CALCULATIONS IN GAUGE FIELD THEORY AT FINITE TEMPERATURE

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Abstract

The beginning of the thesis contains a review of finite temperature field theory using functional methods and the construction of finite temperature Feynman rules. Particular attention is paid to the so-called real and imaginary time formalisms.

The gauge field propagator for a non-abelian gauge theory (pure Yang-Mills) in the Lorentz gauge but with gauge parameter $\alpha \neq 1$ at finite temperature is derived. This is achieved by seeking the most general solution to the equation of motion for the propagator under the constraints of rotational covariance. The propagator is used to compute the one-loop correction to the gauge field propagator, in the real time formalism and arbitrary α.

A well known problem occurs in the real time formalism when finite temperature amplitudes contain products of delta-functions of the same argument. By examining free field theory, we show how one can avoid such constructs, by using derivatives, rather than products, of δ functions. Possible insights on how this can be generalised to interacting field theories are discussed.

Finally, a calculation in coordinate-space showing the temperature independence of the chiral anomaly for an arbitrary gauge group and arbitrary but even dimensional Euclidean space is presented. The properties of field theory in the imaginary time formalism and the methods of Nielsen, Schroer and Crewther are utilised. The temperature independence of the Atiyah-Singer Index theorem is also established.

The original work is introduced in

- Sections 2.2, 2.3, 3.2, 3.4, 4.4, B.2, C.2.
- The finite temperature aspects of the Atiyah-Singer Index theorem in Section 4.5 and of the spin 1 propagator in Section A.1.
- Parts of the table of integrals in Section A.2, particularly those involving derivatives of δ functions.

Declaration

Except where due reference is made, this thesis does not contain any material previously published or written by another person to the best of my knowledge and belief, or any material which has been accepted for the award of any other degree or diploma in any University. I give my consent for this thesis to be made available for photocopying and loan.

Andrew A. Rawlinson.

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Chapter 1

Background to Finite Temperature Field Theory

1.1 Introduction

The purpose of this chapter is two fold

- To present a field theoretic point of view to finite temperature field theory, laying down its mathematical structure and to examine how finite temperature effects are manifested in this approach. Extensions to gauge theories are given particular attention. Care must be exercised to ensure that only the physical degrees of freedom contribute to the quantities that we are interested in calculating.

- To investigate how finite temperature alters the physics of a particular system, with emphasis on the behaviour of quantum field theories. For example, the physics relating to spontaneous symmetry breaking (and phase transitions) and its influence under finite temperature effects will be examined. Also, a brief survey of how finite temperature effects can appear through cross-sections and decay-widths will be presented, using the $\pi^0 \to 2\gamma$ decay width at finite temperature as a specific case. It will seen that some situations are not affected by finite temperature effects - such as the amplitude associated with the chiral anomaly (which forms the subject of Chapter 4).

Conventional quantum field theory only describes fields and their interactions at $T = 0$, absolute zero and when the density of the system is negligible (i.e. many body effects are ignored). For most circumstances this has been sufficient, since many theoretical predictions have been verified experimentally on the basis of $T = 0$ field theory. To some extent, this is not unreasonable since finite temperature effects become important when kT of the system is of the order of magnitude of the masses of the fields. This can seen by noting that thermal energy for a single particle in a gas of fundamental particles at temperature T, is related to the difference of its relativistic and rest mass energies, which for a massive particle of rest mass m_0 is

$$m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)=\frac{3}{2}kT \tag{1.1}$$

where $k = 1.38 \times 10^{-23} JK^{-1}$ is Boltzmann's constant, v is velocity of the particle and c is the speed of light. Hence thermal fluctuations are of the order of m_0c^2 when the factor in the brackets is of O(1) which happens when $v \sim 0.85c$. For electrons (protons) this occurs at a temperature of the order of $T \sim 10^{10}(10^{13})$ K.

Such temperatures are in general beyond the realms of accessibility of present day experiments, mainly due to difficulty in maintaining and controlling such conditions. There are however some exceptions - for example, experiments involving relativistic heavy ion collisions which have been carried out recently, where the dynamics of the resulting quark-gluon plasma can be understood more readily if finite temperature effects are taken into account. Finite density effects are also presumed to play a vital role. Of course nuclear reactions are another instance where finite temperature effects play a role.

On the astrophysical and cosmological level, finite temperature effects play a crucial role. During the early stages of evolution of the universe, the spectrum of particles believed to exist at various epochs can undergo drastic changes (phase transitions) as the universe expands and cools. The dynamics of stellar systems, neutron stars for example, can be sensitive to finite temperature and many body effects.

The main idea is to combine the concepts of statistical mechanics with those of quantum field theory. In some aspects there are some striking similarities between each of these areas. To incorporate finite temperature effects into quantum field theory, various approaches can be used - for example, the real time approach [DJ74,LvW87], imaginary time approach [Be74,We74,LvW87], that of Thermo-Field Dynamics [UMT82] which requires doubling the number of fields, and the closed time path method [Sc61]. We shall be dealing only with the 'real' and 'imaginary' time formalisms.

1.2 Finite Temperature Field Theory

The methods describing gauge theories at finite temperature presented here will follow those of Bernard [Be74], Dolan and Jackiw [DJ74] and Weinberg [We74]. They have given ways for obtaining finite temperature Feynman rules that can be used for perturbation theory. We shall begin with the derivation of the Feynman rules by functional methods, the major reference for this section being Bernard [Be74]. This section constitutes the derivation of the Feynman rules in the imaginary time formalism, ITF.

Suppose the dynamics of any field $\phi(\vec{x}, t)$ in the Heisenberg picture, which could have scalar, vector, spinor etc. properties, and its conjugate momentum field $\pi(\vec{x}, t)$, are governed by the Hamiltonian density $\mathcal{H}(\pi, \phi)$. Letting $\phi(\vec{x}, 0)$ be the Schrödinger picture field, we can define eigenstates, $|\phi_0>$ and $|\phi_1>$, of $\phi(\vec{x},0)$ by

$$\begin{aligned} \phi(\vec{x},0)\,|\phi_0> &= \phi_0(\vec{x})\,|\phi_0>, \\ \phi(\vec{x},0)\,|\phi_1> &= \phi_1(\vec{x})\,|\phi_1>. \end{aligned} \tag{1.2}$$

The transition amplitudes from an initial state $|\phi_0>$ at time $t=0$ to a final state $|\phi_1>$ at $t=t_1$ can be expressed in terms of the Hamiltonian form of the Feynman functional

$$<\phi_1|\,e^{-iHt_1}\,|\phi_0> = N\int \mathcal{D}\pi\,\mathcal{D}\phi\,\exp\left\{i\int_0^{t_1} dt\int d^3x\,\left[\pi\dot{\phi}-\mathcal{H}(\pi,\phi)\right]\right\} \tag{1.3}$$

where the integral over classical fields, $\int\mathcal{D}\phi$, is restricted to field configurations of the form

$$\phi_0(\vec{x}) \text{ at } t=0 \qquad\qquad \text{and} \qquad\qquad \phi_1(\vec{x}) \text{ at } t=t_1 \tag{1.4}$$

and the functional integral over the conjugate momenta $\int\mathcal{D}\pi$ is unrestricted. N is a normalisation constant and the time derivative of the field is defined in the usual way by

$$\dot{\phi}(\vec{x},t)=\frac{\partial\,\phi(\vec{x},0)}{\partial\,t}. \tag{1.5}$$

It is generally understood that momentum integrations are performed before field integrations.

To incorporate finite temperature effects, one lets $i\,t_1=\beta$, where $\beta=1/T$ the inverse temperature (in units such that Boltzmann's constant $k=1$), and performs a change of variable of

$$i\,t=\tau \tag{1.6}$$

in (1.3) to obtain

$$< \phi_1 | \, e^{-H\beta} \, | \phi_0 > = N \int \mathcal{D}\pi \, \mathcal{D}\phi \, \exp \left\{ \int_0^\beta d\tau \int d^3x \, \left[i \, \pi \dot{\phi} - \mathcal{H}(\pi, \phi) \right] \right\} \qquad (1.7)$$

where 'time' differentiation is now defined by $\dot{\phi} = \partial \phi / \partial \tau$.

The most important quantity to compute when dealing with many body systems and statistical mechanics is the partition function $Z = Tr \, e^{-\beta H}$. In principle, one can determine the complete dynamics and behaviour of the system from the partition function - for example, intensive variables such as pressure, specific heat capacitance etc. can be obtained by differentiation of Z by extensive variables such as volume, internal energy, entropy etc.

For the field theoretic case, one can take the statistical mechanical approach for calculating finite temperature Green's functions in terms of thermal averages of τ ordered fields :

$$\langle \, T \, [\phi(\vec{x_1}, \tau_1) \, \phi(\vec{x_2}, \tau_2) \, \ldots] \, \rangle = \frac{Tr \, \left[e^{-\beta H} \, T \, [\phi(\vec{x_1}, \tau_1) \, \phi(\vec{x_2}, \tau_2) \, \ldots] \right]}{Tr \, e^{-\beta H}} \qquad (1.8)$$

where T is the τ ordering symbol. The periodicity properties of the fields can be deduced from the following : let $\phi_a, \phi_b, \ldots$ be any fields, either bosonic or fermionic, keeping in mind that $\tau_i \in [0, \beta]$

$$\begin{aligned} \langle \, T \, [\phi_a(\vec{x_1}, \beta) \, \phi_b(\vec{x_2}, \tau_2) \, \ldots] \, \rangle &= \frac{Tr \, \left[e^{-\beta H} \, \phi_a(\vec{x_1}, \beta) \, T \, [\phi_b(\vec{x_2}, \tau_2) \, \ldots] \right]}{Tr \, e^{-\beta H}} \\ &= \frac{Tr \, \left[T \, [\phi_b(\vec{x_2}, \tau_2) \, \ldots] \, e^{-\beta H} \, \phi_a(\vec{x_1}, \beta) \right]}{Tr \, e^{-\beta H}} \\ &= \frac{Tr \, \left[T \, [\phi_b(\vec{x_2}, \tau_2) \, \ldots] \, \phi_a(\vec{x_1}, 0) e^{-\beta H} \right]}{Tr \, e^{-\beta H}} \\ &= \pm \, \langle \, T \, [\phi_a(\vec{x_1}, 0) \, \phi_b(\vec{x_2}, \tau_2) \, \ldots] \, \rangle \qquad (1.9) \end{aligned}$$

where a $+(-)$ is contributed when ϕ_a is a boson (fermion) operator and we have used

$$e^{-\beta H}\,\phi_a(\vec{x_1},\tau)\,e^{\beta H} = \phi_a(\vec{x_1},\tau+\beta) \tag{1.10}$$

which is just the time translation of a field operator. The cyclic property of the trace has also been used.

The partition function can be derived by allowing the field integration $\int \mathcal{D}\phi$ to be carried over only those classical fields that have the same configuration at $\tau = \beta$ as at $\tau = 0$, i.e. the fields are periodic. So the partition function, Z, is

$$\begin{aligned} Tr\, e^{-\beta H} &= \sum_{\phi} < \phi | e^{-\beta H} | \phi > \\ &= N \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \, \exp\left\{ \int_0^{\beta} d\tau \int d^3x \left[i\,\pi\dot{\phi} - \mathcal{H}(\pi,\phi)\right]\right\}. \end{aligned} \tag{1.11}$$

In most cases of interest, the Hamiltonian density $\mathcal{H}$ is at worst quadratic in the conjugate momenta π - meaning that by completing the squares one can do the π integration. This causes a shift in π, with π being replaced by its value that gives the stationary point of the integrand and is given by

$$i\,\dot{\phi} = \frac{\partial\,\mathcal{H}(\pi,\phi)}{\partial\,\pi} \tag{1.12}$$

which is the prescription that one needs to convert from the Hamiltonian to the effective Lagrangian formalism. The effective Lagrangian $\mathcal{L}_{eff}$ is now a function of $\mathcal{L}(\phi, i\,\dot{\phi})$ where all τ derivatives in $\mathcal{L}$ are understood to be multiplied by i. The partition function now becomes

$$Tr\, e^{-\beta H} = N'(\beta) \int_{\text{periodic}} \mathcal{D}\phi \, \exp\left\{ \int_0^{\beta} d\tau \int d^3x \, \mathcal{L}_{eff}\left(\phi, i\,\dot{\phi}\right)\right\} \tag{1.13}$$

where $N'(\beta)$ is a new, infinite, normalisation constant. Its temperature dependence is due to the π functional integration. A similar infinite normalisation factor also arises in zero temperature field theory and is usually ignored.

Applying this to the case of non-gauge theories, one can obtain finite temperature Feynman rules. As usual, the quadratic part of the effective Lagrangian determines the propagators of the theory, whereas the non-quadratic parts are vertices and hence describe interactions. The change of variable (1.6) is just the well known 'Wick rotation' which converts the theory from Minkowski space to Euclidean space. Thus one does not need the $i\epsilon$ prescription to specify the poles of the propagators. The periodic properties of the field configurations, or equivalently, the range of the τ integration is restricted to $[0, \beta]$, means that from a Fourier transform point of view, energy summations rather than integrations are the norm of finite temperature field theory. For bosons, the energy is given by

$$k_0 \to \omega_n = \frac{2n\pi}{\beta} \tag{1.14}$$

whereas for fermions

$$k_0 \to \omega_n = \frac{(2n+1)\pi}{\beta} \tag{1.15}$$

and n is summed over the range $(-\infty, \infty)$.

Suppose we have a scalar field ϕ governed by the Lagrangian

$$\begin{aligned} \mathcal{L}(\phi, \dot{\phi}) &= \mathcal{L}_{eff}(\phi, \dot{\phi}) \\ &= \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \lambda\phi^4. \end{aligned} \tag{1.16}$$

Then (1.13) becomes

$$\begin{aligned} Tr\, e^{-\beta H} &= N'(\beta)\int_{\text{periodic}} \mathcal{D}\phi \\ &\cdot \exp\left\{\int_0^\beta d\tau \int d^3x \left(\frac{-1}{2}\left[(\partial_0\phi)^2 + (\partial_i\phi)(\partial_i\phi) + m^2\phi^2\right] - \lambda\phi^4\right)\right\}. \end{aligned} \tag{1.17}$$

Since ϕ is periodic in the interval $[0, \beta]$, one can expand the field in its Fourier series

$$\phi(\vec{x}, \tau) = \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \, e^{i\omega_n \tau} \, \phi_n\left(\vec{k}\right) \tag{1.18}$$

where

$$\phi_n\left(\vec{k}\right) = \int d^3x \int_0^\beta d\tau \, e^{-i\vec{k}\cdot\vec{x}} \, e^{-i\omega_{\bar{n}}\tau} \, \phi\left(\vec{x}, \tau\right) \tag{1.19}$$

and

$$\omega_n = \frac{2\pi n}{\beta}. \tag{1.20}$$

The time part of the δ function is given by

$$\int_0^\beta e^{i(\omega_n - \omega_{n'})\tau} = \beta \, \delta_{n\,n'} \tag{1.21}$$

and can be used to show that

$$\begin{aligned} S_0 &= \frac{-1}{2}\frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \left(\omega_n^2 + \vec{k}^2 + m^2\right) \phi_n(\vec{k}) \, \phi_{-n}(-\vec{k}) \\ &= \frac{-1}{2}\frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} D(k) \, \phi_n(\vec{k}) \, \phi_{-n}(-\vec{k}) \end{aligned} \tag{1.22}$$

which comes from the quadratic part of the action

$$S_0 = \int_0^\beta d\tau \int d^3x \left(\frac{-1}{2}\left[(\partial_0 \phi)^2 + (\partial_i \phi)(\partial_i \phi) + m^2\phi^2\right]\right) \tag{1.23}$$

and

$$D(k) = \omega_n^2 + \vec{k}^2 + m^2. \tag{1.24}$$

Thus the Feynman propagator Δ_F in the momentum representation is

$$\Delta_F(\omega_n, \vec{k}) = (D(k))^{-1} = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} \tag{1.25}$$

whereas its coordinate space representation is

$$\Delta_F(\vec{x}-\vec{x}',\tau-\tau') = \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}.(\vec{x}-\vec{x}')+i\omega_n(\tau-\tau')}\, \frac{1}{\omega_n^2+\vec{k}^2+m^2}. \tag{1.26}$$

Interactions can be included just as for the case of zero temperature field theory by expanding the exponential of the interaction term, $\lambda\phi^4$, - the result is perturbation theory. To do this, the following functional formula is used

$$\begin{aligned}&\int \mathcal{D}\phi\; e^{S_0}\, \phi(\vec{x}_1,\tau_1)\, \phi(\vec{x}_2,\tau_2)\, \phi(\vec{x}_3,\tau_3)\, \phi(\vec{x}_4,\tau_4)\, \ldots \\ &\quad = C\; (\det D)^{-1/2}\, \left\{ \underbrace{\phi(\vec{x}_1,\tau_1)\, \phi}(\vec{x}_2,\tau_2)\;\; \underbrace{\phi(\vec{x}_3,\tau_3)\, \phi}(\vec{x}_4,\tau_4) \right\} + \text{permutations}\end{aligned} \tag{1.27}$$

where the contraction of two fields is given by the Feynman propagator

$$\underbrace{\phi(\vec{x}_1,\tau_1)\, \phi}(\vec{x}_2,\tau_2) = \Delta_F(\vec{x}-\vec{x}',\tau-\tau'). \tag{1.28}$$

This means that one can use exactly the same set of diagrams to compute a thermal amplitude as one uses for zero temperature field theory. So, the finite temperature Feynman rules are those at zero temperature with the following modifications

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} &\rightarrow \frac{i}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \\ k_0 &\rightarrow i\,\omega_n \\ (2\pi)^4\,\delta^4(k_1+k_2+\ldots) &\rightarrow \frac{1}{i}(2\pi)^3\,\beta\,\delta_{\omega_{n_1}+\omega_{n_2}+\ldots}\,\delta^3(\vec{k_1}+\vec{k_2}+\ldots) \end{aligned} \tag{1.29}$$

where the factors of i come from performing the Wick rotation to Euclidean space and ω_n is given by (1.14) and (1.15).

The normalisation constant $N'(\beta)$ and $(\det D)^{-1/2}$ also need to be evaluated when calculating the partition function $Tre^{-\beta H}$. Consider free scalar field theory, whose Lagrangian is (1.16) with $\lambda = 0$. In such a case, (1.17) is a simple Gaussian which can be evaluated exactly using (1.27) (noting that there are no fields to contract)

$$\int \mathcal{D}\phi \, e^{S_0} = K \, (\det D)^{-1/2}. \tag{1.30}$$

Now, it is well known that

$$\begin{aligned} \ln Z = \ln Tr \, e^{-\beta H} &= \frac{-1}{2} \ln [\det D] + \ln N'(\beta) + K' \\ &= \frac{-1}{2} Tr \ln D + \ln N'(\beta) + K' \end{aligned} \tag{1.31}$$

where the determinant of an operator is expressed as the product of its eigenvalues, and K and K' are temperature independent constants which are not important to the calculation. Hence, using (1.24) one gets

$$\ln Z = \frac{-1}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln \left(\omega_n^2 + \vec{k}^2 + m^2\right) + \ln N'(\beta). \tag{1.32}$$

To perform the summation, we write

$$\sum_n \ln \left(\omega_n^2 + \vec{k}^2 + m^2\right) = \int_{1/\beta^2}^{\vec{k}^2+m^2} da^2 \sum_n \frac{1}{\omega_n^2 + a^2} + \sum_n \ln(\omega_n^2 + 1/\beta^2) \tag{1.33}$$

the lower limit $1/\beta^2$ has been chosen so as to give no β dependence to the final result.

The second term on the RHS of (1.33) is temperature dependent and infinite, but when the π and ϕ integrations are carried out, its contribution to $\ln Z$ is cancelled by $\ln N'(\beta)$ up to a β independent constant. Putting all this together one finds that $N'(\beta)$, $(\det D)^{-1/2}$ and all diagrams not connected by external lines are cancelled by the denominator of (1.8).

To evaluate the first term on the RHS of (1.33) one can use the Regge trick of introducing a function that has poles at $\omega = 2\pi n/\beta$ with residue 1 and choosing a contour in the complex ω plane which includes all the poles. The most suitable function is

$$\frac{1}{2}\beta \cot\left(\frac{1}{2}\beta\omega\right)$$

and a contour that can be continued into the upper and lower half regions of the complex plane is chosen which will give residues at the poles $\omega = \pm ia$. Bearing this in mind, the $\int da^2$ of (1.33) can then be performed. The partition function becomes

$$\begin{aligned} \ln Z &= \int \frac{d^3k}{(2\pi)^3} \ln\left(\csc\frac{\beta\omega_k}{2}\right) + \beta\,\text{independent constant} \\ &= \int \frac{d^3k}{(2\pi)^3} \left[\frac{-\beta\omega_k}{2} - \ln\left(1 - e^{-\beta\omega_k}\right)\right] \qquad (1.34) \end{aligned}$$

where $\omega_k^2 = \vec{k}^2 + m^2$.

Equation (1.34) is the partition function for an ideal Bose gas. Notice the zero point energy of the vacuum has been included (the first term of the last line of (1.34) - which is expected as using the functional method does not incorporate normal ordering.

1.3 Gauge Theories

Care must be exercised when the techniques outlined in the previous section are applied to gauge theories. Consider QED with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \qquad (1.35)$$

If the Coulomb gauge $\partial_i A_i = 0$ is chosen or an axial gauge, e.g. $A_0 = 0$ or $A_3 = 0$, and one calculates H, then the partition function, Z, is just that which describes a massless Bose gas with two degrees of freedom. However if one chooses the Lorentz gauge, $\partial_\mu A^\mu = 0$ with gauge parameter $\alpha = 1$ (the so-called Feynman gauge), and proceeds to evaluate the Hamiltonian H and the partition function $Z = Tr\, e^{-\beta H}$, the resulting theory describes a Bose gas with three positive and one negative metric states

$$\ln Tr\, e^{-\beta H}\Big|_{\text{Feynman gauge}} = 3\int \frac{d^3k}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2} - \ln\left(1-e^{-\beta\omega_k}\right)\right] + \int \frac{d^3k}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2} - \ln\left(1+e^{-\beta\omega_k}\right)\right] \quad (1.36)$$

where $\omega_k^2 = \vec{k}^2$. The results given by (1.36) are highly suspicious because they include the thermodynamics of spurious degrees of freedom arising from the fact we have chosen the Feynman gauge. In the Coulomb or axial gauge, the photon has two independent degrees of freedom, whereas in the Feynman gauge it has four degrees of freedom. The extra degrees come from the longitudinal and timelike photons, which in reality do not exist. The partition function given in (1.36) includes these two extra degrees of freedom.

The point of this exercise was to show that the partition function $Tr\, e^{-\beta H}$ is not necessarily a physically meaningful construct in all gauges. In some gauges spurious particles are wrongly included as physical degrees of freedom. Such particles are never in thermal equilibrium with the thermal heat bath.

Thus the partition function should be evaluated when a physical gauge is used - by a physical gauge we mean one which has the correct number of degrees of freedom. One can ask whether it is possible to get the same partition function in other, non-physical, gauges by modifying certain functional techniques. In short the answer is yes. To see this, consider a non-abelian gauge theory governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}\, F^{a\mu\nu} \quad (1.37)$$

where

$$F^a_{\mu\nu} = \partial_\mu\, A^a_\nu - \partial_\nu\, A^a_\mu + g\, f^{abc}\, A^b_\mu\, A^c_\nu \quad (1.38)$$

and f^{abc} are the structure constants of the group. Suppose we write the Hamiltonian in the axial gauge, $A_3 = 0$. In this case there are two degrees of freedom for each gauge field, A_1^a and A_2^a. In this gauge, the partition function is

$$\begin{aligned} Z = Tr\, e^{-\beta H}|_{\text{Axial gauge}} &= N \int \Pi_a \mathcal{D}P_1^a \mathcal{D}P_2^a \int_{\text{periodic}} \mathcal{D}A_1^a \mathcal{D}A_2^a \\ &\quad . \exp\left\{ \int_0^\beta d\tau \int d^3x \left[i\, P_j^a \dot{A}_j^a - \mathcal{H}\left(A_j^a, P_j^a\right)\right]\right\} \end{aligned} \tag{1.39}$$

where P_j^a are the conjugate momenta of A_j^a with $j = 1, 2$. The P_j^a integration can be done with the result

$$Z = [N'(\beta)]^{2n} \int_{\text{periodic}} \mathcal{D}A\, \Pi_a\, \delta\left(A_3^a\right) \exp\left[\int_0^\beta d\tau \int d^3x\, \mathcal{L}(A, i\,\dot{A})\right]. \tag{1.40}$$

Apart from the limits of the τ integration, this is the same result as one gets using the Faddeev-Popov method. It is understood that $\mathcal{D}A$ means functional integration over all components of the gauge field. The gauge condition is enforced by the δ functional and $a = 1, \ldots, n$. This can be expressed in a more general form

$$\begin{aligned} Z &= [N'(\beta)]^{m} \int_{\text{periodic}} \mathcal{D}A\, \mathcal{D}\phi\, \exp\left[\int_0^\beta d\tau \int d^3x\, \mathcal{L}(A, \phi, i\,\dot{A}, i\,\dot{\phi})\right] \\ &\quad . \det\left[\frac{\partial F^b}{\partial \omega^c}\right] \Pi_b\, \delta\left(F^b\right) \end{aligned} \tag{1.41}$$

where m denotes the total number of physical particles and polarisation states of the theory. The $\Pi_b\, \delta\left(F^b\right)$ term is the product of gauge fixing conditions which selects a surface in function space which corresponds to a physical gauge. The term $\det\left[\partial F^b / \partial \omega^c\right]$ is the Faddeev-Popov determinant and $\omega^c(x)$ are a set of functions that parametrise gauge transformations. For the case of the axial gauge, the determinant in (1.40) is

$$\det\left(\frac{\partial}{\partial x_3}\right) = \text{constant} \tag{1.42}$$

and is apparently temperature independent since temperature only appears through the x_0 coordinate.

Since we started with a gauge invariant Lagrangian, in principle (1.41) should give the same result for whatever gauge is chosen - including non-physical gauges, provided that the surface in function space formed by the gauge conditions $F^b(x) = 0$ intersects the orbit of any gauge field under gauge transformations only once. This ensures that only contributions from physically distinct fields are allowed for evaluation of the partition function- i.e. those fields that are not connected by a gauge transformation. So, one could choose an unphysical gauge $F^b(x) = 0$, but the partition function should still be the same as that computed in a physical gauge.

By specifying the choice of gauge and boundary conditions, one should be able in principle to compute the partition function. For the finite temperature case, this means the field configurations should vanish at spatial infinity and be periodic in the time coordinate $\phi(\vec{x}, x_0 + \beta) = \pm\phi(\vec{x}, x_0)$. Compare this to the zero temperature case where one has fields that vanish at spatial and temporal infinity.

Thus, it has to be borne in mind that Z given by (1.41) is the same in any gauge but that this is not the case for $Tr\, e^{-\beta H}$.

