p.103, Corollary 2.2.1.)

The most unusual quasi-multiple affine designs to date are those with parameters (36,84,14,62) and (36,126,21,6,3) (see [67] p.296, [93] p.283 and [125]). These are remarkable because no (36,42,7,6,1)-design exists (a design corresponding to these parameters would be an affine plane of order 6 which is known not to exist. See [1], Theorem 6.4, [45], p.156 and [67], pp.175-176.)

If we take a 2-cover of $PG(3,q)$ which consists of two disjoint spreads which are projectively equivalent, then the quasi-2-multiple affine designs arising from construction 4.1.1 are particular examples of those constructed by Jungnickel. With the exception of these, the quasi-2-multiple affine designs arising from Construction 4.1.1 are not isomorphic to any of the designs mentioned above.

The quasi-n-multiple affine designs which we constructed in Section 4.1 from n-covers of $PG(3,q)$ satisfy the four conditions:

A(i) The number of points in the design is q^4, for some prime power q.

A(ii) Two blocks intersect in 0, 1 or q points.

A(iii) Given a pair of distinct points, the n blocks through them meet pairwise in a fixed set of q points.

A(iv) No block is repeated.

The previous examples indicate that to reverse Construction 4.1.1 (with $t = 1$) we cannot in general relax any of the four conditions. Hence the quasi-n-multiple affine designs which we consider from this point on satisfy axioms A(i) to A(iv) (and $n \geq 2$, $q \geq 3$ as mentioned before).

Construction 4.2.10. Let P and Q be two distinct points of the design $D(n)$. By axioms A(ii) and A(iii), the n distinct blocks through P and Q intersect in exactly q points. We define this set of points to be the line (uniquely) defined by P and Q. It follows that the points of the design together with the lines constructed above constitute a finite uniform linear space $\mathcal{L}_{D(n)}$ of order q. $\mathcal{L}_{D(n)}$ is called the *associated linear space* of $D(n)$.

Theorem 4.2.11. Let $D(n)$ be a quasi-n-multiple affine design satisfying axioms A(i) to A(iv), $n \geq 2$ and $q \geq 3$. Then $D(n)$ is naturally embeddable in $AG(4,q)$ if and only if $\mathcal{L}_{D(n)}$ has Δ-bound q^2 and is not generated by 4 points when $q = 3$.

Proof. ($\Rightarrow$) If $D(n)$ is naturally embeddable in $AG(4,q)$, then $\mathcal{L}_{D(n)}$ is identical to $AG(4,q)$ by construction. Hence $\mathcal{L}_{D(n)}$ has Δ-bound q^2 by Remark 4.2.9 (3). Also, if $q = 3$, then $\mathcal{L}_{D(n)}$ is not generated by 4 points.

($\Leftarrow$) If $\mathcal{L}_{D(n)}$ has Δ-bound q^2 and is not generated by 4 points when $q = 3$, then by Theorem 4.2.8, $\mathcal{L}_{D(n)}$ is isomorphic to $AG(m,q)$ for some m. Since $|\mathcal{L}_{D(n)}| = q^4$, we have that $m = 4$. Furthermore, by the construction of $\mathcal{L}_{D(n)}$, each block of $D(n)$ becomes a plane of $\mathcal{L}_{D(n)}$. Combining these facts, we see that $D(n)$ can be naturally embedded in $AG(4, q)$. □

Theorem 4.2.12. Let $D(n)$ be a quasi-n-multiple affine design for which $\mathcal{L}_{D(n)}$ has $\triangle$-bound q^2 and is not generated by 4 points. Then $D(n)$ induces at least one n-cover of $PG(3, q)$.

Proof. By the conditions on $\mathcal{L}_{D(n)}$, $D(n)$ is naturally embeddable in $AG(4, q)$. Let P be a point of $D(n)$ and consider the $n(q^2 + 1)$ blocks of $D(n)$ containing P. These blocks correspond to $n(q^2 + 1)$ planes of $AG(4, q)$ which contain the point P. If we complete $AG(4, q)$ to $PG(4, q)$, then the $n(q^2 + 1)$ planes define $n(q^2 + 1)$ lines of the 3-dimensional space $PG(3, q) = PG(4, q) \backslash AG(4, q)$. Let these lines form the set C_n.

Let R be an arbitrary point of $PG(3, q)$ and let $Q \neq P, R$ be a point on the line $\langle P, R \rangle$. P and Q both represent points of $D(n)$. Hence there are exactly n distinct blocks of $D(n)$ containing them. This implies that there are exactly n distinct planes containing P and Q. Therefore there are exactly n lines of C_n through R. Since R is arbitrary, C_n is an n-cover of $PG(3, q)$. □

Remark 4.2.13. By the method used in Theorem 4.2.12 we have not discounted the fact that the design may give rise to more than one n-cover. However, we conjecture that the n-cover is in fact unique. □

4.3. HALL TRIPLE SYSTEMS AND THE BURNSIDE PROBLEM

Definition 4.3.1. ([64], p.320) Let G be a group generated by r elements such that $g^n = 1_G$ for each element $g \in G$ where n is a fixed positive integer. Then $G = B(n, r)$ is called the Burnside group of order n on r generators. □

Given the above definition of a Burnside group, the Burnside problem is to establish which of the Burnside groups are finite. Despite the simplicity of the statement of the problem, its solution for given values of m and n is in most cases anything but trivial. In [64], Hall demonstrates that $B(2,r)$, $B(3,r)$, $B(4,r)$ and $B(6,r)$ are finite for all r. However, for $n = 5$ it is not even known if $B(5,2)$ is finite (see [62]). The most general result to date (see [62]) is that $B(n,r)$ is infinite for all odd $n \geq 665$ and $r \geq 2$. (NB: Gupta in [62] writes $B(r,n)$ in contrast to Hall who writes $B(n,r)$ in [64].)

Our interest here lies in the Burnside groups $B(3,r)$ with $r \geq 3$. In [68], Hall uses the group $B(3,3)$ to construct the unique non-abelian Hall triple system L_3 on 3^4 points. That this construction technique can be extended to produce non-abelian Hall triple systems of dimension r on $3^{\binom{r}{3}+r}$ points, does not seem to have been mentioned in the literature. Thus the infinite class of Hall triple systems, that we now construct, appears to be new. (The construction follows that of Hall, mutatis mutandis.)

However we include most of the details which are omitted in [68] and we construct the group G (mentioned therein) in a simpler way.

We begin by reviewing several definitions and results relating to groups. (For details, see [64].)

1: Let G be a group and let g_1, g_2 be two arbitrary elements of G. Then the commutator of g_1 and g_2 is defined to be

$$(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2.$$

Furthermore given a third element g_3 of G, the (left) commutator of g_1, g_2 and g_3 is defined to be

$$(g_1, g_2, g_3) = ((g_1, g_2), g_3).$$

The definition can be extended recursively to define a commutator on n elements of G; such a commutator is said to have weight n.

2: Let G be a finite group of order 3^r for some integer $r \geq 1$. Then the commutator subgroup G' of G (that is the subgroup of G generated by the commutators of weight two of G) is abelian.

Also the commutators of weight two satisfy the identities:

(i) $(g_1, g_2)^{-1} = (g_1^{-1}, g_2)$

(ii) $(g_1, g_2)^{-1} = (g_1, g_2^{-1})$

and the commutators of weight three lie in the centre $Z(G)$ of G.

3: Consider now the group $B(3,r)$ with $r \geq 3$. $B(3,r)$, as stated at the beginning of this section, is finite for all $r \geq 1$ and each element of g of $B(3,r)$ satisfies $g^3 = 1$. Therefore the order of $B(3,r)$ is a power of 3, implying that the results in 2 above apply to $B(3,r)$.

Let $\{x_1, x_2, \ldots, x_r\}$ be a set of generators of $B(3,r)$. Then each element of $B(3,r)$ can be uniquely expressed in the form

$$\prod_{i=1}^{r} x_i^{\alpha(i)} \prod_{i<j} (x_i, x_j)^{\beta(i,j)} \prod_{i<j<k} (x_i, x_j, x_k)^{\gamma(i,j,k)}$$

(up to the order of the commutators of weight two and the order of the commutators of weight three) where $\alpha(i)$, $\beta(i,j)$ and $\gamma(i,j,k)$ are congruent to are 0, 1 or 2 modulo 3. From this representation we see that the cardinality of $B(3,r)$ is

$$\begin{aligned} |B(3,r)| &= 3^{\binom{r}{1}} \cdot 3^{\binom{r}{2}} \cdot 3^{\binom{r}{3}} \\ &= 3^{r+\binom{r}{2}+\binom{r}{3}}. \end{aligned}$$

Construction 4.3.2. Consider the Burnside group $B(3,r)$, $r \geq 3$, generated by the elements $\{x_1, \ldots, x_r\}$. For each element g of $B(3,r)$ where

$$g = \prod_{i=1}^{r} x_i^{\alpha(i)} \prod_{i<j} (x_i, x_j)^{\beta(i,j)} \prod_{i<j<k} (x_i, x_j, x_k)^{\gamma(i,j,k)}$$

we define g^* to be the element of $B(3,r)$ with form

$$\prod_{i=1}^{r} (x_i^{-1})^{\alpha(i)} \prod_{i<j} (x_i^{-1}, x_j^{-1})^{\beta(i,j)} \prod_{i<j<k} (x_i^{-1}, x_j^{-1}, x_k^{-1})^{\gamma(i,j,k)}.$$

Clearly g^* is well-defined for each g, $(g^*)^* = g$ and $(g_1g_2)^* = g_1^*g_2^*$.

The aim of this construction is to embed $B(3,r)$ in a larger group in such a way that $B(3,r)$ becomes a normal subgroup of this group. To achieve this, we associate with each element g of $B(3,r)$, a new element denoted by $t(g)$ (if $g = 1_{B(3,r)}$, we simply write t for $t(1_{B(3,r)})$).

Letting the set comprising the elements g and $t(g)$ for all $g \in B(3,r)$ be G, we can equip G with a binary operation $\circ$ which is defined in the following manner (in each instance g_1 and g_2 are arbitrary elements of $B(3,r)$ and juxtaposition represents the group operation of $B(3,r)$):

$$\begin{aligned} g_1 \circ g_2 &= g_1g_2 \\ t(g_1) \circ t(g_2) &= g_1g_2^* \\ t(g_1) \circ g_2 &= t(g_1g_2^*) \\ g_1 \circ t(g_2) &= t(g_1g_2). \end{aligned}$$

It is a routine exercise to show that the operation $\circ$ is associative (this can be done by considering the 2^3 distinct cases). Having established this, we can now easily prove:

Theorem 4.3.3. $(G,\circ)$ is a group and $B(3,r)$ is embedded in G as a normal subgroup.

Proof. Trivially, G is non-empty, and the operation $\circ$ is associative from the above.

The identity element 1 of $B(3,r)$ extends to an identity G because for all $g \in B(3,r)$

$$1 \circ g = 1g = g = g1 = g \circ 1$$

and

$$1 \circ t(g) = t(1g) = t(g) = t(g1^*) = t(g) \circ 1.$$

In addition, each element of G has an inverse. For each $g \in B(3,r)$, the inverse of g in G is its inverse g^{-1} in $B(3,r)$ while the inverse of $t(g)$ is the element $t((g^{-1})^*)$

because

$$t(g) \circ t((g^{-1})^*) = g\left((g^{-1})^*\right)^* = gg^{-1} = 1$$

and

$$t((g^{-1})^*) \circ t(g) = (g^{-1})^* g^* = (g^{-1}g)^* = 1^* = 1.$$

Thus $(G, \circ)$ is a group.

By construction $B(3,r)$ is embedded in G as a subgroup because the two group operations $\circ$ and juxtaposition are equivalent on the elements of $B(3,r)$ in G. Finally, G has twice as many elements as $B(3,r)$. Therefore $[G : B(3,r)] = 2$ and so $B(3,r)$ is a normal subgroup of G. $\square$

Theorem 4.3.4. The centraliser H of t is the subgroup of G generated by t and the commutators (x_i, x_j) with $i < j$.

Proof. Trivially, we have $t^{-1} \circ t \circ t = t$ and so t lies in H.

Before proceeding we note two things: First, since H is a subgroup of G, an element $t(g)$ lies in H if and only if g lies in H because t lies in H and

$$t(g) \circ t = g.$$

Hence, to find the rest of H, it is sufficient to find the elements of $B(3,r)$ which lie in H. Second, since $t^{-1} = t((1^{-1})^*) = t(1) = t$, t is its own inverse (or equivalently t has order two).

