The Bose Representation of $PG(2, q^2)$ and some Associated Varieties

by

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Abstract

In, *On a representation of the Baer subplanes of the Desarguesian plane* $PG(2, q^2)$ *in a projective five dimensional space* $PG(5, q)$, [8], R.C. Bose discusses a representation of $PG(2, q^2)$ in $PG(5, q)$ whereby the points are represented by a 1-spread of $PG(5, q)$. We refer to this as the Bose representation of $PG(2, q^2)$. As we shall see, this representation is closely connected to the representation of $PG(2, q^2)$ in $PG(4, q)$ introduced by André in [1] and developed by Bruck and Bose in [13] and [14] and which we term the Bruck-Bose representation of $PG(2, q^2)$.

In this thesis we discuss the Bose representation in more detail than is done in [8], use it to prove some already known results concerning the Bruck-Bose representation and study some varieties which represent certain substructures of $PG(2, q^2)$ under these representations.

In particular, we find the Bose representation of Baer subplanes, Baer sublines classical unitals and of a general curve of $PG(2, q^2)$. In the first case we see that the structure that Bose described as representing a Baer subplane of $PG(2, q^2)$ is in fact a Segre variety, known as a cubic scroll of planes. From this it follows naturally that, under the Bruck-Bose representation, a classical unital is represented by either a ruled cubic surface or a plane of $PG(4, q)$. Furthermore, we study the ruled cubic surface in detail by considering it as the projection of the Veronese surface in $PG(5, q)$.

Lastly, motivated by Lunardon's paper [36], we look at the image of the 1-spread of $PG(5, q)$ which is the Bose representation of the points of $PG(2, q^2)$, on the Grassmann variety in $PG(14, q)$ which represents the lines of $PG(5, q)$. This gives us a representation of $PG(2, q^2)$ on this variety and we look briefly at this representation, determining the representation of Baer subplanes and classical unitals.

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Chapter 1

Introduction

In this chapter we discuss the background information needed for the rest of the work. As we will be largely concerned with the study of structures in the projective space $PG(n,q)$, we begin by introducing this and the terminology and notation we shall use throughout. Following this we define varieties in $PG(n,q)$ and look at some varieties which will be relevant in subsequent sections. The material contained in sections 1.1 and 1.2 is standard, for more information see [29], [33] and [61].

1.1 Finite Fields

We let $GF(q)$, where $q = p^h$ is a power of a prime number, denote the finite, or Galois, field of q elements. For r a positive integer, the field $GF(q^r)$ is an r-dimensional vector space over $GF(q)$ and we say that $GF(q^r)$ is an **extension** of $GF(q)$. The fact that $GF(q^2)$ is a two dimensional vector space over $GF(q)$ is used often in this work and, in particular, the following representation of $GF(q^2)$.

Let $\xi \in GF(q)$ be such that the quadratic equation

$$f(x) = x^2 + x + \xi$$

is irreducible over $GF(q)$. If we consider $GF(q)$ as a subfield of $GF(q^2)$, then f is reducible over $GF(q^2)$ and if we denote one root of this equation by i then $\{1, i\}$ is a basis for $GF(q^2)$, considered as a two-dimensional vector space over $GF(q)$. Thus we can write $GF(q^2)$ as

$$\{a + ib : a, b \in GF(q)\}$$

If ϕ denotes the automorphism (called the Fröbenius automorphism)

$$\begin{aligned} \phi : \ GF(q^2) &\rightarrow GF(q^2) \\ x &\mapsto x^q \end{aligned}$$

of $GF(q^2)$, fixing $GF(q)$, then the second root of equation 1.1 is $\phi(i) = i^q$, which we write as $\bar{i}$ and call the **conjugate of** i. Similarly, if a is an arbitrary element of $GF(q^2)$ then we denote the image of a under ϕ by $\bar{a}$ and call it the **conjugate** of a.

1.1.1 Polynomials over Finite Fields

Let $\gamma[x_0, x_1, \ldots, x_n]$ denote the ring of polynomials in $x_0, x_1, \ldots, x_n$ over $\gamma = GF(q)$ and $f \in \gamma[x_0, x_1, \ldots, x_n]$ be a homogeneous polynomial in $x_0, x_1, \ldots, x_n$. As f is homogeneous it can be written in the form

$$f = \sum_{i_0+\ldots+i_n=n} a_{i_0\ldots i_n} x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}$$

where $a_{i_0\ldots i_n} \in GF(q)$. The **partial derivative** of f with respect to x_j, denoted $\frac{\partial f}{\partial x_j}$, is defined to be the polynomial

$$\frac{\partial f}{\partial x_j} = \sum_{i_0+\ldots+i_n=n} i_j a_{i_0\ldots i_n} x_0^{i_0} x_1^{i_1} \ldots x_j^{i_j-1} x_n^{i_n}$$

This satisfies the usual properties of derivatives defined by a limiting process except that, as we are working in a field of non-zero characteristic, it should be noted that the partial derivative of f with respect to x_i being zero does not imply that f has no terms containing an x_i. For example if $f = x_0^q x_1^{n-q}$ then

$$\begin{aligned} \frac{\partial f}{\partial x_0} &= q x_0^{q-1} x_1^{n-q} \\ &= 0. \end{aligned}$$

We then define the k-th order partial derivative of f with respect to x_i, $\frac{\partial^k f}{\partial x_i^k}$, inductively by

$$\frac{\partial^k f}{\partial x_i^k} = \frac{\partial}{\partial x_i}\left(\frac{\partial^{k-1} f}{\partial x_i^{k-1}}\right)$$

Similarly we define the second order mixed partial derivative of f with respect to x_i and x_j by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

and the higher order mixed partials are defined appropriately. We shall need the following result.

Theorem 1.1. [61],[32] *(Taylor's theorem) If f is a homogeneous polynomial of degree n in $\gamma[x_0, \ldots, x_n]$ then*

$$f(z_0 + \lambda y_0, \ldots, z_n + \lambda y_n) = \sum_{s=0}^{n} \frac{\lambda^s}{s!} \left(y_0 \frac{\partial}{\partial x_0} + \ldots + y_n \frac{\partial}{\partial x_n} \right)^s f(z_0, \ldots, z_n)$$

where

$$\left(y_0 \frac{\partial}{\partial x_0} + \ldots + y_n \frac{\partial}{\partial x_n} \right)^s f(z_0, \ldots, z_n) = \sum_{i_0+\ldots+i_n=s} y_0^{i_0} \ldots y_n^{i_n} \frac{\partial^s f(z_0, \ldots, z_n)}{\partial x_0^{i_0} \ldots \partial x_n^{i_n}}$$

□

To properly interpret this result the terms

$$\frac{\lambda^n}{s!} \left(y_0 \frac{\partial}{\partial x_0} + \ldots + y_n \frac{\partial}{\partial x_n} \right)^s$$

must be treated as formal expressions. For example, if $f(x_0, x_1) = x_0^2 + x_0 x_1$ then by theorem 1.1

$$\begin{aligned} f(x_0 + \lambda y_0, x_1 + \lambda y_1) &= x_0^2 + x_0 x_1 + \lambda\{y_0(2x_0 + x_1) + y_1 x_0\} + \frac{\lambda^2}{2!}(2y_0^2 + 2y_0 y_1) \\ &= x_0^2 + x_0 x_1 + \lambda(2x_0 y_0 + y_0 x_1 + y_1 x_0) + \lambda^2(y_0^2 + y_0 y_1) \end{aligned}$$

regardless of whether the characteristic of $GF(q)$ is two or not. Noting this, the proof given in Hodge and Pedoe ([32, chapter three]), for fields without characteristic, is valid when considered over a finite field.

1.2 The Projective Space $PG(n, q)$

Consider $V = GF(q^{n+1})$ as an $(n+1)$-dimensional vector space over $GF(q)$. Choose a basis for V so that if $X \in V$ we can write $X = (x_0, \ldots, x_n)$, where $x_i \in GF(q)$. Define a relation on the elements of $V \setminus \{\mathbf{0}\}$, where $\mathbf{0} = (0, \ldots, 0)$, by saying that $X = (x_0, \ldots, x_n)$ and $Y = (y_0, \ldots, y_n)$ are related if and only if $x_i = \rho y_i$ for some $\rho \in GF(q) \setminus \{0\}$. This relation is an equivalence relation, with the equivalence classes being the one-dimensional subspaces of V with $\mathbf{0}$ deleted. We define the **n-dimensional projective space $PG(n, q)$** as the set of all these equivalence classes. The elements of $PG(n, q)$, that is the one-dimensional subspaces of V with $\mathbf{0}$ deleted, are called the **points** of V. We shall identify the coordinates of the point P of $PG(n, q)$ with the coordinate vector $(x_0, \ldots, x_n)$ of any element of V in the one-dimensional subspace of

V corresponding to P. Thus $(x_0, \ldots, x_n)$ and $\rho(x_0, \ldots, x_n)$, for $\rho \in GF(q) \setminus \{0\}$ are coordinates for the same point of $PG(n, q)$.

A **subspace of dimension m**, or **m-space**, of $PG(n, q)$ is a set of points of $PG(n, q)$ for which the corresponding elements of V, together with $\mathbf{0}$, form a subspace of V of dimension $(m + 1)$. A subspace of dimension 1 is called a **line**, a subspace of dimension two is called a **plane** and a subspace of dimension $(n - 1)$ is called a **hyperplane** or, sometimes, a **prime**. A subspace of $PG(n, q)$ of dimension m is naturally isomorphic to $PG(m, q)$. If f is a homogeneous linear polynomial in the variables $x_0, x_1, \ldots, x_n$, with coefficients in $GF(q)$, then the set of points P of $PG(n, q)$ whose coordinates satisfy the equation $f = 0$ form a hyperplane Σ of $PG(n, q)$; we say that Σ and the point Q of $PG(n, q)$ with coordinates $(y_0, \ldots, y_n)$ are **incident** if $f(y_0, \ldots, y_n) = 0$. Any subspace Σ_d of $PG(n, q)$ of dimension d is the intersection of $(n - d)$ hyperplanes $\pi_1, \ldots, \pi_{n-d}$ whose defining equations are linearly independent over $GF(q)$. The point Q of $PG(n, q)$ is incident with Σ_d if it is incident with the hyperplanes $\pi_1, \ldots, \pi_{n-d}$.

Let Σ_r and Σ_s be subspaces of $PG(n, q)$ of dimensions r and s respectively. The subspaces of V corresponding to Σ_r and Σ_s span a subspace of V, corresponding to which there is a subspace Σ_t of $PG(n, q)$. We call Σ_t the **span** of Σ_r and Σ_s and write $\Sigma_t = \langle \Sigma_r, \Sigma_s \rangle$. Similarly, the set of points of $PG(n, q)$ in common to both Σ_r and Σ_s is a subspace of $PG(n, q)$, called the **intersection** of Σ_r and Σ_s, and denoted $\Sigma_r \cap \Sigma_s$. Since the span and intersection of Σ_r and Σ_s correspond to subspaces of a vector space they satisfy Grassmann's Identity;

$$dim\ \Sigma_r + dim\ \Sigma_s = dim(\Sigma_r \cap \Sigma_s) + dim\ \langle \Sigma_r, \Sigma_s \rangle$$

where $dim\ \Sigma_r$ denotes the dimension of the subspace Σ_r.

1.2.1 Collineations

A **collineation** σ of $PG(n, q)$ is a bijection

$$\sigma : \ PG(n, q) \ \rightarrow \ PG(n, q)$$

on the set of points of $PG(n, q)$ which preserves incidence; that is, if the point P is incident with Σ_d, a subspace of dimension d of $PG(n, q)$, then Σ_d^σ is a d-dimensional subspace containing P^σ. The fundamental theorem of projective geometry states that if

σ is a collineation of $PG(n,q)$ then σ acts as follows:

$$\begin{array}{rccl}\sigma: & PG(n,q) & \rightarrow & PG(n,q) \\ & X & \mapsto & X^{\phi}A\end{array}$$

where $\mathbf{A}$ is a non-singular $(n+1)\times(n+1)$ matrix over $GF(q)$ and ϕ is an automorphism of $GF(q)$. The group of all collineations of $PG(n,q)$ is denoted $P\Gamma L(n+1,q)$. A collineation of $PG(n,q)$ for which ϕ is the identity map is called a **projectivity** of $PG(n,q)$ and the group of all projectivities of $PG(n,q)$ is denoted $PGL(n+1,q)$.

If $A_0, A_1, \ldots, A_{n+1}$ are $n+2$ points of $PG(n,q)$ no $n+1$ of which lie in a hyperplane then there is a unique projectivity mapping them to any other given set of $n+2$ points of $PG(n,q)$, no $n+1$ in a hyperplane. Since the $n+2$ points of $PG(n,q)$ which have the coordinates, $(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1), (1,1,\ldots,1)$ satisfy this condition, and can be mapped, by an element of $PGL(n+1,q)$, to any other set of $n+2$ points, no $n+1$ in a hyperplane, we are free to choose any set of $n+2$ points, no $n+1$ in a hyperplane, to have these coordinates.

A bijective map σ between the points and hyperplanes of $PG(n,q)$ which preserves incidence is called a **reciprocity**. If σ has order two, then it is called a **polarity**. If P is a point of $PG(n,q)$ then the hyperplane P^{σ} which is the image of P under a polarity of $PG(n,q)$ is called the **polar** of P. Similarly if σ is a hyperplane of $PG(n,q)$ then its image under σ is a point called the **pole** of σ.

Let P and Q be two points of $PG(n,q)$. If $P \in Q^{\sigma}$ then, as σ is incidence preserving and of order two, it follows that $Q = Q^{\sigma^2} \in P^{\sigma}$. Conversely if $Q \in P^{\sigma}$ then $P \in Q^{\sigma}$. Descriptively, this says that P lies on the polar of Q if and only if Q lies on the polar of P. This is called the **pole-polar** property of a polarity. A point which is incident with its own polar hyperplane is said to be an **absolute** point of the polarity.

Consider the projective space $PG(n,q^2)$. If ϕ denotes the Fröbenius automorphism of $GF(q^2)$ then it induces the collineation σ of $PG(n,q^2)$ defined by

$$\begin{array}{rccl}\sigma: & PG(n,q^2) & \rightarrow & PG(n,q^2) \\ & X=(x_0,\ldots,x_n) & \mapsto & X^{\phi}=(x_0^q,\ldots,x_n^q)\end{array}$$

If Σ_d is a d-dimensional subspace of $PG(n,q^2)$ then the space Σ_d^{σ} is called the **conjugate** of Σ_d and shall often be denoted by $\overline{\Sigma_d}$.

1.2.2 Subgeometries

A **subgeometry** of $PG(n,q)$ is a subset of the points of $PG(n,q)$ which forms a projective space $PG(n,q')$ of dimension n and order q', with respect to the incidence relation of $PG(n,q)$. For example, as $GF(q)$ is a subfield of $GF(q^r)$ for any $r > 1$, $PG(n,q)$ is embedded in $PG(n,q^r)$ in a natural way, and is thus a subgeometry, the natural subgeometry, of $PG(n,q^r)$ of order q. In this respect, we say that $PG(n,q^r)$ is an **extension** of $PG(n,q)$ and when we speak of an extension of $PG(n,q)$ we mean a projective space $PG(n,q^r)$, defined over an extension field $GF(q^r)$ of $GF(q)$. As with fields, we say that $PG(n,q^2)$ is the quadratic extension of $PG(n,q)$ and, more generally, we call any subgeometry of $PG(n,q^2)$ which is isomorphic to $PG(n,q)$ a **Baer subgeometry** of $PG(n,q^2)$.

1.2.3 Baer Subplanes of $PG(2,q^2)$.

Of particular importance to us will be the subplanes of $PG(2,q^2)$ of order q, that is the **Baer subplanes**. If $\mathcal{B}$ is any Baer subplane of $PG(2,q^2)$ then we may choose any four points of $\mathcal{B}$, no three of which are collinear, to have the coordinates $(0,0,1),(0,1,0),(0,0,1)$ and $(1,1,1)$. The points of $\mathcal{B}$ will then be the points of the Baer subplane $PG(2,q)$. For later reference we state this fact as follows;

Theorem 1.2. *The group $PGL(3,q^2)$ is transitive on the Baer subplanes of $PG(2,q^2)$.*

□

Thus, since there are $|PGL(3,q)|$ elements of $PGL(3,q^2)$ fixing a Baer subplane, the total number of Baer subplanes in $PG(2,q^2)$ is

$$\frac{|PGL(3,q^2)|}{|PGL(3,q)|} = q^3(q^3+1)(q^2+1)$$

and we have the following (see [29] for the orders of the groups $PGL(3,q^2)$ and $PGL(3,q)$),

Corollary 1.3. *There are $q^3(q^3+1)(q^2+1)$ Baer subplanes in $PG(2,q^2)$.* □

Since any line of $PG(2,q^2)$ containing 2 points of a Baer subplane $\mathcal{B}$ necessarily intersects $\mathcal{B}$ in $(q+1)$ points and we can show, by counting, that any line of $PG(2,q^2)$ intersects $\mathcal{B}$, all lines of $PG(2,q^2)$ intersect $\mathcal{B}$ in 1 or $(q+1)$ points. If ℓ contains $(q+1)$ points of $\mathcal{B}$ then the points $\ell \cap \mathcal{B}$ are the points of a subgeometry of ℓ of order q, that is a **Baer subline** of ℓ. We sometimes say that ℓ is a line of $\mathcal{B}$.

Lemma 1.4. [11] *If ℓ is a line of $PG(2,q^2)$ and A, B, C are three distinct points of ℓ then there is a unique Baer subline of ℓ containing A, B and C.* ☐

Corollary 1.5. *There are $q(q^2+1)$ Baer sublines of a line of $PG(2,q^2)$.* ☐

Lemma 1.4 is equivalent to the statement that any two distinct Baer sublines of ℓ intersect in either 0,1 or 2 points. The possible forms of the intersection of two Baer subplanes of $PG(2,q^2)$ are also known.

Theorem 1.6. [9] *Two distinct Baer subplanes of $PG(2,q^2)$ intersect in the same number of points and lines. Moreover, this intersection takes one of the following forms:*

1. *The empty set.*
2. *One line and one point, not necessarily incident.*
3. *Two points of a line and a second line through one of those points.*
4. *Three points and three lines forming a triangle configuration.*
5. *$(q+1)$ points of a line and the $(q+1)$ lines through one of those points.*
6. *$(q+2)$ points and $(q+2)$ lines; these being a line ℓ and the $(q+1)$ points of ℓ together with another point of the plane A and the $(q+1)$ lines joining A to the points of ℓ.*

☐

From this we can deduce a result similar to lemma 1.4 but for Baer subplanes.

Lemma 1.7. *Given $P \in PG(2,q^2)$ and three lines ℓ_1, ℓ_2, ℓ_3 through P there is a unique set of $(q-2)$ further lines through P such that these lines together with ℓ_1, ℓ_2, ℓ_3 are the lines of a Baer subplane of $PG(2,q^2)$.*

Proof: If we choose P to be the point $(1,0,0)$ and the non-collinear points $P_i \in \ell_i$ to have the coordinates $(1,0,0), (0,1,0)$ and $(0,0,1)$ respectively we see that the Baer subplane $PG(2,q)$, and the $(q+1)$ lines of this Baer subplane through P satisfy the requirements of the theorem. By theorem 1.6 any other Baer subplane of $PG(2,q^2)$ containing ℓ_1, ℓ_2, ℓ_3 necessarily shares the same set of $(q+1)$ lines through P. ☐

Note that this result does not assert that there is a unique Baer subplane with ℓ_1, ℓ_2, ℓ_3 as lines, merely that any Baer subplane with these lines contains a given set of lines through P.

1.3 Varieties

If f is a homogeneous polynomial in the $(n+1)$ variables $x_0, \dots, x_n$ then the set of points of $PG(n,q)$ with coordinates which satisfy the equation $f(x_0, \dots, x_n) = 0$ form a **hypersurface** (sometimes called a primal) in $PG(n,q)$. We say that $f = 0$ is the defining equation of the hypersurface. If f is irreducible over $GF(q)$ the hypersurface is said to be **irreducible** in $PG(n,q)$ and otherwise is called **reducible**. Further, if f is irreducible in any extension of $GF(q)$ we say that the hypersurface is **absolutely irreducible**. If f is of degree m then the hypersurface $f = 0$ is said to have **order m** and is denoted V_{n-1}^{m}. Once we know the order of a number of hypersurfaces the following result enables to calculate how many points they have in common.

Theorem 1.8. [47] *(Bezout's Theorem) In $PG(n,q)$ n generic*[1] *and irreducible hypersurfaces $V_{n-1}^{r_i}$ of orders r_i have $r_1 \dots r_n$ common points.* □

If $f_1(x_0, \dots, x_n), f_2(x_0, \dots, x_n), \dots$ are homogeneous polynomials over $GF(q)$ then the set of common points of the hypersurfaces they define is called a **variety** in $PG(n,q)$. Note that any variety can be defined as the intersection of a finite number of hypersurfaces (see [32]).

Let V be a variety in $PG(n,q)$. We call V an **irreducible variety of dimension d** if its points are in algebraic one-to-one correspondence with the points of an irreducible hypersurface W_d of a subspace, $PG(d+1,q)$, of dimension $d+1$ in $PG(n,q)$ (see Semple and Roth [47]). More formally that is :

there exists a set of $r+1$ polynomials $f_0, \dots, f_r$, homogeneous and of the same degree in $y_0, \dots, y_{d+1}$, such that for each $Y = (y_0, \dots, y_{d+1}) \in W_d$ the point $X = (x_0, \dots, x_n)$ given by

$$\rho x_i = f_i(y_0, \dots, y_{d+1}), \text{ for some } \rho \in GF(q) \setminus \{0\}$$

is a point of V.

The correspondence is such that a generic point of V arises from only one point of M_d.

---

[1] The term generic is difficult to define and, as it is not central to this exposition, we do not discuss it in detail here and only note that its meaning depends on the context in which it is used and, as in [47], is used to have a meaning roughly equivalent to the phrase *of general position*. We refer the reader to [32] for a precise definition.

The equations

$$f_i(y_0, \dots, y_{d+1}) = \rho x_i \tag{1.1}$$

$$M(y_0, \dots, y_{d+1}) = 0 \tag{1.2}$$

are called the **parametric equations** of V. The parameters are the $(d+1)$ ratios $y_i : y_0$ of which, as $(y_0, \dots, y_{d+1})$ is a point of the hypersurface W_d, only d are independent. An irreducible variety of dimension d is denoted V_d. A variety of dimension one is called a **curve** and a variety of dimension two is called a **surface**.

The Order of a Variety

Let V_d be an irreducible variety of dimension d with parametric equations 1.1, 1.2. We proceed as in Semple and Roth, ([47]). Suppose that Σ_{n-d} is the $n-d$ dimensional subspace of $PG(n,q)$ given by the equations

$$\sum_{i=0}^{n} a_{ij} x_i = 0$$

The variety V_d intersects Σ_{n-d} where these equations have a common solution, and combining these equations we obtain $(d+1)$ hypersurfaces in $PG(d+1,q)$. Therefore, by Bezout's theorem, these hypersurfaces have a finite number of common points (given that they are generic). Thus we have;

Theorem 1.9. [47] *A generic subspace of $PG(n,q)$ of dimension $(n-d)$ intersects V_d in a finite number, k, of points.* □

We call this number k the **order** of the variety V_d and to describe the fact that V_d has order k we write it as V_d^k.

Furthermore, now that we have established the order of V_d^k, we can determine how it intersects a subspace of $PG(n,q)$;

Theorem 1.10. [47] *For $d > h$ a generic subspace of dimension $(n-d+h)$ intersects V_d^k in a V_h^k.* □

The more general problem of determining how V_d^k intersects a variety V_s^r of dimension s and order r is difficult to solve. If the two varieties are generically situated with respect to one another then we may say that

$$V_d^k \cap V_s^r = V_{d+s-n}^{kr}$$

That is, the two varieties intersect in a variety of dimension $(d+s-n)$ and order kr. However, once again, the term generically situated is not easy to define and so for a full explanation of this we refer the reader to an appropriate text, such as [32].

1.4 Curves in the Plane

Let $\mathcal{C}$ denote the curve in $PG(2,q)$ consisting of the points satisfying the equation $f = 0$, where f is a homogeneous polynomial of degree n in x_0, x_1, x_2. As $\mathcal{C}$ is a hypersurface in the plane, its order is given by the number of points that $\mathcal{C}$ has in common with a generic line of the plane. If ℓ is any line of $PG(2,q)$ then the points of ℓ satisfy a linear equation in x_0, x_1, x_2; thus solving this equation simultaneously with that of $\mathcal{C}$ we obtain, in general, n solutions, some of which may be identical and some may lie in an extension of $GF(q)$. Correspondingly, ℓ and $\mathcal{C}$ have n points in common, with, possibly, some of these points being coincident and some lying in an extension of $PG(2,q)$. Therefore a generic line of the plane intersects $\mathcal{C}$ in n points and so $\mathcal{C}$ has order n. Furthermore, if m of these solutions correspond to the point R of $\mathcal{C}$ then we say that ℓ and $\mathcal{C}$ intersect m times at R.

1.4.1 Singular Points

Let $P \in \mathcal{C}$. If all lines of the plane through P intersect $\mathcal{C}$ at least twice at P then we say that P is a **singular point** of $\mathcal{C}$; otherwise P is called a **simple point** of $\mathcal{C}$. If, in addition, all lines of the plane which contain P intersect $\mathcal{C}$ at least m times at P, with some line having precisely m intersections with $\mathcal{C}$ at P then P is said to be a **point of multiplicity m** (or an **m-fold** point) of $\mathcal{C}$. A simple point is thus a point of multiplicity one. We call a point of $\mathcal{C}$ of multiplicity two a **double point**. If P has multiplicity m then a line through P which meets $\mathcal{C}$ more than m times at P is said to be a **tangent** to $\mathcal{C}$ through P. We examine the singular points of $\mathcal{C}$ in more detail.

Suppose that $P \in \mathcal{C}$ has coordinates (p_0, p_1, p_2). Thus the line ℓ joining P to the point $Q = (q_0, q_1, q_2)$ of $PG(2,q)$ contains the points

$$\{(p_0 + \lambda q_0, p_1 + \lambda q_1, p_2 + \lambda q_2) : \lambda \in GF(q) \cup \{\infty\}\}$$

and so meets $\mathcal{C}$ where

$$f(p_0 + \lambda q_0, p_1 + \lambda q_1, p_2 + \lambda q_2) = 0$$

Expanding this as a Taylor series in λ, in the vein of Walker [61], where a similar argument is used, but in the affine case, we obtain

$$\begin{aligned} f(P) + \lambda \left(q_0 \frac{\partial f}{\partial x_0}(P) + q_1 \frac{\partial f}{\partial x_1}(P) + q_2 \frac{\partial f}{\partial x_2}(P) \right) + & \\ \lambda^2 \left(q_0^2 \tfrac{\partial^2 f}{\partial x_0^2}(P) + q_0 q_1 \tfrac{\partial f}{\partial x_0}(P) \tfrac{\partial f}{\partial x_1}(P) + \ldots \right) + \ldots = & \ 0 \end{aligned} \qquad (1.3)$$

which we write as

$$\sum_{i=0}^{n} \frac{\lambda^i}{i!} \left(q_0 \frac{\partial}{\partial x_0} + q_1 \frac{\partial}{\partial x_1} + q_2 \frac{\partial}{\partial x_2} \right)^i f(P) = 0$$

where by $f(P)$ we mean the value of the polynomial f evaluated at $P = (p_0, p_1, p_2)$ and, similarly, by $\frac{\partial f}{\partial x_i}(P)$ we mean $\frac{\partial f}{\partial x_i}$ evaluated at P.

The multiplicity of the root $\lambda = 0$ in equation 1.3 is equal to the number of times in which ℓ meets $\mathcal{C}$ at P. Thus, since $f(P) = 0$, for P to be a simple point of $\mathcal{C}$ we require that the coefficient of λ in this equation is non-zero. Hence P is simple if and only if one of the first partials of f is non-zero at P. In this case the only line through P which meets $\mathcal{C}$ twice at P is the line

$$\frac{\partial f}{\partial x_0}(P)x_0 + \frac{\partial f}{\partial x_1}(P)x_1 + \frac{\partial f}{\partial x_2}(P)x_2 = 0$$

So if P is a simple point of $\mathcal{C}$ then there is a unique tangent line to $\mathcal{C}$ through P.

Conversely, if all the first order partial derivatives of f are zero at P then every line through P has at least two intersections with $\mathcal{C}$ at P and P is a singular point of $\mathcal{C}$. Therefore we have the following result;

Theorem 1.11. *Let $\mathcal{C}$ be a curve of order n in $PG(2,q)$. The point P of $\mathcal{C}$ is singular if and only if*

$$\frac{\partial f}{\partial x_0}(P) = \frac{\partial f}{\partial x_1}(P) = \frac{\partial f}{\partial x_2}(P) = 0$$

□

Now suppose that P is a double point of $\mathcal{C}$. This occurs when the multiplicity of the root $\lambda = 0$ of equation 1.3 is at least two for any point (q_0, q_1, q_2) of $PG(2,q)$, and is exactly two for some point of the plane. This implies that all the first order partials of f are zero at P but some second order partial is non-zero there. Conversely, if all first order partials of f are zero at P and some second order partial is non-zero at P, then P is a double point of $\mathcal{C}$.

In this case the line ℓ intersects $\mathcal{C}$ more than twice at P if and only if

$$\frac{1}{2!} \sum_{i,j=0}^{2} q_i q_j \frac{\partial^2 f}{\partial x_i \partial x_j}(P) = 0$$

This is a quadratic equation in the q_i and hence gives the equation of two lines (possibly coincident or belonging to an extension) through P intersecting $\mathcal{C}$ more than twice at P. Thus there are two tangents to $\mathcal{C}$ through P.

Continuing in this fashion, P is a point of multiplicity k if and only if equation 1.3 has $\lambda = 0$ as a root of multiplicity at least k for any line through P and has $\lambda = 0$ as a root of multiplicity exactly k for some line through P. This happens precisely when all partial derivatives of f of order $k-1$ or less are zero at P but some derivative of order k is non-zero there. In this case the line ℓ intersects $\mathcal{C}$ more than k times at P if and only if

$$\frac{1}{k!}\left(q_0\frac{\partial}{\partial x_0} + q_1\frac{\partial}{\partial x_1} + q_2\frac{\partial}{\partial x_2}\right)^k f(P) = 0$$

This is an equation of degree k in the q_i and thus gives the equation of k lines through P meeting $\mathcal{C}$ more than k times there. Hence there are k tangents to $\mathcal{C}$ through P, some of which may be coincident and some of which may only be lines of an extension of $PG(2,q)$. As f is not identically zero, some derivative of f of order n or less is non-zero and thus P is a point of multiplicity r for some $r \leq n$. To sum up we have;

Theorem 1.12. *Let $\mathcal{C}$, with equation $f = 0$, be a curve of order n in $PG(2,q)$. The point P of $\mathcal{C}$ is of multiplicity k if and only if all derivatives of f of order $k-1$ or less are zero at P, and some derivative of order k is non-zero there. Such a point lies on k tangent lines of $\mathcal{C}$, some of which may coincide and some of which may only be lines of an extension of $PG(2,q)$. Every point of $\mathcal{C}$ is of multiplicity k for some k.*

$\square$

1.4.2 Conics

A curve of order two in $PG(2,q)$ is called a **conic**. Although the structure of conics is well-known, we shall make frequent use of some of their properties and so briefly introduce them here.

Let $\mathcal{C}^2$ be a non-singular conic in $PG(2,q)$. If P is a point of $\mathcal{C}^2$ then, as P is non-singular, $\mathcal{C}^2$ has a unique tangent through P and all other lines through P contain precisely one further point of $\mathcal{C}^2$. It is well-known that up to projectivity there is only one non-singular conic in $PG(2,q)$ and this has the following canonical form (see, for example, [29]).

Theorem 1.13. *A non-singular conic $\mathcal{C}^2$ in $PG(2,q)$ consists of $q+1$ points no three of which are collinear and is projectively equivalent to the conic of equation*

$$x_0^2 + x_1x_2 = 0$$

$\square$

Conversely, by a well-known theorem of B. Segre ([43]), in $PG(2,q)$ with q odd any set of $(q+1)$ points no three of which are collinear (such a set of points is usually called an oval) are the points of a non-singular conic. This result does not hold if q is even. In this case however the following well-known property is important.

Theorem 1.14. *If $\mathcal{C}^2$ is a non-singular conic in $PG(2,q)$ with q even then the $q+1$ tangents to $\mathcal{C}^2$ are concurrent.*

Proof: As $\mathcal{C}^2$ is non-singular we can assume that its equation is $x_0^2 + x_1x_2 = 0$ and thus that it contains the set of points

$$\{(\alpha, 1, \alpha^2) : \alpha \in GF(q) \cup \{\infty\}\}$$

The tangent through the point $P_\alpha = (\alpha, 1, \alpha^2)$ of $\mathcal{C}^2$ thus has the equation

$$2\alpha x_0 + \alpha^2 x_1 + x_2 = 0$$

which, as q is even, is equivalent to

$$\alpha^2 x_1 + x_2 = 0$$

Hence every tangent line to $\mathcal{C}^2$ contains the point $(1,0,0)$. □

The point of concurrency of the $q+1$ tangents to $\mathcal{C}^2$ is called the **nucleus** of the conic.

Suppose now that $\mathcal{C}^2$ is singular with P a singular point. As $\mathcal{C}^2$ is of order two, P is necessarily a double point and every line through P is either contained in $\mathcal{C}^2$ or does not meet $\mathcal{C}^2$ again. Since $\mathcal{C}^2$ is given by a quadratic equation, it thus follows that $\mathcal{C}^2$ consists of two lines through P, which are either distinct lines of $PG(2,q)$, one repeated line of $PG(2,q)$, or two conjugate lines of $PG(2,q^2)$ containing only the point P of $PG(2,q)$. This enables us to prove the following.

Theorem 1.15. [29] *In $PG(2,q)$ a conic is either non-singular or consists of two lines. These lines are either distinct, coincident or are a pair of conjugate lines of $PG(2,q^2)$. The number of conics of each type is;*

- $(q^5 - q^2)$ *non-singular conics.*
- $(q^2+q+1)(q^2+q)/2$ *consisting of two distinct lines of $PG(2,q)$.*
- q^2+q+1 *consisting of one (repeated) line of $PG(2,q)$.*
- $(q^2+q+1)(q^2-q)/2$ *consisting of two conjugate lines of $PG(2,q^2)$.*

Proof: The numbers of conics consisting of two distinct lines or one repeated line are easily calculated. To calculate the number of conics consisting of two conjugate lines of $PG(2, q^2)$, consider a point P of $PG(2, q^2)$ which is also a point of $PG(2, q)$. If ℓ is a line of $PG(2, q^2)$ through P which meets $PG(2, q)$ at P precisely, then its conjugate also contains P (the only point of $PG(2, q)$ on either of these lines) and together they form a conic of $PG(2, q)$ meeting $PG(2, q)$ only at P. As there are $q^2 - q$ lines of $PG(2, q^2)$ through P which are not lines of $PG(2, q)$, there are $(q^2 - q)/2$ conics through P of this type. Since there are $(q^2 + q + 1)$ points in $PG(2, q)$, we obtain the required number of conics consisting of two conjugate lines of $PG(2, q)$.

To conclude, consider the general homogeneous quadratic polynomial in three variables. This polynomial has six coefficients and thus there are $(q^6 - 1)$ such polynomials. Any two represent the same conic if and only if one is a multiple of the other by a non-zero element of $GF(q)$. Thus there are, in total, $(q^6 - 1)/(q - 1) = q^5 + q^4 + \ldots + 1$ conics in the plane. Subtracting from this the number of singular conics we obtain the number of non-singular conics given above. □

1.4.3 Cubic Curves

Let $\mathcal{C}^3$ be an absolutely irreducible cubic curve in $PG(2, q)$ and suppose that $\mathcal{C}^3$ has a singular point P. If P has multiplicity three then any line through P meets $\mathcal{C}^3$ three times there and hence, if it meets the curve again, has four intersections with $\mathcal{C}^3$ and therefore is wholly contained in $\mathcal{C}^3$. However, this contradicts the fact that $\mathcal{C}^3$ is absolutely irreducible; hence P is a double point. Further, if $\mathcal{C}^3$ contains another double point P' then the line PP' is not a line of $\mathcal{C}^3$, yet meets $\mathcal{C}^3$ in more than three points; hence $\mathcal{C}^3$ has at most one double point. Thus we can conclude that $\mathcal{C}^3$ is either non-singular or possesses exactly one singular point which, by necessity, is a double point.

Now suppose that $\mathcal{C}^3$ contains a double point P. There are thus two tangents to $\mathcal{C}^3$ through P which, as they are given by a quadratic equation (see section 1.4), are either both lines of $PG(2, q)$ or both lie in an extension of $PG(2, q)$. In the case that they are distinct and appear in $PG(2, q)$ we call the double point P a **node** of $\mathcal{C}^3$, when they are coincident we call P a **cusp** of $\mathcal{C}^3$ and when they belong to an extension of $PG(2, q)$ we say that P is an **isolated double point** of $\mathcal{C}^3$.

From the above two paragraphs, it follows that if $\mathcal{C}^3$ is an absolutely irreducible cubic curve in $PG(2, q)$ then it is either non-singular, or contains precisely one double point which is either a node, a cusp or an isolated double point. For later use we now state the following result detailing the numbers of each type of these curves in $PG(2,q)$.

Theorem 1.16. [29] *In $PG(2,q)$ there are $q^4(q^3-1)(q^2-1)$ non-singular cubic curves, $q^3(q^3-1)(q^2-1)/2$ absolutely irreducible cubic curves with a node, $q^3(q^3-1)(q+1)$ absolutely irreducible cubic curves with a cusp and $q^3(q^3-1)(q^2-1)/2$ absolutely irreducible cubic curves with an isolated double point.* □

1.5 Normal Rational Curves

The only other type of curves that will be useful in this work are the normal rational curves. The following paraphrases [31] where more detailed information on normal rational curves can be found.

A set of points of $PG(n,q)$ of the form

$$\mathcal{C}^d = \{(f_0(t), f_1(t), \ldots, f_n(t)) : t \in GF(q) \cup \{\infty\}\}$$

where the f_i are polynomials of degree d, with no common factor and at least one being of degree d, is said to form a **rational curve of order d**. If $\mathcal{C}^d$ is not the projection of a rational curve of order d' in $PG(n+1,q)$ then we say that $\mathcal{C}^d$ is a **normal rational curve** in $PG(n,q)$.

If a normal rational curve $\mathcal{C}^d$ exists in $PG(n,q)$ then it is necessary (see [31]) that $q \geq n$, $d = n$ and that the points of $\mathcal{C}^n$ are projectively equivalent to the following set of points of $PG(n,q)$

$$\{(t^n, \ldots, t, 1) : t \in GF(q) \cup \{\infty\}\}$$

Thus, by theorem 1.13, the normal rational curve in $PG(2,q)$ is the non-singular conic. A normal rational curve in $PG(3,q)$ is called a **twisted cubic**. A normal rational curve consists of $(q+1)$ points, no $(n+1)$ in a hyperplane.

1.6 The Tangent Space of a Variety in $PG(n,q)$

In section 1.4 we defined a simple point of a curve in $PG(2,q)$ and the tangent line at such points. Here we extend this to define a simple point of any irreducible variety and the tangent space at such points. We follow the method used in Hodge and Pedoe [32, volume 2].

Consider an irreducible variety V_d^k of dimension d and order k in $PG(n,q)$. Suppose that the homogeneous polynomials

$$f_1(x_0,\ldots,x_n), f_2(x_0,\ldots,x_n),\ldots,f_r(x_0,\ldots,x_n)$$

are linearly independent and are such that the equations

$$f_1(x_0,\ldots,x_n)=0, f_2(x_0,\ldots,x_n)=0,\ldots,f_r(x_0,\ldots,x_n)=0$$

define V_d^k. Define the matrix $\mathbf{T_P}=(\mathbf{t_{ij}})$ by

$$t_{ij}=\frac{\partial f_i}{\partial x_j}(P)$$

Theorem 1.17. [32] *The matrix $\mathbf{T_P}$ has rank at most $n-d$ at any point of V_d^k and there exists at least one point of V_d^k for which the matrix has rank $n-d$ exactly.* □

In light of this result, we call those points P of V_d^k for which the matrix $\mathbf{T_P}$ has rank $n-d$ **simple points** of V_d^k and those for which $\mathbf{T_P}$ has rank less than $n-d$ **singular points** of V_d^k. Note that in the case that $d=1$ and V_1^k is a curve in $PG(2,q)$, defined by the equation $f=0$, $\mathbf{T_P}$ is a (1×3) matrix and this definition says that P is a singular point of the curve if and only if $\mathbf{T_P}$ has rank 0. Or, in other words, if and only if all first order partials of f are zero at P, which is equivalent to the definition of singular points of a curve given earlier.

At a simple point P of V_d^k the equations

$$\sum_{j=0}^{n}\frac{\partial f_i}{\partial x_j}(P)x_j=0 \quad i=1,\ldots,r \tag{1.4}$$

define a subspace of dimension d which we call the **tangent space** to V_d^k at P and which is denoted by the symbol $T_P(V_d^o)$. Again this matches with the definition of a tangent line to a curve in $PG(2,q)$ given in section 1.4.

Note that the definition of simple points and the tangent space at such points is independent of the particular polynomials chosen to define V_d^k (see [32]).

For a hypersurface of equation $f=0$ the point P is simple if and only if $\frac{\partial f}{\partial x_j}(P)\neq 0$ for some $j=1,\ldots,n$ and the tangent space is the hyperplane of equation

$$\frac{\partial f}{\partial x_0}(P)x_0+\ldots+\frac{\partial f}{\partial x_n}(P)x_n=0$$

Theorem 1.18. [32] *If V_d^k is the intersection of the hypersurfaces $V_1,\ldots,V_r$ with equations $f_1=0,\ldots,f_r=0$ then for the simple point P of V_d^k*

$$T_P(V_d^k)=T_P(V_1)\cap\ldots\cap T_P(V_s)$$

where $f_1,\ldots,f_s$ are linearly independent and $V_1,\ldots,V_s$ have simple points at P.

