RELATIVISTIC FLUIDS IN COSMOLOGY

by

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A thesis submitted in accordance with
the requirements of the degree of
Doctor of Philosophy

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May 1980

CONTENTS

ABSTRACT

We attempt a study of the Friedmann cosmological models from the point of view of relativistic thermodynamics. We develope relativistic quantum statistical mechanics within the general relativistic framework for both homogeneous and inhomogeneous situations. Nonequilibrium thermodynamics is also considered and we give some discussion on the behaviour of photons within the models. Arguments show that it is inconsistent to assume both thermal equilibrium and the constancy of photon number. We advance on a programme to explore the full thermodynamical and chemical possibilities within the Friedmann models by first considering very simple models. In particular we detail a model containing Synge gases and black body radiation which mimics the gross features of the standard model and exhibits the behaviour of the photon to baryon ratio. In general the ratio is not constant and it depends on flows of energy and entropy during the expansion. Subsequent analysis considers schemes which may explain the present large value of this ratio. We discuss in some detail the cosmological relevance of recent discoveries within particle physics involving extended spectrums for quarks and leptons. It is suggested that the standard cosmological model is consistent only with a particular view; albeit also a standard one, within modern particle physics. Near and far deviant models are considered in relation to certain persistent issues, the photon to baryon ratio, light element production, within modern cosmology. A new model is presented which attacks the first of these problems in a novel fashion. Subsequently we consider the relevance of the new grand unified theories and a very different approach, that of the bootstrap. Finally we consider useful formalisms for the equations of state and develop efficient numerical schemes needed for the computations.

STATEMENT OF ORIGINALITY

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university. To the best of my knowledge and belief the thesis contains no material previously published or written by another person except where due reference is made in the text of the thesis.

Alan Barnes

ACKNOWLEDGEMENTS

My wholehearted thanks to Peter Szekeres. Without his guidance, support and patience this thesis could not have been completed.

To Cheryl Marshall and Carol Baker who decyphered the manuscript and typed the thesis, also my thanks.

Financial assistance from a Commonwealth Postgraduate Research Award, the University of Adelaide and the Commonwealth Unemployment Service is gratefully acknowledged.

CHAPTER 1

INTRODUCTION

1. Preliminary Remarks

In this thesis we attempt an analysis of some of the major issues which confront contemporary cosmological theorizing.

Since the mid 1960's when the cosmological microwave radiation was discovered and confirmed the background radiation has been a central concern for cosmological model building. A standard model has been developed, commonly called the hot big bang, in which the radiation dominates the early universe. The model has gained considerable approval most notably because of its success in yielding reasonable light element abundances whose synthesis in contemporary or recent objects seems far too low. However this model as it presently stands is deficient in the respect (we will see later how it is deficient in other respects as well) that in assuming the background is a facet of the early universe it does not explain its origin. The background can be characterized by the ratio of the number of photons to the number of baryons - a number henceforth called f_b - which observationally is around 10^8 (the number may be out by an order of magnitude or so depending on the actual present baryon density). This number is much less than other dimensionless numbers occurring in cosmology $\sim 10^{40}$ and is clearly much greater than one and so is in need of explanation. For the standard model the processes which generate the radiation must occur at an epoch previous to that of the nucleosynthesis of the light elements. On the other hand non-standard models have been proposed in which the radiation is produced and thermalized at more recent epochs. The deficit with such models however, is their general inability to naturally produce the observed abundances of these light elements.

The problem of the photon to baryon ratio is our initial motivation for a general study of thermal and chemical evolution within the homogeneous and isotropic Friedmann models. As we will see the problem is part of a more general set of problems which involve the origin of the basic cosmological charges, baryonic, electron lepton, muon lepton ... .

2. A General Outline

We discern in this thesis three main areas of analysis and criticism. The major and central part of the work is concerned with an analysis of cosmological model building.

We begin in Ch. 2 with a general analysis of equilibrium and non-equilibrium thermodynamics in a general relativistic context. The analysis deals with both homogeneous and inhomogeneous models even though in subsequent chapters we deal with only homogeneous situations. We are motivated in this general approach since it brings to a head some important issues in relation to the behaviour of photons within cosmological fluids, we show for example that it is inconsistent to have both thermal equilibrium and photon number conservation, as is commonly assumed. Our formalism is also general enough to include inhomogeneous models since it seems to us that eventually all so-called homogeneous models must be studied as some sort of limit of the inhomogeneous models. This point is related to the surprising degree of across horizon homogeneity as is exemplified by the detailed isotropy of the microwave radiation. Such homogenity has no explanation within the standard model or for that matter any other model and must be analysed from the point of view of inhomogeneous models. Ch. 2 also developes statistical mechanics from a simple starting point and provides the full relativistic quantum statistical equations of state with which the cosmological fluid is to be described. The formalism is set up to allow for chemical reactions between components as well as transfers of energy and entropy.

Subsequent chapters investigate the wide range of models implicit in the description in Ch. 2. We restrict ourselves to the homogeneous and isotropic Friedmann-Robertson-Walker models (without cosmological constant) and from Ch. 3 onwards gradually build models of increasing complexity. We first fix some of the ideas expressed in Ch. 2 in a fully calculable model which mimics the gross behaviour of standard cosmology. The model contains Synge gases of electrons and protons together with black body radiation and allows us to consider models of high and low photon to baryon ratios, since the ratio is a natural parameter at the singularity (i.e. $t = 0$). For initial values of f_b that are not overlarge, f_b can change considerably during the universal expansion. Thus f_b is not in general a constant, it only appears constant for the standard model since it is initially very large. Interestingly for low f_b models Rees has proposed a late epoch photon generating mechanism in which the presently hypothesized missing (or dark) matter is just the matter responsible for producing the present background.

In the next two sections of Ch. 3 we deal with some very simple cosmological models. We deal first with a model containing a single massive gas described by the relativistic quantum statistical equations of state. For degenerate densities of fermions and bosons, that is for models in which the quantum statistics are important, the models exhibit some interesting features. The fermion gas, for example, has a change in behaviour from relativistic degeneracy to non-relativistic degeneracy at a time independent of the extent of that degeneracy. The expansion time previous to the changeover depends only on the fermi energy. In the last section of Ch. 3 we study an expanding fluid which contains only zero mass components. Such models are formally interesting from the point of view of the dynamical behaviour of energies and entropies. The chemical and thermodynamical evolution of the models is trivial. There are no transfers of energy or entropy between the components and the ratio of photons to some conserved number is exactly constant. The equations highlight the complementary character of the chemical potential and the temperature, both fall as the scale parameter and either of them can be used to determine the time in the model. Such models are physically untenable since we must expect pair production of massive particles at some temperature. Nonetheless they do give the high temperature limiting behaviour of any model containing a finite number of elementary species, massive or otherwise. In the same section we give a rather elegant solution for a model containing black body radiation, neutrinos and their antiparticles in thermal and chemical equilibrium.

In Ch. 4 we advance our analysis to include first many species described by quantum statistics in thermal equilibrium but without reactions and subsequently many species participating in the simple reactions of pair production. Most important for us are the flows of entropies and energies between components that are induced by the expansion since they may offer a route to hypothesizing a mechanism which might be responsible for the photon background. In the last section we turn to a discussion of the astrophysical photon generating mechanism of the Rees model.

These chapters interpenetrate with the numerical attack offered in the Appendixes A-D. The work there constitutes our second main area of analysis. The quantum statistical integrals are by no means elementary to calculate and complicated expressions which contain them present some difficulty. In Ch. 2 and Appendixes A and B we present two representations of the integral which depending on the context are expedient to use. The I^{ab} notation allows the most efficient representation of the equations of state and has a natural physical interpretation. However many of the functions that need to be derived contain derivatives of the I^{ab} and the $\{I^{ab}\}$ do not form a closed set under differentiation. Another representation, the Q^n, is presented which does have this latter property and is therefore more useful in calculating the functions we employ. Both I^{ab} and Q^n will be used interchangably and we detail schemes that can rapidly calculate their values in Appendixes A and B. Basically there are 5 regions of calculation; relativistic, non-relativistic, degenerate, non-degenerate and semi-relativistic, semi-degenerate. It is only in the last region where we need resort to actual quadrature of the integrals. Another notational innovation allows us in one stroke to calculate the pair production integrals. Embedded in a fluid of black body radiation the pair producing and annihilating chemical reactions proceed at equilibrium with equal and opposite chemical potentials for particles and antiparticles and one can construct simply linearly related integrals, the J^{ab} and R^n to replace the I^{ab} and Q^n. This is done in Appendix C. Thus the equations developed for many components without pair production in Sec. 2, Ch. 4 can be converted to those for pair production by a simple notational change. Finally in Appendix D we detail a numerical scheme which can tackle the chemical and thermodynamical evolution of the models envisaged in Ch. 4 and previously.

In Ch. 5 in a return to more realistic models we examine the standard model and its near deviants. The standard model as we define it there, is a model containing only one net particle "charge" - the baryonic charge. All other charges - electric, electron lepton, muon lepton, ... are put to zero. The discussion is particularly topical since we seem to be at a unique period in both modern particle physics and cosmological theorizing. The possibility of higher massed leptons beyond the muon, the recently discovered tau particle and more, bring the appropriately modified standard model into confrontation with observation. The great success of the standard model is its "natural" production of a reasonable helium abundance since other models in relation to it seem to require rather "ad hoc" assumptions to produce comparable amounts. However the standard model's helium production is particularly sensitive to the actual number of massless (or small massed) neutrinos types. Each new type increases the energy density during nucleosynthesis and increases net helium production so that the possibility arises that the modified standard model may overproduce helium. The determination of the actual number of lepton types is clearly the province of terrestrial physics however if one is to maintain the standard model scenario then preferred estimates of the present helium abundance limit the number of types to ~ 3-4. In the former case all lepton types have already been discovered. As we will indicate this limit may solve a problem in modern particle physics, for in models which relate lepton and quark flavours there appears no way to limit the actual number of flavours. If however the number of lepton flavours is cosmologically limited to three we have as "fundamental" fermions only the 3 lepton families (6 leptons) and the 6 quarks (u, d, c, s, t, b). Simultaneously we would have mutual confirmation of this simple and popular particle physics makeup and the standard cosmological model itself.

Moreover this is not all. Recent developments in theoretical particle physics have seen the advent of the so-called grand unified theories, wherein weak, electromagnetic and strong interactions are unified under one overarching theoretical framework. One consequence of such theories is the violating of baryon and lepton numbers at very high energies and therefore small cosmological times. As we will see in Ch. 6 the theories may well solve a yet outstanding problem for the standard scenario, the origin of the size of f_b.

On the other hand if higher quark and leptons are in fact discovered we may well be facing a situation in which the standard model is vitiated and quark and leptons become untenable as fundamental particles. The task of particle physics as it seems presently to be conceived would then be to isolate yet smaller building blocks that naturally give the quark and lepton spectrums (Ch. 6).

This brings us to our third major area of analysis; that it behoves us to analyse philosophically these terms: fundamental, natural, ad hoc etc. in relation to cosmological theorizing. We will isolate most of such discussion in this chapter and the last, the reason being that while we consider philosophical analysis as indispensable to any comprehensive understanding of physics and cosmological theorizing it should rather flow from the doing of that physics, than the physics flow from it. Our support is simply the history of physics. However it is precisely the historical way in which physics has internalized certain philosophical positions in an unproblematic way and has subsequently thrown them over in a new and revolutionary physics that demands we isolate part of the philosophical backdrop of contemporary physics. Issues of importance to us here are the very concept of fundamentality and the ability to proffer a complete cosmological analysis. We will look at these among others further below.

Since the standard model has been well studied elsewhere in Ch. 5 we study some of its deviants. These models include photon generating mechanisms, non-zero lepton numbers and exotic particles such as heavy neutrinos. All such models weight the cosmological difficulties still associated with the standard model with differing degrees of importance. Such problems include the origin of the background, the formation of galaxies, the problem of the missing mass as well as the problem of helium generation in the context of an expanded lepton spectrum. In the event that a model can only piece-wise attack some of these problems (and this seems to be the case) comparing the "rightness" of models becomes a thorny task. We thus present a number of deviant models in Sec. 4, Ch. 5 without too much regard for their comparative "correctness". We conclude this chapter with an original model which allows large present densities of neutrinos. Such models have been studied before, but what is novel in our approach is that the present degenerate seas are composed of electron neutrinos and antimuon neutrinos in nearly equal densities. The model can be thought of as symmetric with respect to certain lepton/baryon violating processes. Moreover the equality of the two lepton numbers has some interesting consequences as far as helium production and photon producing mechanisms are concerned. It is possible that the photon background might be generated in a massive lepton cascade when all massive early universe leptons are converted to neutrinos. If this is the case then the very early universe may "naturally" produce reasonable abundances of helium. However for the specific model of the cascade we detail it does not seem that such helium survives to the post-cascade epoch.

In Ch. 6 our concern is the so-called hadron era. In Sec. 2 we discuss the role of the grand unified theories and an alternative approach for particle physics - the bootstrap. The latter as a total physical theory would be opposed to the fundamentality of particles. In Sec. 3 we discuss the hadron era on the basis of a statistical bootstrap. If the presence of large lepton numbers ($\sim f_b$) alters the standard description of the lepton era only very small lepton numbers are needed to alter the consensus view of the hadron era within the statistical bootstrap. In particular if $f_b \leq 10^9$ and the net number of leptons is equal to the baryon asymmetry the very early hadron era is lepton dominated.

3. Atomism, the Cosmological Problem and the Anthropic Principle

(i) Ancient and Modern Atomism

Let us begin historically and trace some of the Hellenic roots of modern atomism. This seems an appropriate place to begin for while the Greeks lacked mathematical and experimental sophistication their theory building was of a more philosophical character and where common ground with modern atomism obtains their thinking may clarify some important contemporary issues.

It is Democritus who is popularly regarded as the initiator of a consistent atomic theory. For him there are only atoms and the void. The logical possibility of the infinite physical divisibility of substance is answered by the indivisibility of atoms. With certain inviolable properties such as size and shape the atoms constitute and determine through their motions all that is not void. Democritus is a determinist "naught happens for nothing, but everything from a ground and of necessity" (Russel 1945 P.66), obsessed with aetiology he wanders half the world acquiring experiences and knowledge since for him the objective world is a world of appearances to be analysed causually in relation to the fundamental atoms. Only the atoms are truly real. Democritus as an empiricist and sceptic in the sensuous world is also the first true mechanistic reductionist. As with all mechanistic approaches however there arises the problem of initial causes, the God given properties and the initial relations of atoms are forever outside the mechanistic framework. The Democritean approach has thus its own cosmological problem. Note that for Russell it is the way that God has been excluded from the day to day running of the world and relegated a role only at its beginning that makes the Democritean approach the closest out of all Greek speculation to that of modern science.

Contrast this to another, and later, Greek proponent of atomism, Epicurus. It is commonly said that Epicurus is a minor revisionist of Democritean doctrine however as the analysis of Marx (1975) exhibits this is hardly the case. For Epicurus it is not necessity but chance that rules, importantly his atoms are atoms for themselves and hence for example cannot be dictated to by the straight line and so must exhibit slight declinations from it (Marx, 1975, P.48). The sensuous world for Epicurus is the real and "science" is of little interest except for debunking superstitions that attribute phenomenae to the agency of the gods. When two naturalistic explanations compete, one is as good as the other since for Epicurus "all senses are heralds of the true" (Marx, 1975, P.39).

Despite his "anti-science" dogmatism it is Epicurus who seems in fact closer to modern physics. His famous declination from the straight line finds analogy in the modern uncertainty principle (Russell 1945, P.246) and the Epicurean doctrine of "minimae partes" bears a direct relationship to the present conception of partons or quarks. The latter has been emphasized by Gaisser and Gaisser (1976). In essence these minima are the smallest parts of something else (the "atoms") and thus can never be isolated (c.f. quark confinement). They are also the lowest level of analysis (i.e. accessible to reason) and the problem of "regressus ad infinitum" is averted. Epicurean philosophy is thus no mere revision of Democritean atomism. The notion of atoms as independent elementary constituents of the world with properties accessible to observation but not accessible to reason (i.e. god given) is replaced by parts with no independent existence which can never be observed and are accessible to reason only. The complementary processes of physical divisibility and mental divisibility, the one halted at the atom the other in the part, are united (to use the language of Marx) in the atom for itself. In Democritus the problem of initial causes lies outside the theory, not so for Epicurus for whom atoms moving with constant speed in the same direction are set in relative motion by declination from the straight line. Epicurus as materialist but not determinist, as the espouser of freedom and the destroyer of Gods is thus the greatest representative of Greek Enlightenment (Marx, 1975, P.73).

The vision of a world structured from fundamental and simple entities has been the mainspring of modern particle physics. However it has been the character of its development that no sooner is a fundamental particle identified that can explain the nature of larger structures than excited states of that particle are also discovered. The first such spectroscopy studied this century, that of the atom, enabled a complete understanding of the atom outside the nucleus. Energies of a few ev are sufficient to excite electrons to higher orbits from which they decay. The resulting quanta are clear indications that the atom is not the fundamental atom of the Greeks. It is not long however after the atom-electron and nucleus are identified as the new fundamental constituents that the nucleus is found to exhibit its own spectroscopy when bombarded by energies of $\sim 10^5$ ev. The details of this spectroscopy become understandable by the postulation of yet smaller entities - protons and neutrons. Again however a third spectroscopy arises (the three have been reviewed by Weiskopff 1968) as the baryons are excited by energies

$\sim 10^8$ ev.

Up to this point the atomism had been in line with the Democritean doctrine, each new group of constituents is hailed enthusiastically as giving the fundamental and observable entities of the world. The last spectroscopy however demands a new twist for though it can be understood by postulating yet smaller entities, the entities require non-integral electric charges and since such charges are not seen in the world the entities must somehow be confined. If this confinement is total (i.e. not violated at some high energy) the historical regress of fundamental to yet more fundamental entity is halted in the Epicurean doctrine of "minimae partes".

It is interesting however that confinement has not stopped the number of quarks needed to explain hadron spectroscopy growing in time. Recent discoveries have already forced the inference of the extended quark spectrum (u, d, c, s, b, t (?)) as well as positive proof of an extended lepton spectrum (e^-, ν_e, μ^-, ν_μ, τ^-, ν_τ(?)). Is this then another spectroscopy which must be explained by yet another level of fundamental entities? Note also the danger of approaching such discoveries from a philosophical point of view, if the electron is in fact a fundamental particle it is hardly a "part" in the Epicurean sense.

There is a further point here that deserves mention. Epicurus as a philosopher of freedom takes freedom as premise and describes atoms as exhibiting objectively uncertain behaviour. However from the modern scientific point of view we have uncertanty not freedom (i.e. the electron does not decide which slit it will go through). For us to establish freedom in the world some means would have to be found whereby "the uncertainty surrounding the physical behaviour of small atomic entities persists uncancelled through the composition of atoms into organisms ... [and] how, in the process of material organisation a selfreflective and self-deterministic principle becomes active". (Margenau 1978, P.280.)

We will look at the consequences of the atomistic approaches further below. However these by no means exhaust the possibilities of consistent approaches within particle physics. One alternate approach would be a conditional elementarity in which certain "elements" appear and exhibit certain properties only within certain energy ranges. A higher energy range merely excites a deeper layer of matter. Another possibility would see hadron physics as not to be understood in terms of particles at all but rather through self consistency. This is the so-called bootstrap theory and is discussed in Ch. 6. Interestingly it also has Hellenic roots this time in the philosophy of Anaxagoras.

(ii) The Cosmological Problem

It is commonly said of the expanding universe that it is evolving. That galaxies and structures form out of earlier, denser, hotter and more homogeneous backgrounds, that distributions of elements evolve with cosmic time etc.. However the word evolution at least as initially used by Jeans (1925) is only a synonym for gradual change and is not to be taken in its biological sense. The change is a unique historical sequence of events within a determinist framework; if the conditions at one epoch are precisely known then conditions at all future epochs can be predicted and those at past epochs retrodicted. If one can conceive of initial conditions then the entire history is already given in them.

This is the usual cosmological prescription and it offers little scope for evolution of the biological type. Selection effects might be important say in the cannibalistic interactions of galaxies but time scales appear too long for similar effects to play a large role in the origin of present structures. Nonetheless both our biosphere and the cosmological arena offer a bewildering diversity of structures and objects often in complex interaction and both seem to have derived from earlier simpler and more "homogeneous" situations. We will return to biological questions when we discuss the anthropic principle further below.

Now the cosmological enterprise is predicated on what we might call a Copernicanism. Terrestrially observed distributions of matter and energy and terrestrially devised physical theory are assumed (usually) to be typical of the whole universe. Importantly however the features we presently see are not typical in time, steady state theory is wrong, the universe is expanding and it is change itself which is typical. Astrophysical observation however is not theory free, our knowledge is extended outward by extrapolating observed regularities (e.g. the standard candles) which at times may be subject to revision (e.g. in Hubble's constant) and interpretations of phenomenae arise which seem never absolutely refutable, (e.g. the local theory of quasars) though they may become too complex to maintain. Observational selection effects may hide some distributions of matter (e.g. under-luminous dust and matter, neutrino backgrounds, burnt out stars etc.) which may only be inferrable by dynamical means (e.g. if the Coma cluster is stable we can calculate the missing mass) yet dynamical stability of larger structures is more the province of cosmology. Another problem arises when the interpretation of a phenomenon is cosmological model dependent. This is particularly true of the microwave radiation. In the standard big bang theory the radiation has free propagated since its separation from the matter at a red shift of some 1500 and very detailed observation of fluctuations may give information on galaxy formation at that time. On the other hand for initially colder models the radiation must be recently produced ($z \sim 10$-100) and in that case detailed observation tells us the nature of the dust and grains needed to thermalize it.

It is clear that each consistent cosmological theory may view the present universe very differently, some model interpreted astrophysical phenomenon may be a resource for one theory and not for another and some phenomenae may be ascribed origins at varying cosmological epochs. The epistemological problem arises as to how we are to consistently compare the "correctness" of cosmological models. Eventually it seems to us theories are judged using meta-scientific criteria such as naturalness, elegance and simplicity. Note that the modern use of natural here does not refer to persistent conjunctions among observed phenomenae but to the way surface phenomenae come to be given "naturally" from an underlying theory containing simple objects. This situation is in many ways analagous to that in atomism; the complexities that are presently observed are assumed to have simple antecedents. The standard model, for example, sees the helium abundance as naturally arising out of the simple assumption of small lepton and baryon asymmetries present at the era of cosmonucleosynthesis. However recent theories of lepton and baryon nonconservation would require that the small asymmetries should also be given naturally from simple antecedents. Yet there is nothing à priori which guarantees that this be the case (though we will see in Ch. 6 that it may be).

The problem of course revolves around the lack of a "complete" physics, whether or not we will ever obtain one (we attempt to discuss these things in Ch. 6) and whether or not the cosmological enterprise is feasible if we do not. One need merely note the way in which Lemaître (1950) is prone to speak in terms of the primeval atom, Gamow (1952) in terms of the primeval nucleus and more recently Hagedorn's (see Ch. 6) approach which might be rendered in terms of the primeval hadron, to emphasize the critical role of atomistic theories and their variants in the cosmological context.

We will return to these issues in the concluding discussion in Ch. 6. Let us now turn to a recently expressed idea very different from the ones we have discussed here.

(iii) The Anthropic Principle and the Problem of the Origin of f_b.

In later chapters we will be concerned with various explanations of the size of f_b. Here we wish to consider a radical idea which in the event of the failure of such attempts may offer an explanation for its size.

The explanation is based on the so-called strong anthropic principle (Carter 1975) which states that the universe must be of such a nature as to admit the evolution of observers within it at some stage. The general use of the anthropic principle has been reviewed in a fascinating paper by Carr and Rees (1979) who show the way crucial events such as the neutrino blow out of supernovae (presumably the dispersal of the heavier elements are necessary to life), standard cosmological helium production etc. are hinged very delicately on relationships between fundamental constants and on the actual physical conditions in the universe (e.g. f_b). In the Carter approach one imagines an ensemble of universes characterized by ranges in values of the fundamental constants h, c, e, G, m_p, m_e, ... and the physical conserved charges, electric, baryon, lepton, ... The universe we are in is characterized by just those values necessary for our existence. The principle has offered explanations for the cosmological coincidences, the size of f_b and may offer many more (see Car and Rees). The idea is a concretization of the Kantian doctrine that the fundamentals of science are not justified by special experience (metaphysical, transcendent etc.) but are the preconditions of all experience. Notice that any "explanation" is never one of necessity for we cannot know that it is necessary that we exist.

The argument for the size of f_b goes like this (also see Sec. 2, Ch. 3). The formation of galaxies within the standard scenario cannot occur until matter has decoupled from radiation and the matter density has become greater than the radiation density, both of these are fulfilled for $f_b < 10^{11}$. If the galaxies are to be of a reasonable size classical Jean's mass arguments require $f_b > 10^6$. That galaxies of appropriate size do exist only shows that f_b takes a value between the limits but since galaxies seem necessary to life (they confine and recycle the heavier elements) the only value that can be seen by observers must lie between the limits. f_b is thus constrained to near $\sim 10^8$. Unknown future arguments may pin this value down rather accurately; we will assume this for the purposes of the following discussion.

We are assuming here in general that though physics may provide necessary relationships among some of the constants and cosmology derive some of the charges some of them must still be assumed at their measured values. If then these values can be shown to have anthropic significance then all measured values must defacto also have anthropic significance. Such a principle is clearly anti-positivistic in its regard for the "consequences" of observers however it is also anti-Copernican since it envisages life as possible only under a very narrow range of conditions. An immediate problem arises since the principle will only explain the order of magnitude of the constants, yet an infinity of universes is possible within that magnitude and there is nothing to say which one we are in. The problem of "grain" might be averted if we were to accept a "strong determinism" for then the ratios and constants take exact values (we just have difficulty in constructing the arguments that accurately pin them down) and only these can give observers. Since terrestrial conditions could never be exactly repeated elsewhere life must occur only on earth. Such a determism however appears distasteful (I could never be allowed to misspel the word I have just written) and includes an unduly anthropocentric concept of life.

In support of the principle it does seem that the existence of terrestrial life is delicately balanced. As Hart (1978, 1979) has shown had earth's orbit been 5% closer to the Sun a runaway greenhouse effect would have produced a venus-like planet, had it been only 1% further away then runaway glaciation would have produced a mars-like planet. His calculations suggest that for stars of the K and M type a continuously habitable zone of the terrestrial kind does not exist. The problem of the origin of life is of crucial importance here, once life has formed it appears very resistant, as bacteria under some extreme conditions have shown. Presumably self-replicating chains of molecules somehow form in the pre-biotic organic soup. No laboratory however has ever succeeded (under presumably "ideal" conditions) in getting self-replicating chains to occur. It is noteworthy that not only electro-magnetic but weak and even superweak forces may have produced effects important in evolution (they may be implicated in the universal asymmetry of biological molecules, see Garay 1978 and the references therein) so that some anthropic connection may exist. It is important also to remember that for an earth which has been around for about 4.5 billion years, life got its start after about 1 billion years yet it still took 3.5 billion years to produce "observers". It may be that observers are harder to produce than life itself.

Now we do not wish to get involved with problems of directed evolution, universality of the genetic code, the role of random processes in evolution, nonetheless these will be of importance to any thorough study of the anthropic principle. Let us conclude this discussion by noting that very few definite constraints can be put on the origin of life. Perhaps the two most basic are the existence of a non-equilibrium situation and conditions which engender the maintenance of the integrity of some coding system. One interesting suggestion here is that of a crystalline physiology (Schneider 1977) a possibility which might occur in the surface of neutron stars. Life under such conditions would put pay to the notion of an anthropic principle.

CHAPTER 2

EQUILIBRIUM AND NON-EQUILIBRIUM THERMODYNAMICS IN A GENERAL RELATIVISTIC SETTING

1. INTRODUCTION

In this chapter we attempt to lay down a quite general framework for relativistic thermodynamics and statistical mechanics for both reversible and irreversible processes. It is within this framework that our discussion of cosmological models will take place. In the interest of generality we include a discussion of thermodynamics in inhomogeneous models even though from Ch.3 onwards we will be concerned only with homogeneous and isotropic models.

Our approach to relativistic thermodynamics is inspired by Synge's treatment of the relativistic gas (Synge 1957) which adopts a statistical mechanical starting point and proceeds via Boltzmans principle. Our treatment is made briefer than his by making use of the partition function and at the same time generalizes his equations to the quantum Fermi-Dirac and Bose-Einstein gases. These equations were originally obtained by Juttner (1911, 1928) and agree with those of Landsberg and Dunning-Davies (1965). In this section (Section 2) we also begin preparation for the numerical calculations, with which the later chapters will be involved, by restructuring these equations in the manner suggested by Guess (1966). The section concludes with a discussion of Bose-Einstein condensation.

The various equations of state to which these thermodynamical functions give rise is the topic of Sec. 3. These equations of state are usually obtained from inversion of the number density relation. However this is possible to do in only a few cases, all of which are presented in this section. One such exact solution is the Synge gas for massive particles and this is presented after a discussion of all the equations of state to which radiation gives rise. Instead of giving the lengthy derivations of the equations for massive components we present only the smallest terms in this section. The derivations and their relevant series approximations are collected together in Appendices A and B. We show there that all equations of state can be expressed in terms of two functions; the modified Bessel functions $K_n(z)$ and a type of generalized zeta function $\phi(z,n)$. Finally we discuss criteria for the onset of degeneracy in a gas, and note that in a cosmological fluid containing black body radiation a natural criterion is the ratio of photons to the number of particles in question.

In Sec. 4 the equations of relativistic thermodynamics are further generalized to gases of several components with chemical reactions between them. These equations are put into a general relativistic setting in Sections 5 and 6 for both local equilibrium and non-equilibrium situations. In Sec. 5 we show the well known fact that a perfect fluid must expand adiabatically, we note however that equilibrium may only be maintained by a considerable flow of heat (and entropy) between components. In Sec. 7 we discuss the non-equilibrium case in some detail and compare our approach to that of Israel (1976).

In the last section we apply these ideas to homogeneous and isotropic cosmological models. For the case of no reactions we discuss solutions to the equations and we note that even if the photon chemical potential is set to zero initially there is nothing in these equations which forces it to remain zero. In fact we show that it is incompatible to assume both thermodynamical equilibrium and conservation of photon number. This is what is commonly done in heuristic epoch-separating discussions of the cosmological dynamics (e.g. Weinberg 1972) but it can only be justified in high photon entropy situations. Our discussion permits a more general study. We present several alternative hypotheses in order to get around this inconsistency. Some of these require the introduction of a photon chemical potential. Finally we present some discussion of the relations between kinetic theory and thermodynamics both for the standard and our non standard models.

2. RELATIVISTIC EQUATIONS OF STATE

The most straight forward, though perhaps not most rigorous method of obtaining the equilibrium equations of state for N particles distributed over a set of energy levels E_i $(i = 1, 2, \ldots)$ is to maximize the number of ways

$$W_\varepsilon(N_1, N_2, \ldots)$$

in which these particles may distribute themselves so that N_i particles have energy E_i. The parameter ε can take values ± 1 or 0 depending on whether the statistics are Fermi-Dirac, Bose-Einstein or classical Boltzman. Standard combinatorial arguments provide the formulae (Sommerfield 1964)

$$W_0 = \prod_i \frac{M_i^{N_i}}{N_i!}$$

$$W_1 = \prod_i \begin{pmatrix} M_i \\ N_i \end{pmatrix}$$

$$W_{-1} = \prod_i \begin{pmatrix} M_i + N_i - 1 \\ N_i \end{pmatrix}$$

where M_i is the degeneracy of the i'th energy level. A constant factor N! has, as usual, been omitted from W_0 to avoid the usual problems associated with the Gibbs paradox. When $\log W_\varepsilon$ is maximized, subject to the constraints

$$\sum N_i = N = \text{const.} \tag{1}$$

$$\sum N_i E_i = E = \text{const.} \tag{2}$$

we find by standard Lagrange multiplier methods and using Stirling's formula that

$$N_i = \frac{-1}{\theta}\frac{\partial Z_\varepsilon}{\partial E_i}, \qquad N = \frac{\partial Z_\varepsilon}{\partial \lambda}, \qquad E = \frac{-\partial Z_\varepsilon}{\partial \theta} \tag{3}$$

where $Z_\varepsilon = Z_\varepsilon(\lambda, \theta, E_i)$ is the partition function

$$Z_0 = \sum_i M_i \, e^{\lambda - \theta E_i} ,$$

$$Z_{\pm 1} = \sum_i M_i \log\left(1 \pm e^{\lambda - \theta E_i}\right) , \tag{4}$$

and λ and θ are Lagrange multipliers associated with constraints (1) and (2) respectively.

We define the entropy by Boltzmann's equation

$$S = k \log W_{max}$$

whence a direct calculation reveals in each case that

$$S = k[\theta E - \lambda N + Z_\varepsilon] . \tag{5}$$

If the energy levels E_i depend on certain external parameters a_μ (e.g. the volume V), and we define the μ-th generalized force to be given by

$$A_i^\mu = -\frac{\partial E_i}{\partial a_\mu} ,$$

then the average generalized force is given by

$$\bar{A}^\mu = \sum_i N_i A_i^\mu = \sum \frac{1}{\theta}\frac{\partial Z_\varepsilon}{\partial E_i}\frac{\partial E_i}{\partial a_\mu} = \frac{1}{\theta}\frac{\partial Z_\varepsilon}{\partial a_\mu} . \tag{6}$$

A slight manipulation of partial derivatives using (3), (5) and (6) leads to the Gibbs relation

$$TdS = dE + \sum \bar{A}^{\mu}\, da_{\mu} - \mu dN \tag{7}$$

where

$$\theta = \frac{1}{kT} \qquad \mu = \lambda kT \tag{8}$$

and T is called the temperature and μ the chemical potential. We also set $a_{\mu} = V$ as the only external parameter.

For an ideal system of N non interacting particles the energy levels E_i and their occupation numbers M_i are obtained by solving the N body wave functions in a box of volume V. In the limit $V \to \infty$ the momentum eigenvalues of this system form a continuum. Thus with one exception (the case of Bose-Einstein condensation) the summation in (4) can be replaced by an integral over phase space and the M_i replaced by $gVh^{-3}d^3p$ where g is the number of internal degrees of freedom.

Equations (4) give

$$Z_0 = \frac{4\pi gV}{h^3}\int_0^{\infty} e^{\lambda-\theta E(p)}p^2dp\ ,$$

$$Z_{\pm 1} = \pm\frac{4\pi gV}{h^3}\int_0^{\infty} \log\left(1 \pm e^{\lambda-\theta E(p)}\right)p^2dp\ .$$

We now find from (3) and (6) that

$$N = \frac{4\pi gV}{h^3}\int_0^{\infty}\frac{p^2dp}{e^{(E(p)-\mu)/kT}+\varepsilon}\ , \tag{9}$$

$$E = \frac{4\pi gV}{h^3}\int_0^{\infty}\frac{p^2\ dE(p)/dp}{e^{(E(p)-\mu)/kT}+\varepsilon}\ , \tag{10}$$

$$P = \frac{4\pi g}{h^3}\int_0^{\infty}\frac{p^3\ dE(p)/dp}{e^{(E(p)-\mu)/kT}+\varepsilon} \tag{11}$$

where the generalized force corresponding to the volume parameter has been defined as the pressure

$$P = \frac{1}{\theta}\frac{\partial Z}{\partial V} = \frac{Z}{\theta} .$$

An integration by parts has been performed to obtain Eq.(11). Finally the entropy is obtained from Eq.(5)

$$S = \frac{1}{T}\,(E + VP - \mu N) , \tag{12}$$

this is merely the traditional equation with the internal energy replaced by the total energy E and the non-relativistic chemical potential $\mu - mc^2$ replaced by the relativistic chemical potential μ (henceforth just "the chemical potential").

It is pleasing to note that both for the non-relativistic case, $E(p) = p^2/2m$, and the relativistic case

$$E(p) = (m^2c^4 + p^4c^2)^{\frac{1}{2}} \tag{13}$$

$d\,E(p)/dp = V_p$, so the equations for total energy and pressure reduce to the "kinetic" formulae

$$E = \int_0^\infty E(p)\;N(p)\;dp, \qquad P = \frac{1}{3}\int_0^\infty N(p)\;V_p p dp$$

where

$$N(p)dp = \frac{4\pi g V\;p^2\;dp}{h^3\left(e^{(E(p)-\mu)/kT} + \varepsilon\right)}$$

is the particle number distribution over momentum intervals.

In the case of relativistic equations of state (13) Eqns. (9-12) can be written in a number of integral representations. The most physically transparent however is

$$n = \frac{N}{V} = A(T)\;I_\varepsilon^{11}\left(\frac{mc^2}{kT}, \frac{\mu}{kT}\right) , \tag{14}$$

$$\rho = \frac{E}{V} = kTA(T)\;I_\varepsilon^{21}\left(\frac{mc^2}{kT}, \frac{\mu}{kT}\right) , \tag{15}$$

$$p = \frac{E}{V} = \frac{kT}{3} A(T)\ I_{\varepsilon}^{03}\left(\frac{mc^2}{kT}, \frac{\mu}{kT}\right) \tag{16}$$

and adopting the useful notation $x = mc^2/kT$, $\lambda = \mu/kT$, the entropy density is

$$s = \frac{S}{V} = \frac{1}{T}(\rho + p - \mu n) ,$$

$$= kA(T)\left[I_{\varepsilon}^{21}(x,\lambda) + \frac{I_{\varepsilon}^{03}(x,\lambda)}{3} - \lambda I_{\varepsilon}^{11}(x,\lambda)\right] \tag{17}$$

where $A(T) = 4\pi g k^3 T^3/h^3c^3$ and

$$I_{\varepsilon}^{ab}(x,\lambda) = \int_x^{\infty} \frac{y^a (y^2 - x^2)^{b/2}}{e^{y-\lambda} + \varepsilon}\, dy , \tag{18}$$

$$= x^{a+b+1} \int_{\chi}^{\infty} \frac{\cosh^a \chi \sinh^{b+1} \chi}{e^{x\cosh\chi - \lambda} + \varepsilon}\, d\chi . \tag{19}$$

In the last equation the substitution $y = x \cosh \chi$ has been made. Since $\sinh^2\chi \le \cosh^2\chi$ we see immediately from (19), (15) and (16) that for $m \neq 0$ the standard inequality $P < \frac{1}{3}\rho$ holds, while for rest mass zero ($x = m = 0$) Eq.(18) results in $p = \rho/3$. Similar equations to the above have been derived by Landsberg and Dunning-Davis (1965) and were originally given by Jüttner (1911, 1928).

The range of variation of the parameters x, λ in the above equations is of particular interest. For a fermion or Boltzman gas we have $0 \le x \le \infty$, $-\infty \le \lambda \le \infty$ and I_{+1}^{11}, I_0^{11} are not bounded from above. In gases of this type the number density at a particular temperature can vary along the entire real line. For a boson gas, however, we have $0 \le x \le \infty$, $-\infty \le \lambda \le x$; values outside this range will lead to negative occupation numbers. Moreover a simple expansion of I_{-1}^{11} (see Sec. 3) shows that it is bounded above by the finite value $I_{max}^{11} = I_{-1}^{11}(mc^2/kT, mc^2/kT)$. Einstein was the first to suggest such a limitation on boson number density was unphysical and resulted from the replacement of the sum over levels in Eq.(4) by an integral. Thus M_i has been replaced by the continuous weight function $gVp^2h^{-3}dp$ which gives the incorrect weight for the ground state M_1 as zero. In the case of fermions the effect is negligible since the Pauli principle allows only one particle per state. Bosons, however, may occupy a state in any numbers and in particular if the density is high enough they may accumulate in the ground state. The effect is known as Bose-Einstein condensation.

Clearly the occupation number of the ground state is (subscript c refers to the condensate)

$$N_1 = N_c = \frac{1}{e^{x-\lambda} - 1} \tag{20}$$

and (14-17) give

$$n = A(T)\ I^{11}_{-1}(x,\lambda) + \frac{1}{V}\frac{1}{e^{x+\lambda} - 1} ,$$

$$\rho = kTA(T)\ I^{21}_{-}(x,\lambda) + \frac{mc^2}{V}\frac{1}{e^{x-\lambda} - 1} ,$$

$$p = \frac{kT}{3}\ A(T)\ I^{03}_{-}(x,\lambda) - \frac{kT}{V}\ \ell n\left(1 - e^{\lambda-x}\right) , \tag{21}$$

$$s = kA(T)\left\{I^{21}_{-1}(x,\lambda) + \frac{I^{03}_{-1}(x,\lambda)}{3} - \lambda I^{11}_{-1}(x,\lambda) + \frac{x - \lambda}{V}\frac{1}{e^{x-\lambda} - 1} - \frac{\ell n(1 - e^{\lambda-x})}{V}\right\} . \tag{22}$$

Huang (1966) shows that in the limit $V \to \infty$ the condensed term should only be included if a finite fraction $\frac{n_c}{n}$ of all particles lie in the ground state. In this case we have

$$n = A(T)\ I^{11}_{-1}(x,x) + n_c \tag{23}$$

$$\rho = kT\ A(T)\ I^{21}_{-1}(x,x) + n_c mc^2 \tag{24}$$

$$p = \frac{kT}{3}\ A(T)\ I^{03}_{-1}(x,x) \tag{25}$$

$$s = kA(T)\left\{I^{21}_{-1}(x,x) + \frac{I^{03}_{-1}(x,x)}{3} - xI^{11}_{-1}(x,x)\right\} \tag{26}$$

and the fluid consists of two different phases, a "gas" phase and a "liquid" phase. Bose-Einstein condensation may be important for certain areas of astrophysics (Sawyer 1973), studies on the evolution of the photon spectrum (Coste et al 1975) or in description of exotic matter in the early universe (Fiore et al 1978). Indeed as we shall show in the next chapter the condensed phase has some interesting cosmological effects. For the moment however we adopt the convention of leaving it out of eqns. (14-17) on the understanding that it must be put in by hand through eqns. (23-26) when the boson density approaches the critical or transitional density

$$n_t = A(T)\ I^{11}_{-1}(x,x)\ .$$

The integral representation (14-18) of the equations of state (9-12) though brief has a number of deficiencies. Many of the quantities which we will find useful later on, specific heats, speed of sound etc. and the numerical problems we will be faced with require differentiation of the equations of state. Differentiation of (14-17) however does not yield the I^{ab} (18).

While still maintaining the use of (14-18) both for notational brevity and physical transparency we adopt a second representation which by increasing the above three integrals to five allows the derivatives of the equations of state to be written in that representation. We define

$$Q^n = \int_0^\infty \frac{\cosh n\chi\ dx}{e^{x\cosh\chi-\lambda} + \varepsilon} \tag{27}$$

where

$$I^{11}_\varepsilon = \frac{x^3}{4}\left(Q^3_\varepsilon - Q^1_\varepsilon\right)\ , \tag{28}$$

$$I_\varepsilon^{21} = \frac{x^4}{8}\left(Q_\varepsilon^4 - Q_\varepsilon^0\right) , \tag{29}$$

$$I_\varepsilon^{03} = \frac{x^4}{8}\left(Q_\varepsilon^4 - 4Q_\varepsilon^2 + 3Q_\varepsilon^0\right) . \tag{30}$$

This and other useful representations have been introduced and discussed at length by Guess (1966). Guess proves the important differential relation

$$d\left(Q_\varepsilon^{n+1} - Q_\varepsilon^{n-1}\right) = 2n\, Q_\varepsilon^n \frac{d\lambda}{x} - \left[(n + 1)Q_\varepsilon^{n+1} + (n - 1)Q_\varepsilon^{n-1}\right]\frac{dx}{x} . \tag{31}$$

This can also be proved from (27) by a lengthy integration by parts.

We can now obtain from the equations of state (9-11) after introducing the useful notation $\beta = 16\pi\zeta(3)(mc/h)^3 = 4\zeta(3)x^3\,A(T)/g$ and dropping unnecessary labels

$$n = \frac{g}{4\zeta(3)}\,\frac{\beta}{x^3}\,I^{11} = \frac{g}{16\zeta(3)}\,(Q^3 - Q^1) , \tag{32}$$

$$\rho = \frac{g}{4\zeta(3)}\,\frac{\beta}{x^3}\,kT\,I^{21} = \frac{g\beta}{32\zeta(3)}\,mc^2(Q^4 - Q^0) , \tag{33}$$

$$p = \frac{g}{12\zeta(3)}\,\frac{\beta}{x^3}\,kT\,I^{03} = \frac{g\beta}{96\zeta(3)}\,mc^2(Q^4 - 4Q^2 + 3Q^0) . \tag{34}$$

The entropy density becomes

$$s = S/V = \frac{g}{4\zeta(3)}\,\frac{\beta}{x^3}\,k\left[I^{21} + \frac{I^{03}}{3} - \lambda I^{11}\right]$$

$$= \frac{g\beta\,k}{96\zeta(3)}\left(4x(Q^4 - Q^2) - 6\lambda(Q^3 - Q^1)\right) \tag{35}$$

and the differential relation gives

$$dn = \frac{g}{16\zeta(3)x}\left(4Q^2 d\lambda - (3Q^3 + Q^1)dx\right) , \tag{36}$$

$$d\rho = \frac{g\beta\,mc^2}{16\zeta(3)x}\left((3Q^3 + Q^1)d\lambda - 2(Q^4 + Q^2)dx\right) , \tag{37}$$

$$dp = \frac{g\beta\ mc^2}{48\zeta(3)x}\left(3(Q^3 - Q^1)d\lambda - 2(Q^4 - Q^2)dx\right) , \tag{38}$$

$$ds = \frac{g\beta\ k}{16\zeta(3)x}\left([(3Q^3 + Q^1)x - 4Q^2\lambda]d\lambda - [2(Q^4 + Q^2)x - (3Q^3 + Q^1)\lambda]dx\right) . \tag{39}$$

Finally let us make a procedural point. In future the word degeneracy will be used to describe both Bose condensation and Fermi degeneracy and since $\lambda \equiv \mu/kT$ determines the extent of that degeneracy we will call it the (relativistic) degeneracy parameter.

3. SPECIAL CASES OF THE EQUATIONS OF STATE

The final form of the equation of state of an ideal gas is usually obtained by the elimination of the chemical potential in the energy and pressure density equations (15) and (16) by use of an inverted form of the number density equation (14). This considerable task has in fact been carried out in full generality by M. Nieto (1969). The resulting equations however, contain difficult double integrals and are not suitable for our later uses of the equations of state. A more traditional approach is to establish series approximations for the integrals involved which are accurate in certain regions of the (x,λ) half-plane. However not all regions can be covered by series approximations and in such cases it becomes necessary to evaluate the integrals through direct numerical calculation.

In Appendix A we detail the appropriate approximation schemes and their regions of validity for both the I^{ab} and Q^n representations. The expansions are given in terms of two fundamental functions, the modified Bessel functions $K_n(z)$ and a generalization of the zeta function $\phi(z,s)$. These functions and the relevant approximations to them are discussed in

Appendix B. In this section we present some of these results (to first order only) and their equations of state. However first we detail the case of radiation (i.e. any m = 0 particle) for which the only known analytic solutions to the integrals exist (exact solutions for $\varepsilon = 0$ are also given but the Boltzman gas is just a limiting case of fermion or boson gas; Boltzman particles do not exist).

(i) Massless particles

In the case of mass zero particles the integrals I^{ab} take a particularly simple form since they no longer depend explicitly upon the temperature. In fact these integrals correspond to our definition of the generalized Riemann zeta functions given in Appendix B, Eq. (6), so we have

$$I^{11}_{\pm 1}(0,\lambda) = \pm\ 2\phi(\pm\ e^{\lambda},\ 3)\ , \tag{40}$$

$$\begin{aligned} I^{21}_{\pm 1}(0,\lambda) &= \pm\ 6\phi(\pm\ e^{\lambda},\ 4) \\ &= I^{03}_{\pm 1}\ . \end{aligned} \tag{41}$$

The behaviour of the functions is studied in detail in Appendix B. We note here that for λ large and negative

$$\pm\ \phi(\pm\ e^{\lambda},\ n) \doteq e^{\lambda}$$

and as $\lambda \to 0$ the functions return the standard Riemann zeta functions. When $\lambda \geq 0$ and for fermions only we have

$$I^{11}_{+1}(0,\lambda) = \frac{\lambda^3}{3} - 4\lambda\phi(-1,\ 2) - 2\phi(-e^{-\lambda},\ 3)\ , \tag{42}$$

$$\begin{aligned} I^{21}_{+}(0,\lambda) &= \frac{\lambda^4}{4} - 6\lambda^2\phi(-1,\ 2) - 12\phi(-1,\ 4) + 6\phi(-e^{-\lambda},\ 4) \\ &= I^{03}_{+1}(0,\lambda). \end{aligned} \tag{43}$$

The case of photons is obtained by setting $\varepsilon = -1$, $g = 2$ if these photons are also blackbody then we set $\mu = 0$ since the particles are assumed to be freely produced in thermal equilibrium and no Lagrange mutliplier arises from the particle number constraint equation (1). The equations of state become

$$n_\gamma = bT^3 \qquad b = 16\pi\zeta(3)\,\frac{k^3}{c^3h^3} = 20.27 \tag{44}$$

$$\rho_\gamma = aT^4 \qquad a = 48\pi\,\frac{\zeta(4)k^3}{c^3h^3} = 7.55 \times 10^{-15} \tag{45}$$

$$= 3p_\gamma$$

where c.g.s. units and degrees kelvin have been used. The entropy density may be expressed via equation (17) as

$$s_\gamma = \frac{1}{T}\,(\rho_\gamma + p_\gamma) = \frac{4}{3}\,aT^3 = 3.6\ kn_\gamma \tag{46}$$

We have used values for the zeta functions given in Appendix B.

For neutrinos one takes $\varepsilon = +1$, $g = 1$ and if we have $\mu_\nu \geq 0$, that is if the neutrinos are degenerate to some extent then

$$n_\nu = \frac{4\pi k^3T^3}{h^3c^3}\left(\frac{\lambda_\nu^3}{3} + \frac{\pi^2}{3}\,\lambda_\nu - 2\phi\left(-e^{-\lambda_\nu},\ 3\right)\right) , \tag{47}$$

$$\rho_\nu = \frac{4\pi k^4T^4}{h^3c^3}\left(\frac{\lambda_\nu^4}{4} + \frac{\pi^2}{2}\,\lambda_\nu^2 + \frac{7\pi^4}{60} + 6\phi\left(-e^{-\lambda_\nu},\ 4\right)\right) \tag{48}$$

$$= 3p_\nu$$

For $\mu_\nu = 0$ these reduce to (Weinberg 1972)

$$n_\nu = \frac{6\pi}{h^3}\,\zeta(3)\,\frac{k^3T^3}{c^3} = \frac{3}{8}\,bT^3 , \tag{49}$$

$$\rho_\nu = \frac{7}{30}\,\frac{\pi^5k^4T^4}{h^3c^3} = \frac{7}{16}\,aT^4 , \tag{50}$$

$$s_\nu = \frac{7}{12}\,aT^3 = 4.20\ kn_\nu . \tag{51}$$

If neutrinos and antineutrinos are in chemical equilibrium with black body photons we may expect $\mu_\nu = -\mu_{\bar\nu}$, which yields the exact relations

$$n_\nu - n_{\bar{\nu}} = \frac{4\pi k^3T^3}{h^3c^3}\left(\frac{\lambda_\nu^3}{3} + \frac{\pi^2}{3}\lambda_\nu\right) , \tag{52}$$

$$p_\nu + p_{\bar{\nu}} = \frac{4\pi k^4T^4}{h^3c^3}\left(\frac{\lambda_\nu^4}{4} + \frac{\pi^2}{2}\lambda_\nu^2 + \frac{7\pi^4}{60}\right) , \tag{53}$$

$$s_\nu + s_{\bar{\nu}} = \frac{4\pi k^4T^3}{h^3c^3}\left(\frac{\pi^2}{3}\lambda_\nu^2 + \frac{7\pi^4}{45}\right) . \tag{54}$$

In the limit of $\mu/kT \to -\infty$ we regain for radiation of both Bose-Einstein and Fermi-Dirac statistics the Boltzman radiation with $\varepsilon = 0$ viz

$$n = 8\pi g \frac{k^3T^3}{h^3c^3} e^{\mu/kT}, \qquad \rho = 3kTn, \qquad p = \frac{\rho}{3} \tag{55}$$

(ii) Massive particles

If the component has a non-zero mass the integrals I^{ab} have for natural regions of approximation depending on whether or not the temperature is high enough for the particles to be relativistic and whether the chemical potential is large enough for degeneracy effects to become important. Clearly the gas will be relativistic for $x = \frac{mc^2}{kT} << 1$ and non-relativistic when $x >> 1$. A measure for degeneracy is the non-relativistic degeneracy parameter $\eta = \lambda - x$. For fermions if η is large and positive the gas is degenerate since to first order only those states with momentum below the fermi momentum will be occupied. If η is large and negative both bosons and fermions will be non-degenerate and in the limit $\eta \to -\infty$ will behave like the relativistic Boltzman gas I_0^{ab}.

a. Synge gas

In this last case $\lambda - x << -1$ the denomination in Eq. (18) can be expanded as a geometric series resulting in an infinite series of modified Bessel functions (see Appendix A and for the Bessel functions Appendix B). The first term of this series dominates for extreme non-degeneracy giving

$$I_0''(x,\lambda) = e^\lambda x^2 k_2(x) ,$$

$$I_0^{21}(x,\lambda) = e^\lambda\left(x^3k_1(x) + 3x^2k_2(x)\right) ,$$

$$I_0^{03}(x,\lambda) = 3e^\lambda x^2k_2(x) .$$

The last relation leads immediately to the classical law

$$p = nkT$$

while the first two yield the equations describing the Synge gas (Synge 1957),

$$\begin{aligned} n &= \frac{4\pi g}{h^3} e^\lambda m^2ckT\ k_2(mc^2/kT) \\ &= 4\pi g \frac{m^3c^3}{h^3}\frac{k_2(x)}{x} e^\lambda \\ &= \frac{g\beta}{4\zeta(3)}\frac{k_2(x)}{x} e^\lambda \end{aligned}$$

where as before $\beta = 16\pi\ \zeta(3)\left(\frac{(mc)^3}{h}\right)$, and

$$\begin{aligned} \rho &= n\left(mc^2\ \frac{k_1(mc^2/kT)}{k_2(mc^2/kT)} + 3kT\right) , \\ &= nmc^2\left(\frac{k_3(x)}{k_2(x)} - \frac{1}{x}\right) , \end{aligned} \tag{58}$$

the last equation following by easily established identities among the modified Bessel functions (Appendix B, Eq. 2). For the Synge gas we should set $g = 1$, since the Pauli exclusion principle plays no role in Boltzman statistics. The entropy density of the Synge gas can now be written as

$$s = kn\left(x\ \frac{k_3(x)}{k_2(x)} - \lambda\right)$$

i.e.

$$s = kn\left(x\ \frac{k_3(x)}{k_2(x)} - \ell n\left(\frac{4\zeta(3)\ nx}{g\beta\ k_2(x)}\right)\right) \tag{59}$$

In the non-relativistic limit we obtain the appropriate Maxwell-Boltzman relations via Eq. (B4)

$$n = (2\pi m\ kT)^{3/2}\ \frac{g}{h^3}\ e^{(\mu - mc^2)/kT} \tag{60}$$

$$\rho = nmc^2 - \frac{3}{2}n\ kT \tag{61}$$

$$s = \frac{3}{2}nk - nk\ \ell n\left(nh^3/g(2\pi m\ kT)^{3/2}\right) \tag{62}$$

This last is known as the Sackur-Tetrode equation (Huang 1966). In the relativistic limit we can use Eq. (B5) to obtain for the Synge gas

$$n = \frac{g\beta}{2\zeta(3)}\ \frac{e^{\lambda}}{x^3}\ , \tag{63}$$

$$\rho = 3nkT = 3p\ , \tag{64}$$

$$s = nk\left\{4 - \ell n\left(\frac{2\zeta(3)}{g\beta}\ nx^3\right)\right\}\ . \tag{65}$$

b. Fermions and Bosons - nonrelativistic and semidegenerate

The expansion in Bessel functions for the nondegenerate gas (B7-10) is appropriate only for $\eta = -x + \lambda$ large and negative, say $\eta \leq -\pi$. Outside this range no single approximation scheme can apply at all temperatures unless η is positive and rather large (fermions), such a degenerate gas is discussed further below. Thus a gas for which $-\pi < \eta < \pi$ we can call semidegenerate. In the nonrelativistic regime the appropriate expansion for the semidegenerate gas is (A13-16) together with truncated expansions for the zeta functions (A27-30 and 35-38). To first order these equations give

$$I''_{\pm} = \pm\left(\frac{\pi x^3}{2}\right)^{\frac{1}{2}}\ \phi\left(\pm e^{+\eta},\ \frac{3}{2}\right), \qquad I^{21}_{\pm} = xI''_{\pm}\ , \qquad I^{03}_{\pm} = \pm\left(\frac{\pi x^3}{2}\right)^{\frac{1}{2}}\ \phi\left(\pm e^{+\eta},\ \frac{5}{2}\right)\cdot 3 \tag{66}$$

where for bosons $\eta \leq 0$ and fermions $\eta < \pi$. In actual fact these expansions are valid in the much wider range $\eta \lesssim x$ for the fermions, we will return to this point later on. As $\eta \to 0$ in the above the generalized zeta functions go to their respective Riemann zeta functions (A8-9).

c. Fermions and Bosons - relativistic

For the relativistic case we discuss and derive two sets of expansions; in the first Eqs. (A17-20) the generalized zeta functions depend only on the relativistic degeneracy parameter λ whilst in the second Eqs. (A35-42) the zeta functions depend on the nonrelativistic degeneracy parameter η. The expansions are equivalent for except for bosons when $\frac{x}{|\eta|} << 1$, in this case it is more accurate to use the second set. To first order we have

$$I_{\pm}^{11} = \pm 2 \cdot \phi(\pm e^{\eta}, 3), \quad I_{\pm}^{21} = \pm 6 \cdot \phi(\pm e^{\eta}, 4) = I_{\pm}^{03} . \tag{67}$$

The truncated expansions (B31-34), (B39-42) give expansions (B31-34), (B39-42) give expansions appropriate for $|\eta| < \pi$ however the zeta functions are given on the wider range $0 < \eta < \infty$ by Eqs. (A15-18) and Eq. (67) is correct here also.

d. Fermions - degenerate

The fermion gas is usually said to be degenerate when $\eta = \lambda - x >> 1$. Expansions appropriate for this case are given in Appendix A by Eq. (A22-32) we have then to first order

$$I^{11} = (\lambda^2 - x^2)^{3/2} / 3, \quad I^{21} = x^4 g\left((\lambda^2/x^2 - 1)\right)/24 + xI^{11},$$
$$I^{03} = \frac{x^4}{8} f\left((\lambda^2/x^{-2})\right) \tag{68}$$

where the functions f and g are given in Appendix A. The gas is degenerate since nearly all the energy levels up to the fermi energy have a high probability of occupation. Clearly at relativistic temperatures we can replace (A22-32) by (A35-41) and (B15-18) however at any temperature as long as $\eta/x \sim \lambda/x >> 1$ we have

$$I^{11} = \lambda^3/3, \quad I^{21} = \lambda^4/4 = I^{03} \tag{69}$$

and the gas is "radiation like" since the equations of state do not depend on the rest mass c.f. Eq. (42-43).