Now

$$\begin{aligned}
(x_i, x_j)^* &= x_i x_j x_i^{-1} x_j^{-1} \\
&= (x_i^{-1}, x_j^{-1}) \\
&= (x_i, x_j^{-1})^{-1} && \text{by 2(i)} \\
&= (x_i, x_j) && \text{by 2(ii).}
\end{aligned}$$

Similarly

$$\left((x_i, x_j)^{-1}\right)^* = (x_i, x_j)^{-1}$$

Also

$$\begin{aligned}
(x_i, x_j, x_k)^* &= \left[(x_i, x_j)^{-1} x_k^{-1} (x_i, x_j) x_k\right]^* \\
&= \left((x_i, x_j)^{-1}\right)^* \left(x_k^{-1}\right)^* (x_i, x_j)^* x_k^* \\
&= (x_i, x_j)^{-1} x_k (x_i, x_j) x_k^{-1} \quad \text{by the two previous parts} \\
&= ((x_i, x_j), x_k^{-1}) \\
&= ((x_i, x_j), x_k)^{-1} \quad \text{by 2(i)} \\
&= (x_i, x_j, x_k)^{-1}.
\end{aligned}$$

Let g be an arbitrary element of $B(3, r)$. Then g lies in H if and only if $t \circ g \circ t = g$. That is, if and only if $(t \circ t(g) = g)$ $g^* = g$. Using the three results above, we have that $g^* = g$ if and only if

$$\begin{aligned}
&\prod_{i=1}^{r} x^{-\alpha(i)} \prod_{i<j} (x_i, x_j)^{\beta(i,j)} \prod_{i<j<k} (x_i, x_j, x_k)^{-\gamma(i,j,k)} \\
&= \prod_{i=1}^{r} x^{\alpha(i)} \prod_{i<j} (x_i, x_j)^{\beta(i,j)} \prod_{i<j<k} (x_i, x_j, x_k)^{\gamma(i,j,k)}.
\end{aligned}$$

The representation of an element of $B(3, r)$ in this form is unique. Hence, it is immediate that $\alpha(i) = 0$ for all i, $\gamma(i, j, k) = 0$ for all i, j, k and $\beta(i, j)$ can be chosen arbitrarily for all i, j.

Thus g lies in H if and only if it is the product of commutators of weight two of the form (x_i, x_j) with $i < j$. Therefore

$$H = \langle t, \{x_i, x_j)\}_{i<j} \rangle.$$

□

Consider the subgroups $\langle t \rangle$ and $\langle \{(x_i, x_j)\}_{i<j} \rangle$ of H. Using the fact that $t \circ (x_i, x_j) \circ t = (x_i, x_j)^* = (x_i, x_j)$ (the latter part of which was proven in Theorem 4.3.4), the commutators of weight two of the form (x_i, x_j) with $i < j$ commute with each other (see 2 on page 141) and that $t \circ t \circ t = t$, it can be proven that both subgroups are normal in H. Furthermore $\langle t \rangle \cap \langle \{(x_i, x_j)\}_{i<j} \rangle = \{1\}$ and $H = \langle t \rangle \langle \{(x_i, x_j)\}_{i<j} \rangle$. Hence

$$H = \langle t \rangle \times \langle \{(x_i, x_j)\}_{i<j} \rangle$$

(see [64], p.33 or [110], p.75).

By the uniqueness of the representation of the elements in $B(3,r)$ we have that the cardinality of $\langle\{(x_i,x_j)\}_{i<j}\rangle$ is $3^{\binom{r}{2}}$. Therefore H has $2.3^{\binom{r}{2}}$ elements, whence $[G:H]=3^{\binom{r}{3}+r}$.

Remark 4.3.5. t is the unique element of order 2 in H because each element g of $B(3,r)$ in H still satisfies $g^3=1$ and for each element $t(g)$, $g\neq 1$ in H we have

$$\begin{aligned} t(g)\circ t(g) &= g\circ t\circ g\circ t \\ &= g\circ t\circ t\circ g \\ &= g\circ(t\circ t)\circ g \\ &= g\circ g \\ &= g^2 \\ &\neq 1. \end{aligned}$$

Hence the number of conjugates of t in G is $[G:H]=3^{\binom{r}{3}+r}$. Each conjugate $a^{-1}\circ t\circ a$ (where $a\in G$) fixes the coset $a^{-1}\circ H$ and permutes the remaining cosets in pairs. Thus labelling H as H_1 and the other cosets arbitrarily as $H_2,\ldots,H_{3^{\binom{r}{3}+r}}$, we appropriately label the conjugates of t as t_i where t_i fixes H_i. □

Let t_i and t_j be two arbitrarily conjugates of t. Since they both have order two, they do not lie in $B(3,r)$. Hence t_i and t_j both lie in $t\circ B(3,r)$, the only proper coset of $B(3,r)$ in G, which implies that $t_i\circ B(3,r)=t_j\circ B(3,r)$ or equivalently that $(t_i\circ t_j)\circ B(3,r)=B(3,r)$. From this, we conclude that

$$(t_i\circ t_j)^3=1.$$

Rearranging this gives

$$t_i\circ t_j\circ t_i=t_j\circ t_i\circ t_j.$$

These two expressions again represent an element of order two, that is a conjugate t_k of t. NB: $t_k\neq t_i\neq t_j\neq t_k$. Furthermore any two of these conjugates uniquely determine the third.

We can now construct a Steiner triple system S as follows:

The points of S are the $3^{\binom{r}{3}+r}$ cosets H_i of H.

The lines of S are the triples $\{H_i, H_j, H_k\}$ for which $t_k = t_i \circ t_j \circ t_i = t_j \circ t_i \circ t_j$, $i < j < k$.

The incidence relation in S is set inclusion.

We now show that S is a Hall triple system.

Let a be an arbitrary element of G and $\{H_i, H_j, H_k\}$ be a line of S. By direct substitution it can be shown that

$$\left\{a^{-1} \circ H_i \circ a,\ a^{-1} \circ H_j \circ a,\ a^{-1} \circ H_k \circ a\right\}$$

is again a line of S. Hence G acts, via conjugation, as a collineation group of S. In particular, S admits the collineation t_i which fixes H_i and interchanges the other two points H_j and H_k of any line of S containing H_i. It follows from Definition 1.6.3 that S is a Hall triple system.

Furthermore, since

$$(x_i^{-1} \circ t \circ x_i)^2 = 1$$

and

$$\begin{aligned} x_i^{-1} \circ t \circ x_i \circ t &= t \circ (x_i^{-1}) \circ t(x_i) \\ &= x_i^{-1} x_i^{*} \\ &= x_i^{-2} \\ &= x_i \end{aligned}$$

for each generator x_i of $B(3, r)$, we can write

$$G = \langle t, x_1, \ldots, x_r \rangle = \langle t, x_1^{-1} \circ t \circ x_1, \ldots, x_r^{-1} \circ t \circ x_r \rangle$$

so G can be generated by $r + 1$ conjugates of t. Therefore the $3^{\binom{r}{3}+r}$ points of S can be generated by $r + 1$ points of S.

If S were isomorphic to an affine space then S would have 3^r points because S has dimension r. However S has $3^{\binom{r}{3}+r} > 3^r$ points. Therefore S is never an affine space.

When $r = 3$, S is isomorphic to L_3, the unique non-abelian Hall triple system on 3^4 points. In accordance with this, we define L_r, $r \geq 3$ to be the Hall triple system of dimension r and size $\binom{r}{3} + r$ constructed above.

Apart from this class of Hall triple systems L_r, there also exists a second class of Hall triple systems which arise from the Burnside groups $B(3, r)$. These systems also turn out to be resolvable. They arise from the same general construction that André used in his description of finite translation planes (see [91], p.2); this construction is described below, after Definition 4.3.6.

Definition 4.3.6. ([91], p.2) Let G be a finite group. A set of subgroups $\{G_i\}$ is said to form a *partition* of G if

(i) $G_i \cap G_j = \{1\}, \quad i \neq j$

and (ii) $G = \bigcup_i G_i$.

□

Construction 4.3.7. ([91], p.2) Let G be a finite group with a partition $\{G_i\}$ and consider the following incidence structure S:

(i) The points of S are the elements of G.

(ii) The lines of S are the left cosets of the subgroups G_i.

(iii) The incidence relation is set inclusion.

It is immediate that G acts via left multiplication as a collineation group of S. In particular, G acts transitively on the point-set of S. This, together with the fact that each element of $G \backslash \{1\}$ lies on a unique line through 1, implies that S is a finite linear space. Finally S is resolvable because we can define a resolution on S, each resolution class of which consists of a subgroup G_i and its cosets in G. □

To apply this technique to the Burnside groups $B(3,r)$, $r \geq 2$, we take the cyclic subgroups of $B(3,r)$ to be the elements of the partition. (These do form a partition because each cyclic subgroup of $B(3,r)$ has order 3 and from Lagrange's theorem, we deduce that two subgroups of order 3 either intersect trivially or they coincide.) The resulting linear space is a resolvable Steiner triple system S_r. Thus each S_r is also a finite Sperner space (see Definition 1.6.7). We now show that each S_r is also a Hall triple system.

Lemma 4.3.8. Let a, b and c be three points of the finite Sperner space S_r. Then a, b and c are collinear in S_r if and only if

$$c = ba^2b.$$

Proof. $\{a,b,c\}$ is a line of S_r

$\Leftrightarrow a\{1, a^2b, a^2c\}$ is a coset in $B(3,r)$

$\Leftrightarrow \{1, a^2b, a^2c\}$ is a (cyclic) subgroup of $B(3,r)$

$$\begin{aligned} \Leftrightarrow \quad a^2c &= (a^2b)^2 \\ \Leftrightarrow \quad a^2c &= a^2ba^2b \\ \Leftrightarrow \quad c &= ba^2b. \end{aligned}$$

□

Lemma 4.3.9. Let a and b be two distinct points of the finite Sperner space S_r. Then the three points a, bab and b^2ab^2 are collinear in S_r.

Proof. Consider the element ba. Since ba lies in $B(3,r)$, it satisfies the identity

$$
\begin{aligned}
& (ba)^3 = 1. \\
\Rightarrow\ & (ba)(ba)(ba) = 1 \\
\Rightarrow\ & (bab)(aba) = 1. \\
\Rightarrow\ & (bab) = (aba)^2 \\
\Rightarrow\ & (ab)(bab)(b) = (ab)(aba)^2(b) \\
\Rightarrow\ & (ab^2)^2 = a(ba)^2(ab)^2. \\
\Rightarrow\ & \{1, a(ba)^2(ab)^2, (ab^2)\} \text{ is a line of } S_r \\
\Rightarrow\ & a^2\{1, a(ba)^2(ab)^2, (ab^2)\} \text{ is a line of } S_r \\
& \text{i.e. } \{a^2, b^2, (ba)^2(ab)^2\} \text{ is a line of } S_r. \\
\Rightarrow\ & (ba)^2(ab)^2 = b^2ab^2 \text{ by Lemma 4.3.8} \\
\Rightarrow\ & (baba)(abab) = b^2ab^2 \\
\Rightarrow\ & b^2ab^2 = (bab)a^2(bab) \\
\Rightarrow\ & \{a, bab, b^2ab^2\} \text{ is a line by Lemma 4.3.8.}
\end{aligned}
$$

□

Theorem 4.3.10. For each $r \geq 2$, the finite Sperner space S_r is a Hall triple system.

Proof. Let a be an arbitrary point of S_r and let σ_a be the symmetry of S_r with fixed point a (see Definition 1.6.2). Let x be a point distinct from a. Then by Lemma 4.3.8, the line containing a and x is $\{a, x, xa^2x\}$. Hence

$$
\begin{aligned}
\sigma_a :\ & a \mapsto a \\
& x \mapsto xa^2x.
\end{aligned}
$$

Now each line of S_r is a coset of a cyclic subgroup of $B(3, r)$. Therefore each line can be written in the form

$$c\{1, b, b^2\} = \{c, cb, cb^2\}$$

for some elements $b, c \in B(3, r)$. The image of this line under the action of σ_a is the set

$$\sigma_a\{c, cb, cb^2\} = \{ca^2c, cba^2cb, cb^2a^2cb^2\}.$$

It remains only to show that this is a line of S_r; we argue as follows.