Proof: Without loss of generality, suppose that $f_1, \ldots, f_t$ are a linearly independent subset of the above polynomials $f_1, \ldots, f_r$. Therefore the equations

$$\sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j}(P)x_j = 0 \qquad i = 1, \ldots, t$$

define $T_P(V_d^k)$. In addition assume that only $f_1, \ldots, f_s$, where $s \leq t$, have a simple point at P. Thus, for any $1 \leq j \leq n$, $\frac{\partial f_i}{\partial x_j}(P) = 0$ where $i = s+1, \ldots, t$. Hence $T_P(V_d^k)$ is given by the equations

$$\sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j}(P)x_j = 0 \qquad i = 1, \ldots, s$$

which is the intersection of the tangent spaces to $V_1, \ldots, V_s$ at P. □

We also will need the following results, the first of which can be found in [29] with a different proof.

Theorem 1.19. [29] *If Π_s is a subspace of $PG(n,q)$ lying on V_d^k and if $P \in \Pi_s$ then*

$$\Pi_s \subset T_P(V_d^k)$$

Proof: Choose coordinates so that Π_s is given by the equations

$$x_{s+1} = \cdots = x_n = 0$$

and

$$f_1(x_0, \ldots, x_n) = 0, f_2(x_0, \ldots, x_n) = 0, \ldots, f_r(x_0, \ldots, x_n) = 0$$

are a set of linearly independent equations defining V_d^k. As Π_s is contained in V_d^k we can write each of the equations f_i in the form (not necessarily in a unique way)

$$f_i(x_0, \ldots, x_n) = x_{s+1}\phi_{s+1}^i + \ldots + x_n\phi_n^i + g^i(x_{s+1}, \ldots, x_n)$$

where the ϕ_j^i are polynomials in $x_0, \ldots, x_n$. Thus

$$\frac{\partial f_i}{\partial x_j}(P) = \begin{cases} 0 & j = 0, 1, \ldots, s \\ \phi_j^i(P) & j = s+1, \ldots, n \end{cases}$$

Hence $T_P(V_d^k)$ is given by the equations

$$\sum_{j=s+1}^{n} \frac{\partial \phi_j^i}{\partial x_j}(P)x_j = 0 \qquad i = 1, \ldots, r$$

and so if $P = (\alpha_0, \ldots, \alpha_s, 0, \ldots, 0)$ is a point of Π_s then P satisfies each of these equations and is thus a point of $T_P(V_d^k)$. □

Lemma 1.20. *If $\Pi_s \cap V_d^k = W$ a variety in Π_s and $P \in \Pi_s$ then*

$$T_P(W) = \Pi_s \cap T_P(V_d^k)$$

Proof: If we choose coordinates so that Π_s has equations

$$x_{s+1} = \cdots = x_n = 0$$

then (with $T_P(V_d^k)$ being given by 1.4) $T_P(W)$ has equations

$$\sum_{j=0}^{s} \frac{\partial f_i}{\partial x_j}(P) x_j = 0 \qquad i = 1, \ldots, r$$

which is $\Pi_s \cap T_P(V_d^k)$. □

1.7 Quadrics

One important class of varieties is the hypersurfaces of order two which are called **quadrics**. Quadrics have been much studied (see, for example, Hirschfeld [29], [30], [31]) and we outline only the key points here.

Being a hypersurface of order two, a quadric in $PG(n,q)$ is the set of points satisfying an equation of the form

$$\sum_{i,j=0}^{n} a_{ij} x_i x_j = 0 \tag{1.5}$$

where the $a_{ij} \in GF(q)$, not all zero. We let $\mathcal{Q}_{n-1}^2$ denote a quadric of $PG(n,q)$.

A quadric is said to be **degenerate** (reducible) if its equation splits into two linear equations in $GF(q)$ or some extension. Otherwise the quadric is called **non-degenerate**. It follows directly from the definition that a degenerate quadric consists of two, possibly coincident, hyperplanes of $PG(n,q^2)$. We note here the following useful lemma.

Lemma 1.21. *Let $\mathcal{Q}_{n-1}^2$ be a quadric in $PG(n,q)$. A subspace Σ_r of $PG(n,q)$ is either contained in $\mathcal{Q}$ or meets it in a quadric of Σ_r.*

Proof: Choose coordinates so that Σ_r is the space $x_{r+1} = x_{r+2} = \ldots = x_n = 0$. It therefore meets the quadric $\mathcal{Q}_{n-1}^2$, with an equation of the form 1.5, in the points of Σ_r which satisfy

$$\sum_{i,j=0}^{r} a_{ij} x_i x_j = 0.$$

Thus, the intersection is a quadric of Σ_r unless $a_{ij} = 0$ $\forall i, j \leq r$, in which case Σ_r is contained in $\mathcal{Q}^2_{n-1}$. □

Corollary 1.22. *A line of $PG(n, q)$ is either contained in a quadric or else meets it in two (possibly coincident) points.* □

1.7.1 Non-Singular Quadrics

From section 1.6, the point P of the quadric $\mathcal{Q}^2_{n-1}$, with equation $f = 0$ of the form 1.5, is a singular point if and only if

$$\frac{\partial f}{\partial x_i}(P) = 0 \qquad \forall i = 1, \dots, n.$$

If $\mathcal{Q}^2_{n-1}$ contains a singular point it is said to be **singular** and otherwise is called **non-singular**. Note that by theorem 1.15 the notions of singularity and degeneracy coincide for conics. However this is not the case for quadrics in higher dimensions where there exists singular quadrics which are not degenerate. The structure of the non-singular quadrics is well-known and we state the main results here.

Theorem 1.23. [29] *The number of projectively distinct non-singular quadrics in $PG(n, q)$ is one or two depending upon whether n is even or odd. They have the following canonical forms;*

- *when n is even; $n = 2s$*

$$x_0^2 + x_1x_2 + \ldots + x_{2s-1}x_{2s} = 0$$

- *when n is odd; $n = 2s - 1$*

$$x_0x_1 + \ldots + x_{2s-2}x_{2s-1} = 0$$

or

$$f(x_0, x_1) + x_2x_3 + \ldots + x_{2s-2}x_{2s-1} = 0$$

where f is a homogeneous polynomial of degree two, irreducible over $GF(q)$. □

The non-singular quadric in $PG(2s, q)$ is called **parabolic** and is denoted $Q(2s, q)$ or $\mathcal{P}_{2s}$. The non-singular quadric in $PG(2s-1, q)$ of the first type is called **hyperbolic** and denoted $Q^+(2s-1, q)$ or $\mathcal{H}_{2s-1}$, while the non-singular quadric in $PG(2s-1, q)$ of the second type is called **elliptic** and denoted $Q^-(2s-1,q)$ or $\mathcal{E}_{2s-1}$. As we can deduce from the canonical forms these quadrics contain many subspaces of $PG(n,q)$. For details of the exact number of each type of subspace on each type as quadric as well as the possible sections of a quadric by a subspace see Hirschfeld and Thas, [31]. Here we note only the following;

Theorem 1.24. [31] *The maximum dimension of a subspace on a non-singular quadric in $PG(n,q)$ is $(n-2)/2$ for $\mathcal{P}_n$, $(n-1)/2$ for $\mathcal{H}_n$ and $(n-3)/2$ for $\mathcal{E}_n$.* □

A subspace of maximum dimension on a quadric is called a **generator**.

Theorem 1.25. [31, theorem 22.5.1] *The number of points on a non-singular quadric is as follows;*

- $\frac{(q^s+1)(q^s-1)}{(q-1)}$ *for the parabolic quadric* $\mathcal{P}_{2s}$
- $\frac{(q^{s-1}+1)(q^s-1)}{(q-1)}$ *for the hyperbolic quadric* $\mathcal{H}_{2s-1}$
- $\frac{(q^s+1)(q^{s-1}-1)}{(q-1)}$ *for the elliptic quadric* $\mathcal{E}_{2s-1}$

□

We shall also need the following result.

Theorem 1.26. [31] *Let $\mathcal{Q}_n$ denote a non-singular quadric in $PG(n,q)$. If q and n are not both even then there is a polarity of $PG(n,q)$ for which the polar of a point P of $\mathcal{Q}_n$ is the tangent space to $\mathcal{Q}_n$ at the point P.* □

1.7.2 The Structure of Singular Quadrics

Let $\mathcal{Q}^2_{n-1}$ be a singular quadric of equation $f=0$ in $PG(n,q)$. Being singular, $\mathcal{Q}^2_{n-1}$ contains a singular point P for which, by definition,

$$\frac{\partial f}{\partial x_j}(P)=0 \qquad \forall j=0,\dots,n$$

Suppose that $\mathcal{Q}^2_{n-1}$ contains another singular point R, distinct from P. A point of the line joining $P=(p_0,\dots,p_n)$ and $R=(r_0,\dots,r_n)$ will have coordinates of the form $(p_0,\dots,p_n)+\lambda(r_0,\dots,r_n)$ for some $\lambda\in GF(q)\cup\{\infty\}$. Now, since f is quadratic, $\frac{\partial f}{\partial x_j}$ is linear and hence

$$\begin{aligned}\frac{\partial f}{\partial x_j}(P+\lambda R) &= \frac{\partial f}{\partial x_j}(P)+\lambda\frac{\partial f}{\partial x_j}(R)\\ &= 0\end{aligned}$$

as P and R are singular points of $\mathcal{Q}^2_{n-1}$. Thus every point on the line joining P and R is singular and we have proved the following well-known result:

Theorem 1.27. *If $\mathcal{Q}^2_{n-1}$ is a quadric in $PG(n,q)$ then the set of all singular points of $\mathcal{Q}^2_{n-1}$ forms a subspace Π_r of $PG(n,q)$.* ☐

The subspace Π_r is called the **vertex** of $\mathcal{Q}^2_{n-1}$.

Lemma 1.28. *Let $\mathcal{Q}^2_{n-1}$ be a quadric in $PG(n,q)$ with vertex Π_r. If P is a non-singular point of $\mathcal{Q}^2_{n-1}$ and Q is a point of Π_r then the line joining P and Q is contained in $\mathcal{Q}^2_{n-1}$.*

Proof: Suppose that P and Q are two points satisfying the hypotheses of the theorem. The line joining P and Q meets $\mathcal{Q}^2_{n-1}$ twice at Q and once at P and is thus contained on $\mathcal{Q}^2_{n-1}$. ☐

We are now in a position to give the structure of a singular quadric in $PG(n,q)$.

Suppose that $\mathcal{Q}^2_{n-1}$ is such a quadric with vertex Π_r and Σ_{n-r-1} is a subspace of dimension $(n-r-1)$ of $PG(n,q)$ which is skew to Π_r. By lemma 1.21, Σ_{n-r-1} intersects $\mathcal{Q}^2_{n-1}$ in a non-singular quadric $\mathcal{Q}^2_{n-r-2}$ of Σ_{n-r-1}. By lemma 1.28, if P is a point of $\mathcal{Q}^2_{n-r-2}$ and Q is a point of Π_r then the line joining P and Q is contained in $\mathcal{Q}^2_{n-1}$. Conversely, if T is a point of $\mathcal{Q}^2_{n-1}$ on neither $\mathcal{Q}^2_{n-r-2}$ nor Π_r then, by lemma 1.28, $\langle T, \Pi_r\rangle$ is an $(r+1)$ dimensional space contained on $\mathcal{Q}^2_{n-1}$. As a space of this dimension necessarily intersects a space of dimension $(n-r-1)$, $\langle T, \Pi_r\rangle$ intersects Σ_{n-r-1} in (at least) a point, R. The line joining R and T thus contains T and joins a point of $\mathcal{Q}^2_{n-r-2}$ to a point of Π_r. Hence we have;

Theorem 1.29. *Let $\mathcal{Q}^2_{n-1}$ be a quadric in $PG(n,q)$ with vertex Π_r. If Σ_{n-r-1} is a space of dimension $(n-r-1)$ which is skew to Π_r then $\mathcal{Q}^2_{n-1}$ consists of the lines joining the points of Π_r to the points of the non-singular quadric $\mathcal{Q}^2_{n-1} \cap \Sigma_{n-r-1}$.* ☐

1.7.3 The Non-Singular Quadrics in $PG(3,q)$

For our later purposes, we shall require more detailed knowledge of the non-singular quadrics in $PG(3,q)$. We look at each in turn.

The Elliptic Quadric in $PG(3,q)$.

If $\mathcal{E}_3$ is an elliptic quadric in $PG(3,q)$ then, by theorem 1.23, it is projectively equivalent to the quadric of equation

$$f(x_0, x_1) + x_2 x_3 = 0$$

where f is an homogeneous quadratic polynomial, irreducible over $GF(q)$. In addition, from theorem 1.24, $\mathcal{E}_3$ contains no lines.

Let P and Q be two points of $\mathcal{E}_3$. As $\mathcal{E}_3$ contains no lines, the line ℓ joining P and Q does not meet $\mathcal{E}_3$ again. Each of the $(q+1)$ planes about the line ℓ meets $\mathcal{E}_3$ in a conic of that plane, containing the two points P and Q, but no line. Thus, by theorem 1.15, each such plane meets $\mathcal{E}_3$ in a non-singular conic. Hence, counting the number of points of $\mathcal{E}_3$ on planes about ℓ, we see that $\mathcal{E}_3$ is a set of $((q+1)(q-1)+2)$ points. Since we know from theorem 1.24 that $\mathcal{E}_3$ contains no lines, no three of these points are collinear. Furthermore each plane of $PG(3,q)$ meets $\mathcal{E}_3$ in either a single point or in a non-singular conic. Those planes which meet $\mathcal{E}_3$ in a conic are called **conic planes**. Note that through the point $P \in \mathcal{E}_3$ there passes $q(q+1)$ conic planes and one plane meeting $\mathcal{E}_3$ only at P, which is necessarily the tangent plane to $\mathcal{E}_3$ at P.

In general, a set of (q^2+1) points of $PG(3,q)$, $q > 2$, no three of which are collinear is called an **ovoid**. Thus the above shows that an elliptic quadric is an example of an ovoid. For q odd it has long been known (see Barlotti [4] and Panella [39]) that the elliptic quadric is the only example of an ovoid. However, for q even there are examples of ovoids which are not elliptic quadrics, (see Tits [57]). Much work has been done on the problem of classifying all ovoids of $PG(3,q)$ and the related problem of characterising ovoids by the nature of their plane sections. For a good survey on this see [38]. We shall make use of the following result of this nature.

Theorem 1.30. [10] *If $\mathcal{O}$ is an ovoid of $PG(3,q)$ containing a conic then $\mathcal{O}$ is an elliptic quadric.* □

The Hyperbolic Quadric in $PG(3,q)$

Once again, we have from theorem 1.23 that a hyperbolic quadric $\mathcal{H}_3$ in $PG(3,q)$ is projectively equivalent to the quadric of equation

$$x_0 x_1 + x_2 x_3 = 0.$$

Thus, if $P = (x_0, x_1, x_2, x_3)$ is a point of $\mathcal{H}_3$ and $x_0 = 0$ then P is of the form $(0, x_1, 0, x_3)$ or $(0, x_1, x_2, 0)$. As each x_i is an element of $GF(q)$ there are $(2q+1)$ points of this form. Alternatively $x_0 \neq 0$ and P is of the form $(x_0, -x_2 x_3, x_2, x_3)$. There are q^2 points on $\mathcal{H}_3$ of this type. Hence we may conclude that $\mathcal{H}_3$ contains (q^2+2q+1) points.

By theorem 1.24 a generator of $\mathcal{H}_3$ is a line. We examine the set of generator lines of $\mathcal{H}_3$. From a consideration of the equation of $\mathcal{H}_3$ it is clear that the lines ℓ_λ given by the equations

$$x_0 + \lambda x_3 = 0, x_2 - \lambda x_1 = 0$$

are generators of $\mathcal{H}_3$. For $\lambda \neq \lambda'$ the two lines ℓ_λ and $\ell_{\lambda'}$ are skew and so the set

$$\mathcal{R}_1 = \{\ell_\lambda : \lambda \in GF(q) \cup \{\infty\}\}$$

consists of $q+1$ skew lines. As $\mathcal{H}_3$ contains (q^2+2q+1) points, it follows that each point of $\mathcal{H}_3$ lies on a unique line of $\mathcal{R}_1$. Any set of $q+1$ skew lines of $\mathcal{H}_3$ is called a **regulus**. Each point of $\mathcal{H}_3$ is contained in a unique line of the regulus.

Similarly, a line m_λ of the form

$$x_0 + \lambda x_3 = 0, x_2 - \lambda x_1 = 0$$

is a generator of $\mathcal{H}_3$ and the set of lines

$$\mathcal{R}_2 = \{m_\lambda : \lambda \in GF(q) \cup \{\infty\}\}$$

also forms a regulus of lines on $\mathcal{H}_3$. We say that $\mathcal{R}_2$ is the **opposite regulus** to $\mathcal{R}_1$. If $\ell_\lambda \in \mathcal{R}_1$ then, as there is a line of $\mathcal{R}_2$ through each point of $\mathcal{H}_3$ and there are $q+1$ lines in $\mathcal{R}_2$, each line of $\mathcal{R}_2$ meets ℓ_λ. That is, each line of regulus meets each line of its opposite regulus.

If $P \in \mathcal{H}_3$ then there is a line of each of $\mathcal{R}_1$ and $\mathcal{R}_2$ through P. The plane generated by these two lines is $T_P(\mathcal{H}_3)$, the tangent plane of $\mathcal{H}_3$ at P and meets $\mathcal{H}_3$ in a quadric of that plane. Therefore, from our knowledge of conics, $T_P(\mathcal{H}_3)$ contains no other points of $\mathcal{H}_3$. Thus, as any line of $\mathcal{H}_3$ containing P is contained in $T_P(\mathcal{H}_3)$ (theorem 1.19), there are no other lines of $\mathcal{H}_3$ through P. Since a regulus of $\mathcal{H}_3$ contains a line through P, $\mathcal{R}_1$ and $\mathcal{R}_2$ are therefore the only reguli on $\mathcal{H}_3$.

The general equation of a quadric in $PG(3,q)$ contains ten coefficients and hence nine points which impose linearly independent conditions on these ten coefficients will generate a unique quadric. If we take three mutually skew lines in $PG(3,q)$ then taking three points from each of these lines we have nine such points. Therefore these three lines define a unique quadric of $PG(3,q)$. As a non-singular hyperbolic quadric is the only quadric of $PG(3,q)$ which contains three mutually skew lines we have;

Theorem 1.31. *Given three mutually skew lines in $PG(3,q)$ there exists a unique hyperbolic quadric for which these lines are generators.* □

Rephrasing this theorem, we say that there is a unique regulus containing three mutually skew lines of $PG(3,q)$. If the lines are ℓ, m, n then we denote the regulus containing these lines by $\mathcal{R}(\ell, m, n)$. A line t of $PG(3,q)$ which intersects each of ℓ, m, n is said to be a **transversal** to ℓ, m and n. Since such a line has three points in common with the hyperbolic quadric with one regulus $\mathcal{R}(\ell, m, n)$ it is a line of this quadric. Thus t is a line of the opposite regulus to $\mathcal{R}(\ell, m, n)$.

1.8 Spreads of $PG(3,q)$

A **spread**, or 1-spread, of $PG(3,q)$ is a set $\mathcal{S}$ of pairwise skew lines of $PG(3,q)$ satisfying the property that each point of $PG(3,q)$ is contained in a unique line of $\mathcal{S}$. As $PG(3,q)$ contains (q^3+q^2+q+1) points, a spread consists of q^2+1 lines and, conversely, any set of q^2+1 pairwise skew lines of $PG(3,q)$ form a spread. If ℓ_1, ℓ_2, ℓ_3 are three skew lines in $PG(3,q)$ then they determine a unique regulus $\mathcal{R}(\ell_1, \ell_2, \ell_3)$ consisting of $q+1$ pairwise skew lines partitioning the points of the hyperbolic quadric defined by the three lines. We call a spread $\mathcal{S}$ of $PG(3,q)$ **regular** if the regulus defined by any three lines of $\mathcal{S}$ is contained in $\mathcal{S}$. The following result is well-known.

Theorem 1.32. *Let $\mathcal{S}$ denote a spread of $PG(3,q)$. $\mathcal{S}$ is regular if and only if the set of lines of $\mathcal{S}$ meeting a line of $PG(3,q)$ which is not in $\mathcal{S}$ form a regulus.* □

1.8.1 A Construction of the Regular Spread

Consider $PG(3,q)$ as a subgeometry of $PG(3,q^2)$. Let ℓ be a line of $PG(3,q^2)$ with no point in $PG(3,q)$ and $\overline{\ell}$ be the conjugate of ℓ with respect to the quadratic extension of $PG(2,q)$. Let $P \in \ell$ and m_P denote the line joining P to its conjugate point $\overline{P} \in \overline{\ell}$. For $P \neq Q$ the lines m_P and m_Q are skew; for otherwise the plane they span would contain ℓ and $\overline{\ell}$, contradicting the fact that these two lines are skew. Since any line m_P contains $q+1$ points of $PG(3,q)$ it follows that the set

$$\mathcal{S} = \{m_P \cap PG(3,q) : P \in \ell\}$$

consists of q^2+1 pairwise skew lines of $PG(3,q)$ and therefore forms a spread of $PG(3,q)$.

If r, s, t are three lines of $\mathcal{S}$ then, as lines of $PG(3,q^2)$, the lines ℓ and $\overline{\ell}$ are transversals to r, s and t. Therefore ℓ and $\overline{\ell}$ are lines of the opposite regulus to $\mathcal{R}(\ell, m, n)$ and thus every line of the regulus $\mathcal{R}(r, s, t)$ (in $PG(3,q^2)$) meets both ℓ and $\overline{\ell}$. Thus, in $PG(3,q)$, all lines of $\mathcal{R}(r, s, t)$ are in $\mathcal{S}$ and so $\mathcal{S}$ is regular. Furthermore it is proved in [12] that any regular spread of $PG(3, q)$ can be constructed in this manner for a unique pair of conjugate lines ℓ and $\overline{\ell}$. Precisely, the following is proved.

Theorem 1.33. [12] *If ℓ is a line of $PG(3, q^2)$ with no point in $PG(3, q)$ then the lines obtained by joining each point P of ℓ to its conjugate point $\overline{P}$ of $\overline{\ell}$ form a regular spread of $PG(3, q)$. Any regular spread of $PG(3, q)$ can be constructed in this way for a unique pair of lines $\ell, \overline{\ell}$ of $PG(3, q^2)$.* □

If $\mathcal{S}$ is a regular spread of $PG(3, q)$ then the lines ℓ and $\overline{\ell}$ of $PG(3, q^2)$ associated with $\mathcal{S}$ as in theorem 1.33 are called the **directrix lines** of $\mathcal{S}$.

1.9 Line Coordinates

As it currently stands, to identify a line ℓ of $PG(n, q)$ we need to express it as the intersection of $(n - 1)$ hyperplanes about ℓ. However there is a way in which we can give homogeneous coordinates to the lines of $PG(n, q)$ and, as we shall find this coordinatisation useful in a subsequent chapter, we present it here. These methods can be extended to give homogeneous coordinates to the m-spaces of $PG(n, q)$ as is done in Hirschfeld, [31], and Hodge and Pedoe, [32]. We use a similar method to these texts, but applied only to the lines of $PG(n, q)$.

Let ℓ be a line in $PG(n, q)$ and $P = (x_0, x_1, \ldots, x_n)$ and $Q = (y_0, y_1, \ldots, y_n)$ be two points of ℓ. Consider the determinants

$$g_{ij}(PQ) = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

We wish to show that the $g_{ij}(PQ)$ are independent of the two points chosen on the line ℓ and hence are suitable to be used as coordinates of ℓ. Note that for any i, j $g_{ij} = -gji$ and $g_{ii} = 0$ so we need only consider the numbers g_{ij} where $i < j$. In other words we need to establish that the $(n+1)n/2$-tuple $(g_{01}(PQ), g_{02}(PQ), \ldots, g_{(n-1)n}(PQ))$ is suitable to be used as homogeneous coordinates of ℓ.

Lemma 1.34. [31] *If R_1 and R_2 are two distinct points of ℓ then $g_{ij}(R_1R_2) = \lambda g_{ij}(PQ)$ for some $\lambda \in GF(q) \setminus \{0\}$ which is independent of i and j.*

Proof: Let $R_1 = p_1P + q_1Q$ and $R_2 = p_2P + q_2Q$, where $p_1, p_2, q_1, q_2 \in GF(q) \cup \{\infty\}$, be any two points of ℓ. Then

$$\begin{vmatrix} p_1x_i + q_1y_i & p_1x_j + q_1y_j \\ p_2x_i + q_2y_i & p_2x_j + q_2y_j \end{vmatrix} = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

Thus $g_{ij}(R_1R_2) = \lambda g_{ij}(PQ)$ where, as R_1 and R_2 are distinct points,

$$\lambda = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \neq 0$$

□

Thus we can use $(g_{00}, g_{01}, \ldots, q_{(n-1)n})$ as homogeneous coordinates for the line ℓ. These homogeneous coordinates for the lines of $PG(n,q)$ are called **Grassmann Coordinates**. When $n = 3$ they are also called **Plücker coordinates**

Theorem 1.35. *If* $(g_{01}, g_{02}, \ldots, g_{(n-1)n})$ *are the coordinates of a line* ℓ *of* $PG(n,q)$ *then the* g_{ij} *satisfy the following conditions;*

$$g_{ij}g_{kl} + g_{il}g_{jk} = g_{ik}g_{jl} \tag{1.6}$$

where $\{i,j,k,l\} \in \{0,1,\ldots,n\}$ *with* $i < j < k < l$.

Proof: Let $X = (x_0, x_1, \ldots, x_n)$ and $Y = (y_0, y_1, \ldots, y_n)$ be two distinct points of ℓ and consider the determinant

$$\begin{vmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \end{vmatrix}$$

where $\{i,j,k,l\} \in \{0,1,\ldots,n\}$ with $i < j < k < l$.

As the determinant contains identical rows it clearly has value 0. However, by simple expansion, the value of this determinant is also $g_{ij}g_{kl} + g_{il}g_{jk} - g_{ik}g_{jl}$. Thus we obtain the required result. □

The coordinate vector of a line of $PG(n,q)$ is a homogeneous $(n+1)n/2$-tuple. Hence, as there are more homogeneous $(n+1)n/2$-tuples than lines in $PG(n,q)$, not all of these $(n+1)n/2$-tuples are the coordinates of lines of $PG(n,q)$. However all the homogeneous $(n+1)n/2$-tuples which satisfy the relations of theorem 1.35 are.

Theorem 1.36. [31] *If* $(g_{01}, g_{02}, \dots, g_{(n-1)n})$ *is a non-zero* $(n+1)n/2$*-tuple of elements of* $GF(q)$ *satisfying the equations 1.6 then* $(g_{01}, g_{02}, \dots, g_{(n-1)n})$ *are the coordinates of a line of* $PG(n, q)$. $\square$

1.10 Grassmann Varieties

The Grassmann coordinates of a line of $PG(n, q)$ are a homogeneous $N = (n+1)n/2$-tuple and so can be considered as a point of $PG(N-1, q)$. Let $\mathcal{L}_{n,q}$ denote the set of lines of $PG(n, q)$ and define the map

$$\gamma : \mathcal{L}_{n,q} \mapsto PG(N-1, q)$$

by $\gamma(\ell) = P$ where the coordinates of the point P are the Grassmann coordinates of the line ℓ. Now let

$$\mathcal{G}_{1,n} = \{P \in PG(N-1, q) : P = \gamma(\ell) \text{ for some line } \ell \text{ of } PG(n, q)\}$$

By theorem 1.35, the Grassmann coordinates of a line of $PG(n, q)$ satisfy the equations 1.6. Hence each such equation defines a quadric of $PG(N-1, q)$ which is contained in $\mathcal{G}_{1,n}$. By theorem 1.36, $\mathcal{G}_{1,n}$ is exactly the intersection of these quadrics. Thus $\mathcal{G}_{1,n}$ is a variety of $PG(N-1, q)$ called the **Grassmann variety** or **Grassmannian** of lines of $PG(n, q)$. In the case $n = 3$ the Grassmannian is a hyperbolic quadric of $PG(5, q)$ called the **Klein quadric**.

Theorem 1.37. [32] *The variety* $\mathcal{G}_{1,n}$ *is absolutely irreducible and non-singular and has dimension* $2(n-1)$ *and order* $\frac{(2(n-1))!}{(n-1)!n!}$. $\square$

We wish to determine the number of subspaces of $PG(N-1, q)$ which are contained on the Grassmannian. To begin, we state the following;

Theorem 1.38. [31] *Let* ℓ *and* ℓ' *be two lines of* $PG(n, q)$ *with* P *and* P' *denoting the points* $\gamma(\ell)$ *and* $\gamma(\ell')$ *of* $\mathcal{G}_{1,n}$*, respectively. The line* PP' *of* $PG(N-1, q)$ *is contained on the Grassmannian if and only if* ℓ *and* ℓ' *intersect. If* ℓ *and* ℓ' *do intersect then the images, under* γ*, of the* $q+1$ *lines of the plane* $\langle \ell, \ell' \rangle$ *which contain the point* $\ell \cap \ell'$ *are the points of the line* PP'. $\square$

Let π be a plane of $PG(n, q)$ and $\mathcal{L}_\pi$ denote the set of lines of π. Suppose that $P_1 = \gamma(\ell_1)$ and $P_2 = \gamma(\ell_2)$ are the points of the Grassmannian corresponding to the lines ℓ_1 and

ℓ_2 of π. By theorem 1.38 the lines of π through the point $\ell_1 \cap \ell_2$ represent a line of the Grassmannian, each point of which is the image under γ of an element of $\mathcal{L}_\pi$. Thus $\gamma(\mathcal{L}_\pi)$ is a subspace of $PG(n,q)$, contained on $\mathcal{G}_{1,n}$. As $\gamma(\mathcal{L}_\pi)$ contains (q^2+q+1) points, this subspace is a plane, called a **Greek plane**. There are as many Greek planes on the Grassmannian $\mathcal{G}_{1,n}$ as there are planes in $PG(n,q)$.

Suppose that α is a Greek plane on $G_{1,n}$ corresponding to the plane π of $PG(n,q)$ and that the 3-space Σ_3 of $PG(N-1,q)$ contains α and is contained in $\mathcal{G}_{1,n}$. Let $P \in \Sigma_3 \setminus \alpha$. As Σ_3 is contained in $\mathcal{G}_{1,n}$ there is a line of $\mathcal{G}_{1,n}$ joining P to every point of α. Thus $\gamma^{-1}(P)$ meets every line of π, which is not possible. Hence any subspace of $PG(N-1,q)$ which contains a Greek plane is not contained on the Grassmannian.

There are, however, other subspaces of $PG(N-1,q)$ on $\mathcal{G}_{1,n}$. Let P be a point of $PG(n,q)$ and $\mathcal{L}_P$ denote the set of lines of $PG(n,q)$ which contain P. As there are $(q^{n-1}+q^{n-2}+\ldots+q+1)$ lines through a point in $PG(n,q)$, $\gamma(\mathcal{L}_P)$ contains $(q^{n-1}+q^{n-2}+\ldots+q+1)$ points of $\mathcal{G}_{1,n}$. Suppose that $P_1=\gamma(\ell_1)$ and $P_2=\gamma(\ell_2)$ are any two such points. In $PG(n,q)$, the $q+1$ lines of the plane $\langle \ell_1, \ell_2 \rangle$ through the point P are mapped by γ to a line of $\mathcal{G}_{1,n}$ contained in $\gamma(\mathcal{L}_P)$. Thus any two points of $\gamma(\mathcal{L}_P)$ are contained in a line of $\gamma(\mathcal{L}_P)$ and so $\gamma(\mathcal{L}_P)$ is a subspace of $PG(N-1,q)$ contained on the Grassmannian. As $\gamma(\mathcal{L}_P)$ contains $(q^{n-1}+q^{n-2}+\ldots+q+1)$ points, it is a subspace of dimension $n-1$. The subspaces of $\mathcal{G}_{1,n}$, which correspond to the points of $PG(n,q)$ in this way are called **Latin spaces** and are $(q^n+\ldots+q+1)$ in number.

As with Greek planes, a Latin space is not contained in any space of higher dimension on the Grassmannian. Suppose that Σ_{n-1} is a Latin space on $\mathcal{G}_{1,n}$ corresponding to the lines of $PG(n,q)$ through the point P. If X is a point of a (possibly larger) space containing Σ_{n-1} then X is collinear, on a line of $\mathcal{G}_{1,n}$, with every point of Σ_{n-1}. Thus $\gamma^{-1}(X)$ intersects $\gamma^{-1}(Y)$ for all points Y of Σ_{n-1}. Thus $\gamma^{-1}(X)$ passes through P and so X is a point of Σ_{n-1}. Summarising, we have;

Theorem 1.39. [29] *The lines of a plane of $PG(n,q)$ correspond to the points of a plane on $\mathcal{G}_{1,n}$, called a Greek plane. The lines through a point of $PG(n,q)$ correspond to the points of a $(n-1)$ dimensional space on $\mathcal{G}_{1,n}$, called a Latin space. Every subspace of $PG(N-1,q)$ contained on $\mathcal{G}_{1,n}$ is either one of these spaces or is a subspace of one of these spaces.* □

1.11 The Ruled Cubic Surface in $PG(4, q)$

Let α be a plane in $PG(4, q)$ containing a non-degenerate conic $\mathcal{C}$ and ℓ be a line of $PG(4, q)$ skew to α. Suppose that the points of ℓ and the points of $\mathcal{C}$ are projectively related (by which we mean that if θ and θ' are non-homogeneous parameters for the points of $\mathcal{C}$ and ℓ, respectively, then $\theta' = \phi(\theta)$, for some $\phi \in PGL(2, q)$). If we join the points of $\mathcal{C}$ and ℓ which correspond to one another under this projective correspondence by lines then the resulting structure V is called a **ruled cubic surface**. $\mathcal{C}$ is known as the **conic directrix**, ℓ as the **line directrix** and the lines joining corresponding points of ℓ and $\mathcal{C}$ are called **generators** of the ruled cubic surface.

Choose coordinates in $PG(4, q)$ so that α is the plane $x_3 = x_4 = 0$, ℓ the line $x_0 = x_1 = x_2 = 0$, $\mathcal{C}$ has equation $x_1^2 = x_0x_2$, $x_3 = x_4 = 0$ and the point $(x^2, xy, y^2, 0, 0)$ of $\mathcal{C}$ corresponds, under the above projective correspondence, to the point $(0, 0, 0, x, y)$ of ℓ. Thus the ruled cubic surface consists of the set of points

$$\{(x^2, xy, y^2, xz, yz)\ : x, y \in GF(q),\ \text{not both zero},\ z \in GF(q) \cup \{\infty\}\} \tag{1.7}$$

Using this formulation we define the following map:

$$\begin{aligned} \sigma : PG(2, q) &\rightarrow \text{V} \\ \sigma : (x, y, z) &\mapsto (x^2, xy, y^2, xz, yz) \end{aligned}$$

Theorem 1.40. *The ruled cubic surface is exactly the intersection of the three quadrics*

$$x_0x_4 - x_1x_3 = 0 \tag{1.8}$$

$$x_2x_3 - x_1x_4 = 0 \tag{1.9}$$

$$x_0x_2 - x_1^2 = 0 \tag{1.10}$$

□

Thus the ruled cubic surface is indeed a variety. To calculate its dimension and order we consider the following map between the plane of $PG(3, q)$ of equation $x_3 = 0$ and V,

$$(x, y, z, 0) \mapsto (x^2, xy, y^2, xz, yz)$$

This defines an algebraic 1-1 correspondence between the points of V and the plane $(x, y, z, 0)$ of $PG(3, q)$. Thus V is an absolutely irreducible variety of dimension two and, moreover, we are justified in calling it a surface.

To determine its order we need to know in how many points a generic plane of $PG(4,q)$ meets V. For this purpose consider the hyperplane $a_0x_0+a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$. This meets the surface in the points with coordinates satisfying

$$a_0x^2+a_1xy+a_2y^2+a_3xz+a_4yz=0 \tag{1.11}$$

Thus, under σ, the points in which a hyperplane intersects V correspond to the points of a conic of $PG(2,q)$ containing the point $(0,0,1)$. Therefore, since a plane of $PG(4,q)$ can be expressed as the intersection of any two hyperplanes containing it, the points in which a plane of $PG(4,q)$ intersects V correspond, under σ, to the points of intersection of two conics of $PG(2,q)$ which both contain $(0,0,1)$. As two such conics generically have three further common points, a generic plane section of V contains three points and the variety has order 3.

Because of this we henceforth denote the ruled cubic surface by V_2^3.

Lemma 1.41. *Any two generators of V_2^3 are skew and no three lie in a common hyperplane.*

Proof: A plane containing two generators necessarily contains ℓ and meets α, the plane of the conic directrix, in a line, contradicting the fact that ℓ and α are skew. Similarly, a hyperplane containing three generators contains both ℓ and α which, once again, contradicts the fact that these spaces are skew. $\square$

Corollary 1.42. *A line of $PG(4,q)$ which is neither a generator nor the line directrix of V_2^3 intersects V_2^3 in 0,1 or 2 points.*

Proof: Suppose the line m (which is neither the line directrix ℓ nor a generator) contains three points of V_2^3. If one of these points is on ℓ then it meets two generators at points not on ℓ and the plane $\langle m,\ell\rangle$ contains the generators through these points which contradicts lemma 1.41. If m is skew to ℓ then $\langle m,\ell\rangle$ is a hyperplane containing the three generators which meet m, in contradiction with lemma 1.41. $\square$

1.12 Segre Varieties

Let $X=(x_0,\dots,x_n)$ be a point of $PG(n,q)$ and $Y=(y_0,\dots,y_m)$ be a point of $PG(m,q)$. Consider the set of points $\mathcal{S}$ of $PG((n+1)(m+1)-1,q)$ of the form

$$\{(x_0y_0,\dots,x_0y_m,x_1y_0,\dots,x_ny_m) : X\in PG(n,q) \text{ and } Y\in PG(m,q)\}$$

Each such point lies on each of the quadrics (for $n < m$)

$$X_i X_{k(n+1)+(j+1)} = X_{j+1} X_{k(n+1)+i}$$

where $i, j = 0, 1, \ldots, n$ with $j \geq i$ and $k = 1, \ldots, m$ and every point common to these quadrics is of the above form. Thus $\mathcal{S}$ is a variety in $PG((n+1)(m+1)-1, q)$ called a **Segre variety**. The Segre variety is absolutely irreducible and non-singular and has dimension nm and order $\frac{(n+m)!}{n!m!}$ (see [31]). We denote it $\mathcal{V}_{n,m}$ and sometimes call it the **Segre product** of $PG(n, q)$ and $PG(m, q)$. The Segre variety was introduced by C.Segre in 1891 (see [46]) where it is discussed over the infinite field. For more information on this case see Hodge and Pedoe, [32]. The Segre variety over a finite field has been studied by Hirschfeld [31] and Casse and O'Keefe [19]. We follow [19].

We discuss the subspaces of $PG((n+1)(m+1)-1, q)$ which are contained on $\mathcal{V}_{n,m}$. Fix $Q = (y_0, \ldots, y_m)$ and consider the set of points

$$\Sigma_n = \{(x_0 y_0, \ldots, x_0 y_m, x_1 y_0, \ldots, x_n y_m) : (x_0, \ldots, x_n) \in PG(m, q)\}$$

The points of Σ_n are spanned by the $(n+1)$ linearly independent points

$$(y_0, \ldots, y_m, 0, \ldots, 0, \ldots, 0, \ldots, 0),$$

$$(0, \ldots, 0, y_0, \ldots, y_m, \ldots, 0, \ldots, 0),$$

$$\vdots$$

$$(0, \ldots, 0, 0, \ldots, 0, \ldots, y_0, \ldots, y_m)$$

and thus Σ_n is a subspace of dimension n of $PG((n+1)(m+1)-1, q)$ which is contained in $\mathcal{V}_{n,m}$. As we vary Q over all points of $PG(m, q)$ we obtain $(q^m + \ldots + q + 1)$ such subspaces, any two of which have no common point.

Similarly, by fixing $R = (x_0, \ldots, x_n)$ we see that the set of points

$$\Sigma_m = \{(x_0 y_0, \ldots, x_0 y_m, x_1 y_0, \ldots, x_n y_m) : (y_0, \ldots, y_m) \in PG(m, q)\}$$

is a subspace of dimension m on $\mathcal{V}_{n,m}$. Again, as there are $(q^n + \ldots + q + 1)$ candidates for R we obtain a set of $(q^n + \ldots + q + 1)$ pairwise skew m-dimensional spaces on $\mathcal{V}_{n,m}$ in this way.