The asymptotic series used in (A22-32) gives difficulties at non-relativistic temperatures for sufficiently small η/x. In this event we return to the series in powers of $1/x$ given by (A13-16) and use the expansions given for the zeta functions (B19). At sufficiently large x rest matter dominates the energy density to give

$$I^{11} = \frac{(2x)^{3/2}}{3}\eta^{3/2}, \qquad I^{21} = xI^{11}, \qquad I^{03} = \frac{2.(2x)^{3/2}}{5}\eta^{5/2}. \tag{70}$$

The average pressure per particle is enhanced over that in a non-degenerate gas by a factor $2\eta/5$ and the ratio of pressure to energy density is small, $2\eta/5x$.

(iii) Criteria for Degeneracy

The foregoing allows us to construct a more physical criterion for the onset of fermion degeneracy or boson condensation in a gas. This is usually done in terms of the number density; degeneracy effects becoming important for

$$n > n_t = A(T)I^{11}_{\pm}(x,x) .$$

For gases undergoing expansion it is better to formulate a condition in a volume independent fashion, we define

$$f = \frac{\beta}{x^3 n} = \frac{4\zeta(3)}{gn}A(T) \tag{71}$$

and the degeneracy condition becomes

$$f > \frac{4\zeta(3)}{gI^{11}_{\pm}(xx)} \tag{72}$$

when $x << 1$

$$f < \frac{2}{g} \qquad \text{for bosons,}$$

$$< \frac{8}{3g} \qquad \text{for fermions,}$$

when $x >> 1$

$$f < \frac{1.5}{gx^{3/2}} \qquad \text{for bosons,}$$

$$< \frac{5.0}{gx^{3/2}} \qquad \text{for fermions.}$$

If one of the components is black body radiation ($\mu_\gamma = 0$) these criteria take a particularly useful form since f becomes the ratio of the number of these photons to the number of the component in question.

Another useful criterion can be given in terms of the specific entropy of a species

$$\frac{s}{kn} = \frac{S}{kN} = \frac{I^{21} + \frac{I^{03}}{3} - \lambda I^{11}}{I^{11}} \,.$$

Ordering in the degenerate state suggests that $s/n \to 0$ as $n \to \infty$ so a criterion for the onset of degeneracy is

$$\frac{s}{kn} < \frac{s_c}{kn_c} = \frac{I^{21}(x,x) + \frac{I^{03}}{3}(x,x) - xI^{11}(x,x)}{I^{11}(x,x)} \,,$$

i.e. when $x << 1$

$$\frac{s}{kn} < 3.60 \qquad \text{for bosons,}$$

$$< 4.20 \qquad \text{for fermions}$$

and when $x >> 1$

$$\frac{s}{kn} < 1.28 \qquad \text{for bosons,}$$

$$< 2.83 \qquad \text{for fermions.}$$

These criteria highlight the fact that for a single component fluid with a conserved particle number undergoing adiabatic expansion, the expansion itself serves to some extent to lift the degeneracy. The effect will be most pronounced for fermions with $2.83 < s/kn < 4.2$ and for bosons with $1.28 < s/kn < 3.6$ since the degenerate states will be removed completely. We will return to the study of such phenomenae in more detail in Ch. 3.

4. MIXTURES AND CHEMICAL REACTIONS AT EQUILIBRIUM

For a mixture of substances (labelled by $A = 1, 2, \ldots, m$) undergoing no chemical reactions

$$N_A = \sum_i N_{A_i} = \text{const.}, \qquad E = \sum_i N_{A_i} E_{A_i} = \text{const.} \tag{74}$$

we clearly have

$$Z = \sum_A Z_A(\lambda_A, \theta) \ ,$$

$$E = \frac{-\partial Z}{\partial \theta} = - \sum_A \frac{\partial Z_A}{\partial \theta} = - \sum E_A \ ,$$

$$N_A = \frac{\partial Z_A}{\partial \lambda_A} \ ,$$

$$S = \sum S_A = k\left(\theta E + Z - \sum \lambda_A N_A\right) \tag{75}$$

and the Gibbs relation (7) becomes

$$T\, ds = dE + \sum \bar{A}^{\mu} da_{\mu} - \sum \mu_A\, dN_A \tag{76}$$

where

$$\mu_A = \lambda_A kT \ , \qquad \theta = \frac{1}{kT} \ .$$

Now if the substances are involved in chemical reactions there will be a smallest and nonreducible set of reactions described by

$$\sum \nu_{\alpha A}\, C_A = 0 \qquad \alpha = 1 \ldots \nu \tag{77}$$

where C_A represents particles of component A and $\nu_{\alpha A}$ are the stoichiometric coefficients of the α^{th} reaction. We may now proceed to calculate the equilibrium rates as before except that the constraints (74), which in differential form read as

$$dN_A = \sum_i dN_{A_i} = 0 \tag{78}$$

must be replaced by

$$dN_1 : dN_2 : \ldots dN_m = \nu_{\alpha_1} : \nu_{\alpha_2} : \ldots : \nu_{\alpha_m} .$$

Again the Lagrange multipliers λ_A ($A = 1 \ldots m$) may be introduced to deal with the problem of maximizing log W, but they are clearly subject to the constraints

$$\sum_A \lambda_A \nu_{\alpha A} = 0 \qquad (\alpha = 1 \ldots . \nu) \tag{79}$$

in order that $\sum_A \lambda_A dN_A = 0$ for arbitrary variations of N_{A_i} subject to (78). Then all formulae of Sec. 2 follow for each component of the gas separately, the temperature T is common to all components, but the chemical potentials $\mu_A = \lambda_A kT$ must, by (79) satisfy

$$\sum_A \mu_A \nu_{\alpha A} = 0 \qquad (\alpha = 1 \ldots \nu). \tag{80}$$

For a mixture of perfect gases we have (14), (16), (16) and (12) replaced by

$$n_A = n_A(\mu_A, T) = A(T)\, I^{11}_{\varepsilon_A}\left(\frac{m_A c^2}{kT}, \frac{\mu_A}{kT}\right) , \tag{81}$$

$$\rho_A = \rho_A(\mu_A, T) = kT\, A(T)\, I^{21}_{\varepsilon_A}\left(\frac{m_A c^2}{kT}, \frac{\mu_A}{kT}\right) , \tag{82}$$

$$P_A = P_A(\mu_A, T) = \frac{1}{3} kT\, A(T)\, I^{03}_{\varepsilon_A}\left(\frac{m_A c^2}{kT}, \frac{\mu_A}{kT}\right) , \tag{83}$$

$$s = \frac{S}{V} = s(\mu_1 \ldots \mu_m, T) = \sum_A s_A(\mu_A, T)$$

$$= \frac{1}{T}\left[\rho + P - \sum \mu_A n_A\right] , \tag{84}$$

where $\rho = \sum_A \rho_A$, $P = \sum_A P_A$ and μ_A are subject to (80).

Using (57) for the case of Boltzman statistics one obtains from (80) the relativistic law of mass action

$$\prod_A n_A^{\nu_{\alpha A}} = \prod_A \left[4\pi g_A ckTm_A^2 h^{-3} K_2\left(\frac{m_A c^2}{kT}\right)\right]^{\nu_{\alpha A}} \qquad \alpha = 1 \ldots n \tag{85}$$

The reason for setting $\mu_\gamma = 0$ for boson radiation can now be clarified. Thus under photon producing reactions such as bremsstrahlung $e^+ \rightleftharpoons e^+ + \gamma$ or pain annihilations $e^+ + e^- \rightleftharpoons 2n\gamma$, with n arbitrary, the constraint equation (80) immediately gives $\mu_\gamma = 0$. It is also clear from this last reaction that for particles and antiparticles in equilibrium with such photons their chemical potentials must be equal and opposite.

5. GENERAL RELATIVISTIC FLUIDS IN LOCAL EQUILIBRIUM

An m-component fluid in general relativity is determined by m equations of state which we will always take to be of the form (81-83). The Einstein field equations are assumed

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{86}$$

where

$$T_{\mu\nu} = \rho u_\mu u_\gamma + ph_{\mu\nu} \tag{87}$$

$$\rho = \sum_A \rho_A(\mu_A, T), \qquad p = \sum_A p_A(\mu_A, T),$$

$$u_\mu u^\mu = -1 ,$$

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu .$$

The conservation laws (Bianchi identities)

$$T^{\mu\nu}_{;\nu} = 0 \tag{88}$$

imply

$$\dot{\rho} + (\rho + P)u^{\nu}_{;\nu} = 0 \tag{89}$$

$$(\rho + P)\dot{u}_{\nu} = -P_{;\mu}h^{\mu}_{\nu} \tag{90}$$

where $\dot{}$ means $_{;\mu}u^{\mu}$. If chemical reactions are occurring then in any small comoving volume V, we have from (78)

$$\dot{N}_1 : \dot{N}_2 : \ldots : \dot{N}_m = \nu_{\alpha 1} : \nu_{\alpha 2} : \ldots : \nu_{\alpha m} \qquad (\alpha = 1 \ldots\ )$$

where $N_A = n_A V$. Using the identity (Synge 1960 p.172)

$$\frac{\dot{V}}{V} = u^{\nu}_{;\nu}$$

we see that

$$\dot{n}_A + n_A u^{\nu}_{;\nu} = \sum_{\alpha} \nu_{\alpha A} r_{\alpha} \qquad (\alpha = 1 \ldots .\ \nu,\ A = 1 \ldots m). \tag{91}$$

where $r_{\alpha} = \dfrac{\dot{N}_A}{\nu_{\alpha A} V}$ is the α-th rate of reaction (= "numbers of reactions"/ unit vol/unit proper time).

The Gibbs relation in V

$$TdS = dE + PdV - \sum \mu_A dN_A$$

may be re-expressed as a relation on the entropy density

$$Td(sV) = d(\rho V) + PdV - \sum_{\alpha} \mu_A\, d(n_A V)$$

and using (84) this gives

$$Tds = d\rho - \sum_{A} \mu_A\, dn_A\ . \tag{92}$$

In terms of proper time derivatives, using (81), (83), (76) and the equilibrium condition (80) this results in

$$T\dot{s} = Ts\, u^{\nu}_{;\nu}$$

or equivalently

$$(Su^{\nu})_{;\nu} = 0 \tag{93}$$

an equation representing the conservation of entropy. Key to this derivation was the conservation identity (88), and in particular its u^{μ} component (89). For individual components we should not expect

$$T^{\mu\nu}_{A;\nu} = 0$$

to hold, so that in general

$$\dot{\rho}_A + (\rho_A + P_A)u^{\nu}_{;\nu} = Q_A \neq 0 .$$

The Q_A represent the heat flux per unit volume to the A^{th} component since

$$dQ_A = d\rho_A + (\rho_A + P_A)dV/V = (dE_A + P_A dV)/V .$$

The equation for the A^{th} entropy current will read

$$(S_A u^{\nu})_{;\nu} = \left\{Q_A - \sum_{\alpha} \mu_A \nu_{\alpha A} r_{\alpha}\right\} / T , \tag{94}$$

the terms on the right hand side represent entropy flux due to heat exchanges between components and due to chemical reactions respectively. Thus in general, even at equilibrium the entropy of individual components is not conserved; however, the total entropy is conserved. The breakdown of the latter phenomenon, i.e. the case of net internal entropy production, is characteristic of irreversible processes (de Groot 1951) which we consider in the next section.

6. NON-EQUILIBRIUM RELATIVISTIC FLUIDS

For a general fluid we postulate an energy stress tensor of the form (Ehlers, 1961)

$$T_{\mu\nu} = \rho u_\mu u_\nu + Ph_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu} \tag{95}$$

where again

$$u_\mu u^\mu = -1, \qquad h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

and

$$q_\mu u^\mu = 0, \qquad \pi_{\mu\nu} = \pi_{\nu\mu}, \qquad \pi_{\mu\nu} u^\mu = 0$$

the q_μ is called the heat flux relative to u^μ and $\pi_{\mu\nu}$ the viscous stresses. Note that the division of stresses into a viscous and non-viscous parts (i.e. the term $Ph_{\mu\nu}$) is not unique at this stage, we shall return to this point later.

The conservation laws (88) give

$$\dot{\rho} + (\rho + P)\theta = -q^\nu_{;\nu} - q_\mu \dot{u}^\mu - \pi^{\mu\nu} u_{\mu;\nu} \tag{96}$$

and

$$(\rho + P)\dot{u}_\mu = -h^\nu_\mu (P_{;\nu} + \dot{q}_\nu + \pi^\rho_{\nu;\rho}) - q_\mu \theta - q^\nu u_{\mu;\nu} \tag{97}$$

where

$$\theta = u^\mu_{;\mu} . \tag{98}$$

Let the fluid consist of several components, u^μ_A being the proper 4-velocity of the A^{th} component, n_{0A} its particle density as measured in a local frame at rest with respect to u^μ_A (i.e. its proper particle density) and set

$$N^\mu_A = n_{0A} u^\mu_A$$

for the A^{th} particle flux vector. A simple decomposition of N^μ_A into parts parallel and orthogonal to u^μ gives

$$N^\mu_A = J^\mu_A + n_A u^\mu \tag{99}$$

where

$$J^\mu_A = N^\nu_A h^\mu_\nu \qquad (J^\mu_A u_\mu = 0)$$

is the A^{th} particle flux relative to u^μ and

$$n_A = -n_{0A} u^\nu_A u_\nu \tag{100}$$

is the particle density as measured in the u^μ frame (this reduces in special relativity to the classical formula $n_A = n_{0A}\,(1 - v_A^2)^{\frac{1}{2}}$ attributing changes in number density as being due to Lorentz contraction). Now just as for Eq. (91) we must have in the particle's proper frame

$$N^\mu_{A;\mu} = \sum_\alpha \nu_{\alpha A} r_\alpha \,,$$

and on substituting Eq. (99) we obtain

$$\dot{n}_A + n_A \theta = -J^\mu_{A;\mu} + \sum_\alpha \nu_{\alpha A} r_\alpha \,. \tag{101}$$

In order to make contact with equilibrium thermodynamics it is usual to postulate that the equilibrium equations of state (81-84) still hold good everywhere, but to drop the constraint equation (80) which is imposed by strict equilibrium. These equations then serve to define the pressure $P = \sum P_A$ and the entropy density as functions of ρ and n_A, by eliminating μ_A and T. This is the way in which the pressure is separated from the viscous stresses. Finally Eqn. (88) holds again since it depends only on the equations of state without imposition of (80), so that one may after some juggling of terms using Eqs. (84, 96) and (101) obtain an entropy production law

$$S^\mu_{;\mu} = \sum \,, \tag{102}$$

where

$$S^\mu = su^\mu + (q^\mu/T) - \sum_A \mu_A J^\nu_A/T \tag{103}$$

is the entropy flux vector and

$$\sum = -\frac{1}{T^2} q^{\nu}(T_{,\nu} + T\dot{u}_{\nu}) - \frac{1}{T}\pi^{\mu\nu}u_{\mu;\nu} - \sum_A \left(\frac{\mu_A}{T}\right)_{,\mu} J^{\mu}_A + \sum_{\alpha} \frac{A_\alpha r_\alpha}{T} \tag{104}$$

is the rate of internal entropy production (/unit volume). In the last term on the R.H.S. we have set

$$A_\alpha = -\sum_A \mu_A \nu_{\alpha A} \tag{105}$$

an expression known as the chemical affinity of the α-th reaction.

The terms on the R.H.S. of Eq. (103) represent the bodily transport of entropy, and flow of entropy induced by heat transfers and particle drifts (diffusion). These terms are all to be regarded as describing the reversible part of the entropy production, i.e. the part that can be accounted for by transport from one place to another. The terms making up $\sum$ cannot be so interpreted and are to be regarded as due to purely internal irreversible processes. The main requirement is the second law of thermodynamics, that

$$\sum \geq 0 \tag{106}$$

and $\sum = 0$ only for equilibrium.

A common way of ensuring that (106) holds is to postulate linear constitutive relations between the "currents" and "forces" on the R.H.S. of (104):

$$q_\mu = -\kappa h^{\nu}_{\mu}(T_{;\nu} + T\dot{u}_{\nu}) \tag{107}$$

$$\pi_{\mu\nu} = -\eta\,\sigma_{\mu\nu} - \beta h_{\mu\nu}\theta \tag{108}$$

$$J_{A_\mu} = -\delta\, h^{\nu}_{\mu}(\mu_A/T)_{;\nu} \tag{109}$$

$$r_\alpha = L_{\alpha\beta}A_\beta \tag{110}$$

with the shear tensor $\sigma_{\mu\nu}$ defined by

$$\sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} = h_{(\mu}{}^{\alpha}h^{\beta}_{\nu)}u_{\alpha;\beta}\ , \quad \sigma_{\mu}{}^{\mu} = 0.$$

In these equations κ, η, β and δ are positive constants known as coefficients of heat conduction, shear viscosity, bulk viscosity and diffusion respectively, and $L_{\alpha\beta}$ is a symmetric positive definite matrix (no confusion should arise with other uses for these symbols which except for the last are restricted to this section). The laws (107-110) then represent relativistic generalizations of Fourier's law, Newton's law of viscosity, Fick's law and Onsager's law. These can be regarded as purely phenomenological relations, or they can be derived from the relativistic kinetic theory of gases (Stewart 1971, Israel 1972). The latter procedure is of course preferable particularly as it gives a handle on the explicit calculation of the various coefficients on the basis of the particle interactions. Substituting in Eq. (105) we see indeed that

$$\begin{aligned} \sum = &+ \frac{1}{T^2}\kappa h^{\mu\nu}(T_{,\mu} + T\dot{u}_{\mu})(T_{,\nu} + T\dot{u}_{\nu}) + \frac{2}{T}\eta\sigma^2 + \frac{1}{3}\beta\theta^2 \\ &+ \delta h^{\mu\nu}\sum_{A}(\mu_A/T)_{,\mu}(\mu_A/T)_{;\nu} + \sum_{\alpha,\beta} h_{\alpha\beta}A_{\alpha}A_{\beta} \\ &\geq 0 \end{aligned} \tag{111}$$

where $\sigma^2 = \frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu}$.

A significant problem concerning these constitutive relations is that they are not frame independent. As pointed out by Israel (1976) the quantities ρ, P, q_{μ}, $\pi_{\mu\nu}$ occurring in (95) depend implicitly on the choice of 4-vector u^{μ} and should be written as $\rho(u)$, $P(u)$ etc.. Under a change of frame

$$u_{\mu} \to u'_{\mu} = \gamma u_{\mu} + v_{\mu}\ , \qquad v_{\mu}u^{\mu} = 0$$

we find

$$\rho \to \rho' = \rho(u') = T_{\mu\nu}u'^{\mu}u'^{\nu} = \gamma^2\rho + (\gamma^2 - 1)P - 2q^{\mu}v_{\mu}\gamma + \pi_{\mu\nu}v^{\mu}v^{\nu}$$

and

$$n_A \to n_A' = n_A\gamma - J_A^{\mu}v_{\mu} \ .$$

Thus there is implicit, via Eqs. (81-82) a complicated transformation of chemical potentials $\mu_A \to \mu_A'$ and temperature $T \to T'$ and associated with them a change $P \to P'$, $s \to s'$, under any such change of frame. On the other hand the heat flux changes as

$$q^{\mu} \to q'^{\mu} = -h'^{\mu}_{\ \rho}T^{\rho\nu}u'_{\nu}$$

and there is nothing to suggest that all these independent transformations will occur in such a way as to preserve the form of Fourier's law (107), except in the most specialized cases.

In particular, if $T_{\mu\nu}u^{\mu}u^{\nu} \geq 0$ for all time-like vectors u (positive energy requirement) then a theorem of Synge (1960) assures us that there is a unique time-like eigenvector of u_E^{μ} of $T_{\mu\nu}$ (the energy frame) for which $q^{\mu}(u_E) = 0$. Eq. (107), if it was to hold in all frames, would give the too restrictive equation

$$(Tu_{E_\nu})^{\cdot} = -T_{,\nu} \ ,$$

and similarly Eq. (109) would provide an over restrictive requirement on the A-particle frame u_A^{μ} for which J_A^{μ} vanishes.

By inclusion of 2nd order terms (assuming u^{μ} and u_A^{μ} depart at most by 1st order from the energy frame u_E^{μ}, and similar smallness assumptions on the ratio $\pi_{\mu\nu}/\rho$) Israel has shown how to modify the constitutive relations (107-110) in order that they are truly frame independent (to second order) at the same time introducing a second order modification to S^{μ} and $\sum$ in order to retain Eqns. (102) and (106).

Notwithstanding this important advance we shall not here adopt

Israel's modified equations. Rather let us work in the energy frame ($q_\mu = 0$) and, since we will from Sec. 7 onwards be dealing mainly with homogeneous situations, we will also impose the condition of no diffusion ($u_{A\mu} = u_\mu$) and shear viscosity ($\eta = 0$). Incidentally, if one specifies the frame as the energy frame, there is no harm in postulating the relations (108-110) in this frame; one must just remember that they may transform to awkward equations in other frames.

In the above analysis all components were assumed to be at the same temperature T. The case where the interaction between the components is not sufficient to produce such an equalization of temperature is easily dealt with at least in the case of zero conduction, diffusion and viscosity ($q^\mu = J^\mu_A = \pi_{\mu\nu} = 0$). We assume then that

$$T_{\mu\nu} = \sum_A T_{A\mu\nu}$$

where

$$T_{A\mu\nu} = \rho_A u_\mu u_\nu + P_A h_{\mu\nu} .$$

From the discussion at the end of Sec. 5 we see that Eq. (94) holds for individual components, whence summing over A gives

$$(su^\rho)_{;\rho} = \sum_A \frac{Q_A}{T_A} - \sum_A \frac{\mu_A}{T_A} \nu_{\alpha A} r_\alpha$$

where T_A is the temperature of the A^{th} component and Q_A is the heat flux/unit volume imparted to the A^{th} component by the others. Since

$$T^{;\nu}_{\mu\nu} = \sum_A T^{;\nu}_{A\mu\nu} = 0$$

we must have

$$\sum_A Q_A = 0 . \qquad (112)$$

Since Q_A depends on the thermalizing processes between different components it is reasonable to expect it to be proportional to the number of particle collisions and the energy differences of particles in such collisions, thus

$$Q_A = \sum_B a_{AB} n_A n_B (T_B - T_A) \tag{113}$$

By Eq. (112) with arbitrary $n_A n_B$ we must have $a_{AB} = a_{BA}$ and clearly we require $a_{AB} \geq 0$ if heat is to pass only from hot to cold. Adopting a modified Onsager law

$$\nu_\alpha = L_{\alpha\beta} A'_\beta \qquad A'_\beta = -\sum_A \mu_A \nu_{\beta A} / T_A$$

we obtain the entropy production

$$(su^\rho)_{;\rho} = \sum_{A<B} n_A n_B a_{AB} \frac{(T_B - T_A)^2}{T_A T_B} + \sum L_{\alpha\beta} A'_\alpha A'_\beta$$

$$\geq 0 \ . \tag{114}$$

7. THERMODYNAMICS OF ROBERTSON-WALKER COSMOLOGIES

Let us now apply these ideas to a cosmological situation. Assumptions of homogeneity and isotropy allow us to adopt the usual Robertson-Walker metric

$$ds^2 = -c^2 dt^2 + R^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right] , \qquad K = \pm 1 \text{ or } 0 \tag{115}$$

In this case we have $\theta = \frac{3\dot{R}}{R}$ and the energy stress tensor must be of the perfect fluid form so that Eqs. (89) and (91) reduce to

$$\dot{\rho} + 3(\rho + P) \dot{R}/R = 0 \tag{116}$$

$$\dot{n}_A + 3n_A \dot{R}/R = \sum \nu_{A\alpha} r_\alpha(t) \tag{117}$$

while the Einstein equations reduce to the single (independent) equation

$$(\dot{R}^2 + Kc^2)/R^2 = 8\pi G\rho/3c^2 \ . \tag{118}$$

In later chapters we consider the time development of this system in some generality, for the moment however let us consider the simplest case of no reactions ($r_\alpha = 0$). Given the initial values at some time $t = t_0$

$$R(t_0) = R_0, \quad \mu_A(t_0) = \mu_{A_0}, \quad T(t_0) = T_0$$

the above equations predict a unique time development. Equivalently of course $n_A(t_0)$ and T_0 could be given and the equation of state (81) solved for the chemical potentials μ_{A_0}. Eq. (117) integrates to give number conservation

$$n_A R^3 = n_{A_0} R_0^3 = \text{constant} \tag{119}$$

and Eq. (116) can be replaced by entropy conservation

$$sR^3 = s_0 R_0^3 = \text{const.} \tag{120}$$

From (81), (84) and (82) we may solve ρ as a function of n_A and s, substituting this into the R.H.S. of Eq. (118) and using Eqs. (119-120) we obtain a single first order differential equation for R from which there results a unique solution

$$R = R(t, R_0, \mu_{A_0}, T_0) \ .$$

The time development of $\mu_A(t)$ and $T(t)$ are then obtained from (119) and (120). Now if it is assumed that the fluid contains as a component black body radiation following the Planck law with $\mu_\gamma = 0$ we see an immediate inconsistency for there is nothing in the equations which implies that an initially zero value of μ_γ is preserved. This problem has several possible resolutions within the context of the relativistic thermodynamics developed in this chapter; we now consider each of these in some detail.

(A) There are photon-producing reactions such as bremsstralung ($e \to e + \gamma$) or pair annihilation (e.g. $e^+ + e^- \to 2\gamma$) and these reactions are proceeding at equilibrium rates.

This latter requirement implies the vanishing of the chemical affinities $A_\alpha = 0$ (i.e. Eq. 80), and since the number of photons produced on the R.H.S. of the various reactions is essentially arbitrary, we must set $\mu_\gamma = 0$. Thus we assume the photons satisfy the normal Planck law, Eq. (45).

If bremsstrahlung is the only reaction then all other particles are conserved

$$n_A(\mu_A, T)R^3 = \text{const.}, \qquad (A \neq \gamma) \tag{121}$$

while

$$\dot{n}_\gamma + 3n_\gamma \dot{R}/R = r_b \tag{122}$$

where r_b is the bremsstrahlung photon production rate. Entropy conservation

$$S(T, \mu_1, \mu_2 \ldots)R^3 = \text{const.} \tag{123}$$

together with Eq. (121) can be solved for T and μ_A in terms of R and T_0, μ_{A0} and when substituted into the R.H.S. of Eq. (118) gives a unique solution for $R(t, R_0, \mu_{A0}, T_0)$. Finally the photon production rate is read off from Eq. (89) using (45)

$$r_b = b(T^3R^3)^{\cdot}/R^3 \tag{116}$$

Details of this calculation for a Boltzman distribution of protons and electrons are carried out in the next chapter. If photon entropy greatly dominates the particle entropy, as appears to be the case in the present universe, then to a good approximation we may set $s \simeq s_\gamma \propto n_\gamma \propto T^3$ (Eq. (46)) whence $T \alpha R^{-1}$, $n_\gamma R^3 \simeq$ const., $r_b \simeq 0$ and we have an effective photon number conservation. If however the radiation and matter entropies are more comparable considerable changes in photon number occur.

The case of pair annihilation is dealt with as follows, in which we only consider electron-positron pairs. From $A_{ea} = 0$ (A_{ea} = electron pair annihilation chemical affinity) we have $\mu_{e^-} = -\mu_{e^+}$ (since $\mu_\gamma = 0$), whence

$$n_{e^-} = n_e(\mu_e, T) \qquad n_{e^+} = n_e(-\mu, T) \ .$$

Furthermore

$$\dot{n}_{e^-} + 3n_{e^-} \dot{R}/R = \dot{n}_{e^+} + 3n_{e^+} \dot{R}/R = r_{ea}$$

whence

$$\left(n_e(\mu_e, T) - n_e(-\mu_e, T)\right) R^3 = \text{const.}.$$

Together with entropy conservation (in which we suppress chemical potentials of other particles such as protons etc.)

$$s(\mu_e, T) R^3 = \text{const.}$$

we can again solve for T and μ_e in terms of R and proceed exactly as above. Such fluids are considered in detail in Chapter 4.

(B) No photon-producing reactions, but matter and radiation interact strongly (e.g. through Compton scattering) to equalize their temperatures.

Clearly with a constraint on photon number we should expect from the general considerations of Sec. 2 that there is a photon chemical potential. Thus the Planck law (44-45) must be abandoned in favour of the more general Bose law (15-16) or (40-41). We may now proceed exactly as above, using Eqns. (119) and (120) to deduce the time evolution of the universe. The photon energy-momentum stress tensor will in general not satisfy a conservation law i.e.

$$\dot{\rho}_\gamma + (\rho_\gamma + p_\gamma)\theta \neq 0$$

with the R.H.S. representing the rate of heat transfer per unit volume imparted to the photon field in order to preserve the thermal equilibrium. It may be calculated if required at the end, after all time-evolutions of physical quantities have been determined. The main drawback of this procedure is the possibility of μ_γ becoming positive leading to divergences in integrals (15-16) and of zero point condensation at low temperatures for $\mu_\gamma < 0$.

(C) All components are at the same temperature, photon-producing reactions occur but not at equilibrium rates.

In this case it is necessary to appeal to the discussion in Sec. 6. One may no longer assume $A_\alpha = 0$, and as a working hypothesis one could consider an Onsager law

$$r_\alpha = \sum_\beta L_{\alpha\beta} A_\beta \ .$$

Substituting this in the R.H.S. of Eq. (112) and using the equations of state (81-83), we may regard (116-118) as a set of simultaneous first order differential equations in μ_A, T and R which will have a unique solution for any initial values μ_{A_0}, T_0 and R_0. Again it is necessary to assume a non-zero chemical potential μ_γ. The main drawback to this suggestion is that there appears to be no ready-made procedure for estimating the Onsager coefficients $L_{\alpha\beta}$, say on the basis of the kinetic theory of reactions.

(D) Components at different temperatures.

This might be appropriate if processes such as Compton scattering are not sufficiently strong to equalize temperatures. We now replace (116) with the equations

$$\dot{\rho}_A + 3(\rho_A + P_A)\dot{R}/R = Q_A = \sum_B a_{AB}n_An_B(T_B - T_A) \qquad (125)$$

and the system of euqations (117), (125) and (118) serve to define a first order system of differential equations which determine a unique time development for μ_A, T_A and R. The coefficients a_{AB} defined by Eq. (113) can in principle be calculated from the various scattering cross-sections among the different particle species.

We should perhaps add another possibility to this list. That the photons are in both thermal and chemical equilibrium with the other components however the number of photons produced in a reaction is not arbitrary. The chemical potential of photons is then negative and must satisfy (80). It is difficult however to imagine a physical situation in which this could occur, the problem being that the very neutrality of photons seems to require that if photons are produced at all they will generally be produced in arbitrary numbers.

The above relativistic thermodynamical approaches can, in the end, only be justified by a detailed study of the microprocesses occurring in the fluid. Such a kinetic approach though more rigorous is also much more complicated and is often shunned as an alternative to the thermodynamical approach. Indeed in the standard cosmological model (Weinberg 1972) kinetic theory need only enter through the calculation of characteristic particle collision times. Thus the expanding fluid is conceived of passing through a sequence of equilibrium or near equilibrium configurations each of which can be tackled by positing the types of particles in thermal and chemical equilibrium and assuming any non-equilibrium processes have little back effect on the distribution functions. Transitions between configurations (e.g. from (A) to (B)) occur instantaneously and at a temperature to be determined by comparing characteristic collision times $t_c \sim \frac{1}{<\sigma n v>}$ (the <> indicates some averaging process) involved in maintaining the initial configuration with the expansion time. Such a transition can often be designated by a unique decoupling temperature, for example, in the standard model neutrinos decouple from the rest of the fluid at $\sim 10^{10}$°K and thereafter do not interact again. Of course if one is interested in the detailed evolution of the distribution function one must return to the kinetic approach (e.g. Kompaneets 1957, Coste and Pegraud 1975).

In later chapters we will be interested in models which differ significantly from the standard model, our approach however will be much the same as above. What this amounts to is using the physics to determine the type of equilibrium configuration and the relativistic thermodynamics above to study its cosmologically important features. The assumption of ideal and therefore collissionless fluids might at first seem to deny the possibility of studying strongly interacting fluids. However, as we will show, there are two important examples where this is not the case. The first exploits the fact that the thermodynamics conceals an interaction in the Pauli principle and that this interaction can effect its own type of equilibrium. The second considers recent models of hadronic matter in which the consequences of the "strong" interactions between particles are almost totally given in terms of the mass spectrum of the particles themselves.

Before we can consider such situations however we must be able to solve for the dynamical and thermodynamical evolution of models of considerable complexity. This technical detail suggests the best route of attack is through the generalization of simple models and this will be the task of Ch. 3 (sec. 2, 3) and Ch. 4. It is sensible however to first fix some of the ideas of this chapter within a physically reasonable model. We thus will devote the first section of the next chapter to the consideration of a cosmological model which mimics the gross features of standard cosmology.

CHAPTER 3

A SYNGE GAS WITH BLACK BODY RADIATION AND SOME VERY SIMPLE QUANTUM MODELS

1. INTRODUCTION

We begin our consideration of relativistic thermodynamics within Robertson-Walker cosmologies with some simple cases. In the next section we outline a model consisting of Synge gases of electrons and protons together with black body radiation. The model has impact on a number of levels. It is realistic in the sense that it imitates some of the gross features of standard cosmology and it has educational value since many of these features can be described analytically with the use of certain functions developed within the section. Most importantly however, it allows a context for discussion of the origin of the photon to baryon ratio since within the models the ratio appears as a natural parameter to be given at $t = 0$.

The ratio which in general is a ratio of black body photon number to a conserved particle number has considerable importance in this work. It has already appeared as a criterion for degeneracy in Sec.3 Ch.2 and as we will show it is intimately connected with transfers of energies and entropies between components. In a sense this is because the photon density acts as a thermometer $\sim T^3$ for the models while the conserved number density $\sim R^{-3}$ acts as a scale. Their ratio is then an integral measure of the amount the expansion disturbs the equilibrium distributions - i.e. if the ratio is constant the shape of the equilibrium distribution is maintained by the expansion while if the ratio is not constant the expansion is pushing the distribution away from its initial equilibrium form. In the latter case we would expect the transfers of energy and entropy mentioned above. The concept of the ratio remains useful even if black body photons are not present within the model.

In Sec. 3 we study a model containing a single massive fluid but with the full quantum statistics of the equations of state. The model in general must be solved numerically for dynamical and chemical evolution but in three particular cases we can provide analytic results. The three cases, nondegeneracy (Synge gas), fermion degeneracy and boson condensation provide interesting contrasts. For example the epoch at which rest mass begins to dominate the dynamics occurs for the Synge gas at $x = 1$, for the fermion degenerate gas when $x >> 1$ and for the boson condensed gas when $x << 1$.

Section 4 completes the chapter with a discussion of models containing many different types of zero rest mass components. These contrast sharply with the evolving matter models of Sec. 3 for it turns out that a single solution of the equations of state gives the chemical composition of the fluid for the whole of the expansion; the chemical composition is frozen in throughout the expansion. For these "collisionless" models the very notion of equilibrium becomes irrelevant as long as we insure that the components begin the expansion with their equilibrium distributions, since the expansion itself preserves these distributions. We give an explicit solution to the dynamics even though in general we need numerical methods to calculate the chemical composition. In some quite complex situations however the latter does not apply and as example we give the complete and elegant solutions to a system of neutrinos and antineutrinos in thermal and chemical equilibrium with black body radiation. These solutions will be of interest for later chapters.

2. SYNGE GAS WITH BLACK BODY RADIATION

Let us consider in this section a simple yet instructive cosmological model which can characterize some of the ideas expressed in the previous chapter. The model contains both matter and radiation in the form of a Synge gas of electrons and protons together with black body radiation. We are thus in situation A (except no pair production) of Sec.7 of Chapter 2.

Motivation for this problem comes from the fact that most considerations of realistic cosmologies are conducted in a heuristic spirit. The evolution of the universe being split into a number of epochs according to whether the matter or radiation dominates the dynamical behaviour and according to what thermalizing forces are at work (e.g. see Peebles, 1971; Weinberg, 1972). The behaviour in each epoch is discussed entirely in terms of the dominant component and transitions between epochs becomes a somewhat uneven and jerky affair. We are saved in this by the observational fact that while the present energy density resides mostly in the matter, the entropy is almost entirely carried by photons whose number on the basis of the 2.7°k black body temperature is some 10^8 times that of the baryons. Thus in any epoch where photons and matter are strongly coupled the thermodynamics is almost entirely dictated by the photon field. In the early universe then, thermodynamics is determined by radiation while dynamics is determined by whichever component dominates the energy density at that time.

However as we have discussed in Chapter 1 the origin of the high photon to baryon ratio is something of a mystery and has been the subject of many "explanations". It is important therefore to investigate this ratio in models which though simple still reproduce the gross features of observational cosmology. The model to be treated below consists of equal numbers of protons and electrons described by the Synge equations of state in thermal equilibrium with black body radiation. The fact that within these models the photon to baryon ratio appears as a natural parameter to be given at $t = 0$ allows us to treat low ratio universes as well. Moreover in general the ratio will not be constant during the expansion; the standard cosmological model being characterized by a ratio so large that it is very nearly constant.

The equations of state have been discussed in Ch.2, Sec. 2 they are (2:56-59)

$$n_i = \frac{\beta_i}{4\zeta(3)} \frac{K_2(x_i)}{x_i} e^{\lambda_i} , \tag{1}$$

$$\rho_i = n_i m_i c^2 \left(\frac{K_3(x_i)}{K_2(x_i)} - \frac{1}{x_i} \right) , \tag{2}$$

$$P_i = n_i kT , \tag{3}$$

where 1 = electrons, 2 = protons and g_i has been set to 1. The entropy density of the massive components is

$$s_i = kn_i \left(x_i \frac{K_3(x_i)}{K_2(x_i)} - \lambda_i \right) \tag{4}$$

and since the number of electrons and protons are equal and conserved we can define the photon number to baryon (electron) number ratio

$$f = \frac{n_\gamma}{n_i} = \frac{\beta}{x^3 n} \tag{5}$$

and rewrite (1) to obtain

$$\lambda_i = \ell n\, 4\zeta(3) - \ell n\, f - \ell n\, k_2(x_i) - 2\, \ell n\, x_i ,$$

where $\beta = 16\pi\zeta(3)\ (mc/h)^3$ and m is the mass of the reference species.

The equation of state for the black body ($\mu_\gamma = 0$) radiation has been give (2:44-45) and the total entropy of the fluid is

$$\frac{s}{kn} = \frac{S}{kN} = \alpha f + 2\, \ell n\, f - \delta + F(x_1) + F(x_2) , \tag{7}$$

where $\alpha = \frac{4a}{3bk} = \frac{2\pi^4}{45\ \zeta(3)} = 3.60157$,

$$\delta = 2\ \ell n\ 4\ \zeta(3) = 3.14066$$

and

$$F(x) = x\,\frac{K_3(x)}{K_2(x)} + \ell n\ k_2(x) + 2\ \ell n\ x\ . \qquad (8)$$

$F(x)$ is a monotonically increasing function of x with the following limit behaviours

$F(x) \sim 4 + \ell n\ 2$ for $x << 1$ (relativistic regime),

$F(x) \sim \frac{3}{2}\ \ell n\ x$ for $x >> 1$ (non-relativistic regime)

(see Appendix B, Eqs. 4, 5).

For temperatures greater than 10^{13}°K, when both electrons and protons are relativistic

$$\frac{s_0}{kn_0} \approx \alpha f + 2\ \ell n\ f + 8 + 2\ \ell n\ 2 - \delta\ , \qquad (9)$$

whence $f \to$ some constant value f_0 as $x \to 0$. f_0 is to be interpreted as the initial photon/baryon ratio in the universe. The entropy equation (7) can then be regarded as giving f as a function of temperature or equivalently of $x = mc^2/kT$ through

$$\alpha f_0 + 2\ \ell n\ f_0 + 8 + 2\ \ell n\ 2 = \alpha f + 2\ \ell n\ f + F(x_1) + F(x_2)\ , \qquad (10)$$

$$x_i = (m_i/m)x\ .$$

The solution to Eq.(10) must be obtained numerically using Newton's method. Results for different values of f_0 are plotted in Fig. 1.

Clearly f is not constant and decreases monotonically with increasing x. However for values of $f_0 \geq 10^2$ the effect is negligible down to at least 10^{3}°k. This constancy of photon number together with the assumptions that the photons are black body has the important

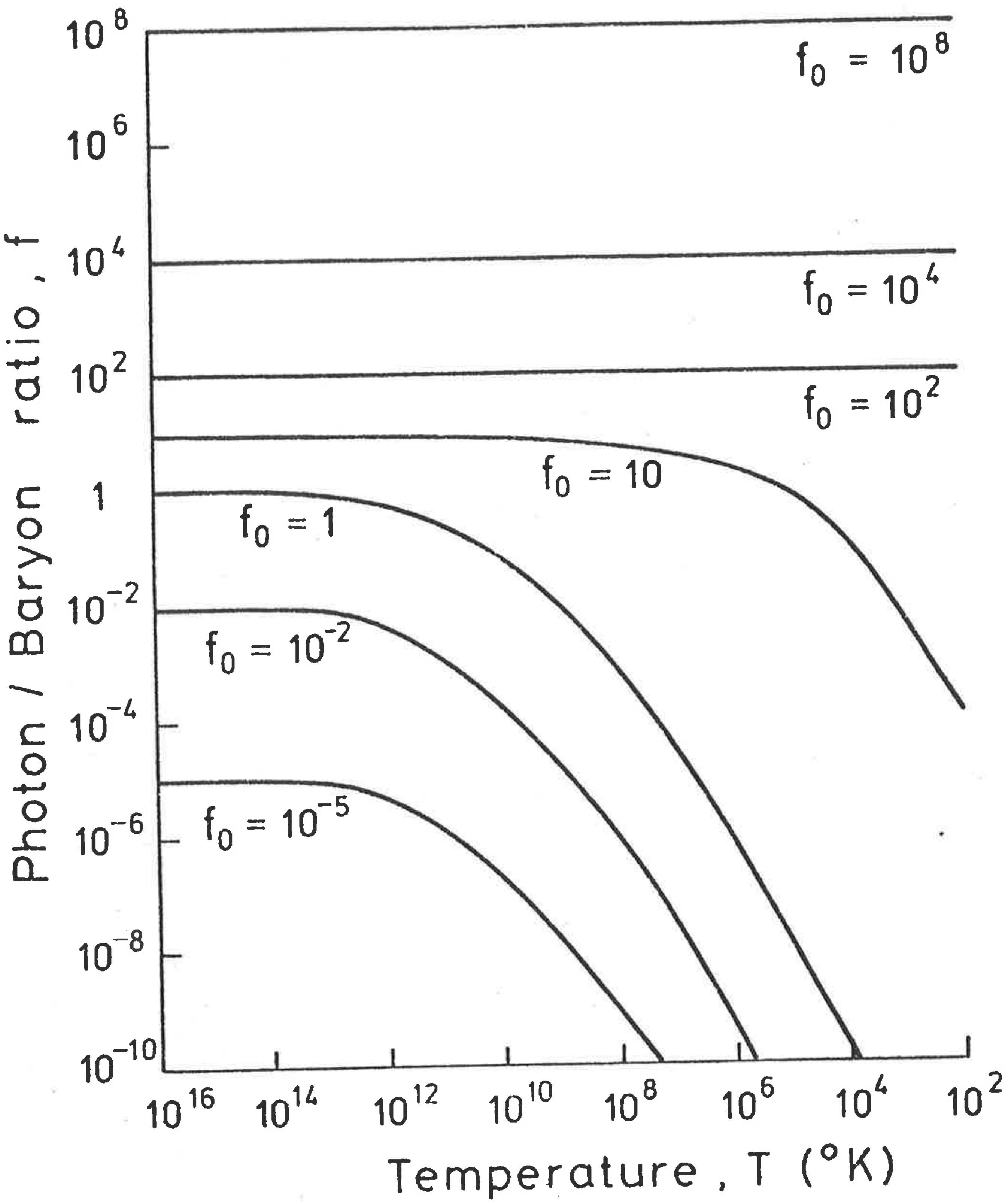

Fig. 1: Ratio of photon to baryon numbers versus temperature for a variety of models.

consequence that the shape of the black body spectrum is preserved by the expansion alone. That is with f constant RT is constant and the frequency of an individual photon ν_γ which falls off as R^{-1} is proportional to T, so that an individual photon need not be subsequently scattered to find itself at a frequency consistent with equilibrium at a new temperature. The net exchange of energy between components is then zero. With smaller values of f_0 however the decrease of f is directly attributable to the flow of energy from the photon to the matter components necessary to maintain equilibrium. In the notation of Sec. 5, Ch. 2 the rate of transfer of energy per unit volume from radiation to matter is

$$-Q = -\left(\dot{\rho}_\gamma + 3(\rho_\gamma + P_\gamma)\frac{\dot{R}}{R}\right) = -\alpha nkT\,\dot{f}\ . \tag{11}$$

The chemical evolution of the matter components is given by Eq. (6). In the relativistic regime the λ_i are constant and equal

$$\lambda_i = \ell n\ 2\ \zeta(3) - \ell n\ f \tag{12}$$

and in the non-relativistic regime

$$\lambda_i = \ell n(4\zeta(3)\sqrt{\pi/2}) - \ell n\ f - \frac{3}{2}\ell n\ x_i + x_i\ . \tag{13}$$

The latter indicates how the degree of non-degeneracy is enhanced by the expansion since $\lambda_i - x_i$ is less than its initial value by $3/2\ \ell n\ x_i$.

The dynamical evolution of the models is obtained by solving the Friedmann expansion equation (2:118) as follows. The number of protons is conserved during the expansion and is given by its present or fiducial value within some conventional volume as

$$N(t_0) = N_0 = n_0 R_0^3 = nR^3$$

(not to be confused with the photon to baryon ratio f_0 at t = 0). From

(5) we obtain R as a function of x

$$R(x) = \left(\frac{n_0 R_0^3}{\beta}\right)^{\frac{1}{3}} x\, f^{\frac{1}{3}}(x)$$

whence

$$\frac{\dot{R}}{R} = \frac{dx}{dt}\frac{d(\ln R)}{dx} = \frac{dx}{dt}\left(\frac{1}{3}\frac{f'(x)}{f(x)} + \frac{1}{x}\right), \qquad f'(x) \equiv df/dx \ . \tag{14}$$

Substituting this on the L.H.S. of Eq. (2:118) and using Eqs. (2, 3) and (2:45) on the R.H.S. and integrating we obtain after some manipulation

$$t = \int_0^{x(t)} \left(\frac{x}{3}\frac{f'(x)}{f(x)} + 1\right)\left[\frac{8\pi G\ mc^2\beta}{3c^2\ f(x)x^2}\left(\sum_i x_i \frac{K_3(x_i)}{K_2(x_i)} - 2 + \frac{3}{4}\alpha f\right) - \frac{Kc^2\ \beta^{2/3}}{n_0^{2/3}\ R_0^2\ f^{2/3}(x)}\right]^{-\frac{1}{2}} dx \ . \tag{15}$$

Equations (10) and (15) generate a family of Friedmann models characterized by three parameters; the initial photon/baryon ratio f_0, the total number of baryons within the fiducial radius $N_0 = n_0R_0^3$, and the curvature coefficient $K = \pm 1$ or 0. In the special case of $K = 0$ models both temperature dependence and dynamics is determined by the sole parameter f_0.

In Fig. 2 we plot some numerical solutions to these equations. The integration is cut off at 10^3 °k since near this temperature the ions recombine to form hydrogen and a separation of the matter and radiation fluids takes place. We have assumed here, and not unreasonably for this early universe, that the curvature term makes no contribution. The change in gradients of the curves indicate the transition from radiation domination of the dynamics (15) to matter domination. A useful tool to characterize the transition is the deacceleration parameter

$$q = -\frac{\ddot{R}R^2}{R\dot{R}^2} = \frac{\rho + 3P}{2\rho - \dfrac{3Kc^4}{4\pi GR^2}} \tag{16}$$

which on ignoring the curvature term gives

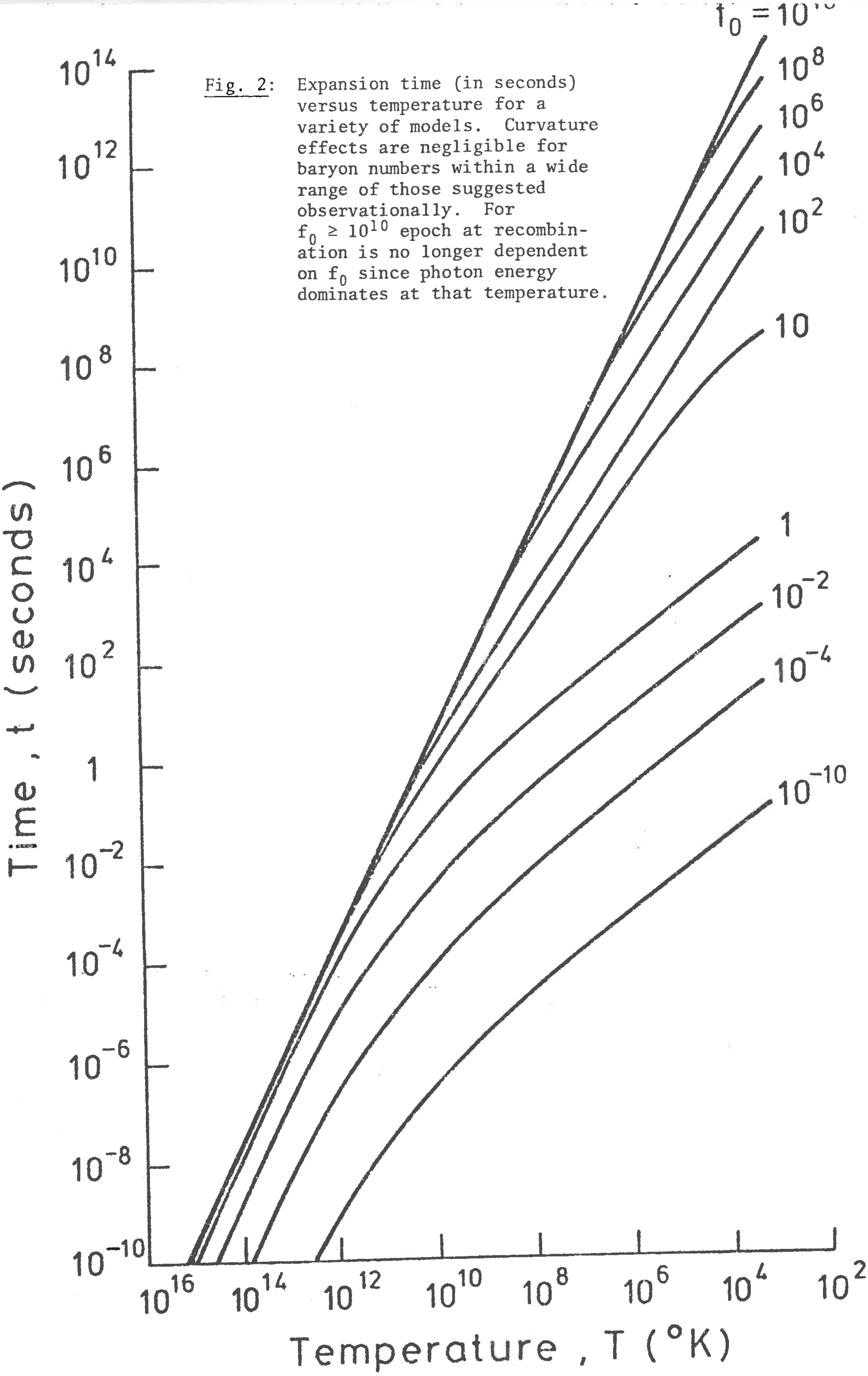

Fig. 2: Expansion time (in seconds) versus temperature for a variety of models. Curvature effects are negligible for baryon numbers within a wide range of those suggested observationally. For $f_0 \geq 10^{10}$ epoch at recombination is no longer dependent on f_0 since photon energy dominates at that temperature.

$$q \approx 1 \quad \text{when photons dominate}$$

$$q \approx \tfrac{1}{2} \quad \text{when matter dominates.}$$

The plots of this function in Fig. 3 suggest that the whole of the non-relativistic regime down to recombination will be dominated by photon behaviour for $f_0 \geq 10^{10}$ or by matter for $f_0 \leq 1$.