By applying Lemma 4.3.9 to the two elements a^2c and b, we have that

$$\{a^2c, ba^2cb, b^2a^2cb^2\}$$

is a line of S_r. Under the collineation of S_r corresponding to left multiplication by c, this line maps to the line

$$\{ca^2c, cba^2cb, cb^2a^2cb^2\}$$

which is identical to $\sigma_a\{c, cb, cb^2\}$.

Since each symmetry of S_r is also a collineation, it follows by definition that S_r is a Hall triple system. □

Having shown that S_r, $r \geq 2$ is a Hall triple system, it is natural to try to determine if it is isomorphic to an existing one. When $r = 2$, the Hall triple system S_2 has 27 points. Thus it is simply isomorphic to $AG(3,3)$ (see [65] for example). Hence in the sequel, we assume that r is at least 3.

As a first step in trying to resolve this problem, we can develop a necessary and sufficient condition for S_r to be a finite affine space. To facilitate this, we consider the 3-CM loop $\mathcal{M}$ associated with S_r (see Section 1.6 for explanations of the terminology used here). By the discussion presented in Section 1.6, the 3-CM loop corresponding to the element 1 in S_r is isomorphic to $\mathcal{M}$. Therefore, we identify these two loops.

Now, by Lemma 4.3.8, three points a, b and c of S_r are collinear if and only if $c = ba^2b$. Hence, letting $\circ$ denote the binary operation of $\mathcal{M}$, we have that

$$\begin{aligned} a \circ b &= (1 \cdot a) \cdot (1 \cdot b) \\ &= (a^2) \cdot (b^2) \\ &= b^2ab^2 \end{aligned}$$

for each pair of elements a, b of S_r. (We note that the Hall triple system arising from

$\mathcal{M}$ is in fact identical to S_r in this example, and not just isomorphic to S_r because

$$\begin{aligned}\{a, b, (a \circ b)^2\} &= \{a, b, (b^2ab^2) \circ (b^2ab^2)\} \\ &= \{a, b, (b^2ab^2)^2(b^2ab^2)(b^2ab^2)^2\} \\ &= \{a, b, (b^2ab^2)^{-1}\} \\ &= \{a, b, ba^2b\}.)\end{aligned}$$

We are now able to prove

Theorem 4.3.11. The Hall triple system S_r, $r \geq 3$, is a finite affine space if and only if the following identify holds in the Burnside group $B(3, r)$:

$$cb^2ab^2c = b^2cacb^2.$$

Proof. By Theorem 1.6.6, S_r is a finite affine space if and only if its associated 3-CM loop $\mathcal{M}$ is associative. This holds if and only if, for all a, b and c in $\mathcal{M}$, we have

$$\begin{aligned} & (a \circ b) \circ c &= \; & a \circ (b \circ c) \\ \Leftrightarrow \; & (b^2ab^2) \circ c &= \; & a \circ (bc^2bc^2) \\ \Leftrightarrow \; & c^2(b^2ab^2)c^2 &= \; & (c^2bc^2)^2a(c^2bc^2)^2 \\ \Leftrightarrow \; & c^2(b^2ab^2)c^2 &= \; & (c^2bc^2)^{-1}a(c^2bc^2)^{-1} \\ \Leftrightarrow \; & c^2(b^2ab^2)c^2 &= \; & cb^2cacb^2c \\ \Leftrightarrow \; & cb^2ab^2c &= \; & b^2cacb^2. \end{aligned}$$

Since a, b and c are also elements of $B(3, r)$, the result is now immediate. □

Unfortunately, despite repeated efforts, we have not been able to ascertain whether or not the identity in Theorem 4.3.11 is valid in $B(3, r)$. Thus the exact nature of S_r is still undetermined. However, no matter what the outcome, there is at least one interesting corollary in each case; we conclude this section with a description of these corollaries.

If S_r, $r \geq 3$ is the finite affine space $AG\left(\binom{r}{3} + \binom{r}{2} + r, 3\right)$, then the general linear group $GL\left(\binom{r}{3} + \binom{r}{2} + r, 3\right)$ has a subgroup G isomorphic to the Burnside group

$B(3,r)$ because, by construction, S_r inherits $B(3,r)$ as a collineation group and in addition, the full collineation group of $AG\left(\binom{r}{3}+\binom{r}{2}+r,3\right)$ is the stated general linear group.

On the other hand, if S_r, $r \geq 3$ is not an affine space, then S_r is also a finite Sperner space which satisfies the property that every Δ-subspace of S_r is an affine plane of order 3 without actually being an affine space. This then shows that the following result of Barlotti can not be generalised to finite Sperner spaces of arbitrary dimension and order 3.

([7], Theorem 1.4.1) Let S be a finite Sperner space of dimension three and order m. If each point of S lies in at least m^2+m distinct affine subplanes of S, then S is the finite affine space $AG(3,m)$. $\square$

4.4. RESOLUTION CLASSES OF BALANCED INCOMPLETE BLOCK DESIGNS AND PARTITIONS OF FINITE AFFINE SPACES

The quasi-n-multiple Sperner designs which we constructed in Section 4.1 are all resolvable, as we have already mentioned in Remark 4.1.3. In the case where the design is an affine plane of order q^2 (that is, when $n=1$ and $v=q^4$), it is well-known that the resolution is unique. However when n is at least 2, it is no longer reasonable to assume that the resolution should be unique. Indeed, we have verified by an exhaustive computer search that the design $D(2)$ constructed in Example 4.1.4 has 248 distinct resolutions (it is not known though, how many of these are isomorphic).

Designs admitting more than one resolution have been studied previously; for example, see [10], [41], [46], [47], [56], [57], [58], [59], [93] and [120]. In most of these papers, the main interest lies in constructing so-called skew resolutions, as these enable the construction of generalised room squares. For further details, we refer the reader to the aformentioned papers. We note, however, in passing that no two of the 248 resolutions of $D(2)$ are skew.

It is apparent, by the technique used for constructing the designs $D(n)$, that each resolution class in a resolution of $D(n)$ corresponds to a partition of the point-set of the appropriate finite affine space by finite affine planes of the same order. Our main aim in this section, therefore, is to briefly examine some of these partitions via considering the slightly more general problem of partitioning the point-set of $AG(2t, q)$ by finite t-dimensional affine subspaces.

Not much seems to have been done regarding such partitions except for the case when $q = 2$ (see [8]). The classical examples of such partitions can be constructed by considering $AG(2t, q)$ to be the vector space $V_{2t}(q)$ of dimension $2t$ over the Galois field $GF(q)$. A subspace V_t of dimension t in $V_{2t}(q)$ and its $q^t - 1$ proper cosets then partition the set of vectors of $V_{2t}(q)$ and this yields a partition of $AG(2t, q)$ by finite t-dimensional affine subspaces.

Alternatively, viewing $AG(2t, q)$ as $PG(2t, q) \backslash PG(2t-1, q)$, such a partition corresponds to a set of q^t t-dimensional projective subspaces which meet $PG(2t-1, q)$ in a single, fixed $(t-1)$-dimensional projective subspace. We shall adopt the latter viewpoint in constructing non-classical partitions of $AG(2t, q)$ and, towards this end, we have

Theorem 4.4.1. Let S be a set of q^t t-dimensional subspaces of $PG(2t, q)$. Then S gives rise to a partition of the point-set of $AG(2t, q)$ by t-dimensional affine subspaces if and only if there exists a hyperplane Σ_{2t-1} of $PG(2t, q)$ for which the following conditions are satisfied:

(i) No element of S lies in Σ_{2t-1},

(ii) The intersection of any two distinct elements of S is wholly contained in Σ_{2t-1}.

Proof. If S gives rise to such a partition of $AG(2t, q)$, then it is immediate that the hyperplane which completes $AG(2t, q)$ to $PG(2t, q)$, satisfies the two conditions stated.

Conversely, assume that Σ_{2t-1} is a hyperplane of $PG(2t, q)$ satisfying the two conditions and consider the $2t$-dimensional affine space $PG(2t,q)\backslash\Sigma_{2t-1}$. By condition (i), no element of S lies in Σ_{2t-1}. Hence, each element of S intersects Σ_{2t-1} in a $(t-1)$-dimensional affine subspace of $PG(2t,q)\backslash\Sigma_{2t-1}$. Furthermore, by condition (ii), these q^t affine subspaces are pairwise disjoint. Hence (slightly abusing notation), $S\backslash\Sigma_{2t-1}$ is a partition of $PG(2t,q)\backslash\Sigma_{2t-1}$ by t-dimensional affine subspaces. □

Example 4.4.2. Let Σ_{2t-2} be a $(2t-2)$-dimensional subspace of $PG(2t,q)$ and arbitrarily label the $q+1$ hyperplanes containing Σ_{2t-2}, Σ^i_{2t-1}, $i \in \{1,\ldots,q,\infty\}$.

Take q arbitrary $(t-1)$-dimensional subspaces of Σ_{2t-2} and label these Σ^i_{t-1}, $i = 1,\ldots,q$.

It is routine to prove that in each Σ^i_{2t-1}, $i=1,\ldots,q$, the number of t-dimensional subspaces containing Σ^i_{t-1} and not lying in Σ_{2t-2} is q^{t-1}. Label these Σ^{ij}_t, $j=1,\ldots,q^{t-1}$ and set

$$S = \left\{\Sigma^{ij}_t \;\middle|\; i=1,\ldots,q;\ j=1,\ldots,q^{t-1}\right\}.$$

Now S is a set of q^t t-dimensional subspaces of $PG(2t,q)$ which are not wholly contained in the hyperplane Σ^{∞}_{2t-1}. Also, by construction, the intersection of any two of these subspaces lies in Σ_{2t-2} which, in turn, lies in Σ^{∞}_{2t-1}.

Hence, by Theorem 4.4.1, S gives rise to a partition of $PG(2t,q)\backslash\Sigma^{\infty}_{2t-1}$ by t-dimensional affine subspaces. Moreover, the partition is classical if and only the subspaces Σ^i_{t-1}, $i=1,\ldots,q$ all coincide. □

Remark 4.4.3. When $t=2=q$ in the previous example, there are essentially two types of partitions of $AG(4,2)$ which result from the construction; one is classical and the other consists of two pairs of planes which correspond to two distinct intersecting lines of $PG(3,2)$. It can be easily argued that the resolutions of any design $D(2)$ arising from a 2-cover of $PG(3,2)$, admit only resolution classes which correspond to either one or both of these partitions. In particular, any $D(2)$ admitting more than one resolution (such as the design in Example 4.1.4) necessarily has some non-classical resolution classes. □

We now consider a third type of partition of the point-set of $AG(2t,q)$ by t-dimensional affine subspaces. However, before presenting its construction, we need

Definition 4.4.4. Let $\mathcal{T}$ be a set of t-dimensional subspaces of $PG(2t,q)$ such that:

(i) The elements of $\mathcal{T}$ intersect pairwise in a single fixed point V.

(ii) The elements of $\mathcal{T}$ induce a partition of the point-set of $PG(2t,q)\backslash\{V\}$. Then $\mathcal{T}$ is called a *t-fan* and the point V is called the *vertex* of $\mathcal{T}$. □

In the Bruck-Bose construction of a translation plane π of order q^t from a $(t-1)$-spread $\mathcal{S}$ of $PG(2t-1,q)$, the set of lines of π through a given point P of π corresponds to the t-fan of $PG(2t,q)$ which has vertex P and elements of the form $\langle P, T_i\rangle$, where, for each $i = 1,\ldots,q^t+1$, T_i is an element of $\mathcal{S}$. We shall show later that all t-fans can be constructed in this manner.

Example 4.4.5. Let Σ_{2t-1} and Σ_{2t-1}^{∞} be two distinct hyperplanes of $PG(2t,q)$ and denote their intersection by Σ_{2t-2}. Let $\mathcal{S}$ be a spread of Σ_{2t-1} and V be a point of $\Sigma_{2t-1}^{\infty}\backslash\Sigma_{2t-2}$. Then, by the discussion preceding this example, the set

$$\mathcal{T} = \left\{\langle V, T_i\rangle \,\middle|\, T_i \in \mathcal{S},\ i = 1,\ldots,q^t+1\right\}$$

is a t-fan of $PG(2t,q)$.