Lemma 1.43. [19] *The Segre variety $\mathcal{V}_{n,m}$ contains two systems of subspaces, the first, Γ_n, consists of $(q^m + \ldots + q + 1)$ pairwise skew n-dimensional spaces and the second, Γ_m consists of $(q^n + \ldots + q + 1)$ pairwise skew m-dimensional spaces. Two subspace from opposite systems have a unique common point and any point of $\mathcal{V}_{n,m}$ is contained in a unique subspace from each system.*

Proof: It only remains to prove to the last sentence. If Σ_n is an element of Γ_n corresponding to the point Q (as in the above discussion) and if Σ_m is an element of Γ_m corresponding to the point R then Σ_n and Σ_m meet exactly at the point $(x_0y_0, \ldots, x_0y_m, x_1y_0, \ldots, x_ny_m)$. Conversely, the point $(x_0y_0, \ldots, x_0y_m, x_1y_0, \ldots, x_ny_m)$ of $\mathcal{V}_{n,m}$ lies only in the elements of Γ_n and Γ_m corresponding to the points Q and R respectively. □

As a direct corollary we have;

Corollary 1.44. *The Segre variety $\mathcal{V}_{n,m}$ contains $(q^n + \ldots + q + 1)(q^m + \ldots + q + 1)$ points.*

□

Theorem 1.45. [19] *The system of n-spaces on the Segre variety $\mathcal{V}_{n,m}$ can be obtained by joining corresponding points of $(n + 1)$ pairwise disjoint and projectively related m-spaces in $PG((n + 1)(m + 1) - 1, q)$. Similarly, the system of m-spaces on $\mathcal{V}_{n,m}$ can be obtained by joining corresponding points of $(m + 1)$ pairwise disjoint and projectively related n-spaces in $PG((n + 1)(m + 1) - 1, q)$.*

Proof: If $\Sigma_0, \ldots, \Sigma_n$ denote $(n + 1)$ pairwise disjoint m-spaces then we may choose coordinates in $PG((n + 1)(m + 1) - 1, q)$ so that they contain the sets of points:

$$\Sigma_0 : \{(x_0, \ldots, x_m, 0, \ldots, 0, \ldots, 0, \ldots, 0) : x_i \in GF(q)\},$$

$$\Sigma_1 : \{(0, \ldots, 0, x_0, \ldots, x_m, \ldots, 0, \ldots, 0) : x_i \in GF(q)\},$$

$$\vdots$$

$$\Sigma_n : \{(0, \ldots, 0, 0, \ldots, 0, \ldots, x_0, \ldots, x_m) : x_i \in GF(q)\}.$$

Furthermore, by an appropriate choice of coordinates, we may assume that the projective correspondence between $\Sigma_0, \ldots, \Sigma_n$ relates points $P_0, \ldots, P_n$ of the form;

$$P_0 : (y_0, \ldots, y_m, 0, \ldots, 0, \ldots, 0, \ldots, 0),$$

$$P_1 : (0, \dots, 0, y_0, \dots, y_m, \dots, 0, \dots, 0),$$

$$\vdots$$

$$P_n : (0, \dots, 0, 0, \dots, 0, \dots, y_0, \dots, y_m)$$

where $y_0, \dots, y_m$ are not all zero. For each choice of $y_0, \dots, y_m$ the points $P_0, \dots, P_n$ span a subspace of $PG((n+1)(m+1)-1, q)$ of dimension n and the set of all such n-spaces are the n-spaces of the Segre variety $\mathcal{V}_{n,m}$. The proof of the second statement is analogous to this. □

Theorem 1.46. [19] *There is a unique Segre variety $\mathcal{V}_{n,m}$ containing any $(n+2)$ m-spaces of $PG((n+1)(m+1)-1, q)$, no $(n+1)$ of which lie in a hyperplane. Similarly, there is a unique Segre variety $\mathcal{V}_{n,m}$ containing $(m+2)$ n-spaces of $PG((n+1)(m+1)-1, q)$, no $(m+1)$ of which lie in a hyperplane.*

Proof: Let $\Sigma_0, \dots, \Sigma_{n+1}$ denote $(n+2)$ pairwise disjoint m-dimensional subspaces of $PG((n+1)(m+1)-1, q)$, no $(n+1)$ in a hyperplane. Note that the condition that no $(n+1)$ of the Σ_i lie in a hyperplane implies that the Σ_i are pairwise disjoint. Now, through a general point of $PG((n+1)(m+1)-1, q)$ there passes a unique n-space meeting each of $(n+1)$ pairwise disjoint m-spaces in a unique point. Therefore, through each of the $(q^m + q^{m-1} + \dots + 1)$ points of Σ_0 there is a unique n-space meeting each of $\Sigma_1, \dots, \Sigma_{n+1}$. The $(q^m + q^{m-1} + \dots + 1)$ n-spaces so obtained are pairwise disjoint and are one system of subspaces of a Segre variety. The proof of the second statement is analogous. □

1.12.1 The Cubic Scroll of Planes in $PG(5, q)$

Consider the Segre product of a plane and a line. This is the set of points

$$\{(y_0z_0, y_0z_1, y_0z_2, y_1z_0, y_1z_1, y_1z_2) : (y_0, y_1) \in PG(1, q), (z_0, z_1, z_2) \in PG(2, q)\} \quad (1.12)$$

of $PG(5, q)$ and is also known as the **cubic scroll of planes** in $PG(5, q)$.

Every point of the cubic scroll of planes lies on each of the 3 quadrics with equations,

$$x_1x_5 - x_2x_4 = 0 \quad (1.13)$$

$$x_2x_3 - x_0x_5 = 0 \quad (1.14)$$

$$x_0x_4 - x_1x_3 = 0 \quad (1.15)$$

and conversely, each point in common to these three quadrics is a point of the cubic scroll. Thus it is a variety. We can infer directly from the coordinatisation given above that the cubic scroll has dimension 3 and we will see shortly that it has order 3. Due to this we denote the cubic scroll by R_3^3.

Alternatively, we can define the cubic scroll as follows. Let π and π' be two skew planes in $PG(5,q)$ and suppose that ω defines a projective correspondence between them. If we join corresponding points under ω by lines then the set of points on these (q^2+q+1) lines are the points of a cubic scroll of planes. (For if we choose coordinates so that the point $(z_0, z_1, z_2, 0, 0, 0)$ of π corresponds to the point $(0, 0, 0, z_0, z_1, z_2)$ of π' under ω then it is evident that we obtain the set of points 1.12; theorem 1.45.)

As it is a Segre variety there are two systems of subspaces contained on the cubic scroll. The first consists of $(q+1)$ skew planes and the second of (q^2+q+1) skew lines. Each point of R_3^3 lies on exactly one space of each system and spaces from different systems intersect in a unique point. Any subspace contained in R_3^3 is either contained in one of these spaces or is one of these spaces.

If π is a plane of R_3^3 and ℓ is a line of π then the set of lines of R_3^3 meeting ℓ meet each plane of the scroll in a line and are the lines of a Segre variety $V_{1,1}$ which is a hyperbolic quadric. There are (q^2+q+1) lines in π and thus (q^2+q+1) hyperbolic quadrics on the scroll.

We discuss the ways in which a hyperplane of $PG(5,q)$ can intersect the cubic scroll. As any two planes of R_3^3 are skew, a hyperplane can contain at most one plane of the scroll and accordingly a hyperplane intersects the scroll in two possible ways.

Let Σ_4 be a hyperplane in $PG(5,q)$ containing the plane π of the scroll. As a hyperplane meets each plane of $PG(5,q)$ in at least a line, Σ_4 meets each of the remaining planes of R_3^3 in a line. Suppose that it meets π' in ℓ'. Since the lines of R_3^3 join projectively related points of π and π' the lines of R_3^3 meeting ℓ' intersect π in a projectively related line, ℓ. Hence these are the lines of a Segre variety $\mathcal{V}_{1,1}$ (theorem 1.45), which is a hyperbolic quadric. That is, these lines constitute one regulus of a hyperbolic quadric and, together with ℓ and ℓ', the lines in which Σ_4 meets the remaining $q-1$ planes of R_3^3 constitute the opposite regulus of this hyperbolic quadric.

Now suppose that Σ_4 contains no plane of the scroll and hence meets every plane of the scroll in a line and, in particular, meets π_1 in ℓ_1 and π_2 in ℓ_2. Again, these planes are projectively related, under projectivity ω, say, and $\omega(\ell_1)$ is a line ℓ_1' of π_2. The line g

of R_3^3 through $\ell_2 \cap \ell_1'$ joins corresponding points under ω and so joins this point to a point of ℓ_1 and is thus contained in Σ_4. Let m be a line of π_1 which does not contain the point $g \cap \pi_1$. The $(q+1)$ lines of the cubic scroll which intersect m are one regulus $\mathcal{R}$ of a three-dimensional hyperbolic quadric $\mathcal{Q}$. Σ_4 does not contain $\mathcal{Q}$, nor any line of $\mathcal{R}$, and thus meets $\mathcal{Q}$ in a non-singular conic $\mathcal{C}$. The intersection of Σ_4 with R_3^3 therefore consists of the $q+1$ lines joining the points of g to the points of $\mathcal{C}$. These points are projectively related so the resulting structure is a ruled cubic surface. Hence we have,

Theorem 1.47. *A hyperplane intersects the cubic scroll of planes in either a ruled cubic surface or in a plane together with a hyperbolic quadric.* □

Thus the cubic scroll of planes is indeed of order 3. In addition, as there are precisely (q^2+q+1) hyperbolic quadrics on R_3^3, there are $(q^2+q+1)(q+1)$ hyperplanes meeting the scroll in a plane and a quadric and hence (q^4-q^2-q) meeting it in a ruled cubic surface.

1.13 Spreads

In section 1.8 we defined and discussed a spread of lines of $PG(3,q)$; we extend this work here. A **t-spread** of $PG(n,q)$ is a set of pairwise disjoint t-spaces of $PG(n,q)$ which partition the points of $PG(n,q)$. A 1-spread is usually called a **spread**. If $\mathcal{S}$ is a t-spread of $PG(n,q)$ then clearly the number of points in $PG(n,q)$ must be a multiple of the number of points in $PG(t,q)$. Thus

$$(q^{t+1}-1)|(q^{n+1}-1)$$

which implies that $(t+1)|(n+1)$. We write

$$(n+1) = (r+1)(t+1)$$

for some integer $r \geq 0$. Moreover, it is well-known (see, for example, [23]) that this condition is sufficient to guarantee the existence of a t-spread of $PG(n,q)$ as the following construction of a t-spread of $PG((r+1)(t+1)-1,q)$ shows. See [18], Thas, and we quote from there.

Let $\Sigma_0, \Sigma_1, \ldots, \Sigma_t$ be $(t+1)$ r-dimensional subspaces of $PG(n,q^{t+1})$ which generate the space and are conjugate with respect to the $(t+1)$-th extension $GF(q^{t+1})$ of $GF(q)$ (that is, if σ denotes the collineation of $PG(n,q^{t+1})$ induced by the automorphism of $GF(q^{t+1})$

mapping x to x^q then $\Sigma_i = \Sigma_0^{\sigma^i}$)). Let $P_0 \in \Sigma_0$ and $P_1, \ldots, P_t$ be the t points conjugate to P_0; so $P_i \in \Sigma_i$. Let Π be the t-dimensional subspace generated by these points. Now Π meets $PG(n,q)$ in a subspace of $PG(n,q)$ of dimension t. The set of all $\frac{(q^{n+1}-1)}{(q^{t+1}-1)}$ such t-spaces forms a t-spread of $PG(n,q)$.

Lemma 1.48. *A necessary and sufficient condition for the existence of a t-spread of $PG(n,q)$ is that*

$$n = (r+1)(t+1) - 1$$

for some integer $r \geq 0$. □

We say that $\mathcal{S}$, a t-spread of $PG((r+1)(t+1)-1,q)$, induces a t-spread in the subspace Σ_d of $PG((r+1)(t+1)-1,q)$ if every element of $\mathcal{S}$ which intersects Σ_d is contained in Σ_d. If $\mathcal{S}$ induces a t-spread in the subspace generated by any two elements of $\mathcal{S}$ then we say that $\mathcal{S}$ is a **geometric spread** (Baer [2]) of $PG((r+1)(t+1)-1,q)$. In some of the literature the term **normal** is used to describe a spread satisfying this property.

From a geometric spread $\mathcal{S}$ of $PG((r+1)(t+1)-1,q)$ we can construct an incidence structure[2] $\mathcal{I}(\mathcal{S})$ as follows. The points of $\mathcal{I}(\mathcal{S})$ are the elements of $\mathcal{S}$, the lines of $\mathcal{I}(\mathcal{S})$ are the subspaces generated by two elements of $\mathcal{S}$ and the incidence is containment; this is as in [45]. There the following results are proved.

Theorem 1.49. [45] *A geometric t-spread of $PG(n,q)$ exists if and only if $(t+1)$ divides $(n+1)$.*

Let $\mathcal{S}$ be a geometric t-spread of $PG((r+1)(t+1)-1,q)$, with $r \geq 0$. The incidence structure $\mathcal{I}(\mathcal{S})$ is isomorphic to the incidence structure formed by the points and lines of $PG(r,q^{t+1})$. □

In the case where $t = 1$ and $r = 2$, $\mathcal{S}$ is a spread of lines of $PG(5,q)$ and the incidence structure $\mathcal{I}(\mathcal{S})$ is isomorphic to $PG(2,q^2)$. This representation of $PG(2,q^2)$ has been studied by R.C.Bose in [8]; we shall study this in later chapters.

1.13.1 Regular t-spreads

Let $\mathcal{S}$ denote a t-spread of $PG((r+1)(t+1)-1,q)$. We define the term regular for such spreads.

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[2] By an incidence structure we mean a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{I}$ are two sets whose elements are called points and lines, respectively, and $\mathcal{I}$ is a subset of $\mathcal{P} \times \mathcal{L}$. For example, a projective space is an incidence structure with respect to the points, lines and natural incidence of that space.

A **t-regulus** in $PG((r+1)(t+1)-1,q)$ is the set of t-spaces of a Segre variety $\mathcal{V}_{r,t}$, ([19]). Again, a 1-regulus is usually called a regulus. As a Segre variety $\mathcal{V}_{1,1}$ is a hyperbolic quadric of $PG(3,q)$, this definition is equivalent with that already given for a regulus of lines of $PG(3,q)$.

Theorem 1.50. [19] *There is a unique t-regulus through any $(r+2)$ t-spaces of $PG((r+1)(t+1)-1,q)$, no $(r+1)$ of which lie in a hyperplane.*

Proof: This is a restatement of theorem 1.46. □

We then define a regular spread of t-spaces of $PG((r+1)(t+1)-1,q)$ as follows: a t-spread $\mathcal{S}$ of $PG((r+1)(t+1)-1,q)$ is said to be **regular** (see [19]) if given any r-space in $PG((r+1)(t+1)-1,q)$ not containing more than one point of any element of $\mathcal{S}$, the $(q^r+\ldots+q+1)$ elements of $\mathcal{S}$ which intersect it form a t-regulus.

In the case of a 1-spread of $PG(3,q)$ this definition says that $\mathcal{S}$ is regular if the lines of $\mathcal{S}$ meeting any line of $PG(3,q)$ not contained in $\mathcal{S}$ form a regulus. Thus, by theorem 1.32, this definition is equivalent to that given earlier for 1-spreads of $PG(3,q)$.

Note that for $q=2$ a t-regulus of $PG(2t+1,2)$ consists of 3 t-spaces and so, by theorem 1.46, every t-spread of $PG(2t+1,2)$ is regular. In some of the literature this situation is avoided by defining a regular spread as one that can be constructed using the construction given immediately prior to the statement of lemma 1.48. To conclude this section we quote the following result.

Theorem 1.51. [19] *A t-spread of $PG((r+1)(t+1)-1,q)$, with $r\geq 2$ is regular if and only if it is geometric.* □

1.14 Unitals

Let $\mathcal{K}$ be a set of k points of $PG(2,q)$. If for every line ℓ of $PG(2,q)$

$$|\ell\cap\mathcal{K}|\in\{n_1,n_2,\ldots,n_r\}$$

where $0\leq n_i\leq(q+1)$ then we say that $\mathcal{K}$ is a **k-arc of type $(n_1,n_2,\ldots,n_r)$**. A line containing n_i points of $\mathcal{K}$ is called an **n_i-secant** of $\mathcal{K}$. If we speak of a k-arc without mentioning its type it is assumed that the k-arc is of type $(0,1,2)$.

Consider a k-arc $\mathcal{K}$ of type (m,n). If $m=0$ such an arc is called a **maximal arc**, of which two trivial examples are the plane $PG(2,q)$ itself (with $n=q+1$) and the set $PG(2,q) \setminus \ell$ for any line ℓ of $PG(2,q)$ (with $n = q$). Excluding the case where $\mathcal{K} = PG(2,q)$, the existence of a maximal arc with n points on a line in $PG(2,q)$ implies that n divides q, as is shown by Cossu in [22]. For q odd this condition is in fact sufficient and examples of maximal arcs with n points on a line in $PG(2,q)$ have been demonstrated (see Denniston [24] and Thas [54]) for every n dividing q. However, when q is odd the opposite is true and besides the trivial examples there are no maximal arcs in $PG(2,q)$ for q odd (see [3]).

Now let $\mathcal{K}$ be a k-arc of type (m,n) with $m \neq 0$, see Tallini-Scafati [53] for a discussion of the elementary properties of such arcs. Here, by considering the number of points of $\mathcal{K}$ on the lines through various points of $PG(2,q)$ it is shown that k satisfies

$$mq + n \leq k \leq (n-1)q + m$$

Arcs satisfying this lower bound are called **minimal with respect to type** and those satisfying the upper bound are called **maximal with respect to type**. Furthermore the following result is proved;

Theorem 1.52. [53] *If $PG(2,q)$ contains a k-arc of type (m,n) with $m \neq 0$ then*

1. *q is a square.*
2. *If $\mathcal{K}$ is maximal with respect to type then it is either the complement of a Baer subplane or a $(q\sqrt{q}+1)$-arc of type $(1, \sqrt{q}+1)$.*
3. *If $\mathcal{K}$ is minimal with respect to type it is either a Baer subplane or the complement of a $(q\sqrt{q}+1)$-arc of type $(1, \sqrt{q}+1)$.*

□

We say that a (q^3+1)-arc of type $(1, q+1)$ in $PG(2,q^2)$ is a **unital**. In some of the literature the term **hermitian arc** is also used.

Let $\mathcal{U}$ denote a unital in $PG(2,q^2)$. Every line of the plane meets $\mathcal{U}$ in one or $(q+1)$ points and we call such lines **tangent** lines and **secant** lines of $\mathcal{U}$ respectively.

Let P be a point of $\mathcal{U}$. There are (q^2+1) lines passing through P and each of the remaining q^3 points of $\mathcal{U}$ lie on a unique such line. Hence, as each line through P is either tangent or secant to the unital, q^2 of these lines are secants of $\mathcal{U}$ and one is a tangent of $\mathcal{U}$. Thus, as $\mathcal{U}$ contains (q^3+1) points it has (q^3+1) tangent lines and $(q^4+q^2+1) - (q^3+1) = (q^4 - q^3 + q^2)$ secants. Similarly, by considering the lines through a point of $PG(2,q^2) \setminus \mathcal{U}$, it is easily shown that of the (q^2+1) lines through such a point $(q+1)$ are tangents of $\mathcal{U}$ and (q^2-q) are secant lines. For future reference we gather these results together.

Lemma 1.53. *Let $\mathcal{U}$ be a unital in $PG(2,q^2)$. In total $\mathcal{U}$ has q^3+1 tangent lines and $(q^4-q^3+q^2)$ secant lines. Through a point of $\mathcal{U}$ there passes a unique tangent and q^2 secants. Through a point of $PG(2,q^2) \setminus \mathcal{U}$ there passes $q+1$ tangents and (q^2-q) secants of $\mathcal{U}$.* □

1.14.1 The Classical Unital

Recall from section 1.2.1 that a bijective correspondence σ between the points and lines of $PG(2,q^2)$ which is incidence preserving and of order two is called a polarity. A point P of $PG(2,q^2)$ for which $P \in P^\sigma$ is called an absolute point of σ and similarly a line ℓ of $PG(2,q^2)$ for which $\ell^\sigma \in \ell$ is called an absolute line of σ.

A polarity of $PG(2,q^2)$ for which the set of absolute points is projectively equivalent to the set of points of the curve defined by the equation

$$x_0^{q+1} + x_1^{q+1} + x_2^{q+2} = 0$$

is called a **unitary polarity**.

The set of absolute points of a unitary polarity σ in $PG(2,q^2)$ form a unital called a **classical unital** (or sometimes an **hermitian curve**). The secant lines of a classical unital are the non-absolute lines of the polarity.

If $P \in \mathcal{U}$ and ℓ is the polar of P then, as P is an absolute point, $P \in \ell$ and hence, as σ is incidence preserving, $\ell^\sigma \in P^\sigma = \ell$ and ℓ is absolute. Therefore the polar of P is the unique tangent to $\mathcal{U}$ through P. Similarly, the pole of a tangent line to $\mathcal{U}$ is the unique point of the unital on that tangent line. If $P \notin \mathcal{U}$ then P is incident with $(q+1)$ tangent lines $\ell_1, \ldots, \ell_{q+1}$ of $\mathcal{U}$ and hence, as σ is incidence preserving,

$$P \in \ell_i \to \ell_i^\sigma \in P^\sigma$$

That is, the $(q+1)$ points of $\mathcal{U}$ $\ell_i^\sigma, \ldots, \ell_{q+1}^\sigma$ are collinear on the polar line of P. Similarly, if ℓ is a secant line of $\mathcal{U}$ the tangent lines of $\mathcal{U}$ intersecting ℓ in a point of $\mathcal{U}$ are concurrent at the pole of ℓ. Furthermore, this property classifies classical unitals among all unitals.

Theorem 1.54. [56] *Let $\mathcal{U}$ be a unital of $PG(2,q^2)$. If the tangents of $\mathcal{U}$ at collinear points of $\mathcal{U}$ are concurrent then $\mathcal{U}$ is classical.* □

The fact that the points of a classical unital are the absolute points of a unitary polarity allows us to deduce other configurational properties of the points of a classical unital.

Let A, B, C, D be four points of the unital $\mathcal{U}$, no three of which are collinear, with a, b, c, d denoting the tangents of $\mathcal{U}$ at these points. We say that $\mathcal{U}$ satisfies the condition of reciprocity if whenever $a \cap b$ is a point of the line joining C and D then $c \cap d$ is a point of the line joining A and B. Now suppose that $\mathcal{U}$ is a classical unital, $X = a \cap b$ and $Y = c \cap d$. As CD is the polar of Y, the condition that $X \in CD$ is equivalent to X being incident with the polar of Y. Thus, by the pole-polar property, this implies that Y is on the polar of X, or, in other words, $Y \in AB$. Hence the classical unital satisfies the condition of reciprocity. Actually, this property also characterises the classical unital.

Theorem 1.55. [53] *A unital is classical if and only if it satisfies the condition of reciprocity.* □

It can also be shown that the points of a secant line of a classical unital are the points of a Baer subline of a line of $PG(2, q^2)$. Conversely, we have the theorem;

Theorem 1.56. [35], [26] *Let $\mathcal{U}$ be a unital in $PG(2, q^2)$, where $q > 2$. If, for every secant line ℓ of $\mathcal{U}$, the points of $\mathcal{U}$ on ℓ are the points of a Baer subline then $\mathcal{U}$ is a classical unital.* □

Equivalently, this can be stated as, if each Baer subline of $PG(2, q^2)$ meets $\mathcal{U}$ in 0,1,2 or $q + 1$ points then $\mathcal{U}$ is classical.

This has been improved to,

Theorem 1.57. [6] *Let $\mathcal{U}$ be a unital in $PG(2, q^2)$, where $q > 2$ and ℓ be a secant line of the unital. If every Baer subline with a point on ℓ meets $\mathcal{U}$ in 0,1,2 or $q + 1$ points then $\mathcal{U}$ is classical.* □

1.15 The Bruck-Bose Representation of Translation Planes

In their papers [13] and [14] Bruck and Bose give a construction for any translation plane with kernel containing $GF(q)$ (see [33] for a definition of these terms). In particular any finite translation plane, and thus $PG(2, q)$, can be constructed in this way. This construction has proved useful in the study of certain structures in translation planes, most notably in the study of unitals. It should be noted that the construction given by Bruck and Bose is equivalent to one given earlier by Andre in [1].

Let Σ_∞ be a $(2t-1)$-space in $\Sigma = PG(2t,q)$ and let $\mathcal{S}$ be a $(t-1)$-spread of Σ_∞. Define an incidence structure $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ as follows.

The points of $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ are of two types. The first being the points of Σ not contained in Σ_∞ and the second being the elements of $\mathcal{S}$.

The lines of $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ are the t-spaces of Σ meeting Σ_∞ exactly in an element of $\mathcal{S}$ and one ideal line ℓ_∞ which consists of the elements of $\mathcal{S}$.

The incidence in $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ is given by natural containment.

Theorem 1.58. [13], [14] *$\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ is a translation plane of order q^t with translation line ℓ_∞ and every translation plane of order q^t which has a kernel containing $GF(q)$ can be constructed in this way.* □

We call the structure in $PG(2t,q)$ associated to a translation plane $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ the **Bruck-Bose representation** of $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$. The map associating $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ to its Bruck-Bose representation is denoted $\boldsymbol{\beta_2}$, and called the **Bruck-Bose map**, so that if A denotes some substructure of $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ its Bruck-Bose representation is denoted $\boldsymbol{\beta_2}(A)$ or $A^{\boldsymbol{\beta_2}}$.

In addition;

Theorem 1.59. [13] *For $q > 2$, the translation plane $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$ is isomorphic to $PG(2, q^t)$ if and only if $\mathcal{S}$ is a regular spread.* □

Note that when $q = 2$ all $(t-1)$ spreads of $PG(2t-1, 2)$ are regular, by the definition given in section 1.13.1, but there are translation planes of order 2^t which are not isomorphic to $PG(2, 2^t)$.

1.15.1 The Representation of the Baer Subplanes of $PG(2, q^2)$

The above construction has had greatest use in the study of the Desarguesian plane $PG(2, q^2)$. Of particular importance is the representation of Baer substructures of $PG(2, q^2)$. To consider these we assume that $t = 2$ so that $\Sigma = PG(4, q)$, Σ_∞ is a 3-space contained in Σ and $\mathcal{S}$ is a regular spread of Σ_∞. Throughout we let ℓ_∞ denote the line of $PG(2, q^2)$ which corresponds to Σ_∞ under the Bruck-Bose map.

A plane π of $PG(4,q)$ meeting Σ_∞ in a line m which is not an element of $\mathcal{S}$ is called a **transversal plane**. The points of $\pi \setminus m$, together with the $q+1$ elements of $\mathcal{S}$ $\ell_1, \ldots, \ell_{q+1}$ which intersect m, are the Bruck-Bose representation of q^2+q+1 points of $PG(2,q^2)$. These points form a Baer subplane of $PG(2,q^2)$ (as is shown in [13]) for which the Baer sublines are represented by the lines of π meeting m. In this respect we say that π is the Bruck and Bose representation of a Baer subplane of $PG(2,q^2)$ meeting ℓ_∞ in $q+1$ points. As any line of $PG(4,q)$ not in Σ_3 is contained in a transversal plane, each of these lines represents a Baer subline with a unique point on ℓ_∞. Conversely, by counting, we can show that the converse of these statements hold. That is, we have;

Theorem 1.60. *A transversal plane of $\mathcal{S}$ is the Bruck-Bose representation of a Baer subplane of $PG(2,q^2)$ meeting ℓ_∞ in $q+1$ points. Conversely, a Baer subplane of $PG(2,q^2)$ containing $(q+1)$ points of ℓ_∞ is represented under the Bruck-Bose map by a transversal plane of $\mathcal{S}$.*

A Baer subline of a line of $PG(2,q^2)$ with a unique point on ℓ_∞ is represented by a line of Σ meeting Σ_∞ in a unique point and, conversely, any such line is the Bruck-Bose representation of a Baer subline of a line of $PG(2,q^2)$ with one point on ℓ_∞. □

Note that the first statement above is true in any translation plane $\pi(\Sigma, \Sigma_\infty, \mathcal{S})$, regardless of whether $\mathcal{S}$ is regular or not but that the converse is not true in general for translation planes other than $PG(2,q^2)$ (see, for example, Freeman [27]). For the case of Baer subplanes meeting ℓ_∞ in a unique point we have;

Theorem 1.61. [21], [9], [59] *In $PG(2,q^2)$ a Baer subplane meeting ℓ_∞ in a unique point is represented in $PG(4,q)$ by a ruled cubic surface with line directrix a line of $\mathcal{S}$ and conic directrix skew to Σ_∞ and lying in a plane meeting Σ_∞ in a (distinct) line of $\mathcal{S}$.* □

For Baer sublines with no point on ℓ_∞ we have;

Theorem 1.62. [35] *A Baer subline of $PG(2,q^2)$ with no point on ℓ_∞ is represented by a conic of Σ with no point in Σ_∞ and contained in a plane meeting Σ_∞ in a line of $\mathcal{S}$.* □

However in this case the converse is not true, that is not all conics of $PG(4,q)$ satisfying the above properties are the Bruck-Bose representation of a Baer subline of a line of $PG(2,q^2)$ (see Metz, [37]). Those that do are called **Baer conics**. Similarly not all ruled cubic surfaces with line directrix a line of $\mathcal{S}$ and conic directrix situated as in result 1.61 are the Bruck-Bose representation of a Baer subplane of $PG(2,q^2)$. Those which are are called **Baer ruled cubics** [21]. In section 5.1.1 we find conditions which characterise the Baer conics and the Baer ruled cubics among all conics and ruled cubics of $PG(4,q)$.

1.15.2 The Bruck-Bose Map and Unitals

Let $\mathcal{U}$ be a unital in $PG(2,q^2)$. If ℓ_∞ is secant to $\mathcal{U}$ then we call $\mathcal{U}$ a hyperbolic unital and if ℓ_∞ is tangent to $\mathcal{U}$ we say that $\mathcal{U}$ is a parabolic unital. In [17] Buekenhout gave the structure of $\boldsymbol{\beta_2}(\mathcal{U}) = \overline{\mathcal{U}}$, the Bruck-Bose representation of $\mathcal{U}$ in the case where $\mathcal{U}$ is a classical unital. There it is shown that if $\mathcal{U}$ is hyperbolic then $\overline{\mathcal{U}}$ is a non-singular quadric of $PG(4,q)$ intersecting Σ_∞ in a regulus (hyperbolic quadric) and, alternatively, if $\mathcal{U}$ is parabolic then $\overline{\mathcal{U}}$ is a quadric cone consisting of the lines joining a point T of Σ_∞ to the points of a three-dimensional non-singular elliptic quadric which has a unique point on Σ_∞. Note that the line of the spread through T is a line of this quadric cone and corresponds under the Bruck-Bose map to the unique point of $\mathcal{U}$ on ℓ_∞.

Conversely, Buekenhout noted that if $\mathcal{Q}$ is a non-singular four-dimensional quadric in $PG(4,q)$ which meets Σ_∞ in a regulus belonging to $\mathcal{S}$ then $\boldsymbol{\beta_2}^{-1}(\mathcal{Q})$ is a hyperbolic unital in $PG(2,q^2)$. Such unitals are known as **Buekenhout hyperbolic unitals**. A classical unital is a Buekenhout hyperbolic unital with respect to any secant line and in [6], by the use of a counting argument, it is shown that any Buekenhout hyperbolic unital is a classical unital.

Similarly, in [17], it was noted that any quadric cone meeting Σ_∞ in a line t of $\mathcal{S}$ and consisting of the lines joining a point of t to the points of an elliptic quadric which meets Σ_∞ exactly in a (distinct) point of t corresponds, under the Bruck-Bose map, to a unital of $PG(2,q^2)$. Furthermore if we replace the elliptic quadric in the above construction with an ovoid (also meeting Σ_∞ precisely in a point of t) then the ovoidal cone so formed corresponds to a unital in $PG(2,q^2)$. Such a unital is called a **Buekenhout-Metz unital**. If T is the point of $PG(2,q^2)$ corresponding to t under the Bruck-Bose map then we sometimes say that $\mathcal{U}$ is Buekenhout-Metz wrt (T,ℓ_∞). Note that ℓ_∞ is necessarily the unique tangent line to $\mathcal{U}$ through T. If the base ovoid of a Buekenhout-Metz unital is not an elliptic quadric then the unital is not classical. Thus, as there exists examples of ovoids which are not elliptic quadrics when q is even, we have non-classical unitals in $PG(2,q^2)$ for q even.

In [37] Metz proved the existence of non-classical unitals in $PG(2, q^2)$ for all q. He did this by showing (roughly) that there are more elliptic quadric cones, positioned as the Buekenhout-Metz unitals with respect to Σ_∞, than classical unitals in $PG(2, q^2)$ and thus not all of these unitals are classical.

Some Characterisations of Buekenhout-Metz Unitals

In the above section it was mentioned that Metz had proved the existence of non-classical Buekenhout-Metz unitals with base ovoid an elliptic quadric. Lefévre-Percsy has given us a condition for determining when a Buekenhout-Metz unital with elliptic quadric base is a classical unital.

Theorem 1.63. [34] *A Buekenhout-Metz unital with elliptic quadric as base is classical, if and only if there exists a secant line of $\mathcal{U}$, not containing the point of $\mathcal{U}$ on ℓ_∞ of $\mathcal{U}$, which intersects $\mathcal{U}$ in a Baer subline.* $\square$

Furthermore, in the same paper, the following was proved.

Theorem 1.64. [34] *Let $\mathcal{U}$ be a unital in $PG(2, q^2)$, where $q > 2$, and ℓ be a tangent line to $\mathcal{U}$. If all Baer sublines having a point on ℓ intersect $\mathcal{U}$ in 0,1,2 or $q + 1$ points then $\mathcal{U}$ is a Buekenhout-Metz unital.* $\square$

Equivalently, this can be stated as:

Let $T \in \mathcal{U}$ and ℓ be the tangent line to $\mathcal{U}$ containing T. If the points of $\mathcal{U}$ on lines through T form Baer sublines and if each Baer subline meeting $\ell \setminus \{T\}$ intersects $\mathcal{U}$ in at most two points then $\mathcal{U}$ is a Buekenhout-Metz unital. This has been improved to,

Theorem 1.65. [6] *Let $\mathcal{U}$ be a unital in $PG(2, q^2)$, where $q > 2$. $\mathcal{U}$ is a Buekenhout-Metz unital if and only if there exists a point T of $\mathcal{U}$ such that the points of $\mathcal{U}$ on each of the q^2 secant lines to $\mathcal{U}$ through T form a Baer subline.* $\square$

Chapter 2

The Veronese Surface

We devote this chapter to a study of a particular variety in $PG(5,q)$, the Veronese surface. Although the definition and properties of the Veronese surface are well-known, and most of those appearing here can be found in [29], the exposition and proofs presented here are, in the main, original.

2.1 Definition of the Veronese Surface

Consider the set of points in $PG(5,q)$

$$V = \{(a_0^2, a_1^2, a_2^2, a_0a_1, a_0a_2, a_1a_2) : a_0, a_1, a_2 \in GF(q), \text{ not all zero}\} \qquad (2.1)$$

Every such point lies on each of the quadrics

$$\begin{array}{ll} f_1 : x_1x_2 - x_5^2 = 0, & f_2 : x_0x_2 - x_4^2 = 0 \\ f_3 : x_0x_1 - x_3^2 = 0, & f_4 : x_2x_3 - x_4x_5 = 0 \\ f_5 : x_1x_4 - x_3x_5 = 0, & f_6 : x_0x_5 - x_3x_4 = 0 \end{array}$$

of $PG(5,q)$ and, conversely, any point incident with all of these quadrics is of the above form, so V is exactly the intersection of these quadrics and is thus a variety.

Note that, by considering the product f_1f_2, we have that $x_0x_1x_2^2 = x_4^2x_5^2$ and thus, using f_3, $x_2^2x_3^2 = x_4^2x_5^2$. For q even this implies that $x_2x_3 = x_4x_5$ which is the equation of the quadric f_4. Similarly, for q even, the equations of f_5 and f_6 can be derived from those of f_1, f_2 and f_3 and so, in this case, V is given by the intersection of f_1, f_2 and f_3.

Using equation 2.1 we can see that the points of V are in one-to-one correspondence with the points of $PG(2, q)$ via the bijective map (also defined in [29])

$$\begin{aligned} \nu : PG(2,q) &\rightarrow PG(5,q) \\ \nu(a_0, a_1, a_2) &= (a_0^2, a_1^2, a_2^2, a_0a_2, a_1a_2, a_1a_2) \end{aligned}$$

and thus V is an absolutely irreducible variety of dimension 2. Hence $V = V_2$ is a surface called the **Veronese surface**. It follows immediately from the above map that the Veronese surface contains $(q^2 + q + 1)$ points. As the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ is of rank 3 at all points of V_2 the Veronese surface is non-singular and has a tangent plane $T_P(V_2)$, at each point P; we discuss these tangent planes in more detail in section 2.4

2.2 Hyperplane Sections of the Veronese Surface

We examine how a hyperplane of $PG(5, q)$ intersects the Veronese surface. Consider the hyperplane Σ_4 of equation

$$\alpha_0x_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 + \alpha_5x_5 = 0$$

It contains the point $P = (a_0^2, a_1^2, a_2^2, a_0a_1, a_0a_2, a_1a_2)$ of V_2 if and only if

$$\alpha_0a_0^2 + \alpha_1a_1^2 + \alpha_2a_2^2 + \alpha_3a_0a_1 + \alpha_4a_0a_2 + \alpha_5a_1a_2 = 0$$

This occurs exactly when

$$\nu^{-1}(P) = (a_0, a_1, a_2)$$

is a point of the conic

$$\alpha_0x^2 + \alpha_1y^2 + \alpha_2z^2 + \alpha_3xy + \alpha_4xz + \alpha_5yz = 0$$

of $PG(2, q)$. Thus the points of a hyperplane section of V_2 correspond to the image under ν of the points of a conic of $PG(2, q)$. Therefore, as a conic in $PG(2, q)$ contains at most $2q + 1$ points, it follows that a hyperplane of $PG(5, q)$ contains at most $2q + 1$ points of V_2. Hence we have,

Lemma 2.1. [29] *No hyperplane of $PG(5, q)$ contains the Veronese surface.* □

Suppose that Σ_4 and Σ_4' are two hyperplanes of $PG(5, q)$. By the above discussion, each meets V_2 in a set of points which is the image under ν of a conic of $PG(2, q)$. Let these conics be denoted by $\mathcal{C}$ and $\mathcal{C}'$ respectively. If Σ_3 is the 3-space which is the intersection of Σ_4 and Σ_4' then Σ_3 intersects V_2 in as many points as $\mathcal{C}$ and $\mathcal{C}'$ have in common. Generically, two conics in the plane meet in four points, so a generic 3-space meets V_2 in four points. Or, in other words, the Veronese surface has order 4. Henceforth we will denote it V_2^4.

An immediate consequence of the fact that the Veronese surface has order four is that a hyperplane intersects it in a curve of degree 4. However, there are many different types of curve of degree four in $PG(4,q)$. To completely elucidate the possible structures of the intersection of a hyperplane with a Veronese surface we need the following lemmas; the first of which appears in [29, lemma 25.1.6], with a different proof, as a special case of a more general result.

Lemma 2.2. [29] *If ω is a projectivity of $PG(2,q)$ then ω defines a permutation of V_2^4. This permutation is induced by a projectivity of $PG(5,q)$.*

Proof: Suppose that ω maps the point X of $PG(2,q)$ to the point XA, where $A = (a_{ij})$ is a non-singular 3×3 matrix over $GF(q)$. Consider the map

$$\begin{array}{rccl} \beta: & PG(2,q) & \rightarrow & V_2^4 \\ & X & \mapsto & \nu(XA) \end{array}$$

Let $P \in V_2^4$, $P = (a_0^2, a_1^2, a_2^2, a_0a_1, a_0a_2, a_1a_2)$, and Q be the point (a_0, a_1, a_2) of $PG(2,q)$ so that $\nu(Q) = P$. Now

$$\begin{aligned} \beta(QA^{-1}) &= \nu(QA^{-1}A) \\ &= \nu(Q) \\ &= P \end{aligned}$$

and so β is surjective. Hence, as $PG(2,q)$ and V_2^4 both contain q^2+q+1 points, β is a bijection. Thus the map

$$\begin{array}{rccl} \tilde{\omega}: & V_2^4 & \rightarrow & V_2^4 \\ & \nu(X) & \mapsto & \nu(XA) \end{array}$$

is a permutation of V_2^4.

We wish to show that there is a projectivity ω' of $PG(5,q)$ such that for $A \in V_2^4$, $\omega'(A) = \tilde{\omega}(A)$

Consider $X = (x_0, x_1, x_2)$, a point of $PG(2,q)$. This is mapped by ω to the point XA

and if we let $f_i(X) = a_{i1}x_0 + a_{i2}x_1 + a_{i3}x_2$ then $XA = (f_1(X), f_2(X), f_3(X))$. Therefore,

$$\begin{aligned}\nu(XA) &= \nu(f_1(X), f_2(X), f_3(X)) \\ &= (f_1(X)^2, f_2(X)^2, f_3(X)^2, f_1(X)f_2(X), f_1(X)f_3(X), f_2(X)f_3(X)) \\ &= (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2)B\end{aligned}$$

where B is a 6×6 matrix over $GF(q)$.

Define,

$$\begin{aligned}\omega' : \; PG(5,q) &\rightarrow PG(5,q) \\ \omega' : \quad X \quad &\mapsto XB\end{aligned}$$

For $P \in V_2^4$, $\omega'(P) = \tilde{\omega}(P)$ and so ω' permutes the points of V_2^4. By lemma 2.1 the points of V_2^4 span $PG(5,q)$, and thus $\omega'(PG(5,q)) = PG(5,q)$ and so B is non-singular and ω' is a projectivity. □

Corollary 2.3. *If ℓ is a line of $PG(2,q)$ then $\nu(\ell)$ is a conic on V_2^4.*

Proof: Consider the line m of $PG(2,q)$ joining the points $(1,0,0)$ and $(0,1,0)$. This contains the set of points

$$\{(1,\lambda,0) : \lambda \in GF(q) \cup \{\infty\}\}$$

which is mapped by ν to the following set of points

$$\{(1,\lambda^2,0,\lambda,0,0) : \lambda \in GF(q) \cup \{\infty\}\}$$

which form a conic of $PG(5,q)$ in the plane $x_2 = x_4 = x_5 = 0$. Hence, by lemma 2.2, a line of $PG(2,q)$ is mapped to a conic on V_2^4. □

Thus V_2^4 contains, at least, (q^2+q+1) conics, namely those corresponding to lines of $PG(2,q)$ under the map ν. Through any point of V_2^4 there will be $q+1$ conics of this sort.