As an example, consider pure electrodynamics (1.35). Let

$$F\,(\vec{x}, \tau) = \partial_\mu\, A^\mu - f\,(\vec{x}, \tau) \tag{1.43}$$

where $f\,(\vec{x}, \tau)$ is an arbitrary but regular function, and $0 \leq \tau \leq \beta$, then the partition function, Z, in Euclidean space is

$$\begin{aligned} Z &= [N'(\beta)]^2 \int \mathcal{D}A \exp\left[\int_0^\beta d\tau \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)\right] \\ &\quad .\det\left(\frac{\delta\,(\partial_\mu\, A^\mu)}{\delta\,\omega}\right) \delta\,(\partial_\mu\, A^\mu - f) \end{aligned} \tag{1.44}$$

where it is noted that there are two $[N'(\beta)]$ factors - the same number as there are of physical states for an electromagnetic (massless) field. Under gauge transformations,

$$\delta A_\mu = -\partial_\mu \omega \qquad (1.45)$$

then

$$\det\left(\frac{\delta\,(\partial_\mu A^\mu)}{\delta\,\omega}\right) = \det\,(-\Box) \qquad (1.46)$$

where $\Box = \partial_\mu \partial^\mu$. As is the case for zero temperature field theory, (1.44) is actually independent of f, so when its RHS is multiplied by

$$\exp\left[-\frac{1}{2\alpha}\int_0^\beta \int d^3x\, f^2\right] \qquad (1.47)$$

and integration over $\mathcal{D}f$ carried out, one gets a β independent normalisation constant which is absorbed into $N'(\beta)$, leading to

$$Z = [N'(\beta)]^2 \det\,(-\Box) \int \mathcal{D}A \exp\left[\int_0^\beta d\tau \int d^3x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right]. \qquad (1.48)$$

At zero temperature, $\det\,(-\Box)$ is just a constant, however at finite temperature it is temperature dependent due to periodic boundary conditions and must be given attention. Choosing the Feynman gauge $\alpha = 1$

$$Z = [N'(\beta)]^2 \det\,(-\Box) \int \mathcal{D}A \exp\left[\int_0^\beta d\tau \int d^3x \left(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu\right)\right]. \qquad (1.49)$$

Performing the functional integration, there are four integrals to do - one for each μ and writing

$$\begin{aligned}\det\,(-\Box) &= \exp\left[Tr\,\ln(-\Box)\right] \\ &= \exp\sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left(\omega_n^2 + \vec{k}^2\right)\end{aligned} \qquad (1.50)$$

where the determinant is defined only on the space of periodic functions, Z becomes

$$\begin{aligned} \ln Z &= 2 \ln N'(\beta) - \sum_n \int \frac{d^3k}{(2\pi)^3} \ln \left(\omega_n^2 + \vec{k}^2 \right) \\ &= 2 \int \frac{d^3k}{(2\pi)^3} \ln \left[-\frac{\beta\omega_k}{2} - \ln \left(1 - e^{-\beta\omega_k} \right) \right] \end{aligned} \tag{1.51}$$

where $\omega_k = \left(\vec{k}^2\right)^{1/2}$. Thus (1.51) now correctly describes a zero mass Bose gas with two polarisation states. This result could not have been arrived at without the assistance of the Faddeev-Popov determinant.

Clearly, for the case of non-abelian gauge theories, the Faddeev-Popov determinant would be more complicated and ghosts could (in some gauges) show their presence. This presents no problem provided that ghosts have Bose-Einstein type Boltzmann factors, the same as photons and gluons. Nevertheless the same principles are involved in the computation of the partition function - or more specifically Z, albeit it will be more complicated.

1.4 Finite Temperature Green's Functions

Converting the results of the previous sections of this chapter, which is a presentation of field theory in the imaginary time formalism, to the real time formalism is non trivial. Dolan and Jackiw [DJ74] have explored how this can undertaken, a summary of which will be presented, using two-point functions as the starting point.

Consider a spin-0 field. The finite temperature two-point function (Green's function) is

$$\begin{aligned} D_\beta(x-y) &= \frac{Tr e^{-\beta H}\, T\left\{\phi(x)\,\phi(y)\right\}}{Tr e^{-\beta H}} \\ &= < T\ \left\{\phi(x)\,\phi(y)\right\} > \end{aligned} \tag{1.52}$$

which for non interacting fields, satisfies

$$\left(\Box_x + m^2\right) D_\beta(x-y) = -i\,\delta(x-y) \tag{1.53}$$

for a given set of boundary conditions. Using the imaginary time formalism (ITF) and letting $0 \le ix_0,\, iy_0 \le \beta$:

$$\begin{aligned} < T\,\{\phi(x)\,\phi(y)\} > &= <\phi(x)\,\phi(y)> \\ &= D^{>}_{\beta}(x-y), \qquad ix_0 \ge iy_0 \\ &= <\phi(y)\,\phi(x)> \\ &= D^{<}_{\beta}(x-y), \qquad iy_0 \ge ix_0. \end{aligned} \tag{1.54}$$

For $ix_0, iy_0 \in [0,\beta]$ the propagator has the property

$$\begin{aligned} D_\beta(x-y)|_{x_0=0} &= D^{<}_{\beta}(x-y)|_{x_0=0} \\ D_\beta(x-y)|_{x_0=-i\beta} &= D^{>}_{\beta}(x-y)|_{x_0=-i\beta} \end{aligned} \tag{1.55}$$

and by using the cyclic property of the trace as well as changes of the field variables under time translations (as outlined in the previous section), one concludes that

$$D_\beta(x-y)|_{x_0=0} = D_\beta(x-y)|_{x_0=-i\beta}. \tag{1.56}$$

The Fourier transform of (1.56), in the ITF, would be just that given in the previous section. However, the propagator can be expressed in terms of real time Fourier integrals. Let

$$\bar{D}^{\{>,<\}}_{\beta}(k) = \int d^4x\, e^{ikx}\, D^{\{>,<\}}_{\beta}(x) \tag{1.57}$$

where the bar denotes Fourier integral transform (as opposed to Fourier sum). Now, writing $\bar{D}_\beta(k)$ for $\bar{D}_\beta(k_0, \vec{k})$, $\rho(k)$ for $\rho(k_0, \vec{k}$ etc., one has

$$
\begin{aligned}
\bar{D}^{<}_{\beta}(k) &= \int d^4x\, e^{i\left(k_0 x_0 - \vec{k}\cdot\vec{x}\right)}\, D^{<}_{\beta}(x_0, \vec{x}) \\
&= \int d^4x\, e^{i\left(k_0 x_0 - \vec{k}\cdot\vec{x}\right)}\, D^{>}_{\beta}(x_0 - i\beta, \vec{x}) \\
&= e^{-\beta k_0} \int d^4x\, e^{ikx}\, D^{>}_{\beta}(x) \\
&= e^{-\beta k_0}\, \bar{D}^{>}_{\beta}(k) \qquad (1.58)
\end{aligned}
$$

where (1.55) and (1.56) have been used.

Equation (1.58) can be rearranged by allowing

$$
\begin{aligned}
\bar{D}^{>}_{\beta}(k) &= [1 + f(k_0)]\, \rho(k) \\
\bar{D}^{<}_{\beta}(k) &= f(k_0)\, \rho(k) \\
f(k_0) &= \frac{1}{e^{\beta k_0} - 1} \\
\rho(k) &= \bar{D}^{>}_{\beta}(k) - \bar{D}^{<}_{\beta}(k). \qquad (1.59)
\end{aligned}
$$

The spectral function $\rho(k)$ defines possible energies for an excitation of momentum $\vec{k}$. The full propagator can be written as

$$
\begin{aligned}
\bar{D}_\beta(k) &= \int d^4x\, e^{ikx} \left[\Theta(x_0) D^{>}_{\beta}(x) + \Theta(-x_0) D^{<}_{\beta}(x)\right] \\
&= i \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi} \left[\frac{\bar{D}^{>}_{\beta}(k_0', \vec{k})}{k_0 - k_0' + i\epsilon} - \frac{\bar{D}^{<}_{\beta}(k_0', \vec{k})}{k_0 - k_0' - i\epsilon}\right] \\
&= i \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi}\, \rho(k_0', \vec{k}) \left[\frac{1 + f(k_0')}{k_0 - k_0' + i\epsilon} - \frac{f(k_0')}{k_0 - k_0' - i\epsilon}\right] \\
&= i \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi}\, \frac{\rho(k_0', \vec{k})}{k_0 - k_0' + i\epsilon} + f(k_0)\, \rho(k). \qquad (1.60)
\end{aligned}
$$

The spectral function can be derived with the assistance of the ITF form of the propagator $D_\beta(k)$

$$D_\beta(\omega_n, \vec{k}) = \int_0^{-i\beta} dx_0\, e^{-2\pi n x_0/\beta} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, D_\beta(x) \tag{1.61}$$

and noting the boundary conditions (1.55) and (1.56) for $x_0 \in [0, -i\beta]$, then

$$\begin{aligned} D_\beta(\omega_n, \vec{k}) &= \int_0^{-i\beta} dx_0\, e^{-2\pi n x_0/\beta} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, D_\beta^{>}(x) \\ &= \int_0^{-i\beta} dx_0\, e^{-2\pi n x_0/\beta} \int_{-\infty}^{\infty} d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \bar{D}_\beta^{>}(x) \\ &= i\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{e^{-\beta k_0} - 1}{k_0 - 2\pi n/(-i\beta)} \left[1 + f(k_0)\right] \rho(k) \\ &= i\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\rho(k_0, \vec{k})}{\omega_n - k_0}. \end{aligned} \tag{1.62}$$

We now extend $D_\beta(\omega_n, \vec{k})$ to a continuous function $D_\beta(k_0, \vec{k})$ and the spectral function will be given by

$$\begin{aligned} \rho(k) &= D_\beta^{>}(k) - D_\beta^{<}(k) \\ &= D_\beta(k_0 + i\epsilon, \vec{k}) - D_\beta(k_0 - i\epsilon, \vec{k}) \end{aligned} \tag{1.63}$$

which in the free field case will be given by

$$\rho(k) = 2\pi\, \epsilon(k_0)\, \delta(k^2 - m^2). \tag{1.64}$$

Thus the free scalar propagator in the real time formalism (RTF) is

$$\bar{D}_\beta(k) = \frac{i}{k^2 - m^2 + i\epsilon} + \frac{2\pi}{e^{\beta E} - 1}\, \delta(k^2 - m^2) \tag{1.65}$$

and satisfies the momentum representation of (1.53)

$$\left(k^2 - m^2\right) \bar{D}_\beta(k) = i \tag{1.66}$$

where $E = (\vec{k}^2 + m^2)^{1/2}$ is the energy and the Feynman prescription $i\epsilon$, is assumed when dealing with the poles in the first term on the RHS of (1.65). Compare this with the ITF form of the propagator

$$D_\beta(k) = \frac{i}{k^2 - m^2} \tag{1.67}$$

For fermions the situation is very similar, although one must now take into account the fact that fermion fields satisfy anti-commutation relations. A very similar procedure is used to derived the RTF form of the fermion propagator $S_\beta(x-y)$

$$\begin{aligned} S_\beta(x-y) &= \frac{Tr\, e^{-\beta H}\, T\, \Psi(x)\bar{\Psi}(y)}{Tr\, e^{-\beta H}} \\ &= < T\, \Psi(x)\bar{\Psi}(y) > \end{aligned} \tag{1.68}$$

which satisfies the following equation

$$(i\, \partial\!\!\!/ - m)\, S_\beta(x-y) = i\, \delta(x-y) \tag{1.69}$$

but with boundary conditions

$$\begin{aligned} S_\beta(x-y)_{x_0=0} &= S_\beta^<(x-y)_{x_0=0} \\ S_\beta(x-y)_{x_0=-i\beta} &= S_\beta^>(x-y)_{x_0=-i\beta} \\ S_\beta(x-y)_{x_0=0} &= -S_\beta(x-y)_{x_0=-i\beta} \end{aligned} \tag{1.70}$$

since in the interval $[0, -i\beta]$

$$\begin{aligned} < T\, \Psi(x)\bar{\Psi}(y) > &= < \Psi(x)\bar{\Psi}(y) > \\ &= S_\beta^>(x-y) \qquad ix_0 \geq iy_0 \\ &= - < \bar{\Psi}(y)\Psi(x) > \end{aligned}$$

$$= \quad S^{<}_{\beta}(x-y) \qquad iy_0 \geq ix_0. \tag{1.71}$$

As for the scalar case, one can write real time Fourier integrals for the fermion propagator

$$\bar{S}^{\{>,<\}}_{\beta}(k) = \int d^4x \, e^{ikx} \, S^{\{>,<\}}_{\beta}(x) \tag{1.72}$$

where we can write

$$\begin{aligned}
\bar{S}^{>}_{\beta}(k) &= [1 - f(k_0)] \, \rho(k) \\
\bar{S}^{<}_{\beta}(k) &= f(k_0) \, \rho(k) \\
f(k_0) &= \frac{1}{e^{\beta k_0} + 1} \\
\rho(k) &= \bar{S}^{>}_{\beta}(k) + \bar{S}^{<}_{\beta}(k).
\end{aligned} \tag{1.73}$$

The spectral function $\rho(k)$ for the fermion is obtained from the ITF propagator

$$\begin{aligned}
S_{\beta}(\omega_n, \vec{k}) &= \int_0^{-i\beta} dx_0 \, e^{-(2n+1)\pi x_0/\beta} \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \, S^{>}_{\beta}(x) \\
&= i \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \, \frac{\rho(k_0, \vec{k})}{\omega_n - k_0}.
\end{aligned} \tag{1.74}$$

where

$$\rho(k_0, \vec{k}) = S_{\beta}(k_0 + i\epsilon, \vec{k}) - S_{\beta}(k_0 - i\epsilon, \vec{k}) \tag{1.75}$$

and the RTF propagator is determined from the spectral function :

$$\begin{aligned}
\bar{S}_{\beta}(k) &= \int d^4x \, e^{ikx} \left[\Theta(x_0) S^{>}_{\beta}(x) - \Theta(-x_0) S^{<}_{\beta}(x)\right] \\
&= i \int_{-\infty}^{\infty} \frac{dk'_0}{2\pi} \, \frac{\rho(k'_0, \vec{k})}{k_0 - k'_0 + i\epsilon} - f(k_0) \, \rho(k)
\end{aligned} \tag{1.76}$$

where for non-interacting fields (with $\not{k} = k_{\mu}\gamma^{\mu}$)

$$\rho(k) \quad = \quad 2\pi \, \epsilon(k_0) \, (\not{k} + m) \, \delta(k^2 - m^2)$$

$$\bar{S}_\beta(k) = \frac{i}{\not{k} - m + i\epsilon} - \frac{2\pi(\not{k}+m)}{e^{\beta E}+1}\delta(k^2 - m^2). \tag{1.77}$$

1.5 Examples

Having presented some of the mathematical machinery of finite temperature field theory in the previous sections, let us present some situations in which finite temperature effects can alter the behaviour of a field theory.

As examples, we shall examine how finite temperature affects the following :

- Spontaneous symmetry breaking.
- Properties of the A_0 component of the gauge field in Wilson lines.
- Cross-sections and widths of any process.

Consider the first case of spontaneous symmetry breaking. Consider a scalar field ϕ described by (1.16). The objects of interest are the effective potential energy V_{eff} which combines V (the zero-loop effective potential), quantum effects and finite temperature effects and where V is

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{1.78}$$

where for $m^2 \geq 0$, the minimum of the potential is

$$V = 0 \qquad \text{at} \qquad \phi = <\phi> = 0. \tag{1.79}$$

The vacuum expectation value (VEV) of ϕ, $<\phi>$, is the value of ϕ which gives the minimum of the potential V. In this case there is no spontaneous symmetry breaking - at the classical level. Spontaneous symmetry breaking can arise due to quantum effects as shown by Coleman and E. Weinberg [CW73].

However, suppose

$$V(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{1.80}$$

and $m^2 > 0$, minima of the potential will now be at some non-zero value of ϕ which can be determined by calculating $\partial V_{eff}/\partial\phi$. The VEV of ϕ for this case is

$$<\phi>= \pm\frac{\sqrt{6}m}{\sqrt{\lambda}} \tag{1.81}$$

and spontaneous symmetry breaking will occur. So, if neither quantum or finite temperature effects are considered, then $V_{eff}(\phi) = V(\phi)$, given by (1.80).

The one-loop effective potential at finite temperature can be obtained by summing all one-particle irreducible diagrams with any number of external legs and in the imaginary time formalism is given by [DJ74,We74,Ka89]:

$$\begin{aligned} V_{eff}(\phi,T)^{1\,\mathrm{loop}} &= \frac{T}{2}\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\left(k^2 - M^2\right) \\ &= \frac{T}{2}\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\left(-4\pi n^2T^2 - E_M^2\right) \\ &= \int\frac{d^3k}{(2\pi)^3}\left[\frac{E_M}{2} + T\ln\left(1 - e^{-\beta E_M}\right)\right] \end{aligned} \tag{1.82}$$

where $M^2 = -m^2 + \lambda\phi^2/2$ and $E_M^2 = \vec{k}^2 + M^2$. Equation (1.82) is actually infinite - due to the first term in the square brackets in the last line. However, the divergence is independent of temperature and can be removed by renormalisation procedures at zero temperature.

Thus, the renormalised effective potential to one-loop order is

$$V_{eff}(\phi,T) = V(\phi) + V_{eff}^{1\,\mathrm{loop}}(\phi,T) \tag{1.83}$$

where $V(\phi)$ is given by (1.80). If $T^2 \sim M^2/\lambda \gg M^2$, then the high temperature

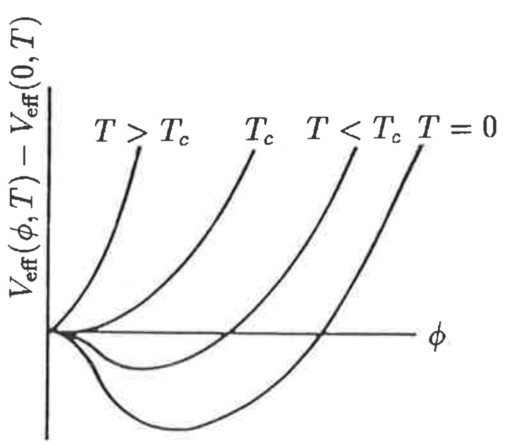


Figure 1.1: The effective potential $V_{eff}(\phi, T) - V_{eff}(0, T)$ for various temperatures.

expansion of (1.83) to order T^4 is

$$V_{eff}(\phi, T) = \frac{\lambda}{4!}\phi^4 + \left(\frac{\lambda T^2}{48} - \frac{m^2}{2}\right)\phi^2 - \frac{\pi^2 T^4}{90} - \frac{m^2 T^2}{24}. \tag{1.84}$$

The shape of $V_{eff}(\phi, T) - V_{eff}(0, T)$ for various temperatures is given in Fig.[1.1]. Note that for $T = 0$ one gets spontaneous symmetry breaking since the minimum of the potential occurs at a non-zero VEV of the scalar field, but as the temperature is increased the depth of the minima decreases and $< \phi >$ approaches zero. Above a certain temperature, the critical temperature T_c, the minimum of the potential is at $< \phi >= 0$ and symmetry is restored.

Applications of this are immediate. The expanding (cooling) of the Universe can induce phase transitions due to the temperature dependence of the effective potential for scalar fields - assuming such fields exist. Particles that were originally massless can attain masses after a phase transition via the Higgs mechanism. The Higgs field is coupled to gauge, fermion or any other fields and the effective Higgs potential is similar to that of (1.82) where, of course, one can include contributions from other fields. Masses of particles can be generated by spontaneous symmetry breaking and are related to the value of $< \phi >$.

Now consider Wilson lines [GJO80,GPY81,Ac84,Man85], which can introduce new properties for the gauge fields. Wilson loops are used to study aspects of confinement. In the ITF, the Euclidean space has cylindrical topology since the fields have periodic boundary conditions. As such, a closed contour cannot necessarily be deformed to a point. Consider the gauge invariant construct $Tr\,\Omega(\vec{x})$, where

$$\Omega(\vec{x}) = P\,\exp\left(i\int_0^\beta dx_0\,A_0(x_0,\vec{x})\right) \tag{1.85}$$

and P is the path ordering symbol and $\beta = T^{-1}$. The free energy of two charges located, at $\vec{x_a}$ and $\vec{x_b}$, is related to the logarithm of the expectation value correlation function

$$\left\langle Tr\,(\Omega^\dagger(\vec{x_b}))\,Tr\,(\Omega(\vec{x_a}))\right\rangle \;\sim\; e^{-K|\vec{x_a}-\vec{x_b}|} \tag{1.86}$$

with the constant K being proportional to temperature. Suppose a Lagrangian $\mathcal{L}$ describes the interaction of gauge fields $A_\mu(x)$ and matter fields $\phi(x)$, the full theory can be summarised by the generating functional

$$Z = \int_{\text{periodic}} \mathcal{D}A_\mu\,\mathcal{D}\phi\,\exp\left(-\int_0^\beta dx_0\int d^3x\,\mathcal{L}(A_\mu,\phi)\right). \tag{1.87}$$

The generating functional Z can be rewritten as [GJO80,RT80] :

$$Z = \int \mathcal{D}\Omega(\vec{x})\,\exp\left(-S_{\text{eff}}(\Omega)\right) \tag{1.88}$$

where $\Omega(x)$ is given by (1.85) and S_{eff} is defined as

$$S_{\text{eff}} = -\ln\int'_{\text{periodic}} \mathcal{D}A_\mu\,\mathcal{D}\phi\,\exp\left(-\int_0^\beta dx_0\int d^3x\,\mathcal{L}(A_\mu,\phi)\right). \tag{1.89}$$

Here, the prime over the functional integral $\int'$ means that the integral is over fields that have the following boundary conditions

$$\begin{aligned} A_i(\beta,\vec{x}) &= A_i^{\Omega}(0,\vec{x}) \equiv \Omega^{-1}(\vec{x})\,A_i(0,\vec{x})\,\Omega(\vec{x}) + \Omega^{-1}(\vec{x})\,\partial_i\,\Omega(\vec{x}) \\ \phi(\beta,\vec{x}) &= \phi^{\Omega}(0,\vec{x}) \equiv \Omega(\vec{x})\,\phi(0,\vec{x}). \end{aligned} \tag{1.90}$$

In other words, the fields at end points of the interval $0 \leq x_0 \leq \beta$ are gauge equivalent.

To choose the gauge condition $A_0 = 0$, a gauge transformation $U(x_0,\vec{x})$ is chosen such that

$$A_0 \rightarrow U^{-1}\,A_0\,U + U^{-1}\,\partial_0\,U = 0 \tag{1.91}$$

where

$$U(x_0,\vec{x}) = P\,\exp\left(i\int_0^{x_0} dx_0'\,A_0(x_0',\vec{x})\right), \tag{1.92}$$

Even after choosing the gauge $A_0 = 0$, the effective action S_{eff} is still invariant when

$$A_0(x) \rightarrow A_0(x) + 2\pi T x_0\, n(\vec{x}) \tag{1.93}$$

which are generated by gauge transformations $U = \exp[ix_0 2\pi T n(\vec{x})]$ where $n(\vec{x})$ is a static integer-valued field. Thus, as Gava et. al. [GJO80] and Rossi and Testa [RT80] point out, one cannot eliminate A_0 completely even though one started with the $A_0 = 0$ gauge.

Note that (1.93) suggests if one compactifies a coordinate, such as x_0 for finite temperature field theory, then the corresponding component of the gauge field A_0 can only take values in the range $-\pi\,T \leq A_0 \leq \pi\,T$, i.e. A_0 is also compact.

Finally, let us examine how temperature affects cross sections or widths of processes, using the $\pi^0 \rightarrow 2\gamma$ decay width as an example [CL88]. For $T = 0$, the pion decay width is

$$\Gamma = (2\pi)^4 \int \frac{d^3k_a}{(2\pi)^3 2k_a^0} \frac{d^3k_b}{(2\pi)^3 2k_b^0} \frac{1}{q_0} \delta^{(4)}(q - k_a - k_b) \sum_{\text{polns}} \left|\mathcal{M}(\pi^0 \to 2\gamma)\right|^2 \quad (1.94)$$

where

$$\sum_{\text{polns}} \left|\mathcal{M}(\pi^0 \to 2\gamma)\right|^2 = \frac{1}{2}\left(\frac{e^2}{4\pi f_\pi}\right)^2 m_\pi^4 \quad (1.95)$$

and a and b label the outgoing photons. After the momentum integration is carried out, we find that

$$\Gamma = \frac{\alpha^2}{64\pi^3}\frac{m_\pi^3}{f_\pi^2} \qquad \text{where} \qquad \alpha = \frac{e^2}{4\pi}. \quad (1.96)$$

Comparing the theoretical result to experiment one gets

$$\Gamma_{\text{theory}} = 8.5 \text{ eV} \qquad\qquad \Gamma_{\text{expt.}} = 7.95 \pm 0.55 \text{ eV}. \quad (1.97)$$

For $T \neq 0$, several things can happen. Finite temperature effects can appear through :

- The amplitude - similar to the amplitude calculated above for spontaneous symmetry breaking. For the case of the $\pi^0 \to 2\gamma$ width, since the anomaly is temperature independent, no finite temperature effects are expected from this sector. However, finite temperature effects appear in another part of the amplitude and contribute to the decay width [CL88].

- Parameters such as mass and coupling constants can depend on temperature. For the pion decay, the temperature dependence of the pion mass m_π^T and pion decay constant f_π^T in the high temperature limit are [LS90,GoL89]

$$\begin{aligned} m_\pi^T &= m_\pi\left(1 + \frac{T^2}{48 f_\pi^2} + \ldots\right) \\ f_\pi^T &= f_\pi\left(1 - \frac{T^2}{12 f_\pi^2} + \ldots\right). \end{aligned} \quad (1.98)$$

where m_π and f_π are $T = 0$ values.

- Phase-space factors of the outgoing photons are modified to :

$$\frac{d^3k_a}{(2\pi)^3 2k_a^0} \to \frac{d^3k_a}{(2\pi)^3 2k_a^0} \frac{1}{1 + n(\beta k_a^0)} \tag{1.99}$$

where the Boltzmann factor is

$$n(\beta k_a^0) = \frac{1}{e^{\beta k_a^0} \mp 1}. \tag{1.100}$$

As usual, the upper sign refers to bosons and the lower to fermions.

Taking into account the finite temperature effects of phase space factors, coupling constants and mass, the contribution of the anomaly to the decay width becomes

$$\Gamma^T = \frac{\alpha^2}{64\pi^3} \frac{\left(m_\pi^T\right)^3}{\left(f_\pi^T\right)^2} \frac{1}{\left(1 - e^{-m_\pi/(2T)}\right)^2}. \tag{1.101}$$

For $T \sim m_\pi$

$$\Gamma^{T \sim m_\pi} = 6.5\,\Gamma \tag{1.102}$$

i.e. width increases with temperature, or equivalently the pions lifetime decreases. For low temperature $T < m_\pi$,

$$\Gamma^T = \frac{\Gamma^{T=0}}{\left(1 - e^{-m_\pi/(2T)}\right)^2} \left(1 + \frac{7}{48}\frac{T^2}{f_\pi^2} + \dots\right) \geq \Gamma^{T=0} \tag{1.103}$$

where $\Gamma^{T=0} = \Gamma$. Thus it appears that the pion lifetime diminishes at finite temperature.

1.6 Conclusion

The functional approach to finite temperature field theory is perhaps one of the more transparent ways of examining finite temperature behaviour of quantum field theories. As we have seen, when gauge theories are analysed, care must be taken when dealing with spurious degrees of freedom of the gauge fields. The same situation occurs for the case of $T = 0$ field theory. The perturbative Feynman rules can then be obtained in either the real or imaginary time formalism.

It is interesting to note how temperature appears in either the real or imaginary formalisms. In the imaginary case, temperature appears through the periodic properties of the fields, and in the fact that energy summations rather than integrations are carried out. For the real time case, finite temperature effects appear through distributions, delta-functions or derivatives thereof, that contribute only when the particle is on mass shell. In the real time situation, one can easily see where finite temperature effects occur since they can be separated into a $T = 0$ part and a finite temperature part. However, in the imaginary time case, the temperature independent part is related to the $n = 0$ term in the energy summation $\sum_n$, but to separate this term out and then perform the summation can be tricky.

Converting the imaginary time results to the real time formalism cannot be achieved by simple analytic continuation. In the imaginary time formalism energies are discrete, and must be converted to a continuum before computing the spectral function which is related to the discontinuity of the imaginary time propagator across the real axis in the complex energy plane. The real time propagators are then related to the spectral functions.

We have seen that cross-sections and decay-widths can be affected at finite temperature through phase space factors of the outgoing photons, temperature dependence of coupling constants, masses and amplitudes.

When one restricts the range, or compactifies, a coordinate x_0, as in the case of finite temperature field theory in the imaginary time formalism, the range of values for the corresponding component of the gauge field A_0 is also compact. When $T \longrightarrow 0$, the range of A_0 is unrestricted.

Having laid the foundations for calculating amplitudes at finite temperature and where finite temperature effects are likely to manifest themselves, we can now apply some of the methods outlined in this chapter to cases of interest.

We begin by looking at the gluon or spin-1 propagator in the real time formalism.

Chapter 2

The Spin-1 Propagator

2.1 Introduction

In this chapter we will use the conventional methods of perturbation theory in the momentum representation - specifically, the real time formalism. The real time formalism (RTF) approach to quantum field theory at finite temperature involves different techniques compared to those used in the imaginary time formalism (ITF). With the ITF one encounters energy summations rather than integrations which can be very difficult, in some cases intractable, to perform beyond one loop diagrams. With the RTF it is possible to separate the amplitude into a temperature dependent and a temperature independent part. It is not as easy to do such a procedure when using the ITF.

The RTF is plagued by its own problems - some of which will be examined in this chapter and discussed in the next chapter. Specifically, in the RTF one encounters δ functions and products of such functions and their derivatives. This may lead to ambiguities for products of δ functions for a particular set of values of momenta for the internal and external lines of a Feynman diagram. These problems are well known throughout the literature and were pointed out by Dolan and Jackiw [DJ74]. Some methods and theories, such as thermo-field theory [UMT82], have been put forward to deal with them.