In Remark 2.1.11, we mentioned that all t-spreads of $PG(2t-1,q)$ are dual. Hence there is a unique element of $\mathcal{S}$ which lies completely in Σ_{2t-2}. Without loss of generality, we can assume that this element is T_{q^t+1}. It is then immediate that

$$S = \left\{\langle V, T_i\rangle \,\middle|\, i = 1,\ldots,q^t\right\}$$

is a set of q^t t-dimensional subspaces of $PG(2t,q)$, none of which lies in Σ_{2t-1}^{∞} and any two of which intersect in the single point V of Σ_{2t-1}^{∞}. Hence, by Theorem 4.4.1, S gives rise to a partition of the point-set of $PG(2t,q)\backslash\Sigma_{t-1}^{\infty}$ by t-dimensional affine subspaces.

Moreover, provided $t > 1$, these partitions are never affinely equivalent to the partitions constructed in Example 4.4.2, because the q^t subspaces of $PG(2t, q)$ giving rise to the partitions intersect differently in the deleted hyperplanes. (When $t = 1$, the partitions are equivalent because there is only one possible resolution of $AG(2, q)$.) □

Remark 4.4.6. It is not known if any designs $D(n)$ arising from n-covers of $PG(3, q)$ admit resolutions with this type of partition as a resolution class. We note, however, that if such a design exists, then $n \geq q^2$ because the planes of $PG(4, q)$ which give rise to the partition meet $PG(3, q)$ in q^2 concurrent lines. □

We conclude this section by examining some of the relationships which exist between t-fans of $PG(2t, q)$ and $(t-1)$-spreads of $PG(2t-1, q)$. It was originally hoped that, by exploiting these relationships, it might have been possible to construct new spreads from ones already known. Although this did not turn out to be the case, we include our results to lend clarity to the situation. We have already indicated one such relationship in the discussion after Definition 4.4.4 (see page 155). The truth of that statement follows from

Theorem 4.4.7. Let $\mathcal{T} = \left\{U_i\right\}_{i=1}^{q^t+1}$ be a t-fan of $PG(2t, q)$ with vertex V and let Σ_{2t-1} be a hyperplane of $PG(2t, q)$ not containing V. Then the set

$$\mathcal{S} = \left\{U_i \cap \Sigma_{2t-1} \,\middle|\, i = 1, \ldots, q^t + 1\right\}$$

is a $(t-1)$-spread of Σ_{2t-1}.

Proof. The $q^t + 1$ subspaces $U_i \cap \Sigma_{2t-1}$ are pairwise disjoint because the elements of $\mathcal{T}$ intersect pairwise in the point V which lies outside of Σ_{2t-1}.

In addition, by the dimension theorem,

$$\begin{aligned} \dim(U_i \cap \Sigma_{2t-1}) &= t + (2t-1) - \dim(U_i \oplus \Sigma_{2t-1}) \\ &= t - 1, \end{aligned}$$

and so the set $\mathcal{S}$ is a spread of Σ_{2t-1}. □

The $(t-1)$-spreads which arise from t-fans in the way described in Theorem 4.4.7 will be referred to as being of type 1 for brevity. Accordingly, the $(t-1)$-spreads which arise from the next result, will be referred to as being of type 2.

Theorem 4.4.8. Let $\mathcal{T} = \left\{U_i\right\}_{i=1}^{q^t+1}$ be a t-fan of $PG(2t,q)$ with vertex V and let δ be a correlation of $PG(2t,q)$. Then $\mathcal{T}^\delta$ is a $(t-1)$-spread of V^δ.

Proof. Under the action of a correlation of $PG(2t,q)$, an n-dimensional subspace is mapped to a $(2t-n-1)$-dimensional subspace and inclusion is reversed.

Hence $\mathcal{T}^\delta = \left\{U_i^\delta\right\}_{i=1}^{q^t+1}$ is a set of $(t-1)$-dimensional subspaces which lie in the hyperplane V^δ. It only remains to show that the elements of $\mathcal{T}^\delta$ are pairwise disjoint.

Assume that, for some i and j, the two subspaces U_i^δ and U_j^δ share a point R. Then U_i and U_j both lie in the hyperplane $R^{\delta^{-1}}$. However, by the dimension theorem,

$$\begin{aligned} 2t-1 = \dim(R^{\delta^{-1}}) &\geq \dim(U_i) + \dim(U_j) - \dim(U_i \cap U_j) \\ &= t + t - 0 \\ &= 2t, \end{aligned}$$

and so we have a contradiction. Hence the elements of $\mathcal{T}^\delta$ are pairwise disjoint.

□

Lemma 4.4.9. Let Σ_{2t-1} be a hyperplane of $PG(2t,q)$. Then there exists a subgroup of $PGL(2t+1,q)$ which fixes Σ_{2t-1} pointwise and which acts transitively on the point-set of $PG(2t,q)\backslash\Sigma_{2t-1}$.

Proof. Since all hyperplanes of $PG(2t,q)$ are projectively equivalent, we can choose Σ_{2t-1} to be the hyperplane with equation $x_{2t}=0$ (with respect to the coordinates $(x_0,x_1,\ldots,x_{2t})$.)

Now for each point $\mathbf{x}$ of $PG(2t,q)\backslash\Sigma_{2t-1}$ define the $(2t+1)\times(2t+1)$ matrix

$$A(\mathbf{x}) = \left[\begin{array}{c|c} I_{2t} & \\ & \mathbf{x}^t \\ \hline \mathbf{0} & \end{array}\right]$$

where the last coordinate of $\mathbf{x}$ is scaled to 1. The matrices of this form generate a subgroup G of $PGL(2t+1,q)$.

It is immediate that each element of G fixes Σ_{2t-1} pointwise. Furthermore the homography with matrix $A(\mathbf{x})$ maps the point $(0,\ldots,0,1)$ to the point $\mathbf{x}$. Hence the orbit of $(0,\ldots,0,1)$ under the action of G is the whole of the point-set $\mathbf{P}$ of $PG(2t,q)\backslash\Sigma_{2t-1}$ that is, G acts transitively on $\mathbf{P}$.

Thus G is a subgroup satisfying the hypotheses of the Lemma. $\square$

Theorem 4.4.10. Let $\mathcal{T} = \left\{U_i\right\}_{i=1}^{q^t+1}$ be a t-fan of $PG(2t,q)$ with vertex V and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two $(t-1)$-spreads of type 1 arising from the hyperplanes Σ^1_{2t-1} and Σ^2_{2t-1} respectively. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are projectively equivalent.

Proof. By Lemma 4.4.9 and the principle of duality, there exists a collineation group G which fixes each hyperplane through V as a whole and which acts transitively on the set of hyperplanes not containing V.

Let σ be an element of G which maps Σ^1_{2t-1} to Σ^2_{2t-1}. Then

$$\begin{aligned} \mathcal{S}_1^\sigma &= \left\{\left(U_i\cap\Sigma^1_{2t-1}\right)^\sigma \;\middle|\; i=1,\ldots,q^t+1\right\} \\ &= \left\{U_i\cap\Sigma^2_{2t-1} \;\middle|\; i=1,\ldots,q^t+1\right\} \\ &= \mathcal{S}_2\,. \end{aligned}$$

Hence $\mathcal{S}_1$ and $\mathcal{S}_2$ are projectively equivalent. $\square$

Theorem 4.4.11. Let $\mathcal{T}$ be a t-fan of $PG(2t,q)$ with vertex V and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two $(t-1)$-spreads of type 2. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are projectively equivalent.

Proof. Let $\mathcal{S}_1 = \mathcal{T}^{\delta_1}$ and $\mathcal{S}_2 = \mathcal{T}^{\delta_2}$ for some correlations δ_1 and δ_2 of $PG(2t,q)$ and let σ be the collineation $\delta_2 \circ \delta_1^{-1}$. Then

$$\begin{aligned}
\mathcal{S}_1^{\sigma} &= \left(\mathcal{T}^{\delta_1}\right)^{(\delta_2 \circ \delta_1^{-1})} \\
&= \left(\left(\mathcal{T}^{\delta_1}\right)^{\delta_1^{-1}}\right)^{\delta_2} \\
&= \mathcal{T}^{\delta_2} \\
&= \mathcal{S}_2 .
\end{aligned}$$

Hence $\mathcal{S}_1$ and $\mathcal{S}_2$ are projectively equivalent. □

Theorems 4.4.10 and 4.4.11 show that two $(t-1)$-spreads of the same type are projectively equivalent. In general, two $(t-1)$-spreads of distinct types need not be projectively equivalent; instead one is projectively equivalent to the dual of the other. This is stated more formally in

Theorem 4.4.12. Let $\mathcal{T} = \left\{U_i\right\}_{i=1}^{q^t+1}$ be a t-fan of $PG(2t,q)$ with vertex V. Let $\mathcal{S}_1$ be a $(t-1)$-spread of type 1 arising from the hyperplane Σ_{2t-1} and let $\mathcal{S}_2$ be a $(t-1)$-spread arising from the correlation δ. Then there exists a correlation from Σ_{2t-1} to V^{δ} which maps $\mathcal{S}_1$ to $\mathcal{S}_2$.

Proof. First, let ϕ be the following mapping.

$$\begin{array}{rccc}
\phi : & \left\{ \begin{array}{l} \text{The pointset} \\ \text{of } \Sigma_{2t-1} \end{array} \right\} & \mapsto & \left\{ \begin{array}{l} \text{The set of lines in} \\ PG(2t,q) \text{ through } V \end{array} \right\} \\
& P & \mapsto & \langle P, V \rangle .
\end{array}$$

ϕ is one-to-one and onto because each line through V meets Σ_{2t-1} in a unique point. Furthermore, ϕ is incidence preserving because

$$\begin{aligned}
\langle P_1, P_2 \rangle^{\phi} &= \langle \langle P_1, P_2 \rangle, V \rangle \\
&= \langle \langle P_1, V \rangle, \langle P_2, V \rangle \rangle \\
&= \langle P_1^{\phi}, P_2^{\phi} \rangle .
\end{aligned}$$

Second, the correlation δ with its domain restricted to the range of ϕ, defines a one-to-one, inclusion reversing mapping onto the set of hyperplanes of V^{δ}.

Hence the composed mapping $\rho = \delta \circ \phi$ is an incidence preserving mapping from Σ_{2t-1} to the dual of V^δ, that is, it is a correlation from Σ_{2t-1} to V^δ.

It remains to show that ρ maps $\mathcal{S}_1$ to $\mathcal{S}_2$. Let $\mathcal{S}_1 = \left\{T_i\right\}_{i=1}^{q^t+1}$ where $T_i = U_i \cap \Sigma_{2t-1}$. Then

$$\begin{aligned} T_i^\rho &= T_i^{(\delta \circ \phi)} \\ &= \langle T_i, V \rangle^\delta \\ &= U_i^\delta \end{aligned}$$

which is an element of $\mathcal{S}_2$.

Hence ρ maps $\mathcal{S}_1$ to $\mathcal{S}_2$. □

Remark 4.4.13. Although two $(t-1)$-spreads of different types are related by a correlation defined between their ambient spaces, it is not clear that they should necessarily be projectively equivalent in general. Indeed, it is not true in projective spaces over countably infinite fields; in the 3-dimensional projective spaces over these fields, there exist examples of spreads whose images under a correlation are no longer spreads. However, there do exist spreads for which this does hold. If $\mathcal{S}$ is a regular spread of $PG(2t-1, F)$ for any field F, then so is its image under a correlation, because correlations preserve reguli (see [27]). It also holds for any symplectic spread $\mathcal{S}$ of $PG(2t-1, q)$ because $\mathcal{S}$ consists of totally isotropic spaces with respect to an appropriate symplectic polarity.

□

CHAPTER V
COMPATIBLE AFFINE PLANES AND NET REPLACEMENT

5.1. ON THE RELATIONSHIP BETWEEN THE EMBEDDING OF A FINITE AFFINE PLANE IN A SECOND AFFINE PLANE OF THE SAME ORDER AND NET REPLACEMENT

In Chapter IV, we described a technique for constructing examples of quasi-n-multiple designs. With $n = 2$ and $t = 1$ these designs become quasi-2-multiple affine designs; our interest lay mainly in showing that these designs are irreducible if and only if the 2-cover used in the construction is proper.

Quasi-2-multiple affine designs have also been studied by Jungnickel in [80]. The construction technique therein involves taking an existing affine plane D and permuting the points of the plane with respect to a permutation π. The images of the lines under π become the lines of a new affine plane D^π (isomorphic to D) defined on the same set of points as D. The union of D with D^π is then a quasi-2-multiple affine design (possibly having some repeated blocks). Using this technique, Jungnickel was able to show that the number of quasi-2-multiple affine designs arising from an affine plane of prime power order q increases without bound as q increases.