Corollary 2.4. *If $\mathcal{C}$ is a non-singular conic in $PG(2,q)$ then $\nu(\mathcal{C})$ is a normal rational curve of degree four on V_2^4.*

Proof: Any non-singular conic in $PG(2,q)$ is projectively equivalent to

$$\mathcal{C} = \{(1,\lambda,\lambda^2) : \lambda \in GF(q) \cup \{\infty\}\}$$

and thus

$$\nu(\mathcal{C}) = \{(1,\lambda^2,\lambda^4,\lambda,\lambda^2,\lambda^3) : \lambda \in GF(q) \cup \{\infty\}\}$$

is a normal rational curve on V_2^4 (in the hyperplane $x_1 = x_4$). Therefore, again by lemma 2.2, any non-singular conic of $PG(2, q)$ is mapped to a normal rational quartic curve on V_2^4. $\square$

Recall that a hyperplane meets the Veronese surface in a set of points which are the image under ν of a conic of $PG(2, q)$. As a conic of $PG(2, q)$ consists of either a single point, a repeated line, two distinct lines or is non-singular, we can summarise the above results as follows;

Theorem 2.5. *A hyperplane intersects V_2^4 in a curve of degree 4. This curve consists of one of,*

- *A point*
- *A conic*
- *Two conics*
- *A normal rational quartic curve* $\square$

As there is a unique conic consisting of the points of a line, the points of two lines or the points of a non-singular conic, the correspondence between the conics of $PG(2, q)$ and the hyperplane sections of V_2^4 is one-to-one in these cases. Therefore the number of hyperplane sections of the last three types above is equal to the number of lines, pairs of distinct lines and non-singular conics of $PG(2, q)$, respectively. However, if $\nu(P)$ is a point of V_2^4 then for any line ℓ of $PG(2, q^2) \setminus PG(2, q)$ containing the point P, the conic consisting of ℓ and $\bar{\ell}$, the conjugate of ℓ, is a conic of $PG(2, q)$ corresponding to a hyperplane meeting V_2^4 solely at the point $\nu(P)$. As there are $(q^2 + 1) - (q + 1)$ lines of $PG(2, q^2)$ through P which are not lines of $PG(2, q)$, there are $\frac{q(q-1)}{2}$ such pairs of conjugate lines and so $\frac{q(q-1)}{2}$ hyperplanes meeting V_2 at $\nu(P)$ precisely. This, together with the fact that there are $(q^5 - q^2)$ non-singular conics in $PG(2, q)$ (theorem 1.15), gives the following result;

Theorem 2.6. *There are $(q^2 + q + 1)q(q - 1)/2$ hyperplanes meeting V_2^4 in a unique point, $(q^2 + q + 1)$ meeting V_2^4 in a conic, $(q^2 + q + 1)(q^2 + q)/2$ meeting V_2^4 in a pair of conics and $(q^5 - q^2)$ hyperplanes meeting V_2^4 in a normal rational quartic curve.* $\square$

2.3 Conics on the Veronese Surface

We saw in corollary 2.3 that V_2^4 contains at least q^2+q+1 conics, namely those which are the images under ν of lines of $PG(2,q)$. We show that these are the only conics on V_2^4.

Lemma 2.7. [29] *Let $\mathcal{C}$ and $\mathcal{C}'$ be two conics on V_2^4 intersecting in the point P. If these conics lie in the planes α and α' respectively then α and α' intersect only at the point P.*

Proof: If α and α' meet in a line then they span a three dimensional subspace Σ_3 of $PG(4,q)$. Thus, for $R \in V_2^4 \setminus \Sigma_3$, $\langle \Sigma_3, R\rangle$ is a hyperplane of $PG(5,q)$ meeting V_2^4 in two conics and the point R, in contradiction with theorem 2.5. Thus α and α' meet only at the point P. □

If $\mathcal{C} = \nu(\ell)$ and $\mathcal{C}' = \nu(\ell')$ are two conics on V_2^4 which are the images under ν of lines of $PG(2,q)$, then as ℓ and ℓ' have a unique common point so too do $\mathcal{C}$ and $\mathcal{C}'$. Because of this, the above lemma has the following corollary.

Corollary 2.8. *If $\mathcal{C}$ is a conic on V_2^4 then $\mathcal{C}$ is the image under ν of a line of $PG(2,q)$.*

Proof: By the above, each pair of the (q^2+q+1) conics on V_2^4 which are the images of lines of $PG(2,q)$ meet in a point and so, by lemma 2.7, define a hyperplane meeting V_2^4 in two conics. We obtain $(q^2+q+1)(q^2+q)/2$ hyperplanes meeting V_2^4 in this way and, by theorem 2.6, these are all of the hyperplanes which intersect V_2^4 in two conics. If $\mathcal{C}$ is another conic on V_2^4 then there will be $q+1$ conics which are the images of a line of $PG(2,q)$ through any point of $\mathcal{C}$. Again by lemma 2.7, $\mathcal{C}$ together with any of these $q+1$ conics defines a hyperplane meeting V_2^4 in two conics contradicting theorem 2.6.

□

For later reference we summarise the above results.

Theorem 2.9. [29]

- *There are (q^2+q+1) conics on the Veronese surface*
- *Any point of the surface is contained on $q+1$ of these conics.*
- *There is a unique conic through any two points of the Veronese surface.*
- *Any two conics on the Veronese surface have a unique common point which is the unique point of intersection of the planes of these conics.* □

Thus the incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ with points being the points of the Veronese surface, lines being the conics of V_2^4 and the inherited incidence from $PG(5,q)$ is a projective plane of order q. As we can coordinatise it with the elements of $GF(q)$ (by using ν) this plane is isomorphic to $PG(2,q)$.

As we now know that any conic on V_2^4 is the image under ν of a line of $PG(2,q)$ it is an easy matter to describe the plane of a conic on V_2^4.

Let $\mathcal{C}$ be a conic of V_2^4 corresponding to the line g of $PG(2,q)$ of equation

$$\ell x + my + nz = 0 \tag{2.2}$$

Multiplying equation 2.2 in turn by x, y and z we obtain the equations

$$\begin{aligned} \ell x^2 + mxy + nxz &= 0 \\ \ell xy + my^2 + nyz &= 0 \\ \ell xz + myz + nz^2 &= 0 \end{aligned}$$

and thus the points of $\mathcal{C} = \nu(\ell)$ belong to the plane defined by the equations

$$\begin{aligned} \ell x_0 + mx_3 + nx_4 &= 0 \\ \ell x_1 + mx_3 + nx_5 &= 0 \\ \ell x_2 + mx_4 + nx_5 &= 0 \end{aligned}$$

We conclude this section by stating a characterisation of V_2^4 in terms of its conic planes.

Theorem 2.10. [51] *Let $\mathcal{R}$ be a set of (q^2+q+1) planes of $PG(5,q)$, with q odd, satisfying the following properties,*

1. *No point belongs to all elements of $\mathcal{R}$.*
2. *Any two distinct elements of $\mathcal{R}$ have a unique common point.*
3. *Any three distinct elements of $\mathcal{R}$ generate $PG(5,q)$.*

The set $\mathcal{R}$ is the set of conic planes of a Veronese surface.

$\square$

2.4 The Tangent Planes of the Veronese Surface

As V_2^4 is a non-singular variety it has a tangent plane at each point. At the point $P = (a_0^2, a_1^2, a_2^2, a_0a_1, a_0a_2, a_1a_2)$ of V_2^4 the tangent plane $T_P(V_2^4)$ has equations

$$x_0\frac{\partial f_i}{\partial x_0}(P) + x_1\frac{\partial f_i}{\partial x_1}(P) + \ldots + x_5\frac{\partial f_i}{\partial x_5}(P) = 0 \quad i = 1, 2, \ldots, 6$$

Thus it is given by the intersection of the six hyperplanes of $PG(5, q)$ defined by the equations,

$$\begin{aligned}
a_2^2x_1 + a_1^2x_2 - 2a_1a_2x_5 &= 0 \\
a_2^2x_0 + a_0^2x_2 - 2a_0a_2x_4 &= 0 \\
a_1^2x_0 + a_0^2x_1 - 2a_0a_1x_3 &= 0 \\
a_0a_1x_2 + a_2^2x_3 - a_1a_2x_4 - a_0a_2x_5 &= 0 \\
a_0a_2x_1 - a_1a_2x_3 + a_1^2x_4 - a_0a_1x_5 &= 0 \\
a_1a_2x_0 - a_0a_2x_3 - a_0a_1x_4 + a_0^2x_5 &= 0
\end{aligned}$$

However, the last three of these are linearly dependent on the first three so the tangent space at P is the plane of $PG(5, q)$ defined by the equations

$$\begin{aligned}
a_2^2x_1 + a_1^2x_2 - 2a_1a_2x_5 &= 0 \\
a_2^2x_0 + a_0^2x_2 - 2a_0a_2x_4 &= 0 \\
a_1^2x_0 + a_0^2x_1 - 2a_0a_1x_3 &= 0
\end{aligned}$$

For q even, the equations defining $T_P(V_2^4)$ become

$$\begin{aligned}
a_2^2x_1 + a_1^2x_2 &= 0 \\
a_2^2x_0 + a_0^2x_2 &= 0 \\
a_1^2x_0 + a_0^2x_1 &= 0
\end{aligned}$$

The properties of the Veronese surface that we have already given enable us to say more about the tangent planes. From result 2.9, the point P is contained in $q + 1$ conics of V_2^4. Each of these conics has a tangent line through P. By lemma 2.7, any two of these tangent lines meet only at P, and each tangent line is contained in the tangent plane at P (see theorem 1.20). Thus $T_P(V_2^4)$ consists exactly of the $q + 1$ lines through P which are tangent to some conic on V_2^4.

Let P_1 and P_2 be two points of V_2^4. By result 2.9, there is a unique conic of V_2^4 containing P_1 and P_2. If $\mathcal{C}$ is this conic, and if t_i is the tangent to $\mathcal{C}$ through P_i then, by the above paragraph, $t_1 \cap t_2$ lies in both $T_{P_1}(V_2^4)$ and $T_{P_2}(V_2^4)$. Furthermore, as two conic planes meet exactly in a point of V_2^4, a tangent line to a conic of V_2^4 through P_1 and a tangent line to a conic of V_2^4 through P_2 only intersect if they are tangents to the same conic ($P_1 \neq P_2$). Thus we have;

Theorem 2.11. [29] *If P_1 and P_2 are distinct points of V_2^4 then $T_{P_1}(V_2^4)$ and $T_{P_2}(V_2^4)$ intersect in a unique point, this point being the intersection of the tangent lines through P_1 and P_2 to the unique conic of V_2^4 through P_1 and P_2.* □

From here the situation varies as q is odd or even.

If q is odd then there is at most two tangents to a conic through a point in the plane of that conic and thus no three tangent planes to V_2^4 have a common point.

If q is even however then the $q+1$ tangent lines to a conic all meet at the nucleus of that conic and thus there are $q+1$ tangent planes to V_2^4 through any such point. Hence we have;

Theorem 2.12. *If q is odd no three tangent planes to V_2^4 have a common point. If q is even then any point on two tangent planes is the nucleus of some conic of V_2^4 and lies on exactly $q+1$ tangent planes of the Veronese surface.* □

Thus there is a fundamental difference in the way that the tangent planes intersect depending upon whether q is even or odd. For q odd, the tangent plane $T_P(V_2^4)$ meets each of the remaining (q^2+q) tangent planes of V_2^4 in distinct points and so P is the only point of $T_P(V_2^4)$ on exactly one tangent plane of V_2^4, and the Veronese surface is the set of points of $PG(5,q)$ which lie on precisely one tangent plane of V_2^4. In his paper [51] Tallini characterised the Veronese Surface, for q odd, as the set of such points on any set of (q^2+q+1) planes of $PG(5,q)$ which intersect in the way that the tangent planes to the Veronese surface intersect. We conclude this section by stating his characterisation.

Theorem 2.13. [51] *In $PG(5,q)$, with q odd, let $\mathcal{T}$ be a set of (q^2+q+1) planes satisfying the following conditions,*

1. *The planes of $\mathcal{T}$ generate PG(5,q).*
2. *Any two planes of $\mathcal{T}$ have a unique common point.*
3. *Any three planes of $\mathcal{T}$ have no common point.*

The set $\mathcal{T}$ is then the set of tangent planes of a Veronese surface V_2^4 and the set of points which lie on precisely one plane of $\mathcal{T}$ are the points of V_2^4. □

2.5 The Nuclear Plane of the Veronese Surface

When q is even, each conic on V_2^4 contains a nucleus. By lemma 2.7, nuclei from distinct conics on V_2^4 are distinct and so there are in all (q^2+q+1) nuclei of conics on V_2^4.

We wish to describe the nucleus of the conic $\mathcal{C} = \nu(g)$. If ℓ has the equation

$$\ell x + my + nz = 0$$

then, as in section 2.3, $\mathcal{C}$ lies in the plane π given by the equations,

$$\begin{aligned} \ell x_0 + mx_3 + nx_4 &= 0 \qquad (2.3)\\ \ell x_1 + mx_3 + nx_5 &= 0 \qquad (2.4)\\ \ell x_2 + mx_4 + nx_5 &= 0 \qquad (2.5)\end{aligned}$$

If $P \in \mathcal{C}$ then the tangent line to $\mathcal{C}$ through P contains the nucleus of $\mathcal{C}$ and is contained in the tangent plane to V_2^4 at P. Recall that if $P = (a_0^2, a_1^2, a_2^2, a_0a_1, a_0a_2, a_1a_2)$ then, as q is even, the tangent plane at P is defined by the equations

$$\begin{aligned} a_2^2x_1 + a_1^2x_2 &= 0 \qquad (2.6)\\ a_2^2x_0 + a_0^2x_2 &= 0 \qquad (2.7)\\ a_1^2x_0 + a_0^2x_1 &= 0 \qquad (2.8)\end{aligned}$$

Since the nucleus of $\mathcal{C}$ is contained on $T_P(V_2^4)$ for all points P of $\mathcal{C}$, and is also a point of π, it follows that it is given by the unique solution of equations (2.3)-(2.8) which is independent of a_0, a_1 and a_2. Thus the nucleus of $\mathcal{C}$ is the point $(0,0,0,n,m,\ell)$. In addition, the nucleus of any conic on V_2^4 lies in the plane $\mathcal{N}$ of $PG(5,q)$ with equation

$$x_0 = x_1 = x_2 = 0$$

As there are q^2+q+1 such nuclei, each point of $\mathcal{N}$ is the nucleus of some conic of V_2^4. We call this plane the **nuclear plane** of V_2^4.

Furthermore the map

$$\phi : PG(2,q) \rightarrow \mathcal{N}$$

which maps the line ℓ of equation $\ell x+my+nz=0$ to the nucleus $(0,0,0,n,m,\ell)$ of $\nu(\ell)$ is a reciprocity. This enables us to deduce various things about the Veronese surface.

For example, it is immediate that the $q+1$ lines through a point of $PG(2,q)$ are mapped to $q+1$ nuclei which are necessarily collinear. Therefore, as lines through a point in $PG(2,q)$ correspond, under ν, to conics through a point on the Veronese surface, we have that the $q+1$ conics through a point of V_2^4 have collinear nuclei. Since the tangent plane at $P \in V_2^4$ consists of the $q+1$ lines tangent to conics of V_2^4 through P we have;

Corollary 2.14. *The nuclear plane of the Veronese surface intersects each tangent plane of the surface in a line.* □

2.6 The Projection of the Veronese Surface

Let P be a point of $PG(5,q)$ and Σ_4 be a hyperplane not containing P. If Q is a point of $PG(5,q)$ distinct to P, then we define the **projection** of Q from P onto Σ_4 to be the point $PQ \cap \Sigma_4$. We call the map γ defined by,

$$\begin{array}{rccc} \gamma: & PG(5,q)\setminus\{P\} & \to & \Sigma_4 \\ & \gamma(Q) & = & PQ\cap\Sigma_4 \end{array}$$

the **projection map** of P onto Σ_4. Note that the map is onto but not one-to-one as the points of any line through P have the same image on Σ_4. Generally, we wish to study the projection of some geometrical object in $PG(5,q)$ onto Σ_4. In particular, if ℓ is a line of $PG(5,q)$ containing P then $\gamma(\ell)$, the projection of ℓ onto Σ_4, is the point $\ell \cap \Sigma_4$. If ℓ does not contain P then $\langle \ell, P\rangle$ is a plane and $\gamma(\ell)$ is the line which is the intersection of this plane with Σ_4. In general, if Σ_d is a d-dimensional subspace of $PG(5,q)$ containing P, then $\gamma(\Sigma_d)$ is a $(d-1)$-dimensional subspace of Σ_4, and if Σ_d does not contain P, then $\gamma(\Sigma_d)$ is a d-dimensional subspace of Σ_4.

We consider the projection of the Veronese surface from the point $P=(0,0,1,0,0,0)$ of the surface onto the hyperplane Σ_4 of $PG(5,q)$ of equation $x_2=0$. Let γ denote the projection map. Thus the point $Q=(a_0^2,a_1^2,a_2^2,a_0a_1,a_0a_2,a_1a_2)$ is projected onto $\gamma(Q)=(a_0^2,a_1^2,0,a_0a_1,a_0a_2,a_1a_2)$ and hence the projection of V_2^4 is the set of points

$$\gamma(V_2^4)=\{(a_0^2,a_1^2,0,a_0a_1,a_0a_2,a_1a_2):(a_0,a_1,a_2)\in PG(2,q)\setminus\{(0,0,1)\}\}$$

We use our knowledge of the Veronese surface to discuss $\gamma(V_2^4)$.

Let $\mathcal{C}$, in the plane α, be a conic of the Veronese surface which contains the point P. As α contains P, its projection is a line g of Σ_4. The points of $\mathcal{C}$ are projected onto q points of this line, with the remaining point of g being the intersection with Σ_4 of the tangent line to $\mathcal{C}$ through P. As there are $q+1$ conics on the Veronese surface which contain the point P, we obtain $q+1$ (skew) lines $g_1, \ldots, g_{q+1}$ in this way. Every point of $\gamma(V_2^4)$ lies on a unique g_i, and every line g_i contains one point R_i which is not in $\gamma(V_2^4)$. Each such point R_i is the intersection with Σ_4 of a line through P which is tangent to a conic of V_2^4 through P. Now $T_P(V_2^4)$ is a plane containing these tangent lines, so we have that the points $R_1, \ldots, R_{q+1}$ are collinear on the line $T_P(V_2^4) \cap \Sigma_4$. Because of this, we define $\gamma(P)$ to be this line so that, with this proviso, $\gamma(V_2^4)$ is the set of points

$$\{(a_0^2, a_1^2, 0, a_0a_1, a_0a_2, a_1a_2) : a_0, a_1 \in GF(q), \text{ not both zero } a_2 \in GF(q) \cup \{\infty\}\}$$

Thus the projection of the Veronese surface from a point of the surface onto a hyperplane not containing that point is the ruled cubic surface as defined in section 1.11. Before proceeding, we note that there is nothing exceptional about projecting from $(0,0,1,0,0,0)$ onto the hyperplane $x_2 = 0$ in that if we project V_2^4 from any of its points, and onto any hyperplane not containing that point, then the resulting projection is a ruled cubic surface.

Chapter 3

The Ruled Cubic Surface

In section 1.11 we defined the ruled cubic surface as the set of points

$$\{(x^2, xy, y^2, xz, yz) \ : x, y \in GF(q), \text{ not both zero, } z \in GF(q) \cup \{\infty\}\} \tag{3.1}$$

of $PG(4,q)$ and gave some elementary properties of the surface. It was shown in section 2.6 that this surface is the projection of the Veronese surface from a point of the surface onto a hyperplane not containing that point (an idea also appearing in [47], in the case of a projective geometry over an infinite field). We use this fact to examine the properties of the ruled cubic surface in more detail. For an alternative discussion of some of the elementary properties of the ruled cubic surface, see [60].

Let $P \in V_2^4$, Σ_4 be a hyperplane of $PG(5,q)$ not containing P and γ denote the projection map from P onto Σ_4. Therefore, from section 2.6, $\gamma(V_2^4) = V_2^3$, the ruled cubic surface, and, in addition, a generator of V_2^3 is the projection of a conic of V_2^4 which contains P. We now turn our attention to finding the projection of the conics of V_2^4 which do not contain P.

Let $\mathcal{C}$, in the plane α, be one such conic. As α does not contain P, its projection $\beta = \gamma(\alpha)$ is a plane of Σ_4 and $\gamma(\mathcal{C})$ is a set of $q+1$ points, no three of which are collinear, in that plane. The lines joining P to the points of $\mathcal{C}$ form a three dimensional quadric cone $\mathcal{Q}$. Therefore $\gamma(\mathcal{C}) = \beta \cap \mathcal{Q}$ is a conic. Hence the projection of a conic of V_2^4 not containing P is a conic of V_2^3. Since two conics on V_2^4 have a unique common point, these conics meet pairwise in a unique point and have one point on each generator. None of these conics has a point on ℓ, the line directrix of V_2^3.

Lemma 3.1. *The planes of any two of the conics of V_2^3, which are the projections of conics of V_2^4, have a unique common point.*

Proof: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conics on $\mathcal{V}_2^4$ in the planes α_1 and α_2, respectively. Let $\beta_i = \gamma(\alpha_i)$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ meet in the point R then $\gamma(R)$ is the unique common point of $\gamma(\mathcal{C}_1)$ and $\gamma(\mathcal{C}_2)$. By way of contradiction, suppose that β_1 and β_2 have another common point T. This implies that the line TP meets α_1 and α_2 in distinct points. Thus P is a point of the hyperplane of $PG(5,q)$ spanned by α_1 and α_2. However this hyperplane intersects $\mathcal{V}_2^4$ in $\mathcal{C}_1$ and $\mathcal{C}_2$ precisely, and therefore no such point T exists. □

Since $\mathcal{V}_2^3$ is a surface of order three, a hyperplane of $PG(4,q)$ meets it in a cubic curve. This curve could have several possible forms but our knowledge of how a hyperplane meets the Veronese surface allows us to describe the curve exactly.

Theorem 3.2. *The hyperplane sections of $\mathcal{V}_2^3$ are of the following types :*

1. *$q(q-1)/2$ meeting $\mathcal{V}_2^3$ in ℓ exactly.*
2. *$q+1$ meeting $\mathcal{V}_2^3$ in ℓ and one generator.*
3. *$q(q+1)/2$ meeting $\mathcal{V}_2^3$ in ℓ and two generators.*
4. *$q^2(q+1)$ meeting $\mathcal{V}_2^3$ in a generator and a conic.*
5. *q^4-q^2 meeting $\mathcal{V}_2^3$ in a twisted cubic curve.*

Proof: The lines through P of a hyperplane of $PG(5,q)$ which contains P meet Σ_4 in the points of a hyperplane of this space. Conversely, if Σ_3 is a hyperplane of Σ_4, then the space spanned by Σ_3 and P is a hyperplane of $PG(5,q)$. Thus we have a one-to-one correspondence between the hyperplanes of $PG(5,q)$, containing P, and the hyperplanes of Σ_4. Further if Σ_4' and Σ_3 are related by this correspondence then,

$$\gamma(\Sigma_4' \cap \mathcal{V}_2^4 \setminus \{P\}) = \Sigma_3 \cap (\mathcal{V}_2^3 \setminus \ell)$$

where $\ell = T_P(\mathcal{V}_2^4) \cap \Sigma_4$ is the line directrix of $\mathcal{V}_2^3$. Thus we can determine the hyperplane sections of $\mathcal{V}_2^3$ by finding the projections on Σ_4 of the intersections with $\mathcal{V}_2^4$ of the hyperplanes of $PG(5,q)$ containing P. From theorem 2.5, there are five possibilities for how Σ_4' may intersect $\mathcal{V}_2^4$. We examine each in turn.

If Σ_4' intersects $\mathcal{V}_2^4$ only at P then Σ_3 contains no point of $\mathcal{V}_2^3 \setminus \ell$. However, Σ_3 necessarily intersects each of the generator lines of $\mathcal{V}_2^3$ and thus does so at ℓ. That is, Σ_3 meets $\mathcal{V}_2^3$ in ℓ exactly. From theorem 2.6, there are $q(q-1)/2$ hyperplanes of $PG(5,q)$ which meet $\mathcal{V}_2^4$ in P exactly and thus we have $q(q-1)/2$ hyperplanes which meet $\mathcal{V}_2^3$ at ℓ exactly.

Next suppose that Σ_4' meets V_2^4 in a conic through P. Thus Σ_3 meets V_2^3 in the projection of this conic, that is a generator g of V_2^3, and contains no other point of $V_2^3 \setminus \{\ell, g\}$. Therefore, again as Σ_3 meets each generator, it meets each at a point of ℓ and thus contains ℓ. Hence, in this case, Σ_3 intersects V_2^3 in ℓ and a generator. Again, as we know that there are $(q+1)$ hyperplanes of $PG(5,q)$ meeting V_2^4 in a conic through P, we have $(q+1)$ hyperplanes meeting V_2^3 in ℓ and a generator.

Similarly, if Σ_4' meets the Veronese surface in two conics through P, then Σ_3 contains two generators of V_2^3 and the line directrix of V_2^3. By theorem 2.6, there are $q(q+1)/2$ hyperplanes meeting V_2^3 in this way.

Alternatively, if Σ_4 contains two conics of V_2^4, of which only one contains P, then Σ_3 contains a generator g and one of the conics of V_2^3 defined above. As this conic has a point on each generator, and Σ_3 contains no generator other than ℓ, Σ_3 meets ℓ only at the point $\ell \cap g$. The number of hyperplanes meeting V_2^3 in this way is $q^2(q+1)$.

Lastly, suppose that Σ_4 intersects V_2^4 in a normal rational curve of order four. The projection of such a curve is a twisted cubic curve in Σ_3'. As Σ_3' intersects ℓ and each generator of V_2^3, this twisted cubic contains a point of each generator and a point of ℓ. By counting, we have already considered the (q^2+q+1) hyperplanes which contain ℓ. Thus this twisted cubic contains a unique point of ℓ, and consequently a unique point of each generator of V_2^3.

The numbers of each type of intersection follow from theorem 2.6. □

Corollary 3.3. *V_2^3 contains exactly q^2 conics, each of which meets every generator in a unique point and is skew to ℓ. The planes of these conics meet pairwise in a point, which is necessarily a point of V_2^3, and are skew to the line directrix of V_2^3.*

Proof: From lemma 3.1, and the discussion immediately preceding it, the projections on V_2^3 of the conics of V_2^4 not containing P form such a set of conics on the ruled cubic. Note that the plane α of such a conic C does not contain a point P of ℓ for, as these conics have a point on each generator, this would imply that α contains the generator through P. There still remains the possibility that there are other conics on the surface.

Suppose that C', in the plane β, is another conic on V_2^3 and A and B are two points of C' which do not lie on a common generator of V_2^3. By theorem 2.9, there is a unique conic of V_2^4 containing the two points $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$. Since A and B are not on a common generator of V_2^3, this conic does not contain P, and hence the projection of this conic C, which lies in the plane α, is one of the conics mentioned in the above paragraph. Now

$\mathcal{C}$ and $\mathcal{C}'$ have the two points A and B in common, and thus α and β span a hyperplane containing two conics of V_2^3, contradicting theorem 3.2. Thus $\mathcal{C} = \mathcal{C}'$ and there are no other conics on the ruled cubic surface.

□

Note that as any conic on V_2^3 has a point on each generator and lies in a plane skew to ℓ, it can be considered as a conic directrix of the ruled cubic surface. Thus the surface has precisely q^2 conic directrices. We call the planes of $PG(4,q)$ which contain a conic of V_2^3 **conic planes** of V_2^3.

Theorem 3.4. *If Q_1 and Q_2 are two points of $V_2^3 \setminus \ell$, which do not lie on a generator line of V_2^3, then there is a unique conic of V_2^3 containing Q_1 and Q_2.*

Proof: By theorem 2.9, there is a unique conic of V_2^4 containing the points $\gamma^{-1}(Q_1)$ and $\gamma^{-1}(Q_2)$. As the points Q_1 and Q_2 do not lie on a common generator of V_2^3, this conic does not contain P and its projection on V_2^3 is the unique conic of V_2^3 containing Q_1 and Q_2.

□

3.1 The Tangent Planes of the Ruled Cubic Surface

The ruled cubic surface is non-singular and of dimension two and therefore there is a tangent space at each point and these tangent spaces are tangent planes. If $Q \in V_2^3$ then we denote the tangent plane at Q by $T_Q(V_2^3)$.

Since the ruled cubic surface is the intersection of the three quadrics,

$$x_0x_4 - x_1x_3 = 0 \tag{3.2}$$
$$x_2x_3 - x_1x_4 = 0 \tag{3.3}$$
$$x_0x_2 - x_1^2 = 0 \tag{3.4}$$

as in theorem 1.40, the tangent plane to the surface at the point $Q = (x^2, xy, y^2, xz, yz)$ is defined by the equations,

$$yzx_0 - xzx_1 - xyx_3 + x^2x_4 = 0 \tag{3.5}$$
$$-yzx_1 + xzx_2 + y^2x_3 - xyx_4 = 0 \tag{3.6}$$
$$y^2x_0 - 2xyx_1 + x^2x_2 = 0 \tag{3.7}$$

However, the third of these equations is linearly dependent on the first two so $T_Q(V_2^3)$ is defined by the equations 3.5 and 3.6.

By theorem 1.19, if Q is a point of V_2^3, then any line of $PG(4,q)$ which lies wholly on V_2^3, and contains Q, is in $T_Q(V_2^3)$. Similarly, by lemma 1.20, if $\mathcal{C}$ is a conic of V_2^3 containing Q, then the tangent line to $\mathcal{C}$ through Q is also a line of $T_Q(V_2^3)$. We use these facts to describe the tangent plane at any point of V_2^3.

Suppose that Q is a point on the generator line g of V_2^3. If Q is the point of intersection of g with the line directrix ℓ then, by the above results, both ℓ and g are lines of the tangent space at Q and so $T_Q(V_2^3) = \langle \ell, g \rangle$. This plane, being a plane about ℓ, contains all of any generator line it intersects and thus, as no two generators are coplanar, $T_Q(V_2^3)$ meets V_2^3 only at g and ℓ. On the other hand, if Q is not a point of ℓ then Q is contained in q conics of V_2^3. Each of these conics has a tangent line through Q and, as conic planes intersect in a point of V_2^3, these lines are distinct. Thus the q tangent lines through Q and g are coplanar on $T_Q(V_2^3)$ and every point of $T_Q(V_2^3)$ lies on one of these lines.

Alternatively, if Q is a point of $V_2^3 \setminus \ell$ then Q is the projection of a point $Q' \neq P$ of the Veronese surface. Thus $T_Q(V_2^3)$ is the plane $\gamma(T_{Q'}(V_2^4))$ and hence consists of the generator line of V_2^3 which is the projection of the unique conic of V_2^4 containing P and Q', and the projections on V_2^3 of the tangent lines through Q' of the other q conics of V_2^4 which contain Q', as above. Note that as $\gamma(P) = T_P(V_2^4) \cap PG(4,q) = \ell$ we cannot use this approach to find the tangent planes of V_2^3 at a point of ℓ.

The Intersection of the Tangent Planes of V_2^3

Let Q_1 and Q_2 be two points of V_2^3. We wish to describe how $T_{Q_1}(V_2^3)$ and $T_{Q_2}(V_2^3)$ intersect.

To begin, we suppose that Q_1 and Q_2 are collinear on a generator line g of V_2^3. Thus $T_{Q_1}(V_2^3)$ and $T_{Q_2}(V_2^3)$ both contain g and, as they are distinct planes, they do not intersect any further.

Now assume that P_1 and P_2 lie on distinct generator lines g_1 and g_2.

If P_1 and P_2 are points of ℓ then their respective tangent planes both contain ℓ and so, as they are distinct planes, intersect exactly in ℓ.

If neither is a point of ℓ then, as the two generators g_1 and g_2 do not intersect, any point of intersection of their tangent planes lies on a tangent line through Q_1 and a tangent line through Q_2. However, any common point of two conic planes is a point of V_2^3 (corollary 3.3), so a tangent line to a conic through Q_1 and a tangent line to a conic through Q_2 intersect if and only if they are tangents to the same conic. By theorem 3.4, there is a unique conic $\mathcal{C}$ of V_2^3 containing Q_1 and Q_2 and thus the unique point of intersection of $T_{Q_1}(V_2^3)$ and $T_{Q_2}(V_2^3)$ is the point of intersection of the tangent lines to $\mathcal{C}$ through Q_1 and Q_2.

Lastly suppose that $Q_1 \in \ell$ and $Q_2 \notin \ell$. This implies that the point $g_2 \cap \ell$ is on both tangent planes. Suppose that $T_{Q_1}(V_2^3)$ also contains the point R of $T_{Q_2}(V_2^3)$. Thus R lies on a line t which is a tangent line to the conic $\mathcal{C}$ of V_2^3 containing Q_2. Let α denote the plane of $\mathcal{C}$. As every conic of V_2^3 contains a point of each generator, $\mathcal{C}$ contains a point of g_1. Thus $T_{Q_1}(V_2^3) = \langle g_1, \ell \rangle$ contains two points of α (it also contains R) and so meets this plane in a line and hence meets ℓ. This is a contradiction, so the two tangent planes have only the point $g_2 \cap \ell$ in common.

3.2 The Nuclear Plane of the Ruled Cubic Surface

When q is even each of the q^2 conics on the ruled cubic surface has a nucleus. In section 2.5 we saw that the nuclei of the conics of the Veronese surface are coplanar on the nuclear plane of Veronese surface. By looking at the projection of this plane onto the ruled cubic, we obtain a similar result for the ruled cubic surface.

Theorem 3.5. *Let V_2^3 be a ruled cubic surface in $PG(4, q)$, with q even. The nuclei of the q^2 conics of V_2^3 are coplanar on a plane which contains the line directrix ℓ of V_2^3.*

Proof: Once more we denote by γ the projection map from $P \in V_2^4$ onto a hyperplane Σ_4 which does not contain P. Let $\mathcal{N}$ denote the nuclear plane of the Veronese surface. Since P is not a point of $\mathcal{N}$ the projection of $\mathcal{N}$ is a plane $\mathcal{M}$ of Σ_4.

If $\mathcal{C}$ is one of the q^2 conics on V_2^4 not containing P then its projection $\mathcal{C}' = \gamma(\mathcal{C})$ is a conic on V_2^3. Let N denote the nucleus of $\mathcal{C}$ and M denote the nucleus of $\mathcal{C}'$. Since the tangents of $\mathcal{C}$ are projected on to the tangents of $\mathcal{C}'$, the projection of the nucleus of $\mathcal{C}$ is the nucleus of $\mathcal{C}'$. Therefore, as $N \in \mathcal{N}$, we have that $M \in \mathcal{M}$. Further, as every conic on V_2^3 is the projection of such a conic of V_2^4, $\mathcal{M}$ contains the nucleus of every conic of V_2^3.

The remaining $(q + 1)$ conics of V_2^4 each contain P and therefore the lines joining P to the nuclei of these conics are the lines of the tangent space to V_2^4 through P. As $T_P(V_2^4) \cap \Sigma_4 = \ell$, the line directrix of V_2^3, these nuclei are projected onto ℓ and so $\mathcal{M}$ is a plane containing ℓ. $\square$

3.3 Curves on the Ruled Cubic Surface

Another useful tool in studying the properties of the ruled cubic surface is the map

$$\begin{aligned}\sigma : PG(2,q) &\rightarrow V_2^3 \\ \sigma : (x,y,z) &\mapsto (x^2, xy, y^2, xz, yz)\end{aligned}$$

defined in section 1.11. In particular, we shall find it a useful aid in studying curves on the ruled cubic surface. Note that σ is bijective as a map between $PG(2,q) \setminus \{(0,0,1)\}$ and $V_2^3 \setminus \{\ell\}$ and dilates $(0,0,1)$ onto the line directrix ℓ of V_2^3. Hence we must be careful when considering the image under σ of a set of points of $PG(2,q)$ containing $(0,0,1)$. On the restricted domain $PG(2,q) \setminus \{(0,0,1)\}$, we let σ^{-1} denote the inverse of σ. To begin, we examine how σ acts on the lines of $PG(2,q)$.

Let m be a line of $PG(2,q)$ through the point $(0,0,1)$. Thus m has an equation of the form

$$ax + by = 0$$

As $ax + by = 0$ for all points of m

$$ax^2 + bxy = 0 \tag{3.8}$$

$$axy + by^2 = 0 \tag{3.9}$$

$$axz + byz = 0 \tag{3.10}$$

for all points (x^2, xy, y^2, xz, yz) of $\sigma(m \setminus \{(0,0,1)\})$. Thus the points of $\sigma(m \setminus \{(0,0,1)\})$ lie on the line which is the intersection of the three hyperplanes

$$ax_0 + bx_1 = 0 \tag{3.11}$$

$$ax_1 + bx_2 = 0 \tag{3.12}$$

$$ax_3 + bx_4 = 0 \tag{3.13}$$

The hyperplanes of $PG(4,q)$ defined by the equations 3.11 and 3.12 both contain the line directrix of V_2^3 and the plane they span intersects α (the conic plane of V_2^3 defined by the equations $x_3 = x_4 = 0$) at the point $((-b/a)^2, -b/a, 1, 0, 0)$, if $a \neq 0$, or $(1,0,0,0,0)$, if $a = 0$. The hyperplane 3.13 contains α and meets the line directrix at the point $(0,0,0,-b/a,1)$, if $a \neq 0$, or $(0,0,0,1,0)$, if $a = 0$. In each case the intersection of the three hyperplanes contains two points of a generator g and thus, as it is a line, is this generator. Therefore the q points of $\sigma(m \setminus \{(0,0,1)\})$ are the q points of g not on ℓ.

As there are $(q+1)$ generators on V_2^3, and $(q+1)$ lines through the point $(0,0,1)$ in $PG(2,q)$, it also follows that, if g is any generator line of V_2^3, then the points of g not on ℓ are mapped by σ^{-1} to the points of a line of $PG(2,q)$ which contains $(0,0,1)$. Under a slight abuse of notation, we shall sometimes write $\sigma(m) = g$ (or $\sigma^{-1}(g) = m$) and say that g is the image of m under σ. Thus we have proved the following;

Theorem 3.6. *The image under σ of a line of $PG(2,q)$ which contains the point $(0,0,1)$ is a generator of the ruled cubic surface. Conversely, σ^{-1} maps a generator of V_2^3 onto such a line.* ☐

Now we let m be a line of $PG(2,q)$ which does not contain the point $(0,0,1)$. Thus the equation of m can be written in the form

$$ax + by + z = 0$$

Again we wish to determine the structure of the set of points $\sigma(m)$.

As $ax + by + z = 0$ for all points of m

$$ax^2 + bxy + xz = 0 \tag{3.14}$$

$$axy + by^2 + yz = 0 \tag{3.15}$$

for all points (x^2, xy, y^2, xz, yz) of $\sigma(m)$. Thus $\sigma(m)$ is given by the intersection of V_2^3 with the plane

$$ax_0 + bx_1 + x_3 = 0 \tag{3.16}$$

$$ax_1 + bx_2 + x_4 = 0 \tag{3.17}$$

The line m meets the line $x = 0$ at the point $(0,1,-b)$ and the line $y = 0$ at the point $(1,0,-a)$. Thus m contains the set of points

$$\{(t, 1, -(at+b)) : t \in GF(q) \cup \{\infty\}\}$$

and so $\sigma(m)$ contains the points

$$\{(t^2, t, 1, -t(at+b), -(at+b)) : t \in GF(q) \cup \{\infty\}\}$$

which form a non-degenerate conic in the above plane. Conversely, as V_2^3 contains exactly q^2 conics, none with a point on ℓ, every conic on V_2^3 is mapped by σ^{-1} to a line of $PG(2,q)$ not through $(0,0,1)$. This gives us the following;

Theorem 3.7. *The map σ associates a line of $PG(2,q)$ which does not contain the point $(0,0,1)$ to a conic of V_2^3. Conversely, every conic of V_2^3 is mapped by σ^{-1} onto a line of $PG(2,q)$ not containing $(0,0,1)$.* $\square$

Using our knowledge of how σ acts on the lines of $PG(2,q)$, we can now establish the image on V_2^3 of a general curve in $PG(2,q)$. To this end, we firstly consider a curve $\mathcal{C}^{2m}$ of order $2m$ with a point of multiplicity m at $(0,0,1)$.

If $f=0$ is an equation defining $\mathcal{C}^{2m}$ then we can write this equation in the form

$$z^{2m}+z^{2m-1}f_1+\ldots+zf_{2m-1}+f_{2m}=0$$

where f_i is a homogeneous polynomial in x and y of degree i. Now, $(0,0,1)$ being a point of multiplicity m implies that all partial derivatives of f, of orders up to and including $(m-1)$, are zero at $(0,0,1)$, but at least one partial derivative of order m is zero there. Hence we can write the equation of $\mathcal{C}^{2m}$ in the form

$$z^m f_m+z^{m-1}f_{m+1}+\ldots+f_{2m}=0$$

where f_i is a homogeneous polynomial in x and y of degree i.

Thus the equation of $\mathcal{C}^{2m}$ is the sum of terms of the form $z^{m-i}x^{m+i-j}y^j$, where $0\le i\le m$, $0\le j\le 2m$ and $m+i-j\ge 0$, which we can write as $(zx)^r(zy)^s x^{m+i-j-r}y^{j-s}$, where $r+s=m-i$, $r\le m+i-j$ and $s\le j$. Now, the condition that $m-i-r=s$ implies that $m+i-r$ and s are either both odd or both even, and hence that $m+i-j-r$ and $j-s$ are either both odd or both even. Thus we can write

$$z^{m-i}x^{m+i-j}y^j=\begin{cases}(zx)^r(zy)^s(x^2)^{\frac{m+i-j-r}{2}}(y^2)^{\frac{j-s}{2}} & \text{if both are even}\\ (zx)^r(zy)^s(x^2)^{\frac{m+i-j-r-1}{2}}(y^2)^{\frac{j-s-1}{2}}(xy) & \text{if both are odd}\end{cases}$$

Hence we can express the equation of $\mathcal{C}^{2m}$ as a (not necessarily unique) polynomial

$$p(x^2,xy,y^2,zx,zy)$$

of degree m in the variables x^2,xy,y^2,zx and zy.

The equation

$$p(x_0,x_1,x_2,x_3,x_4)=0$$

defines a hypersurface S_3^m of order m of $PG(4,q)$ and a point (x^2,xy,y^2,xz,yz) of V_2^3 lies on this hypersurface if and only if $p(x^2,xy,y^2,xz,yz)=0$. Now $S_3^m\cap V_2^3=\mathcal{C}_1^{3m}$ is a curve of degree $3m$ on V_2^3. Thus we have that, for $(x,y,z)\ne(0,0,1)$,

$$\begin{aligned}(x,y,z) \in \mathcal{C}^{2m} &\Leftrightarrow z^m f_m(x,y) + \ldots + f_{2m}(x,y) = 0 \\ &\Leftrightarrow p(x^2, xy, y^2, zx, zy) = 0 \\ &\Leftrightarrow \sigma(x,y,z) \in \mathcal{C}^{3m}\end{aligned}$$

Again under an abuse of notation, we shall write $\sigma(\mathcal{C}^{2m}) = \mathcal{C}^{3m}$ (or $\sigma^{-1}(\mathcal{C}^{3m}) = \mathcal{C}^{2m}$) and say that $\mathcal{C}^{3m}$ is an image of $\mathcal{C}^{2m}$ on V_2^3 under σ. Note that, as the polynomial p above is not unique, there are several different hypersurfaces S_3^m which intersect V_2^3 in the same curve C^{3m}.