Obviously, one hopes that whatever formalism one chooses, the same answers should be obtained. This is not always the case. This can be due to various causes – a major one being the lack of finding, if at all possible, a suitable prescription for dealing with products of two or more δ functions in the RTF, say,

$$\delta(k^2)\delta\left((k+p)^2\right) \tag{2.1}$$

when $\lim p_\mu \to 0$ is taken. In the context of

$$\lim_{p_\mu \to 0} \int_{-\infty}^{\infty} dk_0 \delta(k^2)\delta((k+p)^2) f(k) \sim \delta(0) f(0), \tag{2.2}$$

this is undefined in the sense of normal functions, where p_μ and k_μ are momentum vectors.

At $T = 0$, perturbation theory can be tedious when calculating higher order diagrams, particularly those involving loops. At finite temperature, $T \neq 0$ in the RTF, propagators contain extra terms displaying finite temperature effects, making perturbation theory even more tedious.

In the context of Yang-Mills gauge theories, many calculations are carried out in the Feynman gauge $\alpha = 1$, which in some cases makes it difficult to keep track of gauge dependent terms. Any physical quantities which are evaluated should not depend on the choice of gauge. Clearly, in principle, by making an astute selection of the gauge fixing term, it can greatly assist the ease of calculation of amplitudes containing gauge boson propagators. When results are compared to those derived by other methods, such as those obtained using the ITF or using the RTF but in different gauges, or specific values for the gauge parameter, some discrepancies arise. Thus, it would be convenient to consider the case where not only a gauge fixing term is chosen, but also keeping the gauge parameter arbitrary. This will assist in keeping track of gauge dependent terms, possibly allowing some insight into how gauge dependent and finite temperature effects are related.

As an example to illustrate some of the techniques and problems involved, we shall examine aspects of QCD - $SU(3)$ pure Yang-Mills gauge theory. An important and interesting effect to investigate is the finite temperature dependence of the QCD coupling constant g. Questions asked are :

- To what extent does the (renormalised) coupling constant depend on temperature, i.e. does it become larger or smaller as temperature is increased?
- Do you get asymptotic freedom at high temperatures, i.e. what does the finite temperature β_T function look like?
- Are there any new processes or phenomena incurred by the presence of finite temperature effects?

Baier et. al. [BPS91] and references therein, have given a summary of the various calculations that have been done by many people of the renormalised coupling constant g_R, in finite temperature QCD at one loop order, using the ITF, RTF, and in various gauges. The main object of interest is the thermal β function β_T, defined by

$$\beta_T = T\frac{d}{dT}\left(\frac{g_R(T)^2}{4\pi^2}\right) \tag{2.3}$$

which at one loop order becomes

$$\beta_T = -c(T)\left(\frac{g_R(T)^2}{4\pi^2}\right)^2 \tag{2.4}$$

where

$$g_R(T) = \left(\frac{Z_3^{3/2}(T)}{Z_1(T)}\right) g \tag{2.5}$$

is the running coupling constant, with Z_3 and Z_1 being the gluon field and three-gluon vertex renormalisation constants respectively and $c(T)$ is a temperature dependent coefficient which depends on which gauge is chosen.

One may consider that since the renormalised coupling constant is a physical quantity, it should be gauge-independent.

For $T = 0$, the renormalisation constants (or counterterms) needed to remove the divergences from loop corrections to the three-gluon, four-gluon, ghost-gluon or quark-gluon vertices are not independent - they satisfy constraints called the Slavnov-Taylor identities [Mu87]. It is these identities that ensure the universality of the renormalised coupling constant - i.e. whatever vertex one uses to compute the renormalised coupling constant, the same physical result is obtained if a different vertex is chosen. It should be kept in mind that there is a renormalisation-prescription dependence of the renormalised coupling constant and different prescriptions are related by the renormalisation group equations [CG79].

For $T \neq 0$, the situation is somewhat perplexing. The general understanding of perturbative QCD at finite temperature can be inferred from knowledge of QCD at $T = 0$, particularly the renormalisation group equations, since no new divergences arise from finite temperature effects. One could identify the renormalisation scale Λ_{QCD}, with T in the following way

$$g_R^2(T) \sim \left(\frac{11N}{6} ln\left(T/\Lambda_{QCD}\right)\right)^{-1} \tag{2.6}$$

where N is the number of colours.

However, if a renormalisation scheme at finite temperature is used, then in the RFT, rather than getting the behaviour given by (2.6), one gets, for some scale M,

$$g_R^2(T) \sim \left(\frac{M}{T}\right)^3 \tag{2.7}$$

whereas in the IFT, one obtains

$$g_R^2(T) \sim \frac{M}{T} \tag{2.8}$$

or in some cases g_R may increase with temperature. The main reason for this being the different number of Boltzmann factors one obtains when using either the RTF or ITF.

For the case of finite temperature QCD, high temperature expansions of the coefficient $c(T)$, in various gauges, using different formalisms have been derived and can be summarised as follows [BPS91] :

1. Using RTF (but doubling the number of fields), the three-gluon vertex and the symmetric momentum configuration ($p_0 = q_0 = r_0 = 0, p^2 = q^2 = r^2 = -M^2$) :

$$c(T) \sim \frac{25\pi^2 N}{4}\left(\frac{T}{M}\right)^3. \tag{2.9}$$

2. As above, but using Feynman gauge and the collinear momentum configuration $p = (0,0,0,M) = -q/2 = r$:

$$c(T) \sim \frac{-35\pi^2 N}{24}\left(\frac{T}{M}\right). \tag{2.10}$$

3. Using the thermal Wilson loop (gauge-invariant construct) in the RTF :

$$c(T) \sim \frac{32\pi^2 N}{3}\left(\frac{T}{M}\right)^2. \tag{2.11}$$

4. Using the three-gluon vertex in ITF, symmetric momentum configuration and Feynman gauge :

$$c(T) \sim \frac{-31\pi^2 N}{18}\left(\frac{T}{M}\right). \tag{2.12}$$

Evidently a similar result is obtained using ITF and an axial gauge.

5. A gauge-invariant coupling constant has been derived [La89]:

$$c(T) \sim \frac{-21\pi^2 N}{16}\left(\frac{T}{M}\right). \tag{2.13}$$

As it can be seen, in some cases the β_T function will be positive, i.e. the running coupling constant, g_R, increases with temperature. This is against conventional wisdom, where it is generally expected that asymptotic freedom will result as the temperature of the system is increased.

It would be of interest to see how gauge dependent terms affect amplitudes as well as the β_T function at finite temperature. If we choose the Lorentz gauge, $\partial_\mu A^\mu = 0$, but keep the gauge parameter arbitrary, rather than choosing the Feynman gauge, $\alpha = 1$, then one needs to generalise the propagator given by the authors of ref. [BPS91] to display the gauge dependence of the gluon (spin-1) propagator at finite temperature. This is carried out in the next section.

The new propagator is then used to calculate the one loop correction to the tri-gluon, four-gluon and the quark-gluon vertices. Due to the numerous terms that can arise in such a calculation, it became necessary to use a computer program to handle the unwieldy expressions. The symbolic or algebraic manipulation package *Mathematica* [Wolf] was used. By writing a series of rules to perform algebraic operations it was possible to use *Mathematica* for a major portion of the calculation of amplitudes associated with Feynman diagrams, particularly those at temperature $T = 0$.

However, when one includes finite temperature effects, after performing contractions of 4−vectors with other 4−vectors and tensors etc., the number of terms for some diagrams becomes enormous. For a single diagram for the one loop correction to the tri-gluon vertex using gluons only, the most lengthy expression, the number of terms and hence momentum integrations to be performed was of the order 6000. The number of independent, finite temperature integrals to be evaluated is of the order of a few hundred. At $T = 0$, the total number of terms is of the order a few hundred for the same diagram.

The momentum integrals for the $T = 0$ case can be performed using the well known dimensional regularisation formulae appearing in many books on quantum field theory. In such a case, only a few very basic integrals need be known, others can be obtained by differentiation with respect to a non-integrated momentum vector or a mass parameter of the theory.

Unfortunately, the case for finite temperature integrals is not as simple. When integrals are computed, the advantage of Lorentz covariance, used for the $T = 0$ case, is lost since it is broken by finite temperature effects. At best, one has to resort to rotational covariance. This means that not only the energy and momentum integrations have to be performed separately, but that also if the integrands contain tensorial constructs, then many of the integrations have to be done component by component. The results are then converted into tensorial form with respect to rotational covariance.

When it became apparent that the task of doing the integrations for the case of one loop vertex corrections was massive, it was decided to put this calculation aside. Instead, the same *Mathematica* program was used to investigate the self-energy of the gluon to one loop order, using the new gluon propagator. The results of this calculation, as well as the derivation of the full finite temperature gluon propagator, are presented in the next two sections.

2.2 Derivation of Spin-1 Propagator at Finite Temperature

Given the equation of motion that a field satisfies, the propagator associated with that field can be obtained from a similar equation of motion, but a point source term is introduced. Suppose a field $\Phi(x)$ is a solution of the following equation :

$$\hat{O}\Phi(x) = 0 \tag{2.14}$$

then the propagator $P(x-y)$ satisfies

$$\hat{O}P(x-y) = \delta(x-y) \tag{2.15}$$

where $\hat{O}$ is some operator and $\delta(x-y)$ is a δ function which is the point source term for the field $\Phi(x)$. In order to specify the propagator, boundary conditions must also be given.

As is the case for solving differential equations, in general one would have inhomogeneous and homogeneous solutions and that the full solution is clearly the sum of the two. For the case of field theory at $T = 0$, the propagator does not contain any homogeneous term - if the Feynman prescription is used.

In the ITF, Dolan and Jackiw [DJ74] show that due to periodic boundary conditions of the fields, the propagator itself would also exhibit the same periodicity constraints, further the propagator contains only the inhomogeneous solution.

In the RTF, the situation is different since one can not take the ITF result and then perform a Wick rotation in order to get the RTF result, i.e. naively, one can not do a Wick rotation from Euclidean to Minkowski space for finite temperature field theories as it leads to certain ambiguities. Some of these problems will be discussed in the next chapter.

When using the RTF, finite temperature effects in the propagator appear in the homogeneous part of the solution. We shall see how this arises, using QCD as a typical theory, in 4−dimensional Minkowski spacetime with metric $(+---)$.

Consider an $SU(N)$ non-abelian gauge theory described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + (\partial^\mu \chi^{a*})D^{ab}_\mu \chi^b \tag{2.16}$$

where, as usual

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2.17}$$

is the field strength tensor,

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu, \tag{2.18}$$

f^{abc} are the structure constants of the $SU(N)$ group, χ^a are the Faddeev-Popov ghosts and g is the coupling constant.

Following standard procedures, the Action S, calculated from the Lagrangian (2.16) is

$$S = \int d^4x \mathcal{L}(x) \tag{2.19}$$

which can be rearranged as

$$S = \frac{1}{2}\int d^4x A^{a\mu}\left(\partial_\rho\partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right)A^{a\nu} + \ \ldots \tag{2.20}$$

where the terms quadratic in the gauge fields have been singled out and ... denotes all other remaining terms of $\mathcal{L}$, including interactions and ghosts.

If there were no interactions, then the gauge fields A^a_μ satisfy the following free field equation, which is attained by taking the variation of S with respect to A^a_μ and setting $g = 0$:

$$K^{ab}_{\mu\nu} A^{b\nu} = 0 \tag{2.21}$$

where

$$K^{ab}_{\mu\nu} = \delta^{ab}\left(\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right)\partial_\mu \partial_\nu\right). \tag{2.22}$$

This means that the free gluon propagator, $\Delta^{ab}_{\mu\nu}$, obeys the following equation

$$K^{ab}_{\mu\nu} \Delta^{bc\,\nu\rho}(x,y) = -\delta^{ac}\delta^{\rho}_{\mu}\delta(x-y). \tag{2.23}$$

in the coordinate representation. As is customary for working in perturbation theory, we shall work in the momentum representation, hence the Fourier transform of (2.23) is

$$\tilde{K}^{ab}_{\mu\nu} \tilde{\Delta}^{bc\nu\rho} = -\delta^{ac}\delta^{\rho}_{\mu}. \tag{2.24}$$

where

$$\tilde{K}^{ab}_{\mu\nu} = \delta^{ab}\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right) k_\mu k_\nu\right) \tag{2.25}$$

and $\tilde{\Delta}(k)$ is the Fourier transform of $\Delta(x,y)$.

We find that the most general solution to (2.24) which is rotationally covariant (since Lorentz covariance is broken by finite temperature effects) is

$$\begin{aligned} \tilde{\Delta}^{ab}_{\mu\nu}(k) &= \delta^{ab}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2 + i\epsilon}\right)\frac{1}{k^2 + i\epsilon} \\ &+ \delta^{ab} C\left(g_{\mu\nu}\delta(k^2) + (1-\alpha) k_\mu k_\nu \delta'(k^2)\right) \end{aligned} \tag{2.26}$$

and the constant C is determined by using procedures as outlined in Section 1.4. Without loss of generality, we can set $\alpha = 1$ in (2.26), giving

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \delta^{ab} g_{\mu\nu}\left(\frac{1}{k^2 + i\epsilon} + C\delta(k^2)\right) \tag{2.27}$$

which can be rewritten as

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = i\delta^{ab} g_{\mu\nu} \bar{D}_\beta(k) \tag{2.28}$$

where $\bar{D}_\beta(k)$ is given by (1.65). We then find that

$$C = -2\pi i n_B(|k_0|) \tag{2.29}$$

where

$$n_B(|k_0|) = (\exp[\beta|k_0|] - 1)^{-1} \tag{2.30}$$

is the Boltzmann factor for bosons.

The δ functions have been defined in the following way

$$\delta\left(f(x)\right) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i), \tag{2.31}$$

where the sum $\sum_i$ is over the zeroes of $f(x)$, hence

$$\delta(k^2) = \frac{1}{2|\vec{k}|}\left(\delta(k_0 - \vec{k}) + \delta(k_0 + \vec{k})\right) \tag{2.32}$$

and

$$\begin{aligned} \delta'(k^2) &= \frac{d}{dk^2}\delta(k^2) \\ &= \frac{1}{2k_0}\frac{\partial}{\partial k_0}\delta(k^2). \end{aligned} \tag{2.33}$$

The new feature of (2.26) is the occurrence of a derivative of a δ function $\delta'(k^2)$ instead of unregulated products of δ functions and factors of the form $(k^2 + i\epsilon)^{-1}$. The nearest resemblance to our approach in the literature is a propagator of Kobes and Semenoff [KS85] in a 2×2 matrix formulation. The idea of that formulation is to combat the problem of unregulated products of δ functions and $(k^2+i\epsilon)^{-1}$ terms. Nevertheless they also introduce $\delta'(k^2)$ terms. We shall see in the next section that one-loop calculations are possible for $\alpha \neq 1$ with our propagator. Kobes and Semenoff [KS85] have computed self-energies of the photon and electron in QED in the Feynman gauge $\alpha = 1$, and in this case the $\delta'(k^2)$ terms do not need to be considered.

Terms proportional to $g_{\mu 0}g_{\nu 0}$, $k_0^2 g_{\mu\nu}$, $g_{\mu i}g_{\nu j}$ etc. are allowed by rotational covariance, but the coefficients of such terms are either zero, or when combined with other terms give a term which already exists in (2.26).

If $\alpha = 1$, the derivatives of $\delta(k^2)$ drop out and the propagator becomes

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \delta^{ab} g_{\mu\nu}\left(\frac{1}{k^2+i\epsilon} - \frac{2\pi i}{e^{\beta|k_0|}-1}\delta(k^2)\right) \tag{2.34}$$

which is the same as that given in many references e.g. [BPS90].

From (2.26), it can been seen that in the RTF the finite temperature effects of the propagator appear through the homogeneous solution to (2.24).

The vertices are just those as for the $T = 0$ case. A list of Feynman rules for finite temperature QCD are given in Appendix A.

2.3 One Loop Correction to Gluon Propagator

The calculation of the one loop correction to the gluon propagator, more specifically the polarisation tensor $\Pi^{ab}_{\mu\nu}(p)$ at finite temperature, is very similar to that carried out for the case of $T = 0$. For the finite temperature case, the momentum of the incoming gluon will be set to:

$$\begin{aligned} p^2 &= -m^2 \\ p_\mu &= (0,0,0,m). \end{aligned} \tag{2.35}$$

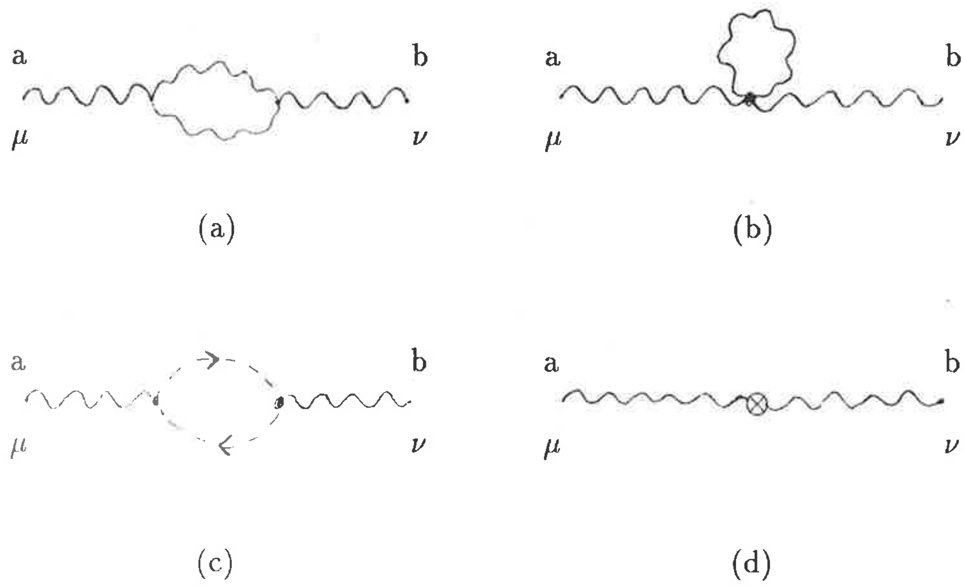


Figure 2.1: Diagrams that contribute to the self energy of the gluon.

Clearly the conditions (2.35) are unphysical, i.e. off-shell, but they simplify both the Boltzmann factors and calculation considerably, particularly for diagrams containing many propagators.

The full gluon propagator $D^{ab}_{\mu\nu}(p)$, can be written as a perturbation series :

$$D^{ab}_{\mu\nu} = \tilde{\Delta}^{ab}_{\mu\nu} + \tilde{\Delta}^{ac}_{\mu\rho}\Pi^{\rho\sigma cd}\tilde{\Delta}^{db}_{\sigma\nu} + \dots \qquad (2.36)$$

where $\tilde{\Delta}^{ab}_{\mu\nu}$ is the free gluon propagator. It is understood that the polarisation tensor and propagator given above are at finite temperature.

Following normal procedures for using Feynman rules, the polarisation tensor is :

$$\Pi^{ab}_{\mu\nu}(p) = \frac{1}{2!}\left\{\text{Fig.[2.1 a]} + \text{Fig.[2.1 b]}\right\} - \text{Fig.[2.1 c]} + \text{Fig.[2.1 d]} \qquad (2.37)$$

where the 1/2! factor is the symmetry factor for Fig.[2.1 a] and Fig.[2.1 b], and the minus sign for Fig.[2.1 c] arises from the ghost loop. Fig.[2.1 d] is the counterterm needed to remove the divergence from the self-energy part.

For $T = 0$ field theory, ghosts are introduced in order to restore unitarity and to give the correct expression for the polarisation tensor. They are spin-0 fields but satisfy anti-commutation relations. To maintain unitarity and the transversality of the polarisation tensor at finite temperature, ghosts must have Bose-Einstein type Boltzmann factors - i.e. the same Boltzmann factors as for gluons. In other words, they obey Bose-Einstein statistics. Naively, one may have expected that as ghosts have anti-commutation relations, then they have Fermi-Dirac type Boltzmann factors and obey Fermi-Dirac statistics, which is not the case.

Using the RTF, the polarisation tensor can be split into a temperature independent and temperature dependent part as follows :

$$\Pi^{ab}_{\mu\nu}(p) = \Pi^{ab}_{\mu\nu}(p)^{T=0} + \Pi^{ab}_{\mu\nu}(p)^{T\neq 0}, \tag{2.38}$$

where [Mu87]

$$\begin{aligned}
\Pi^{ab}_{\mu\nu}(p)^{T=0} &= \frac{g^2\delta^{ab}C_G(-p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)B(2-\epsilon,2-\epsilon)}{1-\epsilon}\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right) \\
&\quad \cdot \left(2(5-3\epsilon) + (1-\alpha)(1-4\epsilon)(3-2\epsilon) + (1-\alpha)^2\frac{\epsilon}{2}(3-2\epsilon)\right) \\
&= \delta^{ab}(p_\mu p_\nu - p^2 g_{\mu\nu})\Pi(p^2).
\end{aligned} \tag{2.39}$$

Dimensional regularisation has been used to obtain equation (2.39) where, after expanding in ϵ, $\Pi(p^2)$ becomes

$$\begin{aligned}
\Pi(p^2) &= \frac{-g_r^2 C_G}{(4\pi)^2}\left[\left(\frac{13}{6} - \frac{\alpha}{2}\right)\left(\frac{1}{\epsilon} - \gamma - ln\left(\frac{-p^2}{4\pi\mu^2}\right) + \frac{31}{9} - (1-\alpha) + \frac{(1-\alpha)^2}{4}\right)\right] \\
&\quad + (Z_3 - 1)
\end{aligned} \tag{2.40}$$

where C_G is related to structure constants f^{acb}

$$f^{acd}f^{bcd} = \delta^{ab}C_G. \tag{2.41}$$

The $(Z_3 - 1)$ term is the counterterm required to remove the divergences of the theory - the value of Z_3, the gluon field or wavefunction renormalisation constant, depends on which renormalisation prescription one chooses. For this specific case the counterterm is temperature independent.

The temperature dependent contribution to the polarisation tensor yields

$$\Pi_{\mu\nu}^{ab}(p)^{T\neq 0} = \frac{g^2 \delta^{ab} C_G}{(2\pi)^4} \left[C_1 \left(g_{\mu\nu} p^2 - p_\mu p_\nu \right) + C_2 g_{\mu 0} g_{\nu 0} p^2 \right] \tag{2.42}$$

where the integrals have been evaluated using the list in Appendix A and the coefficients are

$$\begin{aligned} C_1 &= -\left(2\pi f(1,1) - 4\pi f(1,3) + 2\pi f(1,5) - 3\pi f(3,0) + 5\pi f(3,2) - \pi f(3,4)\right. \\ &\quad \left. -\pi f(3,6) + -3i\pi^2 g(1,0) + -i\pi^2 g(1,2)\right) \\ &\quad -(1-\alpha)\left(-3\pi f(1,-1) + 6\pi f(1,1) - 3\pi f(1,3) + 3\pi f(2,0) - 6\pi f(2,2)\right. \\ &\quad +3\pi f(2,4) + \frac{3\pi f(3,-2)}{2} - \frac{5\pi f(3,0)}{2} + \frac{\pi f(3,2)}{2} + \frac{\pi f(3,4)}{2} - \frac{3\pi f(4,-1)}{2} \\ &\quad + \frac{5\pi f(4,1)}{2} - \frac{\pi f(4,3)}{2} - \frac{\pi f(4,5)}{2} + \frac{3i}{2}\pi^2 g(1,-2) + \frac{i}{2}\pi^2 g(1,0) \\ &\quad \left. + \frac{-3i}{2}\pi^2 m g(2,0) + \frac{-i}{2}\pi^2 m g(2,2)\right) \\ &\quad -(1-\alpha)^2 \left(\pi f(1,-1) - \frac{3\pi f(1,1)}{2} + \frac{\pi f(1,3)}{2} - \pi f(2,0) + 2\pi f(2,2) - \pi f(2,4)\right. \\ &\quad - \frac{\pi f(3,-2)}{2} + \frac{3\pi f(3,0)}{4} - \frac{\pi f(3,4)}{4} + \frac{\pi f(4,-1)}{2} - \pi f(4,1) + \frac{\pi f(4,3)}{2} \\ &\quad + \frac{3i}{8}\pi^2 g(1,-4) + \frac{-i}{4}\pi^2 g(1,-2) + \frac{-i}{8}\pi^2 g(1,0) \\ &\quad \left. + \frac{-i}{4}\pi^2 m g(2,-2) + \frac{i}{4}\pi^2 m g(2,2) + \frac{i}{8}\pi^2 m^2 g(3,0) + \frac{-i}{8}\pi^2 m^2 g(3,2)\right) \end{aligned} \tag{2.43}$$

$$\begin{aligned} C_2 &= -\left(2\pi f(1,1) - 4\pi f(1,3) + 2\pi f(1,5) - \pi f(3,0) + 5\pi f(3,2)\right. \\ &\quad \left. -7\pi f(3,4) + 3\pi f(3,6) + -i\pi^2 g(1,0) + 3i\pi^2 g(1,2)\right) \\ &\quad -(1-\alpha)\left(-\pi f(1,-1) + 2\pi f(1,1) - \pi f(1,3) + \pi f(2,0) - 2\pi f(2,2)\right. \end{aligned}$$

$$
\begin{aligned}
&+\pi f(2,4)+\frac{\pi f(3,-2)}{2}-\frac{\pi f(3,0)}{2}-\frac{\pi f(3,2)}{2}+\frac{\pi f(3,4)}{2} \\
&-\frac{\pi f(4,-1)}{2}+\frac{5\pi f(4,1)}{2}-\frac{7\pi f(4,3)}{2}+\frac{3\pi f(4,5)}{2}+\frac{i}{2}\pi^2 g(1,-2) \\
&+\frac{i}{2}\pi^2 g(1,0)+\frac{-i}{2}\pi^2 m g(2,0)+\frac{3i}{2}\pi^2 m g(2,2)\Big) \\
-(1-\alpha)^2 &\Big(-\pi f(1,-1)+\frac{\pi f(1,1)}{2}-\frac{\pi f(1,3)}{2}+\pi f(2,0) \\
&-3\pi f(2,2)+2\pi f(2,4)+\frac{\pi f(3,-2)}{2}-\frac{3\pi f(3,0)}{4}+\frac{\pi f(3,4)}{4} \\
&-\frac{\pi f(4,-1)}{2}+\pi f(4,1)-\frac{\pi f(4,3)}{2}+\frac{-3i}{8}\pi^2 g(1,-4)+\frac{-i}{8}\pi^2 g(1,0) \\
&+\frac{i}{4}\pi^2 m g(2,-2)+\frac{i}{2}\pi^2 m g(2,0)+\frac{-3i}{4}\pi^2 m g(2,2)+\frac{-i}{8}\pi^2 m^2 g(3,0) \\
&+\frac{3i}{8}\pi^2 m^2 g(3,2)\Big)\,. \qquad (2.44)
\end{aligned}
$$

where the functions $f(a,n)$ and $g(a,n)$ are defined by

$$f(1,n) = \pi\int_0^\infty dx\, \frac{x^n}{(-1+x)^2(1+x)^2}\,\frac{1}{\exp[\beta m x/2]-1} \qquad (2.45)$$

$$f(2,n) = \pi\int_0^\infty dx\, \frac{x^n}{(-1+x)^2(1+x)^2}\,\frac{d}{dx}\frac{1}{\exp[\beta m x/2]-1} \qquad (2.46)$$

$$f(3,n) = \pi\int_0^\infty dx\, \frac{x^n}{(-1+x)^2(1+x)^2}\ln\left(\left|\frac{1+x}{-1+x}\right|\right)\frac{1}{\exp[\beta m x/2]-1} \qquad (2.47)$$

$$f(4,n) = \pi\int_0^\infty dx\, \frac{x^n}{(-1+x)^2(1+x)^2}\ln\left(\left|\frac{1+x}{-1+x}\right|\right)\frac{d}{dx}\frac{1}{\exp[\beta m x/2]-1} \qquad (2.48)$$

$$g(a,n) = \pi\int_1^\infty dx\, x^n\,\frac{d^a}{dm^a}\left(\frac{1}{\exp[\beta m x/2]-1}\right)^2. \qquad (2.49)$$

Notice that the coefficients C_1 and C_2 are complex. The functions $f(1,n), f(2,n)$ etc. arise when only a single δ function or its derivative is present in the momentum integrand and contributes to the real part of the polarisation tensor. The functions $g(1,n), g(2,n)$ etc. however, result from products of two δ functions and/or their derivatives and correspond to the imaginary part of the polarisation tensor. This is primarily due to the fact that the coefficient of the δ function, $2\pi i n_B(|k_0|)$, in the propagator is imaginary, which after multiplying with a similar factor from another propagator will become real.