The speed of sound can also be calculated in these models, Curtis (1954),

$$\frac{Vs^2}{c^2} = \left[\frac{\delta P}{\delta \rho}\right]_{\text{adiabatic}} = \frac{dP/dx}{d\rho/dx} \tag{17}$$

$$= \frac{f + 2xf'/f + 8}{\left(x_1 \dfrac{K_3(x_1)}{K_2(x_1)} + x_2 \dfrac{K_3(x_2)}{K_2(x_2)}\right)\left(x \dfrac{f'}{f} + 3\right) + \alpha f'x + 3\alpha f} \tag{18}$$

after using (10). This reduces to the usual $1/\sqrt{3}$ limit at relativistic temperatures. Associated with the speed of sound is the Jeans mass, which can be given as (e.g. Weinberg (1972))

$$M_J = \frac{4\pi}{3} \frac{N_0}{R^3} (m_1 + m_2) \left(\frac{v_s^2}{G(\rho + P)/c^2}\right)^{3/2} \tag{19}$$

$$= \phi \left(\frac{Vs^2}{c^2} \frac{f^{1/3} x_1^2}{\left(x_1 \dfrac{K_3(x_1)}{K_2(x_1)} + x_2 \dfrac{K_3(x_2)}{K_2(x_2)} + \alpha f\right)}\right)^{3/2}$$

where

$$\phi = \frac{4}{3} \frac{\pi^{5/2} c^3 (m_1 + m_2)}{G^{3/2} \beta^{1/2} m_1^{3/2}} \approx 86 \, M_\odot \, .$$

The Jeans mass provides an order of magnitude estimate to the lower mass limit to the size of growing perturbations. It is clear from the plots of these Jeans masses given in Fig. 4 that low f_0 universes have a critical mass not much larger than a solar mass. If this critical mass is taken to be the mass at which most fluctuations crystallize out into stable lumps of matter, then such a universe may show no tendency to

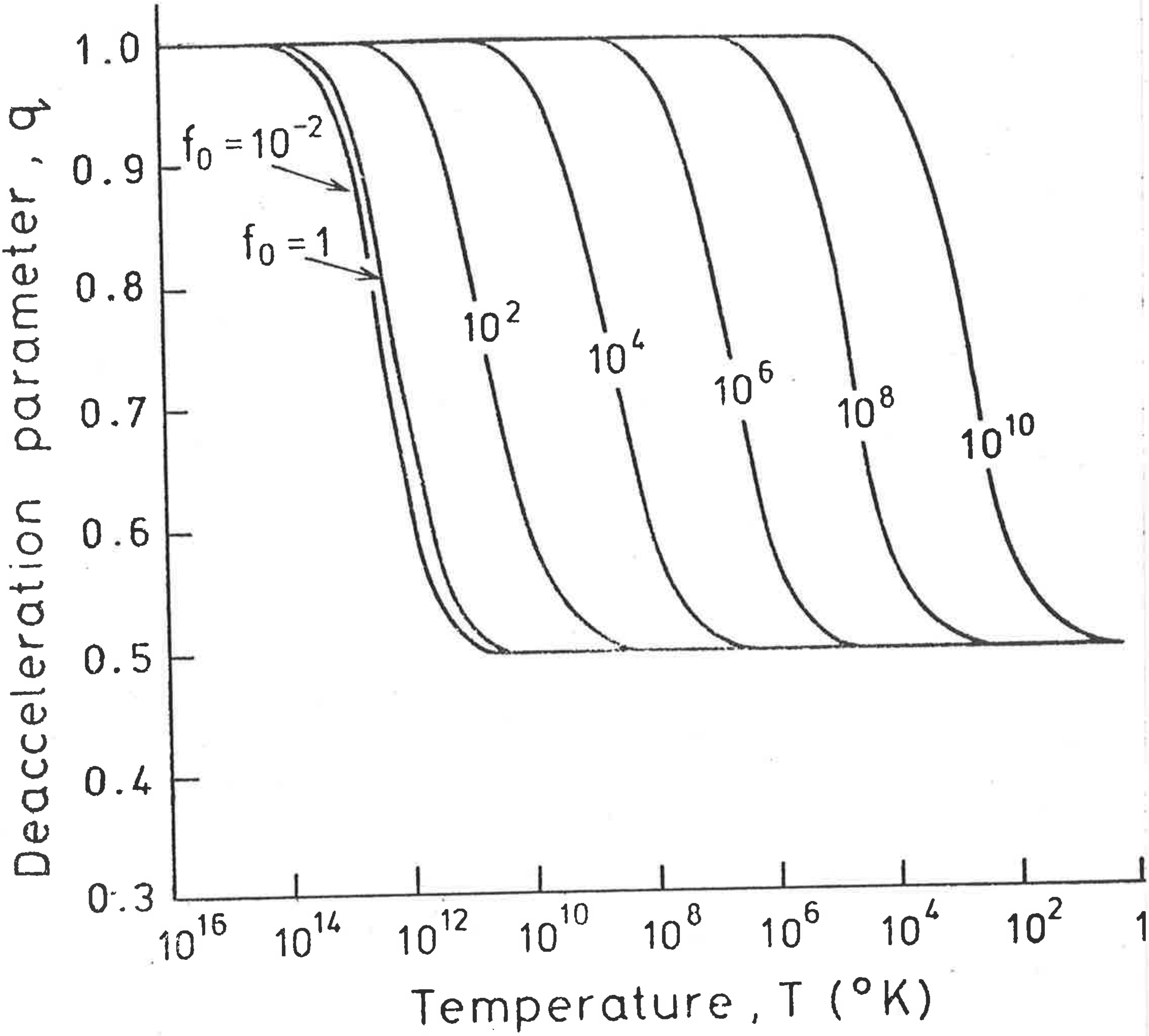

Fig. 3: Transition from photon to matter dominated expansion in terms of the deacceleration parameter versus temperature. Curvature terms are negligible.

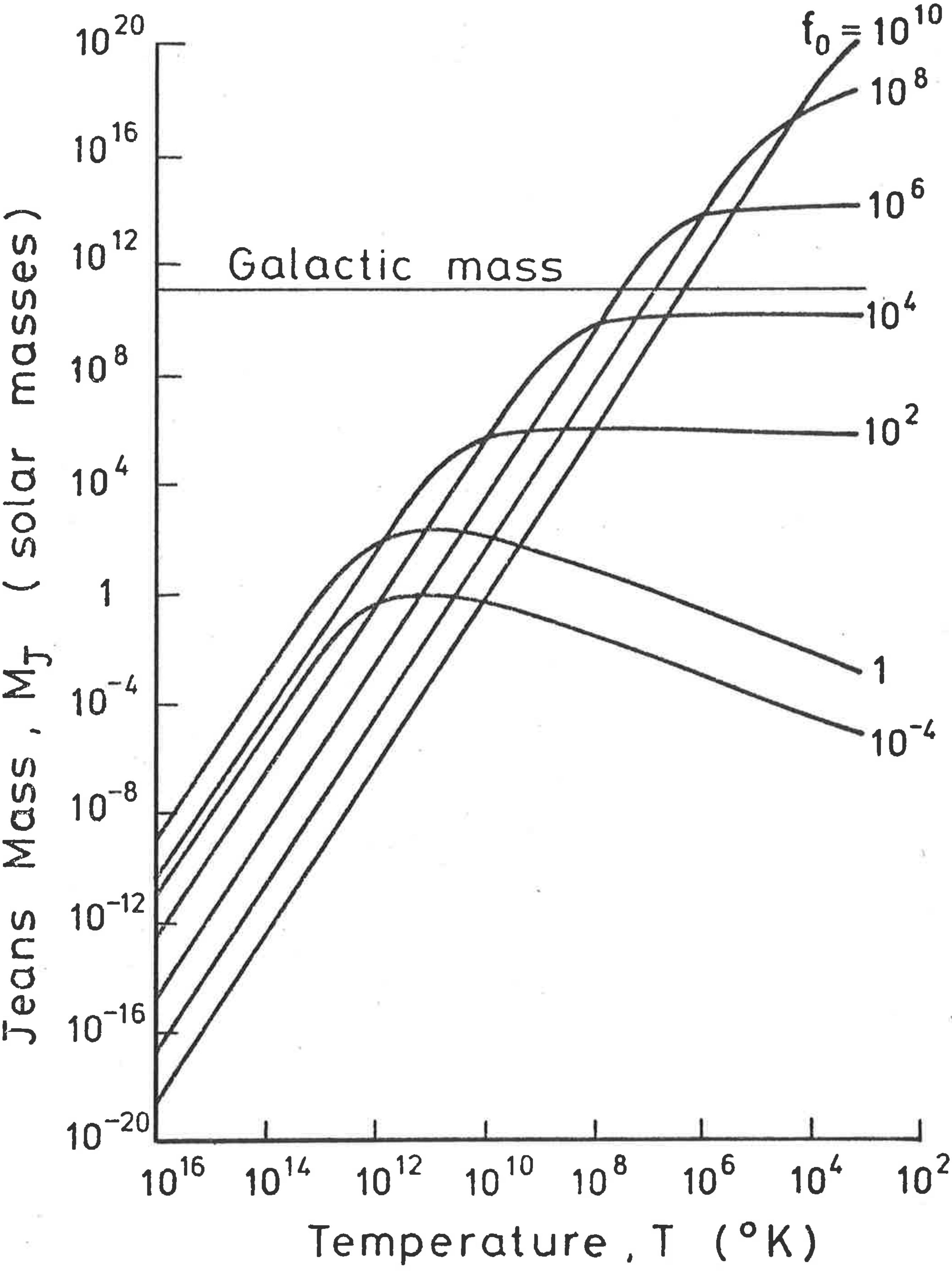

Fig. 4: Jeans mass (in solar mass units) versus temperature. The only growing disturbances are those with mass greater than the Jeans mass. Curves for $f_0 < 10^{-4}$ are similar in shape to $f_0 = 10^{-4}$ but are characterized by smaller maxima.

form agglomerations of more than one or a few stars (and if f_0 is low enough it may show no tendency to form even stars, only cold hydrogen planets).

What seems impressive in these plots is the natural appearance of Jeans masses of the order of a few hundred solar masses for $f_0 \approx 1$. This would strongly implicate the formation of supermassive stars prior to the formation of galaxies just as proposed by Rees (1978). The value $f \approx 10^8$ is then due to thermalized photo production from these sources by various mechanisms which Rees discusses. Our analysis suggests a sharp increase in Jeans' mass after this period of photo production possibly giving rise to the era of galaxy formation. In this way a "natural" basis for the remarkable photon/baryon ratio of 10^8 at galactic epochs may well be found.

However as is clear from Sec. 3, Ch. 2 it is precisely at this initial value of $f_0 \approx 1$ that degenerate effects begin to become important. The Synge gas is only physical when it is an approximation of the non-degenerate behaviour of Fermi or Bose gases. To study the small f models we must therefore return to the full generality of the equations of state (2:14-17). We study cases using the quantum statistics in the next two sections, however the increased complexity of the equations forces a retreat to cosmological models simpler than the one above. Models with 1 massive component are treated in Sec. 3 and models containing only mass zero components (in any number) are treated in Sec. 4. Whilst clearly unrealistic as cosmologies the models do present some interesting features; some of these will prove useful in later more realistic attempts.

3. MODELS WITH ONE MASSIVE COMPONENT

In models containing one massive fluid undergoing adiabatic expansion the particle number is conserved and we have through (2:32) and (5)

$$f^{-1} = \frac{g}{4\zeta(3)} I''_\varepsilon(x,\lambda) \ . \tag{20}$$

As before f can be considered as the ratio of the density of black body photons to that of the component, even though these photons are not present. It is proportional to T^3R^3.

Unlike the Synge gas the variables (x,λ) cannot be separated to give the simple form $f = f(x)$, instead the adiabatic condition can be used together with (20) to eliminate f. Thus the specific entropy (2:5, 2:35)

$$\frac{s}{n} = \frac{kgf}{4\zeta(3)} \left[I^{21}_\varepsilon(x,\lambda) + \frac{I^{03}_\varepsilon(x,\lambda)}{3} - \lambda\ I''_\varepsilon(x,\lambda) \right] = \text{const.} \tag{21}$$

and becomes

$$\frac{s}{nk} = \left[I^{21}_\varepsilon(x,\lambda) + \frac{I^{03}_\varepsilon(x,\lambda)}{3} - \lambda\ I''_\varepsilon(x,\lambda) \right] / \ I''_\varepsilon(x,\lambda) = \text{const.} \tag{22}$$

In the extreme relativistic regime the I^{ab}_ε will show no temperature dependence and go over to those for radiation (2.40-2.41) and

$$f = \frac{4\zeta(3)}{g\ I''_\varepsilon(0,\lambda)} \ , \tag{23}$$

$$\frac{s}{kn} = \frac{4}{3}\,\frac{I^{21}_\varepsilon(0,\lambda)}{I''_\varepsilon(0,\lambda)} - \lambda = \text{const.} \tag{24}$$

The initial data is given by some s/kn or $f(0) \equiv f_0$, the chemical evolution $\lambda = \lambda(x)$ follows from (22) and the thermodynamical evolution $f = f(x)$ from Eq. (20). Once this last is known the dynamical evolution can be given as

$$t = \int_0^{x(t)} \left(\frac{xf'(x)}{3f(x)} + 1 \right) \left[\frac{2\pi G\ mc^2\beta g}{3c^2\zeta(3)x^2}\, I^{21}_\varepsilon(x) - \frac{Kc^2\beta^{2/3}}{n_0^{2/3}\ R_0^2\ f^{2/3}(x)} \right]^{-\frac{1}{2}} dx \ . \tag{25}$$

As in the Synge gas case a family of models is generated by

Eqs. (20, 22, 25) which is characterized by the three parameters, the specific entropy or initial photon to particle ratio f_0, the total number of particles within some fiducial radius $N_0 = n_0 R_0^3$, and the curvature coefficient $K = \pm 1$ or 0. For the zero curvature models ($K = 0$) both temperature dependence and dynamics is determined by the sole parameter f_0.

Use of the alternate notation for the integrals, the $Q_\varepsilon^n(x,\lambda)$, can somewhat simplify the solution of the equations since one need not solve (20, 22) to give explicit forms for $d\lambda/dx$, df/dx. Thus on substitution of (2:28-2:30) into (22), differentiation with respect to x and use of the differential relation (2:31), we obtain

$$\frac{d\lambda}{dx} = \frac{Q^1}{Q^2} + \left\{\frac{Q^3 - Q^1}{Q^2}\right\} \frac{({Q^1}^2 + 3Q^3Q^1 - 4{Q^2}^2)}{\left[\frac{8}{3} Q^2(Q^4 - Q^2) - (3Q^3 + Q^1)(Q^3 - Q^1)\right]} . \tag{26}$$

A similar substitution and differentiation of (21) gives

$$\frac{x}{4} \frac{df}{dx} \frac{1}{f} = \frac{Q^1 - Q^2 \frac{d\lambda}{dx}}{Q^3 - Q^1} \tag{27}$$

and hence

$$\frac{x}{4} \frac{f'}{f} = \frac{{Q^1}^2 + 3Q^3Q^1 - 4{Q^2}^2}{(3Q^3 + Q^1)(Q^3 - Q^1) - \frac{8}{3} Q^2(Q^4 - Q^2)} . \tag{28}$$

Solution of the equations must now proceed by numerical methods, Eq. (22) can be solved by Newton's method for $\lambda = \lambda(x)$ after which (21, 28) can be used in (25) and the integral summed by quadrature.

The Q^n notation also allows direct expression of some of the quantities developed in the previous section. The deacceleration parameter becomes on ignoring the curvature term and using (2:33-2:34)

$$q = \frac{1}{2}\left\{1 + \frac{Q^4 - 4Q^2 + 3Q^0}{Q^4 - Q^0}\right\} = 1 - \frac{2(Q^2 - Q^0)}{(Q^4 - Q^0)} \tag{29}$$

and the speed of sound (17) through the use of (2:37-38) is

$$\frac{v_s^2}{c^2} = \frac{1}{3}\frac{\left[3(Q^3 - Q^1)\frac{d\lambda}{dx} - 2(Q^4 - Q^2)\right]}{\left[(3Q^3 + Q^1)\frac{d\lambda}{dx} - 2(Q^4 + Q^2)\right]} . \tag{30}$$

The Jeans mass then follows simply from (19). We note that for $x << 1$ Eqns. (A35-38) imply $Q^n >> Q^{n-1}$ so that (29) and (30) give the expected limit behaviour $q = 1$ and $v_s^2/c^2 = 1/3$.

It will prove instructive to consider separately special cases of the equations above.

(i) Synge Gas.

In the limiting case of λ large and negative the non-degenerate gas is described by Eqs. (2:56-59) and we have using (20, 21)

$$f^{-1} = \frac{g}{4\zeta(3)} x^2 e^\lambda K_2(x) \tag{31}$$

and

$$\frac{S}{kN} = \ln f + F(x) - \ln\left(4\zeta(3)/g\right) \tag{32}$$

where $F(x)$ is given by (8). It follows immediately that

$$\lambda = \frac{xK_3(x)}{K_2(x)} - \frac{S}{kN} \tag{33}$$

and

$$f = f_0 \; e^{4+\ln 2-F(x)} . \tag{34}$$

Since $xK_3(x)/K_2(x) \to 4$ as $x \to 0$ the fluid can only be non-degenerate for a large S/kN or f_0. In the relativistic regime f is constant and in the non-relativistic regime it is a decreasing function

$$f \sim 109 \cdot f_0 x^{-3/2} \qquad \text{for } x >> 1$$

so that $T \sim 1/R^2$ which is much faster than in the relativistic regime where $T \sim 1/R$. During the expansion the relativistic degeneracy parameter increases from $\lambda \sim \ln\left(2\zeta(3)/g\right) - \ln f_0$ $(x << 1)$ to $\lambda = \ln\left(2\zeta(3)/g\right) - \ln f + x - 11/4$ however the non-relativistic parameter

$\eta = \lambda - x$ decreases by 11/4. Clearly the expansion serves to increase the degree of non-degeneracy of the fluid.

As expected the fluid changes from radiation ($\rho = 3p$) to matter-like ($\rho >> p$) behaviour near $x = 1$, this is indicated by a deacceleration parameter which depends on none of the initial data

$$q = \frac{1}{2}\left(1 + 3 / \left(xK_3(x)/K_2(x) - 1\right)\right) . \tag{35}$$

Similarly the speed of sound depends only on the temperature

$$\frac{v_s^2}{c^2} = \frac{1}{x}\frac{K_2(x)}{K_3(x)}\left(1 - 1/\left(xF'(x) - 3\right)\right) \tag{36}$$

where

$$F'(x) = \frac{dF}{dx} = x\frac{K_3^2(x)}{K_2^2(x)} - 5\frac{K_3(x)}{K_2(x)} - x + \frac{4}{x} \tag{37}$$

and the Jeans mass becomes (19)

$$M_J = M_R\left(\frac{v_s^2}{c^2}\frac{K_2(x)}{K_3(x)} f^{\frac{1}{3}} x\right)^{3/2} \tag{38}$$

where $M_R \simeq 86\ M_\odot$ when the species mass is the proton mass.

(ii) Degenerate Fermi Gas.

For sufficiently large $\eta >> 1$ the I^{ab} are given by (2:68) and using (20) we obtain

$$f = \frac{12\zeta(3)}{g(\lambda^2 - x^2)^{3/2}} . \tag{39}$$

Eq. (22) and (A:33) yield

$$\frac{S}{kN} = \frac{\pi^2\lambda}{\lambda^2 - x^2} \tag{40}$$

and hence

$$\lambda = \frac{\pi^2}{\frac{S}{Nk}}\frac{1}{2}\left(1 + \sqrt{1 + \left(\frac{2Sx}{Nk\pi^2}\right)^2}\right) \tag{41}$$

i.e.

$$\lambda \approx \pi^2 \frac{Nk}{S}, \qquad \text{for } x << \frac{\pi^2 Nk}{2S}, \tag{42}$$

$$\lambda \approx x + \frac{\pi^2 Nk}{2S}, \text{ for } x >> \frac{\pi^2 Nk}{2S}. \tag{43}$$

Initially then η is large $\eta \approx \pi^2/(S/kN)$, f_0 and S/kN are small; η subsequently falls by a factor of 2 during the expansion. As the plots of the deacceleration parameter indicate, the degenerate fermi gas retains a radiation-like behaviour until $x \triangleq \pi^2 Nk/2S$; i.e. well into the non-relativistic ($x >> 1$) region. The dynamics for $x \lesssim \lambda/2$ is on ignoring curvature effects and with μ_F as λkT

$$t = 2^{-1}\left(\frac{8\pi G}{3c^2}\rho\right)^{-\frac{1}{2}} = (\lambda kT)^{-2}\left(\frac{32\pi^2 G}{3c^5h^3}g\right)^{-\frac{1}{2}} = \left(2.95/\mu_{F\ mev}\right)^2. \tag{44}$$

In the last $g = 2$. The changeover thus occurs at a time which is independent of the extent of the degeneracy

$$t = \left(\frac{128\pi^2 Ggm^4c^8}{3c^5h^3}\right)^{-\frac{1}{2}} = \left(\frac{1.48}{mc^2_{\ mev}}\right)^2. \tag{45}$$

For $x >> \lambda/2$ the gas dynamics is dominated by rest mass and

$$t = \frac{2}{3}\left(\frac{8\pi G}{3c^2}\rho\right)^{-\frac{1}{2}} = \frac{2}{3}\left(\frac{8\pi G}{3c^2}nmc^2\right)^{-\frac{1}{2}} = \left(\frac{\eta}{x}\right)^{-\frac{3}{4}}\left(\frac{8\pi^2 G2^{3/2}mc^2}{3c^5h^3}\right)^{-\frac{1}{2}}. \tag{46}$$

As has already been noted in Sec. 2, Ch. 2, if the initial degeneracy is not too extreme the expansion may lift it completely. The effect will occur in models for which $2.83 \le S/kN \le 4.2$.

(iii) Degenerate Bose Gas.

The equations derived at the beginning of this section do not apply and the condensed phase must be put in directly. In the thermodynamical limit the equations of state are given by (2:23-2:26) with the number density of particles in the condensed phase as

$$n_c = n - \frac{g}{4\zeta(3)}\frac{1}{x^3} I^{11}_{-1}(x,x) \tag{47}$$

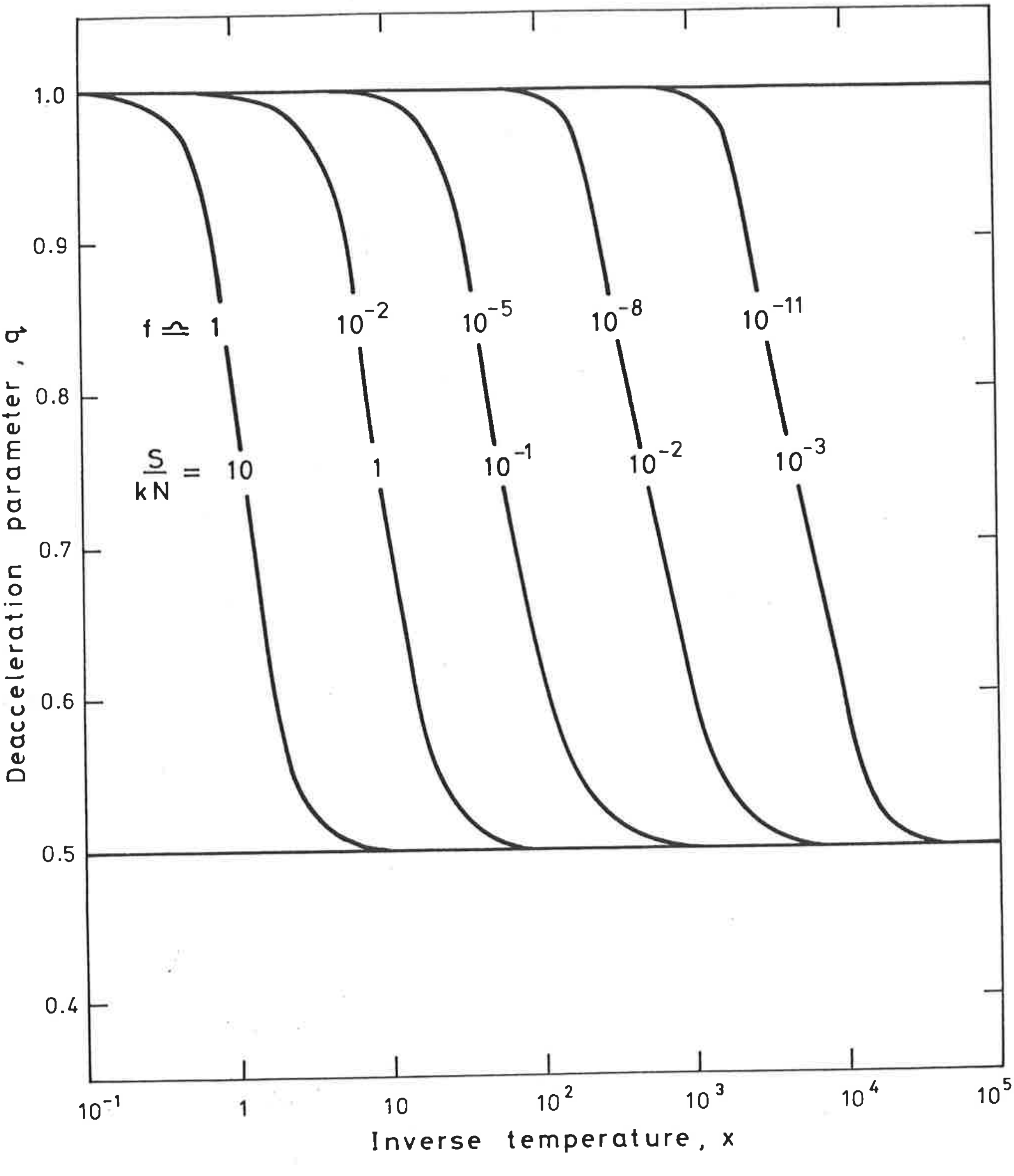

Fig. 5 The behaviour of the deacceleration parameter versus x for fermion degenerate models.

which then takes over the role normally played by the chemical potential which has been set equal to x. Rewriting this equation as

$$f_c^{-1} = f^{-1} - \frac{g}{4\zeta(3)} I_{-1}^{11}(x,x) \tag{48}$$

where f_c represents the ratio of black body photons to the number of particles in the condensate, we see for $x << 1$ and using (A:39, $\eta = 0$) that

$$f_c^{-1} = f^{-1} - g/2 \ . \tag{49}$$

Since the condensate makes no contribution to entropy we have

$$\frac{S}{kN} = \frac{gf}{4\zeta(3)} \left[I_{-1}^{21}(x,x) + \frac{I_{-1}^{03}}{3}(x,x) - xI_{-1}^{11}(x,x) \right] \tag{50}$$

and hence for $x << 1$ by (A:39-41)

$$\frac{S}{kN} = \frac{2\zeta(4)}{\zeta(3)} g\, f_0 \ . \tag{51}$$

We thus obtain

$$f = f_0\ 8\zeta(4) \Bigg/ \left[I_{-1}^{21}(x,x) + \frac{I_{-1}^{03}(x,x)}{3} - xI_{-1}^{11}(x,x) \right] , \tag{52}$$

and through (47) and (48)

$$n_c = \frac{g}{4\zeta(3)} \frac{1}{x^3} \left[\frac{kN}{S} \left\{ I_{-1}^{21}(x,x) + \frac{I_{-1}^{03}}{3}(x,x) - xI_{-1}^{11}(x,x) \right\} - I_{-1}^{11}(x,x) \right], \tag{53}$$

that is, as long as the right hand side of (53) is positive.

As with fermions we note that the degenerate phase can be removed by the expansion, this will occur for $1.28 \lesssim S/kN \lesssim 3.6$. Thus in the most extreme case in which the condensate can be removed by the expansion the condensate contains ~3/4 of the total number of particles. The boson gas does however provide an interesting reversal to the behaviour of the fermion gas, for consider the energy density in the relativistic regime

$$\rho = kT \frac{\beta}{x^3} \frac{3}{2} \frac{\zeta(4)}{\zeta(3)} + n_c mc^2 . \tag{54}$$

The condensate certainly dominates the number density and at any temperature, however (54) indicates that the kinetic energy and not the rest mass energy must dominate the energy density for sufficiently high temperatures, viz

$$\frac{\rho}{n_c mc^2} = \frac{3}{2} \frac{\zeta(4)}{\zeta(3)} \frac{f_c}{x} + 1 \tag{55}$$

is large for $x \ll f_c$, ($\approx S/(kNg\ 1.8)$). For such small values of x the fluid has a radiation-like behaviour with $\rho \approx 3p$, changeover occurs at $x \approx S/(kNg\ 1.8)$ and excepting the special cases mentioned above the rest matter will subsequently dominate the dynamics even though the temperature is still relativistic. The deacceleration parameter (16) is given by

$$q = 1 - \frac{2(Q^2 - Q^0) + h/2}{Q^4 - Q^0 + h} \tag{56}$$

where

$$h \equiv \frac{8}{x^3} \left[\frac{kN}{S} \left(I^{21}_{-1}(x,x) + \frac{I^{03}_{-1}}{3}(x,x) - xI^{11}_{-1}(x,x) \right) - I^{11}_{-1}(x,x) \right] .$$

The parameter is graphed in Fig. 6.

The dynamics of the expanding fluid is given through (2:118)

$$t = \int_0^{x(t)} \left(\frac{xf'}{3f} + 1 \right) \left[\frac{8\pi G}{3c} mc^2 \left(\frac{\beta g}{4\zeta(3)} \frac{I^{21}_{-1}(x,x)}{x^2} + n_c x^2 \right) \right]^{-\frac{1}{2}} dx \tag{57}$$

where

$$\frac{xf'}{4f} = \frac{8Q^2 - 3Q^3 - 5Q^1}{\frac{8}{3}(Q^4 - Q^2) - (Q^3 - Q^1)} \tag{58}$$

which can be obtained through (50) and (2:28-31). It is clear (A:35-38) the term is negligible in the relativistic regime, so that during the period of radiation-like behaviour

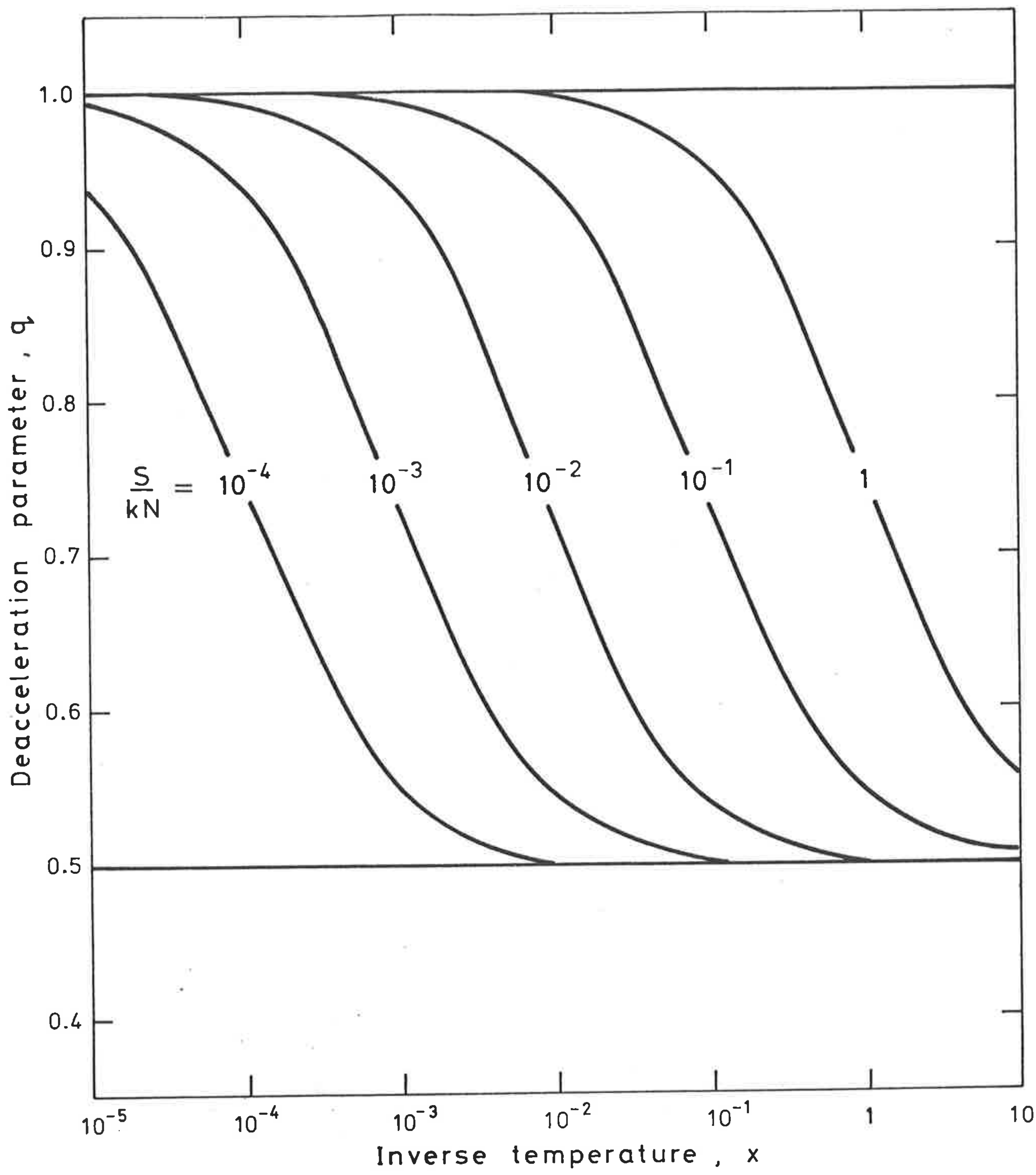

Fig. 6 The behaviour of the deacceleration parameter versus x for boson degenerate models.

$$t = \frac{x^2}{2}\left[\frac{8\pi G}{3c^2} mc^2 \beta g \, \frac{3}{2}\frac{\zeta(4)}{\zeta(3)}\right]^{-\frac{1}{2}} = (kT)^{-2}\left(\frac{256\pi^2 G}{c^5 h^3} g\zeta(4)\right)^{-\frac{1}{2}} . \tag{59}$$

That is the dynamics is independent of the extent of the condensation until $x \approx S/(kNg\ 1.8)$.

The speed of sound can be calculated

$$\frac{v_s^2}{c^2} = \frac{1}{3}\,\frac{\left(3(Q^3 - Q^1) - 2(Q^4 - Q^2)\right)}{(3Q^3 + Q^1) - 2(Q^4 + Q^2) + h'x/2} \tag{60}$$

where $h' = \frac{4kN}{S}(3Q^3 + Q^1 - Q^4 - 3Q^2) - \frac{2}{x}\left(4Q^2 - (3Q^3 + Q^1)\right)$ and hence

$$\frac{v_s^2}{c^2} = \frac{1}{3}\,\frac{\left(3(Q^3 - Q^1) - 2(Q^4 - Q^2)\right)}{\left(3Q^3 + Q^1 - Q^4 - 3Q^2\right)}\,\frac{1}{\left(2 + \frac{2kN}{S}x\right)} . \tag{61}$$

4. A GENERALIZED TOLMAN MODEL

The models containing massive components detailed in the previous section are clearly evolving. All the interesting dimensionless parameters and thermodynamical functions have a dependence on the cosmic time t. That this dynamical structure has its origin in the very mass of the particles can be seen by looking at the contrasting situation; that of zero mass particles.

Consider a fairly general situation in which a number of different types of radiation are in thermal and chemical equilibrium. For example we may have a fluid containing black body photons, neutrinos and antineutrinos expanding in thermal and chemical equilibrium. Such models are not physically feasible due to their near collisionless property (e.g. photons and the principle of superposition of the electromagnetic field) however for the same reason they best approximate the ideal gases of the previous chapter. Moreover they provide the limiting case of matter filled models at relativistic temperatures. We might call them the generalized Tolman models (see Tolman 1931).

The equations of state and entropy are given in (2:14-2:17) and as before the dynamical evolution comes through (2.118). Each of the I_i^{ab} (2:40-41) depend directly on only one parameter λ_i and it is not difficult to see that any inversion of the equations of state and adiabatic condition formed from (2:14-17) to give the set $\{\lambda_i\}$ can only depend on the dimensionless terms n_1/n_2, ρ_1/n_1kT, S/nk ... etc. Such a set of solutions is thus formed only of ratios and sums of the $I_i^{ab}(\lambda_i)$ and cannot depend on the temperature.

In these pure radiation models then, we have the following static situation. There is no chemical evolution and the $\{\lambda_i\}$ are constant throughout the expansion. The specific entropy of each component is individually conserved and there is no heat flow between components. Even if the weak interactions between the particles were to break down e.g. neutrinos decouple, this picture will not change. The precise chemical nature of the gas is thus a frozen feature during the expansion.

The thermodynamical evolution is trivial. The ratio of black body photons to some conserved number is constant and R = const./T.

The reasons for this behaviour are similar to those given in Sec. 2 for a Synge gas dominated by black body photons. Since $R \sim T^{-1}$ the momentum of each radiation particle which scales as R^{-1} is proportional to T. The particle need not be subsequently scattered to find itself at a momentum consistent with equilibrium at the new temperature and the occupation numbers of each of the scaling energy levels of a species remain constant. The chemical potentials scale as the temperature and there can be no net exchange of energy between components.

The integrals I^{ab} are given by (2:40-42), however they are too complex to allow a general solution for the chemical concentrations except for some special cases discussed below. In spite of this the dynamics equation (2:118) is solvable in full generality.

We have using (2:82)

$$\rho = \sum_i 4\pi g_i \frac{k^4T^4}{h^3c^3} I_i^{21}(\lambda_i) \tag{62}$$

$$= \frac{1}{R^4} f_N^{4/3} \frac{N^{4/3}\, hc4\pi}{(16\pi\zeta(3))^{4/3}} \cdot \sum_i g_i\, I_i^{21}(\lambda_i)$$

where

$$f_N = 16\pi\zeta(3)\left(\frac{kTR}{hc}\right)^3 \frac{1}{N} = \text{const.} \tag{63}$$

and N is a reference species. Integrating directly we have

$$t = R^2/[(d_1 - Kc^2R^2)^{\frac{1}{2}} + d_1^{\frac{1}{2}}] \tag{64}$$

with

$$d_1 = \frac{8\pi G}{3c^2} f_N^{4/3} \frac{N^{4/3}\, hc4\pi}{(16\pi\zeta(3))^{4/3}} \sum g_i\, I_i^{21}(\lambda_i) = \text{const.} \tag{65}$$

$$= \frac{8\pi G}{3c^2} \rho R^4$$

giving the temperature dependence as

$$t = T^{-2} \Big/ \left[d_2^{\frac{1}{2}} + \left(d_2 - \frac{Kc^2}{T^2} \frac{(16\pi\zeta(3)k^3)^{2/3}}{f_N N\, h^3c^3} \right)^{\frac{1}{2}} \right] \tag{66}$$

with

$$d_2 = \frac{8\pi G}{3c^2} \cdot \frac{4\pi k^4}{h^3c^3} \sum_i g_i\, I_i^{21}(\lambda_i) = \text{const.}$$

$$= \frac{8\pi G}{3c^2} \rho T^{-4} .$$

In the case of the K = 0 model we have simply

$$t = R^2/2d_1^{\frac{1}{2}}, \qquad t = T^{-2}/2d_2^{\frac{1}{2}}, \qquad t = 2^{-1}(8\pi G\rho/3c^2)^{-1} \tag{67}$$

the natural generalization of Tolman's model for black body photons for which $gI^{21}(0) = 12\zeta(4)$ (2:41) and

$$t = \left(\frac{3c^2}{32\pi G a_s T^4}\right)^{\frac{1}{2}} = \left(\frac{1.52 \times 10^{10}}{T}\right)^2 . \tag{68}$$

The general result is given by (68) with the T scaled by a factor $\left[\sum_i g_i\ I_i^{21}(\lambda_i)/12\zeta(4)\right]^{\frac{1}{4}}$; for example if the model contained only highly degenerate neutrinos in Eq. (68) $T \rightarrow \lambda T/2.68$ ($\lambda >> 1$). In general then

$$t = (1.52 \times 10^{10}/T\ d_3)^2 \tag{69}$$

with

$$d_3 = \left[\sum_i g_i\ I_i^{21}(\lambda_i)\ /\ 12\zeta(4)\right]^{\frac{1}{4}} ,$$

We note that the same results are obtainable through direct use of the Euler equation (2:116) and $\rho = 3p$, this gives $\rho R^4 =$ const., and the equations follow via (2:118).

In the closed models we note that when R attains its maximum $R = d_1^{\frac{1}{2}}/c$ then t attains its maximum $t = d_1^{\frac{1}{2}}/c^2 = R\ \max/c$ and T its minimum

$$T = c\left[\frac{8\pi G}{3c^2}\ \frac{4\pi k^2}{hc}\left(\frac{f_N N}{16\pi\zeta(3)}\right)^{2/3} \sum g_i\ I_i^{21}(\lambda_i)\right]^{-\frac{1}{2}} .$$

Let us now discuss some special cases of these models.

(i) <u>Photon Fluid with Chemical Potential.</u>

This case is of interest in relation to the discussion in Sec. 7 Ch. 2. The number density (2:32, 2:41 and 23) give

$$1 = \frac{f_N}{\zeta(3)}\ \phi(e^\lambda,\ 3) \tag{70}$$

where $\lambda \leq 0$. The equation is trivial when $\lambda = 0$. The entropy equation (2:35) with (2:40-2:41) gives

$$\frac{S}{kN} = 4\ \frac{\phi(e^\lambda,\ 4)}{\phi(e^\lambda,\ 3)} - \lambda \tag{71}$$

Using the expansions (B:33-34) solutions to this equation for various starting values of S/kN can be calculated and are plotted in Fig. 7.

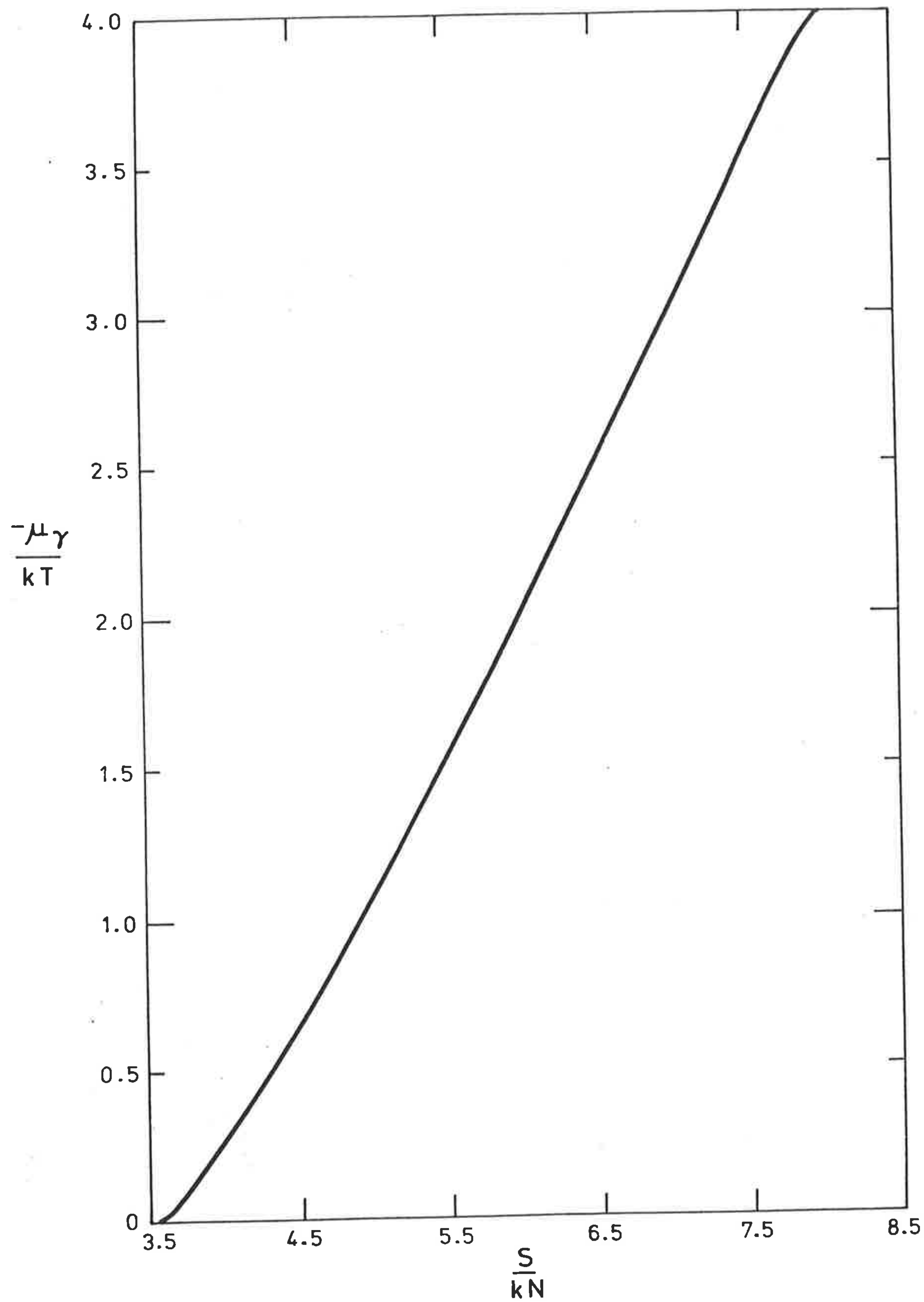

Fig. 7 Chemical potential for photons versus S/kN

Only S/kN > 3.6 is considered since below this value we encounter Bose-Einstein condensation of the photons. The condensation is of little relevance here since following the discussion in Sec. 1, Ch. 2 we set $\lambda = 0$ and release the constraint on photon number. However the condensate is of zero energy and does not contribute to the pressure or the entropy so it cannot deflect the dynamics from its black body behaviour. This is a direct consequence of imposing the thermodynamical limit on Eqs. (2:20-2:22), clearly this is not "necessary" and some recent work Becker and Castell (1977) rejects the assumption. That work is inspired by a suggestion of C.F. Von Weizäcker that the photon condensate be identified with matter and in the treatment the lowest energy is not zero but proportional to the radius of the universe.

(ii) Black Body Photons, Neutrinos and Anti-neutrinos.

Even though the simple model above must be solved numerically more complex models can yield exact solutions. Consider an expanding fluid containing black body photons, electron neutrinos and their anti-particles in thermal and chemical equilibrium. The chemical potential of the neutrinos is equal but opposite to that of its antiparticles and the relevant equations are (2:44-46, 2:52-54). Conservation of electron lepton number $\left((n_\nu - n_{\bar{\nu}})R^3\right)$ gives

$$1 = \frac{f_N}{12\zeta(3)} \lambda(\lambda^2 + \pi^2) \tag{72}$$

where N refers to the net number of neutrinos. The total entropy per net neutrino is

$$\frac{S}{kN} = \frac{f_N}{12\zeta(3)} \pi^2(\lambda^2 + \pi^2) \tag{73}$$

and hence the solutions

$$\lambda = \frac{kN\pi^2}{S}, \qquad f_N = \frac{12\zeta(3)}{\pi^2} \frac{S}{kN} \frac{1}{\left[\pi^2 + \dfrac{k^2N^2\pi^4}{S^2}\right]}. \tag{74}$$

The dynamics can then be obtained simply through (67) since

$$\sum_{i}^{3} g_i \, I_i^{21} = \left(\frac{\lambda^2 + \pi^2}{2}\right)^2 = \left(\pi^2 + \frac{k^2N^2\pi^4}{S^2}\right)^2 \Big/ 4 \tag{75}$$

and for the case of the K = 0 models we have

$$t = \left(\frac{4.08 \times 10^{10}}{T(\lambda^2 + \pi^2)^{\frac{1}{2}}}\right)^2 . \tag{76}$$

CHAPTER 4

MORE COMPLEX MODELS, PAIR PRODUCTION AND THE REES MODEL

1. INTRODUCTION

We continue our investigation of equations of state within Robertson-Walter cosmologies. In Sec.2 we discuss a many component model using quantum statistics but we allow no reactions. We indicate the quickest method of solution and as in Ch.3 we derive equations which exhibit some of the more interesting features of the models. Of particular interest are the energy and entropy flows between the various species which occur when the masses of the particles become important in the equations of state. For example in a fluid containing degenerate (fermi) species, other things being equal, the expansion shifts the total entropy in the direction of the most massive species. In Sec.2 we will not be concerned with any actual calculations since the many component model with no reactions admits an immediate generalization to the many component model with pair production and annihilations.

A scheme is set up so that only a change of notation is necessary to convert any equations in Sec.2 to those for pair production which are the concern of Sec.3. It consists of using some linearly related functions the $J_\varepsilon^{ab}(x,\lambda)$, $R_\varepsilon^n(x,\lambda)$ which replace the $I_\varepsilon^{ab}(x,\lambda)$, $Q_\varepsilon^n(x,\lambda)$ of Sec.2 in a way to be described. The J^{ab} and R^n are discussed in Appendix C and they turn out to be somewhat easier to use than the I^{ab}, Q^n. Numerical schemes capable of providing solutions to the many component models are detailed in Appendix D. In Sec.2 we calculate for some of these models, however in their late stages the models will revert to either a Synge gas with black body radiation or a degenerate single particle gas (as long as there is one mass reasonably greater than all the others).

In Sec.4 we present a qualitative discussion of the Ree's model. This may seem somewhat out of our main line of development, however the Ree's model emphasizes at least one particular thing. That in ignoring the chemical reactions possible within the fluid our descriptive scenarios can differ significantly from a realistic one. The Ree's model of course resorts to chemical reactions in inhomogeneous regions (supermassive stars) within the expanding fluid but its general method of photon generation is paradigmatical of attempts we will in future make. This chapter thus ends with a qualitative discussion of the Ree's mechanism in the following chapter we turn to a quantitative approach to chemical reactions in general.

2. MANY COMPONENT MODELS - NO REACTIONS

In continuing the investigation of the equations of state (2: 32-35) within Robertson-Walker cosmologies we study in this section models in which many species are in thermal equilibrium with themselves and black body radiation. We allow no chemical reactions so that the number (= N_i within some comoving volume) of each species is preserved during expansion. For the moment we consider the number of different species to be finite (= m, not to be confused with the mass of the particles) and small enough so that the number itself will have no thermodynamical consequences; we will relax this assumption in Ch.6. As in Ch.3 we can always set up the equations so that the volume dependence can be separated into a photon to reference particle number ratio (labelled N), they become

$$1 = \frac{fg}{4\zeta(3)} \quad I^{11}_{\varepsilon}(x,\lambda), \tag{1}$$

$$\frac{N_i}{N} = \frac{g_i}{g} \quad \frac{I^{11}_{\varepsilon}(x_i,\lambda_i)}{I^{11}_{\varepsilon}(x_i,\lambda_i)} \quad , \text{ m-1 equations.} \tag{2}$$

In future we will drop all variables and just label the integrals by the number (i) of the component they describe. The total entropy density is given by (2:35, 2:46)

$$s/k = \sum_{i=1}^{m} \frac{g_i}{4\zeta(3)} \quad \frac{\beta_i}{x_i^{\ 3}} \quad \frac{(I^{21}_i + I^{03}_i - \lambda_i I^{11}_i)}{3} \quad + \alpha \frac{\beta}{x^3} \tag{3}$$

where $\beta_i = 16\pi \ \zeta(3) \ (m_i c/h)^3 = \beta x_i^{\ 3}/x^3$ and $\alpha = 4\zeta(4)/\zeta(3) = 2\pi^4/45\zeta(3) = 3.60$. Dividing through by the number density of the reference species gives

$$\frac{s}{kN} = \sum_{i=1}^{m} \frac{N_i}{N} \left(\frac{I^{21}_i + I^{03}_i/3}{I^{11}_i} - \lambda_i \right) + \alpha f \tag{4}$$

which is then the total entropy per reference particle. Equations (1,2,4) constitute m+1 equations in the m+1 functions $\{\lambda_i(x), f(x)\}$. The problem of initial data is made clear by considering these equations in the extreme relativistic region ($\max(x_i) << 1$) the $I^{ab}_{\varepsilon i}$ are then replaced by (2:40 - 41) so that (1,2,4) give

$$1 = - \frac{\varepsilon f g}{2\zeta(3)} \quad \phi(-\varepsilon \ e^{\lambda},3) \tag{5}$$

$$\frac{N_i}{N} = \frac{g_i}{g} \frac{\varepsilon_i}{\varepsilon} \quad \frac{\phi(-\varepsilon_i e^{\lambda_i},3)}{\phi(-\varepsilon \ e^{\lambda},3)} \tag{6}$$

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N} \left(\frac{\phi(-\varepsilon_i e^{\lambda_i},4) \ .4}{\phi(-\varepsilon_i e^{\lambda i},3)} - \lambda_i \right) + \alpha f \tag{7}$$

As in Sec.4, Ch.3 the $\{\lambda_i\}$, are constant and the initial data is given by the m-1 N_i/N and S/kN, or equivalently by the ratio of the photon number to that of each species

$$f_{i0} = f_0 N/N_i \qquad i = 1 \ldots\ldots m \tag{8}$$

where $f(x \to 0) \equiv f_0$. The f_i also supply a criterion for the degeneracy of a species (Sec.2, Ch.2).

If each f_{i0} is large every component is non degenerate in the relativistic regime asymptotically approaching the Synge gas as $f_{i0} \to \infty$. Eqs.(A7-11) give to first order

$$\lambda_{i0} = -\ell n\left(\frac{g_i f_{i0}}{2\zeta(3)}\right) \tag{9}$$

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N}\,(4 + \ell n(g_i/2\zeta(3)) + \ell n\ f_{i0}) + \alpha f_0 . \tag{10}$$

As long as m is not too large the photons will dominate the entropy equation and determine the thermal evolution of the model, Eq.(4) then gives the generalization of the 2 component Synge gas of Sec.2, Ch.3.

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N}\,(F(x_i) + \ell n(g_i/4\zeta(3)) + \ell n\ f_i) + \alpha f \tag{11}$$

where the F(x) are given by (3:8), note that for x>>1, $F(x) \sim 1.5\ \ell n\ x$ so Eq.(11) will be valid up to very high x indeed (degeneracy will eventually occur at an epoch $x \sim e^{f_0}$).