An idea similar to this, but which is more general, inspired the notion of net replacement. In [101] Ostrom states "Given two finite projective or affine planes of the same order, we can take the same objects to be the "points" in both cases. The lines of one plane then generate configurations in the other". In practice, finding replacement nets has usually involved the construction of various configurations such as subplanes for example, in a given affine plane. Consequently, the resulting plane in the majority of cases has been a translation plane exactly when the original plane was a translation plane; this is in spite of the fact that any finite affine plane can be constructed from any other affine plane of the same order via net replacement. Hence, in this chapter we use this fact to consider the problem of constructing new finite affine planes from a pair of existing affine planes of the same order.

To commence, we review the embedding of a finite affine plane in a second affine plane of the same order.

Let π_1 and π_2 be two finite affine planes of the same order n. Since the two planes have the same order, they also have equal numbers of points. Therefore we can construct a bijection ϕ from the point-set of π_2 to the point-set of π_1. Under the action of ϕ, the image of a line of π_2 is the set of images of the points lying on the line. That is, if the line is

$$m = \{P_i \mid i = 1 \text{ to } n\},$$

then

$$\phi(m) = \{\phi(P_i) \mid i = 1 \text{ to } n\}.$$

This embedding technique works for any such bijection ϕ. However, some bijections may give a more useful embedding of π_2 into π_1. For example, we noted in Remark 1.5.5 that all trivial nets of degree k are isomorphic to one another (where $k = 1$ or 2). Hence, if we choose a trivial net of degree k in π_2 and another of degree k in π_1, then we can construct a bijection ϕ between the point-sets of the two planes in such a way that the planes have k parallel classes in common.

It follows from this by identifying π_2 with its image $\phi(\pi_2)$ embedded in π_1, that π_2 can be obtained from π_1 via net replacement where $\pi_1 \backslash \mathcal{N}$ is the replaceable net, $\pi_2 \backslash \mathcal{N}$ is the replacement net and $\mathcal{N}$ is the net consisting of the parallel classes shared by π_1 and π_2. Note: By the preceding paragraph; we can choose ϕ so that $\mathcal{N}$ has at least degree 2. Thus, provided the order of the plane is greater than 3, we can guarantee that the replacement is non-trivial.

In the remainder of this section we shall show that $\pi_1\backslash\mathcal{N}$ and $\pi_2\backslash\mathcal{N}$ can be decomposed into pairs of complementary irreducible replacement nets. First, we introduce some notation and terminology.

Let π_1 and π_2 be two finite affine planes of the same order such that π_2 is embedded in π_1 via a bijection ϕ defined between their point-sets. Henceforth we identify each line m of π_2 with its image $\phi(m)$ in π_1. Also let π_1^* denote the projective completion of π_1 and let ℓ_∞ denote the special line of π_1.

For each pair of distinct points P, Q in π_1^*, let $\langle P, Q\rangle$ denote the unique line of π_1^* containing P and Q. For each line m of π_2 we then define the subset d_m of ℓ_∞ as follows:

$$d_m = \{\ell_\infty \cap \langle P, Q\rangle \mid P, Q \in m,\ P \neq Q\};$$

then for each parallel class $\mathcal{P}^2$ of π_2 we define

$$d(\mathcal{P}^2) = \bigcup_{m\in\mathcal{P}^2} d_m.$$

Note: $d(\mathcal{P}^2)$ is also a subset of ℓ_∞.

Definition 5.1.1. With respect to the notation above, we define d_m to be the *special set of the line* m and $d(\mathcal{P}^2)$ to be the *special set of the parallel class* $\mathcal{P}^2$. □

Using ℓ_∞ and the special sets of the parallel classes of π_2, we can define a finite incidence structure $(\mathbf{P}, \mathbf{B}, \mathbf{I})$ with

$\mathbf{P} =$ the set of points on ℓ_∞

$\mathbf{B} =$ the set of special sets of the parallel classes of π_2

$\mathbf{I} =$ set inclusion.

In turn, using $(\mathbf{P}, \mathbf{B}, \mathbf{I})$ we can define a relation $\sim$ on the set of special sets of the parallel classes of π_2 as follows: Given two parallel classes $\mathcal{P}_1^2$ and $\mathcal{P}_2^2$ of π_2,

$$d(\mathcal{P}_1^2) \sim d(\mathcal{P}_2^2)$$

if and only if $d(\mathcal{P}_1^2)$ and $d(\mathcal{P}_2^2)$ are connected by a chain in $(\mathbf{P}, \mathbf{B}, \mathbf{I})$. It is evident that $\sim$ is an equivalence relation; the equivalence classes with respect to the relation $\sim$ thus partition the set of special sets of the parallel classes of π_2.

Definition 5.1.2. Let E be an equivalence class with respect to the relation $\sim$. Then the subset

$$\mathcal{R} = \bigcup_{d(\mathcal{P}^2) \in E} d(\mathcal{P}^2)$$

of ℓ_∞ is called a *replacement set.* □

Remark 5.1.3. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two replacement sets arising from the equivalence classes $[\mathcal{R}_1]$ and $[\mathcal{R}_2]$. Suppose there exists a point $P \in \mathcal{R}_1 \cap \mathcal{R}_2$. By the manner in which $\mathcal{R}_1$ and $\mathcal{R}_2$ are constructed, it follows that there exists at least one line m of π_2 with $P \in d_m$. Hence, if $\mathcal{P}^2$ is the parallel class of π_2 containing m, then $d(\mathcal{P}^2)$ is a subset of $\mathcal{R}_1 \cap \mathcal{R}_2$ and so $d(\mathcal{P}^2)$ lies in $[\mathcal{R}_1] \cap [\mathcal{R}_2]$. As $[\mathcal{R}_1]$ and $[\mathcal{R}_2]$ are equivalence classes it follows that $[\mathcal{R}_1] = [\mathcal{R}_2]$ and $\mathcal{R}_1 = \mathcal{R}_2$. Therefore the replacement sets also partition the set of points of ℓ_∞. □

The reason for calling the sets $\mathcal{R}$ replacement sets will become apparent in the following construction.

Construction 5.1.4. Continuing with the situation as we have described it, let $\mathcal{R}$ be a replacement set and $[\mathcal{R}]$ be the equivalence class from which it arises. Define $\mathcal{N}^1$ and $\mathcal{N}^2$ as follows:

$\mathcal{N}^1 =$ The set of lines of π_1^* which meet ℓ_∞ in a point of $\mathcal{R}$, plus the set of points of π_1.

$\mathcal{N}^2 =$ The set of lines of π_2 which lie in parallel classes $\mathcal{P}^2$ such that $d(\mathcal{P}^2)$ is an element of $[\mathcal{R}]$, plus the set of points of π_1.

We can then prove:

Theorem 5.1.5. $\mathcal{N}^1$ and $\mathcal{N}^2$ are conjugate replacement nets.

Proof. By construction if a line of π_i $(i = 1$ or $2)$ lies in $\mathcal{N}^i$ then the parallel class containing the line also lies in $\mathcal{N}^i$. Hence, as $\mathcal{N}^i$ is embedded in π_i, we have that $\mathcal{N}_i$ is indeed a net.

It remains only to show that if two points lie on a line of one net, then they also lie on a line of the other net.

Suppose the points P, Q lie on a line ℓ of $\mathcal{N}^1$. Let m be the unique line of π_2 containing P and Q. Let ℓ^* be the line of π_1^* corresponding to ℓ. Then since ℓ^* meets ℓ_∞ in a point of $\mathcal{R}$, it follows that $\ell^* \cap \ell_\infty$ is a point of d_m. Letting $\mathcal{P}^2$ be the parallel class of π_2 containing m, it is then immediate that $\ell^* \cap \ell_\infty$ is also a point of $d(\mathcal{P}^2)$. Thus $d(\mathcal{P}^2)$ lies in $[\mathcal{R}]$ and so m is a line of $\mathcal{N}^2$. Therefore P, Q lie on a line of $\mathcal{N}^2$.

Suppose P, Q lie on a line m of $\mathcal{N}^2$. Let $\langle P, Q\rangle$ be the unique line of π_1^* containing P and Q. If $\mathcal{P}^2$ is the parallel class of π_2 containing m, then $d(\mathcal{P}^2)$ lies in $[\mathcal{R}]$ because m is a line of $\mathcal{N}^2$. Thus $d(\mathcal{P}^2)$ is a subset of $\mathcal{R}$. It follows that

$$\langle P, Q\rangle \cap \ell_\infty \in d_m \subseteq d(\mathcal{P}^2) \subseteq \mathcal{R}.$$

Hence $\langle P, Q\rangle$ is a line of $\mathcal{N}^1$ and so P, Q lie on a line of $\mathcal{N}^1$. □

Lemma 5.1.6. Let $\mathcal{P}^2$ be a parallel class of π_2. Then $\mathcal{P}^2$ is also a parallel class of π_1 if and only if $|d(\mathcal{P}^2)| = 1$.

Proof. Clearly if $\mathcal{P}^2$ is also a parallel class of π_1, then $|d(\mathcal{P}^2)| = 1$. Thus suppose that $|d(\mathcal{P}^2)| = 1$. Assume that $\mathcal{P}^2$ contains a line m of π_2 which is not also a line of π_1. It follows that m contains three points which are not collinear in π_1. The three lines defined by these three points give rise to three distinct points of d_m. Hence

$$3 \leq |d_m| \leq |d(\mathcal{P}^2)|$$

which contradicts the hypothesis. Thus every line of π_2 in $\mathcal{P}^2$ is also a line of π_1 and so $\mathcal{P}^2$ is also a parallel class of π_1. □

Lemma 5.1.7. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be distinct replacement sets. Then the nets $\mathcal{N}_1^1$ and $\mathcal{N}_2^1$ in π_1 corresponding to $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint.

Proof. Assume $\mathcal{N}_1^1$ and $\mathcal{N}_2^1$ are not disjoint. Then there exists at least one line ℓ^* of π_1^* common to $\mathcal{N}_1^1$ and $\mathcal{N}_2^1$. The point $\ell^* \cap \ell_\infty$ is common to both $\mathcal{R}_1$ and $\mathcal{R}_2$. By Remark 5.1.3, it then follows that $\mathcal{R}_1 = \mathcal{R}_2$, which contradicts the hypothesis. Therefore $\mathcal{N}_1^1$ and $\mathcal{N}_2^1$ are disjoint nets. $\square$

Lemma 5.1.8. Let $\mathcal{R}$ be a replacement set with corresponding conjugate nets $\mathcal{N}^1$ and $\mathcal{N}^2$. Then $\mathcal{N}^1$ and $\mathcal{N}^2$ are irreducible with respect to one another.

Proof. Assume that the nets are not irreducible with respect to one another. Then there exist conjugate nets $(\mathcal{N}^1)'$ and $(\mathcal{N}^2)'$ with

$$(\mathcal{N}^1)' \subseteq \mathcal{N}^1 \quad \text{and} \quad (\mathcal{N}^1)' \neq \mathcal{N}^1$$
$$(\mathcal{N}^2)' \subseteq \mathcal{N}^2 \quad \text{and} \quad (\mathcal{N}^2)' \neq \mathcal{N}^2.$$

Let $\mathcal{R}'$ be the subset of $\mathcal{R}$ corresponding to $(\mathcal{N}^1)'$ and let $\{\mathcal{R}'\}$ denote the subset of $[\mathcal{R}]$ which corresponds to the set of special sets of the parallel classes of $(\mathcal{N}^2)'$.

Now $(\mathcal{N}^2)'$ is not equal to $\mathcal{N}^2$; hence there exists a parallel class $\mathcal{P}^2$ of $\mathcal{N}^2$ which does not lie in $(\mathcal{N}^2)'$. The special set $d(\mathcal{P}^2)$ of $\mathcal{P}^2$ is connected by a chain (in $(\mathbf{P}, \mathbf{B}, \mathbf{I})$) to every element of $[\mathcal{R}]$. Therefore there exists a special set $d\big((\mathcal{P}^2)'\big)$ such that

$$d\big((\mathcal{P}^2)'\big) \notin \{\mathcal{R}'\}$$

while

$$d_m \cap \mathcal{R}' \neq \phi$$

for some line m of $(\mathcal{P}^2)'$.