The fact that the coefficients C_1 and C_2 are complex could be interpreted in a couple of ways :

- That absorptive or emissive effects occurred in the process. This is possible in the context of finite temperature field theory since the system is in a thermal heat bath where particles can be absorbed or emitted by the thermal vacuum but keeping overall conservation of energy and momentum.

- Since the coefficient C, (2.29), of the temperature dependent term for the propagator contains a factor i, clearly taking products of various numbers of δ functions are going to give imaginary contributions to the polarisation tensor.

Clearly this calculation is invalid for the case when $p_\mu = 0$ since one would then have products of δ functions of the same argument, which are not well defined.

These results reduce to those obtained by Fujimoto and Yamada [FY87] who do their calculation in the Feynman gauge $\alpha = 1$ using Thermo Field Dynamics, and if the $g(1,n)$ etc. functions are set to zero.

With the conditions (2.35), the Ward identity for the gluon self-energy part

$$p^\mu \Pi^{ab}_{\mu\nu} = 0 \tag{2.50}$$

is satisfied.

One can decompose the polarisation tensor $\Pi^{ab}_{\mu\nu}$ into transverse $\mathcal{T}$ and longitudinal $\mathcal{L}$ projection tensors :

$$\Pi^{ab}_{\mu\nu}(p) = \delta^{ab}\left[A(p)\mathcal{T}_{\mu\nu} + B(p)\mathcal{L}_{\mu\nu}\right] \tag{2.51}$$

which have the following properties

$$\mathcal{T}_{0\mu} = 0$$

$$
\begin{aligned}
\mathcal{T}_{ij} &= \delta_{ij} - \frac{p_i p_j}{\vec{p}^{\,2}} \\
\mathcal{L}_{\mu\nu} &= \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} - \mathcal{T}_{\mu\nu}.
\end{aligned} \tag{2.52}
$$

The projection tensors satisfy

$$
\begin{aligned}
p_\mu \mathcal{T}^{\mu\nu} &= 0 \\
p_\mu \mathcal{L}^{\mu\nu} &= 0.
\end{aligned} \tag{2.53}
$$

Considering the finite temperature part of the polarisation tensor, we find that it can be written as

$$
\Pi^{ab}_{\mu\nu}(p)^{T\neq 0} = \frac{g^2 \delta^{ab} C_G}{(2\pi)^4} \left[A(p)\mathcal{T}_{\mu\nu} + B(p)\mathcal{L}_{\mu\nu}\right]^{T\neq 0} \tag{2.54}
$$

where

$$
\begin{aligned}
A(p) &= -\left(C_1 + C_2\right) p^2 \\
B(p) &= C_1 \vec{p}^{\,2}.
\end{aligned} \tag{2.55}
$$

2.4 Conclusion

Deriving the finite temperature spin-1 propagator in the general Lorentz gauge (with arbitrary gauge parameter α) in the real time formalism (RTF) leads not only to δ functions but also their derivatives. The propagator was obtained by seeking the solution to its equation of motion with a point source term rather than by the usual field theory methods of evaluating the vacuum expectation value of a time ordered product of two gauge fields. Finite temperature effects still appear through the homogeneous solution to the equation of motion for the gluon fields (or equation of motion for any field for that matter). The inhomogeneous solution is unaffected by finite temperature effects - it gives the same expression as one gets at $T = 0$.

The gluon self-energy part has been calculated in the general Lorentz gauge while keeping the gauge parameter α arbitrary, using the RTF. An extra term proportional to $g_{\mu 0} g_{\nu 0}$ in the polarisation tensor arises due to the fact that Lorentz covariance is broken when finite temperature effects are taken into account. The final expression is basically simple despite the enormous amount of work required to calculate the amplitude for a few simple Feynman diagrams. One finds that coefficients of various terms in the polarisation tensor are complex.

Having performed the calculation, one becomes aware that the conventional approach to perturbation theory in the RTF is somewhat ambiguous and that great care must be taken as to what should be done to products of δ functions, particularly when their arguments coincide.

This leads one to consider the role of the Wick expansion at finite temperature and whether the finite temperature propagator (in any gauge) in the RTF containing δ functions and/or their derivatives, should be used directly in perturbation theory. Further discussion of this issue will form the subject of the next chapter.

Chapter 3

Finite Temperature Perturbation Theory in the RTF

3.1 Introduction

The calculation of the one loop correction to the gluon propagator provides a good background for a possible attack on how to approach finite temperature field theory perturbatively in the real time formalism (RTF). Although, one should be open to the possibility that the process may in fact be a nonperturbative phenomenon.

Many authors have put forward theories to circumvent some of the problems concerning the RTF. For example, Thermo Field Dynamics (TFD) [UMT82] has evolved as a result of this. However, while it provides a mechanism for dealing with some of these problems, it is not without its own drawbacks. In TFD, there is a doubling of the number of fields - every field has an additional field, called the tilde field. In this case propagators and vertices become 2×2 matrices, whose properties are such that it allows some cancellations of the undesirable products of δ functions. When a calculation in TFD is carried out, the physics is contained in the $(1,1)$ component of the matrix amplitude. There is still some uncertainty as to what physical meaning, if any, the tilde field represents. It is more of an artefact introduced to get rid of the mathematical problem of dealing with products of δ functions rather than a solution to a physical problem.

As is pointed by Dolan and Jackiw [DJ74], in higher order calculations, one encounters integrals of the form

$$\int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{2\pi i}{\exp[\beta |p_0|] - 1} \delta(p^2 - m^2) \right)^n \tag{3.1}$$

whereas in the imaginary time formalism the analogous objects are

$$\frac{1}{(n-1)!} \left(i \frac{\partial}{\partial m^2} \right)^{n-1} \frac{-1}{i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - m^2}. \tag{3.2}$$

Thus, in the RTF, expressions of the form (3.1) are conventionally understood to mean

$$\frac{1}{(n-1)!} \left(i \frac{\partial}{\partial m^2} \right)^{n-1} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{2\pi i}{\exp[\beta |p_0|] - 1} \delta(p^2 - m^2) \right). \tag{3.3}$$

The imaginary time formalism does not require a doubling in the number of fields in order to carry out a calculation. It seems incongruous that in one case it is necessary to double the number of fields in order to get any sense of the theory, whereas in the other case this is not so.

There does not seem to be any unified consensus as to how one should deal with perturbation theory in the RTF and at the same time making it consistent, as is the case for zero temperature Feynman rules. Many authors seem to have their own prescription for curing the malaise. The 'derivative method' as outlined in this chapter is no exception, where one differentiates with respect to momenta rather than masses. For the purposes of this chapter, unless otherwise explicitly stated, the 'derivative method' is understood to mean differentiation by momenta, not by mass.

3.2 Free Field Theory in the RTF

An ideal way to examine how a theory behaves perturbatively is to find a theory that can be solved exactly - i.e. a theory in which the dynamics is fully understood, and at the same time, a perturbative treatment is possible. Obviously if one knows the exact theory, then it would not be necessary to resort to any of the ideas of perturbation theory. However when interactions are considered it is, in general, very difficult, if not intractable, to solve the full theory exactly. In the case for free field theories it might be possible to pursue such a course of action. Although it might be trivial and an overly simplified situation, it may give some insight into how perturbative effects are manifested in finite temperature field theories, at least in the free field case.

Consider a free scalar field theory governed by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{3.4}$$

which leads to an equation of motion for the ϕ field

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0. \tag{3.5}$$

In the momentum representation, the propagator for the scalar field at finite temperature, $\Delta(p, m)$, is

$$\Delta(p,m) = \frac{1}{p^2 - m^2 + i\epsilon} - \frac{2\pi i}{\exp[\beta|p_0|] - 1}\,\delta(p^2 - m^2). \tag{3.6}$$

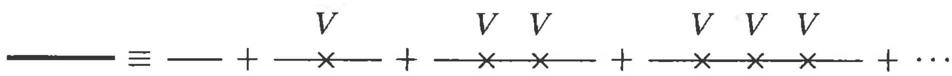


Figure 3.1: The full propagator in terms of two-point interactions.

In this case, both terms of (3.4), being quadratic in the field variables, were used to derive the propagator. Now consider the case where the mass term in (3.4) is treated as an interaction, a two-point interaction, then do a perturbation expansion in m^2. The Feynman rules for this case are, for the propagator $\Delta(p)$

$$\Delta(p) = \frac{1}{p^2 + i\epsilon} - \frac{2\pi i}{\exp[\beta|p_0|] - 1}\,\delta(p^2) \tag{3.7}$$

and the two-point vertex

$$V \sim m^2. \tag{3.8}$$

Now consider the following perturbation series, using the above Feynman rules

$$\begin{aligned} &\Delta(p) + \Delta(p)V\Delta(p) + \Delta(p)V\Delta(p)V\Delta(p) + \ \ldots \\ &= \sum_{n=i}^{\infty} \left(\Delta(p)V\right)^n \Delta(p) \end{aligned} \tag{3.9}$$

which can be represented diagrammatically as in Fig.3.1 where the bold line is $\Delta(p, m)$, the thin lines $\Delta(p)$ and the crosses are interactions V. It is immediately clear that this is undefined because of the products of δ functions. This demonstrates that the propagator of the form (3.6) or (3.7) cannot be directly used for perturbation theory.

However, if a Taylor series expansion of (3.6) about the point $-m^2$ is calculated, one gets

$$\Delta(p,m) \;=\; \sum_{n=0}^{\infty} \frac{(-m^2)^n}{n!} \left[\left(\frac{d}{dp^2}\right)^n \frac{1}{p^2} - \frac{2\pi i}{(\exp[\beta|p_0|] - 1)} \left(\frac{d}{dp^2}\right)^n \delta(p^2) \right]$$

$$= \Delta_0(p,m) + \Delta^T(p,m). \tag{3.10}$$

It is observed that in (3.10), derivatives rather than products of δ functions occur. Derivatives of δ functions are well defined constructs. Part of the task now is to see if this can be converted into a perturbation like expansion in the mass term m^2. Since $p^2 = p_0^2 - \vec{p}^{\,2}$, the derivatives can be written as

$$\frac{d}{dp^2} = -\frac{\partial}{\partial \vec{p}^{\,2}} = -\frac{1}{2|\vec{p}\,|}\frac{\partial}{\partial|\vec{p}\,|}\,, \tag{3.11}$$

then (3.10) can be rearranged as

$$\begin{aligned}\Delta(p,m) &= \sum_{n=0}^{\infty}\frac{(m^2)^n}{n!}\left(\frac{\partial}{\partial\vec{p}^{\,2}}\right)^n\left[\frac{1}{p^2} - \frac{2\pi i}{(\exp[\beta|p_0|]-1)}\delta(p^2)\right]\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(m^2\frac{\partial}{\partial\vec{p}^{\,2}}\right)^n\Delta(p).\end{aligned} \tag{3.12}$$

It can now be seen that (3.12) gives a relation between (3.6) and (3.7).

Let us examine the temperature dependent term of (3.10)

$$\begin{aligned}\Delta^T(p,m) &= -\sum_{n=0}^{\infty}\frac{(-m^2)^n}{n!}\frac{2\pi i}{(\exp[\beta|p_0|]-1)}\left(\frac{d}{dp^2}\right)^n\delta(p^2)\\ &= \frac{-2\pi i}{(\exp[\beta|p_0|]-1)}\left[\delta(p^2) + \frac{d}{dp^2}(-m^2)\,\delta(p^2)\right.\\ &\qquad\left. +\frac{1}{2!}\frac{d}{dp^2}(-m^2)\,\frac{d}{dp^2}(-m^2)\,\delta(p^2) + \ldots\right]\end{aligned} \tag{3.13}$$

Note that the first 'propagator' of the diagram is just the δ function. If there is more than one propagator in the diagram, the extra 'propagators' appear in the form of derivatives with respect to the square of the momentum associated with that 'propagator'. Thus, extra propagators seem to manifest themselves as differential operators acting on a δ function. The Boltzmann factor associated with the δ function is positioned to the left of the differential operators and the energy in its exponential term corresponds to the energy given by the δ function, e.g.

$$\frac{1}{\exp[\beta|a_0 + b_0 + \ldots + c_0|]} \text{ (diff. operators) } \delta\left((a+b+\ldots+c)^2 - m^2\right). \quad (3.14)$$

We shall call the above procedure the 'derivative method'.

Now, consider expanding

$$\int dp_0\, \delta(p^2 - m^2)\, f(|p_0|) = \frac{f(\omega)}{\omega} \quad (3.15)$$

as a Taylor series in $(-m^2)$, where $\omega = \sqrt{|\vec{p}|^2 + m^2}$. For the LHS we get

$$\int dp_0 \sum_{n=0}^{\infty} \frac{(-m^2)^n}{n!}\, \delta^{(n)}(p^2)\, f(|p_0|) \quad (3.16)$$

and for the RHS

$$\frac{f(|\vec{p}|)}{|\vec{p}|} + m^2\left(\frac{-f(|\vec{p}|)}{2|\vec{p}|^3} + \frac{f'(|\vec{p}|)}{2|\vec{p}|^2}\right) + m^4\left(\frac{3f(|\vec{p}|)}{8|\vec{p}|^5} - \frac{3f'(|\vec{p}|)}{8|\vec{p}|^4} + \frac{f''(|\vec{p}|)}{8|\vec{p}|^3}\right) + \ldots. \quad (3.17)$$

The terms of (3.17) can be generated from (3.16) if the derivatives of the δ functions are given by

$$\begin{aligned} \delta^{(n)}(p^2) &= \left(\frac{d}{dp^2}\right)^n \delta(p^2) \\ &= \left(\frac{-1}{2|\vec{p}|}\frac{\partial}{\partial|\vec{p}|}\right)^n \delta(p^2). \end{aligned} \quad (3.18)$$

It is worth noting that the series (3.17) can also be obtained from

$$\int dp_0 \sum_{n=0}^{\infty} \left(\frac{-m^2}{p^2 + i\epsilon}\right)^n \left(\frac{f(|p_0|)}{i\pi}\right) \quad (3.19)$$

where $p^2 + i\epsilon = (p_0 + |\vec{p}| - i\epsilon)(p_0 - |\vec{p}| + i\epsilon)$ is the usual Feynman prescription for handling the poles of the propagator. The δ function has been defined in the following fashion

$$\delta(p^2) = \frac{1}{2|\vec{p}|}\left(\delta(p_0 - |\vec{p}|) + \delta(p_0 + |\vec{p}|)\right) \tag{3.20}$$

where a simple δ function is defined by

$$\delta(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2} = \frac{-1}{2\pi i}\lim_{\epsilon\to 0}\left(\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right). \tag{3.21}$$

From this it is apparent that $\delta(p^2)$ has 4 poles, one in each quadrant in the complex energy plane - compare this to the Feynman prescription which has two poles, one each in the second and fourth quadrants.

Thus, the δ function prevents us from using the naive Wick rotation to go from the real to the imaginary time formalism (or vice versa) as it could lead to some problems as the poles in the first and third quadrants will have to be crossed. In quantum field theory, the Wick rotation is carried out on the assumption that there are no poles in the first and third quadrants. It is possible that the form or regularisation of the δ functions given above is not appropriate for performing calculations in finite temperature field theory in the RTF.

Whether one chooses (3.18) or (3.19) is dependent on what set of boundary conditions are chosen. This is usually determined by taking the Fourier transform and seeing what one needs to do with positive and/or negative energy solutions. One has to keep in mind that in the RTF, the finite temperature part of the propagator corresponds to the homogeneous solution to the equation of motion.

Obviously one would like to include more general interactions, rather than just those of the two-point type, such as three-point interactions and loops etc. We shall examine how the 'derivative' method could be extended to such cases.

As a prelude, we shall now present a very brief outline how perturbative field theory at finite temperature is carried out using Wick's theorem.

Then, we use the 'derivative' method to investigate how quantum effects can cause mass-shifts, denoted by δm^2, of a particle at finite temperature.

3.3 Interacting Field Theory in the RTF

Usually interacting field theories are examined perturbatively, primarily because it is the only method at our disposal for which calculations of amplitudes, decay rates etc. can be carried out. The thermal average of the full propagator or the so-called temperature Green's function, with free Hamiltonian $\hat{H}_0$, interaction Hamiltonian $\hat{H}^I$ and partition function Z, is defined by [FW71],

$$\begin{aligned} \mathcal{G}(\vec{x},\tau,\vec{y},\tau') &= -\frac{1}{Z}\,\mathrm{Tr}\; e^{-\beta\hat{H}_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{\hbar^n n!}\int_0^{\beta\hbar}d\tau_1\int d\vec{x}_1\ldots\int_0^{\beta\hbar}d\tau_n\int d\vec{x}_n \\ &\quad\cdot\, T_\tau\left[\hat{H}^I(\vec{x}_1,\tau_1)\ldots\hat{H}^I(\vec{x}_n,\tau_n)\,\phi(\vec{x},\tau)\phi(\vec{y},\tau')\right], \qquad (3.22) \end{aligned}$$

where the partition function Z is

$$\begin{aligned} Z &= \mathrm{Tr}\; e^{-\beta\hat{H}_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{\hbar^n n!}\int_0^{\beta\hbar}d\tau_1\int d\vec{x}_1\ldots\int_0^{\beta\hbar}d\tau_n\int d\vec{x}_n \\ &\quad\cdot\, T_\tau\left[\hat{H}^I(\vec{x}_1,\tau_1)\ldots\hat{H}^I(\vec{x}_n,\tau_n)\right] \qquad (3.23) \end{aligned}$$

and knowledge of the finite temperature aspects of the time-ordering operator is necessary. This can be done by considering Wick's theorem. When a theory is quantised, it is possible to compute diagrams for n-point functions with the assistance of Wick's theorem - certainly for the case of $T = 0$ field theory. Let us examine the basics of Wick's theorem and how it presents problems when used for perturbation theory at finite temperature.

When computing a time ordered product of field configurations at $T = 0$, Wick's theorem shows how this may be reduced in terms of normal ordered combinations of the fields and their contractions. For example, for a time ordered product of two fields

$$T[\phi(x_1)\phi(x_2)] =: \phi(x_1)\phi(x_2) : + \underbrace{\phi(x_1)\phi}(x_2) \tag{3.24}$$

where the contraction of two fields $\phi(x_1)$ and $\phi(x_2)$ is defined by

$$\begin{aligned}\underbrace{\phi(x_1)\phi}(x_2) &= T[\phi(x_1)\phi(x_2)] - : \phi(x_1)\phi(x_2) : \\ &= < 0|\phi(x_1)\phi(x_2)|0 > \end{aligned} \tag{3.25}$$

and is the propagator for the $\phi(x)$ field, further it is a c-number. Hence when calculating the vacuum expectation value of a time ordered product of fields, the contributions from the normal ordered products vanish.

For the more general case we have

$$\begin{aligned} T[\phi(x_1)\ldots\phi(x_n)] &= : \phi(x_1)\ldots\phi(x_n) : \\ &+ \sum_{k<l} \sigma : \phi(x_1)\ldots\widehat{\phi(x_k)}\ldots\widehat{\phi(x_l)}\ldots\phi(x_n) : < 0|T\phi(x_k)\phi(x_l)|0 > + \ldots \\ &+ \sum_{k_1<k_2<\ldots<k_{2p}} \sigma_P : \phi(x_1)\ldots\widehat{\phi(x_{k_1})}\ldots\widehat{\phi(k_{k_{2p}})}\ldots\phi(x_n) : \\ &\quad \times \sum_P < 0|T[\phi(x_{k_{P_1}})\phi(x_{k_{P_2}})|0 > \ldots < 0|T[\phi(x_{k_{P_{2p-1}}})\phi(x_{k_{P_{2p}}})|0 > \\ &+ \ldots \end{aligned} \tag{3.26}$$

where $\sum_P$ denotes the sum over all possible signed permutations of pairs of fields with σ and σ_P taking account of the sign of the permutation when the fields are (anti-) commuted through the normal ordered term to get the contractions.

Extending this to include finite temperature effects requires very careful consideration. Some ideas and concepts have to be modified, such as the vacuum and the role normal ordering plays in field theories at finite temperature. A number of people have examined this as early as 1955 [Ma55,Th57,Bl58,BD58,Ga60,BM61,FW71]. Let us also examine what happens if this naive approach is used.

Taking the statistical mechanical approach, the thermal average of any operator

A is given by

$$
\begin{aligned}
< A > &= \frac{Tr\left\{e^{-\beta H}A\right\}}{Tre^{-\beta H}} \\
&= \frac{\sum_{N,j} < N\, j|e^{-\beta H}A|N\, j >}{\sum_{N,j} < N\, j|e^{-\beta H}|N\, j >}
\end{aligned}
\tag{3.27}
$$

where the $\sum_{N,j}$ is the sum over the number of particles N, and j represents all possible quantum numbers of the system.

Using this and (3.24), we can see how the thermal average of a time ordered product of fields could be defined

$$
\begin{aligned}
&\frac{Tr\left\{e^{-\beta H}T[\phi(x_1)\phi(x_2)]\right\}}{Tre^{-\beta H}} = \\
&\frac{Tr\left\{e^{-\beta H} : \phi(x_1)\phi(x_2) :\right\}}{Tre^{-\beta H}} + \frac{Tr\left\{e^{-\beta H} < 0|\phi(x_1)\phi(x_2)|0 >\right\}}{Tre^{-\beta H}}.
\end{aligned}
\tag{3.28}
$$

The conventional $T = 0$ vacuum state $|0>$, consisting of no particles is just one of the states in the summation. Recalling that $< 0|\phi(x_1)\phi(x_2)|0 >$ is a c-number, and therefore can be pulled outside the summation $\sum_{N,j}$, the last term of (3.28) becomes

$$
\frac{Tr\left\{e^{-\beta H} < 0|\phi(x_1)\phi(x_2)|0 >\right\}}{Tre^{-\beta H}} =< 0|\phi(x_1)\phi(x_2)|0 > . \tag{3.29}
$$

In general, the normal ordered term would not vanish, since the summation is made over not only the vacuum state, but also non-zero particle number states, so

$$
\frac{Tr\left\{e^{-\beta H} : \phi(x_1)\phi(x_2) :\right\}}{Tre^{-\beta H}} \neq 0 \tag{3.30}
$$

This, in essence, defeats the purpose of normal ordering as understood in the context of $T = 0$ field theory. When one defines the propagator as a vacuum expectation value of a time ordered product of two fields, one cannot simply ignore normal ordered terms or rather, care must be taken when working with the $T = 0$

vacuum, or whether a new vacuum should be introduced for finite temperature field theory. It should be borne in mind that as finite temperature field theory describes a many particle system, then it may be the case that the ground state of the system is not the vacuum state, or rather, not the zero particle state.

When using Wick's theorem, one actually deals with 'contractions' of pairs of fields. In the $T = 0$ field theory, a contraction of two fields, denoted by an underbrace connecting the two fields as defined in (3.25), is just the free propagator $D(x_1 - x_2)$

$$\begin{aligned} \underbrace{\phi(x_1)\phi(x_2)} &= < 0|\phi(x_1)\phi(x_2)|0 > \\ &= D(x_1 - x_2). \end{aligned} \tag{3.31}$$

So, one might modify the definition of contraction of fields when dealing with finite temperature field theory. This was first recognised by Gaudin [Ga60] and also discussed in [FW71]. The contraction in this case is given by

$$\underbrace{\phi(x_1)\phi(x_2)} = \frac{[\phi(x_1), \phi(x_2)]_\mp}{1 \mp e^{\pm\beta E}} \tag{3.32}$$

where $E = E_b - E_a$, the upper sign refers to bosons and the lower to fermions and

$$[\phi(x_1), \phi(x_2)]_\mp = \phi(x_1)\phi(x_2) \mp \phi(x_2)\phi(x_1) \tag{3.33}$$

is the (anti-) commutator if $\phi(x)$ is a (fermionic) bosonic field.

If we denote the creation and annihilation operators by $a_j^\dagger$ and a_j respectively, then contractions between creation and annihilation operators, depending on their order, are

$$\underbrace{a_j^\dagger a_j} = \frac{\left[a_j^\dagger, a_j\right]_\mp}{1 \mp e^{\beta E_j}} = \frac{1}{e^{\beta E_j} \mp 1} = n_j$$

$$\underbrace{a_j a_j^\dagger} = \frac{\left[a_j, a_j^\dagger\right]_\mp}{1 \mp e^{-\beta E_j}} = \frac{1}{1 \mp e^{-\beta E_j}} = 1 \pm n_j \tag{3.34}$$

where

$$n_j = \frac{Tr\left\{e^{-\beta H_0} a_j^\dagger a_j\right\}}{Tr\left\{e^{-\beta H_0}\right\}} = \frac{1}{e^{\beta E_j} \mp 1} \tag{3.35}$$

is the number operator for particles of species j.

Wick's theorem for finite temperature field theory would now give

$$\begin{aligned} Tr\left\{e^{-\beta H}\phi(x_1)\phi(x_2)\phi(x_3)\ldots\phi(x_n)\right\} &= \underbrace{\phi(x_1)\phi(x_2)}Tr\left\{e^{-\beta H}\phi(x_3)\ldots\phi(x_{2n})\right\} \\ &\mp \underbrace{\phi(x_1)\phi(x_3)}Tr\left\{e^{-\beta H}\phi(x_2)\phi(x_4)\ldots\phi(x_{2n})\right\} + \ldots \\ &+ \underbrace{\phi(x_1)\phi(x_{2n})}Tr\left\{e^{-\beta H}\phi(x_2)\phi(x_3)\ldots\phi(x_{2n-1})\right\}. \end{aligned} \tag{3.36}$$

There are striking similarities in Wick's theorem for the finite temperature case when compared to the zero temperature case. Unfortunately, this in itself is not sufficient to allow us to handle the problems associated with the δ functions encountered in finite temperature field theory in the real time formalism, where in the momentum representation, one replaces the contraction of the fields by the propagator (3.6) or (3.7).

We now turn to the case of applying the derivative method to the self-energy of a particle.

3.4 Interactions and the Derivative Method in the RTF

To see how the derivative method in Sect.3.2 can be applied to interactions and loop diagrams, we shall choose the case of the self-energy of a scalar particle - specifically the mass-shift. Our guiding principle is that a mathematical expression

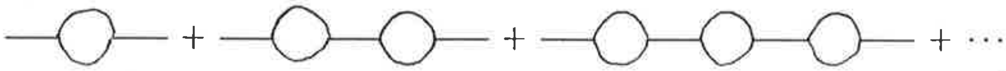


Figure 3.2: Self-energy insertions to the free propagator.

should be well defined at all stages of a calculation.

For simplicity, let us work with ϕ^3 field theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3. \tag{3.37}$$

The free propagator will be given by (3.6). Including self-energy insertions to the free propagator as displayed in Fig. 3.2, would cause a mass-shift in the full propagator.

The full propagator

$$\Delta(p, m, \delta m^2) = \frac{1}{p^2 - m^2 + \delta m^2 + i\epsilon} - \frac{2\pi i}{\exp[\beta|p_0|] - 1}\,\delta(p^2 - m^2 + \delta m^2). \tag{3.38}$$

can be expressed as a series, very similar to (3.10), but this time an expansion in the mass-shift parameter δm^2 is performed instead. The result is

$$\begin{aligned} \Delta(p, m, \delta m^2) &= \sum_{n=0}^{\infty} \frac{(\delta m^2)^n}{n!} \left[\left(\frac{d}{dp^2} \right)^n \frac{1}{p^2 - m^2 + i\epsilon} \right. \\ &\left. - \frac{2\pi i}{(\exp[\beta|p_0|] - 1)} \left(\frac{d}{dp^2} \right)^n \delta(p^2 - m^2) \right] \end{aligned} \tag{3.39}$$

where we note that the derivative can be written as

$$\frac{d}{d(p^2 - m^2)} = \frac{d}{dp^2}. \tag{3.40}$$

Here, we let

$$\delta m^2 = \delta m^2_{T=0} + \delta m^2_T \tag{3.41}$$

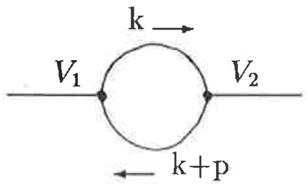


Figure 3.3: The scalar self-energy to order λ^2.

be just the simple one loop self-energy correction to the free scalar propagator as shown in Fig.3.3.