Next we suppose that $\mathcal{C}^n$ is a curve of degree n in $PG(2,q)$, having a point of multiplicity m at $(0,0,1)$, with $n < 2m$. This implies that $\mathcal{C}^n$ together with $(2m-n)$ lines not containing $(0,0,1)$, is a curve of degree $2m$ in $PG(2,q)$ with a point of multiplicity m at $(0,0,1)$. Thus, by the above, its image on V_2^3 is a curve of degree $3m$. This curve contains $(2m-n)$ conics (the images of the lines not containing $(0,0,1)$) and so the image of $\mathcal{C}^n$ on V_2^3 is a curve of degree $3m - 2(2m-n) = 2n - m$.

Finally suppose that $\mathcal{C}^n$ is a curve of degree n in $PG(2,q)$, having a point of multiplicity m at $(0,0,1)$, with $n > 2m$. Hence $\mathcal{C}^n$ together with $(n-2m)$ lines through $(0,0,1)$ is a curve of degree $2(n-m)$ in $PG(2,q)$, with a point of multiplicity $(n-m)$ at $(0,0,1)$. Thus, by the above, its image on V_2^3 is a curve of degree $3(n-m)$ containing $(n-2m)$ generators. Thus the image of $\mathcal{C}^n$ on V_2^3 is a curve of degree $3(n-m)-(n-2m) = (2n-m)$. This gives us the following result;

Theorem 3.8. *If $\mathcal{C}^n$ is a curve of degree n in $PG(2,q)$ with a point of multiplicity m at $(0,0,1)$ then its image on V_2^3 is a curve $\mathcal{C}^{2n-m}$ of degree $(2n-m)$.* □

Once more we write $\sigma(\mathcal{C}^n) = \mathcal{C}^{2n-m}$ and note that by this we mean that, for $(x,y,z) \neq (0,0,1)$, $(x,y,z) \in \mathcal{C}^n \Leftrightarrow \sigma(x,y,z) \in \mathcal{C}^{2n-m}$.

As an example, consider $\mathcal{C}$ a non-degenerate conic of $PG(2,q)$ containing the point $(0,0,1)$. Here we have $n = 2$ and $m = 1$ so, by theorem 3.8, the image of $\mathcal{C}$ on V_2^3 is a curve of degree three. As $\mathcal{C}$ is non-degenerate it contains no lines and so its image under σ on V_2^3 contains neither a generator nor a conic of the ruled cubic. Thus it is irreducible. Furthermore, it follows from theorem 3.2 that no plane of $PG(4,q)$ meets V_2^3 in a cubic curve and so this curve is not contained in a plane. Hence it is a twisted cubic. Conversely, as there are $(q^4 - q^2)$ twisted cubics on V_2^3 and the same number of non-degenerate conics in the plane containing the point $(0,0,1)$, the image under σ of any twisted cubic on V_2^3 is a non-degenerate conic containing $(0,0,1)$. Thus we have;

Theorem 3.9. *The image on V_2^3 of a non-degenerate conic of $PG(2,q)$ which contains the point $(0,0,1)$ is a twisted cubic curve. Conversely, any twisted cubic curve on V_2^3 is the image of such a curve.* □

In general, it is not immediately clear how the image on V_2^3 of a curve in $PG(2,q)$ containing $(0,0,1)$ intersects ℓ. In the above case, as each twisted cubic on V_2^3 is exactly the intersection of a hyperplane with V_2^3, it follows that all of the above twisted cubics have a point on each generator, one of which is on ℓ. Or, in other words, that the image on V_2^3 of a non-degenerate conic $\mathcal{C}$ containing $(0,0,1)$ meets ℓ in a unique point. Now, q of the $(q+1)$ lines of $PG(2,q)$ through $(0,0,1)$ meet $\mathcal{C}$ in one further point, and the other, the tangent line ℓ_t to $\mathcal{C}$ at $(0,0,1)$, meets $\mathcal{C}$ exactly at $(0,0,1)$. Thus, as σ induces a correspondence between the generators of V_2^3 and the lines of $PG(2,q)$ through $(0,0,1)$ and each generator of V_2^3 meets $\mathcal{C}^3 = \sigma(\mathcal{C})$ in a unique point, the point of $\mathcal{C}$ on ℓ is on the generator of V_2^3 which corresponds to ℓ_t under σ. In general, we can prove a similar result which enables us to determine how the image on V_2^3 of an arbitrary curve in $PG(2,q)$ intersects ℓ.

Theorem 3.10. *Let $\mathcal{C}^n$ be a curve of degree n in $PG(2,q)$ with a point of multiplicity m at $(0,0,1)$. If there are $r(\leq m)$ tangents $t_1,\dots,t_r$ to $\mathcal{C}^n$ at $(0,0,1)$ in $PG(2,q)$ and these correspond to the generators $g_1,\dots,g_r$ of V_2^3 then $\sigma(\mathcal{C}^n)$ contains the points $g_1\cap\ell,\dots,g_r\cap\ell$.*

Proof: Firstly we consider the case where $n=2m$, so we are looking at a curve $\mathcal{C}^{2m}$ of degree $2m$ with a point of multiplicity m at $(0,0,1)$. Once again, we can write the equation of $\mathcal{C}^{2m}$ in the form

$$z^m f_m + z^{m-1} f_{m+1} + \ldots + f_{2m} = 0$$

where the f_i are homogeneous polynomials in x and y of degree i.

For some r, $(0 \leq r \leq m)$, the polynomial f_m has r zeros over $GF(q)$ and so splits into

$$f_m = (a_1x + b_1y)\ldots(a_rx+b_ry)f_{m-r}$$

where $a_i, b_i \in GF(q)$ and f_{m-r} is a homogeneous polynomial in x and y of degree $(m-r)$ satisfying $f_{m-r}(a,b) \neq 0$ for $(a,b) \neq (0,0)$.

Now, the equations

$$a_i x + b_i y = 0 \qquad i = 1, \ldots, r$$

are the equations of the lines of $PG(2,q)$ which are tangent lines to $\mathcal{C}^{2m}$ at $(0,0,1)$. Under σ each is mapped to a generator meeting ℓ at the point $x_0 = x_1 = x_2 = a_i x_3 + b_i x_4 = 0$. If we write the equation of f_{m+j} $(0 \leq j \leq m)$ in the form

$$f_{m+j} = \sum_{i=0}^{m+j} a_{ij} x^{m+j-i} y^i$$

then we can write the equation of $\mathcal{C}^{2m}$ as

$$z^m \sum_{i=0}^{m} a_{i0} x^{m-i} y^i + \sum_{j=1}^{m} z^{m-j} \sum_{i=0}^{m+j} a_{ij} x^{m+j-i} y^i = 0$$

Now, as in the preamble to theorem 3.8, the image of $\mathcal{C}^{2m}$ on V_2^3 is given by the intersection of a hypersurface V_3^m with V_2^3. By considering the equation of $\mathcal{C}^{2m}$ in the above form we can see that V_3^m has an equation of the form

$$\sum_{i=0}^{m} a_{i0} x_3^{m-i} x_4^i + \sum_{j=1}^{m} \sum_{i=0}^{m+j} a_{ij} x_0^{r_{ij}} x_1^{s_{ij}} x_2^{t_{ij}} x_3^{u_{ij}} x_4^{v_{ij}} = 0$$

where $r_{ij} + s_{ij} + t_{ij} + u_{ij} + v_{ij} = m + j$. Further, as each term $a_{ij} x_0^{r_{ij}} x_1^{s_{ij}} x_2^{t_{ij}} x_3^{u_{ij}} x_4^{v_{ij}}$ comes from a term of the form $z^{m-j} a_{ij} x^{m+j-i} y^i$ with $j \geq 1$, at least one of r_{ij}, s_{ij} or t_{ij} is non-zero for every fixed (i,j). Each of these coefficients is zero for $x_0 = x_1 = x_2 = 0$ and thus, as ℓ has equation $x_0 = x_1 = x_2 = 0$, V_3^m meets ℓ where

$$\sum_{i=0}^{m} a_{i0} x_3^{m-i} x_4^i = 0$$

As above, we can write this equation in the form

$$(a_1 x_3 + b_1 x_4) \ldots (a_r x_3 + b_r x_4) f_{m-r} = 0$$

where $f_{m-r}(0,0) \neq 0$ and so V_3^m intersects ℓ at the points $x_0 = x_1 = x_2 = a_i x_3 + b_i x_4 = 0$, for $i = 1, \ldots r$. That is, at the points in which the generators of V_2^3 corresponding to the tangent lines of $\mathcal{C}^{2m}$ through $(0,0,1)$ meet ℓ.

For the case in which $n \neq 2m$ we proceed as in the proof of theorem 3.8.

Let $2m > n$ and consider the curve $\mathcal{C}^{2m}$ of order $2m$ consisting of $\mathcal{C}^n$ together with $(2m - n)$ lines not containing $(0,0,1)$. This curve is of order $2m$ with a point of multiplicity m at $(0,0,1)$ and has r tangent lines through $(0,0,1)$, these being the r tangent lines to $\mathcal{C}^n$

through $(0,0,1)$. Thus, by the above, $\sigma(\mathcal{C}^{2m})$ intersects ℓ at the points of intersection with ℓ of the generators of V_2^3 corresponding to these tangents. As the lines not containing $(0,0,1)$ correspond to conics not meeting ℓ, all of these points are on $\sigma(\mathcal{C}^n)$.

Lastly suppose that $n > 2m$ and there is a line s through $(0,0,1)$ not tangent to $\mathcal{C}^n$. Thus $\mathcal{C}^n$ together with the line s repeated $(n-2m)$ times is a curve $\mathcal{C}^{2(n-m)}$ of order $2(n-m)$ with a point of multiplicity $(n-m)$ at $(0,0,1)$. The tangent lines to $\mathcal{C}^{2(n-m)}$ at $(0,0,1)$ are the tangent lines $t_1, \ldots, t_r$ of $\mathcal{C}^n$ at $(0,0,1)$ and the line s. Thus its image on V_2^3 meets ℓ at the points in which the generators $\sigma(t_1), \ldots, \sigma(t_r), \sigma(s)$ meet ℓ. All of these points but $\sigma(s) \cap \ell$ are on $\sigma(\mathcal{C}^n)$.

If all the the lines $t_1, \ldots, t_{q+1}$ through $(0,0,1)$ are tangent to $\mathcal{C}^n$ then, by considering the curve consisting of $\mathcal{C}^n$ together with the line t_1 repeated $(n-2m)$, times we can conclude that all points of ℓ except possibly $\sigma(t_1) \cap \ell$ are on $\sigma(\mathcal{C}^n)$. If we then consider $\mathcal{C}^n$ together with t_2 repeated $(n-2m)$ times we can see that this point is on $\sigma(\mathcal{C}^n)$ too.

$\square$

Note that in some extension $GF(q^k)$ of $GF(q)$ the polynomial f_m in the above proof will split into m linear factors. Thus in $PG(2,q^k)$ the curve $\mathcal{C}^n$ will have m (not necessarily distinct) tangents. Therefore in $PG(4,q^k)$, $\sigma(\mathcal{C}^n)$ will intersect ℓ in m points (some of which may coincide).

3.3.1 Normal Rational Curves on the Ruled Cubic Surface

We have already seen that V_2^3 contains (q^2+q+1) conics, all of which are skew to ℓ, and (q^4-q^2) twisted cubics, all of which meet ℓ in a unique point. To illustrate an application of the theorems in the previous section we now calculate the number of normal rational curves of degree four on V_2^3.

Suppose that $\mathcal{C}$ is a non-degenerate conic of $PG(2,q)$ which does not contain the point $(0,0,1)$. By putting $n=2$ and $m=0$ into theorem 3.8 we have that the image of this conic is a quartic curve on the ruled cubic. As $\mathcal{C}$ is non-degenerate, this quartic contains neither a line nor a conic and so is irreducible. Furthermore it contains $(q+1)$ points (as $\mathcal{C}$ contains $(q+1)$ points of $PG(2,q) \setminus \{(0,0,1)\}$) and, by theorem 3.2, is not contained in a hyperplane of $PG(4,q)$. Thus it is a normal rational curve of degree 4 on V_2^3. As $\mathcal{C}$ does not contain $(0,0,1)$, this curve does not meet ℓ, and each generator meets it in $0, 1$ or 2 points, as the corresponding line through $(0,0,1)$ is an external, tangent or secant line of $\mathcal{C}$.

In this case however these are not the only normal rational curves of degree four on V_2^3. Consider an irreducible cubic curve $\mathcal{C}^3$ of $PG(2,q)$ with a double point at $(0,0,1)$. Putting $n = 3$ and $m = 2$ into theorem 3.8 we see that its image on V_2^3 is a curve $\mathcal{C}^4$ of degree four. Again, as $\mathcal{C}^3$ is irreducible, $\mathcal{C}^4$ contains neither a line nor a conic and so is irreducible. As no hyperplane of $PG(4,q)$ meets V_2^3 in a quartic curve, this curve is properly contained in $PG(4,q)$ and so is a normal rational curve. In addition, since $\mathcal{C}^3$ has a double point at $(0,0,1)$, $\mathcal{C}^3$ has either two tangents through $(0,0,1)$ in $PG(2,q)$ (if the double point is a node), one repeated tangent through $(0,0,1)$ (the double point being a cusp) or no tangents through $(0,0,1)$ in $PG(2,q)$ (the singular point being an isolated double point). Thus, these normal rational curves on V_2^3 meet ℓ in either 2,1 or 0 points.

By theorem 3.8, the only other curve in $PG(2,q)$ for which the image on V_2^3 is a quartic curve is a curve of degree four with a point of multiplicity four at $(0,0,1)$. However the image of such a curve is not a normal rational curve.

We wish to find if these are the only normal rational curves of degree 4 on V_2^3.

We will need the following lemma;

Lemma 3.11. *If $\mathcal{C}^4$ is a normal rational curve of degree 4 on V_2^3 and $\sigma^{-1}(\mathcal{C}^4)$ is a cubic curve $\mathcal{C}^3$ in $PG(2,q)$ then $\mathcal{C}^3$ is irreducible and has a double point at $(0,0,1)$.*

Proof: If $\mathcal{C}^3$ were reducible it would contain a line m. Hence $\mathcal{C}^4$ would contain $\sigma(m)$ which is either a line or a conic contradicting the fact that $\mathcal{C}^4$ is a normal rational curve.

As the image of $\mathcal{C}^3$ under σ is a curve of degree four, $\mathcal{C}^3$ has a double point at $(0,0,1)$ or we obtain a contradiction with theorem 3.8. □

Let $\mathcal{C}^4$ be a normal rational curve of degree 4 on V_2^3 . Thus

$$\mathcal{C}^4 = \{(f_0(t), f_1(t), f_2(t), f_3(t), f_4(t)) : t \in GF(q) \cup \{\infty\}\}$$

where,

- The $\{f_i(t)\}$ are linearly independent polynomials
- The degree of each f_i is at most four

Since any five cubic polynomials are linearly dependent we can infer from these conditions that at least one of the f_i has degree four. Moreover, if a polynomial p, of degree at least one, divides all of the f_i then the five polynomials f_i/p are each of degree at most three and so are linearly dependent. Thus there exists $\lambda_0, \dots, \lambda_4$,not all zero, such that $\sum_{i=0}^4 \lambda_i f_i/p = 0$ and so $\sum_{i=0}^4 \lambda_i f_i = 0$, contradicting the linear independence of the f_i. Thus the polynomials f_i satisfy,

- At least one of the f_i has degree four
- The f_i have no, non-constant, common factor

By theorem 1.40, as $\mathcal{C}^4$ is contained on V_2^3 it lies on each of the three quadrics

$$\begin{aligned} x_0x_4 &= x_1x_3 \\ x_2x_3 &= x_1x_4 \\ x_0x_2 &= x_1^2 \end{aligned}$$

In particular, the third of these equations tells us that

$$f_0f_2 = f_1^2$$

and so, if we write f_0 and f_2 in the form

$$\begin{aligned} f_0 &= (f_0')^2 p_0 \\ f_2 &= (f_2')^2 p_2 \end{aligned}$$

where p_0, p_2 have no repeated factors, then

$$f_1^2 = (f_0')^2(f_2')^2 p_0p_2$$

Thus

$$p_0p_2 = g^2$$

for some polynomial g, and so, as each of p_0 and p_2 have no repeated factors,

$$p_0 = p_2 = g$$

Note that f_0' and f_2' cannot both be of degree zero for then $f_0 = (f_0')^2 g$ and $f_2' = (f_2')^2 g$ would be multiples of each other contradicting the linear independence of the f_i. Also, as each f_i is of degree at most four, each f_i' is of degree at most two and, as at least one is of degree two, g is also of degree at most two.

Hence we can write

$$(f_0, \dots, f_4) = ((f_0')^2 g, f_0'f_2'g, (f_2')^2 g, f_3, f_4)$$

where

- f_i' and g are of degree at most two
- f_0' and f_2' are not both of degree 0

Further note that f_0' and f_2' could possibly have a common factor but that if it is of degree two then $f_0 = f_2$, contradicting the linear independence of the f_i, and so it is of degree at most one.

Imposing the equations of the other two above quadrics on the f_i now gives us the relationship

$$f_0' f_4 = f_2' f_3$$

If we let f_c be the polynomial consisting of any possible common factors of f_0' and f_2' (by above, f_c has degree at most one, with possibly $f_c = 1$) then this relationship implies that

$$\begin{aligned} f_3 &= f_0' h / f_c \\ f_4 &= f_2' h / f_c \end{aligned}$$

for some polynomial h which has no common non-constant factors with either g or f_c.

Thus, we can write $\mathcal{C}^4$ as

$$\{(f_0'(t)^2, f_0'(t)f_2'(t), f_2'(t)^2, f_0'(t)h(t)/f_c(t)g(t), f_2'(t)h(t)/f_c(t)g(t)); t \in GF(q) \cup \{\infty\}\}$$

where, in addition to the above constraints on the degrees of the polynomials f_0', f_2', g and f_c, the fact that f_3 and f_4 are both of degree at most four implies that the degree of h/f_c is no more than three (as f_0' and f_2' are not both of degree 0). Also if deg $f_c = 1$ then deg f_i' is at least two for some i, for otherwise f_0' and f_2' would be multiples of one another, contradicting linear independence. Therefore $\mathcal{C}^4$ is mapped by σ^{-1} to a curve $\mathcal{C}$ containing the points,

$$\begin{aligned} &\{(f_0'(t), f_2'(t), h(t)/f_c(t)g(t)) : t \in GF(q) \cup \{\infty\}\} \\ = \ &\{(g(t)f_c(t)f_0'(t), g(t)f_c(t)f_2'(t), h(t)) : t \in GF(q) \cup \{\infty\}\} \end{aligned}$$

We show that this curve is either a conic not containing $(0, 0, 1)$ or a cubic curve with a double point at $(0, 0, 1)$.

To begin we suppose that g has degree 0.

Additionally, assume that f_c has degree 0 and h has degree at most two. Hence, as at least one of the f_i has degree 4, one of f'_0 or f'_2 is of degree 2. So

$$\sigma^{-1}(\mathcal{C}^4) = \mathcal{C}^2 = \{(f_c(t)f'_0(t), f_c(t)f'_2(t), h(t)); t \in GF(q) \cup \{\infty\}\}$$

is a conic not containing $(0,0,1)$. Note also that this conic is irreducible (as $\mathcal{C}^4$ is) and the corresponding normal rational curve $\mathcal{C}^4$ is skew to ℓ.

Now if f_c has degree 0 and h has degree at least 3 then, as f'_0, f'_2 are not both of degree 0, h is of degree exactly 3. Hence, as f_3 and f_4 are of degree at most 4, each f'_i is of degree at most one, with at least one of degree 1. Thus, $\mathcal{C}^3 = \sigma^{-1}(\mathcal{C}^4)$ is a cubic curve which, by lemma 3.11, is irreducible and has a double point at $(0,0,1)$. The corresponding normal rational curve meets ℓ at a unique point (that for which the parameter $t = \infty$) and so, by theorem 3.10, $\mathcal{C}^3$ has exactly one tangent through $(0,0,1)$ and so the double point is a cusp.

To complete the case where g has degree 0 we let f_c have degree 1. Hence, as was stated above, at least one of f'_0 or f'_2 is of degree 2 and so h is of degree at most 3. So, once again, $\mathcal{C}^3 = \sigma^{-1}(\mathcal{C}^4)$ is a cubic curve and hence is irreducible with a double point at $(0,0,1)$. The corresponding normal rational curve meets ℓ at a unique point (that for which the parameter $t = t_1$, the zero of f_c) and thus there is a unique tangent to $\mathcal{C}^3$ through $(0,0,1)$ and the double point is again a cusp.

Secondly we assume that deg $g = 1$ and so (as f_0, f_1 and f_2 are all of degree four or less) deg $f'_0 \leq 1$ and deg $f'_2 \leq 1$ with at least one of degree 1. Hence deg $f_c = 0$, for otherwise f'_0 and f'_2 would be multiples of one another contradicting linear independence. Now, as each of $f_0, f_1$and f_2 is of degree strictly less than four, one of f_3 or f_4 is of degree 4 and thus we can infer that h has degree 3. Therefore $\mathcal{C}^3 = \sigma^{-1}(\mathcal{C}^4)$ is a cubic curve which, again, is irreducible and has a double point at $(0,0,1)$. Here the corresponding normal rational curve meets ℓ twice (at the points of $\mathcal{C}^4$ for which the parameter t takes the values ∞ and t_1, the zero of $g(t)$) and we see that $\mathcal{C}^3$ has two tangents through $(0,0,1)$, so the double point is a node.

To conclude we assume that deg $g = 2$. As in the previous case this implies that f'_0 and f'_2 are of degree one or less, f_c is of degree 0 and h is of degree three or less. Thus $\mathcal{C}^3 = \sigma^{-1}(\mathcal{C}^4)$ is a cubic curve, which is necessarily irreducible and has a double point at $(0,0,1)$. The point $\mathcal{C}^4(t)$ of the corresponding normal rational curve is on ℓ if and only if t is a zero of g. Therefore, as g is of degree two, $\mathcal{C}^4$ meets ℓ in either 2, 1or 0 points, as g has 2,1 or 0 roots in $GF(q)$, and the double point is either a node, cusp or an isolated double point accordingly.

Hence a normal rational curve of degree four on V_2^3 is mapped by σ onto either a conic not containing $(0,0,1)$ or an irreducible cubic with $(0,0,1)$ as a double point. As we have already shown that any such curve is mapped by σ onto a normal rational curve of degree 4 on V_2^3, we have the following;

Theorem 3.12. *There is a one-to-one correspondence between the set of normal rational curves of degree 4 on V_2^3 and the set consisting of the irreducible conics of $PG(2,q)$ not containing $(0,0,1)$, together with the irreducible cubics of $PG(2,q)$ with a double point at $(0,0,1)$. Additionally, those normal rational curves meeting ℓ correspond to cubics with either a node or a cusp at $(0,0,1)$, and those skew to ℓ correspond to either an irreducible conic of $PG(2,q)$ not containing $(0,0,1)$ or a cubic curve with an isolated double point at $(0,0,1)$.*

□

Corollary 3.13. *The number of normal rational curves of degree four contained on V_2^3 is $q^4(q-1)(q+2)$. Of these $q^3(q-1)(q+1)^2/2$ meet ℓ and $q^3(q-1)(q^2+2q-1)/2$ are skew to ℓ.*

Proof: From theorem 1.16 there are $q^3(q^3-1)(q^2-1)/2$ absolutely irreducible cubic curves with a node and $q^3(q^3-1)(q+1)$ absolutely irreducible cubic curves with a cusp in $PG(2,q)$. Thus the number of each type which contain the point $(0,0,1)$ is $q^3(q-1)(q^2-1)/2$ and $q^3(q-1)(q+1)$ respectively. Hence, by theorem 3.12, there are $q^3(q-1)((q^2-1)/2+q+1)$ normal rational curves of degree four meeting ℓ and contained on V_2^3.

Similarly, by theorem 1.16, there are $q^3(q^3-1)(q^2-1)/2$ absolutely irreducible cubic curves with an isolated double point in $PG(2,q)$. Hence there are $q^3(q-1)(q^2-1)/2$ absolutely irreducible cubics with an isolated double point at $(0,0,1)$. By theorem 1.15 there are (q^5-q^2) non-singular conics in $PG(2,q)$. Hence the number of non-singular conics of $PG(2,q)$ which do not contain the point $(0,0,1)$ is $(q^5-q^2)q^2/(q^2+q+1) = q^4(q-1)$. From theorem 3.12 the total number of normal rational curves of degree four on V_2^3 which do not meet ℓ is the sum of these two numbers and we obtain the stated result.

□

Chapter 4

The Bose Representation of $PG(2, q^2)$

4.1 Definition of the Bose Map

In his paper [8], R.C. Bose discusses a representation of the Desarguesian projective plane $PG(2, q^2)$ in the five dimensional projective space $PG(5, q)$, whereby the points of $PG(2, q^2)$ are represented by a 1-spread of lines of $PG(5, q)$ and the lines of $PG(2, q^2)$ are represented by the 3-spaces of $PG(5, q)$ spanned by pairs of lines of this 1-spread. We describe the this representation of $PG(2, q^2)$, but using a method different to that used in [8].

Consider the bijective map

$$\begin{array}{rccl} \phi : & (GF(q^2))^3 & \rightarrow & (GF(q))^6 \\ \phi : & (a, b, c) & \mapsto & (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \end{array}$$

where

$$\begin{aligned} a &= \alpha_1 + i\alpha_2 \\ b &= \beta_1 + i\beta_2 \\ c &= \gamma_1 + i\gamma_2 \end{aligned}$$

and i is a zero of $x^2 + x + \xi$, $\xi \in GF(q)$, which is irreducible over $GF(q)$. We wish to examine how this map behaves when we consider the triples (a, b, c) as points of $PG(2, q^2)$ and the sextuples $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ as points of $PG(5, q)$.

Lemma 4.1. *$\phi(a, b, c)$ and $\phi(a', b', c')$ are the coordinates of the same point of $PG(5, q)$ if and only if $a' = \delta a$, $b' = \delta b$ and $c' = \delta c$ for some $\delta \in GF(q)$.*

Proof: Let $\phi(a,b,c) = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ and $\phi(a',b',c') = (\alpha_1', \alpha_2', \beta_1', \beta_2', \gamma_1', \gamma_2')$. Then $\phi(a,b,c)$ and $\phi(a',b',c')$ represent the same point of $PG(5,q)$

$$\begin{aligned}
&\Leftrightarrow \quad (\alpha_1', \alpha_2', \beta_1', \beta_2', \gamma_1', \gamma_2') = \delta(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \text{ for some } \delta \in GF(q) \\
&\Leftrightarrow \quad \alpha_1' = \delta\alpha_1, \alpha_2' = \delta\alpha_2, \beta_1' = \delta\beta_1, \beta_2' = \delta\beta_2, \gamma_1' = \delta\gamma_1, \gamma_2' = \delta\gamma_2 \\
&\Leftrightarrow \quad \alpha_1' + i\alpha_2' = \delta(\alpha_1 + i\alpha_2), \beta_1' + i\beta_2' = \delta(\beta_1 + i\beta_2) \text{ and } (\gamma_1' + i\gamma_2') = \delta(\gamma_1 + i\gamma_2) \\
&\Leftrightarrow \quad a' = \delta a, b' = \delta b, c' = \delta c.
\end{aligned}$$

□

If (a,b,c) is considered as a point of $PG(2,q^2)$ then any of the (q^2-1) triples (ka,kb,kc), where k is a non-zero element of $GF(q^2)$, are coordinates of the same point of $PG(2,q^2)$. By lemma 4.1, the $(q-1)$ triples $(\delta a, \delta b, \delta c)$, where δ is a non-zero element of $GF(q)$, are mapped by ϕ to the same point of $PG(5,q)$. Thus, as there are $(q+1)$ equivalence classes in $GF(q^2) \setminus \{0\}$ under the relation $k_1 = \delta k_2$, for $\delta \in GF(q) \setminus \{0\}$, ϕ maps a point of $PG(2,q^2)$ to a set of $(q+1)$ points of $PG(5,q)$. Furthermore, if (a,b,c) and (a',b',c') are coordinates for the point $P \in PG(2,q^2)$ then

$$\begin{aligned}
& a/a' = b/b' \\
\Rightarrow \quad & (\alpha_1 + i\alpha_2)/(\alpha_1' + i\alpha_2') = (\beta_1 + i\beta_2)/(\beta_1' + i\beta_2') \\
\Rightarrow \quad & \alpha_1\beta_1' - \xi\alpha_2\beta_2' - \alpha_1'\beta_1 + \xi\alpha_2'\beta_2 \\
& +i(\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_1'\beta_2 + \alpha_2'\beta_2 - \alpha_2\beta_2' - \alpha_2'\beta_2) = 0
\end{aligned}$$

where $a = \alpha_1 + i\alpha_2$ and $b = \beta_1 + i\beta_2$. Thus if (a,b,c) is any triple representing P, and we let $(x_0, x_1, x_2, x_3, x_4, x_5)$ be coordinates in $PG(5,q)$, then $\phi(a,b,c) = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ satisfies the equations

$$\beta_1 x_0 - \xi\beta_2 x_1 - \alpha_1 x_2 + \xi\alpha_2 x_3 = 0 \tag{4.1}$$

$$\beta_2 x_0 + (\beta_1 - \beta_2)x_1 - \alpha_2 x_2 - (\alpha_1 - \alpha_2)x_3 = 0 \tag{4.2}$$

Similarly $a/a' = b/b'$ and $b/b' = c/c'$ so that $\phi(a,b,c)$ also satisfies the equations

$$\gamma_1 x_0 - \xi\gamma_2 x_1 - \alpha_1 x_4 + \xi\alpha_2 x_5 = 0 \tag{4.3}$$

$$\gamma_2 x_0 + (\gamma_1 - \gamma_2)x_1 - \alpha_2 x_4 - (\alpha_1 - \alpha_2)x_5 = 0 \tag{4.4}$$

$$\gamma_1 x_2 - \xi\gamma_2 x_3 - \beta_1 x_4 + \xi\beta_2 x_5 = 0 \tag{4.5}$$

$$\gamma_2 x_2 + (\gamma_1 - \gamma_2)x_3 - \beta_2 x_4 - (\beta_1 - \beta_2)x_5 = 0 \tag{4.6}$$

Equations 4.5 and 4.6 are linearly dependent on equations 4.1 - 4.4 whereas the other four are linearly independent. Hence these six equations define a line in $PG(5,q)$. Thus the $q+1$ points to which P is mapped to by ϕ form a line in $PG(5,q)$ defined by the equations 4.1 - 4.4.

Lemma 4.2. *The (q^2-1) 3-tuples of elements of $GF(q^2)$ which represent a point of $PG(2,q^2)$ are mapped by ϕ to the points of a line of $PG(5,q)$.* ☐

If the point P corresponds to the line ℓ in this way then we write $\ell = \boldsymbol{\beta_1}(P)$. Let

$$S_1 = \{\boldsymbol{\beta_1}(P) : P \text{ is a point of } PG(2,q^2)\}$$

Lemma 4.3. [8] *S_1 is a 1-spread of $PG(5,q)$.*

Proof: By lemma 4.1, if P and Q are two distinct points of $PG(2,q^2)$ then the lines $\boldsymbol{\beta_1}(P)$ and $\boldsymbol{\beta_1}(Q)$ are skew. Hence S_1 consists of (q^4+q^2+1) skew lines of $PG(5,q)$ and is thus a 1-spread of $PG(5,q)$. ☐

Let ℓ be the line of $PG(2,q^2)$ of equation $ax+by+cz=0$. We examine the relationship between the (q^2+1) lines $\boldsymbol{\beta_1}(P)$, where P is a point of ℓ.

$$\begin{aligned}
(\alpha,\beta,\gamma)\in\ell \quad &\Leftrightarrow \quad a\alpha+b\beta+c\gamma=0\\
&\Leftrightarrow \quad (a_1+ia_2)(\alpha_1+i\alpha_2)+(b_1+ib_2)(\beta_1+i\beta_2)+(c_1+ic_2)(\gamma_1+i\gamma_2)=0\\
&\Leftrightarrow \quad a_1\alpha_1-\xi a_2\alpha_2+b_1\beta_1-\xi b_2\beta_2+c_1\gamma_1-\xi c_2\gamma_2=0\\
&\text{and} \quad a_2\alpha_1+(a_1-a_2)\alpha_2+b_2\beta_1+(b_1-b_2)\beta_2+c_2\gamma_1+(c_1-c_2)\gamma_2=0
\end{aligned}$$

Hence, if $P=(\alpha,\beta,\gamma)$ is a point of ℓ, then the points of the line $\boldsymbol{\beta_1}(P)$ satisfy the equations

$$\begin{aligned}
a_1x_0-\xi a_2x_1+b_1x_2-\xi b_2x_3+c_1x_4-\xi c_2x_5 &= 0\\
a_2x_0+(a_1-a_2)x_1+b_2x_2+(b_1-b_2)x_3+c_2x_4+(c_1-c_2)x_5 &= 0.
\end{aligned}$$

Therefore the 3-space Σ_3 defined by these two equations contains the (q^2+1) lines $\boldsymbol{\beta_1}(P)$, where $P\in\ell$. We say that $\Sigma_3=\boldsymbol{\beta_1}(\ell)$ and let

$$H_3 = \{\boldsymbol{\beta_1}(\ell) : \ell \text{ is a line of } PG(2,q^2)\}$$

As the lines $\boldsymbol{\beta_1}(P)$, for $P\in\ell$, are in S_1, they are skew and therefore form a spread of Σ_3; that is, S_1 induces a spread in Σ_3. Additionally, the elements of H_3 are precisely the spaces spanned by two distinct elements of S_1. Thus,

Theorem 4.4. *S_1 is a geometric spread of $PG(5,q)$.* ☐

Hence, we have, as in theorem 1.49, the following result (which also follows as a consequence of the definitions of the maps ϕ and $\boldsymbol{\beta_1}$);

Theorem 4.5. *The incidence structure* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ *where*

$$\begin{aligned} \mathcal{P} &= S_1 \\ \mathcal{L} &= H_3 \\ \mathcal{I} &= \in \end{aligned}$$

is isomorphic to $PG(2, q^2)$. □

We call this the **Bose representation** of $PG(2, q^2)$. If A is any set of points of $PG(2, q^2)$ then we denote the structure in $PG(5, q)$ which represents the points of A by $\boldsymbol{\beta_1}(A)$ and call it the Bose representation of A. We call $\boldsymbol{\beta_1}$ the Bose map.

If $\mathcal{S}$ is an arbitrary geometric 1-spread of $PG(5, q)$ then, as in theorem 1.49, the incidence structure which has the elements of $\mathcal{S}$ as points, the 3-spaces generated by pairs of elements of $\mathcal{S}$ as lines and with the incidence given by containment, is isomorphic to $PG(2, q^2)$. The Bose representation is identical to this construction but, in addition, gives coordinates for the line of $\mathcal{S}$ which corresponds to the point P of $PG(2, q^2)$. Thus it is helpful in discussing the representation of certain subsets of $PG(2, q^2)$ under the isomorphism of theorem 1.49. In particular, in section 4.4 and chapter 5, we will see that it is useful by finding the Bose representations of curves, Baer subplanes and classical unitals of $PG(2, q^2)$.

4.2 The Directrix Planes

As S_1 is a geometric 1-spread of $PG(5, q)$ and we know, from theorem 1.51, that a geometric spread of $PG(5, q)$ is regular, S_1 is a regular 1-spread of $PG(5, q)$. Hence, by definition, given a line ℓ not in S_1, the $(q + 1)$ lines of S_1 which meet ℓ form a regulus in a three dimensional subspace of $PG(5, q)$. So if Σ_3 is an element of H_3, and ℓ is a line of Σ_3 not in the spread $\mathcal{S}$ of Σ_3 induced by $\mathcal{S}_1$, then the $(q + 1)$ elements of $\mathcal{S}$ which meet ℓ form a regulus in Σ_3. Therefore, by theorem 1.32, we have proved;

Theorem 4.6. *If* $\Sigma_3 \in H_3$ *then the spread of* Σ_3 *induced by* S_1 *is regular.* □

Now consider $PG(5, q)$ as a subgeometry of $PG(5, q^2)$. Since the spread $\mathcal{S}$ of lines of S_1 in $\Sigma_3 \in H_3$ is regular, we know from theorem 1.33 that there are conjugate lines m and $\overline{m}$ of $PG(5, q^2)$ such that $\mathcal{S}$ is the set of lines obtained by joining $P \in m$ to $\overline{P} \in \overline{m}$. As H_3 contains $(q^4 + q^2 + 1)$ 3-spaces we have, in total, $(q^4 + q^2 + 1)$ such pairs of conjugate lines. Furthermore, these lines are the lines of two conjugate planes of $PG(5,q^2)$ and thus the lines of S_1 are the lines which join conjugate points of these planes as we now demonstrate.

Theorem 4.7. *Consider the conjugate planes of $PG(5,q^2)$ defined by the equations*

$$\begin{aligned} \Pi: \;& x_0+ix_1 = x_2+ix_3 = x_4+ix_5 = 0 \\ \overline{\Pi}: \;& x_0+\bar{i}x_1 = x_2+\bar{i}x_3 = x_4+\bar{i}x_5 = 0. \end{aligned}$$

If we join each point P of Π to its conjugate point $\overline{P}$ of $\overline{\Pi}$ then the q^4+q^2+1 lines so obtained are exactly the lines of S_1 extended to $PG(5,q^2)$.

Proof: Noting that (a,b,c) and (ia,ib,ic) are coordinates for the same point P of $PG(2,q^2)$ it follows that the line $\ell = \boldsymbol{\beta}_1(P)$ of S_1 contains the points $\phi(a,b,c) = (\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2)$ and $\phi(ia,ib,ic) = (-\xi\alpha_2,\alpha_1-\alpha_2,-\xi\beta_2,\beta_1-\beta_2,-\xi\gamma_2,\gamma_1-\gamma_2)$ of $PG(5,q)$. Thus this line of S_1 contains the set of points

$$\{(\alpha_1-r\xi\alpha_2,\alpha_2+r(\alpha_1-\alpha_2),\beta_1-r\xi\beta_2,\beta_2+r(\beta_1-\beta_2),\dots) : r\in GF(q)\cup\{\infty\}\}$$

and so, when extended to $PG(5,q^2)$, contains the points

$$\{(\alpha_1-r\xi\alpha_2,\alpha_2+r(\alpha_1-\alpha_2),\beta_1-r\xi\beta_2,\beta_2+r(\beta_1-\beta_2),\dots) : r\in GF(q^2)\cup\{\infty\}\}$$

Therefore it meets the hyperplane $x_0+ix_1=0$ of $PG(5,q^2)$ where

$$r = \frac{(\alpha_1+i\alpha_2)}{(\xi\alpha_2+i(\alpha_2-\alpha_1))} = \bar{r}.$$

That is, at the point

$$P = (\alpha_1-\bar{r}\xi\alpha_2,\alpha_2+\bar{r}(\alpha_1-\alpha_2),\beta_1-\bar{r}\xi\beta_2,\beta_2+\bar{r}(\beta_1-\beta_2),\dots)$$

which also lies on the hyperplanes $x_2+ix_3=0$ and $x_4+ix_5=0$ of $PG(5,q^2)$ and so is a point of Π. Similarly ℓ meets $\overline{\Pi}$ at the point

$$Q = (\alpha_1-s\xi\alpha_2,\alpha_2+s(\alpha_1-\alpha_2),\dots)$$

where

$$s = \frac{(\alpha_1+\bar{i}\alpha_2)}{(\xi\alpha_2+\bar{i}(\alpha_2-\alpha_1))}$$

Note that

$$\begin{aligned} s^q &= \frac{(\alpha_1+\bar{i}\alpha_2)^q}{(\xi\alpha_2+\bar{i}(\alpha_2-\alpha_1))^q} \\ &= \frac{(\alpha_1+i^{q^2}\alpha_2)}{\xi\alpha_2+i^{q^2}(\alpha_2-\alpha_1))} \\ &= \frac{(\alpha_1+i\alpha_2)}{\xi\alpha_2+i(\alpha_2-\alpha_1)} \\ &= \bar{r} \end{aligned}$$

so

$$\begin{aligned}(x_1 - s\xi x_2)^q &= x_1^q - s^q\xi^q x_2^q \\ &= x_1 - \overline{r}\xi x_2\end{aligned}$$

and

$$\begin{aligned}(x_2 + s(x_1 - x_2))^q &= x_2^q + s^q(x_1 - x_2)^q \\ &= x_2 + r(x_1 - x_2)\end{aligned}$$

Hence the points P and Q are conjugates as required. □

Theorem 4.8. *Π and $\overline{\Pi}$ are the unique pair of conjugate planes of $PG(5, q^2)$ such that the lines of S_1 are the lines joining corresponding points of these planes.*

Proof: If $\Sigma_3 \in H_3$ then the lines $\Sigma_3 \cap \Pi$ and $\Sigma_3 \cap \overline{\Pi}$ are directrix lines of the spread $\mathcal{S}$ of Σ_3 induced by S_1. As the directrix lines of this spread are unique, the planes Π and $\overline{\Pi}$ are unique. □

We call Π and $\overline{\Pi}$ the **directrix planes** of S_1.

Suppose P is a point of $PG(2, q^2)$ corresponding to the line $\ell = \boldsymbol{\beta_1}(P)$ of S_1 and that m is a line of $PG(2, q^2)$ corresponding to $\Sigma_m = \boldsymbol{\beta_1}(m)$. It then follows that Σ_m meets Π in a line which contains the point $\Pi \cap \ell$. Thus the map which associates the point $P \in PG(2, q^2)$ with the point $\Pi \cap \ell$ is a collineation between $PG(2, q^2)$ and Π and thus, in effect, the geometry in $PG(2, q^2)$ is mimicked in Π. This means that if we are considering the Bose representation $\boldsymbol{\beta_1}(A)$ of a point-set A in $PG(2, q^2)$, then the lines of S_1 of which $\boldsymbol{\beta_1}(A)$ consists, meet Π in a set of points isomorphic to A. This viewpoint can sometimes be useful when considering the Bose representation.