It follows that $(\mathcal{P}^2)'$ belongs to both $(\mathcal{N}^2)'$ and $(\mathcal{N}^2)\backslash(\mathcal{N}^2)'$ which gives us a contradiction.

Thus $\mathcal{N}^1$ and $\mathcal{N}^2$ are irreducible with respect to one another. $\square$

Theorem 5.1.9. Let π_1 and π_2 be a pair of finite affine planes of the same order. Let π_2 be embedded in π_1 via a bijection ϕ defined between their point-sets. Then (identifying π_2 with $\phi(\pi_2)$) π_2 can be obtained from π_1 by replacing pairwise disjoint nets in π_1 which are irreducible with respect to their replacement nets. Moreover, this net replacement is unique for a given bijection ϕ.

Proof. Assume that π_2 is embedded in π_1. Let $\mathcal{R}_i$, $i = 1$ to k (for some $k \geq 1$) be the replacement sets corresponding to the equivalence classes of the relation $\sim$ which we described prior to Definition 5.1.2. Let $\mathcal{N}_i^1$ and $\mathcal{N}_i^2$ be the nets in π_1 and π_2 arising from $\mathcal{R}_i$.

Consider first those replacement sets with cardinality one.

The degree of the net in π_1 corresponding to such a replacement set is one, implying that the net consists of a single parallel class of π_1. Thus the net in π_2 conjugate to the net in π_1 also consists of a single parallel class $\mathcal{P}^2$ where $\mathcal{P}^2$ satisfies $|d(\mathcal{P}^2)| = 1$. By Lemma 5.1.6 it is immediate that the parallel classes are common to both planes.

Let the set of these parallel classes form the net $\mathcal{N}$. Without loss of generality the other nets in π_1 can be labelled $\mathcal{N}_1^1, \mathcal{N}_2^1, \ldots, \mathcal{N}_m^1$ ($m \leq k$). Thus

$$\pi = \mathcal{N} \cup \mathcal{N}_1^1 \cup \mathcal{N}_2^1 \cup \ldots \cup \mathcal{N}_m^1.$$

Because the nets $\mathcal{N}_i^1$ are pairwise disjoint by Lemma 5.1.7, we can replace them, one at a time, with their conjugate replacement nets. Since these replacement nets lie in π_2 and they are irreducible with respect to the nets they are replacing, we have

$$\pi_2 = \mathcal{N} \cup \mathcal{N}_1^2 \cup \mathcal{N}_2^2 \cup \ldots \cup \mathcal{N}_m^2$$

and π_2 has been obtained in the manner described.

Assume that π_2 can also be obtained from π_1 in the same manner by replacing nets $\mathcal{M}_i^1$ with nets $\mathcal{M}_i^2$, $i = 1$ to m. Given a pair of points P, Q in π_1 there is then a pair of conjugate nets $\mathcal{N}_i^1, \mathcal{N}_i^2$ and also a pair of conjugate nets $\mathcal{M}_j^1, \mathcal{M}_j^2$ such that P, Q lie on a line in each of these nets. Thus $\mathcal{N}_i^1 \cap \mathcal{M}_j^1 \neq \phi$ and $\mathcal{N}_i^2 \cap \mathcal{M}_j^2 \neq \phi$. Hence, by Lemma 5.1.8, $\mathcal{N}_i^1 = \mathcal{M}_j^1$ and $\mathcal{N}_i^2 = \mathcal{M}_j^2$ because $\mathcal{N}_i^1$ and $\mathcal{N}_i^2$ are irreducible with respect to each other.

Thus the pairs of replacement nets in the two representations can be paired off, implying that $m = m'$. It also follows that the nets consisting of parallel classes common to both planes, coincide.

Therefore the net replacement is unique. $\square$

The method employed in proving Theorem 5.1.9 suggests a possible technique for constructing finite affine planes from a given pair of finite affine planes π_1 and π_2 of the same order.

First embed π_2 in π_1 via a bijection ϕ defined between their point-sets and construct the replacement sets of the type defined in Definition 5.1.2. Then we can represent π_1 and π_2 as

$$\pi_1 = \mathcal{N} \cup \mathcal{N}_1^1 \cup \ldots \cup \mathcal{N}_k^1$$

$$\text{and} \quad \pi_2 = \mathcal{N} \cup \mathcal{N}_1^2 \cup \ldots \cup \mathcal{N}_k^2$$

where $\mathcal{N}$ is the net containing the parallel classes common to both planes and $\mathcal{N}_i^1, \mathcal{N}_i^2$ are irreducible with respect to each other.

In performing the net replacements to obtain π_2 from π_1, we also obtain a set of intermediate affine planes by replacing at least one but not all of the replaceable nets. If $\mathcal{I}$ is the set of subscripts of the nets replaced, then denote the resulting plane by

$$\pi_1^{\mathcal{I}}.$$

The number of such intermediate planes is $2^k - 2$ where k is the number of replacement sets of degree greater than one. If $k = 1$, then the number of intermediate planes is zero; as this case is of no interest, we make the following definition.

Definition 5.1.10. Let π_1 and π_2 be two affine planes of the same order. If there exists at least one embedding of π_2 in π_1 which results in at least two distinct replacement sets (of the type defined in Definition 5.1.2) of degree greater than one, then π_1 and π_2 are said to be *compatible.* If no such embedding exists, then the planes are said to be *incompatible.* □

We shall examine the compatibility of some affine planes in Section 5.3.

5.2. RÉDEI BLOCKING SETS, REPLACEMENT NETS AND SWITCHING SETS

Before considering the compatibility of certain pairs of finite affine planes, we need to examine in more detail the structure of the lines in a replacement net and find lower bounds on the cardinalities of replacement sets. The first result in this direction is

Theorem 5.2.1. ([28]) Let π be a finite affine plane of order n with projective completion π^*. Suppose π contains a replaceable net $\mathcal{N}$. If S is the set of special points corresponding to the parallel classes of $\mathcal{N}$, m is a line in the replacement net and m is not also a line of π, then $m \cup S$ is an $(n, |S|)$-blocking set of π^*. □

We note that the blocking set in Theorem 5.2.1 is not necessarily a Rédei blocking set because it may not be irreducible. (In fact, we can deduce from the next theorem that the blocking set of Theorem 5.2.1 is irreducible if and only if $S = d_m$.)

Theorem 5.2.2. Let π be a finite affine plane of order n with projective completion π^*. Suppose π contains a replaceable net $\mathcal{N}$. If m is a line of this replacement net and m is not also a line of π, then $m \cup d_m$ is a Rédei $(n, |d_m|)$-blocking set of π^*.

Proof. We begin by showing that $m \cup d_m$ is an $(n, |d_m|)$-blocking set.

Let ℓ be an arbitrary line of π^*. ℓ either coincides with ℓ_∞ or it meets ℓ_∞ in a point of d_m or it meets ℓ_∞ in a point not in d_m.

Case (i): If $\ell = \ell_\infty$, then $\ell \cap (m \cup d_m) = d_m$. Since $0 < |d_m| < n+1$, it follows that ℓ meets $m \cup d_m$ in at least one point but does not lie completely in $m \cup d_m$.

Case (ii): If ℓ meets ℓ_∞ in a point of d_m, then trivially $\ell \cap (m \cup d_m)$ is non-empty. If ℓ lies entirely in $m \cup d_m$, then each point of m lies on ℓ which implies that m is a line of π, contradicting the hypothesis. Hence ℓ does not lie completely in $m \cup d_m$.

Case (iii): Suppose ℓ meets ℓ_∞ in a point of $\ell_\infty \backslash d_m$. The number of lines of π^* through $\ell \cap \ell_\infty$ other than ℓ_∞ is n. Assume that $\ell \cap m$ is empty. Then one of the lines through $\ell \cap \ell_\infty$ meets m in at least two points because $|m| = n$. Hence $\ell \cap \ell_\infty$ lies in d_m by the definition of d_m. This contradicts the supposition. Thus $|\ell \cap m| = 1$, so ℓ meets $m \cup d_m$ in at least one point but does not lie completely in $m \cup d_m$.

By Cases (i), (ii), (iii) and the fact that the points of d_m are collinear, it follows that $m \cup d_m$ is an $(n, |d_m|)$-blocking set.

To complete the proof, we show that $m \cup d_m$ is irreducible; by Theorem 1.7.3, it is sufficient to show that each point of $m \cup d_m$ lies on a tangent.

Consider the n points of m. If P is any point of $\ell_\infty \backslash d_m$, then the lines through P other than ℓ_∞ meet m in at most one point (otherwise P would lie in d_m and we would have a contradiction). As $|m| = n$ and there are exactly n lines through P other than ℓ_∞, it is immediate then, that each of these lines meets m in exactly one point. No point of d_m lies on any of these lines. Hence each point of m lies on a tangent to $m \cup d_m$.

Let P be a point of d_m. Then there exists a line through P which meets m in at least two points. There are at most $|m| - 2 = n - 2$ other lines through P meeting m. Since d_m is a subset of ℓ_∞ and ℓ_∞ passes through P, it then follows that the maximum number of lines through P required to cover $m \cup d_m$ is n. This leaves at least one line through P which contains no other point of $m \cup d_m$. Hence P lies on a tangent of

$m \cup d_m$.

It follows that the blocking set is also irreducible and so is a Rédei blocking set.

$\square$

In [32], Bruen and Thas have given a detailed treatment of blocking sets of type (n, k) in finite projective planes of order n. Their main theorem on the structure of such a blocking set in $PG(2, q)$ relies on a now well-known result of Rédei on non-linear functions of $GF(q)$ (see [107]). They proved:

Theorem 5.2.3. Let $\mathcal{B}$ be a blocking set of type (q, k) in $PG(2, q)$. Let $q = p^d$ and assume

$$k < \min\left\{\frac{(q+1)}{2}\ ,\ 1 + \left[\frac{(q-1)}{(p^e+1)}\right]\right\},\quad e < \frac{d}{2}\ .$$

Then

$$\mathcal{B} = \pi_0 \cup \ell$$

where π_0 is a Baer subplane of π and ℓ is a set of $k - (\sqrt{q} + 1)$ collinear points which lie on a line of π_0. $\square$

In addition, Bruen has proved:

Theorem 5.2.4. ([30]) Let $\mathcal{B}$ be a blocking set of type (p, k) in $PG(2, p)$, p prime. Then $k \geq \frac{p+3}{2}$. $\square$

Before proving the main theorem of this section we need to briefly consider the affine Baer subplanes of $AG(2, q^2)$. In the representation of a translation plane of order q^2 and of dimension 2 over its kernel by a spread of $PG(3, q)$, it is well-known that each plane of $PG(4, q)$ which meets $PG(3, q)$ in a single line not belonging to the spread represents an affine Baer subplane of the translation plane (see [19]). (Such a plane is referred to as a transversal plane in [121].) While it is not true in general that these are the only affine baer subplanes of an arbitrary translation plane, it is true in the case of $AG(2, q^2)$, that is:

Theorem 5.2.5. ([121]) If π is a Desarguesian affine plane of order q^2, then the affine Baer subplanes of π are represented precisely by the transversal planes of $PG(4,q)\backslash PG(3,q)$. $\square$

We now have the necessary background to prove:

Theorem 5.2.6. Let $\mathcal{N}$ be a replaceable net in $AG(2,q^2)$, $q = p^d$. If the degree of $\mathcal{N}$ is

$$k < \min\left\{\frac{(q^2+1)}{2},\ 1+\left[\frac{(q^2-1)}{(p^e+1)}\right]\right\},\quad 0 \leq e < d,$$

then the replacement net $\mathcal{N}'$ of $\mathcal{N}$ is a translation net and $(AG(2,q^2)\backslash\mathcal{N}) \cup \mathcal{N}'$ is a translation plane.

Proof. Consider an arbitrary line m of $(AG(2,q^2)\backslash\mathcal{N}) \cup \mathcal{N}'$ which is not also a line of $AG(2,q^2)$. By Theorem 5.2.2, $m \cup d_m$ is a Rédei $(q^2, |d_m|)$-blocking set of $PG(2,q^2)$ (the projective completion of $AG(2,q^2)$.)