The zero temperature part of the self energy $\delta m^2_{T=0}$ can be obtained from the conventional $T = 0$ Feynman rules. The finite temperature contribution to the self-energy in the derivative method is

$$\begin{aligned}\Pi(p,m)^T &= \frac{1}{2!}\int\frac{d^4k}{(2\pi)^4}\left[\frac{-2\pi i}{e^{\beta|k_0|}-1}V_2\frac{d}{d\left((k+p)^2-m^2\right)}V_1\,\delta(k^2-m^2)+\right.\\ &\left.\frac{-2\pi i}{e^{\beta|k_0+p_0|}-1}V_1\frac{d}{d\left(k^2-m^2\right)}V_2\,\delta((k+p)^2-m^2)\right] \qquad (3.42)\end{aligned}$$

where 2! is the symmetry factor and V_1 and V_2 are vertex factors which in general will be functions of momentum (e.g. for non-abelian theories). Each term within the large braces is obtained by starting with a δ function for one propagator and then continuing anti-clockwise to get the rest of the expression. For scalar field theory (3.37), $V_1 = V_2 = \lambda$.

The derivative of the first δ function in (3.42) may be rewritten as (using (3.40))

$$\begin{aligned}\frac{d}{d(k+p)^2}\delta(k^2-m^2) &= \frac{dk^2}{d(k+p)^2}\frac{d}{dk^2}\delta(k^2-m^2)\\ &= \left(1-\frac{p\cdot k}{(k+p)^2}-\frac{p^2}{(k+p)^2}\right)\frac{d}{dk^2}\delta(k^2-m^2) \quad (3.43)\end{aligned}$$

and the derivative of the second δ function is treated in a similar fashion.

For simplicity, let us consider a scalar field with $m = 0$ and compute the self-energy for the unphysical external momentum configuration $p_\mu = (0, \vec{p})$ such that

$p^2 = -M^2$. Using the finite temperature integrals in Appendix A (with m replaced by M), $\Pi(p,0)^T$ becomes

$$\begin{aligned}
\Pi(p,0)^T &= \frac{\lambda^2}{2!(2\pi)^4}\int d^4k \left\{\frac{-2\pi i}{e^{\beta|k_0|}-1}\left(1-\frac{p\cdot k}{(k+p)^2}-\frac{p^2}{(k+p)^2}\right)\frac{d}{dk^2}\delta(k^2)\right. \\
&\left. +\frac{-2\pi i}{e^{\beta|k_0|}-1}\left(1+\frac{p\cdot k}{(k-p)^2}-\frac{p^2}{(k-p)^2}\right)\frac{d}{dk^2}\delta(k^2)\right\} \\
&= \frac{2\lambda^2}{2!(2\pi)^4}\left(f(1,-1)-\frac{3f(1,1)}{2}+\frac{f(1,3)}{2}-f(2,0)+2f(2,2)-f(2,4)\right. \\
&\quad -\frac{f(3,-2)}{2}+\frac{5f(3,0)}{4}-f(3,2)+\frac{f(3,4)}{4}+\frac{f(4,-1)}{2} \\
&\quad \left. -f(4,1)+\frac{f(4,3)}{2}\right). \qquad (3.44)
\end{aligned}$$

Thus to order λ^2, the finite temperature contribution to the mass-shift is

$$\delta m_T^2 \sim \Pi(p,0). \qquad (3.45)$$

Let us compare these results with those obtained from the conventional real time approach. The expression for the self-energy $\Pi(p,m)_{\text{conventional}}$ would be

$$\begin{aligned}
\Pi(p,m)_{\text{conventional}} &= \frac{\lambda^2}{2!\,(2\pi)^4}\int d^4k\left(\frac{1}{k^2-m^2+i\epsilon}-\frac{2\pi i}{e^{\beta|k_0|}-1}\delta(k^2-m^2)\right) \\
&\cdot\left(\frac{1}{(k+p)^2-m^2+i\epsilon}-\frac{2\pi i}{e^{\beta|k_0+p_0|}-1}\delta((k+p)^2-m^2)\right) \qquad (3.46)
\end{aligned}$$

where the finite temperature contribution is

$$\begin{aligned}
\Pi(p,m)^T_{\text{conventional}} &= \frac{\lambda^2}{2!\,(2\pi)^4}\int d^4k\left\{\frac{-2\pi i}{(e^{\beta|k_0|}-1)\,((k+p)^2-m^2+i\epsilon)}\delta(k^2-m^2)\right. \\
&-\frac{2\pi i}{(e^{\beta|k_0+p_0|}-1)\,(k^2-m^2+i\epsilon)}\delta((k+p)^2-m^2) \\
&\left.+\frac{(-2\pi i)^2}{(e^{\beta|k_0|}-1)\,(e^{\beta|k_0+p_0|}-1)}\delta(k^2-m^2)\,\delta((k+p)^2-m^2)\right\}. \qquad (3.47)
\end{aligned}$$

Letting $p_\mu = (0,0,0,M)$, $m = 0$ and using the integrals in Appendix A

$$\Pi(p,0)^T_{\text{conventional}} = \frac{\lambda^2}{2!\,(2\pi)^4}\left[i\pi\left(2f(3,0) - 4f(3,2) + 2f(3,4)\right) - 2\pi^2 g(1,0)\right] \quad (3.48)$$

which is simpler than but quite different from the result obtained from the derivative method (3.44).

Some comments regarding the derivative method are in order :

- In the derivative method, the only mass variables appearing in the amplitude are those from the δ functions.
- If the vertex factors V_1 and V_2 have momentum dependence, it is not clear at present whether such factors should be differentiated along with the δ functions or if they should be extracted from the amplitude and placed together with the Boltzmann factor before the differential operators.
- What combinatorial factors one attaches to a particular diagram have still yet to be clarified.

3.5 Conclusion

In principle an amplitude evaluated in the imaginary time formalism (ITF) can be continued to the real time formalism (RTF) using techniques given in Sect. 1.4. Usually the ITF expressions for the free propagators and vertices are continued to the RTF, giving the well known finite temperature Feynman rules in the RTF and then applied to perturbative calculations. More correctly, the calculation for whatever amplitude one has in mind should be done entirely in the ITF and then continued to the RTF. This way, it will be more likely that one gets distributions, which may be quite complicated, instead of products of distributions (such as δ functions). However, performing the energy summations in the ITF can be very difficult, thus other methods for dealing with perturbation in the RTF have to be constructed. The mass derivative and derivative methods try to bridge this gap.

The mass derivative method applies only when the momenta and mass variables in the products of δ functions coincide. Converting from products of δ functions to mass derivatives of δ functions for a specific set of masses and momenta is not a 'smooth' process, in the sense that one has to replace certain functions with an entirely new function, or rather, replace a set of distributions with a new distribution. The derivative method is a 'smooth' process and each step is well defined from a mathematical viewpoint. It reproduces the exact free field theory results, when compared to a theory in which a perturbation in the mass term is carried out.

As it stands, the derivative method is mathematically well defined and at the level of an initial investigation, it appears to be physically legitimate. The procedure needs to be extended to four-scalar (in ϕ^4 theory) or four-gluon (in non-abelian theory) vertices or diagrams of more than one loop. Further development of the derivative method requires a more extensive research program which is beyond the scope of a single thesis.

Chapter 4

The Chiral Anomaly

4.1 Introduction

Since their discovery [St49,Sc51,Su66,Su67,BJ69,Ba69,AB69,Ad69], anomalies have been the subject of much investigation and they have played a crucial role in understanding the nature of quantum field theories. For the case of the $\pi^0 \to 2\gamma$ decay amplitude, the presence of the anomaly allows very good agreement with experimental data, without which the PCAC theory could not account for the large value observed. However in the case of the weak interactions, the Weinberg-Salam-Glashow Model, anomalies presented a very different picture.

At first, they appeared to be a curse or the death knell of certain field theories, making them non-renormalisable. Prior to their discovery, renormalisation procedures did not clash with symmetries that existed at the classical level. However when quantum effects are taken into account, anomalies arise when some symmetries are not preserved. Later it was realised that by choosing certain gauge groups and representations thereof, it was possible and essential to cancel these anomalies. This narrowed the search for theories that might be applicable for describing the various interactions we observe. For the $\pi^0 \to 2\gamma$ case, the anomaly does not affect renormalisability since the axial current associated with it is not a current of the

Standard Model $SU(3) \times SU(2) \times U(1)$. For QED and QCD only vector couplings between the fermions and gauge fields have been observed experimentally, whereas for the case the weak interaction (Weinberg-Salam Model) axial vector couplings also occur.

As the name suggests, an anomaly arises when something 'unexpected' happens to a particular equation under certain circumstances. There are various types of anomalies - chiral, conformal etc. In this chapter we shall only be concerned with the chiral anomaly, which was the first to be discovered and occurs in gauge theories that have gauge fields coupled to chiral fermions.

Specifically for the case of the chiral anomaly, they arise when both chiral and gauge symmetries cannot simultaneously be preserved. A regularisation scheme that satisfies both chiral and gauge invariance cannot be found. In such a case, one must forego either gauge or chiral invariance - usually gauge invariance is retained and chiral invariance is not. This allows the physics of the process to become more apparent since physically relevant quantities are believed to be gauge invariant constructs. However this procedure is by no means unique as one could have just as well chosen to retain chiral invariance but not gauge invariance - this makes it difficult to obtain the physics inherent in the theory. Generally, when theories are quantised, anomalies appear as extra terms in certain equations - it is these terms that break some symmetries that were present at the classical level. In other words, some equations at the classical level are not necessarily maintained at the quantum level.

The chapter will present an introduction to the chiral anomaly in four dimensions - the triangle diagram using the Feynman diagrammatic approach in the momentum representation. Next, a section on the derivation of the chiral anomaly using the Nielsen-Schroer method [NS77] which begins by defining the necessary physics on a hypersphere S^{2n} (a compact manifold), then transforming the results to R^{2n}. Then a section will be given showing the temperature independence of the chiral anomaly by extending the Nielsen-Schroer analysis to include finite temperature effects. A finite temperature delta-function, δ_T, which is anti-periodic in the time coordinate, called either the 'anti-periodic δ function' or the 'temperature δ function' is required. The Nielsen-Schroer method is carried out entirely in the coordinate representation.

Finally, the last section deals with an interesting aspect of the Atiyah-Singer Index theorem which relates the number of zero modes (i.e. massless modes) of the Dirac operator to the topological properties of the gauge field, in particular its winding number. After presenting the $T = 0$ situation, it is shown that the Atiyah-Singer Index is unaffected by finite temperature effects.

4.2 The Triangle Anomaly - Feynman Diagram Approach

Many books on advanced quantum field theory provide a good introduction to the subject of chiral anomalies from a Feynman diagram viewpoint [Ry85,Fr87,IZ85]. The triangle anomaly, also known as the Bell-Adler-Jackiw anomaly, is the chiral anomaly in 4 dimensions. It arises in any gauge theory that has couplings to chiral fermions.

Vector and axial vector couplings of fermions Ψ with gauge fields can be described by the interaction Lagrangian

$$\mathcal{L} = -g_V \bar{\Psi} \gamma_\mu \Psi A^\mu - g_A \bar{\Psi} \gamma_\mu \gamma_5 \Psi Z_A^\mu \tag{4.1}$$

where g_V and g_A are the vector and axial vector coupling constants, and A_μ and Z_μ^A are the gauge fields that couple to the vector and axial vector vertices respectively.

The vector and axial vector currents are

$$J_\mu = \bar{\Psi}\gamma_\mu\Psi \qquad \text{and} \qquad J_\mu^5 = \bar{\Psi}\gamma_\mu\gamma_5\Psi. \tag{4.2}$$

The axial vector coupling consists of the γ_5 matrix defined by

$$\begin{aligned} \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3 \\ &= \frac{1}{4!}\ \mathcal{E}_{\alpha\beta\gamma\delta}\ \gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \end{aligned} \tag{4.3}$$

where the Dirac matrices γ_μ satisfy the Dirac algebra

$$\begin{aligned} [\gamma_\mu, \gamma_\nu]_+ &= 2\, g_{\mu\nu} \\ [\gamma_5, \gamma_\mu]_+ &= 0 \end{aligned} \tag{4.4}$$

and $\mathcal{E}_{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor.

At a classical level (calculating tree-level diagrams only) and using the Dirac equation one finds that the vector current is conserved :

$$\partial^\mu J_\mu = 0 \tag{4.5}$$

but that the axial vector current is not :

$$\partial^\mu J_\mu^5 = 2im\bar{\Psi}\gamma_5\Psi = 2mJ^5 \tag{4.6}$$

where $J^5 = i\bar{\Psi}\gamma_5\Psi$ is the chiral density and m is the mass of the fermion coupling the axial vector vertex.

However, when higher order corrections are included, for example to one loop order, there are two Feynman diagrams, each consisting of a closed, but oppositely directed, fermion loop with two vector and an axial vector vertex shown in Fig.[4.1],

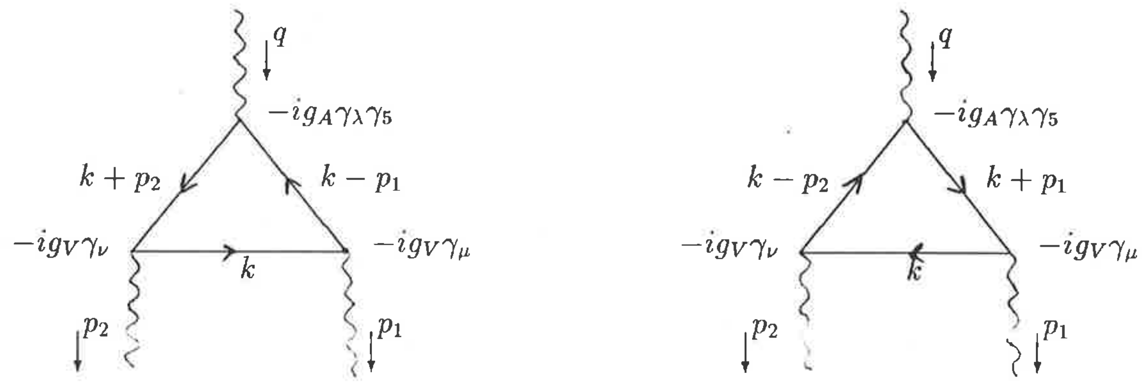


Figure 4.1: Diagrams that contribute to the chiral anomaly.

(4.6) is modified but (4.5) is unaltered. It is these higher order diagrams that contribute to the chiral anomaly. The change to (4.6) is the chiral anomaly.

Contributions from all possible types of fermions circulating the loop are summed over. The amplitude for the amputated diagrams can be written as

$$I_{\mu\nu\lambda}(p_1,p_2,m) = T_{\mu\nu\lambda}(p_1,p_2,m) + T_{\nu\mu\lambda}(p_2,p_1,m) \tag{4.7}$$

where the first (second) term on right hand side of (4.7) corresponds to the first (second) diagram of Fig.[4.1]. Normally the computation of amplitudes is greatly assisted by using the techniques of dimensional regularisation, where the number of spacetime dimensions, d, is analytically continued to the complex plane. However, this method cannot be applied in the case of the chiral anomaly due to the inability of defining a generalised γ_5 type matrix to arbitrary dimensions - it is only defined for spacetimes of even number of dimensions.

Using Feynman rules, we find that

$$\begin{aligned} T_{\mu\nu\lambda} = (i)^3(-ig_A)(-ig_V)^2 \int \frac{d^4k}{(2\pi)^4} \cdot \frac{1}{k^2-m^2+i\epsilon} \\ \cdot \frac{1}{(k-p_1)^2-m^2+i\epsilon} \cdot \frac{1}{(k+p_2)^2-m^2+i\epsilon} \; 4 \; t_{\mu\nu\lambda} \end{aligned} \tag{4.8}$$

where

$$4\ t_{\mu\nu\lambda} = Tr_s\left[(\not k + m)\gamma_\mu(\not k - \not p_1 + m)\gamma_\lambda\gamma_5(\not k + \not p_2 + m)\gamma_\nu\right] \quad (4.9)$$

where Tr_s denotes the spinor trace. Note that $T_{\mu\nu\lambda}$ is Bose symmetric under the interchange $(p_1, \mu) \rightarrow (p_2, \nu)$ and so the contribution of the second diagram to $I_{\mu\nu\lambda}$ gives a factor of 2, so only one diagram needs to be evaluated.

It is important to note that (4.8) is linearly divergent, meaning that shifting the integration variable alters $T_{\mu\nu\lambda}$ by a finite amount. This can be seen by considering the following

$$\begin{aligned} \int d^4k f(k) &= \int d^4k' f(k-a) \\ &= \int d^4k' f(k') - a_\mu \int d^4k \frac{\partial}{\partial k_\mu} f(k) \quad + \ldots \end{aligned} \quad (4.10)$$

where we have used the Taylor Series expansion. Suppose the original integral is linearly divergent, the second term in (4.10) is finite since when it is converted to a surface term by Gauss' theorem, the integrand $f(k) \sim |k|^{-3}$ and the surface area $\sim |k|^3$.

When matter (fermions) couples to gauge fields via vector or axial vector interactions, expressions called the Ward Identities are encountered - one each associated with the vector and axial vector current. The Ward identities are established by considering such a shift in the integration variables described above. Naively, the Ward identities would be

$$\begin{aligned} (p_1 + p_2)^\lambda I_{\mu\nu\lambda} &= 0 & \text{(Axial)} \quad (4.11) \\ p_1^\mu I_{\mu\nu\lambda} &= 0 & \text{(Vector)} \quad (4.12) \\ p_2^\nu I_{\mu\nu\lambda} &= 0 & \text{(Vector).} \quad (4.13) \end{aligned}$$

In reality, when the vector Ward identities (4.12) and (4.13) are imposed, which are essential for charge conservation in quantum electrodynamics, we cannot at the same time satisfy the axial vector Ward identity (4.11). We will present an outline of how this occurs.

As the linear divergent term is independent of the mass m of the fermion, without any loss generality it can be set to zero, thus

$$T_{\mu\nu\lambda} = -g_A g_V^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr_s[\not{k}\gamma_\mu(\not{k}-\not{p}_1)\gamma_\lambda\gamma_5(\not{k}+\not{p}_2)\gamma_\nu]}{k^2(k-p_1)^2(k+p_2)^2}. \tag{4.14}$$

Now consider

$$(p_1+p_2)^\lambda T_{\mu\nu\lambda} = -g_A g_V^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr_s[\not{k}\gamma_\mu(\not{k}-\not{p}_1)(\not{p}_1+\not{p}_2)\gamma_5(\not{k}+\not{p}_2)\gamma_\nu]}{k^2(k-p_1)^2(k+p_2)^2} \tag{4.15}$$

and rewriting $(\not{p}_1+\not{p}_2)\gamma_5$ as

$$(\not{p}_1+\not{p}_2)\gamma_5 = -(\not{k}-\not{p}_1)\gamma_5 - \gamma_5(\not{k}+\not{p}_2) \tag{4.16}$$

leads to

$$\begin{aligned}(p_1+p_2)^\lambda T_{\mu\nu\lambda} = g_A g_V^2 &\int \frac{d^4k}{(2\pi)^4} \frac{Tr_s[\not{k}\gamma_\mu\gamma_5(\not{k}+\not{p}_2)\gamma_\nu]}{k^2(k+p_2)^2} \\ +g_A g_V^2 &\int \frac{d^4k}{(2\pi)^4} \frac{Tr_s[\not{k}\gamma_\mu(\not{k}-\not{p}_1)\gamma_5\gamma_\nu]}{k^2(k-p_1)^2}.\end{aligned} \tag{4.17}$$

Both terms on the right hand side of (4.17) are second rank pseudotensors depending on only one 4−momentum. Naively, there does not exist any tensor that has this property, so it is tempting to draw the conclusion that (4.11) is satisfied.

If we now calculate

$$p_1^\mu T_{\mu\nu\lambda} = -g_A g_V^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr_s[\not{k}\not{p}_1(\not{k}-\not{p}_1)\gamma_\lambda\gamma_5(\not{k}+\not{p}_2)\gamma_\nu]}{k^2(k-p_1)^2(k+p_2)^2} \tag{4.18}$$

and change the integration variable to $k' = (k + p)$ and do the literal substitution, one gets

$$p_1^\mu T_{\mu\nu\lambda} = -g_A g_V^2 \int \frac{d^4 k}{(2\pi)^4} \frac{Tr_s\left[(\not{k}' - \not{p}_2)\not{p}_1(\not{k}' - \not{p}_1 - \not{p}_2)\gamma_\lambda \gamma_5 \not{k}' \gamma_\nu\right]}{k'^2 (k' - p_1 - p_2)^2 (k' - p_2)^2}. \tag{4.19}$$

Again, if we rewrite $p_1 = -(k - p_1 - p_2) + (k' - p_2)$ then (4.19) becomes

$$\begin{aligned} p_1^\mu T_{\mu\nu\lambda} &= g_A g_V^2 \int \frac{d^4 k'}{(2\pi)^4} \frac{Tr_s\left[(\not{k}' - \not{p}_2)\gamma_\lambda \gamma_5 \not{k}' \gamma_\nu\right]}{k'^2 (k' - p_2)^2} \\ &\quad - g_A g_V^2 \int \frac{d^4 k'}{(2\pi)^4} \frac{Tr_s\left[(\not{k}' - \not{p}_1 - \not{p}_2)\gamma_\lambda \gamma_5 \not{k}' \gamma_\nu\right]}{k'^2 (k' - p_1 - p_2)^2} \end{aligned} \tag{4.20}$$

which vanishes for the same reasons as for (4.17).

Similarly, by letting $p_2 = (k'' + p_1 + p_2) - (k'' + p_1)$ we find that

$$\begin{aligned} p_2^\nu T_{\mu\nu\lambda} &= -g_A g_V^2 \int \frac{d^4 k''}{(2\pi)^4} \frac{Tr_s\left[(\not{k}'' + \not{p}_1)\gamma_\mu \not{k}'' \gamma_\lambda \gamma_5\right]}{k''^2 (k'' + p_1)^2} \\ &\quad + g_A g_V^2 \int \frac{d^4 k''}{(2\pi)^4} \frac{Tr_s\left[\gamma_\mu \not{k}'' \gamma_\lambda \gamma_5 (\not{k}'' + \not{p}_1 + \not{p}_2)\right]}{k''^2 (k'' + p_1 + p_2)^2} \end{aligned} \tag{4.21}$$

also vanishes.

Thus it seems that the Ward identities (4.11) - (4.13) are satisfied. However, when we performed the change of variables in calculating the above, we did not consider finite contributions from 'surface terms' as described in (4.10). Let us define $S_{\mu\nu\lambda}$ by

$$T_{\mu\nu\lambda} = -g_A g_V^2 S_{\mu\nu\lambda} \tag{4.22}$$

where the linearly divergent piece is

$$S_{\mu\nu\lambda} = \frac{1}{(2\pi)^4} \int d^4 k \; \frac{Tr_s\left[\not{k}\gamma_\mu \not{k}\gamma_\lambda \gamma_5 \not{k}\gamma_\nu\right]}{k^6}. \tag{4.23}$$

Let us now see what happens to this when the integration variable is changed to

$k' = (k + a)$:

$$S'_{\mu\nu\lambda} = S_{\mu\nu\lambda} + C_{\mu\nu\lambda\rho} a^\rho \tag{4.24}$$

where

$$C_{\mu\nu\lambda\rho} = -\frac{1}{(2\pi)^4} \int d^4k \frac{\partial}{\partial k_\rho} \left[\frac{Tr_s\left[\not{k}\gamma_\mu \not{k}\gamma_\lambda\gamma_5 \not{k}\gamma_\nu\right]}{k^6} \right]. \tag{4.25}$$

Using the cyclic property of the spinor trace Tr_s, and the following expression

$$\begin{aligned} Tr_s\left[\gamma_5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_\epsilon\gamma_\zeta\right] &= -4i\mathcal{E}_{\delta\epsilon\zeta\kappa}\left(g_{\kappa\alpha}g_{\beta\gamma} - g_{\kappa\beta}g_{\alpha\gamma} + g_{\kappa\gamma}g_{\alpha\beta}\right) \\ &\quad +4i\mathcal{E}_{\alpha\beta\gamma\kappa}\left(g_{\kappa\delta}g_{\epsilon\zeta} - g_{\kappa\epsilon}g_{\delta\zeta} + g_{\kappa\zeta}g_{\delta\epsilon}\right) \end{aligned} \tag{4.26}$$

we find that (4.25) becomes

$$C_{\mu\nu\lambda\rho} = -\frac{4i}{(2\pi)^4} \mathcal{E}_{\mu\nu\lambda\epsilon} \int d^4k \frac{\partial}{\partial k_\rho} \left(\frac{k_\epsilon}{k^4}\right). \tag{4.27}$$

Now transform this integral to Euclidean space, i.e. let $k_4 = ik_0$. Note that the integral vanishes when $\rho \neq \epsilon$ since it is odd in k. If $\rho = \epsilon$, then $(k_\mu)^2 = \frac{1}{4}k^2$ (no sum in μ). Converting the 4–dimensional integral to a 3–dimensional surface integral gives, noting that the solid angle for S^3 is $\oint d\Omega = 2\pi^2$:

$$\begin{aligned} C_{\mu\nu\lambda\rho} &= \frac{4}{(2\pi)^4} \mathcal{E}_{\mu\nu\lambda\epsilon} \int d^4k_E \frac{\partial}{\partial k_\rho} \left(\frac{k_\epsilon}{k^4}\right) \\ &= \frac{1}{(2\pi)^4} \mathcal{E}_{\mu\nu\lambda\rho} \int d^4k_E \frac{\partial}{\partial k_\epsilon} \left(\frac{k_\epsilon}{k^4}\right) \\ &= \frac{1}{(2\pi)^4} \mathcal{E}_{\mu\nu\lambda\rho} \oint \left(d^3S_E\right)_\epsilon \frac{k_\epsilon}{k^4} \\ &= \frac{1}{(2\pi)^4} \mathcal{E}_{\mu\nu\lambda\rho} \oint \frac{k_\epsilon}{k} \left(k^3 d\Omega\right) \frac{k_\epsilon}{k^4} \\ &= \frac{1}{8\pi^2} \mathcal{E}_{\mu\nu\lambda\rho}. \end{aligned} \tag{4.28}$$

Therefore by shifting the integration variables $k' = k + p_2$ and $k'' = k - p_1$, we see that (4.12) and (4.13) respectively become

$$p_1^\mu S_{\mu\nu\lambda} = \frac{1}{8\pi^2}\ \mathcal{E}_{\mu\nu\lambda\rho}\ p_2^\rho\ p_1^\mu \tag{4.29}$$

$$p_2^\nu S_{\mu\nu\lambda} = \frac{1}{8\pi^2}\ \mathcal{E}_{\mu\nu\lambda\rho}\ p_1^\rho\ p_2^\nu \tag{4.30}$$

but the axial vector Ward identity (4.11) is still satisfied. This demonstrates that it is impossible to satisfy both the vector and axial vector Ward identities. As it stands, this is unacceptable since (4.29) and (4.30) violate conservation of charge. To cure this, we define :

$$\tilde{S}_{\mu\nu\lambda} = S_{\mu\nu\lambda}(p_1, p_2) + S_{\mu\nu\lambda}(p_2, p_1) + \frac{1}{4\pi^2}\ \mathcal{E}_{\mu\nu\lambda\rho}(p_1 - p_2)^\rho \tag{4.31}$$

then the Ward identities become

$$(p_1 + p_2)^\lambda \tilde{S}_{\mu\nu\lambda} \ = \ \frac{1}{2\pi^2}\ \mathcal{E}_{\mu\nu\lambda\rho}\ p_2^\lambda\ p_1^\rho \qquad \text{(Axial)} \tag{4.32}$$

$$p_1^\mu \tilde{S}_{\mu\nu\lambda} \ = \ 0 \qquad \text{(Vector)} \tag{4.33}$$

$$p_2^\nu \tilde{S}_{\mu\nu\lambda} \ = \ 0 \qquad \text{(Vector).} \tag{4.34}$$

That is, the vector Ward identities (4.33) and (4.34) are satisfied thus preserving conservation of charge, but the axial vector Ward identity (4.32) has an extra term - this is the anomaly. This is endemic in the theory. It can not be circumvented by choosing another regularisation scheme where only one fermion is involved. Different regularisation schemes may put the anomaly somewhere else, but cannot get rid of it.