In the other extreme where each of the f_{i0} is small (consider only fermions) then simple generalizations of the degeneracy equations (3:39 - 40) will give to first order

$$f = \frac{12\zeta(3)}{g(\lambda^2 - x^2)^{3/2}} \quad , \quad \frac{N_i g}{N g_i} = \frac{(\lambda_i^2 - x_i^2)^{3/2}}{(\lambda^2 - x^2)^{3/2}} \tag{12}$$

and

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N} \frac{\pi^2 \lambda_i}{(\lambda_i^2 - x_i^2)} + \alpha f. \tag{13}$$

Clearly the photon contribution is negligible and if all species are relativistic (max $(x_i) << 1$) we have the solutions

$$\lambda = \pi^2 \frac{kN}{S} \sum_{i=1}^{m} \left(\frac{N_i}{N}\right)^{2/3} \left(\frac{g_i}{g}\right)^{1/3} ,$$

$$\lambda_j = \pi^2 \frac{kN}{S} \sum_{i=1}^{m} \frac{N_i^{2/3} N_j^{1/3}}{N} \left(\frac{g_i}{g_j}\right)^{1/3} \tag{14}$$

Notice that the j'th species has a share $1/\sum_i (N_i/N_j)^{4/3}$ $(g_j/g_i)^{1/3}$ of the total energy (see 2:69) and a share $1/\sum_i (N_i/N_j)^{2/3}$ $(g_i/g_j)^{1/3}$ of the total entropy.

For sufficiently low temperatures each x_j will be larger than the R.H.S. of (14) and we have $x_j >> 1$, $\lambda_j \simeq x_j$ with $\lambda_j - x_j = \eta_j >> 1$, as long as each species remains degenerate. Eq.(12) then gives

$$(\lambda_j - x_j)x_j = \eta_j x_j = \left(\frac{N_j g}{N g_j}\right)^{2/3} \eta x$$

$$\text{i.e.} \quad \frac{\eta_j}{\eta} = \frac{m}{m_j} \left(\frac{N_j g}{N g_j}\right)^{2/3} \tag{15}$$

the entropy is

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N} \frac{\pi^2}{2\eta_i} \tag{16}$$

giving

$$\eta = \frac{\pi^2}{2} \frac{kN}{s} \sum_{i=1}^{m} \left(\frac{N_i}{N}\right)^{1/3} \left(\frac{g_i}{g}\right)^{2/3} \frac{x_i}{x}$$

$$\eta_j = \frac{\pi^2}{2} \frac{kN}{S} \sum_{i=1}^{m} \frac{N_i^{1/3} N_j^{2/3}}{N} \left(\frac{g_i}{g_j}\right)^{2/3} \frac{m_i}{m_j} . \tag{17}$$

The expansion has redistributed energy and entropy between the species so that the share of the total energy for the j'th species is now (see 2:70)

$$1/\sum_{i=1}^{m} \frac{g_i\ x_i\ I_i^{11}}{g_j\ x_j\ I_j^{11}} = 1/\sum_{i=1}^{m} \frac{m_i\ N_i}{m_j\ N_j}$$

while the share of the total entropy is

$$1/\sum_{i=1}^{m} \frac{N_i^{1/3}\ g_i^{2/3}\ m_i}{N_j^{1/3}\ g_j^{2/3}\ m_j} .$$

Since this regime is matter dominated the first of these is obvious however the second indicates there will be entropy flows, other things being equal, in the direction of the most massive species. In such a case ($N_i = N_j$, $g_i = g_j$) the most massive component will dominate the entropy $\eta_d = \pi^2 Nk/2S$ if it is much more massive than the next most massive component; note however degeneracy is not lifted since $\lambda_d = m\ \pi^2\ Nk/S$ and we have assumed the number of components, m, is not sufficiently large to change the character of the thermodynamics. Alternatively the least massive component (with $m_{min}/m_{max} \ll 1$) will lose most of its energy and entropy, in particular this is so if the component is massless e.g. neutrinos. Of course in a physically feasible model decoupling of the neutrinos will occur at some stage, thereafter they will expand adiabatically.

As in Ch.3 it proves useful to represent the general equations (1,2,4) in Guess's notation (2: 28-30) giving

$$1 = \frac{fg\ x^3}{16\ \zeta(3)}\ (Q^3-Q^1), \tag{18}$$

$$\frac{N_i}{N} = \frac{g_i\ x_i^3}{g\ \ x^3}\ \frac{(Q_i{}^3-Q_i{}^1)}{(Q^3\ -\ Q^1)}\ , \tag{19}$$

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N} \left(\frac{2}{3}\, x_i \frac{(Q_i{}^4-Q_i{}^2)}{(Q_i{}^3-Q_i{}^1)} - \lambda_i\right) + \frac{64\ \zeta(4)}{g(Q^3-Q^1)x^3}\ . \tag{20}$$

Numerical solution of these equations proceeds as suggested in Appendix D, solution is made all the more faster since we can obtain explicit equations for the derivatives of f and λ_i. Taking the derivative of (16) with respect to x and using the differential relation (2:31) gives

$$\frac{Q^2 d\lambda/dx\ -\ Q^1}{Q^3-Q^1} = \frac{Q_i^2 d\lambda_i/dx_i\ -\ Q_i^1}{Q_i^3\ -Q_i^1} \tag{21}$$

Similarly taking the derivative of (20) and using (21) for each $d\lambda_i/dx_i$ gives

$$\frac{d\lambda}{dx} = \frac{Q^1}{Q^2} + \frac{\dfrac{(Q^3-Q^1)}{Q^2} \displaystyle\sum_{i=1}^{m} \frac{N_i\ x_i\ \left({Q_i^1}^2 + 3Q_i^3\ Q_i^1 - 4{Q_i^2}^2\right)}{N\ \ x\ \ Q_i^2\ \ (Q_i^3-Q_i^1)}}{\left[\dfrac{256\zeta(4)}{g\ x^4(Q^3-Q^1)} - \displaystyle\sum_{i=1}^{m} \frac{N_i\ x_i}{N\ x} \frac{((3Q_i^3+Q_i^1)(Q_i^3-Q_i^1) - \frac{8}{3}\ Q_i^2(Q_i^4-Q_i^2))}{Q_i^2(Q_i^3-Q_i^1)}\right]} \tag{22}$$

The first term in the denominator on the RHS is due to the photon contribution $\simeq f/x$. The derivative of f is found from (18)

$$\frac{xf'}{4f} = \frac{(Q^1-Q^2)}{(Q^3-Q^1)}\ d\lambda/dx \tag{23}$$

i.e. $$\frac{xf'}{4f} = - \frac{\displaystyle\sum_{i=1}^{m} \frac{N_i \, x_i}{N \, x} \frac{({Q_i^1}^2 + 3Q_i^3 \, Q_i^1 - 4{Q_i^2}^2)}{Q_i^2 (Q_i^3 - Q_i^1)}}{\left[\dfrac{256\zeta(4)}{g \, x^4 (Q^3 - Q^1)} - \displaystyle\sum_{i=1}^{m} \frac{N_i x_i}{N \, x} \frac{((3Q_i^3 + Q_i^1)(Q_i^3 - Q_i^1) - \frac{8}{3} Q_i^2 (Q_i^4 - Q_i^2))}{Q_i^2 \, (Q_i^3 - Q_i^1)}\right]} \tag{24}$$

When $x_i >> 1$ and $\eta_i << x_i$ Eqs. (A13, A24-27) show that $Q^n \simeq Q^m$ to $0(x^{-1})$ and calculation shows both numerators under the summation signs in (22) cancel to $0(x^{-2})$ and the contribution of the entire right term on the RHS of (22) is negligible giving $d\lambda/dx \simeq 1$ i.e. $\lambda \simeq x +$ const. Similar calculations based on Appendices A and B for the cases $\lambda_i >> x_i$ and $x_i << 1$ for all i indicate that $d\lambda/dx$ is small.

The speed of sound in these models is given by

$$\frac{v_s}{c^2} = \frac{1}{3}\left(1 - \frac{\displaystyle\sum_{i=1}^{m} g_i x_i \left(8Q_i^2 - 8Q_i^1 d\lambda_i/dx_i\right)}{\left[\dfrac{64\pi^4}{15} - \displaystyle\sum_{i=1}^{m} g_i x_i^4 \left(4(Q_i^4 + Q_i^2) - (6Q_i^3 + 2Q_i^1) d\lambda_i/dx_i\right)\right]}\right) \tag{25}$$

which in the relativistic regime with $d\lambda_i/dx_i \to 0$ and Eqs. (A35-38) return $v_s^2/c^2 = 1/3$. The Jeans mass for the models follows through (3:19).

3. MANY COMPONENT MODELS - PAIR ANNIHILATIONS

If the characteristic thermal energy of particles (~ kT) is greater than those particles' rest energy (mc^2) then pair production of both particles and antiparticles will occur. Assuming the presence of black body radiation and chemical equilibrium between particle and antiparticle species we have

$$\mu = -\bar{\mu}\ ,\ \lambda = -\bar{\lambda}\ , \tag{26}$$

the second follows since we assume thermal equilibrium as well.

The equations of the last section admit a natural generalization from m to 2m species once the antiparticles are included. Instead of the 2m equations which arise together with m equations like (26) we can reduce directly to m equations by use of the notational changes

$$J_\varepsilon^{ab}(x,\lambda) \equiv I_\varepsilon^{ab}(x,\lambda) + (-1)^a\ I_\varepsilon^{ab}(x,-\lambda) \tag{27}$$

$$R_\varepsilon^{n}(x,\lambda) \equiv Q_\varepsilon^{n}(x,\lambda) + (-1)^n Q_\varepsilon^{n}(x,-\lambda) \tag{28}$$

The convention that $\lambda \geq 0$ has been adopted so that J^{ab}, R^n are always positive.

All thermodynamical functions can be expressed in terms of these integrals and the differential relations among the Q^n (2:31) hold for the R^n. Transformation of an n component system of equations to include pair production is achieved by the simple changes $I^{ab} \to J^{ab}$, $Q^n \to R^n$; all such equations in Section 2 now apply for the case of pair production. The new integrals are discussed and approximations schemes given for them in Appendix C. In fact the new integrals are easier to handle than the I^{ab} and Q^n since $\lambda+\bar{\lambda} = 0 \Rightarrow \lambda \geq 0$ and there can be no nondegeneracy at high temperatures ($x \gtrsim 1$). If however the fermions are degenerate at high temperatures i.e. $\lambda >> 1$ at $x << 1$, then the new notation just reduces to the Q^n, I^{ab}.

The $J_+^{ab}(x,\lambda)$ are given in the relativistic regime to at least $0(x^2)$ by (C 30-33, C 18-21, cf. 2:52-54)

$$J_+^{11} = \frac{\lambda^3}{3} + \frac{\lambda\pi^2}{3}\ ,\quad J_+^{21} = J_+^{03} = \frac{\lambda^4}{4} + \frac{\lambda^2\pi^2}{2} + \frac{7\pi^4}{60}\ , \tag{29}$$

with

$$J^{21}_{+} + \frac{J^{03}_{+}}{3} - \lambda J^{11}_{+} = \frac{\lambda^2\pi^2}{3} + \frac{7\pi^4}{45}\,, \tag{30}$$

whereas the J^{ab}_{-} (x,λ) are given by (C: 22-25)

$$J^{11}_{-} = \lambda\frac{2\pi^2}{3}\,, \quad J^{21}_{-} = J^{03}_{-} = \frac{2\pi^4}{15}\,, \quad J^{21}_{-} + \frac{J^{03}_{-}}{3} - \lambda J^{11}_{-} = \frac{8\pi^4}{45}\,. \tag{31}$$

The last two show no parameter dependence to $0(x^2)$ due to the restriction imposed by Bose statistics, $\lambda \leq x$. This however also requires $J^{11}_{-} \to 0$ as $x \to 0$ and if there is a net number of bosons over anti bosons it may imply condensation at high temperatures. We will talk of this further below.

In the non relativistic region the J^{ab}_{i} quickly return the I^{ab}_{i}, as long as f_i is not too large, for example for $f_i \simeq 10^{18}$, by $x_i \simeq 50$ the ratio of the number of particles to antiparticles is $\simeq 10^6$ and increasing exponentially with x. With species having f_i smaller than this, the particles will dominate at smaller x_i's. Since annihilation is such a fast process we will usually be able to employ the equilibrium distributions above with a smooth change over from the J^{ab} to the I^{ab}, the antiparticle density can then be ignored.

(i) Boson Condensation

As long as the net number of bosons is non zero, condensation will occur when

$$n_c = n - \frac{\beta}{x^3}\frac{g}{6\zeta(3)}\lambda \tag{32}$$

is greater than zero. This will be the case when

$$n > \frac{\beta g}{x^2 6\zeta(3)} \quad \text{or } x < \frac{6\zeta(3)\ f^{-1}}{g} \quad , \tag{33}$$

for temperatures greater than this the contribution from the bosons to the total entropy remains the same

$$\frac{S}{k} = \frac{8\pi^4}{45} \frac{g}{4\zeta(3)} \ f\ N \tag{34}$$

as does the pressure factor J_{-}^{03} (we have assumed here that f is large i.e. the bosons are non degenerate at low temperatures). Most bosons are of zero kinetic energy

$$f_c^{-1} = f^{-1} - \frac{g\ x}{6\zeta(3)} \tag{35}$$

so $f = f_c$ and the boson energy density becomes

$$p = n_c mc^2 + \frac{\pi^4}{30} \frac{g}{\zeta(3)} \frac{\beta}{x^3} kT \tag{36}$$

$$= kT \frac{\beta}{x^3} \left(\frac{x}{f} + \frac{\pi^4 g}{30\zeta(3)}\right)$$

with the thermal phase clearly dominating the energy contribution from the condensate. The condensate thus has negligible consequences as far as entropy, pressure and energy are concerned.

(ii) Solutions in the Relativistic Region

The revised version of (3) gives in the region $\max(x_i) << 1$ to $0(x^2)$

$$\frac{s}{k} = \frac{1}{4\ (3)} \frac{\beta}{x^3} \left[\sum_{i=1}^{p} g_i \left(\lambda_i^2 \frac{\pi^2}{3} + \frac{7\pi^4}{45}\right) + \sum_{j=p+1}^{m} g_j \frac{8\pi^4}{45} + \frac{8\pi^4}{45} \right] \tag{37}$$

the final term is due to photons and we have ordered the labelling so that the first p species are fermion and the last m-p boson. The revised versions of (1) and (2) give

$$1 = \frac{fg}{4\zeta(3)} \left(\frac{\lambda^3}{3} + \frac{\lambda\pi^2}{3}\right) \tag{38}$$

$$\frac{N_i}{N} = \frac{g_i}{g} \frac{\lambda_i(\lambda_i^{\,2}+\pi^2)}{\lambda\ (\lambda^2+\pi^2)}\ , \quad \frac{N_j}{N_{p+1}} = \frac{g_j}{g_{p+1}} \frac{\lambda_j}{\lambda_{p+1}}\ ; \tag{39}$$

we assume no condensation. Eq.(37) becomes

$$\frac{S}{kN} = \sum_{i=1}^{p} \frac{N_i}{N} \frac{\pi^2}{\lambda_i} + \gamma f \tag{40}$$

where

$$\gamma = \frac{2\pi^4}{45\zeta(3)} \left(1 + \sum_{j=p+1}^{m} g_j - \sum_{i=1}^{p} g_i\right).$$

Solution follows through use of cubics on (39), it becomes rather simple if the fermion components are degenerate $\lambda \gg \pi$ or "non-degenerate" $x \ll \lambda \ll \pi$. In the case of degeneracy the second term in (40) can be ignored and Eq.(14) gives the solution. In the non-degenerate case then

$$\lambda_\ell = \pi^2 \frac{N_\ell k}{S} \left(\frac{8}{15}\frac{1}{g_\ell} + \sum_{i=1}^{p} \frac{7}{15}\frac{g_i}{g_i} + \sum_{j=p+1}^{m} \frac{8}{15}\frac{g_j}{g_\ell}\right), \quad \ell = 1\text{-}p, \tag{41}$$

$$\lambda_j \simeq 0, \quad j = p\text{-}m$$

and

$$f = \frac{12\zeta(3)}{\pi^4} \frac{S}{Nk} \left(\frac{8}{15} + \frac{7}{15} \sum_{i=1}^{p} g_i + \frac{8}{15} \sum_{j=p+1}^{m} g_j\right)^{-1} \tag{42}$$

Here ℓ runs over fermions and bosons as long as the bosons are non-degenerate. The two restrictions $\lambda_i >> x_i$, $\lambda_j << x_j$ in this case require that N_j/N_i be small for all i,j. The solution will apply for all non-degenerate species when x_j becomes equal to the RHS of (41) and the number densities in the condensates can be calculated in a straight forward manner.

In a realistic situation of course p and m are not fixed but depend on the particle mass spectrum. The pair species will be extant in equal numbers at relativistic temperatures and (41-42) again apply. When all species are again non-relativistic the equations governing the non-degenerate Synge gas described in the last section are appropriate however f has changed from its $x \to 0$ value f_0,

$$f = f \quad (1 + \frac{7}{8} \sum_{i=1}^{p^1} g_i + \sum_{j=p+}^{m^1} g_j) \tag{43}$$

assuming that p^1 and m^1 are finite. Note that the fermions must be described by the degenerate equations if p^1 is so large it is equal to the entropy per fermion (Eq.40). Of course Eq.(43) could be used as an explanation of the largeness of f now ~ 10^8 the universe beginning its expansion with a more sensible value, say ≃ 1. However this shifts the problem to - why does the mass spectrum contain some 10^8 species? We exhibit in the graphs that follow the behaviour of f in some of these models.

4. THE REES MODEL

Let us conclude this chapter by considering the Rees model (see Sec.2, Ch.3) and its near relatives (see Rees' references 1-9) somewhat qualitatively.

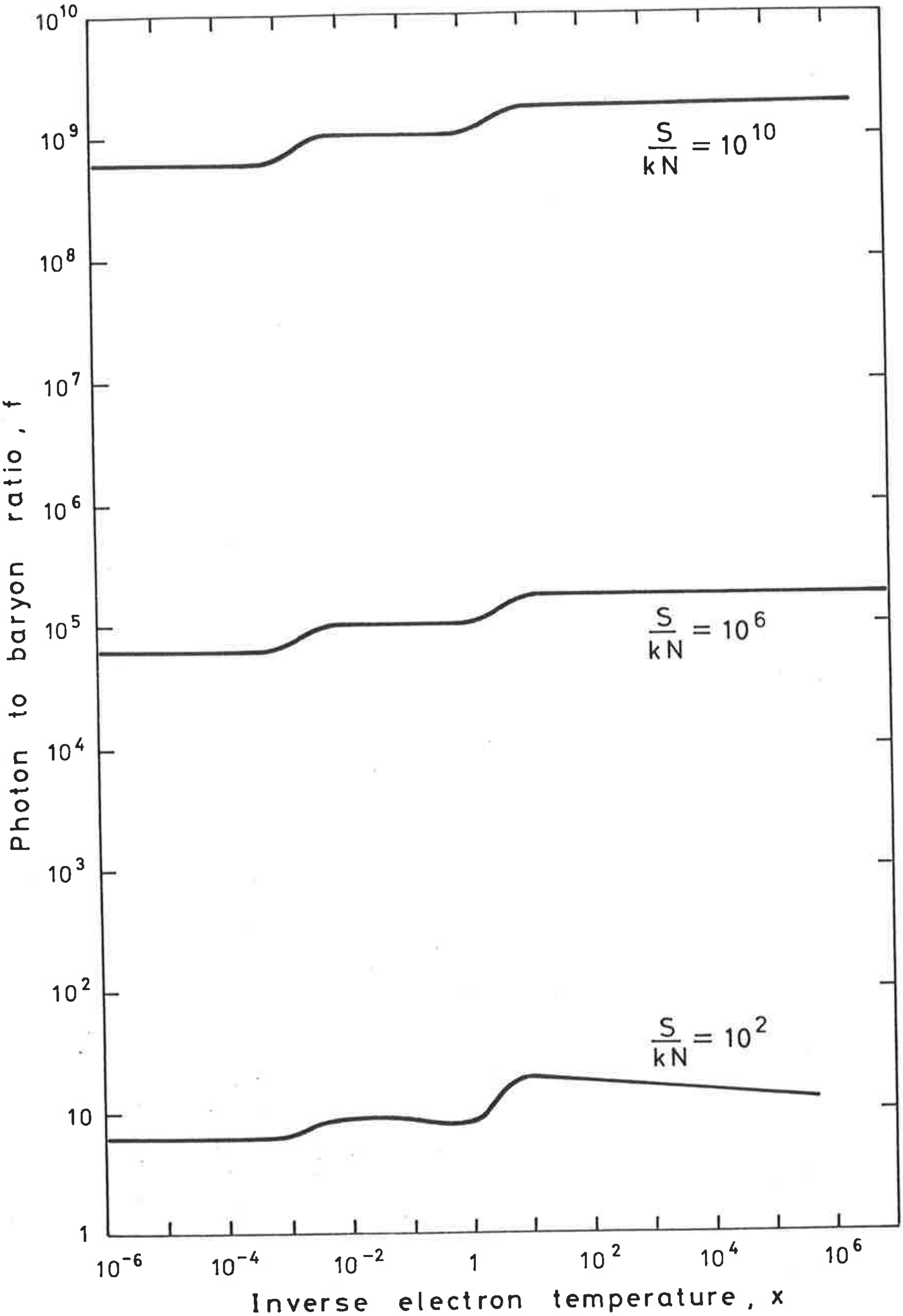

Fig. 1 Behaviour of the photon to baryon ratio for models with equal numbers of photons and electrons and pair production.

The Rees model (Rees 1978) belongs to a family of models which attribute the present microwave background and its near black body shape (see Woody and Richards 1979) to radiation energy production in some specified epoch. This non primordial origin for the black body radiation fits in well with the existence in the present epoch of a considerable "missing mass" (see Weinberg 1972 and Peebles 1971 for reviews). Such mass is underluminous having long since exhausted available energy whose present form is the microwave radiation itself.

Actually one should really distinguish between two levels of missing mass. Thus there is the dynamically missing mass needed to stabilize rich clusters like the coma as well as other groups of galaxies (Geller and Peebles 1973) and even the giant spirals (Ostriker et al. 1974). Such matters however will probably not close the universe (see Tammann 1974). On the other hand there may be a cosmologically missing mass ($q_0 = \pm 1$, $H_0 = 55 \pm 7$ kms^{-1} Mpc^{-1}) which may help bind larger structures.

In the Rees model some 80% of the mass is present as dark matter in galactic haloes and clusters of galaxies. This matter has produced the blackbody background through the production of radiation at an earlier epoch (red shift z) and a temperature T = 2.7 (1 + z) when the radiation to matter density (all matter is in nucleons) is

$$\frac{\rho_\gamma}{\rho_b} = \frac{a_s T^4}{nmc^2} = 6.7 \times 10^{-13} \; f_b (1+z) \qquad (44)$$

where all uncertainties with regard to the actual present energy density have been incorporated into the photon to banyon ratio f.

Clearly at low redshifts typical of (presumed) galaxy formation (z<10) sufficient energy generation is possible but one then requires an implausibly effecient opacity to adequately thermalize the radiation by the present epoch. The problems remain for energy generation at still higher z. Thus Layzer and Hively (1973) have postulated models in which stars of mass 5-10 M_0 release most of their energy at $25 \leq z \leq 50$ however they must also postulate grains with rather bizarre properties (e.g. metallic on the surface, hollow inside) to adequately thermalize the radiation. To its advantage this model may actually provide the heavy elements necessary to the thermalization from a primordial origin. The model is a cold model in which degenerate densities of nucleons, electrons and electron neutrinos determine the primordial light and heavy element production (see Sec.4, Ch.5), an expanding liquid metal shatters into solid (hydrogen metal) fragments, finally forming into stars which lift the degeneracy and supply the microwave background (see D. Layzer 1973 for details and variations).

Alternately high red shifts are also ruled out. If one accepts a maximum energy conversion of 6mev per nucleon characteristic of helium formation then $f(1+z) \leq 10^{10}$ and energy conversion can only occur for $z \leq 10^{2-3}$. In the Rees model an era of pregalactic supermassive star ($m \geq 100$ M_0) formation occurs at $z \geq 100$, rapid hydrogen burning ($\sim 10^7$ years) produces the radiation which is subsequently thermalized by molecules and dust. The active phase $\sim 10^7$ years is over well before galaxies form and the burnt out stars provide the missing mass. The galaxies form in the potential wells of the dark matter.

Using the Rees notation the characteristic time these stars take to burn away a fraction ε of their mass at a rate YLe (Le is the Eddington luminosity) is

$$t \simeq \varepsilon\, t_s\, Y^{-1} \tag{45}$$

where t_s is the characteristic time scale for conversion of rest mass into radiation

$$t_s = \frac{\sigma_T c}{4\pi G m_p} \simeq 4 \times 10^8 \text{ yr.} \tag{46}$$

The expansion at this time is dictated by the rest matter and since the age of the stars is very nearly the age of the universe we have

$$t \simeq \left(\frac{8\pi G}{3c^2}\, \rho_b\right)^{-1}. \tag{47}$$

If some fraction F of the baryonic matter has been included in the stars at time t the radiation density will be

$$\rho_\gamma = \frac{3}{5}\, \varepsilon\, F\, \rho_b \tag{48}$$

and assuming it to be thermalized ($\rho = a_s T^4$) we can calculate its temperature by eliminating ρ_b using (47) and (45). It is then a simple matter to calculate the photon to baryon ratio and give the Rees formulae

$$f_b \simeq \left(\frac{e^2}{G m_p^{\,2}}\right)^{\frac{1}{4}} \left(\frac{m_p}{m_e}\right) \left(\frac{e^2}{\hbar c}\right)^{\frac{3}{4}} Y^{-\frac{1}{2}}\, \varepsilon^{\frac{5}{4}}\, F^{\frac{3}{4}}. \tag{49}$$

The first bracket is the ubiquitous large number ($\simeq 10^{40}$) that appears in the numerical coincidences so that f_b is given by the 4th root of this number multiplied by a set of model dependent factors whose value is somewhat less than one.

Of course the real problem with this surprisingly simple derivation is whether the thermalization actually takes place in the manner required by the detailed black body spectrum and where and when the elements needed to thermalize the radiation are produced.

CHAPTER 5

THE STANDARD MODEL AND SOME DEVIANTS

1. INTRODUCTION

We will now attempt a more realistic approach. The models so far discussed are characterized by few inter-species chemical consequences. The particle-antiparticle annihilations studied in the last chapter provide a rather artificial solution to the origin of f_b problem and if the standard model is to be any guide, chemical reactions are responsible for crucial contemporary observations - ion recombination ~4000°K, light element production ~10^7-10^9°K. In this chapter we study some chemical reaction consequences in a wide range of cosmological models. The temperatures at which we will claim realism are below 10^{12}°K. We will be concerned here basically with the upper range of these temperatures, what is known for the standard model as the lepton era; above 10^{12}°K the description is obscured by the opening up of both lepton and hadron spectrums. Nonetheless consistent with past practice we will assume the models are valid at all temperatures and consider the consequences. It will be up to the next chapter to discuss the conditions at $T \geq 10^{12}$ in a more realistic fashion.

In the next section we set up the equations governing weak interaction equilibrium in the lepton era for the standard model. As the "standard model" we will take to mean - no net lepton numbers, positive net baryon number i.e. lepton symmetric, baryon asymmetric. However we will put our faith in a Carter type approach and claim the right to ask for the cosmological consequences of models with any lepton and baryon asymmetries. From Sec. 4 on the models will become continuously more deviant culminating in Sec. 5 with a model containing degenerate lepton seas but which in the context of certain lepton-baryon non-conserving processes is lepton-baryon symmetric.

The procedure for studying a complex chemically reacting system should be by now quite clear. For the species involved we write down a number of conservation laws for those not involved in reactions, a number of equilibrium occurring reactions and their equations of mass action and a number of conservation laws for those species involved in reactions. These last will be laws conserving certain quantum numbers (charges) associated with extant particles such as electric charge, electron lepton number, muon lepton number, baryon number, strangeness, etc..

Now there are two crucial problems with such an approach. Firstly the number of species is considerably uncertain, phenomenologically we have the newly revised lepton spectrum (e^-, ν_e), (μ^-, ν_μ), $(\tau^-, \nu_\tau?)$ and an expanded hadron sequence $\{p, n, \pi^+, \pi^\circ, \Lambda^\circ, \Sigma^+, \Sigma^\circ, \Sigma^-, \Xi^\circ, \Xi^-, \Omega^-, K^+, K^\circ, \eta^\circ, J/\psi, D\}$, theoretically we might replace the hadron spectron by the quarks $\{u, d, s, c, t, b\}$ (though this can have no cosmological consequences if the quarks are absolutely confined) but as well we would have charged and neutral vector bosons, gluons, Higgs particles, axions etc. and perhaps even exotic possibilities like massive neutral neutrinos. Secondly the established conservation laws of lepton and baryon numbers similar to that for strangeness may be only approximate and dependent on the physical context - for us here whether or not law violating reactions can proceed. Recent theories of grand unification of weak, electromagnetic and strong interactions will violate lepton and baryon laws when the interactions become of equal strength at energies near the grand unification mass $\sim 10^{15}$Gev/c^2. Incidentally one such popular model within SU(5) predicts a desert completely empty of particles between 10^2-10^{15}Gev/c^2.

In the face of such theoretical and experimental advances the interrelations between cosmology and terrestrial particle physics become (to borrow a biological metaphor) symbiotic. Since quantum chromodynamics does not specify the number of flavours (colour interactions are flavour blind) the authority of the standard cosmological model and the sensitivity of its He^4 production to the number of lepton families has, for particle models relating lepton and quark flavours (two flavours per family), been used to limit that number. Popular theories would limit the number of flavours to six with the whole of particle physics given in (e^-, ν_e), (μ^-, ν_μ), (τ^-, ν_τ); (u, d, s, c, t, b) and their related bosons. Clearly such claims require vigorous and detailed scrutiny. In Sec. 4 we attempt such a scrutiny on a number of levels. However we will leave any philosophical arguments about this surge in fundamentalism to Ch. 6. What will be important for us is that the interrelations spoken of above are also back handed. For if there is found another heavy lepton (with its own massless neutrino) beyond the tau the standard model (as it now "stands") may well fall, with it must go some of the faith in the belief that the number of quark flavours are limited and that they are indeed fundamental. The quark "spectrum" will then either become an object for explanation by deeper lying theory (e.g. with "preons") or a mythical structure based on the mistaken belief that the wealth of phenomenological data within particle physics can be explained by fundamental particles that can never be seen.

Now we do not wish here to take sides on these issues nonetheless the standard model has been well described by others so we will direct our attention to its near and far deviants.

The plan of this chapter is therefore as follows. In Sec. 2 we set up the standard model and discuss its features in such a way as to be useful for later deviant model construction. In Sec. 3 we consider some other cosmological problems for the standard picture and discuss some relevant features of its light element production. Section 4 begins by discussing the standard model and its He^4 production as modified by the presence of higher lepton families. We then move on to discuss the relevance of deviant models constructed by including non-standard particles (e.g. massive neutrinos) and non-zero charges.

One such thoroughly deviant model is treated exclusively in Sec. 5. It is a model which addresses the problems of missing energy, light element production and the present size of f_b (the photon to baryon ratio) by postulating that the energy density at the present epoch is dominated by degenerate neutrino seas. While this is no new idea what seems original in the presentation is the assumption that the present seas are of electron neutrinos and anti-muon neutrinos. This lepton symmetry is an attractive feature of the model. We will present in this section a simple analysis of a feature of the model which may well produce the black body background. We also discuss the models light element production however as we will see any light elements formed are destroyed (though only just) during the simply described photon production period. A more complex approach to the photon production era may yet allow reasonable light element abundance survival.

2. THE LEPTON SYMMETRIC STANDARD MODEL

We assume here a family of models with components p^+, n, e^-, μ^-, ν_e, ν_μ, π^+, π^o (labelled by subscripts 1-8) together with their antiparticles and black body radiation. The components are assumed to be in chemical and thermal equilibrium (justification of this is left to specific model analysis) via the pair annihilation reactions

$$p^+ + p^- \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_1 = -\bar{\lambda}_1, \tag{1}$$

$$n + \bar{n} \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_2 = -\bar{\lambda}_2, \tag{2}$$

$$e^- + e^+ \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_3 = -\bar{\lambda}_3, \tag{3}$$

$$\mu^- + \mu^+ \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_4 = -\bar{\lambda}_4, \tag{4}$$

$$\nu_e + \bar{\nu}_e \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_5 = -\bar{\lambda}_5, \tag{5}$$

$$\nu_\mu + \bar{\nu}_\mu \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_6 = -\bar{\lambda}_6, \tag{6}$$

$$\pi^+ + \pi^- \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_7 = -\bar{\lambda}_7, \tag{7}$$

$$\pi^o + \bar{\pi}^o \leftrightarrow \gamma \quad \Rightarrow \quad \lambda_8 = -\bar{\lambda}_8. \tag{8}$$

The components are involved in weak interaction processes (Weinberg 1972 P.534, P.546)

$$e^- + \mu^+ \leftrightarrow \nu_e + \bar{\nu}_\mu \qquad e^+ + \mu^- \leftrightarrow \bar{\nu}_e + \nu_\mu \tag{9}$$

$$\nu_e + \mu^- \leftrightarrow \nu_\mu + e^- \qquad \bar{\nu}_e + \mu^+ \leftrightarrow \bar{\nu}_\mu + e^+ \tag{10}$$

$$\nu_\mu + \mu^+ \leftrightarrow \nu_e + e^+ \qquad \bar{\nu}_\mu + \mu^- \leftrightarrow \bar{\nu}_e + e^- \tag{11}$$

and

$$n + \nu_e \leftrightarrow p^+ + e^-, \qquad n + e^+ \leftrightarrow p^+ + \bar{\nu}_e, \qquad n \leftrightarrow p^+ + e^- + \bar{\nu}_e. \tag{12}$$

When the rates for these processes are fast compared to the expansion rate we have a reduced set of equilibrium reactions and their equations of mass action

$$e^- + \mu^+ \leftrightarrow \nu_e + \bar{\nu}_\mu \qquad \lambda_3 + \bar{\lambda}_4 = \lambda_5 + \bar{\lambda}_6 \tag{13}$$

$$\pi^+ \leftrightarrow e^+ + \nu_e \qquad \lambda_7 = \bar{\lambda}_3 + \lambda_5 \tag{14}$$

$$n + \nu_e \leftrightarrow p + e^- \qquad \lambda_2 + \lambda_5 \leftrightarrow \lambda_1 + \lambda_3 \tag{15}$$

Eq. (14) has been included since π^+ exists as a free particle and may have a non-zero chemical potential.

The reactions conserve the following quantum numbers - electric charge N_q, electron lepton number N_e, muon lepton number N_μ, baryon number N_b - the only assumption we will in general make on these numbers is $N_q \equiv 0$. Compelling reasons for this have been given by Landau and Lifshitz (1963) and Zeldovich (1965), at least in the context of closed models.

Charge conservation is then

$$0 = 2\ J''(x_3,\ \lambda_3) + 2\ J''(x_4,\ \lambda_4) - 2\ J''(x_1,\ \lambda_1) + J''(x_7,\ \bar{\lambda}_7) \tag{16}$$

where the $J^{ab}(x,\lambda)$ are the pair production integrals of Appendix C, we have for the moment dropped the convention that $\lambda \geq 0$. Electron lepton conservation is, after dropping unnecessary symbols

$$4\ \zeta(3)\ f_e^{-1} = 2\ J_3'' + \frac{\lambda_5^3}{3} + \frac{\pi^2}{3}\lambda_5 \tag{17}$$

and muon lepton conservation

$$4\ \zeta(3)\ f_\mu^{-1} = 2\ J_4'' + \frac{\lambda_6^3}{3} + \frac{\pi^2}{3}\lambda_6\ . \tag{18}$$

We have used the black body number to total lepton number ratio and we have assumed the neutrinos are massless ($g_6 = g_5 = 1$) and applied (2:52-53). Baryon conservation is

$$4\ \zeta(3)\ f_b^{-1} = 2\ J_1'' + 2\ J_2''\ . \tag{19}$$

so that the photon to number ratios are related as

$$f_e^{-1} : f^{-1} : f_b^{-1} = N_e : N_\mu : N_b\ . \tag{20}$$

There are no independent conservation laws for the charged pions (π^+, π^-) since the above equations determine their concentrations and since contrary to the representation in Eq. (8) the neutral pion is its own antiparticle

$$\lambda_8 = \bar{\lambda}_8 = 0 \tag{21}$$

and so adopts a black body distribution.

The pair annihilation reactions (1-8) are eliminated by use of the J^{ab} and together with the equation for conservation of total entropy, Eqs. (13-14, 16-19, 21) constitute 9 equations in 9 unknowns $\{\lambda_1 - \lambda_8, f_b\}$ which are to be solved at each temperature in terms of the constants $\{S/N_{bk}, N_e : N_\mu : N_b\}$. Further reduction occurs when λ_5, λ_6, λ_7 are eliminated in (16-18) using the mass action equations (13-15); Eq. (21) is regarded as a solution. The system thus reduces to the unknowns $\{\lambda_1 \ldots \lambda_4, f_b\}$ in the 5 equations (16-19) and

$$\frac{S}{N_{bk}} = \frac{f_b}{4\ (3)} \left\{ \sum_i^8 g_i \left(J_i^{21} + J_i^{03}/3 - \lambda_i J_i'' \right) + \alpha f_b \right\} \tag{22}$$

where as previusly $\alpha = 3.60$.

There is however one constraint on the system imposed due to the Bose statistics ($g = 1$) of the pions

$$|\lambda_7| \leq x_7 \ . \tag{23}$$

If the constants $\{S/N_{bk}, N_e : N_\mu : N_b\}$ are such as to maintain charge balance with a $J_7'' > J''_{max}(x_7, x_7)$ then the equations must be rewritten to include the condensation term. To put this in another way, Eq. (23) places limits on the relative populations within each of the lepton families (14, 13) and the baryons (15). This militates somewhat against the initial reasons for the introduction of the condensation term (Sec. 2, Ch. 2), for even though the pion densities are not limited Eq. (23) does provide real constraints on the relative numbers of other types of particles. We will return to this point later.

The above equations take their simplest form as a charge symmetric model, $N_b = N_\mu = N_e = 0$ and it is immediately clear that if the equations are to be satisfied at all temperatures then $\lambda_{1\to 8} = 0$. In the case of the seriously proposed charge symmetric model (Omnès 1972) two phases of opposite non-zero baryon number separate spatially at temperatures above 10^{12}°K. Restriction to such temperatures will not interest us here.

In the standard cosmological model, $N_q = N_e = N_\mu = 0$ (see Harrison (1973) for an explicit statement) and the number of baryons is such that $f_b \sim 10^8$; i.e. the standard model is hot. In the remainder of this section we note some immediate consequences.

Firstly though the net lepton numbers are zero the net electron (or muon) and net electron (or muon) neutrino numbers are non-zero since to put $\lambda_{3\to 6}$ to zero requires $\lambda_7 = 0$ through (14), $\lambda_1 = \lambda_2$ through (15), hence $\lambda_1 > 0$ through (19), but then charge conservation (16) is violated. On the other hand we cannot put separately $\lambda_7 = 0$ since Eq. (14) then requires both $\lambda_3 = \lambda_5 \geq 0$ in violation of (17) or the argument above.

Secondly we can (tediously) show that $\lambda_7 > 0$ is also excluded. For if $\lambda_7 > 0$ then $\lambda_5 - \lambda_3 > 0$ by (14) however $N_e = 0$ is possible only if either $\lambda_3 > 0$ and $\lambda_5 < 0 \Rightarrow -\lambda_3 + \lambda_5 < 0$ or $\lambda_3 < 0$ and $\lambda_5 > 0 \Rightarrow -\lambda_3 + \lambda_5 > 0$. The first of these is a direct contradiction so $\lambda_7 > 0 \Rightarrow \lambda_3 < 0$ and $\lambda_5 > 0$, a similar argument for muons requires $\lambda_4 < 0$ and $\lambda_6 > 0$. Eq. (15) then requires $\lambda_1 > \lambda_2$ and since we have a baryon asymmetry $\lambda_1 > 0$ through (19). In such a case each positively charged species has a positive chemical potential and charge balance is violated (16).

We thus must have

$$\lambda_3,\ \lambda_4,\ \bar{\lambda}_7,\ \bar{\lambda}_5,\ \bar{\lambda}_6 > 0 \tag{24}$$

by precisely the reverse argument. Charge conservation is then balanced by the protons

$$\lambda_1 > 0 \Rightarrow \lambda_2 > 0 \tag{25}$$

by (15).

We will solve the equations in the relativistic regime below but first we put some restrictions on the degeneracy parameters. Note that the normal convention $J''(x,\lambda) \geq 0$ has been restored in the above equations.

Eq. (25) requires through (19)

$$0 < J''_1 < 2\zeta(3)\ f_b^{-1}$$

since these are the only positively charged particles we have by charge

$$(J''_3,\ J''_4,\ J''_7/2) < 2\zeta(3)\ f_b^{-1}\ .$$

Electron lepton conservation (17) requires

$$0 < \bar{\lambda}_5 < \frac{12\zeta(3)}{\pi^2}\ f_b^{-1}$$

since f_b is large, similarly muon lepton conservation (18) requires

$$0 < \bar{\lambda}_6 < \frac{12\zeta(3)}{\pi^2}\ f_b^{-1}\ .$$

Now as long as the electrons are still relativistic ($T > 10^9$) J''_3 is given approximately by (C:18) so that

$$0 < \lambda_3 \lesssim \frac{6\zeta(3)}{\pi^2}\ f_b^{-1}$$

and so

$$0 < \bar{\lambda}_7 \lesssim \frac{18\zeta(3)}{\pi^2}\ f_b^{-1}$$

and

$$0 < \lambda_4 \lesssim \frac{6\zeta(3)}{\pi^2}\ f_b^{-1}\ .$$

Restricting ourselves to temperatures less than 10^{12}°K requires $x_1,\ x_2,\ x_4,\ x_7 >> 1$ and clearly each species is non-degenerate, they are therefore described by (C:7-9), which gives the Maxwell-Boltzman formulae

$$J''_i = \sqrt{\frac{\pi}{2}}\, x_i^{3/2}\, 2 \sinh \lambda_i e^{-x_i} \qquad i = 1, 2, 4, 7. \tag{26}$$

The contributions of pions and muons are thus vanishingly small at such temperatures and for protons and neutrons we have

$$J''_i = \sqrt{\frac{\pi}{2}}\, x_i^{3/2}\, e^{\lambda_i - x_i} \qquad i = 1, 2, \tag{27}$$

with $0 << \lambda_i < x_i$ since $f_b^{-1} \sim 10^{-8}$, so that $\lambda_1 = \lambda_2$ to within a factor $\sim 10^{-8}$. The equilibrium ratio of the number of neutrons to the total number of baryons is then controlled by the neutron mass difference $m_n c^2 - m_p c^2 = 1.293$ mev and is

$$\chi_n = \frac{n_n}{n_n + n_p} = \left(1 + \exp(x_2 - x_1)\right)^{-1}. \tag{28}$$

Via baryon conservation we then obtain (19)

$$J''_1 = (1 - \chi_n)\, 2\zeta(3)\, f_b^{-1} \tag{29}$$

with the negative charged electrons balancing charge conservation

$$\begin{aligned} \lambda_3 &= (1 - \chi_n)\, \frac{6\zeta(3)}{\pi^2}\, f_b^{-1} \\ &= \frac{\bar{\lambda}_5}{2}\,. \end{aligned} \tag{30}$$

Clearly f_b is so large that consequences of the baryon asymmetry are negligible, the chemical potentials of all species can be safely put to zero and the era $\sim 10^9 - 10^{12}$ is dominated by "black body" densities of electrons, electron neutrinos, muon neutrinos and their antiparticles - it is aptly called the lepton era. Actually in the standard model's more realistic account of this era the neutrinos will decouple from the rest of the fluid at $\sim 10^{10}$°K and subsequently freely propagate. Thus any subsequent heating of the non-neutrino fluid, such as the pair annihilations of electrons will not affect the neutrinos and a temperature difference will occur between the two fluids.

Of course it would have been much simpler to assume the standard model picture and show consistency with (17-19) but notice that up to Eq. (25) the assumption of small f_b^{-1} has not been used. In fact the equations which follow (25) can be simply generalized by use of cubic solutions to account for degeneracy of baryons or leptons.

Consider the consequences of a particularly high baryonic density with say $f_b^{-1} >> x_7^3$. Now Eq. (15) demands

$$\lambda_2 - \lambda_1 = \bar{\lambda}_7 \leq x_7$$

however since the densities are cubic in the degeneracy parameter Eq. (19) requires either λ_1 or $\lambda_2 \gtrsim x_7$ and so

$$\lambda_1 \sim \lambda_2 >> x_7 \; .$$

Thus for a sufficiently great degeneracy, $f_b^{-1} >> x_7^3$, the number of neutrons and protons are constrained to be the same while their density is free. On the other hand leptons (say the electron type) have $\lambda_3 = \bar{\lambda}_7 - \bar{\lambda}_5$ by (14) and $\lambda_3 = 2^{-1/3}\, \bar{\lambda}_5$, $\bar{\lambda}_5 = (2^{1/3} + 1)^{-1}\, \bar{\lambda}_7$ by cubic solution of (17) and their densities reach maxima independent of the actual baryon density when

$$\lambda_3 = (2^{-1/3} + 1)^{-1}\, x_7, \qquad \bar{\lambda}_5 = (2^{1/3} + 1)^{-1}\, x_7.$$

At such densities charge balance is maintained by a condensed density of pions which in the framework used here can be arbitrarily large; nonetheless there is defacto a constraint on lepton densities. Furthermore the constraints are not due to the different number of helicity states available to massive and massless leptons, if the neutrinos had $g = 2$ the $2^{1/3}$ in the above is merely replaced by 1.

As previously we will assume our model to be appropriate at extremely relativistic temperatures, in doing so we obtain a rather interesting result. At such temperatures we have to $O(x^2)$ for fermions (C:18)

$$J''_{+} = \frac{\pi^2}{3}\lambda + \frac{\lambda^3}{3} \tag{31}$$

and for bosons with no condensation (C:38)

$$J''_{-} = \frac{2\pi^2}{3}\lambda . \tag{32}$$

The cubic terms can be ignored and the solutions for f_b large

$$\lambda_1 = 5\lambda_3, \quad \lambda_2 = 8\lambda_3, \quad \lambda_4 = \lambda_3, \quad \bar{\lambda}_5 = 2\lambda_3, \quad \bar{\lambda}_6 = 2\lambda_3, \quad \bar{\lambda}_7 = 3\lambda_3 \tag{33}$$

where

$$\lambda_3 = \frac{6\zeta(3)}{13\pi^2} f_b^{-1}$$

are obtained easily. Entropy conservation gives through (C:21-25)

$$\frac{S}{kN_b} = 42\,\frac{\zeta(4)}{\zeta(3)}\, f_b . \tag{34}$$

In such an idealized model Bose-Einstein condensation of the negative pions will occur if

$$x_{\pi^-} \leq \frac{18\zeta(3)}{13\pi^2} f_b^{-1} . \tag{35}$$

The time evolution is given here by (3:69) with $\sum_i g_i\, I_i^{21}(\lambda_i) = 129\zeta(4)$ so that condensation occurs before

$$t = 1.5 \times 10^{-8} f_b^{-2} . \tag{36}$$

Notice that for an observed ratio of $f_b \sim 10^8$ the time at which such condensation occurs is just $t \lesssim 10^{-23}$ sec. However this is also the time in which the horizon $\sim ct$ comes within the wavelength (size) $h/m_\pi c$ of the pion i.e. $t_\pi = h/m_\pi c^2 \sim 10^{-23}$ sec. A hadron barrier (Bahcall and Frautschi 1971) may be established for times previous to this since the hadron cannot have established contact with other hadrons since the singularity. We will discuss the relevance of the barrier in more detail in Ch. 6. The coincidence is in fact equivalent to

$$f_b \sim \frac{1}{60}\left(\frac{t_\pi}{t_{p\ell}}\right)^{\frac{1}{2}} \tag{37}$$

where $t_{p\ell}$ is the planck time (Harrison 1967)

$$t_{p\ell} = \left(\frac{Gh}{c^5}\right)^{\frac{1}{2}} \sim 10^{-43} \text{ sec} \tag{38}$$

and the value of the bracket in (37) is just the square root of that ubiquitous large number in cosmology $\sim 10^{40}$. This in fact is the same as in the Rees' model. In that model the value f_b is a result of a specified energy production mechanism, however in the above model there seems nothing to indicate the mechanism which would give the above coincidence as consequence.

A realistic cosmological model will of course be completely different from the above at such times; spectrums of the leptons and hadrons will be opened up by the high temperatures $\gtrsim 10^{12}$°K and many species will determine the change balances (16-19). Whether or not an equation similar to (36) will then appear will also depend on the distribution of electric charges within the leptons, mesons and baryons. We will look at such an era in more detail in Ch. 6. Lastly, notice that Eq. (37) has a historical relevance, for had the microwave background been discovered and measured before accelerators had opened up the hadron mass spectrum (~ 1950) then the coincidence would probably have become the basis of an attempted explanation for the photon to baryon ratio. What sort of explanation this would be is far from clear.

3. THE STANDARD MODEL: PROBLEMS AND HELIUM PRODUCTION

The natural prediction within the standard cosmological model of

He^4 abundance of $\sim$ 20-30% is perhaps its greatest achievement. Such is the sensitivity of the cosmological light element production that deviations from the standard picture such as, inhomogeneity and anisotropy, degenerate neutrino backgrounds, expanded spectrums of leptons, etc. are severely limited (see Schramm and Wagoner (1977) for an excellent review). In fact if a present deuterium abundance of $\sim 2 \times 10^{-5}$ $\{n(d)/n(H)\}$ is ascribed a primordial origin within the standard model, then it actually specifies the present baryon density $\lesssim 7 \times 10^{-31}$ gcm^{-3} i.e. $\Omega \lesssim .1$ for $H_0 \sim 60$km sec^{-1} Mpc^{-1}. The universe is then open and all of its mass is associated with the galaxies (Gott et al. 1974).

Of course one could take the opposite view that the light elements have a recent origin (e.g. Fowler 1970) in supermassive stars (Hartquist and Cameron 1977) or other unknown objects. In such a case nucleosynthesis in the early universe must be limited in some way - perhaps by a degenerate electron neutrino background (first suggested by Zeldovich (1962) in the context of cold models) which represses neutron production through reactions (12). The unexpectedly low solar neutrino flux is in fact consistent with a low $\sim$ 10% He^4 content of the original sun (Marx 1975) and determinations for the solar wind and solar cosmic ray abundances have gone as low as $\sim$ 16% (Hundhausen 1972). However a recent origin would suggest a much greater scatter than is observed; for example Searle and Sargent (1972) have investigated two compact blue galaxies whose abundance of heavy elements is at least a factor of ten less than in normal galaxies but the helium abundance is almost the same. Recent work by Piembert (1975) suggests that the primordal value is $.20 \leq Y \leq .25$ with evolved systems (2% of the mass in heavy elements) having about 4-6% more helium than primitive systems (<< 1% of the mass in heavy elements). Arguments for deuterium are somewhat less certain since it may be astrated, He^4 almost certainly can not. There is a strong consensus however for a cosmological production of the light elements and we will accept that consensus here. Detailed arguments can be found in the recent reviews (York 1977, Schramm and Wagoner 1977).

The standard model itself is not without its problems. Apart from little known astrophysical phenomenae such as quasars and black holes which may have considerable cosmological consequences, explanation of some larger scale features of the universe is poor. The emerging picture is one of hierarchy of structures, galaxies, clusters, super-clusters, these last forming a three dimensional cell-like pattern with interior "holes" of ~ 100-150 Mpc (see I.A.U. Symp. No. 79 ed. Longair 1978). Now if we ignore the whirl theory of the origin of this structure - the high anisotropy in the early universe seems inconsistent with cosmological helium production Barrow (1976) - then the structure either grows out of adiabatic perturbations (the matter and the radiation are perturbed) or entropy fluctuations (the protons and electrons are perturbed but the radiation density is constant). The former yields a fragmenting scenario with the proto clusters forming first, gas clouds of galactic mass separating out and with stars finally forming within the galaxies; in the latter stars form first, cluster in galaxies which then gravitationally interact to form larger systems. The fluctuations cannot have a thermal origin (Lifschitz 1946) but the ad hoc assumed spectrum must come from over the horizon. Thus not only must the universe be highly homogeneous and isotropic on the large scale (e.g. microwave fluctuations of less than 10^{-3} indicate density perturbations of less than 10^{-2} on a scale 500 Mpc, see Longair 1978 p.227) but areas which have never been in casual contact must have the same small scale fluctuation spectrum. Note that in the whirl theory the dissipating turbulence must be continuously regenerated by the decay of larger scale eddies coming through the horizon. The missing matter further complicates these issues however it seems more likely it is spread through the clusters rather than totally following the galactic distribution. If it were all in the galaxies greater dynamical friction would be seen between them.

Other problems are better known. The unexplained à prioris of the standard model - the photon to baryon ratio, baryon charge asymmetry, muon and lepton charge symmetry etc. (see Sec. 4). The third of these may not seem a problem at all since it is usually described as a simplicity assumption. Whatever the status of simplicity arguments however, a solution to the baryon charge asymmetry by baryon non-conservation and therefore de facto lepton non-conservation makes the lepton charges a matter for argument.

These problems highlight a difficulty for cosmological reasoning in general. The relative epistemological significance of the problems for model constructing itself is far from clear. Thus in the Rees model the otherwise missing mass is the starting point for a natural explanation of the photon to baryon ratio, but helium production is rendered problematic. This problem might be resolved by meshing the model with a cold model of the Layzer type (as we will see in Sec. 4) but the primordial heavy element abundances are not yet worked out (do they correspond with abundances in the oldest stars?) and many would regard the invoking of just the right lepton number to gain the correct helium abundance as unnatural. Is this then preferable to the standard model's simple and natural explanation of both He^4 and D abundances as well of its lack of explanation of the origin of the photon entropy? We will not pursue the cosmological implications of these words "natural" and "simple" here, though we will return to them in the next chapter, suffice it to say that a Carter-like anthropic argument for the photon entropy within the standard model may well be the most natural explanation of all. The photon to baryon ratio has about this value since it is the only one consistent with us observing it.

What would be preferable with respect to model constructing is to either construct models which tackle many of the problems simultaneously or models which can be brought into confrontation with observational data. The former we will attempt to do in Sec. 5 of this chapter. An example of the latter is the Omnès' charge symmetric model which seems wrong since the antimatter is simply not seen (Steigman 1976). The possibility remains that the standard model's production of He^4, slightly modified for recent advances in particle physics, may also be in conflict with the data. We will investigate this in the next section. First let us discuss the production itself.

The nucleosynthesis consists of three main stages. The formation of neutrons through the weak interaction (Eq. 12), formation of the bridging species deuterium through the electromagnetic interaction and light element production through a chain of two body nuclear reactions occurring once d has reached a sufficient density. The equilibrium neutron abundance (28) is

$$\chi_n = \left(1 + \exp(1.50 \times 10^{10}/T)\right)^{-1}$$

which will be frozen in when the rates of the weak interactions (12) fall below some characteristic expansion rate.