Since

$$|d_m| \leq k < \min\left\{\frac{(q^2+1)}{2},\ 1+\left[\frac{(q^2-1)}{p^e+1)}\right]\right\},\quad e < d,$$

it then follows by Theorem 5.2.3 that $m \cup d_m$ consists of the points of a Baer subplane of $PG(2,q^2)$ with $|d_m| - (q+1)$ collinear points lying on a line of the Baer subplane. (Note: In fact $|d_m| = q+1$ but for our argument this is not important.) Hence m consists of the points of an affine Baer subplane of $AG(2,q^2)$. Thus by Theorem 5.2.5, in the representation of $AG(2,q^2)$ by a regular spread of $PG(3,q)$, m is represented by a transversal plane of $PG(4,q)\backslash PG(3,q)$.

Moreover every line of $(AG(2,q^2)\backslash\mathcal{N}) \cup \mathcal{N}'$ which is also a line of $AG(2,q^2)$ is also represented by a plane of $PG(4,q)\backslash PG(3,q)$.

Therefore $(AG(2,q^2)\backslash\mathcal{N}) \cup \mathcal{N}'$ is naturally embeddable in $AG(4,q)$ (in the sense used in Chapter IV). Hence, as mentioned in the discussion after Remark 4.2.9, $(AG(2,q^2)\backslash\mathcal{N}) \cup \mathcal{N}'$ can be represented by a spread $\mathcal{S}$ of $PG(3,q)$.

Letting $\mathcal{S}_k$ denote the regular spread representing $AG(2,q^2)$, it is immediate that $\mathcal{N}$ is represented by the partial spread $\mathcal{S}_k \backslash \mathcal{S}$ and that $\mathcal{N}'$ is represented by the partial spread $\mathcal{S} \backslash \mathcal{S}_k$. These are conjugate partial spreads, that is they constitute a switching set and so $(AG(2,q^2)\backslash\mathcal{N}) \cup \mathcal{N}'$ is a translation plane. $\square$

Corollary 5.2.7. Let $\mathcal{N}$ be a replaceable net in $AG(2,p^2)$, p prime, which is irreducible with respect to its replacement net $\mathcal{N}'$. If the degree d of $\mathcal{N}$ satisfies the inequality $p+1 < d < \frac{p^2+1}{2}$, then $d \geq \frac{(p+1)(p+3)}{4}$.

Proof. By Theorem 5.2.6, $\mathcal{N}'$ is a translation net because the degree of $\mathcal{N}$ is less than $\frac{p^2+1}{2}$. Hence the net replacement is equivalent to switching two conjugate partial spreads of $PG(3,q)$.

Now the partial spread $\mathcal{S}$ (lying in a regular spread) which represents $\mathcal{N}$ is not a regulus because $|\mathcal{S}| = d < p+1$. Moreover $\mathcal{N}$ is irreducible with respect to $\mathcal{N}'$. Therefore $\mathcal{S}$ contains at least two distinct reguli which meet in at least one line. Hence by Theorem 1.8.16, $\mathcal{S}$ has at least $\frac{(p+1)(p+3)}{4}$ lines and so

$$d \geq \frac{(p+1)(p+3)}{4}.$$

$\square$

5.3. ON THE COMPATIBILITY OF FINITE AFFINE PLANES

In general, given a pair of finite affine planes of the same order, it is not a simple task to determine whether or not they are compatible because of the large number of possible embeddings of the second plane in the first. Therefore, unless otherwise stated, our attention will be restricted to the case in which one of the planes is a Desarguesian affine plane.

In the simplest case where the planes have prime order p, they are always incompatible. This can be seen as follows: Let π_1 be the Desarguesian affine plane and let

π_2 be embedded in π_1. By Theorem 5.1.9, π_2 can be obtained from π_1 by replacing pairwise disjoint nets which are irreducible with respect to their replacement nets. If this amounts to replacing the net π_1 with π_2, then the planes are incompatible with respect to the embedding. Otherwise there exists a line m of π_2 which is not a line of π_1.

In this case, by Theorem 5.2.2, $m \cup d_m$ is a $(p, |d_m|)$-Rédei blocking set of π_1^* (where π_1^* is the projective completion of π_1). By Theorem 5.2.4, we have that $|d_m| \geq \frac{p+3}{2}$. Hence each replaceable net in π_1 has degree at least $\frac{p+3}{2}$ and so π_1 can have at most one replaceable net. Hence the planes are again incompatible with respect to the embedding. As these two cases exhaust the possibilities, it follows that the planes are always incompatible. Note: This result is effectively equivalent to a result obtained by Bruen in [30] (see Theorem 7 therein) although the terminology and the setting differ.

The next simplest case involves affine planes of order p^2 with p a prime. It is well-known that every affine plane of order 4 is isomorphic to the Desarguesian affine plane of that order.

However, for each $p \geq 3$, there exist non-Desarguesian affine planes of order p^2. Thus the first two cases of interest are those for which p is equal to 3 or 5, and it is these cases which we shall focus our attention on in the rest of the section.

In 1989, Clement Lam *et al* verified by computer that there are, up to projective equivalence, exactly four projective planes of order 9. These are the Desarguesian plane, the Hall plane, the Hughes plane and the dual Hall plane. Hence assuming the validity of this result, all affine planes of order 9 are also known. However, we include our results on affine planes of order 9 in the hope that they may provide a suitable setting for an alternative proof cum verification of Lam's findings. These results are based on the following:

Theorem 5.3.1. ([83]) Let π be a finite projective plane of order 9 which possesses at least one Baer subplane. Then π is one of the four known planes. $\square$

Lemma 5.3.2. Let π be a finite affine plane of order 9. If π possesses a replaceable net of degree 4 or 5, then it is a known plane.

Proof. Let $\mathcal{N}$ be the replaceable net and m be a line of the replacement net $\mathcal{N}'$ such that m is not also a line of π. Then, by Theorem 5.2.2, $m \cup d_m$ is a Rédei blocking set of π^* (the projective completion of π).

By the restriction on the degree of $\mathcal{N}$, it follows that $m \cup d_m$ contains either 13 or 14 points. Therefore, by Remark 1.7.7, 13 of the points of $m \cup d_m$ are the points of a Baer subplane of π^*. It is immediate then, by Theorem 5.3.1, that π^* is one of the four known projective planes of order 9 and so π itself is a known plane. $\square$

Theorem 5.3.3. Let π_1 and π_2 be two compatible finite affine planes of order 9. If π_1 is a known plane, then π_2 and all intermediate affine planes are also known planes.

Proof. Since π_1 and π_2 are compatible, they can be decomposed as indicated below:

$$\pi_1 = \mathcal{N} \cup \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_n$$
$$\pi_2 = \mathcal{N} \cup \mathcal{N}_1' \cup \ldots \cup \mathcal{N}_n' ,$$

where $\mathcal{N}_i$ and $\mathcal{N}_i'$ are irreducible conjugate replacement nets and n is at least 2. By Theorem 1.7.4, a Rédei blocking set in a finite projective plane of order 9 is of type $(9, k)$ with $k \geq 4$. Hence, by the theory developed in the previous sections, each of the nets $\mathcal{N}_i$ (and $\mathcal{N}_i'$) has degree at least 4. Furthermore, since n is at least 2, each net $\mathcal{N}_i$ (and $\mathcal{N}_i'$) has degree at most 6. Thus there are essentially four equivalent cases to consider. These are as listed below:

(i) $|\mathcal{N}_1| = 4, \quad |\mathcal{N}_2| = 4$

(ii) $|\mathcal{N}_1| = 4, \quad |\mathcal{N}_2| = 5$

(iii) $|\mathcal{N}_1| = 4, \quad |\mathcal{N}_2| = 6$

(iv) $|\mathcal{N}_1| = 5, \quad |\mathcal{N}_2| = 5.$

In each case, the affine plane π_2 and the intermediate planes $\pi_1^{\{1\}}$ and $\pi_1^{\{2\}}$ all possess replaceable nets of degree 4 or 5. Hence, by Lemma 5.3.2, they are all known planes.

□

Corollary 5.3.4. Let π be a finite affine plane of order 9. If π is not a known plane, then π is incompatible with each of the known planes.

Proof. If π were compatible with at least one known plane, then by Theorem 5.3.3, it would itself be a known plane. As this is not the case, π is incompatible with every known plane. □

We can deduce from Corollary 5.3.4 and Theorem 5.1.9, that if an as yet undiscovered affine plane of order 9 exists, then it is obtainable from each known affine plane π of order 9 by replacing a single net of degree k in π where k satisfies

$$6 \leq k \leq 8.$$

Thus in order to prove or verify Lam's findings, it may be useful to examine the irreducible blocking sets of type $(9, k)$, $6 \leq k \leq 8$ in the known planes of order 9, noting in particular how they intersect.

When p is equal to five, unlike the previous case, not all affine planes of order 25 are known. However, all projectively distinct spreads of $PG(3, 5)$ have been determined (see [97], [6] and [124]). It is reported in [6] and [124] that there are 24 projectively distinct spreads of $PG(3, 5)$, 13 of which possess reguli; however, recent results of C. Charnes suggest that there are at most 21 projectively distinct spreads. Consequently, there are at most 21 non-isomorphic translation planes of order 25.

Consider now $\pi_1 = AG(2, 25)$. By Theorem 5.2.6 and Corollary 5.2.7, any replaceable net in $AG(2, 25)$ which is not a derivable net has degree at least 12. Assume that the affine plane π_2 is compatible with π_1. Using the bounds on the degrees of the replaceable nets in π_1, it follows that π_2 can be obtained from π_1 by replacing at most four pairwise disjoint nets (which are irreducible with respect to their replacement nets) and that the degrees of the nets fall into one of the fourteen possibilities listed below:

(i)	12,6	(viii)	13,13
(ii)	12,6,6	(ix)	14,6
(iii)	12,12	(x)	14,6,6
(iv)	12,13	(xi)	$r, 6$ with $15 \leq r \leq 20$
(v)	12,14	(xii)	6,6,6,6
(vi)	13,6	(xiii)	6,6,6
(vii)	13,6,6	(xiv)	6,6.

In all, there are 66 different intermediate planes to consider. However, many of these arise by replacing translation nets in $AG(2,25)$. Thus the resulting plane in each such instance is a translation plane and so is already known. In addition, a number of intermediate planes can be constructed from π_2 via (multiple) derivation. For example (using the notation established in Section 5.1) in case (vi) where

$$\begin{aligned} AG(2,25) = \pi_1 &= \mathcal{N} \cup \mathcal{N}_1 \cup \mathcal{N}_2 \\ \pi_2 &= \mathcal{N} \cup \mathcal{N}_1' \cup \mathcal{N}_2' \end{aligned}$$

$$|\mathcal{N}_1| = 13 \text{ and } |\mathcal{N}_2| = 6$$

The intermediate plane $\pi_1^{\{1\}} = \mathcal{N} \cup \mathcal{N}_1' \cup \mathcal{N}_2$ can be derived from π_2 by replacing $\mathcal{N}_2'$. Since π_2 is itself a known plane we shall assume that the planes which can be (multiply) derived from π_2 are also known.

Hence, with this understanding, the number of remaining possibilities is reduced to just three. More precisely, we have:

Theorem 5.3.5. Let $\pi_1 = AG(2,25)$ and π_2 be a second affine of order 25 which is compatible with π_1. If at least one of the intermediate planes constructed from π_1 and π_2 is a previously unknown plane, then one of the three situations below occurs:

$$
\begin{array}{llll}
\text{(i)} & \pi_1 = \mathcal{N} \cup \mathcal{N}_1 \cup \mathcal{N}_2, & |\mathcal{N}_1| = 12,\ |\mathcal{N}_2| = 13, & \pi_1^{\{2\}} \text{ is new,} \\
\text{(ii)} & \pi_1 = \mathcal{N}_1 \cup \mathcal{N}_2, & |\mathcal{N}_1| = 12,\ |\mathcal{N}_2| = 14, & \pi_1^{\{2\}} \text{ is new,} \\
\text{(iii)} & \pi_1 = \mathcal{N}_1 \cup \mathcal{N}_2, & |\mathcal{N}_1| = 13 = |\mathcal{N}_2|, & \pi_1^{\{1\}} \text{ and } \pi_2^{\{2\}} \text{ are new.}
\end{array}
$$

□

We have that in each case, a net of degree 13 or 14 is replaced. Thus at least half of the parallel classes in $AG(2, 25)$ need to be replaced in order to construct a new affine plane from $AG(2, 25)$ and a second affine plane π_2 of order 25. It is therefore, unlikely that any new planes will arise from this technique.

A similar type of analysis can be performed on $AG(2, p^2)$ for each $p \geq 7$. In these cases however, not all translation planes of order p^2 are known. In spite of this we can still conclude that a new plane is not likely to be a non-translation plane because by Theorem 5.2.6 it would be necessary to replace at least half of the parallel classes in $AG(2, p^2)$.