This poses a serious threat to quantum field theories since they cannot be renormalised under these circumstances. However, by choosing the gauge group and fermionic representations carefully, it is possible to cancel the anomaly where all contributions from different fermions are summed. It is remarkable that this seems to be the case in observation - provided that each generation of fermions is complete, i.e. each generation consists of an electron, an anti-neutrino and three colours each of up and down quarks, then the anomaly is cancelled. This is where one of the major predictions of gauge theories is made - that of the top quark, as it is needed to complete the third generation of fundamental fermions (at energy levels currently accessible by present day accelerators).

As we calculated only the amputated amplitude to the triangle diagram, if we include the polarisation vectors of the outgoing photons the divergence of the axial current (4.6) is modified to

$$\partial^{\mu} J^5_{\mu} = 2mJ^5 + \frac{g_V^2}{8\pi^2}\tilde{F}_{\mu\nu}F^{\mu\nu} \tag{4.35}$$

where $\tilde{F}_{\mu\nu}$, the dual of $F_{\mu\nu}$, is given by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\mathcal{E}^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{4.36}$$

The second term of the RHS of (4.35) is another way of stating the chiral anomaly. Even in the limit of massless fermions, the axial vector current J^5_{μ} is not conserved. Since the anomaly arose from a fermion loop with an axial vector and 2 vector vertices (in 4−dimensional spacetime), it is thus a quantum rather than classical effect.

Having seen how the chiral anomaly arises in gauge theories in four spacetime dimensions, it is natural to ask whether the anomaly is peculiar to four dimensions - do anomalies exist in higher dimensional field theories ? Many methods can be used to show that anomalies indeed exist in higher dimensional theories. They have been investigated by using the following techniques

- Feynman diagrammatic techniques similar to those given above [FK83].

- Path integrals [Fu80,WN87,Gi86].
- Differential geometry [Zu84,AG84].
- Derivative expansions [Fs85,No84 et al,DK87].
- Point splitting [Sc51,NS77,Ha69,JJ69,Crew].
- Chiral Jacobians [RD85].

To examine the finite temperature aspects of the anomalies, one can make some progress using the methods listed above. For example

- Dolan and Jackiw [DJ74] used finite temperature Feynman rules to show that the anomaly is temperature independent in the Schwinger model (two-dimensional QED). Similar conclusions have been arrived at using derivative expansions [DK87] and chiral Jacobians [RD85].
- Calculating Feynman diagrams for four-dimensional QED [CL88] also shows the temperature independence of the chiral anomaly as does evaluating the determinant of the Dirac operator [LN88].
- The anomaly for four-dimensional QCD is temperature independent [IM83].

From this it is tempting to conclude that there is strong evidence that the chiral anomaly is temperature independent in more general theories. To prove this using the methods above can be very tedious.

At zero temperature the point splitting technique provides a very elegant way of arriving at the chiral anomaly. This has been shown by Schwinger [Sc51] and extended to include four-dimensional QCD by Nielsen and Schroer [NS77]. Crewther [Crew] has further generalised this to arbitrary gauge groups in an even dimensional spacetime. It will transpire that this method is very well suited to investigate finite temperature effects of the chiral anomaly. The zero temperature case will be outlined in the next section, the following section treating the finite temperature situation.

4.3 The Chiral Anomaly - Point Splitting Approach

To provide the tools required for the examination of the chiral anomaly at finite temperature it will be helpful to give an outline of how the anomaly arises from the point splitting method at zero temperature. An advantage of this procedure is that it is gauge invariant and also it is calculated in the coordinate rather than momentum representation. In pursuing this, it is crucial that the zero modes of the Dirac operator in the presence of a background gauge field be identified. This is greatly simplified if the Dirac operator is defined on a compact manifold since its spectrum of eigenvalues is discrete. Choosing a non-compact manifold results in a continuous spectrum making it difficult to separate and count the number of zero modes.

To begin, the Dirac operator is defined on the compact $2n$ dimensional manifold, the hypersphere S^{2n}. This is the surface of a sphere of unit radius in $2n+1$ dimensions. An even dimensional manifold is required so as to define a γ_5 type matrix. It is only in such theories that chiral operators can be defined, i.e. left and right handed components of fermions can only be constructed in an even dimensional spacetime. Further, an Euclidean manifold is chosen - this does not affect the computation of the anomaly, only making it easier to carry through some steps of the calculation. The S^{2n} results are then stereographically projected onto R^{2n} and the anomaly is then derived.

Details of the relation between the Dirac operators in S^{2n} and R^{2n} are given in Appendix B. The Dirac equation after being projected from S^{2n} to R^{2n} is

$$i\not{D}u_\lambda(x) = \frac{2}{1+x^2}\lambda u_\lambda(x) \qquad\qquad x_\mu \in R^{2n} \tag{4.37}$$

Compare this with the well known Dirac equation in R^{2n}

$$i\not{D}\Psi(x) = m\Psi(x) \qquad\qquad x_\mu \in R^{2n} \tag{4.38}$$

where the covariant derivative $D_\mu = \partial_\mu + A_\mu$ acts on a representation R of the gauge group G. The background gauge field is smooth, antihermitean $A_\mu = -A_\mu^\dagger$ and has field strength tensor $F_{\mu\nu} = [D_\mu, D_\nu]$ which is assumed to die off sufficiently quickly for large x.

For each configuration of A_μ, the eigenfunctions $u_\lambda(x)$ have the following orthonormality relations

$$\int d^{2n}x \frac{2}{1+x^2} u_\lambda(x)^\dagger u_{\lambda'}(x) = \delta_{\lambda\lambda'} \tag{4.39}$$

and completeness relations

$$\sum_\lambda u_\lambda(x) u_\lambda(y)^\dagger = \frac{1+x^2}{2}\delta(x-y) \tag{4.40}$$

where the summation Σ_λ includes those eigenfunctions that have degenerate eigenvalues, and $\delta_{\lambda\lambda'}$ vanishes if λ and λ' are distinct.

Now introduce the fermionic propagator, which includes interactions with the gauge field

$$S'(x,y) = {\sum_\lambda}' \lambda^{-1} u_\lambda(x) u_\lambda(y)^\dagger \tag{4.41}$$

where the prime indicates that zero modes are excluded. If there are N distinct zero modes $u_{0i}(x)$,

$$i\not{D} u_{0i}(x) = 0 \qquad\qquad i = 1, \ldots, N \tag{4.42}$$

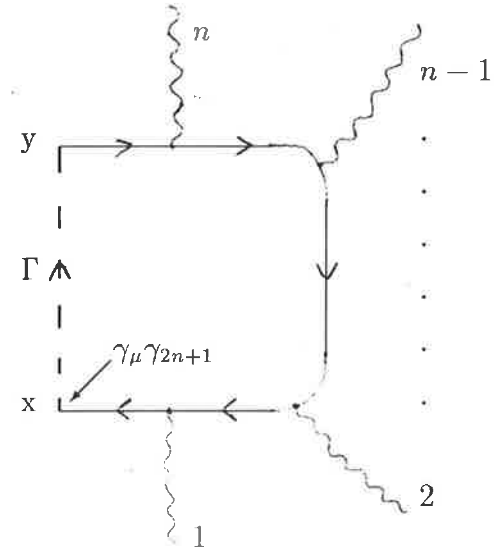


Figure 4.2: The contribution to the chiral anomaly in $2n$ dimensions.

then $S'(x,y)$ obeys the equation

$$i\not{D}_x S'(x,y) = \delta(x-y) - \frac{2}{1+x^2}\sum_{i=1}^{N} u_{0i}(x)u_{0i}(y)^{\dagger}. \tag{4.43}$$

The integer N is related to the 'winding number' of the gauge field.

Gauge-invariant functions may be constructed by Schwinger's method [Sc51]: let the fermion propagate from x to y, parallel transport back to x, and take the trace. This can be represented pictorially by Fig.[4.2]. The gauge fields (wavy lines), labelled by $1, 2, \ldots, n-1, n$, are attached to the fermion propagator $S'(x,y)$ and the dashed line Γ is the path taken by the parallel transport operator $E_\Gamma(y,x)$.

Let

$$G_\mu(x,y) = Tr_g\, \mathcal{G}_\mu(x,y)E_\Gamma(y,x) \tag{4.44}$$

be the gauge-invariant function obtained from the axial-vector projection

$$\mathcal{G}_\mu(x,y) = Tr_s i\gamma_\mu\gamma_{2n+1}S'(x,y) \tag{4.45}$$

where Tr_g, Tr_s are respectively, the gauge and spinor traces. The parallel transport operator is represented by the path ordered exponential

$$\begin{aligned} E_\Gamma(y,x) &= P_\Gamma \exp\left(-\int_x^y dz_\mu A_\mu(z)\right) \\ &= I - \int_x^y dz_\mu\, A_\mu(z) + \int_x^y dz'_\mu \int_x^{z'} dz_\nu\, A_\mu(z')\, A_\nu(z) - \ldots \end{aligned} \tag{4.46}$$

and involves ordering P_Γ along a path Γ running from x to y. As usual γ_μ and γ_{2n+1} are Hermitean $2^n \times 2^n$ matrices and obey the following anticommutation relations :

$$[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu} \qquad [\gamma_\mu, \gamma_{2n+1}]_+ = 0 \qquad \gamma_{2n+1} = (-1)^n \gamma_1 \gamma_2 \ldots \gamma_{2n}. \tag{4.47}$$

Under a gauge transformation U, the fermion propagator transforms as

$$S'(x,y) \to U(x)\, S'(x,y)\, U(y)^\dagger \tag{4.48}$$

and path ordered exponential as

$$E_\Gamma(y,x) \to U(y)\, E_\Gamma(y,x)\, U(x)^\dagger. \tag{4.49}$$

By noting the cyclic property of the trace, it can be seen that (4.44) is gauge invariant.

The anomaly arises from the divergence of G_μ in the limit $y \to x$ in which the path Γ shrinks to a point. In this limit, the ordered exponential $E_\Gamma(y,x)$ is regular,

$$E_\Gamma(y,x) = I + (x-y)_\mu A_\mu\left(\tfrac{1}{2}(x+y)\right) + O\left((x-y)^2\right) \tag{4.50}$$

but the fermion propagator $S'(x,y)$ is singular, with the leading power at $x \sim y$

given by

$$S'(x,y) \sim -i\not{\partial}^{-1}\delta(x-y) = \frac{-i(n-1)!}{2}\frac{(\not{x}-\not{y})}{(\pi(x-y)^2)^n}. \tag{4.51}$$

This requires finding singular terms in the short-distance expansion of $S'(x,y)$ which can be isolated by rewriting (4.43) in the form

$$(\not{\partial}+\not{A})\left(iS'(x,y)-\not{\partial}^{-1}\delta(x-y)\right) = -\not{A}\,\not{\partial}^{-1}\delta(x-y) + \text{regular}. \tag{4.52}$$

where is $\not{\partial}^{-1}$ is defined through the following relation

$$\not{\partial}\left(\not{\partial}^{-1}\delta(x-y)\right) = \delta(x-y) \tag{4.53}$$

which shows that $\not{\partial}^{-1}\delta(x-y)$ is the Green's function for the $\not{\partial}$ operator. Applying $\not{\partial}^{-1}$ to (4.52) gives

$$\begin{aligned}(\not{\partial}+\not{A})&\left(iS'(x,y)-\not{\partial}^{-1}\delta(x-y)+\not{\partial}^{-1}\not{A}\,\not{\partial}^{-1}\delta(x-y)\right)\\ &=\not{A}\,\not{\partial}^{-1}\not{A}\,\not{\partial}^{-1}\delta(x-y) + \text{regular}.\end{aligned} \tag{4.54}$$

Repeating the procedure will eventually make the R.H.S. of (4.54) regular, depending on the number of spacetime dimensions. Thus the propagator can be rearranged as

$$S'(x,y) = -i\left[\not{\partial}^{-1}-\not{\partial}^{-1}\not{A}\,\not{\partial}^{-1}+\not{\partial}^{-1}\not{A}\,\not{\partial}^{-1}\not{A}\,\not{\partial}^{-1}+\ldots\right]\delta(x-y)+\text{regular} \tag{4.55}$$

where the number of terms inside the square brackets is finite. As it stands, this expression is rather restrictive and somewhat cumbersome as gauge covariance is lost.

A much more convenient and elegant method for examining the short-distance expansion of $S'(x,y)$ is to use covariant derivatives rather than partial derivatives.

The singularity behaviour of (4.43) can be isolated by considering

$$S'(x,y) = -i(\not{D}\not{D})_x^{-1}\not{D}_x\left\{\delta(x-y) - \frac{2}{1+x^2}\sum_{i=0}^{N} u_{0i}(x)u_{0i}(y)^\dagger\right\} \tag{4.56}$$

with the inverse of

$$\not{D}\not{D} = D^2 + \tfrac{1}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu} \tag{4.57}$$

expanded as a series in the field-strength tensor $F_{\mu\nu}$:

$$(\not{D}\not{D})^{-1} = \sum_{k=0}^{n-1}\left\{-D^{-2}\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\right\}^k D^{-2} + \left\{-D^{-2}\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\right\}^n(\not{D}\not{D})^{-1}. \tag{4.58}$$

All singularities arise from the δ function $\delta(x-y)$ in (4.51). Contributions from the zero-mode term are regular at $x \sim y$, since N is finite - this was primarily the reason why a compact manifold was chosen. These singularities are either a simple logarithm

$$\sim (\text{constant})\ln(x-y)^2$$

or powers

$$\propto (x-y)^{-p} \qquad\qquad 0 \le p \le 2n-1$$

including direction-dependent terms such as

$$\frac{(x-y)_\alpha(x-y)_\beta}{(x-y)^2}.$$

The effect of each operator D^{-1} is to decrease the power p of the singularity by 1. Consequently all singular terms in the expansion of $S'(x,y)$ are generated by the finite series $\sum_k$ in (4.58) acting on $-i\not{D}\delta(x-y)$ in (4.56).

Because of the identity

$$Tr_s\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_k}\gamma_{\mu_{2n+1}} = \begin{cases} 0 & k < 2n \\ (2i)^n \mathcal{E}_{\mu_1\mu_2\ldots\mu_{2n}} & k = 2n \end{cases} \tag{4.59}$$

the spinor trace in (4.45) eliminates all singular terms except $k = n - 1$:

$$\begin{aligned} \mathcal{G}_\mu(x,y) &= Tr_s\gamma_\mu\gamma_{2n+1}\left\{-D^2\tfrac{1}{4}[\gamma_\alpha,\gamma_\beta]F_{\alpha\beta}\right\}^{n-1} D^{-2}\not{D}_x\delta(x-y) \\ &\qquad + (\text{finite})_{x\sim y} \\ &= 2(-i)^n\mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}\left\{D^{-2}F_{\alpha_1\beta_1}\ldots D^{-2}F_{\alpha_{n-1}\beta_{n-1}}D^{-2}D_\nu\right\}_x \\ &\qquad .\delta(x-y) + (\text{finite})_{x\sim y}. \end{aligned} \tag{4.60}$$

The short-distance limit is to be specified by taking

$$\eta = (y - x) \to 0 \tag{4.61}$$

with the mid-point

$$z = \frac{(x+y)}{2} \tag{4.62}$$

held fixed. Noting that if M and N are operators

$$(M+N)^{-1} = M^{-1} - M^{-1}NM^{-1} + M^{-1}NM^{-1}NM^{-1} - \ldots \tag{4.63}$$

and that

$$D^{-2} = \frac{1}{\partial^2 + 2A\cdot\partial + (\partial\cdot A) + A^2} \tag{4.64}$$

then potentially singular terms in (4.60) arise from the first two terms in the expansions

$$D^{-2} = \partial^{-2} - 2\,\partial^{-2}A\cdot\partial\,\partial^{-2} + O(\partial^{-4})$$

$$F_{\mu\nu}(w) \;=\; F_{\mu\nu}(z) + (w-z)_\rho \partial_\rho F_{\mu\nu}(z) + O((w-z)^2) \qquad (4.65)$$

where for $F_{\mu\nu}$ Taylor's theorem has been used.

The leading $O(\eta^{-1})$ power in $\mathcal{G}_\mu$ is generated by ∂^{-2} acting directly on the delta function in (4.60) :

$$\partial^{-2n}\partial_\nu \delta(\eta) = -\frac{2(-1)^n}{(n-1)!(4\pi)^n}\frac{\eta_\nu}{\eta^2} \qquad (4.66)$$

see Appendix C for details.

Contributions involving $\partial_\gamma F_{\alpha\beta}$ sum to zero (provided the limiting procedure of (4.61) and (4.62) is adopted), but $\mathcal{G}_\mu$ contains non-leading singularities due to the A-dependent terms in (4.65) and of $D_\nu = \partial_\nu + A_\nu$:

$$\left(D^{-2n}D_\nu\right)_x \delta(x-y) = \left(\partial_\nu + A_\nu - 2n\; A\cdot\partial\;\partial_\nu\;\partial^{-2}\right)_x \partial^{-2n}\delta(x) + (\text{finite})_{x\sim y}. \quad (4.67)$$

We can rid ourselves of these A-dependent terms by multiplying (4.67) by (4.50) and using the identity

$$2n\,\partial_\mu\,\partial_\nu\,\partial^{-2(n-1)}\delta(x) = (g_{\mu\nu} + x_\mu\partial_\nu)\,\partial^{-2n}\delta(x) + \text{constant}. \qquad (4.68)$$

where the constant term in (4.68) takes account of the arbitrary constant of integration (subtraction at x_0) needed to define

$$\partial^{-2n}\delta(x) = \frac{(-1)^{n-1}}{(4\pi)^n(n-1)!}\ln\left(x^2/x_0^2\right). \qquad (4.69)$$

The result is :

$$E_\Gamma(y,x)(D^{-2n}D_\nu)_x\delta(x-y) = (\partial_\nu)_x\partial^{-2n}\delta(x-y) + (\text{finite})_{x\sim y}. \qquad (4.70)$$

It is therefore convenient to specify the short-distance behaviour of $\mathcal{G}_\mu$ in the following way :

$$
\begin{aligned}
&E_\Gamma(z+\tfrac{1}{2}\eta, z-\tfrac{1}{2}\eta)\, \mathcal{G}_\mu(z-\tfrac{1}{2}\eta, z+\tfrac{1}{2}\eta) \\
&= \frac{-i^n}{4^{n-1}\pi^n(n-1)!}\, \mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}\, \frac{\eta_\nu}{\eta^2} \left(F_{\alpha_1\beta_1}\ldots F_{\alpha_{n-1}\beta_{n-1}}\right)_z \\
&\quad + R_\mu(z,\eta) \qquad (4.71)
\end{aligned}
$$

where $\lim_{\eta\to 0} R_\mu(z,\eta)$ is finite and independent of the direction of η_ν. There are non-leading singularities or direction-dependent terms such as

$$
\mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}\, \frac{\eta_\nu\eta_\gamma}{\eta^2}\, [D_\gamma, F_{\alpha_1\beta_1}]\; F_{\alpha_2\beta_2}\ldots F_{\alpha_{n-1}\beta_{n-1}}. \qquad (4.72)
$$

This property seems to be peculiar to the axial-vector current in a Yang-Mills field. (For example, in curved space, the analogue of $G_\mu(x,y)$ gives rise to a direction-dependent term at short distances [NSR78].)

Equation (4.71) allows us to define a subtracted amplitude

$$
\tilde{G}_\mu(z) = \lim_{\eta\to 0} Tr_g R_\mu(z,\eta). \qquad (4.73)
$$

Evidently the standard procedure [NS77,Sc51] of averaging over $\pm\eta_\mu$ is also applicable and gives the same answer :

$$
\tilde{G}_\mu(z) = \lim_{\eta\to 0} \frac{1}{2}\left\{G_\mu(z-\tfrac{1}{2}\eta, z+\tfrac{1}{2}\eta) + G_\mu(z+\tfrac{1}{2}\eta, z-\tfrac{1}{2}\eta)\right\}. \qquad (4.74)
$$

The $O(\eta^{-1})$ subtraction in $G_\mu(x,y)$ is conserved as a result of the Bianchi identity,

$$
\begin{aligned}
&\frac{\partial}{\partial z_\mu} Tr_g\, \mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}} \left(F_{\alpha_1\beta_1}\ldots F_{\alpha_{n-1}\beta_{n-1}}\right)_z \\
&= Tr_g \mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}} \left[D_\mu, F_{\alpha_1\beta_1}\ldots F_{\alpha_{n-1}\beta_{n-1}}\right] \\
&= 0 \qquad (4.75)
\end{aligned}
$$

so $\partial \cdot \tilde{G}$ can be obtained as the $\eta \to 0$ limit of the identity

$$\frac{\partial}{\partial z_\mu} Tr_g\, R_\mu(z,\eta) = \left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu}\right) G_\mu(x,y). \qquad (4.76)$$

The divergence (4.76) can be written

$$\left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu}\right) G_\mu(x,y) = T_1 + T_2 \qquad (4.77)$$

with

$$T_1 = Tr_g \left\{ \overrightarrow{D}_\mu (x) \mathcal{G}_\mu(x,y) - \mathcal{G}_\mu(x,y) \overleftarrow{D}_\mu (y) \right\} E_\Gamma(y,x) \qquad (4.78)$$

and

$$T_2 = Tr_g \left\{ \overrightarrow{D}_\mu (y) E_\Gamma(y,x) - E_\Gamma(y,x) \overleftarrow{D}_\mu (x) \right\} \mathcal{G}_\mu(x,y) \qquad (4.79)$$

where for any function $f(x,y)$

$$\overrightarrow{D}_\mu (x)\, f(x,y) = \frac{\partial f(x,y)}{\partial x_\mu} + A_\mu(x)\, f(x,y) \qquad (4.80)$$

and

$$f(x,y)\, \overleftarrow{D}_\mu (x) = -\frac{\partial f(x,y)}{\partial x_\mu} + f(x,y)\, A_\mu(x)\,. \qquad (4.81)$$

Equations (4.43) and (4.45) allow us to write the first term T_1 in terms of zero modes u_{0i} :

$$T_1 = 2\left\{ \left(1+x^2\right)^{-1} + \left(1+y^2\right)^{-1} \right\} \sum_i u_{0i}(y)^\dagger E_\Gamma(y,x) \gamma_{2n+1} u_{0i}(x). \qquad (4.82)$$

For the second term T_2, we need the short-distance result

$$\overrightarrow{D}_\mu (y) E_\Gamma(y,x) - E_\Gamma(y,x) \overleftarrow{D}_\mu (x) = (x-y)_\beta F_{\mu\beta}(z) + O((x-y)^2) \qquad (4.83)$$

which, together with the leading singularity of (4.71), implies

$$T_2 = \frac{-i^n}{4^{n-1}\pi^n(n-1)!}\frac{\eta_\beta\eta_\nu}{\eta^2}\mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}Tr_g F_{\mu\beta}F_{\alpha_1\beta_1}\ldots F_{\alpha_{n-1}\beta_{n-1}} + O(\eta) \quad (4.84)$$

in the $\eta \to 0$ limit (4.61). The leading term of (4.84) seems to be $O(\eta\eta/\eta^2)$, but it is not really direction dependent because

$$\mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}F_{\mu\beta}F_{\alpha_1\beta_1}\ldots F_{\alpha_{n-1}\beta_{n-1}} \quad (4.85)$$

is proportional to $\delta_{\nu\beta}$. This can be seen by supposing that we have an α dimensional spacetime, then

$$\begin{aligned} T_{i\{ji_1\ldots i_\alpha\}} &= \delta_{ij}\,\mathcal{E}_{i_1 i_2\ldots i_\alpha} + \delta_{ii_1}\,\mathcal{E}_{i_2\ldots i_\alpha j} + \delta_{ii_2}\,\mathcal{E}_{i_3\ldots i_\alpha j i_1} + \ldots + \delta_{ii_\alpha}\,\mathcal{E}_{ji_1\ldots i_{\alpha-1}} \\ &= 0 \end{aligned} \quad (4.86)$$

since there does not exist any construct which is antisymmetrised with respect to $\alpha+1$ indices in α dimensions. Note that the indices within braces $\{i_1 \ldots i_\alpha\}$ are antisymmetrised. This means that

$$F_{i_1 i_2}\ldots F_{i_{\alpha-1}i_\alpha}\,T_{i\{ji_1\ldots i_\alpha\}} = 0 \quad (4.87)$$

and

$$F_{ii_2}\ldots F_{i_{\alpha-1}i_\alpha}\,\mathcal{E}_{i_2\ldots i_\alpha j} = \frac{-\delta_{ij}}{\alpha}\,F_{i_1 i_2}\ldots F_{i_{\alpha-1}i_\alpha}\,\mathcal{E}_{i_1\ldots i_\alpha}. \quad (4.88)$$

Consequently the $\eta \to 0$ limit of T_2 can be taken without ambiguity to yield the result

$$\partial_\mu \tilde{G}_\mu(z) = \frac{4}{1+z^2}\sum_i u_{0i}(z)^\dagger \gamma_{2n+1} u_{0i}(z) + \lim_{\eta\to 0} T_2 \quad (4.89)$$

where

$$\lim_{\eta\to 0} T_2 = \frac{i^n}{2^{2n-2}\pi^n n!}\mathcal{E}_{\alpha_1\beta_1\ldots\alpha_n\beta_n}\, Tr_g\, F_{\alpha_1\beta_1}\ldots F_{\alpha_n\beta_n} \tag{4.90}$$

is the anomaly, in agreement with other calculations [Zu84].

4.4 Chiral Anomaly at Finite Temperature

In this section we confirm that the chiral anomaly is temperature independent for an arbitrary gauge group in an even dimensional spacetime. This is achieved by extending the procedure outlined in the previous section to finite temperature using the imaginary time formalism [Be74]. Many of the steps in the procedure are very similar to those carried out above, although care must be taken to ensure that any new properties arising from the finite temperature analysis are given particular attention; for example, the δ function now contains an infinite sum of terms.

Heuristically the anomaly should be temperature independent since it arises from the short-distance behaviour of quantum field theories. A regularisation scheme should be chosen so as to exhibit the finite temperature effects clearly. Finite temperature effects, in the imaginary time formalism [Be74], are manifested through global, rather than local properties of the field theory. No new divergences arise in such theories, the same counterterms used for zero temperature field theories can be used for renormalisation. Thus, one may expect the anomaly to be temperature independent. Further, the anomaly arises as a result of a finite, rather than infinite, effect in the regularisation scheme.

For $T \neq 0$, using the imaginary time formalism, the (fermion) boson fields are (anti-) periodic in the time coordinate, x_1, (we choose units such that Boltzmann's constant $k = 1$):

$$A^T_\mu(x_1 + 1/T, \vec{x}) = A^T_\mu(x_1, \vec{x})$$

$$
\begin{aligned}
&= T \sum_{j=-\infty}^{\infty} A^j_\mu(\vec{x}) \exp[i2\pi jTx_1] \\
u^T_\lambda(x_1 + 1/T, \vec{x}) &= -u^T_\lambda(x_1, \vec{x}) \\
&= T \sum_{j=-\infty}^{\infty} u^j_\lambda(\vec{x}) \exp[i(2j+1)\pi Tx_1]. \qquad (4.91)
\end{aligned}
$$

This imposes a restriction on the x_1 coordinate

$$x_1 \in [0, 1/T]. \qquad (4.92)$$

In $2n$ dimensions this necessitates operators to be defined on $S^1 \times M$, where M is a $(2n-1)$ dimensional manifold and S^1 takes into account the (anti-) periodic properties of the fields in the time coordinate. For our purposes we choose $M = S^{2n-1}$, giving the Dirac operator on $S^1 \times S^{2n-1}$ a discrete spectrum. Stereographic projection from S^{2n-1} to R^{2n-1} is carried out, leaving the S^1 factor unchanged :

$$S^1 \times S^{2n-1} \rightarrow S^1 \times R^{2n-1} \qquad (4.93)$$

Equivalently, one could use

$$R^1 \times S^{2n-1} \rightarrow R^1 \times R^{2n-1} \qquad (4.94)$$

on the understanding that R^1 is a compact manifold whose range is restricted to $[0, 1/T]$ and let the coordinates be

$$x_1 \in R^1 \qquad \vec{x} = (x_2, x_3, \ldots, x_{2n}) \in R^{2n-1}. \qquad (4.95)$$

One can convert from R^1 to S^1 by letting

$$x_1 \rightarrow \frac{1}{2\pi T}\theta \qquad 0 \leq \theta \leq 2\pi\,. \qquad (4.96)$$

Using the results of Appendix B, the stereographically projected Dirac equation from $R^1 \times S^{2n-1}$ to $R^1 \times R^{2n-1}$ is :

$$i\not{D}^T u_\lambda(x) = \frac{2}{1+\vec{x}^2}\lambda u_\lambda(x) \tag{4.97}$$

where the covariant derivative is

$$\not{D}^T = \gamma_1(\partial_1 + A_1^T) + \vec{\gamma}.(\vec{\partial} + \vec{A}^T). \tag{4.98}$$

The orthonormality and completeness relations of the eigenfunctions $u_\lambda(x)$ are modified to

$$\int_0^{1/T} dx_1 \int d^{2n-1}x \, \frac{2}{1+\vec{x}^2} u_\lambda(x)^\dagger u_{\lambda'}(x) = \delta_{\lambda\lambda'} \tag{4.99}$$

$$\sum_\lambda u_\lambda(x) u_\lambda(y)^\dagger = \frac{1+\vec{x}^2}{2}\delta_T(x-y) \tag{4.100}$$

where $\delta_T(x-y)$ is now the temperature dependent δ function.