Without going to the trouble of integrating over the products of number densities and cross sections we can get a crude picture of what is occurring by considering the characteristic rate of the reaction $\nu_e + n \rightarrow p^+ + e^-$

$$\tau^{-1} = \sigma_{wk} n_{\nu_e} c = g^2 64\pi^5 k^5 T^5 / h^7 c^6 \ , \qquad (40)$$

where σ_{wk} is quadratic in neutrino energy in the center of mass frame

$$\sigma_{wk} = g^2k^2T^2/h^4c^4 \ , \qquad (41)$$

the neutrino density is given through (2:49) and the Fermi constant $g = 1.41 \times 10^{-49}$ erg cm^3. The cosmological time is given through (3:69) with $\sum_i g_i J_i^{21}/12\zeta(4) = 9/2$

$$t = 1.09(10^{10}/T)^2 \text{ sec} \ ; \qquad (42)$$

notice that the expansion rate $\mathring{R}/R = (8\pi G\rho/3c^2)^{-\frac{1}{2}}$ has an extra factor of 2. The weak interactions will fall out of equilibrium when $\tau \gtrsim t$ giving a decoupling temperature $\sim 10^{10}$°K. Apart from incorporation into nucleii the neutron abundance frozen-in at around this temperature will change only through free neutron decay.

Nucleosynthesis proceeds by the nonresonant neutron capture

$$\rho + n \leftrightarrow d + \gamma \ . \qquad (43)$$

The cross sections as is well known behave inversely with relative velocity (Chui 1968 p.341) so that the reaction rate per free neutron is proportional to the proton density

$$\tau_d^{-1} = 4.55 \times 10^{-20} n_p = \chi_p n_b 4.55 \times 10^{-20}$$

i.e.

$$\tau_d^{-1} \approx 10^{-18} T^3 \chi_p f_b^{-1} \qquad (44)$$

where we will use the mass fraction of a nuclide atomic number Z_i, nucleon number A_i

$$\chi_i = n_i A_i n_b^{-1} \ . \qquad (45)$$

This is much faster than the characteristic expansion time so that deuterium will appear as its equilibrium density which is given by the Maxwellian form of the equation of mass action (2:85) for (43)

$$X_d \doteq X_n X_p \, e^{x_n + x_p - x_d} \, x^{-3/2} \, f_b^{-1} \left(2\zeta(3)\sqrt{\pi}\right)^{-1} . \tag{46}$$

The binding energy (2.23 Mev) of deuterium is low enough to prevent appreciable densities of the nuclide until near 10^9K. Only then can complex nucleii be built up by the reactions

$$d + d \leftrightarrow He^3 + n, \tag{47}$$

$$d + d \leftrightarrow t + p, \tag{48}$$

$$He^3 + n \leftrightarrow t + p, \tag{49}$$

$$t + d \leftrightarrow He^4 + n. \tag{50}$$

The branching ratio in the first two of these is around 1.

The cross section for these reactions are given in Peebles (1966). For reaction (49) the cross section varies approximately inversely as the relative velocity but is a factor some 10^4 greater than that for (43) giving a rate per free neutron as

$$\tau^{-1} = 10^{-14} T^3 X_{He^3} f_b^{-1} . \tag{51}$$

The cross sections for (47, 48 and 50) have been well studied and take the general form

$$\sigma(E) = E^{-1} S(E) \exp(-bE^{-\frac{1}{2}}) ,$$

Bahcall (1964), Peebles (1966) where the exponential term represents the Gamow penetration factor. The rates are calculated by averaging $<\sigma v>$ over the Maxwell-Boltzmann distribution and are given amongst others by Schramm and Wagoner (1977). The reader is referred there for the details.

What will be important for us about these rates is their proportionality to reactant concentration and inverse dependence on the photon to baryon ratio. The combined rate for (47-48), for example, is approximately

$$\tau^{-1} \simeq 10^{-9}\ T^{7/3}\ \exp(-4.3 \times 10^{3}\ T^{-1/3})\ X_d f_b^{-1} \tag{52}$$

and light element production can only begin when X_d becomes sufficiently large. Once nucleosynthesis begins, as is indicated by the increasing numerical coefficients in Eqs. (44, 51, and 52), it proceeds very rapidly and almost all available neutrons are incorporated into the tightly bound He^4. Higher mass nucleii formation is prevented due to the absence of stable nucleii with A = 5, A = 8 and the coulomb barrier. In this way a helium abundance of between Y ~ .2 - .3 can be built up, the exact value is model dependent i.e. on f_b.

That we get a reasonable amount of helium, and not Y = 1 or 0, is due to two related coincidences. The first is the fact that the weak interactions governing the neutron abundance go out of equilibrium to allow a medium value of the n/p ratio. If the ratio were frozen-in at temperatures a bit higher the ratio would be ~ 1 and we would get Y ~ 1; alternately we would get no helium at all. The second coincidence is that the time between the freezing out of the neutrons and their subsequent incorporation is not too much greater than the free decay time. Too large and Y ~ 0, too small and Y may be greater than observed ranges. It is actually the fact that when the first calculations were made of the nucleosynthesis 11.7 minutes was used for the nucleon half life, subsequent measurements reduced this to 10.61 minutes and most recently Bondarenko et al. (1978) have again reduced it to 10.13 minutes. There is still some uncertainty here (Tayler 1979), but the new value would reduce the helium fraction yet again. This may be a good thing, as we will show in the next section, but it does add some tentativeness to any conclusions drawn on standard model nucleosynthesis.

4. DEVIANT MODELS

The above picture of nucleosynthesis is not qualitatively altered if a model contains components which are neutral in terms of the fundamental charges (16-19). Such debris will accelerate the expansion and can be described by the parameter

$$\zeta^2 \equiv 2\rho/9\rho_\gamma \tag{53}$$

where $9\rho_\gamma/2$ is the energy density in the lepton era for the standard model.

The resulting effects on helium production can be described by a best fit expression (Wagoner 1973)

$$Y = .421 + .380 \log_{10}\zeta - .0195 \log_{10}f_b \tag{54}$$

as long as the baryons are not degenerate. Note that the abundance is not particularly sensitive to the present baryon density - for present density of $\sim 2 \times 10^{-29}\mathrm{gcm}^{-3}$, $f_b \sim 3 \times 10^7$ and $Y \sim .28$ but for a density 100 times less $\sim 2 \times 10^{-31}\mathrm{gcm}^{-3}$, $f_b \sim 3 \times 10^9$ and $Y \sim .24$ (we have used a present photon temperature of 2.7).

The debris might contain relict quarks, massive charged and neutral leptons, axions and other exotic particles left over from processes occurring at higher densities. However the massive species, unless they are in degenerate densities, will make little contribution to ζ^2 with the main contribution coming from light particles (say $mc^2 \lesssim 1$ mev) at black body densities. The massive particles will not affect nucleosynthesis though they may have a large effect on the cosmological energy blaance once the universe becomes matter dominated.

Consider the effect of an expanded lepton spectrum (e, μ, τ, ..., ν_e, ν_μ, ν_τ, ...) where each type has its own light neutrino (experimental mass limits are $m_{\nu_e} \lesssim 60$ ev/c^2, $m_{\nu_\mu} \lesssim .65$ mev/c^2, $m_{\nu_\tau} \lesssim 250$ mev/c^2 Perl (1978)). This is the sequential lepton model. The data it seems is also consistent with the tau being an electron or muon related ortholepton i.e. it takes the same lepton number as the same sign e or μ and there is no extra neutrino. We will adopt the view here that τ is sequential and ν_τ is light. The pair produced thermal densities of the light particles gives (Steigman et al 1977, Yang et al 1978, Schramm 1978)

$$\zeta^2 = 1 + \frac{7}{36}\Delta\ell \tag{55}$$

where $\Delta\ell$ is the number of lepton types greater than two. The change in helium abundance for $\Delta\ell << 36/7$ is then

$$\Delta Y \simeq .016\ \Delta\ell\ . \tag{56}$$

Now if nucleons dominate the present mass density then present galactic dynamics provide the lower limit (Gott et al 1974) $\sim 2 \times 10^{-31}$gcm^{-3} for the nucleon density. Thus for $Y \lesssim .25$ we have $\Delta\ell \lesssim 1$ and for the weaker limit $Y \lesssim .30$, $\Delta\ell \lesssim 5$-6. This situation is not saved much by using the newly determined neutron life time to readjust Eq. (54) though it is in the right direction. The change in Y is around -.01, (Tayler 1979). The first helium abundance limit is suggested by Peimbert's recent arguments (Peimbert 1975) and in such a case perhaps all the leptons have been discovered i.e. e, μ, τ only. Alternately there are yet more sequential leptons to be discovered and either this model and thus the standard model is wrong or the nucleon density does not now provide all the mass density. If the latter is so the missing mass might be provided by massive relict particles, even so the baryon density can hardly be much less than the figure above. We will look at the former possibility below. The second limit is interesting in theories which relate lepton types and quark flavour pairs. The maximum number of quark flavours consistent with asymptotic freedom is 16 (Appelquist et al. 1978), the neutron life time corrected abundance allows neatly for the corresponding 8 lepton types. Actually this is not a firm flavour limit since all that is needed to explain the observed scaling behaviour is a temporary freedom.

For such theories the first limit demands an end to both quark and lepton spectrums and there are just the three fermion families

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}$$
$$\begin{pmatrix} e \\ \nu_e \end{pmatrix} \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \quad \begin{bmatrix} \tau \\ \nu_\tau \end{bmatrix} \tag{57}$$

where the top quark t has not yet been seen (see Close 1979 for a review). If there do turn out to be no more leptons and no more quarks become necessary then the above argument is a most singular meshing of cosmological reasoning and terrestrial physics. The standard model will be literally enthroned and the higher energy particle spectrums will yield few surprises. On the other hand if in the next decade we see even higher lepton flavours then the standard model must fall.

There are still many uncertainties however and not the least of them is that there is even no clear argument demanding the number of flavours be finite (Glashow 1978). Another is non-conservation of the higher lepton numbers. It is well known that the strange quarks couples to the down quark s through a Cabibbo rotation which allows the strangeness changing process $s \to d\gamma$. On the lepton side the same mechanism could be responsible for a muon conversion to an electron. Note however that neutrino mixing can offer no solution to the solar neutrino problem (Bahcall and Frautschi 1969) and that the corresponding neutrinos would need to be massive.

One possibility which has received some attention is that the neutrinos are actually massive. If the higher neutrinos are not light then a large number of types is allowed. The heavier leptons could then be excited states of the fundamental lepton (e^-, ν_e) to which they decay (such decays must involve a number of particles since simple decays e.g. $\mu^+ \to e^+ + \gamma$ are not seen). Astrophysical and cosmological arguments can put stringent restraints on decay times and masses (Cowsik and McClelland 1972, Goldman and Stephenson 1979, Cowsik 1979). For example if the neutrinos are light and longer lived than the universe then their domination of a present density of 2×10^{-29}gcm^{-3} requires $m_\nu \leq 55$ev (Gunn et al 1978). This is much stronger than the experimental limits. Notice that since there are now two helicity states per neutrino the restrictions from helium abundance are even more rigorous. On the other hand if the neutrinos are sufficiently short lived and there is sufficient time for the decay products (ν_e, $\bar{\nu}_e$) of the decoupled ν_μ, $\bar{\nu}_\mu$, ν_τ, $\bar{\nu}_\tau$, ... to thermalize before they too decouple then the limits are just those of the standard model without any extra leptons (Tayler 1978). The existence of a direct interaction between ν_μ and $e^\pm$ via the neutral currents tends to coincide the decouplings and make this possibility unlikely.

(i) Nucleosynthesis with Non-Zero Lepton Numbers

In a series of papers Yahil and Beaudet (1975), Beaudet and Goret (1975) and Beaudet and Yahil (1977) have considered deviant models in which the lepton numbers (17, 18) are not artificially put to zero. Each lepton number is assumed conserved and the higher leptons, τ etc.

are ignored. The left hand sides of Eqs. (17-18) are then non-zero, the lepton degeneracy parameters are no longer infinitesimal and the neutron fraction is different from (28) due to reaction (15) since the electron neutrinos are still coupled at the neutron fraction freeze-in. Thus if the neutrinos are present in large numbers, say λ_5, $\lambda_6 \gtrsim 1$, then the proton, neutron equilibrium densities favour protons. Alternately antineutrinos will favour neutrons. Notice that if the degeneracy parameters are around unity the neutrino densities are much the same as the photon density and the net neutrino to baryon ratio is very large.

For the lepton numbers considered in the above works the situation is quite simple. The first part of Eq. (30) still applies and indicates the maximum values of λ_3 (when $X_n = 0$) is $\sim f_b^{-1}$ so that charge conservation forces the neutrinos to carry most of the electron lepton number. This will also be the case for muon number. Since we are considering temperatures well below 10^{12}°K, reactions analagous to (12) for muons cannot occur and the muon neutrinos will act as debris. The resulting neutron fraction is therefore

$$X_n = \left(1 + \exp(x_2 - x_1 + \lambda_5)\right)^{-1} \tag{58}$$

where

$$\lambda_5 \simeq \frac{12\zeta(3)}{\pi^2} f_e^{-1} .$$

Clearly the small values of $\lambda_5 \sim 1$ assumed in (58) will affect X_n appreciably. Once it is frozen in, X_n will change only by neutron decay and nucleosynthesis can proceed as in Sec. 3. As emphasized by Linde (1979) the constraints derived above on higher massed lepton flavours no longer apply. The muon and electron lepton numbers offer two independent parameters so that the observed abundances of helium and deuterium can be cosmologically produced for a large range of present baryon densities $10^{-31}gcm^{-3}$ - $10^{-28}gcm^{-3}$.

There are opposing tendencies at work here. Thus increasing λ_6 (or $\bar{\lambda}_6$) drives the freeze-out point to higher temperatures so increasing the pre-nucleosynthesis neutron fraction and decreasing neutron decay. At the same time increasing λ_5 forces reaction (15) to the right increasing the proton abundance and decreasing the presynthesis fraction. Thus one can obtain appropriate values for Y. Fortunately reasonable d values can be obtained due to another two tendencies. Thus for the standard model higher present baryon densities prevent d production by increasing the effective burning of d to He^4 (see table 3 Beaudet and Yahil 1977) however the expansion rate can be increased by adding muon leptons allowing a higher survival of d. The abundances can be obtained for electron degeneracy in the range $-.25 \le \lambda_5 \le 1.8$ and muon degeneracy with a large range $0 < |\lambda_6| < 40$. In fact slightly better agreement for other light elements is also obtained. The quite large values of λ_6 indicate we could allow very large numbers of flavours (over 1000 near the top of the range) which would then replace the muon neutrinos as the accelerator.

Some concern has been voiced over the introduction of large or even small lepton numbers into the standard model (Schramm and Wagoner 1977). The introduction seems artificial since the lepton numbers are adjusted to fit the abundances yet no "natural" reasons are given for their size. The standard model however naturally produces the observed abundance of He^4 and d specifies the present baryon density. Of course the problem is that the standard model allows baryon asymmetry but no lepton asymmetry. This small baryon asymmetry might well be attributable to some symmetry breaking schemes which will operate at much higher temperatures than the lepton era (see Ch. 6).

Yet such schemes will not conserve either baryon or individual lepton numbers, in fact it may be baryon minus lepton number which is conserved (Weinberg 1979).

This point has been taken up recently by Schramm and Steigmann (1979) and Dimopoulos and Feinberg (1979) who argue against the large numbers of leptons needed above compared to the baryons. The latter work studies some hypothesized lepton and baryon nonconserving processes and concludes that a large lepton to baryon ratio is unlikely. It remains possible however if certain parameters in their calculations fortuitously cancel. Whatever the case, Linde's observation retains its force since one can always manipulate the charges of the neutrino debris (ν_μ, ν_τ ...) so that some total lepton number (in the sense of the nonconserving processes which has produced it) is arbitrarily small. Again this might seem artificial but it might well be the only model consistent with observation and not inconsistent with theory.

There is another property of these models which it is appropriate to discuss here. If the neutrinos which constitute the accelerator are of a single lepton type then for certain values of the accelerator Bose condensation of the pions becomes necessary.

The papers above deal only with the late lepton era when the density of muons (we assume this is the heavily populated lepton type) is negligible. Consider however the chemical composition near or above 10^{12}°K. At such temperatures λ_7 cannot be large ($\lambda_7 \leq x_7$, Eq. (23)) and chemical equilibrium demands (13, 14)

$$\bar{\lambda}_4 = \lambda_7 + \bar{\lambda}_6 \ .$$

Taking f_μ^{-1} to be large and negative (i.e. positive charged antimuons dominate) then (18) demands either $\bar{\lambda}_6$ or $\bar{\lambda}_4$ is large and so $\bar{\lambda}_4 \sim \bar{\lambda}_6 >> 1$. Thus the muon lepton number at these temperatures is constituted of 2/3 antimuons and 1/3 antimuon neutrinos. Assuming that the electron lepton number is small, $f_e^{-1} \sim 0$ (there is no qualitative change if $f_e^{-1} \sim 1$) then we must have

$$\frac{1}{3}\bar{\lambda}_7 \lesssim \lambda_3 \lesssim \bar{\lambda}_7 \frac{1}{(1 + 2^{1/3})} ,$$

$$\frac{2}{3}\bar{\lambda}_7 \lesssim \bar{\lambda}_5 \lesssim \bar{\lambda}_7 \frac{2^{1/3}}{(1 + 2^{1/3})}$$

where the R.H.S. terms refers to degeneracy for the electron leptons (possible only for $x_7 >> 1$) and the L.H.S. limits to the non-degenerate solution.

The only way that the large antimuon positive electric charge density can be balanced (16) is for the negatively charged pions to be degenerate i.e.

$$\bar{\lambda}_7 = x_7 .$$

For example, a value $\bar{\lambda}_6 \sim 40$ in the late lepton era requires $f_\mu^{-1} \sim -4.5 \times 10^3$ so that if f_μ is constant during previous expansion (this is a critical assumption as we will see in Sec. 5) then $\bar{\lambda}_4 \sim \bar{\lambda}_6 \sim 28 >> x_\pi^-$ at $T \sim 10^{12}$°K. The number of condensed pions needed to balance electric charge is then some 3×10^3 times the photon number or some 3×10^{11} times the net number of bosons.

Indeed any of the models above with a late lepton era characterized by some $|f_e^{-1}|$, $|f_\mu^{-1}|$, $|f_\tau^{-1}|$, ... greater than f_b^{-1} such that

$$|f_e^{-1} + f_\mu^{-1} + f_\tau^{-1} + \ldots| > f_b^{-1}$$

or

$$|N_e + N_\mu + N_\tau + \ldots| > N_b$$

will require consideration of pion condensation at $\gtrsim 10^{12}$°K. Moreover the condensation may extend into the lepton era $< 10^{12}$°K, as would the example suggested above. Interestingly the requirement that no condensation occur after the end of the hadron era ($\geq 10^{12}$°K) is similar to requiring that the net total lepton number, in the sense of processes which regard e^-, μ^-, τ^-, ... as excited states of the one lepton (e^-), be no greater than the net baryonic number.

(ii) Heavy Neutrinos

Another intriguing possibility is the existence of heavy neutral leptons on the basis of the enlarged guage group SU(3) × U(1) (Lee and Weinberg 1977, Gunn et al. 1978). The group was proposed to explain observed trimuon events whose most natural explanation was the production and decay of a charged heavy lepton m^- and a coupling of ν_μ to m^- via a new gauge boson thus making an enlargement of the standard Weinberg-Salam SU(2) × U(1) model necessary (see Gunn et al. (1978) for references and discussion). One consequence of this model is that heavy neutral leptons will exist which will carry a new conserved quantum number. This ensures the lightest of such leptons some degree of stability depending on whether this symmetry group is exact or embedded in a much larger group with a small coupling to the whole group. The other leptons are expected to be quite unstable. Astrophysical consequences of such massive neutrinos have been considered independently by a number of authors (Lee and Weinberg (1977), Discus et al. (1978), Sato et al. (1977), Gunn et al. (1978), Steigman et al. (1978) and Stecker et al. (1978)), we will discuss some relevant consequences below.

Since the leptons have full strength weak couplings the freeze-out temperature will occur when the neutrinos are nonrelativistic (assume $m_L \gg 1 \text{mev}/c^2$) so that their density will be less than the photon density by a factor $\sim \exp(-m_L c^2/kT) \ll 1$. Unstable leptons will rapidly decay and the densities of the neutral leptons ($L°$, $\bar{L}°$) will be too low for any significant effect on nucleosynthesis. If the L°'s are not to provide a mass density now greater than 2×10^{-29}gcm^{-3} then their mass must be

$$M_{L^\circ} \geq 2 \text{ Gev}/c^2 \quad \text{if } M_L >> 1 \text{ Mev}/c^2 .$$

We have already noted above that if neutrinos are light

$$M_\nu \lesssim 55 \text{ ev}/c^2 \quad \text{if } M_\nu < 1 \text{ mev}/c^2 .$$

Note that if the d is of cosmological origin it still specifies the present baryon density.

Other arguments considerably limit both mass and lifetimes and are summarized in Fig. 2 of Gunn et al. (1978). Further constraints have been derived by Lindley (1979) on the basis of photofission of the light elements during the postnucleosynthesis radiation era by energetic photons from the decay $L^\circ \rightarrow \nu\gamma$ (where the ν is a conventional massless neutrino). For M_{L° between 5 and 100 mev the L°'s must decay before nucleosynthesis begins. Such evidence strongly suggests the L° are longer lived than the age of the universe and have a mass greater than 2 Gev. Either that or their lifetimes are so small they have no observable effects at all.

Whatever the initial correlation between L° and baryon perturbations it is expected (Gunn et al. 1978) that in the end they will be correlated. They will collapse by a factor of 2 due to the phenomenon of violent relaxation (Lynden-Bell (1967)) but they will not follow the baryons down to the denser galactic discs and stars since they cannot radiate away their energy and can couple only gravitationally. As diffuse halos around the galaxies or stripped and distributed through clusters of galaxies they could not be better stuff to constitute the dynamical missing mass. However unless they can avoid clustering with the galaxies they will not close the universe. To supply the missing mass at $\Omega \simeq .1$ then $M_{L^\circ} \sim 10$ Gev/c^2 is the appropriate neutral lepton mass.

(iii) Cold Models and Helium Production

Compared to the standard model at a specific nucleon density the cold models are characterised by very much lower temperatures. In relegating the black body background entropy and energy production to a late epoch the lepton symmetric cold models face the problem of over-production of He^4 since at light element producing nucleon densities the neutron proton ratio is close to one.

Zel'dovich (1962) was the first to point out that such over-production could be prevented by the introduction of net lepton numbers greater than the baryon numbers. In his model equal densities of baryons, electrons and neutrinos begin expansion as a zero entropy fluid. When the fermi energy of the baryons is greater than their rest energy the neutron-proton ratio is one, however when the fermi energy is well below the nucleon rest energy, reaction (15) is driven to the right since at equal densities the electron fermi energy is smaller than the neutrino fermi energy by $\sim 2^{-1/3}$ due to the different spin statistics of the particles. All baryons end up as protons and helium formation is neatly eliminated.

More recently Kaufman (1970) has described how the helium abundance might be constrained to its observed value. The models contain degenerate or semidegenerate densities of baryons and leptons and presumably go over in their later stages to models of the Layzer and Rees type which can account for the present black body background. At sufficiently great densities the baryon number will be held by higher massed hyperons since such an arrangement is thermodynamically favoured. Kaufman notes, as have Ambartsumyan and Saakyan (1960) pointed out in the context of neutron star models, that the higher massed hyperons and mesons can only decay if there is sufficient rest energy not only to supply the rest masses of the decay products but also the fermi energies of any fermions produced. Of course a critical assumption here is that hyperon states are independent of other nucleon states as far as the Pauli principle is concerned. In general decay rates and decay routes for the higher states will be affected and subsequent heating, for example in a model degenerate at supernuclear densities, might remove nucleon degeneracy but not electron degeneracy.

Kaufman calculates that as the total lepton to baryon ratio varies from around 1.2 to 1.35 the helium production varies from .25 to 0. The latter ratio is less than the ratio 2 for the Zel'dovich model but again no justification for the ratio's exact value, except the He^4 abundance, is forthcoming. Nonetheless this value will be much less than that for the standard model with nonzero lepton numbers discussed above and may be more likely in terms of lepton and baryon non-conservation.

Another aspect of these cold models is that the densities may be high enough for appreciable heavy element abundances to occur. Three body reactions such as

$$He^4 + He^4 + n \rightleftharpoons He^9 + \gamma \tag{59}$$

may be important in bridging the A = 5 and A = 8 (A is mass number) mass gaps (Kaufman 1975). Such heavy element production would be an advantage for theories of the thermalization of photons produced in a later epoch but it is not clear if it can give the element abundances seen in the oldest stars.

5. A LEPTON CASCADE AND A THOROUGHLY DEVIANT MODEL

In this section we present a cosmological model which represents a radical departure from what is termed the standard model. The model is based on a simple proposition. That the presently missing cosmological energy (for $\Omega \simeq 1$) and perhaps some of the missing dynamical energy is in the form of massless neutrinos. This has been suggested many times before (e.g. Weinberg (1972)) but what is novel in what we propose here is that the neutrinos consist in degenerate seas of electron neutrinos and antimuon neutrinos. The density of the seas is considered the same and so in the context of lepton nonconserving processes but overall additive lepton number conservation the model is lepton symmetric. Also since the electron neutrinos are left handed and the antimuon neutrinos are right handed the model is symmetric with respect to handedness. Actually the lepton number densities here will be so large compared to the baryon density that in the context of lepton and baryon number violating processes the model might be considered totally symmetric. We will look at such processes in more detail in the next chapter.

The model we envisage still contains the observed background radiation in the present epoch so it remains hot for the baryons, for neutrinos however it is very cold. In this sense the model is some average of the cold and hot models of the previous section. However if those models were collected under the rubric "deviant" then the enormous neutrino degeneracies required in this model suggest the description "thoroughly deviant" as being more appropriate. As we will show below the model offers some interesting if peculiar approaches to the problems of the origin of the present black body background and the light elements. The proposal above suggests the black body radiation has its origin in a violent cascade when nearly all massive early universe leptons are converted to their massless neutrinos.

Light elements may be formed in an even earlier era and in a manner in many ways the reverse of the standard model. However for the simple model of the cascade we discuss we do not expect the light elements to survive to the post-cascade era.

We must emphasize here that the arguments below are meant to be no more than suggestive. We discuss the model qualitatively rather than quantitatively since as we will indicate, detailed calculation may face some difficulties. Though such description is hardly adequate, to our knowledge the model has not previously appeared in the literature and for this reason alone it is worth presenting.

(i) The Neutrino Seas in the Present Epoch

A radiative solution to the problem of missing energy alters the normal relationships (i.e. those for a massive universe) between the deacceleration parameter, Hubble's constant etc. (Weinberg 1972). Thus the deacceleration parameter is defined (3:16)

$$q \equiv - \frac{\ddot{R}\ R^2}{\dot{R}^2\ R} = \frac{\rho + 3p}{2\rho - \dfrac{3Kc^4}{4\pi G R^2}} ,$$

however if the energy density is dominated by radiation

$$\rho = 3p$$

then the expansion equation (2:118) gives

$$\frac{Kc^2}{R^2} = (q - 1) \frac{\dot{R}^2}{R^2}$$

or in terms of present values

$$\frac{Kc^2}{R_0^2} = (q_0 - 1) H_0^2 .$$

In terms of the critical energy needed to close the model

$$\rho_c = \frac{3c^2}{8\pi G} H_0^2 \qquad (60)$$

we have

$$\rho_0/\rho_c = q_0 \tag{61}$$

instead of the more usual relation $\rho_0/\rho_c = 2q_0$ for matter dominated models. Thus the critical values for $K = 0$, $\rho_0 = \rho_c$ become $q_0 = 1$ rather than $q_0 = \frac{1}{2}$. Moreover the density derived for observed q_0, H_0 for the above equations is just half that for a matter dominated model.

We adopt here the following presently observed values and their correction coefficients (Tamman 1974)

$$H_0 = 60h, \qquad q_0 = 1\Omega, \qquad T_0 = 2.9\phi \tag{62}$$

the coefficient ϕ has been introduced since $2.7/2.9 = .93$ may find considerable importance in the model. The cosmological energy density for these values is

$$\rho_0 \simeq 7 \times 10^{-30}c^2(\Omega h^2) \text{ erg cm}^{-3} \,. \tag{63}$$

We can take the energy associated with the galaxies to be

$$\rho_G \simeq 2 \times 10^{-31}c^2(h^2) \text{ erg cm}^{-3} \tag{64}$$

which is much greater than that presently associated with the black body radiation $\sim 6 \times 10^{-34}c^2(\phi^4)$. If this galactic density is composed mainly of baryons then $n_b \simeq 1.2 \times 10^{-7}(h^2)$, $f \simeq 4 \times 10^9(h^{-2}\phi^3)$, the ratio of total to galactic density is

$$\rho_0/\rho_G \simeq 35\Omega \,, \tag{65}$$

and the model is radiation dominated now if most of the rest matter is contained in the galaxies. Notice that since $\rho_{rad} \sim R^{-4}$, $\rho_{matt} \sim R^{-3}$ in the far future the model must become matter dominated while in the distant past it is so radiation dominated we should be able to ignore the matter completely.

The age of the universe is given through (3:64) and on omitting curvature effects it is

$$t_0 = \tfrac{1}{2}\left(\frac{8\pi G}{3c^2}\rho_0\right)^{-\frac{1}{2}} = \tfrac{1}{2} H_0^{-1} , \qquad \Omega = 1 \tag{66}$$

for the massive models the factor outside the bracket is 2/3. Using the useful conversion relation $(100 \text{ KmS}^{-1}\text{mpc})^{-1} = 9.78 \times 10^9$ years the age is consistent ($\sim 8.2 \times 10^9\ h^{-1}$yr) with the lower limits set on the ages of the globular clusters: $8 < t_0 < 18$ billion years. A massive model at this density is therefore considerably more safely within the limits. Such a comparison however is misconstrued. That should be compared here is the ages for matter and radiation models as dictated by the observed values of q_0, H_0. Since the density of the matter model is just twice the density of the hypothesized radiation model the actual ratio of ages is just $2/(3\sqrt{2}) : 1/2 \sim 1:1.06$ and the radiation model is in fact marginally older. Thus lower limits provided by measured ages must actually favour a radiation dominated model; just the opposite of which is concluded by some authors (e.g. Gott et al. 1974).

The equations governing the energy density of the two neutrino seas are to a good approximation

$$\rho_0/2 = \rho_{\nu_e} = \rho_{\bar{\nu}_\mu} = \frac{20.27}{16\zeta(3)} \frac{\mu^4}{k^3} \tag{67}$$

i.e.

$$\frac{\mu}{k} = 68.2(\Omega h^2)^{\frac{1}{4}}, \qquad \mu = 5.88 \times 10^{-3}(\Omega h^2)^{\frac{1}{4}} \text{ ev} \tag{68}$$

giving

$$n_{\nu_e} = n_{\bar{\nu}_\mu} = \frac{20.27}{12\zeta(3)}\left(\frac{\mu}{k}\right)^3 = 4.46 \times 10^5(\Omega h^2)^{\frac{3}{4}} \text{ cm}^{-3} , \tag{69}$$

a useful conversion factor here is $k^{-1} = 11604.85$°K/ev. The photon to lepton ration is $f_e = f_\mu = 1.11 \times 10^{-3}(\Omega h^2)^{-\frac{3}{4}} (\phi)^3$ and the lepton to baryon ratio $N_e/N_b = 3.7 \times 10^{12}(\Omega^{3/4}\ h^{1/2})$.

What we cannot infer about these neutrinos is their temperature or to put it another way the entropy of the degenerate seas. Since at present the matter is nearly entirely transparent to such low energy neutrinos, their temperature will depend on the nature of the primordial decoupling of them from the matter and black body radiation. After decoupling the neutrinos can hardly have been reheated so at present we must have $T_\nu \leq T_\gamma$. If in fact the temperatures are the same then the degeneracy of each sea is $\lambda = \mu/kT_\nu \simeq 24(\Omega h^2)\ (\phi^{-1})$. Weinberg (1972) in his discussion of neutrinos dominating the energy density concludes $\lambda \lesssim 45$ (P.543, he considers ν_e, ν_μ not $\bar{\nu}_e$, $\bar{\nu}_\mu$) however his arguments are in the context of the standard model which suggests a reheating of the matter and photons after decoupling due to e^-, e^+ annihilation. This standard calculation heating factor increases the temperature by $(11/4)^{1/3}$; in our thoroughly deviant model there will also be a heating factor but enormously larger than this value.

The relative temperatures of the fermi seas and the photon radiation also indicates how they share in the total entropy. Thus using (2:46) and (2:54) and kepping only higher order terms

$$\frac{S_\gamma}{S_\nu} = \frac{T_\gamma}{T_\nu}\,\frac{8\pi^2}{15(\mu/kT_\gamma)^2}\ , \tag{70}$$

S_ν is the total entropy of one neutrino sea. Clearly for $T_\gamma \simeq T_\nu$ the neutrinos easily dominate the entropy but notice that if the electromagnetic radiation contains as much entropy as the two neutrino seas (weak radiation) $T_\gamma/T_\nu \sim 210(\Omega h^2)^{\frac{1}{2}}\phi^{-2}$ i.e. $\lambda_\nu \simeq 5 \times 10^3(\Omega h^2)^{\frac{1}{4}}\phi^{-3}$. Though there are considerable uncertainties here, let us ask what type of very early universe might correspond to such a large present degeneracy.

(ii) The Very Early Equilibrium

Since the chemical potential of the neutrinos is presently falling inversely as the scale factor their degeneracy parameters have been constant at least as far back as the decoupling. However at sufficiently early times the densities or temperatures must have been high enough for the rates of the weak interaction processes $\nu_e + e^- \rightarrow \nu_e + e^-$, $\bar{\nu}_\mu + p^+ \rightarrow \bar{\nu}_\mu + p^+$ etc. to establish thermal equilibrium among extant species and for the reaction (Eq. (9))

$$\mu^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_\mu \tag{71}$$

to establish chemical equilibrium among the leptons. In such case mass action gives

$$\bar{\mu}_4 + \mu_3 = \mu_5 + \bar{\mu}_6 \tag{72}$$

and the lepton number presently carried by the neutrinos will be shared with their massive partners. In future we will drop all cumbersome bars on the muon and muon neutrinos, as long as it is recognized there are the positively charged muons and their neutrinos (conventionally antimuons) this should cause no confusion.

For $\mu_4 << M_4c^2$ Eqs. (17, 18) give

$$n_e = \frac{20.27}{4\zeta(3)} \left\{ \frac{2}{3}\left(\frac{\mu_3}{k}\right)^3 + \frac{1}{3}\left(\frac{\mu_5}{k}\right)^3 \right\} , \tag{73}$$

$$n_\mu = \frac{20.27}{4\zeta(3)} \left\{ \frac{2}{3}\left(\frac{\mu_4}{k}\right)^3 + \frac{1}{3}\left(\frac{\mu_6}{k}\right)^3 \right\} \tag{74}$$

where subscripts e, μ refer to electron and antimuon lepton numbers respectively. The effect of all other less numerous particles is negligible so charge balance requires

$$\mu_3 = \mu_4 \tag{75}$$

and the equality of lepton numbers

$$\mu_5 = \mu_6 \tag{76}$$

so that the equilibrium solution is simply

$$\mu_3 = \mu_4 = \mu_5 = \mu_6 \; . \tag{77}$$

The total entropy per lepton is given through

$$\frac{S_T}{N_e k} = \frac{\pi^2}{3} kT \sum_i \frac{g_i}{\mu_i} = \frac{2\pi^2}{\lambda_3} \tag{78}$$

so for a universal temperature T we have the degeneracy parameter

$$\lambda_3 = \frac{N_e k}{S_T} 2\pi^2 \equiv \lambda \tag{79}$$

and the massive leptons carry 2/3 of their respective lepton numbers.

Consider the nature of this thermal and chemical equilibrium as the chemical potentials fall with the scale factor (a direct consequence of (73, 74) and conservation of lepton numbers). Thermal equilibrium would be characterized by deviations of less $\sim kT$ between the four fermion seas. As such it can only be established by interacting particles which lie within $\sim kT$ of their fermi surfaces. We can now use the weak interaction rates given in the first part of (40) to establish if thermal equilibrium can be maintained. The relevant rate is

$$\tau_{th}^{-1} = n_{kT} \sigma_{wk} v \tag{80}$$

where n_{kT} is the target number density within $\sim kT$ of the fermi surface v is the velocity of the incoming particle and σ_{wk} is given using the universal fermi coupling constant g

$$\sigma_{wk} \simeq \frac{g^2 \; E^2}{h^4 \; c^4} \; . \tag{81}$$

Here $g = 1.41 \times 10^{-49}$ erg cm^3 and E_ν is the centre of mass neutrino energy. We will accept here, though it will be of little importance, that the weak cross section saturates at around 1 Gev and

$$\sigma_{wk} = \sigma_S \simeq 10^{-38} \text{ cm}^2 \; . \tag{82}$$

The expansion time is given by (3:69) with $\sum g_i \, J_i^{21}(\lambda_i)/12\zeta(4) = \lambda_3^4/8\zeta(4)$,

$$t = (2.6 \times 10^{10}/T\lambda)^2 \; . \tag{83}$$

Notice that if the weak interaction is saturated then $v \sim c$ and with $n_{kT} \simeq n/\lambda_3$ we have

$$t\tau^{-1} \simeq 2.9 \times 10^{-7}\, T \tag{84}$$

i.e. thermal equilibrium for $T \gtrsim 4 \times 10^6$. This is a large overestimate of $t\tau^{-1}$ since we do not expect saturated cross sections at these temperatures, nonetheless at temperatures much higher than this the seas are certainly in thermal equilibrium. For unsaturated centre of mass neutrino energies E_ν and nonrelativistic velocities v we need merely multiply the R.H.S. of (84) by the factor $(E_\nu/1 \text{ Gev})^2\, v/c$.

The situation for chemical equilibrium is somewhat different. As the fermi densities fall and the fermi energies scale as $1/R$ the masses of the muon and the electron will be brought into the equations governing the equilibrium. The muon density is proportional to $(\mu_4^2 - m_4^2c^4)^{3/2}$ and the muon momentum $(\mu_4^2 - m_4c^2)^{1/2}/c$ will eventually become so low that the momentum balance of the interactions that make up the equilibrium (71) is violated. The other three components in (71) will have their momenta given as their respective fermi energies on c. Thus a momentum deficit will occur on the L.H.S. of (71) when the difference in μ_4/c and the actual muon momentum $(\mu_4^2 - m_4^2c^4)^{1/2}/c \sim \mu_4/c - m_4^2c^4/(2\mu_4 c)$ cannot be supplied by the thermal energy kT at the top of the seas. This will occur when $m_4^2c^4/2\mu \sim kT$ or $x_4 \sim (2\lambda_4)^{\frac{1}{2}}$. To see this another way assume the equilibrium for when $kT < m_4^2c^4/2\mu$ and consider the solutions.

Charge balance will require

$$\mu_3 = (\mu_4^2 - m_4^2c^4)^{\frac{1}{2}} \equiv \mu_4(1 - \theta)$$

where

$$\theta \equiv m^2c^4/2\mu^2 = x^2/2\lambda^2$$

lepton equality and mass action then give

$$\mu_5 = \mu_6 = \mu_4(1 - \theta/2) \ .$$

The momentum deficit on the L.H.S. is then at least

$$\left(\mu_3 + \mu_4(1 - \theta) - \mu_5 - \mu_6\right)/c = -\mu_4\theta$$

in clear violation of momentum conservation for $x_4 > (\lambda_4 2)^{\frac{1}{2}}$.

The seas thus decouple chemically but not thermally due to the great number of scattering processes still possible at the tops of the seas. The total entropy is still conserved and the number densities (2/3 n_e, 1/3 n_e respectively for massive and massless leptons) fall as R^{-3}. We are thus in the situation already described in Sec. 2, Ch. 4. The solutions are

$$\mu_5 = \mu_6 = (\mu_3^2 - m_3^2c^4) = (\mu_4^2 - m_4^2c^4) = \left[\frac{4\zeta(3)k^3\ n_e}{20.27}\right]^{1/3} \tag{85}$$

at least to first order. Notice that these solutions neither restore momentum balance nor exaggerate the imbalance.

Eventually however the fermi energies of the neutrinos falls below the rest mass energy of the muons. The fermi energy of the muons must then be calculated more accurately through the use of (A:30). An enormous cascade will then occur as the expansion opens up the neutrino seas to the decay

$$\frac{S_{e,\nu}}{N_e\,k} = \frac{20.27}{4\zeta(3)} T^3_{e,\nu} \frac{2\pi^2}{3}\left(\frac{m_4 c^2}{2kT_{e,\nu}}\right)^2 \Big/ n_e = 4\pi^2 \frac{kT_{e,\nu}}{m_4 c^2} . \tag{92}$$

If we assume the neutrinos are not heated during the cascade then

$$T_{s,\nu} = T_{e,\nu}\left(3^{1/3}\; 2/\sqrt{3}\right) \tag{93}$$

and the entropy contained in the matter and photon radiation (subscript m) is

$$\frac{S_{e,m}}{N_e k} = \frac{kT_{e,\nu}}{m_4 c^2}\,\pi^2\left(\frac{16.3^{1/3}}{3} - 4\right) = \frac{kT_{e,\nu}}{m_4 c^2}\,\pi^2 3.692 . \tag{94}$$

The cascade thus gives the remaining matter and radiation a considerable amount of entropy, the ratio of entropy in matter and radiation to that in all the neutrinos is ~ .92. If the photons are to be the present black body background then the post decoupling degeneracy parameter is $4.5 \times 10^3(\Omega h^2)^{1/4}\ \phi^{-3}$, the end of cascade neutrino temperature is $1.3 \times 10^8(\Omega h^2)^{-3/4}\phi^3\,^\circ K$, and the present ratio of photonic to neutrino temperatures is $193(\Omega h^2)^{1/2}\ \phi^{-2}$ with the present neutrino sea temperature of $1.5 \times 10^{-2}(\Omega h^2)^{-1/2}\ \phi^2$. Of course this assumes no entropy production during the cascade, any entropy production which ended up in the matter and photons will increase the degeneracy parameter.

In the adiabatic cascade scenario energy conversion to neutrinos is very efficient. A simple caclulation of the energy density at onset using the first order terms (A:31) for the muons shows that if one assumes that the energy density during the cascade scales as radiation (i.e. as R^{-4}) then a comparison of the appropriately scaled energy density with the end of cascade neutrino energy density indicates they are equal to the second significant figure.

The many uncertainties here, both in the simple cascade scenario itself and the parameters $\{\Omega, h, \phi\}$ suggest we remain flexible about

$$\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu \ . \qquad (86)$$

The cascade will be completed when $\mu_5 \simeq m_4 c^2/2$ since by then even the least energetic muon's decay products can climb their respective fermi walls. Notice that the time when this cascade occurs is practically independent of the extent of the degeneracy $t \sim \left[2.6 \times 10^{10}/(m_4 c^2/k)\right) \simeq 5 \times 10^{-4}$ s (as we have commented on for the changeover in behaviour for the single component model in Ch. 3) and that this time is comfortably slower than the free muon decay time $\sim 10^{-6}$ s.

(iii) The Lepton Cascade

During this short epoch the fermi equilibrium between the four degenerate seas $\{e^-, \mu^+, \nu_e, \bar{\nu}_\mu\}$ is destroyed. The entire lepton number contained in these seas is converted into two degenerate seas of neutrinos. We will argue here that this violent cascade is responsible for a superheating of the non-neutrino fluid and that the entropy so received presently constitutes the black body radiation. As before we will ignore the behaviour of the baryons, their small numbers can have little effect. We will return to them when we discuss the possibility of light element formation within the models.

For a present neutrino fermi energy given by (68) the chemical potential will be half the muon rest energy at a red shift $z = 9 \times 10^9 (\Omega h^2)^{-\frac{1}{4}}$, the present black body radiation will then have a temperature of $2.6 \times 10^{10} (\Omega h^2)^{-\frac{1}{4}} \, \phi$. The cascade should be complete by this epoch and if it produces the black body radiation the fraction of energy in the photons is very small $\sim 10^{-4}$ so that the cascade must have been very efficient in placing most of the precascade energy in the two neutrino seas. We will assume here that these seas have been "smoothly stacked", this seems necessary to allow use of the standard thermodynamical relations.

Let us take the simple view of the cascade that there has been no appreciable reallignment of the fermi levels and their values are given at onset by the solutions (85). Onset will occur suddenly when

$$\mu_4 = 2\mu_5 + m_3c^2 \,, \tag{87}$$

neglecting the electron mass the first order solution (85) gives

$$\mu_4 = \frac{2}{\sqrt{3}} m_4c^2, \qquad \mu_5 = \mu_6 = \mu_3 = \frac{1}{\sqrt{3}} m_4c^2, \qquad n_e = \frac{20.27}{4\zeta(3)}\left(\frac{m_4c^2}{\sqrt{3}k}\right)^3 . \tag{88}$$

If subsequent reheating is not too high (i.e. $kT << m_4c^2$) as it will be if the cascade produces the present black body background (~ 2.6×10^{10}K) then the last of the muons decay when

$$\mu_5 = \mu_6 = m_4c^2/2 \,. \tag{89}$$

At this stage nearly all the leptons (there will be a small density of electrons to balance nucleonic charge and there may be (e^-, e^+) pairs) will be neutrinos and

$$n_e = \frac{20.27}{4\zeta(3)} \frac{1}{3}\left(\frac{m_\mu c^2}{2k}\right)^3 . \tag{90}$$

The scale factor has thus increased over its onset value by $3^{1/3}\, 2/\sqrt{3} = 1.665$. Assuming adiabatic expansion during the cascade we can relate the entropies at onset (we use subscript s for start) to the entropies at the end (subscript e). The entropy at onset is given by

$$\frac{S_{s,T}}{N_e k} = \frac{20.27}{4\zeta(3)} T_s^3\left\{4\,\frac{\pi^2}{3}\cdot\frac{m_4^2c^4}{k^2T_s^2} + \frac{2\pi^2}{3}\cdot\frac{m_4^2c^4}{k^2T_s^2} + \frac{2\pi^2}{3}\cdot\frac{m_4^2c^4}{k^2T_s^2}\right\} \Big/ n_e$$

$$= \frac{8}{\sqrt{3}} \frac{\pi^2 kT_s}{m_4c^2} \,, \tag{91}$$

where the terms in the bracket are the contributions from muons, electrons and the neutrinos respectively. At the end of the cascade the total neutrino entropy $S_{e,\nu}$ is given by the actual present value of λ. Presumably a detailed calculation would specify the value of λ responsible for the present black body background. We will not do such calculations here for reasons to be discussed below. If however we assume that the cascade does produce the black body radiation we can easily calculate the very early universe (pre-cascade) value of the photon to baryon ratio for various values of the present neutrino degeneracy parameter (equal to $\lambda_{e,\nu}$). We present relevant cascade parameters in Table 1. Postulating the very early universe neutrino degeneracy parameter ($\lambda_{s,\nu}$) we give the cascade onset temperature T_s via Eq. (88) the end temperature T_e via Eq. (93) and the end degeneracy parameter $\lambda_{e,\nu}$ via Eq. (89). The present ratio of photon to neutrino temperatures now is given by $T_{e,\nu}$ divided by the end photon temperature 2.6×10^{10}°K, the end photon to baryon ratio $f_{e,b} = 4 \times 10^9$ and the very early era photon to baryon ratio is calculated through the nucleon density at the start 1.2×10^{-7} $(9 \times 10^9 . 3^{1/3}\ 2/\sqrt{3})^3$ (where the first number is the present nuclear density and the bracket is the scale factor at onset).

Let us discuss the cascade in more detail somewhat qualitatively. For general purposes we will assume a degeneracy parameter $\sim 10^4$. At onset Eq. (84) with its correction term indicates the rates for particles within kT of their fermi energies are slower than the expansion rate (due to $(E_\nu/1\ \text{Gev})^2 \sim 10^{-2}$ and $T_{e,\nu} \sim 10^8$) so that there is neither chemical nor thermal equilibrium when the cascade begins. The cascade is thus initiated by the expansion itself through the muon decay (86). The positron produced in muon decay will annihilate very quickly in the electron sea since it faces the full sea density as targets and its cross-section $\sim \sigma_T \sim 10^{-25}$ cm^2 is very much larger than that for any weak processes which might be in competition. Annihilation channels will be overwhelmingly photonic (e.g. $\nu_e\bar{\nu}_e$ is blocked) and will on

TABLE 1

$\lambda_{s,r}$	T_s	e,r	$T_{e,r}$	(T_γ/T_r)now	(f_b)early era
5×10^4	1.4×10^7	7.2×10^4	8.5×10^6	6.1×10^3	.14
3×10^4	2.4×10^7	4.3×10^4	1.4×10^7	1.8×10^3	.66
10^4	7.1×10^7	1.4×10^4	4.2×10^7	6.1×10^2	1.8×10^1
5×10^3	1.4×10^8	7.2×10^3	8.5×10^7	3.1×10^2	1.4×10^2
3×10^3	2.4×10^8	4.3×10^3	1.4×10^8	1.8×10^2	6.6×10^2

Relevant cascade parameters and the early universe photon to baryon ratio for $T_{e,\gamma} = 2.6 \times 10^{10}$, $f_{e,b} = 4 \times 10^9$ and a present nucleon density 1.2×10^{-7} cm^{-3}. For the adiabatic cascade $\lambda_{e,\nu} \sim 4.5 \times 10^3(\Omega h^2)^{\frac{1}{4}}\ \phi^{-3}$ and (f_b)early era $\sim 6 \times 10^2$ $h^2\ \phi^9$.

on average share the electrons fermi energy. The cross-section for gamma ray energy degredation should also be high, a target density of some $n_e(E_\gamma/E_F)$ (E_γ gamma ray energy, E_F target sea fermi energy) will then ensure the photons very rapidly deposit their energy as heat in the electron and muon seas. Heating of the neutrino seas cannot occur by this route and photodissociation of any extant nucleii is highly unlikely. The annihilation photons may be sufficiently energetic to push electrons or muons well above their fermi energies. In such a case the energetic massive fermions will see a large density of targets and the neutrinos may be produced by reaction (71). The rate of this reaction compared to the expansion rate is not very large at onset and falls very rapidly with the massive lepton fermi levels (as p_F^5 where p_F is the fermi momentum of muons and electrons - they are equal due to charge balance). This constitutes our justification for the assumption of smooth stacking of the energy levels in the neutrino seas since the cascade is driven by the expansion opening up neutrino energy levels to the muon decay (86) whilst the relatively rising neutrino fermi levels ensure that (86) can only just proceed. That is to say, that at each R we have well defined number and energy densities of the changing lepton seas and smooth stacking is confirmed. The thermal energy itself will produce superheating of the matter and photons and the decay (86) will convert it to neutrino fermi energy, whilst fast coulomb interactions should ensure equilibrium among the electrically charged particles.

Detailed calculations for this era present a considerable theoretical task. One cannot proceed and calculate the rates for various processes in a naive way since the leptons are not free. Instead one must return to the matrix elements governing the interactions in question and calculate cross-sections (or $<\sigma n>$) while tending to the fermi blocking factors. Rapidly changing fermi levels, the influence of the muon mass and the existence of charged and neutral currents will further compound the difficulties. This last for example suggests different rates for the scatterings $e^- + \nu_e \rightarrow e^- + \nu_e$ and $\bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + \bar{e}$ which might have important effects near the end of the cascade. Presumably one should revert to the type of approach discussed for relativistic non-equilibrium thermodynamics in Sec. 6, Ch. 2 however one needs first to show in a rigorous way that the assumption of smooth stacking is correct. There is of course a negative feedback mechanism associated with the growing neutrino fermi levels which ensures a "quasi-equilibrium" configuration (i.e. an extended cascade) however the amount of energy which ends up as photons depends crucially on the actual route by which the neutrinos are produced. There is also a further complication which our naive analysis has so far ignored. Fermi blocking of exit channels in Eq. (86) will alter the lifetime of the muon from its free lifetime $\sim 10^{-6}$ sec. The cascade may start later than (88) and end later than

(89), this may have relevance to light element production to be discussed below. Clearly detailed calculations though necessary will be hard won we will therefore leave them for future work.

In spite of these difficulties one feature of the model as a producer of the black body background is a considerable improvement over other photon producing models. As long as the photon radiation is produced with an equivalent temperature above that for electron-positron pair production the problem with the photon spectrum's black body character is resolved. Rapid pair production interactions should rapidly thermalize the spectrum (if it is not already black body).

(iv) Light Element Formation

We assume here that some appropriately described cascade produces the present black body radiation. The very early era, where as we will show the light elements are formed, is then characterized by the low values of f_b given in Table 1. We will also assume that the f_b is not too low so that degeneracy effects will not be important for nucleons. This will be the case if the degeneracy parameter for the leptons is not too large. Since the leptons dominate the entropy in the very early era with relativistic fermi energies f_e is constant and so is f_b. The light elements are formed in a way which is intriguingly in contrast to the standard model's mechanism, it may be however that the following comments are of academic interest only since as we will show any extant light elements will be disrupted during the adiabatic cascade (albeit only "just disrupted"). Other cascade models may well allow for their survival.

Consider the neutron/proton balance for the reaction

$$n + \nu_e \leftrightarrow p^+ + e^- . \tag{95}$$

During the very early equilibrium between the four lepton seas the neutron to proton ratio is constrained to one since the chemical potentials of the electrons and their neutrinos are equal (77) and the rates are much faster than the expansion (84). As the temperature falls below 10^{13}K the electron on the right of (95) must find sufficient energy from thermal sources (non-degenerate and non-relativistic nucleons) to create the heavier neutron and supply enough energy to the neutrino to reach its fermi surface. When the neutron-proton rest energy difference (~ 1.293 mev) becomes greater than this available thermal energy (~ 1-2 kT) neutrons can no longer be produced. This corresponds to a freeze-in temperature of $T_f \sim (.75\text{-}1.5) \times 10^{10}$°K.

Consider now the light element building reactions discussed in Sec. 3. None of these reqctions (43, 47-50) involve leptons so the degenerate seas will not affect their rates. Ignoring mass fractions the slowest of these rates is that for formation of deuterium (44) so that the deviant model ratio of cosmological to reaction times (per free neutron) is, using (83)

$$t\tau_d^{-1} \simeq \frac{10^2 T X_p}{f_b \lambda^2} \;. \tag{96}$$

The deuterium abundance should thus appear in its equilibrium abundance for

$$f_b \lambda^2 << 10^{12} \tag{97}$$

when $T \sim 10^{10}$. This constraint is satisfied by all possibilities listed in Table 1.

In the standard model, deuterium equilibrium also occurs at these temperatures. However its low binding energy and the high photon to baryon ratio prevents its density from becoming sufficiently high to induce the production of the higher massed nuclides. Such a bottleneck does not occur here since X_d, as given by the mass action equation (46), is increased by a factor $\sim 10^{7-9}/f_b$ over the value at corresponding temperatures for the standard model. Since (44) is the slowest of the light element building rates the light elements must appear in their equilibrium densities (we ignore any heavy element production) until the temperature falls to T_f and the abundances are frozen-in. Subsequently all free neutrons are destroyed as it becomes thermodynamically favourable for (87) to proceed to the right. Such a scenario for nucleosynthesis is just the reverse of the standard model's.