4.3 The Sets S_1 and H_3 and Subspaces of $PG(5, q)$

We shall require some information which describes S_1 and H_3 in greater detail and, in particular, how the subspaces of $PG(5, q)$ may intersect these sets. Most of these results appear in [8] but, once again, the exposition is different.

Lemma 4.9. [8] *Any two 3-spaces of H_3 intersect exactly in a line of S_1 and any two lines of S_1 are contained in a unique element of H_3.*

Proof: If $\ell_1 = \boldsymbol{\beta}_1(P_1)$ and $\ell_2 = \boldsymbol{\beta}_1(P_2)$ are lines of S_1 then $\boldsymbol{\beta}_1(P_1P_2)$ is the unique element of H_3 containing them. Let $\Sigma_1 = \boldsymbol{\beta}_1(m_1)$ and $\Sigma_2 = \boldsymbol{\beta}_1(m_2)$ be two elements of H_3. If the point Q of $PG(5,q)$ is contained in both Σ_1 and Σ_2 then, as S_1 induces a spread in elements of H_3, the line of S_1 through Q is in $\Sigma_1 \cap \Sigma_2$ too. Thus, as m_1 and m_2 have a unique common point Q, Σ_1 and Σ_2 intersect precisely in the line $\boldsymbol{\beta}_1(Q)$. □

As a consequence of this we have the following useful result.

Theorem 4.10. [8] *Every hyperplane of $PG(5,q)$ contains a unique element of H_3.*

Proof: Let Σ_3 and Σ_3' be two distinct elements of H_3. If they are contained in a hyperplane Σ_4 of $PG(5,q)$ then they have a plane in common, contradicting lemma 4.9. Therefore each of the $(q+1)$ hyperplanes containing Σ_3 contain no further element of H_3. As there are (q^4+q^2+1) three-dimensional spaces in H_3, we can deduce from this that all of the $(q^4+q^2+1)(q+1)$ hyperplanes of $PG(5,q)$ contain a unique element of H_3.

□

The Subspaces of $PG(5,q)$

By theorem 4.10, every hyperplane of $PG(5,q)$ is situated identically with respect to S_1 and H_3 in that it contains exactly one element Σ_3 of H_3, and meets every other element of H_3 in a (distinct) plane and contains the (q^2+1) lines of S_1 which are in Σ_3, and meets every other line of S_1 in a (distinct) point. For subspaces of other dimensions there are more alternatives. If ℓ is a line of $PG(5,q)$ then ℓ is either in S_1 or not and, accordingly, we divide the lines of $PG(5,q)$ into two classes, those which are in S_1 and those which are not. A line of $PG(5,q)$ is said to be of **type 1** or **type 2** depending upon whether it is a line of S_1 or not. Correspondingly, a 3-space of $PG(5,q)$ is said to be of **type 1** or **type 2** depending upon whether it is an element of H_3 or not, [8].

For planes the situation appears to be more complicated. By lemma 4.9, a plane of $PG(5,q)$ contains at most one line of S_1 and is contained in at most one element of H_3 and thus there appears to be four types of planes. However this is not the case.

Lemma 4.11. [8] *A plane of $PG(5,q)$ either contains exactly one line of S_1 and is contained in one element of H_3 or contains no line of S_1 and is contained in no element of H_3.*

□

Proof: If π is a plane of $PG(5,q)$ contained in $\Sigma_3 \in H_3$ then π meets each of the (q^2+1) lines of S_1 contained in Σ_3. Hence, as these lines are skew, it meets q^2 of them in a unique point and contains the other. Suppose that π is a plane of $PG(5,q)$ containing the line ℓ of S_1. If A is a point of π not on ℓ then let ℓ_A denote the line of S_1 through this point. Hence $\langle \ell, \ell_A \rangle$ is an element of H_3 and contains π. Thus a plane contains a line of S_1 if and only if it contains an element of H_3. As no plane contains two lines of S_1 or is contained in two elements of H_3 the result follows. $\square$

Therefore, there are two types of plane in $PG(5,q)$. Those containing a line of S_1 and contained in an element of H_3, which are said to be of **type 1**, and those containing no line of S_1 and contained in no element of H_3, which are said to be of **type 2**, [8].

Properties of the Subspaces

A line of S_1 is skew to all other lines of S_1 and all the (q^3+q^2+q+1) planes about it are necessarily planes of type 1. It is contained in q^2+1 elements of H_3 and skew to all remaining elements of H_3. A line ℓ of type two has one line of S_1 through each of its points and these $(q+1)$ lines are contained in a common element $\Sigma_3 \in H_3$ and form a regulus in Σ_3. If ℓ_A denotes the line of S_1 through $A \in \ell$ then the plane $\langle \ell, \ell_A \rangle$ is of type 1 and so ℓ is contained in $q+1$ planes of type 1. All the remaining planes containing ℓ are of type 2. Σ_3 is the only element of H_3 which contains ℓ.

Lemma 4.12. [8] *There are* $(q^4+q^3+q^2+q+1)(q^4+q^2+1)$ *lines of type 2.*

Proof: S_1 contains (q^4+q^2+1) lines and subtracting this from the total number of lines in $PG(5,q)$ gives the above number. $\square$

A plane π of type 1 contains one line ℓ of S_1 and (q^2+q) lines of type 2. If $A \in \pi$, and A lies on the line $\ell_A \neq \ell$ of S_1 then, by lemma 4.9, $\langle \ell, \ell_A \rangle$ is the unique element of H_3 containing π. All lines of a plane π of type 2 are of type 2 and so are contained in a unique element of H_3. By lemma 4.9, these are the only elements of H_3 which meet π in a line, and hence all other elements of H_3 meet π in a unique point.

Lemma 4.13. [8] *Of the planes of* $PG(5,q)$, $(q^4+q^2+1)(q^3+q^2+q+1)$ *are of type 1 and* $q^3(q^3+1)(q^3+q^2+q+1)$ *are of type 2.*

Proof: A line of S_1 is contained in (q^3+q^2+q+1) planes, all of type 1. No type 1 plane can contain two lines of S_1, so the total number of type 1 planes is therefore $(q^3+q^2+q+1)(q^4+q^2+1)$. Subtracting this from the total number of planes in $PG(5,q)$ gives the required number of type 2 planes. $\square$

If $\Sigma_3 \in H_3$ then Σ_3 contains (q^2+1) lines of S_1 and thus $(q^2+1)(q+1)$ planes of type 1. As a plane of type 2 is contained in no element of H_3, Σ_3 contains no type 2 planes. Every other element of H_3 meets Σ_3 in a line of S_1.

Let Σ_3 be a 3-space not in H_3. If Σ_4 is a hyperplane containing Σ_3 then, by lemma 4.10, Σ_4 contains a unique element Σ of H_3. Hence Σ_3 intersects Σ in a plane which is necessarily of type 1 (elements of H_3 contain no type 2 planes) and so contains a line ℓ of S_1, which, as 3-spaces containing two lines of S_1 are elements of H_3, is the unique line of S_1 in Σ_3. The $q+1$ planes of Σ_3 about ℓ are all of type 1 and all other planes in Σ_3 are of type 2. Each of these planes is contained in an element of H_3 and, as elements of H_3 contain no type 2 planes, these are the only elements of H_3 which meet Σ_3 in a plane. To sum up we have,

Lemma 4.14. *A 3-space Σ_3 not in H_3 contains a unique line ℓ of S_1 and $q+1$ planes of type 1, $\pi_1, \dots, \pi_{q+1}$, which are the intersections of $q+1$ elements of H_3 with Σ_3. Each point of $\Sigma_3 \setminus \ell$ lies on a unique plane π_i and hence if A is some point-set in $PG(2,q^2)$ then $\Sigma_3 \cap \boldsymbol{\beta}_1(A) = \bigcup(\pi_i \cap \boldsymbol{\beta}_1(A))$.* □

4.4 The Bose Representation of Curves of $PG(2,q)$

Consider $\mathcal{C}^n$, a curve in $PG(2,q^2)$ given by the equation $f(x,y,z) = 0$, where f is a homogeneous polynomial of degree n over $GF(q^2)$. We wish to describe the Bose representation of $\mathcal{C}^n$.

Let

$$f = \sum_{j+k+l=n} a_{jkl} x^j y^k z^l$$

where $a_{jkl} \in GF(q^2)$. If a, b, c are elements of $GF(q^2)$ then we can write $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$ and $a_{jkl} = r_{jkl} + is_{jkl}$, where $a_i, b_i, c_i, r_{jkl}, s_{jkl}$ are all in $GF(q)$ so that

$$f(a,b,c) = \sum_{j+k+l=n} (r_{jkl} + is_{jkl})(a_1 + ia_2)^j (b_1 + ib_2)^k (c_1 + ic_2)^l$$

Thus, using the fact that $i^2 = -i - \xi$, where $\xi \in GF(q)$, we can write this as

$$f(a,b,c) = f_1(a_1, a_2, b_1, b_2, c_1, c_2) + if_2(a_1, a_2, b_1, b_2, c_1, c_2)$$

for some polynomials f_1 and f_2 with coefficients in $GF(q)$.

If f is of degree n then so too are f_1 and f_2. The equations $f_1 = 0$ and $f_2 = 0$ define hypersurfaces V_1 and V_2 of degree n in $PG(5,q)$. Further, since $\boldsymbol{\beta}_1(a,b,c) = (a_1, a_2, b_1, b_2, c_1, c_2)$, the point (a,b,c) of $PG(2,q^2)$ is a point of $\mathcal{C}^n$ if and only if the point

$(a_1, a_2, b_1, b_2, c_1, c_2)$ of $PG(5,q)$ is a point of the hypersurfaces V_1 and V_2. Thus, the Bose representation of $\mathcal{C}^n$ is the intersection of the two hypersurfaces V_1 and V_2.

For example, consider the conic $\mathcal{C}^2$ of $PG(2,q^2)$ defined by the equation

$$x^2 + yz = 0$$

If $a, b, c \in GF(q^2)$ with $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$ and $c = \gamma_1 + i\gamma_2$ then

$$\begin{aligned}
(a,b,c) \in \mathcal{C}^2 &\Leftrightarrow a^2 + bc = 0 \\
&\Leftrightarrow (\alpha_1 + i\alpha_2)^2 + (\beta_1 + i\beta_2)(\gamma_1 + i\gamma_2) = 0 \\
&\Leftrightarrow \alpha_1^2 + \beta_1\gamma_1 + i^2(\alpha_2^2 + \beta_2\gamma_2) + i(2\alpha_1\alpha_2 + \beta_1\gamma_2 + \beta_2\gamma_1) = 0 \\
&\Leftrightarrow \alpha_1^2 + \beta_1\gamma_1 - \xi(\alpha_2^2 + \beta_2\gamma_2) + i(2\alpha_1\alpha_2 + \beta_1\gamma_2 + \beta_2\gamma_1 - \alpha_2^2 - \beta_2\gamma_2) = 0
\end{aligned}$$

Thus, if we let

$$f_1 = x_0^2 + x_2x_4 - \xi(x_1^2 + x_3x_5)$$

and

$$f_2 = 2x_0x_1 + x_2x_5 + x_3x_4 - x_5^2 - x_3x_5$$

then (a,b,c) is a point of $\mathcal{C}^2$ if and only if the point $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ of $PG(5,q)$ is a point of intersection of the hypersurfaces $f_1 = 0$ and $f_2 = 0$. Hence the Bose representation of $\mathcal{C}^2$ is the intersection of these two hypersurfaces of $PG(5,q)$.

In addition, in the general case, we can show that the following relationships are satisfied by the polynomials f_1 and f_2.

Theorem 4.15. *Let $\mathcal{C}^n$ be a curve in $PG(2,q^2)$ defined by the equation $f(x,y,z) = 0$, where f is a homogeneous polynomial of degree n. The Bose representation of $\mathcal{C}^n$ consists of the intersection of two hypersurfaces of $PG(5,q)$ defined by the equations $f_1(x_0, \dots, x_5) = 0$ and $f_2(x_0, \dots, x_5) = 0$ where, in general, f_1 and f_2 both have degree n. In addition, the polynomials f_1 and f_2 satisfy the following relationships;*

$$\frac{\partial f_1}{\partial x_{j+1}} = -\xi \frac{\partial f_2}{\partial x_j} \tag{4.7}$$

$$\frac{\partial f_1}{\partial x_j} = \frac{\partial f_2}{\partial x_{j+1}} + \frac{\partial f_2}{\partial x_j} \tag{4.8}$$

where $j = 0, 2$ or 4 and $x^2 + x + \xi$, $\xi \in GF(q)$, is irreducible over $GF(q)$.

Proof: The first part of the theorem was shown above. For the proof of the last sentence we use induction on n, the degree of f. Suppose that $f = a_0x + a_1y + a_2z$, for $a_i \in GF(q^2)$, is linear. By proceeding as described in the preamble to this theorem, the points of the Bose representation of the line $f = 0$ lie on the 3-space which is the intersection of the hyperplanes of $PG(5, q)$ defined by the equations

$$f_1 = \alpha_1 x_0 + \alpha_2 x_2 + \alpha_3 x_4 - \xi(\beta_1 x_1 + \beta_2 x_3 + \beta_3 x_5)$$

and

$$f_2 = \beta_1 x_0 + (\alpha_1 - \beta_1)x_1 + \beta_2 x_2 + (\alpha_2 - \beta_2)x_3 + \beta_3 x_4 + (\alpha_3 - \beta_3)x_5$$

where $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$ and $c = \gamma_1 + i\gamma_2$ for $\alpha_i, \beta_i, \gamma_i \in GF(q)$. Hence

$$\frac{\partial f_1}{\partial x_0} = \alpha_1, \quad \frac{\partial f_1}{\partial x_1} = -\xi\beta_1, \quad \frac{\partial f_2}{\partial x_0} = \beta_1, \quad \frac{\partial f_2}{\partial x_1} = \alpha_1 - \beta_1$$

and the relationships 4.7 and 4.8 hold.

Suppose that

$$f = \sum_{i+j+k=n} a_{ijk} x^i y^j z^k$$

where $a_{ijk} \in GF(q^2)$, is of degree n and the Bose representation of the curve $f = 0$ is given by the intersection of the two hypersurfaces $f_1 = 0$ and $f_2 = 0$ of $PG(5, q)$. Now, we can write,

$$\begin{aligned} f(x, y, z) &= \sum_{i+j+k=n} a_{ijk} x^i y^j z^k \\ &= x_0 \sum_{i \neq 0} a_{ijk} x^{i-1} y^j z^k + y \sum_{j \neq 0} a_{0jk} y^{j-1} z^k + z a_{00n} z^{n-1} \\ &= x_0 g(x, y, z) + x_1 h(x, y, z) + x_2 k(x, y, z) \end{aligned}$$

where the Bose representations of $g = 0$, $h = 0$ and $k = 0$ are given by the intersections of the hypersurfaces $g_1 = 0$ and $g_2 = 0$, $h_1 = 0$ and $h_2 = 0$ and $k_1 = 0$ and $k_2 = 0$. Thus we can express the polynomials f_1 and f_2 in terms of the g_i, h_i and k_i. Hence

$$f_1 = x_0 g_1 + x_2 h_1 + x_4 k_1 - \xi(x_1 g_2 + x_3 h_2 + x_5 k_2)$$

and

$$f_2 = x_0 g_2 + x_1(g_1 - g_2) + x_2 h_2 + x_3(h_1 - h_2) + x_4 k_2 + x_5(k_1 - k_2)$$

Now, as h and k contain no x terms,

$$\frac{\partial h_i}{\partial x_0} = \frac{\partial h_i}{\partial x_1} = \frac{\partial k_i}{\partial x_0} = \frac{\partial k_i}{\partial x_1} = 0,$$

and the partial derivatives of f_1 and f_2 can be written as,

$$\begin{aligned}
\frac{\partial f_1}{\partial x_0} &= g_1 + x_0 \frac{\partial g_1}{\partial x_0} - \xi x_1 \frac{\partial g_2}{\partial x_0} \\
\frac{\partial f_1}{\partial x_1} &= x_0 \frac{\partial g_1}{\partial x_1} - \xi \left(g_2 + x_1 \frac{\partial g_2}{\partial x_1} \right) \\
\frac{\partial f_2}{\partial x_0} &= g_2 + x_0 \frac{\partial g_2}{\partial x_0} + x_1 \left(\frac{\partial g_1}{\partial x_0} - \frac{\partial g_2}{\partial x_0} \right) \\
\frac{\partial f_2}{\partial x_1} &= x_0 \frac{\partial g_2}{\partial x_1} + (g_1 - g_2) + x_1 \left(\frac{\partial g_1}{\partial x_1} - \frac{\partial g_2}{\partial x_1} \right)
\end{aligned}$$

and we need to show that these expressions satisfy the relationships 4.7 and 4.8. Now,

$$\begin{aligned}
\frac{\partial f_1}{\partial x_0} &= g_1 + x_0 \frac{\partial g_1}{\partial x_0} - \xi x_1 \frac{\partial g_2}{\partial x_0} \\
&= g_1 + (x_0 - x_1) \frac{\partial g_1}{\partial x_0} + x_1 \left(\frac{\partial g_1}{\partial x_0} - \xi \frac{\partial g_2}{\partial x_0} \right)
\end{aligned}$$

and since g is of degree $(n-1)$ and hence, by the inductive hypothesis, satisfies 4.7 and 4.8 this becomes

$$\begin{aligned}
\frac{\partial f_1}{\partial x_0} &= g_1 + (x_0 - x_1) \frac{\partial g_1}{\partial x_0} + x_1 \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_1}{\partial x_0} \right) \\
&= g_1 + (x_0 - x_1) (\frac{\partial g_2}{\partial x_0} + \frac{\partial g_2}{\partial x_1}) + x_1 \left(\frac{\partial g_1}{\partial x_0} + \frac{\partial g_1}{\partial x_0} \right)
\end{aligned}$$

Manipulating this expression we obtain

$$\frac{\partial f_1}{\partial x_0} = \frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_0}$$

Similarly,

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= x_0 \frac{\partial g_1}{\partial x_1} - \xi \left(g_2 + x_1 \frac{\partial g_2}{\partial x_1} \right) \\
&= -\xi g_2 + x_0 \frac{\partial g_1}{\partial x_1} - \xi x_1 \left(\frac{\partial g_1}{\partial x_0} - \frac{\partial g_2}{\partial x_0} \right) \\
&= -\xi g_2 - \xi x_0 \frac{\partial g_2}{\partial x_0} - \xi x_1 \left(\frac{\partial g_1}{\partial x_0} - \frac{\partial g_2}{\partial x_0} \right)
\end{aligned}$$

as g_1 and g_2 satisfy the relationships 4.7 and 4.8. Thus

$$\frac{\partial f_1}{\partial x_1} = -\xi \frac{\partial f_2}{\partial x_0}$$

The proofs of 4.7 and 4.8 in the cases that $j = 2$ and $j = 4$ are analogous. □

4.5 The Relationship with the Bruck-Bose Representation of $PG(2,q^2)$

Recall from section 1.15 that, if Σ_∞ is a hyperplane of $PG(4,q)$ and $\mathcal{S}$ is a spread of Σ_∞, then we can form the projective plane $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$. This plane is isomorphic to $PG(2,q^2)$ and we call it the Bruck-Bose representation of $PG(2,q^2)$. The isomorphism between $PG(2,q^2)$ and $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ was denoted $\boldsymbol{\beta_2}$. As we now demonstrate, this representation is connected in a natural way to the Bose representation of $PG(2,q^2)$.

Let Σ_4 be a hyperplane in $PG(5,q)$. From corollary 4.10, Σ_4 contains a unique element Σ_∞ of H_3 and the q^2+1 lines of S_1 in Σ_∞, which necessarily form a regular spread $\mathcal{S}$ of Σ_∞. Thus, we can construct the Bruck and Bose projective plane $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$. Therefore we have a Bruck-Bose representation of $PG(2,q^2)$ in every hyperplane of Σ_5.

We can say more about the association between the two maps. The hyperplane Σ_4 contains precisely those elements of S_1 which are in Σ_∞ and meets each remaining line of S_1 in a unique point of $\Sigma_4 \setminus \Sigma_\infty$. Conversely, there is a unique line of S_1 through each such point. Similarly, Σ_∞ is the unique element of H_3 in Σ_4 and every other element of H_3 meets Σ_4 in a plane about a line of $\mathcal{S}$. Conversely, there is an element of H_3 containing any plane meeting Σ_∞ in a line of $\mathcal{S}$. Thus, if P is a point of $PG(2,q^2)$ and ℓ a line of $PG(2,q^2)$, then $\boldsymbol{\beta_2}(P) = \boldsymbol{\beta_1}(P) \cap \Sigma_4$ and $\boldsymbol{\beta_2}(\ell) = \boldsymbol{\beta_1}(\ell) \cap \Sigma_4$. We state this as follows;

Theorem 4.16. *Let Σ_4 be a hyperplane of $PG(5,q)$ containing the unique element Σ_∞ of H_3 and $\mathcal{S}$ be the spread of Σ_∞ induced by S_1. If $\boldsymbol{\beta_1}$ denotes the Bose map and $\boldsymbol{\beta_2}$ the Bruck-Bose map between $PG(2,q^2)$ and $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ then*

$$\boldsymbol{\beta_2}(P) = \boldsymbol{\beta_1}(P) \cap \Sigma_4$$

for any point P of $PG(2,q^2)$, and

$$\boldsymbol{\beta_2}(\ell) = \boldsymbol{\beta_1}(\ell) \cap \Sigma_4$$

for any line ℓ of $PG(2,q^2)$. $\square$

Hence if A is any set of points of $PG(2,q^2)$ and we know the structure of the Bose representation of A, $\boldsymbol{\beta_1}(A)$, then we can determine the Bruck and Bose representation $\boldsymbol{\beta_2}(A)$ of A by considering the intersection of $\boldsymbol{\beta_1}(A)$ with a hyperplane Σ_4 of $PG(5,q)$. Note that as Σ_4 intersects the directrix planes Π and $\overline{\Pi}$ of S_1 in the directrices of $\mathcal{S}$

this also tells us how $\boldsymbol{\beta}_{\mathbf{2}}(A)$ intersects the directrices of $\mathcal{S}$ in the quadratic extension $PG(4,q^2)$.

Chapter 5

The Bose Representation of Baer Subplanes and Classical Unitals

The main theme of Bose's paper [8] is that the Bose map gives a concise representation of the Baer subplanes and Baer sublines of $PG(2, q^2)$. In addition, and not discussed in [8], the Bose representation of the classical unitals of $PG(2, q^2)$ is also quite appealing. Here we discuss these representations and, as in section 4.5, use them to elucidate the Bruck-Bose representations of these structures. Furthermore, it is shown that the Bose representation is useful in that it simplifies the proofs of several known results concerning unitals and Baer subplanes.

5.1 The Bose Representation of Baer Subplanes

We discuss the Bose representation of the Baer subplanes of $PG(2, q^2)$ and the Baer sublines of a line of $PG(2, q^2)$, obtaining, from a different perspective, similar results to those obtained in [8].

If $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) = \phi(\alpha, \beta, \gamma)$ (with ϕ as defined in section 4.1) is a point of $PG(5, q)$ then, for any non-zero $k \in GF(q^2)$, the point

$$\phi(kx, ky, kz) = (\kappa_1\alpha_1 - \xi\kappa_2\alpha_2, \kappa_2\alpha_1 + (\kappa_1 - \kappa_2)\alpha_2, \kappa_1\beta_1 - \xi\kappa_2\beta_2, \dots)$$

of $PG(5, q)$, where $k = \kappa_1 + i\kappa_2$ and $\xi \in GF(q)$ is such that the polynomial $x^2 + x + \xi$ is irreducible over $GF(q)$, lies on the same line of S_1. Hence, for any non-zero $k \in GF(q^2)$, with $k = \kappa_1 + i\kappa_2$, the map

$$\begin{array}{rrcl} \psi_k : & PG(5,q) & \to & PG(5,q) \\ \psi_k : & \mathbf{X} & \mapsto & \mathbf{XA} \end{array}$$

where $\mathbf{A}$ is the matrix

$$\begin{pmatrix} \kappa_1 & -\xi\kappa_2 & 0 & 0 & 0 & 0 \\ \kappa_2 & \kappa_1 - \kappa_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_1 & -\xi\kappa_2 & 0 & 0 \\ 0 & 0 & \kappa_2 & \kappa_1 - \kappa_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_1 & -\xi\kappa_2 \\ 0 & 0 & 0 & 0 & \kappa_2 & \kappa_1 - \kappa_2 \end{pmatrix}$$

preserves the lines of S_1.

Lemma 5.1. *1. ψ_k is a projectivity of $PG(5,q)$.*

2. For $k \in GF(q)$, ψ_k is the identity map. For $k \notin GF(q)$, ψ_k fixes no point of $PG(5,q)$ and every line of S_1.

3. $\psi_k = \psi_\ell \Leftrightarrow k = \delta\ell$ for some $\delta \in GF(q)$.

4. There are $(q+1)$ distinct projectivities of this type.

Proof: By simple expansion

$$|\mathbf{A}| = (\kappa_1(\kappa_1 - \kappa_2) + \xi\kappa_2^2)^3$$

so that

$$|\mathbf{A}| = 0 \Leftrightarrow \kappa_1^2 - \kappa_1\kappa_2 + \xi\kappa_2^2 = 0$$

Consider the quadratic $x^2 - x + \xi$. If this has roots x_1 and x_2 then $-x_1$ and $-x_2$ will be roots of $x^2 + x + \xi$, contradicting the irreducibility of this quadratic. Thus, for $\kappa_2 \neq 0$,

$$\begin{aligned} \left(\frac{\kappa_1}{\kappa_2}\right)^2 - \left(\frac{\kappa_1}{\kappa_2}\right) + \xi &= \frac{\kappa_1^2 - \kappa_1\kappa_2 + \xi\kappa_2^2}{\kappa_2^2} \\ &\neq 0 \end{aligned}$$

and so $|\mathbf{A}| \neq 0$.

If $\kappa_2 = 0$ then $|\mathbf{A}| = \kappa_1^6 \neq 0$, as κ_1 and κ_2 are not both zero. Thus $|\mathbf{A}| \neq 0$ and ψ_k is a projectivity.

2. This follows from the definition of ψ_k.

3. Suppose that $\psi_k = \psi_u$, for some $u \in GF(q^2)$ where $u = \mu_1 + i\mu_2$. Then

$$\psi_k(1,0,0,0,0,0) = \psi_u(1,0,0,0,0,0)$$

which implies that $(\kappa_1, \kappa_2, 0, 0, 0, 0)$ and $(\mu_1, \mu_2, 0, 0, 0, 0)$ are coordinates for the same point of $PG(5,q)$. Therefore, $\kappa_1 = \sigma\mu_1$ and $\kappa_2 = \sigma\mu_2$ for some $\sigma \in GF(q)$ and thus $k = \sigma u$.

Conversely, if $k = \sigma u = \sigma(\mu_1 + i\mu_2)$ for some $\sigma \in GF(q)$ and X is any point of $PG(5,q)$, with coordinates $(x_0, x_1, x_2, x_3, x_4, x_5)$, then

$$\begin{aligned}
\psi_k(X) &= (\kappa_1 x_0 - \xi\kappa_2 x_1, \kappa_2 x_0 + (\kappa_1 - \kappa_2)x_1, \dots) \\
&= (\sigma\lambda_1 x_0 - \xi\sigma\lambda_2 x_1, \sigma\lambda_2 x_0 + \sigma(\lambda_1 - \lambda_2)x_1, \dots) \\
&= \sigma(\lambda_1 x_0 - \xi\lambda_2 x_1, \lambda_2 x_0 + (\lambda_1 - \lambda_2)x_1, \dots) \\
&= \psi_u(X)
\end{aligned}$$

Hence $\psi_k = \psi_u$.

4. This follows from 3 as there are $(q^2 - 1)$ non-zero elements in $GF(q^2)$, $(q-1)$ non-zero elements in $GF(q)$ and $(q^2-1)/(q-1) = (q+1)$. □

Consider a line ℓ of $PG(5,q)$ of type 2; that is ℓ is not a line of S_1. As ψ_k is a projectivity, the image of ℓ under ψ_k is a line m which is of type 2. If $P \in \ell$ and lies on the line ℓ_P of S_1 then, as ψ_k fixes lines of S_1, $\psi_k(P)$ is the point of intersection of ℓ_P and m. Thus the lines ℓ and m are projectively related, by ψ_k, in such a way that corresponding points are incident with the same line of S_1. As in theorem 1.45, this correspondence defines a Segre variety $\mathcal{V}_{1,1}$, that is a three-dimensional hyperbolic quadric $\mathcal{Q}$, with one regulus $\mathcal{R}$ being the lines $P\psi_k(P)$ where P is a point of ℓ. Note that the lines of $\mathcal{R}$ are all in S_1. For $k' \in GF(q^2)$ with $k' \neq \sigma k$ for any $\sigma \in GF(q)$, the projectivity $\psi_{k'}$ is distinct to ψ_k. Similarly to above, the projectivity $\psi_{k'}$ acting on ℓ defines a three dimensional hyperbolic quadric $\mathcal{Q}'$. However, as there is a unique line of S_1 through each point of ℓ, $\mathcal{R}$ is a regulus of $\mathcal{Q}'$ too and $\mathcal{Q}'$ is identical to $\mathcal{Q}$. Hence the line $\psi_{k'}(\ell)$ is a line of the opposite regulus to $\mathcal{R}$ and the opposite regulus is made up of the image of ℓ under the $(q+1)$ distinct projectivities described in lemma 5.1.

To describe the Bose representations of the Baer Subplanes of $PG(2,q^2)$ we must examine how ψ_k acts on the planes of $PG(5,q)$.

Let α be a plane of $PG(5,q)$ of type 2. Its image α' under ψ_k is a plane of type 2. Hence the planes α and α' are projectively related (via ψ_k), with, as above, corresponding points being incident with a common line of S_1. Therefore, by theorem 1.45, the planes α and α' generate a cubic scroll of planes with lines in S_1 and with planes being the image of α under the $(q+1)$ distinct projectivities described in lemma 5.1. Each plane of type 2 is contained in a unique cubic scroll of this type and, as each such scroll contains $(q+1)$ planes of type 2, we have, by lemma 4.13, the following result:

Lemma 5.2. *There are $q^3(q^3+1)(q^2+1)$ cubic scrolls of planes where the planes of the scroll are of type 2 and the lines of the scroll are in S_1.* □

Suppose R_3^3 is one such cubic scroll of planes. Let α be a plane of the scroll and B denote the set of (q^2+q+1) lines of S_1 which meet α; that is B is the set of lines of the scroll. In the Bose representation, each line of B represents a point of $PG(2,q^2)$. Let $\mathcal{B}$ denote this set of points. We examine how $\mathcal{B}$ is intersected by the lines of $PG(2,q^2)$.

Let ℓ be a line of $PG(2,q^2)$ corresponding, via the Bose map, to the 3-space $\Sigma_3 \in H_3$. By the definition of a type 2 plane, α is contained in no element of H_3. In particular, Σ_3 does not contain α and thus α and Σ_3 intersect in either a point or a line. Since Σ_3 contains precisely those elements of B which meet α in a point of $\alpha \cap \Sigma_3$ it thus contains either 1 or $(q+1)$ lines of B. Hence ℓ meets $\mathcal{B}$ in either 1 or $(q+1)$ points. Suppose that ℓ and ℓ' are two lines of $PG(2,q^2)$ containing $(q+1)$ points of $\mathcal{B}$ and Σ_3' is the Bose representation of ℓ'. In this case Σ_3 and Σ_3' both intersect α in lines. The line of B through the point of intersection of these lines thus corresponds to a point of $\mathcal{B}$ in common to both ℓ_1 and ℓ_2. Therefore the points of $\mathcal{B}$ are the points of a Baer subplane of $PG(2,q^2)$ and R_3^3 is the Bose representation of this subplane. Furthermore, if ℓ contains $(q+1)$ points of $\mathcal{B}$, and is thus a Baer subline of $\mathcal{B}$, then Σ_3 meets α in a line t of type 2 and so, as above, the lines of B which meet t form a regulus. Therefore the Baer sublines of the Baer subplane $\mathcal{B}$ correspond to reguli of lines of R_3^3.

In addition, from lemma 5.2, there are $q^3(q^3+1)(q^2+1)$ such cubic scrolls which, by corollary 1.3, is the number of Baer subplanes of $PG(2,q^2)$. Similarly, there are $q(q^2+1)$ reguli of lines of S_1 in an element of H_3 and thus $q(q^2+1)(q^4+q^2+1)$ reguli of lines of S_1. By corollary 1.5, this is the number of Baer sublines in $PG(2,q^2)$. We sum up these results in the following.

Theorem 5.3. [8] *In the Bose representation of $PG(2,q^2)$,*

(1) The Baer subplanes are represented by cubic scrolls of planes of type 2, for which the lines of the scroll are in S_1. Conversely, all such cubic scrolls represent Baer subplanes of $PG(2,q^2)$.

(2) A Baer subline of a line of $PG(2,q^2)$ is represented by a regulus of lines of S_1 and, conversely, any regulus of lines of S_1 represents a Baer subline of $PG(2,q^2)$. □

5.1.1 The Bruck-Bose Representation of Baer Subplanes

In section 4.5 we discussed a connection between the Bose and Bruck-Bose representations of $PG(2,q^2)$. As we now know the Bose representation of Baer subplanes and Baer sublines of $PG(2,q^2)$, we can exploit this connection in order to find the Bruck-Bose representation of the Baer subplanes and Baer sublines of $PG(2,q^2)$.

We begin by finding the Bruck-Bose representation of a Baer subline b of a line ℓ of $PG(2,q^2)$. As before, let Σ_4 be a hyperplane of $PG(5,q)$ containing $\Sigma_\infty \in H_3$ and $\mathcal{S}$ denote the regular spread of Σ_∞ induced by S_1. We determine the Bruck-Bose representation of b in the plane $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S}) = PG(2,q^2)$. Let $\ell_\infty = \boldsymbol{\beta_2}(\Sigma_\infty)$ be the Bruck-Bose representation of Σ_∞. The Bose representation of ℓ in $PG(5,q)$, $\boldsymbol{\beta_1}(\ell)$, is an element Σ_3 of H_3 and $\boldsymbol{\beta_1}(b)$ is a regulus $\mathcal{R}$ of lines of S_1 (in Σ_3) (theorem 5.3). Therefore, as in section 4.5, $\boldsymbol{\beta_2}(b) = \mathcal{R} \cap \Sigma_4$. Note that the intersections of $\mathcal{R}$ with the directrix planes of S_1 are points of the directrix lines of $\mathcal{S}$. There are three cases to consider.

Firstly assume that $\Sigma_3 = \Sigma_\infty$ or, equivalently, that b is a Baer subline of ℓ_∞. In this case the lines of $\mathcal{R}$ are contained in Σ_∞ and so $\boldsymbol{\beta_2}(b) = \mathcal{R}$. Conversely, by theorem 5.3, all reguli of lines of S_1 represent Baer sublines of $PG(2,q^2)$.

Alternatively $\Sigma_3 \neq \Sigma_\infty$ (b is not a Baer subline of ℓ_∞) so that Σ_3 meets Σ_4 in a plane α about $t \in S$.

Suppose that t is a line of $\mathcal{R}$ (or, equivalently, that b contains a point of ℓ_∞). This means that $\boldsymbol{\beta_2}(b)$ consists of t together with the q points in which α intersects the lines of $\mathcal{R} \setminus t$. As $\mathcal{R}$ is one regulus of a hyperbolic quadric, it follows that $\alpha \cap \mathcal{R}$ is a degenerate conic consisting of two lines, one of which is t and the other, t', contains these q points. In this case we say that $\boldsymbol{\beta_2}(b) = t'$, although we could also regard it as the degenerate conic containing the lines t and t'. Note that this would imply that, in the extension, $\boldsymbol{\beta_2}(b)$ contains the points $t \cap \Pi$ and $t \cap \overline{\Pi}$ of the directrices of $\mathcal{S}$. Conversely, if m is any line of $PG(4,q)$ not contained in Σ_∞, then the lines of S_1 meeting m form a regulus and so are the Bose representation of a Baer subline. Hence any such line is the Bruck-Bose representation of a Baer subline of $PG(2,q^2)$.

Lastly, suppose that t is not a line of $\mathcal{R}$. Thus $\alpha \cap \mathcal{R}$ consists of $q+1$ points of $\alpha \setminus t$. Hence, as $\alpha \cap \mathcal{R}$ is a quadric in a plane and α does not contain a line of $\mathcal{R}$, $\boldsymbol{\beta_2}(b)$ is a non-degenerate conic $\mathcal{C}$ in α. In this case the converse, namely that every non-degenerate conic in α represents a Baer subline, is not true (see Metz, [37]) and those that do are called Baer conics. We seek to characterise the Baer conics. To this end, consider the quadratic extension $PG(5, q^2)$ of $PG(5, q)$. As, in the extension, the directrices of $\mathcal{S}$ are lines of the opposite regulus to $\mathcal{R}$, the conic $\mathcal{C}$ contains a point of each directrix. Since α, the plane of $\mathcal{C}$, contains the line $t \in \mathcal{S}$ these points are $T = t \cap \Pi$ and $\overline{T} = t \cap \overline{\Pi}$. This is the condition we seek.

Theorem 5.4. *A conic $\mathcal{C}$ in a plane α of $PG(4, q) \setminus \Sigma_\infty$ meeting Σ_∞ in a line t is a Baer conic if and only if it contains the points $T = t \cap \Pi$ and $\overline{T} = t \cap \overline{\Pi}$.*

Proof: That a Baer conic satisfies this property has been shown above and so it only remains to prove that all conics satisfying this condition are Baer conics. Let $\mathcal{C}$ be a conic of $PG(4, q) \setminus \Sigma_\infty$ in the plane α about $t \in S$ and containing the points T and $\overline{T}$. In addition, suppose that A, B, C are three points of $\mathcal{C}$ contained in the lines ℓ_A, ℓ_B, ℓ_C of S_1. As α is a plane of type 1 it lies in a unique 3-space Σ of H_3. The lines ℓ_A, ℓ_B, ℓ_C define a regulus of lines $\mathcal{R}$ in Σ which represents a Baer subline of Σ and so $\alpha \cap \mathcal{R} = \mathcal{C}'$ is a Baer conic. Now, in the extension, $\mathcal{C}'$ contains the five points $A, B, C, T, \overline{T}$ (as it is a Baer conic) and, by assumption, $\mathcal{C}$ does too. Hence the conics $\mathcal{C}$ and $\mathcal{C}'$ have five common points and so coincide. Thus $\mathcal{C} = \mathcal{C}'$ is a Baer conic. □

Summarising, we have;

Theorem 5.5. *In the Bruck-Bose representation $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ of $PG(2, q^2)$ the representation of the Baer sublines is as below.*

1. *A Baer subline of ℓ_∞ is represented by a regulus of lines of $\mathcal{S}$. Conversely, any such regulus represents a Baer subline of ℓ_∞.*
2. *A Baer subline with a unique point on ℓ_∞ is represented by a line of Σ_4 meeting Σ_∞ in a unique point. Conversely, any such line represents a Baer subline with a unique point on ℓ_∞.*
3. *A Baer subline with no point on ℓ_∞ is represented by a non-degenerate conic, with no point in Σ_∞, in a plane meeting Σ_∞ in a line t of $\mathcal{S}$. Such a conic contains the points in which t meets the directrix lines of $\mathcal{S}$. Conversely, any conic satisfying these conditions represents a Baer subline with no point on ℓ_∞.* □

Similarly, we can determine $\boldsymbol{\beta_2}(\mathcal{B})$ where $\mathcal{B}$ is a Baer subplane of $PG(2,q^2)$. Again, we determine the representation of $\mathcal{B}$ in $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ (with $\Sigma_4, \Sigma_\infty, \mathcal{S}$ as defined above). From theorem 5.3, the points of $\boldsymbol{\beta_1}(\mathcal{B})$ are the lines of a cubic scroll R_3^3 of type 2 planes. Consequently $\boldsymbol{\beta_2}(\mathcal{B})$ is given by the intersection $R_3^3 \cap \Sigma_4$. From theorem 1.47 such an intersection is one of

(a) a plane α of the scroll and a quadric Q with one regulus in S_1

or

(b) a ruled cubic surface V_2^3 with line directrix a line of S_1.

Note that, in the extension, the lines of R_3^3 which correspond to points of $\mathcal{B}$, meet each directrix plane in a Baer subplane of that plane (see section 4.2).

There are two cases to consider as $\mathcal{B}$ meets ℓ_∞ in either a Baer subline or a unique point.

Firstly, assume that $\mathcal{B}$ and ℓ_∞ meet in a Baer subline. Therefore, by theorem 5.5, R_3^3 intersects Σ_∞ in a regulus, $\mathcal{R}$. Thus $R_3^3 \cap \Sigma_4$ is an intersection of type (a) above with the plane α meeting Σ_∞ in a line of the opposite regulus to $\mathcal{R}$. In this case we say that $\boldsymbol{\beta_2}(\mathcal{B}) = \alpha$. Conversely, any plane of Σ_4 meeting Σ_∞ in a line not in $\mathcal{S}$ is of type 2. Thus, the lines of S_1 meeting such a plane are the lines of a cubic scroll which is the Bose representation of a Baer subplane. Hence any such plane is the Bruck and Bose representation of a Baer subplane of $PG(2,q^2)$.