Following the previous section, the finite temperature fermion propagator

$$S'_T(x,y) = \sum_\lambda{}' \lambda^{-1} u_\lambda^T(x) u_\lambda^T(y)^\dagger \tag{4.101}$$

obeys the equation

$$i\not{D}_x^T S'_T(x,y) = \delta_T(x-y) - \frac{2}{1+\vec{x}^2}\sum_{i=1}^N u_{0i}^T(x) u_{0i}^T(y)^\dagger. \tag{4.102}$$

Due to the boundary conditions (4.91), the fermion propagator has the property

$$S'_T(x_1 + 1/T, \vec{x}, y) = -S'_T(x_1, \vec{x}, y) \tag{4.103}$$

leading to an anti-periodic or temperature δ_T function

$$\begin{aligned}
\delta_T(x_1+1/T,\vec{x}) &= -\delta_T(x_1,\vec{x}) \\
&= \frac{\Gamma(n)}{2\pi^n}\,\partial_\mu \sum_{j=-\infty}^{\infty}(-1)^j \frac{x_\mu+\delta_{\mu 1}\,j/T}{((x_1+j/T)^2+|\vec{x}|^2)^n}
\end{aligned} \tag{4.104}$$

where $\mu = 1, 2, \ldots, 2n$. A derivation of (4.104) is outlined in Appendix C.

The path-ordered exponential, $E_\Gamma(x,y)$, together with (4.91) has the property

$$E_\Gamma^T\,(x_1+1/T,\,\vec{x},\,y_1+1/T,\,\vec{y}) = E_\Gamma^T\,(x_1,\,\vec{x},\,y_1,\,\vec{y}) \tag{4.105}$$

and as $x \to y$

$$E_\Gamma^T(y,x) = I + (x-y)_\mu A_\mu^T\,(\tfrac{1}{2}(x+y)) + O((x-y)^2). \tag{4.106}$$

Since $x_1, y_1 \in [0, 1/T]$, then at finite temperature (4.44) becomes

$$G_\mu^T(x,y) = Tr_g\;\mathcal{G}_\mu^T(x,y)E_\Gamma^T(y,x) \tag{4.107}$$

where after some work

$$\begin{aligned}
\mathcal{G}_\mu^T(x,y) &= Tr_s i\gamma_\mu\gamma_{2n+1}S'_T(x,y) \\
&= 2(-i)^n\mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}\left\{D^{-2}F_{\alpha_1\beta_1}\ldots D^{-2}F_{\alpha_{n-1}\beta_{n-1}}D^{-2}D_\nu\right\}_x^T \\
&\qquad .\delta_T(x-y) + (\text{finite})_{x\sim y} \\
&= \frac{2\,i^n}{(n-1)!(4\pi)^n}\mathcal{E}_{\mu\nu\alpha_1\beta_1\ldots\alpha_{n-1}\beta_{n-1}}\left(F_{\alpha_1\beta_1}\ldots F_{\alpha_{n-1}\beta_{n-1}}\right)_x^T \\
&\qquad \cdot\sum_{j=-\infty}^{\infty}(-1)^j\frac{(x-y)_\nu-\delta_{\nu 1}\,j/T}{(x_1-y_1+j/T)^2+|\vec{x}-\vec{y}|^2} + (\text{finite})_{x\sim y}.
\end{aligned} \tag{4.108}$$

The zero temperature counterpart to (4.108) is nothing but the $j = 0$ term.

Letting $\eta = (y - x)$, the finite temperature equivalent of T_2, (4.90), is now

$$T_2 = \frac{i^n}{2^{2n-2}\pi^n n!}\mathcal{E}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} Tr_g F_{\alpha_1\beta_1}\ldots F_{\alpha_n\beta_n} \cdot \sum_{j=-\infty}^{\infty}(-1)^j \frac{\eta^2 + \eta_1 j/T}{(\eta_1 + j/T)^2 + |\vec{\eta}|^2}. \tag{4.109}$$

In the limit $\eta \rightarrow 0$, only the $j = 0$ term survives in the summation $\sum_j$ of (4.109), leaving

$$\lim_{\eta\rightarrow 0} T_2 = \frac{i^n}{2^{2n-2}\pi^n n!}\, \mathcal{E}_{\alpha_1\beta_1\ldots\alpha_n\beta_n}\, Tr_g\, (F_{\alpha_1\beta_1}\ldots F_{\alpha_n\beta_n})^T \tag{4.110}$$

representing the anomaly. The form of the anomaly is unchanged by finite temperature effects - the coefficient is independent of temperature. However, global properties of the field configurations may be affected at finite temperature.

4.5 Atiyah-Singer Index

For $T = 0$, Nielsen and Schroer [NS77] show that the Atiyah-Singer Index can obtained by noting that integrating (4.89) with (4.90) over all space gives :

$$\int d^{2n}x\, \partial_\mu\, \tilde{G}_\mu(x) = \oint_S dS_\mu\, \tilde{G}_\mu(x) = 0 \tag{4.111}$$

where S is the surface enclosing the volume of integration. This was obtained by considering the corresponding equation on the hypersphere, which was shown to vanish when integrated over the hyperspherical coordinates. Hence one has

$$2\int d^{2n}x\, \frac{2}{1+z^2}\sum_i u_{0i}(x)^\dagger \gamma_{2n+1} u_{0i}(x) = -\int d^{2n}x\, \mathcal{A}(x) \tag{4.112}$$

where, from (4.90), $\mathcal{A}(x) = \lim_{\eta\rightarrow 0} T_2$ is the anomaly.

The zero modes of the Dirac operator $u_{0i}(x)$ can be expressed in terms of eigenstates of the matrix γ_{2n+1} since $[\gamma_{2n+1}, \not{D}]_+ = 0$:

$$\gamma_{2n+1}\, u_{0i}(x) = \pm\, u_{0i}(x)\,. \tag{4.113}$$

Usually states that have eigenvalue +1 are called right-handed whereas those with eigenvalue −1 are left-handed states. So, (4.112) can be rewritten as

$$2\,(n_R - n_L) = -\int d^{2n}x\, \mathcal{A}(x) = 2\,\nu \tag{4.114}$$

with $n_R\,(n_L)$ being the number of right(left)-handed zero modes and ν is an integer, usually regarded as the Atiyah-Singer Index.

Let us now examine what happens at finite temperature. The integral of the divergence of the axial current is

$$\int_0^\beta dx_1 \int d^{2n-1}x\, \partial_\mu\, \tilde{G}_\mu(x) = \int_0^\beta dx_1 \int d^{2n-1}x\, \partial_1\, \tilde{G}_1(x) + \int_0^\beta dx_1 \int d^{2n-1}x\, \partial_i\, \tilde{G}_i(x). \tag{4.115}$$

The second term on the RHS of (4.115) is zero for the same reasons that lead to the result of (4.111). By noting the finite temperature properties of solutions to the Dirac operator (4.91), and the parallel transport operator E^T_Γ (4.105), one can deduce that $\tilde{G}_\mu(x)$ obeys

$$\tilde{G}_\mu(x_1 + \beta, \vec{x}) = \tilde{G}_\mu(x_1, \vec{x}). \tag{4.116}$$

Thus the first term of (4.115) can be rearranged as

$$\begin{aligned}\int_0^\beta dx_1 \int d^{2n-1}x\, \partial_1\, \tilde{G}_1(x) &= \int d^{2n-1}x\, \left\{\tilde{G}_1(\beta, \vec{x}) - \tilde{G}_1(0, \vec{x})\right\} \\ &= 0\end{aligned} \tag{4.117}$$

because of (4.116). Hence

$$\int_0^\beta dx_1 \int d^{2n-1}x\, \partial_\mu \tilde{G}_\mu(x) = 0 \tag{4.118}$$

which shows that the Atiyah-Singer Index theorem is not affected by finite temperature effects.

4.6 Conclusion

The Nielsen-Schroer technique provides a very elegant way of deriving the chiral anomaly in the coordinate representation. It has transpired that it is also the most convenient method to extend to finite temperature and to show the temperature independence of the anomaly. To do the analysis by Feynman diagrammatic techniques would have been much more tedious. To the best of my knowledge, the coordinate representation of an anti-periodic or temperature δ_T function has not been observed in the literature.

The temperature independence of the anomaly - or more specifically, the amplitude associated with the anomaly, does not mean that the $\pi^0 \to 2\gamma$ decay width Γ, is temperature independent. In fact the width does depend on temperature [CL88], through the phase space factors of the outgoing photons, as outlined in Section 1.5. Other parameters, e.g. the pion decay constant f_π [LS90] and the pion mass m_π [GoL89] also depend on temperature.

After this work was completed, I became aware of a paper by Wang [Wa89] who reaches similar conclusions regarding the temperature independence of the anomaly by using chiral Jacobians.

The Nielsen-Schroer method also allows one to obtain the Atiyah-Singer Index in a relatively simple way. This was well suited to extend to finite temperature, and showed its temperature independence. From a physical point of view, this means that the number of zero or massless modes of the Dirac operator is unchanged as the temperature is varied – provided no phase transitions are encountered.

Appendix A

Feynman Rules

This appendix gives the finite temperature Feynman rules needed to compute the self-energies of the gluon and scalar fields. Lists of finite temperature integrals with δ functions are also given.

A.1 Finite Temperature Feynman Rules

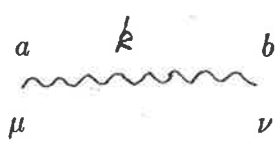
The Lagrangian for an $SU(N)$ non-abelian gauge theory is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + (\partial^\mu \chi^{a*})D^{ab}_\mu \chi^b. \tag{A.1}$$

Using the conventions of Muta [Mu87], Baier et. al. [BPS90] the finite temperature Feynman rules are :

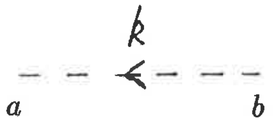
Propagators:

- Gluons A^a_μ



$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \delta^{ab}\left[\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)\frac{1}{k^2+i\epsilon} - \frac{2\pi i}{\exp[\beta|k_0|]-1}\left(g_{\mu\nu}\delta(k^2) + (1-\alpha)\,k_\mu k_\nu \delta'(k^2)\right)\right] \quad \text{(A.2)}$$

- Ghosts χ^a

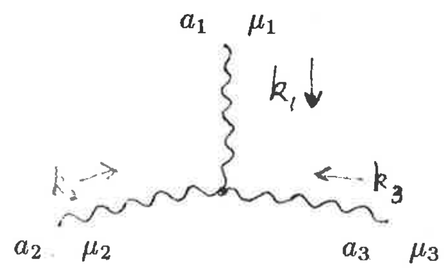


(A.3)

$$\tilde{\Delta}^{ab}(k) = -\delta^{ab}\left(\frac{1}{k^2+i\epsilon} - \frac{2\pi i}{\exp[\beta|k_0|]-1}\delta(k^2)\right) \quad \text{(A.4)}$$

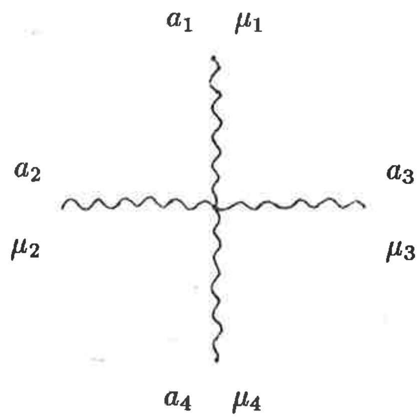
Vertices:

- Tri-gluon vertex



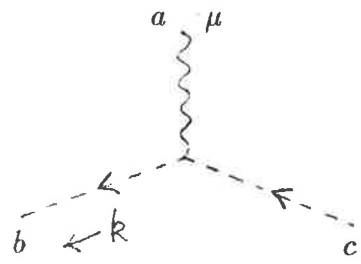
$$-\,igf^{a_1a_2a_3}\left((k_1-k_2)_{\mu_3}\,g_{\mu_1\mu_2} + (k_2-k_3)_{\mu_1}\,g_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2}\,g_{\mu_3\mu_1}\right) \quad \text{(A.5)}$$

- Four-gluon vertex



$$- \quad g^2 \left(\left(f^{13,24} - f^{14,32} \right) g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} + \left(f^{12,34} - f^{14,23} \right) g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} + \left(f^{13,42} - f^{13,24} \right) g_{\mu_1 \mu_4} g_{\mu_3 \mu_2} \right) \tag{A.6}$$

- Ghost-gluon vertex



$$- ig f^{abc} k_\mu \tag{A.7}$$

- Loop integrations : For gluon loops there is a factor $d^4k/((2\pi)^4 i)$ and for ghost loops a factor $-d^4k/((2\pi)^4 i)$.

The structure constants of the group are denoted by f^{abc} and

$$f^{12,34} = f^{a_1 a_2 a} f^{a_3 a_4 a}. \tag{A.8}$$

For scalar fields described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \tag{A.9}$$

the scalar propagator is

$$\Delta(p, m) = \frac{1}{p^2 - m^2 + i\epsilon} - \frac{2\pi i}{e^{\beta |p_0|} - 1} \delta(p^2 - m^2). \tag{A.10}$$

A.2 Finite Temperature Integrals

To evaluate integrals with δ functions or products of δ functions and their derivatives, it is convenient, at first, to use cartesian coordinates to integrate out all the δ functions then use spherical polar coordinates to do all angular integrations. What remains is a single integral over the magnitude of resultant vector from the spherical polar coordinates. Usually the final integral is left untouched, since it can not be put into a simple closed form.

Integrals that are normally encountered can be summarised as follows, where $\beta = 1/T$ and $a, n = 0, 1, 2, \ldots$ as:

$$f(1,n) = \pi \int_0^\infty dx \frac{x^n}{(-1+x)^2(1+x)^2} \frac{1}{\exp[\beta m x/2] - 1} \quad (A.11)$$

$$f(2,n) = \pi \int_0^\infty dx \frac{x^n}{(-1+x)^2(1+x)^2} \frac{d}{dx} \frac{1}{\exp[\beta m x/2] - 1} \quad (A.12)$$

$$f(3,n) = \pi \int_0^\infty dx \frac{x^n}{(-1+x)^2(1+x)^2} \ln\left(\left|\frac{1+x}{-1+x}\right|\right) \frac{1}{\exp[\beta m x/2] - 1} \quad (A.13)$$

$$f(4,n) = \pi \int_0^\infty dx \frac{x^n}{(-1+x)^2(1+x)^2} \ln\left(\left|\frac{1+x}{-1+x}\right|\right) \frac{d}{dx} \frac{1}{\exp[\beta m x/2] - 1} \quad (A.14)$$

$$g(a,n) = \pi \int_1^\infty dx\, x^n \frac{d^a}{dm^a} \left(\frac{1}{\exp[\beta m x/2] - 1}\right)^2. \quad (A.15)$$

Below are the finite temperature integrals used to evaluate the one loop correction to the gluon propagator, given that the momenta of the external lines are set to $p_\mu = (0, \vec{p})$ such that $p^2 = -m^2$.

Integrals involving one δ function :

$$\int d^4k \delta(k^2) n_\beta(|k_0|) = m^2 \left(f(1,1) - 2f(1,3) + f(1,5)\right)$$

$$\int d^4k \frac{1}{(k+p)^2} \delta(k^2) n_\beta(|k_0|) = \frac{-f(3,0)}{2} + f(3,2) - \frac{f(3,4)}{2}$$

$$\int d^4k \frac{1}{((k+p)^2)^2} \delta(k^2) n_\beta(|k_0|) = \frac{f(1,1) - f(1,3)}{m^2}$$

$$\int d^4k k_\mu \frac{1}{(k+p)^2} \delta(k^2) n_\beta(|k_0|) = p_\mu \left(\frac{-f(1,1)}{2} + f(1,3) - \frac{f(1,5)}{2} + \frac{f(3,0)}{4} - \right.$$

$$\frac{f(3,2)}{2} + \frac{f(3,4)}{4}\Bigg)$$

$$\int d^4k k_\mu \frac{1}{((k+p)^2)^2}\delta(k^2)n_\beta(|k_0|) = \frac{p_\mu}{m^2}\left(\frac{-f(1,1)}{2} + \frac{f(1,3)}{2} + \frac{f(3,0)}{4} - \frac{f(3,2)}{2} + \frac{f(3,4)}{4}\right)$$

$$\int d^4k k_\mu k_\nu \frac{1}{(k+p)^2}\delta(k^2)n_\beta(|k_0|) = m^2\left(\frac{f(1,1)}{8} - \frac{f(1,3)}{4} + \frac{f(1,5)}{8} - \frac{f(3,0)}{16} + \frac{3f(3,2)}{16} - \frac{3f(3,4)}{16} + \frac{f(3,6)}{16}\right) g_{\mu\nu} + m^2\left(\frac{-f(1,1)}{8} + \frac{f(1,3)}{4} - \frac{f(1,5)}{8} + \frac{f(3,0)}{16} - \frac{5f(3,2)}{16} + \frac{7f(3,4)}{16} - \frac{3f(3,6)}{16}\right) g_{\mu 0}g_{\nu 0} + \left(\frac{3f(1,1)}{8} - \frac{3f(1,3)}{4} + \frac{3f(1,5)}{8} - \frac{3f(3,0)}{16} + \frac{7f(3,2)}{16} - \frac{5f(3,4)}{16} + \frac{f(3,6)}{16}\right) p_\mu p_\nu$$

$$\int d^4k k_\mu k_\nu \frac{1}{((k+p)^2)^2}\delta(k^2)n_\beta(|k_0|) = \left(\frac{f(1,1)}{4} - \frac{f(1,3)}{2} + \frac{f(1,5)}{4} - \frac{f(3,0)}{8} + \frac{f(3,2)}{4} - \frac{f(3,4)}{8}\right) g_{\mu\nu} + \left(\frac{-f(1,1)}{4} + \frac{3f(1,3)}{4} - \frac{f(1,5)}{2} + \frac{f(3,0)}{8} - \frac{f(3,2)}{4} + \frac{f(3,4)}{8}\right) g_{\mu 0}g_{\nu 0} + \left(\frac{3f(1,1)}{4} - \frac{5f(1,3)}{4} + \frac{f(1,5)}{2} - \frac{3f(3,0)}{8} + \frac{3f(3,2)}{4} - \frac{3f(3,4)}{8}\right) \frac{p_\mu p_\nu}{m^2}$$

$$\int d^4k k_\mu k_\nu k_\rho \frac{1}{((k+p)^2)^2}\delta(k^2)n_\beta(|k_0|) = \left(\frac{-15f(1,1)}{16m^2} + \frac{7f(1,3)}{4m^2} - \frac{13f(1,5)}{16m^2} + \frac{15f(3,0)}{32m^2} - \frac{33f(3,2)}{32m^2} + \frac{21f(3,4)}{32m^2} - \frac{3f(3,6)}{32m^2}\right) p_\mu p_\nu p_\rho + \left(\frac{-3f(1,1)}{16} + \frac{3f(1,3)}{8} - \frac{3f(1,5)}{16} + \frac{3f(3,0)}{32} - \frac{7f(3,2)}{32} + \frac{5f(3,4)}{32} - \frac{f(3,6)}{32}\right)(p_\mu g_{\nu\rho} + p_\nu g_{\rho\mu} + p_\rho g_{\mu\nu}) + \left(\frac{3f(1,1)}{16} - \frac{f(1,3)}{2} + \frac{5f(1,5)}{16} - \frac{3f(3,0)}{32} + \frac{9f(3,2)}{32} - \frac{9f(3,4)}{32} + \frac{3f(3,6)}{32}\right)(p_\mu g_{\nu 0}g_{\rho 0} + p_\nu g_{\rho 0}g_{\mu 0} + p_\rho g_{\mu 0}g_{\nu 0})$$

Integrals involving a derivative of a single δ function :

$$\int d^4k \delta'(k^2)n_\beta(|k_0|) = 2f(1,-1) - 4f(1,1) + 2f(1,3) - 2f(2,0) + 4f(2,2) - 2f(2,4)$$

$$\int d^4k \frac{1}{(k+p)^2}\delta'(k^2)n_\beta(|k_0|) = \frac{1}{m^2}\left(f(1,1) - f(1,3) - f(3,-2) + 2f(3,0) - f(3,2) + f(4,-1) - 2f(4,1) + f(4,3)\right)$$

$$\int d^4k k_\mu \delta'(k^2)n_\beta(|k_0|) = 0$$

$$\int d^4k k_\mu \frac{1}{(k+p)^2}\delta'(k^2)n_\beta(|k_0|) = \frac{p_\mu}{m^2}\left(-f(1,-1) + \frac{3f(1,1)}{2} - \frac{f(1,3)}{2} + f(2,0) - 2f(2,2) + f(2,4) + \frac{f(3,-2)}{2} - \frac{3f(3,0)}{4} + \frac{f(3,4)}{4} - \frac{f(4,-1)}{2} + f(4,1) - \frac{f(4,3)}{2}\right)$$

$$\int d^4k k_\mu k_\nu \delta'(k^2)n_\beta(|k_0|) = m^2\left(\frac{-f(1,1)}{6} + \frac{f(1,3)}{3} - \frac{f(1,5)}{6} + \frac{f(2,2)}{6} - \frac{f(2,4)}{3} + \frac{f(2,6)}{6}\right) g_{\mu\nu} + m^2\left(\frac{-f(1,1)}{3} + \frac{2f(1,3)}{3} - \frac{f(1,5)}{3} - \frac{2f(2,2)}{3} + \frac{4f(2,4)}{3} - \frac{2f(2,6)}{3}\right) g_{\mu 0}g_{\nu 0}$$

$$\int d^4k k_\mu k_\nu \frac{1}{(k+p)^2}\delta'(k^2)n_\beta(|k_0|) = \left(\frac{f(1,-1)}{4} - \frac{f(1,1)}{4} - \frac{f(1,3)}{4} + \frac{f(1,5)}{4} - \frac{f(2,0)}{4} + \frac{f(2,2)}{2} - \frac{f(2,4)}{4} - \frac{f(3,-2)}{8} + \frac{f(3,0)}{4} - \frac{f(3,2)}{8} + \frac{f(4,-1)}{8} - \frac{3f(4,1)}{8} + \frac{3f(4,3)}{8} - \frac{f(4,5)}{8}\right) g_{\mu\nu} + \left(\frac{-f(1,-1)}{4} + \frac{f(1,1)}{4} + \frac{f(1,3)}{2} - \frac{f(1,5)}{2} + \frac{f(2,0)}{4} - \frac{f(2,2)}{2} + \frac{f(2,4)}{4} + \frac{f(3,-2)}{8} - \frac{3f(3,2)}{8} + \frac{f(3,4)}{4} - \frac{f(4,-1)}{8} + \frac{5f(4,1)}{8} - \frac{7f(4,3)}{8} + \frac{3f(4,5)}{8}\right) g_{\mu 0}g_{\nu 0} + \left(\frac{3f(1,-1)}{4} - \frac{3f(1,1)}{4} - \frac{f(1,3)}{2} + \frac{f(1,5)}{2} - \frac{3f(2,0)}{4} + \frac{3f(2,2)}{2} - \frac{3f(2,4)}{4} - \frac{3f(3,-2)}{8} + \frac{f(3,0)}{2} + \frac{f(3,2)}{8} - \frac{f(3,4)}{4} + \frac{3f(4,-1)}{8} - \frac{7f(4,1)}{8} + \frac{5f(4,3)}{8} - \frac{f(4,5)}{8}\right) \frac{p_\mu p_\nu}{m^2}$$

$$\int d^4k k_\mu k_\nu \frac{1}{((k+p)^2)^2}\delta'(k^2)n_\beta(|k_0|) = \frac{g_{\mu\nu}}{m^2}\left(\frac{f(1,-1)}{2} - \frac{3f(1,1)}{4} + \frac{f(1,3)}{4} - \frac{f(2,0)}{2} + f(2,2) - \frac{f(2,4)}{2} - \frac{f(3,-2)}{4} + \frac{3f(3,0)}{8} - \frac{f(3,4)}{8} + \frac{f(4,-1)}{4} - \frac{f(4,1)}{2} + \frac{f(4,3)}{4}\right) + \frac{1}{m^2}\left(\frac{-f(1,-1)}{2} + \frac{f(1,1)}{4} - \frac{f(1,3)}{4} + \frac{f(2,0)}{2} - \frac{3f(2,2)}{2} + f(2,4) + \frac{f(3,-2)}{4} - \frac{3f(3,0)}{8} + \frac{f(3,4)}{8} - \frac{f(4,-1)}{4} + \frac{f(4,1)}{2} - \right.$$

$$\left.\frac{f(4,3)}{4}\right) g_{\mu 0} g_{\nu 0} + \left(\frac{3f(1,-1)}{2} - \frac{7f(1,1)}{4} - \frac{f(1,3)}{4} - \frac{3f(2,0)}{2} + \frac{5f(2,2)}{2} - f(2,4) - \frac{3f(3,-2)}{4} + \frac{9f(3,0)}{8} - \frac{3f(3,4)}{8} + \frac{3f(4,-1)}{4} - \frac{3f(4,1)}{2} + \frac{3f(4,3)}{4}\right) \frac{p_\mu p_\nu}{m^4}$$

$$\int d^4k k_\mu k_\nu k_\rho \frac{1}{(k+p)^2} \delta'(k^2) n_\beta(|k_0|) = \left(\frac{-5f(1,-1)}{8m^2} + \frac{23f(1,1)}{48m^2} + \frac{19f(1,3)}{24m^2} - \frac{31f(1,5)}{48m^2} + \frac{5f(2,0)}{8m^2} - \frac{17f(2,2)}{12m^2} + \frac{23f(2,4)}{24m^2} - \frac{f(2,6)}{6m^2} + \frac{5f(3,-2)}{16m^2} - \frac{11f(3,0)}{32m^2} - \frac{11f(3,2)}{32m^2} + \frac{15f(3,4)}{32m^2} - \frac{3f(3,6)}{32m^2} - \frac{5f(4,-1)}{16m^2} + \frac{13f(4,1)}{16m^2} - \frac{11f(4,3)}{16m^2} + \frac{3f(4,5)}{16m^2}\right) p_\mu p_\nu p_\rho + \left(\frac{f(1,-1)}{8} + \frac{5f(1,1)}{48} - \frac{17f(1,3)}{24} + \frac{23f(1,5)}{48} - \frac{f(2,0)}{8} + \frac{7f(2,2)}{12} - \frac{19f(2,4)}{24} + \frac{f(2,6)}{3} - \frac{f(3,-2)}{16} - \frac{f(3,0)}{32} + \frac{11f(3,2)}{32} - \frac{11f(3,4)}{32} + \frac{3f(3,6)}{32} + \frac{f(4,-1)}{16} - \frac{5f(4,1)}{16} + \frac{7f(4,3)}{16} - \frac{3f(4,5)}{16}\right) (p_\mu g_{\nu 0} g_{\rho 0} + p_\nu g_{\rho 0} g_{\mu 0} + p_\rho g_{\mu 0} g_{\nu 0}) + \left(\frac{-f(1,-1)}{8} + \frac{7f(1,1)}{48} + \frac{f(1,3)}{12} - \frac{5f(1,5)}{48} + \frac{f(2,0)}{8} - \frac{f(2,2)}{3} + \frac{7f(2,4)}{24} - \frac{f(2,6)}{12} + \frac{f(3,-2)}{16} - \frac{3f(3,0)}{32} - \frac{f(3,2)}{32} + \frac{3f(3,4)}{32} - \frac{f(3,6)}{32} - \frac{f(4,-1)}{16} + \frac{3f(4,1)}{16} - \frac{3f(4,3)}{16} + \frac{f(4,5)}{16}\right) (p_\mu g_{\nu\rho} + p_\nu g_{\rho\mu} + p_\rho g_{\mu\nu})$$