Defining

$$\eta = \lambda_p - x_p, \qquad \delta = \lambda_p - \lambda_n \tag{98}$$

the degeneracy parameters for the equilibrium reactions (43, 47-50) are

$$\lambda_d = 2\lambda_p - \delta, \qquad \lambda_t = 3\lambda_p - 2\delta,$$

$$\lambda_{He^3} = 3\lambda_p - \delta, \quad \lambda_{He^4} = 4\lambda_p - 2\delta. \tag{99}$$

All the elements will have a near Maxwellian distribution and close to the freeze-in temperature we have to a good approximation

$$4\zeta(3) f_b^{-1} = \sum_i g_i A_i \sqrt{\frac{\pi}{2}}\, x_i^{3/2}\, e^{\lambda_i - x_i} . \tag{100}$$

Adopting the values $g_{He^4} = 1$, $g_p = g_n = g_t = g_{He^3} = 2$, $g_d = 3$; dividing through (100) by $g_p\sqrt{\pi/2}\, x_p^{3/2}$ and using the relations in Eq. (99) we obtain on writing $T_{10} = T/10^{10}$

$$\exp(-9.837 + \ln T_{10} - \ln f_b) = \exp(\eta) + \exp(\eta + \delta - 1.50\, T_{10}^{-1})$$
$$+ 6.2^{\frac{1}{2}} \exp(2\eta - \delta + 1.08\, T_{10}^{-1}) + 9.3^{\frac{1}{2}} \exp(3\eta - 2\delta + 6.84\, T_{10}^{-1})$$
$$+ 9.3^{\frac{1}{2}} \exp(3\eta - \delta + 7.46\, T_{10}^{-1}) + 16 \exp(4\eta - 2\delta + 29.83\, T_{10}^{-1}) . \tag{101}$$

We have used respectively on the R.H.S.; p, n, d, t, He^3, He^4 and we have used the values (in mev/c): $m_p - m_q = 1.294$, $2m_p - m_d = .9312$, $3m_p - m_t = 5.895$, $3m_p - m_{He^3} = 6.224$, $4m_p - m_{He^4} = 25.7094$.

The precise value of δ is in some doubt but since it equals $\lambda_{\nu_e} - \lambda_{e^-}$ it can hardly be greater in magnitude than one. Detailed calculations using the weak interaction rates is needed to determine both this number and T_f in terms of the model defining parameters $\{f_b, \lambda\}$. The light element abundances are then determined through (101) evaluated at the freeze-in and can be written symbolically as $X_i = f_i^n(f_b, \lambda)$. We shall not attempt such calculations here however, intead we treat δ, T_f as parameters and content ourselves with showing that "reasonable" abundances are possible for "reasonable" values of δ, T_f, and f_b. The degeneracy parameter itself will play no direct role in the abundances equation (101).

In the accompanying figures we plot the frozen-in abundances under the assumptions that none of the light elements are disrupted, that the mass gaps at A = 5 and A = 8 prevent the formation of higher massed nuclides and that there is no post-freez-in addition due to subsequent neutron inclusion. The last may not be the case if the rate of deuterium formation is sufficiently fast to absorb a number of neutrons before they can be protonized through (87).

In Figs. (1-6) we display the abundances as a function of the temperature for the three values $\delta = 1, 0, -1$ and $f_b = 1, 20$. For reasons of presentation we have taken $\delta(T) = \delta = \text{const.}$, this is not particularly relevant however since it is only its value at the freeze-in which is important. In Fig. 7 we indicate how f_b is constrained by the limits $-1 \leq \delta \leq 1$ and $.75 \leq T_{10} \leq 1.5$ if the models are to produce reasonable amounts of He^4, say $\sim .25$. With $\delta = 0$ we must have $.1 \leq f_b \leq 50$, the constraints are more restricting on the upper f_b limit for $\delta = +1$ and less restricting for $\delta = -1$. We must reiterate

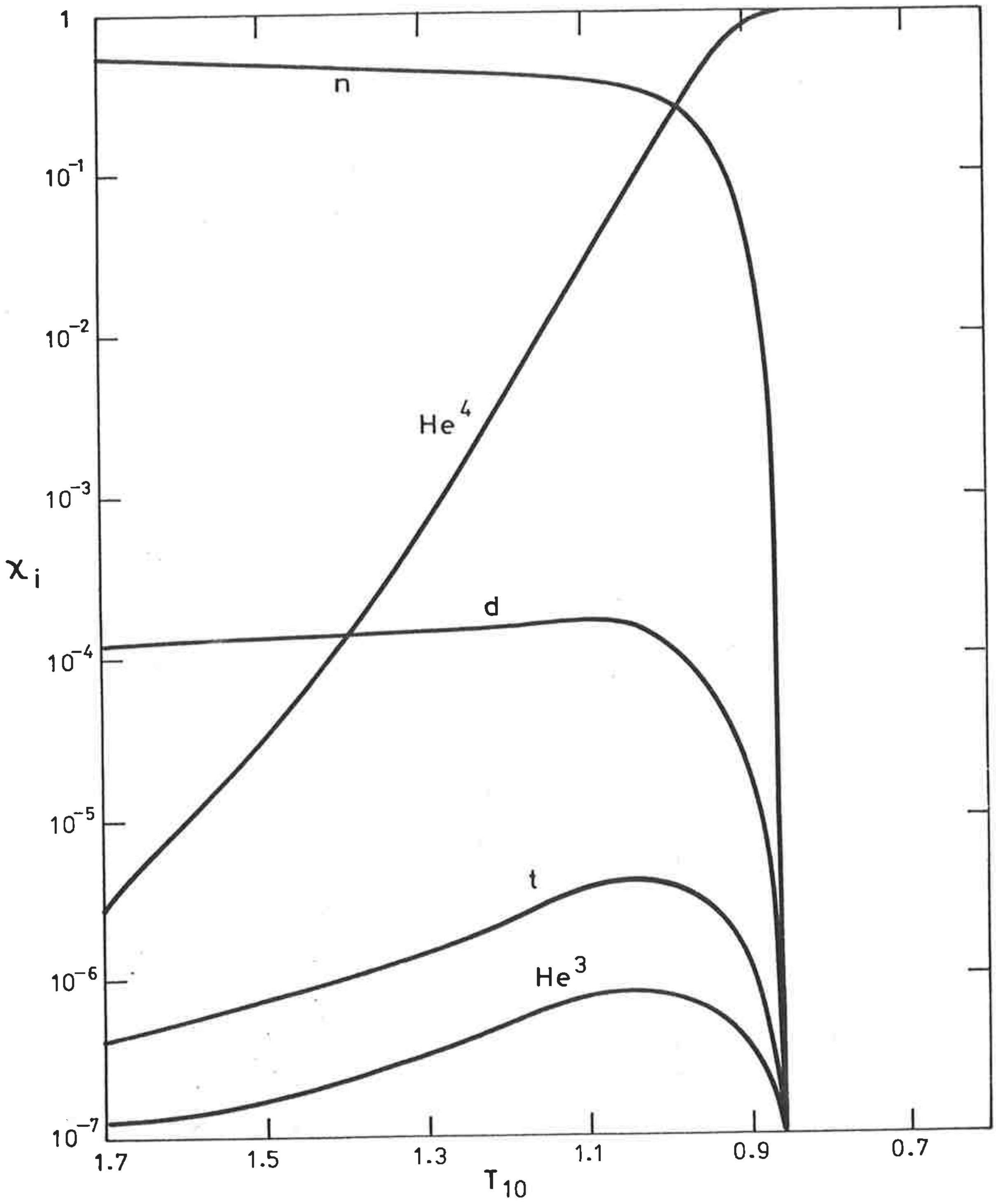

Fig. 1 Abundances for f_b =1.0 , δ = +1.0

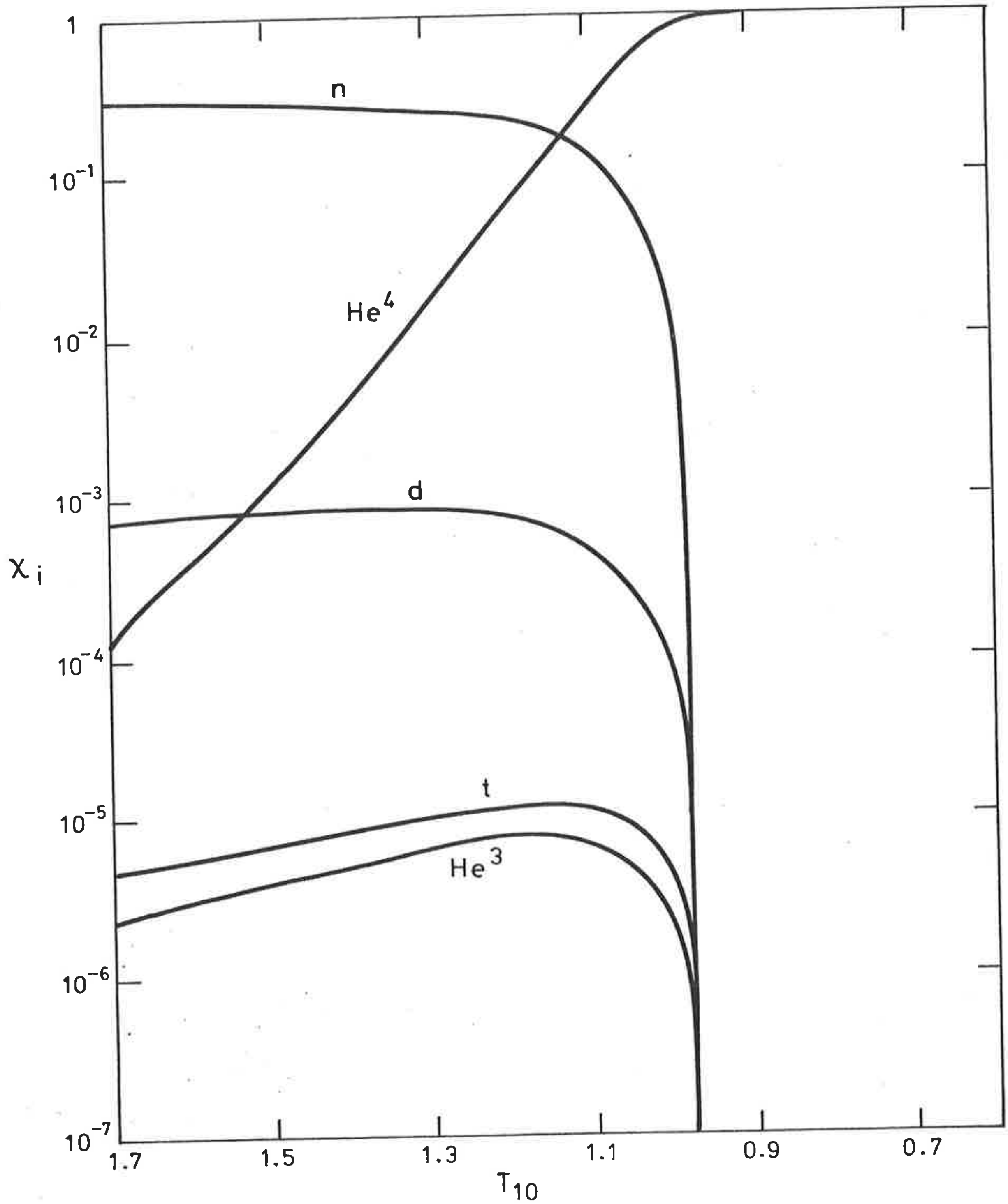

Fig. 2 Abundances for $f_b = 1.0$, $\delta = 0$

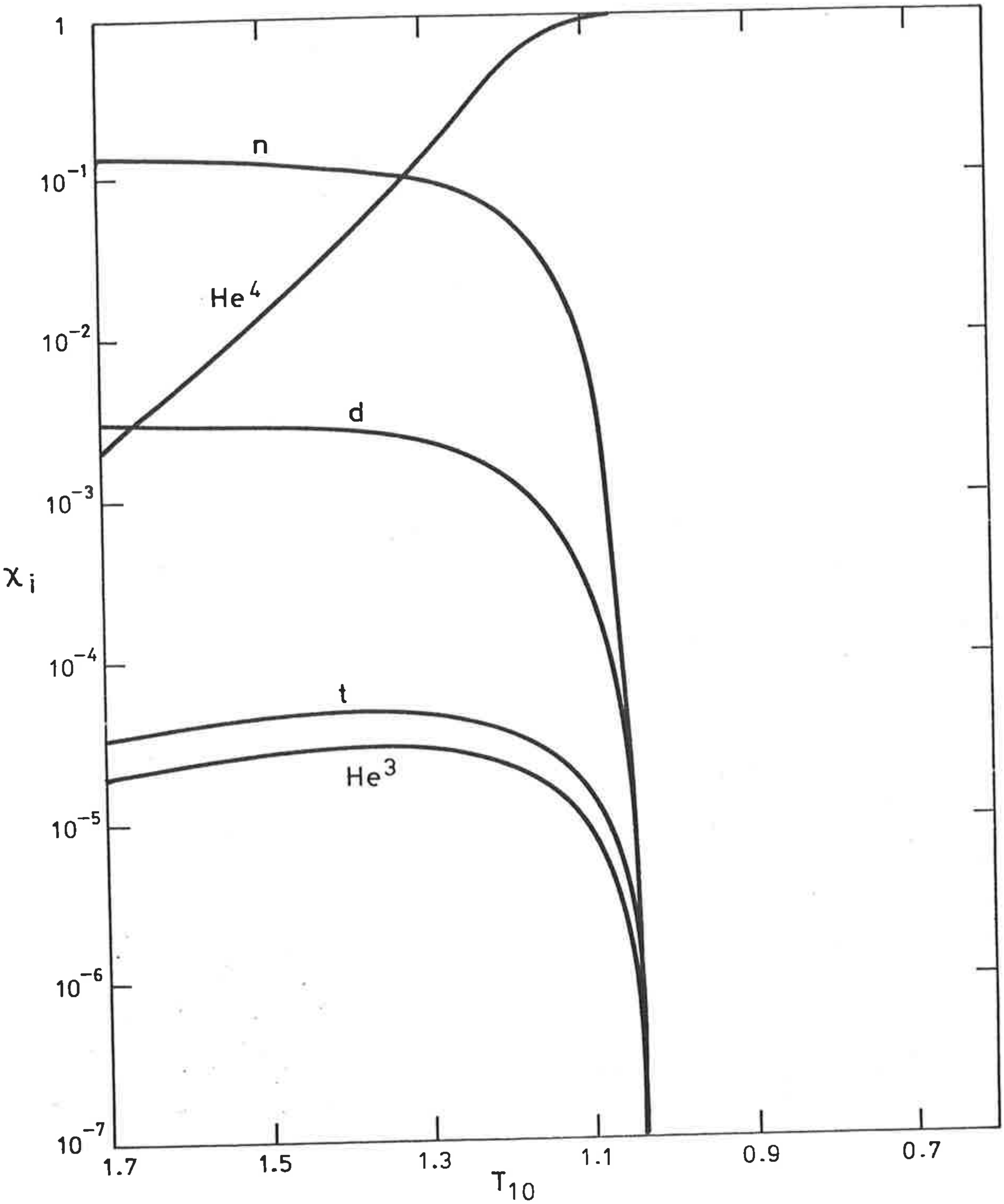

Fig. 3 Abundances for $f_b = 1.0$, $\delta = -1.0$

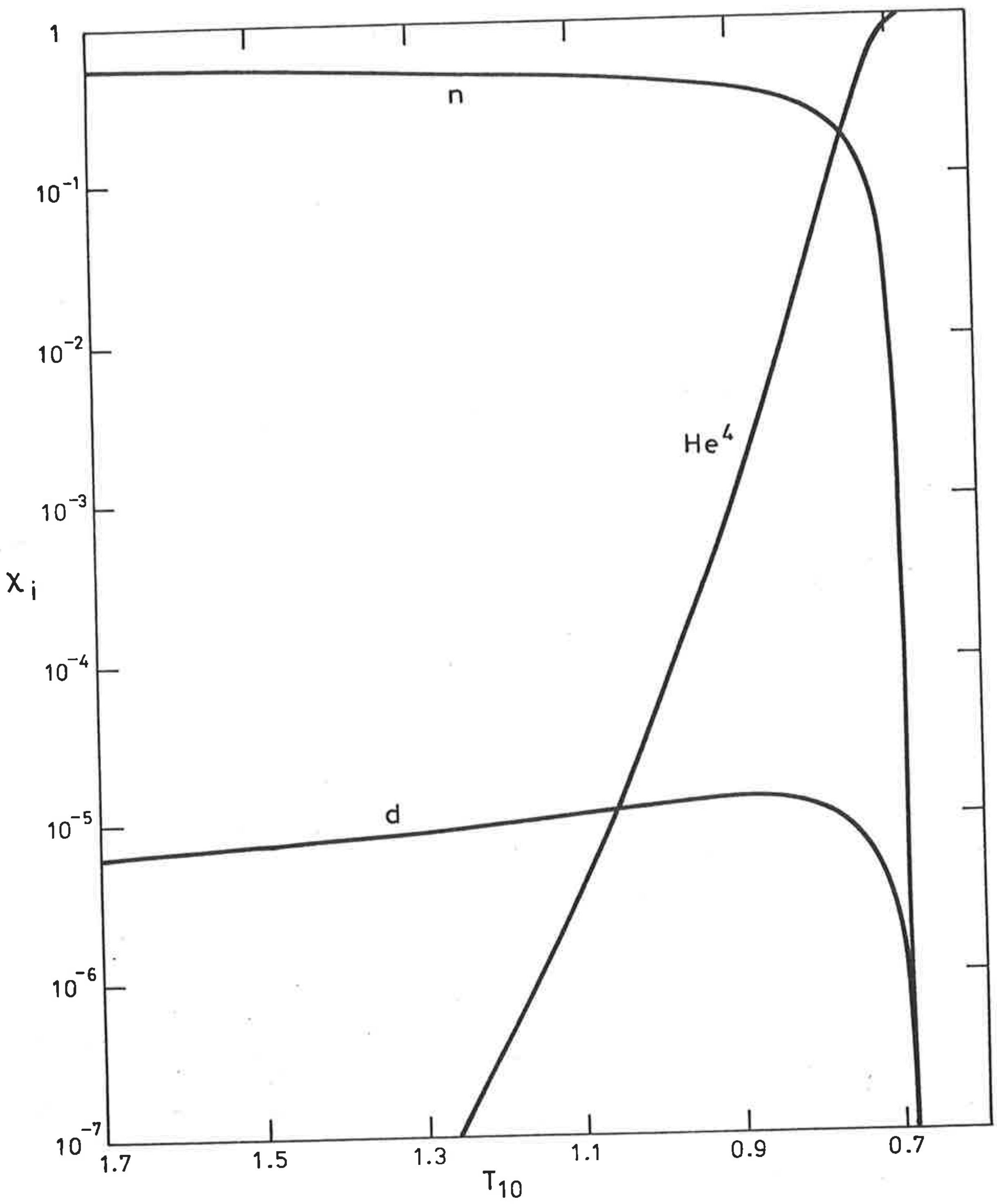

Fig. 4 Abundances for $f_b = 20$, $\delta = +1.0$

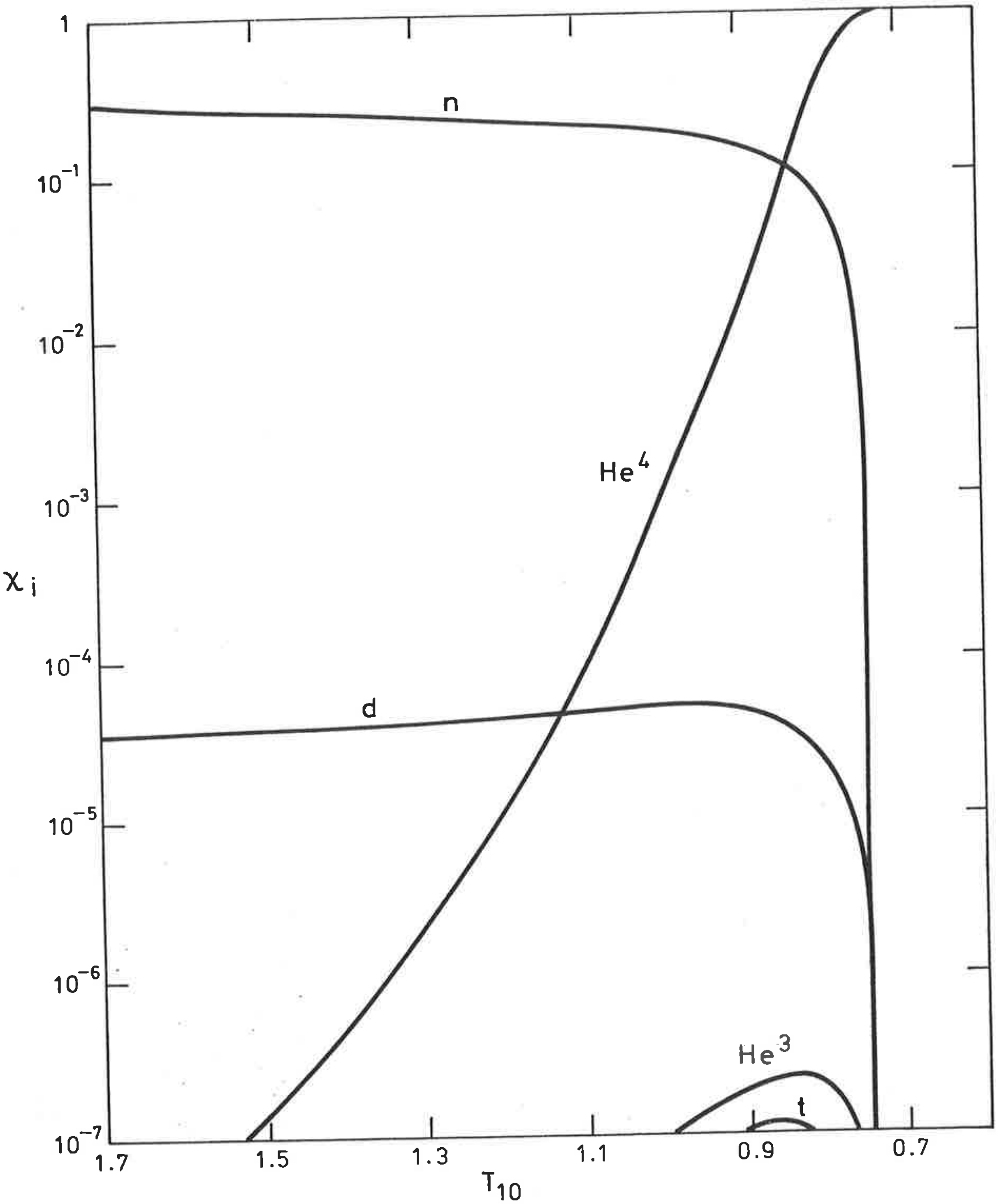

Fig. 5 Abundances for $f_b = 20$, $\delta = 0$

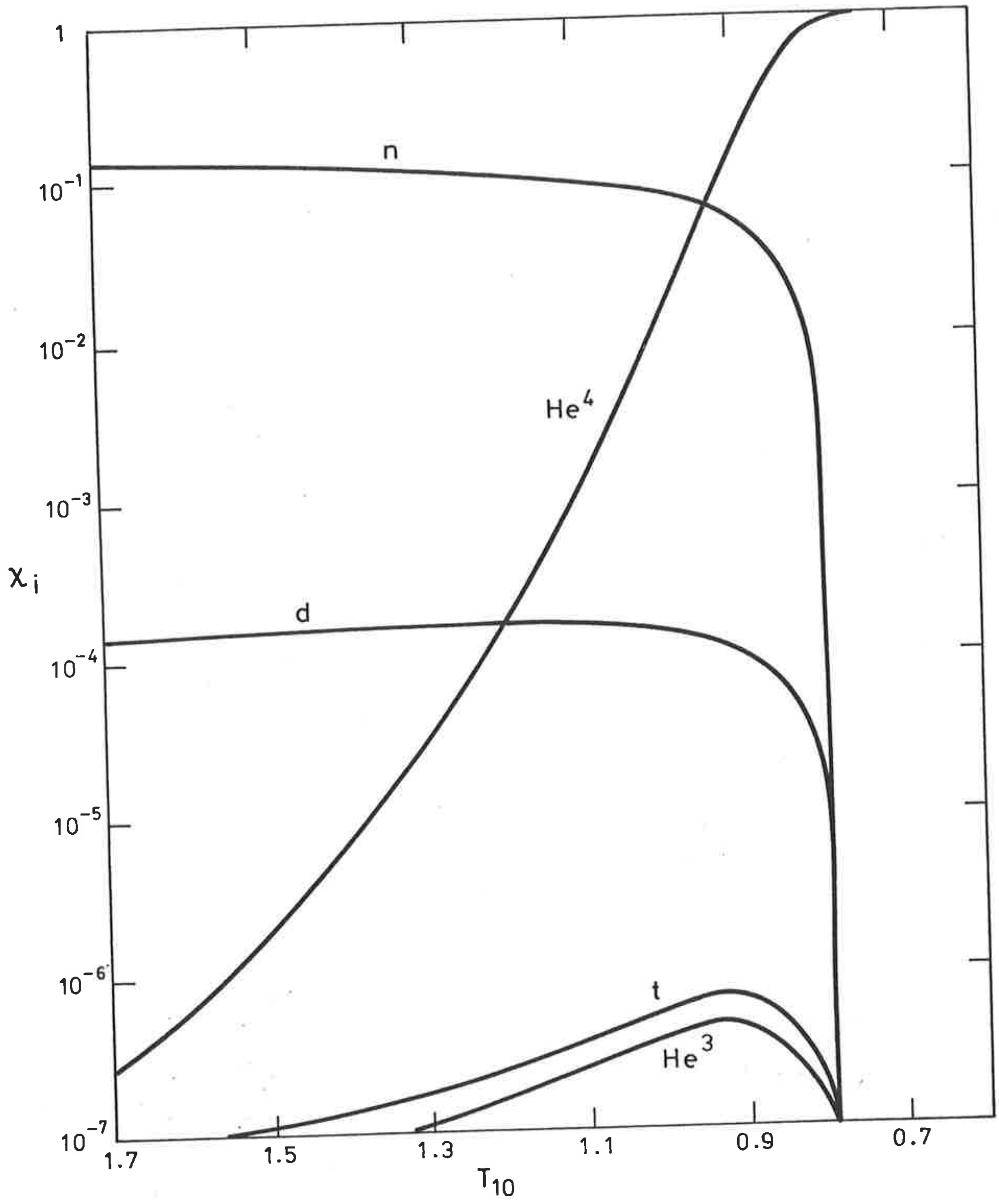

Fig. 6 Abundances for $f_b = 20$, $\delta = -1.0$

however that it is the calculable numbers δ, T_f which determine whether a specific ($\{f_b,\lambda\}$) model is viable in terms of its He^4 production. Moreover as the exponential terms in (101) suggest the value of f_b for the successful model will be given rather accurately since He^4 production rapidly goes to 100%.

Actually the real situation will be somewhat more complicated than intimated above. After freeze-in the light elements will continue to react with each other and some redistribution will take place. Furthermore the increased density compared to the standard model may allow more heavy element production than in that model. Any heavy elements that are produced will be affected by the lepton seas since the rate for any reaction which is exothermic and includes leptons will be very fast. Interestingly most of these reactions, e.g.

$$B^8 + e^- \rightarrow B^8_e + \nu_e \qquad Q \sim 15 \text{ mev}, \tag{102}$$

will be neutronization reactions (see Wagoner et al. 1967 for a list) and may well be reversed by the pure neutrino seas resulting from the cascade.

The values of f_b that allow reasonable light elements should be consistent with the value of f_b implied by a specific value of λ which produces the black body background. It is obvious however that the value implied by the adiabatic cascade $f_b \sim 6 \times 10^2$ is too high for appreciable production. Now we have deliberately taken a worst case scenario here. The actual values of q_0, H_0 could be larger and T_0 smaller moreover the baryon density we have taken $1.2 \times 10^{-7}\ h^2$ is almost certainly a lower limit and the luminous matter density associated in galaxies (64) is uncertain to a factor ~ 3 (Tamman 1974) anyway. We have thus required of our model that it produce the rather large photon to baryon ratio $\sim 4 \times 10^9$; for a smaller present ratio the very early era ratio consistent with the adiabatic cascade is also smaller. Thus for a present temperature of $T_0 = 2.7$ and a nucleon density three times that given in (64) the f_b value for the adiabatic cascade comes well within the ranges given in Fig. 7.

These considerations however are rather academic unless the light elements so produced survive the cascade. As we will now show for the simple adiabatic cascade above this is not the case.

Feasible neutrino induced disruption reactions for the light elements are

$$\nu_e + He^4 \rightarrow d + p + p + e^-, \qquad Q = -25.29 \text{ mev}, \tag{103}$$

$$\nu_e + He^3 \rightarrow 3p + e^-, \qquad Q \sim -6.9 \text{ mev}, \tag{104}$$

$$\nu_e + t \rightarrow d + p + e^-, \qquad Q \sim -5.5 \text{ mev}, \tag{105}$$

$$\nu_e + d \rightarrow p + p + e^-, \qquad Q \sim -1.47 \text{ mev}. \tag{106}$$

The third reaction will be in competition with the protonization of tritium to He^3

$$\nu_e + t \rightarrow He^3 + e^- \qquad Q \sim .01 \text{ mev}, \tag{107}$$

protonization of He^4 to Li^4 will not occur since Li^4 does not exist (Chiu 1968).

Now actual disruption will depend on two factors. Whether, during the cascade, the growing energy difference between the fermi energies of the neutrino and electron seas is sufficient to allow the above endothermic reactions to proceed and whether the rates of the reactions at the first such permissable energy are faster than the expansion rate. For reactions proceeding with the weak cross-section (81) we can adjust (84) to give

$$t\tau^{-1} \simeq 3 \times 10^{-4} \mu_\nu^3 \cdot \frac{\mu_d}{\mu_\nu} \cdot \frac{v}{c} \tag{108}$$

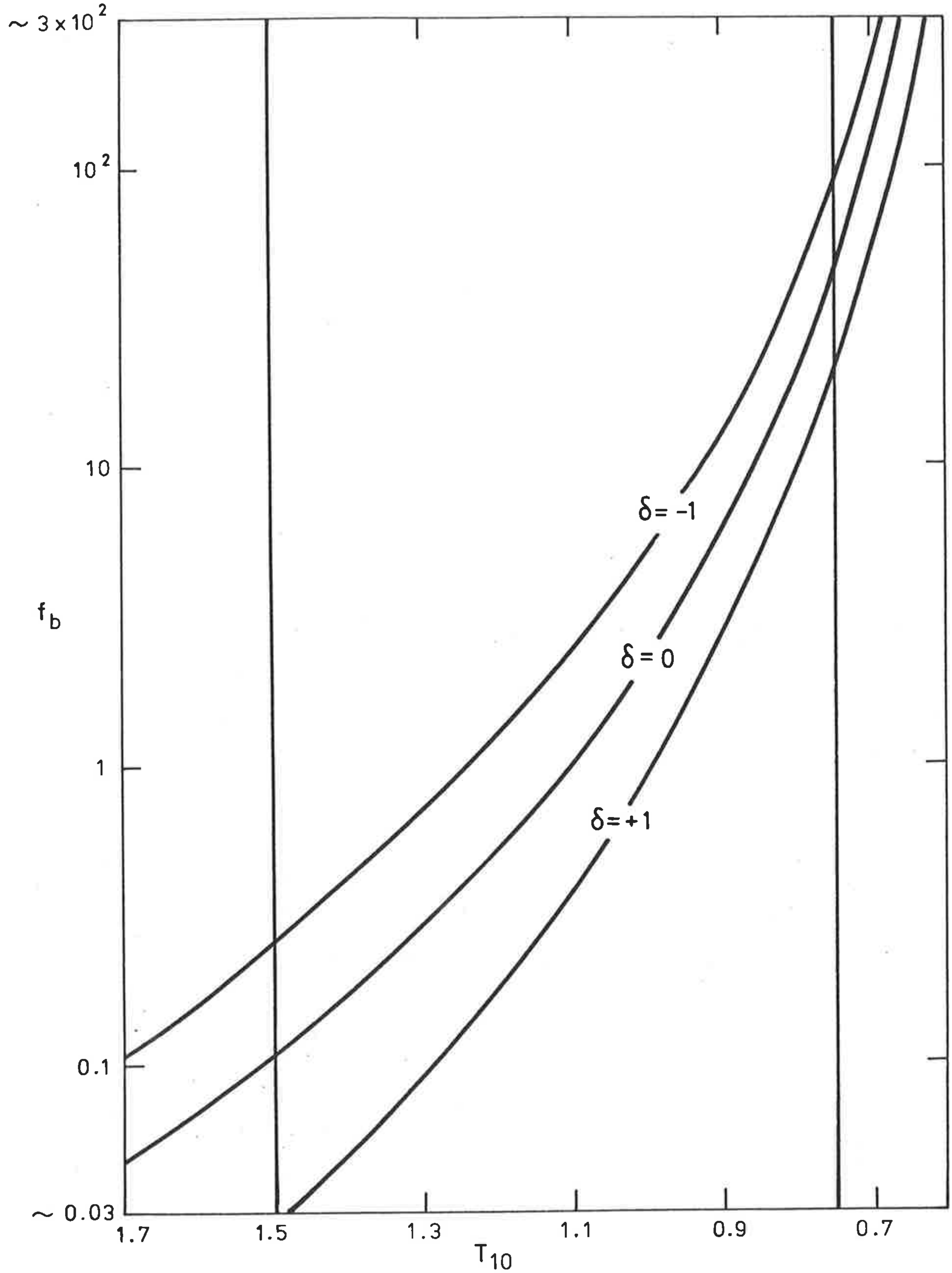

Fig. 7 f_b for models which give $X_{He^4} \simeq 0.25$ versus the freeze-in temperature.

where μ_ν is the fermi energy of the neutrinos in mev, μ_d is the energy difference between neutrino and electron fermi energies in mev, v is the velocity of the nuclide concerned ($v \simeq c/x^{\frac{1}{2}}$) and τ^{-1} is the number of reactions per nuclide per unit time. At onset $\mu_\nu \sim 60$ mev $v/c \sim 10^{-5/2}$, $\mu_d \sim kT$ so that for $\lambda \sim 10^4$ we have $t\tau^{-1} \sim 10^{-5}$ and none of the reactions (103-107) will occur. The rates rise with μ_d but in the case of deuterium for example reaction (106) becomes energetically feasible well before its rate of disruption becomes appreciable. A sensible measure of the survival of light elements is therefore the evaluation of (108) at cascade end $\mu_\nu \sim \mu_d$, $\mu_\nu \sim 53$ mev. Assuming the non-neutrino fluid has then the microwave background temperature 2.6×10^{10}°K we have

$$t\tau^{-1} \simeq 24/A \tag{109}$$

where A is the mass number of the nuclide concerned. On the basis of this equation we can expect no light elements produced in the very early era to survive neutrino induced disruption during and after the cascade. Actually a more accurate calculation would give a smaller coefficient in (109); since most matter is converted to radiation the expansion is speeded up a factor $\sim (3^{1/4} / 3^{1/3})^2$ times (83), appropriate averaging of the cross-section and the density contributes a factor $\sim 3/5$ and removing the effect of dynamically unfeasible reactions for He^4 disruption contributes $\sim 7/8$. Including an increase of $\sqrt{2}$ associated with thermal nuclide speed the coefficient drops to ~ 14. Even though Eq. (109) then gives $t\tau^{-1} \sim 7$ for He^4 and falls off as $T^{-7/2}$ no appreciable quantities of He^4 would survive. Primordial light element production thus seems inconsistent with the simple adiabatic cascade discussed above.

Alternatives to that cascade scenario must be explored through detailed calculations. Indeed it may turn out that the slower decay time for the muons extends the cascade in such a way that (109) is much smaller. The model might then produce sufficient He^4 as well as the black body background. Such speculations aside however, the twin features of a matter-photon fluid heating cascade and the peculiar "histroy" of the light elements suggest the model deserves further attention.

The discussion above is also interesting in relation to another non-zero lepton number model considered in Part (i) of Sec. 4. In that model the standard treatment of nucleosynthesis is modified by the introduction of varying densities of the two lower lepton types (e,μ). We have already noted that pion condensation after the hadron era ($> 10^{12}$°K) is a general feature of these models unless the lepton charge in the muon leptons is actually distributed across higher lepton types. Interestingly if the latter is not the case there may well occur a lepton cascade as the high number densities of massive leptons (say antimuons) lose their lepton numbers to the corresponding neutrinos. Presumably the pion condensate once necessary to balance electric charge will also be destroyed during this period. Again we would expect most change to occur near $x_4 \sim \bar{\lambda}_4$. In the example considered in Sec. 4, $\bar{\lambda}_4 \sim 28$ in the early lepton era and f_μ is a constant during expansion. However if the enormous energies liberated by the loss of all $\{\mu^+, \pi^-\}$ heat the fluid the late lepton era f_μ value may be much larger. For the example mentioned the late lepton era has $\bar{\lambda}_6 \sim 40$, $f_\mu^{-1} \sim 4.5 \times 10^3$; early universe parameters consistent with these values may be much larger.

Limitations of space prevent detailed considerations of such models, we will instead return to them in future work.

(v) Criticism and Discussion

The foremost point, for any model similar to that above, must be whether such neutrino seas are detectable. Weinberg (1972) has reviewed the situation for detection of neutrino seas in general and it seems that the most stringent limits on the fermi levels come from the cosmological density itself, that is Eq. (68). More recently Lucey (1976) has considered various possibilities; apparently the failure of the rates of the decays $K_L^o \rightarrow 2\pi$, $K_S^o \rightarrow 2\pi$ to show degeneracy effects indicates seas must have a fermi level below 10^{-2} ev. This of course does not violate (68). Stodolsky (1975) has also considered some ingenious experimental possibilities. In the context of conventional current-current V-A theory he considers the split in energy which occurs for two helicty states of an electron in motion through a degenerate neutrino background. The effect is proportional to neutrino density but is unfortunately very small $\sim 10^{-24}$ ev. Stodolsky also suggests using neutral currents and nucleons in a similar fashion. For our model however the spin-spin interactions with ν_e and $\bar{\nu}_\mu$ would presumably cancel out.

Astrophysical possibilities may in the end offer the best hope of detection. However for neutrinos with an energy given by Eq. (68) no known object is sufficiently dense to trap them nor would blocking factors cause any significant effects. Recent supernovae models (Schramm 1975) consider the effect of neutrino degeneracy which may well be related via neutral currents to the blow off (Freedman 1974). These neutrinos do not have a cosmological origin and their lower states are depopulated due to a "window" effect produced by the quadratic dependence of the cross-section on neutrino energy. The least energetic neutrinos which are trapped have an energy well above the fermi level of the cosmological sea and it is the slowness of neutrino downscatter (from ~10 mev) due to fermi inhibition which causes degeneracy of the supernovae sea. If however the neutrino cross-section does not fall off quadratically at low energies then there may well be some contemporary and evolutionary effects.

It may turn out that the seas are simply undetectable and can only be inferred through dynamical considerations. If they were to supply most of the dynamically missing "mass" then they would neatly solve the problems with that matter's undetectability. The latter crisis as we have seen has been the subject for some rather exotic solutions (e.g. heavy neutrinos, black holes). The pitfalls of some of these have been reviewed by Gott et al. (1974) and we refer the interested reader there. Presumably the degenerate seas also play a crucial role in determining the larger scale structure of the universe. The work of Mèszáros (1974) has some relevance here. From a study of fluctuation growth for point masses embedded in a smooth radiation fluid of greater energy density he concludes that clusters of galaxies could not form. However in that picture all fluctuations are in the matter and not the radiation fluid i.e. are isothermal and not adiabatic. A more realistic analysis would have to account for fluctuations within the sea densities. We will not even attempt to cover the complex issues involved here, however a couple of points are worth noting. Firstly the neutrino energy density falls as R^{-4} while the matter density falls as R^{-3}, thus for a density enhancement which does not completely participate in the Hubble flow, other things being equal, the neutrino density will build up relative to its surrounds faster than the matter density. Secondly the model may be more hospitable to the growth of fluctuations than the standard model since at comparable red shifts the model will be denser than the standard model for the same present density. Thus at recombination the degenerate seas model is some 1500 times denser than the comparable standard model and the gravitational potential within adiabatic fluctuations is presumably also greater. Notice that such degenerate seas confer a dynamical aspect to large scale structuring; structures formed at large red shifts may well lose stability as the neutrinos flow out.

The qualitative description we have given of the thoroughly deviant model above can hardly justify it as a serious contender as a cosmological model. In relation however to other photon producing models of the Rees or Layzer type photon production has a more cosmological character and problems associated with homogeneity, isotropy and the black body character of the radiation do not arise. Most importantly the peculiar history of the light elements for the adiabatic cascade and the smallness of (109) suggests that a truly detailed calculation might still provide solution to both helium and microwave radiation problems for a specific pair of early universe parameters $\{\lambda, f_b\}$. Accepting this hypothetical success for the moment we see that some difficulties immediately arise. At temperatures $\sim 10^{12}$ apart from an expanded hadron spectrum we should expect excitation of the weak interaction bosons. This is part of the larger question of what earlier epoch description is consistent with the model as discussed above. We shall treat this epoch in some detail in the next chapter. Presumably it is also the onus of such an investigation to explain the values of λ and f_b themselves. This brings up the difficulty of how whatever mechanism which stacks the electron and muon lepton seas resists stacking the higher lepton flavours τ .... A perhaps more pertinent question relates to fluctuations in the microwave radiation caused by the cascade itself. Photon production may turn out to be quite sensitive to primordial fluctuations within the seas as well as to fluctuations during the cascade itself.

Now if this section seems unwarrantably speculative let us note there may be good reason to believe that the earliest epochs are in some way intimately connected with degenerate neutrino seas. As we have discussed in Sec. 3, Ch. 3 the time in relativistic degenerate models depends only on the fermi energy. Thus taking Eq. (83) as typical for an early universe containing degenerate fermion seas we have

$$t \simeq 5\mu_f^{-2} \text{ sec} \tag{110}$$

where μ_f is in mev. Now if we ask for the time at which the characteristic size of the fermion $\sim h/p_f$ is just contained by the horizon $\sim ct$ we obtain

$$t \sim 10^{-44} \text{ sec.} \tag{111}$$

This is just the plack time, the size of the leptons is just the fundamental length $\sim 10^{-33}$ cm and their energy is just the planck energy $\sim 10^{20}$ Gev. There is then one "fundamental" lepton per horizon volume at the planck time.

Of course at these energies we must consider the consequences of the "full" lepton and hadron spectrums. We must also consider the relevance of the new grand unified theories wherein a single scheme incorporates strong, electromagnetic and weak interactions which will have equal strength at energies $\sim 10^{15-16}$ Gev. In this context for the thoroughly deviant model (with $\lambda \sim 10^4$ = const.) above we have the following coincidence. At the planck time all non-leptons have a characteristic energy $\sim \mu_5/\lambda \sim 10^{20}/10^4 \sim 10^{16}$Gev. This is just the grand unification energy. On this basis it would seem that any scheme that could give the parameters $\{\lambda, f_b\}$ is fundamentally one of quantum cosmology.

CHAPTER 6

THE HADRON ERA AND BEFORE

1. INTRODUCTION

In this chapter we will attempt to discuss more realistic models for cosmologies in their hadron eras. By this era we mean an epoch when thermal energies or densities are sufficient to open up the higher lepton and hadron spectrums. That is when the study of cosmology leaves the familiar low energy world of {p, n, e, γ} (or (u, d), (e^-, ν_e)) to consider the consequences of much higher energies, energies which are only phenomenologically known through high energy cosmic ray events, perhaps supernovae and the modern particle accelerators.

So far, apart from minor lapses we have kept away from the very profound and fundamental issues which are involved in cosmological theorizing and particle physics and which began our discussions in Ch. 1. However any discussion of the hadron era must try to do justice to the various theoretical and philosophical positions that characterize modern particle physics. Two distinct approaches will be particularly important here. The first as we have discussed in Ch. 5 would regard the particles ((u, d), (s, c), (t, b), (e^-, ν_e), (μ^-, ν_μ), (τ^-, ν_τ)) as fundamental entities out of which all other states are to be built, while the second would regard no state or being more fundamental than any other. The latter is part of a general physical philosophy called the bootstrap and bears some historical similarities to the philosophy of Leibnitz. The two different outlooks give two very different cosmologies as we will see.

The way in which we have approached cosmological model building since Ch. 2 has disguised a methodological difficulty. Instead of attempting to account in some self-consistent way for the forces and interactions extant at a certain temperature or density, we have taken the much easier route of postulating the types of particles at an epoch and considering the reactions between them. However there is no guarantee that such assumptions are feasible. Certainly we must revise them for energy regimes which change the conserved quantum numbers yet there may be no route to our assumed epoch once earlier epochs are better understood. This must be the case if some of the quantum numbers which characterize our hypothesized epoch are not a feature of the initial singularity, they are then the result of processes occurring during the expansion and their values are not free. We will discuss in this chapter one model which takes this idea to its extreme; the singularity is characterized by no quantum numbers (except perhaps electric charge) and the values of f_b, f_e etc. are derived by calculation. These are the so-called grand unified theories which violate baryon and lepton numbers. The theories, hailed by some as indicating the end to the particle physics theoretical programme, would attempt to give the entire chemical content of the universe and in particular would mesh with the standard model's (as revised for higher lepton flavours) description of the lepton era. We attend to them in the next section.

In the third section we consider a rather different approach, that of the statistical bootstrap. The bootstrap only applies to strongly interacting particles however after some discussion of the relevant literature we attempt to introduce non-zero lepton charges. We do this in a naive way (we merely add numbers of various lepton types to the hadronic fluid) but as we will see the result is far from trivial. In particular if leptons are as common as (net) baryons the consensus view of the hadron era is altered and leptons dominate the energy density for a period after the hadron barrier.

In the final section we round of the thesis with some discussion and observations.

2. QUARKS, GRAND UNIFIED THEORIES AND THE BOOTSTRAP

When considering the hadron era we begin to move outside the realm of contemporary particle physics. Already in Ch. 5 we have seen that a particular theoretical vision of higher lepton and quark spectroscopy is in mutual support with the standard big bang scenario. The two models may either stand or fall together. A simple list of this now "standard" particle description (Harari 1979b) - 3 families of quarks and leptons (two per family, three colours per quark), 12 guage bosons, 8 SU(3) gluons, 4 SU(2) × U(1) electroweak vector bosons - shows just how theoretical such a fundamentalist view has become; apart from the photon only the leptons have been unequivocally seen. In the near future accelerators may reach energies capable of producing the vector bosons but since coloured states have never (yet) been observed coloured particles must somehow be confined. Whether this confinement is total or only partial (perhaps the gluons are massive) will have divergent cosmological consequences.

If quarks are confined no matter what the average particle energies within the cosmological fluid then we need not even consider them in cosmological argument. Their epiphenomenae the hadrons are never ionized into constituents and the multitudes of hadronic resonances will dominate the hadronic equations of state. A theory which might describe these features is discussed in the next section. On the other hand if quarks are not confined completely and are fundamental entities then for sufficiently high temperatures (or when hadron "bags" overlap) the universe will be filled with a number of types of black body radiations given by just the particles of the standard description.

It is interesting that this latter view gives rise to severe constraints on the existence of quarks. Zel'dovich and Pikelner (1965) were the first to note that if quarks are allowed their freedom at some early epoch then imperfect inclusion in hadrons and imperfect quark pair annihilation during subsequent expansion demands that the presently surviving quarks should be as common as gold atoms $n_q/n_p \sim 10^{-10}$. If it is accepted that not all matter outside stars has been cooked and ejected then allowing for both fusion of quarks within stars (and perhaps quasars) and a low primordial element planetary abundance of $\sim 1\%$, the predicted quark abundance is very much greater than the measured limits of $\sim 10^{-20}$ quarks per nucleon (see LaRue et al. 1979 and references therein).

In a recent paper Wagoner and Steigman (1979) have argued that these calculations do not apply for free quark masses $\gtrsim 10$ Gev/c^2. The reasoning has to do with the quark-hadron transition. Since the mass of quarks within nucleons is smaller than 1 Gev a quark fluid at such thermal energies is relativistic and the total quark density is comparable to photon density. During the transition the number of quarks which can interact with the potentially free high energy quarks drops suddenly due to inclusion in hadrons. The gluon field is then unscreened and due to the proportionality of the quark-quark potential to separation the quarks acquire a large mass as they actually become free. Since the free quark density contains an $\exp(-mc^2/kT)$ term it turns out for free masses $\gtrsim 10$ Gev the quarks freeze out during the transition and their density is then sufficiently low to be in accord with observation.

(i) Grand Unified Theories

These theories offer the possibility of unifying the three basic elementary particle forces strong, electromagnetic and weak as different manifestations of the same fundamental interaction. An interaction which involves a single coupling constant, the fine structure constant (Georgi and Glashow 1974). The theories are motivated by the success of the Weinberg-Salaam $SU(2) \times U(1)$ guage theory of weak and electromagnetic interactions and of $SU(3)_c$ colour gauge theory of the strong interactions.

The pattern of generations of quarks and leptons within the standard prescription has two striking independent features. Within each generation the pattern of quarks is very similar to the pattern of leptons and some profound connection between them is suggested. Moreover each generation seems similar to the others so that the old $e - \mu$ problem is generalized to a problem of apparently redundant generations of quarks and leptons which differ only through their masses. Such relationships suggest that both quarks and leptons belong to the same multiplier of a large guage group which contains both $SU(2) \times U(1)$ and $SU(3)_c$ and which therefore unifies electroweak and strong interactions.

One suggestion for such a grand unification group is the simple group SU(5) proposed by Georgi and Glashow (1974). As a simple group the theory will contain only one coupling constant and the group has a maximal subgroup structure of $SU(2) \times U(1) \times SU(3)_c$ so that the Weinberg-Salaam model and quantum chromodynamics are naturally incorporated. If experiments force us beyond $SU(2) \times U(1)$ however then some larger group will be needed. Some possibilities are SO(10) and the exceptional Lie groups E_6 and E_7.

Whatever the unifying group the grand unified theories have in common a number of features of considerable cosmological significance.

Of particular relevance is the existence of a superheavy mass scale at which energies the coupling strength of the three interactions become the same. The convergence is brought about through the (predicted to be continuing) logarithmic decrease of the effective QCD coupling constant and the mass scale can be estimated as $\simeq 10^{15}$ Gev/c^2 (see Appelquist et al. 1978 for a general review; the actual mass scale depends on the grand unified theory, the Weinberg angle etc.). A similar mass scale comes from another independent argument. Since quarks and leptons are placed together in single representations of the unifying gauge group there will be gauge bosons connecting leptons and quarks which lead to a breakdown in lepton and baryon conservation. A lower limit $\sim 10^{30}$ years on the measured proton lifetime (Learned et al. 1979) thus suggests a grand unification mass of $\gtrsim 10^{15}$ Gev/c^2. It may be possible to achieve grand unification at lower energies. However the so-called Weinberg mixing angle which is a measure of the relative strengths of weak and electromagnetic interactions and which is determined by embedding SU(2) $\times$ U(1) in a larger group appears to be too high (Harari 1979b).

Notice that a grand unification mass of $\sim 10^{15}$ Gev/c^2 is well below the Planck mass $\sim 10^{19}$ Gev/c^2 and that this is necessary for "grand unified theories" without gravitation to be possible. Further, for the standard model as discussed in the last chapter the grand unification energies of 10^{15} Gev occur at $T \sim 10^{28}$°K and $t \sim 10^{-35}$ sec which is well after the end of the Planck era $t \sim 10^{-43}$, $T \sim 10^{32}$°K.

The application of these ideas to cosmology has recently received considerable attention (e.g. Weinberg 1979, Ellis et al. 1979, Ellis and Steigman 1979). Such work attempts a natural explanation of the baryon asymmetry, a cosmological feature seemingly made compulsory by the lack of observable antimatter. As well it would attempt to derive the size of the photon to baryon ratio. Interestingly a successful scheme requires three unusual features. Not only must baryon number be violated, say through the decay of the so-called x bosons (mass ~ grand unification mass), but decays must violate CP (and C) invariance and occur when B violating equilibration is imperfect. This last is necessary since perfect B violating eqüilibrium cannot produce any net baryon number (Weinberg 1979) whilst CP non-conservation (hitherto seen only in the $K^0_,\bar{K}^0$ system) ensures T violation and consequently net baryon numbers in the decay products.

Ellis et al. (1979) have considered a baryon generating scenario for SU(5). At temperatures between 10^{19}-10^{15} Gev/K it is expected that the rates of all weak, strong, and electromagnetic forces are in non-equilibrium. However when baryon number violating forces reach equilibrium below 10^{15} Gev any Planck era (à priori) or previously generated baryon excess will be destroyed. Somewhat later these B violating forces again go out of equilibrium, the CP violating decays produce a baryon asymmetry and the final ratio of total entropy to net baryon number gives a photon to baryon ratio.

Ellis et al. calculate the ratio for SU(5) as 10^{10}, rather higher than the expected ratio ~ 10^8 for the standard model. The generated baryon asymmetry is low due to the fact that at lowest order the CP violation is 8th order in the Yukawa coupling constant. It is too early to say that the attempt using SU(5) has failed however it does suggest that other unifying groups should be investigated. Ignatiev et al. (1979) have studied SO(10) which seems to more easily give the correct photon to baryon ratio. This comparitive (and tentative) success of SO(10) over SU(5) highlights a crucial issue for cosmological theorizing and terrestrial physics. As we have seen in Ch. 5, helium production within the standard model is consistent with only a narrow range of feasible particle-theoretical models. It may turn out that the constraint imposed by producing just the right amount of baryon asymmetry severely limits cosmologically viable grand unified theories. Furthermore the very early universe may well be the only "laboratory" capable of testing such theories (an upper limit to the proton life-time or discovery of the elusive Higg's boson would confirm such theories are indicated but would hardly test them). It would be appealing to many if in fact only one such theory could produce the correct photon to baryon ratio. The so-called "end" to the particle physics theoretical programme would then be consummated by the removal of that nagging cosmological difficulty - why f_b takes a value around 10^8.

Such speculations aside, it is quite clear that the consequences of unified theories for cosmology are only beginning to be studied. It is worthwhile noting therefore the ways in which any such analyses may be unsound. The non-conservations discussed above arise from the spontaneous symmetry breaking of the underlying gauge theories. There is thus, presumably, no way to predict the direction of the breaking nor imbed that direction à priori in the initial conditions. That the world is baryonic (or more appropriately kaonic) must be added to such theories as an ad hoc à priori. If this problem relates to the global sign of the asymmetry even more problematic is how there can be a global asymmetry at all. Since the breaking is spontaneous one cannot expect regions which have had no causual contact since the singularity to break in the same direction. In fact there should be no larger than horizon correlations in baryonic sign and one would expect a random pattern of breaking. As Brown and Stecker (1979) have recently noted such a situation leads more naturally to a baryon symmetric cosmology as opposed to an asymmetric cosmology. The complicated phase separating mechanisms proposed for the Omne's model (1972) are thus not necessary.

This point can of course be taken the other way. The absence of any appreciable observable quantities of antimatter is contrary evidence to any spontaneous breaking theory.