Now suppose that $\mathcal{B}$ meets ℓ_∞ in a unique point. Therefore R_3^3 intersects Σ_∞ in a line r of $\mathcal{S}$. As this is the only line of S_1 in Σ_4, the intersection of Σ_4 and R_3^3 is of type (b). We say that $\boldsymbol{\beta_2}(\mathcal{B}) = V_2^3$. Since the conic directrices of V_2^3 represent Baer sublines of $\mathcal{B}$, the plane of each meets Σ_∞ in a line t of $\mathcal{S}$. From theorem 5.5, each such conic contains the points $t \cap \Pi$ and $t \cap \overline{\Pi}$. Since there are (q^2+1) conic directrices on V_2^3 there is a (unique) conic plane of V_2^3 about any line of $\mathcal{S} \setminus r$ and we can deduce that the directrix lines $\Sigma_4 \cap \Pi$ and $\Sigma_4 \cap \overline{\Pi}$ are, in the extension, lines of V_2^3. Once again, as there are conics of $PG(4,q)$ in planes meeting Σ_∞ in lines of $\mathcal{S}$ which do not represent Baer sublines of $PG(2,q^2)$, there are ruled cubic surfaces, with line directrix in $\mathcal{S}$ and conic directrices in planes about lines of $\mathcal{S}$, which do not represent Baer subplanes of $PG(2,q^2)$. Those that do are called **Baer ruled cubics**. We have the following characterisation of the Baer ruled cubics which is also obtained, in a different manner, in [40].

Theorem 5.6. [40] *Let* V_2^3 *be a ruled cubic surface of* $PG(4,q)$ *which meets* Σ_∞ *in exactly its line directrix and has a conic directrix in a plane meeting* Σ_∞ *in a line of* $\mathcal{S}$. V_2^3 *is a Baer ruled cubic if and only if it contains the directrix lines* $\Pi \cap \Sigma_\infty$ *and* $\overline{\Pi} \cap \Sigma_\infty$.

Proof: Again we have already shown that any Baer ruled cubic satisfies this condition. For the converse, suppose that V_2^3 is a ruled cubic surface satisfying the hypotheses of the theorem. Let g_1, g_2 be generator lines of V_2^3 with $A_1, B_1 \in g_1$ and $A_2, B_2 \in g_2$. The lines of S_1 through these four points meet Π in a quadrangle which determines a Baer subplane $\mathcal{B}$ in Π. $\boldsymbol{\beta}_1(\mathcal{B})$ is then an R_3^3 containing the lines of S_1 through A_1, B_1, A_2 and B_2 and meeting Σ_4 in a ruled cubic surface R_2^3 (for otherwise the intersection would consist of a plane and a quadric and the points A_1, A_2, B_1, B_2 would be coplanar, which would imply that g_1 and g_2 are coplanar, a contradiction). Now R_2^3, being a Baer ruled cubic, contains $t, g_1,$ and g_2 as generators, where $t = \Pi \cap \Sigma_\infty$. That is, V_2^3 and R_2^3 have three common generators and so coincide (since a projective correspondence between a line and a conic is completely determined by three pairs of corresponding points, see [48]). That is, $V_2^3 = R_2^3$ is a Baer ruled cubic. □

Summarising;

Theorem 5.7. *In the Bruck-Bose representation $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ of $PG(2, q^2)$ the representation of the Baer subplanes is as given below;*

1. *A Baer subplane with $q + 1$ points on ℓ_∞ is represented by a plane of $PG(4, q)$ meeting Σ_∞ in a line not belonging to $\mathcal{S}$. Conversely, any such plane represents a Baer subplane of $PG(2, q^2)$.*

2. *A Baer subplane with a unique point on ℓ_∞ is represented by a ruled cubic surface. Such a ruled cubic contains the directrices of $\mathcal{S}$. Conversely, any ruled cubic surface which contains the directrices of $\mathcal{S}$ as generators represents a Baer subplane of $PG(2, q^2)$.* □

5.2 The Bose Representation of the Classical Unital

The study of unitals in $PG(2, q^2)$ has been greatly aided by the use of the Bruck-Bose representation of $PG(2, q^2)$. As there is a close connection between this and the Bose representation of $PG(2, q^2)$, it is not surprising that the Bose representation also aids this study. In particular, the Bose representation of the classical unital is more concise than that given by the Bruck-Bose map and so it is often easier to prove theorems about the classical unital in this setting.

Theorem 5.8. *The Bose representation of a classical unital of $PG(2, q^2)$ is an elliptic quadric of $PG(5, q)$ which, in $PG(5, q^2)$, contains the directrix planes of S_1. Conversely,* *any elliptic quadric of $PG(5,q)$ which, when extended to $PG(5,q^2)$, contains the directrix planes of S_1, is the Bose representation of a classical unital of $PG(2,q^2)$.*

Proof: Let $\mathcal{U}_c$ be a classical unital in $PG(2,q^2)$. Thus we can choose coordinates in $PG(2,q^2)$ so that the points of $\mathcal{U}_c$ are the points satisfying the equation

$$xx^q + yy^q + zz^q = 0$$

We want to find the Bose representation of $\mathcal{U}_c$. Suppose that (a,b,c) is a point of $\mathcal{U}_c$ with $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$ and $c = \gamma_1 + i\gamma_2$. Then, since the map $x \mapsto x^q$ is an automorphism of $GF(q^2)$ fixing each element of $GF(q)$,

$$\begin{aligned}(a,b,c) \in \mathcal{U}^c &\Leftrightarrow a\bar{a} + b\bar{b} + c\bar{c} = 0\\ &\Leftrightarrow (\alpha_1 + i\alpha_2)(\alpha_1 + i^q\alpha_2) + (\beta_1 + i\beta_2)(\beta_1 + i^q\beta_2)\\ &\qquad +(\gamma_1 + i\gamma_2)(\gamma_1 + i^q\gamma_2) = 0\\ &\Leftrightarrow \alpha_1^2 + \beta_1^2 + \gamma_1^2 + (i + i^q)(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2)\\ &\qquad +ii^q(\alpha_2^2 + \beta_2^2 + \gamma_2^2) = 0 \qquad (5.1)\end{aligned}$$

where i and i^q are the zeros of $x^2 + x + \xi$, $\xi \in GF(q)$ and so $(i + \bar{i}) = -1$ and $i\bar{i} = \xi$. Thus equation 5.1 becomes

$$\alpha_1^2 - \alpha_1\alpha_2 + \xi\alpha_1^2 + \beta_1^2 - \beta_1\beta_2 + \xi\beta_2^2 + \gamma_1^2 - \gamma_1\gamma_2 + \xi\gamma_2^2 = 0$$

and hence (a,b,c) is a point of $\mathcal{U}_c$ if and only if the point $\boldsymbol{\beta_1}(a,b,c) = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ is a point of the quadric,

$$x_0^2 - x_0x_1 + \xi x_1^2 + x_2^2 - x_2x_3 + \xi x_3^2 + x_4^2 - x_4x_5 + \xi x_5^2 = 0$$

This is the equation of an elliptic quadric in $PG(5,q)$ and, when considered as a quadric of $PG(5,q^2)$, contains the directrix planes.

Conversely, let $\mathcal{Q}$ denote a quadric as in the hypothesis. Suppose that the point P of $PG(5,q)$ on the line ℓ_P of S_1 lies on $\mathcal{Q}$. Then, in $PG(5,q^2)$, $\mathcal{Q}$ contains the points P, $\ell_P \cap \Pi$ and $\ell_P \cap \overline{\Pi}$ of ℓ_P and hence all of ℓ_P. Thus, in $PG(5,q)$, $\mathcal{Q}$ contains any line of S_1 which it meets. Therefore, as an elliptic quadric in $PG(5,q)$ has $(q^3+1)(q+1)$ points, $\mathcal{Q}$ consists of q^3+1 lines of S_1. We need to show that these q^3+1 lines are the Bose representation of a classical unital. To do this we show that the set of q^3+1 points $\mathcal{U}$ in which they meet Π form a classical unital in Π (see section 4.2).

If ℓ is one of the above mentioned q^3+1 lines then, as $\mathcal{Q}$ is an elliptic quadric, any element of H_3 about ℓ which contains another of these lines must contain $q+1$ of them.

Thus, by counting, q^2 of the elements of H_3 about ℓ contain $q+1$ of these lines and the remaining element of H_3 about ℓ meets $\mathcal{Q}$ in ℓ exactly. We call this the tangent three dimensional space to $\mathcal{Q}$ about ℓ and denote it $T_\ell(\mathcal{Q})$. If ℓ intersects Π in the point P then the (q^2+1) elements of H_3 about ℓ intersect Π in the (q^2+1) lines of Π through P. Hence the above implies that of the q^2+1 lines of Π through a point of $\mathcal{U}$, q^2 are $(q+1)$-secant to $\mathcal{U}$ and one is tangent to $\mathcal{U}$. Thus, in total, $\mathcal{U}$ has q^3+1 tangents and $q^2(q^3+1)/(q+1) = q^4-q^3+q^2$ $(q+1)$-secants. Thus all lines of Π are tangent or $q+1$ secant to $\mathcal{U}$ and so $\mathcal{U}$ forms a unital in Π.

We prove that $\mathcal{U}$ is classical by showing that it satisfies the condition of reciprocity (see section 1.14.1). Let A, B, C, D be four points of $\mathcal{U}$, no three of which are collinear, and t_A, t_B, t_C, t_D be the tangents to $\mathcal{U}$ through A, B, C, D respectively with t_A and t_B intersecting at the point X. Suppose that the line CD contains X. We need to show that $t_C \cap t_D = Y$ lies on the line AB. Let ℓ_P denote the line of S_1 through the point P. Thus, equivalently, we must show that $\ell_Y = T_{\ell_C}(\mathcal{Q}) \cap T_{\ell_D}(\mathcal{Q})$ is contained in $\langle \ell_A, \ell_B \rangle$.

Let ρ denote the polarity of $PG(5,q)$ induced by the quadric $\mathcal{Q}$. So, if $P \in \mathcal{Q}$ the polar of P is the tangent space to $\mathcal{Q}$ at P, denoted $T_P(\mathcal{Q})$. By corollary 4.10, this contains a unique element of H_3 which, as it contains P, contains ℓ_P. Hence this element of H_3 is $T_{\ell_P}(\mathcal{Q})$, as all other elements of H_3 about ℓ_P meet $\mathcal{Q}$ in a hyperbolic quadric.

Consider the line ℓ_A. By the above, the polar of any point of ℓ_A contains $T_{\ell_A}(\mathcal{Q})$ so this is the polar of ℓ_A. Similarly, the polar of ℓ_B is $T_{\ell_B}(\mathcal{Q})$, which implies that the polar of $\langle \ell_A, \ell_B \rangle$ is $T_{\ell_A}(\mathcal{Q}) \cap T_{\ell_B}(\mathcal{Q}) = \ell_X$, which is contained in $\langle \ell_C, \ell_D \rangle$. Thus, by the pole-polar property, the polar of $\langle \ell_C, \ell_D \rangle$ is contained in $\langle \ell_A, \ell_B \rangle$. That is, $\ell_Y \subset < \ell_A, \ell_B >$. Hence, by theorem 1.55, $\mathcal{U}$ is a classical unital. □

5.2.1 The Bruck-Bose Representation of the Classical Unital

As we did with Baer subplanes, we may now use this to obtain the Bruck-Bose representation of the classical unital.

Corollary 5.9. *If $\mathcal{U}$ is a classical unital in $PG(2,q^2)$ then the Bruck-Bose representation of $\mathcal{U}$ in $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ is either*

1. *A non-singular parabolic quadric meeting Σ_∞ in $(q+1)$ lines of $\mathcal{S}$, or*
2. *An elliptic quadric cone consisting of the lines joining $P \in \Sigma_\infty$ to the points of an elliptic quadric meeting Σ_∞ in a unique point A, where A and P are collinear on*

a line of the spread, $\mathcal{S}$.

In both cases, the quadrics contain the directrix lines of $\mathcal{S}$. Conversely, any such quadric containing the directrix lines of $\mathcal{S}$ is the Bruck-Bose representation of a classical unital.

Proof: Let Σ_4 be a hyperplane in $PG(5,q)$ containing $\Sigma_\infty \in H_3$ and $\mathcal{S}$ denote the (regular) spread of Σ_∞ induced by S_1. From theorem 5.8 and section 4.5 the Bruck-Bose representation of $\mathcal{U}$ is given by the intersection $\mathcal{Q} \cap \Sigma_4$, where $\mathcal{Q}$ is an elliptic quadric of $PG(5,q)$ containing the directrix planes. There are two cases two consider.

If Σ_4 is a tangent hyperplane of $\mathcal{Q}$ then $\mathcal{Q} \cap \Sigma_4$ is an elliptic cone as given in the theorem.

Otherwise, $\mathcal{Q} \cap \Sigma_4$ is a non-singular parabolic quadric as stated in the theorem.

In both cases Σ_∞ meets each of the directrix planes of S_1 in lines which are thus lines of the quadric.

Conversely, suppose that $\mathcal{P}_4$ is a non-singular parabolic quadric in Σ_4 meeting Σ_∞ in $(q+1)$ lines of $\mathcal{S}$. As the general equation of a quadric of $PG(4,q)$ has 15 coefficients and the general equation of a quadric of $PG(5,q)$ has 21 coefficients, there will be a unique quadric $\mathcal{Q}$ containing $\mathcal{P}_4$ and any 6 additional points of $PG(5,q)$ which impose linearly independent conditions on the coefficients of $\mathcal{P}_4$. Let A, B, C be three points of $\mathcal{P}_4 \setminus \Sigma_\infty$ such that the lines ℓ_A, ℓ_B, ℓ_C of S_1 through these points are not contained in a common element of H_3. Now $\mathcal{P}_4$, together with two points from each of these three lines (not A, B or C), lie in a unique quadric $\mathcal{Q}$ of $PG(5,q)$ which, in $PG(5,q^2)$, contains the directrix planes. If $\mathcal{Q}$ contains $P \in PG(5,q)$ then, as it contains the directrix planes, $\mathcal{Q}$ contains three points of the line of S_1 through P and hence all of this line. Thus $\mathcal{Q}$ consists exactly of the lines of S_1 which intersect Σ_4 in a point of $\mathcal{P}_4$. So, as $\mathcal{P}_4$ contains $(q^2+1)(q^2-1)/(q-1)$ points (theorem 1.25), of which $(q+1)^2$ are the points of $\mathcal{P}_4 \cap \Sigma_\infty$, we have

$$\begin{aligned}
|\mathcal{Q}| &= (q+1)\left[\frac{(q^2+1)(q^2-1)}{(q-1)} - (q+1)^2\right] + (q+1)^2 \\
&= (q+1)^2\left[(q^2+1) - (q+1) + 1\right] \\
&= (q+1)^2(q^2-q+1) \\
&= (q+1)(q^3+1)
\end{aligned}$$

Thus, as a non-singular elliptic quadric is the only quadric of $PG(5,q)$ with this number of points, $\mathcal{Q}$ is such a quadric.

Let $\Pi_0\mathcal{E}_3$ denote an elliptic quadric cone as in the theorem. Let A, B, C be three points of $\Pi_0\mathcal{E}_3 \setminus \Sigma_\infty$ such that the lines ℓ_A, ℓ_B, ℓ_C of S_1 through these points are not contained in a common element of H_3, and such that the element of H_3 generated by any two of these lines does not contain the line ℓ_∞ in which $\Pi_0\mathcal{E}_3$ meets Σ_∞. As above, $\Pi_0\mathcal{E}_3$ together with two points from each of these three lines (not A, B or C) lie in a unique quadric $\mathcal{Q}$ of $PG(5, q)$ which, in $PG(5, q^2)$, contains the directrix planes. As $\mathcal{E}_3$ contains $(q^2 + 1)$ points, $\Pi_0\mathcal{E}_3$ contains $(q(q^2 + 1) + 1)$ points of which $(q + 1)$ are points of Σ_∞. Thus, similarly to above, $\mathcal{Q}$ contains $(q^3(q + 1) + (q + 1))$ points and is thus a non-singular elliptic quadric. □

The representations of the classical unital given in theorem 5.8 and corollary 5.9 also enables us to determine how a classical unital intersects a Baer subplane.

Theorem 5.10. *A classical unital intersects a Baer subplane in one of*

1. *A point of the Baer subplane,*
2. *A line of the Baer subplane,*
3. *Two lines of the Baer subplane,*
4. *A non-singular conic of the Baer subplane.*

Proof: Let $\mathcal{U}$ be a classical unital in $PG(2, q^2)$ with Bose representation $\mathcal{Q}$ and $\mathcal{B}$ be a Baer subplane with Bose representation R_3^3. We show that the lines of S_1 in common to $\mathcal{Q}$ and R_3^3 meet the directrix plane Π in one of the required ways.

Let π be a plane of R_3^3. As a plane meets a quadric in a quadric of that plane, π intersects $\mathcal{Q}$ in either a point, a line, two lines or a non-singular conic. The lines of S_1 in common to $\mathcal{Q}$ and R_3^3 are the lines of S_1 meeting π in a point of $\pi \cap \mathcal{Q}$. Hence $\mathcal{Q}$ intersects all planes of R_3^3 and, in particular, $\Pi \cap R_3^3$, in this way. □

5.3 Buekenhout and Buekenhout-Metz unitals

We can make use of the Bruck-Bose representation of the classical unital given in corollary 5.9 to examine when a Buekenhout unital or a Buekenhout-Metz unital is classical.

Firstly, recall from section 1.15.2 that a Buekenhout parabolic unital is a unital for which the Bruck-Bose representation in $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ is a non-singular parabolic quadric meeting Σ_∞ in a regulus belonging to $\mathcal{S}$. Clearly, a classical unital has this representation with respect to any secant line, but it is not obvious that any such unital is classical. This was shown in [6] using a non-trivial counting argument. However any such unital, as it contains $q+1$ lines of $\mathcal{S}$, contains the directrix lines of $\mathcal{S}$, and thus is classical by corollary 5.9 giving us;

Theorem 5.11. [6] *A Buekenhout unital of* $PG(2,q^2)$ *is a classical unital.* □ ✓

If $\mathcal{U}$ is a Buekenhout-Metz unital, that is the Bruck-Bose representation of $\mathcal{U}$ is an ovoidal cone $\mathcal{O}$ in $PG(4,q)$, then examples for which $\mathcal{U}$ is non-classical are known. As was outlined in section 1.15.2, if $\mathcal{O}$ is not an elliptic quadric then $\mathcal{U}$ is non-classical (see Buekenhout, [17]) and, furthermore, there are examples in which $\mathcal{O}$ is an elliptic quadric and $\mathcal{U}$ is non-classical (see Metz, [37]). In this second case, a restatement of corollary 5.9 gives a condition under which such unitals are classical.

Theorem 5.12. *Let* $\mathcal{U}$ *be a Buekenhout-Metz unital in* $PG(2,q^2)$ *where the associated ovoidal cone* $\mathcal{O}$ *has an elliptic quadric base. The unital* $\mathcal{U}$ *is classical if and only if, in the extension,* $\mathcal{O}$ *contains the directrices of* $\mathcal{S}$. □

An alternative characterisation of Buekenhout-Metz unitals among classical unitals has been given by Lefévre-Percsy in [35]. This result has a simple proof in this setting.

Theorem 5.13. [35] *Let* $\mathcal{U}$ *be a Buekenhout-Metz unital wrt* (T,ℓ_∞) *in* $PG(2,q^2)$ *where the associated ovoidal cone has an elliptic quadric base. If* $\mathcal{U}$ *has a secant line* ℓ *not containing* T *which meets* $\mathcal{U}$ *in a Baer subline then* $\mathcal{U}$ *is a classical unital.*

Proof: As ℓ intersects $\mathcal{U}$ in a Baer subline and does not contain T, the intersection of $\boldsymbol{\beta_2}(\ell)$ and $\boldsymbol{\beta_2}(\mathcal{U})$ is a Baer conic. By theorem 5.4, such a conic intersects the directrices $d, \overline{d}$ of $\mathcal{S}$ in the points P and $\overline{P}$, respectively. Let t be the line of Σ_∞ which is the Bruck-Bose (and Bose) representation of T. As Σ_∞ is tangent to $\boldsymbol{\beta_2}(\mathcal{U})$ and the directrices are lines of Σ_∞ (in the extension), it follows that they each meet $\boldsymbol{\beta_2}(\mathcal{U})$ doubly at $t \cap d$. Hence, counted according to multiplicity, d has three points on $\boldsymbol{\beta_2}(\mathcal{U})$ and so is contained in it. Therefore, by corollary 5.9, $\mathcal{U}$ is a classical unital. □ ✓

Actually the condition that $\mathcal{U}$ has an elliptic quadric as base ovoid is superfluous. Thus we obtain a characterisation of any Buekenhout-Metz unital among classical unitals.

Theorem 5.14. *Let* $\mathcal{U}$ *be a Buekenhout-Metz unital wrt* (T,ℓ_∞) *in* $PG(2,q^2)$. *If* $\mathcal{U}$ *has a secant line not containing* T *which meets* $\mathcal{U}$ *in a Baer subline then* $\mathcal{U}$ *is a classical unital.*

Proof: Suppose that $\mathcal{U}$ is a Buekenhout-Metz unital wrt (T, ℓ_∞). Then, as above, $\beta_2(\mathcal{U})$ contains a Baer conic $\mathcal{C}$ and so, as an ovoid containing a conic is an elliptic quadric (theorem 1.30), it follows that any hyperplane section of $\beta_2(\mathcal{U})$ about $\mathcal{C}$ is an elliptic quadric. Thus, by the above theorem, the unital is classical. □

We also have;

Corollary 5.15. *If $\mathcal{U}$ is a unital in $PG(2,q^2)$ which is Buekenhout Metz wrt (T_1, ℓ_1) and Buekenhout-Metz wrt (T_2, ℓ_2) (with $T_1 \neq T_2$) then $\mathcal{U}$ is classical.*

Proof: If $\mathcal{U}$ is Buekenhout-Metz wrt (T_1, ℓ_1) then a secant line of $\mathcal{U}$ through T_1 is represented in $\pi(\Sigma_4, \Sigma_\infty, \mathcal{S})$ by a line of $PG(4,q)$ meeting Σ_∞ in a unique point. Therefore all secants of $\mathcal{U}$ through T_1 are Baer sublines. Thus $\mathcal{U}$ is Buekenhout-Metz wrt (T_2, ℓ_2) with a Baer subline through $T_1 \neq T_2$. Hence, by theorem 5.14, $\mathcal{U}$ is classical. □

5.4 A Characterisation of the Classical Unital

It is well-known that if $\mathcal{U}$ is a classical unital in $PG(2,q^2)$ and ℓ is a secant of $\mathcal{U}$ then the points of ℓ on $\mathcal{U}$ are the points of a Baer subline of ℓ. In the Bose setting this follows easily; for $\beta_2(\mathcal{U})$ is an elliptic quadric of $PG(5,q)$ and $\beta_2(\ell)$ consists of $(q+1)$ skew lines of this elliptic quadric contained in a three-dimensional subspace Σ_3 of $PG(5,q)$. As Σ_3 meets an elliptic quadric in a quadric of Σ_3, these $(q+1)$ lines are necessarily one regulus of a hyperbolic quadric. By theorem 5.3, they are thus the Bose representation of a Baer subline of a line of $PG(2,q^2)$. That is, we have;

Theorem 5.16. *A secant line ℓ of a classical unital $\mathcal{U}$ of $PG(2,q^2)$ intersects $\mathcal{U}$ in the points of a Baer subline of ℓ.* □

Furthermore, it has been shown by Lefèvre-Percsy in [35] and Faina and Korchmàros in [26] that the converse of the above result is also true. That is, that a unital for which every secant line meets the unital in the points of a Baer subline is a classical unital. Lefèvre-Percsy did this by showing, using a characterisation of Buekenhout-Metz unitals given in [34], that such a unital is Buekenhout-Metz with respect to each of its tangent lines. She then shows that the base ovoid of this Buekenhout-Metz unital is an elliptic quadric so that the result follows by theorem 5.13. Faina and Korchmàros take a different approach and show, using some results from group theory, that such a unital satisfies the condition of reciprocity and hence is classical. Here we use the idea of a quadratic set to show that the Bose representation of such a unital is an elliptic quadric of $PG(5,q)$ containing the directrix planes of S_1 and hence that the unital is classical.

A **quadratic set** ([16]) in $PG(n,q)$ is a set $\mathcal{K}$ of points of $PG(n,q)$ satisfying the following conditions

1. $\mathcal{K}$ is of type $(0,1,2,q+1)$ with respect to the lines of $PG(n,q)$.
2. The union of 1-secants and $(q+1)$-secants through $P \in \mathcal{K}$, together with P itself is either $PG(n,q)$ or a hyperplane. We call this the **tangent space** to $\mathcal{K}$ at P and denote it $T_P(S)$.

Lemma 5.17. *In $PG(2,q)$ there are six types of quadratic sets. They are as follows,*

1. *the empty set*
2. *a point*
3. *a line*
4. *2 lines*
5. *a $(q+1)$-arc*
6. *$PG(2,q)$*

Proof: Let $\mathcal{K}$ be a quadratic set in $PG(2,q)$. If $\mathcal{K}$ does not contain a line then it is a k-arc and, for the tangent space condition to be satisfied, consists of either $0, 1$ or $(q+1)$ points.

If $\mathcal{K}$ consists solely of a line ℓ then it is a quadratic set. If besides ℓ it contains another point P then, for $T_P(\mathcal{K})$ to be either a line or $PG(2,q)$, $\mathcal{K}$ must contain either precisely one more line through P, and so be the union of two lines, or contain all lines through P, whence $\mathcal{K} = PG(2,q)$. $\square$

Lemma 5.18. [16], [30] *Let $\mathcal{K}$ be a quadratic set and Π_s a subspace of dimension s in $PG(n,q)$. Then $\mathcal{K}' = \mathcal{K} \cap \Pi_s$ is a quadratic set in Π_s and $T_P(\mathcal{K}') = T_P(\mathcal{K}) \cap \Pi_s$ for any point P of $\mathcal{K}'$.*

Proof: If ℓ is a line of Π_s then $\ell \cap \mathcal{K}' = \ell \cap \mathcal{K}$ and so $\mathcal{K}'$ is of type $(0,1,2,q+1)$. Let $P \in \mathcal{K}$. Any line of Π_s through P containing 1 or $(q+1)$ points of $\mathcal{K}'$, contains that number of points of $\mathcal{K}$ and so $T_P(\mathcal{K}') \subseteq T_P(\mathcal{K}) \cap \Pi_s$. Conversely, any line of $PG(n,q)$

through P containing 1 or $(q+1)$ points of $\mathcal{K}$, contains that many points of $\mathcal{K}'$ and so $T_P(\mathcal{K}) \cap \Pi_s \subseteq T_P(\mathcal{K}')$. Thus $T_P(\mathcal{K}') = T_P(\mathcal{K}) \cap \Pi_s$. Therefore, as $\mathcal{K}$ is a quadratic set, $T_P(\mathcal{K}')$ is either Π_s or a hyperplane in Π_s. □

Lemma 5.19. [16], [30] *$\mathcal{K}$ is a quadratic set in $PG(n,q)$ if and only if each plane section is a quadratic set.*

Proof: Necessity follows from the previous theorem. Suppose each plane section of a set $\mathcal{K}$ of points of $PG(n,q)$ is a quadratic set. Let ℓ be a line of $PG(n,q)$ and Π a plane containing ℓ. As $\mathcal{K} \cap \ell = (\mathcal{K} \cap \Pi) \cap \ell$ and $\Pi \cap \mathcal{K}$ is a quadratic set, it follows that $\mathcal{K}$ is of type $(0,1,2,q+1)$ and that ℓ is a $(q+1)$-secant (or tangent) of $\mathcal{K}$ if and only if it is a $(q+1)$-secant (or tangent) of $\mathcal{K} \cap \Pi$, for all planes Π containing ℓ. Let $P \in \mathcal{K}$ and ℓ_1, ℓ_2 be two lines either 1-secant or $(q+1)$-secant to $\mathcal{K}$. Let $\Pi = \langle \ell_1, \ell_2 \rangle$. By hypothesis, $\Pi \cap \mathcal{K}$ is a quadratic set with $T_P(\mathcal{K} \cap \Pi) = \Pi$. Thus $\Pi \subset T_P(\mathcal{K})$ and $T_P(\mathcal{K})$ is a subspace. As each plane containing P has a line in $T_P(\mathcal{K})$, it follows that $T_P(\mathcal{K})$ is a hyperplane or $PG(n,q)$. □

Theorem 5.20. [16], [30] *If $\mathcal{K}$ is a quadratic set in $PG(n,q)$, which is not a subspace of $PG(n,q)$, and every plane section of $\mathcal{K}$ is the empty set, a point, a line, two lines or a non-degenerate conic then $\mathcal{K}$ is a quadric.* □

Therefore, to show that a unital of $PG(2,q^2)$ for which all secant lines are Baer sublines is a classical unital, we need to show that its Bose representation is a quadratic set of $PG(5,q)$ for which all plane sections are either empty or a conic (possibly degenerate).

We shall make use of the following result which first appeared in [15]. The proof here is as in [20].

Lemma 5.21. [15] *Let $\mathcal{U}$ be a unital and $\mathcal{B}$ be a Baer subplane of $PG(2,q^2)$. If b_1 denotes the number of lines of $\mathcal{B}$ which, when extended to $PG(2,q^2)$, are tangents to $\mathcal{U}$ and s denotes the points in $\mathcal{B} \cap \mathcal{U}$ then*

$$b_1 + s = 2(q+1).$$

Proof: Counting in two ways the set of ordered pairs (P,ℓ) where P is a point of $\mathcal{U}$ and ℓ is a line of $\mathcal{B}$ containing P, we obtain the equation,

$$s(q+1) + (q^3+1-s).1 = b_1.1 + (q^2+q+1-b_1)(q+1)$$

from which the result follows. □

Let $\overline{\mathcal{U}}$ be a unital in $PG(2, q^2)$ with all secants Baer sublines and let $\mathcal{U} = \beta_1(\overline{\mathcal{U}})$ be the Bose representation of $\overline{\mathcal{U}}$. Thus $\mathcal{U}$ is a set of (q^3+1) lines of S_1 such that every element of H_3 contains either one or $(q+1)$ of these lines, and if $\Sigma \in H_3$ contains $(q+1)$ of these lines, then those lines form a regulus. Those elements of H_3 which contain one line of $\mathcal{U}$ are called **tangents** of $\mathcal{U}$ and those meeting it in a regulus are called **secants** of $\mathcal{U}$. To begin we examine how lines intersect $\mathcal{U}$.

Lemma 5.22. *A line of $PG(5, q)$ contains 0,1,2 or $(q+1)$ points of $\mathcal{U}$.*

Proof: A line of S_1 contains either 0 or $(q+1)$ points of $\mathcal{U}$. Let ℓ be a line not in S_1, which contains the points A, B, C of $\mathcal{U}$ on the lines ℓ_A, ℓ_B, ℓ_C of S_1 respectively. Then $\Sigma = \langle \ell_A, \ell_B \rangle$ is in H_3, and contains ℓ, which implies that C and thus ℓ_C are also in Σ. By hypothesis, $\mathcal{R}(\ell_A, \ell_B, \ell_C)$ consists of $(q+1)$ lines of $\mathcal{U}$ and so, as ℓ is a line of the opposite regulus, it contains $(q+1)$ points of $\mathcal{U}$. □

For the sake of brevity, we call lines of $PG(5, q)$ which contain $(q+1)$ points of $\mathcal{U}$, **lines of $\mathcal{U}$** and the those lines of $\mathcal{U}$ which are of type 2, **transversal lines of $\mathcal{U}$**.

Lemma 5.23. *$\mathcal{U}$ contains $q^5 + q^3 - q^2 + 1$ lines, (q^2+1) through each point of $\mathcal{U}$.*

Proof: There are (q^3+1) lines of $\mathcal{U}$ in S_1. A unital has $(q^4 - q^3 + q^2)$ secants and so $\mathcal{U}$ has $(q^4 - q^3 + q^2)(q+1)$ transversal lines (as elements of H_3 intersect exactly in a line of S_1 no transversal line is contained in two distinct elements of H_3). This gives the required number. If P is a point of $\mathcal{U}$ then the line ℓ_P of S_1 through P is in $\mathcal{U}$. Additionally, the q^2 elements of H_3 about ℓ_P which are secant to $\mathcal{U}$ contain a transversal line through P.

□

By lemma 5.19, to show that $\mathcal{U}$ is a quadratic set we need to show that the intersection of $\mathcal{U}$ with a plane of $PG(5, q)$ is a set of the type given in lemma 5.17. For the type 1 planes this is straightforward.

Lemma 5.24. *A plane of type 1 meets $\mathcal{U}$ in a conic.*

Proof: Let π be a type 1 plane in $PG(5, q)$. From section 4.3, π is contained in an element Σ of H_3 which meets $\mathcal{U}$ in either a line or a regulus. Thus, in the former case, $\pi \cap \mathcal{U}$ is either a point or a line and, in the latter, it is a plane section of a regulus. □

In the case of the type 2 planes the result is more difficult. We need the following lemmas, of which the next seems particularly important.

Lemma 5.25. *A plane containing a line of $\mathcal{U}$ meets $\mathcal{U}$ in either that line exactly or in two lines.*

Proof: If the plane is of type 1 then the result follows from the previous lemma. So suppose that π is a type 2 plane containing the line ℓ of $\mathcal{U}$. As π is of type 2, ℓ is a transversal line of $\mathcal{U}$ and there are (q^2-1) further transversal lines through each point of ℓ. Hence there are $(q^2-1)(q+1)$ transversal lines of $\mathcal{U}$ meeting ℓ. Let t, t' be two such transversals and consider the plane $\langle t, \ell\rangle$. Such a plane is of type 2 and if it meets t' at a point not on ℓ then it contains t'. However, by lemma 5.21, a type 2 plane contains at most $2q+2$ points of $\mathcal{U}$, so this cannot occur. Thus the $(q^2-1)(q+1)$ planes $\langle t, \ell\rangle$, where t is a transversal line meeting ℓ, are distinct and meet $\mathcal{U}$ in exactly two lines. Similarly, each of the $q+1$ planes $\langle \ell_A, \ell\rangle$, where ℓ_A is a line of S_1 meeting ℓ, meets $\mathcal{U}$ in two lines. Every point of $\mathcal{U}$ is in one of these planes so the remaining planes about ℓ meet $\mathcal{U}$ in ℓ exactly. Thus π meets $\mathcal{U}$ in ℓ exactly or in two lines. □

Lemma 5.26. *Let Σ be a three dimensional space of $PG(5,q)$ which is not an element of H_3. If the line ℓ_P of S_1 contained in Σ is also in $\mathcal{U}$, then the intersection $\Sigma \cap \mathcal{U}$ consists of either*

(a) ℓ_P and q transversal lines of $\mathcal{U}$, all meeting ℓ_P in the same point

or

(b) a hyperbolic quadric.

Furthermore, the intersection is of type (b) if and only if each element of H_3 about ℓ_P meeting Σ in a plane is a secant of $\mathcal{U}$.

Proof: Let Σ satisfy the conditions of the theorem. By lemma 4.14 there are $(q+1)$ elements of H_3 meeting Σ in the $(q+1)$ planes about ℓ_P in Σ. Furthermore, if these spaces are $\Sigma_1, \ldots, \Sigma_{q+1}$, meeting Σ in $\pi_1, \ldots, \pi_{q+1}$, respectively, then $\Sigma \cap \mathcal{U} = \bigcup(\pi_i \cap \mathcal{U})$, where $\pi_i \cap \mathcal{U} = \ell_P$ if Σ is tangent to $\mathcal{U}$, and $\pi_i \cap \mathcal{U} = \ell_P$ and a transversal line through ℓ_P if Σ is secant to $\mathcal{U}$. As there is only one tangent line through a point of a unital, at most one of the Σ_i is tangent to $\mathcal{U}$.

Suppose that Σ_{q+1} is tangent to $\mathcal{U}$ and that $\Sigma_i \cap \mathcal{U} = \ell_P \cup m_i$, for $i = 1, \ldots, q$. Let $m_1 \cap \ell_P = A$. Note that as the planes π_i meet only in ℓ_P, the m_i are either skew or meet at a point of ℓ_P. Suppose that $m_2 \cap \ell_P = B \neq A$ and consider a plane β about m_1 in Σ which does not contain ℓ_P. The line m_2 meets β at a point not on m_1 so, by lemma 5.25, β contains two lines of $\mathcal{U}$. However $\beta \cap \mathcal{U}$ consists of, at most, m_1 and the $(q-1)$ points

$m_i \cap \beta$ for $i = 2, \dots, q$ and so cannot contain two lines of $\mathcal{U}$. Thus $m_2 \cap \ell_P = A$ too and all the lines m_i pass through A. Thus Σ and $\mathcal{U}$ intersect as in (a) of the theorem.

Now, if γ is any plane about ℓ_P in Σ_{q+1}, the three dimensional space $\langle \gamma, m_1 \rangle$ is not in H_3, contains ℓ_P and is met by Σ_{q+1} in a plane about ℓ_P. Hence, by the above argument, $\langle \gamma, m_1 \rangle$ meets $\mathcal{U}$ in ℓ_P together with q transversal lines through A, one of which is m_1. There are $(q+1)$ 3-spaces of this kind, they meet pairwise in the plane $\langle m_1, \ell_P \rangle$, between them contain $(q+1)(q-1)+1 = q^2$ distinct transversals (ie all the transversals) of $\mathcal{U}$ through A and are all in the four dimensional space $\langle \Sigma_{q+1}, m_1 \rangle$.

Similarly, the q^2 transversal lines through any other point of ℓ_P are all contained in a hyperplane about Σ_{q+1}. As there are $(q+1)$ hyperplanes about Σ_{q+1}, and $(q+1)$ points on ℓ_P, all of the hyperplanes about Σ_{q+1} meet $\mathcal{U}$ in ℓ_P together with q^2 transversal lines all incident with the same point of ℓ_P. Consequently, if Σ contains two transversal lines of $\mathcal{U}$ through different points of ℓ_P, it is not contained in any hyperplane about Σ_{q+1} and so does not meet Σ_{q+1} in a plane. Thus each element of H_3 meeting Σ in a plane is secant to $\mathcal{U}$ and Σ contains $(q+1)$ transversal lines of $\mathcal{U}$, each meeting ℓ_{q+1} in distinct points. These lines meet ℓ_P in distinct points, for if two of them, say m and n, do not then $\Sigma = \langle \ell_P, m, n \rangle$ is contained in the hyperplane containing ℓ_P, m, n and Σ_{q+1}.

It remains to show that the intersection $\Sigma \cap \mathcal{U}$ is indeed a hyperbolic quadric. Suppose that the $(q+1)$ transversal lines in which Σ meets $\mathcal{U}$ are $\ell_1, \dots, \ell_{q+1}$. No two of these lines intersect, for if they did the plane containing them would also contain ℓ_P, contradicting lemma 5.25. Thus, if $A_1 \in \ell_2 \setminus \ell_P$ then $\langle \ell_1, A \rangle$ is a plane meeting $\mathcal{U}$ in a line and a point and hence, by lemma 5.25, contains two lines of $\mathcal{U}$. Let this second line be g_1. As each line ℓ_i intersects $\langle \ell_1, A \rangle$, g_1 has a point on each ℓ_i. None of these points is on ℓ_P, for otherwise $\langle \ell_1, A \rangle = \langle \ell_1, g_1 \rangle$ would contain three lines of $\mathcal{U}$. Thus g_1 is a transversal to the $(q+1)$ skew lines $\ell_1, \dots, \ell_{q+1}$. Similarly, for $A_2 \in \ell_2 \setminus \ell_P$ $(A_2 \neq A_1)$ we can find a line g_2 which is a transversal to the ℓ_i. Thus $\ell_1, \dots \ell_{q+1}$ are the $q+1$ transversals of the three skew lines g_1, g_2 and ℓ_P (for g_1 and g_2 intersecting would contradict lemma 5.25) and so are one regulus of a hyperbolic quadric. □

Lemma 5.27. *$\mathcal{U}$ is a quadratic set in $PG(5, q)$.*

Proof: By lemma 5.19, we need to show that each plane section of $\mathcal{U}$ is a quadratic set. Let π be a plane in $PG(5, q)$.

If π is of type 1 then, by lemma 5.24, it meets $\mathcal{U}$ in a conic which, by lemma 5.17, is a quadratic set in π.

Let π be a type 2 plane. Since a null set is quadratic, we may assume that π contains the point P of $\mathcal{U}$. Thus $\Sigma = \langle \pi, \ell_P \rangle$, where ℓ_P is the line of S_1 through P, is a three dimensional space not in H_3 (as it contains a type 2 plane) which contains a line in both $\mathcal{U}$ and S_1. Hence $\Sigma \cap \mathcal{U}$ is given by lemma 5.26.

Suppose that this intersection is of the type (a) of lemma 5.26 and that the q transversal lines of $\mathcal{U}$ in Σ all pass through the point A. In this case, if π contains ℓ_P then, by lemma 5.25, it intersects $\mathcal{U}$ in either one or two lines. Likewise, if π doesn't contain ℓ_P but meets it at A, then it meets $\mathcal{U}$ in either A exactly or one or two transversals.

Lastly, if π intersects ℓ_P at the point $B \neq A$ then it contains no transversal (for otherwise it would contain A) but meets each transversal, and so intersects $\mathcal{U}$ in $(q+1)$ points. If three of these points were collinear then the lines of $\Sigma \cap \mathcal{U}$ through these points would be coplanar, contradicting lemma 5.25, and hence these points form a $(q+1)$-arc.

If $\Sigma \cap \mathcal{U}$ is of type (b) of lemma 5.26, that is, a hyperbolic quadric, then Π meets $\mathcal{U}$ in a plane section of a hyperbolic quadric, which is a quadric in a plane.

By lemma 5.17, each of the above possible intersections of π and $\mathcal{U}$ form a quadratic set in a plane. Thus $\mathcal{U}$ is a quadratic set. □

Lemma 5.28. *$\mathcal{U}$ is a quadric of $PG(5, q)$.*

Proof: We know that $\mathcal{U}$ is a quadratic set so, by theorem 5.20, to show it is a quadric we must show that all non-empty plane sections are conics. As no plane intersects $\mathcal{U}$ in more than two lines, $\mathcal{U}$ contains no plane and so it only remains to show that any plane meeting $\mathcal{U}$ in a $(q+1)$-arc meets it in a conic.