Integrals involving two δ functions :

$$\int d^4k \delta(k^2) \delta((k+p)^2) n_\beta(|k_0|)^2 = \frac{g(1,0)}{2}$$

$$\int d^4k k_\mu \delta(k^2) \delta((k+p)^2) n_\beta(|k_0|)^2 = \frac{-p_\mu}{4} g(1,0)$$

$$\int d^4k k_\mu k_\nu \delta(k^2) \delta((k+p)^2) n_\beta(|k_0|)^2 = m^2 \left(\frac{g(1,0)}{16} - \frac{g(1,2)}{16}\right) g_{\mu\nu} + m^2 \left(\frac{-g(1,0)}{16} + \frac{3g(1,2)}{16}\right) g_{\mu 0} g_{\nu 0} + \left(\frac{3g(1,0)}{16} - \frac{g(1,2)}{16}\right) p_\mu p_\nu$$

$$\int d^4k k_\mu k_\nu k_\rho \delta(k^2) \delta((k+p)^2) n_\beta(|k_0|)^2 = \left(\frac{-5g(1,0)}{32} + \frac{3g(1,2)}{32}\right) p_\mu p_\nu p_\rho + \left(\frac{m^2 g(1,0)}{32} - \frac{3m^2 g(1,2)}{32}\right) (p_\mu g_{\nu 0} g_{\rho 0} + p_\nu g_{\rho 0} g_{\mu 0} + p_\rho g_{\mu 0} g_{\nu 0}) +$$

$$\left(\frac{-(m^2g(1,0))}{32}+\frac{m^2g(1,2)}{32}\right)(p_\mu g_{\nu\rho}+p_\nu g_{\rho\mu}+p_\rho g_{\mu\nu})$$

$$\int d^4k\delta(k^2)\delta'((k+p)^2)n_\beta(|k_0|)^2=\frac{g(1,-2)}{2m^2}-\frac{g(2,0)}{2m}$$

$$\int d^4kk_\mu\delta(k^2)\delta'((k+p)^2)n_\beta(|k_0|)^2=\left(\frac{-g(1,-2)}{4}+\frac{g(1,0)}{4}\right)\frac{p_\mu}{m^2}+g(2,0)\frac{p_\mu}{4m}$$

$$\int d^4kk_\mu k_\nu\delta(k^2)\delta'((k+p)^2)n_\beta(|k_0|)^2=\left(\frac{g(1,-2)}{16}-\frac{g(1,0)}{16}-\frac{mg(2,0)}{16}+\frac{mg(2,2)}{16}\right)g_{\mu\nu}+\left(\frac{-g(1,-2)}{16}-\frac{g(1,0)}{16}+\frac{mg(2,0)}{16}-\frac{3mg(2,2)}{16}\right)g_{\mu0}g_{\nu0}+\left(\frac{3g(1,-2)}{16m^2}-\frac{5g(1,0)}{16m^2}-\frac{3g(2,0)}{16m}+\frac{g(2,2)}{16m}\right)p_\mu p_\nu$$

$$\int d^4kk_\mu k_\nu k_\rho\delta(k^2)\delta'((k+p)^2)n_\beta(|k_0|)^2=\left(\frac{-5g(1,-2)}{32m^2}+\frac{3g(1,0)}{8m^2}-\frac{3g(1,2)}{32m^2}+\frac{5g(2,0)}{32m}-\frac{3g(2,2)}{32m}\right)p_\mu p_\nu p_\rho+\left(\frac{g(1,-2)}{32}+\frac{3g(1,2)}{32}-\frac{mg(2,0)}{32}+\frac{3mg(2,2)}{32}\right)\cdot(p_\mu g_{\nu0}g_{\rho0}+p_\nu g_{\rho0}g_{\mu0}+p_\rho g_{\mu0}g_{\nu0})+\left(\frac{-g(1,-2)}{32}+\frac{g(1,0)}{16}-\frac{g(1,2)}{32}+\frac{mg(2,0)}{32}-\frac{mg(2,2)}{32}\right)(p_\mu g_{\nu\rho}+p_\nu g_{\rho\mu}+p_\rho g_{\mu\nu})$$

$$\int d^4kk_\mu k_\nu\delta'(k^2)\delta((k+p)^2)n_\beta(|k_0|)^2=\left(\frac{g(1,-2)}{16}-\frac{g(1,0)}{16}-\frac{mg(2,0)}{16}+\frac{mg(2,2)}{16}\right)g_{\mu\nu}+\left(\frac{-g(1,-2)}{16}-\frac{g(1,0)}{16}+\frac{mg(2,0)}{16}-\frac{3mg(2,2)}{16}\right)g_{\mu0}g_{\nu0}+\left(\frac{3g(1,-2)}{16m^2}+\frac{3g(1,0)}{16m^2}-\frac{3g(2,0)}{16m}+\frac{g(2,2)}{16m}\right)p_\mu p_\nu$$

$$\int d^4kk_\mu k_\nu k_\rho\delta'(k^2)\delta((k+p)^2)n_\beta(|k_0|)^2=\left(\frac{-5g(1,-2)}{32m^2}-\frac{3g(1,0)}{16m^2}+\frac{3g(1,2)}{32m^2}+\frac{5g(2,0)}{32m}-\frac{3g(2,2)}{32m}\right)p_\mu p_\nu p_\rho+\left(\frac{g(1,-2)}{32}+\frac{g(1,0)}{16}-\frac{3g(1,2)}{32}-\frac{mg(2,0)}{32}+\frac{3mg(2,2)}{32}\right)(p_\mu g_{\nu0}g_{\rho0}+p_\nu g_{\rho0}g_{\mu0}+p_\rho g_{\mu0}g_{\nu0})+\left(\frac{-g(1,-2)}{32}+\frac{g(1,2)}{32}+\frac{mg(2,0)}{32}-\frac{mg(2,2)}{32}\right)(p_\mu g_{\nu\rho}+p_\nu g_{\rho\mu}+p_\rho g_{\mu\nu})$$

$$\int d^4kk_\mu k_\nu\delta'(k^2)\delta'((k+p)^2)n_\beta(|k_0|)^2=\left(\frac{3g(1,-4)}{16m^2}-\frac{g(1,-2)}{8m^2}-\frac{g(1,0)}{16m^2}-\frac{g(2,-2)}{8m}+\frac{g(2,2)}{8m}+\frac{g(3,0)}{16}-\frac{g(3,2)}{16}\right)g_{\mu\nu}+\left(\frac{-3g(1,-4)}{16m^2}-\frac{g(1,0)}{16m^2}+\right.$$

$$\frac{g(2,-2)}{8m}+\frac{g(2,0)}{4m}-\frac{3g(2,2)}{8m}-\frac{g(3,0)}{16}+\frac{3g(3,2)}{16}\Bigg)g_{\mu 0}g_{\nu 0}+\Bigg(\frac{9g(1,-4)}{16m^4}-$$

$$\frac{5g(1,0)}{16m^4}-\frac{3g(2,-2)}{8m^3}-\frac{g(2,0)}{4m^3}+\frac{g(2,2)}{8m^3}+\frac{3g(3,0)}{16m^2}-\frac{g(3,2)}{16m^2}\Bigg)p_\mu p_\nu$$

Appendix B

The Dirac Equation

The first section of this appendix details a projection of the Dirac equation from a compact to a non-compact manifold suited for zero temperature field theory. The second section presents a similar analysis but for finite temperature field theory.

B.1 Dirac Equation : $S^{2n} \leftrightarrow R^{2n}$

This section gives an outline of how the stereographic projection of the Dirac equation from S^{2n} to R^{2n} is carried out [NS77,Crew]. This is applicable to the case of zero temperature field theory.

Let $\{x_\mu\}$ and $\{r_\mu, r_{2n+1}\}$ be the coordinates of R^{2n} and S^{2n} respectively ($\mu = 1, \ldots, 2n$). Performing stereographic projection from the south pole, the coordinates are related by

$$r_\mu = \frac{2x_\mu}{1+x^2} \qquad\qquad r_{2n+1} = \frac{1-x^2}{1+x^2} \tag{B.1}$$

where

$$x^2 = x_1^2 + \ldots + x_{2n}^2 \tag{B.2}$$

and the coordinates of S^{2n} satisfy

$$r_1^2 + \ldots + r_{2n}^2 + r_{2n+1}^2 = 1. \tag{B.3}$$

The volume elements of the respective manifolds are related by

$$\left(\frac{2}{1+x^2}\right)^{2n} d^{2n}x = d\Omega \tag{B.4}$$

where $d\Omega$ is the elemental solid angle for the hypersphere.

The Dirac matrices for R^{2n} are denoted by γ_μ while those for S^{2n} are given by Γ_μ and are defined as

$$\Gamma_\mu = -i\,\gamma_\mu\gamma_{2n+1} \qquad\qquad \Gamma_{2n+1} = \gamma_{2n+1} \tag{B.5}$$

where

$$\gamma_{2n+1} = (-1)^n\gamma_1\gamma_2\ldots\gamma_{2n} \tag{B.6}$$

and satisfy the anti-commutation relations

$$[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu} \qquad\qquad [\gamma_\mu, \gamma_{2n+1}]_+ = 0. \tag{B.7}$$

If we let the coordinates of S^{2n} be given by Latin subscripts $a, b = 1, \ldots, 2n, 2n+1$ and those of R^{2n} by Greek subscripts $\mu, \nu = 1, \ldots, 2n$ then using

$$s_{ab} = \frac{1}{4i}[\Gamma_a, \Gamma_b] \tag{B.8}$$

and the angular momentum operator

$$l_{ab} = r_a p_b - r_b p_a \qquad\qquad p_a = -i\frac{\partial}{\partial r_a} \tag{B.9}$$

or equivalently

$$l_{\mu\nu} = -ix_\mu\partial_\nu + ix_\nu\partial_\mu \qquad\qquad l_{2n+1\,\mu} = -ix_\mu x_\nu\partial_\nu - \frac{i}{2}\left(1-x^2\right)\partial_\mu \tag{B.10}$$

one can show that

$$\frac{\left(1+x^2\right)^{2n}}{2} i\gamma_\mu\partial_\mu = \left(1+x^2\right)^{n-1}\left(1-i\gamma_\mu x_\mu\right)\left(s_{ab}l_{ab}+n\right)\left(1+i\gamma_\nu x_\nu\right)\left(1+x^2\right)^{n-1}. \tag{B.11}$$

Equation (B.11) gives a relation between the Dirac operator in R^{2n} to that of S^{2n}

$$i\gamma_\mu\partial_\mu \to s_{ab}l_{ab} + n. \tag{B.12}$$

If gauge fields are included, then the correspondence of the potentials in R^{2n}, A_μ, to those in S^{2n}, $\hat{A}_\mu$ and $\hat{A}_{2n+1}$, is

$$\hat{A}_\mu = \frac{1+x^2}{2}A_\mu - x_\mu x_\nu A_\nu \qquad\qquad \hat{A}_{2n+1} = -x_\mu A_\mu \tag{B.13}$$

and the $\hat{A}_a$ are constrained by

$$r_a\hat{A}_a = 0 \tag{B.14}$$

i.e. the field configurations $\hat{A}_a$ are 'tangential' to the surface of the hypersphere S^{2n}.

Interactions of fermions with gauge fields can be implemented, as is normally done, by minimal substitution to (B.11), i.e.

$$\partial_\mu \to D_\mu = \partial_\mu + A_\mu \tag{B.15}$$

which means that on the hypersphere

$$l_{ab} \to L_{ab} = l_{ab} + ir_a\hat{A}_b - ir_b\hat{A}_a \tag{B.16}$$

where $A_\mu = -A_\mu^\dagger$ are the anti-hermitian gauge potentials of any gauge group.

The normalisations of the fermion fields are

$$\int \hat{\Psi}_1(r)\hat{\Psi}_2(r)d\Omega = \int \Psi_1(x)\Psi_2(x)\frac{2}{1+x^2}d^{2n}x \tag{B.17}$$

where $\hat{\Psi}(r)$ are fermion fields on S^{2n} and $\Psi(x)$ are those on R^{2n} and are related by

$$\Psi(x) = \left(\frac{2}{1+x^2}\right)^n \frac{1}{\sqrt{2}}\left(1 - i\gamma_\mu x_\mu\right)\hat{\Psi}(r) \tag{B.18}$$

B.2 Dirac Equation : $S^{2n-1} \times R^1 \leftrightarrow R^{2n-1} \times R^1$

The treatment of the finite temperature case is analogous to that of the previous section, except that the (anti-) periodic properties of the fields are taken into account. This means that the range of the R^1 coordinate is restricted to $[0, \beta]$. One could have just as well chosen S^1 rather than R^1. The convention regarding indices for the various coordinates used in this section is as follows

- The index $\mu = 1, 2, \ldots, 2n$ labels the coordinates of $R^1 \times R^{2n-1}$
- The indices $a, b = 2, 3, \ldots, 2n+1$ label the coordinates of S^{2n}
- The indices $i, j, k, l, m = 2, \ldots, 2n$ label the coordinates of R^{2n-1}.

Label the coordinates as follows

$$\begin{aligned} R^1 \times R^{2n-1} &\rightarrow R^1 \times S^{2n-1} \\ (x_1, x_2, \ldots, x_{2n}) &\rightarrow (r_1, r_2, \ldots, r_{2n}, r_{2n+1}) \end{aligned} \tag{B.19}$$

where the first coordinate of each set is that of R^1. Stereographically projecting from S^{2n-1} to R^{2n-1} and leaving the R^1 component unchanged, the relations between the coordinates are

$$r_1 = x_1 \qquad r_i = \frac{2x_i}{1+\vec{x}^2} \qquad r_{2n+1} = \frac{1-\vec{x}^2}{1+\vec{x}^2} \tag{B.20}$$

where

$$\vec{x} = (x_2, x_3, \ldots, x_{2n}) \tag{B.21}$$

and

$$r_2^2 + r_3^2 + \ldots + r_{2n}^2 + r_{2n+1}^2 = 1. \tag{B.22}$$

The volume elements are related by

$$dx_1 d^{2n-1}x = dr_1 d\Omega \tag{B.23}$$

where $d\Omega$ is the solid angle element for the hypersphere S^{2n-1} and $dx_1 = dr_1$. The same definitions for l_{ab} and s_{ab} as in (B.9) and (B.8) respectively are used (noting that now $a, b = 2, 3, \ldots, 2n+1$). The matrix γ_{2n+1} is still defined as in (B.6) and satisfies the same anticommutation relations.

The relation between the Dirac operators in the respective manifolds is now

$$\begin{aligned} i\gamma_\mu \partial_\mu &= 2\,\frac{(1 - i\gamma_j x_j)}{(1+\vec{x}^2)} \\ &\cdot \left[\frac{i}{2}(1 + i\gamma_m x_m)\ \gamma_1\ (1 - i\gamma_l x_l)\frac{\partial}{\partial r_1} + s_{ab} l_{ab} + \frac{2n-1}{2}\right] \frac{(1 + i\gamma_k x_k)}{(1+\vec{x}^2)} \end{aligned} \tag{B.24}$$

where $j, k, l, m = 2, 3, \ldots, 2n$ and $\mu = 1, 2, \ldots, 2n$.

Equation (B.24) shows that the relation of the Dirac operator in $R^1 \times R^{2n-1}$ to that of $R^1 \times S^{2n-1}$ is

$$i\gamma_\mu \partial_\mu \to \frac{i}{2}(1 + i\gamma_m x_m)\ \gamma_1\ (1 - i\gamma_l x_l)\frac{\partial}{\partial r_1} + s_{ab} l_{ab} + \frac{2n-1}{2}. \tag{B.25}$$

The relations of the gauge fields are

$$\hat{A}_1 = A_1 \qquad \hat{A}_j = \frac{1+\vec{x}^2}{2}A_j - x_j x_k A_k \qquad \hat{A}_{2n+1} = -x_j A_j \tag{B.26}$$

where the $\hat{A}_a$ are constrained by

$$r_a \hat{A}_a = 0 \tag{B.27}$$

and by performing the minimal substitution as in (B.15) one can include interactions.

The fermion fields are normalised such that

$$\int dr_1\, d\Omega \hat{\Psi}_1(r)\hat{\Psi}_2(r) = \int dx_1\; d^{2n-1}x\; \Psi_1(x)\Psi_2(x)\frac{2}{1+\vec{x}^{\,2}} \tag{B.28}$$

where $\hat{\Psi}(r)$ are fermion fields on $R^1 \times S^{2n}$ and $\Psi(x)$ are those on $R^1 \times R^{2n}$ and are related by

$$\Psi(x) = \left(\frac{2}{1+\vec{x}^{\,2}}\right)^{n-1/2} \frac{1}{\sqrt{2}}\left(1 - i\gamma_k x_k\right)\hat{\Psi}(r). \tag{B.29}$$

Appendix C

Representations of δ Functions

This appendix gives coordinate representations of zero and finite temperature delta functions and other associated functions required for evaluating the short distance behaviour of the fermion propagator.

C.1 The δ Function for R^{2n}

A coordinate representation of a δ function for R^{2n} is [Crew]

$$\delta(x) = \frac{\Gamma(n)}{2\pi^n}\,\partial_\mu \left[\frac{x_\mu}{(x^2)^n}\right]. \tag{C.1}$$

Hence it can be seen that

$$\begin{aligned} \partial_\mu\,\partial^{-2}\,\delta(x) &= \frac{\Gamma(n)}{2\pi^n}\,\frac{x_\mu}{(x^2)^n} \\ &= \frac{(-1)^{n-1}}{2^{2n-1}\,\pi^n\,(n-1)!}\,\left(\partial^2\right)^{n-1}\left[\frac{x_\mu}{x^2}\right], \end{aligned} \tag{C.2}$$

and if this procedure is repeated $n-1$ times one arrives at

$$\partial_\mu\left(\partial^{-2}\right)^n\,\delta(x) = \left(\partial^{-2}\right)^n\,\partial_\mu\,\delta(x) = \frac{(-1)^{n-1}}{2^{2n-1}\,\pi^n\,(n-1)!}\left[\frac{x_\mu}{x^2}\right]. \tag{C.3}$$

The meaning of ∂^{-2} can be inferred from the following

$$\partial^2 \left(\partial^{-2}\,\delta(x)\right) = \delta(x) \tag{C.4}$$

i.e. $\partial^{-2}\,\delta(x)$ is the Green's function for the ∂^2 operator.

Let $g(x,z)$ be the solution of

$$\partial^2\, g(x,z) = \delta(x-z) \tag{C.5}$$

then by using conventional methods, the solution to (as an example)

$$\partial^2\, \left[\partial^{-2}\, F_{\alpha\beta}\,\partial^{-2}\ \not{\partial}\,\delta(x-y)\right] = F_{\alpha\beta}\,\partial^{-2}\ \not{\partial}\,\delta(x-y) \tag{C.6}$$

is

$$\begin{aligned} \partial^{-2}\, F_{\alpha\beta}\,\partial^{-2}\ \not{\partial}\,\delta(x-y) &= \int d^{2n}z\, g(x,z)\, F_{\alpha\beta}(z) \int d^{2n}w\, g(z,w)\,\gamma_\mu \frac{\partial}{\partial w_\mu}\,\delta(w-y) \\ &= F_{\alpha\beta}(x) \int d^{2n}z \int d^{2n}w\, g(x,z)\, g(z,w)\,\gamma_\mu \frac{\partial}{\partial w_\mu}\,\delta(w-y) + \ldots \end{aligned} \tag{C.7}$$

where Taylor's expansion has been used for $F_{\mu\nu}$.

C.2 The Anti-Periodic or Temperature δ Function

The antiperiodic or temperature δ function on $S^1 \times R^{2n-1}$ can be obtained by finding the solution to the following equation

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} f(x,y) = \delta_T(x-y) \tag{C.8}$$

with the boundary condition

$$f(x_1 + 1/T, \vec{x}, y) = -f(x_1, \vec{x}, y). \tag{C.9}$$

This implies

$$\delta_T(x_1 + 1/T, \vec{x}) = -\delta_T(x_1, \vec{x}). \tag{C.10}$$

Since $f(x, y) = f(x - y)$, let

$$f(x) = T \sum_{j=-\infty}^{\infty} \exp[i(2j+1)\pi T x_1] \int \frac{d^{2n-1}k}{(2\pi)^{2n-1}} \exp\left[i\vec{k}.\vec{x}\right] \tilde{f}\left(j, \vec{k}\right) \tag{C.11}$$

where

$$\tilde{f}\left(j, \vec{k}\right) = \frac{-1}{(2j+1)^2\pi^2T^2 + |\vec{k}|^2}. \tag{C.12}$$

To perform the summation we use [Jo61]

$$\sum_{m=1,\ m \text{ odd}}^{\infty} \frac{\cos(m\theta)}{m^2 - a^2} = \frac{\pi \sin(a(\pi/2 - \theta))}{4a\ \cos(a\pi/2)} \tag{C.13}$$

where $0 \leq \theta \leq \pi$ and a may be complex.

Hence

$$\sum_{j=-\infty}^{\infty} \frac{\exp[i(2j+1)\pi T x_1]}{(2j+1)^2\pi^2T^2 + |\vec{k}|^2} =$$
$$\frac{-1}{2|\vec{k}|T} \left\{(-\exp[-|\vec{k}|x_1] + \frac{\exp[|\vec{k}|x_1] + \exp[-|\vec{k}|x_1]}{\exp[|\vec{k}|/T] + 1}\right\}. \tag{C.14}$$

Choosing polar coordinates for R^{2n-1}, the angular integration over $\tilde{f}$ can be performed to yield

$$
\begin{aligned}
f(x) &= \left(\frac{2}{|\vec{x}|}\right)^{n-3/2} \pi^{n-1/2} \int_0^\infty d|\vec{k}|\, |\vec{k}|^{n-3/2} J_{n-3/2}\left(|\vec{k}|\, |\vec{x}|\right) \{\exp[-|\vec{k}|\, x_1] \\
&\quad - \sum_{j=0}^{\infty}(-1)^j \left(\exp[|\vec{k}|\,(x_1 - 1/T - j/T)] + \exp[|\vec{k}|\,(-x_1 - 1/T - j/T)]\right)\} \\
&= \pi^{n-1/2} \frac{\Gamma(2n-2)}{\Gamma(n-1/2)} \sum_{j=-\infty}^{\infty} (-1)^j \frac{1}{((x_1 + j/T)^2 + |\vec{x}|^2)^{n-1}} \qquad \text{(C.15)}
\end{aligned}
$$

where $J_n(z)$ is a Bessel function of the first kind; the solid angle in d dimensions is [tHV72]

$$
\frac{2\pi^{d/2}}{\Gamma(d/2)}
$$

and the following were used [GR80]

$$
\begin{aligned}
&\int_0^\infty \exp[-\alpha z] J_\nu(\beta z)\, z^{\mu-1} dz = \\
&\left(\frac{\beta}{2\alpha}\right)^\nu \frac{\Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} F\left((\nu+\mu)/2, (\nu+\mu+1)/2, \nu+1, -\beta^2/\alpha\right) \qquad \text{(C.16)}
\end{aligned}
$$

where F is the Hypergeometric function and

$$
F(-n, \beta, \beta, -z) = (1+z)^n \quad (\beta \quad \text{arbitrary}). \qquad \text{(C.17)}
$$

Hence

$$
\delta_T(x) = \frac{\Gamma(n)}{2\pi^n}\, \partial_\mu \sum_{j=-\infty}^{\infty} (-1)^j \frac{x_\mu + \delta_{\mu 1}\, j/T}{((x_1 + j/T)^2 + |\vec{x}|^2)^n}. \qquad \text{(C.18)}
$$

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ERRATA

Page 7, Eq.(1.17) uses $\partial_0\phi = \partial\phi/\partial\tau$.

Page 8, Eq.(1.21) should read $\int_0^\beta d\tau\, e^{i(\omega_n-\omega_{n'})\tau} = \beta\delta_{nn'}$.

Page 9, Eq.(1.28): τ and τ' should be replaced by τ_1 and τ_2 respectively.

Page 18, second line : '$\rho(k_0, \vec{k}$ etc.' should read '$\rho(k_0, \vec{k})$ etc.'

Page 24, Eq.(1.84): This result is found in [DJ74], with $m^2 \to -m^2$.

Page 27: f_π in Eq.(1.95) is defined through the following relation [IZ85]

$$< 0|A_\mu(x)|\pi(p) >= ip_\mu f_\pi e^{-ip\cdot x}$$

Page 32, Eq.(2.1) and Eq.(2.2) are actually well defined. The problem in finite temperature field theory in the RTF is to find out how to deal with constructs of the form

$$\left(\delta(k^2)\right)^2 .$$

Page 36, middle: Strictly speaking $\partial_\mu A^\mu = 0$ only when $\alpha = 0$.

Page 42, Sect. 2.3: Computing the self-energy of the gluon at finite temperature is greatly simplified if one sets the energy of the external gluon p_0, to zero. In this case, the Boltzmann factors for each particle in the loop will be identical. Thus, in general, we can set the external momentum of the gluon to $p_\mu = (0, \vec{p})$ and $p^2 = -m^2$ and evaluate the polarisation tensor. By taking advantage of the rotational covariance of the theory, one can then set $\vec{p} = (0, 0, m)$ in order to calculate the coefficients C_1 and C_2, which are functions of p_0. The final result in Eq.(2.42) is specific to the case of $p_\mu = (0, \vec{p})$ and $p^2 = -m^2$. While keeping these conditions in mind, one sees that the Ward identity Eq.(2.50) is satisfied.

Page 44: ϵ in Eq.(2.39) is defined by $\epsilon = (4 - d)/2$ where d is the number of spacetime dimensions.

Page 45, Eqs.(2.43) and (2.44): A comment on the singularity structure of the $f(a, n)$ functions in the coefficients C_1 and C_2. The integrands of the various combinations of the $f(a, n)$ in C_1 and C_2 to order $(1 - \alpha)^0$ and $(1 - \alpha)^1$ are well defined at $x = 1$ since they occur in the form of $f(a, n) - 2f(a, n + 2) + f(a, n + 4)$ which is non-singular at $x = 1$. For order $(1 - \alpha)^2$ terms, singularities arise when a term

$$(1 - \alpha)\,\frac{(k + p)_\mu (k + p)_\nu}{((k + p)^2 + i\epsilon)^2}$$

from the $T = 0$ part of one propagator is combined with

$$(1 - \alpha)\, k_\rho k_\sigma \delta'(k^2)\, \frac{1}{e^{\beta|k_0|} - 1}$$

from the $T \neq 0$ part of the other propagator. The singularity occurs when both $k^2 = 0$ and $(k + p)^2 = 0$ simultaneously. It appears the singularity (a linear divergence) is more severe than that inferred by simple power counting (a logarithmic divergence). Further investigation is required in order to find an appropiate prescription to handle this type of singularity.

The combinations of the integrands as encountered in C_1 and C_2 are well behaved at $x = 0$. Series expansions of the integrands near the point $x = 0$ are, for C_1

$$\frac{8\pi}{\beta m}\left(-1 + (1+\alpha) - \frac{2}{3}(1+\alpha)^2\right) + O(x)$$

and C_2

$$\frac{8\pi}{3\beta m}\left((1+\alpha) - 2(1+\alpha)^2\right) + O(x).$$

Page 46 Eq.(2.49): the following replacement should be made

$$\frac{d^a}{dm^a} \longrightarrow \frac{d^{a-1}}{dm^{a-1}}$$

Page 48: The definiton of $\mathcal{L}_{\mu\nu}$ in (2.52) should read

$$\mathcal{L}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + w\, g_{\mu 0} g_{\nu 0} - \mathcal{T}_{\mu\nu}$$

and (2.55) should be

$$\begin{aligned} A(p) &= B(p) = C_1\, p^2 \\ w &= \frac{C_2}{C_1}. \end{aligned}$$

Page 56, line 5: The δ function is a distribution which arises from the pinching of two singular functions (with poles) - resulting in a real singular function with no poles.
Page 60, after Eq.(3.32): 'where $E = E_b - E_a$' should read 'where E is the energy associated with the propagator'.
Page 62: δm^2 is the momentum dependent self-energy.
Page 71, above Eq.(4.8): A number of generalisations of the γ_5 matrix to complex dimensions d have been put forward. It is not clear which of these generalisations are consistent when applied to particular calculations. Apparently, dimensional regularisation can be applied to γ_5 type problems. In this case, either the commutation relations (4.4) or the cyclicity of the trace has to be abandoned for $\epsilon \neq 0$.
Page 72 Eq.(4.10): $f(k-a)$ should be replaced by $f(k'=a)$.
Page 101 Eq.(A.15): the following replacement should be made

$$\frac{d^a}{dm^a} \longrightarrow \frac{d^{a-1}}{dm^{a-1}}$$