



# The Bose Representation of $PG(2, q^2)$ and some Associated Varieties

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# Abstract

In, *On a representation of the Baer subplanes of the Desarguesian plane  $PG(2, q^2)$  in a projective five dimensional space  $PG(5, q)$* , [8], R.C. Bose discusses a representation of  $PG(2, q^2)$  in  $PG(5, q)$  whereby the points are represented by a 1-spread of  $PG(5, q)$ . We refer to this as the Bose representation of  $PG(2, q^2)$ . As we shall see, this representation is closely connected to the representation of  $PG(2, q^2)$  in  $PG(4, q)$  introduced by André in [1] and developed by Bruck and Bose in [13] and [14] and which we term the Bruck-Bose representation of  $PG(2, q^2)$ .

In this thesis we discuss the Bose representation in more detail than is done in [8], use it to prove some already known results concerning the Bruck-Bose representation and study some varieties which represent certain substructures of  $PG(2, q^2)$  under these representations.

In particular, we find the Bose representation of Baer subplanes, Baer sublines classical unitals and of a general curve of  $PG(2, q^2)$ . In the first case we see that the structure that Bose described as representing a Baer subplane of  $PG(2, q^2)$  is in fact a Segre variety, known as a cubic scroll of planes. From this it follows naturally that, under the Bruck-Bose representation, a classical unital is represented by either a ruled cubic surface or a plane of  $PG(4, q)$ . Furthermore, we study the ruled cubic surface in detail by considering it as the projection of the Veronese surface in  $PG(5, q)$ .

Lastly, motivated by Lunardon's paper [36], we look at the image of the 1-spread of  $PG(5, q)$  which is the Bose representation of the points of  $PG(2, q^2)$ , on the Grassmann variety in  $PG(14, q)$  which represents the lines of  $PG(5, q)$ . This gives us a representation of  $PG(2, q^2)$  on this variety and we look briefly at this representation, determining the representation of Baer subplanes and classical unitals.