For q odd, any $(q+1)$-arc is a conic so there is no more to prove and hence we assume that q is even and that the plane π meets $\mathcal{U}$ in a $(q+1)$-arc, $\mathcal{C}$. By lemma 5.24, we can assume that π is of type 2. As $\mathcal{C}$ is a $(q+1)$-arc, there is a unique line through each point of $\mathcal{C}$ which meets $\mathcal{C}$ in that point exactly. Such a line is said to be a tangent of $\mathcal{C}$. In total, $\mathcal{C}$ has $(q+1)$ tangents and it is well-known (see, for example, [29]) that these lines are concurrent at a point N, called the nucleus of $\mathcal{C}$. Now, lemma 5.21 implies that there are $(q+1)$ elements of H_3, $\Sigma_1, \ldots, \Sigma_{q+1}$ which meet π in lines and are tangent to $\mathcal{U}$. Suppose that each of $\Sigma_1, \ldots, \Sigma_{q+1}$ meet π in lines which are tangent to $\mathcal{C}$. Let ℓ_N be the line of S_1 through N and consider the three dimensional space $\Sigma = \langle \ell_N, \pi \rangle$. As π is of type 2, this is not an element of H_3 and so, by lemma 4.14, $\Sigma \cap \mathcal{U} = \bigcup(\pi_i \cap \mathcal{U})$ where $\pi_i = \Sigma_i \cap \Sigma$. Hence $\Sigma \cap \mathcal{U} = \mathcal{C}$. However this contradicts the fact that Σ meets $\mathcal{U}$ in a quadratic set. Therefore not all the Σ_i meet π in lines tangent to $\mathcal{C}$. Recall that each line of π is contained in a unique element of H_3, and thus we have that one of the tangent lines to $\mathcal{C}$, say the one meeting $\mathcal{C}$ at the point P, is contained in an element of H_3 secant to $\mathcal{U}$. As each other line of π through P meets $\mathcal{C}$ (and hence $\mathcal{U}$) in two points, the elements of H_3 about these lines are also secants of $\mathcal{U}$.

Consider the three dimensional space $\Sigma = \langle \ell_P, \pi \rangle$. By the above paragraph, each element of H_3 about ℓ_P meeting Σ in a plane is secant to $\mathcal{U}$. Hence, by lemma 5.26, $\Sigma \cap \mathcal{U}$ is a hyperbolic quadric. π meets $\mathcal{U}$ in a plane section of this hyperbolic quadric and so meets $\mathcal{U}$ in a conic. Hence, by lemma 5.20, $\mathcal{U}$ is a quadric. □

To complete the proof, we need to demonstrate that $\mathcal{U}$ is indeed an elliptic quadric containing the generator planes. From our knowledge of quadrics in $PG(5,q)$, as $\mathcal{U}$ contains $(q+1)(q^2+1)$ points and no plane, it is a non-singular elliptic quadric. In $PG(5,q^2)$, it contains the (q^3+1) points of Π (the directrix plane) in which the lines of S_1 in $\mathcal{U}$ meet Π. Thus it contains Π and, similarly, $\overline{\Pi}$.

Hence $\mathcal{U}$ is the Bose representation of a classical unital and so we have,

Theorem 5.29. *A unital of $PG(2,q^2)$ for which all secant lines are Baer sublines is a classical unital.* □

Chapter 6

Other Perspectives on the Bose Representation

One difficulty in determining the Bose representation of a certain set of points of $PG(2,q^2)$ is that the relationship between the coordinates of a point P of $PG(2,q^2)$ and the points of the line $\boldsymbol{\beta_1}(P)$, which is the Bose representation of P, is quite complicated. As it currently stands, the point (a,b,c) of $PG(2,q^2)$ corresponds to the line ℓ_P of S_1 of defined by the equations,

$$\beta_1 x_0 - \xi\beta_2 x_1 - \alpha_1 x_2 + \xi\alpha_2 x_3 = 0 \tag{6.1}$$

$$\beta_2 x_0 + (\beta_1 - \beta_2)x_1 - \alpha_2 x_2 - (\alpha_1 - \alpha_2)x_3 = 0 \tag{6.2}$$

$$\gamma_1 x_0 - \xi\gamma_2 x_1 - \alpha_1 x_4 + \xi\alpha_2 x_5 = 0 \tag{6.3}$$

$$\gamma_2 x_0 + (\gamma_1 - \gamma_2)x_1 - \alpha_2 x_4 - (\alpha_1 - \alpha_2)x_5 = 0 \tag{6.4}$$

where $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$, $c = \gamma_1 + i\gamma_2$, $\alpha_i, \beta_i, \gamma_i \in GF(q)$ and i is a zero of $x_2 + x + \xi$, $\xi \in GF(q)$, which is irreducible over $GF(q)$. The relationship between the coordinates of P and the coordinates of the points in which ℓ_P meets the directrix planes of S_1 is also complex, as we saw in the proof of theorem 4.7. In this chapter, we shall find a way of simplifying this relationship so that the coordinate approach can be used more profitably to study the Bose representation. As an application, we use this to discuss the image of the lines of S_1 on the Grassmannian of lines of $PG(5,q)$.

6.1 The Bose Map Reconsidered

For the purpose of simplifying the relationship between the coordinates of a point P of $PG(2,q^2)$ and the coordinates of the line $\beta_1(P)$, we consider the following map.

$$\begin{aligned}\omega : PG(5,q^2) &\rightarrow PG(5,q^2)\\ \omega : \mathbf{X} &\mapsto \mathbf{XA}\end{aligned}$$

where $\mathbf{A}$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0\\ \bar{i} & 0 & 0 & i & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & \bar{i} & 0 & 0 & i & 0\\ 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & \bar{i} & 0 & 0 & i \end{pmatrix}$$

By simple expansion, the determinant of $\mathbf{A}$ is $(\bar{i}-i)^3$ which is non-zero and thus ω is a collineation of $PG(5,q^2)$.

As ω is a collineation, the set of lines $\omega(S_1)$ form a geometric spread of the space $\omega(PG(5,q))$, a subgeometry of $PG(5,q^2)$, isomorphic to $PG(5,q)$. The incidence structure for which the points are the elements of $\omega(S_1)$, the lines are the elements of $\omega(H_3)$ and the incidence is containment is thus isomorphic to $PG(2,q^2)$. We shall see that, under this isomorphism, the coordinates of a point A of $PG(2,q^2)$, and the coordinates of the points of its corresponding line ℓ_A, are related in a less complicated manner.

6.1.1 The Action of ω

We begin by considering how ω acts on the directrix planes Π and $\overline{\Pi}$ of the 1-spread S_1 of $PG(5,q)$. Recall, from section 4.2, that these planes are given by the equations

$$x_0 + ix_1 = x_2 + ix_3 = x_4 + ix_5 = 0$$

and

$$x_0 + \bar{i}x_1 = x_2 + \bar{i}x_3 = x_4 + \bar{i}x_5 = 0$$

respectively. Thus, if P is a point of Π then $P = (-ia_1, a_1, -ia_2, a_2, -ia_3, a_3)$ for some $a_1, a_2, a_3 \in GF(q^2)$. Hence

$$\begin{aligned}\omega(P) &= PA\\ &= ((\bar{i}-i)a_1, (\bar{i}-i)a_2, (\bar{i}-i)a_3, 0, 0, 0)\end{aligned}$$

and $\omega(\Pi)$ is the plane of $PG(5,q^2)$ defined by the equations

$$x_3 = x_4 = x_5 = 0.$$

Similarly, $\omega(\overline{\Pi})$ is the plane defined by the equations

$$x_0 = x_1 = x_2 = 0.$$

With this knowledge of how ω acts on the directrix planes, we can determine the action of ω on the sets S_1 and H_3. If $\ell \in S_1$ then ℓ is the join of the point $P \in \Pi$ to its conjugate point $\overline{P} \in \overline{\Pi}$. Thus $\omega(\ell)$ is the join of $\omega(P)$ and $\omega(\overline{P})$. Now, with P and $\omega(P)$ as above, $\overline{P}$ is the point $(\overline{-ia_1}, \overline{a_1}, \overline{-ia_2}, \overline{a_2}, \overline{-ia_3}, \overline{a_3})$ and thus

$$\begin{aligned} \omega(\overline{P}) &= (0,0,0,\overline{-ia_1} + i\overline{a_1}, \overline{-ia_2} + i\overline{a_2}, \overline{-ia_2} + i\overline{a_2}) \\ &= (0,0,0,(i-\bar{i})\overline{a_1}, (i-\bar{i})\overline{a_2}, (i-\bar{i})\overline{a_3}) \\ &= (0,0,0,\overline{a_1}, \overline{a_2}, \overline{a_3}) \end{aligned}$$

Hence, $\omega(\ell)$ joins $(a_1, a_2, a_3, 0, 0, 0) \in \omega(\Pi)$ to $(0,0,0,\overline{a_1}, \overline{a_2}, \overline{a_3}) \in \omega(\overline{\Pi})$. Thus the lines of $\omega(S_1)$ are the lines of $PG(5,q^2)$ containing sets of points of the form

$$\{(x,y,z,0,0,0) + \lambda(0,0,0,\overline{x},\overline{y},\overline{z}) : \lambda \in GF(q^2) \cup \{\infty\}\}$$

where $x, y, z \in GF(q^2)$, not all zero. We let $\mathcal{S}$ denote the set $\omega(S_1)$. As the elements of H_3 are the 3-spaces of $PG(5,q^2)$ generated by pairs of elements of S_1, $\omega(H_3)$ is the set of 3-spaces of $PG(5,q^2)$ generated by pairs of elements of $\mathcal{S}_1$. We let $\mathcal{H}$ denote the set $\omega(H_3)$. Note that the elements of $\mathcal{H}$ are 3-spaces of the subgeometry $\omega(PG(5,q))$ of $PG(5,q^2)$.

Lastly, we discuss the set $\omega(PG(5,q))$. If $Q = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ is a point of $PG(5,q)$ then

$$\begin{aligned} \omega(Q) &= QA \\ &= (\alpha_1 + \bar{i}\alpha_2, \alpha_3 + \bar{i}\alpha_4, \alpha_5 + \bar{i}\alpha_6, \alpha_1 + i\alpha_2, \alpha_3 + i\alpha_4, \alpha_5 + i\alpha_6) \\ &= (a_1, a_2, a_3, \overline{a_1}, \overline{a_2}, \overline{a_3}) \end{aligned}$$

where $a_1 = \alpha_1 + \bar{i}\alpha_2, a_2 = \alpha_3 + \bar{i}\alpha_4, a_3 = \alpha_5 + \bar{i}\alpha_6$. Thus any point of $\omega(PG(5,q))$ can be written in the form $(a_1, a_2, a_3, \overline{a_1}, \overline{a_2}, \overline{a_3})$. Conversely, $(a_1, a_2, a_3, \overline{a_1}, \overline{a_2}, \overline{a_3})$ and $(b_1, b_2, b_3, \overline{b_1}, \overline{b_2}, \overline{b_3})$ represent the same point of $PG(5,q^2)$ if and only if $a_i = \rho b_i$ for some $\rho \in GF(q) \setminus \{0\}$. Thus the set

$$\{(a,b,c,\overline{a},\overline{b},\overline{c} : a,b,c \in GF(q^2)\}$$

contains $\frac{q^6-1}{q-1} = q^5 + q^4 + \ldots + 1$ elements and so is $\omega(PG(5,q))$.

Alternatively, if ℓ is a line of $\mathcal{S}$ then, by the above, ℓ is of the form

$$\ell = \{(x,y,z,0,0,0) + r(0,0,0,\overline{x},\overline{y},\overline{z}) : r \in GF(q^2) \cup \{\infty\}\}$$

and contains $q+1$ points of $\omega(PG(5,q))$. Thus, from above, these are the points of ℓ which can be written in the form 6.1.1.

Therefore, if $P = (x,y,z,r\overline{x},r\overline{y},r\overline{z})$ is a point of ℓ then it is a point of $\omega(PG(5,q))$ if and only if, for some $t \in GF(q^2) \setminus \{0\}$,

$$(x,y,z,r\overline{x},r\overline{y},r\overline{z}) = t(a,b,c,\overline{a},\overline{b},\overline{c})$$

Thus

$$ta = x \text{ and } t\overline{a} = r\overline{x}$$

which implies that

$$r = t/\overline{t}$$

for some $t \in GF(q^2) \setminus \{0\}$. As $(t/\overline{t})^{q+1} = 1$ for any such t, and there are precisely $(q+1)$ distinct values of $t/\overline{t}$ for $t \in GF(q^2) \setminus \{0\}$, it follows that these are the solutions of the equation $x^{q+1} = 1$ over $GF(q^2)$. Hence the point P is a point of $\omega(PG(5,q))$ if and only if r is a solution of this equation. Thus $\omega(PG(5,q))$ can be expressed as

$$\{(x,y,z,r\overline{x},r\overline{y},r\overline{z}) : (x,y,z) \in PG(2,q^2) \text{ and } r^{q+1} = 1\}$$

and the lines of $\mathcal{S}$ are of the form

$$\{(x,y,z,r\overline{x},r\overline{y},r\overline{z}) : r^{q+1} = 1\}$$

for fixed $(x,y,z) \in PG(2,q^2)$.

For the sake of clarity we state the main result of this section.

Theorem 6.1. *The incidence structure with points being the lines of $\mathcal{S}$, lines being the elements of $\mathcal{H}$ and the incidence being containment is isomorphic to $PG(2,q^2)$. Explicitly, the map*

$$P \mapsto \ell_P$$

associating the point $P = (x,y,z)$ of $PG(2,q^2)$ to the line ℓ_P of $\mathcal{S}$ containing the set of points

$$\{(x,y,z,0,0,0) + \lambda(0,0,0,\overline{x},\overline{y},\overline{z}) : \lambda \in GF(q^2) \cup \{\infty\}\}$$

is an isomorphism. □

We shall also call this the Bose representation of $PG(2,q^2)$ and denote the corresponding map by $\boldsymbol{\beta_1}$.

6.2 A Connection with the Grassmannian $\mathcal{G}_{1,5}$

In section 1.10 we introduced the Grassmann variety $\mathcal{G}_{1,n}$ of lines of $PG(n,q)$. Since S_1 is a spread of lines of $PG(5,q)$, to each line ℓ of S_1 there corresponds a point $\gamma(\ell)$ of $\mathcal{G}_{1,5}$, where γ is the Grassmann map, as defined in section 1.10, which associates a line of $PG(5,q)$ to the point of $PG(14,q)$ with the same coordinates. Thus, as the lines of S_1 correspond to points of $PG(2,q^2)$ via the Bose map, we have a correspondence between the points of $PG(2,q^2)$ and a subset of the points of $\mathcal{G}_{1,5}$. The complicated form of the coordinates of the lines in S_1 makes it difficult to exploit this correspondence but if we consider the Bose representation as given in theorem 6.1, then something can be said. It should be noted that this idea is used also by Lunardon in [36] where it appears as a special case of a more general setting. Here, by considering only this one case, we have been able to simplify the exposition and prove some extra results. For the remainder of this chapter, when we talk of the Bose map or Bose representation we refer to that defined in theorem 6.1.

If $\mathcal{P}$ denotes the set of all points of $PG(2,q^2)$, we are thus interested in the map

$$\begin{aligned} g : \mathcal{P} &\rightarrow G_{1,5} \\ g : P &\mapsto \gamma(\boldsymbol{\beta_1}(P)) \end{aligned}$$

That is, g maps P to the point of $\mathcal{G}_{1,5}$ with coordinates being those of the line of $\mathcal{S}$ which corresponds to P under the Bose map. In particular, we are interested in studying the subset $g(\mathcal{P})$ of $\mathcal{G}_{1,5}$.

As the lines of $\mathcal{S}$ are all lines of the subgeometry $\omega(PG(5,q))$ of $PG(5,q^2)$, the corresponding points of $\mathcal{G}_{1,5}$ will lie in a subgeometry of $PG(14,q^2)$ of order q. Let Σ_q denote this subgeometry.

Explicitly, if $P=(a_0,a_1,a_2)$ is a point of $PG(2,q^2)$ then this corresponds, under the Bose map, to the line $\ell = \boldsymbol{\beta_1}(P)$ of $\mathcal{S}$, where ℓ is the line joining the points $(a_0,a_1,a_2,0,0,0)$ and $(0,0,0,a_0^q,a_1^q,a_2^q)$ of $PG(5,q^2)$. This line has grassmann coordinates

$$(0,0,a_0^{q+1},a_0a_1^q,a_0a_2^q,0,a_1a_0^q,a_1^{q+1},a_1a_2^q,a_2a_0^q,a_2a_1^q,a_2^{q+1},0,0,0)$$

and so is mapped by γ to the point Q of Σ_q with these coordinates. Thus g maps the point (a_0,a_1,a_2) of $PG(2,q^2)$ to the point Q of Σ_q with the above coordinates.

Therefore, $g(\mathcal{P})$ is the set of points

$$\{(0,0,a_0^{q+1},a_0a_1^q,a_0a_2^q,0,a_1a_0^q,a_1^{q+1},a_1a_2^q,a_2a_0^q,a_2a_1^q,a_2^{q+1},0,0,0) : P \in PG(2,q^2)\}$$

where $P=(a_0,a_1,a_2)$. Hence the points of $g(\mathcal{P})$ all lie in the eight dimensional subspace Σ_8^* of $PG(14,q^2)$ defined by the equations

$$x_0 = x_1 = x_5 = x_{12} = x_{13} = x_{14} = 0$$

Further, Σ_q intersects Σ_8^* in a Baer subgeometry Σ_8 of Σ_8^* which contains the points of $g(\mathcal{P})$.

Lemma 6.2. [36] *$g(\mathcal{P})$ is a set of (q^4+q^2+1) points, no three collinear, of an eight dimensional subspace of order q of $PG(14,q^2)$.*

Proof: Since there are (q^4+q^2+1) points in $PG(2,q^2)$, it follows immediately that there are this many points in $g(\mathcal{P})$. As in section 1.10, the subspaces of $PG(14,q^2)$ contained on $\mathcal{G}_{1,5}$ are the images under γ of either the lines through a point in $PG(5,q^2)$, or the lines of a plane in $PG(5,q^2)$. Hence, as no two lines of $\mathcal{S}$ intersect, no two points of $\gamma(\mathcal{S}) = g(\mathcal{P})$ are collinear on a line of $\mathcal{G}_{1,5}$. Further, as $\mathcal{G}_{1,5}$ is the intersection of quadrics (see section 1.10), any line meeting $\mathcal{G}_{1,5}$ in three points is wholly contained in $\mathcal{G}_{1,5}$. Therefore no three points of $g(\mathcal{P})$ are collinear. □

We will find the following result useful.

Lemma 6.3. *If ϕ is a projectivity of $PG(2,q^2)$ then ϕ induces a projectivity of $PG(14,q^2)$ which fixes $g(\mathcal{P})$.*

Proof: Since ϕ is a projectivity we can write it as

$$\begin{aligned}\phi : PG(2,q^2) &\rightarrow PG(2,q^2)\\ \phi : \mathbf{X} &\mapsto \mathbf{XB}\end{aligned}$$

where $\mathbf{B}=(b_{ij})$, $0 \le i,j \le 2$, is a non-singular (3×3) matrix. Note that this projectivity induces a permutation $\tilde{\phi}$ of $g(\mathcal{P})$ whereby the point $g(P)$ of $g(\mathcal{P})$ is permuted to $\tilde{\phi}(g(P)) = g(\phi(P))$.

If $P=(a_0,a_1,a_2)$ is a point of $PG(2,q^2)$ and we let $\phi(P)=(\tilde{a}_0,\tilde{a}_1,\tilde{a}_2)$, where $\tilde{a}_i = a_0b_{0i}+a_1b_{1i}+a_2b_{2i}$, then $g(P)$ is the point

$$(0,0,a_0^{q+1},a_0a_1^q,a_0a_2^q,0,a_1a_0^q,a_1^{q+1},a_1a_2^q,a_2a_0^q,a_2a_1^q,a_2^{q+1},0,0,0)$$

and is permuted by $\tilde{\phi}$ to the point $g(\phi(P))$ with coordinates

$$(0,0,\tilde{a}_0^{q+1},\tilde{a}_0\tilde{a}_1^q,\tilde{a}_0\tilde{a}_2^q,0,\tilde{a}_1\tilde{a}_0^q,\tilde{a}_1^{q+1},\tilde{a}_1\tilde{a}_2^q,\tilde{a}_2\tilde{a}_0^q,\tilde{a}_2\tilde{a}_1^q,\tilde{a}_2^{q+1},0,0,0)$$

where

$$\tilde{a}_i\tilde{a}_j^q = (a_0b_{0i}+a_1b_{1i}+a_2b_{2i})(a_0^qb_{0j}^q+a_1^qb_{1j}^q+a_2^qb_{2i}^q).$$

Consider $(x_0,x_1,\dots,x_8)$ to be coordinates in Σ_8^*. Then the map

$$\begin{aligned}\phi' : \Sigma_8^* &\to \Sigma_8^*\\ \phi' : \mathbf{X} &\mapsto \mathbf{XB}'\end{aligned}$$

where $\mathbf{B}'$ is the (9×9) matrix (with $b_{kl}\mathbf{B}^q = (b_{kl}b_{ij}^q)$)

$$\begin{pmatrix} b_{00}\mathbf{B}^q & b_{01}\mathbf{B}^q & b_{02}\mathbf{B}^q\\ b_{10}\mathbf{B}^q & b_{11}\mathbf{B}^q & b_{12}\mathbf{B}^q\\ b_{20}\mathbf{B}^q & b_{21}\mathbf{B}^q & b_{22}\mathbf{B}^q \end{pmatrix}$$

implements the permutation $\tilde{\phi}$.

Furthermore, the rows of $\mathbf{B}'$ being linearly dependent implies that the rows of the matrix $\mathbf{B}^q$ are linearly dependent which, in turn, implies that the rows of $\mathbf{B}$ are linearly dependent, contradicting the non-singularity of $\mathbf{B}$. Hence $\mathbf{B}'$ is non-singular and thus the map ϕ' is a projectivity of Σ_8^* which fixes $g(\mathcal{P})$. Since we can extend this to a projectivity of $PG(14,q^2)$ (for example, by insisting that all other coordinates are fixed), we have the result. $\square$

Let m be a line of $PG(2,q^2)$. We wish to analyse the set $g(m)$ of (q^2+1) points of $g(\mathcal{P})$. The points of m are mapped, by $\boldsymbol{\beta_1}$, to the (q^2+1) lines of a regular spread of a three-dimensional subspace Σ_3 of $\omega(PG(5,q))$. Let S_m denote this set of lines. Since Σ_3 is a three dimensional projective space of order q, the image, under the grassmann map, of the lines of Σ_3 is a Grassmann variety of the lines of $PG(3,q)$; that is, a five-dimensional hyperbolic quadric over $GF(q)$.

Lemma 6.4. *The points of a line of $PG(2,q^2)$ are mapped by g to the points of a three dimensional elliptic quadric (over $GF(q)$) on $g(\mathcal{P})$.*

Proof: Consider the line m of $PG(2,q^2)$ of equation $z=0$. If $P\in m$, $P=(x,y,0)$, then $\boldsymbol{\beta_1}(P)$ is the line with Grassmann coordinates

$$(0,0,x^{q+1},xy^q,0,0,yx^q,y^{q+1},0,0,0,0,0,0,0)$$

Thus the points of $g(m)$ are points of the hyperbolic quadric $\mathcal{Q}$ defined by the equation

$$x_2x_7 = x_3x_6 \tag{6.5}$$

contained in the 3-space

$$x_0 = x_1 = x_4 = x_5 = x_8 = x_9 = \ldots = x_{14} = 0 \tag{6.6}$$

of $PG(14, q^2)$. However the points of $g(m)$ are also points of Σ_q, which intersects $\mathcal{Q}$ in a three-dimensional elliptic quadric. Since any two lines of $PG(2, q^2)$ are projectively equivalent, the result follows by lemma 6.3. □

Let $\mathcal{Q}_m$ denote the quadric $g(m)$ and Π_m denote the 3-space in which it is contained. Note that as two lines of $PG(2, q^2)$ have a unique common point, the quadrics $\mathcal{Q}_m$ intersect pairwise in a unique point.

6.2.1 Baer Sublines and Baer Subplanes

We examine the image under g of the Baer sublines and Baer subplanes of $PG(2, q^2)$. Let b be the Baer subline

$$\{(x, y, 0) : x, y \in GF(q), \text{ not both zero}\}$$

of the line m of $PG(2, q^2)$ of equation $z = 0$. Thus $g(b)$ is the set of points

$$\{(0, 0, x^{q+1}, xy^q, 0, 0, yx^q, y^{q+1}, 0, 0, 0, 0, 0, 0, 0) : x, y \in GF(q), \text{ not both zero}\}$$

which, as $a^q = a$ for $a \in GF(q)$, can be written as

$$\{(0, 0, x^2, xy, 0, 0, yx, y^2, 0, 0, 0, 0, 0, 0, 0) : x, y \in GF(q), \text{ not both zero}\}$$

Now, $g(m)$ is the quadric given by equations 6.5 and 6.6 and $g(b)$ is the intersection of the plane

$$x_3 = x_6$$

with this quadric. Hence the points of $g(b)$ are the points of a non-singular conic. Thus, from lemma 6.3 we have the first part of the following;

Lemma 6.5. *The points of a Baer subline of a line of* $PG(2, q^2)$ *correspond under* g *to the points of a conic on the Grassmannian. Conversely, every conic section of the quadrics* $\mathcal{Q}_m$ *corresponds to a Baer subline of* $PG(2, q^2)$ *in this way.*

Proof: It only remains to prove the last statement. This follows as there are $q(q^2+1)$ conics on a three dimensional elliptic quadric and the same number of Baer sublines of a line in $PG(2,q^2)$. $\square$

Using a similar method, we can find the image, on the Grassmannian, of a Baer subplane of $PG(2,q^2)$. Consider the Baer subplane of $PG(2,q^2)$

$$\mathcal{B} = \{(\alpha_0, \alpha_1, \alpha_2) : \alpha_i \in GF(q)\}$$

Therefore $g(\mathcal{B})$ is the set of points

$$\{(0,0,\alpha_0^{q+1}, \alpha_0\alpha_1^q, \alpha_0\alpha_2^q, 0, \alpha_1\alpha_0^q, \alpha_1^{q+1}, \alpha_1\alpha_2^q, \alpha_2\alpha_0^q, \alpha_2\alpha_1^q, \alpha_2^{q+1}, 0,0,0) : \alpha_i \in GF(q)\}$$

which, as $a^q = a$ for $a \in GF(q)$, can be written as

$$\{(0,0,\alpha_0^2, \alpha_0\alpha_1, \alpha_0\alpha_2, 0, \alpha_1\alpha_0, \alpha_1^2, \alpha_1\alpha_2, \alpha_2\alpha_0, \alpha_2\alpha_1, \alpha_2^2, 0,0,0) : \alpha_i \in GF(q)\}$$

and so the points of $g(\mathcal{B}) = \gamma(S_{\mathcal{B}})$ lie in the five-dimensional subspace (of order q) of $PG(14,q^2)$ defined by the equations

$$x_0 = x_1 = x_5 = x_{12} = x_{13} = x_{14} = 0, x_3 = x_6, x_4 = x_9, x_8 = x_{10}$$

and form a Veronese surface in this space. As any two Baer subplanes of $PG(2,q^2)$ are projectively equivalent, we can use lemma 6.3 to infer the following:

Theorem 6.6. [36] *If $\mathcal{B}$ is a Baer subplane of $PG(2,q^2)$ then $\gamma(S_{\mathcal{B}})$ is a Veronese surface on $\gamma(\mathcal{S})$.* $\square$

6.2.2 The Quadrics $\mathcal{Q}_m$

If m is a line of $PG(2,q^2)$ then $g(m)$ is a three-dimensional elliptic quadric $\mathcal{Q}_m$ contained in the 3-space Π_m. As there are (q^4+q^2+1) lines in $PG(2,q^2)$, we have (q^4+q^2+1) such elliptic quadrics on $g(\mathcal{P})$. We examine, in greater detail, these quadrics and the 3-spaces which contain them.

Lemma 6.7. *If m_1 and m_2 are two distinct lines of $PG(2,q^2)$ then the spaces Π_{m_1} and Π_{m_2} meet in a unique point, this point being the unique point of intersection of the quadrics $\mathcal{Q}_{m_1}$ and $\mathcal{Q}_{m_2}$.*

Proof: If $P = m_1 \cap m_2$ then the quadrics $\mathcal{Q}_{m_1}$ and $\mathcal{Q}_{m_2}$ both contain the point $g(P)$ and this is the only point of $g(\mathcal{P})$ in common to them.

Suppose $Q \in \Pi_{m_1} \cap \Pi_{m_2}$ with $Q \neq g(P)$. Let α_i, $i = 1, 2$ be a plane about PQ meeting $\mathcal{Q}_{m_i}$ in a conic. Thus $g^{-1}(\alpha_i \cap \mathcal{Q}_{m_i})$ meets $PG(2, q^2)$ in a Baer subline b_i (lemma 6.5). As the α_i each contain $g(P)$, b_1 and b_2 both contain P and thus lie in a Baer subplane $\mathcal{B}$ of $PG(2, q^2)$. Hence $g(\mathcal{B})$ is a Veronese surface for which α_1 and α_2 are conic planes with two common points (Q and $g(P)$), contradicting the way conics of a Veronese surface intersect (lemma 2.7). Hence Π_{m_1} and Π_{m_2} intersect in a unique point. □

Let m be a line of $PG(2, q^2)$. We give equations defining the 3-space Π_m containing the quadric $g(m) = \mathcal{Q}_m$.

Suppose that m is given by the equation

$$ax_0 + bx_1 + cx_2 = 0$$

where $a, b, c \in GF(q^2)$. Multiplying this equation in turn by by x_0^q, x_1^q and x_2^q we obtain the three equations,

$$\begin{aligned}
ax_0^{q+1} + bx_1x_0^q + cx_2x_0^q &= 0\\
ax_0x_1^q + bx_1^{q+1} + cx_2x_1^q &= 0\\
ax_0x_2^q + bx_1x_2^q + cx_2^{q+1} &= 0
\end{aligned}$$

The points of m also satisfy the equation

$$a^qx_0^q + b^qx_1^q + c^qx_2^q = 0$$

and so

$$\begin{aligned}
a^qx_0^{q+1} + b^qx_1^qx_0 + c^qx_2^qx_0 = 0\\
a^qx_0^qx_1 + b^qx_1^{q+1} + c^qx_2^qx_1 = 0\\
a^qx_0^qx_2 + b^qx_1^qx_2 + c^qx_2^{q+1} = 0
\end{aligned}$$

If $P = (a_1, b_1, c_1)$ is a point of m then the point $g(P)$ of $\mathcal{G}_{1,5}$ has coordinates

$$(0, 0, a_1^{q+1}, a_1b_1^q, a_1c_1^q, 0, a_1^qb_1, b_1^{q+1}, c_1^qb_1, a_1^qc_1, b_1^qc_1, c_1^{q+1}, 0, 0, 0)$$

and thus, by the above, every point of $g(m)$ lies on each of the six hyperplanes of $PG(14, q^2)$ given by the equations,

$$\begin{aligned}
ax_2 + bx_6 + cx_9 &= 0\\
ax_3 + bx_7 + cx_{10} &= 0\\
ax_4 + bx_8 + cx_{11} &= 0\\
a^q x_2 + b^q x_3 + c^q x_4 &= 0\\
a^q x_6 + b^q x_7 + c^q x_8 &= 0\\
a^q x_9 + b^q x_{10} + c^q x_{11} &= 0
\end{aligned}$$

Any five of these equations are linearly independent and thus they define the three-dimensional subspace Π_m of Σ_8.

A hyperplane of Σ_8 either contains a given Π_m or meets it in a plane. Since every plane of Π_m meets $\mathcal{Q}_m$ in a unique point or in a (non-singular) conic, we can say that every hyperplane of Σ_8 either contains $\mathcal{Q}_m$, meets it in a unique point or meets it in a conic. We discuss in more detail how a hyperplane of $PG(14, q^2)$ intersects $g(\mathcal{P})$ and, in particular, the number of $\mathcal{Q}_m$ that a given hyperplane can contain.

Lemma 6.8. *Let m_1, m_2, m_3 be three non-concurrent lines of $PG(2, q^2)$ corresponding to the elliptic quadrics $\mathcal{Q}_{m_1}, \mathcal{Q}_{m_2}, \mathcal{Q}_{m_3}$. The three-dimensional spaces $\Pi_{m_1}, \Pi_{m_2}, \Pi_{m_3}$ of Σ_8, which are spanned by the points of $\mathcal{Q}_{m_1}, \mathcal{Q}_{m_2}$ and $\mathcal{Q}_{m_3}$ respectively, do not lie in a common hyperplane of Σ_8.*

Proof: Consider the lines m_1, m_2, m_3 of $PG(2, q^2)$ which have the equations

$$\begin{aligned}
x_0 &= 0\\
x_1 &= 0\\
x_2 &= 0
\end{aligned}$$

respectively. These lines correspond, via g, to the quadrics $\mathcal{Q}_{m_1}, \mathcal{Q}_{m_2}$ and $\mathcal{Q}_{m_3}$ contained in the 3-spaces $\Pi_{m_1}, \Pi_{m_2}, \Pi_{m_3}$ of Σ_8 defined by the equations

$$x_2 = x_3 = x_4 = x_6 = x_9 = 0$$

$$x_3 = x_6 = x_7 = x_8 = x_{10} = 0$$

$$x_4 = x_8 = x_9 = x_{10} = x_{11} = 0$$

respectively, and hence Π_{m_1}, Π_{m_2} and Π_{m_3} span Σ_8. Since, by an appropriate choice of coordinates, we can choose any three non-concurrent lines of $PG(2, q^2)$ to have these equations, the result follows from lemma 6.3. □

Conversely, three concurrent lines of $PG(2,q^2)$ correspond to three Π_m's contained in a hyperplane of Σ_8 and each such hyperplane contains $(q+1)$ of the Π_m's.

Lemma 6.9. *If Q is a point of the Baer subplane $\mathcal{B}$ of $PG(2,q^2)$ then the $(q+1)$ lines of $\mathcal{B}$ through Q correspond to $(q+1)$ spaces Π_m which lie in a common hyperplane of Σ_8. Such a hyperplane contains exactly these $(q+1)$ Π_m's.*

Proof: Suppose that Q is the point$(0,0,1)$ and that $\mathcal{B}$ is the Baer subplane $PG(2,q)$. The $(q+1)$ lines of $\mathcal{B}$ through $(0,0,1)$ have equations of the form

$$ax_0 + bx_1 = 0 \tag{6.7}$$

where $a, b \in GF(q)$, and thus correspond, under g, to the quadrics contained in the three-dimensional spaces defined by the equations

$$\begin{array}{lll} ax_2 + bx_6 = 0 & ax_3 + bx_7 = 0 & ax_4 + bx_8 = 0 \\ ax_2 + bx_3 = 0 & ax_6 + bx_7 = 0 & ax_9 + bx_{10} = 0 \end{array}$$

Subtracting the first of these equations from the fourth we obtain

$$bx_3 - bx_6 = 0$$

and thus each of these 3-spaces is contained in the hyperplane Σ_7 of Σ_8 of equation $x_3 = x_6$ and this hyperplane contains $(q+1)$ of the Π_{m_i}.

By lemma 6.8, if ℓ is a line of $PG(2,q^2)$ not through $(0,0,1)$ then Π_ℓ is not in Σ_7. Suppose that ℓ is a line through $(0,0,1)$, distinct from those described by 6.7, for which Π_ℓ is contained in Σ_7. If r is any line of $PG(2,q^2)$ not containing $(0,0,1)$ then r intersects ℓ and the lines of $PG(2,q)$ in distinct points. Thus there are $(q+2)$ points of $\mathcal{Q}_r$ in Σ_7 and hence $\mathcal{Q}_r$ and thus Π_r are contained in Σ_7, contradicting lemma 6.8.

Since, by theorem 1.2, we can choose any Baer subplane of $PG(2,q^2)$ to be $PG(2,q)$ the result follows from lemma 6.3. □

Corollary 6.10. *If m_1, m_2, m_3 are three concurrent lines of $PG(2,q^2)$, meeting at the point P, then the three-dimensional spaces Π_{m_1}, Π_{m_2} and Π_{m_3} are contained in a common hyperplane of Σ_8. This hyperplane contains precisely $(q+1)$ of the Π_m.*

Proof: Since there is a Baer subplane of $PG(2,q^2)$ containing m_1, m_2 and m_3 (lemma 1.7) this is a direct consequence of lemma 6.9. □

Corollary 6.11. *A hyperplane of Σ_8 containing two of the Π_m's contains $(q+1)$ of the Π_m's.*

Proof: Suppose that Σ_7 is a hyperplane of Σ_8 containing Π_m and $\Pi_{m'}$. By lemma 6.7, Π_m and $\Pi_{m'}$ meet in a unique point and thus span a six-dimensional subspace of Σ_8 about which there are $(q+1)$ hyperplanes of Σ_8.

Let m and m' be the two lines of $PG(2,q^2)$ corresponding to Π_m and $\Pi_{m'}$ and $P = m \cap m'$. If m'' is one of the remaining (q^2-1) lines through P then by, lemma 1.7, there is a Baer subplane of $PG(2,q^2)$ with m, m', m'' as lines of this subplane, and any two Baer subplanes with m, m', m'' as lines has the same set of $(q+1)$ lines through P. Hence, by lemma 6.9, there are $(q^2-1)/(q-1) = (q+1)$ hyperplanes of Σ_8 which contain Π_m and $\Pi_{m'}$ and a further $(q-1)$ of the Π_{m_i}. Therefore all the hyperplanes about $\langle \Pi_m, \Pi_{m'} \rangle$, and in particular Σ_7, contain $(q+1)$ of these spaces. □

Lemma 6.12. *If m is a line of $PG(2,q^2)$ then there is a unique hyperplane of Σ_8 about Π_m which contains no space Π_ℓ, for any line ℓ of $PG(2,q^2)$ distinct from m.*

Proof: About Π_m there are $(q^4 + \ldots + q + 1)$ hyperplanes of Σ_8. By the above lemmas, any hyperplane containing Π_m and another Π_ℓ, for $\ell \neq m$ a line of $PG(2,q^2)$, contains $(q+1)$ of the spaces Π_{m_i} and these correspond to $(q+1)$ lines of a Baer subplane of $PG(2,q^2)$.

Consider $P \in m$. We wish to calculate how many ways we can choose $(q+1)$ lines through P such that they will be the lines of a Baer subplane containing m. By lemma 1.7, choosing any two further lines through P we obtain $(q+1)$ lines of a Baer subplane containing P in a unique way, and thus there are $q^2(q^2-1)/q(q-1) = q(q+1)$ ways of choosing $(q+1)$ such lines. Since there are (q^2+1) points on m, we obtain $q(q+1)(q^2+1)$ such sets of lines as we vary P over m.

Correspondingly, we have

$$q(q+1)(q^2+1) = q^4 + q^3 + q^2 + q$$

hyperplanes about Π_m which contain $(q+1)$ of the Π_m. Thus there is a unique hyperplane of Σ_8 about Π_m which contains no Π_ℓ for $\ell \neq m$. □

Theorem 6.13. *A hyperplane of Σ_8 contains either $0, 1$ or $(q+1)$ of the 3-spaces Π_m. There are $(q^4 + q^2 + 1)$ containing one, $(q^3+q)(q^4+q^2+1)$ containing $(q+1)$ and $q^3(q^5+q^3-q^2-1)$ containing none of the Π_m.*

Proof: That a hyperplane of Σ_8 can contain only $0, 1$ or $(q+1)$ of the Π_m follows from lemma 6.9 and corollary 6.11.

Since there are (q^4+q^2+1) spaces Π_m and, by lemma 6.12, each is contained in a unique hyperplane of Σ_8 containing no further $\Pi_{m'}$, there are (q^4+q^2+1) hyperplanes of Σ_8 containing precisely one Π_m. As there are $(q^4+q^3+q^2+q)$ hyperplanes about a Π_m containing $(q+1)$ of these spaces, there are

$$\frac{(q^4+q^3+q^2+q+1)(q^4+q^2+1)}{(q+1)} = (q^3+q)(q^4+q^2+1)$$

hyperplanes of Σ_8 containing $(q+1)$ of the Π_m. Therefore the remaining

$$(q^8+\ldots+q+1)-(q^3+q+1)(q^4+q^2+1) = q^3(q^5+q^3-q^2-1)$$

of the hyperplanes of Σ_8 contain none of the Π_m. □

6.2.3 Unitals

Let $\mathcal{U}_c$ denote the classical unital in $PG(2,q^2)$. The points of $\mathcal{U}_c$ are projectively equivalent to the points satisfying the equation

$$x_0x_2^q + x_1^{q+1} + x_2x_0^q = 0.$$

We examine the set $g(\mathcal{U}_c)$. We follow a similar method to that used in Lunardon, [36]. If $A=(a_0,a_1,a_2)\in\mathcal{U}_c$ then $a_0a_2^q + a_1^{q+1} + a_2a_0^q = 0$ and so the point $g(A)$ lies on the hyperplane Σ of $PG(14,q^2)$ defined by the equation

$$x_4+x_7+x_9=0$$

Since this hyperplane does not contain Σ_8, it meets it in a hyperplane Σ_7 of Σ_8 which contains the points of $\gamma(\mathcal{U}_c)$. As every line of $PG(2,q^2)$ contains either one or $(q+1)$ points of $\mathcal{U}_c$, the space Σ_7 contains none of the Π_m. Therefore, using lemma 6.3, we have the following result;

Theorem 6.14. [36] *If $\mathcal{U}_c$ is a classical unital of $PG(2,q^2)$ then the points $g(\mathcal{U}_c)$ lie in a hyperplane of Σ_8 which does not contain any of the Π_m.*

Conversely, suppose that the 7-space $\Sigma_7\subset\Sigma_8$ contains none of the Π_m and the points of $g^{-1}(\Sigma_7\cap g(\mathcal{P}))$ are the points of a unital of $PG(2,q^2)$. Then, as $\Sigma_7\cap\mathcal{Q}_m$ is a conic for any Σ_7 not containing Π_m, we have, by lemma 6.5, that every secant line of such a unital is a Baer subline. Thus this unital is classical. Actually we can say more. Since the number of classical unitals in $PG(2,q^2)$ is $q^3(q^5+q^3-q^2-1)$ (see, for example [29]) and, by theorem 6.13, this is the number of hyperplanes of Σ_8 which do not contain a Π_m we have;

Theorem 6.15. *Let* Σ_7 *be a hyperplane of* Σ_8 *containing none of the* Π_m. *The points of* $PG(2, q^2)$ *corresponding to the points of* $\Sigma_7 \cap g(\mathcal{P})$ *form a classical unital in* $PG(2, q^2)$.

□

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