In the case of $AG(2, p^n)$ with $n \geq 3$, it becomes more and more difficult as n increases to state any general results although on the basis of the results in the simplest cases we might be led to conjecture that a non-translation plane of order p^n is incompatible with $AG(2, p^n)$.

5.4. ON BLOCKING SETS IN AFFINE AND PROJECTIVE PLANES

To conclude the chapter, we shall show how Rédei blocking sets in a finite projective plane can be used to construct blocking sets in a finite plane embedded in the projective plane. In particular we construct an infinite class of blocking sets of cardinality $2q - 1$ in $AG(2, q)$. This class is of interest not only because it appears to be new but also because $2q - 1$ is the smallest admissible value for the cardinality of a blocking set in $AG(2, q)$.

However, to begin with, let π be an arbitrary finite projective plane of order n. Assume that π contains a Rédei (n, k)-blocking set $\mathcal{B}$ with $(n+1-k) \geq 4$. Also assume that one of the points P on the k-line ℓ of $\mathcal{B}$ lies on at least two lines tangent to $\mathcal{B}$.

Let m be the set of points of $\mathcal{B}\backslash\ell$ and d_m be the set of points of $\ell \cap \mathcal{B}$. (Note: This notation is in keeping with the notation established in Section 5.1. Treating m as a set of n affine points with respect to the "line at infinity" ℓ, $d_m = \ell \cap \mathcal{B}$ is exactly the set of special points corresponding to m.)

Consider the lines of π through a point R of $\ell\backslash d_m$. Apart from ℓ, each line meets m in at least one point. In fact the number of lines other than ℓ is equal to the number of points of m. Hence each line through R other than ℓ meets m in exactly one point. It follows from the argument that the lines through a point Q of m which meet ℓ in a point of $\ell\backslash d_m$, are all tangent to $\mathcal{B}$ at Q. By construction the number of such lines is $(n+1-k)$. Label these lines arbitrarily as $\ell_\infty, \ell_1, \ell_2, \ldots, \ell_{n-k}$.

Let t_1 and t_2 be two tangents to $\mathcal{B}$ at the point P of d_m. Define $(n-k)$ points P_i as follows:

$$P_1 = t_1 \cap \ell_1,$$
$$P_2 = t_1 \cap \ell_2,$$
$$P_i = t_2 \cap \ell_i, \quad i = 3 \text{ to } (n-k).$$

Finally let $\mathcal{B}'$ be the set of points

$$(\mathcal{B}\backslash\{Q\}) \cup \{P_i\}_{i=1}^{n-k},$$

and let π' be the affine plane $\pi\backslash\ell_\infty$.

Theorem 5.4.1. $\mathcal{B}'$ is a blocking $(2n-1)$-set in π' and $\mathcal{B}'$ contains no line of π'.

Proof. Let $\ell' \neq \ell$ be an arbitrary line of π. Assume ℓ' meets m in at least $k+1$ points. Since m is not a line of π there is a point of m not on ℓ'. Hence the lines through this point and the points of $m \cap \ell'$ define at least $k+1$ distinct points of d_m. However $|d_m| = k$. Thus the assumption is wrong and each line of π meets m in at most k points. It is immediate then that each line of π' also meets m in at most k points.

Also by construction at most $(n-k)-2$ of the points P_i, $i = 1$ to $n-k$ are collinear.

Let ℓ'' be an arbitrary line of π'. If $\ell'' = \ell \backslash (\ell \cap \ell_\infty)$, then ℓ'' meets $\mathcal{B}'$ in exactly the k points of d_m and so ℓ'' does not lie in $\mathcal{B}'$. If $\ell'' \neq \ell \backslash (\ell \cap \ell_\infty)$, then ℓ'' meets d_m in at most one point. By the first paragraph, ℓ'' meets m in at most k points and by the second paragraph it meets the set $\left(P_i\right)_{i=1}^{n+k}$ in at most $(n+k)-2$ points. Thus ℓ'' possesses at most $1 + k + (n+k) - 2 = n-1$ points of $\mathcal{B}'$. Thus ℓ'' does not lie in $\mathcal{B}'$.

To complete the proof, we need to show that each line ℓ'' of π' meets $\mathcal{B}'$ in at least one point. Let ℓ''_π be the line of π containing ℓ''. If ℓ''_π does not contain the point Q, then $\ell''_\pi \cap \mathcal{B} \backslash \{Q\}$ is non-empty. Hence ℓ'' meets $\mathcal{B}'$ in a point of $\mathcal{B} \backslash \{Q\}$. If ℓ''_π meets $\mathcal{B}$ in Q, then ℓ'' meets $\mathcal{B}'$ in one of the points P_i or in a point of d_m. Hence each line of π' meets $\mathcal{B}'$ in at least one point.

Combining the two results, we have that $\mathcal{B}'$ is a blocking set of π'.

Finally,

$$
\begin{aligned}
|\mathcal{B}'| &= (n+k-1) + (n-k) \\
&= 2n - 1.
\end{aligned}
$$

□

In [25], Brouwer and Wilbrink describe a technique for constructing Rédei blocking sets in projective planes arising from spreads of $PG(2t-1, q)$. One such blocking set is constructed by taking a t-dimensional subspace Σ_t of $PG(2t, q)$ which is not contained in $PG(2t-1, q)$ and which meets one of the elements of the spread in a $(t-2)$-dimensional subspace. Every other element of the spread meets Σ_t in at most one point. Thus Σ_t meets exactly $q^{t-1}+1$ elements of the spread. Σ_t also has exactly q^t points in common with $PG(2t, q) \backslash PG(2t-1, q)$, and so Σ_t gives rise to a Rédei $(q^t, q^{t-1}+1)$-blocking set $\mathcal{B}$ in the projective plane arising from the spread. Furthermore it can be shown that $\mathcal{B}$ has at least two $(q^{t-1}+1)$-lines and each line of the plane meets $\mathcal{B}$ in 1, $q+1$ or $q^{t-1}+1$ points.

Now because $\mathcal{B}$ is a Rédei $(q^t, q^{t-1}+1)$-blocking set and $\mathcal{B}$ has at least two $(q^{t-1}+1)$-lines, these two lines will meet in a point P say. Apart from the points on these lines, there are $(q^t - q^{t-1})$ other points of $\mathcal{B}$. Hence the total number of secants of $\mathcal{B}$ at P is at most $2 + (q^t - q^{t-1})$ which is strictly less than $q^t - 1$. Thus there are at least two lines through P tangent to $\mathcal{B}$. Furthermore $(q^t+1)-(q^{t-1}+1) = q^t - q^{t-1} \geq 4$ if $q \geq 3$. Thus for $q \geq 3$, these blocking sets satisfy the conditions stated in the construction at the beginning of the section.

Thus the blocking sets in the projective plane give rise to blocking $(2q^t-1)$-sets in particular affine planes embedded in the projective planes.

As a simple corollary we have:

Corollary 5.4.2. $AG(2, q^t)$, $q \geq 3$ contains blocking $(2q^t-1)$-sets.

Proof. Let $PG(2, q)$, $q \geq 3$ be constructed from a regular spread of $PG(2t-1, q)$. Then by the construction of Brouwer and Wilbrink, $PG(2, q^t)$ contains a Rédei $(q^t, q^{t-1}+1)$-blocking set. Thus as explained above, this gives rise to blocking $(2q^t-1)$-sets of certain of the affine planes of order q^t embedded in $PG(2, q^t)$. All such affine planes are isomorphic to $AG(2, q^t)$ and so the result follows. □

In [76] and [24], it is shown that a blocking set in an affine plane $AG(2, q^t)$ has at least $(2q^t-1)$ points. Hence our examples attain the minimum cardinality. Other examples of blocking $(2q^t-1)$-sets in $AG(2, q^t)$ are described in [12], but none of these has lines of size $q^{t-1}+1$. Thus the class constructed here appears to be new.

As a final remark, we note that while a blocking set in a finite affine plane may contain a line of the plane, the blocking sets constructed here (as shown in Theorem 5.4.1) and in [12] do not contain lines of $AG(2, q^t)$.

CHAPTER VI
CONCLUSION

As we mentioned at the outset, our main aim in this thesis has been to develop the theory of n-covers of $PG(3,q)$ in parallel with the existing theory of spreads of $PG(3,q)$. Thus, in seeking generalisations of the theory of n-covers of $PG(3,q)$ and ideas for further research, it is natural to once again use the theory of spreads as a model.

One obvious way then, in which the theory may be generalised is to define n-covers of higher dimensional projective spaces. An (n,s)-cover of $PG(t,q)$ may be defined as a set of s-dimensional projective subspaces of $PG(t,q)$ which satisfies the property that each point of $PG(t,q)$ is incident with exactly n of the subspaces. (We have in fact implicitly used the notion of $(n,1)$-covers of $PG(4t-1,q)$ in Chapter IV, when we used them to construct quasi-n-multiple Sperner designs.) Many of the results established in this thesis which pertain to n-covers of $PG(3,q)$ would then have direct analogues in this generalised setting.

Another important aspect of the theory of n-covers of $PG(3,q)$ which needs further investigation is that of their construction. The techniques we have described in this thesis, although they are constructive in nature, generally only imply the existence of a proper n-cover. One way of potentially constructing new n-covers is to alter the structures of the existing ones. This may be achieved by adapting the process of switching conjugate partial spreads which was introduced in [20] by Bruck and Bose. More precisely, given an n-cover C_n we can try to find a partial spread $\mathcal{S}$ lying in C_n such that $\mathcal{S}$ has a conjugate partial spread $\mathcal{S}'$. If $\mathcal{S}' \cap C_n = \phi$, then we can replace $\mathcal{S}'$ by $\mathcal{S}$ to form the new n-cover $(C_n \backslash \mathcal{S}) \cup \mathcal{S}'$. Another possible way of finding new n-covers is to consider the images of existing ones under correlations of $PG(3,q)$; by

Theorem 2.1.12, these images will again be n-covers of $PG(3,q)$.

A related matter is that of the classification of the n-covers of $PG(3,q)$. This is certainly a non-trivial problem as it encompasses the classification of the spreads of $PG(3,q)$ which itself is a major task. However, for small values of q and n, the techniques applied in Chapter III may be sufficient. As an aid in classifying the n-covers of $PG(3,q)$, it would also be useful to know the largest integer $b(q)$ for which $PG(3,q)$ possesses a proper $b(q)$-cover. It is not entirely clear, however, if the value of $b(q)$ can be found for a given value of q without first accomplishing the classification.

We now briefly turn to the other material appearing in this thesis.

In Chapter III, we devised a method for determining whether or not a given regular partial packing of $PG(3,q)$ can be extended, by establishing a relationship between these partial packings and blocking sets of $PG(2,q^2)$. This method could be generalised by using the analogous representation of a regular spread of $PG(2t+1,q)$, $t>1$ via a pair of skew conjugate lines of $PG(2t+1,q^{t+1})$; this representation was established by O'Keefe in [98].

In Section 4.3, we constructed two infinite classes of Hall triple systems from the Burnside groups $B(3,r)$, $r \geq 2$. However, we have been unable to ascertain the exact nature of the second class of these systems for $r \geq 3$ because we have no way of proving or disproving that the identity stated in Theorem 4.3.10, is valid in $B(3,r)$. To settle this problem, a more convenient way of representing the Burnside groups $B(3,r)$, $r \geq 3$, needs to be developed because it is generally difficult and time consuming to show that two of their elements with different strings either coincide or are in fact distinct. As a final comment regarding this problem, we note that if the identity does not hold in $B(3,3)$, then the Hall triple system S_3 is not a finite affine space by Theorem 4.3.10 and so neither is S_r for each $r > 3$ because each such S_r contains copies of S_3.

Finally in Chapter V, we discussed the compatibility of finite affine planes. As we have already mentioned, the classification of the affine planes of order 9 follows from classification of the projective planes of order 9 achieved by Lam *et al.* To gain proper insight into the case where the affine planes are of order 25, there are two avenues of investigation which may be taken. One involves actually examining various embeddings of one affine plane into another to try to find an embedding with respect to which the planes are compatible. The other, which may be more profitable and less time consuming, is simply to examine blocking sets of type $(25, k)$ in the corresponding projective plane. In either case, the observations made might perhaps suggest results which could then be proven.

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