These comments should not be taken as too damning. The origin of across horizon homogeneity is perhaps the most perplexing of cosmological problems. Its resolution might provide the key to the random baryonic phases. However the spontaneous breaking occurs well after the Planck era. The deviant model suggested in Sec. 5, Ch. 5 is of interest here since baryon and lepton number generation would occur just after the Planck era.

More work needs to be done on the role of lepton numbers in these approaches. The lepton asymmetry is just as much in need of explanation as the baryon asymmetry. The situation is rather simple for SU(5) since while both B and L are violated B - L is conserved. Dimopoulos and Feinberg (1979) have discussed processes which violate B - L and even just L. If it does turn out that more lepton and quark flavours exist then the grand unified theories may be required to produce large net numbers of leptons (Sec. 4, Ch. 5). While the work of Dimopoulos and Feinberg suggests such numbers are unlikely they consider (for simplicity) only one lepton charge. As we have seen we may well be able to manipulate the sign of the lepton charges within the cosmological fluid so that in the context of specific lepton violating processes the "net" lepton production need not be overlarge.

Finally let us say that while theories of grand unification may answer the quark lepton similarity within a generation they certainly do not explain the pattern of repeating generations (Harari 1979b). One might continue the fundamentalist approach by postulating that quarks and leptons are made up of the same fundamental particles, variously called prequarks, subquarks, muons, alphons, quinks (see

Harari 1979a and the references therein) and preons. Higher generations are then excitations of the first generation. Thus the great success of the quark model in deriving the multitude of hadron states may give way to a simple scheme which gives the quarks and leptons themselves. Harari himself suggests an economical scheme consisting of two fundamental entities called rishons (Hebrew for first), one with a third charge and one neutral. However such a scheme gives no idea as to the quantum number governing generations since as Harari (1979b) indicates, spin, angular momentum and radial excitations seem to be excluded.

Now the particle physics we have attempted to discuss here has been based on a simple "heuristic". That the study of the micro-domain can best proceed by the positing of fundamental entities with categorizable and inviolable properties. A complete analysis of their interactions will then derive all higher order structures as epiphenomenae. Let us now discuss an orientation which at least as far as hadron physics is concerned completely rejects this approach.

(ii) Bootstrap Theory

If mainstream particle physics and its continuing preoccupation with fundamental particles is to be seen as a contemporary equivalent of a Newtonian corpuscularian stance then the bootstrap approach has much in common with the anti-corpuscularian viewpoint of Leibnitz (Gale 1974). The bootstrap philosophy as propounded by Geoffery Chew (1970) "seeks to understand nature not in terms of fundamentals but through self consistency, believing that all physics flows uniquely from the requirement that components be consistent with one another and themselves". There are no fundamental entities with inviolable properties rather all components are considered democratically with none more fundamental than another.

Now by consistency here we mean a "physically comprehensive" consistency. However it is a fact of scientific investigation, especially for particle physics, that the phenomenology and consequent theory building is "local" both in terms of the energy ranges investigated and the components hypothesized. For the bootstrap approach to compete with such "local" theories it must involve itself in a "partial bootstrap".

Bootstrap theory has mostly concentrated on the strong interactions. It is hoped that since all other forces are much weaker than the strong force a theory can be built that concerns itself with consistency within the strong interaction physics only and that this partial bootstrap gives results accurate to at least first order. The S matrix is the language used for expressing the complex interrelationships of the particles and the simplest bootstrap conjecture takes the following form. That there is only one S matrix consistent with the constraints of Poincaré invariance, unitarity and analyticity (Chew 1970). All strong phenomenology including spectrums of particles, their masses, spins and internal quantum numbers as well as coupling constants and all scattering amplitudes would follow from these constraints. Such a conjecture is falsifiable since any S matrix which conformed to these basic requirements and had any adjustable parameters would disprove it.

The conception of hadrons under such a conjecture is very interesting. Each of the hadrons owes their existence to the same forces through which they mutually interact so that the interaction which creates them is also the force of which they are composed. Each hadron is then a constituent of every other hadron. In the next section we will look at the cosmological consequences of such a viewpoint. Fortunately what we do there will depend little on the details of bootstrap physics.

Now we have little competence to give an adequate review of the bootstrap approach, suffice it that we make one last point. The great success of the quark model in deriving the hadron spectrum is also a serious defficiency for the bootstrap approach. Quark quantum numbers are not inconsistent with bootstrap, since an extended family of quarks could be considered on a similar democratic basis. Nonetheless confined quarks are not poles of the S matrix and an explanation of the hadron spectrum should come in another way. A recent attempt in this direction has been initiated by Weissmann (1978) who introduces a new and unobservable concept called "order" through which the hadron states can be derived. Whether in fact the concept explains more than the hadron spectrum is a question for the future.

3. HAGEDORN'S STATISTICAL BOOTSTRAP MODEL

We intend here to discuss the post-nuclear density equations of state for standard and deviant cosmological models. No attempt will be made to do justice to the many attempts that have been made in studying these regions, instead we restrict our attention to one particular approach known as the Hagedorn model.

The model is motivated by the hope that at sufficiently high energies the possible number of hadron channels grows so rapidly that a statistical treatment is feasible. In statistical mechanics since each macroscopic (thermodynamical) state is compatible with many microscopic states the relative probability of two given macroscopic states is determined by the relative number of compatible microstates. Similarly for hadron producing relations

$$A + B \rightarrow \text{hadrons}$$

any analysis of the decay products should come from the statistics of all possible decay channels limited only by the conservation of energy. The microscopic model of hadronic matter used by Hagedorn includes all possible hadronic resonances in a self-consistent way through the "statistical bootstrap"

> "a fireball is an equilibrium of an undetermined number of all fireballs each of which is considered to be a fireball ...". (1)

See the Hagedorn papers, Hagedorn (1965, 1968a,b,c, 1970).

In common with Chew's bootstrap Hagedorn's model conceives of no hadron being fundamental but all as composites (fireballs) of one of the other. As a statistical bootstrap it suffers a number of limitations. It cannot provide absolute predictions on cross-sections and since it refers to physical transition rates and therefore to squared matrix elements nothing can be said about the phases of corresponding production amplitudes. Nonetheless it does provide testable predictions giving branching ratios for various particle numbers etc., see Satz (1974) for a discussion.

The bootstrap (1) is formulated in the following way (Hagedorn 1970). If $N(m)dm$ is the number of different states between mand $m + dm$ (i.e. the hadron mass spectrum) then the partition function can be written down in a straight forward way and on assuming zero chemical potentials and Boltzman statistics we have

$$Z = \frac{4\pi k^3 T^3 V}{h^3 c^3} \int_0^\infty N(m) \left(\frac{mc^2}{kT}\right)^2 K_2\left(\frac{mc^2}{kT}\right) dm \; . \tag{2}$$

On the other hand the "main" fireball (hadronic "glob" or fluid) can always be described as

$$Z = \int_0^\infty \sigma(m,v)\, e^{-\frac{mc^2}{kT}}\, dm \tag{3}$$

where $\sigma(m,v)$ refers to the level density of states. The bootstrap condition demands that in the limit of high energies the two descriptions give the same result so that $\sigma(m) \sim N(m)$ as $m \to \infty$. The only consistent solution is the exponential behaviour

$$N(m) = Am^{-B}\exp(mc^2/kT_h) \tag{4}$$

where the various asymptotic behaviours of $\sigma(m)$ and $N(m)$ give varying values of B. Theoretically the constants A, B, T_h can be measured from actual high energy behaviour or read off from the known hadron spectrum.

A crucial aspect of the model is the temperature constant $T_h \simeq 160$ mev/K. This provides a finite upper maximum temperature since for temperatures higher than this the partition function does not converge. As the temperature increases, more and more kinetic energy is deposited as rest mass in the fluid and it so happens for the exponential spectrum above such conversion is the dominant process near $T \simeq 160$ mev/K and the fluid cannot get any hotter. Now whether T_h is an absolute maximum or indicative of a change in the physical composition of the fluid (i.e. a phase transition) is a mute point. We will not argue support for the Hagedorn model this can be found in Hagedorn's papers quoted above, the more recent works Chaician et al. (1975), Etim and Hagedorn (1977), Hagedorn et al. (1978) and the references that follow. A couple of points however are relevant. One can never be sure that the spectrum is in fact exponential since it may cut off or change at some transitional mass. Nonetheless models of total quark confinement such as the bag models (Chodos et al. (1974)) do lead to a similar spectrum. Incidentally if quarks are not completely confined then the Hagedorn spectrum and present limits on quark density are consistent with a quark masses ≥ 10 Gev/c^2 (Frautschi et al. 1972).

We can write the equations of state given by (2) and (4) as

$$\rho = \phi m_0 c^2 x_0^{B-5} \int x^{3-B} K_3(x) e^{\frac{xT}{T_h}} dx \tag{5}$$

$$p = \phi m_0 c^2 x_0^{B-5} \int x^{2-B} K_2(x) e^{\frac{xT}{T_h}} dx \tag{6}$$

where $\phi = A\beta m_0^{1-B}/(2\zeta(3))$, m_0 is the proton mass and B is in the range $2 < B < 4$.

Since the statistical bootstrap says nothing about the lower states the lower terminal in the integrals is uncertain. However most contributions come from the more massive states and we can integrate

$$\int_y^\infty x^{\nu-B} K_\nu(x) e^{\frac{xT}{T_h}} dx$$

$$= \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{2^n} \frac{1}{n!} \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu - n + \frac{1}{2})} \int_y^\infty x^{\nu-B-n-\frac{1}{2}} e^{-\left(1-\frac{T}{T_h}\right)x} dx$$

$$= \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{2^n} \frac{1}{n!} \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu - n + \frac{1}{2})} \left(1 - \frac{T}{T_h}\right)^{n+B-\nu-\frac{1}{2}} \Gamma\left\{-n - B + \tfrac{1}{2} + \nu, \left(1 - \frac{T}{T_h}\right)y\right\} \tag{7}$$

where the incomplete gamma function

$$\Gamma(\alpha,z) = \Gamma(\alpha) - \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha+k}}{(\alpha + k)k!} .$$

In most cases all terms but the first can be ignored and we obtain a term proportional to $(1 - T/T_h)^{B-\nu-\frac{1}{2}}$ for $B - \nu - \frac{1}{2} < 0$ or $-\ln(1 - T/T_h)$ for $B - \nu - \frac{1}{2} = 0$. If $B - \nu - \frac{1}{2} > 0$ we get a constant.

(i) Zero Net Baryon Number

For a cosmological model containing zero net baryon number we thus obtain for the energy density

$$\begin{aligned} \rho \;&\alpha\; (1 - T/T_h)^{B-7/2} && B < 7/2 \\ &\alpha\; -\ln(1 - T/T_h) && B = 7/2 \\ &\alpha\; \text{const.} && B > 7/2 . \end{aligned} \tag{8}$$

The pressure behaves similarly except that the changeover point is $B = 5/2$ and not $B = 7/2$. The scale factor behaviour as given through entropy conservation is

$$R^3 \propto (1 - T/T_h)^{7/2 - B} \qquad B < 7/2 \tag{9}$$

$$\propto -\ln(1 - T/T_h) \qquad B = 7/2 .$$

The Hagedorn model is clearly very different as a description for the early universe from the standard treatment. Instead of the energy density growing as R^{-4} (all species relativistic) and the pressure as $\simeq \rho/3$, for $B = 5/2$ (the value favoured by the earlier Hagedorn 1970) the energy density grows on approach to the singularity as R^{-3} but the pressure grows only as $|\ln R|$. If $B > 5/2$ the pressure is finite as $T \to T_n$, $R \to 0$.

Actually the model can only be used consistently until a time $\simeq 10^{-23}$ sec. Hagedorn assumes each hadron occupies the volume $V \simeq (h/m_\pi c)^3$; for times earlier than 10^{-23} sec one region of a hadron cannot have communicated with another region of the same hadron since the singularity so that the concept of that particle's elementarity is lost. This is the hadron barrier of Bahcall and Frautschi (1971). Presumably modifications of general relativity are needed to study the physical conditions at times previous to this.

Harrison (1970, 1972) has turned this argument on its head and assumes that only particles of size less than ct can have existed since the singularity. The volume of hadrons is dictated by the mass of the lowest pion allowable at a time t and is given by

$$1 = \frac{4\pi V_h}{(hc)^3} (mc^2)^2 \, kT_h K_2 (mc^2/kT_h) \tag{10}$$

Alexanian (1971), Alexanian and Mejiá-Lira (1975). The model described in these references also yields an exponentially rising mass spectrum but the hadronic boiling point T_h rises as $kT_h \simeq h/t$. Harrison would therefore see no reason to modify general relativity at the "hadron barrier" (the fluctuations $\Delta g/g \simeq 10^{-20}$) because of a failure in our understanding of particle physics and would have a warm $\simeq 10^{12}$°K period from 10^{-4} to 10^{-23} sec previous to which the temperature would have risen steadily to 10^{32}°K at the Planck era $t \simeq 10^{-43}$ sec. Whatever the case with Harrison's suggestion it is clear that a definite change of behaviour takes place at $t \simeq 10^{-23}$ sec.

(ii) Non-Zero Baryon Number

There have been many detailed studies of the Hagedorn model and its implications for cosmology e.g. Huang and Weinberg (1970), Kundt (1971), Frautschi (1971), Carlitz (1972), Stauffer (1972), Carlitz et al. (1973), Sisteró (1973), Dersarkissian (1976); we will not attempt to review these here. Nor will we consider recent efforts to take account of quantum statistics within the statistical bootstrap e.g. Chaichian et al. (1975), Engels et al. (1977), Letessier and Tounsi (1978). Instead we present a simple picture which describes the non-zero baryon case. We will then go on to consider non-zero lepton numbers and display some effects none of the above literature seems to have considered.

One way to proceed is to take the mass spectrum as in (4) and use the pair production quantum statistics integrals, the J^{ab}, with a non-zero chemical to determine the net baryon number i.e.

$$n_b = \phi x_0^{B-4} \int x^{-B} J^{11}(x,\lambda) e^{xT/T_h} dx \; , \tag{11}$$

$$\rho_n = \phi x_0^{B-5} m_0 c^2 \int x^{-B} J^{21}(x,\lambda) e^{xT/T_h} dx \; , \tag{12}$$

$$p_b = \phi x_0^{B-5} m_0 c^2 \int x \frac{J^{03}(x,\lambda)}{3} e^{xT/T_h} dx \; . \tag{13}$$

Huang and Weinberg (1970) and Stauffer (1972) have considered these integrals. However Huang and Weinberg have been concerned with extrapolation to infinitely high densities and to times much less than 10^{-23} sec. (Note, an error in their paper is corrected by Dersarkissian 1976.) Restricting ourselves to times greater than this is equivalent to working in the limit (Stauffer 1972)

$$\lambda << (1 - T/T_h)^{-1} . \tag{14}$$

Using the non-degenerate $J^{ab}(x,\lambda)$ given in (C6-9) we have

$$n_b = \phi x_0^{B-4} \sqrt{\frac{\pi}{2}}\, \Gamma(5/2 - B)(1 - T/T_h)^{B-5/2} \sinh\lambda \qquad B < 5/2 \tag{15}$$

$$\rho_b + p_b = \phi x_0^{B-5}\, m_0c^2\sqrt{\frac{\pi}{2}}\, \Gamma(7/2 - B)(1 - T/T_h)^{B-7/2} \cosh\lambda \qquad B < 7/2 \tag{16}$$

$$p = \phi x_0^{B-5}\, m_0c^2 \cdot \sqrt{\frac{\pi}{2}} \cdot \Gamma(5/2 - B)(1 - T/T_h)^{B-5/2} \cosh\lambda \qquad B < 5/2 \tag{17}$$

When $B = 5/2$ (15) and (17) have the dependence on $(1 - T/Th)$ replaced by $|\ell n(1 - T/Th)|$ and for $B > 5/2$ and $\lambda >> 1$ it is replaced by $1/[\lambda^{B-5/2} (B - 5/2)]$.

In the standard model the temperature is $\simeq 10^{12}$°K ($\simeq$ 160 mev) for $t \simeq 10^{-4}$ sec. At times immediately previous to this the degeneracy parameter is very small and the behaviour is given by the pair production equations (8-9) once the photon radiation is also included. The photon energy density $\simeq a_S T_h^4$ remains constant as R decreases and the photon entropy decreases as R^3. The result is net flows of energy and entropy in the direction of the hadrons such that the hadrons dominate the entropy equation $\left(\alpha(1 - T/Th)^{-1/2}\right)$ for B = 3 at $1 - T/Th \sim 10^{-16}$ and $t \sim 10^{-8}$ sec (for $f_{b,now} \simeq 10^8$). At still earlier times the degeneracy parameter becomes large ($\lambda >> 1$) and the universe is again dominated by non-relativistic baryons, just as it is in the present epoch. The total entropy for this period is

$$S/n_b k \simeq \lambda^{B-5/2}\,(B - 5/2)\,\Gamma(7/2 - B)(1 - T/Th)^{B-7/2} - \lambda, \qquad \text{for } 5/2 < B < 7/2$$

$$\simeq \frac{\sqrt{\pi}}{2}\left\{\frac{\lambda}{1 - T/Th}\right\}^{\frac{1}{2}} - \lambda\,, \qquad \text{for } B = 3. \tag{18}$$

The latter value of B is favoured in recent work e.g. Carlitz et al. (1973). Conservation of baryon number requires that λ rises only slowly $\lambda \sim \ln R^{-3}$ so that the second terms in (18) can be ignored, if the adiabatic expansion is to be maintained we must have $\lambda/(1 - T/Th) \simeq$ const.. The temperature is actually decreasing marginally as λ rises and we have $\lambda \simeq 10^2$, $1 - T/T_h \simeq 10^{-14}$ at the hadron barrier (Stauffer 1972). The limit (14) is satisfied by these numbers. Incidentally Huary and Weinberg (1970) have considered situations in which $\lambda(1 - T/T_h) >> 1$. However in such a case λ is very large and since $t \propto e^{-\lambda/2}$ the time since the singularity is incredibly small, much smaller for example than the Planck era $t \simeq 10^{-43}$ sec.

In this zero lepton number model it is the non-relativistic baryons which dominate the energy density at earliest times in spite of the fact that at much later times it is the photons which dominate the non-relativistic baryons. No detailed account of the size of f_b is forthcoming, it is merely very small at the hadron barrier. It is interesting that Huang and Weinberg in disregarding the barrier and problems of equilibrium at very small times can find an "explanation" for f_b through the fact that the only way their equations can have a solution on approach to the singularity is for the universal temperature to approach a temperature T^1 less than T_h given by $f_{b,now} \simeq (1 - T^1/T_h)^{-1}$. Of course the burden of explanation simply shifts to why the expansion begins with just that temperature.

(iii) Non-Zero Lepton Numbers

Including non-zero lepton numbers in the above scenario immediately produces a difficulty. Assuming weak interaction rates require chemical and thermal equilibrium among the leptons and hadrons, then chemical equilibria in reactions (5:13-15) may demand a non-zero chemical potential for the pions and therefore a chemical potential difference for neutron and proton like baryons. However the statistical bootstrap treats all baryons on an equal footing and the chemical potential variable used above refers only to baryon counting (the pion chemical potential is zero). Furthermore the bootstrap approach says nothing of the distribution of electric charges across the hadrons so that answers to problems like that posed at the end of Sec. 2, Ch. 5 will not be forthcoming within its framework.

This difficulty centres on the fact that the statistical bootstrap is a partial bootstrap. Leptons are simply not included. Whilst the statistical bootstrap is accurate asymptotically at high energies such energies seem low compared to any imaginable opening up of the lepton spectrum. Again the problem of the number of lepton flavours arises.

Incidentally the non-zero lepton number models of the Kaufman-Layzer type suggest another area of difficulty. At high density various particle states may be stabilized against decay due to fermi blocking. One may well ask what effect such factors have on the spectrum itself. Sawyer (1972) has considered such effects for cold neutron star matter; a resonance such as Δ^- excited at supernuclear densities suffers a mass shift $\sim$ 140-400 mev even in the abscence of two body interactions with other baryons. At a given density the higher the excited resonance the larger the expected upward shift. Sawyer concludes that as the density increases the highly excited levels become more gradually populated and that energy shifts may increase at such a rate with density that states beyond the basic baryon octet are absent.

It is by no means clear how to consider all these issues in a cosmological context and we will make no attempt to do so. Nonetheless even small lepton numbers together with rising baryonic spectrums as in (4) do suggest some surprising results. The simplest and probably most naive way to proceed is to assume that each collection of baryons with the same chemical potential is described by a rising number of types as in (4).

Using the notation of Sec. 2, Ch. 5 we have the charged groups

$$q = +1, \qquad \lambda_{p^+} = \lambda + \lambda_7, \tag{19}$$

$$q = 0, \qquad \lambda = \lambda,$$

$$q = -1, \qquad \lambda_{\Sigma^-} = \lambda_7$$

and the equations of mass action

$$\lambda_7 = \bar{\lambda}_3 + \lambda_5 = \lambda_4 + \bar{\lambda}_6 \tag{20}$$

where λ_7 is the degeneracy parameter of the pions π^+ and all similarly charged mesons. In the baryon octet for example we will have for p^+, Σ^+ a degeneracy parameter $\lambda + \lambda_7$, for n, Λ^0 Σ^0 Ξ^0 a parameter λ and Σ^-, Ξ^-, Λ^- a parameter $\lambda - \lambda_7$. All antiparticles will have degeneracy parameters just the opposite of these values. In this way there will be four families of hadrons characterized by the different degeneracy parameters $\lambda + \lambda_7$, λ, $\lambda - \lambda_7$, λ_7 each of which has the rising spectrum Eq. (4). This pattern of chemical potentials is forced simply by the assumption of chemical equilibrium with the leptons.

As with Sec. 2, Ch. 5 we can write down the equations governing the charges which working in the limit (14) are

$$0 = \frac{2}{3}\lambda_3^3 + \frac{2\pi^2}{3}\lambda_3 - 2\, J^{11}(x_4, \bar{\lambda}_4) - 2 \sinh \lambda_7 \cosh \lambda \frac{\phi' x_0^{B-1}}{\beta} \sqrt{\frac{\pi}{2}} \cdot$$

$$\int x^{2-B} K_2(x) e^{xT/T_h}\, dx - \sinh \lambda_7 \frac{\phi' x_0^{B-1}}{\beta} \sqrt{\frac{\pi}{2}} \cdot \int x^{2-B} K_2(x) e^{xT/T_h} dx, \tag{21}$$

$$4\zeta(3)f_e^{-1} = \frac{2}{3}\lambda_3^3 + \frac{2\pi^2}{3}\lambda_3 + \frac{1}{3}\lambda_5^3 + \frac{\pi^2}{3}\lambda_5 \,, \tag{22}$$

$$4\zeta(3)f_{\mu^+}^{-1} = 2\, J^{11}(x_4, \bar{\lambda}_4) + \frac{1}{3}\bar{\lambda}_6^3 + \frac{\pi^2}{3}\bar{\lambda}_6 \,, \tag{23}$$

$$4\zeta(3)f_b^{-1} = \sinh\lambda(2\cosh\lambda_7 + 1)\,\frac{\phi' x_0^{B-1}}{\beta}\sqrt{\frac{\pi}{2}}\int x^{2-B}\,K_2(x)e^{xT/T_h}\,dx \,. \tag{24}$$

The relations are valid for $B \leq 5/2$, for $B > 5/2$ modifications similar to those discussed above apply and for $\lambda >> 1$ a term $\sim \lambda^{B-5/2}$ replaces the integrals. We have altered ϕ to ϕ' to account for each spectrum of hadrons and of the two hadronic contributions to Eq. (21) the first is from the baryons the second from the mesons. Since the temperature is $\sim 10^{12}$°K we have written the electron density in its relativistic form but have used the general integral for the muons. Modification to higher lepton flavours is straight forward. Clearly $\lambda > 0$ if the world is baryonic.

With the limited hadronic spectrum $\{p^+, n, \pi^+, \pi^0\}$ of Sec. 2, Ch. 5 charge balance in the lepton symmetric model ($f_e = f_\mu = 0$) requires $\bar{\lambda}_7 > 0$. Condensation effects can then arise since we may require $\bar{\lambda}_7 = x_7$. For our modified Hagedorn model however the solution $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$ is possible since the net electric charge for all hadrons is then zero (Eq. (21)). The equations collapse to (24) and we have the scenario discussed in Part (ii) above with $\phi' = \phi/3$. In such a model no condensation effects will occur.

As we have consistently argued there is no justification for assuming that the lepton numbers are zero. A post-Planck era scenario like the one suggested in Sec. 2 might be responsible for the baryon asymmetry; the hadron barrier will not apply to the x bosons since their size is $\sim 10^{-27}$ cm, and for SU(5) we might expect an equal lepton asymmetry, say $f_b = f_e$. Even this small lepton number has a considerable effect.

At times subsequent to 10^{-4} sec the electron degeneracy parameter is small ($\simeq 10^{-8}$) however it rises rapidly $\sim R^{-3}$ during the hadron era on approach to the singularity. Flows of energy and entropy to the pair produced hadrons occur as in Part (ii) but the leptons become increasingly degenerate and for $\lambda_3 \gtrsim \pi$, λ_3 rises as R^{-1}. One consequence is that λ_3 and λ_5 become large relative to λ_7 giving $\lambda_3 \simeq \lambda_5 >> 1$ so that charge balance determines λ_7 via the relations

$$\frac{2}{3}\frac{N_e}{N_b} = \frac{2}{3} = \frac{\sinh \lambda_7(2 \cosh \lambda + 1)}{\sinh \lambda(2 \cosh \lambda_7 + 1)}, \qquad B < 5/2 \tag{25}$$

and (24).

For times earlier than 10^{-8} sec (we use $B = 3$ hereon), λ becomes large and rises as $|\ell n\ R^{-3}|$ and the domination of the entropy by the hadrons requires that $\rho_b/n_b kT \simeq 10^8 \simeq$ const. for $S_T/n_b k \simeq 10^8$ (the contribution of degenerate species decreases as λ_3^{-1}). However the total energy density becomes increasingly dominated by the leptons viz.

$$\rho_T/n_b kT = \frac{f_b}{4\zeta(3)}\frac{3\lambda_3^4}{4} + \frac{\rho_b}{n_b kT} \tag{26}$$

$$\simeq \frac{N_e}{N_b}\frac{3}{4}\lambda_3 + 10^8$$

and leptons dominate when $\lambda_3 \gtrsim 10^8$. Using (5.110) for the characteristic expansion time for the model the leptons dominate for $t \lesssim 10^{-20}$ sec ($f_{b,now} \simeq 10^8$) which is well after the hadron barrier.

We can generalize this for a situation with any present epoch electron (or muon, or both) lepton asymmetry given by $f_{e,now}$. The leptons dominate energy density for

$$\lambda_3 \gtrsim 5\ f_{e,now} \qquad \text{i.e. } \mu_3 \gtrsim 700\ f_{e,now}\ \text{mev} \tag{27}$$

and

$$t \lesssim 10^{-5} f_{e,now}^{-2}\ .$$

The result is independent of the details of the particular Hagedorn model, the values of A, B and the naive picture above, and depends only on hadrons being described by the rising spectrum as in (4) and the existence of solutions for all times greater than 10^{-23} sec. Note however that for $f_{e,now} \gtrsim 10^9$ leptons never dominate the energy density after the hadron barrier. Nonetheless pion condensation may be necessary during the hadron era, thus even for $N_e/N_b \simeq 1$ the L.H.S. of Eq. (25) is greater than the R.H.S. for $\lambda >> 1$ and $\lambda_7 = x_7 \simeq 1$. The amount of condensation in this case however is quite small.

For this modified standard model (very small lepton asymmetries) we have the following sequence of epochs. From 10^{-23} sec to a time given by (27) the expansion is dominated by leptons and $t \sim R^{-2}$. The hadron era then extends down to 10^{-4} sec with at first non-relativistic (net) baryons and subsequently non-relativistic baryons, mesons and their antiparticles (in nearly equal numbers) dominating the energy density. The behaviour $t \sim R^{-3/2}$ gives way to $t \sim R^{-2}$ as photons and pair produced leptons dominate the energy density during the "lepton era" 10^{-4} sec $\sim$ 1 min. Photons subsequently dominate through the radiation until around 10^5 years when the lowest lying baryons come to dominate by virtue of their rest mass. The universe is once again matter dominated and $t \sim R^{-3/2}$. Finally we have the present epoch in which baryons, photons and leptons make successively smaller contributions to the observed energy density.

What of times previous to 10^{-23} sec? If we accept Harrison's proposal then the temperature increases toward the singularity as $kT_n \sim h/t$. If leptons dominate after then barrier $\lambda_3 \sim t^{-1/2}$ and subsequently $\lambda_3 \sim t^{-2/3}$ when baryons dominate, previous to the barrier however $\lambda_3 \sim t^{1/2}$ and thus at sufficiently small times the "baryons" again dominate the energy density with $t \sim R^{-3/2}$ and $\lambda_3 \sim t^{1/2}$ on approach to the singularity.

Finally we might consider the case of larger lepton numbers and the relevance of higher massed lepton flavours. Eq. (25) can be generalized to give

$$\frac{2}{3}\sum_{\ell}\frac{N_\ell}{N_b} = \frac{\sinh \lambda_7(2\cosh \lambda + 1)}{\sinh \lambda(2\cosh \lambda_7 + 1)}\,, \qquad B < 5/2 \tag{28}$$

where ℓ refers to a specific lepton flavour and the sum runs over all flavours which give degenerate contributions to the density. Similarly Eq. (26) can be generalized to

$$\frac{\rho_T}{n_b kT} = \frac{3}{4}\sum_{\ell}\frac{|N_\ell|}{N_b}\lambda_\ell + \frac{\rho_b}{n_b kT}\,. \tag{29}$$

For models of the type suggested by Beaudet et al. described in Sec. 4, Ch. 5, $N_e/N_b \simeq 10^8$, $\lambda_e \simeq 1$ while higher lepton flavours may hold some $\sum_{\ell\neq e} |N_b|/N_b \lesssim 10^{12}$ leptons. Such numbers of leptons would certainly dominate the energy density during the hadron era, and unless the "leptons" are distributed across the various lepton types in such a way that the L.H.S. of (28) is small, Bose condensation is also a necessary feature of the region. Note that if Bose condensation is large then the contribution to the energy density by the zero energy mesons may be greater than that of baryons. It will however be less than that of the leptons.

For models of the type envisaged in Sec. 5, Ch. 5 only a small amount of condensation is needed to balance charge since $(N_e - N_\mu)/N_b \simeq 1$. As the temperature approaches 10^{12}°K the density of nucleons approaches nuclear density with $t \simeq 10^{-12}$ sec, $n_b \simeq 10^{37}$ cm^{-3}, the energy density of leptons however is entirely dominant. It is interesting to note that during the expansion it is the hadrons which give their entropy to the leptons during the hadron era and that at a later epoch the leptons may yield up a considerable fraction of this entropy to the photon field.

Also note that for any reasonable model in which the leptons do dominate energy density at the hadron barrier, since the temperature $\simeq 10^{12}$°K the degeneracy parameter is fixed at $\lambda_\ell \simeq 10^9$ at the barrier. For the Harisson model the temperature rises steadily to $T \sim 10^{32}$°K at $t \sim 10^{-43}$ sec and λ_ℓ falls as $t^{1/2}$ and $t^{1/3}$ on approaching the Planck era. If our model contains large lepton numbers of the sizes discussed above then the entire expansion from the Planck era onwards is dominated by leptons.

Finally a word of warning. Little is really known of the equations of state at supernuclear densities. Recent work within the statistical bootstrap, as reviewed by Ilgenfritz et al. (1977), gives evidence of the ongoing development of that area. However only astrophysical features such as hadronic star matter, the big bang itself, may offer the opportunity to apply some of these developments. One possibility of particular importance is that the upper temperature limit indicates a phase transition to a different state of matter at higher density (Carlitz 1972, Hagedorn et al. 1978). Of even more crucial concern is the failure of the statistical bootstrap to produce an observed feature of real collisions - the so-called jets (Etim and Hagedorm 1977).

4. DISCUSSION AND OBSERVATIONS

Since a detailed summary has already been given in Ch. 1 we restrict ourselves here to discussion and some observations.

We have attempted to do some justice to the many thermodynamical and chemical possibilities within the Robertson-Walker cosmologies. We cannot claim to have done this exhaustively nonetheless our study has emphasized some important features.

In comparison to standard methods of treating epochs in terms of their dominant components our detailed thermodynamical analysis seems justified. Equilibrium flows of energy and entropy between various particle species are a general characteristic of the Friedmann models during expansion; such flows being indicated when the product RT departs from its constant value or as we have described it here: when the ratio of black body photons to some conserved number varies. As we have seen in Ch. 3 such energy and entropy flows and changes in the chemical composition of the expanding fluid occur only when some particle's rest mass becomes important in the equations of state.

We have also noted repeatedly that in contrast to the older cosmological descriptions (e.g. Weinberg 1972) leptons play a critical role in the models. The discovery of the tau lepton may have brought the standard model into confrontation with observational data and even small lepton asymmetries modify the consensus view of the hadron era within the statistical bootstrap. Interestingly the leptons once played an important role in model building, however this was prior to the discovery of the microwave background which has demanded we consider hot universes. The discussions in Sec. 4 and particularly Sec. 5 of Ch. 5 have shown we may need to reintroduce large numbers of leptons into cosmological models. The hybrid model in Sec. 5, Ch. 5 is interesting in this regard since the universe is cold for leptons and hot for baryons. Such models show some unusual features and warrant further investigation.

In this last chapter we have brought to a head many issues concerning modern particle physics and cosmology. The way in which cosmological retrodiction has explored conditions at higher and higher energies is similar to the way in which particle physics has explored deeper layers of matter. The question of what a "complete" cosmology looks like (at least from the point of view of thermodynamical and chemical consequences and leaving out [perhaps] across horizon perturbations and other features which may have a gravitational origin) seems answerable only from the basis of a "complete" particle physics (we mean one that does not include gravity). Moreover the various ways in which particle physics can be conceived may give very different cosmologies.

We can see this by contrasting two naive particle physics philosophies. In the first a simple atomism gives the basic entities from which all others are to be constructed, all quantum numbers are contingently conserved (the energy resolution of our experiments) except those of the basis entities which are strictly conserved. In the second there is a never-ending sequence of more basic entities - hadrons, quarks, pre-quarks and so on - wherein the lower reaches of the particle spectrum at one level can explain in rather simple terms many complex features of the level immediately above.

Simple atomism leads to a simple cosmology. We retrodict from presently observed conditions until we reach an epoch when all matter is ionized in terms of its basic units. However from the Planck era to this epoch there is no thermal or chemical evolution (all particles are relativistic) and all particle charges characteristic of this epoch are conditions of the Planck era. While particle physics derives the type and nature of entities that give matter its structure cosmology derives the sizes of the charges of the entities that give the observed universe. For the Laplacian mind then (we assume strict determinism) the world is given by its entities and their numbers. The fact that numbers of particles do not seem to be on the same ontological footing as their properties suggests that they are in need of further explanation and makes ideas such as the anthropic principle (Ch. 1) rather attractive.

On the other hand if the structure of matter is infinite in depth the problem of the origins of presently observed cosmological structures may never be solvable (though perhaps this is not necessary). The best we might do is peel back the layers of matter and simultaneously push back the frontiers of the known universe. This has historically been the case for cosmology and will continue to be so until a "complete" particle physics emerges. If however such complete theories are never to be devised it is difficult to see how rival cosmologies can ever be meaningfully compared since some problems will always remain to be solved. Indeed we may not be able to rule out any expanding cosmology at all.

These abstract considerations underscore a very important issue in the whole cosmological undertaking. This is the notion of retrodicting from complex conditions to simple ones. There is a tendency to define cosmology as the undertaking which can do just that since this is the basis of retrodiction in the first place. Yet it would be naive to expect that some of the features of the present universe are not the integrated result of many contributions each of which require their own explanation. Of course we might need to be convinced that such contributions are necessary (i.e. are indicated by issues relating to other problems) but it is a fallacy to assume that they may not be. The value of f_b might be an integrated effect of this type, the complex sharing of lepton numbers among lepton flavours we suggested in Sec. 4 Ch. 5 - when we introduced large lepton numbers whose net number in the context of lepton/baryon violating processes may be small (i.e. the only way we can save "standard" helium production if we allow many lepton flavours) - may be another. We are not here advocating the rejection of simplicity arguments in adjudicating between rival cosmologies but merely that such arguments can never be ones of necessity.

We have seen in this chapter that we face the possibility of a "complete" particle physics (at least as far as non-gravitational interactions are concerned). The logical possibility arises of a "complete" cosmology (as far as non-gravitational effects are concerned) which has no initial charges (except perhaps electric charge). It is critical both for the grand unified theories and their cosmological relevance that unification takes place below the Planck mass. Furthermore any arguments pertaining to totally unified theories (they are not falsifiable in the sense of Popper, the problem of self-referring statements must arise if the theories are deterministic and all other sciences reduce to them, etc.) are fortunately not yet important. Interestingly for the scenario suggested by Ellis et al. (1979) the chemical content of the universe is derived at an epoch determined by the grand unification mass but the chemical content (i.e. baryon/lepton charges) between this epoch and the Planck era is irrelevant. All memory of those initial charges (if we can define them meaningfully) is destroyed when baryon violating forces come into equilibrium at the grand unification mass.

We can hardly claim to have done justice to all the issues that have been raised in this thesis and we have not even considered other very important areas: black holes, chaotic cosmologies, quantum gravity, kinetic theory etc. Our treatment has been haphazard and often cursory and incomplete. We make no excuse for this other than limited knowledge and limited space.

Let us conclude with a comment on the grand unified theories and the modified standard model's version of the helium production era since these approaches seem to presently be commanding considerable attention. The attention is certainly well deserved since the grand unified theories may derive the correct value of f_b and require only the small lepton asymmetries needed in the modified standard model. However it is just what is presently unsatisfactory about such theories - they do not explain the pattern of repeating generations - which is at the root of the problem regarding helium formation and the number of lepton flavours. One might be excused for speculating therefore, that the problem of the number of flavours is not the province of particle physics but rather of cosmology.

Presumably a cosmological solution to the number of flavours lies in some kind of anthropic connection between the exact helium abundance (and other light element abundances) and the origin or maintenance of life. However we are not even very good at pinning down the exact value of the contemporary light element abundances let alone understanding the evolution of stars and planets sufficiently to calculate the effects of different primordial values. Nonetheless if an anthropic connection were to be found a critical problem with the Carter formulation of the principle (as discussed in Ch. 1) - the problem of "grain" - is solved. The ensemble of universes is then an integer number - the number of generations - each of which is characterized by different exact amounts of light elements only one of which allows life. Even so one could not expect the particle theorists to give up the search for a "better" understanding.

APPENDIX A

1. INTRODUCTION

In this Appendix we discuss approximation schemes to the Fermi-Dirac and Bore-Einstein integrals. The discussion centres on two closely related representations of the integrals which are particularly useful for our purposes. Many other representations have been investigated and approximation schemes developed, thus for example McDougall and Stoner (1938) tackle the non relativistic form of the integrals and more recently Guess (1966) and Bludman and Van Riper (1977) extend previous work on the relativistic form. The work of Guess is of particular relevance here; after discussion of a number of representations he adopts a form which is concise and which allows derivatives of the important thermodynamical functions to be written within it. Approximation schemes to these functions are then developed via complex analysis for the four regions, non-relativistic, extreme relativistic, non degenerate and extreme degenerate, (this last being only possible for the fermions). The central region, semirelativisitic and semidegenerate is tackled by direct numerical analysis.

The work of Guess however, is not totally suitable for our purposes. Thus no continuity is established between that for massive particles and the equations for radiation and the expansions given in the relativistic case become uncertain for bosons with $\lambda \rightarrow x$. Moreover Guess expresses his integrals in terms of functions which have been tabulated to some extent (McDougall and Stoner 1938, Beer et al. 1955) or have been expressed as series expansions in various parts of the literature (Dingle 1957, Van Riper and Bludman 1977, Robinson 1951).

Since we will need these expansions anyway it seems useful to develop a general method of tackling the integrals.

The methods used here to develop approximation schemes are less elegant than those of Guess but proceed in a simpler fashion. As we will show all expansions will be expressible in terms of two functions, the modified Bessel function and a generalized form of Riemann's zeta function. These functions are discussed at length and series approximations given to them in Appendix B.

We define

$$I_{\varepsilon}^{ab}(x_1\lambda) = \int_x^{\infty} \frac{y^a(y^2-x^2)^{b/2}}{e^{y-\lambda}+\varepsilon}\,dy \tag{1}$$

and (Guess 1966)

$$Q_{\varepsilon}^{n}(x,\lambda) = \int_0^{\infty} \frac{\cosh n\chi}{e^{x\cosh\chi-\lambda}+\varepsilon}\,d\chi \tag{2}$$

The relations

$$I^{11} = \frac{x^3}{4}\quad(Q^3-Q^1) \tag{3}$$

$$I^{21} = \frac{x^4}{8}\quad(Q^4-Q^0) \tag{4}$$

$$I^{03} = \frac{x^4}{8}\quad(Q^4-4Q^2+3Q^0) \tag{5}$$

and

$$I^{21} + \frac{I^{03}}{3} - \lambda I^{11} = x^3\left(\frac{x}{6}(Q^4-Q^2) - \frac{\lambda}{4}(Q^3-Q^1)\right)$$

are easily established (in this and the following we drop all unnecessary subscripts and variables) and a differential relation among the Q^n follows from a lengthy integration by parts

$$d(Q^{n+1}-Q^{n-1}) = \frac{2n}{x}\,Q^n d\lambda - ((n+1)Q^n + (n-1)Q^{n-1})\frac{1}{x}\,dx. \tag{6}$$

The form of these integrals immediately suggests the approximation regions, $x \to 0$ relativistic, $x \to \infty$ nonrelativistic, $\lambda >> x$ degenerate, $\lambda << x$ nondegenerate. A minor difficulty arises with the Q^n for massless particles $x = 0$, since the Q^n are not defined in the limit $x \to 0$. The I^{ab} of course are defined and show no explicit x dependence in the limit, they can then be integrated explicitly and are given in Section 3, Chapter 2. Any difficulties with the Q^n are thereby removed by taking the limit $x \to 0$ only at the end of calculations involving the more physical I^{ab}. Clearly no problems with the derivatives can arise since these become explicit also.

In the sections below we derive and discuss the various approximation schemes on the basis of an initial assumption on λ (or $\eta = \lambda - x$). However some of the expressions so produced are valid in a wider range (Sec. 4). It proves useful therefore to include in Sec. 5 a summary of the schemes and a table to illustrate their regions of use.

2. NON DEGENERACY: $\lambda \leq 0$

Consider the case $\eta = \lambda - x \leq 0$. The denominator in the integrals (1) and (2) can be treated as a sum of a geometric series, the summation can be removed to outside the integral and the series integrated term by term with each term corresponding to a modified Bessel function defined by (B1). Relations between these Bessel functions (B2-3) can then be used to derive

$$Q_\varepsilon^n = \sum_{m=1}^{\infty} (-\varepsilon)^{m+1} e^{m\lambda} k_n(mx), \tag{7}$$

$$I_{\varepsilon}^{11} = \sum_{m=1}^{\infty} \frac{(-\varepsilon)^{m+1}}{m} e^{m\lambda} x^2 k_2(mx), \tag{8}$$

$$I_{\varepsilon}^{21} = \sum_{m=1}^{\infty} \frac{(-\varepsilon)^{m+1}}{m} e^{m\lambda} x^3 \left(k_3(mx) - \frac{k_2(mx)}{mx}\right), \tag{9}$$

$$I_{\varepsilon}^{03} = 3 \sum_{m=1}^{\infty} \frac{(-\varepsilon)^{m+1}}{m^2} e^{m\lambda} x^2 k_2(mx), \tag{10}$$

and

$$I_{\varepsilon}^{21} + \frac{I_{\varepsilon}^{03}}{3} - \lambda I_{\varepsilon}^{11} = \sum_{m=1}^{\infty} \frac{(-\varepsilon)^{m+1}}{m} e^{m\lambda} x^2 (xk_3(mx) - \lambda k_2(mx)). \tag{11}$$

In the limit λ large and negative the integrals are I_0^{ab}, Q_0^n and the series above retain only the first term, these are the equations of the Synge gas (2:56-58). In the non relativistic and relativistic regions instead of the series above in the K_n we can use expansions for the K_n and give the above integrals in terms of a generalized zeta function defined by (B6). Since $\eta \le 0$ it will be given by its Dirichlet series

$$\phi(z,s) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad |z| \le 1. \tag{12}$$

(i) Nonrelativistic Region

In the non-relativistic region x is large and the asymptotic expansion (B4) when used in Eq.(7) gives

$$Q_{\pm 1}^{n} = \mp \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \left(\phi(\mp e^{\eta},1/2) + \phi(\mp e^{\eta},3/2)\left(\frac{4n^2-1}{8x}\right) + \phi(\mp e^{\eta},5/2)\left(\frac{(4n^2-1)(4n^2-3^2)}{2!\ 64\ x^2}\right) + \ldots\right). \tag{13}$$

Expansions for the I^{ab} follow easily

$$I_{\pm 1}^{11} = \mp \left(\frac{\pi x^3}{2}\right)^{\frac{1}{2}} \left(\phi(\mp e^{\eta},3/2) + \phi(\mp e^{\eta},5/2)\frac{15}{8x} + \phi(\mp e^{\eta},7/2)\frac{105}{128x^2} + \ldots\right), \tag{14}$$

$$I^{21}_{\pm 1} = \mp \left(\frac{\pi x^2}{2}\right)^{\frac{1}{2}} (\phi(\mp e^{\eta},3/2) + \phi(\mp e^{\eta},5/2)\frac{27}{8x} + \phi(\mp e^{\eta},7/2)\frac{705}{128x^2} + \ldots), \tag{15}$$

$$I^{03}_{\pm 1} = \mp 3\cdot \left(\frac{\pi x^3}{2}\right)^{\frac{1}{2}} (\phi(\mp e^{\eta},5/2) + \phi(\mp e^{\eta},7/2)\frac{15}{8x} + \phi(\mp e^{\eta},9/2)\frac{105}{128x^2} + \ldots), \tag{16}$$

and

$$I^{21}_{\pm} + I^{03}_{\pm}/3 - \lambda I^{11}_{\pm} = \mp \left(\frac{\pi x^3}{2}\right)^{\frac{1}{2}} \left[(x-\lambda)\phi(\mp e^{\eta},3/2) + \left(\frac{35x-15\lambda}{8x}\right)\phi(\mp e^{\eta},5/2) + \left(\frac{945x-105\lambda}{128x^2}\right) \phi(\mp e^{\eta},7/2) + \ldots\right].$$

In general the functions $\phi(\mp e^{\eta},s)$ must be numerically evaluated however calculation is fast for $\eta \ll 0$ by (12) and for $\eta \to 0$, the region in which the Bessel function expansions do give problems, they return the normal Riemann zeta functions. When $\eta > 0$ the relevant expansions are given in Appendix B. It turns out in fact that for fermions the above series are good approximations to the integrals (1,2) as long as $\eta < x$. We will show this in Sec. 4.

Another minor difficulty with the Q^n arises for bosons with $\eta \to 0$. In this case each of the Q^n diverges as $\eta \to 0$ since it contains $\phi(+e^{\eta},\frac{1}{2})$. (see Eq. B27). However as with radiation no real problem arises if the limit $\eta \to 0$ is taken at the end of the calculation. Inspection of (B31-34) indicates (14-16) are convergent and the fact that $d\lambda/dx = 1$ in the limit $\lambda \to x$ causes cancellation of all exploding terms in the derivative equation (6).

(ii) Relativistic Region

In the relativistic region x is small and the series representation of the Bessel functions (B5) can be used to give an expansion in ascending powers of x. Thus (B5) in Eq.(7) gives by the use of Eq.(12)

$$Q^n_{\pm 1} = \mp \sum_{k=0}^{\infty} \frac{1}{2} \left(\frac{2}{x}\right)^{n-2k} (-1)^k \left(\frac{(n-k-1)!}{k!}\right) \phi(\mp e^\lambda, n-2k)$$

$$+ (-1)^{n+1} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{n+2k} \frac{1}{k!(n+k)!} \left[\sum_{m=1}^{\infty} (\mp 1)^{m+1} m^{n+2k} e^{m\lambda} \ell_n\left(\frac{mx}{2}\right) \right.$$

$$\left. \mp \phi(\mp e^\lambda, -n-2k) \left(\sum_{m=1}^{n+k} \frac{1}{2m} + \sum_{m=1}^{k} \frac{1}{2m} - C \right) \right] \qquad (17)$$

where c is Eulers constant, C = .577215.

At relativistic temperatures the first sum is much larger than the second except for Q^0. However since Q^0 goes as ℓnx and will appear only in conjunction with the much larger Q^2 we can ignore it here. The expansions for the I^{ab} thus become

$$I^{11}_{\pm} = \mp \left(2\phi(\mp e^\lambda, 3) - \frac{x^2}{2} \phi(\mp e^\lambda, 1) \ \ldots\right), \qquad (18)$$

$$I^{21}_{\pm} = \mp \left(6\phi(\mp e^\lambda, 4) - \frac{x^2}{2} \phi(\mp e^\lambda, 2) \ \ldots\right), \qquad (19)$$

$$I^{03}_{\pm} = \mp \left(6\phi(\mp e^\lambda, 4) - \frac{3x^2}{2} (\mp e^\lambda, 2) \ \ldots\right). \qquad (20)$$

As in the non-relativistic case we will later (Sec.4) prove these equations valid in a wider range than $\lambda \leq x$.

3. FERMION DEGENERACY: $\eta > 0$

Now consider the case $\eta = \lambda - x > 0$. This is clearly only possible for $\varepsilon = +1$ or fermions which can become degenerate if $\eta = \lambda - x$ is large. When η is large standard techniques (e.g. Chui (1968)) can be used to give asymptotic series for the integrals I^{ab} and Q^n. Since the integrals have the general form

$$T = \int_0^\infty \frac{S(u)}{e^{u-\eta}+1} du$$

we can substitute $u-\eta = v$ and split the range to give

$$T = \int_0^\eta \frac{S(\eta-v)}{e^{-v}+1}\, dv + \int_0^\infty \frac{S(v+\eta)}{e^v+1}\, dv,$$

whence the identity

$$\frac{1}{e^{-v}+1} = 1 - \frac{1}{e^v+1},$$

yields

$$T = \int_0^\eta S(u)du + \int_0^\infty \frac{S(\eta+v)-S(\eta-v)}{e^v+1}\, dv + \int_\eta^\infty \frac{S(\eta-v)}{e^v+1}\, dv.$$

The assumption that $\eta \gg 1$ will in general allow the last integral to be ignored, the numerator in the second integral can then be expanded as a Taylor series in powers of v.

All terms with even derivatives will cancel and we find

$$T = \int_0^\eta S(u)du + 2\left(\overset{(i)}{S}(\eta) \int_0^\infty \frac{v}{e^v+1}\, dv + \frac{1}{3!} \overset{(iii)}{S}(\eta) \int_0^\infty \frac{v^3}{e^v+1}\, dv + \right.$$
$$\left. \frac{1}{5!} \overset{(v)}{S}(\eta) \int_0^\infty \frac{v^3}{e^v+1}\, dv + \ldots\right)$$

which in the notation of the generalized zeta function becomes

$$T = \int_0^\eta S(u)du - 2(\overset{(i)}{S}(\eta)\, \phi(-1,2) + \overset{(iii)}{S}(\eta)\, \phi(-1,4) + \overset{(v)}{S}(\eta)\, \phi(-1,6) + \ldots). \tag{21}$$

It is now a simple but tedious process to establish the expansions to the integrals I^{ab} and Q^η. Since the resulting series are asymptotic we give a considerable number of terms, the last term will give an estimation of the remainder. For some very large x even these terms will not give sufficient accuracy, in this case we can put in the assumption of large x at the beginning, give a series in powers of 1/x and use the above method with an explicit calculation of the remainder. We will return to this point later. We write for convenience in the following

$$w^2 = \lambda^2 - x^2 = \eta(\lambda + x), \tag{22}$$

$$Q^0 = \ln\left(\frac{\lambda}{x} + \sqrt{\frac{\lambda^2}{x^2} - 1}\right) - \frac{\pi^2}{6}\frac{\lambda}{w^3} - \frac{7\pi^4}{120}\frac{\lambda}{w^7}(2\lambda^2 + 3x^2) -$$

$$\frac{31\cdot\pi^6\lambda}{1008\cdot w^{11}}(8\lambda^4 + 40\lambda^2x^2 + 15x^4), \tag{23}$$

$$Q^1 = \frac{w}{x} - \frac{\pi^2}{6}\frac{x}{w^3} - \frac{7\pi^4 x}{120w^7}(4\lambda^2 + x^2) - \frac{31\pi^6 x}{336w}(8\lambda^4 + 12\lambda^2x^2 + x^4), \tag{24}$$

$$Q^2 = \frac{w\lambda}{x^2} + \frac{\pi^2\lambda}{6x^2w^3}(2\lambda^2 - 3x^2) - \frac{7\pi^4\lambda x^2}{24w^7} - \frac{31\pi^6\lambda x^2}{336w^{11}}(14\lambda^2 + 9x^2), \tag{25}$$

$$Q^3 = \frac{w}{3x^3}(4\lambda^2 - x^2) + \frac{\pi^2\cdot}{6x^3w^3}(8\lambda^4 - 12\lambda^2x^2 + 3x^4) - \frac{7\pi^4x^3}{24w^7} -$$

$$\frac{31\ \pi^6x^3}{144\cdot w^{11}}(8\lambda^2 + x^2), \tag{26}$$

$$Q^4 = \frac{\lambda w}{x^4}(2\lambda^2 - x^2) + \frac{\pi^2\lambda}{6x^4w^3}(24\lambda^4 - 40\lambda^2x^2 + 15x^4) + \frac{7\pi^4\lambda}{120x^4w^7}$$

$$(16\lambda^6 - 56\lambda^4x^2 + 70\lambda^2x^4 - 35\ x^6) - \frac{31\pi^6}{16w^{11}}\lambda x^4. \tag{27}$$

For the I^{ab} we also define the standard functions

$$f(z) = z(2z^2-3)(z^2+1)^{\frac{1}{2}} + 3\sinh^{-1}z, \tag{28}$$

$$g(z) = 8z^3((z^2+1)\ -1)^{\frac{1}{2}} - f(z). \tag{29}$$

We then have

$$I^{11} = \frac{w}{3} + \frac{\pi^2}{6}\frac{(2\lambda^2 - x^2)}{w} + \frac{7\pi^4x^4}{120w^5} + \frac{31\pi^6x^4}{1008w^9}(6\lambda^2 + x^4), \tag{30}$$

$$I^{21} = \frac{x^4}{24}g\left(\frac{w}{x}\right) + \frac{xw^3}{3} + \frac{\pi^2}{6}\frac{\lambda}{w}(3\lambda^2 - 2x^2) + \frac{7\pi^4\lambda}{120w^5}(2\lambda^4 - 5\lambda^2x^2 + 4x^4),$$

$$+ \frac{31\pi^6}{1008w^9}\lambda x^4(\lambda^2 + 6x^2) \tag{31}$$

$$I^{03} = \frac{x^4}{8} f\left(\frac{w}{x}\right) + \frac{\pi^2}{2} \lambda w + \frac{7\pi^4 w}{120 w^3} (2\lambda^2 - 3x^2) - \frac{31\pi^6 x^4}{336 w^7} \lambda. \tag{32}$$

The use of these expansions in the evaluation of thermodynamical quantities should proceed with care since in some cases considerable cancellation may occur. Such a case is the entropy density which is proportional to

$$I_+^{21} + \frac{I_+^{03}}{3} - \lambda I_+^{11} = \frac{\pi^2}{3} \lambda \cdot w + \frac{7\pi^4}{360} \frac{\lambda}{w^3} (8\lambda^2 - 13x^2) - \frac{31}{168} \pi^6 \frac{\lambda x^4}{w^9} . \tag{33}$$

4. SEMI-DEGENERACY: η SMALL

Finally it is useful to develop series expansions for the integrals when $\eta = \lambda - x$ is small. The case $\eta \leq 0$ is covered in the non relativistic regime by (13-16) and in the relativistic regime by (17-20) when use is made of the series expansions for the $\phi(z,s)$ with $z \sim 1$ given in Appendix B Eqs. 27-42. However for fermions we still need an expansion valid in the semidegenerate region say $\eta < \pi$ and for bosons the relativistic expansion is not yet in a suitable form. Consider the rearranged integrals

$$I_\pm^{ab} = \int_0^{2x} \frac{(u+x)^a (2ux)^{b/2} (1+\frac{u}{2x})^{b/2}}{e^{u-\eta} \pm 1} du + \int_{2x}^{\infty} \frac{(u+x)^a u^b (1+\frac{2x}{u})^{b/2}}{e^{u-\eta} \pm 1} \tag{34}$$

which can be obtained easily from (1) by the substitution $y = u+x$.

(i) Non relativistic Region

In the non relativistic regime with $x \gg 1$ the second integral is negligible ($\sim O(e^{-x})$) the last term in the numerator of the first integral can be expanded by the binomial theorem whence extending the integral to infinite range we obtain on ignoring another term $\sim O(e^{-x})$

$$I_{\pm}^{11} = \mp \left(\frac{\pi}{2} x^3\right)^{\frac{1}{2}} \left(\phi(\mp e^{\eta},3/2) + \phi(\mp e^{\eta},5/2) \frac{15}{8x} + \phi(\mp e^{\eta},7/2) \frac{105}{128x^2} + \ldots\right)$$

where $\eta < x$ and the $\phi(z,s)$ are defined by Eq.(B6). This equation is identical to Eq.(14) except that the $\phi(z,s)$ are now defined on a wider range. Long and tedious calculations for each of the Q^n and I^{ab} show in general that Eqs. (13-16) are valid for fermions with $x \gg 1$ and $\eta < x$ and for bosons with $x \gg 1$ and $\eta \leq 0$. This result is redundant for bosons but it allows use of tailor made expansions for the leptons. Thus in Eqs. (13-16) for $\eta \leq -\pi$ we use the first few terms of (B7) for $-\pi < \eta < \pi$ the expansions (B35-38) and for $\eta \geq \pi$ we use the series given in (B19). Finally when $\eta \gtrsim x$ we can return to the asymptotic series given in (23-32).

(ii) Relativistic Region

It is a simple matter, using integrals (1 and 2) to establish that the expansions for the relativistic regime Eqs. (17-20) are also valid in a wider range. For fermions the relativistic chemical potential λ can take any values with equations (B15-18, 39-42, 7) applying for λ large and positive, λ small and λ large and negative respectively. The accuracy is $\sim O(x^3)$. For bosons we certainly have $\lambda < x$ so that the expansion (B23) will give $\phi(e^{\lambda},n)$ for $\lambda > 0$ as well as $-\pi < \lambda \leq 0$.

These results serve to reiterate those of Guess (1966) who derives them by more elegant means using Mellin transforms. However the accuracy of the terms given in the boson case becomes questionable as $\lambda \to x$. In fact the remainder becomes large with respect to the last term given when $\frac{|\lambda - x|}{x} = \frac{|\eta|}{x} \ll 1$, and the last term given in Eq.(18) goes as $\sim \ln|\eta| x^2$ which clearly gives difficulties as $\eta \to 0$. It is clearly more sensible to derive a series with terms in η rather than

λ, this can be done by expanding the third term in the numerator of the second integral in Eq.(34) by the binomial theorem. The resulting expansions for both Q^n and I^{ab} are

$$Q^1_{\pm 1} = \mp \frac{1}{x}\,\phi(\mp e^\eta, 1)\ \mp, \tag{35}$$

$$Q^2_{\pm 1} = \mp \left(\frac{2}{x^2}\,\phi(\mp e^\eta, 2) + \frac{2}{x}\,\phi(\mp e^\eta, 1) + \ldots\right), \tag{36}$$

$$Q^3_{\pm 1} = \mp \left(\frac{8}{x^3}\,\phi(\mp e^\eta, 3) + \frac{8}{x^2}\,\phi(\mp e^\eta, 2) + \frac{3}{x}\,\phi(\mp e^\eta, 1) + \ldots\right), \tag{37}$$

$$Q^4_{\pm 1} = \mp \left(\frac{48}{x^4}\,\phi(\mp e^\eta, 4) + \frac{48}{x^3}\,\phi(\mp e^\eta, 3) + \frac{20}{x^2}\,\phi(\mp e^\eta, 2) + \frac{4}{x}\,\phi(\mp e^\eta, 1) \ldots\right), \tag{38}$$

where as before the Q^0 term is smaller than the others and will be ignored,

$$I^{11}_{\pm 1} = \mp \left(2\phi(\mp e^{\eta'}, 3) + 2x\,\phi(\mp e^\eta, 2) + \frac{x^2}{2}\,\phi(\mp e^\eta, 1) \ldots\right), \tag{39}$$

$$I^{21}_{\pm 1} = \mp \left(6\phi(\mp e^\eta, 4) + 6x\,\phi(\mp e^\eta, 3) + \frac{5x^2}{2}\,\phi(\mp e^\eta, 2) + \frac{x^3}{2}\,(\mp e^\eta, 1) \ldots\right), \tag{40}$$

$$I^{03}_{\pm 1} = \mp \left(6\phi(\mp e^\eta, 4) + 6x\,\phi(\mp e^\eta, 3) + \frac{3x^2}{2}\,\phi(\mp e^\eta, 2) - \frac{x^2}{2}\,(\mp e^\eta, 1) \ldots\right). \tag{41}$$

Except for the case of bosons with $\frac{|\eta|}{x} \ll 1$ these last three are accurate to $O(x^4)$ and return Eqs.(17-20) by use of the series expansions of Appendix B. For bosons with $\frac{|\eta|}{x} \ll 1$ we halt the binomial expansion in the second integral of Eq..(34) when the numerator gives the coefficient of the u^0 term, this term can be integrated explicitly while all the higher order terms can have their integrals extended to

$\int_0^\infty$ and the differences $\int_0^{2x}$ collected together with the first integral of Eq.(34) to give the remainder term. We can then show that the remainder must be less than $O(x^{a+b})$. Since a similar analysis can be done for the Q^n we can adopt the convention that when $\frac{|\eta|}{x} << 1$ the last term in all the above equations be replaced by an explicit evaluation. Thus for $\frac{|\eta|}{x} << 1$,

$$\phi(+e^{\eta},1) \to - \ell n(1-e^{-2x+\eta}). \qquad (42)$$

5. SUMMARY

In this appendix we give approximation schemes for the integrals $Q^n_{\pm 1}(x, \ )$, $I^{ab}_{\pm 1}(x,\lambda)$. The schemes operate in tandem with the calculation of two more basic functions, $K_n(x)$, $\phi(\mp e^{\eta},s)$ which are discussed in Appendix B. There are four basic categories of approximation, degenerate, nondegenerate, relativistic and non relativistic; the schemes thus cover all the (λ,x) half-plane except the small region $(\lambda \sim 1,\ x \sim 1)$. In this case the gas is semi-relativistic and semi-degenerate and direct numerical analysis must be used to give the integrals. The regions of approximation overlap sufficiently for rapid calculation and in Table 1 where the schemes are summarized we give conditions on their use and not their validity.

Table 1. Schemes for the $Q^n_{\pm 1}(x,\lambda)$ and $I^{ab}_{\pm}(x,\lambda)$.

Function and Conditions		Eq. n^0 Appendix A	Eq. n^0 Appendix B
$Q^n_{\pm 1}(x,\lambda)$			
bosons			
anyx	$\eta \le -1$	7	1
x>>1	$-1 < \eta \le 0$	13	22 or 27-30
x<<1	$-1 < \eta \le 0$	35-38,42	23 or 31-34
fermions			
anyx	$\eta \le -1$	7	1
$\eta \ge x$	η large	23-27,22,28,29	
x>>1	$-1 < \eta < x$	13	for $\eta<2.5$, 24 or 35-38 for $\eta\ge2.5$, 19
x<<1	any η	35-38 or 17	for $\eta\le-1$, 7 for $-1<\eta<1$, 25 or 39-42 for $\eta\ge1$, 15-18
$I^{ab}_{\pm}(x,\lambda)$			
bosons			
anyx	$\eta \le -1$	8-10	1
x>>1	$-1 < \eta \le 0$	14-16	22 or 27-30
x<<1	$-1 < \eta \le 0$	39-41,42	23 or 31-34
fermions			
anyx	$\eta \le -1$	8-10	1
$\eta \ge x$	η large	30-32,22,28,29	
x>>1	$-1 < \eta < x$	14-16	for $\eta<2.5$,24 or 35-38 for $\eta\ge2.5$,19
x<<1	any η	39-41	for $\eta\le-1$,7 for $-1<\eta<1$,25 or 39-42 for $\eta\ge1$, 15-18

APPENDIX B

1. MODIFIED BESSEL FUNCTIONS - $K_n(z)$

The modified Bessel functions of the second kind can be defined by (Abramowitz and Stegun 1965),

$$K_n(z) = \int_0^\infty e^{-z\cosh t}\cosh nt \, dt. \tag{1}$$

The relations,

$$2nK_n(z) = zK_{n+1}(z) - zK_{n-1}(z), \tag{2}$$

$$2\frac{d}{dz}K_n(z) = -K_{n-1}(z) - K_{n+1}(z), \tag{3}$$

can easily be established and for large z (i.e. non relativistic temperatures) we have the asymptotitc series,

$$K_n(z) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-z} \frac{1}{z^{\frac{1}{2}}} \left(1 + \frac{4n^2-1}{8z} + \frac{(4n^2-1)(4n^2-3^3)}{2!(8z)^2} + \ldots.\right). \tag{4}$$

Clearly at large z considerable care must be taken in the subtraction of the functions. Finally the K_n have a series expansion suitable for small z (relativistic temperatures) viz,

$$K_n(z) = \frac{1}{2}\sum_{=0}^{n-1} \left(\frac{2}{z}\right)^{n-2k} \frac{(-1)^k \; (n-k-1)!}{k!}$$

$$+ (-1)^{n+1}\sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{n+2k} \frac{1}{k!(n+k)!} \left| \ln\frac{z}{2} + C - \sum_{m=1}^{n+k}\frac{1}{2m} - \sum_{m=1}^{k}\frac{1}{2m} \right| \tag{5}$$

where C is Euler's constant and n is a positive integer or zero. Most useful are the approximations

$$K_2(z) \simeq 2/z^2,$$

$$K_3(z) \simeq 8/z^3.$$

2. GENERALIZED ZETA FUNCTIONS - $\phi(z,s)$

Many of the expansions derived in Appendix A involve integrals of the form

$$F^s_{\pm}(\eta) = \int_0^\infty \frac{t^{s-1}}{e^{t-\eta}\pm 1} \, dt.$$

We proceed here to develop series expressions for these integrals. The + sign refers to fermions and η has any value, the - sign refers to bosons and $\eta \leq 0$. The integrals are closely related to Appell's integral and we can set (Truesdell 1945)

$$\phi(z,s) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z} \, dt, \quad s > 0, \; z \neq [1,\infty]. \tag{6}$$

(i) $\eta < 0$

For the case $|z| \leq 1$ we obtain a Dirichlet series

$$\phi(z,s) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \qquad |z| \leq 1. \tag{7}$$

and as $|z| \to 1$ the functions return the Riemann zeta functions given as

$$\phi(+1,s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{8}$$

$$-\phi(-1,s) = \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1-2^{1-s})\,\zeta(s) \tag{9}$$

where the latter should not be confused with the non relativistic degeneracy parameter. The latter can be obtained by use of the identity

$$\phi(z,s) + \phi(-z,s) = 2^{1-s}\,\phi(z^2,s), \quad |z| \neq [1,\infty] \tag{10}$$

and we give for future reference Riemann's functional equation

$$\zeta(1-s) = \Gamma(s)\cos\frac{(s\pi)}{2}\,\zeta(s)\,\frac{2^{1-s}}{\pi^s} \quad . \tag{11}$$

Clearly the function $\phi(z,s)$ in Eq.(6) is in a certain sense a generalization of the zeta function so in future we shall call it the generalized zeta function. The function is closely related to that of Bateman (1965) who defines

$$\phi(z,s,v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s} \quad |z| < 1, \; v \neq 0,-1,-2,\ldots.$$

which takes the integral form

$$\phi(z,s,v) = \frac{1}{z\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{\frac{e^t}{z} - 1}\,dt \tag{12}$$

and gives $\qquad \phi(z,s) = z\phi(z,s,1).$ $\qquad$ (13)

(ii) $\underline{\eta > 0}$

Consider the case of fermions

$$-\phi(-e^\eta,s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-\eta}+1}\,dt. \tag{14}$$

When s is integer and positive the denominator can be expanded in a geometric series the range of the integral split and the series integrated term by term to give for $\eta \geq 0$

$$-\phi(-e^\eta,1) = \ln\,(1+e^\eta), \tag{15}$$

$$-\phi(-e^\eta,2) = \frac{\eta^2}{2} - 2\phi(-1,2) + \phi(-e^{-\eta},2), \tag{16}$$

$$-\phi(-e^{\eta},3) = \left(\frac{\eta^3}{3} - 4\eta\phi(-1,2) - 2\phi(-e^{-\eta},3)\right)/_{\Gamma(3)} \tag{17}$$

$$-\phi(-e^{\eta},4) = \left(\frac{\eta^4}{4} - 6\eta^2\phi(-1,2) - 12\phi(-1,4) + 6\phi(-e^{-\eta},4)\right)/_{\Gamma(4)}. \tag{18}$$

The final term in the last three equations can then be expanded by use of the series Eq.(7).

For s non integral we shall only be concerned with half integral values with $s \geq \frac{1}{2}$. The results are more complicated than those above and require a Taylor expansion with explicit calculation of remainder. We obtain

$$-\phi(-e^{\eta},n+1) = -2\sum_{m=0}^{p} \frac{\phi(-1,2m)\ \eta^{n+\frac{1}{2}-2m}}{\Gamma(n+\frac{3}{2}-2m)} + \tag{19}$$

$$\frac{(-1)^{n+1}.2.\ \Gamma(2p-n+\frac{3}{2})}{\eta^{2p-n+\frac{3}{2}}}\left[C_{2p-n+\frac{1}{2}}(\eta) - \frac{1}{2^{2p+2}}C_{2p-n+\frac{1}{2}}(2\cdot\eta) + \frac{1}{3^{2p+2}}C_{2p-n+\frac{1}{2}}(3\cdot\eta) - \ldots\right].$$

where $p = \simeq \frac{1}{2}(\eta+n-\frac{1}{2})$ and is chosen so that the last term in the summation is the least. The first term in (19) is the standard Sommerfeld expansion (Sommerfeld, 1928) with the $\phi(-1,2m)$ as given in Table 1; we note in particular $\phi(-1,0) = -\frac{1}{2}$. The second term is a calculation of the remainder and was given in this form by Dingle (1957) who provides an exact convergent series for the $C_r(x)$

$$C_r(x) = \frac{-x^2}{r(r-1)}\left[1 + \frac{x^2}{(r-2)(r-3)} + \frac{x^4}{(r-2)(r-3)(r-4)(r-5)} + \ldots\ldots\right]$$

$$+ (-1)^{r+\frac{1}{2}}\frac{\pi x^{r-1}e^{x}}{2.\Gamma(r+1)}. \tag{20}$$

For large x computation of Eq.(20) becomes troublesome due to catastrophic cancellation between the two parts. However for large η we will only need the first of the remainder terms which to an order of magnitude is

$$C_r(x) \simeq \tfrac{1}{2}(x-r+\frac{1}{6}). \tag{21}$$

(iii) $\underline{\eta \simeq 0}$

These equations are appropriate when $\eta>>1$ for the leptons with Eq.(7) being used when $\eta<<0$ for leptons and bosons. In the intermediate range, however the equations are not useful. Bateman (1965) derives other series expansions which show a more appropriate behaviour in this region and moreover allow the $\phi(z,s)$ to be defined in the wider range $|\ln z| < 2\pi$. In our notation these series become

$$\phi(e^w,s) = \Gamma(1-s)(-w)^{s-1} + \sum_{m=0}^{\infty} \zeta(s-m)\frac{w^m}{m!} \qquad s \neq 1,2,3,\ldots\ldots \quad , \tag{22}$$

$$\phi(e^w,n) = \frac{w^{n-1}}{(n-1)!}\left(1 + \frac{1}{2} + \frac{1}{3}\ldots\ldots \frac{1}{n-1} - \ell n|-w|\right) + \sum_{m=0}^{\infty}{}' \zeta(n-m)\frac{w^m}{m!}$$

$$n = 2,3,\ \ldots\ldots \tag{23}$$

The prime on the summation indicates that the term corresponding to $m = n-1$ is omitted and the series are convergent for $|w| < 2\pi$. Use of the identity Eq.(10) now gives

$$\phi(-e^w,s) = \sum_{m=0}^{\infty} \zeta(s-m)\frac{w^m}{m!}\ (2^{1+m-s}-1) = -\sum_{m=0}^{\infty} \eta(s-m)\frac{w^m}{m!} \quad s \neq 1,2,3\ldots\ldots \tag{24}$$

and

$$\phi(-e^w,n) = -\frac{w^{n-1}}{(n-1)!}\ell n(2) - \sum_{m=0}^{1\infty} \eta(n-m)\frac{w^m}{m!} \qquad n = 2,3,\ \ldots\ldots \tag{25}$$

Since the square of z appears in Eq.(10) these last two are convergent for $|w| < \pi$.

Finally in the case of s = 1 we have the simple closed form

$$\phi(\mp e^{w},1) = - \ln (1 \pm e^{w}). \tag{26}$$

(iv) Summary

We give the actual series resulting from (22-26) below. Partial checks can be made on the first group by referring to Robinson (1951) and the last two groups through Bludman & Riper (1977). Accuracy is to at least six decimal places for $|w| \leq 1$. In the case of bosons and fermions with integer s Eqns. (7) and (15-18) provide for rapid calculations outside this range.

The situation for fermions with half integer s is more complicated. The $\phi(-e^{\eta}, n + \frac{1}{2})$ present no problems for $\eta < -1$ (Eq.(7)) but for $\eta > 1$ neither (19) nor the first few terms of (24) can supply the required accuracy. Instead for $\eta \leq 2.5$ we must use many terms of the series (24) (more than 30 for 4 figure accuracy) and for $\eta > 2.5$ Eq.(19) with careful calculation of the remainder.

We summarize the schemes discussed in the appendix in Table 2.

For bosons for $w \leq 1$ we have to at least 6 decimal places

$$\phi(e^{-w},\tfrac{1}{2}) = 1.772454\ w^{-\frac{1}{2}} - 1.460354 + .207886w - 1.27426\ 10^{-2}\ w^2 - 1.41948\ 10^{-3}\ w^3 + 1.85042\ 10^{-4}\ w^4 + 2.57639.10^{-5}\ w^5 - 3.71036\ 10^{-6}\ w^6 - 5.44993\ 10^{-7}\ w^7 \tag{27}$$

$$\phi(e^{-w},\tfrac{3}{2}) = -3.544908\ w^{\frac{1}{2}} + 2.612375 + 1.460354\ w - .103943\ w^2 + 4.2475\ 10^{-3}\ w^3 + 3.54871\ 10^{-4}\ w^4 - 3.70084\ 10^{-5}\ w^5 - 4.29399\ 10^{-6}\ w^6 + 5.30051\ 10^{-7}\ w^7 \tag{28}$$

$$\phi(e^{-w},\tfrac{5}{2}) = 2.363272\ w^{\frac{3}{2}} + 1.341487 - 2.612375\ w - .730177\ w^2 + 3.46477\ 10^{-2}\ w^3 - 1.06188\ 10^{-3}\ w^4 - 7.09742\ 10^{-5}\ w^5 + 6.16807\ 10^{-6}\ w^6 + 6.13426\ 10^{-7}\ w^7 \tag{29}$$

$$\phi(e^{-w},\tfrac{7}{2}) = -.945309\ w^{\frac{5}{2}} + 1.1267338 - 1.3414873\ w + 1.3061877\ w^2 + .243392\ w^3 - 8.66193\ 10^{-3}\ w^4 + 2.12377\ 10^{-4}\ w^5 + 1.18291\ 10^{-5}\ w^6 - 8.81153\ 10^{-7}\ w^7 \tag{30}$$

and

$$\phi(e^{-w},1) = -\ell n(1-e^{-w}) \tag{31}$$

$$\phi(e^{-w},2) = -w(1-\ell nw) + 1.644934 - .25\ w^2 + 1.388889\ 10^{-2}\ w^3 - 6.944444\ 10^{-5}\ w^5 + 7.8735\ 10^{-7}\ w^7 \tag{32}$$

$$\phi(e^{-w},3) = w^2(.75 - .5\ell nw) + 1.202057 - 1.644934\ w + 8.333333\ 10^{-2}\ w^3 - 3.472222\ 10^{-3}\ w^4 + 1.157407\ 10^{-5}\ w^6 \tag{33}$$

$$\phi(e^{-w},4) = -w^3\ (.305556 - .166667\ \ell nw) + 1.082323 - 1.202057\ w + .822467\ w^2 - 2.083333\ 10^{-2}\ w^4 + 6.9444\ 10^{-4}\ w^5 - 1.6534\ 10^{-6}\ w^7. \tag{34}$$

For fermions we have

$$-\phi(-e^{w},\tfrac{1}{2}) = .604899 + .380105\ w + 5.934043\ 10^{-2}\ w^{2} - - 1.464018\ 10^{-2}\ w^{3} - 4.001983\ 10^{-3}\ w^{4} + 1.140178\ 10^{-3}\ w^{5} + 3.321130\ 10^{-4}\ w^{6} - 9.810936\ 10^{-5}\ w^{7} - 2.927207\ 10^{-5}\ w^{8} + 8.799419\ 10^{-6}\ w^{9} + 2.660843\ 10^{-6}\ w^{10} - 8.084684\ 10^{-7}\ w^{11} \tag{35}$$

$$-\phi(-e^{w},\tfrac{3}{2}) = .765147 + .604899\ w + .190052\ w^{2} + 1.978014\ 10^{-2}\ w^{3} - 3.660046\ 10^{-3}\ w^{4} - 8.003967\ 10^{-4}\ w^{5} + 1.900296\ 10^{-4}\ w^{6} + 4.744470\ 10^{-5}\ w^{7} - 1.226367\ 10^{-5}\ w^{8} - 3.252452\ 10^{-6}\ w^{9} + 8.799419\ 10^{-7}\ w^{10} \tag{36}$$

$$-\phi(-e^{w},\tfrac{5}{2}) = .867200 + .765147\ w + .302449\ w^{2} + 6.335080\ 10^{-2}\ w^{3} + 4.945036\ 10^{-3}\ w^{4} - 7.320092\ 10^{-4}\ w^{5} - 1.333994\ 10^{-4}\ w^{6} + 2.714709\ 10^{-5}\ w^{7} + 5.930588\ 10^{-6}\ w^{8} - 1.362630\ 10^{-6}\ w^{9} - 3.252452\ 10^{-7}\ w^{10} \tag{37}$$

$$-\phi(-e^{w},\tfrac{7}{2}) = .927554 + .867200\ w + .382574\ w^{2} + .100816\ w^{3} + 1.583770\ 10^{-2}\ w^{4} + 9.890072\ 10^{-4}\ w^{5} - 1.220015\ 10^{-4}\ w^{6} - 1.905706\ 10^{-5}\ w^{7} + 3.393386\ 10^{-6}\ w^{8} + 6.589543\ 10^{-7}\ w^{9} - 1.362630\ 10^{-7}\ w^{10} \tag{38}$$

and

$$-\phi(-e^{w},1) = \ell n(1+e^{w}) \tag{39}$$

$$-\phi(-e^{w},2) = .693147\ w + .822467 + .25w^{2} + 4.166667\ 10^{-2}\ w^{3} - 1.041667\ 10^{-3}\ w^{5} + 4.960317\ 10^{-5}\ w^{7} - 2.927965\ 10^{-6}\ w^{9} + 1.941538\ 10^{-7}\ w^{11} \tag{40}$$

$$-\phi(-e^{w},3) = .346574\ w^{2} + .901543 + .822467\ w + 8.333333\ 10^{-2}\ w^{3} + 1.041667\ 10^{-2}\ w^{4} - 1.736111\ 10^{-4}\ w^{6} + 6.200397\ 10^{-6}\ w^{8} - 2.927965\ 10^{-7}\ w^{10} \tag{41}$$

$$-\phi(-e^{w},4) = .115525\ w^{3} + .947033 + .901543\ w + .411234\ w^{2} + 2.083333\ 10^{-2}\ w^{4} + 2.083333\ 10^{-3}\ w^{5} - 2.480159\ 10^{-5}\ w^{7} + 6.889333\ 10^{-7}\ w^{9}. \tag{42}$$

Table 1. Riemann Zeta and Related Functions

s	$\zeta(s)$	$\zeta(-s)$	$\eta(s)$	$\eta(-s)$
0	- .5		.5	
1/2	-1.460354	- .207886	.604899	.380105
1	∞	$-8.333333.10^{-2}$	.693147	.25
3/2	2.612375	$-2.548520.10^{-2}$	.765147	.118681
2	1.644934	0.	.822467	0.
5/2	1.341487	$8.516928\ 10^{-3}$	.867200	$-8.78411\ 10^{-2}$
3	1.202057	$8.333333\ 10^{-3}$	.901543	- .125
7/2	1.126734	$4.441011\ 10^{-3}$	.927554	$-9.604760\ 10^{-2}$
4	1.082323	0.	.947033	0.
9/2	1.054708	$-3.091669\ 10^{-3}$	.961484	.136821
5	1.036928	$-3.968253\ 10^{-3}$	.972120	.25
11/2	1.025205	$-2.671458.10^{-3}$	.979897	.239121
6	1.017343	0.	.985551	0.
13/2	1.012006	$2.746768\ 10^{-3}$	.989644	- .494471
7	1.008349	$4.166667\ 10^{-3}$	.992594	-1.062500
15/2	1.005827	$3.269040\ 10^{-3}$	.994714	-1.180250
8	1.004077	0.	.996233	0.
17/2	1.002859	$-4.416033\ 10^{-3}$	.997319	3.193133
9	1.002008	$-7.57577\ 10^{-3}$	.998094	7.750000
19/2	1.001413	$-6.672172\ 10^{-3}$	.998647	9.655665
10	1.000995	$1.114612\ 10^{-3}$	.999040	0.

Note the useful relations $\zeta(2) = \pi^2/6$, $\eta(2) = \pi^2/12$, $\zeta(4) = \pi^4/90$ and $\eta(4) = 7\pi^4/720$.

Table 2. Schemes for the $K_n(x)$ and $\phi(\pm e^{\eta},s)$.

Function	Conditions		Equation
$K_n(x)$			
	$x \gg 1$	any n	4
	$x \ll 1$	" "	5
$\phi(\pm e^{\eta},s)$			
bosons	$\eta \le -1$	any s	7
	$-1 < \eta \le 0$	half integer s	22 or 27-30
	$-1 < \eta \le 0$	integer s	23 or 31-34
fermions	$\eta \le -1$	any s	7
	$-1 < \eta \le 2.5$	half integer s	24 or 35-38
	$-1 < \eta < 1$	integer s	25 or 39-42
	$\eta > 2.5$	half integer s	19
	$\eta \ge 1$	integer s	15-18

APPENDIX C

1. INTRODUCTION

When both particle and antiparticle species are in thermal and chemical equilibrium with black body radiation ($\mu_\gamma = 0$) it proves useful to define new integrals similar to the I^{ab} and Q^n so that one can give directly the thermodynamical functions; $n(x,\lambda) - n(x,-\lambda)$, $\rho(x,\lambda) + \rho(x,-\lambda)$, $p(x,\lambda) + p(x,-\lambda)$ and $s(x,\lambda) + s(x-\lambda)$ (where the convention that $\lambda \geq 0$ is adopted). This can be done consistently by defining the new integrals

$$J_\varepsilon^{ab}(x,\lambda) = I_\varepsilon^{ab}(x,\lambda) + (-1)^a I_\varepsilon^{ab}(x,-\lambda), \tag{1}$$

$$R_\varepsilon^n(x,\lambda) = Q_\varepsilon^n(x,\lambda) + (-1)^n Q_\varepsilon^n(x,-\lambda). \tag{2}$$

The thermodynamical functions for the pair are now given by those for a single species replaced with the above. The relations between the I^{ab} and Q^n Eqs.(A3-5) carry over directly for the J^{ab} and R^n and the differential relation between the Q^n yields the same form for the R^n,

$$d(R_\varepsilon^{n+1} - R_\varepsilon^{n-1}) = 2n\, R_\varepsilon^n \frac{d\lambda}{\lambda} - ((n+1)\, R_\varepsilon^n + (n-1)\, R_\varepsilon^{n-1}) \frac{dx}{x}\,. \tag{3}$$

The task of this appendix is to consider approximation schemes to the J^{ab} and R^n in the same way as Appendix A considers the I^{ab} and Q^n. We note however that the new integrals are defined in a different range; for fermions we have $0 \leq \lambda < \infty$ and bosons $0 \leq \lambda \leq x$. Thus the fluid cannot be non-degenerate at high temperatures.

When λ is large the contributions made by the second integrals in Eqs.(1+2) are clearly negligible and we have

$$J_\varepsilon^{ab} = I_\varepsilon^{ab} \qquad \lambda>>1 \tag{4}$$

$$R_\varepsilon^{n} = Q_\varepsilon^{n} \qquad \lambda>>1. \tag{5}$$

2. NON RELATIVISTIC REGION

For the case of $\lambda \leq x$ we can write with the help of Eqs.(A7-10)

$$R_\varepsilon^n = \sum_{m=1}^{\infty} (-\varepsilon)^{m+1} H^n(m\lambda)\; K_n(mx), \tag{6}$$

$$J_\varepsilon^{11} = \sum_{m=1}^{\infty} (-\varepsilon)^{m+1} x^2 \frac{H^1(m\lambda)}{m} K_2(mx), \tag{7}$$

$$J_\varepsilon^{21} = \sum_{m=1}^{\infty} (-\varepsilon)^{m+1} x^3 \frac{H^2(m\lambda)}{m} (K_3(mx) - K_2(mx)/mx). \tag{8}$$

$$J_\varepsilon^{03} = 3\sum_{m=1}^{\infty} (-\varepsilon)^{m+1} x^2 \frac{H^0(m\lambda)}{m^2} K_2(mx), \tag{9}$$

where the hyperbolic functions

$$H^n(m\ ) = \begin{cases} 2\sinh m\lambda & \text{for n odd} \\ 2\cosh m\lambda & \text{for n even.} \end{cases}$$

Since we are not interested in large λ Eqs.(6-9) provide excellent expansions for the non relativistic region where only the first term is sufficient. The expansions are also useful in the semi-relativistic region $x \simeq 1$ as long as $\frac{x}{(x-\lambda)} \simeq 1$ for then the exponentially decreasing (at large mx) $K_n(mx)$ will ensure rapid convergence.

3. RELATIVISTIC REGION

(i) $\eta < \pi$

In the relativistic regime we can use the expansions in terms of generalized zeta functions given by Eqs. (A35-42). When $\lambda < \pi$ we use the series expansions for the zeta functions Eqs. (B22-26). The equations are much simpler than would be expected from (B31-34, B39-42) due to cancellations between the series.

$$R_+^1 = \frac{\lambda}{x} + \frac{1}{x} \ln \left| \frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})} \right| , \tag{10}$$

$$R_+^2 = \frac{1}{x^2} (3.289868 + \lambda^2) + \frac{1}{x} (2\lambda + 2\ln (\frac{\cosh 3x + \cosh\lambda}{2})) +$$

$$\frac{1}{x} (-.5\,\lambda^2 + 2.083\;10^{-2}\,\lambda^4 - 1.389\;10^{-3}\,\lambda^6 + 1.054\;10^{-4}\,\lambda^8), \tag{11}$$

$$R_+^3 = \frac{1}{x^3} (13.159472\,\lambda + 1.333333\,\lambda^3) + \frac{1}{x} (-\lambda + 3\ln \left| \frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})} \right|)$$

$$+ 1.333\,\lambda - .111\,\lambda^3 + 1.111\;10^{-2}\lambda^5 - 1.124\;10^{-3}\lambda^7, \tag{12}$$

$$R_+^4 = \frac{1}{x^4} (90.915152 + 39.478417\,\lambda^2 + 2\lambda^4) + \frac{1}{x^2} (-6.580 - 2\lambda^2) +$$

$$\frac{1}{x} (4\ln (\frac{\cosh 3x + \cosh\lambda}{2}) - \lambda^2 + 4.167\;10^{-2}\lambda^4 - 2.778\;10^{-3}\lambda^6 +$$

$$2.108\;10^{-4}\lambda^8), \tag{13}$$

and

$$R_-^1 = -\frac{\lambda}{x} - \frac{1}{x} \ln \left| \frac{\sinh(\frac{3x-\lambda}{2})}{\sinh(\frac{3x+\lambda}{2})} \right| , \tag{14}$$

$$R_-^2 = \frac{6.579736}{x^2} - \frac{4}{x} + \frac{2}{x^2} ((x-\lambda)\ln(x-\lambda) + (x+\lambda)\ln(x+\lambda)) -$$

$$\frac{2}{x} \ln(2(\cosh 3x - \cosh\lambda)), \tag{15}$$

$$R_-^3 = \frac{1}{x^3}(26.318945\ \lambda - 1.333333\ \lambda^3) - \frac{8\lambda}{x^2} + \frac{\lambda}{x} - \frac{3}{x}\ \ell n \left|\frac{\sinh(\frac{3x-\lambda}{2})}{\sinh(\frac{3x+\lambda}{2})}\right| +$$

$$\frac{4}{x^3}(x^2 - \lambda^2)\ \ell n\ (\frac{x-\lambda}{x+\lambda}) \ , \tag{16}$$

$$R_-^4 = \frac{1}{x^4}(103.90303 + 78.957\ \lambda^2) + \frac{18.667}{x} - \frac{4}{x}\ \ell n\ (2(\cosh 3x - \cosh\lambda)) -$$

$$\frac{1}{x^4}(8\lambda^3 - 4\lambda x^2 - 4x^3)\ \ell n(x-\lambda) + \frac{1}{x^4}(8\lambda^3 - 4\lambda x^2 + 4x^2)\ \ell n(x+\lambda). \tag{17}$$

We also have

$$J_+^{11} = 3.289868\ \lambda + .333333\ \lambda^3 + x^2(-\frac{\lambda}{2} + \frac{1}{2}\ \ell n \left|\frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})}\right|) +$$

$$x^3(.333\ \lambda - 2.778\ 10^{-2}\lambda^3 + 2.778\ 10^{-3}\lambda^5 - 2.811\ 10^{-4}\lambda^7), \tag{18}$$

$$J_+^{21} = 11.364394 + 4.934802\ \lambda^2 - .8225\ x^2 + \frac{x^3}{2}\ \ell n\ (\frac{\cosh 3x + \cosh\lambda}{2}) +$$

$$.25\ \lambda^4 - .25\ \lambda^2 x^2 + x^3(-.125\ \lambda^2 + 5.208\ 10^{-3}\lambda^4 - 3.472\ 10^{-4}\lambda^6 +$$

$$2.635\ 10^{-5}\lambda^8), \tag{19}$$

$$J_+^{03} = 11.364394 + 4.934802\ \lambda^2 - 2.467\ x^2 - \frac{x^3}{2}\ \ell n\ (\frac{\cosh 3x + \cosh\lambda}{2}) +$$

$$.25\ \lambda^4 - .75\ \lambda^2 x^2 + x^3(.125\ \lambda^2 - 5.208\ 10^{-3}\lambda^4 + 3.472\ 10^{-4}\lambda^6 -$$

$$2.635\ 10^{-5}\lambda^8), \tag{20}$$

and

$$J_+^{21} + \frac{J_+^{03}}{3} - \lambda J_+^{11} = 15.152525 + 3.289868\ \lambda^2 - 1.645\ x^2 - .5\ \lambda^2 x^2 - x^2$$

$$(-\frac{\lambda^2}{2} + \frac{\lambda}{2}\ \ell n \left|\frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})}\right|) + \frac{x^3}{3}\ \ell n\ (\frac{\cosh 3x + \cosh\lambda}{2}) +$$

$$x^3(-.5\ \lambda^2 + 3.125\ 10^{-2}\lambda^4 - 3.009\ 10^{-3}\lambda^6 + 2.987.10^{-4}\lambda^8). \tag{21}$$

Similarly for bosons we have

$$J_-^{11} = 6.579736 \quad - .333\,\lambda^3 + \frac{\lambda x^2}{2} - \frac{x^2}{2}\,\ell n\, \frac{\sinh(\frac{3x-\lambda}{2})}{\sinh(\frac{3x+\lambda}{2})} \quad - 2\lambda x \; +$$

$$(x^2-\lambda^2)\;\ell n\,(\frac{x-\lambda}{x+\lambda})\;, \tag{22}$$

$$J_-^{21} = 12.987879 + 9.870\,\lambda^2 - 1.645\,x^2 - 2\lambda^2 x + .333\,x^2 - .5\,x^3\,\ell n$$
$$(2(\cosh 3x - \cosh\lambda)) - (\lambda^3 - \frac{\lambda x^2}{2} - \frac{x^3}{2})\;\ell n(x-\lambda) \; +$$
$$(\lambda^3 - \frac{\lambda x^2}{2} + \frac{x^3}{2})\;\ell n(x+\lambda), \tag{23}$$

$$J_-^{03} = 12.987879 + 9.870\,\lambda^2 - 4.935\,x^2 - 2\,\lambda^2 x + 2.333\,x^3 - .5\,x^3$$
$$\ell n(2(\cosh 3x - \cosh\lambda)) - (\lambda^3 - \frac{3}{2}\,\lambda x^2 + \frac{x^3}{2}\,)\;\ell n(x-\lambda) \; +$$
$$(\lambda^3 - \frac{3}{2}\,\lambda x^3 - \frac{x^3}{2}\,)\;\ell n(x+\lambda), \tag{24}$$

and

$$J_-^{21} + \frac{J_-^{03}}{3} - \lambda J_-^{11} = 17.317172 + 6.580\,\lambda^2 - 3.290\,x^2 - .667\,\lambda^2 x \; +$$
$$1.111\,x^3 - .333\,x^3\,\ell n(2(\cosh 3x - \cosh\lambda)) + .5\,\lambda x^2\,\ell n \left(\frac{\sinh(\frac{3x-\lambda}{2})}{\sinh(\frac{3x+\lambda}{2})}\right) \; +$$
$$(\frac{x^3-\lambda^3}{3})\;\ell n(x-\lambda) + (\frac{x^3+\lambda^3}{2})\;\ell n(x+\lambda), \tag{25}$$

(ii) Fermions: $\eta \geq \pi$

For fermions in the relativistic region with $\lambda \geq \pi$ we use Eqs. (B15-18) for $\phi(-e^{\lambda-x},n)$ and (B7) for $\phi(-e^{-\lambda-x},n)$. We thus obtain

$$R_+^1 = \frac{\lambda}{x} + \frac{1}{x}\,\ell n\,\frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{\lambda+3x}{2})}\;, \tag{26}$$

$$R_+^2 = \frac{1}{x^2}\,(3.289868 + \lambda^2 - x^2 - 2e^{-\lambda}\,\sinh x + .5e^{-2\lambda}\,\sinh 2x) \; +$$
$$\frac{2}{x}\;\ell n((e^{-\lambda}+e^{-3x}).(1 + e^{-\lambda-3x})), \tag{27}$$

$$R_+^3 = \frac{1}{x^3}(1.333333\,\lambda^3 + 13.159472\,\lambda + 16e^{-\lambda}\sinh x - 2e^{-2\lambda}\sinh 2x) -$$

$$\frac{\lambda}{x} + 2.667 + \frac{3}{x}\,\ell n\left[\frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})}\right] - \frac{16}{x^2}e^{-\lambda}\cosh x + \frac{4}{x^2}e^{-2\lambda}\cosh 2x, \tag{28}$$

$$R_+^4 = \frac{1}{x^4}(2\,\lambda^4 + 39.478417\,\lambda^2 + 90.915152 - 96\,e^{-\lambda}\sinh x + 6e^{-2\lambda}\sinh 2x) +$$

$$\frac{1}{x^3}(96e^{-\lambda}\cosh x - 12e^{-2\lambda}\cosh 2x) + \frac{1}{x^2}(-2\,\lambda^2 - 6.580 - 40e^{-\lambda}\sinh x +$$

$$10e^{-2\lambda}\sinh 2x) + \frac{4}{x}\ell n((e^{-\lambda} + e^{-3x})(1 + e^{-\lambda-3x})), \tag{29}$$

and

$$J_+^{11} = .333333\,\lambda^3 + 3.289868\,\lambda - .5\,\lambda x^2 + 4e^{-\lambda}\sinh x - .5e^{-2\lambda}\sinh 2x -$$

$$4\,xe^{-\lambda}\cosh x + x\,e^{-2\lambda}\cosh 2x + .667x^3 + .5x^2\,\ell n\left[\frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})}\right], \tag{30}$$

$$J_+^{21} = .25\,\lambda^4 + 4.934802\,\lambda^2 + 11.364394 - 12e^{-\lambda}\sinh x + .75e^{-2\lambda}\sinh 2x +$$

$$12x\,e^{-\lambda}\cosh x - 1.5x\,e^{-2\lambda}\cosh 2x - .25\,\lambda^2x^2 - .8225\,x^2 -$$

$$5x^2e^{-\lambda}\sinh x + 1.25x^2e^{-2\lambda}\sinh 2x + .5x^3\ell n((e^{-\lambda}+e^{-3x})(1+e^{-\lambda-3x})), \tag{31}$$

$$J_+^{03} = .25\,\lambda^4 + 4.934802\,\lambda^2 + 11.364394 - 12e^{-\lambda}\sinh x + .75e^{-2\lambda}\sinh 2x +$$

$$12x\,e^{-\lambda}\cosh x - 1.5x\,e^{-2\lambda}\cosh 2x - .75\,\lambda^2x^2 - 2.467x^2 - 3x^2e^{\lambda}\sinh x +$$

$$.75x^2e^{-2\lambda}\sinh 2x - .5x^2\ell n((e^{-\lambda}+e^{-3x})(1+e^{-\lambda-3x})), \tag{32}$$

giving

$$J_+^{21} + \frac{J_+^{03}}{3} - \lambda J_+^{11} = 3.289868\,\lambda^2 + 15.152525 - 1.645\,x^2 - .6667\,\lambda x^3 -$$

$$16e^{-\lambda}\sinh x + e^{-2\lambda}\sinh 2x + 16x\,e^{-\lambda}\cosh x - 2x\,e^{-2\lambda}\cosh 2x -$$

$$4\,\lambda e^{-\lambda}\sinh x + .5\,\lambda e^{-2\lambda}\sinh 2x - 6x^2e^{-\lambda}\sinh x + 1.5x^2e^{-2\lambda}\sinh 2x +$$

$$4\ \lambda x\ e^{-\lambda}\ \cosh x - \lambda x\ e^{-2\lambda}\ \cosh 2x - \frac{\lambda x^2}{2}\ \ell n \left|\frac{\cosh(\frac{3x-\lambda}{2})}{\cosh(\frac{3x+\lambda}{2})}\right| +$$

$$\frac{x^3}{3}\ \ell n((e^{-\lambda}+e^{-x})(1+e^{-\lambda-3x})). \tag{33}$$

4. BOSONS NEAR THE TRANSITIONAL DENSITY: RELATIVISTIC REGION

The behaviour of the boson integrals close to transitional density is of some importance. For λ very close to x i.e. for $x/(x-\lambda)>>1$ we have

$$R^1_- = \frac{1}{x}(.693147 - \lambda), \tag{34}$$

$$R^2_- = \frac{1}{x^2}(6.579736 - 5.3863\ x), \tag{35}$$

$$R^3_- = \frac{1}{x^3}(26.318945\ \lambda - 1.33333\ \lambda^3) - \frac{8\lambda}{x^2} + \frac{\lambda}{x} + \frac{2.079}{x}, \tag{36}$$

$$R^4_- = \frac{1}{x^4}(103.90303 + 78.957\ \lambda^2) + \frac{15.894}{x}, \tag{37}$$

and

$$J^{11}_- = 6.579736\ \lambda - 2\ \lambda x + .347\ x^2 - .333\ \lambda^3 + .5\ \lambda x^2, \tag{38}$$

$$J^{21}_- = 12.987879 + 9.870\ \lambda^2 - 1.645\ x^2 - 2\ \lambda^2 x - 1.32\ 10^{-2}\ x^3, \tag{39}$$

$$J^{03}_- = 12.987879 + 9.870\ \lambda^2 - 4.935\ x^2 - 2\ \lambda^2 x + 2.680\ x^3, \tag{40}$$

giving

$$J^{21}_- + \frac{J^{03}_-}{3} - J^{11}_- = 17.317172 + 6.580\ \lambda^2 - 3.290\ x^2 - 1.013\ \lambda^2 x + .880\ x^3. \tag{41}$$

In the limit $\lambda \rightarrow x$ we thus have

$$R^{1}_{-} = .693147/x - 1, \quad R^{2}_{-} = 6.579736/x^{2} - 5.3863/x,$$

$$R^{3}_{-} = 26.318945/x^{2} - 5.921/x - .333, \quad R^{4}_{-} = 103.90303/x^{4} + 78.957/x^{2} + 15.894/x, \tag{42}$$

and

$$J^{11}_{-} = 6.579736\, x - 1.653\, x^{2} + .1667\, x^{3}, \tag{43}$$

$$J^{21}_{-} = 12.987879 + 8.225\, x^{2} - 2.013\, x^{3}, \tag{44}$$

$$J^{03}_{-} = 12.987879 + 4.935\, x^{2} + .680\, x^{3}, \tag{45}$$

with

$$J^{21}_{-} + \frac{J^{03}_{-}}{3} - J^{11}_{-} = 17.317172 + 3.290\, x^{2} - .133\, x^{3}. \tag{46}$$

5. SUMMARY

We discuss expansions for the pair production functions $J^{ab}_{\varepsilon}(x,\lambda)$, $R^{n}_{\varepsilon}(x,\lambda)$. The degeneracy parameter has the range $0 \le \lambda < \infty$ for fermions, $0 \le \lambda \le x$ for bosons; that is at high temperatures $(x \le 1)$ there can be no non degeneracy.

Whenever λ is large the pair functions go over to their single particle functions. In the non relativistic region Eqs.(6-9) give excellent approximation; the same equations extend into the semi relativistic as long as $x/(x-\lambda)$ is not large (Sec.2). Expansions for the relativistic region are detailed in Sec.3. Bosons and fermions with $0 \le \lambda < \pi$ are discussed in Subsection (i) and fermions with $\lambda \ge \pi$ in Subsection (ii). Finally in Sec.4 we give the functions for the special case of bosons approaching the transitional density, i.e. for $x/(x-\lambda) \gg 1$.

APPENDIX D

The type of numerical problems encountered in this work require speedy techniques for their solution. The notation of Guess has allowed us to set up fast approximating expansions for the integrals it also allows us here to develop convenient numerical schemes based on the many dimensional Newton-Raphson formula (e.g. Isaacson 1966). The Newton-Raphson method is particularly useful since it has the property of quadratic convergence.

Consider the numerical problem presented by an m component gas without reactions. The equations governing the chemical evolution are (c.f. 4.18-4.20)

$$1 = \frac{fg}{16\zeta(3)} \; x^3 \; (Q^3-Q^1), \tag{1}$$

$$\frac{N}{N_i} = \frac{gx^3}{g_i x_i^3} \; \frac{(Q^3-Q^1)}{(Q_i^3-Q_i^1)}, \quad i = 1 \; \ldots \; m, \; m\text{-}1 \text{ equations} \tag{2}$$

$$\frac{S}{kN} = \sum_{i=1}^{m} \frac{N_i}{N} \left[\frac{2x_i}{3} \; \frac{(Q_i^4-Q_i^2)}{(Q_i^3-Q_i^1)} - \lambda_i \right] \tag{3}$$

The numerical problem is to solve the m equations (2-3) given the m constants (g_iN/gN_i, S/kN) for the m chemical potentisl ($\lambda_1 \; \ldots . \; \lambda_m$). The photon to reference particle number ratio f(x) is then given through (1). Choosing the reference component as i = m, Eqs.(2-3) go over to the iterative equations

$$F_j \equiv \frac{x_m^3}{x_j^3} \; \frac{(Q_m^3(x_m,\lambda_m) - Q_m^1(x_m,\lambda_m))}{(Q_j^3(x_j,\lambda_j) - Q_j^1(x_j,\lambda_j))} - \frac{g_jN_m}{gN_j}, \quad j = 1, \ldots ., m\text{-}1 \tag{4}$$

$$G \equiv \sum_{i=1}^{m} \frac{N_i}{N_m} \left[\frac{2}{3} x_i \frac{(Q_i^4(x_i,\lambda_i) - Q_i^2(x_i,\lambda_i))}{(Q_i^3(x_i,\lambda_i) - Q_i^1(x_i,\lambda_i))} - \lambda_i \right] - \frac{S}{kN_m} \tag{5}$$

Solution of these equations proceeds at a given x by the m+1 dimensional Newton-Raphson formula

$$\alpha^1 = \alpha - J^{-1}(H(\alpha))\ H(\alpha) \tag{6}$$

where $\alpha = \begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m \end{pmatrix}$; $\alpha^1 = \begin{pmatrix} \lambda_1^1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_m^1 \end{pmatrix}$, $H(\alpha) = \begin{pmatrix} F_1(\lambda_m,\lambda_1) \\ \vdots \\ F_2(\lambda_m,\lambda_2) \\ \vdots \\ F_{m-1}(\lambda_m,\lambda_{m-1}) \\ \vdots \\ G(\lambda_i \dots \lambda_m) \end{pmatrix}$

and $J^{-1}(H(\alpha))$ is the inverse of the Jacobian of $H(\alpha)$

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial \lambda_1} & & & \frac{\partial F_1}{\partial \lambda_m} \\ & \ddots & & \vdots \\ & & \frac{\partial F_{m-1}}{\partial \lambda_{m-1}} & \frac{\partial F_{m-1}}{\partial \lambda_{m-1}} \\ \frac{\partial G}{\partial \lambda_1} & \cdots & \frac{\partial G}{\partial \lambda_{m-1}} & \frac{\partial G}{\partial \lambda_m} \end{pmatrix} . \tag{7}$$

Most cross terms are absent since F_j depends only (directly) on (λ_m,λ_j). This simplification allows us to give J^{-1} analytically by solving the equivalent m linear equations

$$J(H(\alpha))\ (\alpha'-\alpha) = -H(\alpha)$$

or

$$\frac{\partial F_j}{\partial \lambda_j}(\lambda_j' - \lambda_j) + \frac{\partial F_j}{\partial \lambda_m}(\lambda_m' - \lambda_m) = -F_j \;, \quad j = 1 \ldots m-1,$$

$$\sum_{i=1}^{m} \frac{\partial G}{\partial \lambda_i}(\lambda_i' - \lambda_i) = -G$$

for the m unknowns $\lambda_i' - \lambda_i$. The solution is

$$\lambda_m' - \lambda_m = \frac{-G + \sum_{k=1}^{m-1} F_k \frac{\partial G}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial F_k}}{\left(\frac{\partial G}{\partial \lambda_m} - \sum_{k=1}^{m-1} \frac{\partial G}{\partial \lambda_k} \frac{\partial F_k}{\partial \lambda_m} \frac{\partial \lambda_k}{\partial F_k}\right)} \tag{8}$$

and

$$\lambda_j' - \lambda_j = \frac{-F_j - \frac{\partial F_j}{\partial \lambda_m}(\lambda_m' - \lambda_m)}{\frac{\partial F_j}{\partial \lambda_j}} \;, \quad j = 1 \ldots m-1. \tag{9}$$

The partial derivatives involved can be established through Eq. (A6)

$$\frac{\partial Q^{n+1}}{\partial \lambda} - \frac{\partial Q^{n-1}}{\partial \lambda} = \frac{2n}{x} Q^n \tag{10}$$

giving

$$\frac{\partial F_j}{\partial \lambda_m} = \frac{4x_m^2}{x_j^{\,3}} \frac{Q_m^2}{(Q_j^3 - Q_j^1)} \;, \quad j = 1 \ldots m-1 \tag{11}$$

$$\frac{\partial F_j}{\partial \lambda_j} = -\frac{4x_m^3}{x_j^{\,4}} \frac{(Q_m^3 - Q_m^1)\, Q_j^2}{(Q_j^3 - Q_j^1)^2} \;, \tag{12}$$

$$\frac{\partial F_k}{\partial \lambda_j} = 0 \quad k \neq j, \quad k = 1 \ldots m-1 \tag{13}$$

and

$$\frac{\partial G}{\partial \lambda_i} = \frac{N_i}{N_m}\left\{\frac{4Q_i^3(Q_i^3-Q_i^1) - \frac{8}{3}\,Q_i^2(Q_i^4-Q_i^2}{(Q_i^3-Q_i^1)^2} - 1\right\}, \quad i = 1 \ldots m. \qquad (14)$$

Convergence to solution is rapid since the distance between one iterate and the root and the next iterate decreases as the square of that distance (quadratic convergence). There is however an extreme sensitivity to initial starting values. Such a problem can be overcome by using the exact differentials $\{d\lambda_i/dx\}$ which are always available through equations like (4:22) and adjusting the solution set $\{\lambda_i = \lambda_i(x)\}$ to give the new extrapolated starting values $\{\lambda_i = \lambda_i(x')\}$ at the new $x = x'$. The chemical potentials can then be calculated proceeding from small to large x.

If one of the m components is of mass zero the Q^n are replaced by the first terms of (A35-38); and in the special case of no constraint on that component's number (λ=0) the number of equations reduces to m-1, a term

$$\frac{64\zeta(4)}{g_m x_m^3(Q_m^3-Q_m^1)} = \frac{4\zeta(4)}{\zeta(3)}\; f$$

is added to the right side of (5) and the derivative of this term with respect to λ_m

$$-\frac{256\zeta(4)\quad Q_m^2}{g_m x_m^4\;(Q_m^3-Q_m^1)^2}$$

is added to the right side of the last of (14) i.e. to $\partial G/\partial\lambda_m$.

For the case of pair production, $\lambda_i = -\bar{\lambda}_i$, the Q^n is replaced by the pair production integrals the R^n and the form of the new equations is identical to those above. Massless fermions are a particular case which can be treated quite simply through the use of cubics in the density equations e.g. (2:52). Yet more complex fluids including chemical reactions can be treated in a similar way to the above.

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